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Anomalies in Superconformal Field Theories

by

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Abstract

This thesis presents work related to anomalies in superconformal field theories (SCFTs). Recent holographic computations proved the existence of new supersymmetry anomalies in $\mathcal{N} = 1$ SCFTs with an anomalous R-symmetry. This was also confirmed in the context of the Wess-Zumino (WZ) consistency conditions. Motivated by these results, we provide a comprehensive analysis of the free and massless WZ model in perturbation theory.

The WZ model is classically invariant under the superconformal group $SU(2, 2|1)$, but some of these symmetries are broken by quantum anomalies. There are well known bosonic anomalies associated with R-symmetry and scale invariance, as well as fermionic anomalies associated with the gamma trace of the supercurrent (usually known as S-supersymmetry). We provide the first rigorous loop computation which shows that Q-supersymmetry of conformal supergravity is anomalous at the level of the 4-point correlators, confirming the holographic and WZ consistency conditions computations. In particular, we focus on the Ward identities of the 4-point correlator of two supercurrents and two R-currents $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. The results are verified by two different regulators, namely the cut-off and the Pauli-Villars regularization procedures. We also obtain all the standard anomalies of the WZ model that is coupled to conformal supergravity.

Our results also show that the form of the anomalies and the part of the symmetry they break depend on the multiplet of conserved currents one uses. In particular, the conformal multiplet in the renormalized theory is necessarily anomalous in Q- and S-supersymmetries, while in the Ferrara-Zumino (FZ) multiplet –the minimal massive multiplet– there exists a manifestly non anomalous combination of Q- and S- supersymmetries of the conformal multiplet. We give the counterterm that relates the two multiplets in the correlation functions of interest.

Finally, in the context of the loop computation, we shed light on many subtle issues on the regulators and the derivation of the Ward identities.

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Declaration of Authorship

I, Georgios Katsianis, declare that the thesis entitled *Anomalies in Superconformal Field Theories* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:
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CHAPTER 1

Introduction

Symmetries

Until the 20th century principles of symmetry played little role in theoretical physics. Notions of symmetry such as Galilean invariance or equivalence of inertial frames existed for example in Newton's laws of mechanics. However, conservation laws such as momentum and energy conservation were considered to be a consequence of the dynamical laws of nature rather than the symmetries that underlay these laws.

This approach changed drastically with Einstein. In 1905 Einstein considered the symmetry principle as the most fundamental feature in nature that put constrains on the allowable dynamical laws. Under the assumptions of the principle of special relativity – the requirement that the equations describing the laws of physics have the same form in all inertial frames of reference – and the invariance of the speed of light in vacuum, he was able to derive Lorentz transformations, which were first derived by Lorentz using Maxwell's equations for electromagnetism. Ten years later, generalizing the principle of relativity, i.e. assuming that physical laws are the same in all reference frames – inertial or non-inertial – Einstein constructed the theory of General Relativity which describes the dynamics of gravity. Furthermore, with the development of quantum mechanics in the 1920s symmetry principles came to play an even more fundamental role.

Symmetries of classical theories are encoded by transformations of the dynamical variables that leave the action invariant. There exist discrete symmetries, such as time reversal invariance or mirror reflection, as well as continuous symmetries, namely symmetries which are specified by parameters that can be varied continuously. Examples of such symmetries are spacetime translations and spatial rotations. Symmetries can be global and hold at all points in the spacetime under consideration, or local, which vary from point to point. Local symmetries must be gauged. If $\varepsilon(x)$ is the spacetime dependent parameter of the symmetry transformation, for the invariance of the action we need to enrich the field content and introduce a gauge field $A_\mu(x)$ that transforms under the symmetry of interest as $\delta_\varepsilon A_\mu = g\partial_\mu\varepsilon$, where g is an appropriate constant.

Global vs local symmetries

Global and gauge symmetries are distinct. In particular, a gauge symmetry is not a real symmetry; it can be thought as a redundancy in our description and all states related by a gauge transformation are physically the same state. It is an artefact of the extra degrees of freedom that we use in our theory, thus in principle, any gauge symmetry can be eliminated by going to the classically equivalent description of the system with fewer degrees of freedom. On the contrary, global symmetries are true symmetries of a theory. They do not reduce the degrees of freedom. They are physical, in the sense that states related by them may be considered ‘equivalent’, but these states are not the same. So the key difference between a gauge and a global symmetry is that one is in our theoretical description, while the other is a property of the system. For example, no manipulation can make a point charge have less spherical symmetry (global rotation symmetry), but the electromagnetic gauge symmetry can vanish if we consider the electric and magnetic fields instead of the 4-potential. The price we pay is that Maxwell’s Lagrangian is no longer written in a manifestly Lorentz covariant form.

A very important implication of global continuous symmetries is the existence of conservation laws (conserved currents) associated with them. For instance, spacetime translations lead to conservation of momentum and energy, while spatial rotations imply angular momentum conservation. The connection between global continuous symmetries and conserved currents was made in 1918 by Emmy Noether’s first theorem [1]. A nice review on Noether’s theorem can be found in [2].

Anomalies

The discussion until now concerned symmetries in the context of classical theories. In the quantum regime the situation is slightly more complicated. One quite often encounters infinities in quantum computations. They arise in higher-order perturbative calculations and typical examples are the electron’s self energy and zero point vacuum energy. To have a well defined theory we need to find a consistent way to deal with these divergent quantities. We do that by means of regularization and renormalization. Regularization consists of introducing a new parameter in our theory that encodes all divergences. Common ways to regularize a quantum field theory are to introduce a hard ultraviolet (UV) cut-off R at the integration variable of the Feynman integrals, add in the Lagrangian fictitious massive particles (Pauli-Villars (PV) regulator [3]) or promote the number of spacetime dimensions to a complex number (dimensional regularization [4,5]). In cut-off regularization, the hard cut-off parameter R automatically makes the Feynman integrals convergent. Instead of integrating over all possible values of the loop momenta, we integrate till the value R. The original theory is retrieved in the limit of $R \to \infty$. In the PV regularization, the massive fictitious particles – which come with either opposite from the original fields statistics or with wrong sign kinetic terms – contribute to the quantities of interest in such a way, that they cancel the UV divergences for finite values of the PV masses. The original theory is restored in the limit where the PV masses go to infinity and the fictitious PV fields decouple. The PV mass is the regulator parameter. Moreover, the degree of divergence of a Feynman integral depends on the number of spacetime dimensions we are working on. Promoting the number of dimensions from let us say $d = 4$ to $d = 4 - \epsilon$, makes the integrals convergent. Now the regulator parameter is ϵ and the original theory is given in the limit of $\epsilon \to 0$. After we have found a way to encode all infinities using the regulator parameters, we can introduce counterterms in the action to remove them. The last step is called renormalization.

Generally, regulators tend to break some of the symmetries of the classical theory. For example the PV regulator necessarily breaks classical conformal invariance, since the PV mass introduces a scale to the theory, while the cut-off regulator breaks translation invariance in momentum space. Depending on the computation under consideration some regulators may be more useful than others.

Going back to the discussion of symmetries, the question that arises is how we can generalize the notion of symmetry in quantum theories. The quantum counterparts of Noether's theorem are the Ward identities. These are relations among correlators that involve well defined operators and conserved currents. There are different methods to derive them. One common way is through the path integral formulation of quantum theory, or alternatively we can introduce gauge fields that couple to the conserved currents of the theory; then the Ward identities will be a consequence of the gauge invariance of the action in the presence of these sources.

The Ward identities are still classical equations and one has to compute them in the quantum theory (using Feynman diagrams for example) to verify them. To regulate divergent correlators, as mentioned, we need to include a consistent regulator. If the regulator used respects all symmetries of the classical action, then these symmetries are manifestly satisfied in quantum theory. If the regulator classically violates some of these symmetries, as the PV regulator breaks conformal invariance due to the presence of the PV mass, there going to be some new breaking terms in the original classical Ward identities of interest.

These extra terms have to be computed in the limit where the regulator vanishes (i.e. for $M \to \infty$ in PV regularization, for $\epsilon \to 0$ in dimensional regularization etc). In this limit we expect to get the original theory, however, it turns out that sometimes these extra terms have a non zero contribution. If there are no local counterterms that can be used at the level of the action to remove them, then the original Ward identities receive local contributions at the quantum level. We call the corresponding symmetries anomalous. Thus, for a field transformation to be a symmetry in the quantum theory, leaving the classical action invariant is a necessary but not a sufficient condition. At the same time, there has to exist a regulator that respects this symmetry transformation, or at least there should exist local counterterms to remove possible contributions of the regulators to the classical Ward identities. If none of these are the case, then we have a quantum anomaly.

Computing correlation functions on a flat background and checking whether the Ward identities are satisfied, was the first method that led to the original discovery of anomalies via one-loop triangle diagrams [6,7]. Anomalies also appear as lack of symmetry conservation in the presence of background fields. These are two sides of the same coin. The anomaly can also be shifted around to different symmetries by adding finite local terms in the action. What is crucial is that there exist no counterterms that render all classical symmetries non anomalous. The theories with or without a counterterm that shifts the anomaly are physically distinct, as they preserve different symmetries.

Anomalies are a cornerstone of modern quantum field theories. If a global symmetry is anomalous, classical selection rules are not respected in the quantum theory and classically forbidden processes may occur. This is a feature of the theory and it is linked with observable effects. For example, the axial anomaly explains the π^0 decay and leads to the resolution of the $U(1)$ problem in QCD [6, 7]. On the other hand, anomalies in local (gauge) symmetries lead to inconsistencies, such as lack of unitarity, and they must be cancelled. An important corollary is that anomalous global symmetries cannot be consistently coupled to corresponding local symmetries. The attempt to cancel gauge anomalies, hence build theories consistent with gauge symmetries, often leads to extra constraints on the theories. Reviews on anomalies in quantum field theories may be found in [8–10].

Supersymmetry

Since its discovery in 1970s [11–14], supersymmetry has been a subject of extensive research in the field of high energy physics. Supersymmetry, which relates elementary particles of different quantum nature – bosons with an integer-valued spin and fermions with a half-integer spin – was originally found as a non trivial extension of the Poincaré algebra. It was met with high enthusiasm due to its many appealing features. On the phenomenological side, it gives possible solutions to the hierarchy problem that afflicts the Standard Model, provides candidate particles for dark matter, explains the electroweak symmetry breaking etc ... [15].

Even though there is still no experimental evidence for its existence in nature, supersymmetry is also very useful from a theoretical point of view. If we make supersymmetry local, then we need to introduce a gauge field, the gravitino with a spin 3/2, which is the superpartner of the graviton. Local supersymmetry provides a supersymmetric version of gravity [16–19]. Moreover, supersymmetric quantum field theories in general have a better behaviour in the ultraviolet (UV) limit than their non supersymmetric versions, since many UV divergences cancel between bosonic and fermionic degrees of freedom. These results can be understood clearly in the superspace formalism, where various non-renormalization theorems have been established [20]. As the amount of supersymmetry increases, theories behave even better until we reach maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory, the first 4-dimensional quantum field theory that is UV finite [21, 22].

Extended supersymmetry helps us compute a lot of exact results, such the Seiberg-Witten potential for $\mathcal{N} = 2$ gauge theories [23, 24], the Novikov-Shifman-Vainshtein-Zakharov exact beta functions [25, 26] and many more. Finally, supersymmetry is an essential feature of superstring theory, the leading candidate for a theory of quantum gravity.

Throughout this thesis it is assumed basic knowledge of supersymmetry which can be found in the first chapters of books such as [18, 27, 28].

Anomalies in supersymmetry

Discussion of anomalies in 4d (super)conformal QFT has a long history. It has been known since the 1970s [29, 30] that the trace of the stress tensor $\mathcal{T}^{\mu}_{\mu}$ is anomalous in the presence of a curved background metric $g_{\mu\nu}$ and background source A_μ for a chiral current $\mathcal{J}_\mu$, and the R-current is similarly anomalous. Moreover, there are generally mixed anomalies involving two energy momentum tensors and a chiral current [31, 32].

It has also been known since [33] that the currents sit in a supermultiplet, as do the anomalies. In particular, the trace anomaly and the R-current anomaly are in the same multiplet as the gamma trace of the supercurrent, $\gamma^\mu \mathcal{Q}_\mu$. The latter is an anomaly in the conservation of the special supersymmetry current, $x^\nu \gamma_\nu \mathcal{Q}_\mu$. It follows that special supersymmetry (sometimes also called S-supersymmetry) is anomalous. It was believed however that supersymmetry itself (sometimes called Q-supersymmetry) is preserved, i.e. the conservation of $\mathcal{Q}_\mu$ is non-anomalous.

There have been extensive studies in the past regarding anomalies in supersymmetry. As already mentioned, the existence of anomalies is intimately related to the regularization procedure. In the presence of a manifestly supersymmetric regulator, a theory is free of supersymmetry anomalies. However, the existence of such a regulator is still a matter of debate, especially in supersymmetric gauge theories [34]. Dimensional regularization (DREG) [4, 5] is one of the most widely used regulators. It preserves gauge symmetry and plays a key role in the consistency of the standard model. The prescription in DREG is to promote the number of spacetime dimensions of the theory to a complex number d, something that explicitly breaks supersymmetry. The degrees of freedom of a spinor depend on the number of spacetime dimensions, thus an originally supersymmetric theory in 4 dimensions (i.e. with the same number of bosonic and fermionic degrees of freedom) is not supersymmetric anymore. Moreover, many algebraic manipulations needed to verify supersymmetry invariance, such as Fierz identities, are specific only to 4 dimensions. Another subtle issue is that there is no obvious definition of the Dirac matrix γ_5 in d dimensions. It is known that a totally anticommuting γ_5

$$\{\gamma_5, \gamma^\mu\} = 0 \tag{1.0.1}$$

is inconsistent and does not reproduce the standard chiral anomaly. An *ad hoc* solution to this problem was given by 't Hooft and Veltman by assuming that γ_5 anticommutes with the original 4-dimensional gamma matrices and commutes with everyone else, i.e. the following prescription

$$\begin{aligned} \{\gamma_5, \gamma^\mu\} &= 0\,, \qquad \mu = 0, 1, 2, 3 \\ [\gamma_5, \gamma^\mu] &= 0\,, \qquad \mu > 3, ..., d \end{aligned} \tag{1.0.2}$$

reproduces the usual chiral anomaly.

Initially, there were indications that DREG does not spoil supersymmetry at the quantum level [35,36]. It was shown that the massless and massive Wess-Zumino model does not have supersymmetry anomalies up to two-loop level. These results however, do not contradict our conclusions about the supersymmetry anomalies of the massless Wess-Zumino model. In their analysis, the authors of [35, 36] only considered supersymmetry Ward identities of some 2- and 3-point correlators of elementary fields. On the contrary, we are interested in anomalies that arise in correlators among conserved currents (composite operators) that were not computed there. Moreover, another important technical issue is that in [35, 36] it was assumed that the Dirac matrix γ_5 anticommutes with all gamma matrices in d dimensions in DREG. As we mentioned though, this prescription does not reproduce the correct R-symmetry anomaly of the Wess-Zumino model, something essential in our analysis. Later calculations showed that DREG violates supersymmetry in the quantum regime [37], as expected.

A modified version of DREG was proposed by Siegel [38] based on dimension reduction

(DRED). Siegel continued the spacetime dimensions from 4 to d, where d is less than 4. The momenta are treated in d dimensions, while the gamma matrices and the elementary fields are considered 4-dimensional. This approach avoids the explicit breaking of supersymmetry that arises in DREG. However, as pointed out by Siegel later, the combination of 4-dimensional gamma matrices and d-dimensional metric $g^{\mu\nu}$ leads to inconsistencies, such as the following relation

$$0 = d(d-1)(d-2)(d-3)(d-4), \tag{1.0.3}$$

which cannot be satisfied for non-integer values of d. DRED can be made mathematically 'consistent' by realising the 4-dimensional space that the gamma matrices live as a 'quasi-4-dimensional' space. The 'quasi-4-dimensional' space retains essential 4-dimensional properties but at the same time is also infinite dimensional. This new approach though, makes again the use of Fierz identities invalid. This introduces explicit breakings in supersymmetry, like the case of DREG. Reviews about DRED and its different definitions can be found in [34,39]. There are not many explicit checks of supersymmetry in DRED, however the general belief is that for all practical purposes and for important phenomenological cases there exist versions of DRED that preserve supersymmetry. We expect DRED to violate supersymmetry in 4- and higher order loop computations [40].

It was realised early on [41–49] that one cannot maintain at the quantum level simultaneously $\partial^\mu \mathcal{Q}_\mu = 0$ and $\gamma^\mu \mathcal{Q}_\mu = 0$ and, if the model is a gauge theory, gauge invariance; one of the three conditions must be relaxed. The theory under consideration in this series of papers was the $\mathcal{N} = 1$ supersymmetric Yang-Mills Lagrangian in four spacetime dimensions, which reads as

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} \bar{\psi}^a \gamma^\mu D_\mu \psi^a + \frac{1}{2} C^a C^a, \tag{1.0.4}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{1.0.5}$$

$a = 1, 2, 3$ are colour indices, ψ^a is a massless Majorana spinor, A^c_μ is a gauge field and C^a is an auxiliary field. The covariant derivative is given by

$$D_\mu \psi^a = \partial_\mu \psi^a + g f^{abc} A^b_\mu \psi^c \tag{1.0.6}$$

and the action corresponding to $\mathcal{L}$ is invariant under the following supersymmetry transformations

$$\begin{aligned} \delta_\varepsilon A^a_\nu &= i \bar{\varepsilon} \gamma_\nu \psi^a, \\ \delta_\varepsilon \psi^a &= \frac{i}{2} F^a_{\mu\nu} \gamma^{\mu\nu} \varepsilon + C^a \varepsilon, \\ \delta_\varepsilon C^a &= \bar{\varepsilon} \gamma^\mu D_\mu \psi^a. \end{aligned} \tag{1.0.7}$$

One can compute the 3-point correlator

$$<\mathcal{Q}^\mu(p_1)\bar{\psi}(p_2)A^\nu(p_3)>,$$

where $\mathcal{Q}$ is the supercurrent that corresponds to the above supersymmetry transformations. Q-supersymmetry, S-supersymmetry and gauge invariance imply the following classical equations

$$\begin{aligned} p_{1\mu} <\mathcal{Q}^\mu(p_1)\bar{\psi}(p_2)A^\nu(p_3)> &= 0, \\ \gamma_\mu <\mathcal{Q}^\mu(p_1)\bar{\psi}(p_2)A^\nu(p_3)> &= 0, \\ p_{3\nu} <\mathcal{Q}^\mu(p_1)\bar{\psi}(p_2)A^\nu(p_3)> &= 0. \end{aligned} \tag{1.0.8}$$

As pointed out by the authors of [42], the $<\mathcal{Q}^\mu(p_1)\bar{\psi}(p_2)A^\nu(p_3)>$ correlator is linearly divergent at 1-loop level. Following the same approach with the original derivation of the chiral anomaly [6,7], they showed that there is a loop momentum ambiguity in $<\mathcal{Q}^\mu(p_1)\bar{\psi}(p_2)A^\nu(p_3)>$; by shifting the loop momentum one can change the finite part of the 3-point correlator (in subsection (3.2.2) we present this method for the case of the chiral anomaly). It was shown that there is no choice of momentum routing such that all the three above classical equations are satisfied at the same time in the quantum regime. The same conclusion was reached by different people using various regularization procedures such as point-splitting, DRED, DREG etc. Depending on the regulator one uses, different symmetries are respected, however there is no choice of regulator that satisfies gauge symmetry, Q- and S- supersymmetry simultaneously. One can shift the anomaly between these symmetries with the appropriate counterterms in the action.

It is also interesting to mention the final conclusions of [49] and [48], where both studies used DRED to analyse the Yang-Mills theory (1.0.4). In [49] they concluded that $\partial_\mu \mathcal{Q}^\mu = 0$ and $\gamma_\mu \mathcal{Q}^\mu \neq 0$, which implies that Q-supersymmetry is respected , while the authors of [48] found that $\partial_\mu \mathcal{Q}^\mu \neq 0$ and $\gamma_\mu \mathcal{Q}^\mu = 0$, hence Q-supersymmetry seems to be violated. The apparent contradiction is explained by the fact that the authors of the two papers used different conventions for the gamma matrix algebra in the context of DRED. Furthermore, there are more ambiguities in this regularization procedure since the final result seems to depend on whether one performs the gamma matrix algebra before or after the loop momentum integration. As pointed out by the authors of [49] who found that Q-supersymmetry is conserved ($\partial_\mu \mathcal{Q}^\mu = 0$), the prescription they followed for DRED gives inconsistent results for the usual chiral anomaly. Of course, *ad hoc* solutions to reproduce the standard results were given in that paper, however, the main point we want to emphasize here is that there are many subtle issues within DRED and one has to be careful how to use it and interpret its results.

The common choice for the anomalies we just described, is to arrange for counterterms such that gauge symmetry and Q-supersymmetry are satisfied. This gives the standard superconformal anomaly and is distinct from the anomaly discussed in this thesis. In [43] it was shown that after extending supersymmetry of the model and adding three $\mathcal{N}=1$ chiral multiplets to the Lagrangian (1.0.4), the superconformal anomaly vanishes at 1-loop level. The resulting theory is the 4-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory[1], which is UV finite and free of anomalies.

Another set of studies, reviewed in [50], considers the quantum effective action for elementary fields and examines whether it is invariant under supersymmetry including loop effects; it investigates the conservation of the supercurrent inside correlators of elementary fields. The authors of these studies, rather than using a specific regulator they followed a more formal renormalization program, similar to the BPHZ renormalization scheme. After a systematic subtraction of momentum space integrals which respected fundamental postulates such as Lorentz invariance, unitarity and causality, they renormalized the models under consideration and showed that there is no supersymmetry anomaly. Even though their approach may not be very rigorous in the treatment of infrared divergences (as mentioned by the same authors), the general consensus is that their conclusions should be trusted. Again, this does not contradict the results we present regarding the existence of supersymmetry anomalies: to find the anomaly one should either put the theory on a non-trivial background or consider correlation functions of (classically) conserved currents (composite operators) that were not considered there[2].

Supersymmetry anomalies that have many technical similarities with the anomalies presented in this thesis, were reported in [51–54]. Particularly relevant for us is [52], where the theory under consideration is a chiral matter superfield coupled with a background vector superfield. The analysis is performed in components in the Wess-Zumino gauge, where all the unphysical degrees of freedom of the superfields were put to zero with a supergauge transformation. The only surviving fields are the physical ones. This theory contains the standard $U(1)$ chiral anomalies, which means that the generating functional, $\mathscr{W}$, is non invariant under a $U(1)$ transformation, i.e.

$$\delta_\theta \mathscr{W} \neq 0. \tag{1.0.9}$$

The supersymmetry anomaly was computed through the Wess-Zumino consistency conditions. Very schematically the procedure is the following (a more detailed example on the WZ consistency conditions is given in subsection (4.4.1)): In the presence of anomalies we

[1]The $\mathcal{N}=4$ vector multiplet, in the $\mathcal{N}=1$ language branches into three chiral and one vector multiplets.

[2]To illustrate this point, consider a free fermion in a complex representation in flat spacetime. This theory has a standard axial anomaly originating from the 3-point function of the axial current. However, if one only looks at correlators of elementary fields these are non-anomalous and the axial current inside such correlators is conserved.

have that

$$\delta_i \mathscr{W} = \int d^4x \, e \, \epsilon_i \mathcal{A}_i \,, \tag{1.0.10}$$

where $e \equiv \det(e_\mu^a)$, δ_i denotes the symmetry transformations of the theory under consideration, ϵ_i are the (local) parameters of the transformations and $\mathcal{A}_i$ are the corresponding anomalies. The variations form an algebra, $[\delta_i, \delta_j] = f_{ij}^k \delta_k$, and after using this in (1.0.10) we get the following condition

$$\int d^4x \left(\delta_i (e \, \epsilon_j \mathcal{A}_j) - \delta_j (e \, \epsilon_i \mathcal{A}_i) - f_{ij}^k e \, \epsilon_k \mathcal{A}_k \right) = 0 \,, \tag{1.0.11}$$

namely the WZ consistency condition.

Going back to the analysis of [52], we denote by δ_ε the supersymmetry transformation of the theory in the WZ gauge. Supersymmetry commutes with $U(1)$ symmetry, thus we have

$$[\delta_\varepsilon, \delta_\theta] \mathscr{W} = 0 \Rightarrow \delta_\theta \delta_\varepsilon \mathscr{W} = \delta_\varepsilon \delta_\theta \mathscr{W} \,. \tag{1.0.12}$$

Since the supesymmetry variation of the chiral anomaly is non trivial, $\delta_\varepsilon \delta_\theta \mathscr{W} \neq 0$, we get that

$$\delta_\varepsilon \mathscr{W} \neq 0, \tag{1.0.13}$$

i.e. supersymmetry is anomalous. The above WZ consistency condition can be seen as an equation to determine the form of the supersymmetry anomaly. Arguments that the anomaly cannot be removed with local counterterms can be made using the local algebra that the symmetry variations, δ_i, satisfy. A similar analysis and conclusions were reported in [54]. The key difference between the two studies, is that in [52] the fields of the vector multiplet are non dynamical, hence the chiral symmetry is global and the corresponding anomaly does not render the theory inconsistent. The supersymmetry anomaly induced by the global chiral anomaly is 'physical', in the sense that potentially it can have observable implications. On the contrary, the gauge field of the vector multiplet in [54] is dynamical (propagating) and couples to the chiral current. For the theory to be consistent, the gauge chiral anomaly needs to cancel; this will also make supersymmetry non-anomalous.

At first sight, the above results may seem to be in conflict with superspace calculations which showed that all supersymmetry anomalies are cohomologically trivial [55]. Moreover, a non trivial result is that there exists in superspace a non-vanishing supersymmetric version of the chiral anomaly [56–58]. These facts were often used to argue about the non existence of supersymmetry anomalies. Consider for example the supersymmetry variation in the full superspace, which we denote by δ_ζ, and the variation δ_Ω that gives rise to the supersymmetric chiral anomaly. Now the WZ consistency conditions read as

$$[\delta_\zeta, \delta_\Omega] \mathscr{W} = 0 \Rightarrow \delta_\Omega \delta_\zeta \mathscr{W} = \delta_\zeta \delta_\Omega \mathscr{W} \,. \tag{1.0.14}$$

Since the chiral anomaly is supersymmetric, the above rhs is zero. This means that in contrast to (1.0.12), equation (1.0.14) admits a solution where the effective action is invariant under supersymmetry, i.e.

$$\delta_\zeta \mathscr{W} = 0, \tag{1.0.15}$$

in accordance with the superspace results. So the existence of a supersymmetry anomaly is not needed to satisfy the WZ consistency conditions. Note however, that superspace is an enlarged space with many degrees of freedom and it is supersymmetric by construction. If we want to examine a physical theory [3] that has a superspace formulation, we have to gauge-away all unphysical degrees of freedom in order to match the number of degrees of freedom of the microscopic model. The usual choice is the WZ gauge, where we keep only the physical fields. The WZ gauge though violates supersymmetry explicitly, i.e. after a supersymmetry transformation δ_ζ, we get out of it. However, we can assign a compensating gauge transformation that brings us back to the WZ gauge. The supersymmetry transformation δ_ε in the WZ gauge is given by

$$\delta_\varepsilon = \delta_\zeta + \delta_{\Omega(\zeta)}, \tag{1.0.16}$$

where $\delta_{\Omega(\zeta)}$ is the compensating gauge transformation, with the parameter fixed in such a way that the WZ gauge is restored. The supersymmetry variation of $\mathscr{W}$ in the WZ gauge is equal to

$$\delta_\varepsilon \mathscr{W} = \delta_{\Omega(\zeta)} \mathscr{W} \neq 0. \tag{1.0.17}$$

We see that the whole information about the supersymmetry anomalies of the physical theories is encoded in the supersymmetric version of the chiral anomaly $\delta_\Omega \mathscr{W}$. In the WZ gauge, which is a totally acceptable starting point to examine the properties of a theory, the gauge anomaly of the superspace is transferred to the supersymmetry sector. There is no contradiction anywhere. Superspace and the WZ gauge, is just another way to understand the supersymmetry anomalies that one finds in physical theories.

We should also emphasize here, that if we want to investigate whether a physical model is anomalous or not starting from the superspace results, we must be very careful to gauge away all the non-physical degrees of freedom that can act as compensators for the anomalies. There is always this choice to 'hide' an anomaly using compensating fields (fields that can be put to zero using the gauge freedom). An example of how this can be done is presented in section (3.3). As explained in [59], in $\mathcal{N} = 1$ conformal supergravity one can use a chiral multiplet as a compensator, to hide the anomalies of the R-symmetry, S-supersymmetry and of the scale invariance. If one for example starts from the superspace

[3] By physical theory we mean a model that does not contain extra degrees of freedom which can be put to zero with a gauge transformation.

formalism and keeps this compensating chiral multiplet, the theory would naively look non-anomalous. The anomalies would still be there though, and they would be visible in the Ward identities of the physical theory. Using this compensating chiral multiplet, one can construct the FZ multiplet of old minimal supergravity. The FZ multiplet has more degrees of freedom than the conformal multiplet, so now the auxiliary field of the chiral compensating multiplet cannot be put to zero by a gauge transformation, since it is necessary for the off-shell closure of the supersymmetry algebra; it becomes part of the FZ multiplet, hence it is no longer a compensator. In this case, as it was shown in [59], the theory can be made consistent at the quantum level. This is in total agreement with the loop computations presented in this thesis. Also relevant for us is the discussion in [60] about compensators in supergravity.

A supersymmetry anomaly appears also in theories with gravitational anomalies [61–63], as one may anticipate based on the fact that the energy momentum tensor and the supercurrent are part of the same supermultiplet. Indeed this supersymmetry anomaly sits in the same multiplet as the gravitational anomaly. However, since gravitational anomalies arise only in $4n+2$ dimensions, they are irrelevant for our analysis.

Finally, anomalies associated with correlation functions of conserved currents can be analysed by coupling the currents to external sources, which in our case form an $\mathcal{N}=1$ superconformal multiplet. As such, the anomalies we discuss in this thesis could be related to existing superspace results on anomaly candidates for $D=4$, $\mathcal{N}=1$ supergravity theories [55,64–67] (in particular, in type II anomalies in [66]), though we emphasise that in our case the supergravity fields are external and thus non-dynamical (off-shell). We should also stress that there has never been a loop computation (that we know about) involving the anomalies that we consider.

Holographic anomalies

The anomaly we discuss here was first computed holographically [68]. In holography, given a bulk action, one can use holographic renormalisation [69, 70] to compute the Ward identities and anomalies of the dual QFT. AdS/CFT relates $\mathcal{N}=1$ SCFT in four dimensions to $\mathcal{N}=2$ gauged supergravity in five dimensions. Starting from gauged supergravity in an asymptotically locally AdS_5 spacetime and turning on sources for all superconformal currents one can compute the complete set of superconformal anomalies. This computation is available for holographic CFTs, which in particular means that the central charges should satisfy $a=c$ as $N\to\infty$ [69]. The anomaly for general a, c was obtained in [71] by solving the WZ consistency conditions [72] under the assumption that R-symmetry is only broken by the standard triangle anomaly.

Early attempts to compute the supertrace Ward identity can be found in [73, 74] but these missed contributions to the anomaly involving the R-symmetry current and the Ricci tensor. Following the work of Pestun [75], there was renewed interest in supersymmetric theories on curved spacetimes and their holographic duals. The holographic anomalies for bosonic currents were computed in [76], reproducing (and correcting) known field theory results [77]. The full superconformal anomalies for the $\mathcal{N} = 1$ current multiplet were computed holographically in [68], while [78] obtained the superconformal anomalies in the presence of local supersymmetric scalar couplings. An analogous holographic computation relevant to two-dimensional SCFTs was reported in [79].

Thesis outline

Motivated by the holographic results of [68] and the WZ consistency conditions in [71], we analyse the rigid supersymmetry anomalies for the four dimensional massless Wess-Zumino model in perturbation theory [14], the simplest $\mathcal{N} = 1$ superconformal model. Since this model is free, the 1-loop computation is exact. The supersymmetry anomaly appears for first time in the 4-point function that involves two supercurrents and either two R-symmetry currents or one R-symmetry current and one energy momentum tensor, i.e.

$$\begin{aligned} \partial_\mu <\mathcal{Q}^\mu \bar{\mathcal{Q}}^\nu \mathcal{J}^\kappa \mathcal{J}^\lambda> + \ldots &= \mathcal{A}^{\nu\kappa\lambda}, \\ \partial_\mu <\mathcal{Q}^\mu \bar{\mathcal{Q}}^\nu \mathcal{T}^{\rho\sigma} \mathcal{J}^\lambda> + \ldots &= \mathcal{A}^{\nu\rho\sigma\lambda}, \end{aligned} \tag{1.0.18}$$

where $\mathcal{Q}$, $\mathcal{J}$, $\mathcal{T}$ are the supercurrent, R-symmetry current and energy momentum tensor respectively. The dots denote standard lower order correlators in the supersymmetry Ward identities and $\mathcal{A}^{\nu\kappa\lambda}$, $\mathcal{A}^{\nu\rho\sigma\lambda}$ are the new anomalies. In this thesis we focus on $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

Chapter 2 provides the proof of Noether's theorem and its quantum analogue, the Ward identity, both of which are consequences of the symmetries of a theory. In chapter 3 we discuss regulators and focus on the cut-off and Pauli-Villars regulators. We show how the axial anomaly of the free Dirac fermion arises in both of these regularization procedures. The main aim of chapters 2 and 3 is to provide the necessary technical tools for the analysis of the anomalies of the WZ model.

In chapter 4 we review the massless WZ model. We describe its classical symmetries and derive all the Ward identities of the correlators needed for the computation of the supersymmetry anomalies of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. We also provide consistency condition arguments that show how the standard R-symmetry anomaly induces anomalies in supersymmetry.

All correlators involved in the Ward identities of chapter 4 are divergent, so we need to regulate our theory. A standard way to continue would be to regulate with DRED. However, there exist various prescriptions on how we should handle the gamma matrices within DRED, each one with its own advantages and disadvantages. We are interested in the Ward identities of 4-point correlators and our analysis demands extensive gamma matrix algebra manipulations. We would like to avoid all the possible subtleties of DRED that could arise in that context, thus we proceed using a PV regulator. We find that all diagrams we compute are regulated using as PV fields three massive $\mathcal{N} = 1$ chiral multiplets, one with standard statistics and two with 'wrong' statistics, with masses appropriately correlated. Chapters 5 and 6 provide the description of the aforementioned PV regulator.

Since the regulated action we use is supersymmetric, one may wonder whether this already shows that there is no supersymmetry anomaly. To establish the absence of a supersymmetry anomaly we still need to couple the symmetry currents to sources supersymmetrically. Since the PV fields form massive supermultiplets, they break conformal symmetry, R-symmetry and S-supersymmetry and as such they cannot couple supersymmetrically to conformal supergravity. Instead, one can consistently couple the regulated theory to old minimal supergravity. The supersymmetry of old minimal supergravity can be identified with a field dependent linear combination of the Q- and S-supersymmetry of conformal supergravity. This means that in the presence of the PV regulator, the original R-symmetry, Q- and S-supersymmetry Ward identities of the conformal WZ model contain extra breaking terms that depend only on the PV masses.

In chapter 7 we find from a bottom up perspective (without invoking old minimal supergravity, but using only path integral identities of the flat space theory) the regulated version of the Ward identities of chapter 4. We identify the appropriate local counterterms to restore, whenever possible, the symmetries broken by the regulator, and in the end we confirm the anomalies derived through the WZ consistency conditions [71]. Of course, the supersymmetry of old minimal supergravity is manifestly non anomalous, something that we verify in chapter 8. In chapter 9 we use the cut-off regulator to confirm the Q-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

A series of appendices follow. In appendix A we give our conventions, while in appendix B we write the anomalies of $\mathcal{N} = 1$ SCFTs in the presence of the background sources (which we confirmed via a 1-loop computation). In appendix C we provide a comprehensive analysis on the derivation of the Ward identities. Appendix D contains the expressions of all regulated correlators that we use, and in appendix E we present the explicit computation of integrals that comprise the potential anomalous terms in the classical symmetry Ward identities of the WZ model. Important results are also given in the appendix F. There, we present the renormalized correlators among seagull operators and conserved currents (up to the 3-point function level), which satisfy all their classical Ward identities; we confirm via a 1-loop computation that seagull correlators are non anomalous in the WZ model, as expected. In appendix G we briefly review the symmetries of old minimal supergravity and in appendix H we give the expressions of the correlators in the cut-off regulator.

Most of the work presented in this thesis can be also found in [80, 81].

CHAPTER 2

Symmetries

Symmetries are a concept of great importance in physics. They implement constraints on the structure of the theory and are commonly used to simplify calculations. In a classical field theory, symmetry is defined as any continuous or discrete transformation that leaves the action invariant. Reflection is an example of a discrete symmetry, where there exist one or more than one lines such that, the first half of a physical system is a mirror image of the second half. Examples of continuous symmetry transformations are time and spatial translations, where the physical system has the same properties over a certain time interval and after a change in location. These are part of the Poincaré symmetry group. Another common continuous symmetry, is supersymmetry, which relates two basic classes of elementary particles, bosons, which have an integer-valued spin, and fermions, which have a half-integer spin.

Classical continuous symmetries, are associated with conservation laws according to Noether's theorem [1]. Time translation symmetry implies energy conservation, while spatial translation symmetry implies momentum conservation. The question that arises, is whether these classical symmetries survive in the full quantum theory. It is quite often in quantum computations to encounter divergent quantities. Then, we need to introduce a parameter which helps us deal with them, namely the regulator. For a classical symmetry to survive in the quantum regime, it also needs to respect the regularization procedure. As we will see in the next chapter, this is not always the case.

2.1 Noether's theorem

Let S be the action of an arbitrary classical field $\phi(x)$

$$S = \int d^4x \mathcal{L}(\phi, \partial\phi, x). \tag{2.1.1}$$

Suppose there is an infinitesimal continuous transformation of the field

$$\delta_\varepsilon \phi(x) = \varepsilon \Delta(\phi, \partial\phi, x) \tag{2.1.2}$$

that changes the Lagrangian density $\mathcal{L}$ by a total derivative

$$\delta_\varepsilon \mathcal{L} = \varepsilon \partial_\mu \Lambda^\mu \tag{2.1.3}$$

or equivalently leaves the action invariant up to a boundary term

$$\delta_\varepsilon S = \varepsilon \int d^4x \partial_\mu \Lambda^\mu, \tag{2.1.4}$$

where ε is a small constant parameter. The above continuous global (ε independent of x) field transformation is called a symmetry transformation. The variation of $\mathcal{L}$ under this symmetry transformation is given by:

$$\begin{aligned} \delta_\varepsilon \mathcal{L} &= \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \delta_\varepsilon \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta_\varepsilon \phi \right) \\ &= \varepsilon \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \Delta + \varepsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Delta \right) \end{aligned} \tag{2.1.5}$$

If we consider the Euler-Lagrange equations of motion the first term of the above rhs side vanishes. Using (2.1.3) we get

$$\partial_\mu j^\mu \equiv \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Delta - \Lambda^\mu \right) = 0. \tag{2.1.6}$$

The quantity j^μ is conserved. We can summarize this proof into a theorem, known as Noether's theorem [1]:

Every continuous global transformation of the fields that leaves the action invariant up to surface terms, gives rise to a corresponding conserved current j^μ, such that

$$\partial_\mu j^\mu = 0. \tag{2.1.7}$$

As an example let us consider the action of a complex scalar field

$$S = \int d^4x \left(-\partial_\mu \phi^* \partial^\mu \phi - V(|\phi|^2) \right) . \tag{2.1.8}$$

It is easy to check that the action is invariant under a global $U(1)$ transformation $\phi' \to e^{i\varepsilon}\phi$ which in infinitesimal form is written as:

$$\delta_\varepsilon \phi = i\varepsilon\phi, \qquad \delta_\varepsilon \phi^* = -i\varepsilon\phi^* \tag{2.1.9}$$

Using (2.1.6) we find that the conserved current that corresponds to the $U(1)$ global symmetry is given by:

$$j^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) \tag{2.1.10}$$

Alternatively, one could introduce a background (i.e. non dynamical) gauge field A_μ and replace the partial derivative ∂_μ with the gauge covariant derivative D_μ in (2.1.8)

$$S = \int d^4x \left(-D_\mu\phi^* D^\mu\phi - V(|\phi|^2)\right), \tag{2.1.11}$$

where

$$D_\mu\phi = \partial_\mu\phi + iA_\mu\phi\,, \quad D_\mu\phi^* = \partial_\mu\phi^* - iA_\mu\phi^*\,. \tag{2.1.12}$$

It is straightforward to show that the above action is invariant under the transformation (2.1.9) with a local parameter $\varepsilon(x)$, if at the same time we associate a $U(1)$ transformation of the gauge field of the following form

$$A'_\mu \to A_\mu - \partial_\mu\varepsilon\,. \tag{2.1.13}$$

The Noether's theorem now translates as the invariance of the action under the corresponding local $U(1)$ transformation that acts on the gauge field A_μ. We have that

$$\int d^4x \mathcal{J}^\mu \delta_\varepsilon A_\mu = 0 \Rightarrow \int d^4x \mathcal{J}^\mu(-\partial_\mu\varepsilon) = \int d^4x \left(\partial_\mu \mathcal{J}^\mu\right)\varepsilon = 0, \tag{2.1.14}$$

where in the second equation we integrated by parts. Since ε is arbitrary, the integrand of the last equation must be zero, i.e.

$$\partial_\mu \mathcal{J}^\mu = 0\,. \tag{2.1.15}$$

$\mathcal{J}^\mu$ is the $U(1)$ conserved current in the presence of the background gauge field and is defined as the functional derivative of the action with respect to A_μ

$$\mathcal{J}^\mu \equiv \frac{\delta S[A]}{\delta A_\mu}\,. \tag{2.1.16}$$

Using (2.1.11) we find that

$$\mathcal{J}^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) - 2A^\mu\phi^*\phi\,. \tag{2.1.17}$$

In the limit where $A_\mu \to 0$, the two currents j^μ and $\mathcal{J}^\mu$ coincide.

2.2 Ward identities

The quantum counterparts to Noether's theorem are the Ward identities. These are relations between correlation functions that follow from global or gauge (local) symmetries of the theory. A common way to derive them is through the path integral formulation of quantum theory. A detailed analysis on the Ward identities can be found in [82–84]. Here, we follow a similar procedure with Noether's theorem proof.

Consider an action $S[\phi]$, where we make a transformation of the form $\phi' \to \phi + \varepsilon\delta\phi$ with a local parameter $\varepsilon(x)$. The action changes as

$$\delta_\varepsilon S = \int d^4x \left(-j^\mu \partial_\mu \varepsilon(x) + A(x)\varepsilon(x)\right). \tag{2.2.1}$$

If $A(x)$ is equal to zero, the transformation we performed is a symmetry of the action, and j^μ is the corresponding conserved current. Now consider some local bosonic operators $\mathcal{O}_i[\phi, \partial_\mu\phi]$ that transform as

$$\mathcal{O}_i[\phi'] \to \mathcal{O}_i[\phi + \varepsilon\delta\phi] = \mathcal{O}_i[\phi] + \delta_\varepsilon \mathcal{O}_i, \tag{2.2.2}$$

where to lowest order

$$\begin{aligned}
\delta_\varepsilon \mathcal{O}_i &= \frac{\partial \mathcal{O}_i}{\partial\phi}\varepsilon\delta\phi + \frac{\partial \mathcal{O}_i}{\partial(\partial_\mu\phi)}\partial_\mu(\varepsilon\delta\phi) \Rightarrow \\
\delta_\varepsilon \mathcal{O}_i &= \varepsilon\left(\frac{\partial \mathcal{O}_i}{\partial\phi}\delta\phi + \frac{\partial \mathcal{O}_i}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi\right) + \partial_\mu\varepsilon \frac{\partial \mathcal{O}_i}{\partial(\partial_\mu\phi)}\delta\phi \equiv \varepsilon\delta\mathcal{O}_i + \partial_\mu\varepsilon\delta\mathcal{O}_i^{\prime\mu}.
\end{aligned} \tag{2.2.3}$$

Our aim is to find a relation for the correlation functions of the operators $\mathcal{O}_i$ and the current j^μ. We have

$$\begin{aligned}
&\int [d\phi] e^{iS[\phi]} \prod_{i=1}^n \mathcal{O}_i[\phi(x_i)] = \int [d\phi'] e^{iS[\phi']} \prod_{i=1}^n \mathcal{O}_i[\phi'(x_i)] \Rightarrow \int [d\phi] e^{iS[\phi]} \prod_{i=1}^n \mathcal{O}_i(x_i) \\
&= \int [d\phi] e^{i\left(S[\phi] - \int d^4x (j^\mu \partial_\mu \varepsilon(x) - A(x)\varepsilon(x))\right)} \left[\prod_{i=1}^n \mathcal{O}_i(x_i) + \sum_{i=1}^n \delta_\varepsilon \mathcal{O}_i(x_i) \prod_{j\neq i}^n \mathcal{O}_j(x_j)\right]
\end{aligned} \tag{2.2.4}$$

where $[d\phi]$ is the path integral measure. The equation of the first line is trivial, since we just relabelled the fields. In the second line we used (2.2.1), and the fact that the path integral measure is invariant under the field transformation. Note however, that the invariance of the path integral measure is just an assumption at this point. As we will see in the following sections, at the quantum level this is not always the case. After Taylor expanding the exponential at the rhs of (2.2.4) we get to lowest order in ε

$$-i\int d^4x\varepsilon(x)\left(\partial^x_\mu <j^\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)> + <A(x)\prod_{i=1}^n\mathcal{O}_i(x_i)>\right) = \sum_{i=1}^n <\delta_\varepsilon\mathcal{O}_i(x_i)\prod_{j\neq i}^n\mathcal{O}_j(x_j)>, \tag{2.2.5}$$

where we use the notation

$$\int[d\phi]\mathcal{O}_i(x_i)e^{iS[\phi]} \equiv <\mathcal{O}_i(x_i)> \ . \tag{2.2.6}$$

We write $\delta_\epsilon\mathcal{O}_i(x_i)$ in the following form

$$\begin{aligned}\delta_\varepsilon\mathcal{O}_i(x_i) &= \int d^4x\delta(x-x_i)\left(\varepsilon(x)\delta\mathcal{O}_i(x) + (\partial^x_\mu\varepsilon(x)\delta\mathcal{O}'^\mu_i(x)\right) \Rightarrow \\ \delta_\varepsilon\mathcal{O}_i(x_i) &= \int d^4x\varepsilon(x)\left(\delta(x-x_i)\delta\mathcal{O}_i(x_i) - \partial^x_\mu\left(\delta(x-x_i)\delta\mathcal{O}'^\mu_i(x)\right)\right) \ .\end{aligned} \tag{2.2.7}$$

After substituting (2.2.7) to (2.2.5) we find

$$\begin{aligned}-i\partial^x_\mu <j^\mu(x)\prod_{i=1}^n\mathcal{O}_i(x_i)> -i <A(x)\prod_{i=1}^n\mathcal{O}_i(x_i)> &= \sum_{i=1}^n\delta(x-x_i) <\delta\mathcal{O}_i(x_i)\prod_{j\neq i}^n\mathcal{O}_j(x_j)> \\ &-\sum_{i=1}^n\partial^x_\mu\left(\delta(x-x_i) <\delta\mathcal{O}'^\mu_i(x)\prod_{j\neq i}^n\mathcal{O}_j(x_j)>\right) \ .\end{aligned} \tag{2.2.8}$$

This is the Ward identity that the operators $\mathcal{O}_i$ and the current j^μ satisfy. Note here that $\mathcal{O}_i$ are bosonic (commuting) operators. In case of fermionic (anticommuting) operators, one has to take into account the extra minus signs that come from the change of ordering. In various steps of the above whole analysis we integrated by parts and ignored the boundary terms. The $<\cdot>$ correlators are computed by the standard Wick contractions of the elementary fields that are involved. In the special case where all $\mathcal{O}_i$ operators are equal to one and $A(x) = 0$ (symmetry transformation) we get

$$\partial_\mu <j^\mu(x)> = 0, \tag{2.2.9}$$

which is the conservation law of the classical Noether's current.

Now let us see an example of how we can use (2.2.8) to find the Ward identity of the 3-point function $<j^\mu(x)j^\kappa(y)j^\lambda(z)>$. We are interested in the action (2.1.8) and the $U(1)$ symmetry transformation (2.1.9). The $U(1)$ variation of the current is given by

$$\delta_\varepsilon j^\kappa = -2\partial^\kappa\varepsilon\,\phi^*\phi. \tag{2.2.10}$$

Taking into account that this is a symmetry transformation (i.e. $A(x) = 0$), (2.2.8) gives

$$
\begin{aligned}
&- i\partial_\mu^x <j^\mu(x)j^\kappa(y)j^\lambda(z)> \\
&+ 2\partial_\mu^x \Big(\delta(x-y)\eta^{\mu\kappa} <(\phi^*\phi)(x)j^\lambda(z)> +\delta(x-z)\eta^{\mu\lambda} <(\phi^*\phi)(x)j^\kappa(y)>\Big) = 0\,. \qquad (2.2.11)
\end{aligned}
$$

In the previous section we showed that an alternative way to derive Noether's theorem is to introduce background fields that couple to the conserved currents. The same approach can be applied to the derivation of the Ward identities of correlation functions. In the example under consideration, the appropriate action is again (2.1.11). The gauge invariance of the action implies that

$$\partial_\mu \langle \mathcal{J}^\mu \rangle_s = 0\,. \qquad (2.2.12)$$

This is the analogue of (2.1.15) but we consider it as a 1-point function now. The subscript s denotes that we are in the presence of the background sources. After taking two more functional derivatives with respect to the gauge field, we get

$$\frac{\delta}{\delta A_\lambda(z)}\frac{\delta}{\delta A_\kappa(y)}\partial_\mu^x \langle \mathcal{J}^\mu(x)\rangle_s = \partial_\mu^x \langle \mathcal{J}^\mu(x)\mathcal{J}^\kappa(y)\mathcal{J}^\lambda(z)\rangle_s = 0 \qquad (2.2.13)$$

and in the limit that the background fields go to zero (flat space limit)

$$\partial_\mu^x \langle j^\mu(x)j^\kappa(y)j^\lambda(z)\rangle = 0\,. \qquad (2.2.14)$$

Notice that the Ward identities (2.2.14) and (2.2.11) are not the same. The apparent contradiction comes from the different definitions of the correlators $\langle\cdot\rangle$ and $<\cdot>$. They differ by semilocal correlators that involve functional derivatives of the conserved current with respect to the gauge field, i.e. $\frac{\delta \mathcal{J}^\mu(x)}{\delta A_\kappa(y)}$. As already mentioned, the $<\cdot>$ correlators are the ones computed using Wick contractions/Feynman diagrams. Using (2.1.17), (2.2.14) can be written as

$$
\begin{aligned}
&- i\partial_\mu^x <j^\mu(x)j^\kappa(y)j^\lambda(z)> +4\eta^{\kappa\lambda}\delta(y-z)\partial_\mu^x <j^\mu(x)(\phi^*\phi)(y)> \\
&+ 2\partial_\mu^x \Big(\delta(x-y)\eta^{\mu\kappa} <(\phi^*\phi)(x)j^\lambda(z)> +\delta(x-z)\eta^{\mu\lambda} <(\phi^*\phi)(x)j^\kappa(y)>\Big) = 0. \qquad (2.2.15)
\end{aligned}
$$

The above equation is exactly the same with (2.2.11) besides the second term of the lhs. This term satisfies its own classical Ward identity though, i.e.

$$\partial_\mu^x <j^\mu(x)(\phi^*\phi)(y)>= 0\,. \qquad (2.2.16)$$

We see that the Ward identities derived with functional differentiation and with the path integral formalism coincide. In particular, the functional derivative Ward identities are a linear combination of path integral Ward identities. These are the two most common ways to derive such identities. In particular, the path integral method is referred mainly in textbooks and used in old field theory computations, while the functional derivative method is mainly used in modern computations. In the appendix C we provide a more detailed analysis on the difference between these two methods.

We should emphasize that even though we are talking about correlation functions, all of the above manipulations are classical. The identities derived with both methods are classical identities that arise because of classical (symmetry) transformations. In the quantum regime when one calculates correlators, encounters infinite expressions that need to be dealt with caution. Sometimes the quantization procedure fails to respect the classical Ward identities.

CHAPTER 3

Anomalies

3.1 Regularization

In quantum field theories, we encounter many apparent divergences. Physical quantities though are finite, therefore in renormalizable theories divergences appear only at intermediate stages of calculations. In the end they get cancelled one or the other way. However, these divergences pose technical problems in dealing with them. We need consistent methods to manipulate them and extract finite answers. For that we introduce a new parameter, let us say ϵ, to the divergent quantity O. The quantity is now a function of ϵ, $O(\epsilon)$. For finite values of the parameter ϵ, $O(\epsilon)$ is also finite, i.e. $|O(\epsilon)| < \infty$. Then we say that the divergent quantity O is regularized by the regulator ϵ. At the end of the computation we take the limit where the regulator vanishes and we recover the original theory.

One of the main issues of regularization is that a regulator tends to break certain symmetries of the original theory. Its usefulness depends on what symmetries it retains, how easy it is to deal with, how widely it can be used, etc. In quantum field theories, we see cut-off regularization, Pauli–Villars regularization [3], dimensional regularization [4,5] and many more. In the following subsections we elaborate on the cut-off and Pauli-Villars regulators, which we use later for the analysis of the Wess-Zumino model.

3.1.1 Cut-off regulator

Consider the following integral in Euclidean space

$$\Sigma = \int_0^\infty dk_E \frac{k_E^3}{(k_E^2+m^2)^2}\,, \tag{3.1.1}$$

which is a quite typical quantity that one would have to compute in simple quantum field theories, such as the free boson in four dimensions. By power counting we see that the above integral is logarithmically divergent, so we need to regulate it. The easiest and most naive way to do this, is to introduce an ultraviolet cut-off in the loop momentum k_E, which we denote by the parameter R. The regulated integral will be written as

$$\begin{aligned}\Sigma_{reg} &= \lim_{R\to\infty}\int_0^R dk_E \frac{k_E^3}{(k_E^2+m^2)^2} = \lim_{R\to\infty}\frac{1}{2}\left(-\frac{R^2}{m^2+R^2}+\log[\frac{m^2+R^2}{m^2}]\right)\\ &= -\frac{1}{2}+\frac{1}{2}\lim_{R\to\infty}\log[\frac{m^2+R^2}{m^2}].\end{aligned} \tag{3.1.2}$$

The parameter R enables us to isolate the divergent part of the integral and remove it in a consistent way so we can make predictions within our theory.

In practice, this type of brute force cut-off is not usually preferred, since it breaks almost every symmetry of the original theory, such as translation invariance $k_E \to k_E + a$ in momentum space. However, there are cases when this approach simplifies significantly quantum computations, especially when we are interested in the difference between linearly divergent integrals that are related by a shift of their integration variable.

Shifting the integration variable of a divergent integral is illegitimate and can produce extra finite (or divergent) surface terms. In a Feynman diagram though, the choice of the internal momentum that we are integrating over should not have any physical meaning. Two Feynman diagrams that are related by a shift of the integration variable should correspond to the same physical procedure. If the diagrams are convergent, the shift does not have any effect. If they diverge, they differ by finite or even divergent terms (depending on the degree of divergence). Let us explain how we can compute these extra terms and give a few examples. A detailed analysis of divergent integrals in the cut-off regularization can be found in [85].

Suppose we have the following divergent integral in Minkowski space

$$K = \int d^4k f(k)\,. \tag{3.1.3}$$

We are interested in the quantity

$$\Delta(a) = \int d^4k[f(k+a) - f(k)]$$

$$= \int d^4k[a^\tau \partial_\tau f(k) + \frac{1}{2}a^\tau a^\sigma \partial_\tau \partial_\sigma f(k) + \frac{1}{6}a^\tau a^\sigma a^\rho \partial_\tau \partial_\sigma \partial_\rho f(k) + ...], \tag{3.1.4}$$

where a is a constant 4-vector and $\partial_\tau \equiv \frac{\partial}{\partial k^\tau}$. In the second line we Taylor expanded $f(k+a)$. Each term of (3.1.4) can be computed by integrating over the surface $k^\mu = R^\mu$ using Gauss's theorem. For simplicity the integral is taken over the boundary S^3 at $|R| \to \infty$. $|R|$ is the radius of the hypersphere S^3 and we consider it as the cut-off regulator parameter. Suppose now that K is a superficially linearly divergent integral. All but the first term in (3.1.4) vanish in the limit of $|R| \to \infty$. For the case of four-dimensional Minkowski space we have

$$\Delta(a) = a^\tau \int d^4k \partial_\tau f(k) = 2i\pi^2 a^\tau \lim_{R\to\infty} R^2 R_\tau f(R). \tag{3.1.5}$$

If K was superficially quadratically divergent, we would also have a contribution to $\Delta(a)$ from the second term of (3.1.4). Then, we would have to compute the partial derivative of $f(k)$ with respect to k^σ and integrate using Gauss's theorem, just like in (3.1.5). Let us give now some examples of integrals that appear very often in the computations of chapter 9.

- $a^\tau \int d^4k \partial_\tau (\frac{1}{k^2})$

  The above integral is zero because of odd symmetry with respect to ∂_τ.

- $a^\tau \int d^4k \partial_\tau (\frac{k_\alpha}{k^4})$

$$a^\tau \int d^4k \partial_\tau (\frac{k_\alpha}{k^4}) = 2i\pi^2 a^\tau \lim_{R\to\infty} R^2 R_\tau \frac{R_\alpha}{R^4} = 2i\pi^2 a^\tau \frac{1}{4}\delta_{\alpha\tau} \tag{3.1.6}$$

  For $\alpha \neq \tau$ the integral is zero because of odd symmetry. So we replace $R_\alpha R_\tau = \frac{1}{4}\delta_{\alpha\tau} R^2$, where $\delta_{\alpha\tau}$ is the Euclidean metric.

- $a^\tau \int d^4k \partial_\tau (\frac{k_\alpha}{k^2(k+q)^2})$

  After expanding $\frac{1}{(k+q)^2} = \frac{1}{k^2} - \frac{2k_\lambda q^\lambda}{k^4} + ...$ we get

$$a^\tau \int d^4k \partial_\tau (\frac{k_\alpha}{k^4} - \frac{2k_\lambda k_\alpha q^\lambda}{k^6} + ...)$$

  The second term of the above integral is convergent by power counting (it also vanishes due to odd symmetry), hence it will have no contribution after we use Gauss's theorem. Only the first term survives so

$$a^\tau \int d^4k \partial_\tau (\frac{k_\alpha}{k^2(k+q)^2}) = 2i\pi^2 a^\tau \frac{1}{4}\delta_{\alpha\tau} \tag{3.1.7}$$

- $a^\tau \int d^4k \partial_\tau (\frac{k_\alpha}{k^2})$

$$a^\tau \int d^4k \partial_\tau (\frac{k^\alpha}{k^2}) = 2i\pi^2 a^\tau \lim_{R\to\infty} R^2 R_\tau \frac{R_\alpha}{R^2} = 2i\pi^2 a^\tau \lim_{R\to\infty} \frac{1}{4} R^2 \delta_{\alpha\tau} \tag{3.1.8}$$

- $a^\tau a^\sigma \int d^4k \partial_\tau \partial_\sigma (\frac{k_\alpha k_\rho}{k^4})$

$$\begin{aligned} a^\tau a^\sigma \int d^4k \partial_\tau \partial_\sigma (\frac{k_\alpha k_\rho}{k^4}) &= a^\tau a^\sigma \int d^4k \partial_\tau \left(\frac{\delta_{\alpha\sigma} k_\rho}{k^4} + \frac{\delta_{\rho\sigma} k_\alpha}{k^4} - \frac{4k_\alpha k_\rho k_\sigma}{k^6} \right) \\ &= 2i\pi^2 a^\tau a^\sigma \lim_{R\to\infty} R^2 R_\tau \left(\frac{\delta_{\alpha\sigma} R_\rho}{R^4} + \frac{\delta_{\rho\sigma} R_\alpha}{R^4} - \frac{4R_\rho R_\alpha R_\sigma}{R^4} \right) \\ &= 2i\pi^2 a^\tau a^\sigma \frac{1}{4} \left(\delta_{\alpha\sigma}\delta_{\rho\tau} + \delta_{\rho\sigma}\delta_{\alpha\tau} - \frac{2}{3} \left(\delta_{\rho\alpha}\delta_{\sigma\tau} + \delta_{\rho\sigma}\delta_{\alpha\tau} + \delta_{\rho\tau}\delta_{\alpha\sigma} \right) \right) \end{aligned} \tag{3.1.9}$$

3.1.2 Pauli-Villars regulator

The Pauli-Villars regularization, consists of introducing fictitious massive fields to the original theory. The new Pauli-Villars fields are chosen in such a way, that their contribution cancels the divergent pieces of the physical quantity we need to regulate. One usual prescription mentioned in literature, is to replace the original propagator with the original propagator minus the massive PV propagator. In case of a bosonic propagator, this prescription reads as follows

$$\frac{-i}{k^2+m^2} \to (\frac{-i}{k^2+m^2} - \frac{-i}{k^2+M^2}) = -i \frac{M^2-m^2}{(k^2+M^2)(k^2+m^2)}, \tag{3.1.10}$$

where m is the mass of the original field and M is the PV mass. It is straightforward to see that the regulated propagator has a better UV behaviour. Moreover, since the PV propagator comes with a relative minus sign compared to the original one, the PV field has a wrong-sign kinetic term.

Another common prescription for the regularization of Feynman diagrams, is to add PV fields of opposite statistics with respect to the fields of the original theory. The Feynman diagrams associated with them, will come with an overall minus sign compared to the original diagram, thus capable of cancelling its UV divergences.

