UNIVERSITY OF SOUTHAMPTON
Faculty of Social Sciences
Mathematical Sciences

Equivariant cohomology, lattices, and trees

by

Sam Hughes

A thesis submitted in partial fulfilment for the degree of Doctor of Philosophy

July 2021

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF SOCIAL SCIENCES
MATHEMATICAL SCIENCES

Doctor of Philosophy

by Sam Hughes

This is a 'three paper thesis', the main body of which consists of the following papers:

[1] S. Hughes, *Cohomology of Fuchsian groups and non-Euclidean crystallographic groups*, preprint, available at arXiv:1910.00519 [math.GR], 2019.

[2] S. Hughes, *On the equivariant K- and KO-homology of some special linear groups*, to appear in Algebraic and Geometric topology. Available at arXiv:2004.08199 [math.KT], 2020.

[3] I. Chatterji, S. Hughes and P. Kropholler, *Groups acting on trees and the first ℓ^2-Betti number*, to appear in Proceedings of the Edinburgh Mathematical Society. Available at arXiv:2004.08199 [math.GR], 2020.

[4] S. Hughes, *Graphs and complexes of lattices*, preprint, available at arXiv:2104.13728 [math.GR], 2021.

[5] S. Hughes, *Hierarchically hyperbolic groups, products of* $\mathrm{CAT}(-1)$ *spaces, and virtual torsion-freeness*, preprint, available at arXiv:2105.02847 [math.GR], 2021.

In [1], we compute the cohomology groups of a number of low dimensional linear groups. In particular, for each geometrically finite 2-dimensional non-Euclidean crystallographic group (NEC group), we compute the cohomology groups. In the case where the group is a Fuchsian group, we also determine the ring structure of the cohomology.

In [2], we study K-theoretic properties of arithmetic groups in relation to the Baum–Connes Conjecture. Specifically, we compute the equivariant KO-homology of the classifying space for proper actions of $\mathrm{SL}_3(\mathbb{Z})$, and the Bredon homology and equivariant K-homology of the classifying spaces for proper actions of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ for each prime p. Finally, we prove the Unstable Gromov-Lawson-Rosenberg Conjecture on positive scalar curvature for a large class of groups whose maximal finite subgroups are odd order and have periodic cohomology.

In [3], we generalise results of Thomas, Allcock, Thom-Petersen, and Kar-Niblo to the first ℓ^2-Betti number of quotients of certain groups acting on trees by subgroups with free actions on the edge sets of the graphs.

In [4], we study lattices acting on CAT(0) spaces via their commensurated subgroups. To do this we introduce the notions of a graph of lattices and a complex of lattices giving graph and complex of group splittings of CAT(0) lattices. Using this framework we characterise irreducible uniform $(\text{Isom}(\mathbb{E}^n) \times T)$-lattices by C^*-simplicity and the failure of virtual fibring and biautomaticity. We construct non-residually finite uniform lattices acting on arbitrary products of right angled buildings and non-biautomatic lattices acting on the product of $\mathbb{E}^n$ and a right-angled building. We investigate the residual finiteness, L^2-cohomology, and C^*-simplicity of CAT(0) lattices more generally. Along the way we prove that many right angled Artin groups with rank 2 centre are not quasi-isometrically rigid.

In [5], we prove that a group acting geometrically on a product of proper minimal CAT(-1) spaces without permuting isometric factors is a hierarchically hyperbolic group. As an application we construct, what to the author's knowledge are, the first examples of hierarchically hyperbolic groups which are not virtually torsion-free.

Contents

Research Thesis: Declaration of Authorship

Name: Sam Hughes
Title of thesis: Equivariant cohomology, lattices, and trees

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published works of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself or jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

[1] S. Hughes, *On the equivariant K- and KO-homology of some special linear groups*, to appear in Algebraic and Geometric Topology.

[2] S. Hughes, *Cohomology of Fuchsian groups and non-Euclidean crystallographic groups*, preprint, 2019. arXiv:1910.00519 [math.GR]

[3] I. Chatterji, S. Hughes and P. Kropholler, *Groups acting on trees and the first ℓ^2-Betti number*, to appear in Proceedings of the Edinburgh Mathematical Society.

[4] S. Hughes, *Graphs and complexes of lattices*, preprint, 2021. arXiv:2104.13728 [math.GR]

[5] S. Hughes, *Hierarchically hyperbolic groups, products of* $\mathrm{CAT}(-1)$ *spaces, and virtual torsion-freeness*, preprint, 2021. arXiv:2105.02847 [math.GR]

Signature: ______________________________ Date: ____________________

List of Figures

Acknowledgements

Firstly, I would like to thank my supervisor Professor Ian Leary for his good humoured encouragement, enthusiasm, and support. I would like to thank my second supervisor Dr Ashot Minaysan for his encouragement, numerous helpful comments, and support. I would also like to thank Dr Nick Gill for encouraging me to apply for a PhD.

Mathematically, this thesis would never have happened without conversations and correspondence with a number of people and so I would like to extend my gratitude to: Naomi Andrew, Guy Boyde, Tom Brown, Pierre Emmanuel-Caprace, Jim Davis, Mark Hagen, Susan Hermiller, Jingyin Huang, Ian Leary, Kevin Li, Ashot Minasyan, Harry Petyt, George Simmons, and Matthew Staniforth.

I would like to thank my parents for their unwavering and kind support, encouragement, and motivation.

Matthew, Mike, and Lily, for making 54/9007 objectively the best office. Matt and Lily, for making Maths & Mingle a pleasure to organise. I would also like to extend my thanks to Jelena and the FOS team for their continued help with Maths & Mingle. Amina, Kevin, Matthew, and Vlad, for being excellent co-organisers for PGTC 2020. George, Lily, and Tom, for making the mathematics department during the pandemic an enjoyable place to visit. Guy and Pete, for being excellent housemates and for a near endless list of so-bad-its-good films for Monday nights. I would also like to thank Ashley, Daf, Ellie, Freddie, Gareth, Jack, Jimmy, Otto, and Pedro (Matt).

Finally, I would also like to thank the whole mathematics department in Southampton for being one of the friendliest and welcoming places to conduct research.

Chapter A

Introduction

We will refer to the papers as Paper 1 [Hug19], Paper 2 [CHK20], Paper 3 [Hug20], Paper 4 [Hug21a], and Paper 5 [Hug21b]. Papers 1, 3, 4, and 5 are single-author papers. Paper 2 is joint work with Indira Chatterji and Peter Kropholler. The paper was based on an unfinished project of of Peter and Indira, specifically, they had proved the main result (Theorem 2.A) for graphs of *finite* groups. Peter suggested that I take a look at the project and I found a strategy to extend the result to the class of groups $\mathcal{C}$ and proved the additional computations (Theorem 2.E).

Note that Section B.4 contains a brief summary of joint work with Pierre-Emmanuel Caprace in which we present a sketch of a proof fixing a gap in the main theorem of "Regular elements of CAT(0) groups" by Pierre-Emmanuel Caprace and Gašper Zadnik [CZ13].

Other papers and preprints completed by the author during his PhD studies may be found here [GH21] and [HMPSSn21].

In Paper 1 we compute the group cohomology of lattices in $\mathrm{PGL}_2(\mathbb{R})$ using the equivariant spectral sequence for a Γ-space. The paper is self contained, however, it is expected the reader is familiar with group homology and cohomology. An excellent reference for this is [Bro94].

In Paper 2 we investigate the L^2-cohomology of certain graphs of groups. The techniques rely on basic properties of the L^2-cohomology theory and are developed in the paper. For additional background the reader could consult [Lüc02]. The paper also heavily relies on groups acting on trees, the relevant background here is given in the Section B.1.

In Paper 3 we turn our attention to the equivariant cohomology theory of Bredon and its connection with various conjectures in K-theory. We perform a number of explicit computations of K-groups of arithmetic groups, and prove the Unstable Gromov–Lawson–Rosenberg Conjecture for a large class of groups (Theorem 3.E). Again the paper is essentially self contained, however, the computations of $K_*^\Gamma(\underline{E}\Gamma)$ for $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ use the Bruhat-Tits' tree for Γ. We provide background on the Bruhat-Tits' tree for $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ in Section B.1.2.3.

In Paper 4 we study lattices in more general products of CAT(0) spaces. We introduce the notions of a graph and a complex of lattices and use this to deduce a number of structural properties of lattices in mixed products of either a tree or CAT(0) polyhedral complex and a fairly arbitrary CAT(0) space. The paper relies and builds on work of Caprace–Monod [CM09b] [CM09a] [CM19] and Leary–Minasyan [LM19]. An introduction to CAT(0) groups and spaces is given in Section B.2, an introduction to Leary–Minasyan groups is given in Section B.1.2.2. Finally, the background on CAT(0) lattices in given in Section B.3

In Paper 5 we construct a hierarchically hyperbolic group which is not virtually torsion-free answering a well known folklore question. The paper uses elementary methods and is essentially self contained. However, many of the proofs are streamlined by results in Paper 4 - we highlight the relevant parts in the body of the paper. Indeed, it was whilst writing Paper 4 that the author came up with the ideas for this example.

Finally, I would like to include a few words about how these projects fit together. As the title suggests the overarching theme is "equivariant cohomology, lattices, and trees" - all but one of the papers (Paper 5) features some cohomology calculation, all but one of the papers features lattices acting on CAT(0) spaces (Paper 2), and all of the papers feature groups acting on trees. The project which spurred this was to construct a variant of Leary and Minasyan's groups $\mathrm{LM}(A)$ that acted on the real hyperbolic plane $\mathbb{R}\mathbf{H}^2$, we hoped that this group would be a counterexample to the Flat Closing Conjecture.

The project required an understanding of NEC groups. At some point I realised the cohomology of these NEC groups had not be computed and so Paper 1 was born. I figured that the calculation was profitable, not just because it was interesting in its own right, but because it could be used to give a deeper understanding of the cohomology of these "hyperbolic Leary–Minasyan groups". Once this was computed I became interested in other cohomology theories (L^2 and Bredon) which led to Paper 3 and the joint paper with Indira Chatterji and Peter Kropholler (Paper 2). Paper 3 originally began life as an attempt to compute the Bredon cohomology of all lattices in $\mathrm{PSL}_2(\mathbb{R}) \times T_{p+1}$, where p is a prime and T_{p+1} is the automorphism group of $(p+1)$-regular tree. This was a far too broad problem to tackle, however, the special case of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ was tractable (Theorem 3.C). After learning more about the isomorphism conjectures in K-theory and reading [DP03] I then proved the result on the Unstable Gromov–Lawson–Rosenberg Conjecture (Theorem 3.E).

Returning to the "hyperbolic Leary–Minasyan groups" I soon realised that the structure theory of Caprace–Monod could be extended using more combinatorial methods, this gave rise to Paper 4 and Paper 5. Although "hyperbolic Leary–Minasyan groups" are not counterexamples to the Flat Closing Conjecture by a result of Caprace–Žadnik [CZ13]. However, it transpired that the main result of this paper contains an error (the statement still applies for the above application and we give an ammended statement in Section B.4). This ultimately led to the work with Pierre-Emmanuel Caprace where we hope to fix the paper for the general case.

Chapter B

Background

This chapter provides some background on the papers included in this thesis. The papers are largely self contained, however, we take the opportunity to provide additional information about some of the groups which will appear in this thesis. The cohomology theories we study (ordinary group cohomology, L^2-cohomology, and Bredon cohomology) will be introduced in the relevant papers. However, the reader may wish to consult [Bro94] for a background on group cohomology, [Lüc02] or [Kam19] for a background on L^2-cohomology, and [MV03] for Bredon cohomology.

This entire chapter contains no original content except in Section B.4 where we present a sketch of a proof fixing a gap in the main theorem of [CZ13].

B.1 Groups acting on trees

We shall state some of the definitions and results of Bass-Serre theory. In particular, the action will be on the right. We follow the treatment of Bass [Bas93]. Throughout a *graph* $A = (VA, EA, \iota, \tau)$ should be understood as it is defined by Serre [Ser03], with edges in oriented pairs indicated by $\overline{e}$, and maps $\iota(e)$ and $\tau(e)$ from each edge to its initial and terminal vertices. We will, however, often talk about the geometric realisation of a graph as a metric space. In this case the graph should be assumed to be simplicial (possibly after subdividing) and should have exactly one undirected edge e for each pair $(e, \overline{e})$. We will often not distinguish between the combinatorial and metric notions.

A *tree* $\mathcal{T}$ is a connected non-empty graph without circuits. Let m and n be cardinals. A tree is *n-regular* if each vertex has valence of cardinality n. A tree $\mathcal{T}$ is (n, m)-biregular if the vertices of $\mathcal{T}$ admits a 2-colouring, the vertices of the same colour are not adjacent, vertices of the first colour have valence m, and vertices of the second colour have valence n.

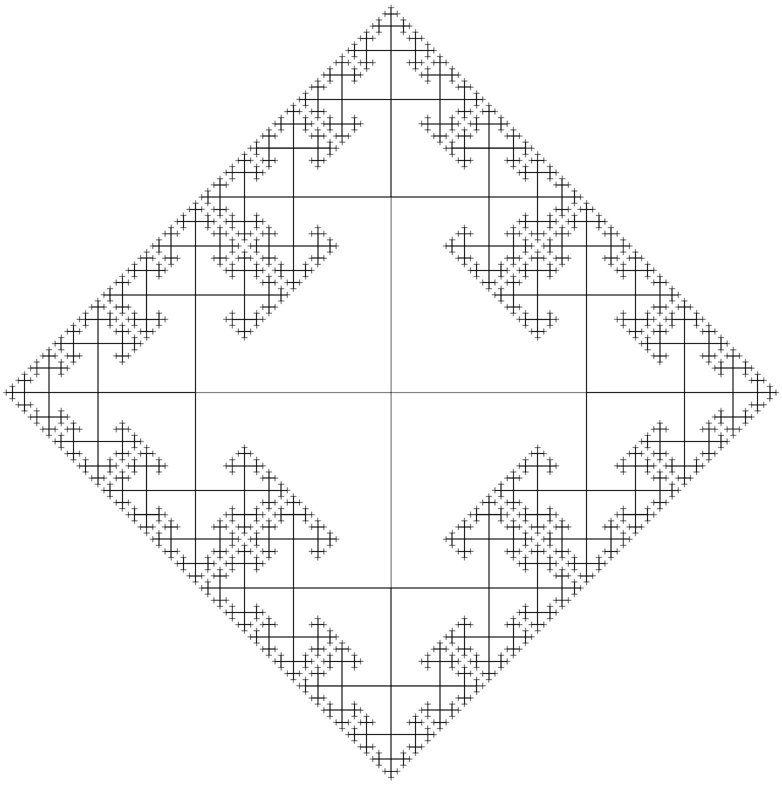

FIGURE B.1: A 4-regular tree.

B.1.1 Graphs of groups

A *graph of groups* $(A, \mathcal{A})$ consists of a graph A together with some extra data $\mathcal{A} = (V\mathcal{A}, E\mathcal{A}, \Phi\mathcal{A})$. This data consists of *vertex groups* $A_v \in V\mathcal{A}$ for each vertex v, *edge groups* $A_e = A_{\overline{e}} \in E\mathcal{A}$ for each (oriented) edge e, and monomorphisms $(\alpha_e : A_e \to A_{\iota(e)}) \in \Phi$ for every oriented edge in A. We will often refer to the vertex and edge groups as *local groups* and the monomorphisms as *structure maps.*

The *path group* $\pi(\mathcal{A})$ has generators the vertex groups A_v and elements t_e for each edge $e \in EA$ along with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\overline{e}} = t_e^{-1}, \\ t_e \alpha_{\overline{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } e \in EA \text{ and } g \in A_e = A_{\overline{e}}. \end{array} \right\}$$

We will often abuse notation and write $\mathcal{A}$ for a graph of groups. The *fundamental group of a graph of groups* can be defined in two ways. Firstly, considering reduced loops based at a vertex v in the graph of groups, in this case the fundamental group is denoted $\pi_1(\mathcal{A}, v)$ (see [Bas93, Definition 1.15]). Secondly, with respect to a maximal or spanning tree of the graph. Let X be a spanning tree for A, we define $\pi_1(\mathcal{A}, X)$ to be the group generated by the vertex groups A_v and elements t_e for each edge $e \in EA$ with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\bar{e}} = t_e^{-1} \text{ for each (oriented) edge } e, \\ t_e \alpha_{\bar{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } g \in A_e, \\ t_e = 1 \text{ if } e \text{ is an edge in } X. \end{array} \right\}$$

Note that the definitions are independent of the choice of basepoint v and spanning tree X and both definitions yield isomorphic groups so we can talk about *the fundamental group* of $\mathcal{A}$, denoted $\pi_1(\mathcal{A})$.

Let G be the fundamental group corresponding to the spanning tree X. For every vertex v and edge e, A_v and A_e can be identified with their images in G. We define a tree with vertices the disjoint union of all coset spaces G/A_v and edges the disjoint union of all coset spaces G/A_e respectively. We call this graph the *Bass-Serre tree* of $\mathcal{A}$ and note that the action of G admits X as a fundamental domain.

Given a group G acting on a tree $\mathcal{T}$, there is a *quotient graph of groups* formed by taking the quotient graph from the action and assigning edge and vertex groups as the stabilisers of a representative of each orbit. Edge monomorphisms are then the inclusions, after conjugating appropriately if incompatible representatives were chosen.

Theorem B.1.1. [Bas93] *Up to isomorphism of the structures concerned, the processes of constructing the quotient graph of groups, and of constructing the fundamental group and Bass-Serre tree are mutually inverse.* □

B.1.2 Examples

We will detail a number of examples groups acting on trees, these will appear in several of the papers later in the thesis.

B.1.2.1 Amalgamated free products and $\mathrm{SL}_2(\mathbb{Z})$

Given groups A, B, and C, and monomorphisms $\alpha_A : C \to A$ and $\alpha_B : C \to B$, we may form the amalgamated free product $\Gamma = A *_C B$. To do this we take the free product $A * B$ and then identify $\alpha_A(C)$ with $\alpha_B(C)$. A corresponding graph of groups for this construction is given as follows: Take a single directed edge e from a vertex v to a vertex w. Assign the vertex group A to v, B to w and the edge group C to e. Now, the monomorphism α_A is assigned to e and the monomorphism α_B is assigned to $\bar{e}$. This is illustrated in Figure B.2 with monomorphisms omitted. It is easy to see that the Bass-Serre tree for the amalgamated free product Γ is the $(|A : C|, |B : C|)$-biregular tree.

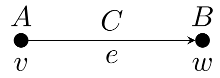

FIGURE B.2: A graph of groups for an amalgamated free product $A *_C B$

Example B.1 ($\mathrm{SL}_2(\mathbb{Z})$)**.** [Ser03, Page 35] The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the real hyperbolic plane $\mathbb{R}\mathbf{H}^2$ by linear fractional transforms. Consider the circular arc e in the upper half plane model starting at $w = e^{i\pi/3}$ and terminating at $v = i$ contained in the circle of radius 1 in $\mathbb{C}$ with origin 0 (this is illustrated in Figure B.3). The $\mathrm{SL}_2(\mathbb{Z})$ orbit of the edge e defines an embedding of the $(2,3)$-biregular tree into $\mathbb{R}\mathbf{H}^2$. It is easy to check that the stabiliser of v is isomorphic to $\mathbb{Z}_4$, the stabiliser of w is isomorphic to $\mathbb{Z}_6$ and the stabiliser of e is the central subgroup isomorphic to $\mathbb{Z}_2$. In particular, $\mathrm{SL}_2(\mathbb{Z})$ splits as an amalgamated free product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

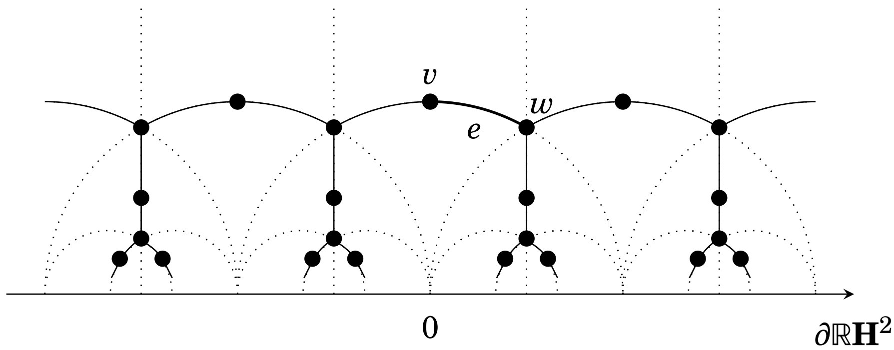

FIGURE B.3: The tree for $\mathrm{SL}_2(\mathbb{Z})$ embedded into $\mathbb{R}\mathbf{H}^2$.

B.1.2.2 HNN extensions, Baumslag–Solitar groups, and Leary–Minasyan groups

Given groups H and A and monomorphisms $i, j : A \to H$, the HNN extension $H*_A$ of H over A is the group defined by the presentation

$$\langle H, t \mid \mathrm{rel}(H),\ ti(a)t^{-1} = j(a)\ \forall a \in A\rangle.$$

HNN extensions arise as the fundamental group of a graph of groups consisting of a single vertex and edge. Here the vertex group is H, the edge group is A, and the edge monomorphisms are i and j. The Bass-Serre tree is the $(|H : i(A)| + |H : j(A)|)$-regular tree.

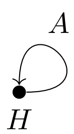

FIGURE B.4: A graph of groups for an HNN extension $H*_A$.

Example B.2 (Baumslag–Solitar groups)**.** The following groups were introduced in [BS62] by Baumslag and Solitar as the first examples of non-Hopfian one-relator groups. Let q and p denote non-zero integers and define the *Baumslag–Solitar group*

$$\mathrm{BS}(p,q) := \langle a, t \mid ta^p t^{-1} = a^q \rangle.$$

The group splits as an HNN-extension $\mathbb{Z}*_{p\mathbb{Z}}$ where the edge groups are $p\mathbb{Z} = \langle a^p \rangle$ and $q\mathbb{Z} = \langle a^q \rangle$. The Bass-Serre tree for $\mathrm{BS}(2,3)$ is depicted in Figure B.5.

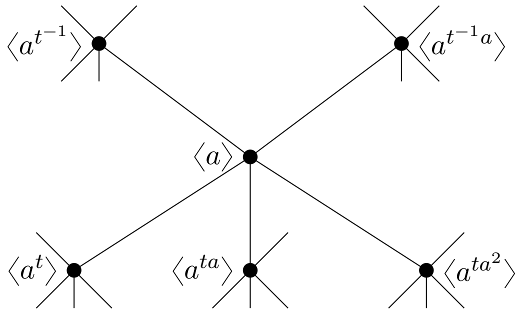

FIGURE B.5: The Bass-Serre tree for $\mathrm{BS}(2,3)$ and the vertex stabilisers.

Example B.3 (Leary–Minasyan groups)**.** The following groups were introduced in [LM19] by Leary and Minasyan as a class of groups containing the first examples of CAT(0) but not biautomatic groups, they were classified up to isomorphism by Valiunas [Val20]. In fact they are not subgroups of any biautomatic group [Val21]. Let $n \geqslant 0$, let $A \in \mathrm{GL}_n(\mathbb{Q})$, and let $L \leqslant \mathbb{Z}^n \cap A^{-1}(\mathbb{Z}^n)$ be a finite index subgroup. The group $\mathrm{LM}(A, L)$ is defined by the presentation

$$\mathrm{LM}(A, L) = \langle x_1, \dots, x_n, t \mid [x_i, x_j] = 1 \text{ for } 1 \leqslant i < j \leqslant n, t\mathbf{x}^{\mathbf{v}}t^{-1} = \mathbf{x}^{A\mathbf{v}} \text{ for } \mathbf{v} \in L \rangle,$$

where we write $\mathbf{x}^{\mathbf{w}} := x_1^{w_1} \cdots x_n^{w_n}$ for $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$. If L is the largest subgroup of $\mathbb{Z}^n$ such that AL is also a subgroup of $\mathbb{Z}^n$, then we denote $\mathrm{LM}(A, L)$ by $\mathrm{LM}(A)$. We refer to the groups $\mathrm{LM}(A, L)$ and $\mathrm{LM}(A)$ as *Leary–Minasyan groups*. The group clearly splits as an HNN extension $\mathbb{Z}^n *_L$.

The Leary–Minasyan groups are in some sense a generalisation of Baumslag-Solitar groups since for $n = 1$, if $L = r\mathbb{Z}$ and $A = s/r \in \mathrm{GL}_1(\mathbb{Q})$ for some non-zero integers r and s, then $\mathrm{LM}(A, L) = \langle x, t \mid tx^r t^{-1} = x^s \rangle = \mathrm{BS}(r, s)$.

As a concrete example, take

$$A = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \text{ and}$$
$$L = \left\langle \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle \text{ so}$$
$$AL = \left\langle \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\rangle.$$

Note that L is index 5 in $\mathbb{Z}^2$ and so must be a maximal subgroup. It follows that

$$\mathrm{LM}(A, L) = \mathrm{LM}(A) = \langle a, b, t \mid [a, b],\ ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2\rangle.$$

Recall that a group G is *residually finite* if for every non-trivial $g \in G$ there exists a finite group F_g and a homomorphism $\phi : G \to F_g$ such that $\phi(g) \neq 1$. Equivalently, the intersection of every finite-index subgroup of G is equal to $\{1\}$. Indeed, since for each $1 \neq g \in G$ we have $\phi(g) \neq 1$, there exists a finite-index normal subgroup $N \trianglelefteq G$ with $g \notin N$. In particular, the intersection of all finite-index normal subgroups, and hence the intersection of all finite-index subgroups is equal to $\{1\}$. The converse is immediate.

Proposition B.1.2. [LM19, Proposition 10.4] *Let $A \in \mathrm{GL}_n(\mathbb{Q})$ and $L \leqslant \mathbb{Z}^n$, then the group $\mathrm{LM}(A, L)$ is residually finite if and only if either $L = \mathbb{Z}^n$, $AL = \mathbb{Z}^n$, or A is conjugate in $\mathrm{GL}_n(\mathbb{Q})$ to a matrix in $\mathrm{GL}_n(\mathbb{Z})$.* □

The Leary–Minasyan groups come equipped with a representation into $\mathrm{AGL}_n(\mathbb{R}) = \mathbb{R}^n \rtimes \mathrm{GL}_n(\mathbb{R})$, here each generator x_i is sent to a basis vector of $\mathbb{R}^n$ identified with $L \otimes \mathbb{R}$ and the element t is sent to the matrix A in $\mathrm{GL}_n(\mathbb{R})$. Using this Leary–Minaysan show the following.

Proposition B.1.3. [LM19, Proposition 7.1] *Each group $\mathrm{LM}(A, L)$ is free-by-abelian-by-cyclic.* □

B.1.2.3 The Bruhat–Tits tree for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$

First we will fix some notation. Our treatment will follow [Ser03, Chapter II]. Let F denote a field with a discrete valuation $v : F \to \mathbb{Z}$, that is, v is a homorphism $F^* \to \mathbb{Z}$ such that for all $x, y \in F$ we have

$$v(x + y) \geqslant \mathrm{Inf}(v(x), v(y))$$

and the convention $v(0) = \infty$. Let $\mathcal{O}$ denote the valuation ring of F and choose $\pi \in F$ such that $v(\pi) = 1$. Note that all ideals in $\mathcal{O}$ are two-sided and let $k = \mathcal{O}/\pi\mathcal{O}$. Let V be a 2-dimensional F-vector space.

A *lattice of V* is any finitely generated $\mathcal{O}$-submodule of V which generates V as an F-vector space. The group F^* acts on the set of lattices $\mathcal{T}$ by right multiplication. We call the elements of $\mathcal{T}/F^*$ the *classes of lattices* and say that two lattices in the same class are *equivalent.*

Let L and L' be two lattices of V. It follows from the Invariant Factor Theorem for modules over a PID that there is a $\mathcal{O}$-basis $\{e_1, e_2\}$ of L and integers a, b such that $\{e_1\pi^a, e_2\pi^b\}$ is a $\mathcal{O}$-basis for L'. Note that $\{a, b\}$ does not depend on the choice of basis for L. Moreover, the integer $d(L, L') := |a - b|$ only depends on the classes Λ and Λ' of L and L'. Thus, we may denote this number by $d(\Lambda, \Lambda')$. We say two classes Λ, Λ' in $\mathcal{T}$ are *adjacent* if $d(\Lambda, \Lambda') = 1$. This endows $\mathcal{T}$ with the structure of a graph.

Theorem B.1.4. [Ser03, Page 70, (Theorem 1)] *With notation as established in this section, the graph $\mathcal{T}$ is a tree.* □

The group $\mathrm{GL}(V)$ acts on the tree $\mathcal{T}$ and we will refer to $\mathcal{T}$ as the *Bruhat-Tits tree of* $\mathrm{GL}(V)$. We will primarily interested in the groups $\mathrm{SL}(V)$ and $\mathrm{PSL}(V)$. Here $\mathrm{SL}(V)$ is the kernel of the *Dieudonné determinant* $\mathrm{GL}(V) \to F^*/(F^*, F^*)$ and $\mathrm{PSL}(V) := \mathrm{SL}(V)/Z(\mathrm{SL}(V))$. Clearly, $\mathrm{SL}(V)$ acts on $\mathcal{T}$ by restricting the action of $\mathrm{GL}(V)$. The kernel of the action of $\mathrm{SL}(V)$ on $\mathcal{T}$ is $Z(\mathrm{SL}(V))$. In particular, $\mathrm{PSL}(V)$ also acts on $\mathcal{T}$.

By [Ser03, Page 78, (Theorem 2)], $\mathrm{SL}(V)$ acts on $\mathcal{T}$ with fundamental domain a single edge and two distinct vertices. Let L and L' be lattices corresponding to two adjacent vertices in $\mathcal{T}$. Clearly, the stabilisers of these vertices can be identified with conjugates of $\mathrm{SL}_2(\mathcal{O})$. Computing their intersection yields the following theorem.

Theorem B.1.5 (Ihara's Theorem)**.** [Ser03, Page 79, (Corollary 1)] *The group* $\mathrm{SL}_2(F)$ *splits as an amalgamated free product* $\mathrm{SL}_2(\mathcal{O}) *_\Gamma \mathrm{SL}_2(\mathcal{O})$ *where*

$$\Gamma := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \pmod{\pi} \right\} \leqslant \mathrm{SL}_2(\mathcal{O}).$$

Let A be a dense subgroup of F, then the group $\mathrm{SL}_2(A)$ is a dense subgroup of $\mathrm{SL}_2(F)$ and we obtain an analogous amalgam splitting. Applying this to the case where $F = \mathbb{Q}$, v is the p-adic valuation, $A = \mathbb{Z}[\frac{1}{p}]$ and $A \cap \mathcal{O} = \mathbb{Z}$ we obtain the following result.

Corollary B.1.6. *If p is a prime number one has* $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{SL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{SL}_2(\mathbb{Z})$ *where*

$$\Gamma_0(p) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \pmod{p} \right\} \leqslant \mathrm{SL}_2(\mathbb{Z}).$$

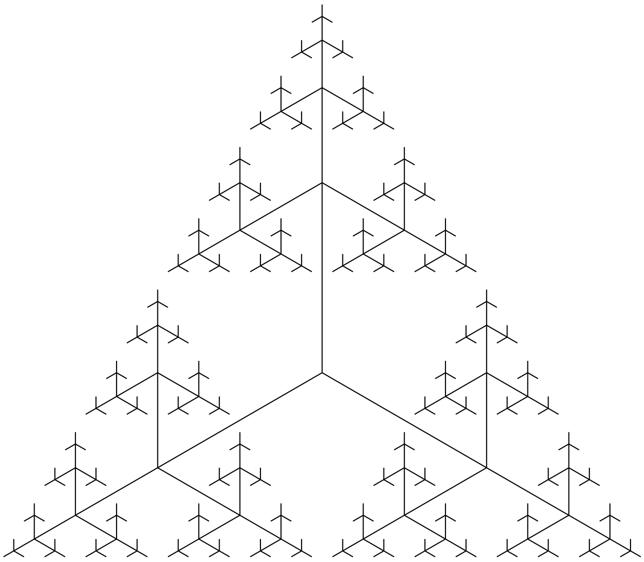

FIGURE B.6: The Bruhat-Tits tree for $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{2}])$.

B.2 CAT(0) spaces and their isometries

In this section we will introduce CAT(0) spaces and their groups of isometries. The main reference for this section is [BH99].

B.2.1 CAT(0) spaces

A geodesic metric space X is CAT(0) if for every geodesic triangle $P = \triangle(p,q,r) \subseteq X$ and comparison triangle $\overline{P} = \triangle(\overline{p},\overline{q},\overline{r})$ in $\mathbb{E}^2$ with the same side lengths as P such that for each pair of points $x, y \in \partial P$ and corresponding pair of points $\overline{x}, \overline{y} \in \partial\overline{P}$ we have

$$d_X(x,y) \leqslant d_{\mathbb{E}^2}(\overline{x},\overline{y}).$$

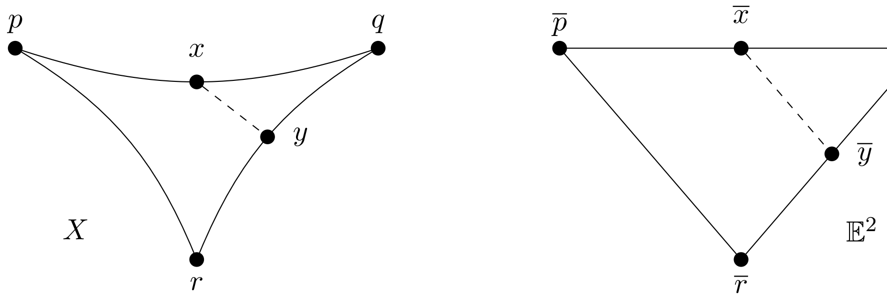

FIGURE B.7: An illustration of the CAT(0) inequality.

Examples B.4. The following spaces are CAT(0).

(i) $\mathbb{E}^n$, that is, Euclidean n-space with the ℓ^2-metric.

(ii) A locally finite tree. This is easily seen since all geodesic triangles are tripods.

(iii) The real hyperbolic plane $\mathbb{R}\mathbf{H}^2$. More generally, any symmetric space of non-compact type [BH99, Theorem II.10.58].

(iv) Euclidean and hyperbolic buildings [Dav98].

(v) The Davis complex of a right-angled Coxeter group and the Salvetti complex of a right-angled Artin group.

We record a number of properties of generic CAT(0) spaces.

Theorem B.2.1. *Let X and Y be* CAT(0) *spaces, then*

(a) [BH99, Proposition II.1.4(1)] *there is a unique geodesic between every two points of X;*

(b) [BH99, Corollary II.1.5] *X is contractible;*

(c) [BH99, Example II.1.15(3)] *$X \times Y$ equipped with the ℓ^2 metric is a* CAT(0) *space;*

(d) [BH99, Proposition II.2.7] *Suppose X is complete. If $C \subset X$ is a bounded subset of radius r, then there exists a unique point $c \in X$, called the centre of C, such that $C \subseteq \overline{B}_r(c)$.* □

A metric space X is *non-positively curved* if it is locally CAT(0), that is, for each $x \in X$ there exists $r_x > 0$ such that the ball $B_{r_x}(x)$ endowed with the induced metric is a CAT(0) space. The following theorem shows the relationship between non-positively curved metric spaces and CAT(0).

Theorem B.2.2 (The Cartan–Hadamard Theorem)**.** [BH99, Theorem II.4.1] *Let X be a complete connected non-positively curved metric space, then the universal cover $\widetilde{X}$ is a* CAT(0) *space.* □

B.2.2 Isometries of CAT(0) spaces

Let X be a metric space and γ an isometry of X. The *displacement function* of γ is the function $d_\gamma : X \to \mathbb{R}_+$ defined by $d_\gamma(x) = d(\gamma x, x)$. The *translation length of* γ, denoted $|\gamma|$, is the infinimum of the image of d_γ. The subset of X where d_γ attains its infinimum will be denoted by $\mathrm{Min}(\gamma)$. For a set of isometries Γ of X, we define $\mathrm{Min}(\Gamma) := \bigcap_{\gamma\in\Gamma} \mathrm{Min}(\gamma)$. An isometry γ of X is called

(i) *elliptic* if γ has a fixed point;

(ii) *hyperbolic* if d_γ attains a strictly positive minimum;

(iii) *parabolic* if d_γ does not attain its minimum;

(iv) *semi-simple* if $\mathrm{Min}(\gamma)$ is non-empty.

Similarly, a group of isometries Γ is *elliptic/hyperbolic/parabolic/semi-simple* if all of its elements are elliptic/hyperbolic/parabolic/semi-simple respectively.

Examples B.5. We will detail three examples.

(a) Let $\mathcal{T}$ be a locally finite tree. We will investigate the min-sets of semi-simple isometries.

  (i) Let γ be an elliptic isometry of $\mathcal{T}$, then $\mathrm{Min}(\gamma)$ is the fixed point set of γ. We claim this is a connected subtree of $\mathcal{T}$. Indeed, if γ fixes two vertices in $\mathcal{T}$, then γ must fix a geodesic between them, but geodesic segments in a tree are unique so we conclude that the fix point set is connected.

  (ii) Let γ be a hyperbolic isometry of $\mathcal{T}$, then $\mathrm{Min}(\gamma)$ is a unique embedded line which we call the *axis* of γ. Indeed, consider a tripod with vertices x, γx, $\gamma^2 x$ and crux v. Let m be the midpoint of the geodesic segment $[x, \gamma x]$ and note that if $d(m, x) \geqslant d(v, x)$, then γ fixes m. This contradicts the fact γ is hyperbolic. Therefore, $d(v, x) > d(m, x)$. If for any point $y \in \mathcal{T}$ we have $d(y, \gamma^2 y) = 2d(y, \gamma y)$ then the γ-translates on $[y, \gamma y]$ will form a γ-invariant line. Thus, it suffices to show that $d(m, \gamma^2 m) = 2d(m, \gamma m)$, and since $v \in [m, \gamma m]$ we need to show that $d(v, \gamma v) = 2d(v, \gamma m)$. But,

$$d(v, \gamma v) = d(\gamma x, \gamma^2 x) - 2d(v, \gamma x) = d(x, \gamma x) - d(x, \gamma x) - 2d(v, \gamma m) = 2d(v, \gamma m)$$

  as required.

(b) Consider the Euclidean n-space $\mathbb{E}^n$. The isometry group $\mathrm{Isom}(\mathbb{E}^n)$ splits as a semi-direct product $\mathbb{R}^n \rtimes \mathrm{O}(n)$. By [BH99, Proposition II.6.5] every isometry of $\mathbb{E}^n$ is semi-simple. Either an isometry γ is elliptic, or there is an integer $0 < k \leqslant n$ such that $\mathrm{Min}(\gamma)$ is an affine subspace E of dimension k. Moreover, if $k < n$, then γ is the product of a non-trivial translation on E and an elliptic isometry on the orthogonal complement $E^\perp$.

(c) Let $\mathbb{R}\mathbf{H}^2$ denote the real hyperbolic plane. The orientation preserving isometry group of $\mathbb{R}\mathbf{H}^2$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. Here an element g can be classified as elliptic, hyperbolic, or parabolic by the trace of a representative lift $\widetilde{g} \in \mathrm{SL}_2(\mathbb{R})$. If $\mathrm{tr}(\widetilde{g}) < 2$ then g is elliptic, if $\mathrm{tr}(\widetilde{g}) > 2$ then g is hyperbolic, and if $\mathrm{tr}(\widetilde{g}) = 2$, then g is parabolic (see for instance [Kat92]). It is easy to see that $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\mathbb{Z}_2$ is generated by elliptic elements and contains parabolic elements.

Many of the results in the previous examples hold more generally. We highlight some important ones here.

Proposition B.2.3. *Let X be a metric space, let γ be an isometry of X, and let Γ be a groups of isometries of X.*

(a) [BH99, Proposition II.6.2] $\mathrm{Min}(\gamma)$ *is* γ *invariant and* $\mathrm{Min}(\Gamma)$ *is* Γ *invariant.*

(b) [BH99, Proposition II.6.2] *If* α *is an isometry of* X*, then* $|\gamma| = |\alpha\gamma\alpha^{-1}|$*, and* $\mathrm{Min}(\alpha\gamma\alpha^{-1}) = \alpha\mathrm{Min}(\gamma)$*. In particular, if* α *commutes with* γ*, then it leaves* $\mathrm{Min}(\Gamma)$ *invariant. if* $N \trianglelefteq \Gamma$*, then* $\mathrm{Min}(N)$ *is* Γ*-invariant.*

Suppose in addition X *is a* $\mathrm{CAT}(0)$ *space.*

(c) [BH99, Proposition II.6.2] d_γ *is convex. Hence,* $\mathrm{Min}(\gamma)$ *is a closed convex set.*

(d) [BH99, Theorem II.6.8] γ *is hyperbolic if and only if there is a geodesic line* $c : \mathbb{E} \to X$ *(i.e. an* axis*) which is translated non-trivially by* γ*. The union of the axes of* γ *equals* $\mathrm{Min}(\gamma)$

Suppose further that X *is a complete* $\mathrm{CAT}(0)$ *space.*

(e) [BH99, Proposition II.6.7] γ *is elliptic if and only if* γ *has a bounded orbit.*

(f) [BH99, Proposition II.6.7] *If* γ^n *is elliptic for some non-zero integer* n *then* γ *is elliptic.*

(g) [BH99, Theorem II.6.8] *If* γ^n *is hyperbolic for some non-zero integer* n *then* γ *is hyperbolic.* □

B.2.3 CAT(0) groups

In this section we will state some well known results about groups acting "geometrically" on CAT(0) space, before we can do this we recall some notions about group actions. Let Γ be a group acting by isometries on a metric space X, the action is

(i) *proper* if for each $x \in X$ there exists a number $r > 0$ such that $\{\gamma \in \Gamma \mid \gamma B_r(x) \cap B_r(x) \neq \varnothing\}$ is finite;

(ii) *cocompact* if there exists a compact subset $K \subseteq X$ such that $\Gamma K = X$.

Suppose a discrete group Γ acts properly cocompactly by isometries on a $\mathrm{CAT}(0)$ space X, then we say Γ is a $\mathrm{CAT}(0)$ *group*. In this case we say that X is a space *realising a* $\mathrm{CAT}(0)$ *structure* on Γ.

