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Bieberbach groups and fibering flat manifolds of diagonal type

by

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Contents

Declaration Of Authorship

I, Ho Yiu Chung, declare that the thesis entitled International Migration Flow Table Estimation and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

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Chapter 1

Introduction

Hilbert's problems are twenty-three problems in mathematics published by German mathematician David Hilbert in 1900. Hilbert's eighteenth problem is related to crystallographic groups. We denote $Isom(\mathbb{R}^n)$ to be the group of all isometries of the n-dimensional Euclidean space $\mathbb{R}^n$. One part of the Hilbert's eighteenth problem is to show that there are only finitely many types of subgroups of $Isom(\mathbb{R}^n)$ with compact fundamental domain. This part of the question is solved by L. Bieberbach in 1910. Those subgroups of $Isom(\mathbb{R}^n)$ with compact fundamental domain are now called crystallographic groups. In the thesis, we will study serval properties of torsion-free crystallographic groups.

In Chapter 2, we first introduce the definition of crystallographic groups. We say Γ is an n-dimensional *crystallographic group* if it is a cocompact and discrete subgroup of $Isom(\mathbb{R}^n) \cong O(n) \ltimes \mathbb{R}^n$. We say Γ is an n-dimensional *Bieberbach group* if Γ is a torsion-free n-dimensional crystallographic group. Next, we will present the first Bieberbach theorem. By the first Bieberbach Theorem (see Section 2.2 for detail), we have the following short exact sequence,

$$0 \to \mathbb{Z}^n \to \Gamma \to G \to 1$$

where $\mathbb{Z}^n$ is a maximal abelian normal subgroup of Γ and G is a finite group. Given such a short exact sequence, it induces a representation $\rho : G \to GL_n(\mathbb{Z})$. The representation ρ is called the *holonomy representation* of Γ and the group G is called the *holonomy group* of Γ. Next, we would give an introduction to group cohomology because it plays a main role in classifying short exact sequences. By using group cohomology, we will state and prove the second and the third Bieberbach theorems. By the three Bieberbach theorems, we will understand the group structure of crystallographic groups. After that, we will discuss the relation between Bieberbach groups and flat manifolds. In the last section of Chapter 2, we will give two ways to classifying all Bieberbach groups namely the Zassenhaus algorithm and the induction method of Calabi. A good reference for the definitions and properties of Bieberbach groups included the three Bieberbach theorems is [31]. A good reference for the introduction to group cohomology is [5].

In Chapter 3, we focus on the below conjecture.

Conjecture 1.0.1 (Dekimpe-Penninckx)**.** Let Γ be an n-dimensional Bieberbach group. Then the minimum number of generators of Γ is less than or equal to n.

The conjecture was solved for some special cases. For example, the conjecture is true if the holonomy group is an odd prime p-group (see [1, Theorem A]), or the holonomy group is an elementary abelian p-group (see [14, Theorem 4.1, Theorem 5.7]). On the other hand, by a computer program namely CARAT, it has been checked that the conjecture is true if the Bieberbach group has dimension less than 7 (see [6]).

There is a connection between the number of generators of Bieberbach group and the number of generators of a finite group that can act freely on an n-torus. By [13, Lemma 6.5.1], if G is a finite group and it acts freely on an n-torus T^n, then the quotient space T^n/G is a manifold with its fundamental group isomorphic to an n-dimensional Bieberbach group. This provides a short exact sequence as below,

$$0 \longrightarrow \pi_1(T^n) \longrightarrow \pi_1(M) \longrightarrow G \longrightarrow 1$$

where $\pi_1(M)$ is an n-dimensional Bieberbach group. Hence if $\pi_1(M)$ can be generated by n elements, then the minimal number of generators of G should not be larger than n. For instance, by [14, Theorem 5.7], we know that $(\mathbb{Z}/2\mathbb{Z})^{n+1}$ cannot act freely on T^n for $n \geq 1$.

We denote $d(G)$ to be the minimal number of generators of the group G. The below three theorems are our main results in Chapter 3.

Theorem A. Let Γ be an n-dimensional crystallographic group with holonomy group isomorphic to $C_m = \langle g \mid g^m = 1 \rangle$ where $m \geq 3$.

(i) If m is divisible by prime larger than 3, then $d(\Gamma) \leq n - 2$.

(ii) If m is not divisible by prime larger than 3 and Γ is torsion-free, then $d(\Gamma) \leq n - 1$.

The idea of the proof of Theorem A(i) is to consider $\Gamma \cap (I_n \times \mathbb{R}^n)$ as a $\mathbb{Z}C_p$-module where p is prime larger than 3. We use the module structure to reduce the number of generators. For Theorem A(ii), we construct a surjective homomorphism from Γ to $\mathbb{Z}$. Then by studying how $\mathbb{Z}$ acts on the kernel of the homomorphism, we can eliminate some redundant generators.

By Theorem A, we get two corollaries. One shows that a general n-dimensional Bieberbach group can be generated by $2n$ elements. The other corollary shows an n-dimensional Bieberbach group with a simple group as holonomy group can be generated by $n - 1$ elements.

Theorem B. Let Γ be an n-dimensional crystallographic group with holonomy group isomorphic to a finite group G.

(i) If the order of G is not divisible by 2 or 3, then $d(\Gamma) \leq n$.

(ii) If the order of G is odd and divisible by 3, then $d(\Gamma) \leq n + 1$.

The idea of the proof of Theorem B is to apply results from [17] to get a relation between the number of generators of the finite group G and its Sylow p-subgroups. Then we apply results from [1] to prove Theorem B.

Theorem C. *Let Γ be an n-dimensional Bieberbach group with 2-generated holonomy group. Then $d(\Gamma) \leq n$.*

The idea of the proof of Theorem C is to consider a Bieberbach subgroup with cyclic holonomy group. Then we apply Theorem A to get the desired bound for generators of the Bieberbach group Γ. Results in Chapter 3 have been published in Geometriae Dedicata (see [8])

In Chapter 4, we will study about an invariant called the diagonal Vasquez invariant. We need to define crystallographic groups of diagonal type before discussing this invariant.

Definition 1.0.2. Let Γ be an n-dimensional crystallographic group and ρ be its holonomy representation. We said Γ is a crystallographic group of *diagonal type* if $im(\rho) \leq D$ where $D = \{A = [a_{ij}] \in GL_n(\mathbb{Z}) \,|\, a_{ij} = 0 \,\text{for}\, i \neq j\}$.

We define Vasquez invariant of diagonal type by modifying the definition of Vasquez invariant of finite groups introduced by A. T. Vasquez in [34]. By [34, Theorem 3.6], we get the below theorem.

Theorem 1.0.3. For any elementary abelian 2-group G, there exists a natural number $x \in \mathbb{N}$ with the property that if Γ is a Bieberbach group of diagonal type where its holonomy group is isomorphic to G, then the lattice subgroup $L \subseteq \Gamma$ contains a normal subgroup N such that Γ/N is a Bieberbach group of dimension at most x.

Definition 1.0.4. Let G be an elementary abelian 2-group and $x \in \mathbb{N}$. We say x has property $\mathcal{S}_d(G)$ if for every Bieberbach group of diagonal type where its holonomy group is isomorphic to G, then its lattice subgroup $L \subseteq \Gamma$ contains a normal subgroup N such that Γ/N is a Bieberbach group of dimension at most x.

Definition 1.0.5. Let G be an elementary abelian 2-group. We define

$$n_d(G) = min\{x \in \mathbb{N} \,|\, x \text{ has property } \mathcal{S}_d(G)\}$$

The natural number $n_d(G)$ is called the *diagonal Vasquez invariant* or *diagonal Vasquez number* of the finite elementary abelian 2-group G.

Our main theorems in this chapter are about the bound or the exact value of diagonal Vasquez invariant of finite groups. In Section 4.2, given a crystallographic group of diagonal type, we will construct a matrix corresponds to it. We calculate the bound or the exact value of diagonal Vasquez invariant of finite groups by using that matrix. In Section 4.4, we will present our two main theorems in this chapter. The below two theorems are our main results about diagonal Vasquez invariant of finite groups.

Theorem D. For $k \geq 2$, the bound of diagonal Vasquez invariant, $n_d(C_2^k)$ is given by

$$5 \cdot 2^{k-3} + 1 \geq n_d(C_2^k) \geq \begin{cases} \frac{k(k+1)}{2} & \text{if } k \geq 2 \text{ is even} \\ \frac{k(k+1)}{2} - 1 & \text{if } k \geq 3 \text{ is odd} \end{cases}$$

Theorem E. For $k \in \{1, 2, 3, 4\}$, the exact value of diagonal Vasquez invariant, $n_d(C_2^k)$ is given by

$$n_d(C_2^k) = \begin{cases} 1 & \text{if } k = 1 \\ 3 & \text{if } k = 2 \\ 5 & \text{if } k = 3 \\ 10 & \text{if } k = 4 \end{cases}$$

Question 1.0.6. We can view the diagonal Vasquez invariant of finite groups as function $f : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ where it maps $k \in \mathbb{N}_{>0}$ to $n_d(C_2^k)$. By Theorem D, we know that the function f is not linear. Is it true that the function f is a quadratic polynomial?

In Chapter 5, we will discuss diffuseness property of Bieberbach groups.

Definition 1.0.7. Let G be a group, $A \subseteq G$ be a subset. We say $a \in A$ is an *extremal point* of A if for all $g \in G/\{1\}$, either $ga \notin A$ or $g^{-1}a \notin A$. Define

$$\Delta(A) = \{a \in A \mid a \text{ is an extremal point}\}$$

We say G is *diffuse* if for any subset $A \subseteq G$ with $2 \leq |A| < \infty$, we have $|\Delta(A)| \geq 2$. We say G is *weakly diffuse* if for any subset $A \subseteq G$ with $1 \leq |A| < \infty$, we have $|\Delta(A)| \geq 1$. We say G is *non-diffuse* if it is not diffuse.

The above definition is introduced by B. Bowditch in [4]. By [22, Proposition 6.2], P. Linnell and D. W. Morris proved that a group is diffuse if and only if it is weakly diffuse. By the above definition, it is clear that if a group has a non-diffuse subgroup, then it is non-diffuse.

Diffuseness of a group is interesting because it related to the Kaplansky's zero divisor conjecture and connectivity. Kaplansky's zero divisor conjecture state that if a group G is torsion-free and R is an integral domain, then the group ring RG has no zero divisor. B. Bowditch discover that the conjecture is true if the group G is diffuse (see [4, Proposition 1.1]). On the other hand, By [12, Theorem 1.2] and [20, Theorem 3.3], we have a Bieberbach group Γ is connective if and only if Γ is diffuse.

Definition 1.0.8. We say Γ is an n-dimensional *generalized Hantzsche-Wendt group* if Γ is an n-dimensional Bieberbach group and its holonomy group is isomorphic to C_2^{n-1}.

Example 1.0.9. The Bieberbach group enumerated in CARAT as "group.32.1.1.194" is a diffuse 4-dimensional generalized Hantzsche-Wendt group. Thus not all generalized Hantzsche-Wendt groups are non-diffuse.

The below two theorems are our main results about diffuseness of Bieberbach groups of diagonal type.

Theorem F. Let Γ be an n-dimensional non-diffuse Bieberbach group of diagonal type with holonomy group isomorphic to C_2^2. Then

$$\Gamma = Z(\Gamma) \oplus (\mathbb{Z}^{n-k-3} \rtimes \Delta_3)$$

where $k = b_1(\Gamma)$, $Z(\Gamma)$ is the center of Γ and Δ_3 is the 3-dimensional non-diffuse Hantzsche-Wendt group (also known as the Promislow group or Passman group).

Theorem G. Let Γ be a non-diffuse Bieberbach group of diagonal type. Then either $\Delta_3 \leq \Gamma$ or there exists $\Gamma' \leq \Gamma$ and exists $\mathbb{Z}^s \trianglelefteq \Gamma$ such that $\Delta_3 \cong \Gamma'/\mathbb{Z}^s$. In additionally, if Γ is a non-diffuse generalized Hantzsche-Wendt group, then $\Delta_3 \leq \Gamma$.

Results from Chapter 4 and Chapter 5 is a preprint (see [9]).

Chapter 2

Bieberbach groups and group cohomology

We have six sections in this chapter. In Section 2.1, we will give an introduction to the group of all isometries of n-dimensional Euclidean space and give the definition of crystallographic groups and Bieberbach groups. In Section 2.2, we will present the first Bieberbach Theorem. By first Bieberbach theorem, crystallographic group is closely related to short exact sequence. Therefore in Section 2.3, we will give an introduction to group cohomology and the relation between group cohomology, group extension and short exact sequence. In Section 2.4, we will give the statement and proof of the second and the third Bieberbach theorems. In Section 2.5, we will discuss the relation between Bieberbach groups and flat manifolds. In Section 2.6, we will present two ways to classify Bieberbach groups.

2.1 Definition of Bieberbach group

Given two arbitrary elements $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{v} = (v_1, ..., v_n)$ in $\mathbb{R}^n$, the n-dimensional Euclidean space. The *norm* of $\mathbf{u}$ is defined as

$$\|\mathbf{u}\| = \sqrt{\sum_{i=1}^{n} u_i^2}$$

It is well known that there exists an unique angle θ between $\mathbf{u}$ and $\mathbf{v}$ where $0 \leq \theta \leq \pi$ such that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|cos\theta \tag{2.1}$$

and the *inner product* in an n-dimensional Euclidean space $\mathbb{R}^n$ is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i = \|\mathbf{u}\|\|\mathbf{v}\|cos\theta$$

In other words, we have

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$$

If we view **u** and **v** as column matrices, then the inner product of elements can consider as matrix multiplication as below

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \mathbf{v}$$

where $\mathbf{u}^T$ is the transpose of the column matrix $\mathbf{u}$. We define the set $\{e_1, ..., e_n\}$ to be the standard orthonormal basis for $\mathbb{R}^n$, where e_k is the k^{th} column of the n-dimensional identity matrix for all $1 \leq k \leq n$. Let M be an element of $GL_n(\mathbb{R})$, we define the (induced or operator) norm of M to be

$$\|M\| = sup\{\|Mx\| \,|\, x \in \mathbb{R}^n, \|x\| = 1\}$$

Definition 2.1.1. An *isometry* of $\mathbb{R}^n$ is an invertible map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that it preserves distance. In other word, we have

$$\|x - y\| = \|f(x) - f(y)\|$$

for any $x, y \in \mathbb{R}^n$.

Remark 2.1.2. It is easy to notice that the set of all isometries of $\mathbb{R}^n$ satisfies all axioms of being a group with respect to composition of maps. We denote that group to be $Isom(\mathbb{R}^n)$.

Definition 2.1.3. Let $a \in \mathbb{R}^n$. A *translation* of $\mathbb{R}^n$ is a map $t_a : \mathbb{R}^n \to \mathbb{R}^n$ defined by the following formula

$$t_a(x) = a + x$$

Remark 2.1.4. Notice that the set of all translation map of $\mathbb{R}^n$ is a normal subgroup of $Isom(\mathbb{R}^n)$ and it is isomorphic to the additive group $\mathbb{R}^n$ by the function $a \mapsto t_a$.

Definition 2.1.5. A linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if for any $x, y \in \mathbb{R}^n$, we have

$$\langle x, y \rangle = \langle A(x), A(y) \rangle$$

We will first present a few propositions and corollaries in order to show that $Isom(\mathbb{R}^n)$ is isomorphic to $O(n) \ltimes \mathbb{R}^n$ where the group $O(n)$ is the *orthogonal group* defined as

$$O(n) = \{M \in GL_n(\mathbb{R}) \,|\, M^T = M^{-1}\}$$

where M^T is the transpose of the matrix M.

Observation 2.1.6. Let $f \in Isom(\mathbb{R}^n)$ which satisfies $f(0) = 0$. Then for any $u \in \mathbb{R}^n$, we have

$$\|f(u)\| = \|f(u) - f(0)\| = \|u - 0\| = \|u\|$$

Proposition 2.1.7. Let $f \in Isom(\mathbb{R}^n)$ with $f(0) = 0$, then

(i) f preserve angles, In other words, the angle between $f(u)$ and $f(v)$ is the same as the angle between u and v for all $u, v \in \mathbb{R}^n$.

(ii) f preserve the inner product. In other words, we have $\langle f(u), f(v)\rangle = \langle u, v\rangle$ for all $u, v \in \mathbb{R}^n$

(iii) f is a linear transformation. In other words, the function f satisfies $f(x + y) = f(x) + f(y)$ and $f(\lambda x) = \lambda f(x)$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Proof. (i): Let u, v be arbitrary elements in $\mathbb{R}^n$. We define θ to be the angle between the two vectors $u, v \in \mathbb{R}^n$ and θ' be the angle between $f(u)$ and $f(v)$ where $0 \leq \theta \leq \pi$ and $0 \leq \theta' \leq \pi$. By (2.1), we have

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|cos\theta$$

and

$$\|f(u) - f(v)\|^2 = \|f(u)\|^2 + \|f(v)\|^2 - 2\|f(u)\|\|f(v)\|cos\theta'$$

Since f is an isometry and it satisfies $f(0) = 0$, by definition of isometry and Observation 2.1.6, it follows that

$$\begin{aligned}\|u - v\|^2 = \|f(u) - f(v)\|^2 &= \|f(u)\|^2 + \|f(v)\|^2 - 2\|f(u)\|\|f(v)\|cos\theta' \\ &= \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|cos\theta'\end{aligned}$$

It forces $cos\,\theta = cos\,\theta'$. Since $0 \leq \theta \leq \pi$ and $0 \leq \theta' \leq \pi$, we conclude that $\theta = \theta'$. Therefore we showed that f preserve angles.

(ii): By part (i), θ is also the angle between $f(u)$ and $f(v)$. Since $f(0) = 0$, by Observation 2.1.6 and definition of inner product, we have

$$\begin{aligned}\langle f(u), f(v)\rangle &= \|f(u)\|\|f(v)\|cos\theta \\ &= \|u\|\|v\|cos\theta \\ &= \langle u, v\rangle\end{aligned}$$

Hence we complete the proof for part (ii).

(ii): Let $\{v_1, ..., v_n\}$ be an orthonormal basis of $\mathbb{R}^n$. In other words, the set $\{v_1, ..., v_n\}$ is a basis of $\mathbb{R}^n$, $\|v_i\| = 1$ for all $1 \leq i \leq n$ and $\langle v_i, v_j\rangle = 0$ for all $i \neq j$. Define $w_i = f(v_i)$. First of all, notice that $\|w_i\| = \|f(v_i)\| = \|v_i\| = 1$. Next, by part (ii), if $i \neq j$, then

$$\langle w_i, w_j\rangle = \langle f(v_i), f(v_j)\rangle = \langle v_i, v_j\rangle = 0$$

Therefore $\{w_1, ..., w_n\}$ is also an orthonormal basis for $\mathbb{R}^n$. Since $\{v_1, ..., v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, for any element $u \in \mathbb{R}^n$, we can express it as

$$u = \sum_{i=1}^{n} \alpha_i v_i$$

where the coefficient $\alpha_i \in \mathbb{R}$ can be determined by the formula $\alpha_i = \langle u, v_i\rangle$. By part (ii), we have

$$\langle u, v_i\rangle = \langle f(u), f(v_i)\rangle = \langle f(u), w_i\rangle \tag{2.2}$$

On the other hand, since $\{w_1, ..., w_n\}$ is also an orthonormal basis, the element $f(u)$ can express as

$$f(u) = \sum_{i=1}^{n} \bar{\alpha}_i w_i \tag{2.3}$$

where the coefficient $\bar{\alpha_i} \in \mathbb{R}$ can be determined by

$$\bar{\alpha_i} = \langle f(u), w_i \rangle \tag{2.4}$$

By combining equation (2.2), (2.3) and (2.4), we have

$$f(u) = \sum_{i=1}^{n} \langle f(u), w_i \rangle w_i = \sum_{i=1}^{n} \langle u, v_i \rangle w_i$$

Let $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have

$$f(x+y) = \sum_{i=1}^{n} \langle x+y, v_i \rangle w_i = \sum_{i=1}^{n} \langle x, v_i \rangle w_i + \sum_{i=1}^{n} \langle y, v_i \rangle w_i = f(x) + f(y)$$

and

$$f(\lambda x) = f(\lambda x) = \sum_{i=1}^{n} \langle \lambda x, v_i \rangle w_i = \sum_{i=1}^{n} \lambda \langle x, v_i \rangle w_i = \lambda f(x)$$

Therefore f is a linear transformation. □

Corollary 2.1.8. Let $f \in Isom(\mathbb{R}^n)$. Then $f : \mathbb{R}^n \to \mathbb{R}^n$ maps $x \in \mathbb{R}^n$ to $g(x) + b$ where g is some linear isometry and $b \in \mathbb{R}^n$.

Proof. Let $b = f(0)$ and set $g(x) = f(x) - b$. We have $g(0) = f(0) - b = 0$. Notice that g is a composition of two isometries and therefore g is an isometry too. Since g is an isometry which satisfies $g(0) = 0$, by Proposition 2.1.7(iii), we can conclude that g is a linear isometry. □

Proposition 2.1.9. There is an one-to-one correspondence between the set of linear isometry and the set of orthogonal matrix $O(n) = \{M \in GL_n(\mathbb{R}) \,|\, M^T = M^{-1}\}$ where M^T is the transpose of the matrix M.

Proof. Let g be an arbitrary linear isometry of $\mathbb{R}^n$. By viewing the element of $\mathbb{R}^n$ as column matrices, we may define g as follow:

$$\begin{aligned} g : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto Ax \end{aligned}$$

where A is an $n \times n$ matrix with entries in $\mathbb{R}$. We want to show that A is indeed an orthogonal matrix. Since g is linear, we have $g(0) = 0$. Consider the (i, j) entry of $A^T A$:

$$\begin{aligned} (A^T A)_{ij} &= (i^{th} \text{ row of } A^T)\,(j^{th} \text{ column of } A) \\ &= \langle (i^{th} \text{ column of } A), (j^{th} \text{ column of } A) \rangle \\ &= \langle g(e_i), g(e_j) \rangle \end{aligned}$$

where $\{e_1, ..., e_n\}$ is the standard orthonormal basis for $\mathbb{R}^n$. By Proposition 2.1.7(ii), we have

$$(A^T A)_{ij} = \langle g(e_i), g(e_j) \rangle = \langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Hence $A^T A$ is an identity matrix. Thus A is an orthogonal matrix.

On the other hand, we need to show that if $A \in O(n)$, then it defines a linear isometry. We claim that the linear map

$$g : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto Ax$$

is an isometry.

For any $u, v \in \mathbb{R}^n$, we have

$$\langle g(u), g(v) \rangle = \langle Au, Av \rangle = (Au)^T (Av) = u^T A^T Av = u^T v = \langle u, v \rangle$$

Therefore by setting $u = v$, we get $\|g(u)\| = \|u\|$. Finally, since g is linear, we have

$$\|g(u) - g(v)\| = \|g(u - v)\| = \|u - v\|$$

Hence g is an isometry. □

By combining Corollary 2.1.8 and Proposition 2.1.9, we get the below result.

Corollary 2.1.10. If $f \in Isom(\mathbb{R}^n)$, then there exists $A \in O(n)$ and $a \in \mathbb{R}^n$ such that $f(x) = a + Ax$.

By Corollary 2.1.10, we can express $Isom(\mathbb{R}^n)$ as a semi-direct product of the group of all translations and the orthogonal group $O(n)$. In other words, we have

$$Isom(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$$

The group operation in $Isom(\mathbb{R}^n)$ is given by

$$(A, a)(B, b) = (AB, a + Ab)$$

and the inverse of an element is given by

$$(A, a)^{-1} = (A^{-1}, -A^{-1}a)$$

for any $(A, a), (B, b) \in Isom(\mathbb{R}^n)$.

Definition 2.1.11. A topological space G that is also a group is called a *topological group* if the group operation

$$f : (x, y) \mapsto xy$$

is continuous in both variables and the inversion mapping

$$g : x \mapsto x^{-1}$$

is also continuous.

Remark 2.1.12. Let $\gamma = (A, a)$ be an arbitrary element in $Isom(\mathbb{R}^n)$. We can express it as an $(n+1)\times(n+1)$ matrix $\begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$, which defines an inclusion $Isom(\mathbb{R}^n) \subset GL_{n+1}(\mathbb{R})$. Thus $Isom(\mathbb{R}^n)$ can be considered as a topological group with topology induced from the Euclidean space $\mathbb{R}^{(n+1)^2}$.

Definition 2.1.13. (i) A subset X of a Euclidean space is *discrete* if for any $x \in X$, there exists an open neighbourhood U_x such that $U_x \cap X = \{x\}$.

(ii) Let Γ be a subgroup of $Isom(\mathbb{R}^n)$. We say Γ is *discrete* if it is a discrete subset of the Euclidean space $\mathbb{R}^{(n+1)^2}$.

(iii) We say Γ acts *properly discontinuously* on $\mathbb{R}^n$ if for any $x \in \mathbb{R}^n$, there exists an open neighbourhood U_x such that the set

$$\{\gamma \in \Gamma \,|\, \gamma U_x \cap U_x \neq \emptyset\}$$

is finite

(iv) We say Γ acts *freely* on $\mathbb{R}^n$ if we have

$$\{\gamma \in \Gamma \,|\, \gamma x = x\} = \{(I_n, 0)\}$$

for any $x \in \mathbb{R}^n$.

Definition 2.1.14. Let Γ be a subgroup of $Isom(\mathbb{R}^n)$. The *orbit space of the action of* Γ *on* $\mathbb{R}^n$ is defined to be the set of Γ-orbits

$$\mathbb{R}^n/\Gamma = \{\Gamma x \,|\, x \in \mathbb{R}^n\}$$

topologized with the quotient topology from $\mathbb{R}^n$.

Now we define the geometric definition of the crystallographic groups.

Definition 2.1.15. Let Γ be a subgroup of $Isom(\mathbb{R}^n)$. We say Γ is a *cocompact* subgroup of $Isom(\mathbb{R}^n)$ if $Isom(\mathbb{R}^n)/\Gamma$ is compact. We say Γ is an n-dimensional *crystallographic group* if it is a discrete cocompact subgroup of $Isom(\mathbb{R}^n)$. Besides, a torsion- free crystallographic group is called a *Bieberbach group*.

For the rest of this section, we will present several lemmas and show that Γ is an n-dimensional crystallographic group if it acts properly discontinuously on $\mathbb{R}^n$ and $\mathbb{R}^n/\Gamma$ is compact. Let M be a metric space, we denote $B_r(x)$ to be an open ball centred at $x \in M$ with radius r. We say M is a *complete metric space* if every Cauchy sequence in M converges in M. In particular, Euclidean space is a complete metric space. Let X be a subset of Euclidean space. We say $\{x_n\}_{n=1}^{\infty}$ is a *convergent sequence in* X *converging in* X if $x_i \in X$ for all $i \in \mathbb{N}$ and the sequence $\{x_n\}_{n=1}^{\infty}$ converges to an element $x \in X$. In other words, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_n - x\| < \epsilon$ for all natural number $n > N$. We say such sequence $\{x_n\}_{n=1}^{\infty}$ is *eventually constant* if there exists $N' \in \mathbb{N}$ such that $x_n = x_{n+1}$ for all $n > N'$.

Lemma 2.1.16. Let X be a subset of Euclidean space. The subset X is discrete if and only if every convergent sequence $\{x_n\}_{n=1}^{\infty}$ in X converging in X is eventually constant.

Proof. First, we suppose X is discrete and let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary convergent sequence in X which converges to $x \in X$. By definition of discreteness, there exists $r > 0$ such that $B_r(x) \cap X = \{x\}$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x, there exists an element $N \in \mathbb{N}$ such that $x_n \in B_r(x)$ for all $n \geq N$. Thus we have $x_n = x$ for all $n \geq N$.

For the reverse direction, we suppose every convergent sequence in X converging in X is eventually constant. Assume by contradiction that X is not discrete. By definition, there exists an element $x \in X$ such that $U_x \cap X \neq \{x\}$ for any open neighbourhood U_x. Therefore we have $\{x\} \subsetneq B_{\frac{1}{n}}(x) \cap X$ for any integer $n > 0$. We choose x_n to be an element in $B_{\frac{1}{n}}(x) \cap X$ which is not equal to x for each integer n. It is clear that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x and it is not eventually constant, which is a contradiction. Therefore X is discrete. □

Lemma 2.1.17. *[31, Lemma 1.2]* Let Γ be a discrete subgroup of $Isom(\mathbb{R}^n)$. Then Γ is closed in $Isom(\mathbb{R}^n)$.

Proof. Suppose by contradiction that Γ is not close. In other words, $Isom(\mathbb{R}^n)/\Gamma$ is not open. Then there exists an element $\gamma \in Isom(\mathbb{R}^n)/\Gamma$ such that all open neighbourhood of γ is not contained in $Isom(\mathbb{R}^n)/\Gamma$. Thus for all $n \in \mathbb{N}$, we can pick γ_n be an element in $B_{\frac{1}{n}}(\gamma) \cap \Gamma$. Consider the sequence $\{\gamma_n\}_{n=1}^{\infty}$. Since the sequence $\{\gamma_n\}_{n=1}^{\infty}$ converges to γ in $Isom(\mathbb{R}^n)$, the sequence $\{\gamma_n\gamma_{n+1}^{-1}\}_{n=1}^{\infty}$ converges to identity in Γ. But the sequence $\{\gamma_n\gamma_{n+1}^{-1}\}_{n=1}^{\infty}$ is not eventually constant. By Lemma 2.1.16, the group Γ is not discrete, which is a contradiction. □

Proposition 2.1.18. *[31, Proposition 1.8]* Let Γ be a discrete subgroup of $Isom(\mathbb{R}^n)$. The group Γ acts freely on $\mathbb{R}^n$ if and only if it is torsion-free.

Proof. Assume Γ is not torsion-free and there exists a non-identity element $\gamma \in \Gamma$ with finite order k. Let x be an arbitrary element in $\mathbb{R}^n$. Define $y = x + \gamma x + \gamma^2 x + \cdots + \gamma^{k-1}x$. It is clear that $\gamma y = y$. Hence Γ cannot act freely on $\mathbb{R}^n$.

For the reverse implication, we assume Γ is torsion-free and we claim that

$$\{\gamma \in \Gamma \,|\, \gamma x = x\} = \Gamma \cap t_x(O(n) \times 0)t_{-x}$$

where $x \in \mathbb{R}^n$. Let $\gamma = (A, a) \in \Gamma$ such that $\gamma x = x$. Therefore we have $a + Ax = x$. It follows that

$$\gamma = (A, a) = (A, x - Ax) = (I_n, x)(A, 0)(I_n, -x)$$

Hence

$$\{\gamma \in \Gamma \,|\, \gamma x = x\} \subseteq \Gamma \cap t_x(O(n) \times 0)t_{-x}$$

On the other hand, let $\gamma \in \Gamma \cap t_x(O(n) \times 0)t_{-x}$. Therefore γ has form $(I_n, x)(A, 0)(I_n, -x)$ for some $A \in O(n)$. We have

$$\gamma x = (I_n, x)(A, 0)(I_n, -x)x = (A, x - Ax)x = x$$

It follows that

$$\Gamma \cap t_x(O(n) \times 0)t_{-x} \subseteq \{\gamma \in \Gamma \,|\, \gamma x = x\}$$

Thus the claim is true. Since the group $O(n)$ is compact and Γ is discrete, we can conclude that the set $\{\gamma \in \Gamma \,|\, \gamma x = x\}$ is finite. Since Γ is torsion-free, it follows that $\{\gamma \in \Gamma \,|\, \gamma x = x\} = \{(I_n, 0)\}$. □

Lemma 2.1.19. *[31, Lemma 1.3]* Let Γ be a discrete subgroup of $Isom(\mathbb{R}^n)$. Then for any $r > 0$, we have

$$\{\gamma \in \Gamma \,|\, \gamma B_r(0) \cap B_r(0) \neq \emptyset\} \subset \Gamma \cap (O(n) \times B_{2r}(0))$$

Proof. Let $\gamma = (A, a) \in \Gamma$ such that $\gamma B_r(0) \cap B_r(0) \neq \emptyset$. There exists $x, x' \in B_r(0)$ such that $\gamma x = a + Ax = x'$. By triangle inequality, we have

$$\|a\| = \|x' - Ax\| \leq \|x'\| + \|Ax\| < 2r$$

Therefore $\gamma \in O(n) \times B_{2r}(0)$. □

Proposition 2.1.20. *[31, Proposition 1.7]* Let Γ be a subgroup of $Isom(\mathbb{R}^n)$. The following conditions are equivalent:

(i) The group Γ acts properly discontinuously on $\mathbb{R}^n$;

(ii) The group Γ is a discrete subgroup of $Isom(\mathbb{R}^n)$;

(iii) For any $x \in \mathbb{R}^n$, the group Γx is a discrete subset of $\mathbb{R}^n$.

Proof. First, we want to prove condition (i) implies condition (ii) and assume Γ acts properly discontinuously on $\mathbb{R}^n$. We first consider a convergent sequence $\{\gamma_n\}_{n=1}^{\infty}$ in Γ which converge to identity. By definition, there exists a neighbourhood U_0 of $0 \in \mathbb{R}^n$ such that the set $\{\gamma \in \Gamma \,|\, \gamma U_0 \cap U_0 \neq \emptyset\}$ is finite. In particular, the set

$$\{\gamma_i \in \{\gamma_n\}_{n=1}^{\infty} \,|\, \gamma_i U_0 \cap U_0 \neq \emptyset\}$$

is finite. Therefore we have $\gamma_i = (I_n, 0)$ for large i and the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is eventually constant. Now, we consider an arbitrary convergent sequence $\{\gamma_n\}_{n=1}^{\infty}$ in Γ which converges to $\gamma \in \Gamma$. Thus the sequence $\{\gamma_n \gamma^{-1}\}_{n=1}^{\infty}$ is a convergent sequence which converges to identity and it is eventually constant. It follows that the convergent sequence $\{\gamma_n\}_{n=1}^{\infty}$ is eventually constant too. By Lemma 2.1.16, we can conclude that Γ is a discrete subgroup of $Isom(\mathbb{R}^n)$.