In both prescriptions, the PV fields are unphysical since they can violate causality or positivity of energy, thus making the regulated theory problematic. However, this is not an issue. At the end of the calculations, after we have dealt with the divergent pieces, we remove the regulator by sending the PV mass to infinity. There, the PV fields decouple and we recover the initial theory. Moreover, it is quite common for people in literature to use both of these prescriptions without considering if they can actually follow from a local Lagrangian. For a consistent computation it is crucial to make sure that this is indeed the case. For example, the modification of the propagator (3.1.10) cannot arise in a Lagrangian without interaction terms. In the rest of this thesis we follow the second prescription, where we add/subtract whole diagrams to regulate divergent quantities. This is the natural approach, since we only examine free theories.

Let us give now an example of how we can use PV regularization to regulate a 2-point correlator. Consider the following action of a free, massive and commuting complex scalar field ϕ,

$$S = \int d^4x \left(-\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \right) , \tag{3.1.11}$$

where we are interested in the 2-point correlator $<(\phi^*\phi)(q)(\phi^*\phi)(-q)>$. In the limit of $q \to 0$, up to a prefactor the 2-point function is equal to Σ of (3.1.1), i.e.

$$<(\phi^*\phi)(0)(\phi^*\phi)(0)> = \frac{2i\pi^2}{(2\pi)^4} \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + m^2)^2} . \tag{3.1.12}$$

The correlator cannot be computed since it diverges logarithmically. We now add in the Lagrangian the anticommuting PV complex scalar field Φ with mass M. We find that the regulated action is given by

$$S_{\text{reg}} = \int d^4x \left(-\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \partial_\mu \Phi^* \partial^\mu \Phi - M^2 \Phi^* \Phi \right) , \tag{3.1.13}$$

while the operator we consider is modified as follows

$$(\phi^*\phi)_{\text{reg}} = (\phi^*\phi) + (\Phi^*\Phi). \tag{3.1.14}$$

The regulated 2-point function is equal to

$$\begin{aligned} &<(\phi^*\phi)_{\text{reg}}(0)(\phi^*\phi)_{\text{reg}}(0)> \equiv <(\phi^*\phi)(0)(\phi^*\phi)(0)> + <(\Phi^*\Phi)(0)(\Phi^*\Phi)(0)> = \\ &\frac{2i\pi^2}{(2\pi)^4} \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + m^2)^2} - \frac{2i\pi^2}{(2\pi)^4} \int_0^\infty dk_E \frac{k_E^3}{(k_E^2 + M^2)^2}. \end{aligned} \tag{3.1.15}$$

The second integral comes with a minus sign as consequence of the anticommuting nature of Φ. It is straightforward to verify this, after performing the Wick contractions. The leading order logarithmic divergences of the above two integrals cancel. Evaluating (3.1.15) for finite values of M we get

$$<(\phi^*\phi)_{\text{reg}}(0)(\phi^*\phi)_{\text{reg}}(0)> = \frac{2i\pi^2}{(2\pi)^4} \frac{1}{2} \log(\frac{M^2}{m^2}). \tag{3.1.16}$$

In the limit where the PV regulator decouples (i.e. $M \to \infty$), the parameter M encodes the logarithmic divergence of the original correlator $<(\phi^*\phi)(0)(\phi^*\phi)(0)>$. It is easy to identify a suitable counterterm to remove this divergence and renormalize the 2-point function.

Recall that in (3.1.2) we computed the integral of (3.1.12) in the cut-off regularization. Comparing that result with (3.1.16) and taking into account the extra prefactor $\frac{2i\pi^2}{(2\pi)^4}$, we see that they do not match. This is not unexpected though. In quantum field theories, it is quite common for various calculations to depend on the specific regulator one uses. However, results obtained from different regulators must be related by local counterterms.

3.2 Chiral anomaly from Feynman diagrams

As explained in the previous section, regularization is an essential step towards a well defined quantum theory. This means that the notion of symmetry should be modified at the quantum level. Invariance of the action is a necessary, but not a sufficient condition for a classical transformation to be a symmetry of the quantum theory. It also needs to respect at the same time the regularization procedure. There are cases when classical symmetries fail to do so. In particular, if there is no regularization procedure that respects a classical symmetry, then this symmetry is violated in the full quantum theory, something that we call an anomaly. Anomalies arise as extra local contributions to the classical symmetry Ward identities.

To elaborate on the mechanisms that the anomalies arise, we examine the chiral anomaly of a massless Dirac fermion. We do that using the two regulators we presented in the previous section, namely, PV and cut-off regularization.

3.2.1 Pauli-Villars

Consider the action of a massless Dirac fermion

$$S = \int d^4x \left(-\bar{\psi}\gamma^\mu \partial_\mu \psi \right) . \tag{3.2.1}$$

This classical action is invariant under the two following field transformations

$$\psi'(x) \to e^{ia}\psi(x), \qquad \psi'(x) \to e^{ib\gamma_5}\psi(x). \tag{3.2.2}$$

a and b are constant parameters and γ_5 is the Dirac matrix. The first is a global $U(1)$ transformation that simply changes the phase of the field. The second is a global chiral transformation that rotates the left-handed and right-handed components of the Dirac field independently. The corresponding to these symmetries conservation laws are

$$\partial_\mu V^\mu(x) \equiv \partial_\mu \left(i\bar{\psi}(x)\gamma^\mu\psi(x) \right) = 0, \qquad \partial_\mu J^\mu(x) \equiv \partial_\mu \left(i\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) \right) = 0 \tag{3.2.3}$$

where V^μ is the vector current and J^μ is the axial current. According to (2.2.8), the Ward identities for the correlation function of two vector and one axial-vector currents are the following:

$$\partial^x_\mu <V^\mu(x)V^\nu(y)J^\lambda(z)>=0, \qquad \partial^z_\mu <V^\mu(x)V^\nu(y)J^\lambda(z)>=0 \tag{3.2.4}$$

After Fourier transforming to momentum space we get

$$q_{1\mu} <V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>=0, \qquad q_{3\lambda} <V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>=0\,. \tag{3.2.5}$$

We want to see whether these conservation laws survive in quantum theory. After substituting the expressions of the currents (3.2.3), we perform all possible Wick contractions of the elementary fields. We find that there are two Feynman diagrams that contribute to the above correlation function.

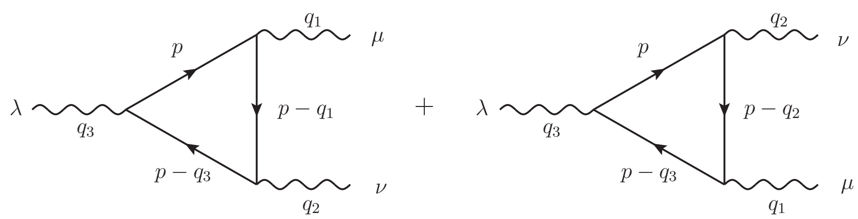

Figure 3.2.1: Feynman diagrams contribution to $<V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>$. The wave lines represent the external axial-vector and vector currents. The straight lines in the loops are fermionic propagators.

The sum of these diagrams is given by

$$<V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>\equiv (2\pi)^4\,\delta(q_1+q_2+q_3)T^{\mu\nu\lambda} \tag{3.2.6}$$

where

$$T^{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(\frac{-i}{i\not{p}}\gamma^\lambda\gamma_5\frac{-i}{i(\not{p}+\not{q}_3)}\gamma^\nu\frac{-i}{i(\not{p}-\not{q}_1)}\gamma^\mu\right)+\left(\begin{array}{ccc} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu\end{array}\right). \tag{3.2.7}$$

Using the relation

$$-\not{q}_3\gamma_5 = \gamma_5\left(\not{p}+\not{q}_3\right)+\not{p}\gamma_5, \tag{3.2.8}$$

we find that

$$q_{3\lambda}T^{\mu\nu\lambda} = A^{\mu\nu}+B^{\mu\nu} \tag{3.2.9}$$

where

$$A^{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(\frac{-i}{i\not{p}}\gamma_5\gamma^\nu\frac{-i}{i\left(\not{p}-\not{q}_1\right)}\gamma^\mu - \frac{-i}{i\left(\not{p}-\not{q}_2\right)}\gamma_5\gamma^\nu\frac{-i}{i\left(\not{p}+\not{q}_3\right)}\gamma^\mu\right), \tag{3.2.10}$$

$$B^{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i}{i\not{p}}\gamma_5\gamma^\mu\frac{-i}{i\left(\not{p}-\not{q}_2\right)}\gamma^\nu - \frac{-i}{i\left(\not{p}-\not{q}_1\right)}\gamma_5\gamma^\mu\frac{-i}{i\left(\not{p}+\not{q}_3\right)}\gamma^\nu\right). \tag{3.2.11}$$

Naively, both $A^{\mu\nu}$ and $B^{\mu\nu}$ vanish upon shifting the integration variable in their second term from p to $p+q_2$ and from p to $p+q_1$ respectively. Thus, the second Ward identity of (3.2.5) seems to be satisfied. However, as we showed in section (3.1.1), such a manipulation is incorrect before regularizing the divergent integrals.

We proceed with Pauli-Villars regularization. We add in our model a massive 'fermion' with opposite statistics (thus missing the minus sign accompanying the fermion loop in the Feynman diagrams). The regulated action will become

$$S_{\text{reg}} = \int d^4x\left(-\bar{\psi}\gamma^\mu\partial_\mu\psi - \bar{\Psi}\gamma^\mu\partial_\mu\Psi - M\bar{\Psi}\Psi\right). \tag{3.2.12}$$

The Pauli-Villars field Ψ violates classically the chiral transformation of (3.2.2). On the contrary the $U(1)$ symmetry is satisfied. The new classical conservation laws of the regulated action are the following

$$\partial_\mu V^\mu_{\text{reg}}(x) \equiv \partial_\mu\left(i\bar{\psi}(x)\gamma^\mu\psi(x) + i\bar{\Psi}(x)\gamma^\mu\Psi(x)\right) = 0, \tag{3.2.13}$$

$$\partial_\mu J^\mu_{\text{reg}}(x) \equiv \partial_\mu\left(i\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) + i\bar{\Psi}(x)\gamma^\mu\gamma_5\Psi(x)\right) = 2iM\bar{\Psi}\gamma_5\Psi. \tag{3.2.14}$$

Chiral symmetry

Since the regulator violates chiral symmetry, there is a possibility that this classical breaking leaves its trace in quantum theory too, even when we take the limit of the PV mass to infinity. This is the limit where the PV fields decouple from our theory, hence the regulator vanishes. To confirm this, one has to compute the regulated version of the correlator $T^{\mu\nu\lambda}$. The extra contribution from the massive PV 'fermion' are two Feynman diagrams similar to figure (3.2.1), but with an overall minus sign. Thus, we get

$$\begin{aligned} T^{\mu\nu\lambda}_{\text{reg}} &= i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i}{i\not{p}}\gamma^\lambda\gamma_5\frac{-i}{i(\not{p}+\not{q}_3)}\gamma^\nu\frac{-i}{i(\not{p}-\not{q}_1)}\gamma^\mu\right) + \begin{pmatrix} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu\end{pmatrix} \\ &- i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i}{i\not{p}+M}\gamma^\lambda\gamma_5\frac{-i}{i(\not{p}+\not{q}_3)+M}\gamma^\nu\frac{-i}{i(\not{p}-\not{q}_1)+M}\gamma^\mu\right) + \begin{pmatrix} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu\end{pmatrix}. \end{aligned} \tag{3.2.15}$$

Notice that for any finite M, $T^{\mu\nu\lambda}_{\text{reg}}$ is convergent and well defined. This is quite straightforward to confirm by expanding $T^{\mu\nu\lambda}_{\text{reg}}$ in the PV mass M

$$T^{\mu\nu\lambda}_{\text{reg}} = T^{\mu\nu\lambda} - T^{\mu\nu\lambda} + M^2\int d^4p\, f^{\mu\nu\lambda}(p) + \ldots = M^2\int d^4p\, f^{\mu\nu\lambda}(p) + \ldots \tag{3.2.16}$$

$\int d^4p\, f^{\mu\nu\lambda}(p)$ is a convergent integral and ... denote higher order terms in M which are also convergent. Because of the gamma matrix algebra there are no linear terms (or any other odd power) in the PV mass in the above expansion.

Using now that

$$-i\not{q}_3\gamma_5 = \gamma_5\left(i(\not{p}+\not{q}_3)+M\right)+\left(i\not{p}+M\right)\gamma_5 - 2M\gamma_5, \tag{3.2.17}$$

we get

$$q_{3\lambda}T^{\mu\nu\lambda}_{\text{reg}} = C^{\mu\nu}+D^{\mu\nu}+E^{\mu\nu}+F^{\mu\nu}+G^{\mu\nu} \tag{3.2.18}$$

where

$$C^{\mu\nu} = i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i}{i\not{p}}\gamma_5\gamma^\nu\frac{-i}{i\left(\not{p}-\not{q}_1\right)}\gamma^\mu - \frac{-i}{i\not{p}+M}\gamma_5\gamma^\nu\frac{-i}{i(\not{p}-\not{q}_1)+M}\gamma^\mu\right), \tag{3.2.19}$$

$$D^{\mu\nu} = i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(-\frac{-i}{i\left(\not{p}-\not{q}_2\right)}\gamma_5\gamma^\nu\frac{-i}{i\left(\not{p}+\not{q}_3\right)}\gamma^\mu + \frac{-i}{i(\not{p}-\not{q}_2)+M}\gamma_5\gamma^\nu\frac{-i}{i(\not{p}+\not{q}_3)+M}\gamma^\mu\right), \tag{3.2.20}$$

$$E^{\mu\nu} = i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i}{i\not{p}}\gamma_5\gamma^\mu\frac{-i}{i\left(\not{p}-\not{q}_2\right)}\gamma^\nu - \frac{-i}{i\not{p}+M}\gamma_5\gamma^\mu\frac{-i}{i(\not{p}-\not{q}_2)+M}\gamma^\nu\right), \tag{3.2.21}$$

$$F^{\mu\nu} = i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(-\frac{-i}{i\left(\not{p}-\not{q}_1\right)}\gamma_5\gamma^\mu\frac{-i}{i\left(\not{p}+\not{q}_3\right)}\gamma^\nu + \frac{-i}{i(\not{p}-\not{q}_1)+M}\gamma_5\gamma^\mu\frac{-i}{i(\not{p}+\not{q}_3)+M}\gamma^\nu\right), \tag{3.2.22}$$

$$G^{\mu\nu} = \int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i}{i\not{p}+M}2iM\gamma_5\frac{-i}{i(\not{p}+\not{q}_3)+M}\gamma^\nu\frac{-i}{i(\not{p}-\not{q}_1)+M}\gamma^\mu\right) + \begin{pmatrix} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu \end{pmatrix}. \tag{3.2.23}$$

The quantities $C^{\mu\nu}$, $D^{\mu\nu}$, $E^{\mu\nu}$ and $F^{\mu\nu}$ converge. Thus, we are allowed to shift their integration variable. By shifting p to $p+q_2$ in $D^{\mu\nu}$ and p to $p+q_1$ in $F^{\mu\nu}$, we find that

$$C^{\mu\nu}+D^{\mu\nu}+E^{\mu\nu}+F^{\mu\nu} = 0. \tag{3.2.24}$$

The only thing that is left to compute is $G^{\mu\nu}$. $G^{\mu\nu}$ captures the extra contribution of the PV mass M to the chiral symmetry Ward identity of $<V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>$. In particular, the broken chiral symmetry identity in the regulated theory is given by

$$q_{3\lambda} <V^\mu_{\text{reg}}(q_1)V^\nu_{\text{reg}}(q_2)J^\lambda_{\text{reg}}(q_3)>= 2M <V^\mu_{\text{reg}}(q_1)V^\nu_{\text{reg}}(q_2)(\bar{\Psi}\gamma_5\Psi)(q_3)>, \tag{3.2.25}$$

where

$$2M <V^\mu_{\text{reg}}(q_1)V^\nu_{\text{reg}}(q_2)(\bar{\Psi}\gamma_5\Psi)(q_3)>\equiv (2\pi)^4\delta(q_1+q_2+q_3)G^{\mu\nu}\,. \tag{3.2.26}$$

What we did in the previous analysis was to see whether the regulated 3-point function $< V^\mu_{\text{reg}}(q_1)V^\nu_{\text{reg}}(q_2)J^\lambda_{\text{reg}}(q_3) >$ satisfies its original chiral symmetry Ward identity (3.2.5). Alternatively, we could have used the Ward identity of the regulated theory (which is satisfied by construction, since all correlators involved in (3.2.25) are properly regulated and convergent) and compute only the breaking term of the rhs, i.e. $G^{\mu\nu}$. Both approaches are equivalent. In the latter is just easier to identify the breaking term.

Using that $\slashed{p}^2 = p^2$ we get

$$G^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}\text{tr}\left(\frac{-i\slashed{p}+M}{p^2+M^2}2iM\gamma_5\frac{-i\left(\slashed{p}+\slashed{q}_3\right)+M}{(p+q_3)^2+M^2}\gamma^\nu\frac{-i\left(\slashed{p}-\slashed{q}_1\right)+M}{(p-q_1)^2+M^2}\gamma^\mu\right)+\left(\begin{array}{ccc} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu \end{array}\right) \tag{3.2.27}$$

and after evaluating the Dirac traces

$$G^{\mu\nu} = \int \frac{d^4p}{(2\pi)^4}\frac{8M^2\epsilon^{\mu\nu\alpha\beta}q_{2\alpha}q_{1\beta}}{(p^2+M^2)\left((p+q_3)^2+M^2\right)\left((p-q_1)^2+M^2\right)}+\left(\begin{array}{ccc} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu \end{array}\right). \tag{3.2.28}$$

To calculate this convergent integral one has to combine the denominators using the standard Feynman parameters. However, in the end we are interested in the limit $M \to \infty$, so we only need the asymptotics of the integral for large M which is obtained by setting $p = Ml$. We have

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{M^2}\int \frac{d^4l}{(2\pi)^4}\frac{8M^2\epsilon^{\mu\nu\alpha\beta}q_{2\alpha}q_{1\beta}}{(l^2+1)\left((l+\frac{q_3}{M})^2+1\right)\left((l-\frac{q_1}{M})^2+1\right)}+\left(\begin{array}{ccc} q_1 & \leftrightarrow & q_2 \\ \mu & \leftrightarrow & \nu \end{array}\right) \\ &= \frac{i}{32\pi^2M^2}16M^2\epsilon^{\mu\nu\alpha\beta}q_{2\alpha}q_{1\beta}. \end{aligned} \tag{3.2.29}$$

We expanded the above integrand in powers of $\frac{q_3}{M}$ and $\frac{-q_1}{M}$ and kept the non vanishing terms in the limit $M \to \infty$. It turns out that only the leading (zeroth) term survives. The extra i comes from the Wick rotation that we did to calculate the integral in Euclidean space. Thus, the final result for $G^{\mu\nu}$ is

$$G^{\mu\nu} = \frac{i}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}q_{2\alpha}q_{1\beta}. \tag{3.2.30}$$

We see that after the removal of the regulator, the chiral Ward identity of (3.2.5) is anomalous

$$q_{3\lambda} < V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3) >= (2\pi)^4\delta(q_1+q_2+q_3)\frac{i}{2\pi^2}\epsilon^{\mu\nu\alpha\beta}q_{2\alpha}q_{1\beta}\,. \tag{3.2.31}$$

An analogous computation shows that the 3-point function of three chiral currents is also anomalous, with $\frac{1}{3}$ the value of the above anomaly, i.e.

$$q_{3\lambda} < J^\mu(q_1)J^\nu(q_2)J^\lambda(q_3) >= (2\pi)^4\delta(q_1+q_2+q_3)\frac{i}{6\pi^2}\epsilon^{\mu\nu\alpha\beta}q_{2\alpha}q_{1\beta}\,. \tag{3.2.32}$$

$U(1)$ symmetry

The PV regulator respects the $U(1)$ transformation (3.2.2), which means that we do not expect any anomalies at the quantum level in the $U(1)$ symmetry Ward identities. In particular, the regulated identity is equal to

$$q_{1\mu} < V^\mu_{\text{reg}}(q_1)V^\nu_{\text{reg}}(q_2)J^\lambda_{\text{reg}}(q_3) >= 0\,, \tag{3.2.33}$$

where as we showed the 3-point function is a well defined convergent integral for finite values of M. Since there is no explicit breaking term on the rhs there is nothing actually to compute, the identity is satisfied for every M, so in the limit $M \to \infty$ too. For completeness though, let us compute the lhs to confirm it.

Using that

$$i\not{q}_1 = \left(i\not{p}+M\right) - \left(i\left(\not{p}-\not{q}_1\right)+M\right) = \left(i\left(\not{p}-\not{q}_2\right)+M\right) - \left(i\left(\not{p}+\not{q}_3\right)+M\right), \tag{3.2.34}$$

we find

$$q_{1\mu}T^{\mu\nu\lambda}_{\text{reg}} = H^{\nu\lambda} + K^{\nu\lambda} \tag{3.2.35}$$

where

$$H^{\nu\lambda} = i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\gamma^\lambda\gamma_5\frac{-i}{i\left(\not{p}+\not{q}_3\right)}\gamma^\nu\frac{-i}{i\left(\not{p}-\not{q}_1\right)}. - \gamma^\lambda\gamma_5\frac{-i}{i\left(\not{p}+\not{q}_3\right)+M}\gamma^\nu\frac{-i}{i\left(\not{p}-\not{q}_1\right)+M}\right) \tag{3.2.36}$$

and

$$K^{\nu\lambda} = -i\int\frac{d^4p}{(2\pi)^4}\text{tr}\left(\gamma^\lambda\gamma_5\frac{-i}{i\left(\not{p}-\not{q}_2\right)}\gamma^\nu\frac{-i}{i\not{p}} - \gamma^\lambda\gamma_5\frac{-i}{i\left(\not{p}-\not{q}_2\right)+M}\gamma^\nu\frac{-i}{i\not{p}+M}\right). \tag{3.2.37}$$

$K^{\nu\lambda}$ and $H^{\nu\lambda}$ are convergent. By shifting $p \to p + q_1$ in $H^{\nu\lambda}$ we see that

$$q_{1\mu} T^{\mu\nu\lambda}_{\text{reg}} = 0. \tag{3.2.38}$$

A similar computation shows that

$$q_{1\mu} < V^{\mu}_{\text{reg}}(q_1) V^{\nu}_{\text{reg}}(q_2) V^{\lambda}_{\text{reg}}(q_3) >= 0\,. \tag{3.2.39}$$

The vector current Ward identities are satisfied, which means that the $U(1)$ classical symmetry is also a symmetry of the quantum theory, as expected.

If we introduce the background sources A_μ, B_μ that couple to the vector and chiral currents respectively, the action of the free Dirac fermion will become

$$S = \int d^4x \left(-\bar{\psi}\gamma^\mu D_\mu \psi \right) \tag{3.2.40}$$

where

$$D_\mu \psi = (\partial_\mu + iA_\mu + i\gamma_5 B_\mu)\psi. \tag{3.2.41}$$

The symmetry transformations of A_μ, B_μ are given by

$$A'_\mu \to A_\mu - \partial_\mu a\,, \qquad B'_\mu \to B_\mu - \partial_\mu b\,, \tag{3.2.42}$$

where a, b are the $U(1)$ and chiral symmetry parameters. One can easily show that in the presence of the background sources the chiral current divergence is modified as

$$\partial_\lambda \langle \mathcal{J}^\lambda \rangle_s = -\frac{1}{(4\pi)^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(A) F_{\rho\sigma}(A) - \frac{1}{3\,(4\pi)^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(B) F_{\rho\sigma}(B), \tag{3.2.43}$$

where

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \qquad F_{\mu\nu}(B) = \partial_\mu B_\nu - \partial_\nu B_\mu\,. \tag{3.2.44}$$

After taking the appropriate functional derivatives with respect to the sources, the above equation reproduces the anomalous $<VVJ>$ and $<JJJ>$ correlators. (3.2.43) tells us that the axial-vector current is not conserved, meaning that the $U(1)$ axial symmetry of the classical action, does not survive the regularization procedure. This is the $U(1)$ axial anomaly and was first obtained by Adler, Bell and Jackiw [6, 7]. Even though for this computation we used Feynman diagrams (which is a perturbative method), the result (3.2.43) is exact and does not receive corrections from higher order terms in interacting theories [86].

These results could have been obtained using a different regularization procedure, such as dimensional regularization or the hard cut-off method. A more elegant approach was given by Fujikawa [87], who interpreted the anomaly as a symptom of the non invariance of the path integral measure under an axial transformation.

3.2.2 Momentum routing

In this section we derive the chiral anomaly of the free and massless Dirac fermion (3.2.1) using another regulator, namely momentum cut-off. In particular, we want to confirm the anomalous Ward identities of the correlators $<VVJ>$ and $<JJJ>$. The 3-point function of the chiral currents is given by

$$<J^\mu(q_1)J^\nu(q_2)J^\lambda(q_3)>\equiv (2\pi)^4\,\delta(q_1+q_2+q_3)\tilde{T}^{\mu\nu\lambda}, \tag{3.2.45}$$

where

$$\tilde{T}^{\mu\nu\lambda}=i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(\frac{-i}{i\not{p}}\gamma^\lambda\gamma_5\frac{-i}{i(\not{p}+\not{q}_3)}\gamma^\nu\gamma_5\frac{-i}{i(\not{p}-\not{q}_1)}\gamma^\mu\gamma_5\right)+\left(\begin{array}{ccc} q_1 & \leftrightarrow & q_2\\ \mu & \leftrightarrow & \nu\end{array}\right). \tag{3.2.46}$$

Notice that using $\{\gamma_5,\gamma^\mu\}=0$, $\tilde{T}^{\mu\nu\lambda}$ is equal to $T^{\mu\nu\lambda}$ of (3.2.6), i.e.

$$<V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>=<J^\mu(q_1)J^\nu(q_2)J^\lambda(q_3)>\ . \tag{3.2.47}$$

We now contract $<VVJ>$ with $q_{1\mu}$ to examine its $U(1)$ symmetry Ward identity. We have

$$q_{1\mu}T^{\mu\nu\lambda}=i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(\frac{-i}{i\not{p}}\gamma^\nu\frac{-i}{i\left(\not{p}+\not{q}_2\right)}\gamma^\lambda\gamma_5-\frac{-i}{i\left(\not{p}+\not{q}_3\right)}\gamma^\nu\frac{-i}{i\left(\not{p}-\not{q}_1\right)}\gamma^\lambda\gamma_5\right)+\left(\begin{array}{ccc} q_2 & \leftrightarrow & q_3\\ \nu & \leftrightarrow & \lambda\end{array}\right). \tag{3.2.48}$$

Similarly with the previous section, we can naively cancel the two terms by shifting the integration variable from $p\to p+q_3$ in the first term and use momentum conservation. The integral is divergent though, so any formal manipulation is illegitimate. We regulate it using a momentum hard cut-off R. Since we deal with integrands that are related by a simple shift of the integration variable, it is more convenient to compute them using Gauss's theorem. By power counting the integral is quadratically divergent, but due to the structure of the gamma matrices the leading order term vanishes. Thus, the integral diverges linearly so we can use (3.1.5). We find that

$$q_{1\mu}T^{\mu\nu\lambda}=\frac{i}{\pi^2}\epsilon^{\rho\sigma\lambda\nu}q_{3\rho}q_{2\sigma}\ . \tag{3.2.49}$$

Similarly, if we contract $T^{\mu\nu\lambda}$ with $q_{2\nu}$ and $q_{3\lambda}$ we get

$$q_{2\nu}T^{\mu\nu\lambda}=\frac{i}{2\pi^2}\epsilon^{\rho\sigma\lambda\mu}q_{3\rho}q_{1\sigma} \tag{3.2.50}$$

and

$$q_{3\lambda}T^{\mu\nu\lambda} = \frac{i}{2\pi^2}\epsilon^{\rho\sigma\nu\mu}q_{2\rho}q_{1\sigma}\,. \tag{3.2.51}$$

From the above equations, we see that after regulating with cut-off all the Ward identities of $<VVJ>$ are violated. However, we have not taken into account yet the fact that $<VVJ>$ is linearly divergent. When we computed it, we chose a specific momentum routing in the loop of the Feynman diagrams. We could have chosen another routing which is related to the original one by a shift. From a physical point of view Feynman diagrams related by a momentum shift are equivalent. Computationally though, they differ by a finite term. Below we write the $<VVJ>$ correlator with an arbitrary momentum routing

$$<V^\mu(q_1)V^\nu(q_2)J^\lambda(q_3)>_{\{a_i\}}\equiv (2\pi)^4\,\delta(q_1+q_2+q_3)T^{\mu\nu\lambda}(a_i), \tag{3.2.52}$$

where

$$T^{\mu\nu\lambda}(a_i) = i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left(\gamma^\mu\frac{-i}{i(\not p+\not a_1)}\gamma^\lambda\gamma_5\frac{-i}{i(\not p+\not a_1+\not q_3)}\gamma^\nu\frac{-i}{i(\not p+\not a_1-\not q_1)}\right)+\begin{pmatrix} q_2 & & q_3\\ \nu & \leftrightarrow & \lambda\\ a_1 & & a_2\end{pmatrix}. \tag{3.2.53}$$

We have that

$$T^{\mu\nu\lambda}(a_i) = T^{\mu\nu\lambda}+\delta T^{\mu\nu\lambda}(a_i) \tag{3.2.54}$$

where using again (3.1.5) we find

$$\delta T^{\mu\nu\lambda}(a_i) = \frac{i}{2\pi^2}\epsilon^{\rho\mu\lambda\nu}\left(a_{1\rho}-a_{2\rho}\right)\,. \tag{3.2.55}$$

The Ward identities of $<VVJ>$ will become

$$q_{1\mu}T^{\mu\nu\lambda}(a_i) = \frac{i}{\pi^2}\epsilon^{\rho\sigma\lambda\nu}q_{3\rho}q_{2\sigma}+q_{1\mu}\delta T^{\mu\nu\lambda}(a_i), \tag{3.2.56}$$

$$q_{2\nu}T^{\mu\nu\lambda}(a_i) = \frac{i}{2\pi^2}\epsilon^{\rho\sigma\lambda\mu}q_{3\rho}q_{1\sigma}+q_{2\nu}\delta T^{\mu\nu\lambda}(a_i), \tag{3.2.57}$$

$$q_{3\lambda}T^{\mu\nu\lambda}(a_i) = \frac{i}{2\pi^2}\epsilon^{\rho\sigma\nu\mu}q_{2\rho}q_{1\sigma}+q_{3\lambda}\delta T^{\mu\nu\lambda}(a_i). \tag{3.2.58}$$

The arbitrary constants a_1 and a_2 will be given by a linear combination of the external momenta q_2 and q_3. All that is left to do is find a way to fix them. Ideally we want to find a choice of a_i so that the rhs of all the above equations is zero at the same time, hence confirm all the Ward identities. However, it is easy to see that there is no such choice. The best we can do is make two of them zero simultaneously. We choose to maintain the vector symmetry Ward identities and violate the chiral symmetry. For that we set

$$a_1 = \frac{q_3-q_2}{2} = -a_2 \tag{3.2.59}$$

and the final result that we get is

$$q_{1\mu} <V^{\mu}(q_1)V^{\nu}(q_2)J^{\lambda}(q_3)>= q_{2\nu} <V^{\mu}(q_1)V^{\nu}(q_2)J^{\lambda}(q_3)>= 0 \tag{3.2.60}$$

and

$$q_{3\lambda} <V^{\mu}(q_1)V^{\nu}(q_2)J^{\lambda}(q_3)>= (2\pi)^4\delta(q_1+q_2+q_3)\frac{i}{2\pi^2}\epsilon^{\rho\sigma\mu\nu}q_{3\rho}q_{2\sigma}\,. \tag{3.2.61}$$

Using momentum conservation, we see that (3.2.61) is equal to (3.2.31). Thus, choosing the appropriate momentum routing so that the $U(1)$ symmetry is satisfied, the cut-off regulator reproduces the chiral anomaly that we found with PV regularization.

The computation of the $<JJJ>$ is identical, since the correlators are the same as we showed before. The chiral symmetry Ward identities of $<JJJ>$ are

$$q_{1\mu}\tilde{T}^{\mu\nu\lambda}(a_i) = \frac{i}{\pi^2}\epsilon^{\rho\sigma\lambda\nu}q_{3\rho}q_{2\sigma} + q_{1\mu}\delta\tilde{T}^{\mu\nu\lambda}(a_i), \tag{3.2.62}$$

$$q_{2\nu}\tilde{T}^{\mu\nu\lambda}(a_i) = \frac{i}{2\pi^2}\epsilon^{\rho\sigma\lambda\mu}q_{3\rho}q_{1\sigma} + q_{2\nu}\delta\tilde{T}^{\mu\nu\lambda}(a_i), \tag{3.2.63}$$

$$q_{3\lambda}\tilde{T}^{\mu\nu\lambda}(a_i) = \frac{i}{2\pi^2}\epsilon^{\rho\sigma\nu\mu}q_{2\rho}q_{1\sigma} + q_{3\lambda}\delta\tilde{T}^{\mu\nu\lambda}(a_i). \tag{3.2.64}$$

Here we want to maintain Bose symmetry so we choose a_1 and a_2 so that the above rhs give the same result. This happens for

$$a_1 = \frac{q_3 - q_2}{6} = -a_2 \tag{3.2.65}$$

and the final result that we get is

$$q_{3\lambda} <J^{\mu}(q_1)J^{\nu}(q_2)J^{\lambda}(q_3)>= (2\pi)^4\delta(q_1+q_2+q_3)\frac{i}{6\pi^2}\epsilon^{\rho\sigma\mu\nu}q_{3\rho}q_{2\sigma}\,. \tag{3.2.66}$$

Again this result confirms the anomaly that we found with the PV regulator. In real space the above equation can be written as

$$\partial^x_\mu <J^{\mu}(x)J^{\nu}(y)J^{\lambda}(z)>= \frac{1}{6\pi^2}\epsilon^{\rho\sigma\nu\lambda}\partial^x_\rho\delta(x-y)\partial^x_\sigma\delta(x-z)\,. \tag{3.2.67}$$

3.3 Anomaly shifting and compensators

Anomaly shifting

As we saw in the previous section, by assigning different momentum routings to the flat space correlator $<VVJ>$ (3.2.52), we are able to shift the anomaly from the $U(1)$ current to the axial current and vice-versa. Shifting around an anomaly between different symmetries of the theory is something that can be done with the use of local counterterms[1].

To be more specific, let us give an example in the case of the free Dirac fermion (3.2.40). Let $\mathcal{V}^\mu$ and $\mathcal{J}^\mu$ be the vector and axial Noether currents in the presence of sources, and A^μ, B^μ the corresponding sources. The partition function $\mathscr{Z}[A,B]$ is classically invariant separately under the transformations (3.2.42), but at one loop order one cannot maintain both invariances. In the context of PV regularization, $U(1)$ symmetry is manifestly respected, while chiral symmetry is anomalous. This means that

$$\mathscr{Z}[A-da,B]=\mathscr{Z}[A,B], \qquad \mathscr{Z}[A,B-db]=\mathscr{Z}[A,B]e^{i\int d^4x b\mathcal{A}}, \tag{3.3.1}$$

where $\mathcal{A}\sim\epsilon^{\mu\nu\rho\sigma}(F_{\mu\nu}(A)F_{\rho\sigma}(A)+\frac{1}{3}F_{\mu\nu}(B)F_{\rho\sigma}(B))$ is the chiral anomaly and $F_{\mu\nu}(A)=\partial_\mu A_\nu-\partial_\nu A_\mu, F_{\mu\nu}(B)=\partial_\mu B_\nu-\partial_\nu B_\mu$. This results in the Ward identities

$$\partial_\mu\langle\mathcal{J}^\mu\rangle_s=\mathcal{A}, \qquad \partial_\mu\langle\mathcal{V}^\mu\rangle_s=0, \tag{3.3.2}$$

where $\langle\cdot\rangle_s$ denotes connected correlation functions in the presence of sources. Since the anomaly is quadratic in the sources, (3.3.2) encodes the fact that triangle diagrams with one or three axial currents are anomalous.

The anomaly can be shifted around by adding finite local terms in the action. For the case of the massless fermion we could consider the partition function,

$$\widetilde{\mathscr{Z}}[A,B]=\mathscr{Z}[A,B]\exp\Big(i\alpha_c\int d^4x\,\epsilon^{\mu\nu\rho\sigma}B_\mu A_\nu F_{\rho\sigma}(A)\Big), \tag{3.3.3}$$

where α_c is a constant. By an appropriate choice of α_c one may arrange for the axial-vector-vector correlator to be conserved on the axial current but then the conservation of the vector current would be anomalous, and the partition function (3.3.3) would be invariant under neither vector nor axial transformations. The theories with or without the counterterm are physically distinct, as they preserve different symmetries. The standard choice is to keep the vector symmetry non-anomalous (in the original context [6,7] the vector symmetry was electromagnetism), but more generally depending on the physics context one may work with either theory. In the context of the AdS/CFT correspondence, the finite counterterms correspond to finite boundary terms that should be specified when defining the bulk theory.

Compensators

Besides shifting the anomaly between symmetries using local counterterms, the anomaly may be 'hidden' by introducing additional (background) fields. In the case of the axial anomaly, for example, we may introduce an external scalar field Φ and modify the partition

[1] In the cut-off regularization, surface terms that arise from different momentum routings at the correlators can always be written in the form of local counterterms.

function as

$$\mathscr{Z}'[A,B,\Phi] = \mathscr{Z}[A,B]\exp\Big(i\int d^4x\,\Phi\mathcal{A}\Big). \tag{3.3.4}$$

Assigning transformations

$$\delta_a\Phi = 0, \qquad \delta_b\Phi = -b, \tag{3.3.5}$$

the partition function $\mathscr{Z}'$ is now gauge invariant under both vector and axial transformations

$$\mathscr{Z}'[A-da,B,\Phi+\delta_a\Phi] = \mathscr{Z}'[A,B,\Phi], \qquad \mathscr{Z}'[A,B-db,\Phi+\delta_b\Phi] = \mathscr{Z}'[A,B,\Phi]. \tag{3.3.6}$$

This does not mean that the anomaly has disappeared; the triangle diagrams are not affected by the new terms in (3.3.4).

One could also use the formulation with Φ to make the Ward identities look as if there is no anomaly. To see this let us rewrite the coupling of Φ in (3.3.4) as

$$\mathscr{Z}'[A,B,\Phi] = \mathscr{Z}[A,B]\exp\Big(i\int d^4x\,\Phi\,\mathcal{O}\Big), \tag{3.3.7}$$

where $\mathcal{O} = \partial_\mu\mathcal{J}^\mu$. In the classical theory $\mathcal{O}$ is proportional to field equations (it is a null operator) and when we regulate the theory, say with Pauli-Villars regularization, $\langle\mathcal{O}\rangle_s$ becomes local (and equal to the anomaly, $\langle\mathcal{O}\rangle = \mathcal{A}$) as the regulator is removed. Working out the Ward identity starting from (3.3.7) one finds,

$$\partial_\mu\langle\mathcal{J}^\mu\rangle_s - \langle\mathcal{O}\rangle_s = 0. \tag{3.3.8}$$

In this form the Ward identity appears non-anomalous and one may be tempted to conclude that the theory would be non-anomalous if we include the coupling to the null operator $\mathcal{O}$. Of course, this is just an illusion: (3.3.8) is equal to (3.3.2). The only thing that happened was that we moved the anomaly from the rhs to the lhs and gave it a different name.

Fields like Φ are called ‘compensators’ because they may be used to restore or compensate for broken symmetries, or ‘gauge-away’ fields because one may set them to zero using gauge transformations. Indeed one may use (3.3.5) to set Φ to zero (and thus also establishing that $\mathscr{Z}'$ is equivalent to $\mathscr{Z}$). Invariance of the partition function under gauge transformations does not by itself imply absence of anomalies, if gauge away fields are present. One must first set to zero all gauge away fields and then check invariance of the partition function. Similarly, one must set all compensators to zero prior to working out the form of the Ward identities. The discussion of compensators is important in supersymmetric theories. Often, supersymmetric Lagrangians have a superspace formulation. Superspace though, is an enlarged space with more degrees of freedom, so if we wish to retrieve the action of the microscopic theory we need to choose a specific gauge (a usual choice is the WZ gauge) by eliminating some of the gauge away fields. A superspace analysis that shows no supersymmetry anomalies, does not automatically result in a non anomalous supersymmetric microscopic (physical) theory. One has to be careful about how the compensators may affect the result, similar to the above analysis.

CHAPTER 4

Free and massless Wess-Zumino model

4.1 Symmetries and the conformal multiplet of conserved currents

In this chapter we consider the free and massless WZ model. We begin with a description of the flat space theory and its classical symmetries, before discussing its coupling to background conformal supergravity. We also derive the classical Ward identities for every correlator involved in the computation of the Q-supersymmetry anomaly of the conformal WZ model. In chapter 6 we will identify a suitable Pauli-Villars regulator, which we use for regulating the 1-loop diagrams. In this chapter we follow closely the relevant discussion of [81].

An off-shell $\mathcal{N} = 1$ chiral multiplet consists of a complex scalar, ϕ, a Grassmann-valued Majorana spinor, χ, and an auxiliary complex scalar, F. The free and massless WZ model for a chiral multiplet in Minkowski space is described by the Lagrangian

$$\hat{\mathscr{L}}_{\rm WZ} = -\partial_\mu \phi^* \partial^\mu \phi - \frac{1}{2}\bar{\chi}\hat{\not{\partial}}\chi + F^* F, \tag{4.1.1}$$

where a hat $\hat{}$ indicates quantities evaluated in a Minkowski background. It will be omitted later on when referring to the corresponding quantities in the presence of background supergravity fields.

4.1.1 Propagators

The momentum space propagators following from the Lagrangian (4.1.1) are

$$\begin{aligned}
\phi(p)\phi^*(p') &= \phi^*(p')\phi(p) = (2\pi)^4\delta(p+p')P_\phi(p), \\
\chi(p)\bar{\chi}(p') &= -\bar{\chi}(p')\chi(p) = (2\pi)^4\delta(p+p')P_\chi(p), \\
F(p)F^*(p') &= F^*(p')F(p) = (2\pi)^4\delta(p+p')P_F(p),
\end{aligned} \tag{4.1.2}$$

where

$$P_\phi(p) = -\frac{i}{p^2}, \qquad P_\chi(p) = -\frac{\not{p}}{p^2}, \qquad P_F(p) = i. \tag{4.1.3}$$

4.1.2 Symmetries

The free and massless Wess-Zumino model is classically invariant under the superconformal group $SU(2,2|1)$ [88–91]. An infinitesimal $SU(2,2|1)$ transformation can be parameterized as

$$\hat{\delta} = a^\mu P_\mu + \ell_{\mu\nu}M^{\mu\nu} + b^\mu K_\mu + \lambda D + \theta_0 R + \bar{\varepsilon}_0 Q + \bar{\eta}_0 S, \tag{4.1.4}$$

where P_μ, $M_{\mu\nu}$, K_μ, D, R, Q and S are respectively the generators of spacetime translations, Lorentz, special conformal, scaling, R-symmetry, Q- and S-supersymmetry transformations. The zero subscript indicates the flat space version of the parameters. The action of these generators on the chiral multiplet fields is given in table 4.1.1.

P_μ	$\delta_a\phi = a^\mu\partial_\mu\phi, \quad \delta_a\chi_L = a^\mu\partial_\mu\chi_L, \quad \delta_a F = a^\mu\partial_\mu F$
K_μ	$\delta_b\phi = b_\mu\big((2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu\phi + 2x^\mu\phi\big)$ $\delta_b\chi_L = b_\mu\big((2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu\chi_L + 3x^\mu\chi_L + x_\nu\gamma^{\mu\nu}\chi_L\big)$ $\delta_b F = b_\mu\big((2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu F + 4x^\mu F\big)$
$M_{\mu\nu}$	$\delta_\ell\phi = \ell_{\mu\nu}x^{[\mu}\partial^{\nu]}\phi, \quad \delta_\ell\chi_L = \ell_{\mu\nu}\big(x^{[\mu}\partial^{\nu]}\chi_L + \frac{1}{4}\gamma^{\mu\nu}\chi_L\big), \quad \delta_\ell F = \ell_{\mu\nu}x^{[\mu}\partial^{\nu]}F$
R	$\delta_{\theta_0}\phi = iq_R\theta_0\phi, \quad \delta_{\theta_0}\chi_L = i(q_R+1)\theta_0\chi_L, \quad \delta_{\theta_0}F = i(q_R+2)\theta_0 F$
D	$\delta_\lambda\phi = \lambda(x^\mu\partial_\mu + 1)\phi, \quad \delta_\lambda\chi_L = \lambda(x^\mu\partial_\mu + \frac{3}{2})\chi_L, \quad \delta_\lambda F = \lambda(x^\mu\partial_\mu + 2)F$
Q	$\delta_{\varepsilon_0}\phi = \frac{\sqrt{2}}{2}\bar{\varepsilon}_{0L}\chi_L, \quad \delta_{\varepsilon_0}\chi_L = \frac{\sqrt{2}}{2}(\not{\partial}\phi\varepsilon_{0R} + F\varepsilon_{0L}), \quad \delta_{\varepsilon_0}F = \frac{\sqrt{2}}{2}\bar{\varepsilon}_{0R}\not{\partial}\chi_L$
S	$\delta_{\eta_0}\phi = -\frac{\sqrt{2}}{2}x^\mu\bar{\eta}_{0R}\gamma_\mu\chi_L, \quad \delta_{\eta_0}\chi_L = \frac{\sqrt{2}}{2}(x^\mu\not{\partial}\phi\gamma_\mu\eta_{0L} + x^\mu F\gamma_\mu\eta_{0R} + 2\phi\eta_{0L})$ $\delta_{\eta_0}F = -\frac{\sqrt{2}}{2}x^\mu\bar{\eta}_{0L}\gamma_\mu\not{\partial}\chi_L$

Table 4.1.1: $SU(2,2|1)$ action on a chiral multiplet of R-charge q_R. Superconformal invariance requires $q_R = -\frac{2}{3}$.

4.1.3 Algebra

The $SU(2,2|1)$ generators satisfy the algebra

$$
\begin{aligned}
&[D,P_\mu]=-P_\mu, \qquad [D,K_\mu]=K_\mu, \qquad [P_\mu,K_\nu]=2(\eta_{\mu\nu}D-2M_{\mu\nu}),\\
&[M_{\mu\nu},P_\rho]=\eta_{\sigma[\mu}\eta_{\nu]\rho}P^\sigma, \qquad [M_{\mu\nu},K_\rho]=\eta_{\sigma[\mu}\eta_{\nu]\rho}K^\sigma,\\
&[M_{\mu\nu},M_{\rho\sigma}]=\eta_{\lambda[\mu}\eta_{\nu]\rho}M^\lambda{}_\sigma-\eta_{\lambda[\mu}\eta_{\nu]\sigma}M^\lambda{}_\rho,\\
&\{Q^\alpha,\bar{Q}_\beta\}=\frac{1}{2}(\gamma^\mu)^\alpha{}_\beta P_\mu, \qquad \{S^\alpha,\bar{S}_\beta\}=-\frac{1}{2}(\gamma^\mu)^\alpha{}_\beta K_\mu,\\
&\{Q^\alpha,\bar{S}_\beta\}=\frac{1}{2}\delta^\alpha{}_\beta D-\frac{1}{2}(\gamma^{\mu\nu})^\alpha{}_\beta M_{\mu\nu}+\frac{3i}{4}(\gamma_5)^\alpha{}_\beta R,\\
&[P_\mu,S]=-\gamma_\mu Q, \qquad [K_\mu,Q]=\gamma_\mu S, \qquad [M_{\mu\nu},Q]=-\frac{1}{4}\gamma_{\mu\nu}Q, \qquad [M_{\mu\nu},S]=-\frac{1}{4}\gamma_{\mu\nu}S,\\
&[D,Q]=-\frac{1}{2}Q, \qquad [D,S]=\frac{1}{2}S, \qquad [R,Q]=i\gamma_5 Q, \qquad [R,S]=-i\gamma_5 S. \qquad (4.1.5)
\end{aligned}
$$

4.1.4 Noether currents and seagull operators

Noether's theorem for $SU(2,2|1)$ invariance results in only three independent current operators, corresponding to the conserved currents associated with translations, R-symmetry and Q-supersymmetry transformations. They comprise the conformal current multiplet of the massless WZ model and are given respectively by

$$
\begin{aligned}
\hat{\mathcal{T}}^\mu{}_\nu &= 2\partial^{(\mu}\phi^*\partial_{\nu)}\phi+\frac{1}{2}\bar\chi\gamma^\mu\partial_\nu\chi-\frac{1}{8}\partial_\rho(\bar\chi\gamma_\nu\gamma^{\rho\mu}\chi+\bar\chi\gamma^\mu\gamma^\rho{}_\nu\chi-\bar\chi\gamma^\rho\gamma^\mu{}_\nu\chi)\\
&\quad-\frac{1}{3}(\partial^\mu\partial_\nu-\eta^\mu_\nu\partial^2)(\phi^*\phi)-\eta^\mu_\nu(\partial_\rho\phi^*\partial^\rho\phi+\frac{1}{2}\bar\chi\partial\!\!\!/\chi-F^*F),\\
\hat{\mathcal{J}}^\mu &= \frac{2i}{3}\Big(\phi^*\partial^\mu\phi-\phi\partial^\mu\phi^*+\frac{1}{4}\bar\chi\gamma^\mu\gamma_5\chi\Big),\\
\hat{\mathcal{Q}}^\mu &= \frac{\sqrt{2}}{2}(\partial\!\!\!/\phi\gamma^\mu\chi_R+\partial\!\!\!/\phi^*\gamma^\mu\chi_L)+\frac{\sqrt{2}}{3}\gamma^{\mu\nu}\partial_\nu(\phi\chi_R+\phi^*\chi_L). \qquad (4.1.6)
\end{aligned}
$$

The currents (4.1.6) satisfy the on-shell conservation laws

$$\partial_\mu\hat{\mathcal{T}}^\mu{}_\nu=0, \qquad \partial_\mu\hat{\mathcal{J}}^\mu=0, \qquad \partial_\mu\hat{\mathcal{Q}}^\mu=0, \qquad (4.1.7)$$

while Lorentz, scale and S-supersymmetry invariance require that

$$\hat{\mathcal{T}}_{[\mu\nu]}=0, \qquad \hat{\mathcal{T}}^\mu_\mu=0, \qquad \gamma_\mu\hat{\mathcal{Q}}^\mu=0. \qquad (4.1.8)$$

The conventions that we follow for symmetrizing and antisymmetrizing the indices of a tensor $T_{a_1\ldots a_n}$, i.e. $T_{(a_1\ldots a_n)}$ and $T_{[a_1\ldots a_n]}$ respectively, are given in (A.0.1).

The stress tensor and the supercurrent in (4.1.6) include suitable improvement terms that do not affect the conservation equations (4.1.7), but ensure that the algebraic constraints (4.1.8) hold on-shell [33]. In particular the divergenceless and traceless term $-\frac{1}{8}\partial_\rho(\bar{\chi}\gamma_\nu\gamma^{\rho\mu}\chi + \bar{\chi}\gamma^\mu\gamma^\rho{}_\nu\chi - \bar{\chi}\gamma^\rho\gamma^\mu{}_\nu\chi)$ of the stress tensor is required for Lorentz invariance, while the divergenceless terms $-\frac{1}{3}(\partial^\mu\partial_\nu - \eta^\mu_\nu\partial^2)(\phi^*\phi)$ of the stress tensor and $\frac{\sqrt{2}}{3}\gamma^{\mu\nu}\partial_\nu(\phi\chi_R + \phi^*\chi_L)$ of the supercurrent are required for scale and S-supersymmetry invariance respectively.

It is convenient to introduce here some well defined operators of this theory, the 'seagull operators',

$$\hat{s}_{(1|0)} = \phi^*\phi, \qquad \hat{s}^\mu_{(2|1)} = \bar{\chi}\gamma^\mu\gamma_5\chi, \qquad \hat{s}_{(3|\frac{1}{2})} = i(\phi^*\chi_L - \phi\chi_R), \qquad \hat{s}^\mu_{(4|1)} = \phi\partial^\mu\phi^*. \tag{4.1.9}$$

As we will see later when we couple the WZ model to conformal background supergravity, the functional derivatives of the conserved currents with respect to the background sources can be expressed only in terms of these seagull operators (C.0.4). Moreover, we also need the following 'null operators'

$$\begin{aligned}
&\hat{n}^{\mu\nu}_{(1|2)} = \bar{\chi}(\gamma^{\mu\nu} - \eta^{\mu\nu})\gamma_5\partial\!\!\!/\chi, \qquad \hat{n}^{\mu\nu}_{(2|2)} = \bar{\chi}(\gamma^{\mu\nu} - \eta^{\mu\nu})\partial\!\!\!/\chi, \qquad \hat{n}_{(3|\frac{1}{2})} = i(\phi^*\partial\!\!\!/\chi_L - \phi\partial\!\!\!/\chi_R),\\
&\hat{n}_{(4|0)} = F^*\phi, \qquad \hat{n}^\mu_{(5|1)} = F^*\partial^\mu\phi, \qquad \hat{n}_{(6|\frac{1}{2})} = i(F^*\chi_L - F\chi_R),
\end{aligned} \tag{4.1.10}$$

which are proportional to the classical equations of motion. The first entry in the subscript $(\cdot|\cdot)$ simply labels the operator, while the second entry indicates its spin. Both seagull and null operators are important in the derivation of the Ward identities with operator insertions in the path integral formalism. The null operators, even though they vanish on-shell, can give a non zero result when put inside correlators. This can be understood heuristically as follows. Consider a null operator inside a 3-point function. This 3-point function can be written in momentum space as a sum of non zero two-point functions, since the terms (of the null operator) proportional to the equations of motion will cancel one propagator in the triangle Feynman diagram. Formally one can find these expressions using the Schwinger-Dyson equations [92, 93], which we present in more detail in the appendix C.

4.1.5 Symmetry transformations of the Noether currents and seagull operators

We are interested in the Ward identities of correlation functions derived using (2.2.8). So we need to evaluate the transformations of the currents and seagull operators with respect to the symmetries of table 4.1.1. In our analysis we only care about symmetries which are explicitly broken by the PV regulator we use later. These are the original Q- and S-supersymmetry of conformal supergravity and the R-symmetry. Note here that the PV regulator satisfies by construction the Q+S supersymmetry of old minimal supergravity.

All the other symmetries such as Lorentz symmetry and diffeomorphisms are manifestly respected by the regulator, thus there is no need to examine them. We just have to be careful to not violate them when we introduce counterterms to restore the broken symmetries.

The R-symmetry transformations of the currents and seagull operators take the form

$$\begin{aligned}
\delta_{\theta_0}\hat{\mathcal{T}}^{\mu}{}_{\nu} &= (\eta^{\mu\rho}\eta^{\sigma}_{\nu} + \eta^{\mu\sigma}\eta^{\rho}_{\nu} - \eta^{\mu}_{\nu}\eta^{\rho\sigma})\hat{\mathcal{J}}_{\rho}\partial_{\sigma}\theta_0 - \frac{i}{6}\bar{\chi}\gamma_{\nu}\gamma_5\chi\partial^{\mu}\theta_0, \\
\delta_{\theta_0}\hat{\mathcal{J}}^{\mu} &= \frac{8}{9}\phi^*\phi\,\partial^{\mu}\theta_0, \\
\delta_{\theta_0}\hat{\mathcal{Q}}^{\mu} &= i\theta_0\gamma_5\hat{\mathcal{Q}}^{\mu} + \frac{i\sqrt{2}}{3}\partial^{\mu}\theta_0(\phi^*\chi_L - \phi\chi_R), \\
\delta_{\theta_0}\hat{s}_{(3|\frac{1}{2})} &= i\theta_0\gamma_5 i\left(\phi^*\chi_L - \phi\chi_R\right),
\end{aligned} \tag{4.1.11}$$

which can be written in terms of the seagull operators (4.1.9) as follows

$$\begin{aligned}
\delta_{\theta_0}\hat{\mathcal{T}}^{\mu}{}_{\nu} &= (\eta^{\mu\rho}\eta^{\sigma}_{\nu} + \eta^{\mu\sigma}\eta^{\rho}_{\nu} - \eta^{\mu}_{\nu}\eta^{\rho\sigma})\hat{\mathcal{J}}_{\rho}\partial_{\sigma}\theta_0 - \frac{i}{6}\hat{s}_{\nu(2|1)}\partial^{\mu}\theta_0, \\
\delta_{\theta_0}\hat{\mathcal{J}}^{\mu} &= \frac{8}{9}\hat{s}_{(1|0)}\,\partial^{\mu}\theta_0, \\
\delta_{\theta_0}\hat{\mathcal{Q}}^{\mu} &= i\theta_0\gamma_5\hat{\mathcal{Q}}^{\mu} + \frac{\sqrt{2}}{3}\partial^{\mu}\theta_0\,\hat{s}_{(3|\frac{1}{2})}, \\
\delta_{\theta_0}\hat{s}_{(3|\frac{1}{2})} &= i\theta_0\gamma_5\,\hat{s}_{(3|\frac{1}{2})}.
\end{aligned} \tag{4.1.12}$$

Similarly, the Q-supersymmetry transformations of the operators we need are

$$\begin{aligned}
\delta_{\varepsilon_0}\hat{\mathcal{J}}^{\mu} =& -i\bar{\varepsilon}_0\gamma_5\hat{\mathcal{Q}}^{\mu} + \frac{\sqrt{2}i}{3}(\phi^*\bar{\chi}_L - \phi\bar{\chi}_R)\partial^{\mu}\varepsilon_0 + \frac{\sqrt{2}i}{3}\bar{\varepsilon}_0\gamma^{\mu}(\phi^*\partial\!\!\!/\chi_L - \phi\partial\!\!\!/\chi_R) \\
&+ \frac{\sqrt{2}i}{6}\bar{\varepsilon}_0\gamma^{\mu}(F^*\chi_L - F\chi_R), \\
\delta_{\varepsilon_0}\hat{\mathcal{Q}}^{\mu} =& \frac{1}{2}\hat{\mathcal{T}}^{\mu}{}_{\nu}\gamma^{\nu}\varepsilon_0 + \frac{i}{8}\partial_{\rho}\left[\hat{\mathcal{J}}_{\sigma}(i\epsilon^{\mu\nu\rho\sigma}\gamma_5 + 2\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\rho\nu}\eta^{\mu\sigma})\gamma_{\nu}\gamma_5\varepsilon_0\right] - \frac{3}{8}\epsilon^{\mu\nu\rho\sigma}\hat{\mathcal{J}}_{\sigma}\gamma_{\nu}\partial_{\rho}\varepsilon_0 \\
&+ \frac{1}{8}(\bar{\chi}\gamma_{\sigma}\gamma_5\chi)\gamma^{\sigma}\gamma_5\partial^{\mu}\varepsilon_0 + \frac{1}{6}\partial_{\rho}(\phi\phi^*)(i\epsilon^{\mu\nu\rho\sigma}\gamma_5 + \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\gamma_{\nu}\partial_{\sigma}\varepsilon_0 \\
&+ \frac{1}{8}(\bar{\chi}(\gamma^{\mu\nu} - \eta^{\mu\nu})\gamma_5\partial\!\!\!/\chi)\gamma_{\nu}\gamma_5\varepsilon_0 - \frac{1}{8}(\bar{\chi}(\gamma^{\mu\nu} - \eta^{\mu\nu})\partial\!\!\!/\chi)\gamma_{\nu}\varepsilon_0 \\
&+ \frac{1}{2}(F^*\partial\!\!\!/\phi\gamma^{\mu}\varepsilon_{0R} + F\partial\!\!\!/\phi^*\gamma^{\mu}\varepsilon_{0L}) + \frac{1}{3}\gamma^{\mu\nu}\partial_{\nu}(\phi F^*\varepsilon_{0R} + \phi^*F\varepsilon_{0L}), \\
\delta_{\varepsilon_0}\hat{s}_{(1|0)} =& -i\frac{\sqrt{2}}{2}\bar{\varepsilon}_0\gamma_5 i\left(\phi^*\chi_L - \phi\chi_R\right) \\
\delta_{\varepsilon_0}\hat{\bar{s}}_{(3|\frac{1}{2})} =& \bar{\varepsilon}_0\left(-i\frac{\sqrt{2}}{4}\gamma^{\sigma}\gamma_5\partial_{\sigma}(\phi^*\phi) - \frac{3\sqrt{2}}{8}\gamma_{\sigma}\hat{\mathcal{J}}^{\sigma} + i\frac{3\sqrt{2}}{16}\gamma_{\sigma}\bar{\chi}\gamma^{\sigma}\gamma_5\chi\right).
\end{aligned} \tag{4.1.13}$$

Using the seagull and null operators we can write the above as

$$\begin{aligned}
\delta_{\varepsilon_0}\hat{\mathcal{J}}^\mu =& -i\bar{\varepsilon}_0\gamma_5\hat{\mathcal{Q}}^\mu+\frac{\sqrt{2}}{3}\,\hat{\bar{s}}_{(3|\frac{1}{2})}\partial^\mu\varepsilon_0+\frac{\sqrt{2}}{3}\bar{\varepsilon}_0\gamma^\mu\,\hat{n}_{(3|\frac{1}{2})}+\frac{\sqrt{2}}{6}\bar{\varepsilon}_0\gamma^\mu\,\hat{n}_{(6|\frac{1}{2})},\\
\delta_{\varepsilon_0}\hat{\mathcal{Q}}^\mu =& \frac{1}{2}\hat{\mathcal{T}}^\mu{}_\nu\gamma^\nu\varepsilon_0+\frac{i}{8}\partial_\rho\big[\hat{\mathcal{J}}_\sigma(i\epsilon^{\mu\nu\rho\sigma}\gamma_5+2\eta^{\mu\nu}\eta^{\rho\sigma}-2\eta^{\rho\nu}\eta^{\mu\sigma})\gamma_\nu\gamma_5\varepsilon_0\big]-\frac{3}{8}\epsilon^{\mu\nu\rho\sigma}\hat{\mathcal{J}}_\sigma\gamma_\nu\partial_\rho\varepsilon_0\\
&+\frac{1}{8}\hat{s}_{\sigma(2|1)}\gamma^\sigma\gamma_5\partial^\mu\varepsilon_0+\frac{1}{6}\partial_\rho(\hat{s}_{(1|0)})(i\epsilon^{\mu\nu\rho\sigma}\gamma_5+\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\gamma_\nu\partial_\sigma\varepsilon_0\\
&+\frac{1}{8}\hat{n}^{\mu\nu}_{(1|2)}\gamma_\nu\gamma_5\varepsilon_0-\frac{1}{8}\hat{n}^{\mu\nu}_{(2|2)}\gamma_\nu\varepsilon_0\\
&+\frac{1}{2}(\hat{n}^\rho_{(5|1)}\gamma_\rho\gamma^\mu\varepsilon_{0R}+\hat{n}^{\rho*}_{(5|1)}\gamma_\rho\gamma^\mu\varepsilon_{0L})+\frac{1}{3}\gamma^{\mu\nu}\partial_\nu(\hat{n}_{(4|0)}\varepsilon_{0R}+\hat{n}^*_{(4|0)}\varepsilon_{0L}),\\
&\delta_{\varepsilon_0}\hat{s}_{(1|0)}=-i\frac{\sqrt{2}}{2}\bar{\varepsilon}_0\gamma_5\,\hat{s}_{(3|\frac{1}{2})}\\
&\delta_{\varepsilon_0}\hat{\bar{s}}_{(3|\frac{1}{2})}=\bar{\varepsilon}_0\left(-i\frac{\sqrt{2}}{4}\gamma^\sigma\gamma_5\partial_\sigma(\hat{s}_{(1|0)})-\frac{3\sqrt{2}}{8}\gamma_\sigma\hat{\mathcal{J}}^\sigma+i\frac{3\sqrt{2}}{16}\gamma_\sigma\,\hat{s}^\sigma_{(2|1)}\right).
\end{aligned}\tag{4.1.14}$$

Let us make a comment here regarding the Q-supersymmetry transformation of the R-current. The zero on-shell term $(\phi^*\bar{\partial\!\!\!/}\chi_L-\phi\bar{\partial\!\!\!/}\chi_R)$ is proportional to the gamma trace of the supercurrent, so the variation of the R-current can be written in the following form

$$\delta_{\varepsilon_0}\hat{\mathcal{J}}^\mu = -i\bar{\varepsilon}_0\gamma_5\hat{\mathcal{Q}}^\mu+\frac{\sqrt{2}i}{3}(\phi^*\bar{\chi}_L-\phi\bar{\chi}_R)\partial^\mu\varepsilon_0+\frac{i}{3}\bar{\varepsilon}_0\gamma_5\gamma^\mu\gamma_\kappa\hat{\mathcal{Q}}^\kappa+\frac{\sqrt{2}i}{6}\bar{\varepsilon}_0\gamma^\mu(F^*\chi_L-F\chi_R).\tag{4.1.15}$$

According to (2.2.8), at the rhs of the Q-superymmetry Ward identity of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ exist the correlators $<\delta_{\varepsilon_0}\mathcal{J}\bar{\mathcal{Q}}\mathcal{J}>$ and $<\delta_{\varepsilon_0}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. Depending on which form we use for $\delta_{\varepsilon_0}\hat{\mathcal{J}}^\mu$, i.e. (4.1.13) or (4.1.15), the correlators of the rhs will have a different form. In particular, using (4.1.13), the Q-supersymmetry Ward identity will take the schematic form

$$\partial<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>=<\delta_{\varepsilon_0}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>+<\delta_{\varepsilon_0}\mathcal{J}\bar{\mathcal{Q}}\mathcal{J}>\ldots\equiv<N_Q\mathcal{J}\mathcal{J}>+<N_J\bar{\mathcal{Q}}\mathcal{J}>+\ldots\tag{4.1.16}$$

while using (4.1.15) we get

$$\partial<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>=<\delta_{\varepsilon_0}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>+<\delta_{\varepsilon_0}\mathcal{J}\bar{\mathcal{Q}}\mathcal{J}>\ldots\equiv<N_Q\mathcal{J}\mathcal{J}>+\gamma<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>+\ldots\tag{4.1.17}$$

N_Q and N_J denote the zero on-shell terms of the Q-supersymmetry transformations of the supercurrent and the R-current respectively, that do not depend on the auxiliary field F. The dots ... denote correlators that are not important for the argument under consideration. As we have already mentioned, correlators that include null (vanishing on-shell) operators are proportional to lower order correlators according to the Schwinger-Dyson equations (C.0.7). Thus, the 3-point functions $<N_Q\mathcal{J}\mathcal{J}>$ and $<N_J\bar{\mathcal{Q}}\mathcal{J}>$ are equal to a sum of 2-point functions. Similarly, $\gamma<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ of (4.1.17) is proportional to a sum of 2-point correlators, as a consequence of the S-supersymmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. Of course, from a computational point of view the rhs of the two above equations are identical, we just grouped the integrals in a different way. The advantage of the symmetry variations (4.1.13), hence (4.1.16), is that it makes easier to reproduce the Q-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ that we get from conformal supergravity (4.2.16). In particular, the correlators $<N_Q\mathcal{J}\mathcal{J}>$ and $<N_J\bar{\mathcal{Q}}\mathcal{J}>$ combine in a way that help us distinguish the 2-point correlators of conserved currents from the 2-point seagull correlators of (4.2.16). On the contrary, if one uses the form (4.1.17) it is not straightforward how to group the 2-point functions in a way that the identity of conformal supergravity is reproduced. The advantage of (4.1.15) though, is that it makes obvious how the introduction of a massive regulator violates Q-supersymmetry. In the massless theory, the term $\frac{i}{3}\bar{\varepsilon}_0\gamma_5\gamma^\mu\gamma_\kappa\hat{\mathcal{Q}}^\kappa$ vanishes on-shell. As we will see in the next chapter, this is not the case in the presence of a non zero mass.