Theorem B.2.4. *If a group* Γ *acts properly cocompactly on a* $\mathrm{CAT}(0)$ *space* X*, then* *(a)* [BH99, Theorem III.Γ.1.1] Γ *is finitely presented;*

(b) [BH99, Proposition II.6.10(2)] *Every element of* Γ *is semi-simple;*

(c) [BH99, Theorem III.Γ.1.1] Γ *has finitely many conjugacy classes of finite subgroups;*

(d) [BH99, Theorem III.Γ.1.1] *Every solvable subgroup of* Γ *is virtually abelian;*

(e) [BH99, Theorem III.Γ.1.1] *Every abelian subgroup of* Γ *is finitely generated;*

(f) [BH99, Theorem III.Γ.1.4] Γ *has solvable word and conjugacy problems.* □

Next we will state the Flat Torus Theorem, we will make use of this theorem in Paper 4, a proof of the theorem can be found in [BH99, Chapter II.7]. An addendum to the theorem was proven by Leary–Minasyan in [LM19, Theorem 6.4], the statement we give combines both versions.

Recall that a *torsor* for an abelian group is a non-empty set on which it acts freely and transitively. An affine space is naturally a torsor for its vector space of translations. By [LM19, Remark 6.2], given a free abelian group L, we may define an inner product $\langle \cdot, \cdot \rangle_L$ on $L \otimes \mathbb{R}$ as follows

$$\langle b, c \rangle_L := \frac{1}{2} \left(d((b+c)x, x)^2 - d(x, bx)^2 - d(x, cx)^2 \right).$$

We are now ready to state the Flat Torus Theorem. Note that items (a), (b), (c), (e), and (g) are from [BH99, Theorem II.7.1], and items (d) and (f) are from [LM19, Theorem 6.4].

Theorem B.2.5 (The Flat Torus Theorem)**.** [BH99, Theorem II.7.1], [LM19, Theorem 6.4] *Let L be a free abelian group of rank n acting properly by semi-simple isometries on a* CAT(0) *space X. Then:*

(a) The min set M for L is non-empty and $M = Y \times \mathbb{E}^n$.

(b) Every $C \in L$ leaves M invariant, respects the product decomposition, and acts trivially on Y and by translation on $\mathbb{E}^n$.

(c) For $y \in Y$, the quotient $(\{y\} \times \mathbb{E}^n)/L$ is an n-torus.

(d) For each $y \in Y$, the subspace $\{y\} \times \mathbb{E}^n$ is a torsor for $L \otimes \mathbb{R}$ under affine extension of the action of L.

(e) If an isometry of X normalises L, then it preserves M and the direct product decomposition.

(f) For any isometry ϕ of X that commensurates L, the image of ϕ in $\mathrm{GL}(L \otimes \mathbb{Q}) \leqslant \mathrm{GL}(L \otimes \mathbb{R})$ *preserves the inner product $\langle \cdot, \cdot \rangle_L$.*

(g) *If a group* Γ *of isometries of* X *normalises* L*, then a finite-index subgroup of* Γ *centralises* L*. If* Γ *is finitely generated, then* ΓL *has a finite-index subgroup containing* L *as a direct factor.* □

The theorem has an important corollary. The *rank* of a CAT(0) space X, denoted $\mathrm{rank}(X)$, is the maximal n such that $\mathbb{E}^n$ isometrically embeds into X.

Corollary B.2.6. *Let* X *be a* CAT(0) *space, then the rank of any free abelian group acting properly by semi-simple isometries on* X*, is at most* $\mathrm{rank}(X)$*.* □

It immediately follows that any abelian subgroup of CAT(0) group Γ, has rank bounded by the rank of the CAT(0) space X realising the CAT(0) structure. The converse to this observation is a famous open problem known as the *Flat Closing Conjecture* we will give a more detailed discussion in Section B.4.

Another related result is the *Algebraic Flat Torus Theorem* which states that abelian subgroups of CAT(0) groups are undistorted. The result is actually true for the more general class of *semihyperbolic groups*, however, this class will not feature in this thesis.

Theorem B.2.7. [BH99, Theorem III.Γ.4.10] *If* Γ *is a* CAT(0) *group and* A *is a finitely generated abelian subgroup, then every monomorphism* $\phi : A \rightarrowtail \Gamma$ *is a quasi-isometric embedding.* □

B.3 Lattices in non-positive curvature

When studying groups geometrically, that is, studying groups acting properly cocompactly by isometries on some metric space X, it is often convenient to study all groups acting geometrically and faithfully on X simultaneously. To do this, we study lattices in the full isometry group $\mathrm{Isom}(X)$. If X is CAT(0) then the structure of $\mathrm{Isom}(X)$ has a rich theory which is reflected by the space X itself. In Section B.3.1 we will recall the definition of a lattice in a locally compact group. In Section B.3.2 we will outline Caprace–Monod's structure theory for the isometry group of a CAT(0) space X. In Section B.3.3 we will look at various notions of irreducibility for lattices acting on products of CAT(0) spaces. Finally, in Section B.3.4 we will detail a number of examples of CAT(0) lattices. The results stated throughout this section will be repeatedly used in Paper 4.

B.3.1 Generalities on lattices

The following definitions may be found in [BL01, Section 1.1]. Let H be a locally-compact group and let μ be a choice of right Haar measure on H. For a measurable subset $U \subset H$

and for all $h \in H$ we have $\mu(Uh) = \mu(U)$ and $\mu(hU) = \mu(U)\Xi(h)$, where $\Xi : H \to \mathbb{R}^\times$ is the *modular character* of H. We say H is *unimodular* if $\Xi = 1$.

Let Γ be a discrete subgroup of H. As explained in [BL01, Section 1.2], H/Γ has an induced measure μ and the projection $H \twoheadrightarrow H/\Gamma$ is locally measure preserving. If $\mu(H/\Gamma) < \infty$ (so Γ has finite covolume), then $\Xi = 1$ and H is unimodular.

A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. A lattice is *uniform* if H/Γ is compact and *non-uniform* otherwise. Let S be a right H-set such that for all $s \in S$, the stabilisers H_s are compact and open, then if $\Gamma \leqslant H$ is discrete the stabilisers are finite (see [BL01, Section 1.5]).

Let X be a locally finite, connected, simply connected simplicial complex. The group $H = \mathrm{Aut}(X)$ of simplicial automorphisms of X naturally has the structure of a locally compact topological group, where the topology is given by uniform convergence on compacta.

Theorem B.3.1 (Serre's covolume formula [Ser71])**.** *Let X be a locally finite simply-connected simplicial complex. Let $\Gamma \leqslant H$ be a lattice with fundamental domain Δ, then there is a nomalisation of the Harr measure μ on H, depending only on X, such that for each discrete subgroup $\Gamma < H$ we have*

$$\mu(H/\Gamma) = \mathrm{Vol}(X/\Gamma) := \sum_{v \in \Delta^{(0)}} \frac{1}{|\Gamma_v|}. \qquad \square$$

B.3.2 Structure theory

We will be primarily interested in lattices in the isometry groups of CAT(0) spaces, we will call these groups CAT(0) *lattices* (note that a uniform CAT(0) lattice is a CAT(0) group). We begin by recording several facts about the structure and isometry groups of general CAT(0) spaces. The definitions and results here are largely due to Caprace and Monod [CM09b] [CM09a] [CM19].

An isometric action of a group H on a CAT(0) space X is *minimal* if there is no non-empty H-invariant closed convex subset $X' \subset X$, the space X is *minimal* if $\mathrm{Isom}(X)$ acts minimally on X. Note that by [CM09b, Proposition 1.5], if X is cocompact and geodesically complete, then it is minimal. The *amenable radical* of a locally compact group H is the largest amenable normal subgroup.

Theorem B.3.2. [CM09b, Theorem 1.6] *Let X be a proper* CAT(0) *space with finite dimensional Tits boundary and assume* $\mathrm{Isom}(X)$ *has no global fixed point in ∂X. There is a canonical closed, convex,* $\mathrm{Isom}(X)$*-stable subset $X' \subseteq X$ such that $G = \mathrm{Isom}(X')$ has a finite index, open, characteristic subgroup $H \trianglelefteq G$ that admits a canonical decomposition*

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q,$$

for some $n, p, q \geqslant 0$, where each S_i is an almost connected simple Lie group with trivial centre and each D_j is a totally disconnected irreducible group with trivial amenable radical. □

Theorem B.3.3. [CM09b, Addendum 1.8] *Let X' and H be as above, then*

$$X' \cong \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each X_i is an irreducible symmetric space and each Y_j is an irreducible minimal CAT(0)*-space.* □

Lemma B.3.4. [CM09a, Lemma 3.4] *Let $A = \mathbb{R}^n \rtimes \mathrm{O}(n)$ and S be a semisimple Lie group without compact factors. Every lattice $\Gamma \leqslant A \times S$ has a finite index subgroup Λ which splits as a direct product $\Lambda_A \times \Lambda'$, where $\Gamma_A := \Gamma \cap (A \times \{1\})$ is a lattice in $(A \times \{1\})$ and Λ' is a lattice in S.* □

The following result from [CM19] is the corrected version of [CM09a, Proposition 3.6], we will make frequent use of this result throughout Paper 4. Recall that G *virtually normalises* a subgroup N if N is a normal subgroup of a finite index subgroup G.

Theorem B.3.5. [CM19, Theorem 2] *Let X be a proper* CAT(0) *space, $H \leqslant \mathrm{Isom}(X)$ a closed subgroup acting cocompactly and minimally, and Γ a lattice in H. Let $E = \mathbb{E}^n$ be the Euclidean de Rham factor of X, where $n \geqslant 0$.*

(a) There exists a free abelian subgroup $A \cong \mathbb{Z}^n$ of Γ, commensurated by Γ, and n is the largest such rank. Moreover, any commensurated abelian subgroup of Γ acts properly on E.

We now assume Γ is finitely generated.

(b) If Γ virtually normalises a free abelian subgroup of rank k, then Γ virtually splits as $\mathbb{Z}^k \times \Gamma'$. Moreover, there is a corresponding invariant decomposition $X \cong \mathbb{E}^k \times X'$ and the projection of $\mathbb{Z}^k$ (resp. Γ') to $\mathrm{Isom}(X')$ is trivial (resp. discrete).

(c) If the projection of Γ to $\mathrm{Isom}(\mathbb{E}^n)$ is virtually abelian, then Γ virtually splits as $\mathbb{Z}^n \times \Gamma'$.

(d) If Γ is residually finite then Γ virtually splits as $\mathbb{Z}^n \times \Gamma'$. □

B.3.3 Irreducibility

Let $X = X_1 \times \cdots \times X_n$ be a product of irreducible proper CAT(0) spaces and let Γ be a lattice in $H = H_1 \times \cdots \times H_n := \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_n)$, with each H_i non-discrete and acting minimally. There are several possible notions of irreducibility for a lattice in H, moreover, in the general setting of CAT(0) groups, they are not necessarily equivalent. In the interest of clarity, we recount each of these and summarise their implications, we follow the treatment in [CM12] [CLB19].

(Irr1) For every $\Sigma \subset \{1, \dots, n\}$, the projection $\pi_\Sigma : \Gamma \to H_\Sigma$ has dense image. Here we say Γ is *topologically irreducible* or an *irreducible lattice.*

(Irr2) The projection to each factor H_i is injective.

(Irr3) For every $\Sigma \subset \{1, \dots, n\}$, the projection $\pi_\Sigma : \Gamma \to H_\Sigma$ has non-discrete image. Here we say Γ is *weakly irreducible* or a *weakly irreducible lattice.*

(Irr4) Γ has no finite index subgroup which splits as a direct product of two infinite subgroups. Here we say Γ is *algebraically irreducible.*

Firstly, if each H_i is a centre-free semisimple algebraic group without compact factors then each of the definitions are equivalent [Mar91]. When each H_i is a non-discrete, compactly generated, tdlc group, then [CLB19, Theorem H] summarises all possible implications. Returning to the setting described above we have that **(Irr2)** $\Rightarrow$ **(Irr3)** $\Rightarrow$ **(Irr4)** and if Γ is finitely generated, then by Theorem B.3.6 we have **(Irr4)** $\Rightarrow$ **(Irr3)**. Note that in general **(Irr4)** $\Rightarrow$ **(Irr2)** fails, unless Γ is residually finite. The following theorem from [CM09a] shows the equivalence of **(Irr3)** and **(Irr4)** for many CAT(0) lattices.

Theorem B.3.6. [CM09a, Theorem 4.2] *Let X be a proper* CAT(0) *space, $H < \text{Isom}(X)$ a closed subgroup acting cocompactly on X, and $\Gamma < H$ a finitely generated lattice.*

(i) *If Γ is irreducible as an abstract group, then for for finite index subgroup $\Gamma_0 < \Gamma$ and any Γ_0-equivariant splitting $X = X_1 \times X_2$ with X_1 and X_2 non-compact, the projection of Γ_0 to both* $\text{Isom}(X_1)$ *and* $\text{Isom}(X_2)$ *is non-discrete.*

(ii) *If in addition the H-action is minimal, then the converse holds.* □

B.3.4 Examples

In this section we will detail a number of examples of CAT(0) lattices. We will pay particular attention to the case of $\text{Isom}(\mathbb{E}^n)$-lattice, tree lattices, lattices in products of trees, and Leary–Minasyan groups. The motivation for this is that in Paper 4 we will introduce a framework for studying lattices in products with a tree factor. In light of the Leary–Minasyan groups a lot of the work Paper 4 will focus on lattices in $\text{Isom}(\mathbb{E}^n) \times T$ where T is the automorphism group of a locally-finite tree.

B.3.4.1 Crystallographic groups

In this section we will investigate *crystallographic groups*, that is, lattices in $\mathrm{Isom}(\mathbb{E}^n)$. The main reference for this section is [Szc12].

Theorem B.3.7 (Bieberbach's First Theorem[1])**.** [Zas48], [Szc12, Theorem 2.1(1)] *Any* $\mathrm{Isom}(\mathbb{E}^n)$*-lattice* Γ *contains a finite index normal subgroup* A *isomorphic to* $\mathbb{Z}^n$ *and the quotient* $P = \Gamma/A$ *is finite.*

Bieberbach's Second Theorem [Szc12, Theorem 2.1(2)] states that there are only finitely many isomorphism classes of $\mathrm{Isom}(\mathbb{E}^n)$-lattices for each n. However, the number of these grows dramatically (Table B.1).

Dimension	1	2	3	4	5	6
Number	2	17	219	4783	222,018	28,927,922

Table B.1: The number of crystallographic groups in low dimensions (data copied from [CS01]).

We deduce the following well known corollary.

Corollary B.3.8. *Every* $\mathrm{Isom}(\mathbb{E}^n)$*-lattice is uniform,* $\mathrm{CAT}(0)$*, virtually free abelian, and for* $n \geqslant 2$ *reducible.*

Proof. That the lattice Γ is uniform and virtually abelian follows from the previous theorem. That the lattice is $\mathrm{CAT}(0)$ follows from the fact that Γ acts properly cocompactly on the quintessential $\mathrm{CAT}(0)$ space $\mathbb{E}^n$. Finally, if $n \geqslant 2$ then Γ is virtually $\mathbb{Z}^n$ and so virtually splits as a direct product of two infinite groups. □

B.3.4.2 Arithmetic groups

Arithmetic groups have already appeared a number of times in this chapter (e.g. $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ or $\mathrm{SL}_2(\mathbb{Z})$). They will play a small but significant role in Paper 4 and are the main examples of lattices acting on symmetric spaces of non-compact type. In this section we will give a very brief overview of the main construction of arithmetic groups. An introductory text can be found here [Mor15] and an in depth study can be found here [Mar91]. We will not survey any structural results about arithmetic groups, instead we remark there are some incredibly deep theorems due to Margulis and refer the reader to [Mar91].

Let k be a number field with ring of integers $\mathcal{O}$. Let $\mathbf{H}$ be a connected non-commutative absolutely simple adjoint k-group. Let V be a set of all inequivalent places of k, denote the subset of Archimedean places by V^∞ and the remaining finite places by V^{fin}. For

[1] Bieberbach was a disgraced German mathematician and Nazi.

each $v \in V$ we denote by k_v the completion of k with respect to v. Let $S \subset V$ be a finite subset of places such that for every $v \in S$, $\mathbf{H}$ is k_v-isotropic.

For $v \in V$ let $\mathbf{H}^+(k_v) < \mathbf{H}(k_v)$ denote the finite index normal subgroup defined in [BFS19, Section 6]. If $v \in V^\infty$ then $\mathbf{H}^+(k_v)$ is the identity component of the real Lie group $\mathbf{H}(k_v)$. Define

$$\mathbf{H}^+_{k,S} := \prod_{v\in S} \mathbf{H}^+(k_v) \quad \text{and} \quad \mathbf{H}_{k,S} := \prod_{v\in S} \mathbf{H}(k_v).$$

Note that the quotient $\mathbf{H}_{k,S}/\mathbf{H}^+_{k,S}$ is finite. The reduction theory of Borel and Harish-Chandra (see for instance [Bor19]) then realises the group $\mathbf{H}(\mathcal{O}[S])$ as a lattice in $\mathbf{H}_{k,S}$ via the diagonal embedding.

Note that $\mathbf{H}_{k,S} = \mathbf{H}_{k,S^\infty} \times \mathbf{H}_{k,S^{\mathrm{fin}}}$ is the splitting of $\mathbf{H}_{k,S}$ into a semisimple real Lie group and a totally disconnected locally compact group.

We call any group commensurable in $\mathbf{H}_{k,S}$ with $\mathbf{H}(\mathcal{O}[S])$ an *arithmetic lattice*. These groups were studied in the seminal work of Margulis [Mar91]. Along with crystallographic groups they are the quintessential examples of CAT(0) lattices. The CAT(0) structure is inherited from the natural action of $\mathbf{H}_{k,S}$ on a product of symmetric spaces of non-compact type and appropriately chosen geometric realisations of Euclidean Bruhat-Tits' buildings [CM09b] [CM09a] (see also [BH99, Chapter II.10]).

We will briefly remark a couple of famous deep results eliciting the structure of arithmetic groups. Firstly, we note that if $H := \mathbf{H}_{k,S}$ is not locally-isomorphic $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ then every lattice in H is arithmetic [Mar91]. Secondly, a lattice Γ in H is arithmetic if and only if $\mathrm{Comm}_H(\Gamma)$ is dense in H [Mar91]. Thirdly, if H is a real semisimple Lie group, then every (non-uniform) lattice in H is finitely presented [GR69].

Example B.6. Concretely, consider $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$, this embeds diagonally as an irreducible non-uniform arithmetic lattice into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$, this is an example of a *Hilbert modular group*. This group acts properly with finite covolume on $\mathbb{R}\mathbf{H}^2 \times \mathbb{R}\mathbf{H}^2$, a product of real hyperbolic planes. One could also consider $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ as an irreducible non-uniform arithmetic lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$. This group acts properly with finite covolume on $\mathbb{R}\mathbf{H}^2 \times \mathcal{T}_{p+1}$.

B.3.4.3 Tree lattices

In this section we introduce tree lattices and edge indexed graphs. The main reference for the content here is the book "Tree lattices" by Bass and Lubotzky [BL01].

Definition B.3.9. Let $\mathcal{G}$ be a graph of groups. For each oriented edge $e \in E\mathcal{G}$ set $i(e) = [G_{\tau(e)} : \phi_{e_0}(G_e)] \in \mathbb{Z} \cup \{\infty\}$. We call the pair $(\mathcal{G}, i)$ the *edge-indexed graph associated to* $\mathcal{G}$. For an edge e we set $\delta(e) = i(\overline{e})/i(e)$ and $\Delta(\gamma) = \delta(e_1)\ldots\delta(e_n)$ for a path $(e_1, \dots, e_n)$ in $\mathcal{G}$. We say that $(\mathcal{G}, i)$ has *bounded denominators* if for some (hence every) $v \in V\mathcal{G}$ the set

$$\left\{ \frac{\Delta(\gamma)}{i(e)} \mid e \in E\mathcal{G},\ \gamma \text{ a path from } v \text{ to } \iota(e) \right\}$$

has bounded denominators. Finally, we say $(\mathcal{G}, i)$ is *unimodular* whenever $\Delta(\gamma) = 1$ for all closed paths γ. Note that bounded denominators implies unimodularity.

FIGURE B.8: Two edge labelled graphs. The loop (left) is unimodular if and only if $m = n$. The two vertices joined by an edge (right) is unimodular for all choices of positive integers p and q

Let $\mathcal{T}$ be a locally finite tree and $T = \mathrm{Aut}(\mathcal{T})$. The group T is a totally disconnected locally compact group with compact open profinite vertex stabilisers. Indeed,

$$T_v = \varprojlim_r T_v|_{B_v(r)}$$

where $B_v(r)$ is the ball of radius r. A subgroup Γ of T is discrete if Γ_v is finite for every vertex in $\mathcal{T}$. Using the volume formula we define

$$\mathrm{Vol}(\mathcal{T}/\Gamma) := \sum_{v \in V\mathcal{T}/\Gamma} \frac{1}{|\Gamma_v|},$$

and we say Γ is a T-*lattice* if this is finite.

Theorem B.3.10. [BK90] *For a faithful finite graph of finite groups $\mathcal{G}$ with Bass-Serre tree $\mathcal{T}$, the group $\Gamma = \pi_1(\mathcal{G})$ is a tree lattice if and only if $(\mathcal{G}, i)$ is unimodular and* $\mathrm{Vol}(\mathcal{T}/\Gamma)$ *is finite.* □

Theorem B.3.11. [BL01, Appendix BCR] *Let T be the automorphism group of a locally-finite tree $\mathcal{T}$ and let H be a closed subgroup of T. There exists an H-lattice if and only if H is unimodular and $\mu(\mathcal{T}//H) < \infty$.*

Example B.7. Recall in Example B.1 we showed that $\mathrm{SL}_2(\mathbb{Z})$ acts on the $(2,3)$-biregular tree $\mathcal{T}_{2,3}$. Indeed, $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. However, the kernel of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{T}_{2,3}$ is the central cyclic group $\mathbb{Z}_2$. It follows that $\mathrm{SL}_2(\mathbb{Z})$ is not a $T_{2,3} := \mathrm{Aut}(\mathcal{T}_{2,3})$-lattice. Now, taking the quotient of $\mathrm{SL}_2(\mathbb{Z})$ by the centre gives the group $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ which acts faithfully on $\mathcal{T}_{2,3}$. In particular, $\mathrm{PSL}_2(\mathbb{Z})$ is a $T_{2,3}$-lattice.

The automorphism group of a tree can also admit non-uniform lattices, these lattices are necessarily infinitely generated and exhibit a wide range of behaviour. Some are arithmetic groups, some are non-residually finite, and some are even simple groups. For a detailed survey see [BL01].

Example B.8 (Nagao rays)**.** This construction is adapted from [BK90, Page 10]. There are non-uniform tree lattices with fundamental domain a ray. The stabilisers are defined as follows, for $n \geqslant 1$, let Γ_n be a strictly increasing sequence of finite groups $\Gamma_1 < \Gamma_2 < \Gamma_3 < \ldots$, let Γ_0^+ be a finite group and let $\Gamma_0 \leqslant \Gamma_0^+ \cap \Gamma_1$ such that $\Gamma_0 < \Gamma_0^+$. The graph of groups $\mathcal{A}$ is shown in Figure B.9 and $\pi_1(\mathcal{A}) = \Gamma_0^+ *_{\Gamma_0} \bigcup_{n\geqslant 0} \Gamma_n$. We call such a lattice a *lattice of Nagao type* and the ray a *Nagao ray*.

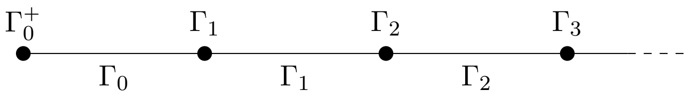

FIGURE B.9: A Nagao ray and its stabilisers.

In some cases this construction has an arithmetic interpretation. Let p be a prime, d a positive integer, and $q = p^d$. The Bruhat-Tits tree of $H = \mathrm{PSL}_2(\mathbb{F}_q(t))$ is the $(q+1)$-regular tree and $\Gamma = \mathrm{PSL}_2(\mathbb{F}_q[t])$ is a non-uniform lattice of Nago type. The local groups are given as follows, $\Gamma_0^+ = \mathrm{PSL}_2(q)$ and

$$\Gamma_n = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{F}_q^\times,\ \deg_t(b) \leqslant n \right\},$$

such that $\bigcup_{n\geqslant 0} \Gamma_n \cong \mathbb{F}_q[t] \rtimes (\mathbb{F}_q^\times/\{\pm 1\})$. This splitting was first proved by Nagao [Nag59] and was generalised by Serre to give a description of the structure of PSL_2 over the coordinate ring of a projective curve [Ser03].

Existence theorems for non-uniform tree lattices are discussed in [BL01] [Car04]. Farb–Hruska investigated the commensurability classes of non-uniform lattices in $T_{m,n}$, the full automorphism group of a biregular tree $\mathcal{T}$ for $m, n \geqslant 3$.

Theorem B.3.12. [FH06] *For each real number $r > 0$, there exist uncountably many commensurability classes of non-uniform lattices in $T_{m,n}$, each having covolume r.*

B.3.4.4 Lattices in products of trees

The study of lattices in products of trees was initiated by Burger and Mozes [BM97] [BM00a] [BM00b]. The authors studied the local actions of the projections of a lattice on each tree and constructed torsion-free simple groups. In Paper 4 we will study lattices acting on products of trees and other CAT(0) spaces. In Paper 5 we modify a Burger–Mozes simple group to obtain the first example of a non-virtually torsion-free hierarchically hyperbolic group. In this section we will provide some background on lattices in products of trees.

Proposition B.3.13. [BM00b, Proposition 1.2] *Let T_1, T_2 be the automorphism groups of locally finite trees $\mathcal{T}_1$ and $\mathcal{T}_2$. For a uniform lattice $\Gamma < T_1 \times T_2$ the following are equivalent:*

(i) *There exists $i \in \{1,2\}$ such that $\pi_{T_i}(\Gamma)$ is discrete;*

(ii) *$\Gamma_1 := \Gamma \cap T_1$, resp. $\Gamma_2 := \Gamma \cap T_2$, are lattices in T_1, resp. T_2, and $\Gamma_1 \times \Gamma_2$ is a finite index subgroup of Γ.* □

The following example is adapted from [BMZ09, Example 1.1.1].

Example B.9. Let $p \neq q$ be odd primes such that $p, q \equiv 1 \pmod 4$, let $H(\mathbb{Q})$ denote the Hamilton quaternion algebra over $\mathbb{Q}$ with basis $\{1, i, j, k\}$ and define

$$Q := \{x \in H(\mathbb{Z})\colon |x| = p^a q^b,\ a, b \in \mathbb{N},\ x \equiv 1 \pmod 2\}.$$

Let $\mathbb{Q}_p$ denote the p-adic integers. Fix $\epsilon_p \in \mathbb{Q}_p$, $\epsilon_q \in \mathbb{Q}_q$ with $\epsilon_p^2 = \epsilon_q^2 = -1$. Let $\Gamma_{p,q}$ denote the image of the homomorphism $\phi : Q \to \mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_q)$ by

$$x_0 + x_1 i + x_2 j + x_3 k \mapsto \left(\begin{bmatrix} x_0 + x_1\epsilon_p & x_2 + x_3\epsilon_p \\ -x_2 + x_3\epsilon_p & x_0 - x_1\epsilon_p \end{bmatrix}, \begin{bmatrix} x_0 + x_1\epsilon_q & x_2 + x_3\epsilon_q \\ -x_2 + x_3\epsilon_q & x_0 - x_1\epsilon_q \end{bmatrix} \right).$$

The group $\Gamma_{p,q} < T_{p+1} \times T_{q+1}$ is a uniform lattice. Indeed, $\Gamma_{p,q}$ is an irreducible uniform arithmetic lattice in $\mathrm{PGL}_2(\mathbb{Q}_p) \times \mathrm{PGL}_2(\mathbb{Q}_q)$. Moreover, $\mathrm{PGL}_2(\mathbb{Q}_p) \rightarrowtail T_{p+1}$ because $\mathcal{T}_{p+1}$ can be identified with the Bruhat-Tits' tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$ (similarly for $\mathrm{PGL}_2(\mathbb{Q}_q)$).

Example B.10. [Rad20, Theorem 5.2] Radu constructed a number of concrete examples of torsion-free simple groups acting on a product of trees. We reproduce one such example here. Define

$$\Gamma := \left\langle a_1, a_2, a_3, b_1, b_2, b_3 \;\middle|\; \begin{array}{c} a_1b_3a_1b_3,\ a_1b_3^{-1}a_1b_3^{-1},\ a_2b_3a_2b_3,\ a_2b_3^{-1}a_3b_3^{-1}, \\ a_3b_1a_3^{-1}b_1^{-1},\ a_3b_2a_3b_3,\ a_3b_2^{-1}a_3b_2^{-1} \end{array} \right\rangle$$

then Γ is uniform lattice in $T_6 \times T_6$. Moreover, Γ has an index 4 subgroup which is simple. Note that in Radu's notation the group Γ is $\Gamma_{6,6,1}$.

B.3.4.5 Leary–Minasyan groups

Many Leary–Minasyan groups are actually uniform CAT(0) lattices in the product of their Bass-Serre tree $\mathcal{T}$ and $\mathbb{E}^n$. A characterisation of this property was given in terms of the matrix A.

Theorem B.3.14. [LM19, Theorem 7.2] *The group $\mathrm{LM}(A, L)$ is a $\mathrm{CAT}(0)$ group if and only if the matrix A is conjugate in $\mathrm{GL}_n(\mathbb{R})$ to an orthogonal matrix.* □

A much more remarkable fact is that many of the CAT(0) Leary–Minasyan groups are actually weakly irreducible lattices in the product of $\mathrm{Isom}(\mathbb{E}^n)$ and $\mathrm{Aut}(\mathcal{T})$. There appears to be some confusion in the literature regarding this property; in particular, [CM09a] claims no such lattices exist. This has been rectified in [CM19].

Theorem B.3.15. [LM19, Theorem 7.5] *Suppose that A has infinite order and is conjugate in $\mathrm{GL}_n(\mathbb{R})$ to an orthogonal matrix. Then $\mathrm{LM}(A, L)$ is a lattice in $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Aut}(\mathcal{T})$ whose projections to the factors are not discrete. In particular, it is weakly irreducible.* □

We will detail the action on $\mathbb{E}^2$ in the case of the Leary–Minasyan group defined in Example B.3 with presentation

$$\Gamma = \langle a, b, t \mid [a, b],\ ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2\rangle.$$

The group Γ has a representation π to $\mathrm{Isom}(\mathbb{E}^n)$ given by $\pi(a) = [1, 0]^T$, $\pi(b) = [0, 1]^T$, and $\pi(t) = A$. The matrix A is a rotation by the irrational amount $\cos^{-1}(3/5)$ and so has infinite order. In particular, Γ is weakly irreducible. Note that the action of Γ on $\mathbb{E}^2$ is pictured in Figure B.10.

Leary and Minasyan proved another remarkable theorem about their groups, namely, that the group $\mathrm{LM}(A)$ is biautomatic if and only if A has finite order. In particular, they constructed the first examples of CAT(0) but not biautomatic groups.

In Paper 4 we will study all weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices (for arbitrary locally-finite trees) simultaneously. We will characterise them in terms of the topology (non-discrete projections), algebraically (abstractly irreducible), geometrically (the action of $\mathcal{T}$ is faithful), analytically (the lattice is C^*-simple), homologically (the lattice does not virtually fibre), and we extend the biautomaticity results of Leary–Minasyan to the whole class.

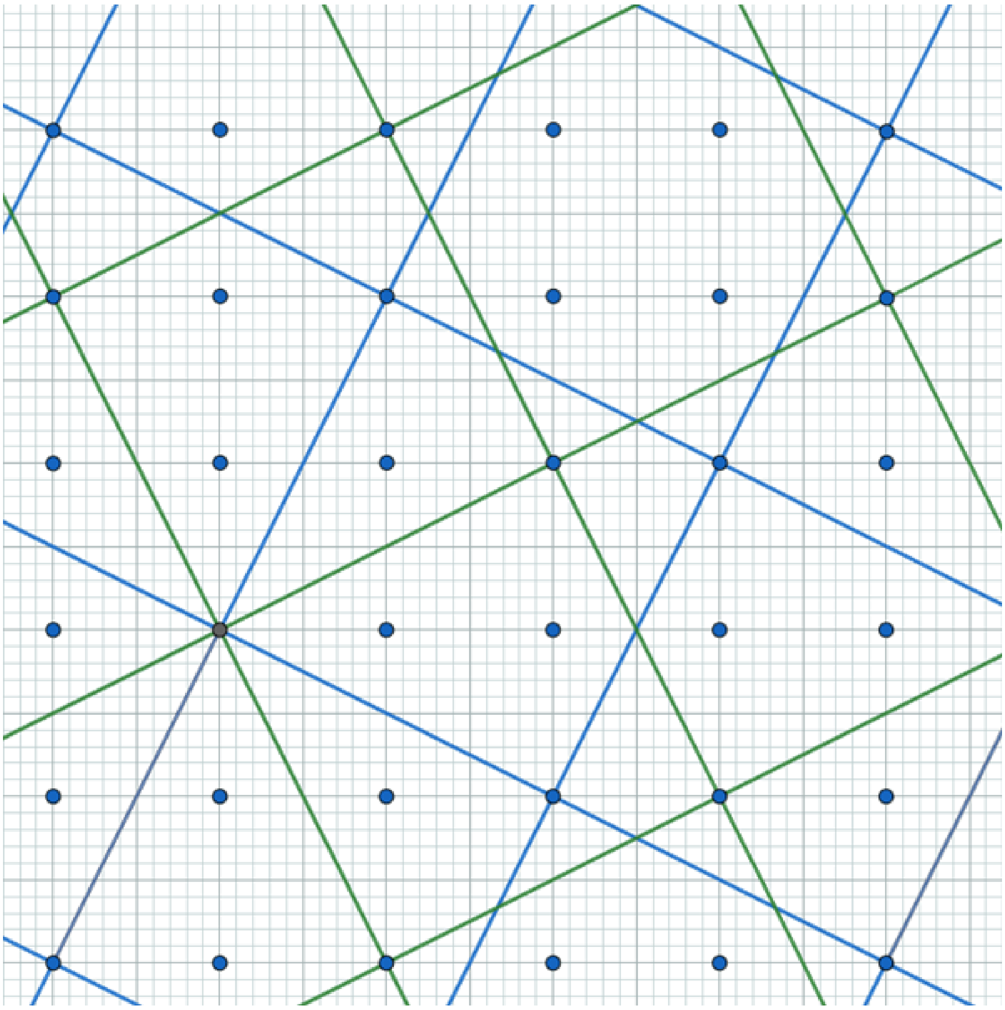

FIGURE B.10: The action of $\mathrm{LM}(A)$ on the Euclidean plane. The blue dots represent elements of $\mathbb{Z}^2$. The green and blue squares represent the finite index subgroups L and L' of $\mathbb{Z}^2$. The action of the stable letter rotates the blue squares to the green squares.

B.4 The Flat Closing Conjecture

One of the most well known and long-standing conjectures regarding $\mathrm{CAT}(0)$ groups is the *flat closing conjecture* [Gro93, Section 6.B_3].

The Flat Closing Conjecture. *Let X be a proper $\mathrm{CAT}(0)$ space and Γ a discrete group acting properly and cocompactly by isometries on X. If X contains a d-dimensional flat, then Γ contains a copy of $\mathbb{Z}^d$.*

The main result of [CZ13], which proves a special case of the flat closing conjecture, is affected by the error in [CM09a]. The next theorem provides a corrected statement of the main result of [CZ13].

Theorem B.4.1. [CZ13, Corollary 1] *Let X be a proper geodesically complete $\mathrm{CAT}(0)$ space and let Γ be a discrete group acting properly cocompactly by isometries on X. Suppose in addition that X is a product of d irreducible factors. If the projection of Γ*

to the isometry group of the Euclidean de Rham factor is discrete, then Γ *contains a subgroup isomorphic to* $\mathbb{Z}^d$. $\square$

Bibliography

[Bas93] Hyman Bass. Covering theory for graphs of groups. *J. Pure Appl. Algebra*, 89(1-2):3–47, 1993. doi:10.1016/0022-4049(93)90085-8.

[BFS19] Uri Bader, Alex Furman, and Roman Sauer. An adelic arithmeticity theorem for lattices in products. *Math. Z.*, 293(3-4):1181–1199, 2019. doi:10.1007/s00209-019-02241-9.

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. doi:10.1007/978-3-662-12494-9.

[BK90] Hyman Bass and Ravi Kulkarni. Uniform tree lattices. *J. Amer. Math. Soc.*, 3(4):843–902, 1990. doi:10.2307/1990905.

[BL01] Hyman Bass and Alexander Lubotzky. *Tree lattices*, volume 176 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2001. doi:10.1007/978-1-4612-2098-5. With appendices by Bass, L. Carbone, Lubotzky, G. Rosenberg and J. Tits.

[BM97] Marc Burger and Shahar Mozes. Finitely presented simple groups and products of trees. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(7):747–752, 1997. doi:10.1016/S0764-4442(97)86938-8.

[BM00a] Marc Burger and Shahar Mozes. Groups acting on trees: from local to global structure. *Inst. Hautes Études Sci. Publ. Math.*, 92:113–150 (2001), 2000.

[BM00b] Marc Burger and Shahar Mozes. Lattices in product of trees. *Inst. Hautes Études Sci. Publ. Math.*, 92:151–194 (2001), 2000.

[BMZ09] Marc Burger, Shahar Mozes, and Robert J. Zimmer. Linear representations and arithmeticity of lattices in products of trees. In *Essays in geometric group theory*, volume 9 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 1–25. Ramanujan Math. Soc., Mysore, 2009.

[Bor19] Armand Borel. *Introduction to arithmetic groups*, volume 73 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2019. doi:10.1090/ulect/073. Translated from the 1969 French original [MR0244260] by Lam Laurent Pham, Edited and with a preface by Dave Witte Morris.

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[BS62] Gilbert Baumslag and Donald Solitar. Some two-generator one-relator non-Hopfian groups. *Bull. Amer. Math. Soc.*, 68:199–201, 1962. doi:10.1090/S0002-9904-1962-10745-9.

[Car04] Lisa Carbone. Non-minimal tree actions and the existence of non-uniform tree lattices. *Bull. Austral. Math. Soc.*, 70(2):257–266, 2004. doi:10.1017/S000497270003447X.

[CHK20] Indira Chatterji, Sam Hughes, and Peter Kropholler. The first ℓ^2-betti number and groups acting on trees, 2020, arXiv:2012.13368 [math.GR].

[CLB19] Pierre-Emmanuel Caprace and Adrien Le Boudec. Bounding the covolume of lattices in products. *Compos. Math.*, 155(12):2296–2333, 2019. doi:10.1112/s0010437x19007644.

[CM09a] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: discrete subgroups. *J. Topol.*, 2(4):701–746, 2009. doi:10.1112/jtopol/jtp027.

[CM09b] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: structure theory. *J. Topol.*, 2(4):661–700, 2009. doi:10.1112/jtopol/jtp026.

[CM12] Pierre-Emmanuel Caprace and Nicolas Monod. A lattice in more than two Kac-Moody groups is arithmetic. *Israel J. Math.*, 190:413–444, 2012. doi:10.1007/s11856-012-0006-3.

[CM19] Pierre-Emmanuel Caprace and Nicolas Monod. Erratum and addenda to "isometry groups of non-positively curved spaces: discrete subgroups", 2019, arXiv:1908.10216 [math.GR].

[CS01] Carlos Cid and Tilman Schulz. Computation of five- and six-dimensional Bieberbach groups. *Experiment. Math.*, 10(1):109–115, 2001. doi:10.1080/10586458.2001.10504433.

[CZ13] Pierre-Emmanuel Caprace and Gašper Zadnik. Regular elements in CAT(0) groups. *Groups Geom. Dyn.*, 7(3):535–541, 2013. doi:10.4171/GGD/195.

[Dav98] Michael W. Davis. Buildings are CAT(0). In *Geometry and cohomology in group theory (Durham, 1994)*, volume 252 of *London Math. Soc. Lecture Note Ser.*, pages 108–123. Cambridge Univ. Press, Cambridge, 1998. doi:10.1017/CBO9780511666131.009.

[DP03] James F. Davis and Kimberly Pearson. The Gromov-Lawson-Rosenberg conjecture for cocompact Fuchsian groups. *Proc. Amer. Math. Soc.*, 131(11):3571–3578, 2003. doi:10.1090/S0002-9939-03-06905-3.

[FH06] Benson Farb and G. Christopher Hruska. Commensurability invariants for nonuniform tree lattices. *Israel J. Math.*, 152:125–142, 2006. doi:10.1007/BF02771979.

[GH21] Nick Gill and Sam Hughes. The character table of a sharply 5-transitive subgroup of the alternating group of degree 12. *International Journal of Group Theory*, 10(1):11–30, 2021. doi:10.22108/ijgt.2019.115366.1531.

[GR69] H. Garland and M. S. Raghunathan. Fundamental domains for lattices in rank one semisimple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 62:309–313, 1969. doi:10.1073/pnas.62.2.309.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[HMPSSn21] Sam Hughes, Eduardo Martínez-Pedroza, and Luis Jorge Sánchez Saldaña. Quasi-isometry invariance of relative filling functions, 2021, arXiv:2107.03355 [math.GR].

[Hug19] Sam Hughes. Cohomology of Fuchsian groups and non-Euclidean crystallographic groups, 2019, arXiv:1910.00519 [math.GR].

[Hug20] Sam Hughes. On the equivariant K- and KO-homology of some special linear groups, 2020, arXiv:2004.08199 [math.KT].

[Hug21a] Sam Hughes. Graphs and complexes of lattices, 2021, arXiv:2104.13728 [math.GR].

[Hug21b] Sam Hughes. Hierarchically hyperbolic groups, products of CAT(-1) spaces, and virtual torsion-freeness, 2021, arXiv:2105.02847 [math.GR].

[Kam19] Holger Kammeyer. *Introduction to ℓ^2-invariants*, volume 2247 of *Lecture Notes in Mathematics*. Springer, Cham, 2019. doi:10.1007/978-3-030-28297-4.

[Kat92] Svetlana Katok. *Fuchsian groups*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992.