Next, we want to prove condition (ii) implies condition (i) and assume Γ is a discrete subgroup of $Isom(\mathbb{R}^n)$. Let $x \in \mathbb{R}^n$ be an arbitrary element in $\mathbb{R}^n$. By Lemma 2.1.19, we have

$$\begin{aligned}\{\gamma \in \Gamma \,|\, \gamma B_r(x) \cap B_r(x) \neq \emptyset\} &= \{\gamma \in \Gamma \,|\, \gamma t_x B_r(0) \cap t_x B_r(0) \neq \emptyset\} \\ &= \{\gamma \in \Gamma \,|\, t_{-x}\gamma t_x B_r(0) \cap t_{-x} t_x B_r(0) \neq \emptyset\} \\ &= \{\gamma \in \Gamma \,|\, t_{-x}\gamma t_x B_r(0) \cap B_r(0) \neq \emptyset\} \\ &\subset t_{-x}\Gamma t_x \cap (O(n) \times B_{2r}(0))\end{aligned}$$

Since Γ is discrete and therefore is closed in $Isom(\mathbb{R}^n)$ by Lemma 2.1.17, the above set is finite. Hence Γ acts properly discontinuously on $\mathbb{R}^n$.

Next, we want to prove condition (ii) implies condition (iii) and assume Γ is a discrete subgroup of $Isom(\mathbb{R}^n)$. We want to prove that Γx is a discrete subset of $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$. Assume by contradiction that there exists $x \in \mathbb{R}^n$ such that Γx is not discrete. It follows that there exists a sequence $\{\gamma_i x = A_i x + a_i\}_{i=1}^{\infty}$ which converges to $y \in \mathbb{R}^n$ and it is not eventually constant. Since the group $O(n)$ is compact, the sequence $\{A_i\}_{i=1}^{\infty}$ converges to some $A \in O(n)$. Notice that the value

$$\|a_i + Ax - y\| \leq \|a_i + A_i x - y\| + \|Ax - A_i x\|$$

can be arbitrarily small for large i. Hence we conclude that the sequence $\{\gamma_i\}_{i=1}^{\infty}$ converges to $\gamma = (A, -Ax + y) \in Isom(\mathbb{R}^n)$. Since Γ is discrete subgroup of $Isom(\mathbb{R}^n)$, by Lemma 2.1.16, the sequence $\{\gamma_i\}_{i=1}^{\infty}$ is an eventually constant convergent sequence. It follows that $\{\gamma_i x\}_{i=1}^{\infty}$ is eventually constant, which is a contradiction.

Last but not least, we want to prove condition (iii) implies condition (ii). Let $\{\gamma_i\}_{i=1}^{\infty}$ be a convergent sequence in Γ. For any $x \in \mathbb{R}^n$, the sequence $\{\gamma_i x\}_{i=1}^{\infty}$ is a convergent sequence in Γx. By assumption, Γx is discrete. By Lemma 2.1.16, the sequence $\{\gamma_i x\}_{i=1}^{\infty}$ is eventually constant. It follows that the sequence $\{\gamma_i\}_{i=1}^{\infty}$ is also eventually constant. This finishes the proof. □

Proposition 2.1.21. Let Γ be a subgroup of $Isom(\mathbb{R}^n)$. Then $\mathbb{R}^n/\Gamma$ is compact if and only if $Isom(\mathbb{R}^n)/\Gamma$ is compact.

Proof. By definition, we have $Isom(\mathbb{R}^n)/O(n) = \mathbb{R}^n$. The group action of Γ acting on $Isom(\mathbb{R}^n)/O(n)$ is given by

$$\begin{aligned}\phi : \Gamma \times Isom(\mathbb{R}^n)/O(n) &\to Isom(\mathbb{R}^n)/O(n) \\ (\gamma, gO(n)) &\mapsto (\gamma g)O(n)\end{aligned}$$

where $\gamma \in \Gamma$ and $g \in Isom(\mathbb{R}^n)$. It is easy to notice that the above action ϕ agrees with a standard action of Γ acts on $\mathbb{R}^n$. Next, we consider the map

$$\begin{aligned}\psi : Isom(\mathbb{R}^n)/\Gamma &\to \mathbb{R}^n/\Gamma = (Isom(\mathbb{R}^n)/O(n))/\Gamma \\ g^{-1}\Gamma &\mapsto \Gamma(gO(n))\end{aligned}$$

Notice that it is a continuous open map and the inverse images of points are compact. Hence $Isom(\mathbb{R}^n)/\Gamma$ is compact if and only if $\mathbb{R}^n/\Gamma$ is compact. □

Remark 2.1.22. By Proposition 2.1.20 and Proposition 2.1.21, we say Γ is an n-dimensional crystallographic group if Γ acts properly discontinuously on $\mathbb{R}^n$ and $\mathbb{R}^n/\Gamma$ is compact.

Definition 2.1.23. Let Γ be a subgroup of $Isom(\mathbb{R}^n)$. An open, connected subset $F \subset \mathbb{R}^n$ is a *fundamental domain* if

$$\mathbb{R}^n = \cup_{\gamma\in\Gamma}\gamma\bar{F}$$

and $\gamma F \cap \gamma' F = \emptyset$ for all $\gamma \neq \gamma' \in \Gamma$.

Remark 2.1.24. Since an n-dimensional crystallographic group is a discrete cocompact subgroup of $Isom(\mathbb{R}^n)$, it has a compact fundamental domain.

2.2 The First Bieberbach Theorem

In this section, we will state and prove the first Bieberbach theorem. Most arguments in this section are extracted from [31, Chapter 2].

Theorem 2.2.1 (The first Bieberbach's Theorem)**.** *[31, Theorem 2.1(1)]* Let $\Gamma \leq Isom(\mathbb{R}^n)$ be an n-dimensional crystallographic group. Then the set of translations of Γ which is $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a torsion-free and finitely generated maximal abelian normal subgroup of rank n with finite index.

Before we prove this theorem, we need several lemmas.

Lemma 2.2.2. *[31, Lemma 2.1]* There exists a neighbourhood of the identity $U \subset O(n)$ such that for any $h \in U$, if $g \in O(n)$ commutes with $[g,h] = ghg^{-1}h^{-1}$, then g commutes with h.

Proof. Define

$$U = \{h \in O(n) \mid \|I_n - h^{-1}\| < \sqrt{2} - 1\}$$

Let h be an arbitrary element in U. Let $g \in O(n)$ such that g commutes with $[g,h]$. Let $\lambda_1, \lambda_2, \cdots, \lambda_r$ be the eigenvalues of the map $g : \mathbb{C}^n \to \mathbb{C}^n$ induced by the an orthogonal matrix $g \in O(n)$ and $\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be the eigenspaces decomposition corresponds to g. Notice that V_i is g-invariant for $i = 1, ..., r$. Since g commutes with $[g,h]$ by assumption, we have

$$g(hg^{-1}h^{-1}) = [g,h] = g^{-1}[g,h]g = (hg^{-1}h^{-1})g$$

Thus the element g commutes with $hg^{-1}h^{-1}$. Moreover, for $i = 1, 2, \cdots, r$ and for all $x \in V_i$, we have $gx = \lambda_i x$. Hence

$$g(hg^{-1}h^{-1})x = (hg^{-1}h^{-1})gx = hg^{-1}h^{-1}\lambda_i x = \lambda_i hg^{-1}h^{-1}x$$

Since h and g are isomorphism, we have $h^{-1}V_i = gh^{-1}V_i$. Thus $h^{-1}V_i$ is also a g-invariant subspace. Therefore we have

$$h^{-1}V_i = (h^{-1}V_i \cap V_1) \oplus (h^{-1}V_i \cap V_2) \oplus \cdots \oplus (h^{-1}V_i \cap V_r)$$

where $h^{-1}V_i \cap V_j = \{x \in h^{-1}V_i \,|\, gx = \lambda_j x\}$. Let $0 \neq x \in (h^{-1}V_i \cap V_j)$ where $i \neq j$. Without loss of generality, we assume $\|x\| = 1$. There exists $y \in V_i$ such that $h^{-1}y = x$. We claim that $\langle x, y\rangle = 0$. Assume by contradiction that $\langle x, y\rangle \neq 0$, we have

$$\lambda_i \langle x, y\rangle = \langle gx, y\rangle = \langle x, g^T y\rangle = \lambda_j \langle x, y\rangle$$

It follows that $\lambda_i = \lambda_j$, which is a contradiction. Thus we conclude that $\langle x, y\rangle = 0$ and therefore we have

$$\sqrt{2} = \|x - y\| = \|h^{-1}y - y\| = \|(h^{-1} - I_n)y\| < \sqrt{2} - 1$$

which is a contradiction. Hence $(h^{-1}V_i \cap V_j) = 0$ if $i \neq j$. Thus we have $h^{-1}V_i = V_i$ for $i = 1, ..., r$. For $x \in V_i$, we have

$$hg(x) = h(\lambda_i x) = \lambda_i h(x) = g(hx)$$

It follows that $gh = hg|_{V_i}$. Since any element of $\mathbb{C}^n$ is a sum of elements from V_i, the orthogonal group element g commutes with h. □

Lemma 2.2.3. *[31, Lemma 2.2]* There exists a neighbourhood of the identity $U \subset O(n)$ such that for any $g, h \in U$, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to the identity, where $x_1 = [g, h]$ and $x_i = [g, x_{i-1}]$ for $i > 1$.

Proof. Define

$$U = \left\{ M \in O(n) \,\middle|\, \|I_n - M\| < \frac{1}{4} \right\}$$

be a neighbourhood of the identity. It is well known that $\|AB\| \leq \|A\|\|B\|$ for any $A, B \in GL_n(\mathbb{R})$ and $\|M\| = 1$ if $M \in O(n)$. For any $M, N \in U$, we have

$$\begin{aligned}\|I_n - [M,N]\| = \|I_n - MNM^{-1}N^{-1}\| &= \|(NM - MN)M^{-1}N^{-1}\| \\ &\leq \|NM - MN\| \\ &= \|(I_n - N)(I_n - M) - (I_n - M)(I_n - N)\| \\ &\leq 2\|I_n - M\|\|I_n - N\|\end{aligned}$$

Since $M, N \in U$, we have

$$\|I_n - [M,N]\| \leq 2\|I_n - M\|\|I_n - N\| < \frac{1}{8} \tag{2.5}$$

Therefore $[M, N] \in U$. For any $g, h \in U$, we define $x_1 = [g, h]$ and $x_i = [g, x_{i-1}]$ for $i > 1$. By above calculation, we can conclude that $x_i \in U$ for all $i > 0$. By (2.5), we have

$$\|I_n - x_1\| \leq 2\|I_n - g\|\|I_n - h\|$$

and

$$\|I_n - x_i\| = \|I_n - [g, x_{i-1}]\| \leq 2\|I_n - g\|\|I_n - x_{i-1}\|$$

for $i > 1$. Hence we get

$$\|I_n - x_i\| \leq 2^i \|I_n - g\|^i \|I_n - h\|$$

for $i > 1$. Since $g, h \in U$, we have $\|I_n - x_i\| < 2^i(\frac{1}{4})^i(\frac{1}{4}) = \frac{1}{2^{i+2}}$ for $i > 1$. Therefore we can conclude that the sequence $\{x_n\}_{n=1}^{\infty}$ converges to identity. □

Lemma 2.2.4. *[31, Lemma 2.3]* Let $G \subset O(n)$ be a connected subgroup and let U be a neighbourhood of the identity. Then $\langle G \cap U\rangle = G$ where $\langle G \cap U\rangle$ is the group generated by the set $G \cap U$.

Proof. It is clear that $\langle G \cap U\rangle \subseteq G$. For the opposite inclusion, define the set $S = G/\langle G \cap U\rangle$. We claim that the sets S and $\langle G \cap U\rangle$ are simultaneously open and closed. Let $x \in \langle G \cap U\rangle$ and $B_\epsilon(x) \subset U$ be an open disk centered at x with radius ϵ. For any $y \in B_\epsilon(x) \cap G$, we have

$$\|yx^{-1} - I_n\| = \|yx^{-1} - xx^{-1}\| = \|y - x\| < \epsilon \tag{2.6}$$

Therefore $yx^{-1} \in B_\epsilon(I_n) \subset G \cap U$ for small enough ϵ. Therefore $y = (yx^{-1})x \in \langle G \cap U\rangle$. It follows that the set $\langle G \cap U\rangle$ is an open set. By definition, the set S is therefore closed. If the set S is an open set, then $G/S = \langle G \cap U\rangle$ is closed. Therefore we remain to show that S is an open set. For any $y \in S$, we assume by contradiction that there exists $x \in B_\epsilon(y) \cap G$ such that $x \in \langle G \cap U\rangle$. By (2.6), we have $y \in \langle G \cap U\rangle$, which is contradiction. Thus our claim is true. Since U is a non-empty set and G is a connected set, we can conclude that $S = \emptyset$ and therefore $G = \langle G \cap U\rangle$. □

Lemma 2.2.5. *[31, Lemma 2.4]* There exists an arbitrary small neighbourhood V of $I_n \in O(n)$ such that for any $g \in O(n)$, we have $gVg^{-1} = V$.

Proof. Let ϵ be a positive number and define $V = B_\epsilon(I_n)$. For any $g \in O(n)$ and $h \in V$, we have

$$\|ghg^{-1} - I_n\| = \|g(h - I_n)g^{-1}\| = \|h - I_n\| < \epsilon$$

Hence we have $gVg^{-1} \subseteq V$ and $g^{-1}Vg \subseteq V$. It follows that $V = g(g^{-1}Vg)g^{-1} \subseteq gVg^{-1}$. Therefore we can conclude that $gVg^{-1} = V$ □

Definition 2.2.6. Let U be a neighbourhood of $I_n \in O(n)$. We say U is a *stable neighbourhood* of identity if it satisfies Lemma 2.2.2, Lemma 2.2.3 and Lemma 2.2.5.

Lemma 2.2.7. *[31, Lemma 2.5]* Let Γ be a crystallographic group and $x \in \mathbb{R}^n$. Then the linear space generated by the set $\{\gamma(x)\}$ where $\gamma \in \Gamma$ is equal to $\mathbb{R}^n$

Proof. Assume the lemma is false that there exists an element $x_0 \in \mathbb{R}^n$ such that the linear space generated by the set $\{\gamma(x_0)\}$ lies in W, a proper linear subspace of $\mathbb{R}^n$. By a new choice of origin in $\mathbb{R}^n$, we may assume that $O(n)$ leaves x_0 fixed. Since $(I_n, 0) \in \Gamma$, we have $(I_n, 0)(x_0) = x_0 \in W$. It follows that for any $\gamma = (A, a) \in \Gamma$, we have $(A, a)(x_0) = a + x_0 \in W$. Hence we must have $a \in W$.

Since Γ is a group, we have $A(W) = W$ for any $A \in p_1(\Gamma)$ where $p_1 : O(n) \ltimes \mathbb{R}^n \to O(n)$ be the projection map. Define

$$W^{\perp} = \{x \in \mathbb{R}^n \,|\, \langle x, y\rangle = 0 \ \ \forall y \in W\}$$

which is the orthogonal complement of W. Let $x \in W^{\perp}$. Then for any $\gamma = (A, a)$, we have

$$\langle \gamma(x), \gamma(x)\rangle = \langle a + Ax, a + Ax\rangle = \langle a, a\rangle + \langle x, x\rangle$$

Hence $\|x\| \leq \|\gamma(x)\|$ for any $\gamma \in \Gamma$. Since Γ is a crystallographic group, it has a compact fundamental domain. It follows that there exists $d > 0$ such that for all $x \in \mathbb{R}^n$, there exists $\gamma \in \Gamma$ such that $\|\gamma(x)\| \leq d$. This is a contradiction. Because by the above calculation, for all $d > 0$, there exists $x \in \mathbb{R}^n$ where $\|x\| > d$ and we have $d < \|x\| \leq \|\gamma(x)\|$ for all $\gamma \in \Gamma$. Therefore Γ cannot have a compact fundamental domain. □

Lemma 2.2.8. *[31, Lemma 2.6]* Let Γ be an n-dimensional abelian crystallographic group, then Γ contains only pure translations.

Proof. Let $(B, b) \in \Gamma$ where $B \neq I_n$. Then we can always choose an origin and a coordinate system in $\mathbb{R}^n$ such that

$$B = \begin{pmatrix} I_r & 0 \\ 0 & B' \end{pmatrix}$$

where I_r is an r-dimensional identity matrix, $B' - I_s$ is a nonsingular $s \times s$ matrix, $r + s = n$ and r can be zero. Moreover, we can assume $b = (b', 0, \cdots, 0)$ where $b' \in \mathbb{R}^r$. By Lemma 2.2.7, there exists an element $\left(C, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) \in \Gamma$ where $t_1 \in \mathbb{R}^r$ and $0 \neq t_2 \in \mathbb{R}^s$. By simple calculation, we have

$$(B, b)\left(C, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \left(BC, b + B\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right) = \left(BC, \begin{pmatrix} b' + t_1 \\ B'(t_2) \end{pmatrix}\right)$$

and

$$\left(C, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}\right)(B, b) = \left(CB, \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + Cb\right) = \left(CB, \begin{pmatrix} * \\ t_2 \end{pmatrix}\right)$$

Since Γ is abelian, we have $B'(t_2) = t_2$ which contradicts that $B' - I_s$ is nonsingular matrix. □

Lemma 2.2.9. *[31, Lemma 2.7]* Let Γ be an n-dimensional crystallographic group and let $p_1 : O(n) \ltimes \mathbb{R}^n \to O(n)$ be a projection map. In other words, we have $p_1((A, a)) = A$ where $(A, a) \in Isom(\mathbb{R}^n)$. Then $(\overline{p_1(\Gamma)})_0$ is an abelian group, where $(\overline{p_1(\Gamma)})_0$ denote the identity component of the closure of $p_1(\Gamma)$ in $O(n)$. In other words, $(\overline{p_1(\Gamma)})_0$ is the connected component of the smallest closed set containing $p_1(\Gamma)$ in $O(n)$ which contain the identity element.

Proof. Let $U = B_\epsilon(I_n)$ be a stable neighbourhood of $I_n \in O(n)$. Let $\gamma_1 = (A_1, a_1)$ and $\gamma_2 = (A_2, a_2)$ be elements in $p_1^{-1}(U) \cap \Gamma$, By recurrence, we define for $i \geq 2$,

$$\gamma_{i+1} = [\gamma_1, \gamma_i]$$

and denote $\gamma_i = (A_i, a_i)$ for all $i > 0$. We now have

$$\begin{aligned}
\gamma_{i+1} &= (A_1, a_1)(A_i, a_i)(A_1^{-1}, -A_1^{-1}a_1)(A_i^{-1}, -A_i^{-1}a_i) \\
&= (A_1A_i, a_1 + A_1a_i)(A_1^{-1}A_i^{-1}, -A_1^{-1}a_1 - A_1^{-1}A_i^{-1}a_i) \\
&= (A_1A_iA_1^{-1}A_i^{-1}, a_1 + A_1a_i - A_1A_iA_1^{-1}a_1 - A_1A_iA_1^{-1}A_i^{-1}a_i) \\
&= ([A_1, A_i]\,, (I_n - A_1A_iA_1^{-1})a_1 + A_1(I_n - A_iA_1^{-1}A_i^{-1})a_i) \\
&= ([A_1, A_i]\,, A_1(I_n - A_i)A_1^{-1}a_1 + A_1A_i(I_n - A_1^{-1})A_i^{-1}a_i)
\end{aligned}$$

Hence we know that $A_{i+1} = [A_1, A_i]$ and $\|a_{i+1}\| \leq \|I_n - A_i\|\|a_1\| + \epsilon\|a_i\|$. By Lemma 2.2.3, we have $lim_{i\to\infty}A_i = I_n$. Hence $lim_{i\to\infty}a_i = 0$. Since Γ is discrete, we have $\gamma_i = (I_n, 0)$ for sufficient large i. Since A_1 commutes with $A_i = [A_1, A_{i-1}]$, by Lemma 2.2.2, we can conclude that A_1 commutes with A_{i-1}. Inductively, we can therefore conclude that A_1 commutes with A_2. It follows that any elements of the set $p_1(\Gamma) \cap U$ commute. Thus the closure of $p_1(\Gamma) \cap U$ is abelian. Since $(\overline{p_1(\Gamma)})_0$ is a connected subgroup, by Lemma 2.2.4, we can conclude that $(\overline{p_1(\Gamma)})_0$ is an abelian group. □

Proof of Theorem 2.2.1. Let $p_1 : O(n) \ltimes \mathbb{R}^n \to O(n)$ be the projection map. Assume first that $\Gamma \cap (I_n \times \mathbb{R}^n)$ is trivial. Then p_1 is an isomorphism of Γ into $O(n)$. Since $O(n)$ is compact, so the closure of $p_1(\Gamma)$ has finite number of components. By lemma 2.2.9, $(\overline{p_1(\Gamma)})_0$ is abelian. Hence Γ contains a subgroup Γ_1 of finite index which is abelian. Notice that Γ_1 has finite index in Γ, it is also a crystallographic group. Hence by lemma 2.2.8, Γ_1 contain of pure translations only, which is a contradiction. Thus we conclude that $\Gamma \cap (I_n \times \mathbb{R}^n)$ is non empty.

Let $W \subseteq \mathbb{R}^n$ be the subspace of $\mathbb{R}^n$ spanned by the pure translations of Γ. Let (I_n, w) be an arbitrary element in the group of all pure translations of Γ and (A, a) be an arbitrary element in Γ, we have

$$(A, a)(I_n, w)(A, a)^{-1} = (I_n, Aw)$$

which is also an element of the group of all pure translations. It follows that we have $p_1(\Gamma)W = W$. We claim that $p_1(\Gamma)|_W$ is a finite group. Assume by contradiction that $p_1(\Gamma)|_W$ is infinite. Let $\{(A_i, a_i)\}_{i=1}^{\infty}$ be an infinite sequence of elements in Γ such that $\lim_{i\to\infty} A_i = I_n$. Define

$$(B_i, b_i) = (I_n, e_k)(A_i, a_i)(I_n, -e_k) = (A_i, (I_n - A_i)e_k + a_i)$$

where $e_k \in \Gamma \cap (I_n \times \mathbb{R}^n)$. Then $\{(B_i, b_i)(A_i^{-1}, -A_i^{-1}a_i)\}_{i=1}^{\infty} = \{(I_n, (I_n - A_i)e_k)\}_{i=1}^{\infty}$ defines a non discrete subset of Γ, which is a contradiction. Moreover, we see that Γ induce a cocompact action on $\mathbb{R}^n/W$. We claim Γ acts properly discontinuously on $\mathbb{R}^n/W$ too. We have decomposition $\mathbb{R}^n = W \oplus W^\perp$ where $W^\perp \cong \mathbb{R}^n/W$. Let $pr_1 : \mathbb{R}^n \to W$ and

$pr_2 : \mathbb{R}^n \to W^{\perp}$ be projections. Since $p_1(\Gamma)|_W$ is finite, we can concentrate on elements $\gamma \in \Gamma$ such that $p_1(\gamma)$ acts as identity on W. By proposition 2.1.20, since Γ is discrete subgroup of $Isom(\mathbb{R}^n)$, the set $\Gamma(0)$ is discrete subset of $\mathbb{R}^n$. We are going to prove our claim by contradiction and assume $pr_2(\Gamma(0))$ is not discrete at $W^{\perp}$ and let $y \in W^{\perp}$ be an accumulation point of $pr_2(\Gamma(0))$. Let $\{pr_2(\gamma_i(0))\}_{i=1}^{\infty}$ be a convergent sequence converges to y where $\gamma_i \in \Gamma$. By using elements from $\Gamma \cap (I_n \times \mathbb{R}^n)$, we can define a new sequence of elements $\bar{\gamma_i} \in \Gamma, i \in \mathbb{N}$ such that $\forall i \in \mathbb{N}$, $pr_1(\bar{\gamma_i}(0)) \subset C \subset W$ where C is a compact set. It is easy to see that $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a cocompact subgroup of W. Notice that the set $\{\bar{\gamma_i}(0)\}, i \in \mathbb{N}$ has an accumulation point at a discrete set $\Gamma(0)$, which is a contradiction. Hence we can conclude that Γ acts properly discontinuously on $\mathbb{R}^n/W$. Therefore Γ is a crystallographic group on $\mathbb{R}^n/W$ with no pure translation. Therefore the dimension of $\mathbb{R}^n/W$ is zero.

Let $(A, a) \in \Gamma$ be an arbitrary element in Γ and (I_n, x) be an arbitrary element in the set of translations subgroup $\Gamma \cap (I_n \times \mathbb{R}^n)$. We have

$$(A, a)(I_n, x)(A, a)^{-1} = (I_n, Ax) \in \Gamma \cap (I_n \times \mathbb{R}^n)$$

Therefore $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a normal subgroup of Γ. Let T be a maximal abelian subgroup of Γ and (A, a) be an arbitrary element of Γ. If (A, a) commutes with any translation of Γ, then we can see that $A = I_n$. Thus the set of translation subgroup $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a maximal abelian normal subgroup of Γ. □

Let Γ be an n-dimensional crystallographic group. By the first Bieberbach theorem, Γ fits into the short exact sequence below,

$$0 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1 \tag{2.7}$$

where G is a finite group, $\iota : \mathbb{Z}^n \hookrightarrow \Gamma$ is an inclusion map which maps e_i to (I_n, e_i) where $e_1, ..., e_n$ are the standard basis of $\mathbb{Z}^n$ and $p : \Gamma \to G$ is a projection map which maps $(A, a) \in \Gamma$ to A. Besides, the group $\mathbb{Z}^n$ is a maximal abelian subgroup. Given such a short exact sequence, it induces a representation $\rho : G \to GL_n(\mathbb{Z})$ given by $\rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$, where $x \in \mathbb{Z}^n$ and $\bar{g}$ is chosen arbitrarily such that $p(\bar{g}) = g$. In this case, we call the group G to be the *holonomy group* and the representation ρ to be the *holonomy representation* of Γ.

Lemma 2.2.10. Using the same notations as above. The induced representation $\rho : G \to GL_n(\mathbb{Z})$ is a faithful representation. In other words, the kernel of ρ is trivial.

Proof. Let $g \in ker(\rho)$. We have $\rho(g) = id_{\mathbb{Z}^n}$. It follows that

$$x = \rho(g)x = \bar{g}\iota(x)\bar{g}^{-1}$$

where $x \in \mathbb{Z}^n$ and $\bar{g}$ is chosen arbitrarily such that $p(\bar{g}) = g$. Thus $\bar{g}$ commutes with any translation of Γ. Since $\mathbb{Z}^n$ is the maximal abelian subgroup, we can conclude that $g = I_n$. Therefore ρ is a faithful representation. □

2.3 Group cohomology and group extension

Let G be a group. The *integral group ring* $\mathbb{Z}G$ is defined to be the free $\mathbb{Z}$-module generated by the elements of G. Therefore an element of $\mathbb{Z}G$ can be uniquely expressed as $\sum_{g\in G} a(g)(g)$ where $a(g) \in \mathbb{Z}$ and $a(g) = 0$ for almost all $g \in G$. We say M is a *G-module* if M is an abelian group and there exists a homomorphism $\phi : G \to Aut(M)$ such that the group G acts on M by $g \cdot m = \phi(g)(m)$. Since all abelian group can be view as a module over $\mathbb{Z}$, a G-module M is the same as a $\mathbb{Z}G$-module. Through out this section, we denote R to be a ring with one.

Definition 2.3.1. Let M_i be R-modules and f_i be R-module homomorphism for all $i > 0$. Consider the below sequence,

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \xrightarrow{f_3} M_4 \to \cdots \tag{2.8}$$

(i) We say the sequence (2.8) is *exact at M_n* if and only if $ker(f_n) = im(f_{n-1})$ for $n > 0$.

(ii) We say the sequence (2.8) is *exact* if and only if it exacts at M_n for all $n > 0$.

Definition 2.3.2. Let M_i be R-module and d_i be R-module homomorphism for $i \geq 0$. Then

$$0 \to M_0 \xrightarrow{d_0}\to M_1 \xrightarrow{d_1} \cdots \xrightarrow{d_n} M_{n+1} \to \cdots$$

is a *cochain complex* if the composition of any two successive maps $d_{n+1} \circ d_n$ is zero map.

Definition 2.3.3. A *short exact sequence* is a 5 terms exact sequence where the first and last term are identity. In other words, the below exact sequence is a short exact sequence

$$0 \to A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0 \tag{2.9}$$

where A,B, C are R-modules. We say the above short exact sequence (2.9) *split* if there is an R-module homomorphism $s : C \to B$ such that $\phi \circ s$ is the identity map on C. In this case, we call the map $s : C \to B$ to be a *splitting homomorphism* for the sequence (2.9) and $B \cong A \oplus C$.

Definition 2.3.4. Let M be an R-module. We say M is a *free module* if there exists a subset $A \subset M$ such that for any non-zero element $x \in M$, there exists unique non-zero elements $r_1, ..., r_n \in R$ and unique $a_1, ..., a_n \in A$ for some $n \in \mathbb{N}$ such that

$$x = \sum_{i=1}^{n} r_i a_i$$

In this case, we say A is a basis or set of generators of M.

Definition 2.3.5. Let P be an R-module. We say P is a *projective module* if P has the following property. For any R-module M and N, if we have a surjection map $\phi : M \to N$, then for every R-module homomorphism from P to N lifts to an R-module homomorphism into M. In other words, given $f \in Hom_R(P, N)$, there exists a lift $F \in Hom_R(P, M)$

making the following diagram commute:

$$\begin{array}{ccccc} & & P & & \\ & \overset{F}{\swarrow} & \downarrow f & & \\ M & \xrightarrow{\phi} & N & \longrightarrow & 0 \end{array}$$

Proposition 2.3.6. Let P be an R-module. P is a projective module if and only if P is a direct summand of a free R-module.

Proof. First, we assume P is a projective module. Notice that P is the quotient of a free module. Thus we always have a short exact sequence

$$0 \to ker(\phi) \to \mathcal{F} \xrightarrow{\phi} P \to 0$$

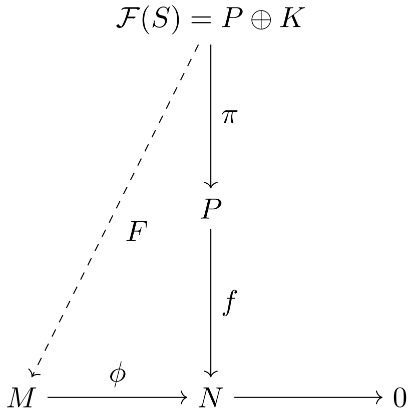
By definition of projective module, the identity map $id : P \to P$ lifts to a homomorphism μ making the following diagram commute,

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & \overset{\mu}{\swarrow} & \downarrow id & & \\ 0 & \longrightarrow & ker(\phi) & \longrightarrow & \mathcal{F} & \xrightarrow{\phi} & P & \longrightarrow & 0 \end{array} \tag{2.10}$$

Since the above diagram commutes, we have $\phi \circ \mu = id$. Thus μ is a splitting homomorphism for the sequence (2.10) and therefore $\mathcal{F} \cong ker(\phi) \oplus P$.

Next, we assume P is a direct summand of a free R-module. Let $\mathcal{F}(S) = P \oplus K$ where $\mathcal{F}(S)$ is a free R-module on some set S and K is R-module. Let M and N are any R-module and $\phi : M \to N$ be surjection. Let $\pi : \mathcal{F}(S) \to P$ be the natural projection and let $f : P \to N$ be any R-module homomorphism. Our aim is to lift the map f to an R-module homomorphism into M. Consider the map $f \circ \pi : \mathcal{F}(S) \to N$. For any $s \in S$, we define $n_s = f \circ \pi(s) \in N$. Since ϕ is surjective, we let $m_s \in M$ be any element of M satisfies $\phi(m_s) = n_s$. By the universal property for free modules (see [15, Section 3, Theorem 6]), there exists a unique R-module homomorphism $F' : \mathcal{F}(S) \to M$ such that

$F'(s) = m_s$. Thus we have the following diagram



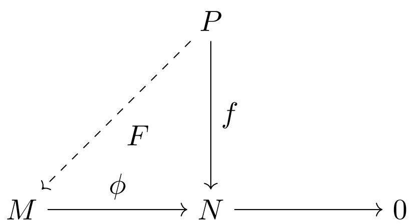
for any $s \in S$, we have

$$\phi \circ F'(s) = \phi(m_s) = n_s = f \circ \pi(s)$$

It follows that $\phi \circ F' = f \circ \pi$. In other words, the above diagram commutes. We define a map $F : P \to M$ where $F(d) = F'((d, 0))$. Since F is a composition of a injection $P \to \mathcal{F}(S)$ and the homomorphism F', the map F is an R-module homomorphism. Then

$$\phi \circ F(d) = \phi \circ F'((d, 0)) = f \circ \pi((d, 0)) = f(d)$$

Thus the below diagram commutes



and we complete the proof. $\square$

By the above proposition, we get the below result.

Corollary 2.3.7. If P is a free module, then P is a projective module.

Definition 2.3.8. Let C^i be R-modules for all $i \geq 0$. Consider the following sequence

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \to \cdots \to C^n \xrightarrow{d^n} C^{n+1} \to \cdots \tag{2.11}$$

where $d^n : C^n \to C^{n+1}$ is homomorphism. We say the sequence (2.11) is a *cochain complex* if composition of any two consecutive maps is the zero map. We define the n^{th} *cohomology group* of that cochain complex to be

$$H^n(\mathcal{C}) = ker(d^n)/im(d^{n-1})$$

where $\mathcal{C}$ is the cochain complex (2.11).

Definition 2.3.9. Let A be an R-module. A *projective resolution* of A is an exact sequence

$$\cdots \to P_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0 \tag{2.12}$$

where P_i are projective R-module for all $i \geq 0$.

Lemma 2.3.10. Let A be an R-module. There always exists a projective resolution of A.

Proof. Choose a free module P_0 with a surjection $d_0 : P_0 \to A$ and define $ker(d_0) = K_0$. Inductively, for $n \geq 1$, we choose a free module P_n with surjection $P_n \to K_{n-1}$ and define K_n to be the kernel of the surjection. We define d_n to be the composition $P_n \to K_{n-1} \to P_{n-1}$. It is clear that $ker(d_n) = ker(P_n \to K_{n-1}) = K_n$. By the above construction, we have a surjection $d_n : P_n \to K_{n-1}$ and $ker(d_n) = im(d_{n+1})$. It follows that the sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to A \to 0$$

is exact and P_i are projective R-module for all $i \geq 0$ by Corollary 2.3.7. □

Given the projective resolution (2.12), we can form a cochain complex by talking homomorphisms of each of the terms into an R-module D. In other words, we apply the functor $Hom_R(-, D)$ to the projective resolution (2.12) and get the below sequence,

$$0 \to Hom_R(A, D) \xrightarrow{d'_0} Hom_R(P_0, D) \xrightarrow{d'_1} Hom_R(P_1, D) \xrightarrow{d'_2} Hom_R(P_2, D) \xrightarrow{d'_3} \cdots \tag{2.13}$$

Let $f \in Hom_R(A, D)$, we define $d'_0(f) = f \circ d_0$. For $n \geq 0$, and let $f \in Hom_R(P_n, D)$, we define $d'_{n+1}(f) = f \circ d_{n+1}$. For any $n \geq 0$ and let $f \in Hom_R(P_{n-1}, D)$ (take $P_{-1} = A$), we have

$$d'_{n+1} \circ d'_n(f) = d'_{n+1}(f \circ d_n) = f \circ d_n \circ d_{n+1}$$

Since the sequence (2.12) is an exact sequence, the composition $d'_{n+1} \circ d'_n$ is zero map for all $n \geq 0$. Therefore the sequence (2.13) is a cochain complex.