Finally, the S-supersymmetry transformations of the R-current and of the supercurrent are

$$\delta_{\eta_0}\hat{\mathcal{J}}^\mu = (\delta_{\varepsilon_0 = x^\mu\gamma_\mu\eta_0} + \widetilde{\delta}_{\eta_0})\hat{\mathcal{J}}^\mu, \qquad \delta_{\eta_0}\hat{\mathcal{Q}}^\mu = (\delta_{\varepsilon_0 = x^\kappa\gamma_\kappa\eta_0} + \widetilde{\delta}_{\eta_0})\hat{\mathcal{Q}}^\mu, \tag{4.1.18}$$

where $\delta_{\varepsilon_0 = x^\kappa\gamma_\kappa\eta_0}$ denotes a Q-supersymmetry transformation with parameter $\varepsilon_0 = x^\kappa\gamma_\kappa\eta_0$ and

$$\begin{aligned}
\widetilde{\delta}_{\eta_0}\hat{\mathcal{J}}^\mu &= \frac{i\sqrt{2}}{3}\bar{\eta}_0\gamma^\mu(\phi^*\chi_L - \phi\chi_R), \\
\widetilde{\delta}_{\eta_0}\hat{\mathcal{Q}}^\mu &= \frac{2}{3}\gamma^{\mu\nu}\partial_\nu(\phi^*\phi\eta_0) + (\phi\bar{\not\partial}\phi^*\gamma^\mu\eta_{0L} + \phi^*\bar{\not\partial}\phi\gamma^\mu\eta_{0R}).
\end{aligned} \tag{4.1.19}$$

4.2 Ward identities in momentum space

In this section we present the classical Ward identities for all correlators of interest. We find it more convenient to write them in momentum space, since in the end we are going to compute them using Feynman diagrams. One of the main goals is to show that the original Q-supersymmetry Ward identity of the correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ in the conformal WZ model is anomalous. In this Ward identity there exists a number of lower order correlation functions, such as $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$, $<\mathcal{T}\mathcal{J}\mathcal{J}>$, $<\mathcal{J}\mathcal{J}>$. It is important to fix these correlators in a way that satisfy all their standard identities and anomalies, before going to $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

To be more specific, for a consistent calculation one has to know the exact renormalization scheme that is used for the lower order correlators in the Ward identities of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. We want to make sure that a non anomalous Q-supersymmetry, does not mean for example a $<\mathcal{T}\mathcal{J}\mathcal{J}>$ correlator with an anomaly in diffeomorphisms. There is always this risk, when we go straight away to the computation of Ward identities of higher order correlators without having fixed the theory in the lower order correlators first. We will see one such example in chapter 9, where we confirm the Q-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ using momentum routing regularization. A naive approach shows that Q-supersymmetry is non anomalous. However, after a more careful consideration we see that in order to prove this we need to use a $<\mathcal{J}\mathcal{J}\mathcal{J}>$ correlator that does not respect Bose symmetry, hence it is non consistent. By Bose symmetry, we mean that $<\mathcal{J}^\mu(p_1)\mathcal{J}^\kappa(p_2)\mathcal{J}^\lambda(p_3)>$ must be invariant under the exchange of the external momenta p_1, p_2, p_3 and the spacetime indices μ, κ, λ. The non Bose-symmetric Feynman diagram $<\mathcal{J}^\mu(p_1)\mathcal{J}^\kappa(p_2)\mathcal{J}^\lambda(p_3)>$, yields different results when contracted with $p_{1\mu}$, $p_{2\kappa}$ or $p_{3\lambda}$, thus it does not reproduce the standard R-symmetry anomaly.

Since we want to examine the massless WZ model as an example of $\mathcal{N}=1$ superconformal field theory, the Ward identities we present in this section are identical to the ones we get after coupling the WZ model to conformal supergravity. To compute them, one has to use the Ward identities at the level of 1-point functions of $\mathcal{N}=1$ conformal supergravity (B.1.1) which were derived in [71] and then further differentiate with respect to the background sources of the conserved currents in order to derive the identities of higher order correlation functions. As already mentioned in section (2.2) and also explained in more detail in the appendix C, the Ward identities of a specific correlator differ when derived with functional differentiation in the presence of sources and when we use operator insertions in the path integral formalism. The functional derivative identities are a sum of path integral identities. In the Ward identities that we write below, from the lower order correlators such as $<\mathcal{J}\mathcal{J}>$ until the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, we group the terms that form their own path integral Ward identities using different colours. In order to do that one has to use (2.2.8), the symmetry transformations of the operators written in the previous section and then perform a Fourier transform to go to momentum space. For example, in (4.2.4),(4.2.5) the (sum of) black, blue and red terms are independently equal to zero as a consequence of the path integral R-symmetry identity.

The aforementioned distinction of the terms that form their own path integral identities -identities which are verified at the quantum level using Feynman diagrams- is useful when one introduces the regulator to deal with the divergent quantities of the theory. The regulator may classically break some of the symmetries of the initial theory, like the Pauli-Villars regulator we introduce in chapter 6 violates R-symmetry and conformal invariance. Using (2.2.8) and the modified symmetry transformations of the operators in the presence of the regulator, one can easily identify and compute the contribution of the regulator to each one of the original classical path integral Ward identities. In fact, following this bottom-up approach we can find all the classical breaking (and potentially anomalous) terms that arise after regularization without the need to know how the regulated theory couples to background supergravity. Of course, one can follow the equivalent approach of first coupling the regulated theory to background supergravity (for the PV regulator in our case that would be old minimal supergravity), then derive the Ward identities of the regulated theory at the 1-point function level, compare them with the original identities of conformal supergravity and deduce the breaking terms. We explain this approach in detail in [81].

Finally, we should also note that we do not include anywhere 1-point functions. As we will show later in the PV regulated theory, the 1-point functions of the Noether currents are zero (see appendix D.2). There are some seagull operators though with non vanishing 1-point functions, such as $<\hat{s}_{(1|0)}>$ (D.2.4). These do not play any role in the computation of anomalies. The correlators inside Ward identities include divergent and finite pieces. The Ward identities must be satisfied independently in both parts. The 1-point functions can contribute only in the divergent parts, since they are tadpole diagrams with no external momentum inside the loop. The only parameter in these diagrams is the regulator that encodes the infinities. On the contrary, the analysis of anomalies cares only about the finite pieces of the correlators.

In this section we are always referring to the flat space operators, but we omit the hat $\hat{}$, to simplify the notation. We also introduce the following quantities to simplify the expressions

$$p_{ij} = p_i + p_j \,, \quad p_{ijk} = p_i + p_j + p_k. \tag{4.2.1}$$

4.2.1 2-point functions

$<\mathcal{J}\mathcal{J}>$

The classical R-symmetry Ward identity for the 2-point function $<\mathcal{J}\mathcal{J}>$ in momentum space is given by

$$p_{3\kappa} <\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>= 0. \tag{4.2.2}$$

$<\mathcal{Q}\bar{\mathcal{Q}}>$

The classical Q- and S-supersymmetry Ward identities for the 2-point function $<\mathcal{Q}\bar{\mathcal{Q}}>$ are given by

$$p_{1\mu} <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)>= 0 \,, \qquad \gamma_\mu <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)>= 0. \tag{4.2.3}$$

4.2.2 3-point functions

$<\mathcal{T}\mathcal{J}\mathcal{J}>$

The 3-point function $<\mathcal{T}\mathcal{J}\mathcal{J}>$ must satisfy Ward identities associated with diffeomorphisms, Lorentz, conformal and R-symmetry. Here we are interested only in the conformal and R-symmetry identities, since these are the ones that are classically violated by the

PV regulator of chapter 6. Diffeomorphisms and Lorentz symmetry are manifestly respected by the regulator, thus they are valid in the quantum theory too. $D_{3R}^{\nu\xi\lambda}$ and $D_{3D}^{\kappa\lambda}$ denote the seagull correlators of the R-symmetry and conformal symmetry Ward identities respectively.

R-symmetry

$$p_{3\kappa} <\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)\mathcal{T}^{\nu\xi}(p_1)> +D_{3R}^{\nu\xi\lambda} = 0 \tag{4.2.4}$$

where

$$\begin{aligned} D_{3R}^{\nu\xi\lambda} &= ip_3^{\lambda}\frac{8}{9} <s_{(1|0)}(p_{34})\mathcal{T}^{\nu\xi}(p_1)> \\ &+ i(\eta^{\nu\rho}\eta^{\sigma\xi} + \eta^{\nu\sigma}\eta^{\rho\xi} - \eta^{\nu\xi}\eta^{\rho\sigma})p_{3\sigma} <\mathcal{J}_{\rho}(p_{13})\mathcal{J}^{\lambda}(p_4)> + \frac{1}{6}p_3^{\nu} <s^{\xi}_{(2|1)}(p_{13})\mathcal{J}^{\lambda}(p_4)> \\ &+ i(\eta^{\nu\lambda}\eta^{\sigma\xi} + \eta^{\nu\sigma}\eta^{\lambda\xi} - \eta^{\nu\xi}\eta^{\lambda\sigma})p_{3\kappa} <\mathcal{J}_{\sigma}(p_{14})\mathcal{J}^{\kappa}(p_3)> + \frac{1}{6}\eta^{\nu\lambda}p_{3\kappa} <s^{\xi}_{(2|1)}(p_{14})\mathcal{J}^{\kappa}(p_3)> . \end{aligned} \tag{4.2.5}$$

The black terms form the classical path integral R-symmetry identity of $< \mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)\mathcal{T}^{\nu\xi}(p_1) >$, while the blue and red terms are also independently equal to zero due to R-symmetry.

Conformal symmetry

$$<\mathcal{T}_{\nu}^{\nu}(p_1)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> +D_{3D}^{\kappa\lambda} = 0 \tag{4.2.6}$$

where

$$\begin{aligned} D_{3D}^{\kappa\lambda} &= -2i <\mathcal{J}^{\kappa}(p_{13})\mathcal{J}^{\lambda}(p_4)> -2i <\mathcal{J}^{\lambda}(p_{14})\mathcal{J}^{\kappa}(p_3)> \\ &+ \frac{1}{6} <s^{\kappa}_{(2|1)}(p_{13})\mathcal{J}^{\lambda}(p_4)> + \frac{1}{6} <s^{\lambda}_{(2|1)}(p_{14})\mathcal{J}^{\kappa}(p_3)> + \frac{8i}{9}\eta^{\kappa\lambda} <\mathcal{T}_{\nu}^{\nu}(p_1)s_{(1|0)}(p_{34})> . \end{aligned} \tag{4.2.7}$$

Similarly, the black terms comprise the conformal symmetry identity of $<\mathcal{T}_{\nu}^{\nu}(p_1)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>$, while the seagull correlator $<\mathcal{T}_{\nu}^{\nu}(p_1)s_{(1|0)}(p_{34})>$ vanishes too, as a consequence of scale invariance.

$<\mathcal{J}\mathcal{J}\mathcal{J}>$

The classical R-symmetry Ward identity for the 3-point function $<\mathcal{J}\mathcal{J}\mathcal{J}>$ in momentum space is given by

$$p_{3\kappa} <\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)\mathcal{J}^{\sigma}(p_1)>= -\frac{8i}{9}p_3^{\lambda} <s_{(1|0)}(p_{34})\mathcal{J}^{\sigma}(p_1)> -\frac{8i}{9}p_3^{\sigma} <s_{(1|0)}(p_{31})\mathcal{J}^{\lambda}(p_4)>$$

$$- \frac{16i}{9}\eta^{\sigma\lambda}p_{3\kappa} < \mathcal{J}^{\kappa}(p_3)s_{(1|0)}(p_{14})> . \tag{4.2.8}$$

$<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$

There are three Ward identities associated with $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$, the ones related to R-symmetry, Q- and S-supersymmetry, which are given below. The quantities $C^{\nu\kappa}_{3Q}$, $C^{\mu\nu}_{3R}$ and $C^{\nu\kappa}_{3S}$ denote the seagull correlators in the Q-supersymmetry, R-symmetry and S-supersymmetry Ward identities respectively.

Q-supersymmetry

$$\begin{aligned} &p_{1\mu} < \mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)> + C^{\nu\kappa}_{3Q} \\ &= ip_{2\mu}B^{\nu\mu\sigma} < \mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)> - \frac{\gamma_\xi}{2} < \mathcal{T}^{\nu\xi}(p_{12})\mathcal{J}^{\kappa}(p_3)> - i\gamma_5 < \mathcal{Q}^{\kappa}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)> \end{aligned} \tag{4.2.9}$$

where

$$\begin{aligned} C^{\nu\kappa}_{3Q} = p_{1\mu}\Big(&\frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi < \mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)> + \frac{i}{8}\eta^{\mu\nu}\gamma^{\sigma}\gamma_5 < s_{\sigma(2|1)}(p_{12})\mathcal{J}^{\kappa}(p_3)> \\ &+ i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} < s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)> + i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} < \mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\ &- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu} - \frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu} + i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi\gamma_5\right) < s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)>\Big) \end{aligned} \tag{4.2.10}$$

and

$$B^{\nu\mu\sigma} = \frac{1}{4}\left(-\frac{1}{2}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi + i\gamma_5\gamma^{\mu}\eta^{\nu\sigma} - i\gamma_5\gamma^{\nu}\eta^{\mu\sigma}\right) . \tag{4.2.11}$$

(4.2.9) is a sum of two Q-supersymmetry path integral Ward identities. The black terms form their own path integral Ward identity, while the blue term contracted with $p_{1\mu}$ is also classically zero.

R-symmetry

Terms of separate colour below, form their own path integral Ward identities. The black terms comprise the classical R-symmetry Ward identity of $<\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)>$ derived using (2.2.8). The blue term contracted with $p_{3\kappa}$ is the R-symmetry identity (4.2.2). Similarly, the red and purple terms contracted with $p_{3\kappa}$ are also classically zero.

$$\begin{aligned} &p_{3\kappa} < \mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)> + C^{\mu\nu}_{3R} \\ &= i\gamma_5 < \mathcal{Q}^{\mu}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)> + i < \mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_{23})> \gamma_5 \end{aligned} \tag{4.2.12}$$

where

$$C_{3R}^{\mu\nu} = p_{3\kappa}\left(i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)> +i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})>\right.$$
$$+\frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi} <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)> +\frac{i}{8}\eta^{\mu\nu}\gamma^{\sigma}\gamma_5 <s_{\sigma(2|1)}(p_{12})\mathcal{J}^{\kappa}(p_3)>$$
$$\left.-p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu}-\frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu}+i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}\gamma_5\right) <s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)>\right). \tag{4.2.13}$$

S-supersymmetry

$$-i\gamma_{\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)> +C_{3S}^{\nu\kappa} = -\frac{3i}{4}\gamma_5 <\mathcal{J}^{\nu}(p_{12})\mathcal{J}^{\kappa}(p_3)> \tag{4.2.14}$$

where

$$C_{3S}^{\nu\kappa} = -i\gamma_{\mu}\left(i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)> +i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})>\right)$$
$$+\frac{i}{2}\gamma^{\nu}\gamma^{\sigma}p_{12\sigma} <s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)> -\frac{2i}{3}\gamma^{\nu\sigma}p_{2\sigma} <s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)>$$
$$+\frac{1}{8}\gamma^{\nu}\gamma_{\sigma}\gamma_5 <s^{\sigma}_{(2|1)}(p_{12})\mathcal{J}^{\kappa}(p_3)> -\frac{3i}{4}\gamma^{\sigma\nu}\gamma_5 <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)> . \tag{4.2.15}$$

The black and blue terms comprise the S-supersymmetry identities of $<\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)>$ and $<\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})>$ respectively.

4.2.3 4-point function

$<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$

Similarly to $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$, there are three identities that $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ should satisfy. The quantities $C_{4Q}^{\nu\kappa\lambda}$, $C_{4R}^{\mu\nu\lambda}$ and $C_{4S}^{\nu\kappa\lambda}$ denote all the seagull correlators in the Q-supersymmetry, R-symmetry and S-supersymmetry Ward identities respectively. Again, terms of separate colour form their own identities. In particular, the black terms comprise the path integral Ward identities (2.2.8) of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, while the coloured terms are the path integral Ward identities of the seagull correlators. To make the expressions below as compact as possible, we have not included terms that involve the correlator $<s_{(1,0)}\mathcal{J}>$, which turns out to be zero in the PV regulated theory due to odd symmetry arguments (D.3.13).

Q-supersymmetry

$$p_{1\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> +C_{4Q}^{\nu\kappa\lambda}$$
$$= ip_{2\mu}B^{\nu\mu\sigma} <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> -\frac{\gamma_{\xi}}{2} <\mathcal{T}^{\nu\xi}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>$$

$$
\begin{aligned}
&- i\gamma_5 <\mathcal{Q}^\kappa(p_{13})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\lambda(p_4)> -i\gamma_5 <\mathcal{Q}^\lambda(p_{14})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)> \\
&+ \gamma_5 B^{\nu\kappa\sigma} <\mathcal{J}_\sigma(p_{123})\mathcal{J}^\lambda(p_4)> +\gamma_5 B^{\nu\lambda\sigma} <\mathcal{J}_\sigma(p_{124})\mathcal{J}^\kappa(p_3)> \qquad (4.2.16)
\end{aligned}
$$

where

$$
\begin{aligned}
C^{\nu\kappa\lambda}_{4Q} \equiv\ & p_{1\mu}\left(\frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi <\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)> +\frac{i}{8}\eta^{\mu\nu}\gamma^\sigma\gamma_5 <s_{\sigma(2|1)}(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>\right. \\
&+ i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\lambda(p_4)> +i\eta^{\mu\lambda}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)> \\
&+ i\frac{8}{9}\eta^{\kappa\lambda} <s_{(1|0)}(p_{34})\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)> +i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{Q}^\mu(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})\mathcal{J}^\lambda(p_4)> \\
&+ i\eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{Q}^\mu(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{24})\mathcal{J}^\kappa(p_3)> +i\eta^{\mu\kappa}\eta^{\nu\lambda}\frac{2}{9} <s_{(3|\frac{1}{2})}(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})> \\
&+ i\eta^{\mu\lambda}\eta^{\nu\kappa}\frac{2}{9} <s_{(3|\frac{1}{2})}(p_{14})\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^\mu - \frac{1}{6}\eta^{\sigma\mu}\gamma^\nu - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi\gamma_5\right) i\frac{8}{9}\eta^{\kappa\lambda} <s_{(1|0)}(p_{12})s_{(1|0)}(p_{34})> \\
&\left. -p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^\mu - \frac{1}{6}\eta^{\sigma\mu}\gamma^\nu - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi\gamma_5\right) <s_{(1|0)}(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>\right) \\
&+ i\frac{8}{9}\eta^{\kappa\lambda}\frac{\gamma_\xi}{2} <\mathcal{T}^{\nu\xi}(p_{12})s_{(1|0)}(p_{34})> -\frac{4\sqrt{2}}{9}\eta^{\kappa\lambda}\gamma_5 <s_{(3|\frac{1}{2})}(p_{134})\bar{\mathcal{Q}}^\nu(p_2)> \\
&- \frac{2\sqrt{2}}{9}\eta^{\kappa\lambda}\gamma_5 <s_{(3|\frac{1}{2})}(p_{134})\bar{\mathcal{Q}}^\nu(p_2)> -\eta^{\nu\lambda}\frac{\sqrt{2}}{3}\gamma_5 <\mathcal{Q}^\kappa(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})> \\
&- \eta^{\nu\kappa}\frac{\sqrt{2}}{3}\gamma_5 <\mathcal{Q}^\lambda(p_{14})\bar{s}_{(3|\frac{1}{2})}(p_{23})> +\frac{1}{24}\eta^{\nu\kappa}\gamma^\sigma <s_{\sigma(2|1)}(p_{123})\mathcal{J}^\lambda(p_4)> \\
&+ \frac{1}{24}\eta^{\nu\lambda}\gamma^\sigma <s_{\sigma(2|1)}(p_{124})\mathcal{J}^\kappa(p_3)> \\
&+ \frac{1}{12}\eta^{\nu\kappa}\gamma^\sigma <s_{\sigma(2|1)}(p_{123})\mathcal{J}^\lambda(p_4)> +\frac{1}{12}\eta^{\nu\lambda}\gamma^\sigma <s_{\sigma(2|1)}(p_{124})\mathcal{J}^\kappa(p_3)> \\
&- \frac{1}{6}\eta^{\nu\kappa}\gamma^\sigma <(s^*_{\sigma(4|1)} - s_{\sigma(4|1)})(p_{123})\mathcal{J}^\lambda(p_4)> -\frac{1}{6}\eta^{\nu\lambda}\gamma^\sigma <(s^*_{\sigma(4|1)} - s_{\sigma(4|1)})(p_{124})\mathcal{J}^\kappa(p_3)> \\
&- \frac{1}{6}\eta^{\nu\kappa}\gamma^\sigma <(s^*_{\sigma(4|1)} - s_{\sigma(4|1)})(p_{123})\mathcal{J}^\lambda(p_4)> -\frac{1}{6}\eta^{\nu\lambda}\gamma^\sigma <(s^*_{\sigma(4|1)} - s_{\sigma(4|1)})(p_{124})\mathcal{J}^\kappa(p_3)> \\
&- \left(-\frac{3}{8}\gamma_\xi\gamma_5\epsilon^{\sigma\xi\nu\kappa} + \frac{i}{2}\eta^{\kappa\sigma}\gamma^\nu - \frac{i}{2}\eta^{\nu\sigma}\gamma^\kappa\right) <\mathcal{J}_\sigma(p_{123})\mathcal{J}^\lambda(p_4)> \\
&- \left(-\frac{3}{8}\gamma_\xi\gamma_5\epsilon^{\sigma\xi\nu\lambda} + \frac{i}{2}\eta^{\lambda\sigma}\gamma^\nu - \frac{i}{2}\eta^{\nu\sigma}\gamma^\lambda\right) <\mathcal{J}_\sigma(p_{124})\mathcal{J}^\kappa(p_3)> . \qquad (4.2.17)
\end{aligned}
$$

R-symmetry

$$
\begin{aligned}
&p_{3\kappa} <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)> +C^{\mu\nu\lambda}_{4R} \\
&= i\gamma_5 <\mathcal{Q}^\mu(p_{13})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\lambda(p_4)> +i <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_{23})\mathcal{J}^\lambda(p_4)> \gamma_5 \qquad (4.2.18)
\end{aligned}
$$

where

$$
C^{\mu\nu\lambda}_{4R} \equiv p_{3\kappa}\left(\frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi <\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)> +\frac{i}{8}\eta^{\mu\nu}\gamma^\sigma\gamma_5 <s_{\sigma(2|1)}(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>\right.
$$

$$
\begin{aligned}
&+ i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\lambda}(p_4)> + i\eta^{\mu\lambda}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)> \\
&+ i\frac{8}{9}\eta^{\kappa\lambda} <s_{(1|0)}(p_{34})\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)> + i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})\mathcal{J}^{\lambda}(p_4)> \\
&+ i\eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{24})\mathcal{J}^{\kappa}(p_3)> + i\eta^{\mu\kappa}\eta^{\nu\lambda}\frac{2}{9} <s_{(3|\frac{1}{2})}(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})> \\
&+ i\eta^{\mu\lambda}\eta^{\nu\kappa}\frac{2}{9} <s_{(3|\frac{1}{2})}(p_{14})\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu} - \frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu} - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}\gamma_5\right) i\frac{8}{9}\eta^{\kappa\lambda} <s_{(1|0)}(p_{12})s_{(1|0)}(p_{34})> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu} - \frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu} - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}\gamma_5\right) <s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>\Bigg) \\
&+ \eta^{\nu\lambda}\gamma_5\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})) + \eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{234}))\gamma_5 \\
&+ \eta^{\mu\lambda}\gamma_5\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{123})\bar{\mathcal{Q}}^{\nu}(p_2)) + \eta^{\mu\lambda}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{Q}}^{\nu}(p_{23}))\gamma_5
\end{aligned} \tag{4.2.19}
$$

S-supersymmetry

$$
- i\gamma_{\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> + C_{4S}^{\nu\kappa\lambda} = -\frac{3i}{4}\gamma_5 <\mathcal{J}^{\nu}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> \tag{4.2.20}
$$

where

$$
\begin{aligned}
C_{4S}^{\nu\kappa\lambda} = &-i\gamma_{\mu}\Bigg(\frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi} <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> + \frac{i}{8}\eta^{\mu\nu}\gamma^{\sigma}\gamma_5 <s_{\sigma(2|1)}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> \\
&+ i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\lambda}(p_4)> + i\eta^{\mu\lambda}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)> \\
&+ i\frac{8}{9}\eta^{\kappa\lambda} <s_{(1|0)}(p_{34})\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)> + i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})\mathcal{J}^{\lambda}(p_4)> \\
&+ i\eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{Q}^{\mu}(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{24})\mathcal{J}^{\kappa}(p_3)> \\
&+ i\eta^{\mu\kappa}\eta^{\nu\lambda}\frac{2}{9} <s_{(3|\frac{1}{2})}(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})> + i\eta^{\mu\lambda}\eta^{\nu\kappa}\frac{2}{9} <s_{(3|\frac{1}{2})}(p_{14})\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu} - \frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu} - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}\gamma_5\right) i\frac{8}{9}\eta^{\kappa\lambda} <s_{(1|0)}(p_{12})s_{(1|0)}(p_{34})> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu} - \frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu} - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}\gamma_5\right) <s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>\Bigg)
\end{aligned} \tag{4.2.21}
$$

4.3 Coupling to background conformal supergravity

Coupling a supersymmetric field theory to off-shell background supergravity allows for a simpler and universal description of the global symmetries and their physical consequences, without reference to a specific model. It also facilitates powerful computational techniques, such as supersymmetric localization [94]. In order to couple a theory to background supergravity we need to gauge the global symmetries by turning on appropriate gauge fields – the supergravity fields – and suppressing their kinetic terms [95]. The local symmetry transformations of off-shell supergravity are universal, which enables the general derivation of the Ward identities and their quantum anomalies, by solving the WZ consistency conditions.

Classically superconformal theories, such as the massless WZ model, can be coupled to conformal supergravity, which facilitates an alternative, more efficient formulation of the Noether procedure. For example, the conserved currents that couple to background supergravity are those satisfying the algebraic constraints (4.1.8), and so this formulation of the Noether procedure leads directly to the improved currents. However, massive theories cannot be consistently coupled to conformal supergravity. Thus, it is not possible to quantize a classically superconformal theory on a conformal supergravity background respecting superconformal symmetry, since any regulator necessarily introduces a mass scale. A suitable background supergravity for massive theories is old minimal supergravity [96], which we will discuss briefly in the appendix G. In the remaining part of this section, we review the coupling of the massless WZ model to background conformal supergravity, focusing on the symmetries of the classical theory.

4.3.1 Symmetries of conformal supergravity

The field content of $\mathcal{N}=1$ conformal supergravity [97–100] consists of the vierbein, e^a_μ, the graviphoton, A_μ, and the gravitino, ψ_μ. Its local symmetries are diffeomorphisms with infinitesimal parameter $\xi^\mu(x)$, Weyl transformations $\sigma(x)$, local Lorentz transformations $\lambda^{ab}(x)$, axial $U(1)$ gauge transformations $\theta(x)$, as well as Q- and S-supersymmetry, parametrized respectively by the local spinors $\varepsilon(x)$ and $\eta(x)$. These local symmetries act on the fields of conformal supergravity as

$$\begin{aligned}
\delta e^a_\mu &= \xi^\lambda\partial_\lambda e^a_\mu + e^a_\lambda\partial_\mu\xi^\lambda - \lambda^a{}_b e^b_\mu + \sigma e^a_\mu - \frac{1}{2}\overline{\psi}_\mu\gamma^a\varepsilon,\\
\delta\psi_\mu &= \xi^\lambda\partial_\lambda\psi_\mu + \psi_\lambda\partial_\mu\xi^\lambda - \frac{1}{4}\lambda_{ab}\gamma^{ab}\psi_\mu + \frac{1}{2}\sigma\psi_\mu + D_\mu\varepsilon - \gamma_\mu\eta - i\gamma_5\theta\psi_\mu,\\
\delta A_\mu &= \xi^\lambda\partial_\lambda A_\mu + A_\lambda\partial_\mu\xi^\lambda + \frac{3i}{4}\overline{\phi}_\mu\gamma_5\varepsilon - \frac{3i}{4}\overline{\psi}_\mu\gamma_5\eta + \partial_\mu\theta,
\end{aligned} \tag{4.3.1}$$

where the covariant derivatives of the spinor parameters, ε and η, are given by

$$\begin{aligned}
D_\mu\varepsilon &\equiv \Big(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} + i\gamma_5 A_\mu\Big)\varepsilon \equiv (\mathscr{D}_\mu + i\gamma_5 A_\mu)\varepsilon,\\
D_\mu\eta &\equiv \Big(\partial_\mu + \frac{1}{2}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} - i\gamma_5 A_\mu\Big)\eta \equiv (\mathscr{D}_\mu - i\gamma_5 A_\mu)\eta,
\end{aligned} \tag{4.3.2}$$

and the spin connection, $\omega_\mu{}^{ab}(e,\psi)$,

$$\omega_{\mu ab}(e,\psi) \equiv \omega_{\mu ab}(e) + \frac{1}{4}(\bar{\psi}_a\gamma_\mu\psi_b + \bar{\psi}_\mu\gamma_a\psi_b - \bar{\psi}_\mu\gamma_b\psi_a). \tag{4.3.3}$$

$\omega_{\mu ab}(e)$ denotes the unique torsion-free part. These transformations close off-shell.

4.3.2 Wess-Zumino model coupled to conformal supergravity

Up to quadratic terms in the gravitino, the coupling of the massless WZ model to conformal supergravity takes the form [88, 101, 102]

$$\begin{aligned} e^{-1}\mathscr{L}_{\rm WZ} = &- D_\mu\phi^* D^\mu\phi - \frac{1}{2}\bar\chi \not{D}\chi + F^*F - \frac{1}{6}\phi^*\phi\, R[\omega(e)] \\ &+ \frac{\sqrt{2}}{2}\bar\psi_\mu(\not\partial\phi\gamma^\mu\chi_R + \not\partial\phi^*\gamma^\mu\chi_L) - \frac{\sqrt{2}}{3}(\phi\bar\chi_R + \phi^*\bar\chi_L)\gamma^{\mu\nu}\mathscr{D}_\mu\psi_\nu + \frac{i\sqrt{2}}{3}A_\mu\bar\psi^\mu(\phi\chi_R - \phi^*\chi_L) \\ &- \frac{1}{6}\partial^\mu(\phi^*\phi)\bar\psi_\nu\gamma^\nu\psi_\mu - \frac{i}{6}\phi^*\phi\,\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_5\gamma_\nu\mathscr{D}_\rho\psi_\sigma \\ &+ \frac{i}{8}\epsilon^{\mu\nu\rho\sigma}\bar\psi_\mu\gamma_\nu\psi_\rho\Big(\phi^*\partial_\sigma\phi - \phi\partial_\sigma\phi^* + \frac{1}{4}\bar\chi\gamma_\sigma\gamma_5\chi\Big) - \frac{1}{16}(\bar\chi\gamma_5\gamma^\nu\chi)(\bar\psi_\mu\gamma_5\gamma_\nu\psi^\mu) + \mathcal{O}(\psi^3), \end{aligned} \tag{4.3.4}$$

where the covariant derivatives act on the chiral multiplet fields as

$$D_\mu\phi = \Big(\partial_\mu + \frac{2i}{3}A_\mu\Big)\phi, \qquad D_\mu\chi = \Big(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} - \frac{i}{3}\gamma_5 A_\mu\Big)\chi. \tag{4.3.5}$$

Under the local symmetries of $\mathcal{N}=1$ conformal supergravity, the WZ fields transform as

$$\begin{aligned} \delta\phi &= \xi^\mu\partial_\mu\phi - \sigma\phi - \frac{2i}{3}\theta\phi + \frac{\sqrt{2}}{2}\bar\varepsilon_L\chi_L, \qquad (4.3.6)\\ \delta\chi_L &= \xi^\mu\partial_\mu\chi_L - \frac{3}{2}\sigma\chi_L + \frac{i}{3}\theta\chi_L - \frac{1}{4}\lambda_{ab}\gamma^{ab}\chi_L + \frac{\sqrt{2}}{2}\Big(\gamma^\mu(D_\mu\phi - \frac{\sqrt{2}}{2}\bar\psi_{\mu L}\chi_L)\varepsilon_R + F\varepsilon_L + 2\phi\eta_L\Big), \\ \delta F &= \xi^\mu\partial_\mu F - 2\sigma F + \frac{4i}{3}\theta F + \frac{1}{2}\bar\varepsilon_R\gamma^\mu\Big(\sqrt{2}D_\mu\chi_L - \gamma^\nu(D_\nu\phi - \frac{\sqrt{2}}{2}\bar\psi_{\nu L}\chi_L)\psi_{\mu R} - F\psi_{\mu L} - 2\phi\,\phi_{\mu L}\Big), \end{aligned}$$

where the gravitino fieldstrength, ϕ_μ is given by

$$\phi_\mu \equiv \frac{1}{3}\gamma^\nu\Big(D_\nu\psi_\mu - D_\mu\psi_\nu - \frac{i}{2}\gamma_5\epsilon_{\nu\mu}{}^{\rho\sigma}D_\rho\psi_\sigma\Big) = -\frac{1}{6}(4\delta_\mu^{[\rho}\delta_\nu^{\sigma]} + i\gamma_5\epsilon_{\mu\nu}{}^{\rho\sigma})\gamma^\nu D_\rho\psi_\sigma. \tag{4.3.7}$$

∇_μ denotes the Levi-Civita connection, while D_μ stands for the spinor covariant derivative, which acts on the gravitino and its fieldstrength as

$$\begin{aligned} D_\mu\psi_\nu &= \Big(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} + i\gamma_5 A_\mu\Big)\psi_\nu - \Gamma^\rho_{\mu\nu}\psi_\rho \equiv (\mathscr{D}_\mu + i\gamma_5 A_\mu)\psi_\nu, \\ D_\mu\phi_\nu &= \Big(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}(e,\psi)\gamma_{ab} - i\gamma_5 A_\mu\Big)\phi_\nu - \Gamma^\rho_{\mu\nu}\phi_\rho = (\mathscr{D}_\mu - i\gamma_5 A_\mu)\phi_\nu. \end{aligned} \tag{4.3.8}$$

Together with the transformations of the supergravity fields in (4.3.1), the above transformations leave the Lagrangian (4.3.4) invariant, up to a total derivative term.

The variation of the WZ Lagrangian (4.3.4) with respect to the background supergravity fields determines the corresponding current operators. This gives

$$\begin{aligned}
\mathcal{T}_a^\mu &= 2D^{(\mu}\phi^* D_{a)}\phi + \frac{1}{2}\bar{\chi}\gamma^\mu D_a\chi - \frac{1}{8}\nabla_\rho(\bar{\chi}\gamma_a\gamma^{\rho\mu}\chi + \bar{\chi}\gamma^\mu\gamma^\rho{}_a\chi - \bar{\chi}\gamma^\rho\gamma^\mu{}_a\chi)\\
&+ \frac{1}{3}e_a^\nu(R_\nu^\mu - \nabla^\mu\nabla_\nu + \delta_\nu^\mu\Box)(\phi\phi^*) - e_a^\mu\Big(D_\nu\phi^* D^\nu\phi + \frac{1}{2}\bar{\chi}\not{D}\chi - FF^* + \frac{1}{6}\phi\phi^* R\Big) + \mathcal{O}(\psi),\\
\mathcal{J}^\mu &= \frac{2i}{3}\Big(\phi^* D^\mu\phi - \phi D^\mu\phi^* + \frac{1}{4}\bar{\chi}\gamma^\mu\gamma_5\chi + \frac{\sqrt{2}}{2}\bar{\psi}^\mu(\phi\chi_R - \phi^*\chi_L)\Big),\\
\mathcal{Q}^\mu &= \frac{\sqrt{2}}{2}(\not{\partial}\phi\gamma^\mu\chi_R + \not{\partial}\phi^*\gamma^\mu\chi_L) + \frac{\sqrt{2}}{3}\gamma^{\mu\nu}\mathscr{D}_\nu(\phi\chi_R + \phi^*\chi_L) + \frac{i\sqrt{2}}{3}A^\mu(\phi\chi_R - \phi^*\chi_L)\\
&\quad - \frac{1}{3}\gamma^{[\mu}\psi_\nu\partial^{\nu]}(\phi\phi^*) - \frac{i}{6}\epsilon^{\mu\nu\rho\sigma}(2\phi^*\phi\,\gamma_5\gamma_\nu\mathscr{D}_\rho\psi_\sigma + \partial_\rho(\phi^*\phi)\gamma_5\gamma_\nu\psi_\sigma)\\
&\quad + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\psi_\rho\Big(\phi^*\partial_\sigma\phi - \phi\partial_\sigma\phi^* + \frac{1}{4}\bar{\chi}\gamma_\sigma\gamma_5\chi\Big) - \frac{1}{8}\gamma_5\gamma_\nu\psi^\mu(\bar{\chi}\gamma_5\gamma^\nu\chi) + \mathcal{O}(\psi^2).
\end{aligned} \tag{4.3.9}$$

Notice that the expression for the R-current is exact to all orders in the gravitino, since the qubic and quartic terms in the gravitino in the WZ action do not involve the gauge field, A_μ. The flat space limit of the currents (4.3.9) coincides with the improved Noether currents (4.1.6).

Using now the equations of motion it can be shown that the currents (4.3.9) satisfy the following classical equations

$$\begin{aligned}
&e_\mu^a\nabla_\nu\mathcal{T}_a^\nu + \nabla_\nu(\bar{\psi}_\mu\mathcal{Q}^\nu) - \bar{\psi}_\nu\overleftarrow{D}_\mu\mathcal{Q}^\nu - F_{\mu\nu}\mathcal{J}^\nu = 0,\\
&\nabla_\mu\mathcal{J}^\mu + i\bar{\psi}_\mu\gamma_5\mathcal{Q}^\mu = 0, \qquad D_\mu\mathcal{Q}^\mu - \frac{1}{2}\gamma^a\psi_\mu\mathcal{T}_a^\mu - \frac{3i}{4}\gamma_5\phi_\mu\mathcal{J}^\mu = 0,\\
&e_\mu^a\mathcal{T}_a^\mu + \frac{1}{2}\bar{\psi}_\mu\mathcal{Q}^\mu = 0, \qquad e_{\mu[a}\mathcal{T}_{b]}^\mu + \frac{1}{4}\bar{\psi}_\mu\gamma_{ab}\mathcal{Q}^\mu = 0, \qquad \gamma_\mu\mathcal{Q}^\mu - \frac{3i}{4}\gamma_5\psi_\mu\mathcal{J}^\mu = 0.
\end{aligned} \tag{4.3.10}$$

These generalize the flat space conservation equations (4.1.7) and algebraic constraints (4.1.8) to a general supergravity background.

If we now consider the above expressions as quantum equations, i.e. as 1-point functions in the presence of background sources, we can easily compute the Ward identities of higher order correlators by taking the appropriate functional derivatives. To find for example the flat space R-symmetry Ward identity for the $\langle\mathcal{T}\mathcal{J}\mathcal{J}\rangle$ correlator, we have to take two derivatives in the first equation of the second line of (4.3.10), one with respect to the vierbein e_μ^a, and one with respect to the gauge field A_μ. Then we take the limit where the background sources go to zero. It is straightforward to show that the Ward identity of $\langle\mathcal{T}\mathcal{J}\mathcal{J}\rangle$ derived with functional differentiation coincides up to seagull correlators with the one computed using operator insertions (4.2.4). Of course, both identities should be exactly the same. In particular, the functional derivatives of the conserved currents with respect to the sources, such as $\frac{\delta\mathcal{J}^\mu}{\delta A_\rho}$, contain all the seagull terms of (4.2.4). In the appendix C, we present a more detailed analysis on the difference between the identities derived through functional differentiation versus operator insertions.

4.4 Consistency conditions

In this section, we also omit the hat $\hat{}$ in the flat space operators to simplify the notation.

4.4.1 Wess-Zumino consistency conditions

The structure of the Q-supersymmetry variation of the supercurrent (4.1.13), implies the possibility of an anomalous Q-supersymmetry Ward identity for the correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. According to (2.2.8), on the rhs of the aforementioned identity exists the 3-point function $<\delta_{\varepsilon_0}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, which among other contains the anomalous $<\mathcal{J}\mathcal{J}\mathcal{J}>$ correlator (4.2.16). Similarly we may anticipate anomalies in the identities of the $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{T}>$ and $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$ correlators due to the anomalous $<\mathcal{J}\mathcal{T}\mathcal{T}>$ of the rhs, where $\mathcal{T}$ is the stress tensor. The relevant Q-supersymmetry identities of these correlators can be written in a schematic form as

$$\partial <\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>=<\delta_{\varepsilon_0}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>+\ldots \equiv<\mathcal{T}\mathcal{T}\mathcal{J}>+<\mathcal{J}\mathcal{T}\mathcal{J}>+\ldots \tag{4.4.1}$$

and

$$\partial <\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{T}>=<\delta_{\varepsilon_0}\bar{\mathcal{Q}}\mathcal{T}\mathcal{T}>+\ldots \equiv<\mathcal{T}\mathcal{T}\mathcal{T}>+<\mathcal{J}\mathcal{T}\mathcal{T}>+\ldots \tag{4.4.2}$$

One has to do the explicit loop computations to see whether the anomalous lower order correlators affect the Q-supersymmetry identities of the 4-point functions. In this thesis we focus only on the loop computation of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. A complete and consistent analysis of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{T}>$ and $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$, demands the regularization of very badly divergent correlators such as $<\mathcal{T}\mathcal{T}>$. The Pauli-Villars regulator we are using is not enough to deal with them, see appendix D. We need to include more regulating PV fields, which will make the computation quite tedious.

The above heuristic arguments about how the R-symmetry anomaly of the WZ model can induce an anomaly in supersymmetry, transform into a concrete proof when the theory is coupled to conformal supergravity, namely the Wess-Zumino consistency conditions.

Let $\mathscr{W}[e, A, \psi]$ be the generating functional of connected graphs. In the presence of anomalies

$$\delta_i \mathscr{W} = \int d^4x \; e \; \epsilon_i \mathcal{A}_i \,, \tag{4.4.3}$$

where $e \equiv \det(e^a_\mu)$, δ_i denotes the superconformal transformations, ϵ_i are the (local) parameters of the transformations and $\mathcal{A}_i$ are the corresponding anomalies. The variations form an algebra, $[\delta_i, \delta_j] = f^k_{ij}\delta_k$, and using this in (4.4.3) we obtain the WZ consistency condition

$$\int d^4x \left(\delta_i(e\;\epsilon_j \mathcal{A}_j) - \delta_j(e\;\epsilon_i\mathcal{A}_i) - f^k_{ij} e\;\epsilon_k \mathcal{A}_k\right) = 0\,. \tag{4.4.4}$$

The local algebra they satisfy is derived in [71]. Below we state the two relevant non vanishing commutators[1]

$$[\delta_\varepsilon, \delta_{\varepsilon'}] = \delta_\xi + \delta_\eta + \delta_\theta\,, \qquad [\delta_\varepsilon, \delta_\eta] = \delta_\sigma + \delta_\eta + \delta_\theta\,. \tag{4.4.5}$$

Assuming the R-symmetry current has the standard triangle anomalies, the WZ consistency conditions (4.4.4) may be viewed as equations to determine the remaining anomalies. This computation is presented in [71] and the results are summarised in the appendix B.

Here we only discuss one of the WZ equations: the one obtained by considering the (vanishing) commutator of R-symmetry (with parameter θ) with Q-supersymmetry (with parameter ε):

$$\int d^4x (\delta_\varepsilon(e\,\theta\mathcal{A}_R) - \delta_\theta(e\,\varepsilon\mathcal{A}_Q)) = 0\,. \tag{4.4.6}$$

Using the explicit form of $\mathcal{A}_R$ (B.1.3), it is easy to see that $\delta_\varepsilon\mathcal{A}_R \neq 0$ and the WZ consistency condition requires that $\mathcal{A}_Q \neq 0$. Considering only the part of $\mathcal{A}_R$ that reproduces the anomaly of the $<\mathcal{J}\mathcal{J}\mathcal{J}>$ correlator, i.e.

$$\mathcal{A}_R = -\frac{1}{48}\frac{1}{54\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu} \tag{4.4.7}$$

we have that

$$\int d^4x\delta_\varepsilon(e\,\theta\mathcal{A}_R) = -\frac{1}{48}\frac{1}{54\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4xe\,\theta\delta_\varepsilon(F_{\rho\sigma}F_{\mu\nu}) = -\frac{4}{48}\frac{1}{54\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4xe\,\theta F_{\mu\nu}\partial_\rho\delta_\varepsilon A_\sigma$$
$$= \frac{i}{48}\frac{1}{18\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4xe\,\partial_\rho\theta F_{\mu\nu}\bar{\varepsilon}\gamma_5\phi_\sigma \neq 0\,. \tag{4.4.8}$$

Using now the following form for $\mathcal{A}_Q$

$$\mathcal{A}_Q = \frac{i}{48}\frac{1}{18\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}A_\rho\gamma_5\phi_\sigma, \tag{4.4.9}$$

we get

$$\int d^4x\delta_\theta(e\,\bar{\varepsilon}\mathcal{A}_Q) = \frac{i}{48}\frac{1}{18\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4xe\,\bar{\varepsilon}\delta_\theta(F_{\mu\nu}A_\rho\gamma_5\phi_\sigma) = \frac{i}{48}\frac{1}{18\pi^2}\epsilon^{\mu\nu\rho\sigma}\int d^4xe\,\partial_\rho\theta F_{\mu\nu}\bar{\varepsilon}\gamma_5\phi_\sigma, \tag{4.4.10}$$

where we used that $F_{\mu\nu}$ and ϕ_σ are invariant under the R-symmetry transformation. We see that the non zero $\mathcal{A}_Q$ (4.4.9) is such that the WZ consistency condition (4.4.6) is satisfied. It is straightforward to show that after 3 functional derivatives in $\mathcal{A}_Q$, two with respect to the gauge field A_μ and one with respect to the gravitino ψ_ν, we get a non zero result even in the flat space limit, i.e. in the limit where the supergravity background sources become zero. Thus, the first non vanishing contribution of $\mathcal{A}_Q$ in flat space, will

[1]The commutators of two diffeomorphisms and two local Lorentz transformations are also non zero and they take a standard form.

be in the Q-supersymmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

Note here that (4.4.9) is just a part of the total Q-supersymmetry anomaly. In the above analysis we considered only the part of the R-anomaly that contributes to the $<\mathcal{J}\mathcal{J}\mathcal{J}>$ correlator. Had we taken into account the part that contributes to the anomaly of $<\mathcal{J}\mathcal{T}\mathcal{T}>$, we would have found the extra terms of $\mathcal{A}_Q$ that would make $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$ anomalous. After three functional derivatives in $\mathcal{A}_Q$ (B.1.3), with respect to the gravitino ψ_ν, the gauge field A_μ, and the vierbein e_μ^a, we get a non zero result in the flat space limit. Lastly, recall that the naive argument we presented in the beginning of this section, which was based on the Q-supersymmetry variation of the supercurrent, implied a possible Q-supersymmetry anomaly in $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{T}>$ (4.4.2). According to (B.1.3) though, we can see that after three functional derivatives in $\mathcal{A}_Q$, one with respect to the gravitino and two with respect to the vierbein, we get a zero contribution in the flat space limit.

A modified version of the above WZ consistency conditions was recently presented in [103]. Relaxing the WZ gauge (in which gauge our analysis is done) and thus restoring the gauge away fields, the authors of [103] showed that the commutator of R-symmetry with Q-supersymmetry in this case is non zero, i.e. $[\widetilde{\delta}_\theta, \widetilde{\delta}_\varepsilon] \neq 0$. $\widetilde{\delta}$ denotes the transformations in the presence of the gauge away fields. This non zero commutator modifies the rhs of (4.4.6) in such a way that the WZ consistency condition is satisfied for $\mathcal{A}_Q$=0, i.e. it appears that there is no supersymmetry anomaly. However, the gauge away fields are present and as explained in section (3.3), it is important to put them equal to zero before examining the microscopic theory.

One may wonder whether this anomaly can be removed by adding a local counterterm $\mathscr{W}_{\text{ct}}$ to the action such that $\mathscr{W}_{\text{ren}} = \mathscr{W} + \mathscr{W}_{\text{ct}}$ is non-anomalous, i.e. $\delta_\varepsilon \mathscr{W}_{\text{ren}} = 0$. Using the commutator of two supersymmetry variations, $[\delta_\varepsilon, \delta_{\varepsilon'}]$ we find

$$(\delta_\xi + \delta_\lambda + \delta_\theta)\mathscr{W}_{\text{ren}} = 0 \quad \Rightarrow \quad (\delta_\xi + \delta_\lambda)\mathscr{W}_{\text{ren}} \neq 0\,, \tag{4.4.11}$$

since $\delta_\theta \mathscr{W}_{\text{ren}} = \mathcal{A}_R \neq 0$. It follows that if one wishes to preserve supersymmetry, $\mathscr{W}_{\text{ct}}$ must break diffeomorphisms and/or local Lorentz transformations.[2] However, this argument refers to the original Q-supersymmetry of conformal supergravity. In chapter 8 we see that there exists a specific linear combination of Q+S supersymmetry that is non anomalous in the WZ model, and at the same time diffeomorphisms and Lorentz transformations remain non anomalous. The price for that is to explicitly break R-symmetry and S-supersymmetry Ward identities at the 3-point function level. This specific Q+S combination is the same with the supersymmetry of old minimal supergravity (see appendix G).

[2] Note that since $\mathcal{A}_R$ is a genuine anomaly it is not possible to set the rhs of the second equation in (4.4.11) to zero using a local counterterm. This implies that there are no further local counterterms that can restore diffeomorphisms and local Lorentz invariance.

After coupling the massless WZ model to conformal supergravity we were able to prove that the standard R-symmetry anomalies induce Q-supersymmetry anomalies in certain correlators in flat space. In chapter 7 we confirm with a loop computation that $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ is indeed anomalous. The PV regulator we use, is not enough to regulate properly all correlators needed for the analysis of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$. However, we expect that a better suited PV regulator will also confirm the Q-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$.

4.4.2 Consistency conditions using correlators

The question that arises is could we have predicted the existence of $\mathcal{A}_Q$ without the insight of background supergravity? Is there a way to make the naive arguments that depend on the variation of the supercurrent into a solid proof? The answer is yes, one could use a similar reasoning with the WZ consistency conditions, but now in terms of correlators. To be more specific, we consider the classical R-symmetry (4.2.18) and Q-supersymmetry (4.2.16) Ward identities of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. We proceed by contracting both sides of (4.2.18) and (4.2.16) with $p_{1\mu}$ and $p_{3\kappa}$ respectively. Then we subtract the two identities. By construction the 4-point functions cancel, so we get

$$\begin{aligned}
&p_{3\kappa}C^{\nu\kappa\lambda}_{4Q} + i\gamma_5 p_{3\kappa} <\mathcal{Q}^\kappa(p_{13})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\lambda(p_4)> + i\gamma_5 p_{3\kappa} <\mathcal{Q}^\lambda(p_{14})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)> \\
&+ \gamma_5 B^{\nu\kappa\sigma}p_{3\kappa} <\mathcal{J}_\sigma(p_{123})\mathcal{J}^\lambda(p_4)> - p_{1\mu}C^{\mu\nu\lambda}_{4R} \\
&+ i\gamma_5 p_{1\mu} <\mathcal{Q}^\mu(p_{13})\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\lambda(p_4)> + ip_{1\mu} <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_{23})\mathcal{J}^\lambda(p_4)> \gamma_5 \\
&+ \frac{\gamma_\xi}{2}p_{3\kappa} <\mathcal{T}^{\nu\xi}(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)> - \gamma_5 B^{\nu\lambda\sigma}p_{3\kappa} <\mathcal{J}_\sigma(p_{124})\mathcal{J}^\kappa(p_3)> \\
&= ip_{2\mu}B^{\nu\mu\sigma}p_{3\kappa} <\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)> . \qquad (4.4.12)
\end{aligned}$$

Classically, the above equation must be satisfied. Although straightforward, this is quite tedious to prove. For that we need to use the R-symmetry and Q-supersymmetry Ward identities for all the 3-point functions involved, including the seagull correlators. We assume that our model has the standard bosonic anomalies. That is exactly the same assumption we made in the analysis of the WZ consistency conditions. This means that the only anomalous correlators in the above expression are $<\mathcal{J}\mathcal{J}\mathcal{J}>$ and $<\mathcal{T}\mathcal{J}\mathcal{J}>$. $<\mathcal{T}\mathcal{J}\mathcal{J}>$ though, has only a trace and not an R-symmetry anomaly, so $p_{3\kappa} <\mathcal{T}^{\nu\xi}(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>$ can be substituted by its classical R-symmetry identity (4.2.4). It can be shown that the lhs is classically zero. Since every correlator of the lhs is non anomalous in the symmetries of interest (by assumption), the lhs must vanish in the quantum level too. So we get

$$0 = ip_{2\mu}B^{\nu\mu\sigma}p_{3\kappa} <\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)> \equiv ip_{2\mu}B^{\nu\mu\sigma}\frac{i}{324\pi^2}\epsilon_\sigma{}^{\lambda\kappa\alpha}p_{3\kappa}p_{4\alpha} \qquad (4.4.13)$$

Had $<\mathcal{J}\mathcal{J}\mathcal{J}>$ been non anomalous, the above expression would be an identity, $0=0$, as it is at the classical level. In the quantum regime though, this implies that either the R-symmetry or the Q-supersymmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ (or both) are anomalous.

We now allow for the existence of possible quantum anomalies $A_{4Q}^{\nu\kappa\lambda}$ and $A_{4R}^{\mu\nu\lambda}$ in the rhs of the identities (4.2.16) and (4.2.18) respectively, i.e.

$$\begin{aligned} p_{1\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> +... &= A_{4Q}^{\nu\kappa\lambda} \\ p_{3\kappa} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> +... &= A_{4R}^{\mu\nu\lambda}, \end{aligned} \tag{4.4.14}$$

where ... denote the lower order correlators. Following the same procedure we find that

$$p_{1\mu}A_{4R}^{\mu\nu\lambda} - p_{3\kappa}A_{4Q}^{\nu\kappa\lambda} = ip_{2\mu}B^{\nu\mu\sigma}p_{3\kappa} <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> . \tag{4.4.15}$$

Since we consider R-symmetry to only have bosonic triangle anomalies (as in the analysis of the WZ conditions), we have that $A_{4R}^{\mu\nu\lambda}=0$, so

$$-p_{3\kappa}A_{4Q}^{\nu\kappa\lambda} = ip_{2\mu}B^{\nu\mu\sigma}p_{3\kappa} <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> \equiv ip_{2\mu}B^{\nu\mu\sigma}\frac{i}{324\pi^2}\epsilon_{\sigma}{}^{\lambda\kappa\alpha}p_{3\kappa}p_{4\alpha}. \tag{4.4.16}$$

The above expression can be used as an equation to determine $A_{4Q}^{\nu\kappa\lambda}$. We find that

$$A_{4Q}^{\nu\kappa\lambda} = p_{2\mu}B^{\nu\mu\sigma}\frac{1}{324\pi^2}\epsilon_{\sigma}{}^{\lambda\kappa\alpha}(p_{4\alpha}-p_{3\alpha}). \tag{4.4.17}$$

Following this alternative consistency condition, we confirm again that the R-symmetry anomaly induces an anomaly in Q-supersymmetry. $A_{4Q}^{\nu\kappa\lambda}$ is equal to the contribution of (4.4.9) to the flat space correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

Since R-symmetry is already broken, one may wonder whether we could save supersymmetry by introducing a new R-anomaly in $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, namely an $A_{4Q}^{\nu\kappa\lambda}=0$ and a non zero $A_{4R}^{\mu\nu\lambda}$. Then, the consistency condition would be

$$p_{1\mu}A_{4R}^{\mu\nu\lambda} = ip_{2\mu}B^{\nu\mu\sigma}\frac{i}{324\pi^2}\epsilon_{\sigma}{}^{\lambda\kappa\alpha}p_{3\kappa}p_{4\alpha}. \tag{4.4.18}$$

This equation though, cannot be solved. $A_{4R}^{\mu\nu\lambda}$ must be an analytic function in the external momenta, which means that the above lhs is equal to zero in the limit $p_1 \to 0$. After some gamma matrix algebra, one can show that the rhs is non zero in the same limit of p_1. Thus, we cannot restore Q-supersymmetry by only introducing a new R-anomaly in $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ and assuming the standard anomalies at the 3-point function level.