[LM19] Ian J. Leary and Ashot Minasyan. Commensurating HNN-extensions: non-positive curvature and biautomaticity, 2019, arXiv:1907.03515 [math.GR].

[Lüc02] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002. doi:10.1007/978-3-662-04687-6.

[Mar91] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. doi:10.1007/978-3-642-51445-6.

[Mor15] Dave Witte Morris. *Introduction to arithmetic groups*. Deductive Press, 2015.

[MV03] Guido Mislin and Alain Valette. *Proper group actions and the Baum-Connes conjecture*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2003. doi:10.1007/978-3-0348-8089-3.

[Nag59] Hirosi Nagao. On GL(2, $K[x]$). *J. Inst. Polytech. Osaka City Univ. Ser. A*, 10:117–121, 1959.

[Rad20] Nicolas Radu. New simple lattices in products of trees and their projections. *Canad. J. Math.*, 72(6):1624–1690, 2020. doi:10.4153/s0008414x19000506. With an appendix by Pierre-Emmanuel Caprace.

[Ser71] Jean-Pierre Serre. Cohomologie des groupes discrets. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 77–169. Ann. of Math. Studies, No. 70, 1971.

[Ser03] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[Szc12] Andrzej Szczepański. *Geometry of crystallographic groups*, volume 4 of *Algebra and Discrete Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. doi:10.1142/8519.

[Val20] Motiejus Valiunas. Isomorphism classification of Leary-Minasyan groups, 2020, arXiv:2011.08143 [math.GR].

[Val21] Motiejus Valiunas. Leary-Minasyan subgroups of biautomatic groups, 2021, arXiv:2104.13688 [math.GR].

[Zas48] Hans Zassenhaus. Über einen Algorithmus zur Bestimmung der Raumgruppen. *Comment. Math. Helv.*, 21:117–141, 1948. doi:10.1007/BF02568029.

Paper 1

COHOMOLOGY OF FUCHSIAN GROUPS AND NON-EUCLIDEAN CRYSTALLOGRAPHIC GROUPS

SAM HUGHES

Abstract. For each geometrically finite 2-dimensional non-Euclidean crystallographic group (NEC group), we compute the cohomology groups. In the case where the group is a Fuchsian group, we also determine the ring structure of the cohomology.

1.1 Introduction

Let Γ be a geometrically finite non-Euclidean crystallographic group (NEC group), i.e. a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R})$ with a finite sided fundamental domain for the action of Γ on the hyperbolic plane $\mathbb{R}\mathbf{H}^2$. Throughout we let $\Lambda(\Gamma)$ denote the *limit set* of Γ. In this paper, we will calculate the cohomology of Γ. In the case where Γ is a Fuchsian group, i.e. Γ is contained in $\mathrm{PSL}_2(\mathbb{R})$, we will also calculate the cohomology ring. Our proof will involve finding a suitable fundamental domain for the action of the group on $\mathbb{R}\mathbf{H}^2 \cup \Lambda(\Gamma)$ and then applying a Cartan-Leray type spectral sequence.

Since $\mathbb{R}\mathbf{H}^2 \cup \Lambda(\Gamma)$ is contractible, the sequence converges to the cohomology of Γ. Using knowledge of the abelianization of Γ, it is easy to compute with the spectral sequence. We will now set the convention that an omission of coefficients in the (co)homology functors should be read as having coefficients in the trivial module $\mathbb{Z}$.

Definition 1.1.1. Let $m_1, \ldots, m_r$ be a set of positive integers each greater than 2. For $j = 1, \ldots, r-1$, let $\hat{t}_j$ be the greatest common divisor of the set of products of $m_1, \ldots, m_r$ taken j at a time. Then, let $t_1 = \hat{t_1}$ and for $j = 2, \ldots, r-1$ let $t_j = \hat{t}_j / \hat{t}_{j-1}$. We define w_j for $j = 1, \ldots, r-1$ by the same process except for we perform the procedure to $2m_1, m_1, \ldots, m_r$ and discard products containing $2m_1 m_1$. Finally, we define w_r to be equal to $2m_1 m_2 \ldots m_r / w_{r-1}$.

Theorem 1.A. *Let Γ be an NEC group of signature*

$$(g, s, \epsilon, [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k}), (), \ldots, ()\}),$$

where the number of empty cycles equals d. Let C_E denote the number of even $n_{i,l}$ and let C_O denote the number of period cycles for which every $n_{i,l}$ is odd.

(a) If $\epsilon = +$ and $d = k = s = 0$ (i.e. Γ is a cocompact Fuchsian group) then

$$H_q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{2g} \oplus \left(\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}\right) & q = 1, \\ \mathbb{Z} & q = 2, \\ \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & q = 2l+1, \text{ where } l \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) If $\epsilon = -$ and $d = k = s = 0$ then

$$H_q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{g-1} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{w_j}\right) & q = 1, \\ \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & q = 2l+1, \text{ where } l \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) If $\epsilon = +$ and $d + k + s > 0$ then

$$H_q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{2g+s+k+d-1} \oplus \mathbb{Z}_2^{C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q = 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} & q = 2p > 0, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) & \\ \quad \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q \equiv 3 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}(q+1)C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q > 1 \text{ and} \\ & q \equiv 1 \pmod 4. \end{cases}$$

(d) If $\epsilon = -$ and $d + k + s > 0$ then

$$H_q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{g+s+k+d-1} \oplus \mathbb{Z}_2^{C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q = 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} & q = 2p > 0, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} \oplus \left(\bigoplus_{i=1}^{k} \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) & \\ \quad \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q \equiv 3 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}(q+1)C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right) & q > 1 \text{ and} \\ & q \equiv 1 \pmod 4. \end{cases}$$

In the case where Γ is a Fuchsian group we also compute the ring structure (Theorem 1.B).

Definition 1.1.2. We will write $\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} = (\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}) \oplus (\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k})$, where the $\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k}$ term is decomposed via the invariant factor decomposition of finite abelian groups. We write $\widetilde{H}^*(X)$ for the reduced cohomology of X, that is the kernel of the map induced by the inclusion of the basepoint. Recall that $H^*(\mathbb{Z}_q) = \mathbb{Z}[x]/(qx)$ where x has degree 2. Define R_q to be the subring of $\widetilde{H}^*(\mathbb{Z}_q)$ generated by x^2 and x^3.

Theorem 1.B. *Let Γ be a Fuchsian group of signature $[g, s; m_1, \ldots, m_r]$.*

(a) If $s = 0$ then $\widetilde{H}^*(\Gamma) \cong \widetilde{H}^*(\Sigma_g) \oplus \left(\bigoplus_{j=1}^{r-1} \widetilde{H}^*(\mathbb{Z}_{t_j})\right) \oplus (\bigoplus_{k=1}^{l} R_{q_k})$.

(b) If $s > 0$ then $\widetilde{H}^*(\Gamma) \cong \widetilde{H}^*(F_{2g+s-1}) \oplus \left(\bigoplus_{j=1}^{r} \widetilde{H}^*(\mathbb{Z}_{m_j})\right)$ *where F_{2g+s-1} is a free group of rank $2g + s - 1$.*

We remark that some of the results have appeared in the literature before. The case where Γ is a cocompact Fuchsian group, so $\epsilon = +$ and $d = k = s = 0$, was considered by Majumdar [15], however, our computation of the ring structure is new. The case $\epsilon = +$ and $d = k = 0$ is a corollary of a result of Huebschmann [11] and the case $\epsilon = -$ and $d = k = s = 0$ was considered by Akhter and Majumdar [1]. Each of these previous results used different methods to the ideas here.

Other interpretations of the cohomology of Fuchsian groups have appeared in the literature. These have primarily dealt with lifting phenomena [16], with Eichler cohomology [5] [6] or with K-theory in relation to the Baum-Connes conjecture [2] [12] [13].

The paper is structured as follows. In Section 1.2 we define the signature of an NEC group. In Section 1.3 we introduce the Cartan Leray type spectral sequence for a Γ-space. Finally, in Section 1.4 we prove Theorem 1.A and Theorem 1.B.

Acknowledgements

I would like to thank my PhD supervisor Professor Ian Leary for his guidance and support. I would also like to thank the anonymous reviewer as their feedback greatly improved the exposition of this paper. This work was supported by the Engineering and Physical Sciences Research Council grant number 2127970.

1.2 Non-Euclidean crystallographic groups

We will first describe Wilkie and Macbeath's NEC signatures [18] [14], then the associated fundamental domain in $\mathbb{R}\mathbf{H}^2 \cup \Lambda(\Gamma)$, and finally we will give a presentation for an NEC group in terms of its signature. For further information on NEC groups the reader should consult [4].

An *NEC signature* consists of a sign $\epsilon = \pm$, and several sequences of integers grouped in the following manner:

(i) Two integers $g, s \geqslant 0$.

(ii) An ordered set of integer *periods* $[m_1, \ldots, m_r]$.

(iii) An ordered set of k *period cycles* $\{C_i := (n_{i,1}, \ldots, n_{i,s_i}) : 1 \leqslant i \leqslant k\}$.

(iv) A further d empty period cycles $(), \ldots, ()$.

The sequences and sign are then combined into the NEC signature, which is written as

$$(g, s, \epsilon, [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k}), (), \ldots, ()\}).$$

We let C_E denote the number of even $n_{i,l}$ and we let C_O denote the number of C_i for which every $n_{i,l}$ is odd.

Associated to each NEC signature is a *surface symbol* describing a fundamental domain for the associated NEC group. The surface symbol is a list of edges travelling around the polygon clockwise. Two edges paired orientably will be indicated by the same letter and a prime. Two edges paired non-orientably will be indicated by the same letter and an asterisk. When $\epsilon = +$, we have the surface symbol

$$\xi_1\xi_1' \ldots \xi_r\xi_r'\epsilon_1\gamma_{1,0} \ldots, \gamma_{1,s_1}\epsilon_1'\epsilon_2 \ldots \epsilon_k\gamma_{k,0} \ldots, \gamma_{k,s_k}\epsilon_k'\alpha_1\beta_1'\alpha_1'\beta_1' \ldots \alpha_g\beta_g'\alpha_g'\beta_g'.$$

When $\epsilon = -$, we have the surface symbol

$$\xi_1\xi_1' \ldots \xi_{r+s}\xi_{r+s}'\epsilon_1\gamma_{1,0} \ldots, \gamma_{1,s_1}\epsilon_1'\epsilon_2 \ldots \epsilon_k\gamma_{k,0} \ldots, \gamma_{k,s_k}\epsilon_k'\alpha_1\alpha_1^* \ldots \alpha_g\alpha_g^*.$$

For $j = 1, \ldots, r$, the period m_j is attached to the vertex v_j common to the edges ξ_j and ξ_j'. For $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant s_i$ the cycle period $n_{i,l}$ is associated with the vertex $w_{i,l}$ common to the edges $\gamma_{i,l-1}$ and $\gamma_{i,l}$. The vertices v_j for $j = r+1, \ldots, r+s$ lie on the boundary $\partial\mathbb{R}\mathbf{H}^2$. For $i = 1, \ldots, d+k$ we label the vertex common to the edges ϵ_i and $\gamma_{i,0}$ or to the edges γ_{i,s_i} and ϵ_i' by $w_{i,0}$. Finally, we label all other vertices v_0. Several examples of fundamental domains are given in Figure 1.1.

For an NEC group Γ we may take the quotient $\mathcal{O} = \mathbb{R}\mathbf{H}^2/\Gamma$. The quotient comes with a natural orbifold structure and many of the geometric-toplogical features of the quotient are reflected in the signature. Indeed, if $\epsilon = +$ then $\mathcal{O}$ is a genus g surface with the disjoint union of s points and $d+k$ open disks removed. We refer to the removed points as the *cusps* of $\mathcal{O}$ and to the boundary of the open disks as the *boundary components* of $\mathcal{O}$. There are r *cone* or orbifold points in the interior $\mathcal{O}$. For the ith boundary component, for $1 \leqslant i \leqslant k$, there are s_i cone or orbifold points on the boundary. The remaining d boundary components do not have any cone points. If $\epsilon = -$ the situation is identical except we begin with a sphere with g cross-caps attached.

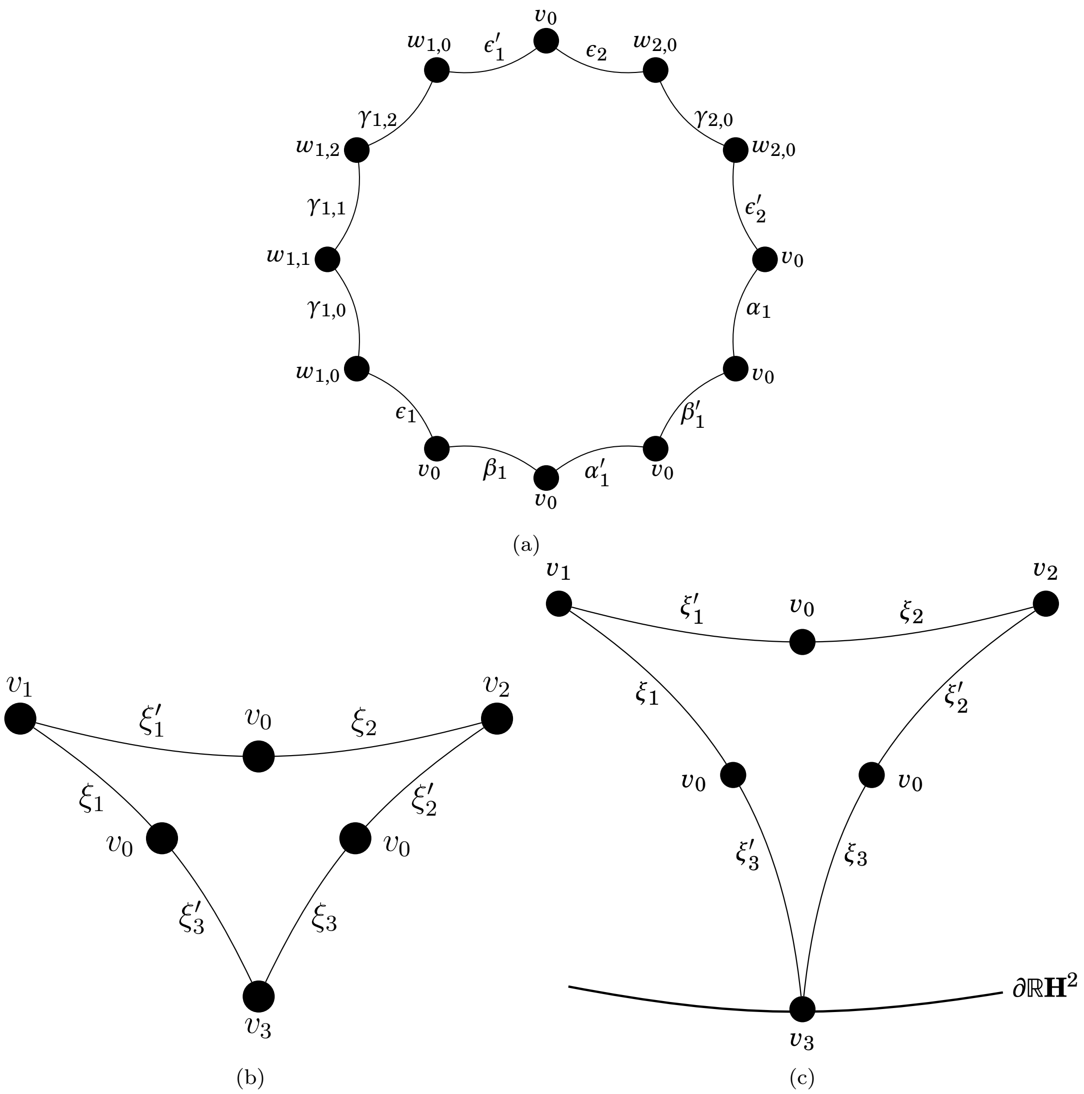

FIGURE 1.1: In (a) we have a fundamental domain for an NEC group of signature $(1, 0, +, [], \{(m, n), ()\})$. The topological quotient of $\mathbb{R}\mathbf{H}^2$ is homeomorphic to a torus with two open discs removed. In the orbifold structure of the quotient we have two cone points on one of the two boundary components. In (b) we have a fundamental domain for a Fuchsian triangle group of signature $(0, 0, +, [p, q, r], \{\}) = [0, 0; p, q, r]$ for $p^{-1} + q^{-1} + r^{-1} < 1$. The topological quotient is homeomorphic to a sphere. In the orbifold structure we have three cone points. In (c) we have a fundamental domain an Fuchsian NEC group of signature $(0, 1, +, [m, n], \{\}) = [0, 1; m, n]$ for $m + n > 4$. The topological quotient is homeomorphic to a punctured sphere. In the orbifold structure we have two cone points in the interior of the punctured sphere.

Under the action of the associated NEC group, for $1 \leqslant j \leqslant r$ the stabiliser of the vertex v_j is a cyclic group of order m_j acting on $\mathbb{R}\mathbf{H}^2$ via rotations. This corresponds exactly to a maximal elliptic subgroup of Γ fixing the point v_j in $\mathbb{R}\mathbf{H}^2$. If v_j lies on $\partial\mathbb{R}\mathbf{H}^2$, that is when $r + 1 \leqslant j \leqslant r + s$, then the stabiliser is isomorphic to $\mathbb{Z}$. This corresponds to a maximal parabolic subgroup of Γ stabilising a cusp.

The stabiliser of the edge $\gamma_{i,l}$ for $1 \leqslant i \leqslant k$ and $0 \leqslant l \leqslant s_k$ or for $k + 1 \leqslant i \leqslant k + d$ and $l = 0$ is a reflection group $\mathbb{Z}_2$. The reflection corresponds to a non-trivial reflection in

Γ reflecting $\mathbb{R}\mathbf{H}^2$ through the geodesic line containing $\gamma_{i,l}$. In the quotient these edges correspond to the edges in the boundary components. The stabiliser of the vertex $w_{i,l}$ for $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant s_k$ is a dihedral group $D_{2n_{i,l}}$ of order $2n_{i,l}$, note the $w_{i,l}$ lies in the ith boundary component. The stabiliser of the vertex $w_{i,l}$ for $1 \leqslant i \leqslant k+d$ and $l = 0$ is a reflection group $\mathbb{Z}_2$. No other points of the polygon are fixed points of the NEC group.

Recall that the rational Euler characteristic of a group Γ of type VF is defined to be $\chi_{\mathbb{Q}}(\Gamma) = \chi(\Gamma')/|\Gamma : \Gamma'|$ where Γ' is a finite index subgroup of type F. Let Γ be an NEC group of signature

$$(g, s, \epsilon, [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k}), (), \ldots, ()\}),$$

if $\epsilon = +$ then

$$\chi_{\mathbb{Q}}(\Gamma) = 2 - 2g - s - r - d - k - \frac{1}{2}\sum_{i=1}^{k} s_i + \sum_{j=1}^{r} \frac{1}{m_i} + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{1}{n_{i,j}}$$

and if $\epsilon = -$ then

$$\chi_{\mathbb{Q}}(\Gamma) = 2 - g - s - r - d - k - \frac{1}{2}\sum_{i=1}^{k} s_i + \sum_{j=1}^{r} \frac{1}{m_i} - \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i} \frac{1}{n_{i,j}}.$$

If $\chi_{\mathbb{Q}}(\Gamma) < 0$ then there exists an NEC group with the corresponding signature, except when $\epsilon = -$ and $s > 0$ where there is no known classification. By the Gauss-Bonnet Theorem we see that the hyperbolic area of a fundamental domain for the NEC group is equal to $-2\pi\chi_{\mathbb{Q}}(\Gamma)$ [17] (see also [4, Theorem 1.1.8]).

For the above equations, there are 17 solutions to $\chi_{\mathbb{Q}}(\Gamma) = 0$, these exactly correspond to the 17 Euclidean wallpaper groups [14, Section 8]. We can now give a presentation for an NEC group. Due to the large number of generators and relations, we detail this in Table 1.1.

Signature element	Generator(s)	Relation(s)
Period m_j for $1 \leqslant j \leqslant r$	x_j	$x_j^{m_j} = 1$
Cycle $(n_{i,1}, \ldots, n_{i,s_i})$ for $1 \leqslant i \leqslant k$ and $0 \leqslant l \leqslant s_i$	e_i $c_{i,0} \ldots c_{i,s_i}$	$c_{i,s_i} = e_i^{-1} c_{i,0} e_i$ $c_{i,l-l}^2 = c_{i,l}^2 = (c_{i,l-1}c_{i,l})^2 = 1$
Cycle $()$ for $k+1 \leqslant i \leqslant k+d$	$e_i,\ c_{i,0}$	$c_{i,0}^2 = 1,\ c_{i,0} = e_i^{-1} c_{i,0} e_i$
s	$x_r, \ldots, x_{r+s}$	See g $\pm$
$g\ +$	$a_1, b_1, \ldots, a_g, b_g$	$\prod_{j=1}^{r+s} x_j \prod_{i=1}^{k+d} e_i \prod_{t=1}^{g} [a_t, b_t] = 1$
$g\ -$	$a_1, \ldots, a_g$	$\prod_{j=1}^{r+s} x_j \prod_{i=1}^{k+d} e_i \prod_{t=1}^{g} a_t^2 = 1$

TABLE 1.1: Generators and relations for an NEC group.

If $d = k = 0$ and $\epsilon = +$, then we write the signature of Γ as $[g, s; m_1, \ldots, m_r]$ and we refer to Γ a *Fuchsian group* (i.e. a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. If $s = 0$, we say that Γ is *cocompact*.

1.3 A Cartan-Leray type spectral sequence

For a more thorough treatment on Γ-equivariant cohomology and related spectral sequences the reader should consult for example [3, Chapter VII]. We will just summarise the theory we need.

Let Γ be a discrete group, X a Γ-complex (in the sense of Brown [3, Chapter I.4]) and M a Γ-module. We define the *Γ-equivariant homology of X with coefficients in M* to be

$$H_*^\Gamma(X; M) := H_*(\Gamma; C_*(X) \otimes M)$$

with diagonal Γ-action on $C_*(X) \otimes M$.

Let $\Omega(p)$ be a set of representatives of Γ-orbits of p-cells in X and let Γ_σ denote the stabiliser of the cell σ. We have a Γ_σ-module $\mathbb{Z}_\sigma$ on which Γ_σ acts on via $\chi_\sigma : \Gamma_\sigma \to \{\pm 1\}$. Note that the action is trivial if Γ_σ fixes σ pointwise. Define $M_\sigma := \mathbb{Z}_\sigma \otimes M$, it follows M_σ is a Γ_σ-module additively isomorphic to M but with the Γ_σ-action twisted by χ_σ. One of the main computational tools is the following spectral sequence.

Theorem 1.3.1. [3, Chapter VII (7.10)] *Let X be a Γ-complex, then there is a spectral sequence*

$$E^1_{pq} := \bigoplus_{\sigma \in \Omega(p)} H_q(\Gamma_\sigma; \mathbb{Z}_\sigma) \Rightarrow H^\Gamma_{p+q}(X; \mathbb{Z}).$$

A description of $d^1_{p,*} : E^1_{p,*} \to E^1_{p-1,*}$ is given in [3, Chapter VII.8], we will summarise it here. Let σ be a p-cell of X and τ a $(p-1)$-cell. Write $\partial_{\sigma\tau} : M_\sigma \to M_\tau$ for the (σ, τ)-boundary component of $C_p(X) \otimes M \to C_{p-1}(X) \otimes M$. Let $\Omega_\sigma = \{\tau : \partial_{\sigma\tau} \neq 0\}$ and note that this is a Γ_σ-invariant set of $(p-1)$-cells. Let $\Gamma_{\sigma\tau} = \Gamma_\sigma \cap \Gamma_\tau$ and let

$$t_{\sigma\tau} : H(\Gamma_\sigma; M_\sigma) \to H(\Gamma_{\sigma\tau}; M_\sigma)$$

denote the transfer map arising from the fact $|\Gamma_\sigma : \Gamma_{\sigma\tau}|$ is finite. Now, $\partial_{\sigma\tau}$ is a $\Gamma_{\sigma\tau}$-map and ∂ is a Γ-map, thus we have a map

$$u_{\sigma\tau} : H_*(\Gamma_{\sigma\tau}; M_\sigma) \to H_*(\Gamma_\tau; M_\tau)$$

induced by $\Gamma_{\sigma\tau} \hookrightarrow \Gamma_\tau$ and ∂. Let τ_0 be a Γ-orbit representative in X and choose $g \in \Gamma$ such that $g(\tau) = \tau_0$. The action of g on $C_{p-1}(X) \otimes M$ induces an isomorphism $M_\tau \to M_{\tau_0}$ which is compatible with the conjugation isomorphism $\Gamma_\tau \to \Gamma_{\tau_0}$ induced by g. It follows there is an isomorphism

$$v_\tau : H_*(\Gamma_\tau; M_\tau) \to H_*(\Gamma_{\tau_0}; M_{\tau_0}).$$

Finally, by [3, Chapter VII (8.1)] up to sign we have

$$d^1_{p,*}|_{H_*(\Gamma_\sigma;M_\sigma)} = \sum_{\tau\in\Omega(p-1)} v_\tau u_{\sigma\tau} t_{\sigma\tau}.$$

1.4 Cohomology

1.4.1 The cocompact Fuchsian case

We will calculate the cohomology of cocompact Fuchsian groups. We note that the proof here is new, except for we calculate the abelianization using Smith normal form in the same way as Majumdar [15].

Proof of Theorem 1.A(a). We will use Theorem 1.3.1. In this case $X = \mathbb{R}\mathbf{H}^2$ endowed with the induced cell structure from the Wilkie-Macbeath polygon. To set up the spectral sequence we observe for each m_j there is a Γ-orbit of 0-cells, where each cell has stabiliser $\mathbb{Z}_{m_j}$. Now, by Theorem 1.3.1 the E^1-page of the spectral sequence has the form given by Figure 1.2.

$$\begin{array}{c|cccc}
3 & \bigoplus_{j=1}^r \mathbb{Z}_{m_j} & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
1 & \bigoplus_{j=1}^r \mathbb{Z}_{m_j} & 0 & 0 & 0 \\
0 & \mathbb{Z}^{r+1} \longleftarrow & \mathbb{Z}^{2g+r} \leftarrow & \mathbb{Z} & 0 \\
\hline
 & 0 & 1 & 2 & 3
\end{array}$$

FIGURE 1.2: The E^1-page of the spectral sequence for a Fuchsian group.

The only non-trivial differentials are along the bottom row. Slightly abusing notation we fix a basis for the chain groups by labelling the chains by the equivariant cells which afford them. Thus, we have a sequence

$$0 \longleftarrow \langle v_0, \dots, v_r\rangle \xleftarrow{d^1_{1,0}} \langle \alpha_i, \beta_i, \xi_1, \dots, \xi_r | i = 1, \dots, g\rangle \xleftarrow{d^1_{2,0}} \langle\gamma\rangle \longleftarrow 0.$$

We have $d^1_{1,0}(\alpha_i) = d^1_{1,0}(\beta_i) = v_0 - v_0 = 0$ for $1 \leqslant i \leqslant g$, $d^1_{1,0}(\xi_j) = v_j - v_0$ for $1 \leqslant j \leqslant r$ and, $d^1_{2,0} = 0$. In particular, $\mathrm{Im}(d^1_{1,0}) \cong \mathbb{Z}^r$, $\mathrm{Ker}(d^1_{1,0}) \cong \mathbb{Z}^{2g}$, $\mathrm{Im}(d^1_{2,0}) = 0$ and $\mathrm{Ker}(d^1_{2,0}) \cong \mathbb{Z}$. From this calculation we deduce the E^2 page is as in Figure 1.3.

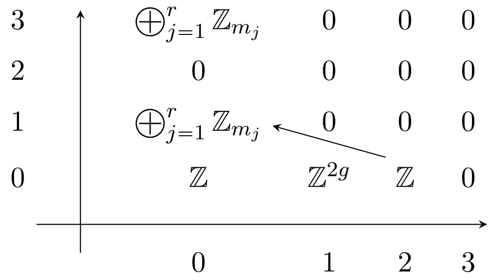

FIGURE 1.3: The E^2-page of the spectral sequence for a Fuchsian group.

The only non-trivial differential is the map drawn in Figure 1.3. Moreover, the spectral sequence clearly collapses after the computation of this differential. We can easily deduce what this differential is using the knowledge of $H_1(\Gamma)$. We will compute the abelianization using the same method as Majumdar [15].

To compute the abelianization we write out the presentation matrix M of Γ and then compute the Smith normal form.

$$M = \begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & & & \ddots & 0 & 0 & & 0 \\ 0 & \cdots & \cdots & 0 & m_r & 0 & & 0 \\ 1 & \cdots & \cdots & 1 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

We find $H_1(\Gamma) = \mathbb{Z}^{2g} \oplus \left(\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}\right)$. The constants t_j (Definition 1.1.1) come from Theorem 6 in Ferrar's book 'Finite Matrices' [8]. In particular, $\prod_{j=1}^{p} t_j$ is equal to the greatest common divisor of the p-rowed minors of M.

It follows from the calculation of the abelianization of Γ that the map $d^2_{2,0}$ is a surjection onto the factor $\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k}$ from the decomposition $\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} = (\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}) \oplus (\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k})$. The result now follows from the fact all extension problems are trivial. □

Corollary 1.4.1. *Let Γ be a cocompact Fuchsian group of signature $[g; m_1, \ldots, m_r]$, then*

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{2g} & q = 1, \\ \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r-1} \mathbb{Z}_{t_j}\right) & q = 2, \\ \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & q = 2l, \text{ where } l \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

1.4.2 Non-orientable NEC groups with no cusps or boundary components

Proof of Theorem 1.A(b). Let $X = \mathbb{R}\mathbf{H}^2$ and let Γ be an NEC group with signature $(g, 0, -, [m_1, \dots, m_r], \{\})$. In this case our E^1-page has the form given in Figure 1.2. The only non-trivial differentials are along the $q = 0$ row. Keeping the same notation as before we now have a sequence

$$0 \longleftarrow \langle v_0, \dots, v_r \rangle \xleftarrow{d^1_{1,0}} \langle \alpha_1, \dots, \alpha_g, \xi_1, \dots, \xi_r \rangle \xleftarrow{d^1_{2,0}} \langle \gamma \rangle \longleftarrow 0.$$

We have $d^1_{1,0}(\alpha_i) = v_0 - v_0 = 0$ for $1 \leqslant i \leqslant g$, $d^1_{1,0}(\xi_j) = v_j - v_0$ for $1 \leqslant j \leqslant r$ and, $d^1_{2,0}(f) = \sum_{i=1}^{g} 2\alpha_i$. In particular, $\mathrm{Im}(d^1_{1,0}) \cong \mathbb{Z}^r$, $\mathrm{Ker}(d^1_{1,0}) \cong \mathbb{Z}^{2g}$, $\mathrm{Im}(d^1_{2,0}) = 2\mathbb{Z}$ and $\mathrm{Ker}(d^1_{2,0}) = 0$. It follows that $E^2_{0,0} = \mathbb{Z}$, $E^2_{1,0} = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$ and $E^2_{2,0} = 0$, the remaining entries are unchanged. Thus, by dimension considerations the spectral sequence collapses.

The result now follows from resolving the extension problem in $H_1(\Gamma)$. Instead we compute the abelianization of Γ from the presentation matrix

$$M = \begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & & & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & m_r & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & 1 & 2 & \cdots & 2 \end{bmatrix} \sim \begin{bmatrix} m_2 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & & & \vdots & \vdots \\ \vdots & & \ddots & & \vdots & \vdots \\ \vdots & & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & m_r & 0 \\ m_1 & \cdots & \cdots & m_1 & m_1 & 2m_1 \end{bmatrix} = M'$$

We find $H_1(\Gamma) = \mathbb{Z}^{g-1} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{w_j} \right)$. The constants w_j (Definition 1.1.1) come from Theorem 6 in Ferrar's book 'Finite Matrices' [8]. In particular, $\prod_{j=1}^{p} w_j$ is equal to the greatest common divisor of the p-rowed minors of M'. □

Corollary 1.4.2. *If Γ is an NEC group with signature $(g, 0, -, [m_1, \dots, m_r], \{\})$ then,*

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & \text{for } q = 0, \\ \mathbb{Z}^{g-1} & \text{for } q = 1, \\ \bigoplus_{j=1}^{r} \mathbb{Z}_{w_j} & \text{for } q = 2, \\ \bigoplus_{j=1}^{r} \mathbb{Z}_{m_j} & \text{for } q = 2l, \text{ where } l \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

1.4.3 Orientable NEC groups with at least one cusp or boundary component

The remaining proofs will use the homology of finite dihedral groups. We record them here for the convenience of the reader.

Theorem 1.4.3. [10] *Let D_{2n} denote a dihedral group of order $2n$. In the case n is odd we have*

$$H_q(D_{2n};\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}_2 & q \equiv 1 \pmod 4, \\ \mathbb{Z}_{2n} & q \equiv 3 \pmod 4, \\ 0 & \textit{otherwise.} \end{cases}$$

$$H_q(D_{2n};\mathbb{Z}_2) = \mathbb{Z}_2 \textit{ for } q \geqslant 0.$$

In the case n is even we have

$$H_q(D_{2n};\mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}_2^{\frac{1}{2}(q+3)} & q \equiv 1 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}q} & q > 0 \textit{ is even}, \\ \mathbb{Z}_2^{\frac{1}{2}(q+1)} \oplus \mathbb{Z}_n & q \equiv 3 \pmod 4. \end{cases}$$

$$H_q(D_{2n};\mathbb{Z}_2) = \mathbb{Z}_2^{q+1} \textit{ for } q \geqslant 0.$$

We will now compute the cohomology of an NEC group with orientable quotient space with at least one boundary component or cusp.

Proof of Theorem 1.A(c). First, assume that $k + d = 0$, so $s > 0$. In this case it is easy to see that we can rearrange the presentation of Γ so that $\Gamma \cong F_{s-1} * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_r}$ where F_{s-1} is a free group of rank $s-1$. The result now follows from a straightforward application of the homology Mayer-Vietoris sequence.

We now treat the case with boundary, let $k, d, s \geqslant 0$ such that $k + d > 0$ and let $\epsilon = +$. We will use Theorem 1.3.1; here our space X is $\mathbb{R}\mathbf{H}^2 \cup \Lambda(\Gamma)$ endowed with the induced cell structure from the Wilkie-Macbeath polygon. To set up the sequence, observe that the stabiliser of a marked point v_j in the interior of the quotient space is a cyclic group $\mathbb{Z}_{m_j}$. If the vertex v_j lies on $\partial\mathbb{R}\mathbf{H}^2$ then the stabiliser is $\mathbb{Z}$. The stabiliser of a marked point $w_{i,l}$ on the boundary of the quotient space is a dihedral group $D_{2n_{i,l}}$, and edges along the boundary are stabilised by reflection groups isomorphic to $\mathbb{Z}_2$. Since the face stabilisers are trivial, vertices have a canonical orientation, and the edges being stabilised by $\mathbb{Z}_2$ are fixed pointwise, all of the orientation characters are trivial. It follows that the E^1-page consists of modules with trivial Γ-action and has the form given in Figure 1.4.

	0		1		2	3
5	$\mathbb{Z}_2^{k+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} H_5(D_{2n_{i,l}})\right)$	$\longleftarrow$	$\mathbb{Z}_2^{k+d+\sum_{i=1}^k s_i}$		0	0
4	$\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} H_4(D_{2n_{i,l}})$		0		0	0
3	$\mathbb{Z}_2^{k+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} H_3(D_{2n_{i,l}})\right)$	$\longleftarrow$	$\mathbb{Z}_2^{k+d+\sum_{i=1}^k s_i}$		0	0
2	$\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} H_2(D_{2n_{i,l}})$		0		0	0
1	$\mathbb{Z}^s \oplus \mathbb{Z}_2^{k+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} H_1(D_{2n_{i,l}})\right)$	$\longleftarrow$	$\mathbb{Z}_2^{k+d+\sum_{i=1}^k s_i}$		0	0
0	$\mathbb{Z}^{1+r+s+k+d+\sum_{i=1}^k s_i}$	$\longleftarrow$	$\mathbb{Z}^{2g+r+s+2k+2d+\sum_{i=1}^k s_i}$	$\longleftarrow$	$\mathbb{Z}$	0

FIGURE 1.4: The E^1-page of the spectral sequence for an orientable NEC group with cusps and boundary.

We will first deal with the differentials $d^1_{*,0}$. By slightly abusing notation and labelling our basis elements for each chain group by the equivariant cells which afford them, we have a sequence

$$0 \leftarrow \left\langle v_j, w_{i,l} \;\middle|\; \begin{array}{c} 0 \leqslant j \leqslant r+s \\ 1 \leqslant i \leqslant k+d \\ 0 \leqslant l \leqslant s_i \end{array} \right\rangle \xleftarrow{d^1_{1,0}} \left\langle \alpha_t, \beta_t, \xi_j, \gamma_{i,l}, \epsilon_i \;\middle|\; \begin{array}{c} 1 \leqslant t \leqslant 2g, \\ 1 \leqslant j \leqslant r+s \\ 1 \leqslant i \leqslant k+d, \\ 0 \leqslant l \leqslant s_i \end{array} \right\rangle$$

$$\langle f \rangle \xleftarrow{\quad} 0, \qquad \langle f \rangle \xrightarrow{d^1_{2,0}} \left\langle \alpha_t, \beta_t, \xi_j, \gamma_{i,l}, \epsilon_i \right\rangle.$$

Computing the image of the differential $d^1_{2,0}$ on the $\mathbb{Z}$-basis element f, we obtain that up to sign

$$f \mapsto \sum_{i=1}^{k} \sum_{l=0}^{s_i} \gamma_{i,l}.$$

So, we find $\mathrm{Im}(d^1_{2,0}) = \mathbb{Z}$ and $E^2_{2,0} = 0$. In light of the description of the fundamental domain in Section 1.2, for $d^1_{1,0}$ we have the following

$$\begin{array}{lcll} \alpha_t & \mapsto & v_0 - v_0 = 0 & \text{for } 1 \leqslant t \leqslant 2g; \\ \beta_t & \mapsto & v_0 - v_0 = 0 & \text{for } 1 \leqslant t \leqslant 2g; \\ \xi_j & \mapsto & v_j - v_0 & \text{for } 1 \leqslant j \leqslant r+s; \\ \gamma_{i,l} & \mapsto & w_{i,(l+1 \pmod{s_i})} - w_{i,l} & \text{for } 1 \leqslant i \leqslant k, \text{ and } 0 \leqslant l \leqslant s_i; \\ \gamma_{i,0} & \mapsto & w_{i,0} - w_{i,0} = 0 & \text{for } k+1 \leqslant i \leqslant k+d; \\ \epsilon_i & \mapsto & w_{i,0} - v_0 & \text{for } 1 \leqslant i \leqslant k+d. \end{array}$$

In particular, we have $\mathrm{Im}(d^1_{1,0}) = \mathbb{Z}^{r+s+k+\sum_{i=1}^{k} s_i}$ and $\mathrm{Ker}(d^1_{1,0}) = \mathbb{Z}^{2g+k+d}$. It then follows that $E^2_{1,0} = \mathbb{Z}^{2g+k+d-1}$ and $E^2_{0,0} = \mathbb{Z}$. At this point, it is easy to see that the spectral sequence will collapse trivially once we have computed the differentials $d^1_{1,*}$.

We will begin with the differential $d^1_{1,q}$ where $q \equiv 1 \pmod 4$. Since the edges connected to the vertices corresponding to the $\mathbb{Z}_{m_j}$ summands have trivial stabilisers, the $\mathbb{Z}_{m_j}$ summands will survive to the E^2-page. In the case $q = 1$, the $\mathbb{Z}$ summands also survive by the same reasoning.

We now draw our focus to the other summands. Let each $D_{2n_{i,l}}$ be generated by a reflection $r_{i,l}$ and a rotation $t_{i,l}$ of order $n_{i,l}$. We have that $H_1(D_{2n_{i,l}})$ is generated by $r^1_{i,l}, t^1_{i,l}$, the images of $t_{i,l}$ and $r_{i,l}$ under the abelianization map. For $q > 1$ there will be extra generators whenever an $n_{i,l}$ is even; we will suppress this from the notation. Note that $t^1_{i,l} = 0$ if n is odd. For each $q \equiv 1 \pmod 4$ we now have a sequence (modulo the extra classes arising from dihedral groups where $n_{i,l}$ is even and when $q > 1$)

$$0 \longleftarrow \left\langle w^q_{i,0}, w^q_{p,0}, r^q_{i,l}, t^q_{i,l} \;\middle|\; \begin{array}{c} 1 \leqslant i \leqslant k \\ 1 \leqslant l \leqslant s_i \\ 1 \leqslant p \leqslant d \end{array} \right\rangle \xleftarrow{d^1_{1,q}} \left\langle \gamma^q_{i,l}, \gamma^q_{p,0} \;\middle|\; \begin{array}{c} 1 \leqslant i \leqslant k \\ 0 \leqslant l \leqslant s_i \\ 1 \leqslant p \leqslant d \end{array} \right\rangle$$

We will break the map $d^1_{1,q}$ into several cases depending on the adjacent edges in the fundamental domain and the cycle type of the boundary component. First, we will consider each 'end' of the ith boundary component with a non-empty period of cycles (i.e. $1 \leqslant i \leqslant k$), the reader should keep Figure 1.1(a) in mind. Here we have

$$(\psi_{i,0})_q : H_q(\langle r_{i,0} \rangle) \hookrightarrow H_q(D_{2n_{i,l}}) \oplus H_q(\mathbb{Z}_2) \text{ by } \gamma^q_{i,0} \mapsto t^q_{i,1} - w^q_{i,0}$$

and

$$(\psi_{i,s_i})_q : H_q(\langle r_{i,s_i} t_{i,s_i} \rangle) \hookrightarrow H_q(\mathbb{Z}_2) \oplus H_q(D_{2n_{i,l}}) \text{ by } \gamma^q_{i,0} \mapsto w^q_{i,0} - t^q_{i,s_i} - r^q_{i,s_i}.$$

For the intermediary edges we have

$$(\psi_{i,l})_q : H_q(\langle r_{i,l} t_{i,l} \rangle) \hookrightarrow H_q(D_{2n_{i,l+1}}) \oplus H_q(D_{2n_{i,l}}) \text{ by } \gamma^q_{i,0} \mapsto t^q_{i,l+1} - t^q_{i,l} - r^q_{i,l}.$$

In each case we are suppressing from the image a possible sum of order 2 classes (distinct from $t^q_{i,l}$ and $r^q_{i,l}$) arising from even dihedral groups. The reason for this is that provided at least one of the $n_{i,l}$ are even, the images of the maps $\psi_{i,l}$ for $0 \leqslant l \leqslant s_i$ are already linearly independent. Of course if all of the $n_{i,l}$ for $0 \leqslant l \leqslant s_i$ are odd, then the classes do not exist.