We define

$$Ext^n_R(A, D) = ker(d'_{n+1})/im(d'_n)$$

for $n \geq 1$ and $Ext^n_R(A, D) = ker(d'_1)$.

For $n \geq 0$, the n^{th} *cohomology group* of group G with R-module M as coefficient is defined as

$$H^n(G, M) = Ext^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

We define the *standard resolution* of $\mathbb{Z}$ as

$$\cdots \to F_n \xrightarrow{d_n} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0 \tag{2.14}$$

where F_n is defined to be $\mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$ where there are $n+1$ copies of $\mathbb{Z}G$ for $n \geq 0$. Notice that F_n is a G-module where the G-action is given by $g \cdot (g_0 \otimes \cdots \otimes g_n) = (gg_0) \otimes g_1 \otimes \cdots \otimes g_n$. Notice that F_n is a free $\mathbb{Z}G$-module of rank $|G|^n$ and the set $\{1 \otimes g_1 \otimes \cdots \otimes g_n \mid g_i \in G \text{ for } 1 \leq i \leq n\}$ is a set of basis of F_n. We denote the basis element

$1 \otimes g_1 \otimes \cdots \otimes g_n$ to be $(g_1, \cdots, g_n)$. We define the map $d_1(1 \otimes g) = g - 1$ and for $n \geq 2$, we define

$$d_n(g_1, ..., g_n) = g_1 \cdot (g_2, ..., g_n) + \sum_{i_1}^{n-1} (-a)^i (g_1, ..., g_{i-1}, g_i g_{i+1}, g_{n+2}, ..., g_n) + (-1)^n (g_1, ..., g_{n-1})$$

Now, we apply the functor $Hom_{\mathbb{Z}G}(-, M)$ to the sequence (2.14) and obtain the below cochain complex

$$0 \to Hom_{\mathbb{Z}G}(\mathbb{Z}, M) \xrightarrow{\epsilon} Hom_{\mathbb{Z}G}(F_0, M) \xrightarrow{d_1'} Hom_{\mathbb{Z}G}(F_1, M) \xrightarrow{d_2'} Hom_{\mathbb{Z}G}(F_2, M) \to \cdots \quad (2.15)$$

Notice that the elements of $Hom_{\mathbb{Z}G}(F_n, M)$ can be uniquely determined by their values on the $\mathbb{Z}G$ basis elements of F_n. In other words, the group $Hom_{\mathbb{Z}G}(F_n, M)$ can be identify with the set of functions from $G \times \cdots \times G$ (n copies of G) to M and $Hom_{\mathbb{Z}G}(F_0, M) = M$. Now, we can give a definition of cohomology of group G with coefficient M as follow.

Definition 2.3.11. Let G be a finite group and M be a G-module. Define $C^0(G, M) = M$, $C^n(G, M)$ to be the collection of all function from G^n to M for $n \geq 1$ and $C^n(G, M) = 0$ for $n < 0$. We define the *coboundary operator* $\delta^n : C^n(G, M) \to C^{n+1}(G, M)$ as below,

$$\begin{aligned} \delta^n f(g_0, g_1, ..., g_n) = & \, g_0 \cdot f(g_1, ..., g_n) \\ & + \sum_{j=1}^{n} (-1)^j f(g_0, ..., g_{j-2}, g_{j-1} g_j, g_{j+1}, ..., g_n) \\ & + (-1)^{n+1} f(g_0, ..., g_{n-1}) \end{aligned}$$

for $n \geq 1$, $\delta^0 m(g_1) = g_1 \cdot m - m$ and $\delta^n = 0$ for $n < 0$. Then

$$0 \to C^0(G, M) \xrightarrow{\delta^0} C^1(G, M) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^n} H^{n+1}(g, M) \to \cdots$$

is a cochain complex. We define $Z^n(G, M) = ker(\delta^n)$ and the elements of $Z^n(G, M)$ are called *n-cocycles*. We define $B^n(G, M) = im(\delta^{n-1})$ and the elements of $B^n(G, M)$ are called *n-coboundaries*. We define the n^{th} *cohomology group* of G with coefficients in M to be

$$H^n(G, M) = Z^n(G, M)/B^n(G, M)$$

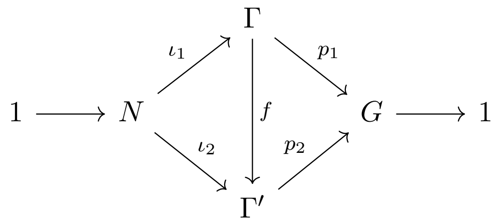
Remark 2.3.12. Using the notation as in (2.15) If G is a finite group and M is a finitely generated G-module. Then $Hom_{\mathbb{Z}G}(F_n, M)$ is a finitely generated abelian group for all $n \geq 0$. Therefore $H^n(G, M)$ is a finitely generated abelian group for all $n \geq 0$.

Now, we discuss the relation between cohomology of groups and extension of groups.

Definition 2.3.13. Let N, G, Γ be groups. We say Γ is an *extension* G *by* N if it fits in the below short exact sequence,

$$1 \to N \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1 \quad (2.16)$$

Let Γ and Γ' are both extension of G by N. We say two extensions are *equivalent via* f if there exists a homomorphism $f : \Gamma \to \Gamma'$ such that the following diagram commute



Lemma 2.3.14. Using the same notations as above, the homomorphism f is indeed an isomorphism.

Proof. Let $\gamma \in ker(f)$. Since $p_2 \circ f(\gamma) = p_2(1) = 1$ and the diagram commutes, we have $p_1(\gamma) = p_2 \circ f(\gamma) = 1$. Thus $\gamma \in ker(p_1)$. By exactness at Γ, there exists $x \in N$ such that $\iota_1(x) = \gamma$. Hence $\iota_2(x) = f \circ \iota_1(x) = f(\gamma) = 1$. Since ι_1 is injective, $x = 1$ and therefore $\gamma = \iota_1(x) = 1$. It follows that f is injective.

Let $\gamma' \in \Gamma'$. Since p_1 is surjective, there exists $\gamma \in \Gamma$ such that $p_1(\gamma) = p_2(\gamma')$. We have $p_2 \circ f(\gamma) = p_1(\gamma) = p_2(\gamma')$ and therefore $p_2(\gamma'(f(\gamma))^{-1}) = 1$. By exactness at Γ', there exists $x \in N$ such that $\iota_2(x) = \gamma'(f(\gamma))^{-1}$. It follows that $f(\iota_1(x)\gamma) = f(\iota_1(x))f(\gamma) = \iota_2(x)f(\gamma) = \gamma'(f(\gamma))^{-1}f(\gamma) = \gamma'$. Thus f is surjective. Therefore f is an isomorphism. □

Lemma 2.3.15. Given the below short exact sequence

$$0 \to N \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1 \tag{2.17}$$

where N is an abelian group. Then it induces an G-action on N. In other words, we can view N as a G-module.

Proof. Since N is an abelian normal subgroup in Γ, G acts on N by conjugation. Explicitly, let $g \in G$, $x \in N$ and pick $\bar{g}$ be an element such that $p(\bar{g}) = g$. We define the action as below,

$$\iota(g \cdot x) = \bar{g}\iota(x)\bar{g}^{-1} \tag{2.18}$$

Let $\bar{g}'$ be another element such that $p(\bar{g}') = g$. Since $\Gamma/N \cong G$, there exists $x_1 \in N$ such that $\bar{g}' = \bar{g}\iota(x_1)$. Since N is an abelian group, we have

$$\bar{g}'\iota(x)\bar{g}'^{-1} = \bar{g}\iota(x_1)\iota(x)\iota(x_1)^{-1}\bar{g}^{-1} = \bar{g}\iota(x)\bar{g}^{-1}$$

Hence the action is independent of choice of $\bar{g}$. Therefore the action given by (2.18) is well defined G-action on N □

Lemma 2.3.16. Equivalent extensions of G by N define the same G-module structure on N.

Proof. Let Γ and Γ' be equivalent extensions and consider the below commutative diagram.

$$\begin{array}{ccccccc} & & & \Gamma & & & \\ & & \overset{\iota_1}{\nearrow} & & \overset{p_1}{\searrow} & & \\ 0 \longrightarrow & N & & \Big\downarrow f & & G & \longrightarrow 1 \\ & & \underset{\iota_2}{\searrow} & & \underset{p_2}{\nearrow} & & \\ & & & \Gamma' & & & \end{array} \tag{2.19}$$

Let $g \in G$ be an arbitrary element of G. let $\bar{g}$ be an element such that $p_1(\bar{g}) = g$. The G-module structure on N induce from Γ is

$$\iota_1(g \cdot x) = \bar{g}\iota_1(x)\bar{g}^{-1}$$

where $x \in N$. Let $\bar{g}' = f(\bar{g})$. Since the diagram (2.19) is a commutative diagram, we have $p_2(\bar{g}') = p_1(\bar{g}) = g$. Thus the G-module structure on N induce from Γ' is

$$\iota_2(g \cdot x) = \bar{g}'\iota_2(x)\bar{g}'^{-1}$$

where $x \in N$. Since ι_1, ι_2 and f are injective. We have

$$f(\bar{g}\iota_1(x)\bar{g}^{-1})) = \bar{g}'\iota_2(x)\bar{g}'^{-1} = \iota_2(g \cdot x)$$

Therefore equivalent extension of G by N defines the same G-action on N. □

Let G be a group and A be a G-module. We would like to studying the relation between $H^2(G, A)$ and group extension of G by A. Roughly speaking, Given a group extension of G by A, we would like to define a class in $H^2(G, A)$. Next, we are going to reverse the procedure. Given a class in $H^2(G, A)$ and we want to construct a group extension of G by A correspond to that given class. Therefore we conclude that there is a bijection between the set of all extension of G by A and group $H^2(G, A)$.

Let G be a group and A be a G-module. We first want to show that the below extension

$$0 \to A \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1 \tag{2.20}$$

defines a 2-cocycle in $Z^2(G, A)$. We want to study the above exact sequence by choosing a set-theoretic cross-section $s : G \to \Gamma$ such that $ps : G \to G$ is an identity map. We call the map s to be a *cross-section* of p. We say the map s is *normalized* or we say s satisfies the *normalization condition* if it satisfies the below condition

$$s(1) = 1 \tag{2.21}$$

In general, s is not necessary a homomorphism. We would like to define a function $f : G \times G \to A$ to measure the failure of s to be a homomorphism. Since for any $g_1, g_2 \in G$, the elements $s(g_1g_2) \in \Gamma$ and $s(g_1)s(g_2) \in \Gamma$ both map to $g_1g_2 \in G$, they differ by an element $\iota(a)$ for some $a \in A$. Therefore we define $f : G \times G \to A$ by the below equation

$$s(g_1)s(g_2) = \iota(f(g_1, g_2))s(g_1g_2) \tag{2.22}$$

In particular, for any $g \in G$, we can view $s(g)$ as a set of coset representative for $\iota(A)$ in Γ. Thus each element of Γ can be written uniquely in the form $\iota(a)s(g)$ for some $a \in A$ and $g \in G$. Besides, we say f is *normalized* if it satisfies the below condition

$$f(g,1) = f(1,g) = 0 \tag{2.23}$$

for all $g \in G$. It is easy to check that if s is normalized, then f is also normalized. We call the function f to be the *factor set* associated to the short exact sequence (2.20) and the section s.

Next, we are going to show that f is an element in $Z^2(G, A)$. Let $\iota(a_1)s(g_1)$ and $\iota(a_2)s(g_2)$ be two arbitrary elements of Γ. By the relation (2.18), we have

$$\iota(g_1 \cdot a_2) = s(g_1)\iota(a_2)s(g_1)^{-1}$$

By the definition of f given by (2.22), we have

$$\begin{aligned} \iota(a_1)s(g_1)\iota(a_2)s(g_2) &= \iota(a_1)\iota(g_1 \cdot a_2)s(g_1)s(g_2) \\ &= \iota(a_1 + g_1 \cdot a_2)\iota(f(g_1, g_2))s(g_1g_2) \\ &= \iota(a_1 + g_1 \cdot a_2 + f(g_1, g_2))s(g_1g_2) \end{aligned}$$

Next, we compute the triple product $[s(g_1)s(g_2)]s(g_3)$ and $s(g_1)[s(g_2)s(g_3)]$, we have

$$[\iota(a_1)s(g_1)\iota(a_2)s(g_2)]\iota(a_3)s(g_3) = \iota(a_1+g_1\cdot a_2+f(g_1, g_2)+(g_1g_2)\cdot a_3+f(g_1g_2, g_3))s(g_1g_2g_3)$$

and

$$\iota(a_1)s(g_1)[\iota(a_2)s(g_2)\iota(a_3)s(g_3)] = \iota(a_1+g_1\cdot a_2+(g_1g_2)\cdot a_3+g_1\cdot f(g_2, g_3)+f(g_1, g_2g_3))s(g_1g_2g_3)$$

Since Γ is group and therefore it satisfies the associative law, f does satisfy the following condition

$$g_1 \cdot f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0$$

Using the same notations in Definition 2.3.11, $f \in ker(\delta^2)$. Thus f is a 2-cocycle. Therefore we can conclude that the factor set f associated to the extension 2.20 and a choice of section s is an element of $Z^2(G, A)$. Let f' be a factor set associated to the extension 2.20 and a different choice of section s'. We are going to show that f and f' differ by a 2-coboundary. For all $g \in G$, the element $s(g)$ and $s'(g)$ lie in the same coset Ag. Thus there exists a function $\phi : G \to A$ such that $s'(g) = \iota(\phi(g))s(g)$ for all $g \in G$. For arbitrary elements $\iota(a_1)s'(g_1)$ and $\iota(a_2)s'(g_2)$ in Γ, we have

$$\begin{aligned} s'(g_1)s'(g_2) &= \iota(f'(g_1, g_2))s'(g_1g_2) \\ &= \iota(f'(g_1, g_2))\iota(\phi(g_1g_2))s(g_1g_2) \\ &= \iota(f'(g_1, g_2) + \phi(g_1g_2))s(g_1g_2) \end{aligned}$$

and

$$\begin{aligned} s'(g_1)s'(g_2) &= \iota(\phi(g_1))s(g_1)\iota(\phi(g_2))s(g_2) \\ &= \iota(\phi(g_1) + g_1 \cdot \phi(g_2) + f(g_1, g_2))s(g_1g_2) \end{aligned}$$

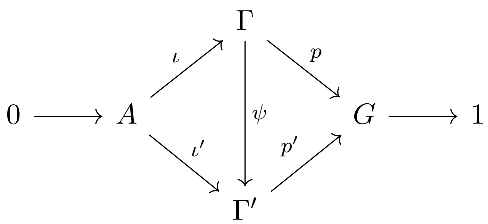
Therefore we have

$$f'(g_1, g_2) = f(g_1, g_2) + \phi(g_1) + g_1 \cdot \phi(g_2) - \phi(g_1g_2) \tag{2.24}$$

It follows that f and f' differ by the 2-coboundary ϕ. Thus we can conclude that the factor sets associated to the extension 2.20 corresponding to a different choices of section give a 2-cocycle in $Z^2(G, A)$ that differ by a coboundary in $B^2(G, A)$. Hence associated to the extension 2.20 is a well defined cohomology class in $H^2(G, A)$ determined by the factor set in 2.22 for any choice of section s.

Remark 2.3.17. In particular, if the extension 2.20 is a split extension, then there is a homomorphism section $s : G \to \Gamma$. Therefore the factor set satisfies $f(g_1, g_2) = 0$ for all $g_1, g_2 \in G$. Hence the trivial cohomology class in $H^2(G, A)$ defined a split extension. In other words, $\Gamma = A \rtimes G$.

Next, we want to prove that equivalent extensions define the same cohomology class in $H^2(G, A)$. Let Γ and Γ' are two equivalent group extensions of G by A. Consider the below commutative diagram



Let s be a section of p, then $s' = \psi \circ s$ is a section of p'. Let $f : G \times G \to A$ be a factor set of the extension correspond to Γ and section s. Recall that f satisfies the condition

$$s(g_1)s(g_2) = \iota(f(g_1, g_2))s(g_1g_2) \tag{2.25}$$

for all $g_1, g_2 \in G$. Applying ψ to (2.25), we have

$$\begin{aligned} s'(g_1)s'(g_2) = \psi(s(g_1))\psi(s(g_2)) &= \psi(\iota(f(g_1, g_2)))\psi(s(g_1g_2)) \\ &= \iota'(f(g_1, g_2))s'(g_1g_2) \end{aligned}$$

for all $g_1, g_2 \in G$. It follows that the factor set for Γ' associated to s' is the same as the factor set for Γ associated to s. Thus equivalent extensions define the same cohomology class in $H^2(G, A)$.

Next, we want to show that given a class in $H^2(G, A)$, we could construct an extension E_f such that its corresponding factor set is in the given class in $H^2(G, A)$. Using the notations in Definition 2.3.11. Let $f \in Z^2(G,A) \subseteq C^2(G,A)$ be a 2-cocycle. Define $f_1 \in C^1(G,A)$ which maps $g \in G$ to $f(1,1)$ for all $g \in G$. We claim that $f - \delta^1(f_1)$ is a normalized 2-cocycle. By definition of δ^1, we have

$$\delta^1(f_1)(g,1) = g \cdot f_1(1) - f_1(g) + f_1(g) = g \cdot f(1,1) \tag{2.26}$$

and

$$\delta^1(f_1)(1,g) = 1 \cdot f_1(g) - f_1(g) + f_1(g) = f(1,1) \tag{2.27}$$

for all $g \in G$. Since f is a 2-cocycle, we have

$$f(g,h) + f(gh,k) = gf(h,k) + f(g,hk) \tag{2.28}$$

for all $g,h,k \in G$. By setting $g = h = 1$ in 2.28, we have

$$f(1,k) - f(1,1) = 0 \tag{2.29}$$

for all $k \in G$. By combining 2.27 and 2.29, we have

$$f(1,g) - \delta^1(f_1)(1,g) = 0 \tag{2.30}$$

On the other hand, by setting $h = k = 1$ in 2.28, we have

$$f(g,1) - g \cdot f(1,1) = 0 \tag{2.31}$$

By combining 2.26 and (2.31), we get

$$f(g,1) - \delta^1(f_1)(g,1) = 0 \tag{2.32}$$

for all $g \in G$. It follows that

$$(f - \delta^1(f_1))(g,1) = (f - \delta^1(f_1))(1,g) = 0$$

Thus, we can conclude that $f - \delta^1(f_1)$ is a normalized 2-cocycle.

Let f be a cohomology class representative in $H^2(G,A)$ where f is a normalized 2-cocycle. Define E_f be the set $A \times G$ with a binary operation on E_f as below

$$(a_1,g_1)(a_2,g_2) = (a_1 + g_1 \cdot a_2 + f(g_1,g_2), g_1g_2)$$

where $(a_1,g_1),(a_2,g_2) \in A \times G$. We claim that E_f is indeed a group. Since f is normalized 2-cocycle, we have

$$(a,g)(0,1) = (a + f(g,1), g) = (a,g)$$

and

$$(0,1)(a,g) = (a + f(1,g), g) = (a,g)$$

Thus (0,1) is a 2-sided identity. Next, we check for associativity. By simple calculation, we have

$$[(a_1,g_1)(a_2,g_2)](a_3,g_3) = (a_1 + g_1 \cdot a_2 + f(g_1,g_2) + (g_1g_2) \cdot a_3 + f(g_1g_2,g_3), g_1g_2g_3)$$

and

$$(a_1, g_1)[(a_2, g_2)(a_3, g_3)] = (a_1 + g_1 \cdot a_2 + (g_1 g_2) \cdot a_3 + g_1 \cdot f(g_2, g_3) + f(g_1, g_2 g_3), g_1 g_2 g_3)$$

Since f satisfies 2.28, we have $[(a_1, g_1)(a_2, g_2)](a_3, g_3) = (a_1, g_1)[(a_2, g_2)(a_3, g_3)]$. Thus the operation satisfies the associativity law. By simple calculation, we get

$$(0, g)[(0, g^{-1})(0, g)] = (g \cdot f(g^{-1}, g) + f(g, 1), g)$$

and

$$[(0, g)(0, g^{-1})](0, g) = (f(g, g^{-1}), g)$$

Since f is normalized and the binary operation on $A \times E$ satisfies associativity law, we have

$$g \cdot f(g^{-1}, g) = f(g, g^{-1})$$

Besides, by simple calculation, we get

$$(a, g)(-g^{-1}a - g^{-1}f(g, g^{-1}), g^{-1}) = (0, 1) = (-g^{-1}a - g^{-1}f(g, g^{-1}), g^{-1})(a, g)$$

Thus for any $(a, g) \in A \times G$, it exists an inverse

$$(a, g)^{-1} = (-g^{-1}a - g^{-1}f(g, g^{-1}), g^{-1})$$

Thus E_f is a group. Define

$$A' = \{(a, 1) \mid a \in A\} \tag{2.33}$$

Since f is a normalized 2-cocycle, A' is a subgroup of E_f, and the map $\iota' : a \mapsto (a, 1)$ is an isomorphism from A to A'. It is clear that A' is a normal subgroup of E_f and the map $p' : (a, g) \mapsto g$ is a surjective homomorphism from E_f to G with kernel A'. Thus we have

$$0 \to A \xrightarrow{\iota'} E_f \xrightarrow{p'} G \to 1 \tag{2.34}$$

By simple calculation, we check that the action of G on A by conjugation in the above extension is the module action specified in determining the 2-cocycle $f \in H^2(G, A)$. The extension (2.34) has a normalized section $s : G \to E_f$ which maps $g \in G$ to $(0, g) \in E_f$ whose corresponding normalized factor set is f. Thus we can conclude that every normalized 2-cocycle arises as the normalized factor set of some extension.

Finally, suppose f' is another normalized 2-cocycle in the same cohomology class in $H^2(G, A)$ as f and let $E_{f'}$ be the corresponding extension Since f and f' are in the same cohomology class, they differ by the coboundary $f_1 : G \to A$. Explicitly, for all $g, h \in G$, we have

$$f(g, h) - f'(g, h) = g \cdot f_1(h) - f_1(gh) + f_1(g)$$

By setting $g = h = 1$, we get $f_1(1) = 0$. Define $\phi : E_f \to E_{f'}$ given by

$$\phi((a, g)) = (a + f_1(g), g)$$

It is clear that ϕ is a bijection. Next, we want to show that ϕ is a homomorphism.

$$\begin{aligned}
\phi((a_1, g_1)(a_2, g_2)) &= \phi((a_1 + g_1 \cdot a_2 + f(g_1, g_2), g_1 g_2)) \\
&= (a_1 + g_1 \cdot a_2 + f(g_1, g_2) + f_1(g_1 g_2), g_1 g_2)) \\
&= (a_1 + f_1(g_1) + g_1 \cdot (a_2 + f_1(g_2)) + f'(g_1, g_2), g_1 g_2) \\
&= (a_1 + f_1(g_1), g_1)(a_2 + f_1(g_2), g_2) \\
&= \phi((a_1, g_1))\phi((a_2, g_2))
\end{aligned}$$

for all $(a_1, g_1), (a_2, g_2) \in E_f$. It follows that ϕ is an isomorphism. Consider the restriction of ϕ to A, we have

$$\phi((a, 1)) = (a + f_1(1), 1) = (a, 1)$$

for all $a \in A$. Therefore $\phi|_A$ is the identity map on A. Similarly, ϕ is the identity map on the second component of (a, g), so ϕ induces the identity map on the quotient of G. It follows that ϕ defines an equivalent between the extensions E_f and $E_{f'}$. This shows that the equivalence class of the extension E_f depends only on the cohomology class of $f \in H^2(G, A)$.

We summarize all the above discussion in the following theorem.

Theorem 2.3.18. Let G be a group and A be a G-module. Let $\mathcal{E}(G, A)$ be the set of equivalence classes of extensions of G by A giving rise to the given action of G on A. Then there is a bijection between the set $\mathcal{E}(G, A)$ and the group $H^2(G, A)$.

Remark 2.3.19. Let G be group and A be a G-module. The trivial class $[0] \in H^2(G, A)$ is correspond to a the below split extension

$$0 \to A \to A \rtimes G \to G \to 1$$

We will present some properties of group cohomology in the rest of this section.

Proposition 2.3.20. *[21, Proposition 5.3, page 117]* Let G be a group and A be a G-module. If $|G| = k$, then every element of $H^n(G, A)$ has order divisible by k for $n > 0$.

Proof. Let $f \in C^n(G, A)$ be an arbitrary n-cochain. Define

$$g(x_1, ..., x_{n-1}) = \sum_{x \in G} f(x_1, ..., x_{n-1}, x)$$

By definition of δ^n for $n > 0$, we have

$$\begin{aligned}
\delta^{n-1} g(x_1, ..., x_{n-1}, x_n) =& x_1 g(x_2, ..., x_{n-1}, x_n) \\
&+ \sum_{j=2}^{n} (-1)^{j+1} g(x_1, ..., x_{j-2}, x_{j-1} x_j, x_{j+1}, ..., x_{n-1}, x_n) \\
&+ (-1)^n g(x_1, ..., x_{n-1})
\end{aligned}$$

and

$$\begin{aligned}\sum_{x\in G}[\delta^n f(x_1,...,x_n,x)] =& \sum_{x\in G}[x_1 f(x_2,...,x_n,x)] \\ &+ \sum_{x\in G}\sum_{j=2}^{n}(-1)^{j+1} f(x_1,...,x_{j-1}x_j,x_{j+1},...,x_n,x) \\ &+ \sum_{x\in G}(-1)^n f(x_1,...,x_{n-1},x_n x) + \sum_{x\in G}(-1)^{n+1} f(x_1,...,x_n)] \\ =& x_1 g(x_2,...,x_n) \\ &+ \sum_{j=2}^{n}(-1)^{j+1} g(x_1,...,x_{j-2},x_{j-1}x_j,x_{j+1},...,x_{n-1},x_n) \\ &+ (-1)^n g(x_1,...,x_{n-1}) + |G|(-1)^{n+1} f(x_1,...,x_n)\end{aligned}$$

It follows that

$$\sum_{x\in G}\delta^n f(x_1,...,x_n,x) = \delta^{n-1} g(x_1,...,x_{n-1},x_n) + |G|(-1)^{n+1} f(x_1,...,x_n)$$

If $f \in Z^n(G,A)$, then we have $\delta^n f = 0$. Hence we have

$$\delta^{n-1} g(x_1,...,x_{n-1},x_n) = \pm|G| f(x_1,...,x_n)$$

Thus the order of f is divisible by k. □

By combining Remark 2.3.12 and the above proposition, we get the below corollary.

Corollary 2.3.21. Let G be a group and $\mathbb{Z}^n$ be a G-module for any $n \geq 1$. If G is finite, then so is $H^2(G,\mathbb{Z}^n)$.

Proposition 2.3.22. *[31, Proposition 2.1]* let G be a group and M be a G-module. If $|G| = m$ is invertible in M, then $H^n(G,M) = 0$ for all $n > 0$.

Proof. Using same notations as in Definition 2.3.11, Let $\phi : C^q(G,M) \to C^q(G,M)$ be a homomorphism that sends $f \in C^q(G,M)$ to $m \cdot f$. It suffices to show that the induced homomorphism $\phi^q : H^q(G,M) \to H^q(G,M)$ is the trivial homomorphism for $q \geq 1$. In other words, we want to show $\phi^q(H^q(G,M)) = 0$. By Proposition 2.3.20, we know that all q-cocycle has order divisible by k. Hence $\phi^q(H^q(G,M)) = 0$. □

2.4 The Second and Third Bieberbach theorems

In this section, we will state and prove the second and the third Bieberbach theorems. Most arguments in this section are extract from [31, Chapter 2]. Before proving the second and third Bieberbach theorems, we want to present a theorem from Zassenhaus and give a algebraic definition of crystallographic group.

Theorem 2.4.1. *[31, Theorem 2.2]* A group Γ is isomorphic to an n-dimensional crystallographic group if and only if Γ has a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of Γ.

Proof. First, we assume Γ is an n-dimensional crystallographic group. By the first Bieberbach theorem, the group of translation $\Gamma \cap (I_n \times \mathbb{R}^n)$ is a normal, free abelian subgroup of finite index which is a maximal abelian subgroup of Γ.

For the reverse direction, Let Γ has a normal, free abelian subgroup $\mathbb{Z}^n$ of finite index which is a maximal abelian subgroup of Γ. In other words, we have

$$0 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1$$

where G is a finite group and $\mathbb{Z}^n$ is a maximal abelian subgroup of Γ. Given such a short exact sequence, it induces a representation $h_\Gamma : G \to GL_n(\mathbb{Z})$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, by Lemma 2.2.10, the representation h_Γ is a faithful representation. We can view the free abelian group $\mathbb{Z}^n$ as a subgroup of $\mathbb{R}^n$. Thus we have an inclusion map $\iota' : \mathbb{Z}^n \to \mathbb{R}^n$. Consider the diagram below,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^n & \xrightarrow{i} & \Gamma & \xrightarrow{p} & G & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i'} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \Gamma' & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle h_\Gamma} & & \\
0 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \rtimes GL_n(\mathbb{R}) & \longrightarrow & GL_n(\mathbb{R}) & \longrightarrow & 0
\end{array}$$

The group Γ' is defined to be the pustout of the monomorphisms $i : \mathbb{Z}^n \to \Gamma$ and $i' : \mathbb{Z}^n \to \mathbb{R}^n$. Notice that all vertical arrows are monomorphism. By Proposition 2.3.22, we have $H^2(G, \mathbb{R}^n) = 0$. By Remark 2.3.19, Γ' is isomorphic to $G \ltimes \mathbb{R}^n$ where the group action of G on $\mathbb{R}^n$ is given by h_Γ. By [11, page 256], any finite subgroup of $GL_n(\mathbb{R})$ is conjugate to a finite subgroup of $O(n)$. Therefore we can conclude that Γ is an n-dimensional crystallographic group. □

Theorem 2.4.2. *[31, Remark 3]* Let Γ be a torsion-free group. The group Γ is isomorphic to a Bieberbach group if and only if there exists $A \leq \Gamma$ such that A is an abelian finitely generated subgroup with finite index.

Proof. Let

$$\Delta(\Gamma) = \{x \in \Gamma \,|\, |\Gamma : C_\Gamma(x)|\} < \infty$$

where $C_\Gamma(x) = \{\gamma \in \Gamma \,|\, \gamma x = x\gamma\}$. We claim that $\Delta(\Gamma) \leq \Gamma$ is a maximal, normal, free abelian subgroup $\mathbb{Z}^n$ of finite index. If so, By Theorem 2.4.1, we can concluded that Γ

is an n-dimensional Bieberbach group. Since $A \subset \Delta(\Gamma)$, we can see that $\Delta(\Gamma)$ is finitely generated. Let $\{x_1, ..., x_n\}$ be a set of generators of $\Delta(\Gamma)$. By [26, Lemma 1.5], we have

$$C_\Gamma(\Delta(\Gamma)) = \bigcap_{i=1}^{n} C_\Gamma(x_i)$$

and $|\Gamma : C_\Gamma(\Delta(\Gamma))| < \infty$. Since $Z(\Delta(\Gamma)) = \Delta(\Gamma) \cap C_\Gamma(\Delta(\Gamma))$ where $Z(\Delta(\Gamma)) = \{\gamma \in \Delta(\Gamma) \,|\, \gamma'\gamma = \gamma\gamma' \text{ for all} \gamma' \in \Delta(\Gamma)\}$, we see that $\Delta(\Gamma)$ has a central subgroup $Z(\Delta(\Gamma))$ of finite index. Let $y_1, ..., y_k$ be coset representatives for $Z(\Delta(\Gamma))$ in $\Delta(\Gamma)$ and set $c_{ij} = y_i y_j y_i^{-1} y_j^{-1}$. It is easy to see that they are all elements $[\Delta(\Gamma), \Delta(\Gamma)]$. Hence we have $[\Delta(\Gamma), \Delta(\Gamma)]$ is finite. Let $\bar{A}$ be a maximal abelian subgroup of Γ which contains A. It is clear that $\bar{A} \subset \Delta(\Gamma)$. Since $\Delta(\Gamma)$ is an abelian group, we have $\bar{A} = \Delta(\Gamma) = \mathbb{Z}^n$ for some $n \in \mathbb{N}$. Therefore we have complete the proof. □

By Theorem 2.4.1, we can introduce the algebraic definition for crystallographic groups.

Definition 2.4.3. A group Γ is an *n-dimensional crystallographic group* if it can express as the following short exact sequence

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

where G is a finite group and $\mathbb{Z}^n$ is a maximal abelian subgroup of Γ.

Theorem 2.4.4 (The second Bieberbach's theorem)**.** *[31, Theorem 2.1(2)]* For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.

Before present the proof of the second Bieberbach's theorem, we need a few propositions and theorem to prove that the number of conjugacy classes of finite subgroups of $GL_n(\mathbb{Z})$ is finite.

Proposition 2.4.5. *[31, Proposition 2.3]* For any natural number $n > 0$, the number of isomorphism classes of finite subgroup of $GL_n(\mathbb{Z})$ is finite.

Proof. Let p be any odd prime. Consider the natural homomorphism

$$\phi : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/p\mathbb{Z})$$

We claim the kernel of the above homomorphism is torsion-free. Assume by contradiction that there exists an element $A \neq I_n \in ker\phi$ such that $A^q = I_n$ where q is a prime. Since $A \in ker(\phi)$, we have $A = I_n + pB$ where B is an $n \times n$ matrix. It follows that

$$I_n = A^q = (I_n + pB)^q = I_n + pqB + \binom{q}{2} p^2 B^2 + \cdots + p^q B^q$$

After rearranging we have

$$\binom{q}{2} pB^2 + \binom{q}{3} p^2 B^3 + \cdots + p^{q-1} B^q = -qB$$

where 0 denotes the $n \times n$ zero matrix. Let α be the maximal number such that p^α divides all entries of B. Thus any entry of pB^2 is divisible by $p(p^\alpha)^2 = p^{2\alpha+1}$. It follows that all entries of qB are divisible by $p^{2\alpha+1}$. If $p \neq q$, then from the maximality of α, we get $2\alpha + 1 \leq \alpha$, which is impossible. Thus $p = q$. Hence $2\alpha \leq \alpha$ and therefore $\alpha = 0$. Since $p = q$, we have

$$B + \binom{p}{2} B^2 + \binom{p}{3} p^2 B^3 + \cdots + p^{p-2} B^q = 0 \tag{2.35}$$

Notice that $\binom{p}{2} = \frac{p(p-1)}{2}$ and p is an odd prime, we concluded that $\binom{p}{2}$ is divisible by p. Thus p divides all entries of B. Therefore we have $\alpha \geq 1$ which is a contradiction. This proves our claim is true.