In practice, the above procedure is not very efficient. It is tedious and one has to perform the same analysis to every correlator of interest. On the contrary, coupling the theory to background supergravity and using the WZ consistency conditions makes our life quite easier. In a few lines we can find the possible quantum anomalies at the level of 1-point correlators, written in a compact form as a function of the background supergravity fields (B.1.3). Further differentiation with respect to the sources gives the anomalies of higher order correlators. However, the main point we would like to make here, is that we can arrive at the same results even from the flat space theory. One may think that we still need the insight from conformal supergravity, since after all, we used the Ward identities of section (4.2). These identities, contain the extra coloured seagull correlators that we get after coupling the WZ model to background supergravity. One of the assumptions of the analysis though, is that all seagull correlators are non anomalous and the anomalies exist only in correlators among conserved currents. All coloured (seagull) correlators of section (4.2) are irrelevant for the consistency conditions of this subsection, since they satisfy their own classical path integral Ward identities that are valid in the quantum theory too (by assumption)[3]. We only need the black terms, which are the ones we get from the flat space theory and the path integral identities (2.2.8).

[3]In the appendix F, we verify that the PV regulated seagull correlators are indeed non anomalous.

CHAPTER 5

Free and massive Wess-Zumino model

5.1 Symmetries and the Ferrara-Zumino current multiplet

Before introducing the Pauli-Villars regulator in the next chapter, we find it instructive to discuss the standard massive WZ Lagrangian

$$\hat{\mathscr{L}}_{\text{WZ}} = -\partial_\mu \phi^* \partial^\mu \phi - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi + F^* F - \frac{m}{2}\bar{\chi}\chi + m(\phi F + \phi^* F^*), \tag{5.1.1}$$

as a reference. Integrating out the auxiliary field F using its equation of motion, $F = -m\phi^*$, the massive WZ Lagrangian (5.1.1) becomes

$$\hat{\mathscr{L}}_{\text{WZ}} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi - \frac{m}{2}\bar{\chi}\chi. \tag{5.1.2}$$

The above Lagrangian is invariant only under a subset of the superconformal symmetries in table 4.1.1, in particular the Poincaré symmetries and Q-supersymmetry. After integrating out the auxiliary field, Q-supersymmetry acts as

$$\begin{aligned}
\delta_{\varepsilon_0}\phi &= \frac{\sqrt{2}}{2}\bar{\varepsilon}_{0L}\chi, \qquad \delta_{\varepsilon_0}\phi^* = \frac{\sqrt{2}}{2}\bar{\varepsilon}_{0R}\chi, \\
\delta_{\varepsilon_0}\chi &= \frac{\sqrt{2}}{2}(\partial\!\!\!/\phi\varepsilon_{0R} + \partial\!\!\!/\phi^*\varepsilon_{0L} - m\phi\varepsilon_{0R} - m\phi^*\varepsilon_{0L}), \\
\delta_{\varepsilon_0}\bar{\chi} &= -\frac{\sqrt{2}}{2}(\bar{\varepsilon}_{0L}\partial\!\!\!/\phi^* + \bar{\varepsilon}_{0R}\partial\!\!\!/\phi + m\bar{\varepsilon}_{0R}\phi + m\phi^*\bar{\varepsilon}_{0L}).
\end{aligned} \tag{5.1.3}$$

Poincaré and Q-supersymmetry invariance results in a conserved and symmetric stress tensor, $\hat{\tilde{\mathcal{T}}}^{\mu}{}_{\nu}$, and a conserved supercurrent, $\hat{\tilde{\mathcal{Q}}}^{\mu}$, i.e.

$$\partial_\mu \hat{\tilde{\mathcal{T}}}^{\mu}{}_{\nu} = 0, \qquad \hat{\tilde{\mathcal{T}}}_{[\mu\nu]} = 0, \qquad \partial_\mu \hat{\tilde{\mathcal{Q}}}^{\mu} = 0, \tag{5.1.4}$$

where the tilde indicates that these are operators of a massive theory and, as in the previous chapter, the hat denotes quantities evaluated in Minkowski space.

Although R-symmetry is no longer a symmetry of the massive WZ Lagrangian, there still exists a non conserved R-current $\hat{\tilde{\mathcal{J}}}^{\mu}$. Moreover the complex scalar operator $\hat{\tilde{\mathcal{O}}}_M$ and its complex conjugate $\hat{\tilde{\mathcal{O}}}_{M^*}$ that we define below are part of the Ferrara-Zumino (FZ) multiplet [33], the simplest multiplet that is used for coupling the massive theory to supergravity.

For the massive WZ model (5.1.1), the FZ multiplet operators take the form

$$\begin{aligned}
\hat{\tilde{\mathcal{T}}}^{\mu}{}_{\nu} &= 2\partial^{(\mu}\phi^*\partial_{\nu)}\phi + \frac{1}{2}\bar{\chi}\gamma^{\mu}\partial_\nu\chi - \frac{1}{8}\partial_\rho(\bar{\chi}\gamma_\nu\gamma^{\rho\mu}\chi + \bar{\chi}\gamma^{\mu}\gamma^{\rho}{}_{\nu}\chi - \bar{\chi}\gamma^{\rho}\gamma^{\mu}{}_{\nu}\chi) \\
&\quad - \frac{1}{3}(\partial^\mu\partial_\nu - \eta^\mu_\nu\partial^2)(\phi^*\phi) - \eta^\mu_\nu(\partial_\rho\phi^*\partial^\rho\phi + \frac{1}{2}\bar{\chi}\partial\!\!\!/\chi - F^*F + \frac{m}{2}\bar{\chi}\chi - m(\phi F + \phi^* F^*)), \\
\hat{\tilde{\mathcal{J}}}^{\mu} &= \frac{2i}{3}\Big(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^* + \frac{1}{4}\bar{\chi}\gamma^\mu\gamma_5\chi\Big), \\
\hat{\tilde{\mathcal{Q}}}^{\mu} &= \frac{\sqrt{2}}{2}(\partial\!\!\!/\phi\gamma^\mu\chi_R + \partial\!\!\!/\phi^*\gamma^\mu\chi_L) + \frac{\sqrt{2}}{3}\gamma^{\mu\nu}\partial_\nu(\phi\chi_R + \phi^*\chi_L) + \frac{\sqrt{2}}{2}m(\phi\gamma^\mu\chi_L + \phi^*\gamma^\mu\chi_R), \\
\hat{\tilde{\mathcal{O}}}_M &= \frac{m}{2}\phi^2, \qquad \hat{\tilde{\mathcal{O}}}_{M^*} = \frac{m}{2}\phi^{*2}.
\end{aligned} \tag{5.1.5}$$

Using the equations of motion following from (5.1.1),

$$\Box\phi = m^2\phi, \qquad \partial\!\!\!/\chi = -m\chi, \qquad F = -m\phi^*, \tag{5.1.6}$$

it is straightforward to verify that the stress tensor and the supercurrent satisfy the identities (5.1.4). Moreover, we find that

$$\hat{\tilde{\mathcal{T}}}^{\mu}_{\mu} = -\frac{m}{2}\bar{\chi}\chi - 2m^2\phi^*\phi, \qquad \partial_\mu\hat{\tilde{\mathcal{J}}}^{\mu} = \frac{im}{3}\bar{\chi}\gamma_5\chi, \qquad \gamma_\mu\hat{\tilde{\mathcal{Q}}}^{\mu} = \sqrt{2}m(\phi\chi_L + \phi^*\chi_R), \tag{5.1.7}$$

which reflect the explicit breaking of scale invariance, R-symmetry and S-supersymmetry respectively.

5.2 Symmetry transformation of currents

Restoring the auxiliary field F we determine that the R-symmetry transformations of the stress tensor and of the supercurrent are

$$\delta_{\theta_0}\hat{\hat{\mathcal{T}}}^\mu{}_\nu = (\eta^{\mu\rho}\eta^\sigma_\nu + \eta^{\mu\sigma}\eta^\rho_\nu - \eta^\mu_\nu\eta^{\rho\sigma})\hat{\mathcal{J}}_\rho\partial_\sigma\theta_0 - \frac{i}{6}\bar{\chi}\gamma_\nu\gamma_5\chi\partial^\mu\theta_0 - \theta_0\eta^\mu_\nu\frac{im}{3}\bar{\chi}\gamma_5\chi,$$
$$\delta_{\theta_0}\hat{\hat{\mathcal{Q}}}^\mu = i\theta_0\gamma_5\hat{\hat{\mathcal{Q}}}^\mu - \frac{i}{3}\theta_0\gamma_5\gamma^\mu\sqrt{2}m(\phi\chi_L + \phi^*\chi_R) + \frac{i\sqrt{2}}{3}\partial^\mu\theta_0(\phi^*\chi_L - \phi\chi_R). \quad (5.2.1)$$

Similarly, the off shell Q-supersymmetry transformations of the R-current and of the supercurrent take respectively the form

$$\begin{aligned}\delta_{\varepsilon_0}\hat{\hat{\mathcal{J}}}^\mu =& -i\bar{\varepsilon}_0\gamma_5\hat{\hat{\mathcal{Q}}}^\mu + \frac{i\sqrt{2}}{3}\bar{\varepsilon}_0\gamma_5\gamma^\mu m(\phi\chi_L + \phi^*\chi_R) + \frac{\sqrt{2}i}{3}(\phi^*\bar{\chi}_L - \phi\bar{\chi}_R)\partial^\mu\varepsilon_0\\ &+ \frac{\sqrt{2}i}{3}\bar{\varepsilon}_0\gamma^\mu(\phi^*(\partial\!\!\!/\chi_L + m\chi_R) - \phi(\partial\!\!\!/\chi_R + m\chi_L))\\ &+ \frac{\sqrt{2}i}{6}\bar{\varepsilon}_0\gamma^\mu((F^* + m\phi)\chi_L - (F + m\phi^*)\chi_R),\\ \delta_{\varepsilon_0}\hat{\hat{\mathcal{Q}}}^\mu =& \frac{1}{2}\hat{\hat{\mathcal{T}}}^\mu{}_\nu\gamma^\nu\varepsilon_0 + \frac{i}{8}\partial_\rho\big[\hat{\hat{\mathcal{J}}}_\sigma(i\epsilon^{\mu\nu\rho\sigma}\gamma_5 + 2\eta^{\mu\nu}\eta^{\rho\sigma} - 2\eta^{\rho\nu}\eta^{\mu\sigma})\gamma_\nu\gamma_5\varepsilon_0\big] - \frac{3}{8}\epsilon^{\mu\nu\rho\sigma}\hat{\hat{\mathcal{J}}}_\sigma\gamma_\nu\partial_\rho\varepsilon_0\\ &+ \frac{1}{8}(\bar{\chi}\gamma_\sigma\gamma_5\chi)\gamma^\sigma\gamma_5\partial^\mu\varepsilon_0 + \frac{1}{6}\partial_\rho(\phi^*\phi)(i\epsilon^{\mu\nu\rho\sigma}\gamma_5 + \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma})\gamma_\nu\partial_\sigma\varepsilon_0\\ &+ \frac{1}{8}(\bar{\chi}(\gamma^{\mu\nu} - \eta^{\mu\nu})\gamma_5(\partial\!\!\!/ + m)\chi)\gamma_\nu\gamma_5\varepsilon_0 - \frac{1}{8}(\bar{\chi}(\gamma^{\mu\nu} - \eta^{\mu\nu})(\partial\!\!\!/ + m)\chi)\gamma_\nu\varepsilon_0\\ &+ \frac{1}{2}((F^* + m\phi)\partial\!\!\!/\phi\gamma^\mu\varepsilon_{0R} + (F + m\phi^*)\partial\!\!\!/\phi^*\gamma^\mu\varepsilon_{0L})\\ &+ \frac{1}{3}\gamma^{\mu\nu}\partial_\nu(\phi(F^* + m\phi)\varepsilon_{0R} + \phi^*(F + m\phi^*)\varepsilon_{0L}).\end{aligned} \quad (5.2.2)$$

The Q-supersymmetry variation of the R-current can be also written as

$$\begin{aligned}\delta_{\varepsilon_0}\hat{\hat{\mathcal{J}}}^\mu =& -i\bar{\varepsilon}_0\gamma_5\hat{\hat{\mathcal{Q}}}^\mu + \frac{i}{3}\bar{\varepsilon}_0\gamma_5\gamma^\mu\gamma_\rho\hat{\hat{\mathcal{Q}}}^\rho + \frac{\sqrt{2}i}{3}(\phi^*\bar{\chi}_L - \phi\bar{\chi}_R)\partial^\mu\varepsilon_0\\ &+ \frac{\sqrt{2}i}{6}\bar{\varepsilon}_0\gamma^\mu((F^* + m\phi)\chi_L - (F + m\phi^*)\chi_R).\end{aligned} \quad (5.2.3)$$

5.3 Massless vs Massive WZ model

In subsection (4.1.5), we derived all the necessary symmetry transformations of the Noether currents and seagull operators that one needs to reproduce the classical path integral Ward identities of section (4.2). The goal is to regulate the theory using an appropriate Pauli-Villars regulator and see whether these identities hold at quantum level. The PV Lagrangian we are using in the next chapter consists of a sum of massive WZ models, with standard and 'wrong' statistics. We want to examine how the massive WZ model classically violates some of the symmetries of the massless and conformal WZ model, thus giving the possibility for anomalous contributions. The main core of the arguments we present here are still true even in the case of fields with 'wrong' statistics, since the form of the equations (i.e. variation of currents and conservation laws) remain the same.

According to (2.2.8), in order to derive the Ward identities (that correspond to a specific transformation of the elementary fields) for some local operators, we need the following: The variation of the local operators with respect to the field transformation, and, we have to know if this transformation is a symmetry of the theory, i.e. whether $A(x)$ of (2.2.8) is zero or not.

Comparing (4.1.7) and (4.1.8) with (5.1.7), it is obvious that the massive WZ model breaks classically R-symmetry, S-supersymmetry and scale invariance. The corresponding $A(x)$ for these symmetries is non zero, something that generates new, possibly anomalous terms in the classical Ward identities of the massless WZ model. Next, we have to see if there are anomalous contributions that come from the symmetry variation of the operators. In the previous section (5.2), we wrote the R-symmetry and Q-supersymmetry variations of the current operators. We need to compare these, with the corresponding ones from section (4.1.5). Note here, that variations of operators that are not included in (5.2), remain the same in the massive theory.

We now compare the R-symmetry variation of the stress tensor and supercurrent of (5.2.1) with (4.1.11). Notice that the term $\theta_0\eta^\mu_\nu\frac{im}{3}\bar{\chi}\gamma_5\chi$ in the transformation of the stress tensor, and the term $\frac{i}{3}\theta_0\gamma_5\sqrt{2}m(\phi\chi_L + \phi^*\chi_R)$ in the transformation of the supercurrent of (5.2.1), do not exist in (4.1.11). These terms give extra anomalous contributions to the R-symmetry Ward identities of correlators that involve the supercurrent or the stress tensor.

Let us consider Q-supersymmetry now. We claim that the original Q-supersymmetry Ward identities of the conformal WZ model, are classically violated by the massive Lagrangian. This is a subtle point that was recently criticized [103,104]. The main reason for that is that the massive WZ model is manifestly invariant under Q-supersymmetry, which results in the conservation law of the supercurrent (5.1.4). However, as we emphasized, one has to also examine whether the symmetry variations of the operators remain the same in the massive theory. After comparing (5.2.2) with (4.1.13) we see that $\delta_{\varepsilon_0}\hat{\hat{\mathcal{Q}}}^\mu$ is of the same form with $\delta_{\varepsilon_0}\hat{\mathcal{Q}}^\mu$. There is an apparent difference in the null operators (i.e. operators that vanish on shell). For example, operators of the form $\not{\partial}\chi$ in (4.1.13) are replaced by $(\not{\partial}+m)\chi$ in (5.2.2). This is of course expected, since in the massive theory the equations of motion change. Both of these operators, $\not{\partial}\chi$ and $(\not{\partial}+m)\chi$, have the same effect in the Feynman diagrams. They cancel massless and massive propagators respectively.

The Q-supersymmetry variation of the R-current though, differ in the two theories. Notice the term $\frac{i\sqrt{2}}{3}\bar{\varepsilon}_0\gamma_5\gamma^\mu m(\phi\chi_L+\phi^*\chi_R)$ of (5.2.2) which is absent from (4.1.13). This term that depends on the mass, violates the Q-supersymmetry Ward identities of correlators that involve the R-current. In the case of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, the massive WZ Lagrangian will introduce an extra 3-point functions at the rhs, of the following form

$$\frac{i\sqrt{2}}{3}\gamma_5\gamma^\lambda m <(\phi\chi_L+\phi^*\chi_R)\tilde{\bar{\mathcal{Q}}}^\nu\tilde{\mathcal{J}}^\kappa>. \tag{5.3.1}$$

This term comprises the potential anomaly of Q-supersymmetry at the regulated level in the 4-point function of interest, and we need to evaluate it.

Had we compared equations (5.2.3) with (4.1.15), we would see that the form of the Q-supersymmetry variation of the R-current is the same in the massless and massive theory. However, (5.2.3) and (4.1.15) would contribute the following 3-point correlator at the rhs of the Q-supersymmetry identity of the 4-point function

$$\frac{i}{3}\gamma_5\gamma^\lambda\gamma_\mu <\tilde{\mathcal{Q}}^\mu\tilde{\bar{\mathcal{Q}}}^\nu\tilde{\mathcal{J}}^\kappa>. \tag{5.3.2}$$

This correlator is proportional to the gamma-trace of the supercurrent, so we need to replace it using the S-supersymmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. According to (5.1.7) and (4.1.8), the gamma-trace of the supercurrent in the massive theory contains an extra mass dependent term compared to the massless theory. This means that the term (5.3.2) will have an anomalous contribution to Q-supersymmetry, equal to (5.3.1). As expected, whether we use the form (5.2.2) or (5.2.3) for the variation of the R-current, we find that Q-supersymmetry is classically violated with the same mass dependent term.

CHAPTER 6

Pauli-Villars regularization

The 1-loop diagrams that determine the correlation functions of conserved currents in the free and massless WZ model (4.1.1) suffer from UV divergences that must be regulated and renormalized. In this chapter we present a supersymmetric Pauli-Villars (PV) regulator that suffices for removing the 1-loop UV divergences from all correlation functions that appear in the Ward identities we examine in section 4.2. We follow closely the relevant discussion of [81].

6.1 Setup

Consistency of any PV regulator requires that its contributions to the 1-loop diagrams follow from a local Lagrangian. A supersymmetric PV regulator further demands that this Lagrangian preserves supersymmetry and hence must involve a number of $\mathcal{N} = 1$ multiplets. The PV regulator we use consists of three massive chiral multiplets, one with standard statistics and two with 'wrong' statistics. The corresponding PV Lagrangian is a standard massive WZ model, except that terms involving the multiplets with wrong statistics are appropriately modified. The PV Lagrangian we consider takes a similar form to (5.1.2)

$$\begin{aligned}-\hat{\mathscr{L}}_{\text{PV}} &= \partial_\mu\varphi_2^*\partial^\mu\varphi_2 + m_2^2\varphi_2^*\varphi_2 + \frac{1}{2}\bar{\lambda}_2\partial\!\!\!/\lambda_2 + \frac{m_2}{2}\bar{\lambda}_2\lambda_2 \\ &\quad + \partial_\mu\varphi_1^*\partial^\mu\varphi_1 + m_1^2\varphi_1^*\varphi_1 + \partial_\mu\vartheta_1\partial^\mu\vartheta_1^* + m_1^2\vartheta_1\vartheta_1^* + \overline{\lambda}_1\partial\!\!\!/\lambda_1 + m_1\overline{\lambda}_1\lambda_1,\end{aligned} \tag{6.1.1}$$

where (φ_2, λ_2) is a standard massive WZ multiplet, consisting of a commuting complex scalar, φ_2, and an anticommuting Majorana spinor, λ_2, while φ_1, ϑ_1 are anticommuting complex scalars and λ_1 is a commuting Dirac spinor. Here, we should emphasize the distinction between the Dirac (e.g. $\overline{\lambda_1}$) and Majorana (e.g. $\bar{\lambda}_2$) conjugates of a spinor, both of which are discussed in appendix A.

In fact, the fields $(\varphi_1, \vartheta_1, \lambda_1)$ form two chiral multiplets with 'wrong' statistics. This can be made manifest by means of the field redefinition

$$\lambda_+ \equiv \frac{1}{2}(\lambda_1 + \lambda_1^C), \quad \lambda_- \equiv \frac{1}{2i}(\lambda_1 - \lambda_1^C), \quad \varphi_+ \equiv \frac{\varphi_1 + \vartheta_1}{2}, \quad \varphi_- \equiv \frac{\varphi_1 - \vartheta_1}{2i}, \tag{6.1.2}$$

where the Majorana conjugate, λ_1^C, is defined in (A.0.10), $\varphi_\pm$ are anticommuting complex scalars and $\lambda_\pm$ are commuting Majorana spinors. The two chiral multiplets correspond to (φ_+, λ_+) and (φ_-, λ_-). The advantage of this parameterization is that it greatly simplifies the discussion of the symmetries preserved by the PV regulator. However, once expressed in terms of the fields (6.1.2), the PV Lagrangian (6.1.1) contains non diagonal terms for the fields of wrong statistics, namely

$$\begin{aligned} -\hat{\mathscr{L}}_{\text{PV}} = &\partial_\mu \varphi_2^* \partial^\mu \varphi_2 + m_2^2 \varphi_2^* \varphi_2 + \frac{1}{2}\bar{\lambda}_2 \partial\!\!\!/ \lambda_2 + \frac{m_2}{2}\bar{\lambda}_2 \lambda_2 \\ &+ 2i(\partial_\mu \varphi_+^* \partial^\mu \varphi_- - \partial_\mu \varphi_-^* \partial^\mu \varphi_+) + 2im_1^2(\varphi_+^* \varphi_- - \varphi_-^* \varphi_+) \\ &+ i(\bar{\lambda}_+ \partial\!\!\!/ \lambda_- - \bar{\lambda}_- \partial\!\!\!/ \lambda_+) + im_1(\bar{\lambda}_+ \lambda_- - \bar{\lambda}_- \lambda_+). \end{aligned} \tag{6.1.3}$$

Notice that the PV Lagrangian contains two independently supersymmetric parts, namely the standard WZ action for the massive chiral multiplet with canonical statistics, and the remaining terms for the two massive chiral multiplets of wrong statistics. The non diagonal terms in (6.1.3) imply that the latter do not preserve supersymmetry independently – it is not possible to write down a supersymmetric Lagrangian for a single massive chiral multiplet of wrong statistics. Supersymmetry does not impose any relation between the mass of the standard WZ multiplet and that of the two chiral multiplets with wrong statistics. However, in appendix D we show that cancellation of the UV divergences requires that $m_2^2 = 2m_1^2$. The regulator is removed when the mass parameter is sent to infinity, where the PV fields decouple from the original theory.

Propagators

It is most convenient to express the propagators of the PV fields in diagonal form using the parameterization (6.1.1). Paying attention to the statistics of the various fields, they take the form

$$\begin{aligned} \overbrace{\varphi_2(p)\varphi_2^*}(p') &= \overbrace{\varphi_2^*(p')\varphi_2}(p) = (2\pi)^4 \delta(p+p') P_{\varphi_2}(p), \\ \overbrace{\lambda_2(p)\bar{\lambda}_2}(p') &= -\overbrace{\bar{\lambda}_2(p')\lambda_2}(p) = (2\pi)^4 \delta(p+p') P_{\lambda_2}(p), \end{aligned}$$

$$\begin{aligned}
\varphi_1(p)\varphi_1^*(p') &= -\varphi_1^*(p')\varphi_1(p) = (2\pi)^4\delta(p+p')P_{\varphi_1}(p),\\
\vartheta_1^*(p)\vartheta_1(p') &= -\vartheta_1(p')\vartheta_1^*(p) = (2\pi)^4\delta(p+p')P_{\vartheta_1}(p),\\
\lambda_1(p)\bar\lambda_1(p') &= \bar\lambda_1(p')\lambda_1(p) = (2\pi)^4\delta(p+p')P_{\lambda_1}(p),
\end{aligned} \tag{6.1.4}$$

where

$$\begin{aligned}
&P_{\varphi_2}(p) = -\frac{i}{p^2+m_2^2}, \qquad P_{\lambda_2}(p) = -i\frac{(-i\not p + m_2)}{p^2+m_2^2},\\
&P_{\varphi_1}(p) = -\frac{i}{p^2+m_1^2}, \qquad P_{\vartheta_1}(p) = -\frac{i}{p^2+m_1^2}, \qquad P_{\lambda_1}(p) = -i\frac{(-i\not p + m_1)}{p^2+m_1^2}.
\end{aligned} \tag{6.1.5}$$

6.2 Symmetries and conserved currents

Just like the standard massive WZ model (5.1.1), the PV Lagrangian (6.1.1) or (6.1.3) is invariant only under Poincaré symmetries and Q-supersymmetry. Q-supersymmetry acts on the PV fields as

$$\begin{aligned}
\delta_{\varepsilon_0}\varphi_2 &= \frac{\sqrt2}{2}\bar\varepsilon_{0L}\lambda_2, \qquad \delta_{\varepsilon_0}\varphi_2^* = \frac{\sqrt2}{2}\bar\varepsilon_{0R}\lambda_2,\\
\delta_{\varepsilon_0}\lambda_2 &= \frac{\sqrt2}{2}(\not\partial\varphi_2\varepsilon_{0R} + \not\partial\varphi_2^*\varepsilon_{0L} - m_2\varphi_2\varepsilon_{0R} - m_2\varphi_2^*\varepsilon_{0L}),\\
\delta_{\varepsilon_0}\bar\lambda_2 &= -\frac{\sqrt2}{2}(\bar\varepsilon_{0L}\not\partial\varphi_2^* + \bar\varepsilon_{0R}\not\partial\varphi_2 + m_2\bar\varepsilon_{0R}\varphi_2 + m_2\varphi_2^*\bar\varepsilon_{0L}),\\
\delta_{\varepsilon_0}\varphi_\pm &= \frac{\sqrt2}{2}\bar\varepsilon_{0L}\lambda_\pm, \qquad \delta_{\varepsilon_0}\varphi_\pm^* = -\frac{\sqrt2}{2}\bar\varepsilon_{0R}\lambda_\pm,\\
\delta_{\varepsilon_0}\lambda_\pm &= \frac{\sqrt2}{2}\Big(\not\partial\varphi_\pm^*\varepsilon_{0L} - \not\partial\varphi_\pm\varepsilon_{0R} - m_1\varphi_\pm^*\varepsilon_{0L} + m_1\varphi_\pm\varepsilon_{0R}\Big),
\end{aligned} \tag{6.2.1}$$

where any sign differences in the transformations of $(\varphi_\pm, \lambda_\pm)$ relative to the standard transformations of (φ_2, λ_2) are due to the different statistics.

Using the field redefinition (6.1.2), we find that the fields $(\varphi_1, \vartheta_1, \lambda_1)$ transform as

$$\begin{aligned}
\delta_{\varepsilon_0}\vartheta_1^* &= -\frac{\sqrt2}{2}\bar\varepsilon_{0L}\lambda_1 = -\frac{\sqrt2}{2}\bar\varepsilon_{0R}\lambda_1, \qquad \delta_{\varepsilon_0}\vartheta_1 = -\frac{\sqrt2}{2}\bar\lambda_1\varepsilon_{0L} = \frac{\sqrt2}{2}\bar\varepsilon_{0L}\lambda_1^C,\\
\delta_{\varepsilon_0}\varphi_1 &= \frac{\sqrt2}{2}\bar\varepsilon_{0R}\lambda_1 = \frac{\sqrt2}{2}\bar\varepsilon_{0L}\lambda_1, \qquad \delta_{\varepsilon_0}\varphi_1^* = \frac{\sqrt2}{2}\bar\lambda_1\varepsilon_{0R} = -\frac{\sqrt2}{2}\bar\varepsilon_{0R}\lambda_1^C,\\
\delta_{\varepsilon_0}\lambda_1 &= \frac{\sqrt2}{2}(\not\partial\vartheta_1^*\varepsilon_{0L} - \not\partial\varphi_1\varepsilon_{0R} - m_1\vartheta_1^*\varepsilon_{0L} + m_1\varphi_1\varepsilon_{0R}),\\
\delta_{\varepsilon_0}\bar\lambda_1 &= -\frac{\sqrt2}{2}(\bar\varepsilon_{0R}\not\partial\vartheta_1 - \bar\varepsilon_{0L}\not\partial\varphi_1^* + m_1\bar\varepsilon_{0R}\vartheta_1 - m_1\bar\varepsilon_{0L}\varphi_1^*).
\end{aligned} \tag{6.2.2}$$

Once again, in these expressions one must be careful to distinguish between the Dirac and Majorana conjugates of a spinor. In particular, for a Majorana spinor, χ, the Dirac conjugate, $\overline{\chi}$, and the Majorana conjugate, $\bar{\chi}$, coincide, i.e. $\overline{\chi} = \bar{\chi}$. Moreover, the Dirac and Majorana conjugates of Weyl spinors are related as $\overline{\chi}_{L,R} = \bar{\chi}_{R,L}$, while those of Dirac spinors are unrelated. For example, using the decomposition of the Dirac spinor λ_1 into two Majorana spinors $\lambda_\pm$ as in (6.1.2), we have

$$\lambda_1 = \lambda_+ + i\lambda_-, \qquad \overline{\lambda}_1 = \overline{\lambda}_+ - i\overline{\lambda}_- = \bar{\lambda}_+ - i\bar{\lambda}_-. \tag{6.2.3}$$

The current operators of the PV Lagrangian are given by

$$\begin{aligned}
\hat{\bar{\mathcal{T}}}^\mu{}_\nu|_{\rm PV} &= 2\partial^{(\mu}\varphi_2^*\partial_{\nu)}\varphi_2 - \frac{1}{3}(\partial^\mu\partial_\nu - \eta^\mu_\nu\partial^2)(\varphi_2^*\varphi_2) - \eta^\mu_\nu(\partial_\rho\varphi_2^*\partial^\rho\varphi_2 + m_2^2\varphi_2^*\varphi_2) \\
&+ \frac{1}{2}\bar{\lambda}_2\gamma^\mu\partial_\nu\lambda_2 - \frac{1}{2}\eta^\mu_\nu(\bar{\lambda}_2\not{\partial}\lambda_2 + m_2\bar{\lambda}_2\lambda_2) - \frac{1}{8}\partial_\rho(\bar{\lambda}_2\gamma_\nu\gamma^{\rho\mu}\lambda_2 + \bar{\lambda}_2\gamma^\mu\gamma^\rho{}_\nu\lambda_2 - \bar{\lambda}_2\gamma^\rho\gamma^\mu{}_\nu\lambda_2) \\
&+ 2\partial^{(\mu}\varphi_1^*\partial_{\nu)}\varphi_1 - \frac{1}{3}(\partial^\mu\partial_\nu - \eta^\mu_\nu\partial^2)(\varphi_1^*\varphi_1) - \eta^\mu_\nu(\partial_\rho\varphi_1^*\partial^\rho\varphi_1 + m_1^2\varphi_1^*\varphi_1) \\
&+ 2\partial^{(\mu}\vartheta_1\partial_{\nu)}\vartheta_1^* - \frac{1}{3}(\partial^\mu\partial_\nu - \eta^\mu_\nu\partial^2)(\vartheta_1\vartheta_1^*) - \eta^\mu_\nu(\partial_\rho\vartheta_1\partial^\rho\vartheta_1^* + m_1^2\vartheta_1\vartheta_1^*) \\
&+ \overline{\lambda}_1\gamma^\mu\partial_\nu\lambda_1 - \eta^\mu_\nu(\overline{\lambda}_1\not{\partial}\lambda_1 + m_1\overline{\lambda}_1\lambda_1) - \frac{1}{4}\partial_\rho(\overline{\lambda}_1\gamma_\nu\gamma^{\rho\mu}\lambda_1 + \overline{\lambda}_1\gamma^\mu\gamma^\rho{}_\nu\lambda_1 - \overline{\lambda}_1\gamma^\rho\gamma^\mu{}_\nu\lambda_1), \\
\hat{\bar{\mathcal{J}}}^\mu|_{\rm PV} &= \frac{2i}{3}\Big(\varphi_2^*\overleftrightarrow{\partial^\mu}\varphi_2 + \frac{1}{4}\bar{\lambda}_2\gamma^\mu\gamma_5\lambda_2\Big) + \frac{2i}{3}\Big(\varphi_1^*\overleftrightarrow{\partial^\mu}\varphi_1 - \vartheta_1^*\overleftrightarrow{\partial^\mu}\vartheta_1 + \frac{1}{2}\overline{\lambda}_1\gamma^\mu\gamma_5\lambda_1\Big), \\
\hat{\bar{\mathcal{Q}}}^\mu|_{\rm PV} &= \frac{\sqrt{2}}{2}(\not{\partial}\varphi_2\gamma^\mu\lambda_{2R} + \not{\partial}\varphi_2^*\gamma^\mu\lambda_{2L}) + \frac{\sqrt{2}}{3}\gamma^{\mu\nu}\partial_\nu(\varphi_2\lambda_{2R} + \varphi_2^*\lambda_{2L}) + \frac{\sqrt{2}}{2}m_2(\varphi_2\gamma^\mu\lambda_{2L} + \varphi_2^*\gamma^\mu\lambda_{2R}) \\
&+ \frac{\sqrt{2}}{2}(\not{\partial}\vartheta_1\gamma^\mu\lambda_{1R} - \not{\partial}\varphi_1^*\gamma^\mu\lambda_{1L}) + \frac{\sqrt{2}}{3}\gamma^{\mu\nu}\partial_\nu(\vartheta_1\lambda_{1R} - \varphi_1^*\lambda_{1L}) + \frac{\sqrt{2}}{2}m_1(\vartheta_1\gamma^\mu\lambda_{1L} - \varphi_1^*\gamma^\mu\lambda_{1R}) \\
&+ \frac{\sqrt{2}}{2}(\not{\partial}\vartheta_1^*\gamma^\mu\lambda^C_{1L} - \not{\partial}\varphi_1\gamma^\mu\lambda^C_{1R}) + \frac{\sqrt{2}}{3}\gamma^{\mu\nu}\partial_\nu(\vartheta_1^*\lambda^C_{1L} - \varphi_1\lambda^C_{1R}) + \frac{\sqrt{2}}{2}m_1(\vartheta_1^*\gamma^\mu\lambda^C_{1R} - \varphi_1\gamma^\mu\lambda^C_{1L}), \\
\hat{\bar{\mathcal{O}}}_M|_{\rm PV} &= \frac{m_2}{2}\varphi_2^2 + m_1\varphi_1\vartheta_1, \qquad \hat{\bar{\mathcal{O}}}_{M^*}|_{\rm PV} = \frac{m_2}{2}\varphi_2^{*2} + m_1\vartheta_1^*\varphi_1^*.
\end{aligned} \tag{6.2.4}$$

We emphasize that the supercurrent for the PV fields remains an anticommuting Majorana fermion, which is essential for coupling the theory to background supergravity. Its Majorana conjugate is

$$\begin{aligned}
\hat{\bar{\bar{\mathcal{Q}}}}^\mu|_{\rm PV} &= \frac{\sqrt{2}}{2}(\bar{\lambda}_{2R}\gamma^\mu\not{\partial}\varphi_2 + \bar{\lambda}_{2L}\gamma^\mu\not{\partial}\varphi_2^*) - \frac{\sqrt{2}}{3}\partial_\nu(\varphi_2\bar{\lambda}_{2R} + \varphi_2^*\bar{\lambda}_{2L})\gamma^{\mu\nu} - \frac{\sqrt{2}}{2}m_2(\varphi_2\bar{\lambda}_{2L} + \varphi_2^*\bar{\lambda}_{2R})\gamma^\mu \\
&+ \frac{\sqrt{2}}{2}\overline{\lambda}_1^C(P_R\gamma^\mu\not{\partial}\vartheta_1 - P_L\gamma^\mu\not{\partial}\varphi_1^*) - \frac{\sqrt{2}}{3}\partial_\nu(\vartheta_1\overline{\lambda}_1^C P_R - \varphi_1^*\overline{\lambda}_1^C P_L)\gamma^{\mu\nu} - \frac{\sqrt{2}}{2}m_1\overline{\lambda}_1^C(\vartheta_1 P_L - \varphi_1^* P_R)\gamma^\mu \\
&+ \frac{\sqrt{2}}{2}\overline{\lambda}_1(P_L\gamma^\mu\not{\partial}\vartheta_1^* - P_R\gamma^\mu\not{\partial}\varphi_1) - \frac{\sqrt{2}}{3}\partial_\nu(\vartheta_1^*\overline{\lambda}_1 P_L - \varphi_1\overline{\lambda}_1 P_R)\gamma^{\mu\nu} - \frac{\sqrt{2}}{2}m_1\overline{\lambda}_1(\vartheta_1^* P_R - \varphi_1 P_L)\gamma^\mu.
\end{aligned} \tag{6.2.5}$$

The FZ multiplet operators of the full theory, comprising the massless WZ model and the PV fields, are the sum of the conformal currents (4.1.6) and the PV operators in (6.2.4), and will be denoted by $\hat{\bar{\mathcal{T}}}^\mu{}_\nu$, $\hat{\bar{\mathcal{J}}}^\mu$, $\hat{\bar{\mathcal{Q}}}^\mu$ and $\hat{\bar{\mathcal{O}}}_M$ in the following. The FZ multiplet operators satisfy the on-shell identities (5.1.4), while the explicit classical breaking of scale invariance, R-symmetry and S-supersymmetry is reflected respectively in the relations

$$\begin{aligned}
\hat{\bar{\mathcal{T}}}^{\mu}_{\mu} &= -2m_2^2\varphi_2^*\varphi_2 - \frac{m_2}{2}\bar{\lambda}_2\lambda_2 - 2m_1^2(\varphi_1^*\varphi_1 + \vartheta_1\vartheta_1^*) - m_1\overline{\lambda}_1\lambda_1 \equiv \hat{\mathcal{B}}_W, \\
\partial_\mu \hat{\bar{\mathcal{J}}}^{\mu} &= \frac{im_2}{3}\bar{\lambda}_2\gamma_5\lambda_2 + \frac{2im_1}{3}\overline{\lambda}_1\gamma_5\lambda_1 \equiv \hat{\mathcal{B}}_R, \\
\gamma_\mu \hat{\bar{\mathcal{Q}}}^{\mu} &= \sqrt{2}m_2(\varphi_2\lambda_{2L} + \varphi_2^*\lambda_{2R}) + \sqrt{2}m_1(\vartheta_1\lambda_{1L} - \varphi_1^*\lambda_{1R}) + \sqrt{2}m_1(\vartheta_1^*\lambda_{1R}^C - \varphi_1\lambda_{1L}^C) \equiv \hat{\mathcal{B}}_S.
\end{aligned} \tag{6.2.6}$$

We have introduced the notation $\hat{\mathcal{B}}_W$, $\hat{\mathcal{B}}_R$ and $\hat{\mathcal{B}}_S$ for the quantities on the rhs of these identities for later convenience. Notice that these quantities, as well as the scalar operator $\mathcal{O}_M$, receive contributions only from the PV fields. The potential anomalies of the classical Ward identities –which arise due to the PV regulator– are given by correlators that involve the operators $\hat{\mathcal{B}}_W$, $\hat{\mathcal{B}}_R$ and $\hat{\mathcal{B}}_S$. The analysis that shows that the correlation functions of interest are properly regulated is given in the appendix D.

CHAPTER 7

Anomalies of the Wess-Zumino model

In this chapter we evaluate the Ward identities of section (4.2) in the regulated theory. The aim is to examine whether they are satisfied in the quantum regime or not. The first step towards that was to identify a suitable regulator that removes all UV divergences from the correlators of interest. We used the PV Lagrangian (6.1.1), and the analysis of the regulated correlators is presented in the appendix D. If the regulator respects all classical symmetries, then the form of the regulated Ward identities remains the same, hence there are no anomalies. If the regulator violates some of the symmetries of the classical theory, like the PV regulator breaks R-symmetry invariance, then there is a possibility of anomalous contributions to the original Ward identities. In the latter case, as already mentioned, there are two equivalent approaches.

The first one is to put the regulated correlators into the original classical identities and after an explicit computation identify the terms that violate them. Then we calculate these terms in the limit where the regulator is removed, and see whether they vanish or not. In the second approach one has to compute the new classically broken Ward identities of the regulated theory. These identities include some extra terms compared to the original ones, that depend only on the regulator. The Ward identities of the regulated theory are satisfied by construction, so we only need to evaluate the new extra terms in the limit where the regulator vanishes. In both approaches the breaking terms that we find are necessarily exactly the same.

The advantage of the first approach is that we do not have to find the new Ward identities of the regulated theory, while the advantage of the second approach is that after computing the new Ward identities, we can immediately deduce the extra breaking anomalous terms without further manipulations of the regulated correlators. In the case of non vanishing breaking terms, one has to examine whether they can be removed by a local counterterm in the action. We find the second approach much more convenient, so we are going to follow it in the analysis of the next sections.

As we stressed before, the main goal is to examine the Q-supersymmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. However, if the PV Lagrangian (6.1.1) we are using is a consistent regulator, it has to reproduce the standard Ward identities and anomalies of the correlators among bosonic operators. After all, if we forget about supersymmetry, the original massless WZ Lagrangian (4.1.1) we are considering, can be seen as the sum of a free fermion and a free boson, which we already know contain trace and R-symmetry anomalies. In particular, according to (B.1.3), the correlator $<\mathcal{T}\mathcal{J}\mathcal{J}>$ has a trace anomaly, while $<\mathcal{J}\mathcal{J}\mathcal{J}>$ has an R-symmetry anomaly. We are going to confirm these in the following sections. There are also anomalies in the Ward identities of $<\mathcal{T}\mathcal{T}\mathcal{J}>$ and $<\mathcal{T}\mathcal{T}\mathcal{T}>$, however, these correlators are not involved in the series of identities necessary for the analysis of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, thus we do not examine them. Moreover, we cannot compute $<\mathcal{T}\mathcal{T}\mathcal{T}>$, since the PV Lagrangian we are using is not sufficient to remove its logarithmic divergences. For that, one has to include more PV fields.

In the following sections we drop the hat $\hat{}$ from the flat space operators, to simplify the notation.

7.1 Bosonic correlators

In this section we examine the bosonic correlators $<\mathcal{J}\mathcal{J}>$, $<\mathcal{T}\mathcal{J}\mathcal{J}>$ and $<\mathcal{J}\mathcal{J}\mathcal{J}>$. Moreover, $<\mathcal{T}\mathcal{J}>$ is equal to zero (D.3.14). Using the results from chapters 5 and 6, we compute the extra contributions of the PV regulator to the corresponding Ward identities of section (4.2). Then, whenever possible we find local counterterms to restore the broken by the regulator symmetries. Since we are interested in coupling the WZ model to conformal supergravity, the counterterms we identify depend only on the background sources of conformal supergravity, i.e. the gravitino ψ_μ, the gauge field A_μ and the vierbein e^a_μ. The explicit computation of integrals is presented in the appendix E.

7.1.1 $<\mathcal{J}\mathcal{J}>$

In the regulated theory, the presence of the PV masses introduce a new term at the rhs of the classical R-symmetry Ward identity of $<\mathcal{J}\mathcal{J}>$ (4.2.2). Using (6.2.6), within the path integral formalism one can find that the new (broken) R symmetry Ward identity is equal to

$$p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>= -i <\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> . \tag{7.1.1}$$

In order to restore R symmetry, this extra term has to vanish or be removed by a local counterterm. In the limit that PV masses go to infinity we find that

$$-i <\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>= -\frac{2i\pi^2}{(2\pi)^4} p_3^{\lambda}\left(\frac{4}{9}\, m_1^2 \log 2 - \frac{p_3^2}{27}\right). \tag{7.1.2}$$

If we renormalize the 2-point function as follows

$$<\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}=<\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> + \frac{2i\pi^2}{(2\pi)^4}\eta^{\kappa\lambda}\left(\frac{4}{9}\, m_1^2 \log 2 - \frac{p_3^2}{27}\right) \tag{7.1.3}$$

we get that the R symmetry Ward identity is satisfied, i.e.

$$p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}= 0 \tag{7.1.4}$$

At the level of the action, the renormalization of the 2-point function $<\mathcal{J}\mathcal{J}>$ can be achieved by the following counterterm,

Counterterm

$$I_{RR} = \left(\frac{2i\pi^2}{(2\pi)^4}\frac{i}{54}A^{\rho}\nabla^{\xi}\nabla_{\xi}A_{\rho} + \frac{2i\pi^2}{(2\pi)^4}\frac{2i}{9}m_1^2 \log 2\, A^{\rho}A_{\rho}\right). \tag{7.1.5}$$

Note here that in the above counterterm, instead of the Levi-Civita connection ∇_{ξ}, we could have just used the partial derivative ∂_{ξ}. The contribution to the 2-point correlator $<\mathcal{J}\mathcal{J}>$ would be the same. We choose though a covariant counterterm, in order to not break diffeomorphisms.

Alternative approach

To examine the classical R-symmetry identity of $<\mathcal{J}\mathcal{J}>$, we evaluated in the large PV mass limit the breaking term at the rhs of the regulated Ward identity (7.1.1). We are allowed to do that, since $<\mathcal{J}\mathcal{J}>$ is properly regulated (see appendix D). Now we are going to compute this identity by following the alternative approach, i.e. first evaluate in the large PV mass limit the $<\mathcal{J}\mathcal{J}>$ correlator, then contract it with the external momentum $p_{3\kappa}$ and see whether it vanishes or not. The reason we do this, is first, to illustrate the role of the 1-point functions, and secondly, to explain in more detail the renormalized correlators, which may cause some confusion.

We mentioned that in the whole analysis we do not include any 1-point functions, since they are irrelevant for the computation of anomalies. They only contribute at the divergent parts of the Ward identities, i.e. terms that depend on the PV mass. We want to justify this by giving a simple example. Moreover, in (7.1.3) we renormalized $<\mathcal{J}\mathcal{J}>$ so that the R-symmetry identity is satisfied. However, note that this renormalization does not render the 2-point function finite. Usually in literature, the terminology 'renormalized correlator' is used to denote correlators which are free of divergences. What we do in (7.1.3), is a partial renormalization of $<\mathcal{J}\mathcal{J}>$, that only suffices for restoring the broken symmetry. $<\mathcal{J}\mathcal{J}>$ also contains logarithmic divergences that satisfy the classical R-symmetry identity, thus there is no need to remove them. From now on, whenever we write renormalized correlators, we will refer to the partially renormalized correlators we just described, unless mentioned otherwise.

Now let us try to make the above discussion a bit more clear. Taking into account the 1-point functions, the classical R-symmetry Ward identity of $<\mathcal{J}\mathcal{J}>$ is given by

$$p_{3\kappa} <\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> + \frac{8i}{9} p_3^{\lambda} <s_{(1|0)}>= 0. \tag{7.1.6}$$

We want to see if the above lhs is zero in the regulated theory. Using (D.3.9), in the large PV mass limit we find that

$$\begin{aligned} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>= &-\frac{2i\pi^2}{(2\pi)^4}\frac{4}{3}\eta^{\kappa\lambda}m_1^2\log 2 + \frac{2i\pi^2}{(2\pi)^4}\frac{p_3^{\lambda}p_3^{\kappa}}{27} \\ &- \frac{2i\pi^2}{(2\pi)^4}\frac{1}{9}\left(p_3^{\kappa}p_3^{\lambda} - p_3^2\eta^{\kappa\lambda}\right)\left(\log m_1^2 - \log\left(2p_3^2\right) + \frac{8}{3}\right) \end{aligned} \tag{7.1.7}$$

Notice that the second line is proportional to the projection operator, hence it vanishes when contracted with $p_{3\kappa}$ or $p_{4\lambda}$ (momentum conservation implies that $p_3 = -p_4$). We see that the logarithmic divergence of $<\mathcal{J}\mathcal{J}>$ does not contribute to the classical R-symmetry identity (7.1.6), so there is no need to remove it. Of course, if we wish to fully renormalize the theory under consideration, we have to introduce local counterterms to remove all divergences, the logarithmic ones as well. However, if we only care about the validity of specific symmetry identities, we just have to include counterterms that remove the breaking terms that arise due to regularization. Using now (D.2.4), we find that the lhs of (7.1.6) in the regulated theory is equal to

$$p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> + \frac{8i}{9} p_3^{\lambda} <\widetilde{s}_{(1|0)}>= -\frac{2i\pi^2}{(2\pi)^4}\frac{4}{9}p_3^{\lambda}m_1^2\log 2 + \frac{2i\pi^2}{(2\pi)^4}\frac{p_3^{\lambda}p_3^2}{27}. \tag{7.1.8}$$

As expected, after computing the 2-point function in the large PV mass limit and then contracting it with the external momentum, we find that the classical R-symmetry Ward identity (7.1.6) is not satisfied. In particular, notice that the above rhs is exactly the same with the contribution of the breaking term (7.1.2). Effectively, what we did here was to confirm the R-symmetry identity of the regulated theory (7.1.1). The contribution of the 1-point function $<\widetilde{s}_{(1|0)}>$ was crucial, so that lhs and rhs match at the divergent pieces, i.e. the terms that depend on m_1^2. These terms however, are not relevant for the computations of anomalies (which are finite), which is why we neglect from our analysis all 1-point functions. The partially renormalized (but still divergent) correlator $<\mathcal{J}\mathcal{J}>$ that satisfies the classical identity (7.1.6) is given by

$$<\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}= -\frac{2i\pi^2}{(2\pi)^4}\frac{8}{9}\eta^{\kappa\lambda}m_1^2\log 2 + \frac{2i\pi^2}{(2\pi)^4}\frac{1}{27}(p_3^{\lambda}p_3^{\kappa} - p_3^2\eta^{\kappa\lambda})$$
$$- \frac{2i\pi^2}{(2\pi)^4}\frac{1}{9}\left(p_3^{\kappa}p_3^{\lambda} - p_3^2\eta^{\kappa\lambda}\right)\left(\log m_1^2 - \log(2p_3^2) + \frac{8}{3}\right) \quad (7.1.9)$$

Having obtained the explicit form of $<\mathcal{J}\mathcal{J}>$, it is rather trivial to introduce counterterms to remove the logarithmic divergences. However, to find the explicit expressions for the higher order correlators is extremely tedious. On the contrary, if we focus only on the breaking terms of the classical Ward identities that are introduced by the PV regulator, we greatly simplify the computation. This is the approach that we follow in the next sections. We compute the breaking terms, and then introduce counterterms to restore symmetries whenever possible. When we examine the Ward identities of higher order correlators, we must be careful to take into account the contribution to these identities of the counterterms that we used to restore symmetries in the lower order correlation functions.

7.1.2 $<\mathcal{J}\mathcal{J}\mathcal{J}>$

In the regulated theory, the classical R-symmetry Ward identity for the 3-point function $<\mathcal{J}\mathcal{J}\mathcal{J}>$ (4.2.8) becomes

$$p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{J}}^{\sigma}(p_1)>= -i <\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{J}}^{\sigma}(p_1)> . \quad (7.1.10)$$

We have used that the 2-point function $<\mathcal{J}s_{(1|0)}>$ is zero (see appendix D). The term on the rhs is the potential anomaly of the 3-point function, that arises due to the non invariance of the PV Lagrangian (6.1.1) under an R-symmetry transformation. In the large PV mass limit we find that

$$-i <\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{J}}^{\sigma}(p_1)>= \frac{i}{324\pi^2}\epsilon^{\sigma\lambda\beta\alpha}p_{3\beta}p_{4\alpha}, \quad (7.1.11)$$

hence,

$$p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{J}}^{\sigma}(p_1)>= \frac{i}{324\pi^2}\epsilon^{\sigma\lambda\beta\alpha}p_{3\beta}p_{4\alpha}. \quad (7.1.12)$$

The R-symmetry Ward identity of $<\mathcal{J}\mathcal{J}\mathcal{J}>$ is anomalous, since the breaking term does not vanish when we remove the regulator. The anomaly cannot be cancelled by a local and gauge invariant counterterm. This 1-loop computation confirms the R-anomaly of (B.1.3) for the specific values $c = 2a = \frac{1}{24}$ of the WZ model.

7.1.3 $<\mathcal{TJJ}>$

The following renormalized correlator $<\mathcal{TJJ}>$, respects diffeomorphisms, Lorentz symmetry, R-symmetry and reproduces the standard trace anomaly of (B.1.3).

$$<\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\rm ren}=<\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> -\frac{2i\pi^2}{(2\pi)^4}A^{\nu\xi\kappa\lambda}_{TJJ}(p_1,p_3,p_4) \quad (7.1.13)$$

Due to the PV masses, there exist potential anomalous terms in the R-symmetry and conformal symmetry Ward identities of the regulated correlator $<\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>$. On the contrary, the PV regulator we are using respects diffeomorphisms and Lorentz symmetry, which means that $<\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>$ manifestly satisfies the Ward identities associated with these two symmetries. Thus, the term $A^{\nu\xi\kappa\lambda}_{TJJ}(p_1,p_3,p_4)$ we use to renormalize $<\mathcal{TJJ}>$, respects on its own Lorentz symmetry, i.e.

$$A^{[\nu,\xi]\kappa\lambda}_{TJJ}(p_1,p_3,p_4)=0 \quad (7.1.14)$$

and diffeomorphisms. The latter is satisfied since $A^{\nu\xi\kappa\lambda}_{TJJ}(p_1,p_3,p_4)$ can arise from the following covariant counterterm

Counterterm

$$I_{TJJ}=e\left(\frac{2i\pi^2}{(2\pi)^4}\frac{i}{54}A^{\rho}\nabla^{\xi}\nabla_{\xi}A_{\rho}+\frac{2i\pi^2}{(2\pi)^4}\frac{2i}{9}\log 2\, m_1^2 A^{\rho}A_{\rho}-\frac{2i\pi^2}{(2\pi)^4}\frac{i}{108}g^{\alpha\beta}R_{\alpha\beta}A_{\rho}A^{\rho}\right), \quad (7.1.15)$$

where

$$-\frac{2i\pi^2}{(2\pi)^4}A^{\nu\xi\kappa\lambda}_{TJJ}(x,y,z)=i^2\frac{\delta}{\delta A_{\lambda}(z)}\frac{\delta}{\delta A_{\kappa}(y)}\frac{\delta}{\delta e_{\nu\xi}(x)}I_{TJJ}. \quad (7.1.16)$$

Note here that $A^{\nu\xi\kappa\lambda}_{TJJ}(p_1,p_3,p_4)$ is a local polynomial in the external momenta, which was initially derived after the loop computation that examined all symmetry identities associated with $<\mathcal{TJJ}>$. The exact expression of $A^{\nu\xi\kappa\lambda}_{TJJ}(p_1,p_3,p_4)$ is complicated so we just state its real space version $A^{\nu\xi\kappa\lambda}_{TJJ}(x,y,z)$, which is given by the above equation. Notice that the last term of (7.1.15) does not contribute to the flat space correlator $<\mathcal{JJ}>$, so (7.1.15) has the same contribution to this correlator as the counterterm (7.1.5). Thus, (7.1.15) can be used to renormalize at the same time $<\mathcal{TJJ}>$ and $<\mathcal{JJ}>$. Taking into account the difference in the normalization of the gauge fields, the same counterterm with (7.1.15) was found in [105], where the trace Ward identity of the Weyl fermion was examined using also a PV regulator.

Now let us take a closer look at the Ward identities that are classically violated by the PV regulator, namely the R-symmetry and conformal symmetry identities.

R-symmetry

Taking into account the renormalized correlators, the R-symmetry Ward identity which is manifestly satisfied in the regulated theory is the following

$$\begin{aligned}
&p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{T}}^{\nu\xi}(p_1)>_{\text{ren}} + \widetilde{D}^{\nu\xi\lambda}_{3R} \\
&= -i <\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{T}}^{\nu\xi}(p_1)> - \eta^{\nu\xi} <\mathcal{B}_R(p_{13})\widetilde{\mathcal{J}}^{\lambda}(p_4)> \\
&- (\eta^{\nu\lambda}\eta^{\sigma\xi} + \eta^{\nu\sigma}\eta^{\lambda\xi} - \eta^{\nu\xi}\eta^{\lambda\sigma}) <\widetilde{\mathcal{J}}_{\sigma}(p_{14})\mathcal{B}_R(p_3)> + \frac{i}{6}\eta^{\nu\lambda} <\widetilde{s}^{\xi}_{(2|1)}(p_{14})\mathcal{B}_R(p_3)> \\
&- \frac{2i\pi^2}{(2\pi)^4} p_{3\kappa} A^{\nu\xi\kappa\lambda}_{TJJ}(p_1, p_3, p_4).
\end{aligned} \tag{7.1.17}$$

Comparing with the classical R-symmetry identity (4.2.4), we see that the above rhs comprises the potential R-anomaly of $<\mathcal{T}\mathcal{J}\mathcal{J}>$. The black correlators are the breaking terms in the R-symmetry path integral identity of $<\mathcal{T}\mathcal{J}\mathcal{J}>$. In particular, the 3-point function $<\mathcal{B}_R\mathcal{J}\mathcal{T}>$ comes from the non conserved regulated R-current (6.2.6), while the black $<\mathcal{B}_R\mathcal{J}>$ is a consequence of the modified R-symmetry variation of the stress tensor in the regulated theory (5.2.1). The coloured correlators, are the breaking terms in the R-symmetry path integral identities of the corresponding coloured seagull correlators of (4.2.4). $A^{\nu\xi\kappa\lambda}_{TJJ}(p_1, p_3, p_4)$ comes from the renormalization of $<\mathcal{T}\mathcal{J}\mathcal{J}>$.

In the large PV mass limit we find that the above rhs vanishes, thus the R-symmetry identity is satisfied

$$p_{3\kappa} <\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)\widetilde{\mathcal{T}}^{\nu\xi}(p_1)>_{\text{ren}} + \widetilde{D}^{\nu\xi\lambda}_{3R} = 0. \tag{7.1.18}$$

Conformal symmetry

Similarly, the conformal symmetry identity of $<\mathcal{T}\mathcal{J}\mathcal{J}>$ in the presence of the PV regulator is given by

$$\begin{aligned}
&<\widetilde{\mathcal{T}}^{\nu}_{\nu}(p_1)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}} + \widetilde{D}^{\kappa\lambda}_{3D} \\
&= <\mathcal{B}_W(p_1)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> + \frac{8i}{9}\eta^{\kappa\lambda} <\mathcal{B}_W(p_1)\widetilde{s}_{(1|0)}(p_{34})> - \frac{2i\pi^2}{(2\pi)^4}\eta_{\nu\xi} A^{\nu\xi\kappa\lambda}_{TJJ}(p_1, p_3, p_4).
\end{aligned} \tag{7.1.19}$$

Comparing with (4.2.6) we find that the whole rhs is the potential conformal anomaly. $\eta_{\nu\xi} A^{\nu\xi\kappa\lambda}_{TJJ}(p_1, p_3, p_4)$ comes from the renormalization of $<\mathcal{T}\mathcal{J}\mathcal{J}>$, while the black and blue correlators of the rhs are a consequence of the breaking of scale invariance (6.2.6) in the Ward identities of $<\mathcal{T}\mathcal{J}\mathcal{J}>$ and the blue $<\mathcal{T}s_{(1|0)}>$ of (4.2.6) respectively.

In the large PV mass limit we find that

$$\begin{aligned}&<\mathcal{B}_W(p_1)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)> +\frac{8i}{9}\eta^{\kappa\lambda} <\mathcal{B}_W(p_1)\widetilde{s}_{(1|0)}(p_{34})> -\frac{2i\pi^2}{(2\pi)^4}\eta_{\nu\xi}A^{\nu\xi\kappa\lambda}_{TJJ}(p_1,p_3,p_4)\\&=\frac{1}{36\pi^2}\left(p_3^\lambda p_4^\kappa - p_3\cdot p_4\eta^{\kappa\lambda}\right), \qquad (7.1.20)\end{aligned}$$

which exactly confirms the conformal anomaly of (B.1.3) for $<\mathcal{T}\mathcal{J}\mathcal{J}>$.

7.2 Fermionic correlators

In this section we examine correlation functions that include fermionic operators. We compute the symmetry identities of $<\mathcal{Q}\bar{\mathcal{Q}}>$, $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ and $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ in the regulated theory and confirm the anomalies (B.1.3) that were derived through the WZ consistency conditions. In particular, we find that $<\mathcal{Q}\bar{\mathcal{Q}}>$ is free of anomalies, $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ respects Q-supersymmetry and R-symmetry but is anomalous in S-supersymmetry, while $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ has Q- and S-supersymmetry anomalies but no R-symmetry anomaly.

7.2.1 $<\mathcal{Q}\bar{\mathcal{Q}}>$

In the regulated theory, the classical Q- and S-supersymmetry Ward identities (4.2.3) for the 2-point function $<\mathcal{Q}\bar{\mathcal{Q}}>$ become

$$p_{1\mu} <\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>= 0\,, \qquad \gamma_\mu <\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>=<\mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>\,. \qquad (7.2.1)$$

Q-supersymmetry is preserved, while S-supersymmetry is broken by the PV masses (6.2.6).