If the boundary component i only contains odd cycles, then $\gamma^q_{i,s_i} = \sum_{l=0}^{s_i-1} \gamma^q_{i,l}$, so we have an order 2 element in the kernel of $d^1_{1,q}$. If the boundary component has an empty period of cycles, then we have exactly one edge $\gamma_{i,0}$ with vertex $w_{i,0}$ at each end. In particular $\gamma^q_{i,0} \mapsto w^q_{i,0} - w^q_{i,0} = 0$. From this analysis we deduce that $\mathrm{Ker}(d^1_{1,q}) = \mathbb{Z}_2^{C_O+d}$ and $\mathrm{Im}(d^1_{1,q}) \cong \mathbb{Z}_2^{k+\sum_{i=1}^k s_i - C_O}$. It then follows from a simple calculation that $E^2_{1,q} = \mathbb{Z}_2^{C_O+d}$ and $E^1_{0,q} \cong \mathbb{Z}_2^{\frac{1}{2}(q+1)C_E + C_O + d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j} \right)$ for $q \equiv 1 \pmod 4$, $q > 1$. When $s > 0$ we have an additional $\mathbb{Z}^s$ summand in $E^2_{0,1}$.

An alternative way of considering these maps is as follows. Let C_{E_i} denote the number of even periods in the ith period cycle. Observe that each period cycle contributes $\frac{1}{2}(q+1)C_{E_i} - 1$ summands of $\mathbb{Z}_2$ to $E^2_{0,q}$. The C_O summands of $\mathbb{Z}_2$ contained in $\mathrm{Ker}(d^1_{1,q})$ cause an additional C_O summands of $\mathbb{Z}_2$ to survive to $E^2_{0,q}$. From, above we then have that

$$k + \sum_{i=1}^{k} \left(\frac{1}{2}(q+1)C_{E_i} - 1 \right) + C_O = k + \frac{1}{2}(q+1)C_E - k + C_O = \frac{1}{2}(q+1)C_E + C_O.$$

We now need to compute the maps $d^1_{1,q}$ for $q \equiv 3 \pmod 4$. We have essentially the same cases and proof as when $q \equiv 1 \pmod 4$ except that $\mathrm{Coker}(d^1_{1,q})$ contains a summand $\mathbb{Z}_{n_{i,l}}$ for each $n_{i,l}$.

Claim For $q \equiv 3 \pmod 4$, $1 \leqslant i \leqslant k$ and $1 \leqslant l \leqslant s_i$ the term $E^2_{1,q} = \mathrm{Coker}(d^1_{1,q})$ contains a summand $\mathbb{Z}_{n_{i,l}}$.

Proof of claim: When $n_{i,l}$ is odd this is immediate. Let $n := n_{i,l}$ be even and consider $H^{q+1}(D_{2n};\mathbb{Z})$ where $q \equiv 3 \pmod 4$. There is an element of order n in $H^{q+1}(D_{2n};\mathbb{Z})$ that corresponds to a power of the second Chern class of the faithful 2-dimensional linear representation ρ of $D_{2n} = \langle r, t\rangle$. Restricting ρ to the subgroup $\langle rt\rangle$ gives the regular representation of $\mathbb{Z}_2 \cong \langle rt\rangle$. Now, the total Chern class of $\mathbb{Z}_2$ is equal to 0 in degree 4. It follows that the map $H^{q+1}(D_{2n};\mathbb{Z}) \to H^{q+1}(\langle rt\rangle)$ has kernel containing a $\mathbb{Z}_n$ summand. Dualizing back to homology, it follows the map $H_q(\langle rt\rangle) \to H_q(D_{2n})$ has cokernel containing a $\mathbb{Z}_n$ summand. This yields the claim. ♦

We conclude the description of E^2 as follows. First, when $q \equiv 3 \pmod 4$ we have $\mathrm{Ker}(d^1_{1,q}) \cong \mathbb{Z}_2^{C_O+d}$ and $\mathrm{Im}(d^1_{1,q}) \cong \mathbb{Z}_2^{k+\sum_{i=1}^k s_i - C_O}$. It follows $E^2_{1,q} \cong \mathbb{Z}^{C_O+d}$ and $E^2_{0,q} \cong \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) \oplus \left(\bigoplus_{j=1}^k \mathbb{Z}_{m_j}\right)$. Every other entry on the E^2-page is 0 trivially.

The theorem follows from resolving the extension problems $0 \to E^2_{1,q-1} \to H_q(\Gamma) \to E^2_{0,q} \to 0$, where $q > 0$ is even. To resolve the extension problems, we will compute the homology of Γ with $\mathbb{Z}_2$ coefficients and then compare the $\mathbb{Z}_2$-rank of $H_q(\Gamma;\mathbb{Z}_2)$ with the $\mathbb{Z}_2$-rank of $(E^2_{1,q-1} \oplus E^2_{0,q}) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}(E^2_{0,q-1},\mathbb{Z}_2)$. Note that the latter is equal to $(q+1)C_E + 2C_O + 2d$. If the ranks are equal, then the extension will split.

Recall that $H_n(\mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2$ for $n \geqslant 0$. Combining this with the $\mathbb{Z}_2$-homology groups of the Dihedral groups (Theorem 1.4.3) and the Γ-equivariant spectral sequence (Theorem 1.3.1), we can set up a spectral sequence calculation. To simplify things, note we are only interested in the maps $d^1_{1,q}$ for $q > 0$.

Let $q > 0$ and let C_T denote the number of odd cycles, so $C_T + C_E = \sum_{i=1}^k s_i$. We then have a sequence

$$0 \longleftarrow \mathbb{Z}_2^{(q+1)C_E+C_T+d+k} \xleftarrow{d^1_{1,q}} \mathbb{Z}_2^{C_E+C_T+d+k} \longleftarrow 0.$$

By essentially using the same calculations as above we have that $\mathrm{Im}(d^1_{1,q}) \cong \mathbb{Z}_2^{C_E+C_T+k-C_O}$. From this we conclude that $E^2_{0,q} = \mathbb{Z}_2^{(q+1)C_E+C_O+d}$ and that $E^2_{1,q} = \mathbb{Z}_2^{C_O+d}$. This gives a $\mathbb{Z}_2$-rank of $(q+1)C_E + 2C_O + 2d$. Thus, the extension splits. □

Corollary 1.4.4. *Let $d + k + s > 0$. If Γ is an NEC group with signature*

$$(g, s, +, [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k}), (), \ldots, ()\})$$

then,

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{2g+s+k+d-1} & q = 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q \equiv 2 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} & q = 2p+1 \text{ where } p \geqslant 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) & \\ \quad \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q > 0 \text{ and } q \equiv 0 \pmod 4. \end{cases}$$

1.4.4 Non-orientable NEC groups with at least one cusp or boundary component

We will now compute the cohomology of an NEC group with non-orientable quotient space and at least one cusp or boundary component. The proof is almost exactly the same as the proof of Theorem 1.A(c) so we will only provide a brief sketch and highlight the differences.

Proof of Theorem 1.A(d) (sketch). First assume that $k + d > 0$. The key differences between the orientable (Figure 1.4) and non-orientable cases is the $E^1_{1,0}$ term and the map $d^1_{2,0}$. The $E^1_{1,0}$ now contains a $\mathbb{Z}^g$ summand instead of a $\mathbb{Z}^{2g}$ summand. The map $d^1_{2,0}$ now sends the generator to the sum of boundary components plus 2 times each generator of the aforementioned $\mathbb{Z}^g$ summand. More precisely (with the same notation as in the proof of Theorem 1.A(c)) we have,

$$f \mapsto \sum_{i=1}^{k} \sum_{l=0}^{s_i} \gamma_{i,l} + 2 \sum_{t=1}^{g} \alpha_t$$

In particular, $E^2_{1,0} = \mathbb{Z}^{g+k+d-1}$. The proof goes through identically from here.

Now assume $g > 0$ and $k + d = 0$, so $s > 0$. We still have that $E^1_{1,0}$ contains a $\mathbb{Z}^g$ summand instead of a $\mathbb{Z}^{2g}$ summand. However, with notation as before,

$$d^1_{2,0}(f) = 2 \sum_{t=1}^{g} \alpha_t.$$

In particular, $E^2_{1,0} = \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$. The remainder of the proof is identical, except we now have an extension problem to determine $H_1(\Gamma)$. We instead resolve this by computing the abelianisation from the presentation matrix

$$M = \begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & & \ddots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & m_r & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & \cdots & 1 & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 \end{bmatrix}.$$

Clearly, M can be reduced to an $(r+1) \times (r+g+s)$ matrix with the only non-zero entries equal to $m_1, \ldots, m_r, 1$ on the leading diagonal. The result follows.

The final case when $g = k = d = 0$ and $s > 0$ follows an almost identical argument to the case $k = d = 0$, $s > 0$ and $\epsilon = +$, so we will not recreate it here.

□

Corollary 1.4.5. *Let $d + k + s > 0$. If Γ is an NEC group of signature*

$$(g, s, -, [m_1, \ldots, m_r], \{(n_{1,1}, \ldots, n_{1,s_1}), \ldots, (n_{k,1}, \ldots, n_{k,s_k}), (), \ldots, ()\})$$

then

$$H^q(\Gamma) = \begin{cases} \mathbb{Z} & q = 0, \\ \mathbb{Z}^{g+s+k+d-1} & q = 1, \\ \mathbb{Z}_2^{C_E+C_O+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q = 2, \\ \mathbb{Z}_2^{\frac{1}{2}(q-1)C_E+C_O+d} & q = 2p+1 \text{ where } p \geqslant 1, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{i=1}^k \bigoplus_{l=1}^{s_i} \mathbb{Z}_{n_{i,l}}\right) & \\ \quad \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q > 0 \text{ and } q \equiv 0 \pmod 4, \\ \mathbb{Z}_2^{\frac{1}{2}qC_E+C_O+d} \oplus \left(\bigoplus_{j=1}^r \mathbb{Z}_{m_j}\right) & q > 2 \text{ and } q \equiv 2 \pmod 4. \end{cases}$$

1.4.5 The ring structure

We will now deal with the computation of the ring structure. Recall from Definition 1.1.2 that R_q is the subring of $\widetilde{H}^*(\mathbb{Z}_q)$ generated by x^2 and x^3, where x is the degree 2 generator of $H^*(\mathbb{Z}_q)$.

Proof of Theorem 1.B. We first prove the result when $s > 0$. Let Γ be a Fuchsian group of signature $[g, s; m_1, \ldots, m_r]$ such that $s > 0$. We may rearrange the presentation of Γ so that $\Gamma \cong F_{s-1} * \mathbb{Z}_{m_1} * \cdots * \mathbb{Z}_{m_r}$ where F_{s-1} is a free group of rank $s-1$. The result is now an easy application of the Mayer-Vietoris sequence for cohomology.

	0	1	2	3
3	$\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}$	0	0	0
2	0	0	0	0
1	$\mathbb{Z}^s \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}_{m_j}\right)$	0	0	0
0	$\mathbb{Z}^{r+s+1} \longleftarrow$	$\mathbb{Z}^{2g+r+s} \leftarrow$	$\mathbb{Z}$	0

FIGURE 1.5: The E^2-page of the cohomological spectral sequence for a cocompact Fuchsian group. Here the element x_j is additive torsion of order m_j.

Now, let Γ be a Fuchsian group of signature $[g, s; m_1, \ldots, m_r]$ such that $s = 0$. We instead consider the equivariant cohomology spectral sequence for Γ. Armed with our calculation for homology, it is clear that E^2-page has the form given in Figure 1.5 (here $m_j x_j = 0$). Now, there is an extension problem for $H^2(\Gamma)$ which we resolved when computing the homology of Γ. Dualising to cohomology via the universal coefficient theorem we see that, up to a change of basis of the x_js, the extension problem kills a subset of this new basis which generate an abelian group isomorphic to $\bigoplus_{k=1}^{l} \mathbb{Z}_{q_k}$. Since the spectral sequence preserves cup products the result follows. □

1.5 Closing remarks

We end with three remarks. Firstly, the author was asked by Professor Gareth Jones whether the same results hold for the 17 wallpaper groups if one takes X to be the Euclidean plane. We confirm here it does, however the cohomology computations of these groups are well known so we will not elaborate on this. Secondly, the results in this paper are consistent with Gaboriau's result that L^2-Betti numbers of lattices in a Lie group are proportional to their covolume [9]. As such one deduces the well known result that for an NEC group Γ the first L^2-Betti number $b_1^{(2)}(\Gamma) = -\chi_{\mathbb{Q}}(\Gamma)$ and all other L^2-Betti numbers vanish. Finally, Fuchsian groups are not determined by their cohomology. Indeed, the groups with signatures $[g, s; 3, 10]$ and $[g, s; 5, 6]$ have isomorphic cohomology rings but the groups are not isomorphic.

References

[1] Akhter, N. and Majumdar, S.: *Determination of the homology and the cohomology of a few groups of isometries of the hyperbolic plane*, GANIT **36**, 65–77 (2016). doi: 10.3329/ganit.v36i0.32774

[2] Berkove, E., Juan-Pineda, D. and Pearson, K.: *A geometric approach to the lower algebraic K-theory of Fuchsian groups*, Topology Appl. **119** (3), 269–277 (2002). doi: 10.1016/S0166-8641(01)00068-2

[3] Brown, K. S.: *Cohomology of groups*, Springer-Verlag, New York (1994).

[4] Bujalance, E., Cirre, F.J., Gamboa, J.M., Gromadzki, G.: *Symmetries of compact Riemann surfaces*, Springer-Verlag, Berlin Heidelberg (2010).

[5] Curran, P.: *Cohomology of F-groups*, Trans. Amer. Math. Soc. **152**, 609–621 (1970). doi: 10.2307/1995590

[6] Eichler, M.: *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67**, 267–298 (1957). doi: 10.1007/BF01258863

[7] Ellis, G. and Williams, G.: *On the cohomology of generalized triangle groups*, Comment. Math. Helv. **80** (3), 571–591 (2005). doi: 10.4171/CMH/26

[8] Ferrar, W. L.: *Finite matrices*, Oxford, at the Clarendon Press (1951).

[9] Gaboriau, D.: *L^2-invariants of equivalence and group relations*, Publ. math., Inst. Hautes Étud. Sci. **95**, 93–150 (2002). doi: 10.1007/s102400200002

[10] Handle, D.: *On products in the cohomology of the dihedral groups*, Tohoku Math. J. (2) **45** (1), 13–42 (1993). doi: 10.2748/tmj/1178225952

[11] Huebschmann, J.: *Cohomology theory of aspherical groups and of small cancellation groups*, J. Pure Appl. Algebra **14** (2), 137–143 (1979). doi: 10.1016/0022-4049(79)90003-3

[12] Hughes, S.: *On the equivariant K- and KO-homology of some special linear groups*, preprint (2020). arXiv: 2004.08199

[13] Lück, W. and Stamm, R.: *Computations of K- and L-theory of cocompact planar groups*, K-Theory **21** (3), 249–292 (2000). doi: 10.1023/A:1026539221644

[14] Macbeath, A. M.: *The classification of non-euclidean plane crystallographic groups*, Canadian J. Math. **19**, 1192–1205 (1967). doi: 10.4153/CJM-1967-108-5

[15] Majumdar, S.: *A free resolution for a class of groups*, J. London Math. Soc. (2) **2**, 615–619 (1970). doi: 10.1017/S0013091500016886

[16] Patterson, S. J.: *On the cohomology of Fuchsian groups*, Glasgow Math. J. **16** (2), 123–140 (1975). doi: 10.1017/S0017089500002615

[17] Singerman, D.: *On the structure of non-Euclidean crystallographic groups*, Proc. Camb. Phil. Soc. **76**, 233–240 (1974). doi: 10.1017/S0305004100048891

[18] Wilkie, H. C.: *On non-Euclidean crystallographic groups*, Math. Z. **9**, 87–102 (1966). doi: 10.1007/BF01110157

Paper 2

THE FIRST ℓ^2-BETTI NUMBER AND GRAPHS OF GROUPS

INDIRA CHATTERJI, PETER KROPHOLLER, AND SAM HUGHES

Abstract. We generalise results of Thomas, Allcock, Thom–Petersen, Kar–Niblo on presentations to the first ℓ^2-Betti number of quotients of certain graphs of groups by subgroups with free actions on the edge sets of the graphs.

2.1 Introduction

The ℓ^2-Betti numbers were introduced by Atiyah as dimensions of heat kernels of certain operators on Riemannian manifolds. The modern formulation assigns ℓ^2-Betti numbers $b_i^{(2)}(G)$ to arbitrary groups G. We refer the reader to Lück's account where the history can be found in the introduction of [11]. Technical results about ℓ^2-Betti numbers that we need can be found in chapters 6 and 8 of loc. cit. The ℓ^2-Euler characteristic $\chi^{(2)}(G)$ is defined to be the alternating sum of these Betti numbers when this series is absolutely convergent. Let $\mathcal{C}$ denote the class of groups F such that

- $\sum_{i\geqslant 0} b_i^{(2)}(G)$ is finite (this being the condition for absolute convergence),
- $b_1^{(2)}(F) = b_2^{(2)}(F) = 0$, and
- either $\chi^{(2)}(F) = 0$ or F is finite.

Note that that $\mathcal{C}$ contains all ℓ^2-acyclic groups (i.e. the groups for which $b_i^{(2)} = 0$ for all $i > 0$) and in particular it contains all amenable groups. Relevant background on ℓ^2-cohomology is included in Section 2.2. In this note we prove the following theorem.

Theorem 2.A. *Let F be a group acting simplicially and cocompactly on a simplicial tree, with vertex and edge stabilisers in $\mathcal{C}$, let N be a subgroup normally generated by m elements, intersecting the vertex stabilisers trivially, and let G denote F/N. Then $\chi^{(2)}(F)$ is defined and setting $k := \chi^{(2)}(F) + m$ the following conclusions hold:*

(i) *If $k \leqslant 0$, then G is infinite.*

(ii) *If $k < 0$, then $b_1^{(2)}(G) \geqslant -k > 0$.*

(iii) *If G is finite, then $k > 0$ and $|G| \geqslant \frac{1}{k}$.*

Note that the hypotheses of this theorem guarantee that N acts freely on the specified tree and in particular N is necessarily a free group. Note also that, according to [2, Corollary 1.4], if $b_1^{(2)}(G) > 0$ then G has no commensurated infinite amenable subgroup and according to [4, Corollary 6] does not have property (T). If we also have $b_2^{(2)}(G) = 0$, then G is in the class $\mathcal{D}_{\text{reg}}$ by [14, Lemma 2.8]. We refer the reader to [3] for background on property (T) and to [14, Definition 2.6] for the definition of the class $\mathcal{D}_{\text{reg}}$. Acylindrically hyperbolic groups form a large class of groups admitting coarsely proper actions on hyperbolic metric spaces. The class is a generalisation of relative hyperbolicity including many Artin groups, mapping class groups, and $\text{Out}(F_n)$. The main result of Osin's paper [12] states that indicable groups with positive first ℓ^2-Betti number are acylindrically hyperbolic. In particular, we have the following corollary.

Corollary 2.B. *Let G, F and N be as in Theorem 2.A. Assume that G is finitely presented, (virtually) indicable and that $\chi^{(2)}(F)+m < 0$. Then G is (virtually) acylindrically hyperbolic.*

The simplest way in which the indicability hypothesis may arise is through *stable letters*: Let T denote the F-tree of Theorem 2.A. Let K denote the (necessarily normal) subgroup generated by the vertex stabilisers. Then there is a subgroup $E \leqslant F$ that complements K and all such subgroups are free of uniquely determined rank. Such a subgroup may be referred to as *a subgroup of stable letters of the action*. The group G has an infinite cyclic quotient when $N \cap E$ has infinite index in E, in other words when there is a stable letter that is faithfully represented in G, and in this case G is indicable.

Recall that a group G is *C^*-simple* if the reduced group C^*-algebra, denoted $C_r^*(G)$, has exactly two norm closed 2-sided ideals, 0, and the algebra $C_r^*(G)$ itself. By [5, Corollary 6.7] we obtain the following.

Corollary 2.C. *With G, F and N as before, G is C^*-simple if and only if it has trivial amenable radical.*

Theorem 2.A has some historical pedigree. It originally began life as a result about quotients of free groups due to Thomas (see Theorem 2.D((i))) and was proved using combinatorial methods [15]. The result was generalised by Allcock to incorporate a bound on the rank of the abelianisation of the quotient group [1]. The introduction of ℓ^2-cohomology came when Peterson–Thom [13, Theorem 3.6] and Kar–Niblo [10] independently linked the inequality of Thomas to the first ℓ^2-Betti number. These discoveries are summarized in the following result.

Theorem 2.D (Thomas [15], Allcock [1], Peterson–Thom [13], Kar–Niblo [10])**.** *Let G be a group with a presentation*

$$\langle x_1, \dots, x_n |\ r_1^{k_1}, \dots, r_m^{k_m} \rangle$$

in which the elements r_i have order k_i when interpreted in G.

(i) *If* $n - \sum_{i=1}^{m} \frac{1}{k_i} \geqslant 1$ *then* G *is infinite.*

(ii) *If* G *is finite then* $|G| \geqslant \frac{1}{1-n+\sum_{i=1}^{m} k_i}$.

(iii) *If* $n - \sum_{i=1}^{m} \frac{1}{k_i} > 1$ *then* G *is non-amenable.*

Deduction of Theorem 2.D from Theorem 2.A. Let G be a group with a presentation

$$G = \langle x_1, \ldots, x_n | \; r_1^{k_1}, \ldots, r_m^{k_m} \rangle.$$

Adding m fresh generators $y_1, \ldots, y_m$, we can give the following alternative presentation of the same group:

$$G = \langle x_1, \ldots, x_n, y_1, \ldots, y_m | \; y_1^{k_1}, \ldots, y_m^{k_m}, r_1 y_1^{-1}, \ldots, r_m y_m^{-1} \rangle.$$

Let F be the group with presentation

$$F = \langle x_1, \ldots, x_n, y_1, \ldots, y_m | \; y_1^{k_1}, \ldots, y_m^{k_m} \rangle$$

and let N be the subgroup of F normally generated by $r_1 y_1^{-1}, \ldots, r_m y_m^{-1}$. Then F is a free product of cyclic groups: in particular it is virtually free and has Euler characteristic $\chi(F) = \sum_{i=1}^{m} \frac{1}{k_i} - n - m + 1$. The condition that the r_i have order k_i in the original presentation ensures that N does not meet any of the finite subgroups of F and so is torsion-free. Applying Theorem 2.A with these choices of F, N, G yields Theorem 2.D. □

Throughout this paper for a group or subgroup G we will adopt the convention that $\frac{1}{|G|}$ be interpreted as zero if G is infinite.

Finally, we also provide a computation of the first ℓ^2-Betti number for certain groups acting on trees. This generalises a result of Lück [7], which covers the case of an amalgamated free product, and a result of Tsouvalas [16, Corollary 3.7]. Tsouvalas assumes the vertex stabilisers are either residually finite or virtually torsion-free and the edge stabilisers are finite. Here we replace both of these assumptions with Lück's less restrictive assumption that the first ℓ^2-Betti numbers of the edge stabilisers vanish. So, for example, the theorem applies to fundamental groups of graphs of $\mathcal{C}$-groups.

Theorem 2.E. *Let* F *be a group acting simplicially on a simplicial tree and let* V *and* E *denote sets of representatives of* F*-orbits of vertices and edges. Assume for each* $e \in E$ *that* $b_1^{(2)}(F_e) = 0$*, then we have*

$$b_1^{(2)}(F) = \sum_{v \in V} \left(b_1^{(2)}(F_v) - \frac{1}{|F_v|} \right) + \sum_{e \in E} \frac{1}{|F_e|} + \frac{1}{|F|}.$$

Acknowledgements

The second author wishes to thank Ian Leary for his tireless and good humoured role as thesis adviser. The second author was supported by the Engineering and Physical Sciences Research Council grant number 2127970. We are also indebted to Kevin Li and to the anonymous referee for a number of helpful comments and corrections.

2.2 Background on ℓ^2-homology

Let G be a group. Then both G and the complex group algebra $\mathbb{C}G$ act by left multiplication on the Hilbert space $\ell^2 G$ of square-summable sequences. The group von Neumann algebra $\mathcal{N}G$ is the ring of G-equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}G$ can be identified with the *ring of affiliated operators*, and is denoted by $\mathcal{U}G$. One has the inclusions $\mathbb{C}G \subseteq \mathcal{N}G \subseteq \ell^2 G \subseteq \mathcal{U}G$ and it is also known that $\mathcal{U}G$ is a self-injective ring which is flat over $\mathcal{N}G$. For more details concerning these constructions we refer the reader to [11] and especially to Theorem 8.22 of §8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [11, §8.3]. Now let Y be a G-CW complex as defined[1] in [11, Definition 1.25 of §1.2]. The ℓ^2-homology groups of Y are then defined to be the equivariant homology groups $H_i^G(Y;\mathcal{U}G)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^G(Y;\mathcal{U}G).$$

The ℓ^2-Betti numbers of a group G are then defined to be the ℓ^2-Betti numbers of EG, that is to say

$$b_i^{(2)}(G) := b_i^{(2)}(EG). \tag{2.1}$$

By [11, Theorem 6.54(8)], the zeroeth ℓ^2-Betti number of G is equal to $1/|G|$. Moreover, if G is finite then $b_n^{(2)}(G) = 0$ for $n \geqslant 1$.

Let $C_*(Y;\mathcal{U}G)$ denote the standard cellular chain complex of Y with coefficients in $\mathcal{U}G$. We have the formula

$$\dim_{\mathcal{U}G} C_i(Y;\mathcal{U}G) = \sum_\sigma \frac{1}{|G_\sigma|}$$

where σ runs through a set of orbit representatives of i-dimensional cells in Y. Suppose that the ℓ^2-Euler characteristic of Y is defined. Standard arguments of homological algebra give the connection between two Euler characteristic computations (for the details see [11, Lemma 6.80(1)]):

$$\chi^{(2)}(Y) = \sum_i (-1)^i b_i^{(2)}(Y) = \sum_i (-1)^i \dim_{\mathcal{U}G} C_i(Y;\mathcal{U}G) = \sum_i (-1)^i \sum_\sigma \frac{1}{|G_\sigma|}. \tag{2.2}$$

[1] This is the usual definition.

We will need the following lemma for the proofs in the next section. One should think of it as a mild generalisation of Theorem 6.54(2) in [11]

Lemma 2.2.1 (Comparison with the Borel construction up to rank)**.** *Let X be a G-CW complex. Suppose for all $x \in X$ the isotropy group G_x is finite or $b_p^{(2)}(G_x) = 0$ for all $0 \leqslant p \leqslant n$, then*

$$b_p^{(2)}(X) = b_p^{(2)}(EG \times X) \quad \textit{for } 0 \leqslant p \leqslant n.$$

Proof. It suffices to prove that the von Neumann dimensions of the kernel and cokernel of the map

$$\mathrm{pr}_p : H_p^G(EG \times X; \mathcal{U}G) \to H_p^G(X; \mathcal{U}G)$$

induced by the projection $EG \times X \to X$ are zero for $0 \leqslant p \leqslant n$. Here $EG \times X$ carries the diagonal action of G. By an identical argument to [11, Theorem 6.54(2)] it suffices to prove for each isotropy subgroup $H \leqslant G$ and $0 \leqslant p \leqslant n$ the kernel and cokernel of the map $\mathrm{pr}_p : H_p^H(EH; \mathcal{U}H) \to H_p^H(*; \mathcal{U}H)$ have dimension equal to zero. If H is finite this follows from [11, Theorem 6.54(8a)], and is immediate if $b_p^{(2)}(H) = 0$ for all $0 \leqslant p \leqslant n$. □

2.3 The Main Theorem

To prove Theorem 2.A, one needs the following method of computing the ℓ^2-Euler characteristic of a group acting on a tree analogous to Chiswell's result [9] for rational Euler characteristic.

Proposition 2.3.1 (Chatterji–Mislin [8])**.** *Let F be a group acting on a tree and let V and E denote sets of representatives of F-orbits of vertices and edges. If the ℓ^2-Euler characteristic of each vertex and edge group is finite, then*

$$\chi^{(2)}(F) = \sum_{v \in V} \chi^{(2)}(F_v) - \sum_{e \in E} \chi^{(2)}(F_e).$$

Proof of Theorem 2.A. There is a cocompact action of F on a tree T with vertex and edge stabilisers in the class $\mathcal{C}$. Let V and E denote the vertex and edge sets. Let $\overline{T}$ denote the quotient graph T/N and write $\overline{V}$ and $\overline{E}$ for its vertex and edge sets. Now $G = F/N$ acts cocompactly on $\overline{T}$ with vertex and edge stabilisers in $\mathcal{C}$. The augmented chain complex of T is the short exact sequence

$$0 \to \mathbb{Z}E \to \mathbb{Z}V \to \mathbb{Z} \to 0$$

of $\mathbb{Z}F$-modules. Restricting to the action of N this short exact sequence leads to a long exact sequence for the homology of N. It is straightforward to identify $H_0(N; \mathbb{Z}V)$ with

$\mathbb{Z}\overline{V}$ and $H_0(N;\mathbb{Z}E)$ with $\mathbb{Z}\overline{E}$, so that the tail end of the sequence takes the form

$$H_1(N;\mathbb{Z}) \to \mathbb{Z}\overline{E} \to \mathbb{Z}\overline{V} \to \mathbb{Z} \to 0. \tag{2.3}$$

Let $\{r_i : i = 1,..m\}$ denote a normal generating set for N. Choose a vertex v_0 in T to be a fixed basepoint. For $1 \leqslant i \leqslant m$ consider the geodesic from v_0 to $v_0 r_i$. In the quotient graph $\overline{T}$ this geodesic descends to a loop because v_0 and $v_0 r_i$ become identified in $\overline{T}$. Now 2-discs can be glued to each loop. By adjoining free G-orbits of 2-discs equivariantly we can build a 2-complex Y with an action of G, whose 1-skeleton is $\overline{T}$, and which has augmented cellular chain complex

$$\mathbb{Z}G^m \to \mathbb{Z}\overline{E} \to \mathbb{Z}\overline{V} \to \mathbb{Z} \to 0. \tag{2.4}$$

By construction the map $\mathbb{Z}G^m \to \mathbb{Z}\overline{E}$ factors through a surjection $\mathbb{Z}G^m \to H_1(N;\mathbb{Z})$. Therefore, the exactness of (2.3) ensures the exactness of (2.4). It follows that Y is 1-acyclic.

Let V_0 and E_0 be sets of orbit representatives of vertices and edges in Y. Now, applying Proposition 2.3.1 then (2.2), we have that

$$\begin{aligned}\chi^{(2)}(F) + m &= \sum_{v\in V_0} \frac{1}{|G_v|} - \sum_{e\in E_0} \frac{1}{|G_e|} + m \\ &= b_0^{(2)}(Y) - b_1^{(2)}(Y) + b_2^{(2)}(Y).\end{aligned}$$

Lemma 2.2.1 with $n = 2$, yields

$$\begin{aligned}\chi^{(2)}(F) + m &= b_0^{(2)}(EG\times Y) - b_1^{(2)}(EG\times Y) + b_2^{(2)}(EG\times Y) \\ &\geqslant b_0^{(2)}(EG\times Y) - b_1^{(2)}(EG\times Y).\end{aligned}$$

Applying [11, Theorem 6.54(1a)] to the projection $f : EG \times Y \to EG$ with $n = 2$ (note that we are using the fact Y is 1-acyclic), we obtain $b_i^{(2)}(EG\times Y) = b_i^{(2)}(EG)$ for $i = 0, 1$. Recalling (2.1), we therefore have

$$\chi^{(2)}(F) + m \geqslant b_0^{(2)}(G) - b_1^{(2)}(G).$$

Let $k = \chi^{(2)}(F) + m$. If $k \leqslant 0$, then $b_0^{(2)}(G) - b_1^{(2)}(G) \leqslant 0$ and so G is infinite, this proves (i). Now, assume $k < 0$. In this case G is infinite and therefore $b_0^{(2)}(G) = 0$. It follows that $b_1^{(2)}(G) \geqslant -k > 0$, this proves (ii).

If G is finite, then $b_0^{(2)}(G) = \frac{1}{|G|}$, $b_1^{(2)}(G) = 0$, and $k > 0$. In particular, $k \geqslant \frac{1}{|G|} > 0$ and (iii) follows. □

2.4 On the ℓ^2-invariants for certain groups acting on trees

Proof of Theorem 2.E. Let V and E denote sets of representatives of F-orbits of vertices and edges for the action of F on the tree. We consider the relevant part of the E^1-page for the F-equivariant spectral sequence (see Chapter VII.9 of [6]) applied to the tree:

$$\begin{array}{c|cc}
1 & \bigoplus_{v\in V} H_1^F(F\times_{F_v} EF_v;\mathcal{U}F) & 0 \\
0 & \bigoplus_{v\in V} H_0^F(F\times_{F_v} EF_v;\mathcal{U}F)) \xleftarrow[d^1]{} & \bigoplus_{e\in E} H_0^F(F\times_{F_e} EF_e;\mathcal{U}F)) \\
\hline
 & 0 & 1
\end{array}$$

If F is finite then $b_1^{(2)}(F)=0$, so d^1 is injective and $E^2_{1,0}=0$. The result follows from the fact $E^1_{0,1}=0$.

Now, assume F is infinite, then d^1 is surjective since $b_0^{(2)}(F)=0$. Thus,

$$\dim_{\mathcal{U}F}(\mathrm{Ker}(d^1)) = \sum_{e\in E} b_0^{(2)}(F_e) - \sum_{v\in V} b_0^{(2)}(F_v).$$

Now, the spectral sequence obviously collapses on the E^2-page and $E^1_{0,1}=E^2_{0,1}$. Since von Neumann dimension is additive over short exact sequences, we have

$$\begin{aligned} b_1^{(2)}(F) &= \dim_{\mathcal{U}F}(\mathrm{Ker}(d^1)) + \dim_{\mathcal{U}F}(E^2_{0,1}) \\ &= \left(\sum_{e\in E} b_0^{(2)}(F_e) - \sum_{v\in V} b_0^{(2)}(F_v)\right) + \sum_{v\in V} b_1^{(2)}(F_v), \end{aligned}$$

and the result follows. □

References

[1] D. Allcock. Spotting infinite groups. *Math. Proc. Cambridge Philos. Soc.*, 125(1):39–42, 1999.

[2] U. Bader, A. Furman, and R. Sauer. Weak notions of normality and vanishing up to rank in L^2-cohomology. *Int. Math. Res. Not. IMRN*, (12):3177–3189, 2014.

[3] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[4] M. E. B. Bekka and A. Valette. Group cohomology, harmonic functions and the first L^2-Betti number. *Potential Anal.*, 6(4):313–326, 1997.

[5] E. Breuillard, M. Kalantar, M. Kennedy, and N. Ozawa. C^*-simplicity and the unique trace property for discrete groups. *Publ. Math. Inst. Hautes Études Sci.*, 126:35–71, 2017.

[6] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982.

[7] N. P. Brown, K. J. Dykema, and K. Jung. Free entropy dimension in amalgamated free products. *Proc. Lond. Math. Soc. (3)*, 97(2):339–367, 2008. With an appendix by Wolfgang Lück.

[8] I. Chatterji and G. Mislin. Hattori-Stallings trace and Euler characteristics for groups. In *Geometric and cohomological methods in group theory*, volume 358 of *London Math. Soc. Lecture Note Ser.*, pages 256–271. Cambridge Univ. Press, Cambridge, 2009.

[9] I. M. Chiswell. Exact sequences associated with a graph of groups. *J. Pure Appl. Algebra*, 8(1):63–74, 1976.

[10] A. Kar and G. Niblo. Some non-amenable groups. *Pub. Mat.*, 56(1):255–259, 2012.

[11] W. Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[12] D. Osin. On acylindrical hyperbolicity of groups with positive first ℓ^2-Betti number. *Bull. Lond. Math. Soc.*, 47(5):725–730, 2015.

[13] J. Peterson and A. Thom. Group cocycles and the ring of affiliated operators. *Invent. Math.*, 185:561–592, 2011.

[14] A. Thom. Low degree bounded cohomology and L^2-invariants for negatively curved groups. *Groups Geom. Dyn.*, 3(2):343–358, 2009.

[15] R. M. Thomas. Cayley graphs and group presentations. *Math. Proc. Cambridge Philos. Soc.*, 103(3):385–387, 1988.

[16] K. Tsouvalas. Euler characteristics on virtually free products. *Comm. Algebra*, 46(8):3397–3412, 2018.

Paper 3

ON THE EQUIVARIANT K- AND KO-HOMOLOGY OF SOME SPECIAL LINEAR GROUPS

SAM HUGHES

ABSTRACT. We compute the equivariant KO-homology of the classifying space for proper actions of $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$. We also compute the Bredon homology and equivariant K-homology of the classifying spaces for proper actions of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ for each prime p. Finally, we prove the Unstable Gromov-Lawson-Rosenberg Conjecture for a large class of groups whose maximal finite subgroups are odd order and have periodic cohomology.

3.1 Introduction

There has been considerable interest in the Baum-Connes conjecture, which states that for a group Γ a certain 'assembly map', from the equivariant K-homology of the classifying space for proper actions $\underline{E}\Gamma$ to the topological K-theory of the reduced group C^*-algebra, is an isomorphism [3]. The Baum-Connes conjecture is known to hold for several families of groups, including word-hyperbolic groups, CAT(0)-cubical groups and groups with the Haagerup property. An excellent survey can be found in [2].

The Baum–Connes Conjecture. *Let Γ be a discrete group, then the following assembly map is an isomorphism*

$$\mu : K_*^\Gamma(\underline{E}\Gamma) \to K_*^{\mathrm{top}}(C_r^*(\Gamma)).$$

There is also a 'real' Baum-Connes conjecture which predicts that an assembly map from the equivariant KO-homology of $\underline{E}\Gamma$ to the topological K-theory of the real group C^*-algebra is an isomorphism. It is known that the Baum-Connes Conjecture implies the Real Baum-Connes Conjecture [4].

The Real Baum–Connes Conjecture. *Let Γ be a discrete group, then the following assembly map is an isomorphism*

$$\mu_{\mathbb{R}} : KO_*^\Gamma(\underline{E}\Gamma) \to KO_*^{\mathrm{top}}(C_r^*(\Gamma)).$$

In spite of the interest, to date there have been very few computations of K^Γ- and KO^Γ-homology. Indeed, for K^Γ-homology there are complete calculations for one relator groups [35], NEC groups [32], some Bianchi groups and hyperbolic reflection groups [30] [38] [39], some Coxeter groups [16] [46] [45], Hilbert modular groups [44], $\mathrm{SL}_3(\mathbb{Z})$ [47], and $\mathrm{PSL}_4(\mathbb{Z})$ [10]. Explicit assembly maps have also been computed for solvable Baumslag-Solitar groups [37], lamplighter groups of finite groups [18] and certain wreath products

[36] [31]. For KO^Γ-homology the author is aware of two complete computations; the first, due to Davis and Lück, on a family of Euclidean crystallographic groups [14], and the second, due to Mario Fuentes-Rumí, on simply connected graphs of cyclic groups of odd order and of some Coxeter groups [17].

In this paper we compute the equivariant K-homology of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ and the equivariant KO-homology of $\mathrm{SL}_3(\mathbb{Z})$. We give the relevant background and the connection to Bredon homology in Section 3.2.

The calculation for KO^Γ-homology is of particular interest because it is (to the author's knowledge) the first computation of KO^Γ_* for a property (T) group. For background on property (T) the reader may consult the monograph [5]. This interest stems from the fact that property (T) is a strong negation of the Haagerup property which implies Baum-Connes [21]. Moreover, the (real) Baum-Connes conjecture is still open for $\mathrm{SL}_n(\mathbb{Z})$ when $n \geqslant 3$. We note that there are counterexamples for the Baum-Connes conjecture for groupoids constructed from $\mathrm{SL}_3(\mathbb{Z})$ and more generally a discrete group with property (T) for which the assembly map is known to be injective [22].

Theorem 3.A (Theorem 3.3.2)**.** *Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^\Gamma(\underline{E}\Gamma) = \mathbb{Z}^8, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^8, \quad 0, \quad \mathbb{Z}^8, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Applying a Künneth type theorem [46, Theorem 3.6] to the isomorphism $\mathrm{GL}_3(\mathbb{Z}) \cong \mathrm{SL}_3(\mathbb{Z}) \times \mathbb{Z}_2$ on the level of Bredon homology, we obtain the following result for $\mathrm{GL}_3(\mathbb{Z})$.

Corollary 3.B (Theorem 3.3.3)**.** *Let $\Gamma = \mathrm{GL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^\Gamma(\underline{E}\Gamma) = \mathbb{Z}^{16}, \quad \mathbb{Z}_2^{16}, \quad \mathbb{Z}_2^{16}, \quad 0, \quad \mathbb{Z}^{16}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

We also consider $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ or $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$, for p a prime, computing the equivariant K-homology groups $K_n^\Gamma(\underline{E}\Gamma)$. There has been considerable interest in determining homological properties of the groups $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{m}])$ and groups related to them [1] [9] [23]. It appears, however, that even with computer based methods the problem of determining the cohomology of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{m}])$ for m a product of 3 primes is out of reach [9]. In Lemma 3.5.4 we give a short proof of the Baum-Connes conjecture for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ and so we obtain the topological K-theory of the reduced group C^*-algebra of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ as well.

Theorem 3.C (Theorem 3.5.6)**.** *Let p be a prime and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then $K_n^\Gamma(\underline{E}\Gamma)$ is a free abelian group with rank as given in Table 3.1. Moreover, since the Baum-Connes Conjecture holds for Γ we have $K_*^\Gamma(\underline{E}\Gamma) \cong K_*^{top}(C_r^*(\Gamma))$.*

$p \equiv$	2	3	1 (mod 12)	5 (mod 12)	7 (mod 12)	11 (mod 12)
$n=0$	7	6	$4+\frac{1}{6}(p-7)$	$6+\frac{1}{6}(p+1)$	$5+\frac{1}{6}(p-1)$	$7+\frac{1}{6}(p+7)$
$n=1$	0	0	3	1	2	0

TABLE 3.1: $\mathbb{Z}$-rank of the equivariant K-homology of the classifying space for proper actions of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for p prime.