For any finite subgroup $G \leq GL_n(\mathbb{Z})$, we have $G \cap ker(\phi)$ is trivial because $ker(\phi)$ is torsion-free. It follows that G is isomorphic to some finite subgroup of $GL_n(\mathbb{Z}/p\mathbb{Z})$. Hence the number of isomorphism classes of finite subgroups of $GL_n(\mathbb{Z})$ is finite. □

Proposition 2.4.6. For any $n > 0$, there exists a positive integer $\upsilon(n)$ such that for any finite subgroup $F \leq O(n)$ has an abelian normal subgroup $A(F)$ such that $|F : A(F)| < \upsilon(n)$.

Proof. Fix $n > 0$ and F be a finite subgroup of $O(n)$. Let $U_\epsilon(I_n)$ be a stable neighbourhood of $I_n \in O(n)$. Define $U' = B_{\frac{\epsilon}{2}}(I_n)$. Let μ denote the Haar measure on $O(n)$ such that $\mu(O(n)) = 1$. We choose $\upsilon(n)$ to be a positive integer such that $\upsilon(n) > \frac{1}{\mu(U)'}$. Define

$$A(F) = \langle F \cap U \rangle$$

By Lemma 2.2.5, $A(F)$ is a normal subgroup. By Lemma 2.2.9, $A(F)$ is an abelian subgroup. We remain to show $A(F)$ has index less that $\upsilon(n)$ in F. Since F is finite group. Let $\{f_1, ..., f_m\}$ be a set of coset representatives of the elements of $F/A(F)$. In other words, we have

$$F/A(F) = \{[f_1], [f_2], \cdots, [f_m]\}$$

where $f_1, ..., f_m \in F$. By definition, if $[f_i] \neq [f_j]$, then $f_iU' \cap f_jU' = \emptyset$. Hence we have

$$m\mu(U') = \sum_{i=1}^{m} \mu(f_iU') \leq \mu(O(n)) = 1$$

It follows that $|F/A(F)| = m \leq \frac{1}{\mu(U')} < \upsilon(n)$. □

Theorem 2.4.7. Let G_l, $l = 1, ..., k$, be the set of finite subgroups of $O(n)$ which can be expressed as integer matrices with determinant ± 1 in $GL_n(\mathbb{R})$. Then k is finite.

Proof. Let A_l be the normal abelian subgroup of G_l described in the above proposition. Since the order of G_l/A_l is bounded, there exists only a finite number of distinct groups of the form G_l/A_l,$l = 1, 2, ..., k$. If we can show there exists only a finite number of A_l, we will have proven our assertion as then the group extensions must also be finite. As a finite abelian subgroup of $O(n)$, A_l is diagnozable over the complex numbers. Hence it has a generating set consisting of at most n elements. Thus we must show that there are only a finite number of possibilities for the order of an element $g \in O(n)$ which is conjugate in $GL_n(\mathbb{R})$ to an integer matrix. For this we observe that the coefficients of the characteristic polynomial of g are integers which are elementary symmetric functions in the eigenvalues $e^{2\pi i\lambda}$ of g. This completes the proof. □

By the above theorem, we have the below result.

Theorem 2.4.8. *[11, Theorem 79.1, Jordan-Zassenhaus Theorem]* For any $n \geq 1$, the number of conjugacy classes of finite subgroups of $GL_n(\mathbb{Z})$ is finite.

Proof of Theorem 2.4.4. For any finite group G, by Theorem 2.4.8, there are finite many non isomorphic G-module $\mathbb{Z}^n$. Notice that for any given G-module $\mathbb{Z}^n$, the number of short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to G \to 1$$

is bounded by the order of $H^2(G, \mathbb{Z}^n)$, which is finite by Corollary 2.3.21. Thus by Theorem 2.4.1, we can conclude that there are finite many number of isomorphism classes of n-dimensional crystallographic groups. □

Theorem 2.4.9 (The third Bieberbach's theorem)**.** *[31, Theorem 2.1(3)]* Two n-dimensional crystallographic groups are isomorphic if and only if they are conjugate in the group $A(n)$, where $A(n) = \mathbb{R}^n \rtimes GL_n(\mathbb{R})$. In other words, Let Γ_1 and Γ_2 be n-dimensional crystallographic groups. They are isomorphic if and only if, then there exists an element $\alpha \in A(n)$ such that $\Gamma_1 = \alpha\Gamma_2\alpha^{-1}$.

Proof. Let $h : \Gamma_1 \to \Gamma_2$ be an isomorphism of n-dimensional crystallographic group. The restriction $h|_{\Gamma_1 \cap (I_n \times \mathbb{R}^n)}$ to the subgroup of translation defines a linear map $x \mapsto Mx$ where $M \in GL_n(\mathbb{R})$. Let $(A, a) \in \Gamma_1$ and $h(A, a) = (B, b) \in \Gamma_2$. For any $i = 1, ..., n$, we have

$$h((A, a)(I_n, e_i)(A, a)^{-1}) = (B, b)(I_n, Me_i)(B, b)^{-1} = (I_n, BMe_i)$$

and

$$h((A, a)(I_n, e_i)(A, a)^{-1}) = h(I_n, Ae_i) = (I_n, MAe_i)$$

Hence we have $MAe_i = BMe_i$ for $i = 1, ..., n$. Therefore $B = MAM^{-1}$. We can conjugate h by some suitable matrix from $GL_n(\mathbb{R})$ such that the matrix M will be the identity. In other words, we define $h' : \Gamma_1 \to \Gamma_2$ as $h'(\Gamma) = (M, 0)^{-1}h(\gamma)(M, 0)$. Let $h'(A, a) = (A, a_A) \in \Gamma_2$. We claim that there exists $x_0 \in \mathbb{R}^n$ such that

$$h'(\gamma) = (I_n, x_0)\gamma(I_n, x_0)^{-1}$$

Define $\bar{h} : \Gamma_1 \to A(n)$ which maps (A, a) to $(A, a - a_A)$. We claim that $\bar{h}$ is homomorphism. Let $(A, a), (B, b) \in \Gamma_1$, we have

$$h'(AB, a+Ab) = h'((A, a)(B, b)) = h'((A, a))h'((B, b)) = (A, a_A)(B, b_B) = (AB, a_A + Ab_B)$$

Thus

$$\begin{aligned}\bar{h}((A,a)(B,b)) &= \bar{h}((AB, a+Ab))\\ &= (AB, a+Ab-a_A-Ab_B)\\ &= (A, a-a_A)(B, b-b_B)\\ &= \bar{h}((A,a))\bar{h}((B,b))\end{aligned}$$

Hence $\bar{h}$ is a homomorphism. It is clear that $ker(\bar{h}) = \Gamma_1 \cap \mathbb{R}^n$. Notice that $\bar{h}(\Gamma) \cong \Gamma_1/ker\bar{h} \cong \Gamma_1/\Gamma_1 \cap (I_n \times \mathbb{R}^n)$ which is a finite group by the first Bieberbach theorem. By Proposition 2.1.18, there is a fixed point $x_0 \in \mathbb{R}^n$ of the action of the finite group $\bar{h}(\Gamma_1)$. Thus we have

$$x_0 = (A, a-a_A)(x_0) = Ax_0 + a - a_A$$

Hence, $a = x_0 - Ax_0 + a_A$. Finally, for any $x \in \mathbb{R}^n$, we get

$$\begin{aligned}(I_n, x_0)(A, a_A)(I_n, -x_0)x &= (I_n, x_0)(A, a_A - Ax_0)x\\ &= (A, x_0 + a_A - Ax_0)x\\ &= (A, a)x\end{aligned}$$

Thus we have $h'(\gamma) = (I_n, x_0)^{-1}\gamma(I_n, x_0)$. Hence

$$\begin{aligned}h(\gamma) &= (M,0)h'(\gamma)(M,0)^{-1}\\ &= (M,0)(I_n,x_0)^{-1}\gamma(I_n,x_0)(M,0)^{-1}\\ &= (M,-Mx_0)\gamma(M^{-1},x_0)\\ &= (M,-Mx)\gamma(M,-Mx)^{-1}\end{aligned}$$

Therefore we completed our proof. □

2.5 Flat manifolds and Bieberbach groups

In this section, we will discuss the relation between Bieberbach groups and flat manifolds.

Definition 2.5.1. A *differential n-dimensional manifold* is a separable Hausdorff topological space M together with a family $\{(U_\alpha, u_\alpha)\}_{a\in A}$ such that it satisfies the following properties.

(*i*) $\{U_\alpha\}_{\alpha\in A}$ is a covering of M by open sets;

(*ii*) u_α is a homeomorphism of U_α onto an open subset of n-dimensional Euclidean space;

(*iii*) if $\alpha, \beta \in A$, then the composition

$$u_\beta \circ u_\alpha^{-1} : u_\alpha(U_\alpha \cap U_\beta) \to u_\beta(U\alpha \cap U_\beta)$$

is an infinitely differentiable (i.e. *smooth*) map; and

(*iv*) $\{U_\alpha, u_\alpha)\}_{a\in A}$ is maximal for the first three properties.

Using the same notations as above, a function $f : U \to \mathbb{R}$ where U is an open set in M is said to be smooth if $(f|_{U\cap U_\alpha}) \circ u_\alpha^{-1}$ is smooth for all $\alpha \in A$. We denote the vector space of smooth functions on U to be $C^\infty(U)$.

If $x \in U_\alpha$, then $u_\alpha(x) = (u_\alpha^1(x), ..., u_\alpha^n(x)) \in \mathbb{R}^n$. The $u_\alpha^i(x)$ are called the *local coordinates of x with respect to U_α*. The pair (U_α, u_α) is called a *local coordinate system.*

Let M be an n-dimensional differentiable manifold. Let $\sigma : I \to M$ be a smooth curve where I is an open interval. If $t \in I$ and f is a real-valued differentiable function on a neighbourhood of $\sigma(t)$, then we define

$$[\sigma'(t)](f) = \lim_{h\to 0} \frac{1}{h}(f(\sigma(t+h)) - f(\sigma(t)))$$

If (U, u) is a local coordinate system with $\sigma(t) \in U$, then we can view f as a function $f(z) = f(u^1(z), ..., u^n(z))$ on a subset of $u(U)$, we denote $u^i(\sigma(t))$ by $u^i(t)$ for $i = 1, ..., n$. By chain rule for derivatives, we have

$$[\sigma'(t)](f) = \sum_{i=1}^{n} \frac{du^i}{dt} \cdot \frac{\partial f}{\partial u^i}\bigg|_{u^i=u^i(t)} \tag{2.36}$$

We called $\sigma'(t)$ to be the *tangent vector to σ at $\sigma(t)$*.

Fix a point $x \in M$ and consider all smooth curves $\sigma : I \to M$ such that $0 \in I$ and $\sigma(0) = x$. Given two such curves σ and τ, we write $\sigma \sim \tau$ if $\sigma'(0) = \tau'(0)$. By 2.36, we have $\sigma \sim \tau$ if and only if

$$\frac{du^i(\sigma(t))}{dt}\bigg|_{t=0} = \frac{du^i(\tau(t))}{dt}\bigg|_{t=0}$$

for all i, where (U, u) is a local coordinate system with $x \in U$. We can see that $\sim$ is an equivalence relation. A *tangent vector to M at x* is an equivalence class of curves. We often identity the tangent vector at x represented by a curve σ, with the operation $\sigma'(0)$ on functions differentiable in a neighbourhood of x. We can observe that the set of all tangent vectors to M at x is the n-dimensional real vector space. We denoted that space to be $T_x(M)$ and is called the *tangent space to M at x*.

Definition 2.5.2. Let M be an n-dimensional differentiable manifold. Let U be an open set in M. A *vector field* V on U is a map which send $x \in U$ to $V_x \in T_x(M)$ of a tangent vector at every point of U such that if $f \in C^\infty(U)$, then the map $x \mapsto V_x(f)$ is also smooth.

Definition 2.5.3. Let $x \in M$. A *connection* ∇ at x is a map which sends (U_x, V) where $U_x \in T_x(M)$ and V is a vector field defined near x to a vector $\nabla_{U_x} V \in T_x(M)$. The map ∇ satisfies the following properties.

(i) the map ∇ is bilinear, and

(ii) if f is smooth near x, then

$$\nabla_{U_x}(f \cdot V) = U_x(f) \cdot V_x + f(x) \cdot \nabla_{U_x} V$$

A connection on M is a map which assigns to each $x \in M$ a connection at x such that if U and V are vector fields, the map $x \mapsto \nabla_{U_x} V$ is a vector field.

Definition 2.5.4. Suppose X and Y are differential manifolds. A map $F : X \to Y$ is said to be *smooth* if $f \in C^\infty(Y)$ then $f \circ F \in C^\infty(X)$. If F is a homeomorphism and both F and F^{-1} are smooth, we say F is a *diffeomorphism.*

Let $x \in X$ and $\sigma \in T_x(X)$, then we define $dF_x(\sigma) \in T_{F(x)}(Y)$ by

$$[dF_x(\sigma)](f) = \sigma(f \circ F)$$

for $f \in C^\infty(U)$. We called dF_x to be the *differential of* F *at* x.

Definition 2.5.5. Suppose that ∇ is a connection of Y and $F : X \to Y$ is locally a diffeomorphism. We get a connection $F^*(\nabla) = \nabla^*$ on X by setting

$$\nabla^*_U(V) = \nabla_{dF(U)}(dF(V))$$

where U and V are vector fields on X and $dF(U)$ is the vector field on $F(X) \subset Y$ that sends f to $[dF_x(U_x)](f)$ where f is smooth near $F(x)$. We call ∇^* to be the *induced connection.* If X has a connection $\tilde{\nabla}$ such that $\nabla^* = \tilde{\nabla}$ and F is a diffeomorphism, we say F is an *affine equivalence.*

Definition 2.5.6. Let M be an n-dimensional differentiable manifold. Let U and V be vector fields on M. Then we define $[U, V]$ be a vector field defined by

$$[U, V](f) = U(V(f)) - V(U(f))$$

for all $f \in C^\infty(M)$.

Definition 2.5.7. Let U and V be vector fields on a manifold M with connection ∇. Define a transformation $R(U, V)$ of vector fields to vector fields by

$$R(U, V)W = -\nabla_U(\nabla_V W) - \nabla_V(\nabla_U W) + \nabla_{[U,V]}W$$

for any vector field W. The transformation R is called the *curvature* of M.

Definition 2.5.8. Let M be a manifold. A Riemannian structure on M is a map which assigns to each $x \in M$ a positive definite inner product $\langle , \rangle_x$ on $T_x(M)$ such that if U and V are vector fields on M, the function $x \mapsto \langle U_x, V_x \rangle$ is a smooth function. We say M with $\langle , \rangle$ is a *Riemannian manifold.* We say M is a flat manifold if its riemannian connection has identically zero curvature.

Theorem 2.5.9. *[35, Corollary 2.4.10]* Let M be an n-dimensional Riemannian manifold where $n \geq 2$. Then M is complete, connected flat manifold if and only if it is isomorphic to $\mathbb{R}^n/\Gamma$ where Γ is subgroup of $Isom(\mathbb{R}^n)$ and it acts freely and properly discontinuously on $\mathbb{R}^n$.

By definition, Γ is a crystallographic group if and only if its acts properly discontinuously and with a compact quotient on $\mathbb{R}^n$. By Proposition 2.1.18, Γ is torsion-free if and only if Γ

acts freely on $\mathbb{R}^n$. By the above theorem, if M is an n-dimensional compact, connected flat manifold if and only if it is isomorphic to $\mathbb{R}^n/\Gamma$ where Γ is a Bieberbach group. Therefore, we can state the Bieberbach Theorems in the context of flat manifolds.

Theorem 2.5.10. *[35, Theorem 3.3.1]* (i) If M is a flat compact connected n-dimensional Riemannian manifold, then M admits a normal Riemannian covering by a flat n-dimensional torus.

(ii) For any natural number n, there are only finitely many affine equivalence classes of flat compact connected n-dimensional Riemannian manifold.

(iii) Two flat compact connected Riemannian manifolds are affinely equivalent if and only if their fundamental groups are isomorphic.

2.6 Classification of crystallographic groups

In this section, we are going to present two classification methods that classify all Bieberbach group. We need to introduce a theorem which give a simple criterion for recongnizing whether a cohomology class is defining a Bieberbach group, the concept of first Betti number and Calabi construction.

Theorem 2.6.1. *[31, Theorem 3.1]* Let Γ be an n-dimensional crystallographic group and let the second cohomology class $\alpha \in H^2(G, \mathbb{Z}^n)$ is defining such Γ. Then Γ is torsion-free if and only if the image of the restriction homomorphism $res_H(\alpha) \in H^2(H, \mathbb{Z}^n)$ is not zero for all prime order cyclic subgroup H of G.

Proof. Let Γ be torsion-free. By first Bieberbach theorem, it fits in the below short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{\phi} \langle g \rangle \to 1 \tag{2.37}$$

Assume by contradiction that there exists an element $g \in G$ of prime order p such that $\alpha' = res_{\langle g \rangle}\alpha = 0$. Hence the below short exact sequence splits

$$0 \to \mathbb{Z}^n \to \phi^{-1}(\langle g \rangle) \to \langle g \rangle \to 0 \tag{2.38}$$

By definition, there exists a section

$$s : \langle g \rangle \to \phi^{-1}(\langle g \rangle)$$

such that the composition $\phi s : \langle g \rangle \to \langle g \rangle$ is an identity map. Since $\langle g \rangle$ is a cyclic subgroup of order p, the group Γ has a torsion element, which is a contradiction. For the reverse direction, we assume Γ has torsion element $\gamma \in \Gamma$, we want to show that there exists a cyclic subgroup $H \leq G$ of prime order such that $res_H(\alpha) = 0$. Assume γ has order pm where p is prime, then consider the element $\gamma^m \in \Gamma$. We have $H = \langle \phi(\gamma^m) \rangle$ is cyclic group of prime order p. We claim that $res_H(\alpha) = 0$. In order to do that, we want to show the below short exact sequence

$$0 \to \mathbb{Z}^n \to \phi^{-1}(H) \to H \to 0 \tag{2.39}$$

is a splits short exact sequence. We define a section $s : H \to \phi^{-1}(H)$ by $s(\phi(\gamma^m)) = \gamma^m$. It is clear that $\phi s : H \to H$ is an identity map. Hence the above short exact sequence is a splits short exact sequence. □

Definition 2.6.2. Let M be a flat manifolds with fundamental group Γ. The rank of the abelian group $H_1(M, \mathbb{Z}) = \Gamma/[\Gamma, \Gamma]$ is the *first Betti number* of Γ. We denoted it to be $b_1(\Gamma)$.

Next, we want to present some results about first Betti number.

Lemma 2.6.3. *[19, Corollary 1.3]* Let Γ be an n-dimensional Bieberbach group. Then we have

$$b_1(\Gamma) = rk((\mathbb{Z}^n)^G)$$

where G is the holonomy group and G and $(\mathbb{Z}^n)^G = \{z \in \mathbb{Z}^n \mid zg = gz \text{ for all } g \in G\}$ (the G-action is given by holonomy representation).

Proof. By the first Bieberbach's theorem, Γ fits in the below short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to G \to 1 \tag{2.40}$$

Consider the $E_2^{1,0}$-term of the Hochschild-Serre spectral sequence associated to the above short exact sequence. We have

$$H^1(G, H^0(\mathbb{Z}^n, \mathbb{Z})) = H^1(G, \mathbb{Z})$$

Since G is a finite group, we have

$$H^1(G, H^0(\mathbb{Z}^n, \mathbb{Z})) = H^1(G, \mathbb{Z}) = Hom(G/[G, G], \mathbb{Z}) = 0$$

The only remaining term on the $p+q = 1$ line is $E_2^{0,1} = H^0(G, H^1(\mathbb{Z}^n, \mathbb{Z})) = H^1(\mathbb{Z}^n, \mathbb{Z})^G$. Since the differential $d_2 : E_2^{0,1} \to E_2^{2,0} = H^2(G, \mathbb{Z})$ maps to a finite group, we have

$$rk(H^1(\Gamma, \mathbb{Z})) = rk(H^1(\mathbb{Z}^n, \mathbb{Z})^G)$$

where $\mathbb{Z}$ is a trivial Γ-module and $rk(A)$ of an abelian group A is the $\mathbb{Q}$-dimension of $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence we have

$$b_1(\Gamma) = rk(H_1(\Gamma, \mathbb{Z})) = rk(H^1(\Gamma, \mathbb{Z})) = rk(H^1(\mathbb{Z}^n, \mathbb{Z})^G) = rk((\mathbb{Z}^n)^G)$$

□

Next, we present the theorem proved by E. Calabi related to Calabi construction.

Theorem 2.6.4. *[31, Proposition 3.1]* Let Γ be an n-dimensional Bieberbach group. If there exists an epimorphism $f : \Gamma \to \mathbb{Z}$, then the group $ker(f) = \Gamma'$ is a Bieberbach group of dimension $n - 1$.

Proof. Consider the below diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma' \cap \mathbb{Z}^n & \xrightarrow{\iota|_{\Gamma' \cap \mathbb{Z}^n}} & \mathbb{Z}^n & \xrightarrow{f|_{\mathbb{Z}^n}} & \mathbb{Z}^n/(\Gamma' \cap \mathbb{Z}^n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow g & & \\
0 & \longrightarrow & \Gamma' & \xrightarrow{\iota} & \Gamma & \xrightarrow{f} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow p & & \downarrow & & \\
0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G/G' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

The exactness of the middle vertical sequence of the above diagram is followed from the definition of Γ. Since $\Gamma' \cap \mathbb{Z}^n \subset \mathbb{Z}^n$, it follows that $\Gamma' \cap \mathbb{Z}^n$ is a free abelian group. It is clear that $|\Gamma' : \Gamma' \cap \mathbb{Z}^n|$ is finite. By Theorem 2.4.2, we can conclude that Γ' is a Bieberbach group. We remain to show that the dimension of Γ' is equal to $n-1$. It is clear that the dimension of Γ is at most n. First, we assume the dimension of Γ' is less than $n-1$. Since the rank of $\Gamma' \cap \mathbb{Z}^n$ is at most $n-2$, the rank of $\mathbb{Z}^n/(\Gamma' \cap \mathbb{Z}^n)$ is at less 2. But this is impossible because the map $g : \mathbb{Z}^n/(\Gamma' \cap \mathbb{Z}^n)$ is an injective map. Next, we assume the dimension of Γ' is equal to n. By similar calculation, we have the group $\mathbb{Z}^n/(\Gamma' \cap \mathbb{Z}^n)$ is either a finite group of a trivial group. Since g is an injection map, the group $\mathbb{Z}^n/(\Gamma' \cap \mathbb{Z}^n)$ cannot be a finite group. Next, notice that f is an surjection map, we know $\mathbb{Z}^n/(\Gamma' \cap \mathbb{Z}^n)$ cannot be a trivial group. Hence Γ' is an $(n-1)$ dimensional Bieberbach group. □

Corollary 2.6.5. Let Γ be an n-dimensional Bieberbach group. If there exists an epimorphism $f : \Gamma \to \mathbb{Z}^k$, then the group $ker(f) = \Gamma'$ is a Bieberbach group of dimension $n-k$.

Proof. We proceed by induction on k. By Theorem 2.6.4, the statement is true for $k=1$. Assume the statement is true for $t = k-1$. Consider the case where $t = k$. Suppose we have a epimorphism $f : \Gamma \to \mathbb{Z}^k$. Consider the elements of $\mathbb{Z}^k$ as k-tuple. We define $p_1 : \mathbb{Z}^k \to \mathbb{Z}^{k-1}$ be a projection map which map $(x_1, ..., x_k) \in \mathbb{Z}^k$ to $(x_1, x_2, ..., x_{k-1}, 0) \in \mathbb{Z}^k$ and define $p_2 : \mathbb{Z}^k \to \mathbb{Z}$ be a projection map which maps $(x_1, ..., x_k) \in \mathbb{Z}^k$ to $x_k \in \mathbb{Z}$. Consider the epimorphism $p_1 \circ f : \Gamma \to \mathbb{Z}^{k-1}$. By induction hypothesis, we have $\Gamma' := ker(p_1 \circ f)$ is an $(n-k+1)$-dimensional Bieberbach group. Observe that $ker(f) = ker(\Gamma' \xrightarrow{f} \mathbb{Z}^k \xrightarrow{p_2} \mathbb{Z})$. Thus by Theorem 2.6.4, we know that $ker(f)$ is an $n-k$-dimensional Bieberbach group. □

Remark 2.6.6. Let Γ be an n-dimensional Bieberbach group with $b_1(\Gamma) = k \neq 0$. By Lemma 2.6.3, we have $b_1(\Gamma) = rk(\mathbb{Z}^n)^G$ where G is the holonomy group of Γ. Since Bieberbach group is defined up to conjugation of affine elements, we assume all elements of Γ can express as

$$\begin{pmatrix} A & B & x \\ 0 & I_k & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $A \in GL_{n-k}(\mathbb{Z})$, B is an integral matrix of dimension $(n-k) \times k$, $x \in \mathbb{Q}^{n-k}$ and $y \in \mathbb{Q}^k$. Then we can define a surjection map $f : \Gamma \to \mathbb{Z}^k$ as

$$f\left(\begin{pmatrix} A & B & x \\ 0 & I_k & y \\ 0 & 0 & 1 \end{pmatrix}\right) = y$$

Hence by Corollary 2.6.5, we can conclude that if Γ is a n-dimensional Bieberbach group with $b_1(\Gamma) = k \geq 1$, then there exists a surjection map $f : \Gamma \to \mathbb{Z}^k$ such that $ker(f)$ is a $(n-k)$-dimensional Bieberbach group.

Finally, we will state the two ways of classification. The first way is called the *Zassenhaus algorithm*. The algorithm is given by the following steps.

1. Classify all finite subgroups of $GL_n(\mathbb{Z})$.
2. Classify all G-module $\mathbb{Z}^n$ where the representation given by the G-action is faithful.
3. Calculate the second cohomology group $H^2(G, \mathbb{Z}^n)$ for all finite subgroup $G \leq GL_n(\mathbb{Z})$ from step 1 and G-module from step 2.
4. Recongnize which crystallographic groups from step 3 are isomorphic.

The second way of classification is called the *induction method of Calabi*. This classification method is only suitable for Bieberbach group. The algorithm is given by the following steps.

1. Classify all Bieberbach group of dimension less than n.
2. Describe all Bieberbach group of dimension n with trivial first Betti number.
3. Describe all Bieberbach group of dimension n, Γ, defined by the below short exact sequence

   $$0 \to \Gamma_{n-1} \to \Gamma_n \to \mathbb{Z} \to 0$$

   where Γ_{n-1} is any Bieberbach group of dimension $n-1$.

Chapter 3

Number of generators of Bieberbach groups

In this chapter, we focus on the Conjecture 1.0.1. We mainly consider the cases where the holonomy group of the Bieberbach groups is cyclic group or is generated by 2 elements. The results in this chapter have been published in Geometriae Dedicata (see [8]).

3.1 Background

Let $\Gamma \leq O(n) \ltimes \mathbb{R}^n$ be an n-dimensional Bieberbach group with G as holonomy group. Recall that Γ will induce the holonomy representation $\rho : G \to GL_n(\mathbb{Z})$. Therefore we can consider $\Gamma \cap (I_n \times \mathbb{R}^n) \cong \mathbb{Z}^n$ as a $\mathbb{Z}G$-module. Let G be a group and M be a $\mathbb{Z}G$-module. We denote $d(G)$ to be the minimal number of generators of the group G and denote $rk_G(M)$ to be the minimal number of generators of M as a $\mathbb{Z}G$-module. This chapter contains three sections. In Section 3.1, we give some basic definitions and some related properties of crystallographic groups. In Section 3.2, we discuss the number of generators of $\mathbb{Z}C_m$-module, where C_m is a cyclic group of order m. In Section 3.3, we present our three main theorems in this chapter. Let G be a cyclic group with generator g and let $\rho : G \to GL_n(\mathbb{Z})$ where $g \mapsto M \in GL_n(\mathbb{Z})$ be its matrix holonomy representation. For convenience, in this chapter, we denote element $(g, a) \in \Gamma$ to be (M, a) and denote the $\mathbb{Z}G$-module $\mathbb{Z}^n$ to be $\mathbb{Z}^n_M$ to specify that the G-action is given by the matrix M. We will denote I_n to be the identity matrix of dimension n and C_m to be a cyclic group of order m.

Remark 3.1.1. Let Γ be an n-dimensional crystallographic group where its holonomy group is isomorphic to G. Let the holonomy group G is generated by m elements namely $a_1, ..., a_m$. By first Bieberbach theorem, we have the below short exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1$$

where ι and p is defined as in (2.7). We can therefore view $\mathbb{Z}^n$ as a $\mathbb{Z}G$-module and we have the following two observations,

(i) $d(\Gamma) \leq rk_G(\mathbb{Z}^n) + d(G)$.

(ii) $\{\iota(e_1), ..., \iota(e_n), \alpha_1, ..., \alpha_m\}$ can be a generating set of Γ where $e_1, ..., e_n$ are the standard basis of $\mathbb{Z}^n$ and α_i is chosen arbitrarily such that $p(\alpha_i) = a_i$ for all $i = 1, ..., m$.

Definition 3.1.2. Let G be a group. Let $\rho : G \to GL_n(\mathbb{Z})$ and $\phi : G \to GL_n(\mathbb{Z})$ be two group representations of G. We say ρ and ϕ are $\mathbb{Z}$-*equivalent* if there exists $S \in GL_n(\mathbb{Z})$ such that $\rho(g) = S^{-1}\phi(g)S$ for all $g \in G$.

Definition 3.1.3. Let G be a group, M be a $\mathbb{Z}G$-module and $\rho : G \to GL_m(\mathbb{Z})$ be the representation correspond to the $\mathbb{Z}G$-module M.

(i) We say N is a *submodule* of M if N is a subgroup of M which is closed under the action of ring elements.

(ii) We say M is *decomposable* if M is the direct sum of submodules. M is *indecomposable* if M is not decomposable.

(iii) We say M is $\mathbb{Z}$-*reducible* if ρ is $\mathbb{Z}$-equivalent to $\phi : G \to GL_n(\mathbb{Z})$ where $\phi(g)$ has form $\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix}$ for all $g \in G$, where P, Q and R are integral matrices. We say M is $\mathbb{Z}$-*irreducible* if M is not $\mathbb{Z}$-reducible.

Now, we are going to give a short introduction to the properties of holonomy representation. Let $M_1, ..., M_k$ be square matrices with entries in $\mathbb{Z}$, we denote $tri(M_1, ..., M_k)$ to be matrix of form as below,

$$tri(M_1, ..., M_k) := \begin{pmatrix} M_1 & & & * \\ & M_2 & & \\ & & \ddots & \\ \text{\Large 0} & & & M_k \end{pmatrix}$$

Let Γ be an n-dimensional Bieberbach group with cyclic holonomy group and let $\rho : C_m \to GL_n(\mathbb{Z})$ be its faithful holonomy representation. Since ρ is defined up to isomorphism, we are able to conjugate it by a suitable invertible matrix and assume $\rho(g) = tri(A_1, ..., A_t)$ for some $t \in \mathbb{N}$ and $A_1, ..., A_t$ are square matrices such that $\mathbb{Z}_{A_1}^{dim(A_1)}$,..., $\mathbb{Z}_{A_t}^{dim(A_t)}$ are $\mathbb{Z}$-irreducible modules and $\sum_{j=1}^{t} dim(A_j) = n$.

Lemma 3.1.4. Let $M = tri(A_1, ..., A_t) \in GL_n(\mathbb{Z})$ where $A_1, ..., A_t$ are square matrices. Denote the order of A_i to be a_i for $i = 1, ..., t$ and m to be the order of M. Then the least common multiple of $a_1, ..., a_t$ equals to m. In particular, m is divisible by a_i for $i = 1, ..., t$.

Proof. We denote the least common multiple of $a_1, ..., a_t$ to be $L.C.M(a_1, ..., a_t)$. Since M has finite order, the order of A_i are all finite for all $i = 1, ..., t$. By simple calculation, we have

$$I_n = M^m = tri(A_1^m, ..., A_t^m)$$

Thus A_i^m are all identity matrix for all $i = 1, ..., t$. If follows that m is divisible by all a_i's. Hence $L.C.M(a_1, ..., a_t) \leq m$. On the other hand, if $L.C.M(a_1, ..., a_t) = l < m$, then we have

$$M^l = tri(A_1^l, ..., A_t^l) := \begin{pmatrix} I & & & * \\ & I & & \\ & & \ddots & \\ 0 & & & I \end{pmatrix}$$

Since M has finite order, it force $M^l = I_n$. It contradicts that the order of M is m. Hence $L.C.M(a_1, ..., a_t) = m$. □

3.2 Generators of $\mathbb{Z}C_m$-module

Let Γ be an n-dimensional crystallographic group with holonomy group isomorphic to C_m. We can consider $\Gamma \cap (I_n \times \mathbb{R}^n) \cong \mathbb{Z}^n$ as a $\mathbb{Z}C_m$-module. Since we can restrict the C_m-action to be a C_k-action as long as m is divisible by k, we can also view $\mathbb{Z}^n$ as a $\mathbb{Z}C_k$-module. It is clear that $rk_{C_m}(\mathbb{Z}^n) \leq rk_{C_k}(\mathbb{Z}^n)$. The below lemma and proposition are on the number of generators of $\mathbb{Z}C_m$-module.