In the large PV mass limit we find

$$<\mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>= -\frac{2i\pi^2}{(2\pi)^4}\gamma_\mu\gamma_\sigma\gamma_5\epsilon^{\nu\rho\mu\sigma}p_{2\rho}\left(\frac{1}{6}\,m_1^2\log 2 - \frac{p_2^2}{72}\right). \qquad (7.2.2)$$

After renormalizing the 2-point function as follows

$$<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>_{\text{ren}}=<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)> +\frac{2i\pi^2}{(2\pi)^4}\gamma_\sigma\gamma_5\epsilon^{\nu\rho\mu\sigma}p_{2\rho}\left(\frac{1}{6}\,m_1^2\log 2\, - \frac{p_2^2}{72}\right), \qquad (7.2.3)$$

we get that

$$p_{1\mu} <\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>_{\text{ren}}= 0\,, \qquad \gamma_\mu <\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>_{\text{ren}}= 0. \qquad (7.2.4)$$

The last term of (7.2.3) that we use to remove the S-supersymmetry anomaly, is zero when contracted with $p_{1\mu}$ or $p_{2\nu}$, thus it does not spoil the Q-supersymmetry identity. The 2-point function $<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>$ is invariant under charge conjugation and the exchange of $\mu \leftrightarrow \nu$ and $p_1 \leftrightarrow p_2$. Charge conjugation is defined as $C(...)^T C^{-1}$, where C is the charge conjugation matrix and T denotes the transpose matrix. So we have that

$$C(<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)>)^T C^{-1} =<\widetilde{\mathcal{Q}}^{\nu}(p_2)\bar{\widetilde{\mathcal{Q}}}^{\mu}(p_1)> . \quad (7.2.5)$$

The term we used in (7.2.3) for renormalizing $<\mathcal{Q}\bar{\mathcal{Q}}>$ also respects this charge conjugation symmetry. This is important so that the renormalized 2-point function is free of anomalies also when contracted with γ_ν, i.e. in the S-supersymmetry identity related with $\bar{\mathcal{Q}}^\nu$. Below is the counterterm that we need to add to the action.

Counterterm

$$I_{QQ} = \left(-\frac{2i\pi^2}{(2\pi)^4}\frac{1}{144}\epsilon^{\alpha\beta\rho\sigma}\bar{\psi}^{\alpha}\gamma_\sigma\gamma_5\mathscr{D}_\rho\mathscr{D}^\xi\mathscr{D}_\xi\psi_\beta - \frac{2i\pi^2}{(2\pi)^4}\epsilon^{\beta\rho\alpha\sigma}\frac{1}{12}m_1^2\log 2\,\bar{\psi}_\alpha\gamma_\sigma\gamma_5\mathscr{D}_\rho\psi_\beta\right) \quad (7.2.6)$$

Like (7.1.5), we use here the covariant derivative $\mathscr{D}_\xi$ instead of a partial derivative, in order to not break diffeomorphisms.

7.2.2 $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$

If we renormalize the regulated 3-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ as follows, it satisfies the classical Q-supersymmetry and R-symmetry Ward identities (4.2.9), (4.2.12), and the S-supersymmetry anomaly of (B.1.3) is reproduced.

$$<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\rm ren}=<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> -\frac{2i\pi^2}{(2\pi)^4}A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3). \quad (7.2.7)$$

$A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)$ must respect the same charge conjugation symmetry as $<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>$, thus we have the following identity

$$C(A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3))^T C^{-1} = A^{\nu\mu\kappa}_{QQJ}(p_2,p_1,p_3). \quad (7.2.8)$$

In the next subsections we examine the Q- and S-supersymmetry identities of $<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>$ related to the supercurrent $\widetilde{\mathcal{Q}}^\mu$. The identity (7.2.8) is important so that the same analysis is still valid in the supersymmetry identities related to $\bar{\widetilde{\mathcal{Q}}}^\nu$. The counterterm that we need to add to the action to get the above renormalization is the following

Counterterm

$$\begin{aligned}I_{QQJ} = &\frac{2i\pi^2}{(2\pi)^4}\frac{2}{108}A^\rho\left(\partial^{[\beta}\bar{\psi}^{\alpha]}\gamma_\alpha\gamma_5\partial_\rho\psi_\beta + \partial_{[\rho}\bar{\psi}^{\alpha]}\gamma_\alpha\gamma_5\partial_\beta\psi^\beta + \partial^{[\alpha}\bar{\psi}^{\beta]}\gamma_\rho\gamma_5\partial_\beta\psi_\alpha\right.\\
&\left.+\partial^{[\alpha}\bar{\psi}^{\beta]}\gamma_\alpha\gamma_5\partial_\beta\psi_\rho + 2\partial_\beta\partial^{[\rho}\bar{\psi}^{\alpha]}\gamma_\alpha\gamma_5\psi_\beta\right)\\
&+\frac{2i\pi^2}{(2\pi)^4}\frac{i}{108}A_\rho\left(\frac{7}{4}\epsilon^{\rho\xi\alpha\beta}\partial_\alpha\bar{\psi}^\sigma\gamma_\xi\partial_\beta\psi_\sigma + \frac{1}{2}\epsilon^{\rho\sigma\xi\alpha}\partial_\alpha\bar{\psi}_\sigma\gamma_\xi\partial_\beta\psi^\beta + \frac{7}{2}\epsilon^{\rho\sigma\xi\alpha}\partial_\beta\bar{\psi}_\sigma\gamma_\xi\partial_\alpha\psi^\beta\right.\end{aligned}$$

$$+\frac{7}{4}\epsilon^{\rho\xi\alpha\beta}\partial_\sigma\bar{\psi}_\alpha\gamma_\xi\partial^\sigma\psi_\beta+\epsilon^{\rho\sigma\xi\alpha}\partial_\beta\partial_\alpha\bar{\psi}_\sigma\gamma_\xi\psi^\beta$$
$$-\frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}\partial_\alpha\bar{\psi}_\sigma\gamma_\xi\partial_\beta\psi^\xi-\frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}\partial_\beta\bar{\psi}_\sigma\gamma_\xi\partial^\xi\psi_\alpha\Big), \tag{7.2.9}$$

where we have that

$$-\frac{2i\pi^2}{(2\pi)^4}A^{\mu\nu\kappa}_{QQJ}(x_1,x_2,x_3)=i^2\frac{\delta}{\delta A_\kappa(x_3)}\frac{\delta}{\delta\bar{\psi}_\mu(x_1)}I_{QQJ}\frac{\overleftarrow{\delta}}{\delta\psi_\nu(x_2)}. \tag{7.2.10}$$

$A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)$ is a local polynomial in the external momenta and it was first computed through the one loop calculation. Since it has a quite lengthy and complicated form, we only write here its real space version $A^{\mu\nu\kappa}_{QQJ}(x_1,x_2,x_3)$ which is given by the above equation.

Now let us examine each one of the three Ward identities associated with $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$.

Q-supersymmetry

The PV regulator modifies the classical Q-supersymmetry identity (4.2.9) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ as follows

$$p_{1\mu}<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)>_{\rm ren}+\widetilde{C}^{\nu\kappa}_{3Q}$$
$$=ip_{2\mu}B^{\nu\mu\sigma}<\widetilde{\mathcal{J}}_\sigma(p_{12})\widetilde{\mathcal{J}}^\kappa(p_3)>_{\rm ren}-i\gamma_5<\widetilde{\mathcal{Q}}^\kappa(p_{13})\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>_{\rm ren}$$
$$+\frac{2i\pi^2}{(2\pi)^4}\Big(A^{\nu\kappa}_{3Q}(p_1,p_2,p_3)-p_{1\mu}A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)\Big)-\frac{i}{3}\gamma^\kappa\gamma_5<\mathcal{B}_S(p_{13})\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)> \tag{7.2.11}$$

where we used that $<\mathcal{T}\mathcal{J}>=0$. The last term of (7.2.11) is the new breaking term that we get since the Q-supersymmetry variation of the R-current is changed in the regulated theory (5.2.2). The above identity is satisfied by construction at the regulated level. Notice however that in (7.2.11) we have used the renormalized, and not the regulated 2-point functions. The reason for that is that we have already fixed our theory at the level of the 2-point functions by adding courterterms that restore the symmetries broken by the regulator. These counterterms contribute to the above identity, in particular, $A^{\nu\kappa}_{3Q}$ is the consequence of the renormalization of $<\mathcal{Q}\bar{\mathcal{Q}}>$ (7.2.6) and $<\mathcal{J}\mathcal{J}>$ (7.1.5). $p_{1\mu}A^{\mu\nu\kappa}_{QQJ}(q_1,q_2,q_3)$ comes from the renormalization of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$.

The last line of (7.2.11) is the potential Q-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. The 2-point function $<\mathcal{B}_S\bar{\mathcal{Q}}>$ was computed in (7.2.2). One just needs to make the appropriate changes in the external momenta and use the result in (7.2.11). In the limit where the PV regulator is removed we find that

$$\frac{2i\pi^2}{(2\pi)^4}\Big(A^{\nu\kappa}_{3Q}(p_1,p_2,p_3)-p_{1\mu}A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)\Big)-\frac{i}{3}\gamma^\kappa\gamma_5<\mathcal{B}_S(p_{13})\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>=0, \tag{7.2.12}$$

thus, there is no Q-supersymmetry anomaly in $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$.

R-symmetry

Taking into account the renormalized correlators, the R-symmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ that is manifestly satisfied in the regulated theory is given by

$$\begin{aligned}
&p_{3\kappa} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}} + \widetilde{C}^{\mu\nu}_{3R} - i\gamma_5 <\widetilde{\mathcal{Q}}^{\mu}(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)>_{\text{ren}} \\
&- i <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_{23})>_{\text{ren}} \gamma_5 = -i <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\mathcal{B}_R(p_3)> + \frac{3}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi <\widetilde{\mathcal{J}}_\sigma(p_{12})\mathcal{B}_R(p_3)> \\
&+ \frac{1}{8}\eta^{\mu\nu}\gamma_\sigma\gamma_5 <\widetilde{s}^{\sigma}_{(2|1)}(p_{12})\mathcal{B}_R(p_3)> + \frac{i}{3}\gamma^{\mu}\gamma_5 <\mathcal{B}_S(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)> + \frac{i}{3} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\mathcal{B}}_S(p_{23})> \gamma^{\nu}\gamma_5 \\
&+ \frac{2i\pi^2}{(2\pi)^4}\Big(A^{\mu\nu}_{3R}(p_1,p_2,p_3) - p_{3\kappa}A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)\Big). \qquad (7.2.13)
\end{aligned}$$

Comparing this identity with (4.2.12), we see that the above rhs comprises the potential R-symmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$.

The 3-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{B}_R>$ is there due to the fact that the R-current is no longer conserved in the presence of the PV masses (6.2.6). The 2-point functions that contain $\mathcal{B}_S$ arise because the R-symmetry variation of the supercurrent is modified in the regulated theory (5.2.1), compared to the massless model (4.1.11). Finally the red and blue terms are the contribution to the potential R-anomaly of the corresponding coloured seagull correlators of (4.2.12). Recall that (4.2.12) is the R-symmetry identity one gets from conformal supergravity. This identity is a linear combination of path integral Ward identities of the 3-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ and other seagull correlators. The presence of the PV mass classically violates the R-symmetry identities of the seagull correlators too. $A^{\mu\nu}_{3R}$ is the contribution of the renormalized $<\mathcal{Q}\bar{\mathcal{Q}}>$ correlators (7.2.6), while $A^{\mu\nu\kappa}_{QQJ}$ is there because of the renormalized 3-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$.

In the large PV mass limit we find that the whole rhs vanishes, hence R-symmetry is respected by $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$, i.e.

$$\begin{aligned}
&p_{3\kappa} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}} + \widetilde{C}^{\mu\nu}_{3R} \\
&- i\gamma_5 <\widetilde{\mathcal{Q}}^{\mu}(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)>_{\text{ren}} - i <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_{23})>_{\text{ren}} \gamma_5 = 0. \qquad (7.2.14)
\end{aligned}$$

S-supersymmetry

Following the same procedure, we find the S-supersymmetry Ward identity of the regulated theory

$$
\begin{aligned}
&-i\gamma_\mu < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3) >_{\text{ren}} + \widetilde{C}^{\nu\kappa}_{3S} + \frac{3i}{4}\gamma_5 < \widetilde{\mathcal{J}}^\nu(p_{12})\widetilde{\mathcal{J}}^\kappa(p_3) >_{\text{ren}} = \\
&-i < \mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3) > + \frac{\sqrt{2}}{3}\eta^{\nu\kappa} < \mathcal{B}_S(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23}) > \\
&+ \frac{2i\pi^2}{(2\pi)^4}\left(A^{\nu\kappa}_{3S}(p_1,p_2,p_3) + i\gamma_\mu A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)\right).
\end{aligned}
\tag{7.2.15}
$$

The rhs of this identity is the potential S-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. The second line is a consequence of the fact that the gamma trace of the supercurrent is not zero in the regulated theory (6.2.6). In particular the 3-point function $<\mathcal{B}_S\bar{\mathcal{Q}}\mathcal{J}>$ is the breaking term of the S-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$, while the blue correlator is the breaking term of the S-supersymmetry identity of the blue 2-point seagull correlator in (4.2.14). $A^{\nu\kappa}_{3S}$ and $A^{\mu\nu\kappa}_{QQJ}$ are the contributions of the renormalized $<\mathcal{J}\mathcal{J}>$ and $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ correlators.

In the large PV mass limit, i.e. for $m_1 \to \infty$ we find that

$$
\begin{aligned}
&-i < \mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3) > + \frac{\sqrt{2}}{3}\eta^{\nu\kappa} < \mathcal{B}_S(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23}) > \\
&+ \frac{2i\pi^2}{(2\pi)^4}\left(A^{\nu\kappa}_{3S}(p_1,p_2,p_3) + i\gamma_\mu A^{\mu\nu\kappa}_{QQJ}(p_1,p_2,p_3)\right) \\
&= \frac{1}{576\pi^2}\Big(2i\epsilon^{\kappa\nu\alpha\beta}p_{1\alpha}p_{2\beta} + 4\gamma_5\left(p_2^\kappa p_2^\nu + p_2^\kappa p_1^\nu - \eta^{\kappa\nu}p_1\cdot p_2 - \eta^{\kappa\nu}p_2^2\right) \\
&+ \gamma_{\alpha\beta}\Big(i\eta^{\kappa\nu}\epsilon^{\alpha\beta\rho\sigma}p_{1\rho}p_{2\sigma} - ip_1^\nu\epsilon^{\kappa\alpha\beta\rho}p_{2\rho} + ip_2^\kappa\epsilon^{\nu\alpha\beta\rho}p_{1\rho} + ip_1\cdot p_2\epsilon^{\kappa\nu\alpha\beta} \\
&+ ip_2^2\epsilon^{\kappa\nu\alpha\beta} - ip_2^\nu\epsilon^{\kappa\alpha\beta\rho}p_{2\rho} + ip_2^\kappa\epsilon^{\nu\alpha\beta\rho}p_{2\rho}\Big)\Big),
\end{aligned}
\tag{7.2.16}
$$

which exactly confirms the S-supersymmetry anomaly (B.1.3) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ in the massless and classically conformal WZ model.

7.2.3 $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$

The renormalized 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ that reproduces the Q- and S-supersymmetry anomalies (B.1.3) and at the same time is free of R-symmetry anomalies is given by

$$
\begin{aligned}
&< \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4) >_{\text{ren}} \\
&= < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4) > - \frac{2i\pi^2}{(2\pi)^4}A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4).
\end{aligned}
\tag{7.2.17}
$$

$A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4)$ is a local polynomial in the external momenta and it was derived by the loop computation. It respects the same charge conjugation symmetry with

$<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>$ and is invariant under the exchange of $\kappa \leftrightarrow \lambda$ and $p_3 \leftrightarrow p_4$, like the 4-point function. We have that

$$C(A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4))^T C^{-1} = A^{\nu\mu\kappa\lambda}_{QQJJ}(p_2,p_1,p_3,p_4) \tag{7.2.18}$$

and

$$A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4) = A^{\mu\nu\lambda\kappa}_{QQJJ}(p_1,p_2,p_4,p_3). \tag{7.2.19}$$

The counterterm we need in order to get the above renormalized 4-point correlator is the following

Counterterm

$$\begin{aligned}
I_{QQJJ} = (-i)\Bigg(&-\frac{2i\pi^2}{(2\pi)^4}\Bigg(\frac{1}{162}A_\alpha A^\beta \partial_\xi\bar{\psi}^\alpha\gamma^\xi\psi_\beta - \frac{5}{648}A_\beta A^\beta\partial_\xi\bar{\psi}^\alpha\gamma^\xi\psi_\alpha + \frac{1}{108}\partial_\alpha A^\xi A^\beta\bar{\psi}^\alpha\gamma_\xi\psi_\beta\\
&+\frac{1}{162}A_\alpha A^\beta\bar{\psi}^\alpha\gamma^\xi\partial_\beta\psi_\xi - \frac{1}{216}A_\beta A^\beta\partial_\alpha\bar{\psi}^\alpha\gamma^\xi\psi_\xi - \frac{5}{648}A_\beta A^\beta\bar{\psi}^\alpha\gamma^\xi\partial_\alpha\psi_\xi\\
&-\frac{1}{108}\partial_\xi A_\alpha A_\beta\bar{\psi}^\alpha\gamma^\xi\psi^\beta - \frac{1}{54}A^\xi A^\alpha\partial_\alpha\bar{\psi}^\beta\gamma_\xi\psi_\beta + \frac{1}{54}A^\xi A^\beta\partial_\alpha\bar{\psi}^\alpha\gamma_\xi\psi_\beta + \frac{1}{36}\partial_\alpha A^\beta A^\xi\bar{\psi}^\alpha\gamma_\xi\psi_\beta\Bigg)\\
&-\frac{2i\pi^2}{(2\pi)^4}\Bigg(\frac{5i}{1296}A_\xi A_\sigma\epsilon^{\sigma\alpha\beta\tau}\bar{\psi}_\alpha\gamma^\xi\gamma_5\partial_\tau\psi_\beta + \frac{i}{216}\partial_\beta A_\sigma A_\tau\epsilon^{\sigma\alpha\xi\tau}\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi^\beta\\
&+\frac{23i}{1296}A^\sigma A_\tau\epsilon^{\tau\alpha\beta\xi}\partial_\sigma\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi_\beta + \frac{i}{432}\partial_\beta A_\sigma A_\tau\epsilon^{\tau\sigma\xi\beta}\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi^\alpha\\
&-\frac{5i}{1296}A^\sigma A_\tau\epsilon^{\tau\alpha\xi\beta}\partial_\beta\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi_\sigma - \frac{23i}{1296}A_\tau A_\sigma\epsilon^{\tau\alpha\xi\beta}\bar{\psi}_\alpha\gamma_\xi\gamma_5\partial_\beta\psi^\sigma\Bigg)\Bigg),
\end{aligned} \tag{7.2.20}$$

where we have that

$$-\frac{2i\pi^2}{(2\pi)^4}A^{\mu\nu\kappa\lambda}_{QQJJ}(x_1,x_2,x_3,x_4) = i^3\frac{\delta}{\delta A_\lambda(x_4)}\frac{\delta}{\delta A_\kappa(x_3)}\frac{\delta}{\delta\bar{\psi}_\mu(x_1)}I_{QQJJ}\frac{\overleftarrow{\delta}}{\delta\psi_\nu(x_2)}. \tag{7.2.21}$$

$A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4)$ is the momentum space version of $A^{\mu\nu\kappa\lambda}_{QQJJ}(x_1,x_2,x_3,x_4)$. Next we examine the symmetry identities associated with $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. In particular, we analyse the R-symmetry and the supersymmetry identities that correspond to the currents $\widetilde{\mathcal{J}}^{\kappa}$ and $\widetilde{\mathcal{Q}}^{\mu}$ respectively.

Q-Supersymmetry

In the regulated theory the classical Q-supersymmetry Ward identity (4.2.16) is modified as follows

$$\begin{aligned}
&p_{1\mu}<\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}} + \widetilde{C}^{\nu\kappa\lambda}_{4Q}\\
&= ip_{2\mu}B^{\nu\mu\sigma}<\widetilde{\mathcal{J}}_{\sigma}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> - \frac{\gamma_\xi}{2}<\widetilde{\mathcal{T}}^{\nu\xi}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}\\
&- i\gamma_5<\widetilde{\mathcal{Q}}^{\kappa}(p_{13})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}} - i\gamma_5<\widetilde{\mathcal{Q}}^{\lambda}(p_{14})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}}\\
&+ \gamma_5 B^{\nu\kappa\sigma}<\widetilde{\mathcal{J}}_{\sigma}(p_{123})\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}} + \gamma_5 B^{\nu\lambda\sigma}<\widetilde{\mathcal{J}}_{\sigma}(p_{124})\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{i}{3}\gamma^{\kappa}\gamma_5 <\mathcal{B}_S(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)> -\frac{i}{3}\gamma^{\lambda}\gamma_5 <\mathcal{B}_S(p_{14})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> \\
&+\eta^{\nu\lambda}\frac{\sqrt{2}}{9}\gamma^{\kappa}\gamma_5 <\mathcal{B}_S(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})> +\eta^{\nu\kappa}\frac{\sqrt{2}}{9}\gamma^{\lambda}\gamma_5 <\mathcal{B}_S(p_{14})\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\
&+\frac{2i\pi^2}{(2\pi)^4}\left(A^{\nu\kappa\lambda}_{4Q}(p_1,p_2,p_3,p_4) - p_{1\mu}A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4)\right). \qquad (7.2.22)
\end{aligned}
$$

The last three lines of the above rhs form the potential Q-supersymmetry anomaly. The presence of the correlators that involve the operator $\mathcal{B}_S$ is a consequence of the modified Q-supersymmetry variation of the R-current in the regulated theory (5.2.2). In particular, the 3-point correlators $<\mathcal{B}_S\bar{\mathcal{Q}}\mathcal{J}>$ comprise the breaking terms of the path integral Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, while the coloured 2-point functions are the breaking terms in the path integral Q-supersymmetry Ward identities of the corresponding coloured seagull correlators of (4.2.16). $A^{\nu\kappa\lambda}_{4Q}$ is the contribution of the renormalized lower order correlators of the rhs. We have already computed the 3-point function $<\mathcal{B}_S\bar{\mathcal{Q}}\mathcal{J}>$ in the regulated S-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. For large PV masses we confirm the Q-supersymmetry anomaly (B.1.3) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, i.e.

$$
\begin{aligned}
&-\frac{i}{3}\gamma^{\kappa}\gamma_5 <\mathcal{B}_S(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)> -\frac{i}{3}\gamma^{\lambda}\gamma_5 <\mathcal{B}_S(p_{14})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> \\
&+\eta^{\nu\lambda}\frac{\sqrt{2}}{9}\gamma^{\kappa}\gamma_5 <\mathcal{B}_S(p_{13})\bar{s}_{(3|\frac{1}{2})}(p_{24})> +\eta^{\nu\kappa}\frac{\sqrt{2}}{9}\gamma^{\lambda}\gamma_5 <\mathcal{B}_S(p_{14})\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\
&+\frac{2i\pi^2}{(2\pi)^4}\left(A^{\nu\kappa\lambda}_{4Q}(p_1,p_2,p_3,p_4) - p_{1\mu}A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4)\right) \\
&=\frac{2i\pi^2}{(2\pi)^4}\frac{-2i}{81}\epsilon^{\rho\kappa\lambda}{}_{\sigma}(p_{4\rho}-p_{3\rho})p_{2\mu}\left(-\frac{1}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}+\frac{i}{4}\gamma_5\gamma^{\mu}\eta^{\nu\sigma}-\frac{i}{4}\gamma_5\gamma^{\nu}\eta^{\mu\sigma}\right). \qquad (7.2.23)
\end{aligned}
$$

R-symmetry

The R-symmetry identity at the regulated level is given by

$$
\begin{aligned}
&p_{3\kappa} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}} +\widetilde{C}^{\mu\nu\lambda}_{4R} \\
&-i\gamma_5 <\widetilde{\mathcal{Q}}^{\mu}(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}} -i <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_{23})\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}\gamma_5 = \\
&-i <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\frac{2i\pi^2}{(2\pi)^4}\left(A^{\mu\nu\lambda}_{4R}(p_1,p_2,p_3,p_4) - p_{3\kappa}A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4)\right) \\
&+\frac{i}{3}\gamma^{\mu}\gamma_5 <\mathcal{B}_S(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\frac{i}{3} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\mathcal{B}}_S(p_{23})\widetilde{\mathcal{J}}^{\lambda}(p_4)>\gamma^{\nu}\gamma_5 \\
&+\frac{3}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi} <\widetilde{\mathcal{J}}_{\sigma}(p_{12})\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\frac{1}{8}\eta^{\mu\nu}\gamma_{\sigma}\gamma_5 <\widetilde{s}^{\sigma}_{(2|1)}(p_{12})\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> \\
&+\eta^{\mu\lambda}\frac{\sqrt{2}}{3} <\widetilde{s}_{(3|\frac{1}{2})}(p_{14})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\mathcal{B}_R(p_3)> -\eta^{\mu\lambda}\frac{\sqrt{2}}{9} <\widetilde{s}_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{B}}_S(p_{23})>\gamma^{\nu}\gamma_5 \\
&+\eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\mathcal{B}_R(p_3)> -\eta^{\nu\lambda}\frac{\sqrt{2}}{9}\gamma^{\mu}\gamma_5 <\mathcal{B}_S(p_{13})\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})> \qquad (7.2.24)
\end{aligned}
$$

The whole rhs is the potential R-symmetry anomaly. Correlators that involve $\mathcal{B}_R$ are a consequence of the non conserved regulated R-current (6.2.6), while correlators that involve $\mathcal{B}_S$ arise because the R-symmetry variation of the supercurrent is modified in the regulated theory. The black correlators of the rhs, form the breaking terms of the path integral identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, while the coloured correlators are the breaking terms of the path integral R-symmetry identities of the corresponding coloured seagull correlators of (4.2.18). $A^{\mu\nu\lambda}_{4R}$ is the contribution of the renormalized correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. In the large PV mass limit we get that the rhs vanishes, hence the R-symmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ is satisfied

$$\begin{aligned}&p_{3\kappa} <\tilde{\mathcal{Q}}^\mu(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>_{\rm ren} + \tilde{C}^{\mu\nu\lambda}_{4R}\\ &- i\gamma_5 <\tilde{\mathcal{Q}}^\mu(p_{13})\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\lambda(p_4)>_{\rm ren} - i <\tilde{\mathcal{Q}}^\mu(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_{23})\tilde{\mathcal{J}}^\lambda(p_4)>_{\rm ren}\gamma_5 = 0. \qquad (7.2.25)\end{aligned}$$

S-supersymmetry

The S-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ that is manifestly satisfied in the regulated theory is given by

$$\begin{aligned}&- i\gamma_\mu <\tilde{\mathcal{Q}}^\mu(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>_{\rm ren} + \tilde{C}^{\nu\kappa\lambda}_{4S} + \frac{3i}{4}\gamma_5 <\tilde{\mathcal{J}}^\nu(p_{12})\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>\\ &= \frac{2i\pi^2}{(2\pi)^4} i\gamma_\mu A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4) - i <\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>\\ &+ \eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23})\tilde{\mathcal{J}}^\lambda(p_4)> + \eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\tilde{\mathcal{J}}^\kappa(p_3)>\\ &+ \frac{8}{9}\eta^{\kappa\lambda} <\tilde{s}_{(1|0)}(p_{34})\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)> . \qquad (7.2.26)\end{aligned}$$

The rhs is the potential S-supersymmetry anomaly. Correlators that contain $\mathcal{B}_S$ are there, since the gamma trace of the supercurrent is no longer zero (6.2.6). The coloured correlators are the breaking terms in the S-supersymmetry identities of the coloured seagull correlators of (4.2.20). In the limit where the regulator vanishes and the PV fields decouple from the original massless model, we find that

$$\begin{aligned}&\frac{2i\pi^2}{(2\pi)^4} i\gamma_\mu A^{\mu\nu\kappa\lambda}_{QQJJ}(p_1,p_2,p_3,p_4) - i <\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>\\ &+ \eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23})\tilde{\mathcal{J}}^\lambda(p_4)> + \eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\tilde{\mathcal{J}}^\kappa(p_3)>\\ &+ \frac{8}{9}\eta^{\kappa\lambda} <\tilde{s}_{(1|0)}(p_{34})\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)>\\ &= \frac{1}{288\pi^2}\left(-i\gamma^{\nu\alpha}p_{3\alpha}\eta^{\kappa\lambda} + i\gamma^{\lambda\alpha}p_{3\alpha}\eta^{\kappa\nu} + i\gamma^{\kappa\lambda}p_3^\nu - i\gamma^{\kappa\nu}p_3^\lambda + 2i\eta^{\kappa\nu}p_3^\lambda - 2i\eta^{\kappa\lambda}p_3^\nu + \gamma_5\epsilon^{\kappa\lambda\nu\alpha}\frac{p_{3\alpha}}{3}\right)\\ &+ \begin{pmatrix} p_3 & \leftrightarrow & p_4\\ \kappa & \leftrightarrow & \lambda\end{pmatrix}. \qquad (7.2.27)\end{aligned}$$

This results confirms the S-supersymmetry anomaly (B.1.3) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

7.3 Total counterterm

The total counterterm that we need to add in the PV regulated theory to retrieve the symmetry identities and anomalies of conformal supergravity in the correlators we analysed, is the following:

$$I = I_{\text{div}} + I_{\text{fin}}, \tag{7.3.1}$$

where I_{div} denotes the divergent part

$$I_{\text{div}} = \frac{2i\pi^2}{(2\pi)^4}\left(\frac{2i}{9}m_1^2 \log 2\, A^\rho A_\rho - \epsilon^{\beta\rho\alpha\sigma}\frac{1}{12}\, m_1^2 \log 2\, \bar{\psi}_\alpha \gamma_\sigma \gamma_5 \mathscr{D}_\rho \psi_\beta\right) \tag{7.3.2}$$

and I_{fin} denotes the finite part

$$\begin{aligned}
I_{\text{fin}} = &-\frac{2i\pi^2}{(2\pi)^4}\frac{1}{144}\epsilon^{\alpha\beta\rho\sigma}\bar{\psi}^\alpha\gamma_\sigma\gamma_5\mathscr{D}_\rho\mathscr{D}^\xi\mathscr{D}_\xi\psi_\beta + \frac{2i\pi^2}{(2\pi)^4}\frac{i}{54}A^\rho\nabla_\mu\nabla^\mu A_\rho - \frac{2i\pi^2}{(2\pi)^4}\frac{i}{108}g^{\alpha\beta}R_{\alpha\beta}A_\rho A^\rho \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{2}{108}A^\rho\Big(\partial^{[\beta}\bar{\psi}^{\alpha]}\gamma_\alpha\gamma_5\partial_\rho\psi_\beta + \partial_{[\rho}\bar{\psi}^{\alpha]}\gamma_\alpha\gamma_5\partial_\beta\psi^\beta + \partial^{[\alpha}\bar{\psi}^{\beta]}\gamma_\rho\gamma_5\partial_\beta\psi_\alpha \\
&+\partial^{[\alpha}\bar{\psi}^{\beta]}\gamma_\alpha\gamma_5\partial_\beta\psi_\rho + 2\partial_\beta\partial^{[\rho}\bar{\psi}^{\alpha]}\gamma_\alpha\gamma_5\psi_\beta\Big) \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{i}{108}A_\rho\Big(\frac{7}{4}\epsilon^{\rho\xi\alpha\beta}\partial_\alpha\bar{\psi}^\sigma\gamma_\xi\partial_\beta\psi_\sigma + \frac{1}{2}\epsilon^{\rho\sigma\xi\alpha}\partial_\alpha\bar{\psi}_\sigma\gamma_\xi\partial_\beta\psi^\beta + \frac{7}{2}\epsilon^{\rho\sigma\xi\alpha}\partial_\beta\bar{\psi}_\sigma\gamma_\xi\partial_\alpha\psi^\beta \\
&+ \frac{7}{4}\epsilon^{\rho\xi\alpha\beta}\partial_\sigma\bar{\psi}_\alpha\gamma_\xi\partial^\sigma\psi_\beta + \epsilon^{\rho\sigma\xi\alpha}\partial_\beta\partial_\alpha\bar{\psi}_\sigma\gamma_\xi\psi^\beta \\
&-\frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}\partial_\alpha\bar{\psi}_\sigma\gamma_\xi\partial_\beta\psi^\xi - \frac{1}{2}\epsilon^{\rho\sigma\alpha\beta}\partial_\beta\bar{\psi}_\sigma\gamma_\xi\partial^\xi\psi_\alpha\Big) \\
&- i\Bigg(-\frac{2i\pi^2}{(2\pi)^4}\Big(\frac{1}{162}A_\alpha A^\beta\partial_\xi\bar{\psi}^\alpha\gamma^\xi\psi_\beta - \frac{5}{648}A_\beta A^\beta\partial_\xi\bar{\psi}^\alpha\gamma^\xi\psi_\alpha + \frac{1}{108}\partial_\alpha A^\xi A^\beta\bar{\psi}^\alpha\gamma_\xi\psi_\beta \\
&+ \frac{1}{162}A_\alpha A^\beta\bar{\psi}^\alpha\gamma^\xi\partial_\beta\psi_\xi - \frac{1}{216}A_\beta A^\beta\partial_\alpha\bar{\psi}^\alpha\gamma^\xi\psi_\xi - \frac{5}{648}A_\beta A^\beta\bar{\psi}^\alpha\gamma^\xi\partial_\alpha\psi_\xi \\
&- \frac{1}{108}\partial_\xi A_\alpha A_\beta\bar{\psi}^\alpha\gamma^\xi\psi^\beta - \frac{1}{54}A^\xi A^\alpha\partial_\alpha\bar{\psi}^\beta\gamma_\xi\psi_\beta + \frac{1}{54}A^\xi A^\beta\partial_\alpha\bar{\psi}^\alpha\gamma_\xi\psi_\beta + \frac{1}{36}\partial_\alpha A^\beta A^\xi\bar{\psi}^\alpha\gamma_\xi\psi_\beta\Big) \\
&- \frac{2i\pi^2}{(2\pi)^4}\Big(\frac{5i}{1296}A_\xi A_\sigma\epsilon^{\sigma\alpha\beta\tau}\bar{\psi}_\alpha\gamma^\xi\gamma_5\partial_\tau\psi_\beta + \frac{i}{216}\partial_\beta A_\sigma A_\tau\epsilon^{\sigma\alpha\xi\tau}\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi^\beta \\
&+ \frac{23i}{1296}A^\sigma A_\tau\epsilon^{\tau\alpha\beta\xi}\partial_\sigma\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi_\beta + \frac{i}{432}\partial_\beta A_\sigma A_\tau\epsilon^{\tau\sigma\xi\beta}\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi^\alpha \\
&-\frac{5i}{1296}A^\sigma A_\tau\epsilon^{\tau\alpha\xi\beta}\partial_\beta\bar{\psi}_\alpha\gamma_\xi\gamma_5\psi_\sigma - \frac{23i}{1296}A_\tau A_\sigma\epsilon^{\tau\alpha\xi\beta}\bar{\psi}_\alpha\gamma_\xi\gamma_5\partial_\beta\psi^\sigma\Big)\Bigg).
\end{aligned} \tag{7.3.3}$$

Since I was deduced through a loop computation, it is a subset of the complete counterterm that relates the FZ multiplet to the superconformal one. In principle, performing a similar analysis involving (a suitable set of) other correlators one should be able to construct the full counterterm, though this would be tedious in practice. In the counterterm I we use partial derivatives, something that will introduce diffeomorphism anomalies in higher order correlation functions. We can always covariantize the above expression to remove the explicit breaking in diffeomorphisms. This will not affect the correlation functions we analysed in this chapter.

CHAPTER 8

Non anomalous Q+S supersymmetry

Using the PV Lagrangian (6.1.1), we regularized the massless and conformal WZ model (4.1.1). The classical symmetry Ward identities of section (4.2) were modified in the regulated theory and contained extra breaking terms depending on the PV regulator. These terms do not vanish, even in the limit where the PV masses are sent to infinity and the regulator decouples from the original model. By adding the local counterterm (7.3.1), we can restore some of the broken symmetries by the regulator, and reproduce the anomalies (B.1.3) of conformal supergravity that were derived through the WZ consistency conditions. In particular, the counterterm (7.3.1) renders all 2-point functions non anomalous, and it contributes so that $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ has only an S-supersymmetry anomaly, $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ has Q-and S-supersymmetry anomalies, while $<\mathcal{T}\mathcal{J}\mathcal{J}>$ has the standard trace anomaly.

In chapter 4, based on the commutator of two supersymmetry variations (4.4.11) and the presence of the genuine R-symmetry anomaly, we argued that one cannot remove the Q-supersymmetry anomaly of conformal supergravity without breaking diffeomorphisms and/or Lorentz symmetry. Although this is true, there still exists a manifestly non anomalous linear combination of the Q- and S-supersymmetry of conformal supergravity. It is straightforward to find this combination using the results and Ward identities of the previous chapter. As we will see in the appendix G, the non anomalous combination of Q+S supersymmetry we state here, is the supersymmetry of old minimal supergravity.

2- and 3-point functions

At the 2-point function level, according to (7.2.1) there is a non anomalous supersymmetry in the regulated theory, which coincides with Q-supersymmetry of conformal supergravity. At the 3-point function level now, we are considering the Q-supersymmetry Ward identity (7.2.11) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ and the S-supersymmetry identity (7.2.1) of $<\mathcal{Q}\bar{\mathcal{Q}}>$. Note however, that we are considering these only at the regulated level, i.e. we do not renormalize the correlators, thus $A^{\nu\kappa}_{3Q}$ and $A^{\mu\nu\kappa}_{QQJ}$ are absent from (7.2.11). (7.2.11) and (7.2.1) are manifestly satisfied and both of them contain the breaking correlators $<\mathcal{B}_S\bar{\mathcal{Q}}>$. We can combine these two identities in a way that the breaking terms cancel between each other. The manifestly non anomalous (i.e. with no breaking terms that depend on the PV regulator) identity that we get is the following

$$p_{1\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> +\widetilde{C}^{\nu\kappa}_{3Q} = ip_{2\mu}B^{\nu\mu\sigma} <\widetilde{\mathcal{J}}_{\sigma}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)> -i\gamma_5 <\widetilde{\mathcal{Q}}^{\kappa}(p_{13})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)> \\ -\frac{i}{3}\gamma^{\kappa}\gamma_5\gamma_{\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_{13})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)>, \tag{8.0.1}$$

where $\widetilde{C}^{\nu\kappa}_{3Q}$ is the regulated version of $C^{\nu\kappa}_{3Q}$ (4.2.10) and $B^{\nu\mu\sigma}$ is given by (4.2.11).

4-point function

Similarly, at the 4-point function level we can use the Q-supersymmetry identity (7.2.22) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ and the S-supersymmetry identity (7.2.15) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. Again we are considering these identities at the regulated level only. We do not renormalize any correlators. Combining (7.2.22) and (7.2.15) so that their breaking correlators cancel, we find the non anomalous identity

$$\begin{aligned}
&p_{1\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\widetilde{C}^{\nu\kappa\lambda}_{4Q} \\
&= ip_{2\mu}B^{\nu\mu\sigma} <\widetilde{\mathcal{J}}_{\sigma}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> -\frac{\gamma_{\xi}}{2} <\widetilde{\mathcal{T}}^{\nu\xi}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> \\
&- i\gamma_5 <\widetilde{\mathcal{Q}}^{\kappa}(p_{13})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)> -i\gamma_5 <\widetilde{\mathcal{Q}}^{\lambda}(p_{14})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> \\
&+ \gamma_5 B^{\nu\kappa\sigma} <\widetilde{\mathcal{J}}_{\sigma}(p_{123})\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\gamma_5 B^{\nu\lambda\sigma} <\widetilde{\mathcal{J}}_{\sigma}(p_{124})\widetilde{\mathcal{J}}^{\kappa}(p_3)> \\
&+ \frac{1}{3}\gamma^{\lambda}\gamma_5\left(-i\gamma_{\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_{14})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> +\widetilde{C}^{\nu\kappa}_{3S} + \frac{3i}{4}\gamma_5 <\widetilde{\mathcal{J}}^{\nu}(p_{124})\widetilde{\mathcal{J}}^{\kappa}(p_3)>\right) \\
&+ \frac{1}{3}\gamma^{\kappa}\gamma_5\left(-i\gamma_{\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_{13})\bar{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\widetilde{C}^{\nu\lambda}_{3S} + \frac{3i}{4}\gamma_5 <\widetilde{\mathcal{J}}^{\nu}(p_{123})\widetilde{\mathcal{J}}^{\lambda}(p_4)>\right),
\end{aligned} \tag{8.0.2}$$

where $\widetilde{C}^{\nu\kappa\lambda}_{4Q}$, $\widetilde{C}^{\nu\kappa}_{3S}$ are the regulated versions of $C^{\nu\kappa\lambda}_{4Q}$ (4.2.17) and $C^{\nu\kappa}_{3S}$ (4.2.15) respectively.

The fact that there exists a manifestly non anomalous combination of Q+S supersymmetry (at least at the correlation functions we are considering), and that combination is exactly the supersymmetry of old minimal supergravity (see appendix G), means that the regulated theory has a non anomalous supercurrent. This supercurrent though, belongs to the Ferrara-Zumino supermultiplet (which couples to old minimal supergravity), rather than to the superconformal current multiplet. After sending the PV mass to infinity in order to retrieve conformal supergravity, Q-supersymmetry is necessarily anomalous.

CHAPTER 9

Q-supersymmetry anomaly with momentum routing

In the previous chapters we provided a thorough analysis of the Ward identities and anomalies in the free and massless WZ model. We regularized the theory with the PV Lagrangian (6.1.1) that classically violates R-symmetry, Q- and S-supersymmetry of conformal supergravity. After removing the regulator and adding the local counterterm (7.3.1), we were able to reproduce for the correlation functions of interest the superconformal anomalies (B.1.3) which were derived through the WZ consistency conditions.

We could, however, follow another approach for regularizing the classical Ward identities of section (4.2). The simplest and most naive way to do that would be to introduce a hard cut-off in the integration variable of the Feynman integrals. We briefly sketched this regularization procedure in subsection (3.1.1). Even though we use another regulator here, the strategy to compute the Q-supersymmetry anomaly of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ remains the same. We first need to fix the theory at the lower order correlators, in a way that they satisfy their standard symmetry identities and anomalies. When we obtain all the partially renormalized 2- and 3-point functions that are involved in the Q-supersymmetry identity (4.2.16), only then we should examine the possibility of a Q-supersymmetry anomaly in $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. The results of the analysis in this chapter were presented in [80].

The computation with momentum cut-off, is significantly more complicated and tedious than the corresponding calculation with PV regularization. In the latter, the result that we get after computing a correlator is unambiguous. All Feynman integrals in PV regularization are properly defined. Different choices of momentum routing at the Feynman diagrams (or equivalently translations at the integration variable in the Feynman integrals) yield the same result. The only thing we can do to renormalize a correlator is to add the appropriate counterterms in the Lagrangian. In cut-off regularization though, besides the counterterms, one has the freedom to choose an arbitrary momentum routing for the Feynman diagrams that correspond to the correlator under consideration. See the analysis of (3.2.2). Divergent correlators with different momentum routing, differ by finite (or even divergent) surface terms. This greatly complicates the analysis of the Ward identities. If we want to examine whether a potential anomalous term can be removed, we have to take into account at the same time, the combination of the allowable counterterms and the extra terms that arise after an arbitrary shift at the momentum routing of the Feynman diagrams. The number of equations one has to solve, makes it almost impossible to fix the theory up to the 4-point function level.

Below we present the analysis for the Q-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}>$. The aim is to show how one can start fixing the theory from the 2-point function level using cut-off. This is done for completeness though, and the calculations regarding $<\mathcal{Q}\bar{\mathcal{Q}}>$ will not play any role in the analysis of the Q-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, which we compute next. We also want to elaborate on what happens if one naively computes Ward identities of higher order correlators without having first fixed the lower order correlators.

9.1 Q-supersymmetry of $<\mathcal{Q}\bar{\mathcal{Q}}>$

We are interested in the classical Q-supersymmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}>$, i.e.

$$p_{1\mu} <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)>= 0. \tag{9.1.1}$$

After performing the Wick contractions, we find that

$$<\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)>= \int \frac{d^4q}{(2\pi)^4}\frac{1}{2}C_0^\mu(q)\frac{-i}{i(\not{p}_1-\not{q})}A_0^\nu(-q)\frac{-i}{q^2}, \tag{9.1.2}$$

where

$$C_0^\mu(q) = i\not{q}\gamma^\mu + \frac{i}{3}\left[\gamma^\mu,\gamma^\rho\right]q_{1\rho}\,, \qquad A_0^\nu(q) = i\gamma^\nu\not{q} - \frac{i}{3}\left[\gamma^\nu,\gamma^\rho\right]q_{2\rho}\,. \tag{9.1.3}$$

The above integral is by power counting cubically divergent, so we regulate with a momentum cut-off. The correlation function that we examine has also the following symmetry

$$<\mathcal{Q}^\nu(p_2)\bar{\mathcal{Q}}^\mu(p_1)>= C(<\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)>)^T C^{-1}\,, \tag{9.1.4}$$

where C is the charge conjugation matrix and T denotes the transpose matrix. After a small and straightforward computation it is easy to see that the choice of momentum routing that we made in (9.1.2) respects the aforementioned symmetry.

We now contract the 2-point function with one of the external momenta and get

$$p_{1\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)>= \int \frac{d^4q}{(2\pi)^4}\frac{1}{2}\left(g^{\nu}(q-p_1)-g^{\nu}(q)\right), \tag{9.1.5}$$

where

$$g^{\nu}(q) = i\gamma_{\alpha}\left(\frac{-6q^{\alpha}q^{\nu}-3q^{\alpha}p_1^{\nu}+(q\cdot p_1)\eta^{\nu\alpha}-q^{\nu}p_1^{\alpha}}{3q^2}\right)+\gamma_{\sigma}\gamma_5\left(\frac{\epsilon^{\sigma\alpha\beta\nu}q_{\alpha}p_{1\beta}}{3q^2}\right). \tag{9.1.6}$$

Had the integral in (9.1.5) been convergent, the Q-supersymmetry Ward identity for the 2-point function would have been satisfied after a simple shift at the integration variable of the term $g^{\nu}(q-p_1)$. As we explained in subsection (3.1.1), these kind of shifts produce surface terms in divergent integrals in the cut-off regularization. We need to Taylor expand $g^{\nu}(q-p_1)$ for small p_1 and compute with Gauss's theorem all the relevant terms that do not vanish. The cut-off parameter that we use is the radius of the hypersphere where we perform Gauss's theorem. Using all the identities and integrals of the subsection (3.1.1) we find that

$$p_{1\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)>= \frac{1}{2304\pi^2}p_1^2\left(4p_1^{\alpha}p_1^{\nu}-p_1^2\eta^{\alpha\nu}\right)\gamma_{\alpha}. \tag{9.1.7}$$

Since the 2-point function (9.1.2) is very badly divergent, one would expect that the contraction with $p_{1\mu}$ would have not only finite but also divergent contributions. It turns out that all divergent parts are proportional to projection operators and vanish identically when contracted with the external momenta. The finite rhs of (9.1.7) does not imply the existence of a Q-supersymmetry anomaly, since we can find a local counterterm to remove it. The counterterm will take the following covariant form

$$J_{QQ} = \frac{1}{2304\pi^2}\left(\mathscr{D}_{\alpha}\bar{\psi}^{\alpha}\gamma^{\rho}\mathscr{D}^2\psi_{\rho}-\frac{5}{2}\mathscr{D}_{\alpha}\bar{\psi}^{\alpha}\not{\mathscr{D}}\mathscr{D}_{\rho}\psi^{\rho}\right), \tag{9.1.8}$$

where ψ is the gravitino, the supercurrent's background source. This counterterm is only relevant for removing the finite parts of the 2-point function that violate the Q-supersymmetry identity. It does not render $<\mathcal{Q}\bar{\mathcal{Q}}>$ finite. If we had chosen a different routing in (9.1.5), for example $q \to q+p_1$, then the rhs of (9.1.7) would be different, and that would also affect the counterterm that we used. So if we need to use this 2-point function in the Ward identities of higher order correlators, we have to use the specific choice of momentum routing of (9.1.5) along with the contribution of the counterterm (9.1.8). A different choice of routing means that we need to change accordingly the counterterm (9.1.8).

9.2 Q-supersymmetry of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$

The classical path integral Q-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ is given by

$$\begin{aligned}
&p_{1\mu} <\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>_{\{a_i\}} + \Omega^{\nu\kappa\lambda}_{\{b_i\}} \\
&= ip_{2\mu}B^{\nu\mu\sigma} <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>_{\{c_i\}} - \frac{\gamma_\xi}{2} <\mathcal{T}^{\nu\xi}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>_{\{d_i\}} \\
&- i\gamma_5 <\mathcal{Q}^{\kappa}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\lambda}(p_4)>_{\{e_i\}} - i\gamma_5 <\mathcal{Q}^{\lambda}(p_{14})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)>_{\{f_i\}} \\
&+ \gamma_5 B^{\nu\kappa\sigma} <\mathcal{J}_{\sigma}(p_{123})\mathcal{J}^{\lambda}(p_4)>_{\{g_i\}} + \gamma_5 B^{\nu\lambda\sigma} <\mathcal{J}_{\sigma}(p_{124})\mathcal{J}^{\kappa}(p_3)>_{\{h_i\}},
\end{aligned} \tag{9.2.1}$$

where

$$\begin{aligned}
\Omega^{\nu\kappa\lambda} \equiv\ & p_{1\mu}\Bigg(\frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi <\mathcal{J}_{\sigma}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> + \frac{i}{8}\eta^{\mu\nu}\gamma^{\sigma}\gamma_5 <s_{\sigma(2|1)}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)> \\
&+ i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\lambda}(p_4)> + i\eta^{\mu\lambda}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu} - \frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu} - i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi\gamma_5\right) <s_{(1|0)}(p_{12})\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>\Bigg) \\
&- \frac{2\sqrt{2}}{9}\eta^{\kappa\lambda}\gamma_5 <s_{(3|\frac{1}{2})}(p_{134})\bar{\mathcal{Q}}^{\nu}(p_2)> + \frac{1}{24}\eta^{\nu\kappa}\gamma^{\sigma} <s_{\sigma(2|1)}(p_{123})\mathcal{J}^{\lambda}(p_4)> \\
&+ \frac{1}{24}\eta^{\nu\lambda}\gamma^{\sigma} <s_{\sigma(2|1)}(p_{124})\mathcal{J}^{\kappa}(p_3)> \\
&- \frac{1}{6}\eta^{\nu\kappa}\gamma^{\sigma} <(s^{*}_{\sigma(4|1)} - s_{\sigma(4|1)})(p_{123})\mathcal{J}^{\lambda}(p_4)> - \frac{1}{6}\eta^{\nu\lambda}\gamma^{\sigma} <(s^{*}_{\sigma(4|1)} - s_{\sigma(4|1)})(p_{124})\mathcal{J}^{\kappa}(p_3)> \\
&- \left(-\frac{3}{8}\gamma_\xi\gamma_5\epsilon^{\sigma\xi\nu\kappa} + \frac{i}{2}\eta^{\kappa\sigma}\gamma^{\nu} - \frac{i}{2}\eta^{\nu\sigma}\gamma^{\kappa}\right) <\mathcal{J}_{\sigma}(p_{123})\mathcal{J}^{\lambda}(p_4)> \\
&- \left(-\frac{3}{8}\gamma_\xi\gamma_5\epsilon^{\sigma\xi\nu\lambda} + \frac{i}{2}\eta^{\lambda\sigma}\gamma^{\nu} - \frac{i}{2}\eta^{\nu\sigma}\gamma^{\lambda}\right) <\mathcal{J}_{\sigma}(p_{124})\mathcal{J}^{\kappa}(p_3)>,
\end{aligned} \tag{9.2.2}$$

and $B^{\nu\mu\sigma}$ is given by (4.2.11).

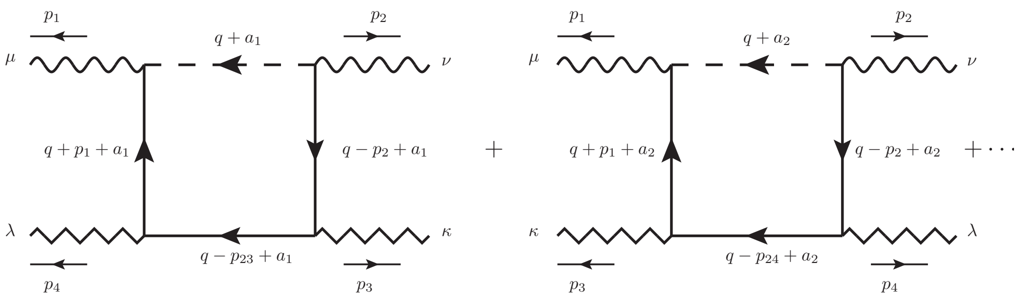

Figure 9.2.1: Part of the Feynman diagrams that contribute to $<\mathcal{Q}^{\mu}(p_1)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>_{\{a_i\}}$. The wave lines represent the external supercurrents and the zig-zag lines denote the external R-currents. Straight and dashed lines in the loop denote fermionic and bosonic propagators respectively. p_i are the external momenta while q is the loop momentum. $\{a_i\}$ denotes collectively the arbitrary momentum routing parameters. For generality we assign a different parameter to each Feynman diagram that corresponds to a correlator.

The subscripts $\{a_i\}$, ... $\{h_i\}$ denote the arbitrary choice of momentum routing that we made in the Feynman integrals of the correlators. The correlators involved in (9.2.1) are given in the appendix H. Now let us make a few comments here.

First, note that in all seagull correlators $\Omega^{\nu\kappa\lambda}$, we assigned for simplicity the same parameters $\{b_i\}$. Of course, we could have chosen the more general case where the routing of every correlator in $\Omega^{\nu\kappa\lambda}$ is independent from the others. This however, would not change the core of the argument we want to make. Secondly, notice that the Q-supersymmetry identity we examine in this section is not the same with (4.2.16). Recall that (4.2.16) is a sum of path integral identities, and is equivalent to the Q-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ that we find from the coupling to background conformal supergravity. Here on the other hand, we are interested in the bottom-up approach. If one wanted to examine about possible quantum anomalies in the Ward identities of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ with no insight from background supergravity, then (9.2.1) would be the natural identity to compute. After all, the coloured seagull correlators of (4.2.16) form their own path integral Q-supersymmetry identities and they can be calculated separately. Moreover, we do not expect any anomalies in the aforementioned identities, since they do not include any known anomalous correlators. In the appendix F we verify this claim with a PV regulator. On the contrary, (9.2.1) contains the anomalous $<\mathcal{J}\mathcal{J}\mathcal{J}>$ diagram, so if there is a Q-supersymmetry anomaly, it has to exist in the classical path integral identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. Of course, if we want to couple the regularized with cut-off WZ model to background supergravity, the analysis of the identities of the coloured seagull correlators is unavoidable. This however, will not change the final result about the existence or not of a Q-supersymmetry anomaly.

Now let us return to the calculation of (9.2.1). This identity is at the regulated level, and we have not renormalized/fixed the lower order correlation functions that contains. After a long and tedious computation we can show that (9.2.1) is satisfied for the following values of the arbitrary constants

$$a_i = b_i = c_i = d_i = e_i = f_i = g_i = h_i = 0. \tag{9.2.3}$$

The original choice of routing that we made in the integrals of the appendix H (i.e. putting the arbitrary constants equal to zero) is such that (9.2.1) is satisfied. To prove this we only need to use Fierz identities and that

$$p_{1\mu}C_0^{\mu}(q) = -i\not{q} + q^2\frac{-i(\not{p}_1 - \not{q})}{(p_1 - q)^2}. \tag{9.2.4}$$

We do not need to perform any illegitimate manipulations in the divergent integrals, such as shift of the integration variable. Therefore, what we show is that Q-supersymmetry is non anomalous at least at the level of the 4-point correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. However, we have not examined at all what happens at the lower order correlators of (9.2.1). It would be totally naive to argue at this point and with this information, that there exists a free, massless and at the same time consistent WZ model with no anomaly in Q-supersymmetry. Maybe the choice of routing that we made in the $<\mathcal{T}\mathcal{J}\mathcal{J}>$ correlator is not compatible with diffeomorphisms or Lorentz symmetry. One has to do the explicit computation, and examine all the symmetry identities for every correlator involved in (9.2.1), including the seagull correlators. After all, from a bottom-up perspective the notion of a seagull operator does not have much significance. In (9.2.1) there are correlators of well defined operators and one needs to compute all of them. So, a non anomalous Q-supersymmetry, may mean a non consistent WZ model with anomalies in diffeomorphisms and in Lorentz symmetry.

Here, we show that the specific choice (9.2.3) that we made to satisfy (9.2.1), makes the 3-point correlator $<\mathcal{J}\mathcal{J}\mathcal{J}>$ inconsistent. In particular $<\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>_0$ does not respect Bose symmetry, hence it does not reproduce the standard R-symmetry anomaly of (B.1.3). The values of $\{c_i\}$ that make $<\mathcal{J}\mathcal{J}\mathcal{J}>$ consistent (i.e. Bose symmetric and with the correct R-symmetry anomaly) are the following

$$c_1 = -\frac{p_4 - p_3}{6}\,, \qquad c_2 = \frac{p_4 - p_3}{6}. \tag{9.2.5}$$

If we use in (9.2.1) the consistent $<\mathcal{J}\mathcal{J}\mathcal{J}>$ correlator instead, the previously non anomalous Q-supersymmetry identity will acquire a new contribution at the rhs equal to

$$\begin{aligned} &ip_{2\mu}B^{\nu\mu\sigma}(-<\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>_0 + <\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>_{\{c_i\}}) = \\ &\frac{2i\pi^2}{(2\pi)^4}(\frac{-2i}{81})\epsilon^{\rho\kappa\lambda}{}_\sigma(p_{4\rho}-p_{3\rho})p_{2\mu}\left(-\frac{1}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi + \frac{i}{4}\gamma_5\gamma^\mu\eta^{\nu\sigma} - \frac{i}{4}\gamma_5\gamma^\nu\eta^{\mu\sigma}\right). \end{aligned} \tag{9.2.6}$$

We computed the above lhs using (3.1.5). This is exactly the same anomaly that we found using the PV regulator (7.2.23), which confirms the results (B.1.3).

The above analysis is the proof for the existence of a Q-supersymmetry anomaly in the conformal WZ model. As we stressed before, a more rigorous approach would be to actually compute all the lower order correlators of (9.2.1). It could happen for example, that the $<\mathcal{T}\mathcal{J}\mathcal{J}>$ correlator we use in (9.2.1) has a diffeomorphism anomaly, and after we include an appropriate counterterm to remove it, this counterterm at the same time cancels the term (9.2.6) and renders (9.2.1) non anomalous again. Of course, we expect this not to be the case, since that would contradict the results of the WZ consistency conditions (B.1.3). We expect that the choice of routing that we made for the other correlators in (9.2.1) is such, that every one of their symmetry identities is satisfied. Probably for that we also need to include appropriate counterterms, but these will be Q-supersymmetric, thus not affecting the anomalous term (9.2.6). We should also note, that the anomaly (9.2.6) is non zero in the limit of $p_1 \to 0$, thus it cannot be removed by a different choice at the routing of the Feynman diagrams of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$.

Although the analysis with the PV regulator in the previous chapters is the most rigorous one, since there we actually computed all correlators involved in the symmetry identities we were interested in, the cut-off regulator provides an alternative and independent proof for the Q-supersymmetry anomaly of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. An interesting point of the cut-off approach, is that the Q-supersymmetry anomaly is only a consequence of the ambiguous surface terms of the $<\mathcal{J}\mathcal{J}\mathcal{J}>$ diagram that give rise to the R-symmetry anomaly, which is what happens in the analysis of the WZ consistency conditions.

CHAPTER 10

Discussion

In this thesis we presented a comprehensive analysis of the free and massless WZ model in perturbation theory. As for any $\mathcal{N} = 1$ SCFT in four dimensions, the renormalized theory admits both a conformal and a Ferrara-Zumino multiplet of currents, with the latter inherited from the regulated theory.[1] The two multiplets are related by a set of local counterterms that shift the anomalies between different symmetries. The conformal multiplet possesses the standard superconformal anomalies of $\mathcal{N} = 1$ SCFTs. In particular, R-symmetry is anomalous because of the standard triangle diagram, but both Q- and S-supersymmetry are necessarily also anomalous. On the contrary, the Poincaré supersymmetry of the FZ multiplet (which corresponds to a specific field-dependent linear combination of the Q- and S-supersymmetry of the conformal multiplet) is non anomalous, but R-symmetry and S-supersymmetry are explicitly broken.

We verified the above statements with a loop computation using a supersymmetric PV regulator. Our analysis focused on the 4-point correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$, which is the minimal one receiving a contribution from the Q-supersymmetry anomaly in flat space. We also examined all the lower order correlators necessary for the analysis of the 4-point function. Furthermore, using a cut-off regulator we confirmed the existence of a Q-supersymmetry anomaly in $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. The counterterm (7.3.1) that we found, is a subset of the complete

[1]Other multiplets such as the R- and S-multiplets are also admissible but were not considered here. The supersymmetry anomaly of the R-multiplet was studied recently in [106,107]. See also [59]. Like the FZ multiplet, the S-multiplet does not suffer from a supersymmetry anomaly.

counterterm that relates the conformal and FZ multiplets.