Theorem 3.D (Theorem 3.5.7)**.** *Let p be a prime and $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Then $K_n^\Gamma(\underline{E}\Gamma)$ is additively isomorphic to the direct sum of two copies of the corresponding equivariant K-homology group of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.*

Finally, we will give a proof of the Unstable Gromov-Lawson-Rosenberg Conjecture for positive scalar curvature for a large class of groups whose torsion subgroups have periodic cohomology. The statement and background concerning this conjecture is given in Section 3.6. However, we will introduce the following notation now before the theorem statement. We say a group Γ satisfies:

(M) If every finite subgroup is contained in a unique maximal finite subgroup.

(NM) If M is a maximal finite subgroup of Γ, then the normaliser $N_\Gamma(M)$ of M is equal to M.

(BC) If Γ satisfies the Baum-Connes conjecture.

(PFS) If all maximal finite subgroups of Γ are odd order and have periodic cohomology.

A large number of arithmetic groups satisfy the following theorem including many finite index subgroups $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$ and Hilbert modular groups. We will detail a number of additional examples in Section 3.6.

Theorem 3.E (Theorem 3.6.2)**.** *Let Γ be a group satisfying (BC), (M), (NM) and (PFS). If $\underline{B}\Gamma$ is finite and has dimension at most 9, then the Unstable Gromov-Lawson-Rosenberg Conjecture holds for Γ.*

In Section 3.2 we give the relevant background on equivariant K and KO-homology. In Section 3.3 we give the computations of the equivariant KO-homology for $\mathrm{SL}_3(\mathbb{Z})$ and $\mathrm{GL}_3(\mathbb{Z})$. In Section 3.4 we provide auxiliary computations of the equivariant K-homology of Fuchsian groups. In Section 3.5 we compute the equivariant K-homology of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Finally, in Section 3.6 we prove the results about the Unstable Gromov-Lawson-Rosenberg Conjecture and give a number of examples of groups satisfying the conjecture.

Acknowledgements

This paper contains material from the author's PhD thesis. The author would like to thank his PhD supervisor Professor Ian Leary for his guidance and support. He would also like to thank Jim Davis for pointing out Remark 3.6.3, Naomi Andrew, Guy Boyde, and Kevin Li for helpful conversations and the anonymous reviewer whose feedback greatly improved the exposition of this paper. This work was supported by the Engineering and Physical Sciences Research Council grant number 2127970.

3.2 Preliminaries

In this section we introduce the relevant background from Bredon homology and its interactions with equivariant K- and KO-homology. We follow the treatment given in Mislin's notes [35].

3.2.1 Classifying spaces for families

Let Γ be a discrete group. A Γ-CW complex X is a CW-complex equipped with a cellular Γ-action. We say the Γ action is proper if all of the cell stabilisers are finite.

Let $\mathcal{F}$ be a family of subgroups of Γ which is closed under conjugation and finite intersections. A *model* for the classifying space $E_{\mathcal{F}}\Gamma$ for the family $\mathcal{F}$ is a Γ-CW complex such that all cell stabilisers are in $\mathcal{F}$ and the fixed point set of every $H \in \mathcal{F}$ is contractible. This is equivalent to the following universal property: For every Γ-CW complex Y there is exactly one Γ-map $Y \to E_{\mathcal{F}}\Gamma$ up to Γ-homotopy.

In the case where $\mathcal{F} = \mathcal{FIN}$, the family of all finite subgroups of Γ, we denote $E_{\mathcal{FIN}}(\Gamma)$ by $\underline{E}\Gamma$. We call such a space, the classifying space for proper actions of Γ. Note that if Γ is torsion-free then $\underline{E}\Gamma = E\Gamma$.

3.2.2 Bredon homology

Let Γ be a discrete group and $\mathcal{F}$ be a family of subgroups. We define the *orbit category* $\mathbf{Or}_{\mathcal{F}}(\Gamma)$ to be the category with objects given by left cosets Γ/H for $H \in \mathcal{F}$ and morphisms the Γ-maps $\phi : \Gamma/H \to \Gamma/K$. A morphism in the orbit category is uniquely determined by its image $\varphi(H) = \gamma K$ and $\gamma H \gamma^{-1} \subseteq K$; conversely, each such $\gamma \in \Gamma$ defines a G-map.

A *(left) Bredon module* is a covariant functor $M : \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of Abelian groups. Consider a Γ-CW complex X and a family of subgroups $\mathcal{F}$

containing all cell stabilisers. Let M be a Bredon module and define the *Bredon chain complex with coefficients* in M as follows:

Let $\{c_\alpha\}$ be a set of orbit representatives of the n-cells in X and let Γ_α denote the stabiliser of the cell α. The nth chain group is then

$$C_n := \bigoplus_\alpha M(\Gamma/\Gamma_{c_\alpha}).$$

If $\gamma c'$ is an $(n-1)$-cell in the boundary of c, then $\gamma^{-1}\Gamma_c\gamma \subseteq \Gamma_{c'}$. This defines a Γ-map $\varphi : \Gamma/\Gamma_c \to \Gamma/\Gamma_{c'}$, which in turn gives an induced homomorphism $M(\varphi) : M(\Gamma/\Gamma_c) \to M(\Gamma/\Gamma_{c'})$. Therefore, we obtain a differential $\partial : C_n \to C_{n-1}$. Taking homology of the chain complex (C_*, ∂) gives the *Bredon homology groups* $H_n^{\mathcal{F}}(X; M)$. A right Bredon module and Bredon cohomology are defined analogously using contravariant functors.

3.2.3 Equivariant K-homology

The original definition of equivariant K-homology used Kasparov's KK-theory [3]. There is also homotopy theoretic approach using spaces and spectra over the orbit category due to Davis-Lück [12]. We will highlight the details we need.

Let Γ be a discrete group. In the context of the Baum-Connes conjecture we are specifically interested in the case where $X = \underline{E}\Gamma$, $\mathcal{F} = \mathcal{FIN}$ and $M = \mathcal{R}_\mathbb{C}$ the complex representation ring. We consider $\mathcal{R}_\mathbb{C}(-)$ as a Bredon module in the following way: For $\Gamma/H \in \mathbf{Or}_\mathcal{F}(\Gamma)$ set $\mathcal{R}_\mathbb{C}(\Gamma/H) := \mathcal{R}_\mathbb{C}(H)$, the ring of complex representations of the finite group H. Morphisms are then given by induction of representations.

We note that $\mathcal{R}_\mathbb{C}(\Gamma) := H_0^{\mathcal{F}}(\Gamma) = \operatorname{colim}_{\Gamma/H \in \mathbf{Or}(\Gamma)} R_\mathbb{C}(H)$. In the case that Γ has finitely many conjugacy classes of finite subgroups, $\mathcal{R}_\mathbb{C}(\Gamma)$ is a finitely generated quotient of $\bigoplus \mathcal{R}_\mathbb{C}(H)$, where H runs over conjugacy classes of finite subgroups.

We now exhibit the connection between Bredon homology and $K_*^\Gamma(\underline{E}\Gamma)$, the equivariant K-homology of the classifying space for proper actions. Indeed, for each subgroup $H \leqslant \Gamma$ equivariant K-homology satisfies

$$K_n^\Gamma(\Gamma/H) = K_n^{\text{top}}(C_r^*(H)).$$

In the case H is a finite subgroup we have $C_r^*(H) = \mathbb{C}H$, $K_0^\Gamma(\Gamma/H) = K_0^{\text{top}}(\mathbb{C}H) = \mathcal{R}_\mathbb{C}(H)$, and $K_1^\Gamma(\Gamma/H) = K_1^{\text{top}}(\mathbb{C}H) = 0$. The remaining K^Γ groups are given by 2-fold Bott periodicity. This allows us to view $K_n^\Gamma(-)$ as a Bredon module over $\mathbf{Orb}_{\mathcal{FIN}}(\Gamma)$.

We may use an equivariant Atiyah-Hirzebruch spectral sequence to compute the K^Γ-homology of a proper Γ-CW-complex X from its Bredon homology.

Theorem 3.2.1. [35, Page 50] *Let Γ be a group and X a proper Γ-CW complex, then* *there is an Atiyah-Hirzebruch type spectral sequence*

$$E^2_{p,q} := H^{\mathcal{FIN}}_p(X; K^\Gamma_q(-)) \Rightarrow K^\Gamma_{p+q}(X).$$

3.2.4 Equivariant KO-homology

In this section we summarise the material from [14, Section 9] which we will require for our calculations. Again fixing a discrete group Γ and $\mathcal{F} = \mathcal{FIN}$, we introduce two more Bredon modules, the real representation ring $\mathcal{R}_\mathbb{R}(-)$, and the quaternionic representation ring $\mathcal{R}_\mathbb{H}\mathbf{H}(-)$. These are defined on $\mathbf{Or}_\mathcal{F}(\Gamma)$ in exactly the same way as the complex representation ring. We have natural transformations between the functors. Indeed for a finite subgroup $H \leqslant \Gamma$ we have a diagram:

$$\mathcal{R}_\mathbb{R}(H) \underset{\rho}{\overset{\nu}{\rightleftarrows}} \mathcal{R}_\mathbb{C}(H) \underset{\eta}{\overset{\sigma}{\rightleftarrows}} \mathcal{R}_\mathbb{H}\mathbf{H}(H).$$

Note that the diagram does not commute. For instance let $\mathbf{1} \in \mathcal{R}_\mathbb{R}(H)$ denote the trivial representation, then $\rho\nu(\mathbf{1}) = 2 \cdot (\mathbf{1})$.

For a real representation ψ, the *complexification* is $\nu(\psi) = \psi \otimes \mathbb{C}$. For a complex representation ϕ, the *symplectification* is $\sigma(\phi) = \phi \otimes \mathbb{H}\mathbf{H}$. Going the other way, for an n-dimensional quaternionic representation ω, the *complexification* is $\eta(\omega) = \eta$ considered as $2n$-dimensional complex representation. Similarly, for an n-dimensional complex representation ϕ, the *realification* is $\rho(\phi) = \phi$ considered as a $2n$-dimensional real representation. Note that any composition of the x-ification natural transformations with the same source and target is necessarily not the identity.

The situation for the equivariant KO-homology, denoted $KO^\Gamma_*(-)$, is similar to the equivariant K-homology but more complicated. For a subgroup $H \leqslant \Gamma$ we set $KO^\Gamma_n(\Gamma/H) = KO^{\text{top}}_n(C^*_r(H))$. By [8, Section 1.2], in the case that H is a finite subgroup we have that

$$KO^\Gamma_n(\Gamma/H) = KO^{\text{top}}_n(C^*_r(H)) = \begin{cases} \mathcal{R}_\mathbb{R}(H) & n = 0, \\ \mathcal{R}_\mathbb{R}(H)/\rho(\mathcal{R}_\mathbb{C}(H) & n = 1, \\ \mathcal{R}_\mathbb{C}(H)/\eta(\mathcal{R}_\mathbb{H}\mathbf{H}(H)) & n = 2, \\ 0 & n = 3, \\ \mathcal{R}_\mathbb{H}\mathbf{H}(H) & n = 4, \\ \mathcal{R}_\mathbb{H}\mathbf{H}(H)/\sigma(\mathcal{R}_\mathbb{C}(H)) & n = 5, \\ \mathcal{R}_\mathbb{C}(H)/\nu(\mathcal{R}_\mathbb{R}(H)) & n = 6, \\ 0 & n = 7, \end{cases}$$

with the remaining groups given by 8-fold Bott-periodicity. For X a proper Γ-space, the Atiyah-Hirzebruch spectral sequence from before now takes the form

$$E^2_{p,q} := H^{\mathcal{FIN}}_p(X; KO^{\Gamma}_q(-)) \Rightarrow KO^{\Gamma}_{p+q}(X).$$

3.2.5 Spectra and homotopy

The section gives an alternative Γ-equivariant homotopy theoretic viewpoint. Now, we consider Γ-equivariant homology theories as functors $\mathbf{E}\colon \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Spectra}$. Technically, to avoid functorial problems one must take composite functors through the categories C^*-**Cat** and **Groupoids**. We do not concern ourselves with this complication and refer the reader to [12] and [15].

Instead we will take for granted that there is a composite functor

$$\mathbf{KO}\colon \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Spectra}$$

which satisfies $\pi_n \mathbf{KO}(\Gamma/H) = KO^{\text{top}}_n(C^*_r(H))$. When $\mathcal{F} = \mathcal{FIN}$ this perspective gives a homotopy theoretic construction of the (real) Baum-Connes assembly map. Indeed, we have maps

$$B\Gamma_+ \wedge \mathbf{KO} \simeq \operatorname*{hocolim}_{\mathbf{Or}_{\mathcal{TRV}}(\Gamma)} \mathbf{KO} \to \operatorname*{hocolim}_{\mathbf{Or}_{\mathcal{F}}(\Gamma)} \mathbf{KO} \to \operatorname*{hocolim}_{\mathbf{Or}_{\mathcal{ALL}}(\Gamma)} \mathbf{KO} \simeq \mathbf{KO}(C^*_r(\Gamma;\mathbb{R})).$$

The assembly map $\mu_{\mathbb{R}}$ is then π_n applied to the composite.

3.2.6 Group C^*-algebras and KK-theory

In this section we give a brief outline of Kasparov's KK-theory, the material here will not be used elsewhere in the paper. The theory was introduced by Kasparov in [26] [27] in relation to the Novikov Conjecture. The original formulation of the Baum-Connes Conjecture using KK-theory was given in [3].

For a C^*-algebra A define $\mathbf{M}_\infty(A)$ to be the direct limit of sets of $(n \times n)$-matrices over A as $n \to \infty$. Similarly, define $\mathrm{GL}_\infty(A)$ to be the direct limit of groups of invertible $(n \times n)$-matrices over A.

Topological or operator K-theory is a 2-periodic homology theory of unital C^*-algebras denoted $K^{\text{top}}_*(-)$. The zeroth K-group of a unital C^*-algebra A is defined to be the Grothendieck group of the set of projections in $\mathbf{M}_\infty(A)$ up to Murray von Neumann equivalence. The first K-group is defined to be $\mathrm{GL}_\infty(A)/\mathrm{GL}_\infty(A)_0$, where $\mathrm{GL}_\infty(A)_0$ is the path component of the identity.

An alternative formulation is given by Kasparov's bifunctor $KK(-,-)$. For any two C^*-algebras A and B there is an abelian group $KK(A,B)$. An element of $KK(A,B)$ is a homotopy class of (A,B)-Fredholm bimodules (see [2, Section 3] for the precise definition). The zeroth K-group of A from before is recovered as $KK(\mathbb{C},A)$ and the first K-group is recovered as $KK(C_0(\mathbb{R}),A)$.

Let Γ be a discrete group. The *reduced C^*-algebra* of Γ, denoted $C_r^*(\Gamma)$, is the norm closure of the algebra of bounded operators on $\ell^2(\Gamma)$ by the left regular representation of Γ. The algebra and its K-groups are intimately related with the theory of elliptic operators on manifolds M with fundamental group Γ. For more information the reader should consult the survey [2] and the references therein.

3.3 Equivariant KO-homology of $\mathrm{SL}_3(\mathbb{Z})$

3.3.1 A classifying space for proper actions

A model for $X = \underline{E}\mathrm{SL}_3(\mathbb{Z})$ can be constructed as a $\mathrm{SL}_3(\mathbb{Z})$-equivariant deformation retract of the symmetric space $\mathrm{SL}_3(\mathbb{R})/O(3)$. This construction has been detailed several times in the literature ([50, Theorem 2], [20, Theorem 2.4] or [47, Theorem 13]), so rather than detailing it again here, we simply extract the relevant cell complex and cell stabilisers. Specifically, we follow the notation of Sánchez-García [47] and collect the information in Table 3.2.

3.3.2 Proof of Theorem 3.A

The calculation of the equivariant KO-groups will require the following proposition and an analysis of the representation theory of the finite subgroups of $\mathrm{SL}_3(\mathbb{Z})$. We remark that one could prove a dozen subtle variations on the theme of the following proposition. However, rather than do this we offer the slogan: "Computations with coefficients in $KO_n^\Gamma(-)$ can be greatly simplified by looking for chain maps to the Bredon chain complex with coefficients in $\mathcal{R}_\mathbb{C}(-)$."

Proposition 3.3.1. *Let Γ be a discrete group, $\mathcal{F} = \mathcal{FIN}$ and suppose X is a proper Γ-CW complex with finitely many Γ orbits of cells in each dimension. Assume that for every cell stabiliser the real, complex and quarternionic character tables are equal, then the Atiyah-Hirzebruch spectral sequence converging to $KO_*^\Gamma(X)$ has E^2-page isomorphic to*

$$E^2_{p,q} = H_p^{\mathcal{FIN}}(X; K_0^\Gamma) \otimes KO_q(*) \oplus \mathrm{Tor}_1^\mathbb{Z}[H_{p-1}^{\mathcal{FIN}}(X; K_0^\Gamma), KO_q(*)]$$

where for $q = 0, \ldots, 7$ we have

$$KO_q(*) = \mathbb{Z}, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2, \quad 0, \quad \mathbb{Z}, \quad 0, \quad 0, \quad 0$$

Dimension	Cell	Boundary	Stabiliser
3	T_1	$-t_1 + t_2 - t_3 + t_4 - t_5$	$\{1\}$
2	t_1	$e_1 - e_2 - e_4$	$\mathbb{Z}_2$
	t_2	$e_4 - e_5 + e_6$	$\{1\}$
	t_3	$e_6 - e_7 + e_8$	$\mathbb{Z}_2^2$
	t_4	$e_1 - e_3 + e_5 + e_8$	$\mathbb{Z}_2$
	t_5	$e_2 - e_3 + e_6 - e_6 + e_7$	$\mathbb{Z}_2$
1	e_1	$v_1 - v_2$	$\mathbb{Z}_2^2$
	e_2	$v_3 - v_1$	D_3
	e_3	$v_5 - v_1$	D_3
	e_4	$v_3 - v_2$	$\mathbb{Z}_2$
	e_5	$v_4 - v_2$	$\mathbb{Z}_2$
	e_6	$v_4 - v_3$	$\mathbb{Z}_2^2$
	e_7	$v_5 - v_3$	D_4
	e_8	$v_5 - v_4$	D_4
0	v_1	-	$\mathrm{Sym}(4)$
	v_2		D_6
	v_3		$\mathrm{Sym}(4)$
	v_4		D_4
	v_5		$\mathrm{Sym}(4)$

TABLE 3.2: Cell structure and stabilisers of a model for $\underline{E}\mathrm{SL}_3(\mathbb{Z})$.

and the remaining groups are given by 8*-fold Bott-periodicity.*

Note that the Tor terms vanish except possibly when $q = 1$ or 2.

Proof. Since the three character tables are equal, the complexification from $\nu : \mathcal{R}_{\mathbb{R}} \to \mathcal{R}_{\mathbb{C}}$ and the symplectification from $\sigma : \mathcal{R}_{\mathbb{C}} \to \mathcal{R}_{\mathbb{H}}\mathbf{H}$ are isomorphisms. In the other direction, the complexification from $\eta : \mathcal{R}_{\mathbb{H}}\mathbf{H} \to \mathcal{R}_{\mathbb{C}}$ and the realification from $\rho : \mathcal{R}_{\mathbb{C}} \to \mathcal{R}_{\mathbb{R}}$ correspond to multiplication by 2. We will now compute each row of the spectral sequence in turn.

$\mathbf{q = 0}$: We have $E^2_{p,0} = H_p^{\mathcal{FIN}}(X; KO_0^{\Gamma})$ which is exactly equal to $H_p^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{R}})$, the result follows from the isomorphism $H_p^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{R}}) \cong H_p^{\mathcal{FIN}}(X; K_0^{\Gamma}) \otimes \mathbb{Z}$ and the vanishing of the Tor group.

$\mathbf{q = 1}$: The realification $\rho : \mathcal{R}_{\mathbb{C}} \to \mathcal{R}_{\mathbb{R}}$ is multiplication by 2, thus the cokernel of the map

$$\rho_* : C_*^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{C}}) \to C_*^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{R}})$$

is the modulo 2 reduction of $C_*^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{R}})$. Consider $C_*^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{R}})$ as a chain complex of abelian groups. The result follows from the Universal Coefficient Theorem in homology with $\mathbb{Z}_2$ coefficients applied to the homology of the chain complex $C_*^{\mathcal{FIN}}(X; \mathcal{R}_{\mathbb{R}})$.

$\mathbf{q=2}$: The complexification $\eta : \mathcal{R}_{\mathbb{H}}\mathbf{H} \to \mathcal{R}_{\mathbb{C}}$ is multiplication by 2 and $C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{C}})$ is isomorphic to $C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{R}})$. The result now follows as in the case $q=1$.

$\mathbf{q=3}$: Immediate since $KO_3^{\Gamma}(-)=0$.

$\mathbf{q=4}$: Since ν and σ are both isomorphisms, their composition gives an isomorphism of Bredon chain complexes $C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{R}}) \cong C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{H}}\mathbf{H})$. The result now follows as in the $q=0$ case.

$\mathbf{q=5}$: Since σ is an isomorphism, the cokernel of the map

$$\sigma_* : C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{C}}) \to C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{H}}\mathbf{H})$$

vanishes. The result follows.

$\mathbf{q=6}$: Since ν is an isomorphism, the cokernel of the map

$$\nu_* : C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{R}}) \to C_*^{\mathcal{FIN}}(X;\mathcal{R}_{\mathbb{C}})$$

vanishes. The result follows.

$\mathbf{q=7}$: Immediate since $KO_7^{\Gamma}(-)=0$. □

Theorem 3.3.2 (Theorem 3.A)**.** *Let* $\Gamma = \mathrm{SL}_3(\mathbb{Z})$*, then for* $n=0,\ldots,7$ *we have*

$$KO_n^{\Gamma}(\underline{E}\Gamma) = \mathbb{Z}^8, \quad \mathbb{Z}_2^8, \quad \mathbb{Z}_2^8, \quad 0, \quad \mathbb{Z}^8, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8*-fold Bott-periodicity.*

Proof. Let $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $\mathcal{F} = \mathcal{FIN}$ and $X = \underline{E}\mathrm{SL}_3(\mathbb{Z})$. We can now complete the calculation for the equivariant KO-homology groups. First, we recap the calculation of the Bredon chain complex with complex representation ring coefficients due to Sánchez-García. We have a chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z}^{11} \xrightarrow{\partial_2} \mathbb{Z}^{28} \xrightarrow{\partial_1} \mathbb{Z}^{26} \longrightarrow 0$$

where

$$\partial_3 \sim \begin{bmatrix} 1 & \mathbf{0}_{1\times 10} \end{bmatrix}, \quad \partial_2 \sim \begin{bmatrix} I_{10} & \mathbf{0}_{10\times 18} \\ \mathbf{0}_{10\times 1} & \mathbf{0}_{1\times 18} \end{bmatrix}, \quad \text{and} \quad \partial_1 \sim \begin{bmatrix} I_{18} & \mathbf{0}_{18\times 8} \\ \mathbf{0}_{10\times 18} & \mathbf{0}_{10\times 8} \end{bmatrix}.$$

Therefore, the homology groups of the chain complex are isomorphic to $\mathbb{Z}^8$ in dimension 0 and to 0 in every other dimension.

Now, the cell stabiliser subgroups of $\mathrm{SL}_3(\mathbb{Z})$ acting on X are isomorphic to $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, D_3, D_4, $\mathrm{Sym}(4)$ and D_6. Each of these satisfies the conditions of the proposition above. This is easily checked by computing the Schur indicators of each of the irreducible characters of each group. Since the Schur indicator equals 1 in every case we conclude the three character tables for each group are equal (see for instance [19, Exercise 3.38]). Applying this to the previous calculation we obtain a single non-trivial column when $p = 0$ in the Atiyah-Hirzebruch spectral sequence and so it collapses trivially. □

Corollary 3.3.3 (Corollary 3.B). *Let $\Gamma = \mathrm{GL}_3(\mathbb{Z})$, then for $n = 0, \dots, 7$ we have*

$$KO_n^{\Gamma}(\underline{E}\Gamma) = \mathbb{Z}^{16}, \quad \mathbb{Z}_2^{16}, \quad \mathbb{Z}_2^{16}, \quad 0, \quad \mathbb{Z}^{16}, \quad 0, \quad 0, \quad 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Proof. First, note that the direct product of $\mathbb{Z}_2$ with any of the cell stabiliser subgroups of $\mathrm{SL}_3(\mathbb{Z})$ still satisfies the conditions of the Proposition 3.3.1. Now, we may compute the E^2-page of the associated Atiyah Hirzebruch spectral sequence by applying the Künneth formula [46, Theorem 3.6] to the calculation of each row of the E^2-page for $\underline{E}\mathrm{SL}_3(\mathbb{Z})$. Since the spectral sequence is concentrated in a single column we have isomorphisms

$$KO_n^{\mathrm{GL}_3(\mathbb{Z})}(\underline{E}\mathrm{GL}_3(\mathbb{Z})) \cong KO_n^{\mathrm{SL}_3(\mathbb{Z})}(\underline{E}\mathrm{SL}_3(\mathbb{Z})) \otimes KO_n^{\mathbb{Z}_2}(*),$$

from which the result is immediate. □

3.4 Equivariant K-homology of Fuchsian groups

In this section we compute the equivariant K-homology of every finitely generated Fuchsian group, that is, a finitely generated discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. The reason for this apparent detour is that we will later split the groups $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ as amalgamated free products of certain Fuchsian subgroups. Thus, we can use a Mayer-Vietoris type argument to compute their K-homology.

Note that Theorem 3.4.1(a) was computed in [32] along with a more general result for cocompact NEC groups. Moreover, their integral cohomology was determined by the author in [23]. An introduction to Fuchsian groups is provided by [28].

The computation is made easier by the fact that every finitely generated Fuchsian group is described by piece of combinatorial data called a *signature* [28, Chapter 4.3]. Indeed, a Fuchsian group of *signature* $[g, s; m_1, \dots, m_r]$ has a presentation with generators

$$a_1, \dots, a_{2g}, c_1, \dots, c_s, d_1, \dots, d_r$$

and relations

$$\prod_{i=1}^{g}[a_i, a_{g+i}] \prod_{j=1}^{r} d_j \prod_{k=1}^{s} c_k = d_1^{m_1} = \cdots = d_r^{m_r} = 1,$$

and acts on the hyperbolic plane $\mathbb{R}\mathbf{H}^2$ with a $4g+2s+2r$ sided fundamental polygon. The tessellation of the polygon under the group action has $1+s+r$ orbits of vertices, s of which are on the boundary $\partial\mathbb{R}\mathbf{H}^2$, $2g+s+r$ orbits of edges and 1 orbit of faces. All edge and face stabilisers are trivial. All vertex stabilisers are trivial except for r orbits of vertices, each of which is stabilised by some $\mathbb{Z}_{m_j}$. Note that if $s=0$ we say Γ is *cocompact.*

The signature also describes a quotient 2-orbifold which is homeomorphic to a genus g surface with s points removed. The orbifold data is then given by the r marked points, each corresponding to one of the m_j, or equivalently a maximal conjugacy of finite subgroups.

If $r=0$, we do not write any m_j in the signature. In which case Γ has signature $[g,s;]$, is torsionfree, and isomorphic to either the fundamental group of a genus g surface, or a free group of rank $2g+s-1$.

Theorem 3.4.1. *Let Γ be a Fuchsian group of signature $[g,s;m_1,\dots,m_r]$.*

(a) If $s=0$ then,

$$K_n^\Gamma(\underline{E}\Gamma) = K_n(C_r^*(\Gamma)) = \begin{cases} \mathbb{Z}^{2-r+\sum_{j=1}^r m_j} & n \text{ even,} \\ \mathbb{Z}^{2g} & n \text{ odd.} \end{cases}$$

(b) If $s>0$ then,

$$K_n^\Gamma(\underline{E}\Gamma) = K_n(C_r^*(\Gamma)) = \begin{cases} \mathbb{Z}^{1-r+\sum_{j=1}^r m_j} & n \text{ even,} \\ \mathbb{Z}^{2g+s-1} & n \text{ odd.} \end{cases}$$

Proof of (a). Let Γ be a Fuchsian group of signature $[g,s;m_1,\dots,m_r]$ with $s=0$ and $\mathcal{F}=\mathcal{FIN}$. Since Γ satisfies the Baum-Connes conjecture [21] it is enough to compute the equivariant K-homology. The hyperbolic plane with the induced cell structure of the Γ action is a model for $\underline{E}\Gamma$ (see for instance [33]). Recall that the cell structure has $r+1$ orbits of vertices, $2g+r$ orbits of edges and exactly 1 orbit of 2-cells. One vertex v_0 is stabilised by the trivial group and for $j=1,\dots,r$ the vertex v_j is stabilised by $\mathbb{Z}_{m_j}$. Thus, we have a Bredon chain complex

$$0 \longleftarrow \mathbb{Z}\oplus\left(\bigoplus_{j=1}^r \mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j})\right) \xleftarrow{\partial_1} \mathbb{Z}^{2g+r} \xleftarrow{\partial_2} \mathbb{Z} \longleftarrow 0,$$

and substituting in $\mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j})=\mathbb{Z}^{m_j}$ we obtain

$$0 \longleftarrow \mathbb{Z}\oplus\left(\bigoplus_{j=1}^r \mathbb{Z}^{m_j}\right) \xleftarrow{\partial_1} \mathbb{Z}^{2g+r} \xleftarrow{\partial_2} \mathbb{Z} \longleftarrow 0.$$

We fix the following basis for each chain group: In degree 0 we have generators $x_{j,l}$, for $j = 1, \dots, r$ and $l = 1, \dots, m_j$, and the generator z. In degree 1 we have $a_1, \dots, a_{2g}$ and $y_1, \dots, y_r$, and in degree 2, the generator w. An easy calculation yields that $\partial_2(w) = 0$, $\partial_1(a_i) = 0$, and $\partial_1(y_j) = \sum_{l=1}^{m_j} x_{j,l} - z$. Thus,

$$H_n^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) = \begin{cases} \mathbb{Z}^{1+\sum_{j=1}^{r}(m_j-1)} & \text{if } n = 0; \\ \mathbb{Z}^{2g} & \text{if } n = 1; \\ \mathbb{Z} & \text{if } n = 2; \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence given in [35, Theorem 5.27] and we obtain $K_0^{\Gamma}(\underline{E}\Gamma) = H_0^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) \oplus H_2^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$ and $K_1^{\Gamma}(\underline{E}\Gamma) = H_1^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$. □

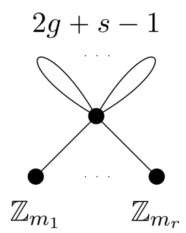

FIGURE 3.1: A graph of groups for a non-cocompact Fuchsian group.

Proof of (b). Let Γ be a Fuchsian group of signature $[g, s; m_1, \dots, m_r]$ with $s > 0$ and let $\mathcal{F} = \mathcal{FIN}$. In this case we can rearrange the presentation of Γ such that we have a splitting of Γ as an amalgamated free product $\Gamma \cong \mathbb{Z}^{s-1} * \mathbb{Z}_{m_1} * \dots * \mathbb{Z}_{m_r}$. Now, Γ splits as a finite graph of finite groups (Figure 3.1) and it is easy to see that the Bass-Serre tree of Γ is a model for $\underline{E}\Gamma$.

We will first compute the Bredon homology $H_*^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$ with coefficients in the representation ring, then apply the equivariant Atiyah-Hirzebruch spectral sequence. We have a Bredon chain complex

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r} \mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j}) \right) \xleftarrow{\partial} \mathbb{Z}^{2g+s-1} \longleftarrow 0,$$

substituting in $\mathcal{R}_{\mathbb{C}}(\mathbb{Z}_{m_j}) = \mathbb{Z}^{m_j}$ we obtain

$$0 \longleftarrow \mathbb{Z} \oplus \left(\bigoplus_{j=1}^{r} \mathbb{Z}^{m_j} \right) \xleftarrow{\partial} \mathbb{Z}^{2g+s-1} \longleftarrow 0.$$

Let the first non-zero term have generating set $\langle x_{j,l}, z \mid j = 1, \dots, r, \ l = 1, \dots, m_j \rangle$ and the second term $\langle a_1, \dots, a_{2g}, c_1, \dots, c_{s-1}, d_1, \dots, d_r \rangle$. It is easy to see the differential ∂ is given by $\partial(a_i) = \partial(b_i) = \partial(c_k) = 0$ and $\partial(d_j) = \sum_{l=1}^{m_j} x_{j,l} - z$. It follows that $H_0^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) = \mathbb{Z}^{1+\sum_{j=1}^{r}(m_j-1)}$, $H_1^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}}) = \mathbb{Z}^{2g+s-1}$ and 0 otherwise. The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence [35, Theorem 5.27]. In particular, we have $K_n^{\Gamma}(\underline{E}\Gamma) = H_n^{\mathcal{F}}(\underline{E}\Gamma; \mathcal{R}_{\mathbb{C}})$ for $n = 0, 1$. □

3.5 Computations for $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$

3.5.1 Preliminaries

In an abuse of notation, throughout this section we will denote the image $\{\pm A\}$ of a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ in $\mathrm{PSL}_2(\mathbb{R})$ by the matrix A. Recall that for p a prime we have $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$, where $\Gamma_0(p)$ is the level p Hecke principle congruence subgroup (see for instance Serre's book "Trees" [48]). The amalgamation is specified by two embeddings of the congruence subgroup $\Gamma_0(p)$ into $\mathrm{PSL}_2(\mathbb{Z})$. The first is given by

$$\Gamma_0(p) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \colon c \equiv 0 \pmod{p} \right\}$$

and the second via

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} a & pb \\ p^{-1}c & d \end{bmatrix}.$$

In light of this we will collect some facts about each of the groups in the amalgamation. We begin by recording (Table 3.3) the Fuchsian signatures and the associated Bredon homology for each of the groups $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z})$. Note that when $p \equiv 11 \pmod{12}$ the group $\Gamma_0(p)$ is free.

Lemma 3.5.1. *The signatures and Bredon homology groups of $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z})$ are given in Table 3.3.*

$\Gamma_0(p)$	Signature	$H_0^{\mathcal{F}}(\Gamma; R_{\mathbb{C}})$	$H_1^{\mathcal{F}}(\Gamma; R_{\mathbb{C}})$
$p = 2$	$[0, 2; 2]$	$\mathbb{Z}^2$	$\mathbb{Z}$
$p = 3$	$[0, 2; 3]$	$\mathbb{Z}^3$	$\mathbb{Z}$
$p \equiv 1 \pmod{12}$	$[0, \frac{1}{6}(p-7)+1; 2, 2, 3, 3]$	$\mathbb{Z}^7$	$\mathbb{Z}^{\frac{1}{6}(p-7)}$
$p \equiv 5 \pmod{12}$	$[0, \frac{1}{6}(p+1)+1; 2, 2]$	$\mathbb{Z}^3$	$\mathbb{Z}^{\frac{1}{6}(p+1)}$
$p \equiv 7 \pmod{12}$	$[0, \frac{1}{6}(p-1)+1; 3, 3]$	$\mathbb{Z}^5$	$\mathbb{Z}^{\frac{1}{6}(p-1)}$
$p \equiv 11 \pmod{12}$	$[0, \frac{1}{6}(p+7)+1;]$	$\mathbb{Z}$	$\mathbb{Z}^{\frac{1}{6}(p+7)}$
$\mathrm{PSL}_2(\mathbb{Z})$	$[0, 1; 2, 3]$	$\mathbb{Z}^4$	0

TABLE 3.3: Fuchsian signatures and Bredon homology groups of $\Gamma_0(p)$ for p prime.

Proof. Let $[0, s; m_1, \dots, m_r]$ be the signature of $\Gamma_0(p)$. We will first compute the ordinary cohomology groups of $\Gamma_0(p)$, then using these we will deduce the signatures, finally the Bredon homology may then be read off of Theorem 3.4.1(b). Our computation of the cohomology will be near identical to the computation in [1, Section 2]. The key difference is that the modules in [1] are for the lifts of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{R})$ whereas we are always working with the projectivised groups (see the discussion after [1, Proposition 2.2]). Note that the fact the signature of $\mathrm{PSL}_2(\mathbb{Z})$ is $[0, 1; 2, 3]$ is well known.

Let $G = \mathrm{PSL}_2(p)$ and let Q be the subgroup of equivalence classes of matrices with lower left hand entry equal to zero. Clearly, $Q \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{1}{2}(p-1)}$ (unless $p = 2, 3$ where $Q \cong \mathbb{Z}_p$). Each $\Gamma_0(p)$ fits into a short exact sequence with normal subgroup a congruence subgroup $\Gamma(p)$ isomorphic to a free group and quotient Q.

Now, recall [48, Example 4.2(c)] that $\mathrm{PSL}_2(\mathbb{Z})$ acts on a tree $\mathcal{T}$ with fundamental domain an edge. Moreover, G acts on $\mathcal{T}/\Gamma(p)$. The stabiliser subgroups for both actions are $\mathbb{Z}_2$ and $\mathbb{Z}_3$ for the vertices and trivial for the edges. It follows $\Gamma_0(p)$ acts freely on $EQ \times \mathcal{T}$ and so $EQ \times_Q \mathcal{T}/\Gamma_0(p)$ is a model for $B\Gamma_0(p)$.

Let C^* denote the cellular cochains on the B-CW complex $\mathcal{T}/\Gamma(p)$, then by [1, Section 2] we have the following isomorphisms of B-modules

$$C^0 := \mathbb{Z}[G/\mathbb{Z}_2] \mid_B \oplus \mathbb{Z}[G/\mathbb{Z}_3] \mid_B \quad \text{and} \quad C^1 := \mathbb{Z}[G] \mid_B .$$

As in [1, Section 2] we have a long exact sequence

$$\cdots \to H^n(\Gamma_0(p); \mathbb{Z}) \to H^n(B; C^0) \to H^n(B; C^1) \to H^{n+1}(\Gamma_0(p); \mathbb{Z}) \to \cdots$$

Since C^0 is a permutation module, $H^1(B; C^0) = 0$. Calculating ranks yields that $H^1(\Gamma_0(p); \mathbb{Z}) = \mathbb{Z}^{N(p)}$ where

$$N(p) = 1, \quad 1, \quad \frac{1}{6}(p-7), \quad \frac{1}{6}(p+1), \quad \frac{1}{6}(p-1), \quad \frac{1}{6}(p+7),$$

ordered as in Table 3.3. In [23, Theorem 1.4(b)] it is shown that a Fuchsian group Γ of signature $[0, s; m_1, \ldots, m_r]$ has $H^1(\Gamma; \mathbb{Z}) \cong \mathbb{Z}^{s-1}$. In particular, we deduce the signature of $\Gamma_0(p)$ must have the form $[0, N(p) + 1; m_1, \ldots, m_r]$.

Now, C^1 is a free B-module and so we have an isomorphism $H^{2n}(\Gamma_0(p); \mathbb{Z}) \cong H^2(B; C^0)$ for all $n \geqslant 1$. As in [1, Proposition 2.3] we obtain that

$$H^{2n}(\Gamma_0(p); \mathbb{Z}) \cong \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_6^2, \quad \mathbb{Z}_2^2, \quad \mathbb{Z}_3^2, \quad 0,$$

ordered as in Table 3.3. The result now follows from the following three facts. Firstly, for a Fuchsian group Γ of signature $[0, s; m_1, \ldots, m_r]$, each m_j corresponds to a conjugacy class of maximal finite cyclic subgroups $\mathbb{Z}_{m_j}$. Secondly, by the proof of [23, Theorem 1.4(b)], each maximal conjugacy class of finite cyclic subgroups $\mathbb{Z}_{m_j}$ contributes a $\widetilde{H}^*(\mathbb{Z}_{m_j}; \mathbb{Z})$ summand to $\widetilde{H}^*(\Gamma; \mathbb{Z})$. Thirdly, $\mathrm{PSL}_2(\mathbb{Z})$ and hence $\Gamma_0(p)$ has no elements of order 6. □

Remark 3.5.2. Let $\widetilde{\Gamma_0}(p)$ denote the lift of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{R})$. An alternative computation of $H^*(\Gamma_0(p);\mathbb{Z})$ can be achieved by back solving the Lyndon-Hochschild-Serre spectral sequence (see for instance [7, Chapter VII.6]) for the group extension $\mathbb{Z}_2 \rightarrowtail \widetilde{\Gamma_0}(p) \twoheadrightarrow \Gamma_0(p)$ which takes the form

$$E_2^{*,*} = H^*(\Gamma_0(p); H^*(\mathbb{Z}_2;\mathbb{Z})) \Rightarrow H^*(\widetilde{\Gamma_0}(p);\mathbb{Z})$$

using the cohomology calculations for $\widetilde{\Gamma_0}(p)$ in [1].

We shall also record the conjugacy classes of finite order elements of $\Gamma_0(p)$ and $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Note that the only conjugacy classes of finite subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ are one class of groups isomorphic to $\mathbb{Z}_2$ and one to $\mathbb{Z}_3$ since $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. The conjugacy classes of finite subgroups of $\Gamma_0(p)$ can be read off of the signature, there is exactly one of order m_j for each $j = 1, \dots, r$.

Lemma 3.5.3. *The number of conjugacy classes of finite order elements in* $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ *are those given in Table 3.4.*

$p \equiv$	2	3	1 (mod 12)	5 (mod 12)	7 (mod 12)	11 (mod 12)
Identity	1	1	1	1	1	1
Order 2	1	2	1	1	2	2
Order 3	4	2	2	4	2	4
Total	6	5	4	6	5	7

Table 3.4: Number of conjugacy classes of finite order elements of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for p prime.