Lemma 3.2.1. Let $\rho : C_p \to GL_n(\mathbb{Z})$ be a faithful representation and $\mathbb{Z}^n$ be the correspondence $\mathbb{Z}C_p$-module, where p is prime. Then

$$rk_{C_p}(\mathbb{Z}^n) \leq \begin{cases} n - p + 2 & \text{if } p \leq 19 \\ n - p + 3 & \text{if } p > 19 \end{cases}$$

Proof. Let g be the generator of C_p. Assume $\rho(g) = tri(A_1, ..., A_k)$ where $\mathbb{Z}_{A_1}^{dim(A_1)}, \cdots, \mathbb{Z}_{A_k}^{dim(A_k)}$ are $\mathbb{Z}$-irreducible $\mathbb{Z}C_p$-modules. By Lemma 3.1.4, there exists $i \in \{1, ..., k\}$ such that A_i has order p. By [11, Theorem 74.3], A_i has dimension $p-1$ and the module $\mathbb{Z}_{A_i}^{dim(A_i)}$ is isomorphic to an ideal in $\mathbb{Z}[\zeta]$ where ζ is a primitive p-root of unity. If $p \leq 19$, by [29, Section 29.1.3], the class number of $\mathbb{Z}[\zeta]$ is 1. Therefore the module $\mathbb{Z}_{A_i}^{dim(A_i)}$ is a principle ideal and it is isomorphic to $\mathbb{Z}[\zeta]$. Hence $rk_{C_p}(\mathbb{Z}_{A_i}^{dim(A_i)}) = 1$. Now assume $p > 19$. Since $\mathbb{Z}[\zeta]$ is a Dedekind domain. By [29, Section 7.1-2], every ideal in a Dedekind domain can be generated by two elements. Hence $rk_{C_p}(\mathbb{Z}_{A_i}^{dim(A_i)}) \leq 2$. Therefore we have

$$\begin{aligned} rk_{C_p}(\mathbb{Z}^n) &\leq n - dim(A_i) + rk_{C_p}(\mathbb{Z}_{A_i}^{dim(A_i)}) \\ &= n - p + 1 + rk_{C_p}(\mathbb{Z}_{A_i}^{dim(A_i)}) \\ &\leq \begin{cases} n - p + 2 & \text{if } p \leq 19 \\ n - p + 3 & \text{if } p > 19 \end{cases} \end{aligned}$$

This finishes the proof of this lemma. □

Remark 3.2.2. The above lemma is obtained by using the idea of [1, Lemma 2.3].

Proposition 3.2.3. Let $\rho : C_m \to GL_n(\mathbb{Z})$ be a faithful representation and $\mathbb{Z}^n$ be the correspondence $\mathbb{Z}C_m$-module of ρ, where $m \geq 3$.

(i) If m is divisible by prime larger than 3, then $rk_{C_m}(\mathbb{Z}^n) \leq n - 3$.

(ii) If m is not divisible by prime larger than 3, then $rk_{C_m}(\mathbb{Z}^n) \leq n-1$.

Proof. Let $m = p_1^{s_1} \cdots p_t^{s_t}$ be the prime decomposition of m and assume $p_1 < \cdots < p_t$. Let g be the generator of C_m.

(i): Consider $H = \langle g^{m/p_t} \rangle \cong C_{p_t}$, a subgroup of C_m. We can view $\mathbb{Z}^n$ as a $\mathbb{Z}C_{p_t}$-module where the C_{p_t}-action is given by $\rho|_H$. Since $\rho|_H$ is a faithful representation, by Lemma 3.2.1, we have

$$rk_{C_{p_t}}(\mathbb{Z}^n) \leq \begin{cases} n - p_t + 2 & \text{if } p_t \leq 19 \\ n - p_t + 3 & \text{if } p_t > 19 \end{cases}$$

Since m is divisible by prime larger than 3, we have $rk_{C_m}(\mathbb{Z}^n) \leq n - 5 + 2 = n - 3$.

(ii): We observe that m is either divisible by 3 or 4. If m is divisible by 3, we consider $\mathbb{Z}^n$ as $\mathbb{Z}C_3$-module. By Lemma 3.2.1, we have $rk_{C_3}(\mathbb{Z}^n) \leq n-1$. Hence $rk_{C_m}(\mathbb{Z}^n) \leq n-1$. Now we assume m is divisible by 4. Consider $H' = \langle g^{m/4} \rangle \cong C_4$, a subgroup of C_m. We can view $\mathbb{Z}^n$ as a $\mathbb{Z}C_4$-module by restricting the C_m-action to a C_4-action, where the C_4-action is given by $\rho|_{H'}$. We assume $\rho|_{H'}(g^{m/4}) = tri(M_1, ..., M_k)$ and $\mathbb{Z}_{M_1}^{dim(M_1)}, ..., \mathbb{Z}_{M_k}^{dim(M_k)}$ are $\mathbb{Z}$-irreducible $\mathbb{Z}C_4$-modules. By Remark 3.1.4, there exists $i \in \{1, ..., k\}$ such that M_i is a matrix of order 4. Let $\phi : C_4 \to GL_n(\mathbb{Z})$ be the corresponding representation of $\mathbb{Z}_{M_i}^{dim(M_i)}$. By [2, Section 5, page 10], there is only one faithful integral $\mathbb{Z}$-irreducible C_4-representation up to equivalence. Hence we assume $\phi(g^{m/4})$ is $\mathbb{Z}$-equivalent to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Therefore we have $dim(M_i) = 2$. Let $y_1 = (1, 0) \in \mathbb{Z}^2$ and $y_2 = (0, 1) \in \mathbb{Z}^2$ be the standard basis of $\mathbb{Z}_{M_i}^2$. We have $\phi(g^{m/4})y_2 = y_1$. Hence $\mathbb{Z}_{M_i}^2$ can be generated by y_2 as a $\mathbb{Z}C_4$-module. Thus we have $rk_{C_4}(\mathbb{Z}_{M_i}^{dim(M_i)}) = 1$. Since $rk_{C_4}(\mathbb{Z}_{M_z}^{dim(M_z)}) \leq dim(M_z)$ for all $z = 1, ..., k$, we have

$$rk_{C_m}(\mathbb{Z}^n) \leq \sum_{z=1}^{k} rk_{C_4}(\mathbb{Z}_{M_z}^{dim(M_z)}) \leq n - dim(M_i) + rk_{C_4}(\mathbb{Z}_{M_i}^{dim(M_i)}) \leq n - 1$$

□

3.3 Proofs of Theorem A, B and C

Theorem A. Let Γ be an n-dimensional crystallographic group with holonomy group isomorphic to $C_m = \langle g \,|\, g^m = 1 \rangle$ where $m \geq 3$.

(i) If m is divisible by prime larger than 3, then $d(\Gamma) \leq n-2$.

(ii) If m is not divisible by prime larger than 3 and Γ is torsion-free, then $d(\Gamma) \leq n-1$.

Proof. (i): By Remark 3.1.1(i), we have $d(\Gamma) \leq rk_{C_m}(\mathbb{Z}^n) + 1$. Since m is divisible by prime larger than 3, by Proposition 3.2.3, we have $rk_{C_m}(\mathbb{Z}^n) \leq n-3$. Therefore we have $d(\Gamma) \leq n-2$.

(ii): By Remark 3.1.1(ii), let $\Gamma = \langle \iota(e_1), ..., \iota(e_n), \alpha \rangle$, where $e_1, ..., e_n$ are the standard basis of $\mathbb{Z}^n$ and $p(\alpha) = g$. By Lemma 2.6.3, we have $b_1(\Gamma) = rk((\mathbb{Z}^n)^{C_m})$. It is well known that

$b_1(\Gamma) \neq 0$ (see [31, Example 4.1]). Let $k = b_1(\Gamma) > 0$. Without loss of generality, every element of Γ can be expressed as $(tri(M, I_k), a)$ where $a \in \mathbb{R}^n$ and $M \in GL_{n-k}(\mathbb{Z})$. In particular, let $\alpha = (tri(A, I_k), x)$ where $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $A \in GL_{n-k}(\mathbb{Z})$ which do not fix any non-trivial elements. In other words, $Au = u$ if and only if $u = 0$ for $u \in \mathbb{R}^{n-k}$. First we assume $x_{n-k+1} = \cdots = x_n = 0$. Let $v := (x_1, ..., x_{n-k}) \in \mathbb{R}^{n-k}$. By simple calculations, we get $\alpha^m = \left(I_n, (\sum_{s=0}^{m-1} A^s v, 0, ..., 0)\right)$. Since $A(\sum_{s=0}^{m-1} A^s v) = \sum_{s=0}^{m-1} A^s v$, we have $\sum_{s=0}^{m-1} A^s v = 0$. There is a contradiction because $\alpha^m = (I_n, 0)$. Therefore there exists $i \in \{n-k+1, ..., n\}$ such that $x_i = \frac{q}{z} \neq 0 \in \mathbb{Q}$. Define $f : \Gamma \to \mathbb{Z}$ where it maps $(tri(M, I_k), (y_1, ..., y_n)) \in \Gamma$ to $zy_i \in \mathbb{Z}$. Hence we have $f(\alpha) = q$, $f(\iota(e_i)) = z$ and $f(\iota(e_j)) = 0$ for all $j \neq i$. We claim that f is a surjective homomorphism. Let $\gamma_1 = (tri(M_1, I_k), (m_1, ..., m_n)) \in \Gamma$ and $\gamma_2 = (tri(M_2, I_k), (m'_1, ..., m'_n)) \in \Gamma$. By simple calculation, we get

$$\gamma_1\gamma_2 = (tri(M_1M_2, I_k), (*, ..., *, m_{n-k+1} + m'_{n-k+1}, \; ... \;, m_n + m'_n))$$

Hence we have $f(\gamma_1) + f(\gamma_2) = f(\gamma_1\gamma_2)$. Therefore f is a homomorphism. Notice that q and z are coprime, there exists integers σ and τ such that $\sigma q + \tau z = 1$. Hence we have $f(\alpha^\sigma \iota(e_i)^\tau) = 1$. Therefore f is surjective. Observe that

$$ker(f) = \langle \iota(e_1), ..., \iota(e_{i-1}), \iota(e_{i+1}) ..., \iota(e_n) \rangle \cong \mathbb{Z}^{n-1}$$

We have the below short exact sequence

$$0 \longrightarrow ker(f) \cong \mathbb{Z}^{n-1} \longrightarrow \Gamma \xrightarrow{\quad f \quad} \mathbb{Z} \longrightarrow 0 \tag{3.1}$$

By Lemma 2.3.15, such short exact sequence will induce a representation $\rho : \mathbb{Z} \to GL_{n-1}(\mathbb{Z})$ given by $\rho(x)e_j = \bar{x}\iota(e_j)\bar{x}^{-1}$ where $f(\bar{x}) = x$ for all $j \neq i$. Consider the restriction $\bar{\rho} := \rho|_{q\mathbb{Z}} : q\mathbb{Z} \to GL_{n-1}(\mathbb{Z})$. We claim that $ker(\bar{\rho}) = mq\mathbb{Z}$. Let $qx \in ker(\bar{\rho})$ for any $x \in \mathbb{Z}$. We have $e_j = \bar{\rho}(qx)e_j = \alpha^x \iota(e_j)\alpha^{-x} = p(\alpha^x)e_j$ for all $j \neq i$. Hence $p(\alpha^x)$ needs to be an identity matrix. Therefore x is multiple of m or $x = 0$. Hence $ker(\bar{\rho}) \subseteq mq\mathbb{Z}$. Since $p(\alpha^m)$ is an identity matrix, $\bar{\rho}(mqx)(e_j) = \alpha^{mx}\iota(e_j)\alpha^{-mx} = p(\alpha^{mx})e_j = e_j$ for all $j \neq i$ and $x \in \mathbb{Z}$. Hence $mq\mathbb{Z} \subseteq ker(\bar{\rho})$. Therefore we have $ker(\bar{\rho}) = mq\mathbb{Z}$. Now we can obtain a faithful representation $\phi : q\mathbb{Z}/mq\mathbb{Z} \to GL_{n-1}(\mathbb{Z})$ given by $\phi(\bar{x}) = \bar{\rho}(x)$ where x is the representative of $\bar{x} \in q\mathbb{Z}/mq\mathbb{Z}$. Hence we can view $\mathbb{Z}^{n-1}$ as a $\mathbb{Z}C_m$-module with faithful C_m-representation. By Proposition 3.2.3, $\mathbb{Z}^{n-1}$ can be generated by $n-2$ elements. By 3.1, we have $d(\Gamma) \leq rk_{C_m}(\mathbb{Z}^{n-1}) + 1 \leq n-1$. □

The corollary below gives the general bound on the number of generators of general Bieberbach groups.

Corollary 3.3.1. Let Γ be an n-dimensional Bieberbach group with holonomy group G. Then $d(\Gamma) \leq 2n$.

Proof. Let $|G| = p_1^{s_1} \cdots p_k^{s_k}$ be the prime decomposition of order of G. By [17, Theorem A], we have

$$d(G) \leq \max_{1 \leq i \leq k} d(P_i) + 1$$

where P_i is the Sylow p_i-subgroup of G for $i = 1, ..., k$. We fix $j \in \{1, ..., k\}$ such that $d(P_j) = \max_{1 \leq i \leq k} d(P_i)$. We first assume $p_j \geq 3$. We can consider $\Gamma \cap (I_n \times \mathbb{R}^n) \cong \mathbb{Z}^n$ as a $\mathbb{Z}P_j$-module. By [1, Theorem A], we have $d(P_j) + rk_{P_j}(\mathbb{Z}^n) \leq n$. Hence we have

$$d(\Gamma) \leq d(G) + rk_{P_j}(\mathbb{Z}^n) \leq d(P_j) + 1 + rk_{P_j}(\mathbb{Z}^n) \leq n + 1$$

Now we assume $p_j = 2$. If G is a 2-group, then by [1, Theorem A], we have $d(\Gamma) \leq 2n$. If G is not a 2-group, then there exists $g \in G$ such that g has order $p \geq 3$. Hence we can consider $\mathbb{Z}^n$ as a $\mathbb{Z}C_p$-module. By Lemma 3.2.1, we have $rk_{C_p}(\mathbb{Z}^n) \leq n - 1$. By [1, Proposition 2.2], we have $d(P_j) \leq n$. Hence we have

$$d(\Gamma) \leq d(G) + rk_{C_p}(\mathbb{Z}^n) \leq d(P_j) + 1 + rk_{C_p}(\mathbb{Z}^n) \leq 2n$$

□

Corollary 3.3.2. Let Γ be an n-dimensional Bieberbach group with holonomy group G, where G is a simple group but not C_2. Then $d(\Gamma) \leq n - 1$.

Proof. By Remark 3.1.1(i), we have $d(\Gamma) \leq d(G) + rk_G(\mathbb{Z}^n)$. If G is a cyclic group of odd prime order, then by Theorem A, we have $d(\Gamma) \leq n - 1$. Now, we assume G is not a cyclic group of prime order. It is well known that G is a simple abelian group if and only if G is a cyclic group of prime order and every finite non-abelian simple group is not solvable. By Burnside's Theorem, [15, Page 886], there exists a prime $p \geq 5$ such that the order of G is divisible by p. So we can view $\mathbb{Z}^n$ as a $\mathbb{Z}C_p$-module. By Lemma 3.2.1, we have $rk_{C_p}(\mathbb{Z}^n) \leq n - 3$. By [3, Theorem B], we have $d(G) \leq 2$. Hence we have $d(\Gamma) \leq d(G) + rk_G(\mathbb{Z}^n) \leq 2 + rk_{C_p}(\mathbb{Z}^n) \leq n - 1$. □

The rest of this chapter will present the proof of Theorem B and Theorem C.

Theorem B. Let Γ be an n-dimensional crystallographic group with holonomy group isomorphic to a finite group G.

(i) If the order of G is not divisible by 2 or 3, then $d(\Gamma) \leq n$.

(ii) If the order of G is odd and divisible by 3, then $d(\Gamma) \leq n + 1$.

Proof. Let $|G| = p_1^{s_1} \cdots p_k^{s_k}$ be the prime decomposition of the order of G, where $p_1 < \cdots < p_k$.

(i): First, we want to calculate the number of generators of the holonomy group G. By [17, Theorem A], we have

$$d(G) \leq \max_{1 \leq i \leq k} d(P_i) + 1$$

where P_i is the Sylow p_i-subgroup of G for $i = 1, ..., k$. We fix $j \in \{1, ..., k\}$ such that $d(P_j) = \max_{1 \leq i \leq k} d(P_i)$. Let $\rho : G \to GL_n(\mathbb{Z})$ be the holonomy representation for Γ.

By definition, ρ is a faithful representation. Therefore P_i acts faithfully on $\mathbb{Z}^n$. By [1, Proposition 2.2], we have

$$d(G) \leq \frac{n - rk\left((\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + 1$$

Now, we consider the lattice part. We can view $\Gamma \cap (\mathbb{R}^n \times I) \cong \mathbb{Z}^n$ as a $\mathbb{Z}P_j$-module. By [1, Proposition 2.5], we have

$$rk_{P_j}(\mathbb{Z}^n) \leq \frac{(a-1)\left(n - rk(\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j}$$

where $a = 2$ if $p_j \geq 19$, otherwise $a = 3$. Therefore we have

$$\begin{aligned} d(\Gamma) \leq d(G) + rk_{P_j}(\mathbb{Z}^n) &\leq \frac{n - rk\left((\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + 1 + \frac{(a-1)\left(n - rk(\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} \\ &= \frac{a\left(n - rk(\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} + 1 \end{aligned}$$

We need to show

$$\frac{a\left(n - rk(\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} + 1 \leq n$$

We have

$$\begin{aligned} &\frac{a\left(n - rk(\mathbb{Z}^n)^{P_j}\right)}{p_j - 1} + rk(\mathbb{Z}^n)^{P_j} + 1 \leq n \\ \iff & an - a \cdot rk(\mathbb{Z}^n)^{P_j} + (p_j - 1)rk(\mathbb{Z}^n)^{P_j} + p_j - 1 \leq n(p_j - 1) \\ \iff & (p_j - 1 - a)rk(\mathbb{Z}^n)^{P_j} \leq (p_j - 1 - a)n - (p_j - 1) \\ \iff & rk(\mathbb{Z}^n)^{P_j} \leq n - \frac{p_j - 1}{p_j - 1 - a} = n - 1 - \frac{a}{p_j - 1 - a} \end{aligned}$$

If $5 \leq p_j \leq 19$, we have $\frac{a}{p_j-1-a} = \frac{2}{p_j-3} \leq 1$. If $p_j > 19$, we have $\frac{a}{p_j-1-a} = \frac{3}{p_j-4} < 1$. Therefore we can conclude that if $rk(\mathbb{Z}^n)^{P_j} \leq n-2$, then $d(\Gamma) \leq n$. By Cauchy's Theorem [15, Page 93, Theorem 11], P_j has an element $x \in P_j$ with order p_j. Let C_{p_j} be a cyclic subgroup of P_j generated by x. Consider $(\mathbb{Z}^n)^{C_{P_j}}$, where C_{P_j} acts faithfully on $\mathbb{Z}^n$ via $\rho|_{C_{P_j}} : C_{P_j} \to GL_n(\mathbb{Z})$. By [11, Theorem 74.3], the degree of a faithful indecomposable C_{p_j}-representation is either $p_j - 1$ or p_j. If the degree is $p_j - 1$, then it has trivial fix point set. If the degree is p_j, then the fix point set is 1-dimensional. Observe that $rk(\mathbb{Z}^n)^{C_{p_j}}$ has maximum value when $\rho|_{C_{P_j}}$ is a direct sum of one faithful indecomposable sub-representation and all others are trivial sub-representations. Therefore $rk(\mathbb{Z}^n)^{C_{p_j}} \leq n - p_j + 1 \leq n - 4$. Hence we have $rk(\mathbb{Z}^n)^{P_j} \leq n-4$. Therefore we can conclude $d(\Gamma) \leq n$.

(ii): By [1, Theorem A], we can assume G is not a p-group. By [17, Theorem A], we have

$$d(G) \leq \max_{1 \leq i \leq k} d(P_i) + 1$$

where P_i is the Sylow p_i-subgroup of G for $i = 1, ..., k$. If $\max_{1 \leq i \leq k} d(P_i) \neq d(P_1)$, then by part (i), we have $d(\Gamma) \leq n$. Therefore we assume $\max_{1 \leq i \leq k} d(P_i) = d(P_1)$. We can consider the lattice part as a $\mathbb{Z}P_1$-module. Since P_1 is a Sylow 3-subgroup, by [1, Theorem A], we have $d(P_1) + rk_{P_3}(\mathbb{Z}^n) \leq n$. Hence we can conclude that $d(\Gamma) \leq d(P_3) + rk_{P_3}(\mathbb{Z}^n) + 1 = n + 1$. □

Theorem C. Let Γ be an n-dimensional Bieberbach group with 2-generated holonomy group. Then $d(\Gamma) \leq n$.

Proof. Let G be the holonomy group of Γ. Let x and y be the generators of G. They have order a and b respectively. If either $a = 1$ or $b = 1$, then G is a cyclic group. By [14, Theorem 5.7] and Theorem A, $d(\Gamma) \leq n$. Next, consider cases where $a \geq 3$ or $b \geq 3$. It is sufficient to consider only the case where $a \geq 3$. By Remark 3.1.1(ii), let $\Gamma = \langle \iota(e_1), ..., \iota(e_n), \alpha, \beta \rangle$, where $e_1, ..., e_n$ are the standard basis for $\mathbb{Z}^n$, $p(\alpha) = x$ and $p(\beta) = y$. Define $\Gamma' = \langle \iota(e_1), ..., \iota(e_n), \alpha \rangle$. Notice that Γ' is an n-dimensional Bieberbach subgroup of Γ with holonomy group C_a. Since $a \geq 3$, by Theorem A, $d(\Gamma') \leq n-1$. Hence we have $d(\Gamma) \leq n$. Finally, we assume $a = b = 2$. Consider element $xy \in G$. Since G is finite, xy has finite order. If xy is of order 1 (i.e. $xy = 1$), then $x = y$. So $G \cong C_2$. By [14, Theorem 5.7], $d(\Gamma) \leq n$. If xy is of order 2 (i.e. $xyxy = 1$), then $xy = yx$. Hence $G \cong C_2 \times C_2$. By [14, Theorem 5.7], we have $d(\Gamma) \leq n$. Lastly, we assume xy is of order k, where $k \geq 3$. We can rewrite the generating set of Γ to be $\{\iota(e_1), ..., \iota(e_n), \alpha\beta, \beta\}$. Define $\Gamma'' = \langle \iota(e_1), ..., \iota(e_n), \alpha\beta \rangle$, which is an n-dimensional Bieberbach subgroup of Γ with holonomy group isomorphic to C_k. By Theorem A, $d(\Gamma'') \leq n-1$. Therefore $d(\Gamma) \leq n$. $\square$

By [1], [14] and the three main theorems in this chapter, the Conjecture 1.0.1 is still open for certain cases of holonomy group where the minimal number of generators has at least three elements. For example, the case where the holonomy group is a 2-group or the order of holonomy group is even. By Corollary 3.3.1, the corresponding n-dimensional Bieberbach group can be generated by $2n$ elements. Another case is when the order of holonomy group is odd and divisible by 3. In this case, by Theorem B, the corresponding n-dimensional Bieberbach group can be generated by $n+1$ elements. In order to prove this conjecture fully, we believe further study is needed in the key case where the holonomy group is a 2-group.

Chapter 4

Bieberbach groups of diagonal type and Vasquez invariant

4.1 Bieberbach groups of diagonal type

Let Γ be an n-dimensional crystallographic group of diagonal type. As an immediate consequence of diagonality of the holonomy representation, it follows that the holonomy group of Γ is isomorphic to C_2^k for some $k \geq 1$. By the first Bieberbach Theorem, it fits in the below short exact sequence

$$0 \to \mathbb{Z}^n \xrightarrow{\iota} \Gamma \xrightarrow{p} C_2^k \to 0 \tag{4.1}$$

where C_2^k acts diagonally on $\mathbb{Z}^n$. By the third Bieberbach Theorems, we can assume $\rho(g)$ is a diagonal matrix with all diagonal entries equal to 1 or -1 for all $g \in C_2^k$ by conjugating Γ with suitable element in $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$. We denote $diag(a_1, ..., a_n)$ to be the diagonal matrix where the entries starting in the upper left corner are $a_1, ..., a_n$. Let $\alpha \in H^2(C_2^k, \mathbb{Z}^n)$ be the second cohomology class defining the short exact sequence (4.1). Since C_2^k acts diagonally on $\mathbb{Z}^n$, we can express the C_2^k-module $\mathbb{Z}^n$ as direct sum of n copies of C_2^k-module $\mathbb{Z}$. Hence we have the isomorphism

$$H^2(C_2^k, \mathbb{Z}^n) \cong H^2(C_2^k, M_1 \oplus \cdots \oplus M_n)$$

where $M_j \cong \mathbb{Z}$ for $j = 1, ..., n$. Thus we have $\alpha \cong \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$, where $\alpha_j \in H^2(C_2^k, M_j)$ for $j = 1, ..., n$. An obvious action defines C_2^k-modules $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ and a short exact sequence,

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

By [31, Proposition 2.2], we have $H^2(C_2^k, M_j) \cong H^1(C_2^k, \mathbb{R}/M_j)$ and $\alpha_j \cong \alpha'_j \in H^1(C_2^k, \mathbb{R}/M_j)$ for all $j = 1, ..., n$.

Lemma 4.1.1. Using the same notations as above, Consider $[\alpha'_j] \in H^1(C_2^k, \mathbb{R}/M_j)$ where $j \in \{1, ..., n\}$, we can pick a representative $\beta_j \in [\alpha'_j]$ such that $\beta_j(g) \in \{0, \frac{1}{2}\}$ for all $g \in C_2^k$.

Proof. Fix $j \in \{1, ..., n\}$. We define

$$Der(C_2^k, \mathbb{R}/M_j) = \{f : C_2^k \to \mathbb{R}/M_j \,|\, \forall x, y \in C_2^k,\ f(xy) = x \cdot f(y) + f(y)\} \tag{4.2}$$

and

$$P(C_2^k, \mathbb{R}/M_j) = \{f : C_2^k \to \mathbb{R}/M_j \,|\, \exists m \in \mathbb{R}/\mathbb{Z}, \forall x \in C_2^k, f(x) = x \cdot m - m\} \tag{4.3}$$

By [31, Page 19], we have

$$H^1(C_2^k, \mathbb{R}/M_j) \cong Der(C_2^k, \mathbb{R}/M_j)/P(C_2^k, \mathbb{R}/M_j)$$

We could assume $\alpha'_j(1) = 0$. First, we assume C_2^k acts trivially on $\mathbb{R}/M_j$. It follows that $P(C_2^k, \mathbb{R}/M_j)$ is trivial. For any $g \in C_2^k$, by (4.2), we have

$$0 = \alpha'_j(1) = \alpha'_j(gg) = g \cdot \alpha'_j(g) + \alpha'_j(g) = 2\alpha'_j(g) \tag{4.4}$$

It follows that $\alpha'_j(g) \in \{0, \frac{1}{2}\}$ for all $g \in C_2^k$. Next, we assume C_2^k acts non trivially on $\mathbb{R}/M_j$. Let $g_1, ..., g_k$ be generators of C_2^k and assume without loss of generality that g_1 acts non-trivially on $\mathbb{R}/M_j$ and g_i acts trivially on $\mathbb{R}/M_j$ for all $i = 2, ..., n$. Define $\beta'_j \in Der(C_2^k, \mathbb{R}/M_j)$ such that $\beta'_j(g_1) = 0$ and $\beta'_j(g_i) = \alpha'_j(g_i)$ for all $i \in \{2, ..., n\}$. For all $g \in C_2^k$ that acts trivially on $\mathbb{R}/M_j$, by definition of $Der(C_2^k, \mathbb{R}/M_j)$, we have

$$0 = \beta'_j(1) = \beta'_j(gg) = g \cdot \beta'_j(g) + \beta'_j(g) = 2\beta'_j(g)$$

and

$$\beta'_j(g_1 g) = g_1 \cdot \beta'_j(g) + \beta'_j(g_1) = -\beta'_j(g)$$

It follows that $\beta'_j(g) \in \{0, \frac{1}{2}\}$ for all $g \in C_2^k$. We remain to show that β'_j and α'_j are in the same cohomology class. For all $g \in \langle g_2, ..., g_k \rangle$, we have

$$\beta'_j(g) = \alpha'_j(g)$$

and

$$\begin{aligned}
\beta'_j(g_1 g) - \alpha'_j(g_1 g) &= g_1 \cdot \beta'_j(g) + \beta'_j(g_1) - g_1 \cdot \alpha'_j(g) - \alpha'_j(g_1) \\
&= -\beta'_j(g) + \beta'_j(g_1) + \alpha'_j(g) - \alpha'_j(g_1) \\
&= -\alpha'_j(g_1)
\end{aligned}$$

Thus we have

$$(\beta'_j - \alpha'_j)(g) = \begin{cases} 0 & \text{if } g \text{ acts trivially on } \mathbb{R}/M_j \\ -\alpha'_j(g_1) & \text{if } g \text{ acts non-trivially on } \mathbb{R}/M_j \end{cases}$$

Hence we have $(\beta'_j - \alpha'_j)(g) = g \cdot (\frac{\alpha'_j(g_1)}{2}) - \frac{\alpha'_j(g_1)}{2}$ for all $g \in C_2^k$. It follows that $\beta'_j - \alpha'_j \in P(C_2^k, \mathbb{R}/M_j)$. We can conclude that we can always pick a representative $\beta'_j \in [\alpha'_j]$ such that $\beta'_j(g) \in \{[0], [\frac{1}{2}]\}$ for all $g \in C_2^k$. $\square$

By the above lemma, we could assume $\alpha'_j(g) \in \{0, \frac{1}{2}\}$ for all $j \in \{1, ..., n\}$ and for all $g \in C_2^k$. It follows that α'_j has order 2 if it is not a trivial class.

Let $p : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the natural homomorphism. We say $s : C_2^k \to \mathbb{R}^n$ is a *vector system* for Γ if $ps \in \oplus_{1 \le j \le n} \alpha'_j$. By [23, Section 3], we get an isomorphism

$$\Gamma \cong \left\{ \begin{pmatrix} \rho(g) & s(g)+z \\ 0 & 1 \end{pmatrix} \middle| \, g \in C_2^k, \, z \in \mathbb{Z}^n \right\}$$

and

$$\Gamma \cong \left\langle \begin{pmatrix} \rho(g) & s(g) \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} I_n & e_i \\ 0 & 1 \end{pmatrix} \middle| \, g \in C_2^k, \, i \in \{1, ..., n\} \right\rangle \tag{4.5}$$

where I_n is the n-dimensional identity matrix and e_i is the i^{th} column of I_n. For convenience, we may express the matrix $\begin{pmatrix} \rho(g) & s(g)+z \\ 0 & 1 \end{pmatrix}$ in the form $(\rho(g), s(g)+z)$.

By Lemma 4.1.1, we could take $\alpha'_1 \oplus \cdots \oplus \alpha'_n : C_2^k \to \{0, \frac{1}{2}\}^n$ be a vector system. Thus for an arbitrary element $\gamma \in \Gamma$, we can express it as $\gamma = (diag(a_a, ..., a_n), (x_1, ..., x_n))$ where $a_1, ..., a_n \in \{-1, 1\}$ and $x_1, ..., x_n \in \frac{1}{2} + \mathbb{Z}$. Besides, the set $\{\iota(e_1), ..., \iota(e_n), \gamma_1, ..., \gamma_k\}$ is a generating set of Γ where $\gamma_i = (\rho(g_i), \alpha'(g_i))$. We called $\{\gamma_1, ..., \gamma_k\}$ to be a set of *non-lattice generators* of Γ.

4.2 Characteristic matrix for crystallographic group

In this section, for each n-dimensional crystallographic groups of diagonal type where its holonomy group is isomorphic to C_2^k, we define a $((2^k - 1) \times n)$-matrix which gives a combinatorial description of the crystallographic group of diagonal type.

Let S^1 be the unit circle in $\mathbb{C}$. We consider the elements $g_i \in Aut(S^1)$ given by

$$g_0(z) = z, \;\; g_1(z) = -z, \;\; g_2(z) = \bar{z}, \;\; g_3(z) = -\bar{z}$$

for all $z \in S^1$.

Equivalently, we can identity S^1 with $\mathbb{R}/\mathbb{Z}$. For any $[t] \in \mathbb{R}/\mathbb{Z}$, we have

$$g_0([t]) = [t]\,, \;\; g_1([t]) = \left[t + \frac{1}{2}\right], \;\; g_2([t]) = [-t]\,, \;\; g_3([t]) = \left[-t + \frac{1}{2}\right]$$

Let $\mathcal{D} = \langle g_i \,|\, i = 0, 1, 2, 3\rangle$. It is easy to see that

$$g_3 = g_1 g_2, \;\; g_i^2 = g_0 \text{ and } g_i g_0 = g_0 g_i = g_i \tag{4.6}$$

for $i = 1, 2, 3$. Notice that $\mathcal{D}$ is isomorphic to the Klein four-group. We define an action of $\mathcal{D}^n$ on T^n by

$$(t_1, ..., t_n)(z_1, ..., z_n) = (t_1 z_1, ..., t_n z_n)$$

for $(t_1, ..., t_n) \in \mathcal{D}^n$ and $(z_1, ..., z_n) \in T^n = S^1 \times \cdots \times S^1$. Any subgroup $\mathbb{Z}_2 \subseteq \mathcal{D}^n$ defines a $(1 \times n)$-row matrix with entries in $\mathcal{D}$, which in turn defines a row matrix entries in the set $\{0, 1, 2, 3\}$ under the identification $i \leftrightarrow g_i$ for $0 \le i \le 3$.

Let Γ be an n-dimensional crystallographic group and let $\alpha \in H^2(C_2^k, \mathbb{Z}^n)$ be the cohomology class corresponds to Γ. As mentioned at Section 4.1, we have

$$\alpha \cong \alpha' \cong \alpha'_1 \oplus \alpha'_2 \oplus \cdots \oplus \alpha'_n$$

where $\alpha' \in H^1(C_2^k, \mathbb{R}/M_1 \oplus \cdots \oplus \mathbb{R}/M_n)$ and $\alpha'_j \in H^2(C_2^k, \mathbb{R}/M_j)$ for $j = 1, ..., n$. Let $g \in C_2^k$ be a non identity element and $\rho : C_2^k \to \mathbb{Z}^n$ be the holonomy representation of Γ. We have $\rho(g) = diag(X_1, ..., X_n)$ and $\alpha'(g) = (\alpha'_1(g), ..., \alpha'_n(g))^T = (x_1, ..., x_n)^T$ where $X_j \in \{1, -1\}$ and $x_j \in \left\{0, \frac{1}{2}\right\}$ for $j = 1, ..., n$. The corresponding element of $\mathcal{D}^n$ is an n-tuple $(t_1, ..., t_n) \in \mathcal{D}^n$ defined by

$$t_j([t]) = [X_j t + x_j]$$

where $t \in \mathbb{R}$ and $j \in \{1, ..., n\}$. We define $A_\Gamma(g, M_j) = t_j \in \{0, 1, 2, 3\}$ where $j \in \{1, ..., n\}$ under the identification $i \leftrightarrow g_i$ for $0 \leq i \leq 3$. In other words, we have

$$A_\Gamma(g, M_j) = \begin{cases} 0 & \text{if } X_j = 1 \text{ and } x_j = 0 \\ 1 & \text{if } X_j = 1 \text{ and } x_j = \frac{1}{2} \\ 2 & \text{if } X_j = -1 \text{ and } x_j = 0 \\ 3 & \text{if } X_j = -1 \text{ and } x_j = \frac{1}{2} \end{cases}$$

Fix $h_1, ..., h_{2^k - 1}$ be all non identity elements of C_2^k. We define a $((2^k - 1) \times n)$ matrix A_Γ as $(A_\Gamma)_{i,j} = A_\Gamma(h_i, M_j)$. We called the matrix A_Γ to be a *characteristic matrix* of Γ. Note that given a crystallographic group Γ, the matrix A_Γ is not unique since we could re-index the holonomy group elements h_i's and the module M_i's.