The FZ multiplet is more natural if one wishes to view the massless model as the zero mass limit of a massive WZ model, while the conformal multiplet is more natural if one wishes to view the massless WZ model as an example of an $\mathcal{N} = 1$ SCFT. Indeed, in the context of the AdS/CFT correspondence only the conformal multiplet is available and it is in this context that the anomaly was first discovered [68].

The presence of a supersymmetry anomaly in the conformal and R-multiplet [106,107] is an important caveat one should keep in mind in the context of supersymmetric localisation, especially when the results are compared with holographic computations, as noted in [68,71,78,79] in relation to the analysis of [108]. In particular, in the presence of anomalies physical observables depend on the choice of current multiplet and one should make sure that only results specific to a given multiplet are compared. Given that different multiplets are often used for field theory and holographic computations, failing to do so may result in a superficial mismatch. We anticipate that a local counterterm analogous to that relating the conformal and FZ multiplets interpolates between the R-multiplet, which couples to new minimal supergravity, and the S-multiplet, corresponding to 16+16 supergravity, enabling one to remove the supersymmetry anomaly of the R-multiplet. Determining this counterterm would be particularly interesting for supersymmetric localization applications.

Since current multiplets describing SCFTs are related by finite local counterterms, such counterterms can be used to match the computation of physical observables using different multiplets. Indeed, it was through the identification of a non-covariant local counterterm (specific to a class of rigid supersymmetric backgrounds) that the authors of [108] managed to reconcile their holographic computation with the expected field theory result. Understanding the general structure of supersymmetry anomalies in different multiplets allows one to explicitly determine the local counterterms that interpolate between them. Of course, such counterterms are not unique since it is always possible to add further ‘trivial’ local counterterms that preserve all the symmetries. In particular, the local counterterms interpolating between the conformal and FZ multiplets that we determined here through the 1-loop computation may agree with the superspace results in [104] up to such trivial terms.

Another interesting computation would be the complete analysis of the $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$ correlator and the confirmation of its Q-supersymmetry anomaly in the conformal multiplet. For that, one needs to include more regulating PV fields in order to cancel the divergences of correlators such as $<\mathcal{T}\mathcal{T}>$, $<\mathcal{T}\mathcal{T}\mathcal{T}>$ etc. Such a regulator would be sufficient to properly regulate all possible FZ and conformal multiplet correlators. The analysis of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$ would also help us construct the full counterterm that interpolates between conformal and old minimal supergravity. However, if one is interested in the general form of this counterterm, the most efficient way to deduce it would be from the structure of the solution of the WZ consistency conditions.

Our loop calculation with Pauli-Villars and cut-off regulators, along with the analysis of the WZ consistency conditions, established without a doubt the existence of anomalies in the Q-supersymmetry Ward identities of conformal supergravity. Knowing this result, it would be quite interesting to perform the same analysis using dimensional reduction (DRED) as a regularization procedure, and see which one of the different prescriptions of DRED reproduces the correct Q-supersymmetry anomaly. DRED is supposed to respect supersymmetry in all practical applications, thus explaining how and why the anomaly arises in this context, it could provide us with a better understanding on supersymmetric regularization.

It would also be interesting to investigate the existence of such an anomaly in other dimensions and/or extended supersymmetry. In the case of extended supersymmetry, there is more freedom and more possible combinations for the anomalies to cancel. We expect similar cancellations with the case of supersymmetric Yang-Mills theories, where the S-supersymmetry anomaly of the $\mathcal{N} = 1$ model vanishes in the maximally supersymmetric $\mathcal{N} = 4$ theory [43]. Supersymmetry anomalies in gauge theories in dimensions different than four have been examined for example in [52, 54]. In particular, in [52] there were given expressions for $d = 6$. Based on the WZ consistency conditions argument (4.4.6), the supersymmetry anomaly is a consequence of the R-symmetry anomaly. However, taking into account our whole analysis and the discussion of the conformal and FZ multiplets, (classical) conformal symmetry is also an essential condition for the existence of the Q-supersymmetry anomaly in $d = 4$. Having the explicit form of the abelian and non-abelian chiral anomalies in $2n$-dimensional spacetimes [109], in principle, one can use the WZ consistency conditions to compute the supersymmetry anomalies in $d \neq 4$. If conformal symmetry is in general a necessary condition, then we can have an analogue to our result only up to $d = 6$ (this is what we expect). If for $d \neq 4$, R-anomaly is capable of inducing a Q-supersymmetry anomaly in models which are not classically conformal, then we can have supersymmetry anomalies up to $d = 10$. Recall that the R-anomaly can be also present in massive theories, such as the free massive Dirac fermion [2].

Moreover, in this thesis supergravity was viewed as non-dynamical and it would be interesting to extend the analysis to include dynamical supergravity. Our results imply that only the FZ multiplet can be consistently coupled to (old minimal) dynamical supergravity, since the Poincaré supersymmetry of old minimal supergravity is non anomalous. This is in line with earlier work [59] where it was argued that quantum anomalies in the matter sector require the use of old minimal supergravity. However, the conformal multiplet that

[2] The R-symmetry anomaly arises from the UV behaviour of Feynman diagrams and a small finite Dirac mass is irrelevant at this limit.

suffers from a supersymmetry anomaly may still be coupled to dynamical supergravity in the context of effective field theory [110–114]. In that context, the anomalies are cancelled either by fields with a mass above the cut-off through a generalized Green-Schwarz mechanism, or more generally by supersymmetric anomaly inflow. It would be very interesting to examine the exact mechanisms of such cancellations.

Finally, in the context of loop calculations, the whole analysis and methods presented in this thesis could be used for the computation of anomalies in different theories. Recently, there has been a debate about the existence of CP-odd terms in the trace anomaly of Weyl fermions, whose coefficients are purely imaginary [105,115–119]. This anomaly cannot be present in the massless WZ model (4.1.1), since we use a Majorana fermion there, which is symmetric under charge conjugation. However, the massless WZ Lagrangian can also be written using Weyl fermions as follows

$$\mathscr{L}_{\mathrm{WZ}} = -\partial_\mu \phi^* \partial^\mu \phi - \overline{\chi}_L \not{\partial} \chi_L. \tag{10.0.1}$$

According to [116], the first correlator that receives a contribution from the parity-odd trace anomaly in the flat space is $<\mathcal{TTT}>$, where $\mathcal{T}$ is the stress tensor of the Weyl fermion. It would be interesting to regulate the Lagrangian (10.0.1) and apply our methods to see if $<\mathcal{TTT}>$ has a parity-odd trace anomaly or not. In case of a positive answer, one could examine whether this has implications on supersymmetry.

Appendices

APPENDIX A

Spinor conventions and identities

We largely follow the spinor conventions of [18]. We use the Minkowski metric $\eta = \text{diag}\,(-1, 1, 1, 1)$ and the Levi-Civita symbol $\varepsilon_{\mu\nu\rho\sigma} = \pm 1$ satisfies $\varepsilon_{0123} = 1$. This is related to the Levi-Civita *tensor* as $\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g}\;\varepsilon_{\mu\nu\rho\sigma} = e\;\varepsilon_{\mu\nu\rho\sigma}$, where $e \equiv \det(e_\mu^a)$ is the determinant of the vierbein. We also use the convention that complex conjugation reverses the order of Grassmann fields (spinors or scalars), e.g. $(AB)^* \equiv B^* A^*$. The symmetrization and antisymmetrization of a tensor $T_{a_1 \ldots a_n}$ with respect to its indices is given respectively by

$$T_{(a_1 \ldots a_n)} = \frac{1}{n!} \sum_p T_{a_{p(1)} \ldots a_{p(n)}}\,, \qquad T_{[a_1 \ldots a_n]} = \frac{1}{n!} \sum_p \delta_p T_{a_{p(1)} \ldots a_{p(n)}}, \tag{A.0.1}$$

where the sum is taken over all permutations, p, of $1, \ldots, n$ and δ_p is $+1$ for even permutations and -1 for odd permutations.

Gamma matrices The gamma matrices satisfy the Hermiticity properties

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0, \qquad \gamma_5^\dagger = \gamma_5, \tag{A.0.2}$$

where the chirality matrix in four dimensions is given by

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{A.0.3}$$

The antisymmetrized products of gamma matrices are defined as

$$\gamma^{\mu_1\mu_2\cdots\mu_n} \equiv \gamma^{[\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n]}. \tag{A.0.4}$$

The following is a list of identities in d dimensions that the antisymmetrized products of gamma matrices satisfy, several of which we use repeatedly in this thesis (see also section 3 of [18]):

$$\begin{aligned}
\gamma^{\mu\nu\rho} &= \frac{1}{2}\{\gamma^\mu, \gamma^{\nu\rho}\}, \\
\gamma^{\mu\nu\rho\sigma} &= \frac{1}{2}[\gamma^\mu, \gamma^{\nu\rho\sigma}], \\
\gamma^{\mu\nu}\gamma_{\rho\sigma} &= \gamma^{\mu\nu}{}_{\rho\sigma} + 4\gamma^{[\mu}{}_{[\sigma}\delta^{\nu]}{}_{\rho]} + 2\delta^{[\mu}{}_{[\sigma}\delta^{\nu]}{}_{\rho]}, \\
\gamma_\mu\gamma^{\nu_1\ldots\nu_p} &= \gamma_\mu{}^{\nu_1\ldots\nu_p} + p\delta_\mu^{[\nu_1}\gamma^{\nu_2\ldots\nu_p]}, \\
\gamma^{\nu_1\ldots\nu_p}\gamma_\mu &= \gamma^{\nu_1\ldots\nu_p}{}_\mu + p\gamma^{[\nu_1\ldots\nu_{p-1}}\delta_\mu^{\nu_p]}, \\
\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} &= \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}, \\
\gamma^{\mu\nu\rho\sigma}\gamma_{\tau\lambda} &= \gamma^{\mu\nu\rho\sigma}{}_{\tau\lambda} + 8\gamma^{[\mu\nu\rho}{}_{[\lambda}\delta^{\sigma]}{}_{\tau]} + 12\gamma^{[\mu\nu}\delta^{\rho}{}_{[\lambda}\delta^{\sigma]}{}_{\tau]}, \\
\gamma^{\mu\nu\rho}\gamma_{\sigma\tau\lambda} &= \gamma^{\mu\nu\rho}{}_{\sigma\tau\lambda} + 9\gamma^{[\mu\nu}{}_{[\tau\lambda}\delta^{\rho]}{}_{\sigma]} + 18\gamma^{[\mu}{}_{[\lambda}\delta^{\nu}{}_{\tau}\delta^{\rho]}{}_{\sigma]} + 6\delta^{[\mu}{}_{[\lambda}\delta^{\nu}{}_{\tau}\delta^{\rho]}{}_{\sigma]}, \\
\gamma^{\mu_1\ldots\mu_r\nu_1\ldots\nu_s}\gamma_{\nu_s\ldots\nu_1} &= \frac{(d-r)!}{(d-r-s)!}\gamma^{\mu_1\ldots\mu_r}, \\
\gamma^{\mu\rho}\gamma_{\rho\nu} &= (d-2)\gamma^\mu{}_\nu + (d-1)\delta^\mu_\nu, \\
\gamma^{\mu\nu\rho}\gamma_{\rho\sigma} &= (d-3)\gamma^{\mu\nu}{}_\sigma + 2(d-2)\gamma^{[\mu}\delta^{\nu]}{}_\sigma, \\
\gamma_{\mu\nu}\gamma^{\nu\rho\sigma} &= (d-3)\gamma_\mu{}^{\rho\sigma} + 2(d-2)\delta_\mu^{[\rho}\gamma^{\sigma]}, \\
\gamma^{\mu\nu\lambda}\gamma_{\lambda\rho\sigma} &= (d-4)\gamma^{\mu\nu}{}_{\rho\sigma} + 4(d-3)\gamma^{[\mu}{}_{[\sigma}\delta^{\nu]}{}_{\rho]} + 2(d-2)\delta^{[\mu}{}_{[\sigma}\delta^{\nu]}{}_{\rho]}, \\
\gamma_{\mu\rho}\gamma^{\rho\sigma\tau}\gamma_{\tau\nu} &= (d-4)^2\gamma_\mu{}^\sigma{}_\nu + (d-4)(d-3)\big(\gamma_\mu\delta^\sigma_\nu - \gamma^\sigma g_{\mu\nu}\big) \\
&\quad + (d-3)(d-2)\delta^\sigma_\mu\gamma_\nu - (d-3)\gamma^\sigma\gamma_{\mu\nu}, \\
\gamma_\rho\gamma^{\mu_1\mu_2\ldots\mu_p}\gamma^\rho &= (-1)^p(d-2p)\gamma^{\mu_1\mu_2\ldots\mu_p}.
\end{aligned} \tag{A.0.5}$$

For $d=4$ specifically, we have the gamma matrix identities

$$\begin{aligned}
&\gamma^\rho\gamma^\mu\gamma^\sigma + \gamma^\sigma\gamma^\mu\gamma^\rho = 2(g^{\mu\rho}\gamma^\sigma + g^{\mu\sigma}\gamma^\rho - g^{\rho\sigma}\gamma^\mu), \\
&\gamma^\rho\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu\gamma^\rho = 2\gamma^{\rho\mu\sigma}, \\
&\gamma^\rho\gamma^\mu\gamma^\sigma = g^{\mu\rho}\gamma^\sigma + g^{\mu\sigma}\gamma^\rho - g^{\rho\sigma}\gamma^\mu + \gamma^{\rho\mu\sigma} \\
&\gamma^{\mu\rho}\gamma^\sigma = \gamma^{\mu\rho\sigma} + \gamma^\mu g^{\rho\sigma} - \gamma^\rho g^{\mu\sigma} \\
&\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]} = -i\epsilon^{\mu\nu}{}_{\rho\sigma}\gamma_5 + 2g_\rho^{[\mu}g_\sigma^{\nu]}, \\
&\gamma^{\mu\nu\rho\sigma} = -i\epsilon^{\mu\nu\rho\sigma}\gamma_5, \\
&\gamma^{\mu\rho\sigma} = i\epsilon^{\mu\rho\sigma\nu}\gamma_\nu\gamma_5, \\
&\gamma^{\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\gamma_{\rho\sigma}\gamma_5,
\end{aligned} \tag{A.0.6}$$

as well as the trace relations

$$\begin{aligned}
&\mathrm{tr}\,(\text{any odd number of gamma matrices}) = 0,\\
&\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma_5) = 0,\\
&\mathrm{tr}\,(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu},\\
&\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}),\\
&\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma}.
\end{aligned} \tag{A.0.7}$$

Dirac conjugate The Dirac conjugate of a Dirac spinor, χ, is defined as

$$\overline{\chi} \equiv i\chi^\dagger\gamma^0, \tag{A.0.8}$$

and is denoted with a thick overbar in order to distinguish it from the Majorana conjugate.

Majorana conjugate and spinors The Majorana conjugate of a Dirac spinor, χ, is defined as

$$\bar{\chi} \equiv \chi^T C, \tag{A.0.9}$$

where C is the charge conjugation matrix (see section 3.1.8 of [18]). A spinor χ is said to be Majorana if it equals its charge conjugate, or equivalently, if its Dirac and Majorana conjugates coincide, i.e.

$$\chi^C \equiv B^{-1}\chi^* = \chi \quad \Leftrightarrow \quad \overline{\chi} = \bar{\chi}, \tag{A.0.10}$$

where the unitary matrix B is related to the charge conjugation matrix, C, as in eq. (3.47) of [18].

Dirac spinor bilinears involving Majorana conjugation in four dimensions satisfy the identity

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-1)^p\bar{\chi}\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda, \qquad \text{[eq. (3.53) in [18]]}. \tag{A.0.11}$$

which also implies that

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\gamma_5\chi = (-1)^p\bar{\chi}\gamma_5\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda = \bar{\chi}\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\gamma_5\lambda. \tag{A.0.12}$$

Majorana fermion bilinears possess in addition the reality property

$$(\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda)^* = \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda, \qquad \text{[eq. (3.82) in [18]]}. \tag{A.0.13}$$

Chirality projectors and Weyl spinors The Weyl projections of a generic Dirac spinor, χ, are defined as

$$\chi_L \equiv P_L \chi \equiv \frac{1}{2}(1+\gamma_5)\chi, \qquad \chi_R \equiv P_R \chi \equiv \frac{1}{2}(1-\gamma_5)\chi. \tag{A.0.14}$$

Notice that, since there are no Majorana-Weyl spinors in four dimensions, the Weyl projection of a Majorana spinor is Weyl but not Majorana. Another potential source of confusion we should emphasize is the following relation between the Dirac and Majorana conjugates of Weyl spinors:

$$\overline{\chi}_L \equiv i\chi_L^\dagger \gamma^0 = i\chi^\dagger P_L^\dagger \gamma^0 = \overline{\chi} P_R = \bar{\chi}_R. \tag{A.0.15}$$

Fierz identities Finally, we make extensive use of the following Fierz relations in four dimensions

$$\begin{aligned} \chi_L \bar{\lambda}_L &= -\frac{1}{2} P_L (\bar{\lambda}\chi_L) + \frac{1}{8} P_L \gamma^{\mu\nu} (\bar{\lambda}\gamma_{\mu\nu}\chi_L), \\ \chi_L \bar{\lambda}_R &= -\frac{1}{2} P_L \gamma^\mu (\bar{\lambda}\gamma_\mu \chi_L). \end{aligned} \tag{A.0.16}$$

APPENDIX B

Conformal multiplet Ward identities and anomalies

The superconformal Ward identities can be formulated independently of any specific SCFT in terms of the a and c anomaly coefficients, whose values depend on the specific theory. The current multiplet of $\mathcal{N} = 1$ superconformal theories consists of the stress tensor, $\mathcal{T}_a^\mu$, the R-current, $\mathcal{J}^\mu$, and the supercurrent, $\mathcal{Q}^\mu$, which is a Majorana spinor in our conventions. These couple respectively to the vierbein, e_μ^a, the graviphoton, A_μ, and the gravitino, ψ_μ, which comprise the field content of $\mathcal{N} = 1$ conformal supergravity [97–100], which we briefly review in section (4.3). The consistent (as opposed to the covariant [120]) current operators are defined accordingly as

$$\langle \mathcal{T}_a^\mu \rangle_s \equiv e^{-1} \frac{\delta \mathscr{W}}{\delta e_\mu^a}, \qquad \langle \mathcal{J}^\mu \rangle_s \equiv e^{-1} \frac{\delta \mathscr{W}}{\delta A_\mu}, \qquad \langle \mathcal{Q}^\mu \rangle_s \equiv e^{-1} \frac{\delta \mathscr{W}}{\delta \bar{\psi}_\mu}, \tag{B.0.1}$$

where $e \equiv \det(e_\mu^a)$, $\mathscr{W}[e, A, \psi]$ is the generating function of renormalized connected current correlators, and the notation $\langle \cdot \rangle_s$ denotes connected correlation functions in the presence of sources. In particular, further derivatives of these one-point functions result in higher-point functions.

The current operators (B.0.1) are defined independently of whether there exists a Lagrangian description of the theory. If a Lagrangian description exists, $\mathscr{W}[e, A, \psi]$ is given by

$$\mathscr{W}[e, A, \psi] = -i \log \mathscr{Z}[e, A, \psi], \tag{B.0.2}$$

where $\mathscr{Z}[e, A, \psi]$ is obtained from the path integral

$$\mathscr{Z}[e, A, \psi] = \int [\mathrm{d}\{\Phi\}] e^{iS[\{\Phi\};e,A,\psi]}\Big|_{\text{ren}}, \tag{B.0.3}$$

over the microscopic fields $\{\Phi\}$, after renormalization.

B.1 Ward identities for 1-point functions with arbitrary sources

The superconformal Ward identities and anomalies for arbitrary $\mathcal{N} = 1$ SCFTs in four dimensions were derived in [71], using the local symmetries of $\mathcal{N} = 1$ conformal supergravity and the associated WZ consistency conditions. They take the form

$$\begin{aligned}
&e^a_\mu \nabla_\nu \langle \mathcal{T}^\nu_a \rangle_s + \nabla_\nu(\bar\psi_\mu \langle \mathcal{Q}^\nu \rangle_s) - \bar\psi_\nu \overleftarrow{D}_\mu \langle \mathcal{Q}^\nu \rangle_s - F_{\mu\nu} \langle \mathcal{J}^\nu \rangle_s \\
&\qquad + A_\mu (\nabla_\nu \langle \mathcal{J}^\nu \rangle_s + i\bar\psi_\nu \gamma_5 \langle \mathcal{Q}^\nu \rangle_s) - \omega_\mu{}^{ab}\Big(e_{\nu[a} \langle \mathcal{T}^\nu_{b]} \rangle_s + \frac{1}{4} \bar\psi_\nu \gamma_{ab} \langle \mathcal{Q}^\nu \rangle_s \Big) = 0, \\
&e^a_\mu \langle \mathcal{T}^\mu_a \rangle_s + \frac{1}{2} \bar\psi_\mu \langle \mathcal{Q}^\mu \rangle_s = \mathcal{A}_W, \\
&e_{\mu[a} \langle \mathcal{T}^\mu_{b]} \rangle_s + \frac{1}{4} \bar\psi_\mu \gamma_{ab} \langle \mathcal{Q}^\mu \rangle_s = 0, \\
&\nabla_\mu \langle \mathcal{J}^\mu \rangle_s + i\bar\psi_\mu \gamma_5 \langle \mathcal{Q}^\mu \rangle_s = \mathcal{A}_R, \\
&D_\mu \langle \mathcal{Q}^\mu \rangle_s - \frac{1}{2} \gamma^a \psi_\mu \langle \mathcal{T}^\mu_a \rangle_s - \frac{3i}{4} \gamma_5 \phi_\mu \langle \mathcal{J}^\mu \rangle_s = \mathcal{A}_Q, \\
&\gamma_\mu \langle \mathcal{Q}^\mu \rangle_s - \frac{3i}{4} \gamma_5 \psi_\mu \langle \mathcal{J}^\mu \rangle_s = \mathcal{A}_S.
\end{aligned} \tag{B.1.1}$$

The spin connection is given by (4.3.3) while the gravitino fieldstrength, ϕ_μ by (4.3.7).

Since the gravitino and the supercurrent have opposite R-charge, the covariant derivative acts on the supercurrent as

$$D_\mu \langle \mathcal{Q}^\nu \rangle = \Big(\partial_\mu + \frac{1}{4} \omega_\mu{}^{ab}(e, \psi) \gamma_{ab} - i\gamma_5 A_\mu \Big) \langle \mathcal{Q}^\nu \rangle + \Gamma^\nu_{\mu\rho} \mathcal{Q}^\rho \equiv (\mathscr{D}_\mu - i\gamma_5 A_\mu) \langle \mathcal{Q}^\nu \rangle. \tag{B.1.2}$$

The superconformal anomalies on the rhs of the Ward identities (B.1.1) are local functions of the background conformal supergravity fields and take the form

$$\begin{aligned}
\mathcal{A}_W =& \frac{c}{16\pi^2} \Big(W^2 - \frac{8}{3} F^2 \Big) - \frac{a}{16\pi^2} E + \mathcal{O}(\psi^2), \\
\mathcal{A}_R =& \frac{(5a - 3c)}{27\pi^2} \widetilde{F} F + \frac{(c - a)}{24\pi^2} \mathcal{P}, \\
\mathcal{A}_Q =& -\frac{(5a - 3c)i}{9\pi^2} \widetilde{F}^{\mu\nu} A_\mu \gamma_5 \phi_\nu + \frac{(a - c)}{6\pi^2} \nabla_\mu (A_\rho \widetilde{R}^{\rho\sigma\mu\nu}) \gamma_{(\nu} \psi_{\sigma)} - \frac{(a - c)}{24\pi^2} F_{\mu\nu} \widetilde{R}^{\mu\nu\rho\sigma} \gamma_\rho \psi_\sigma + \mathcal{O}(\psi^3), \\
\mathcal{A}_S =& \frac{(5a - 3c)}{6\pi^2} \widetilde{F}^{\mu\nu} \Big(D_\mu - \frac{2i}{3} A_\mu \gamma_5 \Big) \psi_\nu + \frac{ic}{6\pi^2} F^{\mu\nu} (\gamma_\mu{}^{[\sigma} \delta^{\rho]}_\nu - \delta^{[\sigma}_\mu \delta^{\rho]}_\nu) \gamma_5 D_\rho \psi_\sigma
\end{aligned}$$

$$+\frac{3(2a-c)}{4\pi^2}P_{\mu\nu}g^{\mu[\nu}\gamma^{\rho\sigma]}D_\rho\psi_\sigma+\frac{(a-c)}{8\pi^2}\Big(R^{\mu\nu\rho\sigma}\gamma_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}g^{\mu[\nu}\gamma^{\rho\sigma]}\Big)D_\rho\psi_\sigma+\mathcal{O}(\psi^3). \tag{B.1.3}$$

The antisymmetric tensor $F_{\mu\nu}=2\partial_{[\mu}A_{\nu]}$ denotes the fieldstrength of the graviphoton, while the dual fieldstrength, $\widetilde{F}_{\mu\nu}$, is given by

$$\widetilde{F}_{\mu\nu}\equiv\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}. \tag{B.1.4}$$

These are the building blocks of the two independent quadratic curvature invariants

$$F^2\equiv F_{\mu\nu}F^{\mu\nu},\qquad F\widetilde{F}\equiv\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{B.1.5}$$

The geometric curvature invariants are built out of the Riemann tensor and its dual

$$\widetilde{R}_{\mu\nu\rho\sigma}\equiv\frac{1}{2}\epsilon_{\mu\nu}{}^{\kappa\lambda}R_{\kappa\lambda\rho\sigma}, \tag{B.1.6}$$

(which is not symmetric under exchange of the first and second pair of indices) as well as the Schouten tensor, $P_{\mu\nu}$, which in four dimensions is given by

$$P_{\mu\nu}\equiv\frac{1}{2}\Big(R_{\mu\nu}-\frac{1}{6}Rg_{\mu\nu}\Big). \tag{B.1.7}$$

The independent quadratic curvature invariants are the square of the Weyl tensor, W^2, the Euler density, E, and the Pontryagin density, $\mathcal{P}$, defined respectively as

$$\begin{aligned}W^2&\equiv W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-2R_{\mu\nu}R^{\mu\nu}+\frac{1}{3}R^2,\\ E&=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2,\\ \mathcal{P}&\equiv\frac{1}{2}\epsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\rho\sigma}R_{\mu\nu}{}^{\rho\sigma}=\widetilde{R}^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}.\end{aligned} \tag{B.1.8}$$

Finally, the anomaly coefficients a and c are normalized as in [77], so that for N_χ free chiral multiplets and N_v free vector multiplets they take the form

$$a=\frac{1}{48}(N_\chi+9N_v),\qquad c=\frac{1}{24}(N_\chi+3N_v). \tag{B.1.9}$$

In particular, for the WZ model we have

$$c=2a=\frac{1}{24}. \tag{B.1.10}$$

APPENDIX C

Functional differentiation versus operator insertions

In this appendix we discuss the relation between correlation functions defined through functional differentiation and those obtained by path integral operator insertions. To emphasise the difference we use different notation for the two types of correlator. Throughout this thesis correlation functions defined through functional differentiation are denoted by wide brackets, $\langle \cdot \rangle$, while those involving operator insertions by $<\cdot>$.

Let J denote the source for an operator $\mathcal{O}$. Then the two definitions of n-point functions are

$$\langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \mathscr{W}[J] \Big|_{J=0}, \tag{C.0.1}$$

$$< \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) > = \frac{1}{\mathscr{Z}} \int [\mathrm{d}\{\Phi\}] \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) e^{iS[\{\Phi\}]}, \tag{C.0.2}$$

where $\{\Phi\}$ denotes collectively all elementary fields and $\mathscr{W}[J] = -i \log \mathscr{Z}[J]$. The functional derivatives defining the wide bracket correlators in (C.0.1) are taken with the operators kept fixed. In contrast, in the correlators defined in (C.0.2) one takes the functional derivatives in the path integral keeping fixed the operator at $J = 0$. The chain rule in this case produces additional semi-local correlators involving $\delta \mathcal{O}(x_i)/\delta J(x_j)$ – the so-called 'seagull terms'. We should emphasize that this dependence of the operator $\mathcal{O}$ on the source J that arises through the classical coupling of the theory to background sources is local and should not be confused with the generically non-local dependence of the 1-point function $\langle \mathcal{O} \rangle_J$ that is obtained by performing the path integral in the presence of sources. The seagull terms are theory dependent and their contribution drops out when all insertions are at separated points. However, since our purpose is to discuss anomalies, which are local contributions to the Ward identities, we cannot ignore the contribution of seagull terms. The distinction between the two definitions of correlators is important, because the structure of ultraviolet divergences is different in the two cases. In particular, only the ultraviolet divergences of current multiplet correlators defined through functional differentiation can be cancelled by counterterms that depend on the background supergravity fields. We should also mention that Feynman diagram computations, which we use for evaluating the Ward identities at the quantum level, result in correlation functions involving operator insertions.

From (C.0.1) and (C.0.2), it follows that the two definitions of current multiplet correlation functions of the conformal WZ model that are relevant for our analysis are related as

$$\langle \mathcal{J}^\mu(x)\mathcal{J}^\nu(y)\rangle = \, i <\mathcal{J}^\mu(x)\mathcal{J}^\nu(y)> + <\frac{\delta \mathcal{J}^\mu(x)}{\delta A_\nu(y)}>,$$

$$\begin{aligned}\langle \mathcal{J}^\mu(x)\mathcal{J}^\nu(y)\mathcal{J}^\kappa(z)\rangle = \, & i^2 <\mathcal{J}^\mu(x)\mathcal{J}^\nu(y)\mathcal{J}^\kappa(z)> \\ & + i <\frac{\delta \mathcal{J}^\mu(x)}{\delta A_\kappa(z)}\mathcal{J}^\nu(y)> + i <\mathcal{J}^\mu(x)\frac{\delta \mathcal{J}^\nu(y)}{\delta A_\kappa(z)}> + i <\frac{\delta \mathcal{J}^\mu(x)}{\delta A_\nu(y)}\mathcal{J}^\kappa(z)>,\end{aligned}$$

$$\langle \mathcal{T}^\mu_\nu(x)\mathcal{J}^\rho(y)\rangle = \, i <\mathcal{T}^\mu_\nu(x)\mathcal{J}^\rho(y)> + <\frac{\delta \mathcal{T}^\mu_\nu(x)}{\delta A_\rho(y)}>,$$

$$\begin{aligned}\langle \mathcal{T}^\mu_\nu(x)\mathcal{J}^\rho(y)\mathcal{J}^\sigma(z)\rangle = \, & i^2 <\mathcal{T}^\mu_\nu(x)\mathcal{J}^\rho(y)\mathcal{J}^\sigma(z)> \\ & + i <\frac{\delta \mathcal{T}^\mu_\nu(x)}{\delta A_\rho(y)}\mathcal{J}^\sigma(z)> + i <\frac{\delta \mathcal{T}^\mu_\nu(x)}{\delta A_\sigma(z)}\mathcal{J}^\rho(y)> + i <\mathcal{T}^\mu_\nu(x)\frac{\delta \mathcal{J}^\rho(y)}{\delta A_\sigma(z)}> + <\frac{\delta^2 \mathcal{T}^\mu_\nu(x)}{\delta A_\rho(y)\delta A_\sigma(z)}>,\end{aligned}$$

$$\langle \mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\rangle = \, i <\mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)> + <\frac{\delta \mathcal{Q}^\mu(x)}{\delta \psi_\nu(y)}>,$$

$$\begin{aligned}\langle \mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\mathcal{J}^\rho(z)\rangle = \, & i^2 <\mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\mathcal{J}^\rho(z)> \\ & + i <\frac{\delta \mathcal{Q}^\mu(x)}{\delta \psi_\nu(y)}\mathcal{J}^\rho(z)> + i <\frac{\delta \mathcal{Q}^\mu(x)}{\delta A_\rho(z)}\bar{\mathcal{Q}}^\nu(y)> + i <\mathcal{Q}^\mu(x)\frac{\delta \bar{\mathcal{Q}}^\nu(y)}{\delta A_\rho(z)}>,\end{aligned}$$

$$\begin{aligned}\langle \mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\mathcal{J}^\rho(z)\mathcal{J}^\sigma(w)\rangle = \, & i^3 <\mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\mathcal{J}^\rho(z)\mathcal{J}^\sigma(w)> \\ & + i^2 <\frac{\delta \mathcal{Q}^\mu(x)}{\delta \psi_\nu(y)}\mathcal{J}^\rho(z)\mathcal{J}^\sigma(w)> + i^2 <\frac{\delta \mathcal{Q}^\mu(x)}{\delta A_\rho(z)}\bar{\mathcal{Q}}^\nu(y)\mathcal{J}^\sigma(w)> + i^2 <\frac{\delta \mathcal{Q}^\mu(x)}{\delta A_\sigma(w)}\bar{\mathcal{Q}}^\nu(y)\mathcal{J}^\rho(z)> \\ & + i <\frac{\delta \mathcal{Q}^\mu(x)}{\delta \psi_\nu(y)}\frac{\delta \mathcal{J}^\rho(z)}{\delta A_\sigma(w)}> + i <\frac{\delta \mathcal{Q}^\mu(x)}{\delta A_\rho(z)}\frac{\delta \bar{\mathcal{Q}}^\nu(y)}{A_\sigma(w)}> + i <\frac{\delta \mathcal{Q}^\mu(x)}{\delta A_\sigma(w)}\frac{\delta \bar{\mathcal{Q}}^\nu(y)}{\delta A_\rho(z)}>\end{aligned}$$

$$+ i^2 <\mathcal{Q}^\mu(x)\frac{\delta\bar{\mathcal{Q}}^\nu(y)}{\delta A_\rho(z)}\mathcal{J}^\sigma(w)> +i^2 <\mathcal{Q}^\mu(x)\frac{\delta\bar{\mathcal{Q}}^\nu(y)}{\delta A_\sigma(w)}\mathcal{J}^\rho(z)> +i^2 <\mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\frac{\mathcal{J}^\rho(z)}{\delta A_\sigma(w)}> . \tag{C.0.3}$$

Analogous expressions can be derived for any other current multiplet correlator. For the conformal current multiplet correlators of the massless WZ model we discuss in chapter 4, the relation between these two definitions simplifies because several operator derivatives vanish. In particular, using the form of the currents in (4.1.6) and (4.3.9) when the theory is coupled to background supergravity, we determine that the only non zero operator derivatives are

$$\begin{aligned}
\frac{\delta\mathcal{T}^{\mu\nu}(x)}{\delta A_\rho(y)} &= -(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\rho\sigma})\delta(x,y)\mathcal{J}_\sigma+\frac{i}{6}\eta^{\mu\rho}\delta(x,y)\hat{s}^\nu_{(2|1)},\\
\frac{\delta\mathcal{T}^{\mu\nu}(x)}{\delta A_\rho(y)\delta A_\sigma(z)} &= \frac{8}{9}(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\rho\sigma})\delta(x,y)\delta(x,z)\hat{s}_{(1|0)},\\
\frac{\delta\mathcal{J}^\mu(x)}{\delta A_\nu(y)} &= -\frac{8}{9}\eta^{\mu\nu}\delta(x,y)\hat{s}_{(1|0)},\\
\frac{\delta\mathcal{J}^\mu(x)}{\delta\bar{\psi}_\nu(y)} &= \frac{\delta\mathcal{Q}^\mu(x)}{\delta A_\nu(y)} = -\frac{\sqrt{2}}{3}\eta^{\mu\nu}\delta(x,y)\hat{s}_{(3|\frac{1}{2})},\\
\frac{\delta\mathcal{Q}^\mu(x)}{\delta\psi_\nu(y)} &= \frac{3}{8}\epsilon^{\mu\nu\rho\sigma}\hat{\mathcal{J}}_\rho\gamma_\sigma\delta(x,y)+\frac{1}{8}\hat{s}^\rho_{(2|1)}\gamma_5\gamma_\rho\eta^{\mu\nu}\delta(x,y)\\
&\quad+\frac{1}{6}\partial^\rho\hat{s}_{(1|0)}(2\eta^{[\mu}_\rho\eta^{\nu]}_\sigma+i\epsilon^{\mu\nu}{}_{\rho\sigma}\gamma_5)\gamma^\sigma\delta(x,y)+\frac{i}{3}\epsilon^{\mu\nu\rho\sigma}\hat{s}_{(1|0)}\gamma_5\gamma_\sigma\partial^x_\rho\delta(x,y).
\end{aligned}\tag{C.0.4}$$

Clearly, the difference between correlation functions defined via functional differentiation and operator insertions affects the form of the Ward identities. In section (4.2) we present the Ward identities in terms of correlators defined using operator insertions, while in (4.3.10) we present the corresponding identities at the 1-point function level, in terms of correlation functions defined through functional differentiation. The latter form of the Ward identities is universal and follows directly from the symmetries of the background supergravity the current multiplet couples to. To derive the Ward identities of higher order correlators we need to further differentiate with respect to the appropriate background sources. When expressed in terms of correlators defined through operator insertions, however, these identities contain the additional seagull terms which can be seen explicitly in the path integral Ward identities of section (4.2). In fact, as already mentioned in section (2.2), the universal form of the Ward identities in terms of correlators obtained by functional differentiation can be expressed as a linear combination of path integral Ward identities involving operator insertions.

To illustrate this point, let us consider the R-symmetry Ward identity of the 3-point correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$. We compute this identity by taking two functional derivatives with respect to the gravitini in the R-symmetry Ward identity that one finds after coupling the WZ model to conformal supergravity (4.3.10). We find that

$$\nabla_\kappa \langle \mathcal{J}^\kappa(z)\mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(y)\rangle + \delta(z-x)i\gamma_5\langle \mathcal{Q}^\mu(z)\bar{\mathcal{Q}}^\nu(y)\rangle + \delta(z-y)i\langle \mathcal{Q}^\mu(x)\bar{\mathcal{Q}}^\nu(z)\rangle\gamma_5 = 0. \tag{C.0.5}$$

Using now the relations between the correlators (C.0.3) and the explicit expressions for the derivatives of the currents (C.0.4) in the massless WZ model, we find that the above identity in momentum space and in the flat space limit is modified as follows

$$\begin{aligned}
&p_{3\kappa} <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)> \\
&+ p_{3\kappa}\Bigg(i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <s_{(3|\frac{1}{2})}(p_{13})\bar{\mathcal{Q}}^\nu(p_2)> + i\eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{Q}^\mu(p_1)\bar{s}_{(3|\frac{1}{2})}(p_{23})> \\
&+ \frac{3i}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi <\mathcal{J}_\sigma(p_{12})\mathcal{J}^\kappa(p_3)> + \frac{i}{8}\eta^{\mu\nu}\gamma^\sigma\gamma_5 <s_{\sigma(2|1)}(p_{12})\mathcal{J}^\kappa(p_3)> \\
&- p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^\mu - \frac{1}{6}\eta^{\sigma\mu}\gamma^\nu + i\frac{1}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi\gamma_5\right) <s_{(1|0)}(p_{12})\mathcal{J}^\kappa(p_3)>\Bigg) \\
&= i\gamma_5 <\mathcal{Q}^\mu(p_{13})\bar{\mathcal{Q}}^\nu(p_2)> + i <\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_{23})> \gamma_5.
\end{aligned} \tag{C.0.6}$$

We can easily see that this expression is identical to the Ward identity (4.2.12), which as explained there, is a linear combination of path integral Ward identities. In particular, using the symmetry variations of the currents and seagull operators written in subsection (4.1.5) and the path integral Ward identities (2.2.8), it is straightforward to show that terms of the same colour in (C.0.6) form their own path integral Ward identities. Here we also justify the claim that we made in section (4.2), that the Ward identities presented there are the same with the ones we get after coupling the theory to background conformal supergravity.

Schwinger–Dyson equations

The Ward identity is a relation between correlation functions that involve conserved currents and follow from global symmetries of the theory. Using the path integral formalism, we derived its general form (2.2.8). One can find similar identities for operators that vanish on-shell (i.e. operators proportional to equations of motion), such as $\bar{\chi}\bar{\partial}\!\!\!/\chi$ in the massless WZ model (4.1.1), where the equations of motion imply that $\bar{\partial}\!\!\!/\chi = 0$.

Consider the action $S[\phi]$, and the normal ordered composite operators $\mathcal{O}[\phi]$, $G[\phi]\frac{\delta S[\phi]}{\delta\phi}$, where ϕ is an elementary field of the theory and $\frac{\delta S[\phi]}{\delta\phi}$ give the equations of motion. $G[\phi]\frac{\delta S[\phi]}{\delta\phi}$ is the vanishing on-shell operator we are interested in. For example, in the case of $\bar{\chi}\bar{\partial}\!\!\!/\chi$, we identify $G[\phi] \to \bar{\chi}$ and $\frac{\delta S[\phi]}{\delta\phi} \to -\bar{\partial}\!\!\!/\chi$. We have

$$\int [d\phi] G(x)\frac{\delta S}{\delta\phi(x)}\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)e^{iS} = \int [d\phi] G(x)\frac{\delta e^{iS}}{i\delta\phi(x)}\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)$$

$$= i\int [d\phi] G(x)\frac{\delta\mathcal{O}(x_1)}{\delta\phi(x)}\cdots\mathcal{O}(x_n)e^{iS} + i\int [d\phi]\mathcal{O}(x_1)\cdots G(x)\frac{\delta\mathcal{O}(x_n)}{\delta\phi(x)}e^{iS}$$

$$\Rightarrow -i < G(x)\frac{\delta S}{\delta\phi(x)}\mathcal{O}(x_1)\cdots\mathcal{O}(x_n) > = < G(x)\frac{\delta\mathcal{O}(x_1)}{\delta\phi(x)}\cdots\mathcal{O}(x_n) > + < \mathcal{O}(x_1)\cdots G(x)\frac{\delta\mathcal{O}(x_n)}{\delta\phi(x)} > . \tag{C.0.7}$$

To go from the first to second line we integrated by parts in the field space. (C.0.7) is the Schwinger–Dyson equation for correlators that involve zero on-shell operators. Notice that the $(n+1)$-point correlator of the lhs is given by a sum of n-point functions. To derive the Schwinger–Dyson equation, first we need to identify the $G(x)$ part of the vanishing operator under consideration, and then compute the normal ordered operators $G(x)\frac{\delta\mathcal{O}(x_n)}{\delta\phi(x)}$.

The Schwinger–Dyson equations are important, if one wishes to match the Q-supersymmetry Ward identities derived through the path integral formalism (2.2.8), with the corresponding ones from conformal supergravity. In particular, consider the Q-supersymmetry identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. According to (2.2.8), since at the lhs we have a 4-point correlator, at the rhs we are going to have only 3-point functions. Notice however, that in the Q-supersymmetry identity of conformal supergravity (4.2.16), there exist a number of 2-point correlators at the rhs. This can be explained by the fact that the Q-supersymmetry variations of the supercurrent and the R-current (4.1.13) contain zero on-shell terms. As we explained schematically in (4.1.16), there exist some 3-point correlators at the rhs of the Q-supersymmetry identity, that involve these vanishing on-shell operators. One can substitute them using the Schwinger–Dyson equations (C.0.7), and retrieve in this way the 2-point correlators of the conformal supergravity identity (4.2.16). However, since we already had the conformal supergravity Ward identities, there was no need to use the Schwinger–Dyson equations in our analysis for the anomalies of the WZ model.

APPENDIX D

Correlators

D.1 Regularization

In this appendix we write all correlation functions necessary for the analysis of the Ward identities of the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. These correlators receive contributions from the original massless WZ model (4.1.1) and the PV Lagrangian (6.1.1). Although the contribution from each Lagrangian does not converge separately, the total sum is free from UV divergences, if the following condition on PV masses is met

$$m_2 = \sqrt{2} m_1. \tag{D.1.1}$$

All n-point correlators of the following sections, which after suppressing the spacetime indices we denote here by $<n>$, can be written in the following form

$$<n>= \sum_i d_i N(m_i) \equiv N(0) + N(m_2) - 2N(m_1), \tag{D.1.2}$$

where $i = 0, 1, 2$, $d_0 = 1$, $d_2 = 1$, $d_1 = -2$ and $m_0 = 0$. $N(0)$ is the contribution to the correlator of the original WZ model (4.1.1), $N(m_2)$ is the contribution of the PV fields (φ_2, λ_2) that comprise a standard massive chiral multiplet, and $-2N(m_1)$ is the contribution of the fields $(\varphi_1, \vartheta_1, \lambda_1)$. The factor of 2 in front of $N(m_1)$ is to be expected, since the fields $(\varphi_1, \vartheta_1, \lambda_1)$ form two chiral multiplets with wrong statistics. For the regularization of $<n>$, it is crucial that the contribution of $(\varphi_1, \vartheta_1, \lambda_1)$ comes with a relative minus sign with respect to the original fields. It is quite common in literature to assume that fields of opposite statistics contribute to the correlation functions with a relative minus sign compared to the fields with standard statistics. However, this is not always so obvious. Here we have confirmed that this is indeed true, after performing all possible Wick contractions of the elementary fields.

The quantities $N(m_i)$ are divergent integrals. The degree of divergence depends on the specific correlator we are examining. The most badly divergent correlators by power counting, are the 2-point functions $<\mathcal{Q}\bar{\mathcal{Q}}>$ (D.3.6) and $<\mathcal{T}\mathcal{J}>$ (D.3.14) which diverge cubically, i.e. in the large momentum limit they behave as $\sim \int_0^\infty dq q^2$. Higher order correlators have a better behaviour in the UV limit due to the increased number of propagators that lower the superficial degree of divergence. For example, the 3-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ (D.4.1) is quadratically divergent while the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ (D.5.1) diverges linearly. Our aim is to show that the rhs of (D.1.2) is well defined and convergent. For that we Taylor expand $N(m_i)$ in powers of m_i. We get

$$N(m_i) = N(0) + m_i^2 N_2 + m_i^4 N_4 + ... \tag{D.1.3}$$

where the dots ... denote higher order terms. Notice that in the above expansion there are no odd powers in the mass m_i. It can be shown, after an explicit computation using the expressions for the correlators written in this appendix, that all these terms vanish due to the structure of the gamma matrices. In particular, the correlators where fermionic operators are involved, such as $<\mathcal{Q}\bar{\mathcal{Q}}>$ (D.3.6), take the form $P_R(...)P_L$, where P_R and P_L are the projection operators (A.0.14). This means that in the dots (...), only terms with an odd number of gamma matrices survive. Similarly, the odd powers of the mass m_i in the correlators of bosonic operators, such as $<\mathcal{J}\mathcal{J}>$ (D.3.9), vanish after using the trace identities.

If the integral $N(0)$ has a superficial degree of divergence d[1], then the corresponding degree of divergence of the quantities N_2 and N_4 will be $d-2$ and $d-4$ respectively. Using (D.1.3) in (D.1.2) we get

$$<n>= (N(0) + N(0) - N(0) - N(0)) + (m_2^2 - 2m_1^2)N_2 + (m_2^4 - 2m_1^4)N_4 + ... \tag{D.1.4}$$

Taking into account that the most badly divergent correlators we are interested in have a $d = 3$, we see that N_2 is at most linearly divergent ($d = 1$), while N_4 is always convergent ($d = -1$). All the other higher order terms in the above expansion are of course convergent. Using the condition (D.1.1) we find that the two leading order pieces of (D.1.4) vanish, so

$$<n>= 2m_1^4 N_4 + ... \tag{D.1.5}$$

[1]A negative d denotes a convergent integral, $d = 0$ and $d = 1$ denote logarithmically and linearly divergent integrals etc.

We have proven that the PV Lagrangian (6.1.1) suffices for removing all the UV divergences from the correlation functions necessary for the analysis of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. The two main technical elements for this proof were the fact that the most divergent correlators were cubically divergent, and that there were no odd powers in the PV mass expansion of the correlators. The first can be seen after examining the superficial degree of divergence of the correlators and the second after explicit computations using the expressions for the correlators.

According to the anomalies (B.1.3) derived using the WZ consistency conditions, the 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$ has a Q-supersymmetry anomaly too. We made the claim in chapter 4 that our PV regulator is not appropriate for the analysis of the aforementioned correlator. One of the reasons for that, is that among others, we need to use the 2-point function $<\mathcal{T}\mathcal{T}>$ which cannot be regulated by the Lagrangian (6.1.1). $<\mathcal{T}\mathcal{T}>$ is quartically divergent by power counting ($d = 4$), hence N_4 is superficially logarithmically divergent. After an explicit computation, one can show that in the corresponding expression (D.1.5) for $<\mathcal{T}\mathcal{T}>$, there exists indeed a non regulated logarithmic divergence which needs more PV fields in order to be cancelled.

D.2 1-point functions

The 1-point functions of the supercurrent and of the scalar operators $\mathcal{O}_M$, $\mathcal{O}_{M^*}$ vanish trivially due to the absence of possible self contractions. The R-current 1-point function is given by

$$<\widetilde{\mathcal{J}}^{\mu}(p)>= \frac{2i}{3}\int \frac{d^4q}{(2\pi)^4}\Big(2iq^{\mu}(P_{\phi}(q) + P_{\varphi_2}(q) - P_{\varphi_1}(q) - P_{\vartheta_1}(q)) - \frac{1}{4}\mathrm{tr}\left[\gamma^{\mu}\gamma_5(P_{\chi}(q) + P_{\lambda_2}(q) - 2P_{\lambda_1}(q))\right]\Big) = 0, \tag{D.2.1}$$

where the first line vanishes due to parity and in the second line we have used the trace identities (A.0.7).

It is straightforward to show that the stress tensor 1-point function vanishes too. We have

$$<\widetilde{\mathcal{T}}^{\mu\nu}(p)>= \int \frac{d^4q}{(2\pi)^4}\Big(2q^{\mu}q^{\nu}(P_{\phi}(q) + P_{\varphi_2}(q) - P_{\varphi_1}(q) - P_{\vartheta_1}(q)) + i\eta^{\mu\nu}(1+1-1-1) - \frac{1}{2}q^{\nu}\mathrm{tr}\left[i\gamma^{\mu}(P_{\chi}(q) + P_{\lambda_2}(q) - 2P_{\lambda_1}(q))\right] - \frac{i}{2}\eta^{\mu\nu}\mathrm{tr}\,(1+1-2)\Big) = 0, \tag{D.2.2}$$

where the last equality follows from the relation between the scalar and spinor propagators

$$P_\chi(p) = -i\not{p} P_\phi(p), \qquad P_{\lambda_1}(p) = (-i\not{p} + m_1) P_{\varphi_1}(p), \qquad P_{\lambda_2}(p) = (-i\not{p} + m_2) P_{\varphi_2}(p), \tag{D.2.3}$$

and the trace identities (A.0.7).

Lastly, the non zero 1-point function of the seagull operator $s_{(1|0)}$ is given by

$$<\widetilde{s}_{(1|0)}(p)> = \int \frac{d^4 q}{(2\pi)^4} \left(P_\phi(q) + P_{\varphi_2}(q) - P_{\varphi_1}(q) - P_{\vartheta_1}(q) \right) = \frac{1}{8\pi^2} m_1^2 \log 2. \tag{D.2.4}$$

Here we have used the relation $m_2 = \sqrt{2} m_1$ between the two masses of the PV regulator.

D.3 2-point functions

In this and in the following two sections, we write the exact expressions that one gets for every correlation function, after performing all possible Wick contractions of the elementary fields. We have not made any other manipulations in the integrals below. Every correlator is a sum of three integrals, one that comes from the original WZ model, one that comes from the massive chiral multiplet of (6.1.1) with standard statistics, and there is one contribution from the two chiral multiplets of opposite statistics of (6.1.1). The i is summed over the values $i = 0, 1, 2$, and d_i take the values $d_0 = 1$, $d_1 = -2$ and $d_2 = 1$. To simplify the expressions we also define the following quantities:

$$\lambda_0 \equiv \chi \,, \qquad \varphi_0 \equiv \phi, \tag{D.3.1}$$

$$p_{ij} = p_i + p_j, \tag{D.3.2}$$

$$C_i^\mu(k) = i\not{k}\gamma^\mu + m_i \gamma^\mu + \frac{i}{3} [\gamma^\mu, \gamma^\nu] \, q_{1\nu}, \tag{D.3.3}$$

$$A_i^\nu(k) = i\gamma^\nu \not{k} - m_i \gamma^\nu - \frac{i}{3} [\gamma^\nu, \gamma^\rho] \, q_{2\rho}, \tag{D.3.4}$$

$$G_i^{\mu\nu} = (q_1 - k)^\mu k^\nu + (q_1 - k)^\nu k^\mu - \eta^{\mu\nu} (q_1 - k) \cdot k + \frac{1}{3} \left(\eta^{\mu\nu} q_1^2 - q_1^\nu q_1^\mu \right) + m_i^2 \eta^{\mu\nu}, \tag{D.3.5}$$

where $m_0 = 0$.

The cubically divergent 2-point function $<\mathcal{Q}\bar{\mathcal{Q}}>$, linearly divergent $<s_{(3|\frac{1}{2})}\bar{s}_{(3|\frac{1}{2})}>$ and quadratically divergent $<\mathcal{Q}\bar{s}_{(3|\frac{1}{2})}>$ are given by:

$$<\widetilde{\mathcal{Q}}^\mu(p_1) \widetilde{\bar{\mathcal{Q}}}^\nu(p_2)> = \sum_{i=0}^{2} \int \frac{d^4 q}{(2\pi)^4} \frac{d_i}{2} P_R C_i^\mu(q) P_{\lambda_i}(p_1 - q) A_i^\nu(-q) P_L P_{\varphi_i}(q) + (P_R \leftrightarrow P_L) \,, \tag{D.3.6}$$

$$<\widetilde{s}_{(3|\frac{1}{2})}(p_1)\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_2)>= -\sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4} d_i \gamma_5 P_R P_{\lambda_i}(p_1-q)P_L\gamma_5 P_{\varphi_i}(q) + (P_R \leftrightarrow P_L)\,, \tag{D.3.7}$$

$$<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_2)>= \sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4} d_i \frac{i\sqrt{2}}{2} P_R C_i^\mu(q) P_{\lambda_i}(p_1-q)P_L\gamma_5 P_{\varphi_i}(q) + (P_R \leftrightarrow P_L)\,. \tag{D.3.8}$$

The quadratically divergent bosonic correlators $<\mathcal{J}\mathcal{J}>$, $<\mathcal{J}s_{(2|1)}>$, $<s_{(2|1)}s_{(2|1)}>$ and $<\mathcal{T}s_{(1|0)}>$ take the form

$$<\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>= \sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4}\, d_i\, \frac{4}{9}(2q+p_3)^\kappa(2q+p_3)^\lambda P_{\varphi_i}(q)P_{\varphi_i}(q+p_3)$$
$$+\sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4}\, d_i\, \frac{1}{18}\mathrm{tr}\left(\gamma^\kappa\gamma_5 P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\right), \tag{D.3.9}$$

$$-6i<\widetilde{s}^\kappa_{(2|1)}(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=<\widetilde{s}^\kappa_{(2|1)}(p_3)\widetilde{s}^\lambda_{(2|1)}(p_4)>$$
$$= -\sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4} 2\, d_i\, \mathrm{tr}\left(\gamma^\kappa\gamma_5 P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\right), \tag{D.3.10}$$

and

$$<\widetilde{\mathcal{T}}^{\mu\nu}(p_1)\widetilde{s}_{(1|0)}(p_3)>= -\sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4} d_i G_i^{\mu\nu}(q)P_{\varphi_i}(q)P_{\varphi_i}(q+p_3), \tag{D.3.11}$$

while the logarithmically divergent $<s_{(1|0)}s_{(1|0)}>$ is equal to

$$<\widetilde{s}_{(1|0)}(p_3)\widetilde{s}_{(1|0)}(p_4)>= \sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4}\, d_i\, P_{\varphi_i}(q)P_{\varphi_i}(q+p_3). \tag{D.3.12}$$

The following two correlators, $<\mathcal{J}s_{(1|0)}>$ and $<\mathcal{T}\mathcal{J}>$, turn out to be zero, i.e.

$$<\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{s}_{(1|0)}(p_4)>= \sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4}\, d_i\, \frac{2}{3}(2q+p_3)^\kappa P_{\varphi_i}(q)P_{\varphi_i}(q+p_3) = 0, \tag{D.3.13}$$

$$<\widetilde{\mathcal{T}}^{\mu\nu}(p_1)\widetilde{\mathcal{J}}^\lambda(p_4)>= \sum_{i=0}^{2}\int \frac{d^4q}{(2\pi)^4} d_i \left(\frac{-i}{6}\mathrm{tr}\left(\gamma_\sigma P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\right)(iq^\nu\eta^{\mu\sigma} - iq^\sigma\eta^{\mu\nu})\right.$$
$$\left. -\frac{-i}{6}\mathrm{tr}\left(P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\right) M_i\eta^{\mu\nu}\right)$$

$$+\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\frac{1}{24}\mathrm{tr}\left(\gamma_\sigma\gamma_5 P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\right)\epsilon^{\mu\nu\rho\sigma}ip_{1\rho}=0. \quad \text{(D.3.14)}$$

It is straightforward to show that (D.3.13) vanishes, after the transformation $q\to -q-p_3$ at the integration variable. In (D.3.14) we have not included the integrals that come from the Wick contractions of the elementary fields ϕ, φ_2, φ_1, ϑ_1 and their complex conjugates, since they are identically zero due to odd symmetry. Moreover, in the large PV mass limit, the two sums of integrals in (D.3.14) cancel each other.

Finally, in order to justify our claim that the PV Lagrangian (6.1.1) is not enough to regulate $<\mathcal{T}\mathcal{T}>$ (which is important for the complete analysis of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{T}\mathcal{J}>$), we write the 2-point function $<\mathcal{B}_W\mathcal{B}_W>$, which is necessary for the analysis of the Ward identities of $<\mathcal{T}\mathcal{T}>$. Using (6.2.6) we find that

$$\begin{aligned}<\mathcal{B}_W(p_1)\mathcal{B}_W(p_2)>=&\int\frac{d^4q}{(2\pi)^4}\left(4m_2^4P_{\varphi_2}(q)P_{\varphi_2}(q+p_2)-8m_1^4P_{\varphi_1}(q)P_{\varphi_1}(q+p_2)\right)\\ &+\int\frac{d^4q}{(2\pi)^4}\left(-\frac{m_2^2}{2}\mathrm{tr}\left(P_{\lambda_2}(q)P_{\lambda_2}(q+p_2)\right)+m_1^2\mathrm{tr}\left(P_{\lambda_1}(q)P_{\lambda_1}(q+p_2)\right)\right). \quad \text{(D.3.15)}\end{aligned}$$

Using the relation of the PV masses (D.1.1), we find that in the above first line there exists a logarithmic divergence which is equal to $-\frac{2i\pi^2}{(2\pi)^4}8m_1^4\int\frac{dq}{q}$. This divergence is not cancelled by the divergent terms of the integral of the second line. Thus, we see the failure of the PV Lagrangian (6.1.1) to regulate $<\mathcal{B}_W\mathcal{B}_W>$.