Proof. The result follows from the following observation: If there is a conjugacy of elements of order 2 (resp. 3) in $\Gamma_0(p)$, then each of class of elements of order 2 (resp. 3) in $\mathrm{PSL}_2(\mathbb{Z})$ fuses in $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. To see this, consider an element in the first copy of $\mathrm{PSL}_2(\mathbb{Z})$, conjugate it to an element in $\Gamma_0(p)$, and then conjugate it to an element in the other copy of $\mathrm{PSL}_2(\mathbb{Z})$. □

Lemma 3.5.4. *Both* $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ *and* $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ *satisfy the Baum-Connes Conjecture.*

Proof. Since $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$, the Bass-Serre tree of the amalgamation is a locally-finite 1-dimensional contractible $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$-CW complex. Moreover, each of the stabilisers Γ_c have $\mathrm{cd}_{\mathbb{Q}}(\Gamma_c) = 1$, being a graph of finite groups. Now, we apply [35, Corollary 5.14] to see that the stabilisers satisfy Baum-Connes and [35, Theorem 5.13] to see that $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ does. The proof is identical for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. □

3.5.2 Computations

There is a long exact Mayer-Vietoris sequence for computing the Bredon homology of an amalgamated free product.

Theorem 3.5.5. [35, Corollary 3.32] *Let $\Gamma = H *_L K$ and let M be a Bredon module. There is a long exact Mayer-Vietoris sequence:*

$$\begin{array}{ccccc} \cdots \longrightarrow H_n^{\mathcal{FIN}}(L;M) & \longrightarrow & H_n^{\mathcal{FIN}}(H;M) \oplus H_n^{\mathcal{FIN}}(K;M) \\ & & \downarrow \\ \cdots \longleftarrow H_{n-1}^{\mathcal{FIN}}(L;M) & \longleftarrow & H_n^{\mathcal{FIN}}(\underline{E}\Gamma;M) \end{array}$$

We are now ready to compute the K-theory of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.

Theorem 3.5.6 (Theorem 3.C)**.** *Let p be a prime and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then $K_n^\Gamma(\underline{E}\Gamma)$ is a free abelian group with rank as given in Table 3.1. Moreover, since the Baum-Connes Conjecture holds for Γ we have $K_*^\Gamma(\underline{E}\Gamma) \cong K_*^{top}(C_r^*(\Gamma))$.*

Proof. There are 6 cases to consider, the two cases when $p = 2, 3$ and the four cases given by $p \equiv 1, 5, 7, 11 \pmod{12}$. Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ and $\mathcal{F} = \mathcal{FIN}$. In each case we have the following long exact Mayer-Vietoris sequence

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ H_2^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) & \longrightarrow & H_1^{\mathcal{F}}(\Gamma_0(p);\mathcal{R}_{\mathbb{C}}) & \longrightarrow & \left(H_1^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z});\mathcal{R}_{\mathbb{C}})\right)^2 \\ & & & & \downarrow \\ \left(H_0^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z});\mathcal{R}_{\mathbb{C}})\right)^2 & \longleftarrow & H_0^{\mathcal{F}}(\Gamma_0(p);\mathcal{R}_{\mathbb{C}}) & \longleftarrow & H_1^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) \\ \downarrow & & & & \\ H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) & \longrightarrow & 0. & & \end{array}$$

We have computed the Bredon homology groups of $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ in Table 3.3. Thus, we can separate the above sequence into two sequences. Indeed, $H_1^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z});\mathcal{R}_{\mathbb{C}})) = 0$, so it follows that $H_2^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) \cong H_1^{\mathcal{F}}(\Gamma_0(p);\mathcal{R}_{\mathbb{C}})$. The other sequence is then given by the remaining terms.

We will treat the case $p = 2$, the other cases proceed identically. We have $H_2^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) = \mathbb{Z}$ and an exact sequence

$$0 \longrightarrow H_1^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z}^8 \longrightarrow H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) \longrightarrow 0.$$

We now compute the colimit $H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) = \operatorname{colim}_{\Gamma/H \in \mathbf{Or}_{\mathcal{F}}(\Gamma)} \mathcal{R}_{\mathbb{C}}(H)$. Since we have a complete description of the conjugacy classes of finite subgroups of Γ and the only inclusions are given by $\{1\} \hookrightarrow \mathbb{Z}_2$ and $\{1\} \hookrightarrow \mathbb{Z}_3$, it follows that $H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) = \mathbb{Z}^6$. Moreover, for the sequence to be exact, it follows the map $\mathbb{Z}^2 \to \mathbb{Z}^8$ must be an isomorphism onto the kernel of the first map. In particular $H_1^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) = 0$.

The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence given in [35, Theorem 5.27] and we obtain $K_0^{\Gamma}(\underline{E}\Gamma) = H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) \oplus H_2^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}})$ and $K_1^{\Gamma}(\underline{E}\Gamma) = H_1^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}})$. We record the Bredon homology groups for the remaining cases in Table 3.5, the reader can easily verify these. Note that they are always torsion-free and so are completely determined by their $\mathbb{Z}$-rank. □

$p \equiv$	2	3	1 (mod 12)	5 (mod 12)	7 (mod 12)	11 (mod 12)
$n = 0$	6	5	4	6	5	7
$n = 1$	0	0	3	1	2	0
$n = 2$	1	1	$\frac{1}{6}(p-7)$	$\frac{1}{6}(p+1)$	$\frac{1}{6}(p-1)$	$\frac{1}{6}(p+7)$

TABLE 3.5: $\mathbb{Z}$-rank of the Bredon homology of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for p prime.

The computation for $\mathrm{SL}_2(\mathbb{Z}[1/p])$ is almost entirely analogous. We highlight the differences below.

Theorem 3.5.7 (Theorem 3.D)**.** *Let p be a prime and $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Then $K_n^{\Gamma}(\underline{E}\Gamma)$ is additively isomorphic to the direct sum of two copies of the corresponding equivariant K-homology group of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$.*

Proof (Sketch). Let $\mathcal{F} = \mathcal{FIN}$. First, we must compute the Bredon homology of the lifts of $\mathrm{PSL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ to $\mathrm{SL}_2(\mathbb{R})$. For this we use the graph of groups in Figure 3.1 and note that now every edge group is the same central copy of $\mathbb{Z}_2$ and the vertex groups change as follows: The vertices with trivial vertex group now have vertex group the same central copy of $\mathbb{Z}_2$. The vertex groups isomorphic to $\mathbb{Z}_2$ are now $\mathbb{Z}_4$ and the vertex groups isomorphic to $\mathbb{Z}_3$ are now $\mathbb{Z}_6$ (each extended by the central $\mathbb{Z}_2$). Computing the Bredon homology we find that in each case it is isomorphic to the direct sum of two copies of the corresponding Bredon homology group in the projective case.

Now, we apply the long exact Mayer-Vietoris sequence to the amalgamated free product decomposition of $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. Since $H_1^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z});\mathcal{R}_{\mathbb{C}}) = 0$, like in the projective case, the sequence splits into two exact sequences. The computation of $H_2^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])))$ is immediate as before and is additively isomorphic to the direct sum of two copies of $H_2^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]);\mathcal{R}_{\mathbb{C}})$.

To compute the zeroth and first homology groups we will again use the colimit isomorphism $H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) = \operatorname{colim}_{\Gamma/H \in \mathbf{Or}_{\mathcal{F}}(\Gamma)} \mathcal{R}_{\mathbb{C}}(H)$. To use this we obtain a count of the total number of conjugacy classes of finite order elements in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$. To do this use a near identical argument to Lemma 3.5.3 that takes into account the central $\mathbb{Z}_2$ subgroup. It follows that the number of conjugacy classes of elements of finite order in $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ is equal to twice the number for the corresponding projective group. It follows that the colimit computation for $H_0^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]);\mathcal{R}_{\mathbb{C}})$ is additively isomorphic to the direct sum of two copies of $H_0^{\mathcal{F}}(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]);\mathcal{R}_{\mathbb{C}})$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$.

From here one computes $H_1^{\mathcal{F}}(\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}]);\mathcal{R}_{\mathbb{C}})$ in an identical manner to the projective case. The resulting groups are isomorphic to the direct sum of two copies of the corresponding Bredon homology groups in the projective case. The result now follows from the collapsed Atiyah-Hirzebruch spectral sequence given in [35, Theorem 5.27] and we obtain $K_0^{\Gamma}(\underline{E}\Gamma) = H_0^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}}) \oplus H_2^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}})$ and $K_1^{\Gamma}(\underline{E}\Gamma) = H_1^{\mathcal{F}}(\Gamma;\mathcal{R}_{\mathbb{C}})$. □

3.6 The Unstable Gromov-Lawson-Rosenberg Conjecture

Given a smooth closed n-manifold M a classical question is to ask whether M admits a Riemannian metric of positive scalar curvature. In a vast generalisation of the Atiyah-Singer index theorem, Rosenberg [41] exhibits a class in $KO_n^{\mathrm{top}}(C_r^*(\pi_1(M)))$ which is an obstruction to M admitting a metric of positive scalar curvature.

More precisely, let M be a closed spin n-manifold and $f\colon M \to B\Gamma$ be a continuous map for some discrete group Γ. Let $\alpha\colon \Omega_n^{\mathrm{Spin}}(B\Gamma) \to KO_n^{\mathrm{top}}(C_r^*(\Gamma))$ be the index of the Dirac operator. If M admits a metric of positive scalar curvature, then $\alpha[M,f] = 0 \in KO_n^{\mathrm{top}}(C_r^*(\Gamma))$

The Unstable Gromov–Lawson–Rosenberg Conjecture. *Let M be a closed spin n-manifold and $\Gamma = \pi_1(M)$. If $f\colon M \to B\Gamma$ is a continuous map which induces the identity on the fundamental groups, then M admits a metric positive scalar curvature if and only if $\alpha[M,f] = 0 \in KO_n^{\mathrm{top}}(C_r^*(\Gamma))$.*

The conjecture has been verified in the case of some finite groups [51] [42] [29] [40], when the group has periodic cohomology, torsion-free groups for which the dimension of $B\Gamma$ is less than 9 [24], and cocompact Fuchsian groups [15]. However, there are counterexamples: The first is due to Schick [49] who disproves the conjecture for the direct product $\mathbb{Z}^4 \times \mathbb{Z}_n$ when n is odd; while other instances have been constructed in [24]. For more information on the Unstable GLR Conjecture the reader should consult [24] and the references therein.

3.6.1 Proof of Theorem 3.E

We will now prove the conjecture for a large class of groups. Our proof is structurally similar to the proof by Davis-Pearson [15] so we will summarise their method and highlight any differences.

Let ko be the connective cover of KO with covering map p and let D be the ko-orientation of spin bordism. The map α (from above) is obtained by the following composition

$$\Omega_n^{\mathrm{Spin}}(B\Gamma) \xrightarrow{D} ko_n(B\Gamma) \xrightarrow{p} KO_n(B\Gamma) \xrightarrow{\mu_{\mathbb{R}}} KO_n^{\mathrm{top}}(C_r^*(\Gamma))$$

We note that $ko_n(*) = 0$ for $n < 0$ and that p is an isomorphism for $n \geqslant 0$ on the one point space.

Recall from the introduction that a group Γ satisfies:

(M) If every finite subgroup is contained in a unique maximal finite subgroup.

(NM) If M is a maximal finite subgroup of Γ, then the normaliser $N_\Gamma(M)$ of M is equal to M.

(BC) If Γ satisfies the Baum-Connes conjecture.

(PFS) If all maximal finite subgroups of Γ are odd order and have periodic cohomology.

Proposition 3.6.1. *Let Γ be a group satisfying (BC), (M), and (NM). Let Λ be a set of conjugacy classes of maximal finite subgroups of Γ. There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc}
\widetilde{KO}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n(BH) & \longrightarrow & \widetilde{KO}_n(B\Gamma) & \longrightarrow & \widetilde{KO}_n(\underline{B}\Gamma) \\
\downarrow id & & \downarrow \mu_{\mathbb{R}} & & \downarrow \mu_{\mathbb{R}} & & \downarrow id \\
\widetilde{KO}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n^{\mathrm{top}}(C_r^*(H)) & \longrightarrow & \widetilde{KO}_n^{\mathrm{top}}(C_r^*(\Gamma)) & \longrightarrow & \widetilde{KO}_n(\underline{B}\Gamma).
\end{array}$$

Proof. First, since Γ satisfies (BC), (M) and (NM) by either [13, Corollary 3.13] or the proof of [13, Theorem 4.1] for any constant functor $\mathbf{E}_c\colon \mathbf{Or}_{\mathcal{F}}(\Gamma) \to \mathbf{Spectra}$ by $\Gamma/H \mapsto \mathbf{E}$ there are long exact sequences

$$\cdots \to \bigoplus_{(H)\in\Lambda} H_n(BH;\mathbf{E}) \longrightarrow \left(\bigoplus_{(H)\in\Lambda} \pi_n(\mathbf{E})\right) \oplus H_n(B\Gamma;\mathbf{E}) \longrightarrow H_n(\underline{B}\Gamma;\mathbf{E}) \to \cdots$$

and

$$\cdots \longrightarrow \bigoplus_{(H)\in\Lambda} \widetilde{H}_n(BH;\mathbf{E}) \longrightarrow \widetilde{H}_n(B\Gamma;\mathbf{E}) \longrightarrow \widetilde{H}_n(\underline{B}\Gamma;\mathbf{E}) \longrightarrow \cdots$$

We perform a diagram chase exactly as in [15, Proposition 4], taking $\mathbf{E} = \mathbf{KO}$ and the isomorphism

$$\pi_n\left(\operatorname*{hocolim}_{\mathbf{Or}_{\mathcal{F}}(\Gamma)}(\mathbf{E}_c)\right) \cong H_n(\underline{B}\Gamma;\mathbf{E}),$$

the result follows. □

Theorem 3.6.2 (Theorem 3.E)**.** *Let Γ be a group satisfying (BC), (M), (NM) and (PFS). If $\underline{B}\Gamma$ is finite and has dimension at most 9, then the Unstable Gromov-Lawson-Rosenberg Conjecture holds for Γ.*

Proof. Let **ko** be the spectrum of the connective cover of **KO**. Via the cover we obtain a natural transformation $p : \mathbf{ko}_c \to \mathbf{KO}_c$ of constant $\mathbf{Or}_{\mathcal{F}}(\Gamma)$-**Spectra**. From the previous proposition we obtain a commutative diagram

$$\begin{array}{ccccccc}
\widetilde{ko}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{ko}_n(BH) & \longrightarrow & \widetilde{ko}_n(B\Gamma) & \longrightarrow & \widetilde{ko}_n(\underline{B}\Gamma) \\
\downarrow p & & \downarrow \mu_{\mathbb{R}}\circ p & & \downarrow \mu_{\mathbb{R}}\circ p & & \downarrow p \\
\widetilde{KO}_{n+1}(\underline{B}\Gamma) & \longrightarrow & \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n^{\mathrm{top}}(C_r^*(H)) & \longrightarrow & \widetilde{KO}_n^{\mathrm{top}}(C_r^*(\Gamma)) & \longrightarrow & \widetilde{KO}_n(\underline{B}\Gamma).
\end{array} \qquad (\dagger)$$

By Joachim-Schick [24, Lemma 2.6] p is an isomorphism for $n \geqslant 6$ and an injection for $n = 5$. Now, suppose that $n \geqslant 5$ so we are in the setting of the GLR conjecture. Consider an element

$$\beta \in K := \mathrm{Ker}\,(\mu_{\mathbb{R}} \circ p : ko_n(B\Gamma) \to KO_n(C_r^*(\Gamma; \mathbb{R}))$$

and note that $K \cong \mathrm{Ker}(\mu_{\mathbb{R}} \circ p : \widetilde{ko}_n(B\Gamma) \to \widetilde{KO}_n^{\mathrm{top}}(C_r^*(\Gamma))$. Combining the diagram ($\dagger$) with the isomorphism p: $\widetilde{ko}_n(\underline{B}\Gamma) \to \widetilde{KO}_n^{\mathrm{top}}(\underline{B}\Gamma)$ for $n \geqslant 6$ (injection for $n = 5$), we can deduce that there exists

$$\gamma \in \mathrm{Ker}\left(\bigoplus_{(H)\in\Lambda} ko_n(BH) \to \bigoplus_{(H)\in\Lambda} KO_n(BH)\right)$$

which maps to β.

For a group L let $ko_n^+(BL)$ be the subgroup of $ko_n(BL)$ given by $D[M, f]$ where M is a positively curved spin manifold and f is a continuous map. In [6] the authors prove for any finite group of odd order with periodic cohomology H, that $ko_n^+(BH) = \mathrm{Ker}(\mu_{\mathbb{R}} \circ p : ko_n(BH) \to KO_n^{\mathrm{top}}(C_r^*(H))$. Thus, we have $\gamma \in ko_n^+(BH)$ and $\beta \in ko_n^+(B\Gamma)$. Now, in [52] it is proven that if $D[M, f] \in ko_n^+(BG)$, then M admits a metric of positive scalar curvature. In particular, we are done. □

Remark 3.6.3. It is unclear whether the assumption that the finite subgroups having odd order can be dropped. Indeed, it was pointed out to the author by J.F. Davis that the statement of [6, Corollary 2.2] contains a misprint. One should instead (in the notation of [6]) define $\mathcal{Y}_n(B\pi)$ to be the kernel of $A \circ p$ restricted to the subgroup $D(\Omega^n(B\pi)) \subseteq ko_n(B\pi)$. After making this correction, the statements and proofs in the paper are correct. This error caused a mistake in the main theorem of [15] which is only correct if one restricts to Fuchsian groups whose torsion only has odd order.

The dimension bound on $\underline{B}\Gamma$ is an artefact of the proof and arises from the lemma of Joachim–Schick [24, Lemma 2.6]. Specifically **ko** is the -3-connected cover of **KO** and so if $\dim \underline{B}\Gamma \leqslant 9$ the natural transformation of the corresponding Atiyah–Hirzebruch spectral sequences $E^*_{p,q} \to F^*_{p,q}$ is an isomorphism for $p+q \geqslant 6$ and an injection when $p+q=5$. The following corollary shows that one always has a bound relating $\dim M$ to $\dim \underline{B}\Gamma$ such that the α-invariant can determine the existence of a metric of positive scalar curvature on M (provided $\pi_1(M)$ satisfies the other hypothesis of Theorem 3.6.2).

Corollary 3.6.4. *Let M be a connected closed spin n-manifold and let $\Gamma = \pi_1(M)$ be a group satisfying (M), (NM) and (PFS). Suppose the assembly map $\mu_{\mathbb{R}}$ is injective and $\underline{B}\Gamma$ is finite of dimension N. If $n \geqslant \max\{5, N-4\}$, then M admits a metric with positive scalar curvature if and only if $\alpha[M,f]=0$.*

3.6.2 Some examples

In this section we will detail some applications of Theorem 3.E to various families of groups. These results are new whenever the groups involved are infinite and have torsion.

3.6.2.1 Graphs of groups

In [43, Theorem 3.1] it is shown that the fundamental groups of graphs of groups with vertex groups satisfying (M) and (NM) and with torsion-free edge groups, satisfy (M) and (NM). It follows that we have the following combination theorem:

Corollary 3.6.5. *The Γ be a finitely presented fundamental group of a graph of groups such that the vertex groups satisfy (BC), (M) and (NM) and the edge groups are torsion-free and satisfy (BC). If the vertex groups satisfy (PFS) and $\underline{B}\Gamma$ has dimension at most 9, then Γ satisfies the Unstable GLR Conjecture.*

3.6.2.2 3-manifold groups

In [43, Section 3.3] it is shown that 3-manifold groups satisfy (M) and (NM) and there is a well known classification of finite subgroups of orientable connected 3-manifold groups. These are exactly the groups which act freely on the 3-sphere and so have periodic cohomology by [7, Chapter VI.9]. In [34] it is shown 3-manifold groups satisfy (BC). Applying Theorem 3.E we obtain the following result (which is new whenever Γ is infinite and contains torsion):

Corollary 3.6.6. *Let M be a closed orientable connected 3-manifold with fundamental group Γ. If Γ has no elements of order 2 then Γ satisfies the Unstable GLR Conjecture.*

3.6.2.3 One-relator groups

In [13, Page 32] it is shown that one-relator groups satisfy (BC), (M) and (NM) and admit a two dimensional model for $\underline{E}\Gamma$. By [25, Theorem 3] every element of finite order in Γ is conjugate to a power of w. Hence, every finite order subgroup is cyclic of odd order and so satisfies (PFS). Applying Theorem 3.E we obtain the following result (which to the authors knowledge is new whenever Γ is infinite and contains torsion):

Corollary 3.6.7. *Let $\Gamma = \langle X \mid w \rangle$ be a finitely generated one-relator group and suppose w has odd order when interpreted in Γ, then Γ satisfies the Unstable GLR Conjecture.*

3.6.2.4 Hilbert modular groups

Let k be a totally real number field of degree n and $\mathcal{O}_k$ be its ring of integers. The Hilbert modular group of k is defined to be $\mathrm{PSL}_2(\mathcal{O}_k)$ and is a lattice in $\mathrm{PSL}_2(\mathbb{R})^n$. Note that if $k = \mathbb{Q}$ then we recover the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$. Properties (BC), (M) and (NM) are given in [11, Lemma 4.3]. Applying Theorem 3.E we obtain the following new result:

Corollary 3.6.8. *Let k be a totally real number field with degree less than or equal to 4. Let $\Gamma \leqslant \mathrm{PSL}_2(\mathcal{O}_k)$ be finitely presented. If all finite subgroups of Γ are cyclic of odd order then Γ satisfies the Unstable GLR Conjecture.*

Proof. This follows from the fact every finite subgroup of $\mathrm{PSL}_2(\mathbb{R})^n$ is a product of finite cyclic groups. □

3.6.2.5 Subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$

In this section we will prove the result that many subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$ satisfy the conditions of Theorem 3.E. The result is new whenever the subgroup is infinite, has torsion, and is not isomorphic to a Fuchsian group. We will also compute the KO-theory of $C_r^*(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]);\mathbb{R})$.

Corollary 3.6.9. *Let $p \equiv 11 \pmod{12}$ and let $\Gamma < \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ be finitely presented. If Γ has no elements of order 2, then Γ satisfies the Unstable GLR Conjecture.*

There are many finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ for $p \equiv 11 \pmod{12}$ satisfying the hypothesis of the corollary. Indeed, by the amalgamated free product decomposition, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is generated by the two subgroups isomorphic to $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Thus, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is a four generated group $\langle a, b, c, d \rangle$ where a and c have order 2, and b and d have order 3. The kernel of the homomorphism $\phi : \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \to \mathbb{Z}_2$ by $a, c \mapsto 1$ and $b, d \mapsto 0$ is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ with no 2-torsion.

Proof. The proof follows from applying Theorem 3.E to the observation that every finite subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ is cyclic and hence has periodic cohomology and the following lemma. Note that Γ is necessarily a proper subgroup since $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ always contains elements of order 2. □

Lemma 3.6.10. *Let p be a prime, then $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfies (M). Moreover, if $p \equiv 11 \pmod{12}$ then $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfies (NM).*

Proof. Since each non-trivial finite subgroup of Γ is of order 2 or 3 it is obvious that Γ satisfies (M). Now, assume $p \equiv 11 \pmod{12}$ and note that $\mathrm{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ satisfies (NM). Recall the amalgamated free product decomposition, $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]) = \mathrm{PSL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathrm{PSL}_2(\mathbb{Z})$. The amalgamated subgroup $\Gamma_0(p)$ is torsion-free so we may apply [43, Theorem 3.1]. □

An alternative direct proof of the calculations of the K-groups of $C_r^*(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$ when $p \equiv 11 \pmod{12}$ is as follows. Note that this bypasses the computation of the Bredon homology but does not give us a way to compute either invariant for $\mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$.

Lemma 3.6.11. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then $\underline{B}\Gamma \simeq \bigvee_{b_1(\Gamma_0(p))} S^2$*

Proof. Let $X = \underline{E}\mathrm{PSL}_2(\mathbb{Z}) \times_{\mathrm{PSL}_2(\mathbb{Z})} \Gamma$ and $Y = \underline{E}\Gamma_0(p) \times_{\Gamma_0(p)} \Gamma$. We have

$$\begin{aligned} \underline{B}\Gamma &\simeq \operatorname*{hocolim}_{\mathbf{Top}} (X \leftarrow Y \rightarrow X)/\Gamma, \\ &\simeq \operatorname*{hocolim}_{\mathbf{Top}} (X/\Gamma \leftarrow Y/\Gamma \rightarrow X/\Gamma). \end{aligned}$$

Since $\underline{E}\mathrm{PSL}_2(\mathbb{Z})/\mathrm{PSL}_2(\mathbb{Z})$ is an interval and $\underline{E}\Gamma_0(p)/\Gamma_0(p)$ is a finite graph, we have

$$\underline{B}\Gamma \simeq \operatorname*{hocolim}_{\mathbf{Top}} \left(I \leftarrow \bigvee_{b_1(\Gamma_0(p))} S^1 \rightarrow I \right),$$

but I is contractible, so up to homotopy this becomes a suspension of a wedge of circles. In particular, $\underline{B}\Gamma \simeq \bigvee_{b_1(\Gamma_0(p))} S^2$. □

Theorem 3.6.12. *Let $p \equiv 11 \pmod{12}$ and let $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, then*

$$K_0^{\mathrm{top}}(C_r^*(\Gamma)) = \mathbb{Z}^{7+\frac{1}{6}(p+7)} \quad \text{and} \quad K_1^{\mathrm{top}}(C_r^*(\Gamma)) = 0.$$

Proof. Let Λ be a set of representatives of finite subgroups of Γ. By [13, Theorem 4.1(a)] we have a short exact sequence

$$0 \rightarrow \bigoplus_{(H)\in\Lambda} \widetilde{K}_n^{\mathrm{top}}(C_r^*(H)) \rightarrow K_n^{\mathrm{top}}(C_r^*(\Gamma)) \rightarrow K_n(\underline{B}\Gamma) \rightarrow 0.$$

The only nontrivial part now is computing $K_n(\underline{B}\Gamma)$, but we have already shown that $\underline{B}\Gamma$ is homotopy equivalent to a wedge of $\bigvee_{b_1(\Gamma_0(p))} S^2$, i.e. a wedge of 2-spheres. Thus, we can simply apply the homological Atiyah-Hirzebruch spectral sequence (which collapses trivially) to obtain that $K_0(\underline{B}\Gamma) = \mathbb{Z}^{\frac{1}{6}(p+7)+1}$ and $K_1(\underline{B}\Gamma) = 0$. □

A near identical argument can be used to compute the KO-groups of $C_r^*(\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}]))$ when $p \equiv 11 \pmod{12}$.

Theorem 3.6.13. *Let $p \equiv 11 \pmod{12}$ be a prime and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$. Except for an extension problem in dimensions congruent to $1, 3$ and 4 modulo 8, we have for $n = 0, \dots, 7$ that*

$$KO_n^{\mathrm{top}}(C_r^*) = \mathbb{Z}^5,\ \ \mathbb{Z}_2^3,\ \ \mathbb{Z}^{2+\frac{1}{6}(p+7)} \oplus \mathbb{Z}_2^3,\ \ \mathbb{Z}_2^{\frac{1}{6}(p+7)},\ \ \mathbb{Z}^5 \oplus \mathbb{Z}_2^{\frac{1}{6}(p+7)},\ \ 0,\ \ \mathbb{Z}^{2+\frac{1}{6}(p+7)},\ \ 0$$

and the remaining groups are given by 8-fold Bott-periodicity.

Proof. Let Λ be a set of representatives of finite subgroups of Γ. As before, by [13, Theorem 4.1(a)] we have a short exact sequence

$$0 \to \bigoplus_{(H)\in\Lambda} \widetilde{KO}_n^{\mathrm{top}}(C_r^*(H)) \to KO_n^{\mathrm{top}}(C_r^*(\Gamma)) \to KO_n(\underline{B}\Gamma) \to 0.$$

Now, $KO_n(\underline{B}\Gamma) \cong KO_n(*) \oplus KO_{n-2}(*)^{\frac{1}{6}(p+7)}$ and the groups $KO_n^{\mathbb{Z}_m}(*) \cong KO_n^{\mathrm{top}}(C_r^*(\mathbb{Z}_m))$ are given in [17, Section 2.1]. □

Our methods leave open the following.

Question 3.6.14. Let p be a prime, then does $\mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$ satisfy the Unstable GLR Conjecture? What about a lattice in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{Q}_p)$?

References

[1] A. Adem and N. Naffah, *On the Cohomology of* $\mathrm{SL}_2(\mathbb{Z}[1/p])$, in Geometry and Cohomology in Group Theory (London Mathematical Society Lecture Note Series no. 252), 1–9, Cambridge University Press. doi: 10.1017/CBO9780511666131.002

[2] M.P.G. Aparicio, P. Julg and A. Valette *The Baum-Connes conjecture: an extended survey.* In: Chamseddine A., Consani C., Higson N., Khalkhali M., Moscovici H., Yu G. (eds) Advances in Noncommutative Geometry (2019). Springer, Cham. doi: 10.1007/978-3-030-29597-4_3

[3] P. Baum, A Connes, and N. Higson, *Classifying spaces for proper actions and K-theory of group C^*-algebras*, Contemporary Mathematics **167** (1994), 241-–291.

[4] P. Baum and M. Karoubi, *On the Baum-Connes conjecture in the real case*, Quarterly Journal of Mathematics, **55** (2004), no. 3, 231–235. doi: 10.1093/qmath/hag051

[5] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T) (New Mathematical Monographs)*. Cambridge: Cambridge University Press 2008. doi: 10.1017/CBO9780511542749

[6] B. Botvinnik, P.B. Gilkey and S. Stolz, *The Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology*, Journal of Differential Geometry, **46** (1997), 374–405. doi: 10.4310/jdg/1214459973

[7] K.S. Brown, *Cohomology of Groups*. Graduate Texts in Mathematics 87. New York: Springer-Verlag 1982. doi: 10.1007/978-1-4684-9327-6

[8] R.R. Bruner and J.P.C. Greenlees, *Connective Real K-Theory of Finite Groups*, *Mathematical Surveys and Monographs* **169**. American Mathematical Society, Providence, RI 2010.

[9] A.T. Bui and G. Ellis, *The homology of* $\mathrm{SL}_2(\mathbb{Z}[1/m])$ *for small* m, Journal of Algebra, **408** (2014), 102–108. doi: 10.1016/j.jalgebra.2013.11.002.

[10] A.T. Bui, A.D. Rham, and M. Wendt, *The Farrell–Tate and Bredon homology for* $\mathrm{PSL}_4(\mathbb{Z})$ *via cell subdivisions*, Journal of Pure and Applied Algebra, **223** (2019), no. 7, 2872—2888. doi 10.1016/j.jpaa.2018.10.002

[11] M. Bustamante and L.J.S. Saldaña, *On the algebraic K-theory of the Hilbert modular group*, Algebraic and Geometric Topology, **16** (2016), no. 4, 2107–2125. doi: 10.2140/agt.2016.16.2107

[12] J. Davis and W. Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-Theory, **15** (1998), 201—252. doi: 10.1023/A:1007784106877

[13] J. Davis and W. Lück, *The p-chain spectral sequence*, K-Theory, **30** (2003), no. 1, 71–104. doi: 10.1023/B:KTHE.0000015339.41953.04

[14] J. Davis and W. Lück, *The topological K-theory of certain crystallographic groups*, Journal of Noncommutative Geometry, **7** (2013), 373–431. doi: 10.4171/JNCG/121

[15] J. Davis and K. Pearson, *The Gromov-Lawson-Rosenberg conjecture for cocompact Fuchsian groups*, Proceedings of the American Mathematical Society, **131** (2003), 3571–3578. doi:10.1090/S0002-9939-03-06905-3

[16] D. Degrijse and I.J. Leary, *Equivariant vector bundles over classifying spaces for proper actions*, Algebraic & Geometric Topology, **17** (2017), no. 1, 131–156. doi: 10.2140/agt.2017.17.131

[17] M. Fuentes-Rumi, *The equivariant K- and KO-theory of certain classifying spaces via an equivariant Atiyah-Hirzebruch spectral sequence*, preprint (2020). arXiv:1905.02972 [math.KT]

[18] R. Flores, S. Pooya and A. Valette, *K-theory and K-homology for the lamplighter groups of finite groups*, Proceedings of the London Mathematical Society, **115** (2017), no. 3, 1207–1226. doi: 10.1112/plms.12061

[19] W. Fulton and J. Harris, *Representation theory. A first course*. Graduate Texts in Mathematics 129. New York: Springer-Verlag 1991. doi: 10.1007/978-1-4612-0979-9

[20] H.-W. Henn, *The cohomology of* $\mathrm{SL}(3,\mathbb{Z}[1/2])$, K-Theory, **16** (1999), no. 4, 299–359.

[21] N. Higson and G. Kasparov, *Operator K-theory for groups which act properly and isometrically on Hilbert space*, Electronic Research Announcements of the Amererican Mathematical Society, **3** (1997), 131-–142. doi: 10.1090/S1079-6762-97-00038-3

[22] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum–Connes Conjecture*, Geometric and Functional Analysis, **12** (2002), no. 2, 330–354. doi: 10.1007/s00039-002-8249-5

[23] S. Hughes, *Cohomology of Fuchsian Groups and Non-Euclidean Crystallographic Groups*, preprint (2019). arXiv:1910.00519 [math.GR]

[24] M. Joachim and T. Schick, *Positive and negative results concerning the Gromov-Lawson-Rosenberg conjecture*, Geometry and topology: Aarhus (1998), Contemporary Mathematics, vol. 258, American Mathathematical Society, Providence, RI, 2000, pp. 213-–226. doi:10.1090/conm/258/04066

[25] A. Karrass, W. Magnus, and D. Solitar, *Elements of finite order in groups with a single defining relation*, Communications on Pure and Applied Mathematics, **13** (1960), 57–66. doi:10.1002/cpa.3160130107

[26] G.G. Kasparov, *Operator K-theory and its applications*, Journal of Soviet Mathematics, **37** (1987), no. 6, 1373-–1396. doi: 10.1007/BF01103851

[27] G.G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Inventiones Mathematicae, **91** (1988), no. 1, 147–201. doi: 10.1007/BF01404917

[28] S. Katok, *Fuchsian Groups*, The University of Chicago Press, Chicago, 1992.

[29] S. Kwasik and R. Schultz, *Positive scalar curvature and periodic fundamental groups*, Commentarii Mathematici Helvetici, **65** (1990), 271-–286. doi: 10.1007/BF02566607

[30] J.-F. Lafont, I.J Ortiz, A.D. Rham and R. Sánchez-García, *Equivariant K-homology for hyperbolic reflection groups*, The Quarterly Journal of Mathematics, **69** (2018), no. 4, 1475—1505. 10.1093/qmath/hay030

[31] X. Li, *K-theory for generalized Lamplighter groups*, Proceedings of the American Mathematics Society, **147** (2019), 4371–4378. doi: 10.1090/proc/14619

[32] W. Lück and R. Stamm, *Computations of K- and L-theory of cocompact planar groups*, K-Theory, **21** (2000), no. 3, 249–292. doi:10.1023/A:1026539221644

[33] A.M. Macbeath, *The classification of non-euclidean plane crystallographic groups*, Canadian Journal of Mathematics. Journal Canadien de Mathématiques, **19** (1967), 1192–1205. doi: 10.4153/CJM-1967-108-5

[34] M. Matthey, H. Oyono-Oyono and W. Pitsch, *Homotopy invariance of higher signatures and 3-manifold groups*, Bulletin de la Société Mathématique de France, **136** (2008), no. 1, 1—25. doi: 10.24033/bsmf.2547

[35] G. Mislin and A. Valette, *Proper group actions and the Baum-Connes conjecture*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, 2003.

[36] S. Pooya, *K-theory and K-homology of the wreath products of finite with free groups*, Illinois Journal of Mathematics, **63** (2019) no. 2, 317–334. doi: 10.1215/00192082-7768735

[37] S. Pooya and A. Valette, *K-theory for the C^*-algebras of solvable Baumslag-Solitar groups*, Glasgow Mathematical Journal, **60** (2018) no. 2, 481–486. doi:10.1017/S0017089517000210

[38] A.D. Rham, *Homology and K-theory of the Bianchi groups*, Comptes Rendus Mathematique, **349** (2011), no. 11–12, 615–619. doi: 10.1016/j.crma.2011.05.014.

[39] A.D. Rham, *On the equivariant K-homology of PSL_2 of the imaginary quadratic integers*, Annales de l'Institut Fourier, **66** (2016), no. 4, 1667–1689. doi: 10.5802/aif.3047

[40] J. Rosenberg, *C^*-algebras, positive scalar curvature, and the Novikov conjecture II*, in Geometric Methods in Operator Algebras (Kyoto 1983), H. Araki and E. G. Effros, eds., Pitman Research Notes in Math, no. 123, Longman/Wiley 1986, 341—374.

[41] J. Rosenberg, *C^*-algebras, positive scalar curvature, and the Novikov conjecture, III*, Topology, **25** (1986), 319-–336.

[42] J. Rosenberg and S. Stolz, *Manifolds of positive scalar curvature*, in Algebraic Topology and its Applications, MSRI Publications 27, Springer 1994, 241-–267.

[43] L.J.S. Saldaña, (2020). *On the dimension of groups that satisfy certain conditions on their finite subgroup*, Glasgow Mathematical Journal, ahead of press. doi:10.1017/S0017089520000531

[44] L.J.S. Saldaña and M. Velásquez, *The algebraic and topological K-theory of the Hilbert modular group*, Homology, Homotopy and Applications, **20** (2018) no. 2, 377–402. doi: 10.4310/HHA.2018.v20.n2.a19

[45] R. Sánchez-García, *Equivariant K-homology of the classifiying space for proper actions*, Ph.D. thesis, University of Southampton, 2005.

[46] R. Sánchez-García, *Equivariant K-homology for some Coxeter groups*, Journal of the London Mathematical Society, **75** (2007), no. 3, 773–790. doi: 10.1112/jlms/jdm035

[47] R. Sánchez-García, *Bredon homology and equivariant K-homology of* $\mathrm{SL}(3,\mathbb{Z})$, Journal of Pure and Applied Algebra, **212** (2008), no. 5, 1046–1059. doi: 10.1016/j.jpaa.2007.07.019

[48] J.-P. Serre, *Trees*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation. doi: 10.1007/978-3-642-61856-7

[49] T. Schick, *A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture*, Topology, **37** (1998), 1165-–1168. doi: 10.1016/S0040-9383(97)00082-7

[50] C. Soulé, *The cohomology of* $\mathrm{SL}_3(\mathbb{Z})$, Topology **17** (1978), 1–22. doi: 10.1016/0040-9383(78)90009-5

[51] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Annals of Mathematics, **136** (1992), 511-–540. doi: 10.2307/2946598

[52] S. Stolz, *Positive scalar curvature metrics - existence and classification questions*, Proceedings of the International Congress of Mathematicians (Zurich 1994), Birkhäuser, 1995, 625-636. doi: 10.1007/978-3-0348-9078-6_56

Paper 4

GRAPHS AND COMPLEXES OF LATTICES

SAM HUGHES

Abstract. We study lattices acting on CAT(0) spaces via their commensurated subgroups. To do this we introduce the notions of a graph of lattices and a complex of lattices giving graph and complex of group splittings of CAT(0) lattices. Using this framework we characterise irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices by C^*-simplicity and the failure of virtual fibring and biautomaticity. We construct non-residually finite uniform lattices acting on arbitrary products of right angled buildings and non-biautomatic lattices acting on the product of $\mathbb{E}^n$ and a right-angled building. We investigate the residual finiteness, L^2-cohomology, and C^*-simplicity of CAT(0) lattices more generally. Along the way we prove that many right angled Artin groups with rank 2 centre are not quasi-isometrically rigid.

4.1 Introduction

Let H be a locally compact group with Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. We say the lattice *uniform* is H/Γ is cocompact and *non-uniform otherwise*. We say a lattice Γ in a product $H_1 \times H_2$ is *weakly irreducible* if the projection of Γ to each factor is non-discrete, otherwise we say Γ is *reducible*. Given a pair of locally compact groups H_1 and H_2 there are a number of basic questions one can ask:

(Q1) Does $H_1 \times H_2$ contain weakly irreducible lattices?

(Q2) What are the generic properties of a weakly irreducible lattice?

In the classical setting of lattices in semisimple Lie groups and linear algebraic groups over local fields these questions are well studied. Indeed, there are deep theorems such as the Margulis normal subgroup theorem, super-rigidity theorem, and the arithmeticity theorem [47].

The non-classical setting is more complicated and was initiated by studing lattices in the full automorphism group of a locally-finite polyhedral complex. A striking example of the non-classical setting is given by the work of Burger and Mozes [12] [13] [14]. The authors constructed torsion-free simple groups which could be realised as cocompact irreducible lattices in a product of automorphism groups of locally-finite trees.

Thus, one should find a class of spaces which contain the exciting phenomena to be found in products of polyhedral complexes whilst enjoying a strong geometric grounding. The answer was to be found in the notion of non-positive curvature or CAT(0) spaces. The theory encompasses symmetric spaces, non-positively curved manifolds, Euclidean and hyperbolic buildings, and more [6]. The reader is referred to [6] for a comprehensive introduction to the theory.

Assumption 4.1.1. *Throughout this paper, all actions of groups on graphs or polyhedral complexes are assumed to* admissible. *That is, each element of a group fixes pointwise each cell it preserves.*

A systematic study of the full isometry groups of CAT(0) spaces and their lattices was undertaken by Caprace and Monod [24] [23] [26]. The authors showed in [24, Theorem 1.6], that under mild hypotheses on a CAT(0) space X, there is finite index subgroup of $H \leqslant \mathrm{Isom}(X)$ which splits as

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q, \tag{4.1}$$

for some $n, p, q \geqslant 0$, where each S_i is an almost connected simple Lie group with trivial centre and each D_j is a totally disconnected irreducible group with trivial amenable radical. Moreover by [24, Addendum 1.8], X itself splits as

$$X = \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q \tag{4.2}$$

where each X_i is an irreducible symmetric space of non-compact type and each Y_j is an irreducible minimal CAT(0)-space.

Taking these decompositions as a starting point motivates a new approach towards CAT(0) groups, that is, understanding the lattices in each of the factors individually and then how the factors interact. The later question is the central goal of this paper: To provide a combinatorial framework for studying lattices in products of irreducible CAT(0) spaces and deduce properties of the weakly irreducible lattices. To this end we introduce the notion of a *graph of lattices* (Definition 4.3.2) with fixed locally-finite Bass-Serre $\mathcal{T}$ (we will also assume that the tree is unimodular and its automorphism group is non-discrete, these are essentially non-degeneracy conditions so that there are tree lattices). Note that in the case of a product of two trees a similar construction was considered by Benakli and Glasner [5].