Let $r_1 = (a_1\, a_2 \cdots a_n)$ and $r_2 = (b_1\, b_2 \cdots b_n)$ be rows of A_Γ. We denote $\star$ to be the group multiplication in the Klein four-group corresponds to (4.6). In other words, we have $a_1 \star b_1 = c_1$ if and only if $g_{a_1} g_{b_1} = g_{c_1}$ and $r_1 \star r_2 = (a_1 \star b_1 \ \cdots \ a_n \star b_n)$.

Lemma 4.2.1. Using the same notations as above and assume $h_{s_1} = h_{s_2} h_{s_3}$ for some $s_1, s_2, s_3 \in \{1, ..., 2^k - 1\}$. Then the ($s_1^{th}$ row of A_Γ) = (s_2^{th} row of A_Γ) $\star$ (s_3^{th} rows of A_Γ.)

Proof. Let $\rho(h_{s_2}) = diag(X_1, ..., X_n)$, $\alpha'(h_{s_2}) = (x_1, ..., x_n)$, $\rho(h_{s_3}) = diag(Y_1, ..., Y_n)$ and $\alpha'(h_{s_3}) = (y_1, ..., y_n)$. we have

$$\begin{aligned} &(diag(X_1, ..., X_n), (x_1, ..., x_n))(diag(Y_1, ..., Y_n)(y_1, ..., y_n)) \\ =&(diag(X_1Y_1, ..., X_nY_n), (X_1y_1 + x_1, ..., X_ny_n + x_n)) \end{aligned}$$

Thus the corresponding element of $A_\Gamma(h_{s_1}, M_j)$ for any $j \in \{1, ..., n\}$ is given by

$$g[t] = [X_jY_jt + X_jy_j + x_j] = [X_j(Y_jt + y_j) + x_j]$$

where $g \in \mathcal{D}$ and $t \in \mathbb{R}$. Therefore, we have

$$A_\Gamma(h_{s_1}, M_j) = A_\Gamma(h_{s_2}, M_j) \star A_\Gamma(h_{s_3}, M_j)$$

It follows that (s_1^{th} row of A_Γ) = (s_2^{th} row of Γ) $\star$ (s_3^{th} rows of Γ). □

Remark 4.2.2. Using the same notations as above and assume the holonomy group of Γ which is C_2^k is generated by elements $h_1, ..., h_k$. If we know the value of $A_\Gamma(h_i, M_j)$ for all $i \in \{1, ..., k\}$ and for all $j \in \{1, ..., n\}$, then by Lemma 4.2.1, we can workout the matrix A_Γ. Define a $(k \times n)$-matrix A such that $A_{i,j} = A_\Gamma(h_i, M_j)$ where $1 \leq i \leq k$ and $1 \leq j \leq n$. The matrix A is the same as the matrix constructed in [24, Section 2].

Next, we would like to reverse the above construction. Given a characteristic matrix, we are going to define a crystallographic group.

Let A_Γ be a $(2^k - 1) \times n$ dimensional characteristic matrix such that $(A_\Gamma)_{ij} = A_\Gamma(g_i, M_j)$ where $g_1, ..., g_{2^k-1}$ are all non identity element of C_2^k and $M_1, ..., M_n \cong \mathbb{Z}$. Without loss of generality, we assume $g_1, ..., g_k$ are the generators of C_2^k. First, we need to define a representation $\rho : C_2^k \to GL_n(\mathbb{Z})$. For any $1 \leq i \leq k$, we define $\rho(g_i) = diag(X_1, ..., X_n)$ where

$$X_j = \begin{cases} 1 & \text{if } A_\Gamma(g_i, M_j) \in \{0, 1\} \\ -1 & \text{if } A_\Gamma(g_i, M_j) \in \{2, 3\} \end{cases}$$

for all $1 \leq j \leq n$. Next, we are going to define a cohomology class $\alpha' \in H^1(C_2^k, \mathbb{R}/M_1 \oplus \cdots \oplus \mathbb{R}/M_n)$ where the C_2^k-module structure of $\mathbb{R}/M_1 \oplus \cdots \oplus \mathbb{R}/M_n$ is given by ρ. For any $1 \leq i \leq k$, we define $\alpha'(g_i) = (s_1, ..., s_n)$ where

$$s_j = \begin{cases} 0 & \text{if } A_\Gamma(g_i, M_j) \in \{0, 2\} \\ \frac{1}{2} & \text{if } A_\Gamma(g_i, M_j) \in \{1, 3\} \end{cases}$$

for all $1 \leq j \leq n$. Since we have a cohomology class α', we could define a n-dimensional crystallographic group Γ. By Section 4.1, Γ is generated by $\{\iota(e_1), ..., \iota(e_n), \gamma_1, ..., \gamma_k\}$ where

$$\iota(e_j) = \begin{pmatrix} I_n & e_j \\ 0 & 1 \end{pmatrix}$$

and e_j is the j^{th} column of the n-dimensional identity matrix for $1 \leq j \leq n$ and

$$\gamma_i = \begin{pmatrix} \rho(g_i) & \alpha'(g_i) \\ 0 & 1 \end{pmatrix}$$

for $1 \leq i \leq k$. By the construction, we can see that the characteristic matrix of Γ equals to A_Γ. Notice that the holonomy group of Γ is not necessary isomorphic to C_2^k because the representation $\rho : C_2^k \to GL_n(\mathbb{Z})$ is not necessary faithful.

If two characteristic matrices define isomorphic crystallographic groups, we will say that they are *equivalent*. In particular, matrix obtained from swapping rows or columns of A_Γ equivalents to A_Γ.

Example 4.2.3. Let Γ be the Bieberbach group enumerated in CARAT as "min.19.1.1.7". Let

$$\gamma_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

be non-lattice generators of Γ. The holonomy group of Γ is C_2^2 which is generated by $h_1 = p(\gamma_1)$ and $h_2 = p(\gamma_2)$ where $p : \Gamma \to C_2^2$ be the surjection map defined at (4.1). Using the same notations as Remark 4.2.2, We have

$$A = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$$

We get the third row of A_Γ by using Lemma 4.2.1. We have

$$\begin{pmatrix} 2 & 2 & 1 & 3 \end{pmatrix} \star \begin{pmatrix} 1 & 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 & 1 \end{pmatrix}$$

Thus

$$A_\Gamma = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 1 & 0 & 2 & 2 \\ 3 & 2 & 3 & 1 \end{pmatrix}$$

By simple calculation, we have

$$\gamma_1\gamma_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & \frac{1}{2} - 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore we check that the third row of A_Γ is indeed equals to (3 2 3 1).

Example 4.2.4. Let

$$A_{\Gamma'} = \begin{pmatrix} 2 & 1 & 2 & 3 \\ 1 & 2 & 0 & 2 \\ 3 & 3 & 2 & 1 \end{pmatrix}$$

be a characteristic matrix such that $(A_{\Gamma'})_{ij} = A_\Gamma(g_i, M_j)$ where g_1, g_2, g_3 are non identity elements of C_2^2 and $M_1, M_2, M_3, M_4 \cong \mathbb{Z}$. Given such a matrix, we are going to define a 4-dimensional crystallographic group. Notice that (3^{rd} row of $A_{\Gamma'}$) = (1^{st} row of $A_{\Gamma'}$) $\star$ (2^{nd} rows of $A_{\Gamma'}$). Thus g_1, g_2 are the generators of C_2^2. First, we define a representation $\rho : C_2^2 \to GL_4(\mathbb{Z})$ where

$$\rho(g_1) = diag(-1, 1, -1, -1) \quad \text{and} \quad \rho(g_2) = diag(1, -1, 1, -1)$$

Next, we define a cohomology class $\alpha' \in H^1(C_2^2, \mathbb{R}^4/\mathbb{Z}^4)$ where

$$\alpha'(g_1) = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right) \quad \text{and} \quad \alpha'(g_2) = \left(\frac{1}{2}, 0, 0, 0\right)$$

Thus the characteristic matrix $A_{\Gamma'}$ defines a 4-dimensional crystallographic group Γ' where its non lattice generators are

$$\gamma_1' = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2' = \begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Compare the group Γ' with Γ defined in Example 4.2.3. Observe that we have

$$\gamma_i' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \gamma_i \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for $i = 1, 2$ where γ_i are the elements defined in Example 4.2.3. By the third Bieberbach theorem, we have $\Gamma \cong \Gamma'$. Thus A_Γ and $A_{\Gamma'}$ define isomorphic crystallographic group and therefore $A_{\Gamma'}$ is equivalent to A_Γ. This result is not surprising because we observe that $A_{\Gamma'}$ can be obtained by swapping the 2^{nd} and the 3^{rd} column of A_Γ.

Next, we are going to derive some properties of the characteristic matrix A_Γ.

Lemma 4.2.5. Using the same notations as above. $A_\Gamma(h, M_j) = 1$ if and only if $0 \neq res_{\langle h \rangle}\alpha_j' \in H^1(C_2, \mathbb{R}/M_j)$.

Proof. Let Γ' be the group corresponding to $res_{\langle h \rangle}\alpha_i$. Notice that Γ' can be expressed as

$$0 \to M_i \cong \mathbb{Z} \to \Gamma' \xrightarrow{p} \langle g \rangle \cong C_2$$

Therefore $res_{\langle h \rangle}\alpha_i \neq [0]$ if and only if $\Gamma' \cong \mathbb{Z}$. Recall that $\rho : C_2^k \to GL(\mathbb{Z}^n)$ is the holonomy representation of Γ and $\alpha_1' \oplus \cdots \oplus \alpha_n' : C_2^k \to \{0, \frac{1}{2}\}^n$ is the vector system of Γ. Let $\rho(h) = diag(X_1, ..., X_n)$. By (4.5), $\Gamma' \cong \mathbb{Z}$ if and only if

$$\Gamma' \cong \left\langle \begin{pmatrix} X_j & \alpha_j'(h) \\ 0 & 1 \end{pmatrix} \right\rangle$$

Thus $\Gamma' \cong \mathbb{Z}$ if and only if $(X_j, \alpha_j'(h)) = (1, \frac{1}{2})$. Hence, we conclude that $\Gamma' \cong \mathbb{Z}$ if and only if $A_\Gamma(h, M_j) = 1$. □

Lemma 4.2.6. Let A_Γ be a $(2^k - 1) \times n$ dimensional characteristic matrix which defines an n-dimensional crystallographic group Γ.

(i) Γ has torsion element if and only if A_Γ exists a row where all its entries are not equals to 1.

(ii) The holonomy group of Γ is not C_2^k if and only if A_Γ exists a row where all its entries are equals to 0 or 1 only.

Proof. By construction, the characteristic matrix A_Γ defines a cohomology class $\alpha = \bigoplus_{1\le j\le n} \alpha_j \in \bigoplus_{1\le j\le n} H^1(C_2^k, \mathbb{R}/M_j)$ where $M_j \cong \mathbb{Z}$ for all $j = 1, ..., n$. By [31, Theorem 3.1], Γ has torsion element if and only if there exists $g \in C_2^k$ such that

$$res_{\langle g\rangle}\alpha = res_{\langle g\rangle}\alpha_1 \oplus \cdots \oplus res_{\langle g\rangle}\alpha_n = 0$$

Hence Γ has torsion element if and only if $res_{\langle g\rangle}\alpha_i = 0$ for all $i = 1, ..., n$. By Lemma 4.2.5, we can conclude that Γ has torsion element if and only if $A_\Gamma(g, M_i) \neq 1$ for all $i = 1, ..., n$, which complete the prove of part (i).

Next, the holonomy group of Γ is not C_2^k if and only if there exists $g \in C_2^k$ such that g acts trivially on $\mathbb{R}^n/\mathbb{Z}^n$. By construction, g acts trivially on $\mathbb{R}^n/\mathbb{Z}^n$ if and only if $A_\Gamma(g, M_i) \in \{0, 1\}$ for all $i = 1, ..., n$. Hence we complete the prove of part (ii). □

From the above Lemma, we have

Corollary 4.2.7. Let Γ be an n-dimensional Bieberbach group of diagonal type where its holonomy group isomorphic to C_2^k. The $(2^k - 1) \times n$ dimensional characteristic matrix A_Γ satisfies the below two properties,

(i) For every row of A_Γ, there exists an entry equal to 1.

(ii) For every row of A_Γ, there exists an entry equal to either 2 or 3.

Proof. It follows immediately from Lemma 4.2.6. □

Definition 4.2.8. Let $\alpha \in H^2(C_2^k, \mathbb{Z})$. We define $\mathcal{R}(\alpha) = \{g \in C_2^k \mid res_{\langle g\rangle}\alpha \neq 0\}$.

Remark 4.2.9. Let Γ be an n-dimensional crystallographic group and its holonomy group isomorphic to C_2^k. Let $\alpha \in H^2(C_2^k, \mathbb{Z}^n)$ be the cohomology class corresponds to Γ. Notice that we have $\alpha \cong \alpha_1 \oplus \cdots \oplus \alpha_n$ where $\alpha_j \in H^2(C_2^k, M_j)$ where $j \in \{1, ..., n\}$ and $M_j \cong \mathbb{Z}$. For any $j \in \{1, ..., n\}$, by Lemma 4.2.5, we have $A_\Gamma(g, M_j) = 1$ if and only if $g \in \mathcal{R}(\alpha_j)$.

Proposition 4.2.10. Let $0 \neq \alpha \in H^2(C_2^k, \mathbb{Z})$ where C_2^k acts trivially on $\mathbb{Z}$. Then we have $|\mathcal{R}(\alpha)| = 2^{k-1}$.

Proof. By Lemma 4.1.1, we have $\alpha \cong \alpha' \in H^1(C_2^k, \mathbb{R}/\mathbb{Z})$ and we can assume $\alpha'(g) \in \{0, \frac{1}{2}\}$. Thus

$$|\mathcal{R}(\alpha)| = \left|\left\{g \in C_2^k \,|\, \alpha'(g) = \frac{1}{2}\right\}\right|$$

Since $\alpha \neq 0$, there exists $g \in C_2^k$ such that $\alpha'(g) = \frac{1}{2}$. Let $C_2^k = C_2^{k-1} \sqcup gC_2^{k-1}$ where $C_2^{k-1} \le C_2^k$. For any $h \in C_2^{k-1}$, we have

$$\alpha'(gh) = \alpha'(g) + \alpha'(h) = \begin{cases} 0 & \text{if } \alpha'(h) = \frac{1}{2} \\ \frac{1}{2} & \text{if } \alpha'(h) = 0 \end{cases}$$

Thus $|\mathcal{R}| = |C_2^{k-1}| = 2^{k-1}$ □

Proposition 4.2.11. Let $0 \neq \beta \in H^2(C_2^k, \mathbb{Z})$ where C_2^k acts non-trivially on $\mathbb{Z}$ via $\rho : C_2^k \to GL(\mathbb{Z})$. Then $|\mathcal{R}(\beta)| = 2^{k-2}$.

Proof. Define $\alpha = res_{ker(\rho)}\beta \in H^2(ker(\rho), \mathbb{Z})$. Since $\beta \neq 0$ and $H^2(\langle g \rangle, \mathbb{Z}) = 0$ for all $g \notin ker(\rho)$, it follows that $\alpha \neq 0$ and

$$|\mathcal{R}(\beta)| = |\{g \in C_2^k \mid res_{\langle g \rangle}\beta \neq 0\}| = |\{h \in ker(\rho) \cong C_2^{k-1} \,|\, res_{\langle h \rangle}\alpha \neq 0\}| = |\mathcal{R}(\alpha)|$$

By Proposition 4.2.10, we have $|\mathcal{R}(\beta)| = 2^{k-2}$. □

Remark 4.2.12. Let Γ be an n-dimensional Bieberbach group of diagonal type where its holonomy group isomorphic to C_2^k. An arbitrary column of A_Γ corresponds to a cohomology class $\alpha \in H^2(C_2^k, \mathbb{Z})$. By Proposition 4.2.10 and Proposition 4.2.11, we have $|\mathcal{R}(\alpha)| = 2^{k-2}$ or 2^{k-1}. By Lemma 4.2.5, we can conclude that in every column of A_Γ, there exists at least 2^{k-2} entries are equal to 1.

Proposition 4.2.13. Let $\alpha \in H^2(C_2^k, \mathbb{Z})$ where C_2^k acts non-trivially on $\mathbb{Z}$ via $\rho : C_2^k \to GL(\mathbb{Z})$. If $\mathcal{T} \subseteq \mathcal{R}(\alpha)$ with $|\mathcal{T}| \geq 2^{k-3} + 1$, then $\langle \mathcal{T} \rangle \cong ker(\rho)$.

Proof. Since $\mathcal{T} \subseteq \mathcal{R}(\alpha) \subseteq ker(\rho)$, we have $\langle \mathcal{T} \rangle \leq ker(\rho)$. We assume by contradiction that $\langle \mathcal{T} \rangle \lneqq ker(\rho)$. Since $|\mathcal{T}| \geq 2^{k-3} + 1$, we have $\langle \mathcal{T} \rangle \cong C_2^{k-2}$. Consider $\alpha' = res_{\langle \mathcal{T} \rangle}\alpha \in H^2(C_2^{k-2}, \mathbb{Z})$. Recall that $\mathcal{R}(\alpha') = \{h \in \langle \mathcal{T} \rangle \cong C_2^{k-2} | res_{\langle h \rangle}\alpha' \neq 0\}$. By Proposition 4.2.10, we have $|\mathcal{R}(\alpha')| = 2^{k-3}$. Since $\mathcal{T} \subseteq \mathcal{R}(\alpha')$, we have

$$2^{k-3} + 1 \leq |\mathcal{T}| \leq |\mathcal{R}(\alpha')| = 2^{k-3}$$

which is a contradiction. □

Corollary 4.2.14. Let Γ be an n-dimensional Bieberbach group of diagonal type with its holonomy group is isomorphic to C_2^k. Let

$$\alpha_1 \oplus \cdots \oplus \alpha_n \in H^2(C_2^k, M_1) \oplus \cdots \oplus H^2(C_2^k, M_n)$$

be the cohomology class corresponding to standard extension of Γ where $M_z \cong \mathbb{Z}$ for $z = 1, ..., n$. Let $\rho_z : C_2^k \to GL(M_z)$ be the representations given by the C_2^k-action on M_z and let $\mathcal{R}(\alpha_z) = \{g \in C_2^k \,|\, res_{\langle g \rangle}\alpha_z \neq [0]\}$ for all $z = 1, ..., n$. If there exists $i, j \in \{1, ..., n\}$ such that ρ_i and ρ_j are non-trivial representations and there exists a subset $\mathcal{T} \subseteq \mathcal{R}(\alpha_i) \cap \mathcal{R}(\alpha_j)$ such that $|\mathcal{T}| \geq 2^{k-3} + 1$, then $\mathcal{R}(\alpha_i) = \mathcal{R}(\alpha_j)$.

Proof. Let $i, j \in \{1, ..., n\}$ such that ρ_i and ρ_j are non-trivial representation and there exists a subset $\mathcal{T} \subseteq \mathcal{R}(\alpha_i) \cap \mathcal{R}(\alpha_j)$ such that $|\mathcal{T}| \geq 2^{k-3} + 1$. By Proposition 4.2.13, we have $ker(\rho_i) \cong \langle \mathcal{T} \rangle \cong ker(\rho_j)$. Since $\mathcal{R}(\alpha_i) \subseteq ker(\rho_i) \cong \langle T \rangle$ and $\mathcal{R}(\alpha_j) \subseteq ker(\rho_j) \cong \langle \mathcal{T} \rangle$, every element inside $\mathcal{R}(\alpha_i) \cup \mathcal{R}(\alpha_j)$ can be expressed as a combination of elements of $\mathcal{T}$. Let $x = t_1 \cdots t_s \in \mathcal{R}(\alpha_i)$ where $t_1, ..., t_s \in \mathcal{T}$. We have $A_\Gamma(x, M_i) = \bigstar_{z=1}^{s} A_\Gamma(t_z, M_i)$ and

$A_\Gamma(x, M_j) = \bigstar_{z=1}^{s} A_\Gamma(t_z, M_j)$. Since $\mathcal{T} \subseteq \mathcal{R}(\alpha_i) \cap \mathcal{R}(\alpha_j)$, we have $A_\Gamma(t, M_i) = A_\Gamma(t, M_j) = 1$ for any $t \in \mathcal{T}$. Hence we have

$$A_\Gamma(x, M_i) = \bigstar_{z=1}^{s} A_\Gamma(t_z, M_i) = \bigstar_{z=1}^{s} A_\Gamma(t_z, M_j) = A_\Gamma(x, M_j)$$

By Remark 4.2.9, we have $x \in \mathcal{R}(\alpha_i)$ if and only if $x \in \mathcal{R}(\alpha_j)$. Hence $\mathcal{R}(\alpha_i) = \mathcal{R}(\alpha_j)$. □

4.3 Vasquez invariant of diagonal type

By Section 2.5 and [7, Chapter II], we know that the fundamental group of compact flat Riemannian manifold is a Bieberbach group. Vasquez invariant allows one to determine whether given flat Riemannian manifold fibers over a lower dimensional flat Riemannian manifold with fibers flat tori [34].

Let M be a closed flat Riemannian manifold with the fundamental group $\pi_1(M) = \Gamma$. Let $T^k = \mathbb{R}^n/\mathbb{Z}^n$ be a flat torus where Γ acts on it by isometries. Then Γ also acts on the space $\widetilde{M} \times T^k$ by isometries, where $\widetilde{M}$ is the universal cover of M. It is easy to show that the space $(\widetilde{M} \times T^k)/\Gamma$ is a flat manifolds (see [34, Section 2]). $(\widetilde{M} \times T^k)/\Gamma$ is called the *flat toral extension* of the manifolds M. We shall make the convention that a point is the 0-dimensional torus, and hence any flat manifold can be a flat toral extension of itself.

We first give the definition of Vasquez invariant introduced by A. T. Vasquez in [34].

Theorem 4.3.1. *[34, Theorem 3.6]* For any finite group G, there exists a natural number $x \in \mathbb{N}$ with the property that if Γ is a Bieberbach group where its holonomy group is isomorphic to G, then the lattice subgroup $L \subseteq \Gamma$ contains a normal subgroup N such that Γ/N is a Bieberbach group of dimension at most x.

Definition 4.3.2. Let G be a finite group and $x \in \mathbb{N}$. We say x has property $\mathcal{S}(G)$ if for every Bieberbach group Γ where its holonomy group is isomorphic to G, then its lattice subgroup $L \subseteq \Gamma$ contains a normal subgroup N such that Γ/N is a Bieberbach group of dimension at most x. We define

$$n(G) = \min\{x \in \mathbb{N} \,|\, x \text{ has property } \mathcal{S}(G)\}$$

The number $n(G)$ is called the *Vasquez invariant* or *Vasquez number* of the finite group G.

We can reformulate the statement of Theorem 4.3.1 geometrically.

Theorem 4.3.3. *[34, Theorem 4.1]* For any finite group G, there exists a natural number $x \in \mathbb{N}$ with the property that if M is any compact flat Riemannian manifolds with holonomy group G, then M is a flat toral extension of some compact flat Riemannian manifolds of dimension at most x.

Definition 4.3.4. Let $n \geq 1$ be a natural number. A $\mathbb{Z}G$*-lattice* is any G-module isomorphic to a free abelian group $\mathbb{Z}^n$. Let M be a $\mathbb{Z}G$-lattice where $\{e_1, ..., e_n\}$ is the generating set of M. We say M is a diagonal $\mathbb{Z}G$-lattice if $g \cdot e_i = \pm e_i$ for all $g \in G$ and $i \in \{1, ..., n\}$.

By [32, Theorem 3], there is another way to define Vasquez invariant of finite groups.

Definition 4.3.5. Let G be a finite group and let L be $\mathbb{Z}G$-lattice. An element $\alpha \in H^2(G, L)$ is said to be *special* if its extension defines a Bieberbach group. $\mathbb{Z}G$-lattice L has *property* $\mathbb{S}$ if for any $\mathbb{Z}G$-lattice M and any special element $\alpha \in H^2(G, M)$, there exists a G-homomorphism $f : M \to L$ such that $f_* : H^2(G, M) \to H^2(G, L)$ sends α to another special element $f_*(\alpha) \in H^2(G, L)$. The Vasquez invariant of a finite group G is then

$$n(G) = \min\{rank_{\mathbb{Z}}(L) \mid L \text{ is } \mathbb{Z}G\text{-lattice with property } \mathbb{S}\}.$$

Let G be a finite group. By [10, Theorem 1], we have $n(G) \leq \sum_{C \in \mathcal{X}} |G : C|$ where $\mathcal{X}$ is the set of conjugacy classes of G of prime order. In particular, by [10, Theorem 2], we have $n(G) = \sum_{C \in \mathcal{X}} |G : C|$ if G is a p-group.

Remark 4.3.6. We get Theorem 1.0.3 by adapting Theorem 4.3.1 to the special case of Bieberbach groups of diagonal type.

Remark 4.3.7. It is clear that $1 \leq n_d(G) \leq n(G)$. Hence by [10, Theorem 2], we have $n_d(C_2) = 1$.

Lemma 4.3.8. Let Γ be a n-dimensional Bieberbach group of diagonal type where its holonomy group is isomorphic to G and let $\alpha \in H^2(G, \mathbb{Z}^n)$ be the corresponding cohomology class. Let $f : \mathbb{Z}^n \to M$ be a G-homomorphism such that $f_*(\alpha)$ is special. Then $f_*(\alpha)$ defines a Bieberbach group of diagonal type.

Proof. Since Γ is a Bieberbach group of diagonal type, let $\{e_1, ..., e_n\}$ be a basis of $\mathbb{Z}^n$ such that $g \cdot e_i = \pm e_i$ for all $g \in G$ and $i = 1, ..., n$. It follows that M is generated by $f(e_1), ..., f(e_n)$. The holonomy representation of the Bieberbach group defined by $f_*(\alpha)$ is given by the G-action on M. Since f is a module homomorphism, for all $g \in G$ and for all $i \in \{1, ..., n\}$ we have

$$g \cdot f(e_i) = f(g \cdot e_i) = f(\pm e_i) = \pm f(e_i)$$

Hence $f_*(\alpha)$ defines a Bieberbach group of diagonal type. $\square$

By Lemma 4.3.8 and Theorem 4.3.1, we can reformulate Theorem 1.0.3 as follows.

Theorem 4.3.9. For any elementary abelian 2-group G, there exists a natural number $x \in \mathbb{N}$ with the property that if Γ is a Bieberbach group of diagonal type where its holonomy group is isomorphic to G, then the lattice subgroup $L \subseteq \Gamma$ contains a normal subgroup N such that Γ/N is a Bieberbach group of diagonal type with dimension at most x.

Remark 4.3.10. Let G be an elementary abelian 2-group and let L be a diagonal faithful $\mathbb{Z}G$-lattice (G acts faithfully on L). Let $\alpha \in H^2(G, L)$ be cohomology class defines Γ, a Bieberbach group of diagonal type where its holonomy group is isomorphic to G. By Theorem 4.3.9, there exists a normal subgroup $N \trianglelefteq L$ such that Γ/N is a Bieberbach group of diagonal type with dimensional at most $n_d(G)$. In other words, we can define a G-homomorphism $f : L \to L/N$ such that $f_* : H^2(G, L) \to H^2(G, L/N)$ sends α to another special element $f_*(\alpha)$ which defines a Bieberbach group of diagonal type with dimension at most $n_d(G)$. Besides, L/N is a diagonal $\mathbb{Z}G$-lattice of rank at most $n_d(G)$.

Definition 4.3.11. Let G be an elementary abelian 2-group and let L be a diagonal $\mathbb{Z}G$-lattice. An element $\alpha \in H^2(G, L)$ is said to be a *diagonal special* element if its extension defines a Bieberbach group of diagonal type. We say a diagonal $\mathbb{Z}G$-lattice L has *property* $\mathbb{S}_d$ if for any diagonal $\mathbb{Z}G$-lattice M and any diagonal special element $\alpha \in H^2(G, M)$, there exists a G-homomorphism $f : M \to L$ such that $f_* : H^2(G, M) \to H^2(G, L)$ sends α to another diagonal special element $f_*(\alpha)$.

Theorem 4.3.12. Let G be an elementary abelian 2-group. Define

$$n'_d(G) = \min\{rank_{\mathbb{Z}}(L) \mid L \text{ is a diagonal } \mathbb{Z}G\text{-lattice with property } \mathbb{S}_d\}$$

Then we have $n_d(G) = n'_d(G)$.

Proof. By definition, it is clear that $n_d(G) \leq n'_d(G)$. Now we want to prove that $n'_d(G) \leq n_d(G)$. Let L be a diagonal $\mathbb{Z}G$-lattice of minimal rank with property $\mathbb{S}_d$. In other words, $rank_{\mathbb{Z}}(L) = n'_d(G)$. Let M be any diagonal $\mathbb{Z}G$-lattice and $\alpha \in H^2(G, M)$ be any diagonal special element. Since L has property $\mathbb{S}_d$ and by Definition 4.3.11, there exists a G-homomorphism $g : M \to L$ such that $g_*(\alpha) \in H^2(G, L)$ is a diagonal special element.

First, we assume L is a faithful diagonal $\mathbb{Z}G$-lattice (G acts faithfully on L). Since L is faithful, by Remark 4.3.10, there exists a diagonal $\mathbb{Z}G$-lattice K with $rank_{\mathbb{Z}}(K) \leq n_d(G)$ and a G-homomorphism $h : L \to K$ such that $h_*(g_*(\alpha))$ is a special element defining a Bieberbach group of diagonal type. Hence K is a diagonal $\mathbb{Z}G$-lattice with property $\mathbb{S}_d$. It follows that $n'_d(G) \leq rank_{\mathbb{Z}}(K)$. Therefore we have $n'_d(G) \leq rank_{\mathbb{Z}}(K) \leq n_d(G)$.

Now assume L is not a faithful diagonal $\mathbb{Z}G$-lattice. Let P be a faithful diagonal $\mathbb{Z}G$-lattice. Consider the faithful diagonal $\mathbb{Z}G$-lattice $L \oplus P$. We have $g_*(\alpha) \oplus 0 \in H^2(G, L) \oplus H^2(G, P)$ is a diagonal special element. By Remark 4.3.10, there exists a diagonal $\mathbb{Z}G$-lattice N with $rank_{\mathbb{Z}}(N) \leq n_d(G)$ and a G-homomorphism $f : L \oplus P \to N$ such that $f_*(g_*(\alpha) \oplus 0) \in H^2(G, N)$ is special. Since

$$Hom_G(L \oplus P, N) \cong Hom_G(L, N) \oplus Hom_G(P, N),$$

we can let $f = f_1 \oplus f_2$ where $f_1 \in Hom_G(L, N)$ and $f_2 \in Hom_G(P, N)$. Therefore, $f_*(g_*(\alpha) \oplus 0) = (f_1)_*(g_*(\alpha))$. Thus N is a diagonal $\mathbb{Z}G$-lattice with property $\mathbb{S}_d$. It follows that $n'_d(G) \leq rank_{\mathbb{Z}}(N)$. Hence we have $n'_d(G) \leq n_d(G)$. □

4.4 Proofs of Theorems D and E

In this section, given a Bieberbach group Γ of diagonal type, we will analyse the characteristic matrix A_Γ to determine whether there exists a normal subgroup such that the quotient is still a Bieberbach group.

Definition 4.4.1. Let Γ be an n-dimensional Bieberbach group of diagonal type. The characteristic matrix A_Γ is said to be *col-reducible* (by i^{th} column) if after removing a column (i^{th} column) from A_Γ, there still exists an entry equals to 1 in every row. We say A_Γ is *col-irreducible* if it is not col-reducible.

Lemma 4.4.2. Let $f : M_1 \oplus \cdots \oplus M_n \to M_1 \oplus \cdots \oplus M_n$ be a C_2^k-homomorphism where $M_1, \ldots, M_n$ are all C_2^k-lattices of rank one. Let e_i be the generator of M_i and $\rho_i : C_2^k \to GL(M_i)$ be the representation defining the C_2^k-action on M_i for all $1 \le i \le n$. For any $i \in \{1, ..., n\}$, if there exists $t \ge 2$ and $i_1, ..., i_t \in \{1, ..., n\}$ such that $f(e_i) = a_{i_1}e_{i_1} + \cdots a_{i_t}e_{i_t}$ where $a_{i_1}, ..., a_{i_t} \neq 0$, then $ker(\rho_{i_1}) = \cdots = ker(\rho_{i_t})$.