D.4 3-point functions

The fermionic 3-point correlators that we are interested in are the quadratically divergent $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ and the linearly divergent $<\mathcal{Q}\bar{s}_{(3|\frac{1}{2})}\mathcal{J}>$, $<\mathcal{Q}\bar{\mathcal{Q}}s_{(1|0)}>$. They take the following form:

$$\begin{aligned}&<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)>=\\ &\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\left(\frac{i}{6}P_{\varphi_i}(q)P_RC_i^\mu(q)P_{\lambda_i}(p_1-q)\gamma^\kappa\gamma_5P_{\lambda_i}(-q-p_2)A_i^\nu(-q)P_L\right.\\ &\left.+\frac{1}{3}(2q+p_3)^\kappa P_{\varphi_i}(q)P_{\varphi_i}(q+p_3)P_R\gamma_5C_i^\mu(q)P_{\lambda_i}(p_1-q)A_i^\nu(-q-p_3)P_L\right)+(P_R\leftrightarrow P_L),\end{aligned} \quad \text{(D.4.1)}$$

$$\begin{aligned}&<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)>=\\ &\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\left(\frac{-\sqrt{2}}{6}P_{\varphi_i}(q)P_RC_i^\mu(q)P_{\lambda_i}(p_1-q)\gamma^\kappa\gamma_5P_{\lambda_i}(-q-p_2)P_L\gamma_5\right.\end{aligned}$$

$$+\frac{\sqrt{2}i}{3}(2q+p_3)^\kappa P_{\varphi_i}(q)P_{\varphi_i}(q+p_3)P_R\gamma_5 C_i^\mu(q)P_{\lambda_i}(p_1-q)P_L\gamma_5\Big)+(P_R\leftrightarrow P_L), \quad \text{(D.4.2)}$$

$$<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{s}_{(1|0)}(p_3)>=$$
$$\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\frac{1}{2}P_{\varphi_i}(q)P_{\varphi_i}(q+p_3)P_R C_i^\mu(q)P_{\lambda_i}(p_1-q)A_i^\nu(-q-p_3)P_L+(P_R\leftrightarrow P_L). \quad \text{(D.4.3)}$$

Similarly, the bosonic 3-point correlators relative for our analysis are the quadratically divergent $<\mathcal{T}\mathcal{J}\mathcal{J}>$ and the linearly divergent $<\mathcal{J}\mathcal{J}\mathcal{J}>$. They are given below:

$$<\widetilde{\mathcal{T}}^{\mu\nu}(p_1)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\frac{-4}{9}G_i^{\mu\nu}(q)(2q+p_4-p_1)^\kappa(2q+p_4)^\lambda$$
$$P_{\varphi_i}(q-p_1)P_{\varphi_i}(q)P_{\varphi_i}(q+p_4)+\begin{pmatrix}p_3 & \leftrightarrow & p_4\\ \kappa & \leftrightarrow & \lambda\end{pmatrix}$$
$$+\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\left(\frac{1}{18}\mathrm{tr}\left(\gamma_\sigma P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\gamma^\kappa\gamma_5 P_{\lambda_i}(q-p_1)\right)(iq^\nu\eta^{\mu\sigma}-iq^\sigma\eta^{\mu\nu})\right.$$
$$\left.-\frac{1}{18}\mathrm{tr}\left(P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\gamma^\kappa\gamma_5 P_{\lambda_i}(q-p_1)\right)M_i\eta^{\mu\nu}\right)+\begin{pmatrix}p_3 & \leftrightarrow & p_4\\ \kappa & \leftrightarrow & \lambda\end{pmatrix}$$
$$+\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\frac{i}{72}\mathrm{tr}\left(\gamma_\sigma\gamma_5 P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\gamma^\kappa\gamma_5 P_{\lambda_i}(q-p_1)\right)\epsilon^{\mu\nu\rho\sigma}ip_{1\rho}+\begin{pmatrix}p_3 & \leftrightarrow & p_4\\ \kappa & \leftrightarrow & \lambda\end{pmatrix}, \quad \text{(D.4.4)}$$

$$<\widetilde{\mathcal{J}}^\mu(p_1)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=$$
$$\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i\frac{i}{54}\mathrm{tr}\left(\gamma^\mu\gamma_5 P_{\lambda_i}(q)\gamma^\lambda\gamma_5 P_{\lambda_i}(q+p_4)\gamma^\kappa\gamma_5 P_{\lambda_i}(q-p_1)\right)+\begin{pmatrix}p_3 & \leftrightarrow & p_4\\ \kappa & \leftrightarrow & \lambda\end{pmatrix}. \quad \text{(D.4.5)}$$

D.5 4-point function

The linearly divergent 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ is given by

$$<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=\sum_{i=0}^{2}\int\frac{d^4q}{(2\pi)^4}d_i$$
$$\Big(-\frac{1}{18}P_{\varphi_i}(q)P_R C_i^\mu(q)P_{\lambda_i}(p_1-q)\gamma^\kappa\gamma_5 P_{\lambda_i}(-q-p_{24})\gamma^\lambda\gamma_5 P_{\lambda_i}(-q-p_2)A_i^\nu(-q)P_L$$
$$+\frac{2}{9}(2q+p_3)^\kappa(2q+2p_3+p_4)^\lambda P_{\varphi_i}(q)P_{\varphi_i}(q+p_3)P(q+p_{34})P_R C_i^\mu(q)P_{\lambda_i}(p_1-q)A_i^\nu(-q-p_{34})P_L$$
$$+\frac{2i}{18}(2q+p_3)^\kappa P_{\varphi_i}(q)P_{\varphi_i}(q+p_3)\gamma_5 P_R C_i^\mu(q)P_{\lambda_i}(p_1-q)\gamma^\lambda\gamma_5 P_{\lambda_i}(-q-p_{23})A_i^\nu(-q-p_3)P_L\Big)$$

$$+ (P_R \leftrightarrow P_L) + \begin{pmatrix} p_3 & \leftrightarrow & p_4 \\ \kappa & \leftrightarrow & \lambda \end{pmatrix}. \tag{D.5.1}$$

APPENDIX E

Correlators with insertions of $\mathcal{B}_W$, $\mathcal{B}_R$ and $\mathcal{B}_S$

In this appendix we evaluate all 1-loop correlators that involve the symmetry breaking operators $\mathcal{B}_W$, $\mathcal{B}_R$ and $\mathcal{B}_S$ and constitute the potential anomalies in the symmetry Ward identities of chapter 7. Since these operators depend on the PV fields only, all such correlators are pure contact terms once the PV masses are sent to infinity.

The aim of this appendix is to follow the analysis and structure of chapter 7, and provide all the intermediate steps that lead to the final results presented there. We also define the following quantities to simplify the expressions of the correlators:

$$\begin{aligned}
\Delta_i^{(1)}(p) &= m_i + p^2(1-y)y \\
\Delta_i^{(2)}(p,q) &= m_i + p^2 y + q^2 z - (py - qz)^2 \\
\Delta_i^{(3)}(p,q,k) &= m_i + p^2 y + q^2 z + k^2 t - (py - qz - kt)^2.
\end{aligned} \tag{E.0.1}$$

In the following, we only state the finite pieces of the correlators after the PV mass is sent to infinity, which are the relevant ones for the computation of the anomalies.

E.1 Bosonic correlators

E.1.1 $<\mathcal{J}\mathcal{J}>$

Below we compute the PV contribution to the R-symmetry Ward identity of $<\mathcal{J}\mathcal{J}>$. We have

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=\int\frac{d^4q}{(2\pi)^4}\Big(\frac{-i}{9}m_2\,\mathrm{tr}\left(\gamma_5P_{\lambda_2}(q)\gamma^\lambda\gamma_5P_{\lambda_2}(q-p_3)\right)$$
$$-\frac{-2i}{9}m_1\,\mathrm{tr}\left(\gamma_5P_{\lambda_1}(q)\gamma^\lambda\gamma_5P_{\lambda_1}(q-p_3)\right)\Big). \tag{E.1.1}$$

After using the Feynman parameters, we set $\ell^\mu=q^\mu-p_3^\mu y$ and ignore terms with odd power of ℓ

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=\int_0^1 dy\int d\ell_E\frac{2i\pi^2}{(2\pi)^4}\frac{8}{9}m_1^2p_3^\lambda$$
$$\left(\frac{\ell_E^3}{\left(\ell_E^2+2m_1^2+p_3^2(y-y^2)\right)^2}-\frac{\ell_E^3}{\left(\ell_E^2+m_1^2+p_3^2(y-y^2)\right)^2}\right) \tag{E.1.2}$$

For large PV masses we find that

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=-\frac{2i\pi^2}{(2\pi)^4}p_3^\lambda\left(\frac{4}{9}m_1^2\log 2-\frac{p_3^2}{27}\right). \tag{E.1.3}$$

E.1.2 $<\mathcal{J}\mathcal{J}\mathcal{J}>$

The PV contribution to the R-symmetry Ward identity of $<\mathcal{J}\mathcal{J}\mathcal{J}>$ is given by the following correlator

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)\widetilde{\mathcal{J}}^\sigma(p_1)>=\int\frac{d^4q}{(2\pi)^4}\left(\frac{1}{27}m_2\mathrm{tr}\left(\gamma^\sigma\gamma_5P_{\lambda_2}(q)\gamma^\lambda\gamma_5P_{\lambda_2}(q+p_4)\gamma_5P_{\lambda_2}(q-p_1)\right)\right.$$
$$+\frac{1}{27}m_2\mathrm{tr}\left(\gamma^\sigma\gamma_5P_{\lambda_2}(q)\gamma_5P_{\lambda_2}(q+p_3)\gamma^\lambda\gamma_5P_{\lambda_2}(q-p_1)\right)$$
$$-\frac{2}{27}m_1\mathrm{tr}\left(\gamma^\sigma\gamma_5P_{\lambda_1}(q)\gamma^\lambda\gamma_5P_{\lambda_1}(q+p_4)\gamma_5P_{\lambda_1}(q-p_1)\right)$$
$$\left.-\frac{2}{27}m_1\mathrm{tr}\left(\gamma^\sigma\gamma_5P_{\lambda_1}(q)\gamma_5P_{\lambda_1}(q+p_3)\gamma^\lambda\gamma_5P_{\lambda_1}(q-p_1)\right)\right). \tag{E.1.4}$$

After using Feynman parameters and trace identities we get

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)\widetilde{\mathcal{J}}^\sigma(p_1)>=$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{-4i}{27}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_4,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_4,p_1))^3}\right)$$

$$\mathrm{tr}\left(\gamma^\sigma\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma_5\right)(2p_4\,y-2p_1\,z+p_4)_\alpha p_{3\beta}$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{-4i}{27}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_3,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_3,p_1))^3}\right)$$
$$\mathrm{tr}\left(\gamma^\sigma\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma_5\right)(2p_3\,y-2p_1\,z-p_1)_\alpha p_{3\beta}. \tag{E.1.5}$$

We integrate over ℓ_E and find

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)\widetilde{\mathcal{J}}^\sigma(p_1)>=$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-i}{27}m_1^2\left(\frac{1}{\Delta_2^{(2)}(p_4,p_1)}-\frac{1}{\Delta_1^{(2)}(p_4,p_1)}\right)\mathrm{tr}\left(\gamma^\sigma\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma_5\right)(2p_4\,y-2p_1\,z+p_4)_\alpha p_{3\beta}$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-i}{27}m_1^2\left(\frac{1}{\Delta_2^{(2)}(p_3,p_1)}-\frac{1}{\Delta_1^{(2)}(p_3,p_1)}\right)\mathrm{tr}\left(\gamma^\sigma\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma_5\right)(2p_3\,y-2p_1\,z-p_1)_\alpha p_{3\beta}. \tag{E.1.6}$$

In the limit where the PV mass m_1 goes to infinity we have that

$$\lim_{m_1\to\infty}m_1^2\left(\frac{1}{\Delta_2^{(2)}(p_4,p_1)}-\frac{1}{\Delta_1^{(2)}(p_4,p_1)}\right)$$
$$=\lim_{m_1\to\infty}m_1^2\left(\frac{1}{\Delta_2^{(2)}(p_3,p_1)}-\frac{1}{\Delta_1^{(2)}(p_3,p_1)}\right)=-\frac{1}{2}, \tag{E.1.7}$$

so we get

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)\widetilde{\mathcal{J}}^\sigma(p_1)>=$$
$$\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{i}{54}\mathrm{tr}\left(\gamma^\sigma\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma_5\right)p_{3\beta}(2p_4\,y-2p_1\,z+p_4-2p_3\,y+2p_1\,z+p_1)_\alpha$$
$$=\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{2i}{54}\mathrm{tr}\left(\gamma^\sigma\gamma^\alpha\gamma^\lambda\gamma^\beta\gamma_5\right)p_{3\beta}p_{4\alpha}\,y=\frac{i}{324\pi^2}\epsilon^{\sigma\lambda\beta\alpha}p_{3\beta}p_{4\alpha}. \tag{E.1.8}$$

This is the standard R-symmetry anomaly of the 3-point function $<\mathcal{J}\mathcal{J}\mathcal{J}>$.

E.1.3 $<\mathcal{T}\mathcal{J}\mathcal{J}>$

R-symmetry

The PV contribution to the potential R-symmetry anomaly of $<\mathcal{T}\mathcal{J}\mathcal{J}>$ is given by the following sum of correlators

$$-i<\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)\widetilde{\mathcal{T}}^{\nu\xi}(p_1)>-\eta^{\nu\xi}<\mathcal{B}_R(p_{13})\widetilde{\mathcal{J}}^\lambda(p_4)>$$
$$-(\eta^{\nu\lambda}\eta^{\sigma\xi}+\eta^{\nu\sigma}\eta^{\lambda\xi}-\eta^{\nu\xi}\eta^{\lambda\sigma})<\widetilde{\mathcal{J}}_\sigma(p_{14})\mathcal{B}_R(p_3)>+\frac{i}{6}\eta^{\nu\lambda}<\widetilde{s}^\xi_{(2|1)}(p_{14})\mathcal{B}_R(p_3)>. \tag{E.1.9}$$

We have that

$$\frac{i}{6} < \tilde{s}^{\xi}_{(2|1)}(p_{14})\mathcal{B}_R(p_3) > = < \tilde{\mathcal{J}}^{\xi}(p_{14})\mathcal{B}_R(p_3) > \tag{E.1.10}$$

and since the 2-point correlator $< \mathcal{J}\mathcal{B}_R >$ was computed in (E.1.3), we only need to compute the 3-point function $<\mathcal{B}_R\mathcal{J}\mathcal{T}>$. We have

$$\begin{aligned}
&- i < \mathcal{B}_R(p_3)\tilde{\mathcal{J}}^{\lambda}(p_4)\tilde{\mathcal{T}}^{\nu\xi}(p_1) > = \int \frac{d^4q}{(2\pi)^4}\frac{1}{9}m_2\left(\eta^{\nu\sigma}q^{\xi} - q^{\sigma}\eta^{\nu\xi}\right) \\
&\Big[\mathrm{tr}\left(\gamma_\sigma P_{\lambda_2}(q)\gamma^\lambda\gamma_5 P_{\lambda_2}(q+p_4)\gamma_5 P_{\lambda_2}(q-p_1)\right) + \mathrm{tr}\left(\gamma_\sigma P_{\lambda_2}(q)\gamma_5 P_{\lambda_2}(q+p_3)\gamma^\lambda\gamma_5 P_{\lambda_2}(q-p_1)\right)\Big] \\
&+ \int \frac{d^4q}{(2\pi)^4}\frac{i}{9}m_2^2\eta^{\nu\xi}\Big[\mathrm{tr}\left(P_{\lambda_2}(q)\gamma^\lambda\gamma_5 P_{\lambda_2}(q+p_4)\gamma_5 P_{\lambda_2}(q-p_1)\right) \\
&+ \mathrm{tr}\left(P_{\lambda_2}(q)\gamma_5 P_{\lambda_2}(q+p_3)\gamma^\lambda\gamma_5 P_{\lambda_2}(q-p_1)\right)\Big] \\
&+ \int \frac{d^4q}{(2\pi)^4}\frac{i}{36}m_2\epsilon^{\nu\xi\rho\sigma}p_{1\rho} \\
&\Big[\mathrm{tr}\left(\gamma_\sigma\gamma_5 P_{\lambda_2}(q)\gamma^\lambda\gamma_5 P_{\lambda_2}(q+p_4)\gamma_5 P_{\lambda_2}(q-p_1)\right) + \mathrm{tr}\left(\gamma_\sigma\gamma_5 P_{\lambda_2}(q)\gamma_5 P_{\lambda_2}(q+p_3)\gamma^\lambda\gamma_5 P_{\lambda_2}(q-p_1)\right)\Big] \\
&+ \int \frac{d^4q}{(2\pi)^4}\frac{-2}{9}m_1\left(\eta^{\nu\sigma}q^{\xi} - q^{\sigma}\eta^{\nu\xi}\right) \\
&\Big[\mathrm{tr}\left(\gamma_\sigma P_{\lambda_1}(q)\gamma^\lambda\gamma_5 P_{\lambda_1}(q+p_4)\gamma_5 P_{\lambda_1}(q-p_1)\right) + \mathrm{tr}\left(\gamma_\sigma P_{\lambda_1}(q)\gamma_5 P_{\lambda_1}(q+p_3)\gamma^\lambda\gamma_5 P_{\lambda_1}(q-p_1)\right)\Big] \\
&+ \int \frac{d^4q}{(2\pi)^4}\frac{-2i}{9}m_1^2\eta^{\nu\xi}\Big[\mathrm{tr}\left(P_{\lambda_1}(q)\gamma^\lambda\gamma_5 P_{\lambda_1}(q+p_4)\gamma_5 P_{\lambda_1}(q-p_1)\right) \\
&+ \mathrm{tr}\left(P_{\lambda_1}(q)\gamma_5 P_{\lambda_1}(q+p_3)\gamma^\lambda\gamma_5 P_{\lambda_1}(q-p_1)\right)\Big] \\
&+ \int \frac{d^4q}{(2\pi)^4}\frac{-2i}{36}m_1\epsilon^{\nu\xi\rho\sigma}p_{1\rho} \\
&\Big[\mathrm{tr}\left(\gamma_\sigma\gamma_5 P_{\lambda_1}(q)\gamma^\lambda\gamma_5 P_{\lambda_1}(q+p_4)\gamma_5 P_{\lambda_1}(q-p_1)\right) + \mathrm{tr}\left(\gamma_\sigma\gamma_5 P_{\lambda_1}(q)\gamma_5 P_{\lambda_1}(q+p_3)\gamma^\lambda\gamma_5 P_{\lambda_1}(q-p_1)\right)\Big].
\end{aligned} \tag{E.1.11}$$

Following the standard procedure with the Feynman parameters, in the large PV mass limit we find that

$$\begin{aligned}
&- i < \mathcal{B}_R(p_3)\tilde{\mathcal{J}}^{\lambda}(p_4)\tilde{\mathcal{T}}^{\nu\xi}(p_1) > = \frac{2i\pi^2}{(2\pi)^4}\frac{i}{27}\Big(p_4^2 p_3^\xi\eta^{\lambda\nu} - p_3^2 p_4^\xi\eta^{\lambda\nu} - p_4^2 p_3^\lambda\eta^{\xi\nu} \\
&+ p_3\cdot p_4 p_3^\xi\eta^{\lambda\nu} - p_3\cdot p_4 p_3^\lambda\eta^{\xi\nu} - p_3\cdot p_4 p_4^\xi\eta^{\lambda\nu} + p_3^2 p_3^\xi\eta^{\lambda\nu} - p_4^\lambda p_3^\nu p_3^\xi + p_3^\lambda p_3^\nu p_4^\xi \\
&+ p_4^\lambda p_3^\nu p_4^\xi + p_3^\lambda p_4^\nu p_4^\xi + \frac{1}{2}p_{1\rho}p_{3\kappa}p_{4\alpha}\epsilon_\beta{}^{\lambda\kappa\alpha}\epsilon^{\nu\xi\rho\beta}\Big).
\end{aligned} \tag{E.1.12}$$

Conformal symmetry

The PV contribution to the trace anomaly of $<\mathcal{T}\mathcal{J}\mathcal{J}>$ is given by the following sum of correlators

$$<\mathcal{B}_W(p_1)\tilde{\mathcal{J}}^{\kappa}(p_3)\tilde{\mathcal{J}}^{\lambda}(p_4)> + \frac{8i}{9}\eta^{\kappa\lambda} <\mathcal{B}_W(p_1)\tilde{s}_{(1|0)}(p_{34})> . \tag{E.1.13}$$

We have that

$$
\begin{aligned}
&<\mathcal{B}_W(p_1)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>= \\
&\int \frac{d^4q}{(2\pi)^4}\left(\frac{-8}{9}m_2^2(2q+p_4)^\lambda(2q+p_4-p_1)^\kappa P_{\varphi_2}(q)P_{\varphi_2}(q+p_4)P_{\varphi_2}(q-p_1)\right. \\
&+\left.\frac{16}{9}m_1^2(2q+p_4)^\lambda(2q+p_4-p_1)^\kappa P_{\varphi_1}(q)P_{\varphi_1}(q+p_4)P_{\varphi_1}(q-p_1)\right)+\begin{pmatrix} p_3 & \leftrightarrow & p_4 \\ \kappa & \leftrightarrow & \lambda\end{pmatrix} \\
&+\int \frac{d^4q}{(2\pi)^4}\left(\frac{-m_2}{18}\mathrm{tr}\left(P_{\lambda_2}(q)\gamma^\lambda\gamma_5 P_{\lambda_2}(q+p_4)\gamma^\kappa\gamma_5 P_{\lambda_2}(q-p_1)\right)\right. \\
&\left.+\frac{m_1}{9}\mathrm{tr}\left(P_{\lambda_1}(q)\gamma^\lambda\gamma_5 P_{\lambda_1}(q+p_4)\gamma^\kappa\gamma_5 P_{\lambda_1}(q-p_1)\right)\right)+\begin{pmatrix} p_3 & \leftrightarrow & p_4 \\ \kappa & \leftrightarrow & \lambda\end{pmatrix}. \qquad \text{(E.1.14)}
\end{aligned}
$$

In the limit where the PV masses are sent to infinity we find

$$<\mathcal{B}_W(p_1)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=\frac{1}{108\pi^2}\left(3p_1^2\eta^{\kappa\lambda}+p_3^\kappa p_4^\lambda+2p_4^\kappa p_3^\lambda-3\eta^{\kappa\lambda}p_3\cdot p_4\right). \qquad \text{(E.1.15)}$$

We now compute the 2-point correlator. Similarly, in the large PV mass limit we have

$$
\begin{aligned}
&<\mathcal{B}_W(p_1)\widetilde{s}_{(1|0)}(p_{34})>=\int\frac{d^4q}{(2\pi)^4}\left(-2m_2^2P_{\varphi_2}(q)P_{\varphi_2}(q-p_1)+4m_1^2P_{\varphi_1}(q)P_{\varphi_1}(q-p_1)\right) \\
&=\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int_0^1 dy\, 4m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(1)}(p_1))^2}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(1)}(p_1))^2}\right) \\
&=\frac{2i\pi^2}{(2\pi)^4}\frac{p_1^2}{6}. \qquad \text{(E.1.16)}
\end{aligned}
$$

E.2 Fermionic correlators

E.2.1 $<\mathcal{Q}\bar{\mathcal{Q}}>$

The potential S-supersymmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}>$ is given by

$$
\begin{aligned}
<\mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>=&\int\frac{d^4q}{(2\pi)^4}\left(-2P_Lm_1P_{\lambda_1}(p_1-q)P_{\varphi_1}(q)A_1^\nu(-q)P_L\right. \\
&\left.+P_Lm_2P_{\lambda_2}(p_1-q)P_{\varphi_2}(q)A_2^\nu(-q)P_L\right)+(P_R\leftrightarrow P_L). \qquad \text{(E.2.1)}
\end{aligned}
$$

We use Feynman parameters, set $\ell^\mu=q^\mu-p_1^\mu y$, ignore terms with odd power of ℓ and find

$$
\begin{aligned}
&<\mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>=\int_0^1 dy\int d\ell_E\frac{2i\pi^2}{(2\pi)^4}2m_1^2\left(-i\gamma^\nu\not{p}_1\,y+i(\not{p}_1-\not{p}_1\,y)\gamma^\nu-\frac{i}{3}\left[\gamma^\nu,\gamma^\rho\right]p_{2\rho}\right) \\
&\left(\frac{\ell_E^3}{\left(\ell_E^2+m_1^2+p_1^2(y-y^2)\right)^2}-\frac{\ell_E^3}{\left(\ell_E^2+2m_1^2+p_1^2(y-y^2)\right)^2}\right). \qquad \text{(E.2.2)}
\end{aligned}
$$

For large PV masses we get that

$$<\mathcal{B}_S(p_1)\bar{\tilde{\mathcal{Q}}}^\nu(p_2)>= -\frac{2i\pi^2}{(2\pi)^4}\gamma_\mu\gamma_\sigma\gamma_5\epsilon^{\nu\rho\mu\sigma}p_{2\rho}\left(\frac{1}{6}\,m_1^2\log 2-\frac{p_2^2}{72}\right). \tag{E.2.3}$$

E.2.2 $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$

R-symmetry

The contribution of the PV fields to the R-symmetry Ward identity of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ is given by the following sum of correlators

$$\begin{aligned}&-i<\tilde{\mathcal{Q}}^\mu(p_1)\bar{\tilde{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)> +\frac{3}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi <\tilde{\mathcal{J}}_\sigma(p_{12})\mathcal{B}_R(p_3)>\\&+\frac{1}{8}\eta^{\mu\nu}\gamma_\sigma\gamma_5 <\tilde{s}^\sigma_{(2|1)}(p_{12})\mathcal{B}_R(p_3)> +\frac{i}{3}\gamma^\mu\gamma_5 <\mathcal{B}_S(p_{13})\bar{\tilde{\mathcal{Q}}}^\nu(p_2)> +\frac{i}{3} <\tilde{\mathcal{Q}}^\mu(p_1)\bar{\mathcal{B}}_S(p_{23})> \gamma^\nu\gamma_5.\end{aligned} \tag{E.2.4}$$

Since we have already computed $<\mathcal{B}_S(p_1)\bar{\tilde{\mathcal{Q}}}^\nu(p_2)>$ in the previous subsection, we can easily find $\frac{i}{3}\gamma^\mu\gamma_5 <\mathcal{B}_S(p_{13})\bar{\tilde{\mathcal{Q}}}^\nu(p_2)>$ after we make the appropriate changes in the external momenta. Moreover, if we take its charge conjugate (i.e. $C(...)^TC^{-1}$) and change $p_1 \leftrightarrow p_2$ and $\mu\leftrightarrow\nu$ we get the third term of the second line of (E.2.4).

We also have that

$$\frac{i}{6} <\tilde{s}^\sigma_{(2|1)}(p_{12})\mathcal{B}_R(p_3)>=<\tilde{\mathcal{J}}^\sigma(p_{12})\mathcal{B}_R(p_3)>, \tag{E.2.5}$$

and since the 2-point function $<\tilde{\mathcal{J}}\mathcal{B}_R>$ has already been computed in (E.1.3), we only need to compute the following

$$\begin{aligned}&-i<\tilde{\mathcal{Q}}^\mu(p_1)\bar{\tilde{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)>\\&=\int\frac{d^4q}{(2\pi)^4}\left(\frac{m_2}{3}P_RC_2^\mu(q)P_{\lambda_2}(p_1-q)\gamma_5P_{\lambda_2}(-p_2-q)A_2^\nu(-q)P_LP_{\varphi_2}(q)\right.\\&\left.-\frac{2m_1}{3}P_RC_1^\mu(k)P_{\lambda_1}(p_1-q)\gamma_5P_{\lambda_1}(-p_2-q)A_1^\nu(-q)P_LP_{\varphi_1}(q)\right)+(P_R\leftrightarrow P_L)\,.\end{aligned} \tag{E.2.6}$$

After using Feynman parameters we find that

$$\begin{aligned}-i<\tilde{\mathcal{Q}}^\mu(p_1)\bar{\tilde{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)>&= \frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int_0^1 dy\frac{-2}{3}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(1)}(p_2))^2}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(1)}(p_2))^2}\right)\\&\left(\gamma_5\not{\epsilon}\gamma^\mu\gamma^\nu-\gamma_5\gamma^\mu\gamma^\nu\not{\epsilon}+\gamma_5\gamma^\mu\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}-\gamma_5\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\nu\frac{p_{1\sigma}}{3}\right)\end{aligned}$$

$$+\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dy\,dz\frac{4}{3}m_1^2\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_1))^3}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_1))^3}\right)$$
$$\left(-\gamma_5\gamma^\xi\gamma^\mu\gamma_\xi\gamma^\kappa\gamma^\nu+\gamma_5\gamma^\xi\gamma^\mu\gamma^\kappa\gamma^\nu\gamma_\xi+\gamma_5\gamma^\mu\gamma^\xi\gamma^\kappa\gamma^\nu\gamma_\xi\right)\frac{p_{3\kappa}}{4}$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dy\,dz\frac{4}{3}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_1))^3}\right)$$
$$\Big(-\gamma_5\not{a}\gamma^\mu(\not{p}_1+\not{a})\not{p}_3\gamma^\nu+\gamma_5\not{a}\gamma^\mu\not{p}_3\gamma^\nu\not{a}-\gamma_5\not{a}\gamma^\mu\not{p}_3\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}$$
$$+\gamma_5\gamma^\mu(\not{p}_1+\not{a})\not{p}_3\gamma^\nu\not{a}-\gamma_5\gamma^\mu(\not{p}_1+\not{a})\not{p}_3\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}$$
$$+\gamma_5\left[\gamma^\mu,\gamma^\sigma\right](\not{p}_1+\not{a})\not{p}_3\gamma^\nu\frac{p_{1\sigma}}{3}-\gamma_5\left[\gamma^\mu,\gamma^\sigma\right]\not{p}_3\gamma^\nu\not{a}\frac{p_{1\sigma}}{3}+\gamma_5\left[\gamma^\mu,\gamma^\sigma\right]\not{p}_3\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{1\sigma}\,p_{2\rho}}{9}\Big)$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dy\,dz\frac{4}{3}m_1^4\gamma_5\gamma^\mu\not{p}_3\gamma^\nu\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_1))^3}\right),\tag{E.2.7}$$

where

$$c=p_2\,y\,,\quad a=p_2\,y-p_1\,z.\tag{E.2.8}$$

Note that we also used symmetric integration, i.e. we substituted in the integrand $l^\alpha l^\beta\to l^2\frac{\eta^{\alpha\beta}}{4}$. Finally, we find that

$$<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\widetilde{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)>$$
$$=-\frac{2i\pi^2}{(2\pi)^4}\Bigg(\gamma_\kappa\left(\frac{1}{36}\epsilon^{\mu\kappa\rho\sigma}p_{1\rho}p_{2\sigma}p_1^\nu-\frac{1}{108}\epsilon^{\nu\kappa\rho\sigma}p_{1\rho}p_{2\sigma}p_1^\mu+\frac{1}{54}\epsilon^{\nu\kappa\rho\sigma}p_{1\rho}p_{2\sigma}p_2^\mu\right.$$
$$\left.+\frac{1}{108}\epsilon^{\mu\nu\kappa\rho}p_{1\rho}p_2^2+\frac{1}{54}\epsilon^{\mu\nu\kappa\rho}p_{2\rho}p_1^2-\frac{1}{108}\epsilon^{\mu\nu\kappa\rho}p_{1\rho}\,p_2\cdot p_1+\frac{1}{108}\epsilon^{\mu\nu\kappa\rho}p_{1\rho}p_1^2+\frac{1}{108}\epsilon^{\mu\nu\kappa\rho}p_{2\rho}p_2^2\right)$$
$$-\not{p}_1\gamma_5\left(\frac{i}{36}\eta^{\mu\nu}p_1\cdot p_2+\frac{i}{54}\eta^{\mu\nu}p_1^2+\frac{i}{54}\eta^{\mu\nu}p_2^2+\frac{i}{108}p_2^\mu p_1^\nu-\frac{i}{108}p_1^\mu p_2^\nu-\frac{i}{54}p_1^\mu p_1^\nu\right)$$
$$-\not{p}_2\gamma_5\left(\frac{i}{36}\eta^{\mu\nu}p_1\cdot p_2+\frac{i}{54}\eta^{\mu\nu}p_1^2+\frac{i}{54}\eta^{\mu\nu}p_2^2+\frac{i}{108}p_2^\mu p_1^\nu-\frac{i}{108}p_1^\mu p_2^\nu-\frac{i}{54}p_2^\mu p_2^\nu\right)$$
$$-\gamma^\mu\gamma_5\left(\frac{i}{108}p_1^2p_2^\nu+\frac{i}{108}p_2^2p_1^\nu+\frac{i}{54}p_1\cdot p_2p_1^\nu+\frac{i}{36}p_1^2p_1^\nu\right)$$
$$-\gamma^\nu\gamma_5\left(\frac{i}{108}p_2^2p_1^\mu+\frac{i}{108}p_1^2p_2^\mu+\frac{i}{54}p_2\cdot p_1p_2^\mu+\frac{i}{36}p_2^2p_2^\mu\right)-\frac{1}{108}\not{p}_1\epsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{2\sigma}\Bigg).\tag{E.2.9}$$

S-supersymmetry

The PV breaking terms in the S-supersymmetry identity are the following

$$-\,i<\mathcal{B}_S(p_1)\widetilde{\widetilde{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)>+\frac{\sqrt{2}}{3}\eta^{\nu\kappa}<\mathcal{B}_S(p_1)\widetilde{\widetilde{s}}_{(3|\frac{1}{2})}(p_{23})>.\tag{E.2.10}$$

We begin by computing the 3-point correlator. We have

$$
\begin{aligned}
& -i <\mathcal{B}_S(p_1)\tilde{\tilde{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)>= \\
& \int \frac{d^4q}{(2\pi)^4}\Big(\frac{m_2}{3}P_{\varphi_2}(q)P_L P_{\lambda_2}(p_1-q)\gamma^\kappa\gamma_5 P_{\lambda_2}(-q-p_2)A_2^\nu(-q)P_L \\
& +\frac{2im_2}{3}(2q+p_3)^\kappa P_{\varphi_2}(q)P_{\varphi_2}(q+p_3)P_L\gamma_5 P_{\lambda_2}(p_1-q)A_2^\nu(-q-p_3)P_L \\
& -\frac{2m_1}{3}P_{\varphi_1}(q)P_L P_{\lambda_1}(p_1-q)\gamma^\kappa\gamma_5 P_{\lambda_1}(-q-p_2)A_1^\nu(-q)P_L \\
& -\frac{4im_1}{3}(2q+p_3)^\kappa P_{\varphi_1}(q)P_{\varphi_1}(q+p_3)P_L\gamma_5 P_{\lambda_1}(p_1-q)A_1^\nu(-q-p_3)P_L\Big) \\
& +(P_R\leftrightarrow P_L).
\end{aligned} \tag{E.2.11}
$$

The above can be written as

$$
\begin{aligned}
& -i <\mathcal{B}_S(p_1)\tilde{\tilde{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)>= \\
& +\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{i}{3}m_1^2\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_1))^3}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_1))^3}\right) \\
& (\gamma^\alpha\gamma^\kappa\gamma_5\gamma_\alpha\gamma^\nu+\gamma^\alpha\gamma^\kappa\gamma_5\gamma^\nu\gamma_\alpha+\gamma^\kappa\gamma_5\gamma^\alpha\gamma^\nu\gamma_\alpha) \\
& +\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{8i}{6}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_1))^3}\right) \\
& \Big(\gamma^\alpha\gamma^\kappa\gamma_5\gamma^\beta\gamma^\nu(a+p_1)_\alpha(a-p_2)_\beta+\gamma^\alpha\gamma^\kappa\gamma_5\gamma^\nu\gamma^\beta(a+p_1)_\alpha a_\beta-\gamma^\alpha\gamma^\kappa\gamma_5\left[\gamma^\nu,\gamma^\rho\right](a+p_1)_\alpha\frac{p_{2\rho}}{3} \\
& -\gamma^\kappa\gamma_5\gamma^\alpha\left[\gamma^\nu,\gamma^\rho\right](a-p_2)_\alpha\frac{p_{2\rho}}{3}+\gamma^\kappa\gamma_5\gamma^\beta\gamma^\nu\gamma^\alpha(a-p_2)_\beta a_\alpha\Big) \\
& +\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{8i}{6}m_1^4\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_1))^3}\right)\gamma_5\gamma^\kappa\gamma^\nu \\
& +\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{8i}{3}m_1^2\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(2)}(p_3,p_1))^3}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(2)}(p_3,p_1))^3}\right)\gamma_5\eta^{\nu\kappa} \\
& +\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{-8i}{3}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_3,p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_3,p_1))^3}\right) \\
& \left(\gamma_5\gamma^\alpha\gamma^\nu(-2b+p_3)^\kappa(p_1+b)_\alpha+\gamma_5\gamma^\nu\gamma^\alpha(-2b+p_3)^\kappa(-p_3+b)_\alpha+\frac{1}{3}(2b-p_3)^\kappa p_{2\rho}\left[\gamma^\nu,\gamma^\rho\right]\gamma_5\right),
\end{aligned} \tag{E.2.12}
$$

where

$$
a=p_2\,y-p_1\,z\,,\quad b=p_3\,y-p_1\,z. \tag{E.2.13}
$$

Finally we find

$$
<\mathcal{B}_S(p_1)\tilde{\tilde{\mathcal{Q}}}^\mu(p_2)\tilde{\mathcal{J}}^\nu(p_3)>=-\frac{2i\pi^2}{(2\pi)^4}\Big(i\gamma_{\kappa\lambda}\Big(\frac{1}{216}\eta^{\mu\nu}\epsilon^{\kappa\lambda\rho\sigma}p_{1\rho}p_{2\sigma}-\frac{1}{432}\epsilon^{\kappa\lambda\mu\rho}p_{1\rho}p_2^\nu+\frac{1}{432}\epsilon^{\kappa\lambda\mu\rho}p_{2\rho}p_1^\nu
$$

$$
\begin{aligned}
&+\frac{1}{144}\epsilon^{\kappa\lambda\nu\rho}p_{1\rho}p_2^\mu+\frac{1}{432}\epsilon^{\kappa\lambda\nu\rho}p_{2\rho}p_1^\mu+\frac{1}{108}\epsilon^{\kappa\lambda\mu\nu}p_1\cdot p_2\\
&+\frac{1}{108}\epsilon^{\kappa\lambda\mu\nu}p_2^2-\frac{1}{54}\epsilon^{\kappa\lambda\mu\rho}p_{2\rho}p_2^\nu+\frac{1}{108}\epsilon^{\kappa\lambda\nu\rho}p_{2\rho}p_2^\mu\Big)\\
&-i\gamma^\mu{}_\kappa\frac{1}{54}\epsilon^{\nu\kappa\rho\sigma}p_{1\rho}p_{2\sigma}+i\gamma_\kappa{}^\sigma\frac{1}{216}\epsilon^{\mu\nu\kappa\rho}p_{1\rho}p_{2\sigma}+i\gamma_\kappa{}^\sigma\frac{1}{72}\epsilon^{\mu\nu\kappa\rho}p_{2\rho}p_{1\sigma}-\frac{i}{108}\epsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{2\sigma}\\
&+\gamma_5\left(\frac{5}{108}\eta^{\mu\nu}p_1\cdot p_2+\frac{1}{12}\eta^{\mu\nu}p_1^2+\frac{7}{108}\eta^{\mu\nu}p_2^2+\frac{1}{36}p_2^\mu p_1^\nu+\frac{1}{108}p_1^\mu p_2^\nu+\frac{1}{18}p_1^\mu p_1^\nu-\frac{1}{27}p_2^\mu p_2^\nu\right)\Bigg).
\end{aligned}
\tag{E.2.14}
$$

We now compute $\frac{2}{3}\eta^{\nu\kappa} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23})>$. We have

$$
\begin{aligned}
&\frac{\sqrt{2}}{3}\eta^{\nu\kappa} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23})>\\
&=\int\frac{d^4q}{(2\pi)^4}\left(\frac{2i}{3}m_2^2\eta^{\nu\kappa}\gamma_5P_{\varphi_2}(q)P_{\varphi_2}(p_1-q)-\frac{4i}{3}m_1^2\eta^{\nu\kappa}\gamma_5P_{\varphi_1}(q)P_{\varphi_1}(p_1-q)\right)\\
&=\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int_0^1 dy\frac{-4i}{3}\eta^{\nu\kappa}m_1^2\gamma_5\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(1)}(p_1))^2}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(1)}(p_1))^2}\right)\\
&=-\frac{2i\pi^2}{(2\pi)^4}\eta^{\nu\kappa}i\gamma_5\frac{p_1^2}{18}
\end{aligned}
\tag{E.2.15}
$$

E.2.3 $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$

R-symmetry

The PV contribution to the potential R-symmetry anomaly of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ is given by the following sum of correlators

$$
\begin{aligned}
&-i<\tilde{\mathcal{Q}}^\mu(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>\\
&+\frac{i}{3}\gamma^\mu\gamma_5<\mathcal{B}_S(p_{13})\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\lambda(p_4)>+\frac{i}{3}<\tilde{\mathcal{Q}}^\mu(p_1)\bar{\mathcal{B}}_S(p_{23})\tilde{\mathcal{J}}^\lambda(p_4)>\gamma^\nu\gamma_5\\
&+\frac{3}{8}\epsilon^{\nu\xi\mu\sigma}\gamma_\xi<\tilde{\mathcal{J}}_\sigma(p_{12})\mathcal{B}_R(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>+\frac{1}{8}\eta^{\mu\nu}\gamma_\sigma\gamma_5<\tilde{s}^\sigma_{(2|1)}(p_{12})\mathcal{B}_R(p_3)\tilde{\mathcal{J}}^\lambda(p_4)>\\
&+\eta^{\mu\lambda}\frac{\sqrt{2}}{3}<\tilde{s}_{(3|\frac{1}{2})}(p_{14})\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)>-\eta^{\mu\lambda}\frac{\sqrt{2}}{9}<\tilde{s}_{(3|\frac{1}{2})}(p_{14})\bar{\mathcal{B}}_S(p_{23})>\gamma^\nu\gamma_5\\
&+\eta^{\nu\lambda}\frac{\sqrt{2}}{3}<\tilde{\mathcal{Q}}^\mu(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\mathcal{B}_R(p_3)>-\eta^{\nu\lambda}\frac{\sqrt{2}}{9}\gamma^\mu\gamma_5<\mathcal{B}_S(p_{13})\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})>.
\end{aligned}
\tag{E.2.16}
$$

In the second line the two terms are related by charge conjugation (i.e. $C(...)^TC^{-1}$)and the exchange $\mu\leftrightarrow\nu$ and $p_1\leftrightarrow p_2$. We have already computed the correlator $<\mathcal{B}_S\bar{\mathcal{Q}}\mathcal{J}>$ in the previous subsection. In the third line we have that $<\mathcal{J}\mathcal{B}_R\mathcal{J}>=\frac{i}{6}<s_{(2|1)}\mathcal{B}_R\mathcal{J}>$ and we have already computed these correlators in (E.1.8). The last two lines are related with charge conjugation and the exchange $\mu \leftrightarrow \nu$ and $p_1 \leftrightarrow p_2$. So we need to compute $\eta^{\nu\lambda}\frac{\sqrt{2}}{3} < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})\mathcal{B}_R(p_3) > -\eta^{\nu\lambda}\frac{\sqrt{2}}{9}\gamma^\mu\gamma_5 < \mathcal{B}_S(p_{13})\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24}) >$ and $-i < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\overline{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>$. We have

$$
\begin{aligned}
&\frac{\sqrt{2}}{3} < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})\mathcal{B}_R(p_3)>= \\
&\int \frac{d^4q}{(2\pi)^4}\left(\frac{-2}{9}m_2 P_{\varphi_2}(q)P_R C_2^\mu(q)P_{\lambda_2}(p_1-q)\gamma_5 P_{\lambda_2}(-q-p_{24})P_L\gamma_5\right. \\
&\left.-\frac{-4}{9}m_1 P_{\varphi_1}(q)P_R C_1^\mu(q)P_{\lambda_1}(p_1-q)\gamma_5 P_{\lambda_1}(-q-p_{24})P_L\gamma_5\right) \\
&+(P_R\leftrightarrow P_L)
\end{aligned} \tag{E.2.17}
$$

and

$$
\begin{aligned}
&-\frac{\sqrt{2}}{9}\gamma^\mu\gamma_5 < \mathcal{B}_S(p_{13})\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})>= \int\frac{d^4q}{(2\pi)^4}\left(-\gamma^\mu\frac{2i}{9}m_2^2 P_{\varphi_2}(q)P_{\varphi_2}(p_{13}-q)\right. \\
&\left.+\gamma^\mu\frac{4i}{9}m_1^2 P_{\varphi_1}(q)P_{\varphi_1}(p_{13}-q)\right).
\end{aligned} \tag{E.2.18}
$$

Using that

$$
P_{\lambda_2}(p_1-q)\gamma_5 P_{\lambda_2}(-q-p_{24}) = -iP_{\varphi_2}(q+p_{24})\gamma_5 + P_{\varphi_2}(q+p_{24})P_{\lambda_2}(p_1-q)i\not{p}_3\gamma_5 \tag{E.2.19}
$$

We get

$$
\begin{aligned}
&\frac{\sqrt{2}}{3} < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})\mathcal{B}_R(p_3)> -\frac{\sqrt{2}}{9}\gamma^\mu\gamma_5 < \mathcal{B}_S(p_{13})\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})>= \\
&\int\frac{d^4q}{(2\pi)^4}\left(\frac{-2i}{9}m_2 P_{\varphi_2}(q)P_{\varphi_2}(q+p_{24})P_R C_2^\mu(q)P_{\lambda_2}(p_1-q)\not{p}_3 P_L\right. \\
&\left.-\frac{-4i}{9}m_1 P_{\varphi_1}(q)P_{\varphi_1}(q+p_{24})P_R C_1^\mu(q)P_{\lambda_1}(p_1-q)\not{p}_3 P_L\right) \\
&+(P_R\leftrightarrow P_L) \\
&=\frac{2i\pi^2}{(2\pi)^4}\int d\ell_E\int dydz\frac{8i}{9}m_1^2\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_{24},p_1))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_{24},p_1))^3}\right) \\
&\left(-\gamma^\alpha\gamma^\mu\not{p}_3(p_{24}\,y-p_1\,z)_\alpha-\gamma^\mu\gamma^\alpha\not{p}_3(p_{24}\,y-p_1\,z+p_1)_\alpha+\frac{1}{3}\left[\gamma^\mu,\gamma^\sigma\right]\not{p}_3 p_{1\sigma}\right).
\end{aligned} \tag{E.2.20}
$$

After integrating over ℓ_E and then taking the limit of m_1 to infinity we find

$$
\frac{\sqrt{2}}{3} < \widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})\mathcal{B}_R(p_3)> -\frac{\sqrt{2}}{9}\gamma^\mu\gamma_5 < \mathcal{B}_S(p_{13})\widetilde{\overline{s}}_{(3|\frac{1}{2})}(p_{24})>
$$

$$= \frac{2i\pi^2}{(2\pi)^4} \int dydz \frac{-i}{9} \left(-\gamma^\alpha \gamma^\mu \not{p}_3 (p_{24}\, y - p_1\, z)_\alpha - \gamma^\mu \gamma^\alpha \not{p}_3 (p_{24}\, y - p_1\, z + p_1)_\alpha + \frac{1}{3} \left[\gamma^\mu, \gamma^\sigma\right] \not{p}_3 p_{1\sigma} \right). \tag{E.2.21}$$

Integrating over the Feynman parameters we get

$$\frac{\sqrt{2}}{3} < \widetilde{\mathcal{Q}}^\mu(p_1) \widetilde{\bar{s}}_{(3|\frac{1}{2})}(-p_1 - p_3) \mathcal{B}_R(p_3) > - \frac{\sqrt{2}}{9} \gamma^\mu \gamma_5 < \mathcal{B}_S(p_1 + p_3) \widetilde{\bar{s}}_{(3|\frac{1}{2})}(-p_1 - p_3) >$$
$$= -\frac{2i\pi^2}{(2\pi)^4} \left(\frac{1}{54} \gamma_\nu \gamma_5 \epsilon^{\mu\nu\rho\sigma} p_{1\rho} p_{3\sigma} - \frac{i}{54} \gamma^\mu p_1 \cdot p_3 + \frac{i}{54} \not{p}_1 p_3^\mu + \frac{i}{54} p_1^\mu \not{p}_3 + \frac{i}{27} p_3^\mu \not{p}_3 \right). \tag{E.2.22}$$

Next we compute the contribution to the potential R-anomaly that comes from the 4-point function:

$$\begin{aligned}
&- i < \widetilde{\mathcal{Q}}^\mu(p_1) \widetilde{\bar{\mathcal{Q}}}^\nu(p_2) \mathcal{B}_R(p_3) \widetilde{\mathcal{J}}^\lambda(p_4) > = \\
&\sum_{j=1}^{2} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{9}\, d_j\, m_j P_{\varphi_j}(q) \\
&P_R C_j^\mu(q) P_{\lambda_j}(p_1 - q) \gamma_5 P_{\lambda_j}(-q - p_{24}) \gamma^\lambda \gamma_5 P_{\lambda_j}(-q - p_2) A_j^\nu(-q) P_L \\
&+ \sum_{j=1}^{2} \int \frac{d^4 q}{(2\pi)^4} \frac{i}{9}\, d_j\, m_j P_{\varphi_j}(q) \\
&P_R C_j^\mu(q) P_{\lambda_j}(p_1 - q) \gamma^\lambda \gamma_5 P_{\lambda_j}(-q - p_{23}) \gamma_5 P_{\lambda_j}(-q - p_2) A_j^\nu(-q) P_L \\
&+ \sum_{j=1}^{2} \int \frac{d^4 q}{(2\pi)^4} \frac{2}{9}\, d_j\, m_j (2q + p_4)^\lambda P_{\varphi_j}(q) P_{\varphi_j}(q + p_4) \\
&\gamma_5 P_R C_j^\mu(q) P_{\lambda_j}(p_1 - q) \gamma_5 P_{\lambda_j}(-q - p_{24}) A_j^\nu(-q - p_4) P_L \\
&+ (P_R \leftrightarrow P_L),
\end{aligned} \tag{E.2.23}$$

where $j = 1, 2$ and $d_1 = -2$, $d_2 = 1$. The above 4-point function is a sum of three sums of integrals. The first and second sums of integrals are related by charge conjugation (i.e. $C(...)^T C^{-1}$) and the exchange $\mu \leftrightarrow \nu$, $p_1 \leftrightarrow p_2$. Using again

$$P_{\lambda_2}(p_1 - q) \gamma_5 P_{\lambda_2}(-q - p_{24}) = -i P_{\varphi_2}(q + p_{24}) \gamma_5 + P_{\varphi_2}(q + p_{24}) P_{\lambda_2}(p_1 - q) i \not{p}_3 \gamma_5 \tag{E.2.24}$$

and following the usual procedure with the Feynman parameters and shifting of the integration variable we find

$$- i < \widetilde{\mathcal{Q}}^\mu(p_1) \widetilde{\bar{\mathcal{Q}}}^\nu(p_2) \mathcal{B}_R(p_3) \widetilde{\mathcal{J}}^\lambda(p_4) > = (\Xi_1^{\mu\nu\lambda} + \text{Ch.C.}) + (\Xi_2^{\mu\nu\lambda} + \text{Ch.C.}) + \Theta_1^{\mu\nu\lambda} + \Theta_2^{\mu\nu\lambda}, \tag{E.2.25}$$

where

$$\Theta_1^{\mu\nu\lambda} = \frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-4i}{9}m_1^2\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(2)}(p_4,p_{13}))^3}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(2)}(p_4,p_{13}))^3}\right)$$
$$\left(\gamma^\lambda\gamma^\mu\gamma^\nu-\gamma^\mu\gamma^\nu\gamma^\lambda\right)$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-8i}{9}m_1^2\int d\ell_E\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_4,p_{13}))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_4,p_{13}))^3}\right)$$
$$\Big(\gamma^\alpha\gamma^\mu\gamma^\nu(2w-p_4)^\lambda w_\alpha-\gamma^\mu\gamma^\nu\gamma^\alpha(2w-p_4)^\lambda(w-p_4)_\alpha$$
$$+\gamma^\mu\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}(2w-p_4)^\lambda-\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\nu\frac{p_{1\sigma}}{3}(2w-p_4)^\lambda\Big), \tag{E.2.26}$$

$$\Theta_2^{\mu\nu\lambda} = \frac{2i\pi^2}{(2\pi)^4}\int dydzdt\frac{-24i}{9}m_1^4\int d\ell_E\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(3)}(p_4,p_1,p_{13}))^4}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(3)}(p_4,p_1,p_{13}))^4}\right)$$
$$\gamma^\mu\not{p}_3\gamma^\nu(-2e+p_4)^\lambda$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydzdt\frac{-24i}{9}m_1^2\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(3)}(p_4,p_1,p_{13}))^4}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(3)}(p_4,p_1,p_{13}))^4}\right)$$
$$\Big(-\gamma_\beta\gamma^\mu\gamma_\alpha\not{p}_3\gamma^\nu\left(-\frac{1}{2}\eta^{\lambda\alpha}e^\beta-\frac{1}{2}\eta^{\lambda\beta}(e+p_1)^\alpha+\frac{1}{4}\eta^{\alpha\beta}(-2e+p_4)^\lambda\right)$$
$$+\gamma_\alpha\gamma^\mu\not{p}_3\gamma^\nu\gamma_\beta\left(-\frac{1}{2}\eta^{\lambda\alpha}(e-p_4)^\beta-\frac{1}{2}\eta^{\lambda\beta}e^\alpha+\frac{1}{4}\eta^{\alpha\beta}(-2e+p_4)^\lambda\right)$$
$$+\gamma^\mu\gamma_\alpha\not{p}_3\gamma^\nu\gamma_\beta\left(-\frac{1}{2}\eta^{\lambda\alpha}(e-p_4)^\beta-\frac{1}{2}\eta^{\lambda\beta}(e+p_1)^\alpha+\frac{1}{4}\eta^{\alpha\beta}(-2e+p_4)^\lambda\right)$$
$$+\gamma^\lambda\gamma^\mu\not{p}_3\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{6}+\gamma^\mu\gamma^\lambda\not{p}_3\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{6}-\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\lambda\not{p}_3\gamma^\nu\frac{p_{1\sigma}}{6}+\left[\gamma^\mu,\gamma^\sigma\right]\not{p}_3\gamma^\nu\gamma^\lambda\frac{p_{1\sigma}}{6}\Big), \tag{E.2.27}$$

$$\Xi_1^{\mu\nu\lambda} = \frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-4i}{9}m_1^2\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_{13}))^3}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_{13}))^3}\right)$$
$$\left(\gamma^\lambda\eta^{\mu\nu}-\gamma^\mu\eta^{\lambda\nu}\right)$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-4i}{9}m_1^2\int d\ell_E\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_{13}))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_{13}))^3}\right)$$
$$\Big(\not{n}\gamma^\mu\gamma^\lambda(\not{n}-\not{p}_2)\gamma^\nu+\not{n}\gamma^\mu\gamma^\lambda\gamma^\nu\not{n}-\not{n}\gamma^\mu\gamma^\lambda\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}+\gamma^\mu\gamma^\lambda(\not{n}-\not{p}_2)\gamma^\nu\not{n}$$
$$+\gamma^\mu\gamma^\lambda(-\not{n}+\not{p}_2)\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}+\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\lambda(-\not{n}+\not{p}_2)\gamma^\nu\frac{p_{1\sigma}}{3}-\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\lambda\gamma^\nu\not{n}\frac{p_{1\sigma}}{3}$$
$$+\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\lambda\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}p_{1\sigma}}{9}\Big)$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydz\frac{-4i}{9}m_1^4\int d\ell_E\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_2,p_{13}))^3}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_2,p_{13}))^3}\right)\left(-\gamma^\mu\gamma^\lambda\gamma^\nu\right), \tag{E.2.28}$$

$$\Xi_2^{\mu\nu\lambda} = \frac{2i\pi^2}{(2\pi)^4}\int dydzdt\frac{12i}{9}m_1^4\int d\ell_E\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(3)}(p_2,p_1,p_{13}))^4}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(3)}(p_2,p_1,p_{13}))^4}\right)$$
$$\Big(\not{u}\gamma^\mu\not{p}_3\gamma^\lambda\gamma^\nu+\gamma^\mu(\not{p}_1+\not{u})\not{p}_3\gamma^\lambda\gamma^\nu-\gamma^\mu\not{p}_3\gamma^\lambda(\not{p}_2-\not{u})\gamma^\nu+\gamma^\mu\not{p}_3\gamma^\lambda\gamma^\nu\not{u}$$
$$-\gamma^\mu\not{p}_3\gamma^\lambda\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}-\left[\gamma^\mu,\gamma^\sigma\right]\not{p}_3\gamma^\lambda\gamma^\nu\frac{p_{1\sigma}}{3}\Big)$$
$$+\frac{2i\pi^2}{(2\pi)^4}\int dydzdt\frac{3i}{9}m_1^2\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(3)}(p_2,p_1,p_{13}))^4}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(3)}(p_2,p_1,p_{13}))^4}\right)$$
$$\Big(\gamma^\alpha\gamma^\mu\gamma_\alpha\not{p}_3\gamma^\lambda(-\not{u}+\not{p}_2)\gamma^\nu+\gamma^\alpha\gamma^\mu(-\not{u}-\not{p}_1)\not{p}_3\gamma^\lambda\gamma_\alpha\gamma^\nu-\not{u}\gamma^\mu\gamma^\alpha\not{p}_3\gamma^\lambda\gamma_\alpha\gamma^\nu-\gamma^\alpha\gamma^\mu\gamma_\alpha\not{p}_3\gamma^\lambda\gamma^\nu\not{u}$$
$$+\gamma^\alpha\gamma^\mu(-\not{u}-\not{p}_1)\not{p}_3\gamma^\lambda\gamma^\nu\gamma_\alpha-\not{u}\gamma^\mu\gamma_\alpha\not{p}_3\gamma^\lambda\gamma^\nu\gamma^\alpha+\gamma^\alpha\gamma^\mu\gamma_\alpha\not{p}_3\gamma^\lambda\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}-\gamma^\alpha\gamma^\mu\not{p}_3\gamma^\lambda\gamma_\alpha\gamma^\nu\not{u}$$
$$+\gamma^\alpha\gamma^\mu\not{p}_3\gamma^\lambda(-\not{u}+\not{p}_2)\gamma^\nu\gamma_\alpha-\not{u}\gamma^\mu\not{p}_3\gamma^\lambda\gamma^\alpha\gamma^\nu\gamma_\alpha+\gamma^\alpha\gamma^\mu\not{p}_3\gamma^\lambda\gamma_\alpha\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}-\gamma^\mu\gamma^\alpha\not{p}_3\gamma^\lambda\gamma_\alpha\gamma^\nu\not{u}$$
$$+\gamma^\mu\gamma^\alpha\not{p}_3\gamma^\lambda\gamma_\alpha\left[\gamma^\nu,\gamma^\rho\right]\frac{p_{2\rho}}{3}+\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\alpha\not{p}_3\gamma^\lambda\gamma_\alpha\gamma^\nu\frac{p_{1\sigma}}{3}+\gamma^\mu\gamma^\alpha\not{p}_3\gamma^\lambda(-\not{u}+\not{p}_2)\gamma^\nu\gamma_\alpha$$
$$+\gamma^\mu(-\not{u}-\not{p}_1)\not{p}_3\gamma^\lambda\gamma^\alpha\gamma^\nu\gamma_\alpha+\left[\gamma^\mu,\gamma^\sigma\right]\gamma^\alpha\not{p}_3\gamma^\lambda\gamma^\nu\gamma_\alpha\frac{p_{1\sigma}}{3}+\left[\gamma^\mu,\gamma^\sigma\right]\not{p}_3\gamma^\lambda\gamma^\alpha\gamma^\nu\gamma_\alpha\frac{p_{1\sigma}}{3}\Big)\,,\tag{E.2.29}$$

and

$$w=p_4y-p_{13}z\,,\quad n=p_2y-p_{13}z\,,\quad e=p_4y-p_{13}t-p_1z\,,\quad u=p_2y-p_{13}t-p_1z.\tag{E.2.30}$$

Note here that we have not included in the above expressions integrals that vanish in the large PV mass limit, such as integrals of the form $\int d\ell_E\left(\frac{m_1^2\ell_E^3}{(\ell_E^2+\Delta_2^{(3)}(p_4,p_1,p_{13}))^4}-\frac{m_1^2\ell_E^3}{(\ell_E^2+\Delta_1^{(3)}(p_4,p_1,p_{13}))^4}\right)$.