Roughly a graph of lattices is a graph of groups such that all local groups are finite-by-commensurable-H-lattices equipped with a morphism to H. We use this to study lattices in the product of $T := \mathrm{Aut}(\mathcal{T})$ and closed subgroups H of the isometry group of a fairly generic CAT(0) space. We prove a structure theorem for $(H \times T)$-lattices. That is, we show every $(H \times T)$-lattice gives rise to a graph of H-lattices and conversely, we give necessary and sufficient conditions for a graph of H-lattices to be an $(H \times T)$-lattice.

Theorem 4.A (Theorem 4.3.3). *Let X be a finite dimensional proper* CAT(0) *space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(A, \mathcal{A}, \psi)$ be a graph of H-lattices* *with locally-finite unimodular non-discrete Bass-Serre tree $\mathcal{T}$, and fundamental group Γ. Suppose $T = \mathrm{Aut}(\mathcal{T})$ admits a uniform lattice.*

(i) *Assume A is finite. If for each local group A_σ the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(H \times T)$-lattice and hence a $\mathrm{CAT}(0)$ group. Conversely, if Λ is a uniform $(H \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.*

(ii) *Under the same hypotheses as (i), Γ is quasi-isometric to $X \times \mathcal{T}$.*

(iii) *Assume X is a $\mathrm{CAT}(0)$ polyhedral complex. Let μ be the normalised Haar measure on H. If for each local group A_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$ and the sum $\sum_{\sigma \in VA} \mu(A_\sigma)/|K_\sigma|$ converges, then Γ is a $(H \times T)$-lattice. Conversely, if Λ is a $(H \times T)$-lattice, then Λ splits as a graph of H-lattices with Bass-Serre tree $\mathcal{T}$.*

We also introduce an analogous construction we call a *complex of lattices* (Definition 4.6.1) by replacing the tree with a $\mathrm{CAT}(0)$ polyhedral complex and then prove an analogous structure theorem (Theorem 4.6.2). In the process we deduce some consequences about commensurated subgroups of $\mathrm{CAT}(0)$ groups.

We study of various properties of $(H \times T)$-lattices providing answers to (Q2). In Section 4.4.1 we investigate the L^2-Betti numbers of $(H \times T)$-lattices and some closely related groups. We also compute the rational homological dimension of S-arithmetic lattices in characteristic $p > 0$ (Theorem 4.4.5). The author expects this latter result is well known however he could not find a reference in the literature. We investigate C^*-simplicity (Section 4.4.2), virtual fibring (Section 4.4.3) and autostackability (Section 4.4.4) of $(H \times T)$-lattices in terms of the properties of H-lattices. We will give the necessary background for each property in the relevant section.

In Section 4.5 we detail a number of constructions and examples of $(H \times T)$-lattices using elementary Bass-Serre theory answering (Q1). The constructions are reminiscent of the "universal covering trick" of Burger and Mozes [13] and so we provide a comparison in Section 4.5.3.

Until Leary and Minasyan's examples of $\mathrm{CAT}(0)$ but not virtually biautomatic groups in [45] there were no known examples of lattices where the projection to $\mathrm{Isom}(\mathbb{E}^n)$ is non-discrete. In light of this we begin a study of weakly irreducible lattices with non-trivial Euclidean de Rham factor. We adapt the biautomaticity criterion given in [45, Theorem 1.2] to apply to arbitrary $\mathrm{CAT}(0)$ lattices in the presence of a Euclidean de Rham factor (Theorem 4.7.7).

For T the automorphism group of a locally-finite tree we give constructions of many more $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices. We then prove the following characterisation of uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices eliciting a number of generic properties of such lattices eleciting a strong answer to (Q2). Note the following theorem is optimal in the sense that irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattices are always non-residually finite and not virtually biautomatic, however, there also exist non-residually finite reducible uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattices for $n \geqslant 3$ (this can be seen by taking the direct product of an irreducible $(\mathrm{Isom}(\mathbb{E}^2)\times T)$-lattice with $\mathbb{Z}^{n-2}$, then applying Theorem 4.7.7).

Theorem 4.B (Theorem 4.7.13)**.** *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice. The following are equivalent:*

(i) *Γ is a weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice;*

(ii) *Γ is irreducible as an abstract group;*

(iii) *Γ acts on $\mathcal{T}$ faithfully;*

(iv) *Γ does not virtually fibre;*

(v) *Γ is C^*-simple;*

(vi) *and if $n = 2$, Γ is non-residually finite and not virtually biautomatic.*

In Section 4.8 we adapt a construction of Horbez and Huang [38] to extend actions from a regular tree to the universal cover of a Salvetti complex $\widetilde{S}_L$ with defining graph L. In particular, from a graph of lattices, one obtains a complex of lattices. With a mild hypothesis on the graph L, we use this construction to obtain weakly irreducible non-biautomatic uniform lattices acting on $\widetilde{S}_L \times \mathbb{E}^n$ for $n \geqslant 2$ (Example 11) answering (Q1). We also deduce a consequence about quasi-isometric rigidity of right angled Artin groups with centre containing $\mathbb{Z}^2$.

Corollary 4.C (Example 11 and Corollary 4.8.5)**.** *Let L be a finite simplicial graph on vertices $V = \{v_1, \dots, v_m\}$ and let $W = \{v_1, \dots, v_5\}$. Suppose $A_W < A_L$ is a free subgroup and that $\mathrm{Sym}(W) \leqslant \mathrm{Aut}(L)$. If A_L is irreducible, then there exists a weakly irreducible uniform lattice in $\mathrm{Aut}(\widetilde{S}_L)\times\mathrm{Isom}(\mathbb{E}^n)$ which is not virtually biautomatic nor residually finite. In particular, $A_L \times \mathbb{Z}^2$ is not quasi-isometrically rigid.*

In [58], Thomas constructs a functor from graphs of groups covered by a fixed biregular tree $\mathcal{T}$ to complexes of groups covered by a fixed "sufficiently symmetric" right-angled building X with parameters determined by the valences of $\mathcal{T}$. We will give the relevant definitions in Section 4.9.1. However, note that by [40] a right-angled building X is uniquely specified by a flag complex L and a set of positive integer parameters $\{q_i\}$, if all of the q_i equal q then we say X has uniform thickness q. In Theorem 4.9.4 we show that Thomas' functor theorem takes a graph of lattices to a complex of lattices and in particular $(H \times T)$-lattices to $(H \times A)$-lattices, where $T = \mathrm{Aut}(\mathcal{T})$, $A = \mathrm{Aut}(X)$, and H is a closed subgroup of the isometry group of a CAT(0) space (under mild hypothesis).

As consequences we construct more CAT(0) groups which are not virtually biautomatic (Corollary 4.9.5) and both uniform and non-uniform weakly irreducible lattices in products of fairly arbitrary hyperbolic and Euclidean buildings (Corollary 4.9.9) answering (Q1). We highlight one special case here:

Corollary 4.D (Special case of Corollary 4.9.5)**.** *Let X be the right-angled building of a regular m-gon of uniform thickness $10n$ and let $A = \mathrm{Aut}(X)$. For each $n \geqslant 2$ there exists a weakly irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice which is not virtually biautomatic nor residually finite. In particular, if Y is irreducible, then the direct product of a uniform A-lattice with $\mathbb{Z}^2$ is not quasi-isometrically rigid.*

4.1.1 Structure of the paper

In Section 4.2 we give the relevant background on lattices acting on CAT(0) spaces. In Section 4.3 we give the relevant background on graphs of groups, define graphs of lattices, and prove the structure theorem (Theorem 4.3.3). In Section 4.4 we investigate L^2-cohomology, C^*-simplicity, virtual fibring, and autostackability of $(H \times T)$-lattices. We also compute the rational homological dimension of group schemes over function fields in positive characteristic. In Section 4.5 we provide a number of constructions and explicit examples of $(H \times T)$-lattices. In Section 4.6 we give the relevant background on complexes of groups, define complexes of lattices, and prove the structure theorem (Theorem 4.6.2). In Section 4.7 we study CAT(0) lattices acting on spaces with non trivial Euclidean de Rham factor. We prove the non-biautomaticity criterion for general CAT(0) groups and prove the characterisation of $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices. In Section 4.8 we adapt the construction of Horbez and Huang. In Section 4.9 we give the relevant background on right-angled buildings and Thomas' functor theorem. We then prove our functor theorem (Theorem 4.9.4) and deduce a number of consequences. Finally, in Section 4.10 we record a few questions and conjectures.

Acknowledgements

This paper contains material from the author's PhD thesis. The author would like to thank his PhD supervisor Ian Leary for his guidance and support. The author would like to thank Motiejus Valiunas for sharing his preprint [60] and Ashot Minasyan for helpful comments on an earlier draft of this paper. Additionally, the author would like to thank Naomi Andrew, Pierre Emmanuel-Caprace, Mark Hagen, Susan Hermiller, Jingyin Huang, and Harry Petyt for helpful correspondence and conversations.

4.2 Preliminaries

4.2.1 Lattices and covolumes

Let H be a locally compact topological group with right invariant Haar measure μ. A discrete subgroup $\Gamma \leqslant H$ is a *lattice* if the covolume $\mu(H/\Gamma)$ is finite. A lattice is *uniform* if H/Γ is compact and *non-uniform* otherwise. Let S be a right H-set such that for all $s \in S$, the stabilisers H_s are compact and open, then if $\Gamma \leqslant H$ is discrete the stabilisers are finite.

Let X be a locally finite, connected, simply connected simplicial complex. The group $H = \mathrm{Aut}(X)$ of simplicial automorphisms of X naturally has the structure of a locally compact topological group, where the topology is given by uniform convergence on compacta.

Theorem 4.2.1 (Serre's covolume formula [55])**.** *Let X be a locally finite simply-connected simplicial complex. Let $\Gamma \leqslant H$ be a lattice with fundamental domain Δ, then there is a nomalisation of the Harr measure μ on H, depending only on X, such that for each discrete subgroup $\Gamma < H$ we have*

$$\mu(H/\Gamma) = \mathrm{Vol}(X/\Gamma) := \sum_{v \in \Delta^{(0)}} \frac{1}{|\Gamma_v|}. \qquad \square$$

Note that T the automorphism group of a locally finite tree $\mathcal{T}$ admits lattices if and only if the group T is unimodular (that is the left and right Haar measures coincide). In this case we say $\mathcal{T}$ is unimodular.

4.2.2 Non-positive curvature

We will be primarily interested in lattices in the isometry groups of CAT(0) spaces, we will call these groups CAT(0) *lattices* (note that a uniform CAT(0) lattice is a CAT(0) group). We begin by recording several facts about the structure and isometry groups of general CAT(0) spaces. The definitions and results here are largely due to Caprace and Monod [24] [23] [26].

An isometric action of a group H on a CAT(0) space X is *minimal* if there is no non-empty H-invariant closed convex subset $X' \subset X$, the space X is *minimal* if $\mathrm{Isom}(X)$ acts minimally on X. Note that by [24, Proposition 1.5], if X is cocompact and geodesically complete, then it is minimal. The *amenable radical* of a locally compact group H is the largest amenable normal subgroup. We can now state Caprace and Monod's group and space decomposition theorems mentioned in the introduction.

Theorem 4.2.2. [24, Theorem 1.6] *Let X be a proper* CAT(0) *space with finite dimensional Tits' boundary and assume* $\mathrm{Isom}(X)$ *has no global fixed point in ∂X. There is a* *canonical closed, convex,* $\mathrm{Isom}(X)$*-stable subset* $X' \subseteq X$ *such that* $G = \mathrm{Isom}(X')$ *has a finite index, open, characteristic subgroup* $H \trianglelefteqslant G$ *that admits a canonical decomposition*

$$H \cong \mathrm{Isom}(\mathbb{E}^n) \times S_1 \times \cdots \times S_p \times D_1 \times \cdots \times D_q,$$

for some $n, p, q \geqslant 0$*, where each* S_i *is an almost connected simple Lie group with trivial centre and each* D_j *is a totally disconnected irreducible group with trivial amenable radical.* □

Theorem 4.2.3. [24, Addendum 1.8] *Let* X' *and* H *be as above, then*

$$X' \cong \mathbb{E}^n \times X_1 \times \cdots \times X_p \times Y_1 \times \cdots \times Y_q$$

where each X_i *is an irreducible symmetric space and each* Y_j *is an irreducible minimal* $\mathrm{CAT}(0)$*-space.* □

4.2.3 Irreducibility

Let $X = X_1 \times \cdots \times X_n$ be a product of irreducible proper $\mathrm{CAT}(0)$ spaces and let Γ be a lattice in $H = H_1 \times \cdots \times H_n := \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_n)$, with each H_i non-discrete and acting minimally. There are several possible notions of irreducibility for a lattice in H, moreover, in the general setting of $\mathrm{CAT}(0)$ groups, they are not necessarily equivalent. In the interest of clarity, we recount each of these and summarise their implications, we follow the treatment in [25] [22].

(Irr1) For every $\Sigma \subset \{1, \ldots, n\}$, the projection $\pi_\Sigma : \Gamma \to H_\Sigma$ has dense image. Here we say Γ is *topologically irreducible* or an *irreducible lattice.*

(Irr2) The projection to each factor H_i is injective.

(Irr3) For every $\Sigma \subset \{1, \ldots, n\}$, the projection $\pi_\Sigma : \Gamma \to H_\Sigma$ has non-discrete image. Here we say Γ is *weakly irreducible* or a *weakly irreducible lattice.*

(Irr4) Γ has no finite index subgroup which splits as a direct product of two infinite subgroups. Here we say Γ is *algebraically irreducible.*

Firstly, if each H_i is a centre-free semisimple algebraic group without compact factors then each of the definitions are equivalent [47]. When each H_i is a non-discrete, compactly generated, tdlc group, then [22, Theorem H] summarises all possible implications. Returning to the setting described above we have that **(Irr2)** $\Rightarrow$ **(Irr3)** $\Rightarrow$ **(Irr4)** and if Γ is finitely generated, then by Theorem 4.2.4 we have **(Irr4)** $\Rightarrow$ **(Irr3)**. Note that in general **(Irr4)** $\Rightarrow$ **(Irr2)** fails, unless Γ is residually finite. The following theorem from [23] shows the equivalence of **(Irr3)** and **(Irr4)** for many $\mathrm{CAT}(0)$ lattices.

Theorem 4.2.4. [23, Theorem 4.2] *Let X be a proper* CAT(0) *space, $H < \mathrm{Isom}(X)$ a closed subgroup acting cocompactly on X, and $\Gamma < H$ a finitely generated lattice.*

(i) *If Γ is irreducible as an abstract group, then for for finite index subgroup $\Gamma_0 < \Gamma$ and any Γ_0-equivariant splitting $X = X_1 \times X_2$ with X_1 and X_2 non-compact, the projection of Γ_0 to both $\mathrm{Isom}(X_1)$ and $\mathrm{Isom}(X_2)$ is non-discrete.*

(ii) *If in addition the H-action is minimal, then the converse holds.* □

Finally, we restate a result of Caprace-Monod which we can use as criterion to determine non-residual finiteness of lattices in products.

Theorem 4.2.5. [23, Theorem 4.10] *Let X be a proper* CAT(0) *space such that $G = \mathrm{Isom}(X)$ acts cocompactly and minimally. Let $\Gamma < \mathrm{Isom}(X)$ be a finitely generated algebraically irreducible lattice. Let $\Gamma' = \Gamma \cap H$, where H is given in Theorem 4.2.2. If the projection of Γ' to an irreducible factor of X has non-trivial kernel, then Γ is not residually finite.* □

4.3 Graphs of lattices

In this section we will review Bass-Serre theory, graphs of spaces and tree lattices. These tools will be fundamental to us in the following chapters. We will then define a graph of lattices and prove the structure theorem for $(H \times T)$-lattices.

4.3.1 Graphs of groups

We shall state some of the definitions and results of Bass-Serre theory. In particular, the action will be on the right. We follow the treatment of Bass [2]. Throughout a graph $A = (VA, EA, \iota, \tau)$ should be understood as it is defined by Serre [56], with edges in oriented pairs indicated by $\overline{e}$, and maps $\iota(e)$ and $\tau(e)$ from each edge to its initial and terminal vertices. We will, however, often talk about the geometric realisation of a graph as a metric space. In this case the graph should be assumed to be simplicial (possibly after subdividing) and should have exactly one undirected edge e for each pair $(e, \overline{e})$. We will often not distinguish between the combinatorial and metric notions.

A *graph of groups* $(A, \mathcal{A})$ consists of a graph A together with some extra data $\mathcal{A} = (V\mathcal{A}, E\mathcal{A}, \Phi\mathcal{A})$. This data consists of *vertex groups* $A_v \in V\mathcal{A}$ for each vertex v, *edge groups* $A_e = A_{\overline{e}} \in E\mathcal{A}$ for each (oriented) edge e, and monomorphisms $(\alpha_e : A_e \to A_{\iota(e)}) \in \Phi$ for every oriented edge in A. We will often refer to the vertex and edge groups as *local groups* and the monomorphisms as *structure maps*.

The *path group* $\pi(\mathcal{A})$ has generators the vertex groups A_v and elements t_e for each edge $e \in EA$ along with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\overline{e}} = t_e^{-1}, \\ t_e \alpha_{\overline{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } e \in EA \text{ and } g \in A_e = A_{\overline{e}}. \end{array} \right\}$$

We will often abuse notation and write $\mathcal{A}$ for a graph of groups. The *fundamental group of a graph of groups* can be defined in two ways. Firstly, considering reduced loops based at a vertex v in the graph of groups, in this case the fundamental group is denoted $\pi_1(\mathcal{A}, v)$ (see [2, Definition 1.15]). Secondly, with respect to a maximal or spanning tree of the graph. Let X be a spanning tree for A, we define $\pi_1(\mathcal{A}, X)$ to be the group generated by the vertex groups A_v and elements t_e for each edge $e \in EA$ with the relations:

$$\left\{ \begin{array}{c} \text{The relations in the groups } A_v, \\ t_{\overline{e}} = t_e^{-1} \text{ for each (oriented) edge } e, \\ t_e \alpha_{\overline{e}}(g) t_e^{-1} = \alpha_e(g) \text{ for all } g \in A_e, \\ t_e = 1 \text{ if } e \text{ is an edge in } X. \end{array} \right\}$$

Note that the definitions are independent of the choice of basepoint v and spanning tree X and both definitions yield isomorphic groups so we can talk about *the fundamental group* of $\mathcal{A}$, denoted $\pi_1(\mathcal{A})$.

Let G be the fundamental group corresponding to the spanning tree X. For every vertex v and edge e, A_v and A_e can be identified with their images in G. We define a tree with vertices the disjoint union of all coset spaces G/A_v and edges the disjoint union of all coset spaces G/A_e respectively. We call this graph the *Bass-Serre tree* of $\mathcal{A}$ and note that the action of G admits X as a fundamental domain.

Given a group G acting on a tree $\mathcal{T}$, there is a *quotient graph of groups* formed by taking the quotient graph from the action and assigning edge and vertex groups as the stabilisers of a representative of each orbit. Edge monomorphisms are then the inclusions, after conjugating appropriately if incompatible representatives were chosen.

Theorem 4.3.1. [2] *Up to isomorphism of the structures concerned, the processes of constructing the quotient graph of groups, and of constructing the fundamental group and Bass-Serre tree are mutually inverse.* □

Let $(A, \mathcal{A})$ and $(B, \mathcal{B})$ be graphs of groups. A *morphism of graphs of groups* $\phi : (A, \mathcal{A}) \to (B, \mathcal{B})$ consists of:

(i) A graph morphism $f : A \to B$.

(ii) Homomorphisms of local groups $\phi_v : A_v \to B_{f(v)}$ and $\phi_e = \phi_{\overline{e}} : A_e \to B_{f(e)}$.

(iii) Elements $\gamma_v \in \pi_1(\mathcal{B}, f(v))$ for each $v \in VA$ and $\gamma_e \in \pi(\mathcal{B})$ for each $e \in EA$ such that if $v = i(e)$ then

- $\delta_e := \gamma_v^{-1}\gamma_e \in \mathcal{B}_{f(v)}$;
- $\phi_a \circ \alpha_e = \mathrm{Ad}(\delta_e) \circ \alpha_{f(e)} \circ \phi_e$.

4.3.2 A Structure theorem

In this section we will define a graph of lattices and prove the structure theorem for $(H \times T)$-lattices. We say L is *covirtually an H-lattice* if there exists a finite normal subgroup $F \lhd L$ such that L/F is isomorphic to an H-lattice.

Definition 4.3.2 (Graph of lattices)**.** Let H be a locally compact group with Haar measure μ. A *graph of H-lattices* $(A, \mathcal{A}, \psi)$ is a graph of groups $(A, \mathcal{A})$ equipped with a morphism of graphs of groups $\psi : \mathcal{A} \to H$ where H is considered as a graph of groups of groups over a single vertex such that:

(i) Each local group $A_\sigma \in \mathcal{A}$ is covirtually an H-lattice and the image $\psi(A_\sigma)$ is an H-lattice;

(ii) The local groups are commensurable in $\Gamma = \pi_1(\mathcal{A})$ and their images are commensurable in H;

(iii) For each $e \in EA$ the element t_e of the path group $\pi(\mathcal{A})$ is mapped under ψ to an element of $\mathrm{Comm}_H(\psi_e(A_e))$.

Theorem 4.3.3 (The Structure Theorem - Theorem 4.A)**.** *Let X be a finite dimensional proper* CAT(0) *space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(A, \mathcal{A}, \psi)$ be a graph of H-lattices with locally-finite unimodular non-discrete Bass-Serre tree $\mathcal{T}$, and fundamental group Γ. Suppose $T = \mathrm{Aut}(\mathcal{T})$ admits a uniform lattice.*

(i) *Assume A is finite. If for each local group A_σ the kernel $\mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$, then Γ is a uniform $(H\times T)$-lattice and hence a* CAT(0) *group. Conversely, if Λ is a uniform $(H \times T)$-lattice, then Λ splits as a finite graph of uniform H-lattices with Bass-Serre tree $\mathcal{T}$.*

(ii) *Under the same hypotheses as (i), Γ is quasi-isometric to $X \times \mathcal{T}$.*

(iii) *Assume X is a* CAT(0) *polyhedral complex. Let μ be the normalised Haar measure on H. If for each local group A_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{A_\sigma})$ acts faithfully on $\mathcal{T}$ and the sum $\sum_{\sigma\in VA} \mu(A_\sigma)/|K_\sigma|$ converges, then Γ is a $(H\times T)$-lattice. Conversely, if Λ is a $(H \times T)$-lattice, then Λ splits as a graph of H-lattices with Bass-Serre tree $\mathcal{T}$.*

We will divert the majority of the proof to the proof of Theorem 4.6.2 due to the similarity of the theorem statement and arguments involved in the proof. The minor difference arises from the fact that the category of graphs of groups is not equivalent to the category of 1-complexes of groups (see [58, Proposition 2.1]) due to the difference in morphisms. We highlight the key differences below.

Proof. We first prove (i). The "if direction" is the same as Theorem 4.6.2(i). For the converse note that an $(H \times T)$-lattice Γ splits as a graph of groups $(A, \mathcal{A})$. Indeed, Γ acts on the tree $\mathcal{T}$ through the projection π_T, now we may apply the fundamental theorem of Bass-Serre theory. The projection to H induces a morphism of graphs of groups $\pi_H : \mathcal{A} \to H$. The same argument as Theorem 4.6.2(i) implies that the local groups are commensurable covirtually commensurable H-lattices. In particular, the images of the elements $t_e \in \pi(\mathcal{A})$ for $e \in EA$ are contained in $\mathrm{Comm}_H(\pi_H(A_\sigma))$ for every local group A_σ. ◆

We now prove (ii). By (i) Γ acts properly discontinuously cocompactly on $X \times \mathcal{T}$. The result follows from the Švarc-Milnor Lemma [6, I.8.19]. ◆

The proof of (iii) is almost identical to (i) we will highlight the differences. Since X is a CAT(0) polyhedral complex, it follows that $X \times \mathcal{T}$ is. Now, we may apply Serre's Covolume Formula to $\Gamma = \pi_1(\mathcal{A})$. Let Δ be a fundamental domain for Γ acting on $X \times \mathcal{T}$, then the covolume of Γ may be computed as

$$\sum_{\sigma \in \Delta^0} \frac{1}{|\Gamma_\sigma|} = \sum_{\sigma \in \pi_{\mathcal{T}}(\Delta^0)} \sum_{\tau \in \pi_{\mathcal{T}}^{-1}(\sigma)} \frac{1}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_{\mathcal{T}}(\Delta^0)} \frac{1}{|K_\sigma|} \sum_{\tau \in \pi_{\mathcal{T}}^{-1}(\sigma)} \frac{|K_\sigma|}{|\Gamma_\tau|} = \sum_{\sigma \in \pi_{\mathcal{T}}(\Delta^0)} \frac{\mu(\pi_H(\Gamma_\sigma))}{|K_\sigma|}.$$

Since $\pi_{\mathcal{T}}(\Delta^0)$ can be identified with VA and the later sum converges by assumption, it follows as before that Γ acts faithfully properly discontinuously and isometrically with finite covolume on $X \times Y$. For the converse we proceed as in Theorem 4.6.2(iii). ◆ □

4.3.3 Reducible lattices

Let X be a proper minimal CAT(0) space and $H = \mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. We will now characterise reducible uniform $(H \times T)$-lattices by both their projections to H and T, and by the separability of the vertex stabilisers in the projection to T. Moreover, if H is linear, we will show that all such lattices are linear, and thus, residually finite. We say that a subgroup $\Lambda \leqslant \Gamma$ is *separable* if it is the intersection of finite-index subgroups of Γ, *virtually normal* if Λ contains a finite index subgroup N such that $N \trianglelefteq \Gamma$, and *weakly separable* if it is the intersection of virtually normal subgroups of Γ.

Proposition 4.3.4. *Let X be a proper minimal* CAT(0) *space and $H = \mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be* *a uniform* $(H \times T)$*-lattice equipped with projections* π_H *and* π_T *to* H *and* T *respectively, then the following are equivalent:*

(i) $\pi_H(\Gamma)$ *is an* H*-lattice;*

(ii) $\pi_T(\Gamma)$ *is a* T*-lattice;*

(iii) *For every vertex* $v \in \mathcal{T}$*, the projection of the vertex stabiliser* $\pi_T(\Gamma_v)$ *is separable in* $\pi_T(\Gamma)$*;*

(iv) *There is a vertex* $v \in \mathcal{T}$ *such that the projection of the vertex stabiliser* $\pi_T(\Gamma_v)$ *is weakly separable in* $\pi_T(\Gamma)$*;*

(v) Γ *is a reducible* $(H \times T)$*-lattice.*

Proof. First, we will show that (i) implies (ii), our proof for this case largely follows [14, Proposition 1.2]. Assume $\pi_H(\Gamma)$ is an H-lattice, then $\Gamma \cdot T$ is closed and so $\Gamma \cap T$ is a uniform T-lattice. Now, $\pi_T(\Gamma)$ normalises $\Gamma \cap T$ and hence by [13, 1.3.6] is discrete. Thus, $\pi_T(\Gamma)$ is discrete and so is a lattice in T.

Next, we will show that (ii) implies (i). Assume $\pi_T(\Gamma)$ is a lattice in T and consider the kernel K of the action of Γ on $\mathcal{T}$. We will show that K is a finite index subgroup of $\pi_H(\Gamma)$. Assume that K has infinite index, then $\pi_H(\Gamma)/K \leqslant \pi_T(\Gamma)$ is an infinite subgroup of the vertex stabiliser, a profinite group, and so cannot be discrete. Thus, K has finite index in $\pi_H(\Gamma)$. Since K acts trivially on $\mathcal{T}$ we see that $K = \Gamma \cap H$. Since $\Gamma \cdot H$ is closed it follows K is an H-lattice. Thus, $\pi_H(\Gamma)$ is virtually a lattice in H and therefore an H-lattice.

Clearly, (v) implies (i) and (ii). We will now prove that (i) and (ii) imply (v). By the previous paragraph we have $K \trianglelefteq \pi_H(\Gamma)$ finite index. Let $\Gamma_T = \{\gamma \mid (e, \gamma) \in \Gamma\}$, we want to show that Γ_T is a uniform T-lattice. Since all uniform T-lattices are commensurable Γ_T will be a finite index subgroup of $\pi_T(\Gamma)$. By the first paragraph we see Γ_T is a uniform lattice. Thus, $K \times \Gamma_T$ is a finite index subgroup of Γ and so Γ is reducible.

Now, evidently (iii) implies (iv). To see that (iv) implies (v) we apply [20, Corollary 30] to $\pi_T(\Gamma)$, noting that a cocompact action on a leafless tree does not preserve any subtree, in particular, $\pi_T(\Gamma)$ is discrete. Finally, we show that (v) implies (iii). Observe that $\pi_T(\Gamma)$ is a virtually free T-lattice which splits as a finite graph of finite groups. Since $\pi_T(\Gamma)$ is a finite graph of finite groups, the vertex stabilisers are separable subgroups. □

One immediate consequence of the theorem is that we can determine whether a lattice is irreducible simply by considering the projections to either H or T. Also, note that if H is the automorphism group of a unimodular leafless tree then we recover [14, Proposition 1.2] and and [20, Corollary 32].

We also have the following observations about the linearity and residual finiteness of reducible lattices.

Proposition 4.3.5. *With the same notation as before, assume H is linear (or lattices in H are residually finite). If Γ is a uniform reducible $(H \times T)$-lattice, then Γ is linear (resp. residually finite).*

Proof. If Γ is reducible, then Γ is virtually a direct product of a linear (resp. residually finite) group with a virtually free group. In particular, Γ is virtually a direct product of linear (resp. residually finite) groups and therefore linear (resp. residually finite). □

Corollary 4.3.6. *With H and T as before, assume H is linear. If Γ is a finitely generated uniform $(H \times T)$-lattice, then exactly one of the following holds:*

(i) Γ *is reducible and therefore linear (hence residually finite);*

(ii) Γ *is irreducible and linear (hence residually finite);*

(iii) Γ *is irreducible and non-residually finite.*

Moreover, if H is a connected centre-free semisimple linear algebraic group without compact factors and Γ is irreducible and linear, then Γ is arithmetic and just-infinite.

Proof. The first case follows from the previous proposition. Now, assume Γ is irreducible and $\pi_H(\Gamma)$ is injective, then π_H is a faithful linear representation of Γ and we are in the second case. Since Γ is linear, π_T must be injective otherwise Γ would contradict Theorem 4.2.5. Now, if either of π_T or π_H are not injective, then by Theorem 4.2.5 we see that Γ is not residually finite. Note that π_T not being injective necessarily implies that π_H is not injective because otherwise Γ would admit a faithful linear representation, contradicting non-residual finiteness. To prove the moreover note that Γ is just-infinite follows from the Bader-Shalom Normal Subgroup Theorem [18] applied to the closure of Γ in $H \times T$. The arithmeticity of Γ follows from [4]. □

Let $vb_p(\Gamma)$ denote the *pth virtual Betti number* of Γ which is defined to be the maximum of the pth Betti number over all finite index subgroups of Γ, or ∞ if the set is unbounded.

Proposition 4.3.7. *With H and T as before, assume H is a connected centre-free semisimple linear algebraic group without compact factors. Let Γ be a finitely generated uniform irreducible $(H \times T)$-lattice. If $vb_1(\Gamma) > 0$, then Γ is not residually finite. In particular, if $b_1(\mathcal{T}/\Gamma) > 0$, then Γ is not residually finite.*

Proof. Since Γ is irreducible, by the previous corollary, either Γ is linear and just-infinite, or Γ is not residually finite. Now, if the virtual Betti number of Γ is greater than zero, then a finite index subgroup Γ' of Γ admits $\mathbb{Z}$ as a quotient and so cannot be just infinite. Hence, Γ' is not residually finite and so neither is Γ.

The quotient space $\mathcal{T}/\Gamma$ gives rise to a graph of groups splitting of Γ with Bass-Serre tree $\mathcal{T}$. An easy application of the Mayer-Vietoris sequence applied to $\mathcal{T}$ shows that $b_1(\Gamma) \geqslant b_1(\mathcal{T}/\Gamma)$. □

4.4 Properties of $(H \times T)$-lattices

In this section we will investigate the L^2-cohomology, C^*-simplicity, virtual fibring, and autostackability of $(H \times T)$-lattices in terms of properties of H-lattices. We remark that in each case the proofs are relatively elementary but depend in an essential way on the structure theorem (Theorem 4.3.3).

4.4.1 L^2–cohomology and dimension

Let Γ be a group. Both Γ and the complex group algebra $\mathbb{C}\Gamma$ act by left multiplication on the Hilbert space $\ell^2\Gamma$ of square-summable sequences. The *group von Neumann algebra* $\mathcal{N}\Gamma$ is the ring of Γ-equivariant bounded operators on $\ell^2 G$. The regular elements of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}\Gamma$ can be identified with the *ring of affiliated operators* $\mathcal{U}\Gamma$.

There are inclusions $\mathbb{C}\Gamma \subseteq \mathcal{N}\Gamma \subseteq \ell^2\Gamma \subseteq \mathcal{U}\Gamma$ and it is also known that $\mathcal{U}\Gamma$ is a self-injective ring which is flat over $\mathcal{N}\Gamma$. For more details concerning these constructions we refer the reader to [46] and especially to Theorem 8.22 of Section 8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [46, Section 8.3].

Let Y be a Γ-CW complex as defined in [46, Definition 1.25]. The ℓ^2-homology groups of Y are defined to be the equivariant homology groups $H_i^\Gamma(Y;\mathcal{U}\Gamma)$, and we have

$$b_i^{(2)}(Y) = \dim_{\mathcal{U}G} H_i^\Gamma(Y;\mathcal{U}\Gamma).$$

The ℓ^2-Betti numbers of a group Γ are then defined to be the ℓ^2-Betti numbers of $E\Gamma$. By [46, Theorem 6.54(8)], the zeroth ℓ^2-Betti number of Γ is equal to $1/|\Gamma|$ where $1/|\Gamma|$ is defined to be zero if Γ is infinite. Moreover, if Γ is finite then $b_n^{(2)}(G) = 0$ for $n \geqslant 1$.

In this section we will compute the L^2-Betti numbers of $(H \times T)$-lattices for a very general choice of H and T. Our primary tool will be Gaboriau's invariance of L^2-Betti numbers under measure equivalence.

Two countable groups Γ and Λ are said to be *measure equivalent* if there exist commuting, measure-preserving, free actions of Γ and Λ on some infinite Lebesgue measure space (Ω, m), such that the action of each of the groups Γ and Λ admits a finite measure fundamental domain. The key examples of measure equivalent groups are lattices in the same locally-compact group [34].

Theorem 4.4.1. *Let H be a unimodular locally compact group with lattices and $\mathcal{T}$ be a locally-finite unimodular tree with automorphism group T. Assume H-lattices do not have two consecutive non-zero L^2-Betti numbers. Let Γ be an $(H \times T)$-lattice and let V and E be a representative set of orbits of vertices and edges respectively for the action of Γ on $\mathcal{T}$. We have*

$$b_n^{(2)}(\Gamma) = \sum_{e \in E} b_{n-1}^{(2)}(\Gamma_e) - \sum_{v \in V} b_{n-1}^{(2)}(\Gamma_v).$$

Proof. Let Λ be a reducible $(H \times T)$-lattice and assume Λ splits as $L \times F_n$ where L is an H-lattice. Using the Künneth formula we see that the L^2-Betti numbers of Λ are non-vanishing in the dimensions precisely 1 higher than the non-vanishing L^2-Betti numbers of L. Both Λ and Γ are measure equivalent, since they both lattices in $(H \times T)$. By Gaboriau's theorem on the invariance of L^2-Betti numbers under measure equivalence [30, Theorem 6.3], the L^2-Betti numbers of Γ are non-vanishing in the same degrees as Λ.

Now, we apply the Γ-equivariant cohomology Mayer-Vietoris ([17, Chapter VII.9]) sequence with $\mathcal{U}\Gamma$ coefficients to the filtration of $E\Gamma$ given by the cell structure of the Bass-Serre tree $\mathcal{T}$. Since the vertex and edge stabilisers of the action on $\mathcal{T}$ do not have two sequential non-zero L^2-Betti numbers, neither does Γ. Thus, the sequence degenerates into short exact sequences

$$0 \to \bigoplus_{e \in E} H^n_\Gamma(\Gamma_e; \mathcal{U}\Gamma) \to \bigoplus_{v \in V} H^n_\Gamma(\Gamma_v; \mathcal{U}\Gamma) \to H^{n+1}_\Gamma(\Gamma; \mathcal{U}\Gamma) \to 0$$

and the result follows from the additivity of von Neumann dimension. □

As an immediate corollary we recover the following well known result.

Corollary 4.4.2. *Let Γ be a tree lattice, then all L^2-Betti numbers of Γ vanish, except*

$$b_1^{(2)}(\Gamma) = \sum_{e \in E} \frac{1}{|\Gamma_e|} - \sum_{v \in V} \frac{1}{|\Gamma_v|}.$$

The assumption of not having two sequential non-zero L^2-Betti numbers turns out to not be very restrictive as [46, Theorem 5.12] and [49, Theorem 1.6] demonstrate. For arbitrary CAT(0) lattices, the presence of the Euclidean de Rham factor causes the L^2-Betti numbers to vanish.

Proposition 4.4.3. *Let X be a proper* CAT(0) *space with non-trivial Euclidean de Rham factor and $H \leqslant \mathrm{Isom}(X)$ be a closed subgroup acting minimally and cocompactly. If Γ is an H-lattice, then the L^2-Betti numbers of Γ vanish.*

Proof. By [26, Theorem 2(i)] Γ has a commensurated free abelian subgroup A and so $b_p^{(2)}(A) = 0$ for all $p \geqslant 0$. Now, we apply [3, Corollary 1.4]. □

Remark 4.4.4. More generally, let X be a proper CAT(0) space with canonical closed convex $\mathrm{Isom}(X)$-stable subset $X' \subseteq X$ such that $X' = M \times X_1 \times \cdots \times X_n$, where M is a symmetric space of non-compact type and each X_i is irreducible and minimal. Assume $\mathrm{rank}_{\mathbb{C}}(\mathrm{Isom}(M)) - \mathrm{rank}_{\mathbb{C}}(\mathrm{Isom}(M)) = 0$, let $H \leqslant \mathrm{Isom}(X')$ be a closed subgroup acting cocompactly and minimally and let Γ be an H-lattice. By measure rigidity and repeat applications of the Künneth theorem we have $b_p^{(2)}(\Gamma) = 0$ for $p < \frac{1}{2}\dim(M) + \sum_{i=1}^n b_i$, where b_i is the smallest dimension such that an $\mathrm{Isom}(X_i)$-lattice has a non-vanishing L^2-Betti number. In particular, if either the L^2-cohomology of an $\mathrm{Isom}(X_i)$-lattice vanishes or $\text{f-rk}(M) > 0$ (see [46, Theorem 5.12]), then the L^2-cohomology of Γ vanishes.

4.4.1.1 Rational homological dimension of group schemes over function fields

Let k be the function field of an irreducible projective smooth curve C defined over a finite field $\mathbb{F}_q$. Let S be a finite non-empty set of (closed) points of C. Let $\mathcal{O}_S$ be the ring of rational functions whose poles lie in S. For each $p \in S$ there is a discrete valuation ν_x of k such that $\nu_p(f)$ is the order of vanishing of f at p. The valuation ring $\mathcal{O}_p$ is the ring of functions that do not have a pole at p, that is

$$\mathcal{O}_S = \bigcap_{p \notin S} \mathcal{O}_p.$$

Let $\bar{k}$ denote the algebraic closure of k. Let $\mathbf{G}$ be an affine group scheme defined over $\bar{k}$ such that $\mathbf{G}(\bar{k})$ is almost simple. For each $p \in S$ there is a completion k_p of k and the group $\mathbf{G}(k_p)$ acts on the Bruhat-Tit's building X_p. Thus, we may embed $\mathbf{G}(\mathcal{O}_S)$ into the product $\prod_{p\in S} \mathbf{G}_p$ as an arithmetic lattice.

In [31] it is shown that $\mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p\in S} \dim X_p$. In light of this Ian Leary asked the author what is $\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S))$? The author suspects the result is well-known, however, it does not seem to appear in the literature. It may be possible to obtain an alternative proof using a result of Roman Sauer [54].

Theorem 4.4.5. *Let* $\mathbf{G}$ *be a simple simply connected Chevalley group. Let* k *and* $\mathcal{O}_S$ *be as above, then*

$$\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p\in S} \dim X_p.$$

Proof. We first note that the group $\Gamma := \mathbf{G}(\mathcal{O}_S)$ is measure equivalent to the product $\prod_{p\in S} \mathbf{G}(\mathbb{F}_q[t_p])$ for some suitably chosen $t_p \in \mathcal{O}_p$. By [49, Theorem 1.6] the group $\mathbf{G}(\mathbb{F}_q[t_p])$ has one non-vanishing L^2-Betti number in dimension $\dim(X_p)$. Hence, by the Künneth formula $\mathbf{G}(\mathbb{F}_q[t_p])$ has one non-vanishing L^2-Betti number in dimension

$d = \prod_{p \in S} \dim X_p$ Thus, by Gaboriau's theorem [30], the group Γ has exactly one non-vanishing L^2-Betti number in dimension d. It follows that $\mathrm{hd}_{\mathbb{Q}}(\Gamma) \geqslant d$. The reverse inequality follows from the fact that Γ acts properly on the d-dimensional space $\prod_{p \in S} \dim X_p$. □

4.4.2 C^*–simplicity

Let Γ be a discrete group. The *reduced C^*-algebra* of Γ, denoted $C^*_r(\Gamma)$, is the norm closure of the algebra of bounded operators on $\ell^2(\Gamma)$ by the left regular representation of Γ. We say Γ is *C^*-simple* if $C^*_r(\Gamma)$ has exactly two norm-closed two-sided ideals 0 and $C^*_r(\Gamma)$ itself. A C^*-simple group Γ enjoys a number of properties including having trivial amenable radical, the infinite conjugacy class (icc) property, the unique trace property [11, Theorem 1.3], and having a free action on its Furstenberg boundary $\partial_F \Gamma$ [43].

In 1975 Powers proved that the free group F_2 is C^*-simple [48]. Since this result it has been a major open problem to classify C^*-simple groups, we refer the reader to [27] for a general survey and [11] for a number of recent developments. In the setting of CAT(0) groups there is a characterisation of C^*-simple CAT(0) cubical groups [44] and of linear groups [11, Theorem 1.6]. In this section we will consider the C^*-simplicity of $(H \times T)$-lattices.