Proof. Let $i \in \{1, ..., n\}$ such that there exists $t \ge 2$ and $i_1, ..., i_t \in \{1, ..., n\}$ such that $f(e_i) = a_{i_1}e_{i_1} + \cdots a_{i_t}e_{i_t}$ where $a_{i_1}, ..., a_{i_t} \neq 0$. For any $g \in C_2^k$ that acts trivially on M_i, since f is a C_2^k-homomorphism, we have

$$\sum_{1 \le z \le t} a_{i_z}(g \cdot e_{i_z}) = g \cdot \sum_{1 \le z \le t} a_{i_z}e_{i_z} = g \cdot f(e_i) = f(g \cdot e_i) = f(e_i) = \sum_{1 \le z \le t} a_{i_z}e_{i_z}$$

Thus $g \cdot e_{i_z} = e_{i_z}$ for all $z \in \{1, ..., t\}$. It follows that $g \in ker(\rho_{i_z})$ for all $z \in \{1, ..., t\}$. For each $h \in C_2^k$ that acts non-trivially on M_i, by similar calculation, we get

$$\sum_{1 \le z \le t} a_{i_z}(h \cdot e_{i_z}) = h \cdot \sum_{1 \le z \le t} a_{i_z}e_{i_z} = h \cdot f(e_i) = f(h \cdot e_i) = f(-e_i) = \sum_{1 \le z \le t} -a_{i_z}e_{i_z}$$

It follows that $h \cdot e_{i_z} = -e_{i_z}$ for all $z \in \{1, ..., t\}$. Therefore $h \notin ker(\rho_{i_z})$ for all $z \in \{1, ...t\}$. Hence we can conclude that $ker(\rho_{i_1}) = \cdots = ker(\rho_{i_t})$. □

Corollary 4.4.3. Let Γ be an n-dimensional Bieberbach group of diagonal type where its holonomy group is isomorphic to C_2^k and $\alpha \in H^2(C_2^k, \oplus_{1 \le i \le n} M_i)$ be the corresponding cohomology class where $M_i \cong \mathbb{Z}$. Let $\rho_i : C_2^k \to GL(M_i)$ be the representation given by the C_2^k-action on M_i for all $1 \le i \le n$. If $ker(\rho_i) \neq ker(\rho_j)$ for all $i \neq j$ and A_Γ is col-irreducible, then there does not exist a C_2^k-homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^s$ where $s < n$ such that $f_*(\alpha)$ is special.

Proof. Assume by contradiction that there exists a C_2^k-homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^s$ where $s < n$ such that $f_*(\alpha)$ is special. Assume $\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n$ where $\alpha_i \in H^2(C_2^k, M_i)$ for $1 \le i \le n$. We have $f_*(\alpha) = f_*(\alpha_1) \oplus \cdots \oplus f_*(\alpha_n)$. Let e_i be the generator of M_i for $1 \le i \le n$. For any $i \in \{1, ..., n\}$, since f is a module homomorphism, there exists $t \ge 1$, $i_1, ..., i_t \in \{1, ..., n\}$ and $a_{i_1}, ..., a_{i_t} \in \mathbb{Z}$ such that $f(e_i) = a_{i_1}e_{i_1} + \cdots + a_{i_t}e_{i_t}$. By Lemma 4.4.2, if $t \ge 2$, we have $ker(\rho_{i_1}) = \cdots = ker(\rho_{i_t})$, which contradicts that the kernels of ρ_i are all distinct for $1 \le i \le n$. Therefore we can assume the homomorphism f has form $f(e_i) = a_{i'}e_{i'}$ for $1 \le i \le n$ where $i' \in \{1, ..., n\}$ and $a_{i'} \in \mathbb{Z}$. By Lemma 4.1.1, we have

$$f_*(\alpha_i) = a_{i'}\alpha_{i'} = \begin{cases} \alpha_{i'} & \text{if } a_{i'} \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

Let $S = \{x \in \{1, ..., n\} \mid a_{x'} \text{ is odd}\} = \{s_1, ..., s_r\}$ for some $r \in \mathbb{Z}$. We have $r \geq 1$, otherwise $f_*(\alpha) = 0$ which is a contradiction. We have $f_*(\alpha) = \alpha_{s'_1} \oplus \cdots \oplus \alpha_{s'_r}$. Let Γ' be the Bieberbach group defined by the cohomology class $f_*(\alpha)$. We can construct the matrix $A_{\Gamma'}$ by defining the i^{th} column of the matrix $A_{\Gamma'}$ equals to the $(s'_i)^{th}$ column of A_Γ. Since Γ' is Bieberbach group, by Lemma 4.2.6, there exists an entry equals to 1 in every row of $A_{\Gamma'}$. Since $r \leq s < n$, there exists $i \in \{1, .., n\}$ such after removing the i^{th} column of A_Γ, there still exists an entry equals to 1 in every row. It contradicts that A_Γ is col-irreducible. □

Remark 4.4.4. Using the same notation as above, consider the characteristic matrix A_Γ. Define a map $\phi : \{0, 1, 2, 3\} \to \{a, b\}$ such that $\phi(0) = \phi(1) = a$ and $\phi(2) = \phi(3) = b$. Define $\psi(A_\Gamma)$ be a $(2^k - 1) \times n$ matrix such that $[\psi(A_\Gamma)]_{ij} = \phi([A_\Gamma]_{ij})$. Observe that $ker(\rho_s) = ker(\rho_t)$ if and only if the s^{th} column of $\psi(A_\Gamma)$ is equal to the j^{th} column of $\psi(A_\Gamma)$.

Proposition 4.4.5. Let Γ be an n-dimensional Bieberbach group with holonomy group isomorphic to C_2^k and let A_Γ be a characteristic matrix . The matrix A_Γ is col-irreducible if and only if A_Γ is equivalent to $\begin{pmatrix} X \\ N \end{pmatrix}$ where X is an $n \times n$ matrix such that all diagonal entries are equal to 1 and other entries are not equal to 1.

Proof. First, we want to prove that $\begin{pmatrix} X \\ N \end{pmatrix}$ is col-irreducible. For any $i \in \{1, ..., n\}$, if we remove the i^{th} column of $\begin{pmatrix} X \\ N \end{pmatrix}$, then the i^{th} row of the of new matrix do not have entries equal to 1. Hence we conclude that $\begin{pmatrix} X \\ N \end{pmatrix}$ is col-irreducible. Now, we assume A_Γ is col-irreducible. For any $i \in \{1, ..., n\}$, we consider the i^{th} column of A_Γ. By definition of col-irreducible, if we remove the i^{th} column of A_Γ, there exists $r_i \in \{1, ..., 2^k - 1\}$ such that the r_i^{th} row of the new matrix do not have entries equal to 1. By Corollary 4.2.7, the r_i^{th} row of A_Γ has at least one entry equal to 1. Therefore we can conclude that $(A_\Gamma)_{r_i,i} = 1$ and $(A_\Gamma)_{r_i,s} \neq 1$ for all $s \neq i$. Notice that we have $r_i \neq r_j$ for any $j \in \{1, ..., n\}$ where $i \neq j$. We define a new matrix A'_Γ as follow. We define the i^{th} row of A'_Γ to be the r_i^{th} row of A_Γ. Since we have $A'_\Gamma = \begin{pmatrix} X \\ N \end{pmatrix}$, we conclude that A_Γ is equivalent to $\begin{pmatrix} X \\ N \end{pmatrix}$. □

Lemma 4.4.6. Let Γ be an n-dimensional Bieberbach group of diagonal type with its holonomy group is isomorphic to C_2^k and let $\Gamma \cap (I_n \times \mathbb{R}^n) = \langle e_1, ..., e_n \rangle \cong \mathbb{Z}^n$. If A_Γ is col-reducible by i^{th} column, then $\Gamma/\langle e_i \rangle$ is a Bieberbach group of diagonal type.

Proof. We define a C_2^k-homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ such that $f(e_i) = 0$ and $f(e_j) = e_j$ for all $j \neq i$. Let

$$\alpha = \oplus_{1 \leq z \leq n} \alpha_z \in H^2(C_2^k, \oplus_{1 \leq z \leq n} M_z)$$

be the cohomology class defining Γ. We have

$$f_*(\alpha) = \alpha_1 \oplus \cdots \oplus \alpha_{i-1} \oplus \alpha_{i+1} \oplus \cdots \oplus \alpha_n$$

and $f_*(\alpha)$ is defining the Bieberbach group $\Gamma/\langle e_i \rangle$. The characteristic matrix corresponding to $\Gamma/\langle e_i \rangle$ can be obtained by removing the i^{th} column of A_Γ. Since A_Γ is col-reducible by i^{th} column, every row of $A_{\Gamma/\langle e_i \rangle}$ has at least one entry equal to 1. Therefore $\Gamma/\langle e_i \rangle$ is a Bieberbach group. By Lemma 4.3.8, $\Gamma/\langle e_i \rangle$ is a Bieberbach group of diagonal type. □

Corollary 4.4.7. Let $k \geq 1$, we have $n_d(C_2^k) \leq 2^k - 1$.

Proof. Assume by contradiction that there exists Γ, an n-dimensional Bieberbach group of diagonal type with its holonomy group isomorphic to C_2^k where $n \geq 2^k$ such that Γ/N is not a Bieberbach group for all normal subgroup $N \leq \Gamma \cap \mathbb{R}^n$. Consider the characteristic matrix A_Γ of Γ. By Proposition 4.4.5, A_Γ cannot be col-irreducible. Thus A_Γ is col-reducible. By Lemma 4.4.6, there exists a normal subgroup $N \leq \Gamma \cap \mathbb{R}^n$ such that Γ/N is a Bieberbach group, which is a contradiction. □

Proposition 4.4.8. Let Γ be an n-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to C_2^k. If $n \geq 5 \cdot 2^{k-3} + 2$ where $k \geq 3$, then A_Γ is col-reducible.

Proof. Let $\alpha = \oplus_{1 \leq z \leq n} \alpha_z \in H^2(C_2^k, \oplus_{1 \leq z \leq n} M_z)$ be the cohomology class corresponding to Γ, where $M_i \cong \mathbb{Z}$ for all $1 \leq i \leq n$. Assume by contradiction that A_Γ is col-irreducible. By Proposition 4.4.5, A_Γ is equivalent to $\begin{pmatrix} X \\ N \end{pmatrix}$ where X is an $n \times n$ matrix such that all diagonal entries are equal to 1 and other entries are not equal to 1 and N is a matrix with $2^k - 1 - n$ rows. Since C_2^k is acting faithfully on $\mathbb{Z}^n$, there exists $i, j \in \{1, ..., n\}$ such that C_2^k acts non-trivially on both M_i and M_j where $i \neq j$. Consider the i^{th} and j^{th} columns of N. By Proposition 4.2.11, the i^{th} and j^{th} columns of N has $2^{k-2} - 1$ entries equal to 1. Since $k \geq 3$, we ensure that the i^{th} and j^{th} columns of N has at least one entry equal to 1. Define

$$z = |\{m \in \{1, ..., 2^k - 1 - n\} \mid N_{m,i} = N_{m,j} = 1\}|$$

and observe that we have

$$2^{k-2} - 1 - z = |\{m \in \{1, ..., 2^k - 1 - n\} \mid N_{m,i} = 1,\, N_{m,j} \neq 1\}|$$

and

$$2^{k-2} - 1 - z = |\{m \in \{1, ..., 2^k - 1 - n\} \mid N_{m,i} \neq 1,\, N_{m,j} = 1\}|$$

Since N has $2^k - 1 - n$ rows, we have

$$2(2^{k-2} - 1 - z) + z \leq 2^k - 1 - n$$

By re-arranging the above inequality, we get

$$n - 2^{k-1} - 1 \leq z$$

Since $n \geq 5 \cdot 2^{k-3} + 2$, it follows that

$$z \geq 5 \cdot 2^{k-3} + 2 - 2^{k-1} + 1 = 2^{k-3} + 1$$

Thus

$$|\{g \in C_2^k \mid A_\Gamma(g, M_i) = A_\Gamma(g, M_j) = 1\}| \geq 2^{k-3} + 1$$

By Corollary 4.2.14, we have

$$\{g \in C_2^k \mid res_{\langle g \rangle}\alpha_i \neq 0\} = \{g \in C_2^k \mid res_{\langle g \rangle}\alpha_j \neq 0\}$$

It contradicts that X is a matrix where all diagonal entries are equal to 1 and other entries are not equal to 1. □

In the next two propositions, we obtain upper and lower bounds for the diagonal Vasquez number.

Proposition 4.4.9. For $k \geq 3$, the upper bound of diagonal Vasquez invariant is given by $n_d(C_2^k) \leq 5 \cdot 2^{k-3} + 1$.

Proof. We proceed by induction. First, consider the base case where $k = 3$. Let Γ be an n-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to C_2^3 where $n \geq 7$. Let $\alpha \in H^2(C_2^3, \mathbb{Z}^n)$ be the cohomology class defining Γ. By Corollary 4.4.7, we have $n_d(C_2^3) \leq 7$. Hence there exists a C_2^3-homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^m$ where $m \leq 7$ such that $f_*(\alpha)$ is special. By Lemma 4.3.8, $f_*(\alpha)$ defines a Bieberbach group of diagonal type. We remain to show that if Γ' is a 7-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to C_2^3 and corresponds to cohomology class $\beta \in H^2(C_2^3, \mathbb{Z}^7)$, then there exists a C_2^3-homomorphism $g : \mathbb{Z}^7 \to \mathbb{Z}^s$ where $s \leq 6$ such that $g_*(\beta)$ is special. We assume by contradiction that there does not exist such homomorphism g. By Lemma 4.4.6, $A_{\Gamma'}$ is col-irreducible. By Proposition 4.4.5, $A_{\Gamma'}$ is equivalent to the 7-dimensional square matrix such that all diagonal entries equal to 1 and all other entries are not equal to 1. This contradicts Remark 4.2.12. Hence we can conclude that there always exists a C_2^3-homomorphism $g : \mathbb{Z}^7 \to \mathbb{Z}^s$ where $s \leq 6$ such that $g_*(\beta)$ is special. Therefore we have $n_d(C_2^3) \leq 6$.

We assume the statement is true for $k \leq t - 1$. Now we consider the case where $k = t$.

Let Γ be an n-dimensional Bieberbach group with its holonomy group isomorphic to C_2^t where $n \geq 5 \cdot 2^{t-3} + 2$. Let

$$\alpha = \alpha_1 \oplus \cdots \oplus \alpha_n \in H^2(C_2^t, M_1 \oplus \cdots \oplus M_n)$$

be the cohomology class corresponding to the standard extension of Γ. We want to show that there exists a C_2^t-homomorphism f such that $f_*(\alpha)$ corresponds to a Bieberbach group of dimension at most $5 \cdot 2^{t-3} + 1$. Since $n \geq 5 \cdot 2^{t-3} + 2$, by Proposition 4.4.8, A_Γ is col-reducible. By Lemma 4.4.6, there exists a module homomorphism $f_1 : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ such that $(f_1)_*(\alpha)$ defines a Bieberbach group of diagonal type Γ_1. If $\dim(\Gamma_1) \geq 5 \cdot 2^{t-3} + 2$

and holonomy group of Γ is isomorphic to C_2^t, then by Proposition 4.4.8 and Lemma 4.4.6, there exists $f_2 : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-2}$ such that $(f_2 \circ f_1)_*(\alpha)$ defines a Bieberbach group of diagonal type. Inductively, there exists $r \in \mathbb{Z}$ and we can define $f_i : \mathbb{Z}^{n-i+1} \to \mathbb{Z}^{n-i}$ for $i = 1, ..., r$ such that $(f_r \circ \cdots \circ f_1)_*(\alpha)$ defines a Bieberbach group of diagonal type Γ_r, either $\dim(\Gamma_r) \leq 5 \cdot 2^{t-3} + 1$ or the holonomy group of Γ_r is isomorphic to $G \subsetneqq C_2^t$. We remain to consider the second case where the holonomy group of Γ_r is isomorphic to a proper subgroup of C_2^t. By induction hypothesis, there exists a module homomorphism $g : \mathbb{Z}^{dim(\Gamma_r)} \to \mathbb{Z}^s$ such that $(g \circ f_r \circ \cdots \circ f_1)_*(\alpha)$ defines a Bieberbach group of dimension at most $5 \cdot 2^{t-4} + 1$. Hence we conclude that there exists a C_2^t-homomorphism f such that $f_*(\alpha)$ defines a Bieberbach group of dimension at most $5 \cdot 2^{t-3} + 1$. □

Proposition 4.4.10. Let Γ be a Bieberbach group of diagonal type with holonomy group isomorphic to C_2^k where $k \geq 2$. Then

$$n_d(C_2^k) \geq \begin{cases} \frac{k(k+1)}{2} & \text{if } k \text{ is even} \\ \frac{k(k+1)}{2} - 1 & \text{if } k \text{ is odd} \end{cases}$$

Proof. First, we assume k is even. We are going to construct a matrix A_Γ and show that it defines Γ, a $\frac{k(k+1)}{2}$-dimensional Bieberbach group of diagonal type such that there does not exist a C_2^k-homomorphism f such that $f_*(\alpha)$ defines a smaller dimensional Bieberbach group where α is the cohomology class defining Γ.

Define a $(k \times k)$-matrix Q such that

$$Q_{ij} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i \neq j \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq k$. Let $S = \{(a, b) \in \{1, ..., k\} \times \{1, ..., k\} \mid a < b\}$. It is easy to see that $|S| = \frac{k(k-1)}{2}$. Let $s_j = (s_j^{(1)}, s_j^{(2)})$ for $1 \leq j \leq \frac{k(k-1)}{2}$ be all elements of S. Define a $(k \times \frac{k(k-1)}{2})$-matrix N such that

$$N_{ij} = \begin{cases} 2 & \text{if } i = s_j^{(1)} \\ 3 & \text{if } i = s_j^{(2)} \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leq i \leq k$ and $1 \leq j \leq \frac{k(k-1)}{2}$. In other words, fix $j \in \{1, ..., \frac{k(k-1)}{2}\}$ and consider the j^{th} column of N. The $(s_j^{(1)})^{th}$ entry of the j^{th} column of N is equal to 2, the $(s_j^{(2)})^{th}$ entry of the j^{th} column of N is equal to 3 and all other entries of the j^{th} column of N is equal to 0.

Define a $(k \times \frac{k(k+1)}{2})$-matrix $A = \begin{pmatrix} Q & N \end{pmatrix}$ by combining Q and N together. Let $g_1, ..., g_k$ be generators of C_2^k and $M_z \cong \mathbb{Z}$ for all $1 \leq z \leq \frac{k(k+1)}{2}$. We define $A_\Gamma(g_i, M_j) = A_{i,j}$ for $1 \leq i \leq k$ and $1 \leq j \leq \frac{k(k+1)}{2}$. By Remark 4.2.2, we can construct a $((2^k - 1) \times \frac{k(k+1)}{2})$-matrix A_Γ.

For example, we assume $k = 2$. We have $S = \{s_1 = (1,2)\}$. We define Q, N and A as below,

$$Q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \ , \ N = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

and the third row of A_Γ can be calculated by "adding" the first two row together. We get

$$\begin{matrix} r_1 \\ r_2 \\ r_1 \star r_2 \end{matrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{pmatrix} = A_\Gamma$$

We denote the i^{th} row of A to be r_i.

Now we are going to show that A_Γ defines a $\frac{k(k+1)}{2}$-dimensional Bieberbach group of diagonal type Γ by using Lemma 4.2.6. Let r be an arbitrary row of A_Γ. There exists $m \in \{1, ..., k\}$ and $1 \leq i_1 < ... < i_m \leq k$ such that the row can be expressed as $r = r_{i_1} \star \cdots \star r_{i_m}$. Notice that the j^{th} column of the row r equals to $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j)$ and

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j) = \bigstar_{1 \leq z \leq m} A_\Gamma(g_{i_z}, M_j)$$

We claim that there exists $c_1, c_2 \in \{1, ..., \frac{k(k+1)}{2}\}$ such that $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_1}) = 1$ and $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_1}) \in \{2, 3\}$. In other words, we claim that there exists an entry on the row equals to 1 and there exists an entry on the row equals to 2 or 3. By Lemma 4.2.6, we can conclude that A_Γ defines a $\frac{k(k+1)}{2}$-dimensional Bieberbach group of diagonal type where its holonomy group isomorphic to C_2^k.

We are going to prove the claim now. First, it is clear the claim is true for $m = 1$.

Now, we assume $2 \leq m \leq k$. There exists $j \in \{1, ..., \frac{k(k-1)}{2}\}$ such that $s_j = (i_1, i_2) \in S$. Then we have

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{k+j}) = \bigstar_{z=1}^{m} A_\Gamma(g_{i_z}, M_{k+j}) = 0 \star \cdots \star 2 \star 0 \star \cdots \star 0 \star 3 \star 0 \star \cdots \star 0 = 1$$

Next, we are going to show there exists an entry equals to 2 or 3 in row r. If m is odd, then there exists $i \in \{1, ..., k\} - \{i_1, ..., i_m\}$. Thus we have

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_i) = \bigstar_{z=1}^{m} A_\Gamma(g_{i_z}, M_i) = \overbrace{2 \star \cdots \star 2}^{\text{odd copies}} = 2$$

If m is even, then we have

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{i_1}) = \bigstar_{z=1}^{m} A_\Gamma(g_{i_z}, M_{i_1}) = \overbrace{2 \star \cdots \star 2 \star 1 \star 2 \star \cdots \star 2}^{\text{odd copies of 2}} = 3$$

Thus, our claim is true. By Lemma 4.2.6, the matrix A_Γ defines a $\frac{k(k+1)}{2}$-dimensional Bieberbach group of diagonal type where its holonomy group isomorphic to C_2^k. Next, notice that A_Γ is equivalent to

$$\begin{pmatrix} r_1 \\ \vdots \\ r_k \\ r_{s_1^{(1)}} \star r_{s_1^{(2)}} \\ \vdots \\ r_{s^{(1)}_{\frac{k(k-1)}{2}}} \star r_{s^{(2)}_{\frac{k(k-1)}{2}}} \\ P \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix}$$

where X is a $(\frac{k(k+1)}{2} \times \frac{k(k+1)}{2})$ matrix such that the diagonal entries all equals to 1 and all other entries are not equal to 1. By Proposition 4.4.5, we can conclude that A_Γ is col-irreducible. Let $\rho_i : C_2^k \to GL(M_i)$ be the representation given by the C_2^k-action on M_i for all $1 \leq i \leq \frac{k(k+1)}{2}$. By Remark 4.4.4, observe that columns of $\psi(A_\Gamma)$ are all distinct. Therefore $\rho_i \neq \rho_j$ for all $i \neq j$. by Corollary 4.4.3, there does not exist an C_2^k-homomorphism f such that $f_*(\alpha)$ defines a smaller dimensional Bieberbach group where α is the cohomology class defining Γ. Hence we have $n_d(C_2^k) \geq \frac{k(k+1)}{2}$ if k is even.

Now, we assume k is odd. We are going to construct a matrix A_Γ and show that it defines Γ, a $(\frac{k(k+1)}{2} - 1)$-dimensional Bieberbach group of diagonal type such that there does not exist a C_2^k-homomorphism f such that $f_*(\alpha)$ defines a smaller dimensional Bieberbach group where α is the cohomology class defining Γ.

Define a $(k \times k)$-matrix Q where

$$Q_{ij} = \begin{cases} 1 & \text{if } i = j \\ 2 & \text{if } i \neq j \end{cases}$$

for $1 \leq i \leq k$ and $1 \leq j \leq k$. Let $S' = \{(a,b) \in \{1,...,k\} \times \{1,...,k\} | a < b\} - \{(1,2),(1,3)\}$. It is easy to see that $|S'| = \frac{k(k-1)}{2} - 2$. Let $s'_j = (s_j'^{(1)}, s_j'^{(2)})$ for $1 \leq j \leq \frac{k(k-1)}{2} - 2$ be all elements of S'. Define a $(k \times (\frac{k(k-1)}{2} - 2))$-matrix N where

$$N_{ij} = \begin{cases} 2 & \text{if } i = s_j'^{(1)} \\ 3 & \text{if } i = s_j'^{(2)} \\ 0 & \text{otherwise} \end{cases}$$

Define a $k \times (\frac{k(k+1)}{2} - 1)$ matrix A such that

$$A = \begin{pmatrix} & & 2 \\ & & 3 \\ & & 3 \\ Q & N & 0 \\ & & \vdots \\ & & 0 \end{pmatrix}.$$

Let $g_1, ..., g_k$ be generators of C_2^k and $M_z \cong \mathbb{Z}$ for $1 \le z \le \frac{k(k+1)}{2} - 1$. We define $A_\Gamma(g_i, M_j) = A_{i,j}$. By Remark 4.2.2, we can construct a $((2^k - 1) \times (\frac{k(k+1)}{2} - 1))$-matrix A_Γ. For example, if $k = 3$, we have

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 & 2 \\ 2 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 3 & 3 \end{pmatrix}$$

and

$$\begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_1 \star r_2 \\ r_1 \star r_3 \\ r_2 \star r_3 \\ r_1 \star r_2 \star r_3 \end{matrix} \begin{pmatrix} 1 & 2 & 2 & 0 & 2 \\ 2 & 1 & 2 & 2 & 3 \\ 2 & 2 & 1 & 3 & 3 \\ 3 & 3 & 0 & 2 & 1 \\ 3 & 0 & 3 & 3 & 1 \\ 0 & 3 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} = A_\Gamma$$

We denote the i^{th} row of A to be r_i.

Now we are going to show that A_Γ defines a $(\frac{k(k+1)}{2} - 1)$-dimensional Bieberbach group of diagonal type Γ by using Lemma 4.2.6. Let r be an arbitrary row of A_Γ. There exists $m \in \{1, ..., k\}$ and $1 \le i_1 < ... < i_m \le k$ such that the row can be expressed as $r = r_{i_1} \star \cdots \star r_{i_m}$. Notice that the j^{th} column of the row r equals to $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j)$ and

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_j) = \bigstar_{1 \le z \le m} A_\Gamma(g_{i_z}, M_j)$$

We claim that there exists $c_1, c_2 \in \{1, ..., \frac{k(k+1)}{2} - 1\}$ such that $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_1}) = 1$ and $A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{c_1}) \in \{2, 3\}$. In other words, we claim that there exists an entry on the row r equals to 1 and there exists an entry on the row r equals to 2 or 3. By Lemma 4.2.6, we can conclude that A_Γ defines a $(\frac{k(k+1)}{2} - 1)$-dimensional Bieberbach group of diagonal type where its holonomy group isomorphic to C_2^k.

We are going to prove the claim now. First, it is clear that the claim is true for $m = 1$.

Next, we assume $2 \le m \le k - 1$. If $(i_{m-1}, i_m) \in \{(1, 2), (1, 3)\}$, then $m = 2$ and we have

$$A_\Gamma(g_{i_1} g_{i_2}, M_{\frac{k(k+1)}{2} - 1}) = A_\Gamma(g_{i_1}, M_{\frac{k(k+1)}{2} - 1}) \star A_\Gamma(g_{i_2}, M_{\frac{k(k+1)}{2} - 1}) = 2 \star 3 = 1$$

If $(i_{m-1}, i_m) \notin \{(1, 2), (1, 3)\}$, then there exists $j \in \{1, ..., \frac{k(k-1)}{2} - 2\}$ such that $s'_j = (i_{m-1}, i_m) \in S'$. Then we have

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{k+j}) = \bigstar_{z=1}^m A_\Gamma(g_{i_z}, M_{k+j}) = 0 \star \cdots \star 2 \star 0 \star \cdots \star 0 \star 3 \star 0 \star \cdots \star 0 = 1$$

Next, we are going to show there exists an entry equals to 2 or 3 in row r. If m is even, we have

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_{i_1}) = \bigstar_{z=1}^k A_\Gamma(g_{i_z}, M_{i_1}) = \overbrace{2 \star \cdots \star 2 \star 1 \star 2 \star \cdots \star 2}^{\text{odd copies of 2}} = 3$$

If $m = k$, then

$$A_\Gamma(g_1 \cdots g_k, M_{\frac{k(k+1)}{2}-1}) = \bigstar_{z=1}^k A_\Gamma(g_z, M_{\frac{k(k+1)}{2}-1}) = 2 \star 3 \star 3 \star 0 \star \cdots \star 0 = 2$$

If m is odd and $m \neq k$, then there exists $i \in \{1, ..., k\} - \{i_1, ..., i_m\}$ and we have

$$A_\Gamma(g_{i_1} \cdots g_{i_m}, M_i) = \bigstar_{z=1}^k A_\Gamma(g_{i_z}, M_i) = \overbrace{2 \star \cdots \star 2}^{\text{odd copies}} = 2$$

Thus, our claim is true. By Lemma 4.2.6, A_Γ defines a $(\frac{k(k+1)}{2}-1)$-dimensional Bieberbach group of diagonal type Γ where its holonomy group isomorphic to C_2^k.

Next, notice that A_Γ is equivalent to

$$\begin{pmatrix} r_1 \\ \vdots \\ r_k \\ r_{s_1'(1)} \star r_{s_1'(2)} \\ \vdots \\ r_{s'_{\frac{k(k-1)}{2}-2}(1)} \star r_{s'_{\frac{k(k-1)}{2}-2}(2)} \\ r_1 \star r_2 \\ P \end{pmatrix} = \begin{pmatrix} X \\ P \end{pmatrix}$$

where X is a $((\frac{k(k+1)}{2} - 1) \times (\frac{k(k+1)}{2} - 1))$ matrix such that the diagonal entries all equals to 1 and all other entries are not equal to 1. By Proposition 4.4.5, we can conclude that A_Γ is col-irreducible. Let $\rho_i : C_2^k \to GL(M_i)$ be the representation given by the C_2^k-action on M_i for all $1 \leq i \leq \frac{k(k+1)}{2} - 1$. By Remark 4.4.4, observe that columns of $\psi(A_\Gamma)$ are all distinct. Therefore $\rho_i \neq \rho_j$ for all $i \neq j$. by Corollary 4.4.3, there does not exist an C_2^k-homomorphism f such that $f_*(\alpha)$ defines a smaller dimensional Bieberbach group where α is the cohomology class defining Γ. Hence we have $n_d(C_2^k) \geq \frac{k(k+1)}{2} - 1$ if k is odd.

□

By combining Proposition 4.4.9 and Proposition 4.4.10, we get Theorem D.

Proof of Theorem E. By Remark 4.3.7, the theorem holds for $k = 1$.

Now assume $k = 2$. By Theorem D, we have $n_d(C_2^2) \geq 3$. By Corollary 4.4.7, we have $n_d(C_2^2) \leq 2^2 - 1 = 3$ and thus $n_d(C_2^2) = 3$.

Now we assume $k = 3$. By Theorem D, we have $5 \leq n_d(C_2^3) \leq 6$. It remains to show that if Γ' is a 6-dimensional Bieberbach group of diagonal type with holonomy isomorphic to C_2^3 where $\alpha' \in H^2(C_2^3, \mathbb{Z}^6)$ is the corresponding cohomology class, then there exists $f : \mathbb{Z}^6 \to \mathbb{Z}^5$ such that $f_*(\alpha')$ is special. Assume by contradiction that there does not exist such f and hence we assume $A_{\Gamma_1'}$ is col-irreducible. By Proposition 4.4.5, A_Γ is equivalent to $\begin{pmatrix} X \\ N \end{pmatrix}$ where the diagonal entries of X is the only entries equal to 1 and N is a row matrix. By Remark 4.2.12, each column of A_Γ has at least 2 entries equal 1. It forces N is a row matrix with all entries equal to 1. Hence the element of the holonomy group corresponds to that row acts trivially on $\mathbb{Z}^6$ which contradicts that the holonomy representation of Γ is faithful. We conclude that $n_d(C_2^3) = 5$.

Now we assume $k = 4$. By Theorem D, we have $10 \leq n_d(C_2^4) \leq 11$. We remain to show that if Γ' is a 11-dimensional Bieberbach group of diagonal type with holonomy group isomorphic to C_2^4 where $\alpha' \in H^2(C_2^4, M_1 \oplus \cdots \oplus M_{11})$ where $M_j \cong \mathbb{Z}$ for $j = 1, ..., 11$ is the corresponding cohomology class, then there exists a C_2^4-homomorphism $f : \mathbb{Z}^{11} \to \mathbb{Z}^{10}$ such that $f_*(\alpha')$ is special. Assume by contradiction that there does not exist such f. By Lemma 4.4.6, the characteristic matrix $A_{\Gamma'}$ is col-irreducible. By Proposition 4.4.5, we assume $A_{\Gamma'}$ is equivalent to $\begin{pmatrix} X \\ N \end{pmatrix}$ where the diagonal entries of X is the only entries equal to 1 and N is a row matrix and N is matrix with four rows. By Remark 4.2.12, each column of $A_{\Gamma'}$ has either 4 or 8 entries equal to 1. It forces each column of N has 3 entries equal to 1. Since N has eleven columns, there exists $i, j \in \{1, ..., 11\}$ such that

$$|\{g \in C_2^4 \mid A_{\Gamma'}(g, M_i) = A_{\Gamma'}(g, M_j) = 1\}| = 3.$$

By Corollary 4.2.14, we have

$$\{g \in C_2^4 \mid A_{\Gamma'}(g, M_i) = 1\} = \{g \in C_2^4 \mid A_{\Gamma'}(g, M_j) = 1\}$$

It follows that $A_{\Gamma'}$ is col-reducible, which is a contradiction. Hence $n_d(C_2^4) = 10$. □

Chapter 5

Diffuseness of Bieberbach groups

5.1 introduction

In this chapter, we will discuss diffuseness property of Bieberbach groups. First, we recall the definition of diffuseness.

Let G be a group, $A \subseteq G$ be a subset. We define

$$\Delta(A) = \{a \in A \,|\, \text{for all } \gamma \in \Gamma, \text{either } \gamma a \notin A \,\text{or}\, \gamma^{-1} a \notin A\}$$

We say G is *diffuse* if for any subset $A \subseteq G$ with $2 \leq |A| < \infty$, we have $|\Delta(A)| \geq 2$. We say G is *weakly diffuse* if for any subset $A \subseteq G$ with $1 \leq |A| < \infty$, we have $|\Delta(A)| \geq 1$. We say G is *non-diffuse* if it is not diffuse.

Definition 5.1.1. Let G be a group and $A \subseteq G$ be a non-empty finite subset. We say A is a *ravel* if $\Delta(A) = \emptyset$.

Remark 5.1.2. By definition, we can see that all finite subgroup is non-diffuse. Let Γ be a group and let $N \leq \Gamma$. By definition of diffuseness, if N is non-diffuse, then Γ is non-diffuse. Thus if Γ is diffuse, then Γ is torsion-free.

Let Γ be a group and let X be a set. We say X is a Γ-*set* if there is an action on X by the group Γ. Given $x \in X$, we define

$$\Gamma(x) = \{\gamma \in \Gamma \,|\, \gamma x = x\}$$

Suppose X is a Γ-set. Given any subset $A \subseteq X$, we define

$$\Delta_\Gamma(A) = \{a \in A \,|\, \text{if } \gamma \in \Gamma \text{ satisfies } \gamma a, \gamma^{-1} \in A, \text{ then } \gamma a = a\}$$

We say X is *diffuse as a* Γ-*set* if given any finite subset $A \subseteq X$ with $2 \leq |A| < \infty$, then $|\Delta_\Gamma(A)| \geq 2$

Lemma 5.1.3. Let Γ be a group. We view Γ as a Γ-set where the structure of the Γ-set is given by left multiplication. We have Γ is diffuse if and only if Γ is diffuse as Γ-set.