Finally, we find that

$$<\widetilde{\mathcal{Q}}^\mu(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\mathcal{B}_R(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>=i\times\frac{2i\pi^2}{(2\pi)^4}\Big(i\not{p}_3\Big(-\frac{1}{81}p_1^\nu\eta^{\lambda\mu}-\frac{1}{36}p_2^\nu\eta^{\lambda\mu}-\frac{1}{18}p_3^\nu\eta^{\lambda\mu}$$
$$+\frac{1}{36}p_1^\mu\eta^{\lambda\nu}+\frac{1}{81}p_2^\mu\eta^{\lambda\nu}+\frac{1}{18}p_3^\mu\eta^{\lambda\nu}+\frac{1}{81}p_1^\lambda\eta^{\nu\mu}-\frac{1}{81}p_2^\lambda\eta^{\nu\mu}\Big)$$
$$+i\not{p}_2\left(\frac{1}{108}p_1^\nu\eta^{\lambda\mu}+\frac{1}{162}p_2^\nu\eta^{\lambda\mu}-\frac{1}{108}p_3^\nu\eta^{\lambda\mu}-\frac{1}{324}p_1^\mu\eta^{\lambda\nu}-\frac{1}{162}p_3^\mu\eta^{\lambda\nu}-\frac{1}{81}p_2^\lambda\eta^{\mu\nu}-\frac{1}{108}p_3^\lambda\eta^{\nu\mu}\right)$$
$$-i\not{p}_1\left(\frac{1}{108}p_2^\mu\eta^{\lambda\nu}+\frac{1}{162}p_1^\mu\eta^{\lambda\nu}-\frac{1}{108}p_3^\mu\eta^{\lambda\nu}-\frac{1}{324}p_2^\nu\eta^{\lambda\mu}-\frac{1}{162}p_3^\nu\eta^{\lambda\mu}-\frac{1}{81}p_1^\lambda\eta^{\mu\nu}-\frac{1}{108}p_3^\lambda\eta^{\nu\mu}\right)$$
$$+i\gamma^\lambda\Big(\frac{1}{162}p_1^\mu p_1^\nu+\frac{5}{162}p_3^\mu p_1^\nu-\frac{1}{162}p_2^\mu p_2^\nu+\frac{1}{54}p_3^\mu p_2^\nu-\frac{1}{54}p_1^\mu p_3^\nu-\frac{5}{162}p_2^\mu p_3^\nu-\frac{1}{162}p_1^2\eta^{\mu\nu}$$
$$-\frac{1}{81}p_1\cdot p_3\eta^{\mu\nu}+\frac{1}{162}p_2^2\eta^{\mu\nu}+\frac{1}{81}p_2\cdot p_3\eta^{\mu\nu}\Big)$$
$$+i\gamma^\nu\Big(\frac{1}{81}p_1^\mu p_1^\lambda-\frac{1}{162}p_2^\lambda p_1^\mu+\frac{1}{108}p_1^\mu p_3^\lambda-\frac{1}{162}p_2^\mu p_2^\lambda+\frac{1}{108}p_2^\mu p_3^\lambda+\frac{1}{162}p_3^\mu p_1^\lambda-\frac{1}{81}p_2^\lambda p_3^\mu$$
$$-\frac{7}{324}p_1^2\eta^{\mu\lambda}-\frac{1}{36}p_2\cdot p_1\eta^{\mu\lambda}-\frac{2}{81}p_3\cdot p_1\eta^{\mu\lambda}-\frac{5}{108}p_2\cdot p_3\eta^{\mu\lambda}-\frac{11}{324}p_2^2\eta^{\mu\lambda}-\frac{1}{36}p_3^2\eta^{\mu\lambda}\Big)$$
$$-i\gamma^\mu\Big(\frac{1}{81}p_2^\nu p_2^\lambda-\frac{1}{162}p_1^\lambda p_2^\nu+\frac{1}{108}p_2^\nu p_3^\lambda-\frac{1}{162}p_1^\nu p_1^\lambda+\frac{1}{108}p_1^\nu p_3^\lambda+\frac{1}{162}p_3^\nu p_2^\lambda-\frac{1}{81}p_1^\lambda p_3^\nu$$

$$
-\frac{7}{324}p_2^2\eta^{\nu\lambda} - \frac{1}{36}p_1\cdot p_2\eta^{\nu\lambda} - \frac{2}{81}p_3\cdot p_2\eta^{\nu\lambda} - \frac{5}{108}p_1\cdot p_3\eta^{\nu\lambda} - \frac{11}{324}p_1^2\eta^{\nu\lambda} - \frac{1}{36}p_3^2\eta^{\nu\lambda}\Big)
$$

$$
+\gamma^\nu\gamma_5\left(\frac{1}{162}\epsilon^{\lambda\mu\rho\sigma}p_{2\rho}p_{3\sigma} - \frac{11}{648}\epsilon^{\lambda\mu\rho\sigma}p_{1\rho}p_{3\sigma}\right) + \gamma^\mu\gamma_5\left(\frac{1}{27}\epsilon^{\lambda\nu\rho\sigma}p_{1\rho}p_{3\sigma} - \frac{7}{648}\epsilon^{\lambda\nu\rho\sigma}p_{2\rho}p_{3\sigma}\right)
$$

$$
+\gamma^\lambda\gamma_5\left(\frac{13}{1296}\epsilon^{\mu\nu\rho\sigma}p_{2\rho}p_{3\sigma} - \frac{5}{1296}\epsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{3\sigma}\right) + \not{p}_3\gamma_5\left(\frac{5}{1296}\epsilon^{\lambda\mu\nu\rho}p_{2\rho} - \frac{13}{1296}\epsilon^{\lambda\mu\nu\rho}p_{1\rho}\right)
$$

$$
+\not{p}_1\gamma_5\frac{1}{81}\epsilon^{\lambda\mu\nu\rho}p_{3\rho} - \not{p}_2\gamma_5\frac{1}{162}\epsilon^{\lambda\mu\nu\rho}p_{3\rho} + \gamma_\kappa\gamma_5\Big(\frac{1}{162}\epsilon^{\lambda\mu\nu\kappa}p_1^2 + \frac{1}{1296}\epsilon^{\lambda\mu\nu\kappa}p_1\cdot p_3
$$

$$
-\frac{1}{162}\epsilon^{\lambda\mu\nu\kappa}p_2^2 - \frac{1}{1296}\epsilon^{\lambda\mu\nu\kappa}p_2\cdot p_3 - \frac{11}{648}\eta^{\mu\nu}\epsilon^{\lambda\kappa\rho\sigma}p_{1\rho}p_{3\sigma} + \frac{5}{648}\eta^{\mu\nu}\epsilon^{\lambda\kappa\rho\sigma}p_{2\rho}p_{3\sigma}
$$

$$
+\frac{7}{1296}\eta^{\lambda\nu}\epsilon^{\mu\kappa\rho\sigma}p_{1\rho}p_{3\sigma} + \frac{1}{432}\eta^{\lambda\nu}\epsilon^{\mu\kappa\rho\sigma}p_{2\rho}p_{3\sigma} + \frac{17}{432}\eta^{\lambda\mu}\epsilon^{\nu\kappa\rho\sigma}p_{1\rho}p_{3\sigma} + \frac{23}{1296}\eta^{\lambda\mu}\epsilon^{\nu\kappa\rho\sigma}p_{2\rho}p_{3\sigma}
$$

$$
-\frac{1}{324}\epsilon^{\lambda\mu\kappa\rho}p_{1\rho}p_2^\nu - \frac{17}{1296}\epsilon^{\lambda\mu\kappa\rho}p_{1\rho}p_3^\nu - \frac{1}{108}\epsilon^{\lambda\mu\kappa\rho}p_{2\rho}p_1^\nu - \frac{1}{162}\epsilon^{\lambda\mu\kappa\rho}p_{2\rho}p_2^\nu - \frac{1}{1296}\epsilon^{\lambda\mu\kappa\rho}p_{2\rho}p_3^\nu
$$

$$
-\frac{7}{648}\epsilon^{\lambda\mu\kappa\rho}p_{3\rho}p_1^\nu - \frac{1}{162}\epsilon^{\lambda\nu\kappa\rho}p_{1\rho}p_1^\mu - \frac{1}{108}\epsilon^{\lambda\nu\kappa\rho}p_{1\rho}p_2^\mu + \frac{31}{1296}\epsilon^{\lambda\nu\kappa\rho}p_{1\rho}p_3^\mu - \frac{1}{324}\epsilon^{\lambda\nu\kappa\rho}p_{2\rho}p_1^\mu
$$

$$
-\frac{17}{1296}\epsilon^{\lambda\nu\kappa\rho}p_{2\rho}p_3^\mu - \frac{2}{81}\epsilon^{\lambda\nu\kappa\rho}p_{3\rho}p_1^\mu - \frac{7}{648}\epsilon^{\lambda\nu\kappa\rho}p_{3\rho}p_2^\mu + \frac{1}{81}\epsilon^{\mu\nu\kappa\rho}p_{1\rho}p_1^\lambda + \frac{1}{162}\epsilon^{\mu\nu\kappa\rho}p_{1\rho}p_3^\lambda
$$

$$
-\frac{1}{81}\epsilon^{\mu\nu\kappa\rho}p_{2\rho}p_2^\lambda + \frac{1}{162}\epsilon^{\mu\nu\kappa\rho}p_{2\rho}p_3^\lambda - \frac{19}{1296}\epsilon^{\mu\nu\kappa\rho}p_{3\rho}p_1^\lambda + \frac{1}{432}\epsilon^{\mu\nu\kappa\rho}p_{3\rho}p_2^\lambda\Big)\Big)\,. \tag{E.2.31}
$$

S-supersymmetry

The PV contribution to the S-supersymmetry identity is given by the sum of the following correlators

$$
\begin{aligned}
&- i <\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)> \\
&+ \eta^{\nu\kappa}\frac{\sqrt{2}}{3} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23})\tilde{\mathcal{J}}^\lambda(p_4)> + \eta^{\nu\lambda}\frac{\sqrt{2}}{3} <\mathcal{B}_S(p_1)\tilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\tilde{\mathcal{J}}^\kappa(p_3)> \\
&+ \frac{8}{9}\eta^{\kappa\lambda} <\tilde{s}_{(1|0)}(p_{34})\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)>\,.
\end{aligned} \tag{E.2.32}
$$

We start by computing the 4-point function. We have

$$
\begin{aligned}
&- i <\mathcal{B}_S(p_1)\tilde{\bar{\mathcal{Q}}}^\nu(p_2)\tilde{\mathcal{J}}^\kappa(p_3)\tilde{\mathcal{J}}^\lambda(p_4)> = \sum_{j=1}^{2}\int\frac{d^4q}{(2\pi)^4}\,d_j\,m_j \\
&\Big(\frac{i}{9}P_{\varphi_j}(q)P_L P_{\lambda_j}(p_1-q)\gamma^\kappa\gamma_5 P_{\lambda_j}(-q-p_{24})\gamma^\lambda\gamma_5 P_{\lambda_j}(-q-p_2)A_j^\nu(-q)P_L \\
&+ \frac{-4i}{9}(2q+p_3)^\kappa(2q+2p_3+p_4)^\lambda P_{\varphi_j}(q)P_{\varphi_j}(q+p_3)P_{\varphi_j}(q+p_{34}) \\
&P_L P_{\lambda_j}(p_1-q)A_j^\nu(-q-p_{34})P_L \\
&+ \frac{-2}{9}(2q+p_3)^\kappa P_{\varphi_j}(q)P_{\varphi_j}(q+p_3)\gamma_5 P_R P_{\lambda_j}(p_1-q)\gamma^\lambda\gamma_5 P_{\lambda_j}(-q-p_{23})A_j^\nu(-q-p_3)P_L\Big) \\
&+ (P_R\leftrightarrow P_L) + \begin{pmatrix} p_3 & \leftrightarrow & p_4 \\ \kappa & \leftrightarrow & \lambda \end{pmatrix}
\end{aligned} \tag{E.2.33}
$$

where $j = 1, 2$ and $d_1 = -2\,, d_2 = 1$. Using the Feynman parameters we can rewrite it as

$$
\begin{aligned}
&- i <\mathcal{B}_S(p_1)\widetilde{\widetilde{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>= \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{-48}{9}m_1^2\int dydzdt\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(3)}(p_{34},-p_3,p_1))^4}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(3)}(p_{34},-p_3,p_1))^4}\right) \\
&\Bigg(\left(\eta^{\kappa\lambda}(-a-p_1)^\beta+\frac{\eta^{\kappa\beta}}{2}(-2a+2p_3+p_4)^\lambda+\frac{\eta^{\lambda\beta}}{2}(-2a+p_3)^\kappa\right)\gamma_\beta\gamma^\nu \\
&+\left(\eta^{\kappa\lambda}(-a+p_{34})^\beta+\frac{\eta^{\kappa\beta}}{2}(-2a+2p_3+p_4)^\lambda+\frac{\eta^{\lambda\beta}}{2}(-2a+p_3)^\kappa\right)\gamma^\nu\gamma_\beta \\
&+\eta^{\kappa\lambda}\frac{p_{2\rho}}{3}\left[\gamma^\nu,\gamma^\rho\right]\Bigg) \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{-24}{9}m_1^2\int dydzdt\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(3)}(p_4,p_{13},p_1))^4}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(3)}(p_4,p_{13},p_1))^4}\right) \\
&\Bigg(\left(\frac{\eta^{\lambda\alpha}}{2}(-b-p_{13})^\beta+\frac{\eta^{\lambda\beta}}{2}(-b-p_1)^\alpha+\frac{\eta^{\alpha\beta}}{4}(-2b+p_4)^\lambda\right)\gamma_\alpha\gamma^\kappa\gamma_\beta\gamma^\nu \\
&+\left(\frac{\eta^{\lambda\alpha}}{2}(-b+p_4)^\beta+\frac{\eta^{\lambda\beta}}{2}(-b-p_1)^\alpha+\frac{\eta^{\alpha\beta}}{4}(-2b+p_4)^\lambda\right)\gamma_\alpha\gamma^\kappa\gamma^\nu\gamma_\beta \\
&+\frac{\eta^{\lambda\alpha}}{2}\frac{p_{2\rho}}{3}\gamma_\alpha\gamma^\kappa\left[\gamma^\nu,\gamma^\rho\right]-\frac{\eta^{\lambda\alpha}}{2}\frac{p_{2\rho}}{3}\gamma^\kappa\gamma_\alpha\left[\gamma^\nu,\gamma^\rho\right] \\
&-\left(\frac{\eta^{\lambda\alpha}}{2}(-b+p_4)^\beta+\frac{\eta^{\lambda\beta}}{2}(-b-p_{13})^\alpha+\frac{\eta^{\alpha\beta}}{4}(-2b+p_4)^\lambda\right)\gamma^\kappa\gamma_\alpha\gamma^\nu\gamma_\beta\Bigg) \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{-24}{9}m_1^4\int dydzdt\int d\ell_E\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(3)}(p_4,p_{13},p_1))^4}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(3)}(p_4,p_{13},p_1))^4}\right) \\
&(-2b+p_4)^\lambda\gamma^\kappa\gamma^\nu \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{12}{9}\frac{1}{4}m_1^2\int dydzdt\int d\ell_E\left(\frac{\ell_E^5}{(\ell_E^2+\Delta_2^{(3)}(p_2,p_{13},p_1))^4}-\frac{\ell_E^5}{(\ell_E^2+\Delta_1^{(3)}(p_2,p_{13},p_1))^4}\right) \\
&\Big(-\left(\eta^{\alpha\beta}(-c+p_2)^\xi+\eta^{\beta\xi}(-c-p_1)^\alpha+\eta^{\alpha\xi}(-c-p_{13})^\beta\right)\gamma_\alpha\gamma^\kappa\gamma_\beta\gamma^\lambda\gamma_\xi\gamma^\nu \\
&-\left(-\eta^{\alpha\beta}c^\xi+\eta^{\beta\xi}(-c-p_1)^\alpha+\eta^{\alpha\xi}(-c-p_{13})^\beta\right)\gamma_\alpha\gamma^\kappa\gamma_\beta\gamma^\lambda\gamma^\nu\gamma_\xi \\
&+\left(-\eta^{\alpha\beta}c^\xi+\eta^{\beta\xi}(-c-p_1)^\alpha+\eta^{\alpha\xi}(-c+p_2)^\beta\right)\gamma_\alpha\gamma^\kappa\gamma^\lambda\gamma_\beta\gamma^\nu\gamma_\xi \\
&-\left(-\eta^{\alpha\beta}c^\xi+\eta^{\beta\xi}(-c-p_{13})^\alpha+\eta^{\alpha\xi}(-c+p_2)^\beta\right)\gamma^\kappa\gamma_\alpha\gamma^\lambda\gamma_\beta\gamma^\nu\gamma_\xi \\
&-\eta^{\alpha\beta}\frac{p_{2\rho}}{3}\gamma_\alpha\gamma^\kappa\gamma_\beta\gamma^\lambda\left[\gamma^\nu,\gamma^\rho\right]+\eta^{\alpha\beta}\frac{p_{2\rho}}{3}\gamma_\alpha\gamma^\kappa\gamma^\lambda\gamma_\beta\left[\gamma^\nu,\gamma^\rho\right]-\eta^{\alpha\beta}\frac{p_{2\rho}}{3}\gamma^\kappa\gamma_\alpha\gamma^\lambda\gamma_\beta\left[\gamma^\nu,\gamma^\rho\right]\Big) \\
&+ \frac{2i\pi^2}{(2\pi)^4}\frac{12}{9}m_1^4\int dydzdt\int d\ell_E\left(\frac{2\ell_E^3}{(\ell_E^2+\Delta_2^{(3)}(p_2,p_{13},p_1))^4}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(3)}(p_2,p_{13},p_1))^4}\right) \\
&\Big((c+p_1)^\alpha\gamma_\alpha\gamma^\kappa\gamma^\lambda\gamma^\nu+(-c-p_{13})^\alpha\gamma^\kappa\gamma_\alpha\gamma^\lambda\gamma^\nu \\
&+(c-p_2)^\alpha\gamma^\kappa\gamma^\lambda\gamma_\alpha\gamma^\nu+c^\alpha\gamma^\kappa\gamma^\lambda\gamma^\nu\gamma_\alpha-\frac{p_{2\rho}}{3}\gamma^\kappa\gamma^\lambda\left[\gamma^\nu,\gamma^\rho\right]\Big) \\
&+\begin{pmatrix} p_3 & \leftrightarrow & p_4 \\ \kappa & \leftrightarrow & \lambda \end{pmatrix}, \qquad \text{(E.2.34)}
\end{aligned}
$$

where

$$a = p_{34}y + p_3z - p_1t\,, \quad b = p_4y - p_{13}z - p_1t\,, \quad c = p_2y - p_{13}z - p_1t. \tag{E.2.35}$$

Note that again we do not include in the above expression integrals that vanish in the large PV mass limit. In the end we find that

$$\begin{aligned}
&<\mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)\widetilde{\mathcal{J}}^\kappa(p_3)\widetilde{\mathcal{J}}^\lambda(p_4)>= i \times \frac{2i\pi^2}{(2\pi)^4}\left(\frac{11}{162}\gamma^{\nu\rho}\eta^{\kappa\lambda}p_{1\rho} + \frac{1}{27}\gamma^{\lambda\rho}\eta^{\kappa\nu}p_{1\rho} + \frac{1}{27}\gamma^{\kappa\rho}\eta^{\lambda\nu}p_{1\rho}\right.\\
&+ \frac{25}{324}\gamma^{\nu\rho}\eta^{\kappa\lambda}\left(p_{3\rho} + p_{4\rho}\right) + \gamma^{\lambda\nu}\frac{1}{54}\left(p_1^\kappa + p_3^\kappa\right) + \gamma^{\kappa\nu}\frac{1}{54}\left(p_1^\lambda + p_4^\lambda\right) + \gamma^{\kappa\lambda}\frac{1}{36}\left(p_3^\nu - p_4^\nu\right)\\
&+ \gamma_5\left(\frac{i}{108}\epsilon^{\kappa\lambda\nu\rho}p_{4\rho} - \frac{i}{108}\epsilon^{\kappa\lambda\nu\rho}p_{3\rho}\right) - \frac{1}{9}p_1^\nu\eta^{\kappa\lambda} + \frac{1}{9}p_1^\lambda\eta^{\kappa\nu} + \frac{1}{9}p_1^\kappa\eta^{\nu\lambda} + \frac{1}{18}p_3^\nu\eta^{\kappa\lambda} + \frac{1}{27}p_3^\lambda\eta^{\kappa\nu}\\
&\left.+\frac{1}{9}p_3^\kappa\eta^{\nu\lambda} + \frac{1}{18}p_4^\nu\eta^{\kappa\lambda} + \frac{1}{27}p_4^\kappa\eta^{\lambda\nu} + \frac{1}{9}p_4^\lambda\eta^{\nu\kappa}\right).
\end{aligned} \tag{E.2.36}$$

Now we compute the two following quantities:

$$\begin{aligned}
&\frac{\sqrt{2}}{3}<\mathcal{B}_S(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\widetilde{\mathcal{J}}^\kappa(p_3)>=\\
&\sum_{j=1}^{2}\int\frac{d^4q}{(2\pi)^4}\,d_jm_j\left(\frac{-2}{9}P_{\varphi_j}(q)P_LP_{\lambda_j}(p_1-q)\gamma^\kappa\gamma_5P_{\lambda_j}(-q-p_{24})P_L\gamma_5\right.\\
&\left.+\frac{-4i}{9}(2q+p_3)^\kappa P_{\varphi_j}(q)P_{\varphi_j}(q+p_3)P_L\gamma_5P_{\lambda_j}(p_1-q)P_L\gamma_5\right) + (P_R \leftrightarrow P_L)
\end{aligned} \tag{E.2.37}$$

and

$$\begin{aligned}
&<\widetilde{s}_{(1|0)}(p_{34})\mathcal{B}_S(p_1)\widetilde{\bar{\mathcal{Q}}}^\nu(p_2)>=\\
&\sum_{j=1}^{2}\int\frac{d^4q}{(2\pi)^4}\,d_jm_jP_{\varphi_j}(q)P_{\varphi_j}(q+p_{34})P_LP_{\lambda_j}(p_1-q)A_j^\nu(-q-p_{34})P_L\\
&+(P_R \leftrightarrow P_L).
\end{aligned} \tag{E.2.38}$$

We can rewrite the above as

$$\begin{aligned}
&\frac{\sqrt{2}}{3}<\mathcal{B}_S(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{24})\widetilde{\mathcal{J}}^\kappa(p_3)>=\\
&\frac{2i\pi^2}{(2\pi)^4}\frac{-8}{9}m_1^2\int dydz\int d\ell_E\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_1,p_{24}))^2} - \frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_1,p_{24}))^2}\right)\\
&(\gamma^\alpha\gamma^\kappa(p_1+d)_\alpha + \gamma^\kappa\gamma^\alpha(p_{24}-d)_\alpha)\\
&+\frac{2i\pi^2}{(2\pi)^4}\frac{16}{9}m_1^2\int dydz\int d\ell_E\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_1,p_3))^2} - \frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_1,p_3))^2}\right)(-2f+p_3)^\kappa
\end{aligned} \tag{E.2.39}$$

and

$$<\widetilde{s}_{(1|0)}(p_{34})\mathcal{B}_S(p_1)\widetilde{\widetilde{\mathcal{Q}}}^{\nu}(p_2)>=$$
$$\frac{2i\pi^2}{(2\pi)^4}\frac{-8}{2}m_1^2\int dydz\int d\ell_E\left(\frac{\ell_E^3}{(\ell_E^2+\Delta_2^{(2)}(p_1,p_{43}))^2}-\frac{\ell_E^3}{(\ell_E^2+\Delta_1^{(2)}(p_1,p_{43}))^2}\right)$$
$$\left(\gamma^\alpha\gamma^\nu(p_1+e)_\alpha+\gamma^\nu\gamma^\alpha(e-p_{34})_\alpha-\frac{p_{2\rho}}{3}\left[\gamma^\nu,\gamma^\rho\right]\right), \quad (E.2.40)$$

where

$$d=p_{24}z-p_1y\,,\quad f=p_3z-p_1y\,,\quad e=p_{43}z-p_1y. \quad (E.2.41)$$

In the large PV mass limit we get

$$<\mathcal{B}_S(p_1)\widetilde{\widetilde{s}}_{(3|\frac{1}{2})}(-p_1-p_3)\widetilde{\mathcal{J}}^{\kappa}(p_3)>=-\frac{2i\pi^2}{(2\pi)^4}\sqrt{2}\left(\frac{1}{9}p_1^\kappa+\frac{5}{36}p_3^\kappa+\frac{1}{18}\gamma^{\kappa\rho}p_{1\rho}+\frac{1}{36}\gamma^{\kappa\rho}p_{3\rho}\right) \quad (E.2.42)$$

and

$$<\mathcal{B}_S(p_1)\widetilde{\widetilde{\mathcal{Q}}}^{\nu}(p_2)\widetilde{s}_{(1|0)}(-p_1-p_2)>=\frac{2i\pi^2}{(2\pi)^4}\left(\frac{1}{6}p_1^\nu+\frac{1}{12}p_2^\nu+\frac{1}{12}\gamma^{\nu\rho}p_{2\rho}\right). \quad (E.2.43)$$

APPENDIX F

Ward identities of seagull correlators

In this thesis we examined the Ward identities of correlators that involve conserved currents, namely, the stress tensor, the R-current and the supercurrent. One of the main goals was to see whether Q-supersymmetry was anomalous or not. The reason we focused on correlators of conserved currents and not of seagull operators, was that only the former couple to the background sources of conformal supergravity. Of course in the identities that we calculated there were seagull correlators too, but the main correlators which their symmetries we were examining, contained only conserved currents. From a bottom up perspective though, if one wishes to look for supersymmetry anomalies in the massless WZ model, there is no reason not to consider correlators that involve seagull operators. In this appendix we present up to the 3-point function level, the renormalized seagull correlators that satisfy their classical Ward identities.

The method we followed to derive them was similar to the analysis of chapters 4 and 7. First, we had to find the classical path integral identities of the seagull correlators using the symmetry variations of the subsection (4.1.5). Then, we identified the PV breaking terms in the classical Ward identities and calculated them in the limit of large PV masses. Finally, we partially renormalized the seagull correlators in momentum space to restore the broken symmetries. We do not present the explicit computation of integrals here, since we have already done a thorough analysis on similar integrals in the appendix E. We only state the final results for the partially renormalized correlators in momentum space. As expected, we confirm that there are no anomalies associated with these correlators. If one wishes to renormalize the seagull correlators at the level of the action, then additional sources must be introduced. The fact that the finite renormalizations in this appendix are all analytic functions in the external momenta, ensures that in principle this can be done.

F.1 2-point functions

F.1.1 $<s_{(2|1)}\mathcal{J}>$

$$<\widetilde{s}^{\kappa}_{(2|1)}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}=<\widetilde{s}^{\kappa}_{(2|1)}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)> +\frac{2i\pi^2}{(2\pi)^4}\frac{6i}{27}\eta^{\kappa\lambda}p_3^2. \tag{F.1.1}$$

The renormalized correlator satisfies the following classical R-symmetry identity

$$p_{4\lambda} <\widetilde{s}^{\kappa}_{(2|1)}(p_3)\widetilde{\mathcal{J}}^{\lambda}(p_4)>_{\text{ren}}= 0 \tag{F.1.2}$$

F.1.2 $<\mathcal{Q}\bar{s}_{(3|\frac{1}{2})}>$

$$<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>_{\text{ren}}=<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)> -\frac{2i\pi^2}{(2\pi)^4}\frac{i3\sqrt{2}}{108}\left(\gamma_5\gamma^{\mu}p_1^2-\gamma_5\not{p}_1p_1^{\mu}\right) \tag{F.1.3}$$

The renormalized correlator satisfies the classical Q- and S-supersymmetry Ward identities

$$p_{1\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>_{\text{ren}}= 0\,, \qquad \gamma_{\mu} <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>_{\text{ren}}= 0. \tag{F.1.4}$$

F.1.3 $<\mathcal{T}s_{(1|0)}>$

$$<\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{s}_{(1|0)}(p_2)>_{\text{ren}}=<\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{s}_{(1|0)}(p_2)> -\frac{2i\pi^2}{(2\pi)^4}\frac{1}{18}\left(\eta^{\nu\xi}p_1^2-p_1^{\nu}p_1^{\xi}\right) \tag{F.1.5}$$

The renormalized $<\mathcal{T}s_{(1|0)}>$ satisfies the following diffeomorphisms, scale and Lorentz symmetry Ward identities

$$p_{1\nu} <\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{s}_{(1|0)}(p_2)>_{\text{ren}}= 0\,, \eta_{\nu\xi} <\widetilde{\mathcal{T}}^{\nu\xi}(p_1)\widetilde{s}_{(1|0)}(p_2)>_{\text{ren}}= 0\,, <\widetilde{\mathcal{T}}^{[\nu,\xi]}(p_1)\widetilde{s}_{(1|0)}(p_2)>_{\text{ren}}= 0. \tag{F.1.6}$$

F.2 3-point functions

F.2.1 $<\mathcal{Q}\bar{\mathcal{Q}}s_{(1|0)}>$

$$
\begin{aligned}
&<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{s}_{(1|0)}(p_3)>_{\text{ren}}=<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{s}_{(1|0)}(p_3)> \\
&-\frac{2i\pi^2}{(2\pi)^4}\frac{1}{36}\left(\eta^{\mu\nu}(\not{p}_2-\not{p}_1)+\gamma^{\mu}\left(2p_1^{\nu}+p_2^{\nu}\right)-\gamma^{\nu}\left(2p_2^{\mu}+p_1^{\mu}\right)\right)
\end{aligned}
\tag{F.2.1}
$$

The renormalized 3-point function satisfies its classical Q- and S-supersymmetry Ward identities. Note here that in the following identities we use the renormalized lower order correlators, and also we do not include the 2-point function $<\mathcal{J}s_{(1|0)}>$ which is zero in the regulated theory.

Q-supersymmetry

$$
\begin{aligned}
&p_{1\mu}<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{s}_{(1|0)}(p_3)>_{\text{ren}} \\
&-p_{1\mu}p_{12\sigma}\left(\frac{1}{6}\eta^{\sigma\nu}\gamma^{\mu}-\frac{1}{6}\eta^{\sigma\mu}\gamma^{\nu}-\frac{i}{6}\epsilon^{\nu\xi\mu\sigma}\gamma_{\xi}\gamma_5\right)<(\widetilde{s}_{(1|0)}(p_{12})\widetilde{s}_{(1|0)}(p_3)> \\
&=-\frac{\gamma_{\xi}}{2}<\widetilde{\mathcal{T}}^{\nu\xi}(p_{12})\widetilde{s}_{(1|0)}(p_3)>_{\text{ren}}-i\gamma_5\frac{\sqrt{2}}{2}<\widetilde{s}_{(3|\frac{1}{2})}(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)>_{\text{ren}}
\end{aligned}
\tag{F.2.2}
$$

S-supersymmetry

$$
\gamma_{\mu}<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{s}_{(1|0)}(p_3)>_{\text{ren}}-\frac{1}{2}\gamma^{\nu}\gamma^{\sigma}p_{12\sigma}<\widetilde{s}_{(1|0)}(p_{12}))\widetilde{s}_{(1|0)}(p_3)>=0
\tag{F.2.3}
$$

F.2.2 $<\mathcal{Q}\bar{s}_{(3|\frac{1}{2})}\mathcal{J}>$

$$
\begin{aligned}
&<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}}=<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)> \\
&-\frac{2i\pi^2}{(2\pi)^4}\frac{3\sqrt{2}}{2}\left(-\frac{1}{54}\gamma^{\mu}(p_1^{\kappa}+p_3^{\kappa})-\frac{1}{54}\eta^{\kappa\mu}\not{p}_3+\frac{i}{54}\gamma_{\xi}\gamma_5\epsilon^{\kappa\mu\xi\rho}p_{1\rho}\right).
\end{aligned}
\tag{F.2.4}
$$

Similarly, the above renormalized correlator satisfies all of its classical symmetry Ward identities. In the following we do not include $<\mathcal{J}s_{(1|0)}>$ and we use the renormalized lower order correlators.

Q-supersymmetry

$$p_{1\mu}\left(<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}} + i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <\widetilde{s}_{(3|\frac{1}{2})}(p_{13})\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>\right)$$
$$= -\frac{3\sqrt{2}}{8}\gamma_{\sigma} <\widetilde{\mathcal{J}}^{\sigma}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}} + \frac{i3\sqrt{2}}{16}\gamma_{\sigma} <\widetilde{s}^{\sigma}_{(2|1)}(p_{12})\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}}$$
$$- i\gamma_5 <\widetilde{\mathcal{Q}}^{\kappa}(p_{13})\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>_{\text{ren}} \tag{F.2.5}$$

R-symmetry

$$p_{3\kappa}\left(<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}} + i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <\widetilde{s}_{(3|\frac{1}{2})}(p_{13})\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>\right)$$
$$= i\gamma_5 <\widetilde{\mathcal{Q}}^{\mu}(p_{13})\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>_{\text{ren}} + i <\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_{23})>_{\text{ren}} \gamma_5 \tag{F.2.6}$$

S-supersymmetry

$$\gamma_{\mu}\left(<\widetilde{\mathcal{Q}}^{\mu}(p_1)\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)>_{\text{ren}} + i\eta^{\mu\kappa}\frac{\sqrt{2}}{3} <\widetilde{s}_{(3|\frac{1}{2})}(p_{13})\widetilde{\bar{s}}_{(3|\frac{1}{2})}(p_2)>\right) = 0 \tag{F.2.7}$$

APPENDIX G

Symmetries of old minimal supergravity

In this appendix, we discuss the symmetries of old minimal supergravity. The main aim is to find its supersymmetry Ward identities and show that they are manifestly non anomalous at the quantum level. We follow closely the relevant discussion of [81].

The off-shell supergravity that sources the FZ multiplet is old minimal supergravity [96, 121, 122]. It can be obtained from $\mathcal{N} = 1$ conformal supergravity by adding a compensating superconformal chiral multiplet, $(\widetilde{\phi}, \widetilde{\chi}, \widetilde{F})$, and suitable gauge fixing [101, 123, 124]. The compensating chiral multiplet sources a subset of operators in the FZ multiplet [33, 125] that form a chiral multiplet. This chiral multiplet comprises a complex scalar operator (x in the notation of [125]) that is sourced by $\widetilde{F}$, the gamma trace of the supercurrent sourced by $\widetilde{\chi}$, and the trace of the stress tensor and the divergence of the R-current, which are sourced by the real and imaginary parts of the complex scalar $\widetilde{\phi}$.

The local symmetry transformations of the compensating multiplet are exactly those of the chiral multiplet in (4.3.6), namely

$$\begin{aligned}
\delta\widetilde{\phi} &= \xi^\mu\partial_\mu\widetilde{\phi} - \sigma\widetilde{\phi} - \frac{2i}{3}\theta\widetilde{\phi} + \frac{\sqrt{2}}{2}\bar{\varepsilon}_L\widetilde{\chi}_L, \qquad \text{(G.0.1)}\\
\delta\widetilde{\chi}_L &= \xi^\mu\partial_\mu\widetilde{\chi}_L - \frac{3}{2}\sigma\widetilde{\chi}_L + \frac{i}{3}\theta\widetilde{\chi}_L - \frac{1}{4}\lambda_{ab}\gamma^{ab}\widetilde{\chi}_L + \frac{\sqrt{2}}{2}\Big(\gamma^\mu(D_\mu\widetilde{\phi} - \frac{\sqrt{2}}{2}\bar{\psi}_{\mu L}\widetilde{\chi}_L)\varepsilon_R + \widetilde{F}\varepsilon_L + 2\widetilde{\phi}\eta_L\Big),\\
\delta\widetilde{F} &= \xi^\mu\partial_\mu\widetilde{F} - 2\sigma\widetilde{F} + \frac{4i}{3}\theta\widetilde{F} + \frac{1}{2}\bar{\varepsilon}_R\gamma^\mu\Big(\sqrt{2}D_\mu\widetilde{\chi}_L - \gamma^\nu(D_\nu\widetilde{\phi} - \frac{\sqrt{2}}{2}\bar{\widetilde{\psi}}_{\nu L}\chi_L)\psi_{\mu R} - \widetilde{F}\psi_{\mu L} - 2\widetilde{\phi}\,\phi_{\mu L}\Big).
\end{aligned}$$

The compensating multiplet allows us to redefine the supergravity fields so that they are invariant under Weyl, S-supersymmetry and axial gauge transformations. Using the conformal supergravity transformations in (4.3.1), it is straightforward to verify that the redefined fields

$$e'^a_\mu = |\widetilde{\phi}|e^a_\mu, \qquad \psi'_\mu = \frac{1}{|\widetilde{\phi}|}(\widetilde{\phi}P_R + \widetilde{\phi}^* P_L)^{3/2}\psi_\mu + \frac{\sqrt{2}}{2|\widetilde{\phi}|}\gamma_\mu(\widetilde{\phi}P_L + \widetilde{\phi}^* P_R)^{1/2}\widetilde{\chi},$$
$$A'_\mu = A_\mu - \frac{3i}{4}|\widetilde{\phi}|^{-2}\Big(\widetilde{\phi}^*\partial_\mu\widetilde{\phi} - \widetilde{\phi}\partial_\mu\widetilde{\phi}^* + \frac{1}{4}\bar{\widetilde{\chi}}\gamma_\mu\widetilde{\chi} - \frac{\sqrt{2}}{2}\bar{\psi}_\mu(\widetilde{\phi}^*\widetilde{\chi}_L - \widetilde{\phi}\widetilde{\chi}_R)\Big). \tag{G.0.2}$$

do not transform under Weyl, S-supersymmetry and axial gauge transformations [101]. Having defined the invariant supergravity fields $(e'^a_\mu, \psi'_\mu, A'_\mu)$, we fix the gauge by setting

$$\widetilde{\phi} = 1, \qquad \widetilde{\chi} = 0. \tag{G.0.3}$$

This gauge choice eliminates the sources of the gamma trace of the supercurrent, the trace of the stress tensor and the divergence of the R-current in the FZ multiplet, all of which are redundant. In this gauge, the field redefinition (G.0.2) reduces to the identity, so that $(e'^a_\mu, \psi'_\mu, A'_\mu) = (e^a_\mu, \psi_\mu, A_\mu)$. However, only a subset of the local transformations of conformal supergravity preserve this gauge. From the transformations (G.0.1) of the compensator multiplet we see that the gauge (G.0.3) is preserved if and only if the local symmetry parameters satisfy the conditions

$$\sigma = 0, \qquad \theta = 0, \qquad \eta = \frac{i}{3}\not{A}\varepsilon - \frac{1}{2}(\widetilde{F}\varepsilon_L + \widetilde{F}^*\varepsilon_R). \tag{G.0.4}$$

The surviving local symmetries are those of old minimal Poincaré supergravity with

$$\delta^{\rm om}_\varepsilon = \delta_\varepsilon + \delta_{\eta(\varepsilon)}, \qquad \eta(\varepsilon) = \frac{i}{3}\not{A}\varepsilon - \frac{1}{2}(\widetilde{F}\varepsilon_L + \widetilde{F}^*\varepsilon_R), \tag{G.0.5}$$

where δ_ε, δ_η are the Q- and S-supersymmetry transformations of $\mathcal{N} = 1$ conformal supergravity.

The field content of old minimal supergravity, therefore, consists of that of $\mathcal{N} = 1$ conformal supergravity, as well as the auxiliary complex scalar, $\widetilde{F}$, of the compensator multiplet, which is not fixed by the gauge fixing conditions (G.0.3). Adopting standard notation [28], we denote this field by M in the following.[1] The supersymmetry transfor-

[1] In fact, M here is related to $M_{\rm WB}$ in [28] as $M = -3M^*_{\rm WB}$.

mations of old minimal supergravity are

$$\begin{aligned}
\delta_\varepsilon^{\rm om} e_\mu^a &= -\frac{1}{2}\overline{\psi}_\mu\gamma^a\varepsilon,\\
\delta_\varepsilon^{\rm om}\psi_\mu &= D_\mu\varepsilon - \gamma_\mu\Big(\frac{i}{3}\not{A}\varepsilon - \frac{1}{2}(M\varepsilon_L + M^*\varepsilon_R)\Big),\\
\delta_\varepsilon^{\rm om} A_\mu &= \frac{3i}{4}\overline{\phi}_\mu\varepsilon - \frac{3i}{4}\overline{\psi}_\mu\Big(\frac{i}{3}\not{A}\varepsilon - \frac{1}{2}(M\varepsilon_L + M^*\varepsilon_R)\Big),\\
\delta_\varepsilon^{\rm om} M &= -\bar{\varepsilon}_R\gamma^\mu\Big(\frac{i}{3}\not{A}\psi_{\mu R} + \frac{1}{2}M\psi_{\mu L} + \phi_{\mu L}\Big).
\end{aligned} \tag{G.0.6}$$

Ward identities

Given the field content and local symmetry transformations of old minimal supergravity, we can define the corresponding current multiplet operators and determine the Ward identities they satisfy. A variation of the generating function of (regulated) connected correlation functions takes the form

$$\delta\widetilde{\mathscr{W}}[e, A, \psi, M] = \int d^4x\, e\big(\langle\widetilde{\mathcal{T}}_a^\mu\rangle_s\delta e_\mu^a + \langle\widetilde{\mathcal{J}}^\mu\rangle_s\delta A_\mu + \delta\bar{\psi}_\mu\langle\widetilde{\mathcal{Q}}^\mu\rangle_s + \langle\widetilde{\mathcal{O}}_M\rangle_s\delta M + \langle\widetilde{\mathcal{O}}_{M^*}\rangle_s\delta M^*\big), \tag{G.0.7}$$

where the local operators defined by

$$\begin{aligned}
&\langle\widetilde{\mathcal{T}}_a^\mu\rangle_s = e^{-1}\frac{\delta\widetilde{\mathscr{W}}}{\delta e_\mu^a}, \qquad \langle\widetilde{\mathcal{J}}^\mu\rangle_s = e^{-1}\frac{\delta\widetilde{\mathscr{W}}}{\delta A_\mu}, \qquad \langle\widetilde{\mathcal{Q}}^\mu\rangle_s = e^{-1}\frac{\delta\widetilde{\mathscr{W}}}{\delta\bar{\psi}_\mu},\\
&\langle\widetilde{\mathcal{O}}_M\rangle_s = e^{-1}\frac{\delta\widetilde{\mathscr{W}}}{\delta M}, \qquad \langle\widetilde{\mathcal{O}}_{M^*}\rangle_s = e^{-1}\frac{\delta\widetilde{\mathscr{W}}}{\delta M^*},
\end{aligned} \tag{G.0.8}$$

comprise the FZ current multiplet [33, 125]. Like the currents (B.0.1) defined from conformal supergravity, this definition of the FZ multiplet is independent of the specific theory and applies even to non Lagrangian theories.

The local symmetries of old minimal supergravity consist of diffeomorphisms, local frame rotations, as well as the local supersymmetry transformations (G.0.6). The algebra of these transformations closes off-shell [96, 121]. Since there exist no gravitational or Lorentz anomalies in four dimensions, diffeomorphisms and local frame rotations are preserved at the quantum level. Whether the old minimal supersymmetry transformations (G.0.6) are anomalous can be determined using the associated WZ consistency conditions. We will not perform such an analysis here, but we will show that the four-point functions of currents in the free and massless WZ model are compatible with a non anomalous old minimal supersymmetry.[2]

Inserting the local symmetry transformations of old minimal supergravity in the varia-

[2] The claim that there exists no supersymmetry anomaly in old minimal supergravity is implicit in [55, 65–67, 126]. However, these works concern the superspace formulation of old minimal supergravity, which contains additional fields that may act as compensators.

tion (G.0.7) and invoking the invariance of $\widetilde{\mathscr{W}}[e, A, \psi, M]$ leads to the three Ward identities

$$
\begin{aligned}
&e^a_\mu \nabla_\nu \langle \widetilde{\mathcal{T}}^\nu_a \rangle_s + \nabla_\nu(\bar{\psi}_\mu \langle \widetilde{\mathcal{Q}}^\nu \rangle_s) - \bar{\psi}_\nu \overleftarrow{D}_\mu \langle \widetilde{\mathcal{Q}}^\nu \rangle_s - F_{\mu\nu} \langle \widetilde{\mathcal{J}}^\nu \rangle_s - \partial_\mu M \langle \widetilde{\mathcal{O}}_M \rangle_s - \partial_\mu M^* \langle \widetilde{\mathcal{O}}_{M^*} \rangle_s \\
&\qquad + A_\mu (\nabla_\nu \langle \widetilde{\mathcal{J}}^\nu \rangle_s + i \bar{\psi}_\nu \langle \widetilde{\mathcal{Q}}^\nu \rangle_s) - \omega_\mu{}^{ab} \Big(e_{\nu[a} \langle \widetilde{\mathcal{T}}^\nu_{b]} \rangle_s + \frac{1}{4} \bar{\psi}_\nu \gamma_{ab} \langle \widetilde{\mathcal{Q}}^\nu \rangle_s \Big) = 0, \\
&e_{\mu[a} \langle \widetilde{\mathcal{T}}^\mu_{b]} \rangle_s + \frac{1}{4} \bar{\psi}_\mu \gamma_{ab} \langle \widetilde{\mathcal{Q}}^\mu \rangle_s = 0, \\
&D_\mu \langle \widetilde{\mathcal{Q}}^\mu \rangle_s - \frac{1}{2} \gamma^a \psi_\mu \langle \widetilde{\mathcal{T}}^\mu_a \rangle_s - \frac{3i}{4} \phi_\mu \langle \widetilde{\mathcal{J}}^\mu \rangle_s \\
&+ \frac{1}{2} \Big(M P_L + M^* P_R - \frac{2i}{3} \not{A} \Big) \Big(\gamma_\mu \langle \widetilde{\mathcal{Q}}^\mu \rangle_s - \frac{3i}{4} \psi_\mu \langle \widetilde{\mathcal{J}}^\mu \rangle_s \Big) \\
&+ \gamma^\mu \Big(\frac{i}{3} \not{A} \psi_{\mu R} + \frac{1}{2} M \psi_{\mu L} + \phi_{\mu L} \Big) \langle \widetilde{\mathcal{O}}_M \rangle_s + \gamma^\mu \Big(- \frac{i}{3} \not{A} \psi_{\mu L} + \frac{1}{2} M^* \psi_{\mu R} + \phi_{\mu R} \Big) \langle \widetilde{\mathcal{O}}_{M^*} \rangle_s = 0.
\end{aligned}
\tag{G.0.9}
$$

Since diffeomorphisms and local frame rotations are preserved at the quantum level, the first two Ward identities hold also in the quantum theory.

Now let us examine the third Ward identity, which is associated with supersymmetry. In particular, we are interested in the supersymmetry identities of $\langle \widetilde{\mathcal{Q}} \bar{\widetilde{\mathcal{Q}}} \rangle$, $\langle \widetilde{\mathcal{Q}} \bar{\widetilde{\mathcal{Q}}} \widetilde{\mathcal{J}} \rangle$ and $\langle \widetilde{\mathcal{Q}} \bar{\widetilde{\mathcal{Q}}} \widetilde{\mathcal{J}} \widetilde{\mathcal{J}} \rangle$. After taking the appropriate functional derivatives, we go to the flat space limit and find the following in momentum space:

2-point function

$$
p_{1\mu} \langle \widetilde{\mathcal{Q}}^\mu(p_1) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \rangle = 0, \tag{G.0.10}
$$

3-point function

$$
\begin{aligned}
&p_{1\mu} \langle \widetilde{\mathcal{Q}}^\mu(p_1) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \widetilde{\mathcal{J}}^\kappa(p_3) \rangle = i p_{2\mu} B^{\nu\mu\sigma} \langle \widetilde{\mathcal{J}}_\sigma(p_{12}) \widetilde{\mathcal{J}}^\kappa(p_3) \rangle - i \gamma_5 \langle \widetilde{\mathcal{Q}}^\kappa(p_{13}) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \rangle \\
&- \frac{i}{3} \gamma^\kappa \gamma_5 \gamma_\mu \langle \widetilde{\mathcal{Q}}^\mu(p_{13}) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \rangle,
\end{aligned}
\tag{G.0.11}
$$

4-point function

$$
\begin{aligned}
&p_{1\mu} \langle \widetilde{\mathcal{Q}}^\mu(p_1) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \widetilde{\mathcal{J}}^\kappa(p_3) \widetilde{\mathcal{J}}^\lambda(p_4) \rangle \\
&= i p_{2\mu} B^{\nu\mu\sigma} \langle \widetilde{\mathcal{J}}_\sigma(p_{12}) \widetilde{\mathcal{J}}^\kappa(p_3) \widetilde{\mathcal{J}}^\lambda(p_4) \rangle - \frac{\gamma_\xi}{2} \langle \widetilde{\mathcal{T}}^{\nu\xi}(p_{12}) \widetilde{\mathcal{J}}^\kappa(p_3) \widetilde{\mathcal{J}}^\lambda(p_4) \rangle \\
&- i \gamma_5 \langle \widetilde{\mathcal{Q}}^\kappa(p_{13}) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \widetilde{\mathcal{J}}^\lambda(p_4) \rangle - i \gamma_5 \langle \widetilde{\mathcal{Q}}^\lambda(p_{14}) \bar{\widetilde{\mathcal{Q}}}^\nu(p_2) \widetilde{\mathcal{J}}^\kappa(p_3) \rangle \\
&+ \gamma_5 B^{\nu\kappa\sigma} \langle \widetilde{\mathcal{J}}_\sigma(p_{123}) \widetilde{\mathcal{J}}^\lambda(p_4) \rangle + \gamma_5 B^{\nu\lambda\sigma} \langle \widetilde{\mathcal{J}}_\sigma(p_{124}) \widetilde{\mathcal{J}}^\kappa(p_3) \rangle
\end{aligned}
$$

$$+\frac{1}{3}\gamma^{\lambda}\gamma_5\left(-i\gamma_\mu\langle\widetilde{\mathcal{Q}}^{\mu}(p_{14})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\kappa}(p_3)\rangle+\frac{3i}{4}\gamma_5\langle\widetilde{\mathcal{J}}^{\nu}(p_{124})\widetilde{\mathcal{J}}^{\kappa}(p_3)\rangle\right)$$
$$+\frac{1}{3}\gamma^{\kappa}\gamma_5\left(-i\gamma_\mu\langle\widetilde{\mathcal{Q}}^{\mu}(p_{13})\widetilde{\bar{\mathcal{Q}}}^{\nu}(p_2)\widetilde{\mathcal{J}}^{\lambda}(p_4)\rangle+\frac{3i}{4}\gamma_5\langle\widetilde{\mathcal{J}}^{\nu}(p_{123})\widetilde{\mathcal{J}}^{\lambda}(p_4)\rangle\right). \quad \text{(G.0.12)}$$

The above are the supersymmetry Ward identities of the aforementioned correlators in the old minimal supergravity. As usual, we have not included the 1-point functions. Moreover, we omitted correlators involving the operators $\mathcal{O}_{M^*}$, $\mathcal{O}_M$, since they vanish trivially due to the absence of non zero Wick contractions. If one wishes to compute these identities for the WZ model using Feynman diagrams, it is necessary to couple the WZ model to old minimal supergravity, in order to find the theory dependent seagull correlators. We will not perform this analysis here.

After comparing the current correlators of (G.0.11) and (G.0.12) with (8.0.1) and (8.0.2), we find that they match exactly. As we proved, the Q+S supersymmetry identities of chapter 8, are manifestly satisfied in the quantum regime. These identities are identical with the supersymmetry identities of old minimal supergravity, up to seagull correlators. This means that the old minimal supergravity is non anomalous, as we claimed. The coupling of the WZ model to old minimal supergravity and the complete analysis which shows that the mentioned identities are the same including the seagull correlators is given in [81].

APPENDIX H

Correlators with arbitrary momentum routing

In this appendix, we write the explicit form of the correlators we use in the path integral Q-supersymmetry identity (9.2.1) of $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$. Since we are working with cut-off regularization, the momentum routing we choose for the integrals that correspond to a correlator is important. In general, one has to choose a specific routing for the Feynman diagrams of the correlators, and then introduce an extra parameter that encodes the freedom that we have for an arbitrary shift in the loop momentum. From a physical point of view, every choice of routing should correspond to the same physical procedure. For convergent correlators, this is indeed true. In divergent correlators though, different choices of routing result in integrals that differ by surface terms. Then, one has to fix these surface terms (using a different routing, or counterterms) so that the correlator under consideration is consistent and satisfies its classical symmetries.

In principle, a correlator is given by a sum of Feynman diagrams/integrals. For generality, we assign a different routing parameter for every one of the integrals that comprise the correlator. This is what the index i represents in the parameters $\{a_i\}$, ... $\{h_i\}$. For example, the 4-point correlator $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ consists of six integrals (H.0.1), thus we have six different routing parameters a_1, ... a_6.

The initial choice of routing that we made in the following correlators (i.e. when the routing parameters are equal to zero) is such, that the symmetry identity (9.2.1) is satisfied without any illegitimate manipulations at the integrals. This choice however, as mentioned in chapter 9, renders the 3-point correlator $<\mathcal{J}\mathcal{J}\mathcal{J}>$ inconsistent. In particular, it does not respect Bose symmetry, thus it does not reproduce the standard R-symmetry anomaly.

$<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$

The linearly divergent 4-point function $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}\mathcal{J}>$ is given by:

$$
\begin{aligned}
&<\mathcal{Q}^\mu(p_1)\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)\,\mathcal{J}^\lambda(p_4)>_{\{a_i\}}= \int \frac{d^4q}{(2\pi)^4}\Bigg\{\Big[\frac{2}{9}(2q+p_4+2a_1)^\kappa \\
&(2q-p_3+2a_1)^\lambda P_\phi(q+\frac{p_3+p_4}{2}+a_1)P_\phi(q-\frac{p_3+p_4}{2}+a_1)P_\phi(q+\frac{p_4-p_3}{2}+a_1) \\
&C_0^\mu(-q-\frac{p_3+p_4}{2}-a_1)P_\chi(q-p_2-\frac{p_3+p_4}{2}+a_1)A_0^\nu(q-\frac{p_3+p_4}{2}+a_1)\Big]+\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ a_1 & \longleftrightarrow & a_2 \end{pmatrix}\Bigg\} \\
&+\int \frac{d^4q}{(2\pi)^4}\Bigg\{\Big[\frac{2i}{18}(2q+2a_3)^\kappa \\
&P_\phi(q+\frac{p_3}{2}+a_3)P_\phi(q-\frac{p_3}{2}+a_3)C_0^\mu(-q-\frac{p_3}{2}-a_3)P_\chi(q-p_2-p_4-\frac{p_3}{2}+a_3) \\
&\gamma^\lambda\gamma_5 P_\chi(q-p_2-\frac{p_3}{2}+a_3)A_0^\nu(q-\frac{p_3}{2}+a_3)\Big]+\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ a_3 & \longleftrightarrow & a_4 \end{pmatrix}\Bigg\} \\
&+\int \frac{d^4q}{(2\pi)^4}\Bigg\{\Big[-\frac{1}{18}P_\phi(q+\frac{p_2-p_1}{2}+a_5) \\
&C_0^\mu(-q-\frac{p_2-p_1}{2}-a_5)P_\chi(q+\frac{p_2+p_1}{2}+a_5)\gamma^\lambda\gamma_5 P_\chi(q-\frac{p_2+p_1}{2}-p_3+a_5) \\
&\gamma^\kappa\gamma_5 P_\chi(q-\frac{p_2+p_1}{2}+a_5)A_0^\nu(q+\frac{p_2-p_1}{2}+a_5)\Big]+\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ a_5 & \longleftrightarrow & a_6 \end{pmatrix}\Bigg\}
\end{aligned}
\tag{H.0.1}
$$

where $C_0^\mu(q)$, $A_0^\nu(q)$ are given in (9.1.3).

$<\mathcal{T}\mathcal{J}\mathcal{J}>$

The quadratically divergent $<\mathcal{T}\mathcal{J}\mathcal{J}>$ is given by

$$
\begin{aligned}
<\mathcal{T}^{\mu\nu}(p_1+p_2)\mathcal{J}^{\kappa}(p_3)\mathcal{J}^{\lambda}(p_4)>_{\{d_i\}} = & \int \frac{d^4q}{(2\pi)^4}\Big\{\Big[\frac{-4}{9}G_0^{\mu\nu}(q+d_1-\frac{p_3+p_4}{2}) \\
& (2q+p_4+2d_1)^{\kappa}(2q-p_3+2d_1)^{\lambda}P_{\phi}(q+\frac{p_3+p_4}{2}+d_1) \\
& P_{\phi}(q-\frac{p_3+p_4}{2}+d_1)P_{\phi}(q+\frac{p_4-p_3}{2}+d_1)\Big] \\
& +\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ d_1 & \longleftrightarrow & d_2 \end{pmatrix}\Big\} \\
& +\int \frac{d^4q}{(2\pi)^4}\Big\{\Big[\frac{i}{18}\,\mathrm{tr}\Big(\gamma_\sigma P_\chi(q+\frac{p_2+p_1}{2}+d_3)\gamma^\lambda\gamma_5 P_\chi(q+\frac{p_2+p_1}{2}+p_4+d_3)\gamma^\kappa\gamma_5 \\
& P_\chi(q-\frac{p_2+p_1}{2}+d_3)\Big) \\
& \Big((q+\frac{p_1+p_2}{2}+d_3)^\nu\eta^{\mu\sigma}-(q+\frac{p_1+p_2}{2}+d_3)^\sigma\eta^{\nu\mu}\Big)\Big] \\
& +\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ d_3 & \longleftrightarrow & d_4 \end{pmatrix}\Big\} \\
& +\int \frac{d^4q}{(2\pi)^4}\Big\{\Big[\frac{i}{72}\,\mathrm{tr}\Big(\gamma_\sigma\gamma_5 P_\chi(q+\frac{p_2+p_1}{2}+d_5)\gamma^\lambda\gamma_5 P_\chi(q+\frac{p_2+p_1}{2}+p_4+d_5)\gamma^\kappa\gamma_5 \\
& P_\chi(q-\frac{p_2+p_1}{2}+d_5)\Big)\,\epsilon^{\mu\nu\rho\sigma}ip_{1\rho}\Big] \\
& +\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ d_5 & \longleftrightarrow & d_6 \end{pmatrix}\Big\}.
\end{aligned}
\tag{H.0.2}
$$

$<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$

The quadratically divergent $<\mathcal{Q}\bar{\mathcal{Q}}\mathcal{J}>$ is equal to

$$
\begin{aligned}
<\mathcal{Q}^{\kappa}(p_1+p_3)\bar{\mathcal{Q}}^{\nu}(p_2)\mathcal{J}^{\lambda}(p_4)>_{\{e_i\}} = & \int \frac{d^4q}{(2\pi)^4}\Big\{\Big[\frac{i}{6}P_\phi(q+\frac{p_2-p_1}{2}+e_1) \\
& C_0^\kappa(-q-\frac{p_2-p_1}{2}-e_1)P_\chi(q+p_3+\frac{p_2+p_1}{2}+e_1)\gamma^\lambda\gamma_5 \\
& P_\chi(q-\frac{p_2+p_1}{2}+e_1)A_0^\nu(q+\frac{p_2-p_1}{2}+e_1)\Big] \\
& +\int \frac{d^4q}{(2\pi)^4}\Big\{\Big[\frac{1}{3}(-2q-2e_2+p_1-p_2+p_4)^\lambda\,P_\phi(q+\frac{p_2-p_1}{2}+e_2) \\
& P_\phi(q+\frac{p_2-p_1}{2}-p_4+e_2)\gamma_5 C_0^\kappa(-q-\frac{p_2-p_1}{2}-e_2) \\
& P_\chi(q+\frac{p_2-p_1}{2}+e_2)A_0^\nu(q+\frac{p_2+p_1}{2}+p_3+e_2-p_4)\Big].
\end{aligned}
\tag{H.0.3}
$$

The 3-point function $<\mathcal{Q}^\lambda(p_1+p_4)\bar{\mathcal{Q}}^\nu(p_2)\mathcal{J}^\kappa(p_3)>_{\{f_i\}}$ is given by (H.0.3) after we make the exchanges $(\kappa\,,p_3\,,e_i)\leftrightarrow(\lambda\,,p_4\,,f_i)$.

$<\mathcal{J}\mathcal{J}\mathcal{J}>$

The linearly divergent $<\mathcal{J}\mathcal{J}\mathcal{J}>$ is given by

$$<\mathcal{J}_\sigma(p_1+p_2)\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>_{\{c_i\}}=$$
$$\int\frac{d^4q}{(2\pi)^4}\frac{i}{54}\left\{\mathrm{tr}\left(\gamma_\sigma\gamma_5P_\chi(q+\frac{p_2+p_1}{2}+c_1)\gamma^\lambda\gamma_5P_\chi(q+\frac{p_2+p_1}{2}+p_4+c_1)\gamma^\kappa\gamma_5P_\chi(q-\frac{p_2+p_1}{2}+c_1)\right)\right.$$
$$\left.+\begin{pmatrix}p_3&\longleftrightarrow&p_4\\ \kappa&\longleftrightarrow&\lambda\\ c_1&\longleftrightarrow&c_2\end{pmatrix}\right\}. \tag{H.0.4}$$

$<\mathcal{J}\mathcal{J}\mathcal{J}>$ respects Bose symmetry for

$$c_1=-\frac{p_4-p_3}{6}\,,\qquad c_2=\frac{p_4-p_3}{6}. \tag{H.0.5}$$

$<\mathcal{J}\mathcal{J}>$

The quadratically divergent 2-point correlator $<\mathcal{J}\mathcal{J}>$ has the following form

$$<\mathcal{J}^\sigma(-p_4)\mathcal{J}^\lambda(p_4)>_{\{g_i\}}=\int\frac{d^4q}{(2\pi)^4}\frac{4}{9}(2q-p_4+2g_1)^\sigma(2q-p_4+2g_1)^\lambda P_\phi(q+g_1)P_\phi(q-p_4+g_1)$$
$$+\int\frac{d^4q}{(2\pi)^4}\frac{1}{18}\mathrm{tr}\left(\gamma^\sigma\gamma_5P_\chi(q+g_2)\gamma^\lambda\gamma_5P_\chi(q+p_4+g_2)\right). \tag{H.0.6}$$

$\Omega^{\nu\kappa\lambda}$

We now give the expressions for the seagull correlators $\Omega^{\nu\kappa\lambda}$ (9.2.2).

$$<\mathcal{Q}^\mu(p_1)\bar{s}_{(3|\frac{1}{2})}(p_2+p_4)\mathcal{J}^\kappa(p_3)>_{\{b_i\}}=$$
$$\int\frac{d^4q}{(2\pi)^4}\left(\frac{-\sqrt{2}}{6}P_\phi(q+\frac{p_2+p_4-p_1}{2}+b_1)\right.$$
$$C_0^\mu(q+\frac{p_2+p_4-p_1}{2}+b_1)P_\chi(p_1-q-\frac{p_2+p_4-p_1}{2}-b_1)\gamma^\kappa\gamma_5P_\chi(-q-p_2-p_4-\frac{p_2+p_4-p_1}{2}-b_1)\gamma_5$$
$$+\frac{\sqrt{2}i}{3}(2q-2p_1+2b_2)^\kappa P_\phi(q+\frac{p_2+p_4-p_1}{2}+b_2)P_\phi(q+\frac{p_2+p_4-p_1}{2}+b_2+p_3)$$
$$\left.\gamma_5C_0^\mu(q+\frac{p_2+p_4-p_1}{2}+b_2)P_\chi(p_1-q-\frac{p_2+p_4-p_1}{2}-b_2)\gamma_5\right), \tag{H.0.7}$$

$$<s_{(1|0)}(p_1+p_2)\mathcal{J}^\kappa(p_3)\mathcal{J}^\lambda(p_4)>_{\{b_i\}}=\int\frac{d^4q}{(2\pi)^4}\Big\{$$
$$\Big[\frac{4}{9}(2q+p_4+2b_1)^\kappa(2q-p_3+2b_1)^\lambda P_\phi(q+\frac{p_3+p_4}{2}+b_1)P_\phi(q-\frac{p_3+p_4}{2}+b_1)P_\phi(q+\frac{p_4-p_3}{2}+b_1)\Big]$$
$$+\begin{pmatrix} p_3 & \longleftrightarrow & p_4 \\ \kappa & \longleftrightarrow & \lambda \\ b_1 & \longleftrightarrow & b_2 \end{pmatrix}\Bigg\}\,, \tag{H.0.8}$$

$$<\mathcal{Q}^\mu(-p_2)\bar{s}_{(3|\frac{1}{2})}(p_2)>_{\{b_i\}}=\int\frac{d^4q}{(2\pi)^4}\frac{i\sqrt{2}}{2}C_0^\mu(q+b_1)P_\chi(-p_2-q-b_1)\gamma_5P_\phi(q+b_1), \tag{H.0.9}$$

$$<(s^*_{\sigma(4|1)}-s_{\sigma(4|1)})(-p_4)\mathcal{J}^\lambda(p_4)>_{\{b_i\}}=$$
$$-\int\frac{d^4q}{(2\pi)^4}\frac{4}{9}(2q-p_4+2b_1)_\sigma(2q-p_4+2b_1)^\lambda P_\phi(q+b_1)P_\phi(q-p_4+b_1), \tag{H.0.10}$$

and

$$<s^\sigma_{(2|1)}(-p_4)\mathcal{J}^\lambda(p_4)>_{\{b_i\}}=\int\frac{d^4q}{(2\pi)^4}\frac{-i}{3}\mathrm{tr}\left(\gamma^\sigma\gamma_5P_\chi(q+b_1)\gamma^\lambda\gamma_5P_\chi(q+p_4+b_1)\right). \tag{H.0.11}$$

Every other correlator in $\Omega^{\nu\kappa\lambda}$ (9.2.2) can be deduced from the above integrals and some symmetry arguments.

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