Proof. It is sufficient to show that $\Delta(A) = \Delta_\Gamma(A)$ for any finite subset $A \subseteq X$ with $2 \leq |A| < \infty$. Let $x \in \Delta(A)$. If $\gamma \in \Gamma$ satisfies $\gamma x, \gamma^{-1}x \in A$, then γ is the identity element because $x \in \Delta(A)$. Thus we have $\gamma x = x$. Hence $\Delta(A) \subseteq \Delta_\Gamma(A)$. For the reverse direction, we assume by contradiction that $x \notin \Delta(A)$ and $x \in \Delta_\Gamma(A)$. Since $x \notin \Delta(A)$, there exists $\gamma \in \Gamma/\{1\}$ such that $\gamma x, \gamma^{-1}x \in A$. Since $x \in \Delta_\Gamma(A)$ and $\gamma x, \gamma^{-1}x \in A$, we have $\gamma x = x$. Thus $\gamma = 1$ which is a contradiction. Hence we have $\Delta_\Gamma(A) \subseteq \Delta(A)$. Therefore $\Delta(A) = \Delta_\Gamma(A)$. □

A *morphism* between two Γ-sets X and Y is a map $f : X \to Y$ such that $f(\gamma x) = \gamma f(x)$ for all $x \in X$ and $\gamma \in \Gamma$. Note that if $y \in Y$, then $f^{-1}y$ has the structure of $\Gamma(y)$-set.

Lemma 5.1.4. *[4, Lemma 2.1]* Let Γ be a group and suppose X and Y are Γ set. Let $f : X \to Y$ be a morphism between two Γ-set. If Y is diffuse as a Γ-set and that $f^{-1}y$ is diffuse as $\Gamma(y)$-set for all $y \in Y$, then X is diffuse as a Γ-set.

Proof. Let A be an arbitrary finite subset of X such that $2 \leq |A| < \infty$ and define $B = f(A) \subseteq Y$. Our aim is to show that $|\Delta_\Gamma(A)| \geq 2$.

First, we assume $|B| \geq 2$. Since Y is diffuse as Γ-set, we have $|\Delta_\Gamma(B)| \geq 2$. Let $b \in \Delta_\Gamma(B)$. Suppose $|A \cap f^{-1}b| \geq 2$. Since $A \cap f^{-1}b \subseteq f^{-1}b$ and $f^{-1}b$ is diffuse as $\Gamma(b)$-set, we have $|\Delta_{\Gamma(b)}(A \cap f^{-1}b)| \geq 2$. Let $a \in \Delta_{\Gamma(b)}(A \cap f^{-1}b)$, we claim that $a \in \Delta_\Gamma(A)$. If $\gamma \in \Gamma$ satisfies $\gamma a, \gamma^{-1}a \in A$, then $f(\gamma a) = \gamma f(a) = \gamma b \in B$ and $f(\gamma^{-1}a) = \gamma^{-1}b \in B$. Since $b \in \Delta_\Gamma(B)$, we have $\gamma b = b$. Thus $\gamma \in \Gamma(b)$ and $\gamma a, \gamma^{-1}a \in A \cap f^{-1}b$. It follows that $\gamma a = a$ because $a \in \Delta_{\Gamma(b)}(A \cap f^{-1}b)$. Therefore we have $a \in \Delta_\Gamma(A)$. By similar argument, if $f^{-1} = \{a\}$ for some $a \in A$. then $a \in \Delta_\Gamma(A)$. In either case, we obtain $\Delta_\Gamma(A) \cap f^{-1}b \neq \emptyset$. Since $|\Delta_\Gamma(B)| \geq 2$, we have $|\Delta_\Gamma(A)| \geq 2$.

Next, we assume $B = \{b\}$ for some $b \in B$. We have $A = f^{-1}b$. We see that $\Delta_\Gamma(A) = \Delta_{\Gamma(b)}(A)$. Since $A = f^{-1}b$ is diffuse as $\Gamma(b)$-set, we have $|\Delta_{\Gamma(b)}(A)| \geq 2$. It follows that $|\Delta_\Gamma(A)| \geq 2$. Thus X is diffuse as a Γ-set. □

Theorem 5.1.5. *[4, Theorem 1.2 (1)]* Let Γ be a torsion-free group. Suppose $N \trianglelefteq \Gamma$ and both N and Γ/N are diffuse. Then Γ is diffuse.

Proof. Let $f : \Gamma \to \Gamma/N$ be quotient map. Notice that f is a morphism of Γ-sets. By Lemma 5.1.3, Γ/N is diffuse as (Γ/N)-set. Hence Γ/N is diffuse as a Γ-set too. Also, if $y \in \Gamma/N$, then $\Gamma(y)$ is conjugate to N in Γ. Thus $\Gamma(y)$ is isomorphic to N. Besides, $f^{-1}y$ is isomorphic to $\Gamma(y) \cong N$ viewed as an $\Gamma(y)$-set. Since N is diffuse, $f^{-1}y$ is diffuse for all $y \in \Gamma/N$. By Lemma 5.1.4, Γ is diffuse. □

Theorem 5.1.6. *[20, Lemma 3.4]* Let Γ be an n-dimensional crystallographic group. If $b_1(\Gamma) = 0$, then Γ is non-diffuse.

Proof. Let G be the holonomy group of Γ. We claim that the set $S = \{\gamma \in \Gamma \mid \|\gamma(0)\| \leq r\}$ for sufficiently large $r > 0$ is a ravel. Let $\gamma' = (I, u) \in S \cap (I_n \times \mathbb{R}^n)$. Our aim is to find an element $\gamma \in \Gamma$ such that $\|\gamma\gamma'(0)\| \leq r$ and $\|\gamma^{-1}\gamma'(0)\| \leq r$. By Lemma 2.6.3, we have $rk(\mathbb{Z}^n)^G = 0$. Thus the group G acts on $\mathbb{R}^n$ without non-trivial fixed point. Hence there exists a real number $\delta < 1$ and there exists $g \in G$ such that

$$\|gu + u\| \leq 2\delta\|u\|$$

Pick $\gamma_0 \in \Gamma$ such that $p(\gamma_0) = g$. Define $w_0 = \gamma_0(0)$. In other words, we have $\gamma_0 = (g, w_0) \in \Gamma$. Fix $r_0 > 0$ such that for all $v \in \mathbb{R}^n$, there exists $t \in T$ such that $\|u - t\| \leq r_0$. Observe that for all $v_1, v_2 \in \mathbb{R}^n$ where $\|v_1 - v_2\| = d$, there exists $x \in w_0 + T$ with

$$max(\|v_1 - x\|, \|v_2 - x\|) \leq r_0 + \frac{d}{2}$$

Apply the above calculation to $v_1 = u$ and $v_2 = -gu$ to find such $x = w_0 + t$. We have

$$d = \|gu + u\| \leq 2\delta\|u\| \leq 2\delta r$$

Define $\gamma = (I_n, t)\gamma_0$. By simple calculation, we get

$$\|\gamma\gamma'(0)\| = \|(I_n, t)(g, w_0)(I_n, u)(0)\| = \|w_0 + gu + t\| = \|gu + x\| \leq r_0 + \frac{d}{2} \leq r_0 + \delta r$$

and

$$\|\gamma^{-1}\gamma'(0)\| = \|(g^{-1}, -g^{-1}w_0)(I_n, -t)(I_n, u)(0)\| = \| - g^{-1}x + g^{-1}u\| = \|u - x\| \leq r_0 + \delta r$$

As $\delta < 1$, for all sufficiently large r, we have $\|\gamma\gamma'(0)\| \leq r$ and $\|\gamma^{-1}\gamma'(0)\| \leq r$. Therefore the set S is ravel. □

Definition 5.1.7. A finite group G is *holonomy diffuse* if every Bieberbach group Γ with holonomy group G is diffuse. The group G is *holonomy anti-diffuse* if every Bieberbach group Γ with holonomy group G is non-diffuse. Otherwise we say that G is *holonomy mixed.*

Next, we state a theorem given by S. Kionke, J. Raimbault in [20] that give an algebraic characterisation of the above three classes of finite group.

Theorem 5.1.8. *[20, Theorem 3.5]* Let G be a finite group.

(i) The group G is holonomy anti-diffuse if and only if it is not solvable.

(ii) The group G is holonomy diffuse if and ony if every Sylow subgroup is cyclic.

(iii) The group G is holonomy mixed if and only if it is solvable and has a non-cyclic Sylow subgroup.

Next, we are going to introduce a way to determine whether a given Bieberbach group Γ is diffuse or not as follow.

Let Γ be n-dimensional Bieberbach group. By Theorem 5.1.6, if $b_1(\Gamma) = 0$, then Γ is a non-diffuse group. We now assume $b_1(\Gamma) = k > 0$. By Corollary 2.6.5, we get an epimorphism

$$f : \Gamma \to \mathbb{Z}^k$$

such that $ker(f)$ is an $(n-k)$-dimensional Bieberbach group. In other words, we have the below short exact sequence

$$0 \to ker(f) \to \Gamma \to \mathbb{Z}^k \to 0$$

Since $\mathbb{Z}^k$ is diffuse, By Proposition 5.1.5, Γ is diffuse if and only if $ker(f)$ is diffuse. Hence we can reduced the question to whether $ker(f)$ is diffuse or not. If the first Betti number of ker(f) is non-zero then we can apply the above steps and obtain another epimorphism such that the kernel of such epimorphism is a Bieberbach group of smaller dimension. We can apply the above steps inductively until either you get a Bieberbach subgroup of trivial first Betti number or the kernel of the last epimorphism is $\mathbb{Z}$ and hence Γ is indeed diffuse.

We want to study the classification of diffuseness of Bieberbach group with holonomy mixed group as holonomy group. In this chapter, we will consider Bieberbach group of diagonal type, which the holonomy group is elementary 2-group.

In next section, we will first consider a simpler case where the holonomy group of Bieberbach groups is $C_2 \times C_2$. Finally, we will present Theorem F and G.

5.2 Proof of Theorem F and G

Lemma 5.2.1. Let Γ be an n-dimensional Bieberbach group and suppose there exists $N \trianglelefteq \Gamma \cap \mathbb{R}^n$ such that $\Gamma' = \Gamma/N$ is still a Bieberbach group. If $b_1(\Gamma) = 0$, then $b_1(\Gamma') = 0$.

Proof. Consider the below short exact sequence,

$$0 \to N \to \Gamma \to \Gamma' \to 1$$

By [5, Chapter 7, Corollary 6.4] we have the following exact sequence

$$\cdots \to H_1(\Gamma, \mathbb{Z}) \to H_1(\Gamma', \mathbb{Z}) \to 0$$

Since $b_1(\Gamma) = 0$, we have $rk(H_1(\Gamma, \mathbb{Z})) = 0$. It forces $rk(H_1(\Gamma', \mathbb{Z})) = 0$. Therefore we have $b_1(\bar{\Gamma}) = 0$. □

By [20, Section 3.4], it presents all non-diffuse Bieberbach groups less than dimension 5. All Bieberbach groups in 2-dimensional is diffuse and there is only one non-diffuse Bieberbach group in 3-dimensional. We denote that group to be Δ_3. It has the below presentation

$$\Delta_3 = \langle x, y \,|\, x^{-1}y^2xy^2 = y^{-1}x^2yx^2 = 1\rangle$$

Δ_3 is indeed a 3-dimensional generalized Hantzsche-Wendt group (also known as Promislow group or Passman group). The following proposition and lemma will tell us why Δ_3 is an important non-diffuse group.

Proposition 5.2.2. If Γ be an n-dimensional Bieberbach group of diagonal type with holonomy group C_2^2 such that $b_1(\Gamma) = 0$. Then $\Delta_3 \leq \Gamma$.

Proof. Since Γ is Bieberbach group of diagonal type and $b_1(\Gamma) = 0$, without loss of generality, we let

$$\alpha = (diag(X_1, ..., X_n), (x_1, ..., x_n)) \text{ and } \beta = (diag(Y_1, ..., Y_n), (y_1, ..., y_n))$$

where $X_i, Y_i \in \{1, -1\}$ and $x_i, y_i \in \{0, \frac{1}{2}\}$ for all $i \in \{1, ..., n\}$ be the non lattice generators of Γ. There exists $i, j \in \{1, ..., n\}$ such that $(X_i, x_i) = (1, \frac{1}{2})$ and $(Y_j, y_j) = (1, \frac{1}{2})$. Otherwise, $\alpha \in \Gamma$ or $\beta \in \Gamma$ is an element of order 2, which contradicts the fact that Γ is torsion-free. There exists $k \in \{1, ..., n\}$ such that $(X_k, Y_k) = (-1, -1)$ and $(x_k, y_k) \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0)\}$ otherwise $\alpha\beta \in \Gamma$ has order 2. By the third Bieberbach's Theorem, we may assume

$$\alpha = \begin{pmatrix} I_s & 0 & 0 & a_1 \\ 0 & -I_p & 0 & b_1 \\ 0 & 0 & -I_q & c_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} -I_s & 0 & 0 & a_2 \\ 0 & I_p & 0 & b_2 \\ 0 & 0 & -I_q & c_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $s, p, q \in \mathbb{Z}^+$, $a_1, a_2 \in \{0, \frac{1}{2}\}^s$, $b_1, b_2 \in \{0, \frac{1}{2}\}^p$ and $c_1, c_2 \in \{0, \frac{1}{2}\}^q$. Besides, a_1, b_2, and c_2 are non-zero.

By a simple calculation, we checked that α and β satisfy the below relation

$$\alpha^{-1}\beta^2\alpha\beta^2 = \beta^{-1}\alpha^2\beta\alpha^2 = 1$$

Since

$$\Delta_3 = \langle x, y \,|\, x^{-1}y^2xy^2 = y^{-1}x^2yx^2 = 1 \rangle$$

there exists a normal subgroup $N \trianglelefteq \Delta_3$ such that $\langle \alpha, \beta \rangle \cong \Delta_3/N$. Let $\bar{\Gamma} = \langle \alpha, \beta \rangle$. Since α^2, β^2 and $(\alpha\beta)^2$ are three linearly independent elements inside the lattice $\bar{\Gamma} \cap \mathbb{R}^n$ and thus $\dim(\bar{\Gamma}) \geq 3$. This implies that N has rank zero and is therefore trivial. Hence we have $\Delta_3 \cong \langle \alpha, \beta \rangle \leq \Gamma$. □

Remark 5.2.3. Since a Bieberbach group with trivial first betti number is a torsion-free metabelian group with a finite commutator subgroup, the above proposition is a special case of [18, Theorem 1].

Lemma 5.2.4. Let Γ be an n-dimensional non-diffuse Bieberbach group of diagonal type with holonomy group C_2^2. Then there exists $\mathbb{Z}^{n-3} \triangleleft \Gamma$ such that $\Gamma/\mathbb{Z}^{n-3} \cong \Delta_3$.

Proof. By Theorem E, we have $n_d(C_2^2) = 3$. So, there exists $\mathbb{Z}^s \triangleleft \Gamma$ such that $\Gamma/\mathbb{Z}^s = \bar{\Gamma}$ is a Bieberbach group with $\dim(\bar{\Gamma}) \leq 3$. By Proposition 5.1.5, $\bar{\Gamma}$ is non-diffuse. Since Δ_3 is the only non-diffuse Bieberbach group below 4-dimensional, we can conclude that $s = n - 3$ and $\bar{\Gamma} \cong \Delta_3$. □

Proof of Theorem F. Let $\{\alpha, \beta\}$ be a set of non-lattice generators of Γ. Since $b_1(\Gamma) = k$, without loss of generality, assume

$$\alpha = (diag(x_1, ..., x_n), (a_1, ..., a_n)) \text{ and } \beta = (diag(y_1, ..., y_n), (b_1, ..., b_n))$$

where $a_j, b_j \in \{0, \frac{1}{2}\}$ for $j \in \{1, ..., n\}$, $(x_i, y_i) \in \{(1,-1), (-1,1)\}$ for $i \in \{1, ..., n-k\}$ and $(x_i, y_i) = (1,1)$ for $i \in \{n-k+1, ..., n\}$.

Given an arbitrary element $\gamma = (diag(z_1, ..., z_{n-k}, 1, ..., 1), (s_1, ..., s_n)) \in \Gamma$ where $z_i \in \{1, -1\}$ for $i \in \{1, ..., n-k\}$ and $(s_1, ..., s_n) \in \mathbb{Q}^n$. By Corollary 2.6.5, there exists a homomorphism $f : \Gamma \to \mathbb{Z}^k$ which maps γ to $(2s_{n-k+1}, ..., 2s_n) \in \mathbb{Z}^k$ and the kernel of the homomorphism is an $(n-k)$-dimensional Bieberbach group.

We claim that $a_i = 0$ for all $i \in \{n-k+1, ..., n\}$. Assume by contradiction that there exists $j \in \{n-k+1, ..., n\}$ such that $a_j \neq 0$. We have $\alpha \notin ker(f)$. Then the holonomy group of $ker(f)$ will either be identity or cyclic group of order two. By [20, Theorem 3.5], $ker(f)$ is diffuse. Since $ker(f)$ and $\mathbb{Z}^k$ are both diffuse, by Proposition 5.1.5, Γ is diffuse, which is a contradiction. Hence $a_i = 0$ for all $i \in \{n-k+1, ..., n\}$. By similar argument, we get $b_i = 0$ for all $i \in \{n-k+1, ..., n\}$. Therefore $\Gamma = Z(\Gamma) \oplus \bar{\Gamma}$, where $Z(\Gamma)$ is the center of Γ and $\bar{\Gamma} = ker(f)$. By Lemma 5.2.4, we have

$$0 \longrightarrow \mathbb{Z}^{n-k-3} \xrightarrow{\iota} \bar{\Gamma} \xrightarrow{\phi} \Delta_3 \longrightarrow 1 \tag{5.1}$$

Notice that $b_1(\bar{\Gamma}) = 0$, otherwise $b_1(\Gamma) > k$. By Proposition 5.2.2, we have $\Delta_3 \leq \bar{\Gamma}$. By restricting the domain of ϕ, we have

$$0 \longrightarrow H \xrightarrow{\iota} \Delta_3 \xrightarrow{\phi|_{\Delta_3}} G \longrightarrow 1$$

where H is a subgroup of $\mathbb{Z}^{n-k-3}$ and G is the image of the map $\phi|_{\Delta_3}$. We claim that $\phi|_{\Delta_3}$ is an isomorphism. Since H is a subgroup of $\mathbb{Z}^{n-k-3}$, H is a diffuse group. By Proposition 5.1.5, G is non-diffuse, otherwise it contradicts that Δ_3 is a non-diffuse group. Besides, G is a quotient of Δ_3. Hence G is a non-diffuse Bieberbach group of dimension less than or equal to three. Thus $G \cong \Delta_3$ and hence $\phi|_{\Delta_3}$ is an isomorphism. Therefore (5.1) is a split short exact sequence. □

Lemma 5.2.5. Let Γ be an n-dimensional non-diffuse Bieberbach group with $b_1(\Gamma) > 0$, then there exists $\Gamma' \leq \Gamma$ such that $b_1(\Gamma') = 0$.

Proof. We proceed by induction on n. Since all Bieberbach group is diffuse if $n \leq 2$, we first consider the base case where $n = 3$. Notice that Δ_3 is the only 3-dimensional non-diffuse Bieberbach group and $b_1(\Delta_3) = 0$. Thus the statement is true for $n = 3$. Assume the statement is true for $n = k$ and consider the case where $n = k+1$. Let Γ be an $(k+1)$-dimensional non-diffuse Bieberbach group with $b_1(\Gamma) = k > 0$. By Theorem 2.6.4, there exists an epimorphism $f : \Gamma \to \mathbb{Z}$ such that $ker(f)$ is a k-dimensional Bieberbach group. By Proposition 5.1.5, $ker(f)$ is non-diffuse. If $b_1(ker(f)) = 0$, then we are done. Assume $b_1(ker(f)) > 0$. By induction hypothesis, there exists $\Gamma' \leq ker(f) \leq \Gamma$ such that $b_1(\Gamma') = 0$. □

Given a non-diffuse Bieberbach group with non trivial center, by the above lemma, there exists a non-diffuse Bieberbach subgroup with trivial center. Therefore before proving Theorem G, we need the below two propositions to consider a simpler case where the Bieberbach group has trivial center.

Proposition 5.2.6. Let Γ be an n-dimensional Bieberbach group of diagonal type, $b_1(\Gamma) = 0$ and its holonomy group is isomorphic to C_2^k. Let $p : \Gamma \to C_2^k$ be the projection map as in (4.1). If $n < 2^k - 1$, then there exists $C_2^{k-1} \leq C_2^k$ such that $p^{-1}(C_2^{k-1})$ is an n-dimensional Bieberbach group with holonomy group isomorphic to C_2^{k-1} and $b_1(p^{-1}(C_2^{k-1})) = 0$.

Proof. First note that there are $2^k - 1$ subgroups in C_2^k isomorphic to C_2^{k-1}. Let $A_1, ..., A_{2^k-1}$ denote these subgroups. Assume by contradiction that $p^{-1}(A_i)$ is Bieberbach group with non-trivial first betti number for all $i \in \{1, ..., 2^k - 1\}$. Hence we have $(\mathbb{Z}^n)^{C_2^k} = 0$ and $(\mathbb{Z}^n)^{A_i} \neq 0$ for all $i \in \{1, ..., 2^k - 1\}$ where $\mathbb{Z}^n \cong \Gamma \cap \mathbb{R}^n$. For any $i \in \{1, ..., 2^k - 1\}$, let $z_i \in (\mathbb{Z}^n)^{A_i}$ and $e_1, ..., e_n$ be the standard basis of $\mathbb{Z}^n$ such that C_2^k acts diagonally on $\{e_1, ..., e_n\}$. We have $z_i = c_{i_1}e_1 + ... + c_{i_n}e_n$ where $c_{i_1}, ..., c_{i_n} \in \mathbb{Z}$. For each $g \in A_i$, we have

$$z_i = g \cdot z_i = c_{i_1}(g \cdot e_1) + ... + c_{i_n}(g \cdot e_n)$$

Thus there exists $t_i \in \{1, ..., n\}$ such that $e_{t_i} \in (\mathbb{Z}^n)^{A_i}$. We conclude that for any $i \in \{1, ..., 2^k - 1\}$, there exists $t_i \in \{1, ..., n\}$ such that $e_{t_i} \in (\mathbb{Z}^n)^{A_i}$. Notice that $t_i \neq t_j$ for all $i \neq j$, otherwise $e_{t_i} \in (\mathbb{Z}^n)^{A_i} \cap (\mathbb{Z}^n)^{A_j} = (\mathbb{Z}^n)^{C_2^k}$, contradicts that $b_1(\Gamma) = 0$. Thus we have $n \geq 2^k - 1$ which is a contradiction. □

Proposition 5.2.7. Let Γ be an n-dimensional Bieberbach group of diagonal type with $b_1(\Gamma) = 0$. Let $\Gamma \cap \mathbb{R}^n = \langle e_1, ..., e_n \rangle$ such that the holonomy group acts diagonally on $\{e_1, ..., e_n\}$. Then either $\Delta_3 \leq \Gamma$ or there exists $\Gamma' \leq \Gamma$ and $\mathbb{Z}^s = \langle e_{i_1}, ..., e_{i_s} \rangle \trianglelefteq \Gamma$ such that $\Gamma'/\mathbb{Z}^s \cong \Delta_3$, where $1 \leq i_1 < \cdots < i_s \leq n$.

Proof. Let C_2^p where $p \geq 1$ be the holonomy group of Γ. We proceed by induction on p. By Theorem F, we know the statement holds for $p = 2$. Assume that it is true for all $p \leq k - 1$. Let Γ be an n-dimensional Bieberbach group of diagonal type with $b_1(\Gamma) = 0$ and the holonomy group is isomorphic to C_2^k.

First we consider the case where $\dim(\Gamma) \leq n_d(C_2^k) < 2^k - 1$. By Proposition 5.2.6, there exists $\Gamma' \leq \Gamma$ such that $b_1(\Gamma') = 0$ and the holonomy group of Γ' isomorphic to C_2^{k-1}. Notice that $\Gamma' \cap \mathbb{R}^n = \Gamma \cap \mathbb{R}^n = \langle e_1, ..., e_n \rangle$. By induction hypothesis, either $\Delta_3 \leq \Gamma'$ in which case $\Delta_3 \leq \Gamma$, or there exists $\Gamma'' \leq \Gamma' \leq \Gamma$ and $\mathbb{Z}^s = \langle e_{i_1}, ..., e_{i_s} \rangle$ where $1 \leq i_1 < \cdots < i_s \leq n$ such that $\Gamma''/\mathbb{Z}^s \cong \Delta_3$. Since C_2^k is acting diagonally on $\mathbb{Z}^s$, we have $\mathbb{Z}^s = \langle e_{i_1}, ..., e_{i_s} \rangle \trianglelefteq \Gamma$.

Next, we assume $\dim(\Gamma) > n_d(c_2^k)$. By Theorem D, without loss of generality, we can assume that there exists $\langle e_1, ..., e_t \rangle \cong \mathbb{Z}^t$ such that $\bar{\Gamma} = \Gamma/\mathbb{Z}^t$ is a Bieberbach group of diagonal type with dimension at most $n_d(C_2^k) < 2^k - 1$. Notice that $\bar{\Gamma} \cap \mathbb{R}^{n-t} = \langle e_{t+1}, ..., e_n \rangle$. By Lemma 5.2.1, we have $b_1(\bar{\Gamma}) = 0$. By previous calculation and the induction hypothesis, we have either $\Delta_3 \leq \bar{\Gamma}$, in which case, $\Delta_3 \cong A/\mathbb{Z}^t$ where $A \leq \Gamma$, or there exists $\bar{\Gamma}'' \leq \bar{\Gamma}$ and $\mathbb{Z}^s = \langle e_{i_1}, ..., e_{i_s} \rangle$ where $t+1 \leq i_1 < \cdots < i_s \leq n$ such that $\bar{\Gamma}''/\mathbb{Z}^s \cong \Delta_3$. Consider the later case, since $\bar{\Gamma}'' \leq \bar{\Gamma} = \Gamma/\mathbb{Z}^t$, we have $\bar{\Gamma}'' \cong B/\mathbb{Z}^t$ where $B \leq \Gamma$. It follows that $B/\mathbb{Z}^t \oplus \mathbb{Z}^s \cong \Delta_3$. □

Proof of Theorem G. Let Γ be an n-dimensional non-diffuse Bieberbach group of diagonal type and $\Gamma \cap \mathbb{R}^n = \langle e_1, ..., e_n \rangle$. By Lemma 5.2.5, there exists $\Gamma_1 \leq \Gamma$ such that $b_1(\Gamma_1) = 0$. By Proposition 5.2.7, either $\Delta_3 \leq \Gamma_1 \leq \Gamma$ or there exists $\Gamma_2 \leq \Gamma_1 \leq \Gamma$ and $\mathbb{Z}^s \trianglelefteq \Gamma_1$ such that $\Gamma_2/\mathbb{Z}^s \cong \Delta_3$. Since the holonomy group of Γ is acting diagonally on $\mathbb{Z}^s$, we have $\mathbb{Z}^s \trianglelefteq \Gamma$.

Now, assume that Γ is a non-diffuse generalized Hantzsche-Wendt group. By [30, Theorem 3.1], Γ is a Bieberbach group of diagonal type. Let the holonomy group of Γ be C_2^p. We proceed by induction on p to show that $\Delta_3 \leq \Gamma$. The base case $p = 2$ is clear. Assume that the statement is true for all $p \leq k-1$ and consider $p = k$. If $b_1(\Gamma) = 0$, then by [30, Proposition 8.2], we have $\Delta_3 \leq \Gamma$. Hence we could assume $b_1(\Gamma) > 0$. By [30, Proposition 4.1], there exists $f : \Gamma \to \mathbb{Z}$ such that $ker(f)$ is an $(n-1)$-dimensional generalized Hantzsche-Wendt group. Since Γ is non-diffuse, by Proposition 5.1.5, $ker(f)$ is non-diffuse. Hence by induction hypothesis, we have $\Delta_3 \leq ker(f) \leq \Gamma$. □

Example 5.2.8. In this example, we point out that there exists a non-diffuse 7-dimensional Bieberbach group Γ of diagonal type with holonomy group isomorphic to C_2^3 which does not contain Δ_3 as subgroup. Therefore Theorem G cannot be improved. Define $A, B, C \in \Gamma$ as below.

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $\{A, B, C\}$ be set of non lattice generators of Γ. Let $p : \Gamma \to C_2^3$ be the projection map as in (2.7). The holonomy group is generated by $p(A), p(B)$ and $p(C)$. Assume by contradiction that $\Delta_3 \leq \Gamma$. Then there exists $C_2^2 \cong \langle x, y \rangle \leq \langle p(A), p(B), p(C) \rangle$ such that

$\langle \bar{x}, \bar{y} \rangle \cong \Delta_3$ for some $p(\bar{x}) = x$ and $p(\bar{y}) = y$. The holonomy group C_2^3 has seven distinct C_2^2 subgroups. Hence we need to consider all seven cases one by one. The argument for all seven cases are similar, we will present one of them and assume $C_2^2 = \langle p(A), p(B) \rangle$. In this case, we have

$$\bar{x} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 + 1/2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & x_2 + 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & x_3 + 1/2 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & x_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & x_7 + 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \bar{y} := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & y_2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & y_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & y_4 + 1/2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & y_5 + 1/2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & y_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & y_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $x_i, y_i \in \mathbb{Z}$ for $i = 1, ..., 7$. By [25, Lemma 1], since $\bar{x}$ and $\bar{y}$ are the standard generators of Δ_3, they must satisfy the relation $a^{-1}waw = I_8$ where $w = (\bar{x}\bar{y})^2$. By simple calculation, we have

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4x_1 + 4y_1 + 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It is clear that the solution of the equation $4x_1 + 4y_1 + 2 = 0$ is never integral which is a contradiction.

Bibliography

[1] A. Adem, K. Dekimpe, N. Petrosyan, and B. Putrycz, 2012, On generators of crystallographic groups and actions on flat orbifolds. Journal of Group Theory 15(4), 553-561

[2] A. Adem, J. Ge, J. Pan and N. Petrosyan, Compatible actions and cohomology of crystallographic groups. J. Algebra 320 (2008), no. 1, 341–353

[3] M. Aschbacher, R. Guralnick, Some Applications of the First Cohomology Group. Journal of Algebra 90, 446-460 (1984)

[4] B.H. Bowditch, A variation on the unique product property, J. London Math. Soc. (2), 62(3):813-826,(2000)

[5] K.S, Brown, Cohomology of groups, Springer, New York, 1982

[6] CARAT, Crystallographic AlgoRithms And Tables. (2006). Lehrstuhl B fur Mathematik, Aachen. Available at: http://wwwb.math.rwth-aachen.de/carat/

[7] L. S. Charlap, Bieberbach Groups and Flat Manifolds, Universitext, SpringerVerlag, Berlin, (1986)

[8] H. Y. Chung. Generators of Bieberbach groups with 2-generated holonomy group. Geom Dedicata **202**, 203–211 (2019). https://doi.org/10.1007/s10711-018-0409-3

[9] H. Y. Chung. Fibering flat manifolds of diagonal type and their fundamental groups. Preprint

[10] G. Cliff, A. Weiss, Torsion Free Space Groups and Permutation Lattices for Finite Groups, Contemp. Math. **93** (1989), 123-132

[11] W. C. Curtis, I. Reiner, Representation theory of finite groups and associative algebras, Wiley, New York, 1962

[12] M. Dadarlat, E. Weld, Connective Bieberbach groups, 2019, https://arxiv.org/pdf/1906.00942.pdf

[13] K. Dekimpe, Almost-Bieberbach Groups: Affine and Polynomial Structures, Lecture Notes in Mathematics, Springer-Verlag: Berlin Heidelberg, Germany, 1996, Vol. 1639

[14] K. Dekimpe and P. Penninckx, On the number of generators of a Bieberbach group, Communications in Algebra **37**, no. 12, (2009), 4332-4345

[15] David S. Dummit, Richard M. Foote, Abstract Algebra, Third edition, Wiley, 2003

[16] D. L. Goncalbes, J. P. Vieira, Infra-abelian groups and free actions of finite groups on the n-torus. (2002) Comm. Algebra 30(6):2791-2803

[17] Robert M. Guralnick, On the number of generators of a finite group, Arch. Math., Vol. 53, 521-523 (1989)

[18] N. Gupta, S. Sidki, On torsion-free metabelian groups with commutator quotients of prime exponent, International journal of algebra and computation, **9** (1999), 493-520

[19] H. Hiller, C. H. Sah, Holonomy of flat manifolds with b_1=0, Quarterly J. Math. **37** (1986), 177-187

[20] S. Kionke, J. Raimbault, On geometric aspects of diffuse groups, Doc. Math., 21:873-915, (2016) With an appendix by Nathan Dunfield.

[21] Saunders Mac Lane, Homology, Springer-Verlag, 1963

[22] P. Linnell, D. W. Morris, Amenable groups with a locally invariant order are locally indicable, 2013, https://arxiv.org/pdf/1202.4716.pdf

[23] R. Lutowski. Finite outer automorphism groups of crystallographic groups. Exp. Math. 22(2013), no. 4, 456-464

[24] R. Lutowski, N. Petrosyan, J. Popko, A. Szczepanski, Spin structures of flat manifolds of diagonal type, 2016, https://arxiv.org/pdf/1602.08585.pdf

[25] R. Lutowski, A. Szczepanski and A. Gasior, A short note about diffuse Bieberbach groups

[26] D. Passman, The algebraic structure of group rings, Intersciencce, 1997

[27] A. Prasad, Counting subspaces of a finite vector space, Resonance, vol. 15; no. 11, pages 977-987 and no. 12, pages 1074-1083, 2010

[28] I. Reiner, Integral representations of cyclic groups of prime order, Proc. Amer. Math. Soc., (1957), 142

[29] P. Ribenboim, Classical theory of algebraic numbers, Springer-Verlag, Heidelberg (2001)

[30] J. P. Rossetti, A. Szczepanski, Generalized Hantzsche-Wendt flat manifolds - Rev. Mat. Iveroamericana **21** (2005), no.3, 1053-1070

[31] A. Szczepanski, Geometry of crystallographic groups, World Scientific, 2012

[32] A. Szczepanski, Decomposition of flat manifolds, Mathematika **44** (1997), 113-119

[33] K. I. Tahara, On the finite subgroups of GL(3,Z), Nagoya Math. J. Vol 41 (1971), 169-209

[34] A. T. Vasquez, Flat Riemannian manifolds, J. Diff. Geom. **4** (1970), 367-382

[35] J. A. Wolf. Space of constant curvature, third edition, Berkeley, 1974