The C^*-simplicity of graphs of groups has been considered before [28], however, the methods developed there are not applicable to $(H \times T)$-lattices because the vertex and edge groups are all commensurable. Instead, we will apply the machinery developed in [11] to prove the C^*-simplicity of $(H \times T)$-lattices via properties of either H or the action on $\mathcal{T}$.

Let Γ be a group. We say a subgroup H is *normalish* if for every $n \geqslant 1$ and $t_1 \ldots, t_n$ the intersection $\bigcap_{i=1}^n H^{t_i}$ is infinite.

Proposition 4.4.6. *Let Γ be the fundamental group of a (possibly infinite) graph of finite groups with leafless Bass-Serre tree $\mathcal{T}$ not isometric to $\mathbb{R}$. If Γ is infinite, not virtually cyclic and acts faithfully on $\mathcal{T}$, then Γ is C^*-simple.*

Proof. As Γ is not finite or virtually cyclic Γ has a positive (possibly infinite) first L^2-Betti number. Indeed, the chain complex of the Bass-Serre tree $C_*(\mathcal{T}; \mathcal{U}\Gamma)$, which is concentrated in dimension 0 and 1, may be used to compute the L^2-homology. As Γ is infinite the boundary map is surjective and so the L^2-homology is concentrated in degree 1. We may pair each orbit of 0-cells v with an orbit of 0-cells e contained in the boundary of e, in each case the dimension of the $\mathcal{U}\Gamma$-module is $1/|\Gamma_v|$ or $1/|\Gamma_e|$, and $1/|\Gamma_e| - 1/|\Gamma_v| \geqslant 0$. Since Γ is non-trivial and not virtually cyclic some of these inequalities must be strict. In particular, we conclude Γ has a (possibly infinite) non-trivial first L^2-Betti number equal to the sum of these partial sums plus extra terms

$1/|\Gamma_e|$ for any orbit of edges not accounted for. Since Γ has a trivial amenable radical and a non-trivial L^2-Betti number we may apply [11, Theorem 6.5] to deduce that Γ is C^*-simple.

Alternatively, we first note that any normalish subgroup of Γ contains a free subgroup since Γ is a faithful graph of finite groups and is not virtually cyclic. Now, we apply [11, Theorem 6.2] to deduce that Γ is C^*-simple. □

The following theorem and corollary give a partial answer to two questions of de la Harpe [27] and consider the more general case of an arbitrary graph of groups. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. The theorem implies the following lattices are C^*-simple:

- H is a semisimple Lie group with trivial centre and Γ is a graph of S-arithmetic lattices. This new whenever Γ is not residually finite. To see this, apply (i) and (i);
- Γ is a lattice in a product of trees. To see this, apply (iii);
- Γ is the fundamental group of a graph of lattices where each vertex and edge group acts on the universal cover of a Salvetti complex corresponding to a right-angled Artin group with trivial centre. To see this, apply (i) and (i) to [11, Theorem 1.6];
- H is the automorphism group of an affine building with no irreducible factor isometric to $\mathbb{E}^n$ and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (i);
- H is the automorphism group of a hyperbolic building and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (i);
- H is a product of the above and Γ is an irreducible $(H \times T)$-lattice. To see this, apply (i);
- $\mathrm{Isom}(\mathbb{E}^n)$ and Γ is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Note this characterises irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices and will follow from (ii) (see Theorem 4.7.13).

The results in this list are new whenever the $(H \times T)$-lattices in question are not cubical or linear groups.

Theorem 4.4.7. *Let $X = X_1 \times \cdots \times X_k$ be a product of proper minimal cocompact* $\mathrm{CAT}(0)$*-spaces each not isometric to $\mathbb{R}$ and let $H = \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_k)$ act without fixed point at infinity. Let $\mathcal{T}$ be a locally-finite non-discrete unimodular leafless tree and $T = \mathrm{Aut}(\mathcal{T})$. Let $n \geqslant 0$ and $\Gamma < \mathrm{Isom}(\mathbb{E}^n) \times H \times T$ be a finitely generated lattice.*

(i) *Assume Γ is reducible and $n = 0$, then Γ is C^*-simple if and only if $\Gamma \cap H$ is C^*-simple, and Γ has the icc property.*

(ii) *Assume* Γ *is weakly irreducible. If one of the following holds:*

(i) H*-lattices have no normalish amenable subgroups;*

(ii) $\mathrm{Ker}(\pi_T)$ *is trivial and* $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)\times H)})$ *is infinite;*

(iii) H*-lattices have a non-zero* L^2*-Betti number and trivial amenable radical;*

then Γ *is* C^**-simple.*

Proof. In the reducible case Γ virtually splits as $F_n \times \Gamma_H$. The result follows from the following three observations [27, Proposition 19 (i,iii,iv)], a direct product of two C^*-simple groups is C^*-simple, finite index subgroups of C^*-simple groups are simple, and a virtually C^*-simple group is C^*-simple if and only if it satisfies the icc property.

Now, assume Γ is irreducible. We will show that (i) implies C^*-simplicity. Since Γ is finitely generated $\mathcal{G} = \mathcal{T}/\Gamma$ is finite. We first show that any amenable normalish subgroup N of Γ must fix a vertex of $\mathcal{T}$. Let $g \in \Gamma$ act as a hyperbolic element on $\mathcal{T}$, choose any other element $h \in \Gamma$ acting hyperbolically on $\mathcal{T}$ with an axis not equal to g, then any normalish subgroup N containing g contains the free group $\langle g, h\rangle$ and so cannot be amenable. Thus, N fixes a vertex of $\mathcal{T}$. Now, by Theorem 4.3.3 every vertex and edge stabiliser of Γ is a finite-by-H-lattice group. Since by assumption H-lattices do not contain any normalish amenable subgroups, neither does Γ. It remains to verify that Γ has no finite normal subgroups, but Γ has trivial amenable radical by [23, Corollary 2.7]. In particular the result now follows from [11, Theorem 6.2].

We next prove (ii) implies C^*-simplicity. Let $K = \mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)\times H)})$, we have that Γ is an extension of K by $\pi_{\mathrm{Isom}(\mathbb{E}^n)\times H)}(\Gamma)$. Now, K is a (possibly infinite) graph of finite groups acting faithfully on $\mathcal{T}$. Indeed, restricting $\pi := \pi_{\mathrm{Isom}(\mathbb{E}^n)\times H}$ to a vertex stabiliser $\Gamma_v < \Gamma$ of the action on $\mathcal{T}$, by Theorem 4.3.3 we see $\mathrm{Ker}(\pi|_{\Gamma_v})$ is finite. Every finite subgroup of Γ, and hence K, is conjugate to a finite subgroup of some vertex stabiliser. Thus, the graph of groups decomposition is given by $\mathcal{T}/K$.

We claim K is not virtually infinite cyclic. Indeed, if K was virtually cyclic, then there exists a commensurated infinite cyclic subgroup $Z < K < \Gamma$. By [26, Theorem 2(i)] Z acts properly on $\mathbb{E}^n$ in the decomposition of X. But $Z < K$, a contradiction.

It follows the group K is C^*-simple by Proposition 4.4.6. Because $\mathrm{Ker}(\pi_T)$ is trivial, every element acts non-trivially on $\mathcal{T}$ and so the centraliser $C_\Gamma(K)$ is trivial. Now, we apply [11, Theorem 1.4] to prove the result.

Finally, we will prove (iii) implies C^*-simplicity. We apply the cohomology Γ-equivariant Mayer-Vietoris sequence with $\mathcal{U}\Gamma$ coefficients arising from filtering $E\Gamma$ by the Bass-Serre tree [17, Chapter VII.9]. Since $\mathcal{T}$ is not a quasi-line there is a vertex v connected to an edge e such that the stabilisers satisfy $|\Gamma_v : \Gamma_e| \geqslant 3$, thus the L^2-Betti numbers of Γ_e are at least 3 times the L^2-Betti number of Γ_v. Now, additivity of von Neumann dimension over exact sequences and a simple counting argument implies every $(H \times T)$-lattice must have a non-trivial L^2-Betti number. Alternatively, we note that every $(H \times T)$-lattice is measure equivalent to $L \times F_r$ where L is an H-lattice and F_r is a free group. Now, an application of the Kunneth formula yields that $L \times F_r$ has a non-trivial L^2-Betti number and so by Gaboriau's theorem [30, Theorem 6.3] so does every $(H \times T)$-lattice. By [23, Corollary 2.7] every $(H \times T)$-lattice has trivial amenable radical, the result follows from [11, Theorem 6.5]. □

A near identical proof to that of (i) yields the following corollary.

Corollary 4.4.8. *Let Γ be the fundamental group of a finite graph of groups. Assume, that for each edge and vertex that are incident that the intersection of the corresponding edge group and the vertex group does not contain either a normalish amenable subgroup or a non-trivial finite normal subgroup. If Γ is irreducible as an abstract group, then Γ is C^*-simple.*

4.4.3 Fibring

Recall that a group Γ is said to *algebraically fibre* if there is a non-trivial homomorphism $\phi : \Gamma \to \mathbb{Z}$ such that $\mathrm{Ker}(\phi)$ is finitely generated. If Γ has a finite index subgroup which algebraically fibres, then we say Γ *virtually fibres*.

Fix a finite generating set S for Γ. A character $0 \neq \phi \in H^1(\Gamma;\mathbb{R}) = \mathrm{Hom}(\Gamma,\mathbb{R})$ lies in the *first Bieri-Neumann-Strebel-Renz (BNSR) invariant* $\Sigma^1(\Gamma)$ if and only if the full subgraph of $\mathrm{Cay}(\Gamma, S)$ spanned by $\{g \in \Gamma \mid \phi(g) \geqslant 0\}$ is connected. The relevance of the BNSR invariant is due to the following classical theorem of Bieri-Neumann-Strebel.

Theorem 4.4.9. [15, Theorem B1] *Let Γ be a finitely generated group and let $\phi : \Gamma \to \mathbb{Z}$ be non-trivial, then $\mathrm{Ker}(\phi)$ is finitely generated if and only if $\{\phi, -\phi\} \subseteq \Sigma^1(\Gamma)$.* □

Theorem 4.4.10. *Let X be a finite dimensional proper $\mathrm{CAT}(0)$ space and let $H = \mathrm{Isom}(X)$ be cocompact and minimal. Let $\mathcal{T}$ be a locally finite unimodular leafless tree not isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose $H^1(L;\mathbb{R}) = 0$ for all H-lattices L, then every $(H \times T)$-lattice Γ does not virtually fibre.*

Note that the hypothesis that $H^1(L;\mathbb{R}) = 0$ for all H-lattices L is satisfied for instance when H is a higher rank semi-simple Lie group. It is unclear whether the hypothesis can be weakened whilst working with such a general choice of Γ and H.

Proof. Let Γ be an $(H \times T)$-lattice, then Γ splits as a graph of H-lattices A. In particular, every vertex and edge group is finite-by-H-lattice and so has trivial first cohomology.

Now, we apply the Mayer-Vietoris sequence of the graph of groups decomposition (see [17, Chapter VI.9]) to obtain an exact sequence

$$0 \longrightarrow H^0(\Gamma) \longrightarrow \bigoplus_{v\in VA} H^0(\Gamma_v) \longrightarrow \bigoplus_{e\in EA} H^0(\Gamma_e) \longrightarrow H^1(\Gamma) \longrightarrow 0.$$

Where the ending 0 is due to the fact $\bigoplus_{v\in VA} H^1(\Gamma_v) = 0$. It follows that $H^1(\Gamma;\mathbb{R}) = H^1(\mathcal{T}/\Gamma;\mathbb{R})$.

Claim: Γ splits as a reduced graph of groups and is not an ascending HNN extension.

We may assume the graph of groups is reduced by contracting any edges with a trivial amalgam $L *_L K$. Note that these contractions do not change the vertex and edge stabilisers, but may change the Bass-Serre tree (the tree will still not be quasi-isometric to $\mathbb{R}$ since there are necessarily other vertices of degree at least 3).

Now for Γ to be an ascending HNN-extension A must consist of a single vertex and edge. Let t be the stable letter of Γ, then t acts as an isometry on X. In particular, by considering covolumes of H-lattices acting on X, the two embeddings of the edge group Γ_e into the vertex group Γ_v must have the same index. Now, since $\mathcal{T}$ is not a quasi-line, these embeddings must have index at least 2 yielding the claim. ◆

Now, $H^1(\Gamma;\mathbb{R}) = \mathrm{Hom}(\Gamma,\mathbb{R})$ and so every character $\phi \in \mathrm{Hom}(\Gamma,\mathbb{R})$ vanishes on every vertex and edge group of the graph of groups decomposition A. Moreover, we may assume A is reduced by contracting any edges of the from $B *_C C$. Thus, we may apply [21, Proposition 2.5] to deduce $\phi \notin \Sigma(\Gamma)$. As this is true for every $(H \times T)$-lattice, it follows Γ does not virtually fibre. □

4.4.4 Autostackability

In this section we will discuss *autostackability* of $(H \times T)$-lattices in terms of H-lattices. The property was introduced by Brittenham, Hermiller and Holt in [7] to simultaneously generalise automatic groups and groups with finite rewriting systems - we will not define the property here since our proofs do not require the definition and are elementary. The class of autostackable groups is broad, including all automatic groups, 3-manifold groups [9], Thompson's group F [19], the Baumslag-Gersten group [39], and some groups not of type FP_3 [8]. In spite of this, it appears to be unknown if every group with solvable word problem is autostackable. Moreover, autostackability properties of the class of CAT(0) groups have largely gone unstudied. In light of Leary and Minasyan's examples of CAT(0) groups which are not biautomatic [45] it would be desirable to determine the autostackability properties of these and related groups.

Theorem 4.4.11. *Let X be a finite dimensional proper* CAT(0) *space and $H = \mathrm{Isom}(X)$. Let $\mathcal{T}$ be a locally finite unimodular tree and let $T = \mathrm{Aut}(\mathcal{T})$. If uniform H-lattices are (auto)stackable, then uniform $(H \times T)$-lattices are (auto)stackable. Moreover, if X is*

CAT(0) *polyhedral complex and finitely presented* H*-lattices are (auto)stackable, then finitely presented* $(H \times T)$*-lattices are (auto)stackable.*

Proof. In either case, by Theorem 4.3.3 we see Γ splits as a graph of H-lattices. In particular, every local group is a commensurable finite-by-H-lattice. Now, by [8, Theorem 3.3] (auto)stackable groups are closed under extension, so we see the local groups are (auto)stackable. By [9, Proposition 4.2] (see also [8, Theorem 3.4]), a group is (auto)stackable with respect to any finite index subgroup. Finally, [9, Theorem 3.5] states that the fundamental group of a graph of groups whose vertex groups are (auto)stackable with respect to the edge groups is (auto)stackable. In particular, Γ is (auto)stackable. ☐

The following corollary follows by induction on the number of trees n with the base case given by the previous theorem. The inductive step is given by applying previous theorem to deduce the result holds for n trees after assuming the result holds for $n-1$ trees. As an example the corollary applies whenever X is CAT(-1).

Corollary 4.4.12. *Let* X *and* H *be as above. Let* $\prod_{i=1}^n \mathcal{T}_i$ *be a product of trees and let* $T = \prod_{i=1}^n \mathrm{Aut}(\mathcal{T}_i)$. *If uniform* H*-lattices are (auto)stackable, then uniform* $(H \times T)$*-lattices are (auto)stackable. Moreover, if* X *is* CAT(0) *polyhedral complex and finitely presented* H*-lattices are (auto)stackable, then finitely presented* $(H \times T)$*-lattices are (auto)stackable.*

In Theorem 4.7.13 we will prove that all irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices are not virtually biautomatic, generalising the result of Leary and Minasyan [45]. However, the following corollary proves that all of these lattices are in fact (auto)stackable.

Corollary 4.4.13. *Uniform* $(\mathrm{Isom}(\mathbb{E}^n) \times T)$*-lattices are (auto)stackable. In particular, the Leary-Minasyan groups are (auto)stackable.*

Proof. A free abelian group is automatic and hence (auto)stackable. As (auto)stackability is closed under finite extensions it follows $\mathrm{Isom}(\mathbb{E}^n)$-lattices are (auto)stackable. Now, we apply the previous theorem. ☐

4.5 Constructions and examples

In this section we will detail a number of constructions and explicit examples of lattices in products of CAT(0) spaces and trees.

4.5.1 Residual finiteness and amalgams

For each symmetric space X of non-compact type with associated Lie group H we will construct infinitely many non-residually finite irreducible $(H \times T)$-lattices, where T is the automorphism group of an appropriate Bass-Serre tree. More generally the construction applies whenever there are upper bounded chains in the poset $(\mathrm{Lat}(H), \leqslant)$.

Theorem 4.5.1. *Let X be a* CAT(0) *space, let $H = \mathrm{Isom}(X)$ act cocompactly and minimally. Let A, B be commensurable uniform H-lattices such that $A \neq B$. Let $C \leqslant_{f.i.} A \cap B$ and $\Gamma = A *_C B$. Let $\mathcal{T}$ be the Bass-Serre tree of Γ and $T = \mathrm{Aut}(\mathcal{T})$. Assume $\mathcal{T}$ is unimodular, then Γ is a $(H \times T)$-lattice. Moreover,*

(i) *If $\langle A, B \rangle < H$ is not an H-lattice, then Γ is an irreducible $(H \times T)$-lattice.*

(ii) *If Γ is irreducible and C is a proper subgroup of $A \cap B$, then Γ is not residually finite.*

Proof. The fact that Γ is a lattice follows from Theorem 4.3.3. Now, (i) follows from Theorem 4.3.4, since if $\langle A, B \rangle$ is not a lattice, then $\pi_H(\Gamma)$ is not a lattice and so Γ is not reducible and hence irreducible. To prove (ii), consider an element γ in $(A \cap B) - C$ and words γ_a and γ_b representing γ in the generating sets of copies of A and B in Γ. Since, $\gamma_a \gamma_b^{-1}$ is not contained in the copy of C in Γ, the element acts non-trivially on $\mathcal{T}$, and so is non-trivial. However, $\pi_H(\gamma_a) = \pi_H(\gamma_b)$, so $\pi_H(\gamma_a \gamma_b^{-1}) = 1_H$. But Γ is irreducible and $\pi_H(\Gamma)$ has a non-trivial kernel so we can apply Caprace and Monod's criteria (Theorem 4.2.5). □

The following lemma is immediate, but combined with the previous theorem, it implies that we can construct non-residually finite groups out of uniform lattices in each Lie group corresponding to a symmetric space of non-compact type.

Lemma 4.5.2. *Let H be a locally compact group with Haar measure μ. If there exists a bound ϵ on the minimal μ-covolume of lattices in H and the set of possible covolumes of H-lattices is discrete, then the poset* $\mathrm{Lat}(H)$ *has maximal elements.*

Example 1. Let X be a symmetric space of non-compact type and H the associated Lie group. Let A and B be commensurable maximal H-lattices such that $A \neq B$. Let C be a finite index proper subgroup of $A \cap B$, then $\Gamma = A *_C B$ is a non-residually finite $(H \times T)$-lattice. Such examples exist by considering arithmetic lattices Γ in H. Indeed, Margulis' commensurator criterion states that $\mathrm{Comm}_H(\Gamma)$ is dense in H and so there exist lattices commensurable to Γ which are not contained in Γ.

In the more general setting of CAT(0)-spaces we have the following corollary.

Corollary 4.5.3. *Assume $\Gamma = A *_C B$ is a uniform $(H \times T)$-lattice such that $A \neq B$ and neither $A < B$ nor $B < A$. If A or B is the upper bound of a chain in $(Lat, \leqslant)$, then Γ is irreducible. Moreover, if C is a proper subgroup of $A \cap B$, then Γ is non-residually finite.*

Proof. Assume without loss of generality that A is the upper bound, then $\langle A, B\rangle$ cannot be a lattice because it would contain A, contradicting the maximality of A. Thus, we can apply Theorem 4.5.1. □

Example 2 (Change of tree)**.** Given an edge transitive but not vertex transitive irreducible $(H \times T_{k,\ell})$-lattice Γ one may construct a non-residually finite irreducible $(H \times T_{mk,n\ell})$-lattice for all $m, n \geqslant 2$ as follows:

Firstly, note Γ splits as a graph of H-lattices. Indeed, $\Gamma = A *_C B$ where A, B and C are covirtually H-lattices. We may assume that A stabilises a vertex of valence k and B stabilises a vertex of valence ℓ. Let N_A and N_B be finite groups of order m and n respectively and pick split extensions $\widetilde{A} = N_A \rtimes A$ and $\widetilde{B} = N_B \rtimes B$. We may construct a graph of lattices by considering the graph of groups corresponding to $\widetilde{A} *_C \widetilde{B}$. The representations of $\widetilde{A}$ and $\widetilde{B}$ are the given by the composites $\widetilde{A} \twoheadrightarrow A \to H$ and $\widetilde{B} \twoheadrightarrow B \to H$. The resulting fundamental group $\widetilde{\Gamma}$ acts on the $(mk, n\ell)$-regular tree, the lattice is irreducible and non-residually finite by Theorem 4.5.1.

This technique gives the following partial solution to the problem of realising lattices in every possible tree for H a rank one real Lie group with trivial centre.

Example 3. Let $H = \mathbf{H}(\mathbb{R})$ be a rank one real Lie group with trivial centre and $H_p = \mathbf{H}(\mathbb{Q}_p)$ denote the same group scheme over the p-adic numbers for some prime p. Let X be the rank-one symmetric space associated to H. The Bruhat-Tits' building for H_p is a tree of valence given by some function f of the prime p. In particular, there is an edge transitive but not vertex transitive S-arithmetic lattice acting on $X \times \mathcal{T}_{f(p)}$. By the previous example we may construct irreducible non-residually finite lattices acting on $X \times \mathcal{T}_{mf(p),nf(p)}$ for all $m, n \geqslant 2$.

These groups are C^*-simple by Theorem 4.4.7, austostackable by Theorem 4.4.11, and if X is $2n$-dimensional, then the groups have a non-trivial L^2-Betti number in dimension $n+1$ by Theorem 4.4.1. If X is odd-dimensional, then the L^2-cohomology vanishes.

Concretely, in the case of $H = \mathrm{PSL}_2(\mathbb{R})$, the function f is given by $f(p) = p + 1$, so we obtain irreducible lattices acting on the $(m(p+1), n(p+1))$-regular tree for all primes p and integers $m, n \geqslant 2$.

4.5.2 Vertex transitive lattices

In this section we will detail some constructions for lattices in a product of a CAT(0) space and a tree such that the lattices act vertex transitively on the tree.

Proposition 4.5.4. *Let $L \leqslant H$ be groups and $t \in \mathrm{Comm}_H(L)$, then there exist finite-index subgroups $J, K \leqslant L$ such that $J^t = K$*

Proof. By definition $K = L \cap L^t$ has finite index in L. Now, set $J = K^{t^{-1}}$, this clearly also has finite index in L. □

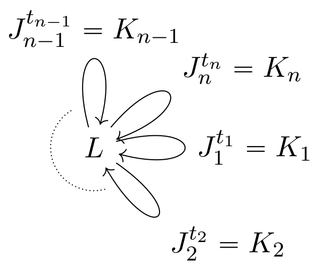

FIGURE 4.1: A single vertex graph of groups.

Let X be a metric space and let $H = \mathrm{Isom}(X)$. Let L be a H-lattice and let $t_1, \dots, t_n \in \mathrm{Comm}_H(L)$. Assume that t_i conjugates a finite-index subgroup $J_i \leqslant L$ to a finite-index subgroup $K_i \leqslant L$ (existence of H_i and K_i is given in the next proposition). In light of Proposition 4.5.4, whilst slightly abusing notation, we can construct a single vertex graph of groups $\mathcal{G}$ where all of the edges are loops (Figure 4.1). We now define $\Gamma = \mathcal{G}(L, \{(J_1, t_1), \dots, (J_n, t_n)\}) := \pi_1(\mathcal{G})$. We can associate to Γ the Bass-Serre tree $\mathcal{T}$ of the graph of groups $\mathcal{G}$. Note that $\mathcal{T}$ is an infinite, locally finite, $(\sum_{i=1}^n |\Gamma : J_i| + |\Gamma : K_i|)$-regular, simplicial tree.

Lemma 4.5.5. *Let Γ be a lattice in a rank-one Lie group H with symmetric space X of non-compact type. Let t be an infinite order elliptic element of H, then*

$$L := \bigcap_{n \in \mathbb{Z}} \Gamma^{t^n}$$

has infinite index in Γ.

Proof. Assume L has finite index, then by Garland and Raghunathan [32] [33], the quotient X/L has finitely many cusps with bounded intersection. Let p be the fixed point of t and consider a Dirichlet domain $\Delta = \Delta_p(L)$ for L at p. Since X/Γ has finitely many cusps, Δ has finitely many sectors (each with bounded intersection) going to infinity. The $\langle t \rangle$-orbit of such a sector is unbounded (indeed it traces out a copy of S^1 in ∂X), but this contradicts Garland-Raghunathan and so we conclude L must have infinite index. □

Theorem 4.5.6. *Let X be a rank-one symmetric space of non-compact type and let H be the associated Lie group. Let L be an H-lattice, $t_1, \dots, t_k \in \mathrm{Comm}_H(\Gamma)$ and let $\Gamma := \mathcal{G}(L, \{(J_i, t_i)\})$ with Bass-Serre tree $\mathcal{T}$. Let $T = \mathrm{Aut}(\mathcal{T})$. If $\pi_H\langle t_1, \dots, t_k\rangle$ contains an infinite order elliptic element t, then Γ is a weakly irreducible $(H \times T)$-lattice.*

Proof. Clearly, the projection of Γ to the group H is not discrete because Γ contains an infinite order elliptic element. Now, the vertex stabilisers of the action of Γ on $\mathcal{T}$ are conjugates of $L < G$ Thus, the kernel of the action is equal to $\mathrm{Core}(\Gamma, L)$, but by the previous lemma this is infinite index in L. It follows that the image of Γ is an infinite subgroup of the vertex stabiliser in T (a compact profinite group) and so cannot have discrete image. □

Example 4. Let H be a non-compact simple Lie group and $\mathcal{O}$ the ring of integers of some number field k. Assume that either $H(\mathcal{O})$ is either an irreducible uniform lattice or rank-one. Now, choose an infinite order elliptic element $t \in \mathrm{Comm}_H(H(\mathcal{O}))$ and construct the group $\Gamma = \mathcal{G}(H(\mathcal{O}), t)$ with Bass-Serre tree $\mathcal{T}$. Let $T = \mathrm{Aut}(\mathcal{T})$. By Theorem 4.3.3 we conclude that Γ is a lattice in $G = (\prod_{\sigma \in S^\infty} H(K^\sigma) \times T)$. Moreover, if t is irreducible, then Γ is a weakly and algebraically irreducible lattice. To see Γ is weakly irreducible, note that the projection of Γ to any sub-product of G is clearly non-discrete. Now, we apply Theorem 4.2.4 to see Γ is algebraically irreducible.

In the next example we will present an explicit presentation of a non-residually finite, irreducible, vertex and edge transitive $(\mathrm{PSL}_2(\mathbb{R}) \times T_{60})$-lattice.

Example 5. Consider the following matrices in $\mathrm{SL}_2(\mathbb{R})$ given by

$$a = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{3}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}, \qquad b = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}(\sqrt{2}-1) \\ \frac{1}{2}(-3\sqrt{2}-3) & \frac{1}{2} \end{bmatrix},$$

$$c = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}(\sqrt{2}+1) \\ \frac{1}{2}(-3\sqrt{2}+3) & \frac{1}{2} \end{bmatrix}, \qquad t = \begin{bmatrix} \frac{1}{5} & \frac{2}{5}\sqrt{2} \\ -\frac{6}{5}\sqrt{2} & \frac{1}{5} \end{bmatrix}.$$

The projectivisation of the matrices a, b and c in $\mathrm{PSL}_2(\mathbb{R})$ generate a Fuchsian group of signature $[0; 2, 2, 3, 3]$ with presentation $L = \langle a, b, c \mid a^2 = b^3 = c^3 = (c^{-1}ab^{-1})^2 = 1\rangle$. The conjugate of L by the infinite order elliptic element t in $\mathrm{PSL}_2(\mathbb{R})$ yields an isometric Fuchsian group $L^t = \langle \alpha, \beta, \gamma\rangle$. The intersection is generated by

$$\begin{aligned} K =& \langle acb^{-1}a,\ cac^{-1},\ b^{-1}acb^{-1},\ c^{-1}bca,\ bcabc^{-1},\ b^{-1}cbc^{-1}ab^{-1},\ b^{-1}c^{-1}bc^{-1}b^{-1},\ c^{-1}acab^{-1} \\ & ababc^{-1}ba,\ abacb^{-1}ab^{-1},\ babac^{-1}b^{-1}c^{-1},\ babcac^{-1}b,\ b^{-1}cabc^{-1}b^{-1}c^{-1}\rangle. \end{aligned}$$

We also find that K is index 30 and has signature $[5; 2, 2, 2, 2]$. Since K is contained in L, to complete our construction we simply need to find $J := t^{-1}(K)$, which will also be contained in Γ. A lengthy calculation yields

$$\begin{aligned} J =&\langle c^{-1}abab^{-1},\ b^{-1}ab,\ cab^{-1}c,\ acb^{-1}abac,\ cabac^{-1}ab,\ c^{-1}acab^{-1}a,\ babac^{-1}bac^{-1}b^{-1}, \\ &ac^{-1}ba,\ bacb^{-1}acb^{-1}c^{-1}ab^{-1}ac^{-1},\ bac^{-1}bc^{-1},\ cbc^{-1}ac^{-1}ab,\ cb^{-1}abcac, \\ &c^{-1}ac^{-1}ac^{-1}b^{-1}acb^{-1}\rangle. \end{aligned}$$

The group $\Gamma = \langle a,b,c,t \mid a^2 = b^3 = c^3 = (c^{-1}ab^{-1})^2 = 1,\ J^t = K\rangle$ is a non-residually finite irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})\times T_{60}$. By Theorem 4.4.1 the only non-vanishing L^2-Betti number of Γ is in dimension 2 and is equal to $-\frac{1}{3}-(-10) = \frac{29}{3}$. By Theorem 4.4.7 Γ is C^*-simple, by Theorem 4.4.11 Γ is autostackable, and by the same argument as in the proof of Theorem 4.4.10, Γ does not algebraically fibre. Moreover, if Γ has first virtual Betti number equal to 1, then Γ does not virtually fibre.

Example 6 (Mixed products)**.** Consider a uniform weakly irreducible lattice in $\mathrm{PSL}_2(\mathbb{R})\times T_{60}$ constructed as a single vertex graph of groups $\mathcal{G}(\Gamma,t)$, assume that the stable letter t acts on $\mathbb{R}\mathbf{H}^2$ as an infinite order elliptic rotation. Similarly, consider a uniform weakly irreducible lattice in $\mathrm{Isom}(\mathbb{E}^2)\times T_{10}$ constructed as a single vertex graph of groups $\mathcal{G}(\mathbb{Z}^2,s)$, assume that the stable letter s acts on $\mathbb{E}^2$ as an infinite order elliptic rotation (such examples were considered by Leary and Minasyan in [45]).

We will now construct a uniform lattice in $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{Isom}(\mathbb{E}^2) \times T_{300}$. Let $\Lambda := \mathcal{G}(\Gamma \times \mathbb{Z}^2, r)$, where r acts as t on $\mathbb{R}\mathbf{H}^2$ and as s on $\mathbb{E}^2$. We claim the projections to each sub-product of the factors are non-discrete and so Λ is not commensurable with any reducible lattice. Thus, by Theorem 4.3.4, Λ is an weakly irreducible lattice.

To prove the claim we investigate each projection in turn. Clearly, the projections to $\mathrm{PSL}_2(\mathbb{R})$, $\mathrm{Isom}(\mathbb{E}^2)$ and $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{Isom}(\mathbb{E}^2)$ are non-discrete. Moreover, by Theorem 4.5.6 or [45, Theorem 7.5] it is easy to see the projection to T_{300} is non-discrete. In fact more is true, the projection is faithful. In light of this it is easy to see the projections to $\mathrm{PSL}_2(\mathbb{R}) \times T_{300}$ and $\mathrm{Isom}(\mathbb{E}^2) \times T_{300}$ are non-discrete.

Note that the choices of the ambient groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{Isom}(\mathbb{E}^2)$ were arbitrary. Indeed, the reader can pick any combination of symmetric spaces of non-compact (and Euclidean) type, or any irreducible proper minimal CAT(0) space which contains lattices which have a non-discrete commensurator and construct a weakly irreducible lattice in the product of the automorphism group of the Bass-Serre tree and the associated real simple Lie groups (and $\mathrm{Isom}(\mathbb{E}^n)$) and the isometry group of the CAT(0) space. This is markedly different to the arithmetic setting where the Lie groups must be isogenous.

Example 7 (Non-uniform lattices in products of trees)**.** Fix a prime p. Consider the linear algebraic group $H = \mathrm{PSL}_2(\mathbb{F}_p((t)))$ and the non-uniform lattice $L = \mathrm{PSL}_2(\mathbb{F}_p[t]) < H$. The Bruhat-Tits' building for H is a $(p+1)$-regular tree $\mathcal{T}$ and L acts with finite covolume and fundamental domain an infinite ray. Let $t \in \mathrm{Comm}_H(L)$ be infinite order and elliptic. By Proposition 4.5.4 there exist finite index subgroups $J, K \leqslant L$ such that

$J^t = K$. Let $n \geqslant 1$ and consider the HNN-extension Γ of L^n over finite index subgroups J^n and K^n where each copy of J is mapped to the corresponding copy of K by t. The group Γ is non-uniform lattice acting on $\mathcal{T}^n_{p+1} \times \mathcal{T}_{2k^n}$ where $k = |L : J|$. Moreover, it is easy to see that Γ is a weakly irreducible lattice.

More generally by [1] non-uniform tree lattices of 'Nagao type' have a dense commensurator in the full automorphism group of the universal covering tree. The construction can be easily adapted to this setting.

4.5.3 The universal covering trick

In this section we will compare the notion of a graph of lattices with the "universal covering trick" of Burger–Mozes [13, Section 1.8] and generalised by Caprace–Monod [23, Section 6.C]. In particular, we will show how in many cases one can obtain a graph of lattices from the universal covering trick. We take the opportunity to point out that many of the groups constructed in the previous sections cannot be obtained from universal covering trick.

Example 8 (The universal covering trick)**.** Let A be the geometric realisation of a locally finite graph (not reduced to a single point) and let $Q < \mathrm{Isom}(A)$ be a vertex transitive closed subgroup. Let C be an infinite profinite group acting level transitively on a locally finite rooted tree $\mathcal{T}_0$. Let B be the 1-skeleton of the square complex $A \times \mathcal{T}_0$ and let $\mathcal{T}$ be the universal cover. Define D to be the extension $1 \rightarrowtail \pi_1(B) \rightarrowtail D \twoheadrightarrow C \times Q \twoheadrightarrow 1$. By [23, Proposition 6.8], there exists a CAT(0) space Y such that $D \rightarrowtail \mathrm{Isom}(Y)$ is a closed subgroup, and D acts cocompactly and minimally without fixed point at infinity.

The classical situation where this is applied is as follows: Let Q be a product of p-adic Lie groups, H be a product of real Lie groups and $\Gamma < H \times Q$ to be an S-arithmetic irreducible lattice. Let A be the 1-skeleton of the Bruhat-Tit's building for X, let $\mathcal{T}$ be the universal cover of A and let $T = \mathrm{Aut}(\mathcal{T})$. Now, Γ lifts to a weakly irreducible lattice $\widetilde{\Gamma} < H \times Q \times T$ and the corresponding graph of lattices is obtained by considering the graph A/Γ equipped with local groups given by the stabilisers of the action of Γ on A.

4.6 Complexes of lattices

In this section we will introduce the notion of a complex of H-lattices. We will then prove a structure theorem analogous to Theorem 4.3.3 for these complexes of H-lattices.

4.6.1 Complexes of groups

The definitions in this section are adapted from [58, Section 1.4] and [36] [37]. Throughout this section if X is a polyhedral complex then X' is its first barycentric subdivision. This is a simplicial complex with vertices VX' and edges EX'. Each $e \in EX'$ corresponds to cells $\tau \subset \sigma$ of X and so we may orient them from σ to τ. We will write $i(e) = \sigma$ and $t(e) = \tau$. We say two edges e and f of X' are *composable* if $i(e) = t(f)$, in which case there exists an edge $g = ef$ of X' such that $i(c) = i(e)$ and $t(c) = t(f)$, and e, f and g form the boundary of a 2-simplex in X. We denote the set of composable edges by E^2X'.

A *complex of groups* $G(X) = (G_\sigma, \psi_e, g_{e,f})$ over a polyhedral complex X is given by the following data:

(i) For each vertex σ of VX', a group G_σ called the *local group at* σ.

(ii) For each edge e of EX', a monomorphism $\psi_e : G_{i(e)} \to G_{t(e)}$ called the *structure map*.

(iii) For each pair of composable edges e and f, an element $g_{e,f} \in G_{t(e)}$ called the *twisting element*. We require these elements to satisfy the following conditions:

  (i) For $(e, f) \in EX'$, we have $\mathrm{Ad}(g_{e,f})\psi_{ef} = \psi_e\psi_f$.

  (ii) For each triple of composable edges a, b and c we have a *cocycle condition* $\psi_a(g_{b,a}) = g_{c,b}g_{cb,a}$.

We say $G(X)$ is *simple* if each of the twisting elements $g_{e,f}$ are the identity.

Some complexes of groups arise from actions on polyhedral complexes. Let G be a group acting without inversions on a polyhedral complex Y. Let $X = Y/G$ with natural projection $p : Y \to X$. For each $\sigma \in VX'$, choose a lift $\widetilde{\sigma} \in VY'$ such that $p\widetilde{\sigma} = \sigma$. The local group G_σ is the stabiliser of $\widetilde{\sigma}$ in G, and the structure maps and twisting elements are given by further choices. The resulting complex of groups $G(X)$ is unique up to isomorphism. A complex of groups isomorphic to a complex of groups arising from a group action is called *developable*.

Let $G(X)$ be a complex of groups over a polyhedral complex X. Let T be a maximal tree in the 1-skeleton of X' and fix a basepoint σ in T. The *fundamental group* of $G(X)$, denoted $\pi_1(G(X), \sigma_0)$, is generated by the set

$$\coprod_{\sigma \in VX'} G_\sigma \coprod \{e^+, e^- : e \in EX'\}$$

subject to the relations

$$\left\{\begin{array}{l}\text{the relations in the groups } G_\sigma, \\ (e^+)^{-1} = e^- \text{ and } (e^-)^{-1} = e^+, \\ e^+ f^+ = g_{e,f}(ef)^+,\ \forall (e,f) \in E^2X', \\ \psi_e(g) = e^+ g e^-,\ \forall g \in G_{i(e)}, \\ e^+ = 1,\ \forall e \in T. \end{array}\right\}$$

If $G(X)$ is developable, then it has a *universal cover* $\widetilde{G(X)}$. This is a simply connected polyhedral complex, equipped with an action of $G = \pi_1(G(X), \sigma_0)$ such that the complex of groups given by $\widetilde{G(X)}/G$ is isomorphic to $G(X)$.

Let $G(X) = (G_\sigma, \psi_e)$ and $H(Y) = (H_\tau, \psi_f)$ be complexes of groups over polyhedral complexes X and Y. Let $f : X' \to Y'$ be a simplicial map sending vertices to vertices and edges to edges. A *morphism* $\Phi : G(X) \to H(Y)$ over f consists of:

(i) A homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in VX'$.

(ii) For each $e \in EX'$ an element $g_e \in H_{t(f(e))}$ such that

  (i) $\mathrm{Ad}(g_e)\psi_{f(e)}\phi_{i(e)} = \phi_{t(e)}\psi_e$;

  (ii) For all $(a,b) \in E^2X'$ we have $\phi_{t(a)}(g_{a,b})g_{ab} = g_e\psi_{f(a)}(g_b)g_{f(a),f(b)}$.

4.6.2 Complexes of lattices

In this section we introduce complexes of lattices in analogy with the graphs of lattices we defined previously.

Definition 4.6.1 (Complex of lattices)**.** Let H be a locally compact group with Haar measure μ. A *complex of H-lattices* $(G(X), \psi)$ is a developable complex of groups equipped with a morphism ψ to H such that:

(i) For each $\sigma \in VX'$, the local group G_σ is covirtually an H-lattice and the image $\psi(G_\sigma)$ is an H-lattice;

(ii) The local groups are commensurable in $\Gamma = \pi_1(G(X), \sigma)$ and their images are commensurable in H.

(iii) For each $e \in EX'$, the elements e^+ and e^- in Γ are mapped to elements of $\mathrm{Comm}_H(\psi(G_\sigma))$.

The analogous structure theorem is given as follows.

Theorem 4.6.2. *Let X be a finite dimensional proper* CAT(0) *space and let $H = \mathrm{Isom}(X)$ contain a uniform lattice. Let $(G(Z), \psi)$ be a complex of H-lattices over a polyhedral complex Z, with universal cover Y, and fundamental group Γ. Suppose $A = \mathrm{Aut}(Y)$ admits a uniform lattice.*

(i) *Assume Z is finite and Y is a* CAT(0) *space. If for each local group G_σ the kernel $\mathrm{Ker}(\psi|_{G_\sigma})$ acts faithfully on Y, then Γ is a uniform $(H \times A)$-lattice and hence a* CAT(0) *group. Conversely, if Λ is a uniform $(H \times A)$-lattice, then Λ splits as a finite complex of uniform H-lattices with universal cover Y.*

(ii) *Under the same hypotheses as (i), Γ is quasi-isometric to $X \times Y$.*

(iii) *Assume X is a* CAT(0) *polyhedral complex and Y is a* CAT(0) *space. Let μ be the normalised Haar measure on H. If for each local group G_σ the kernel $K_\sigma = \mathrm{Ker}(\psi|_{G_\sigma})$ acts faithfully on Y and the sum $\sum_{\sigma \in VZ} \mu(G_\sigma)/|K_\sigma|$ converges, then Γ is a $(H \times A)$-lattice. Conversely, if Λ is a $(H \times A)$-lattice, then Λ splits as a finite complex of H-lattices with universal cover Y.*

Note that by definition we are assuming all complexes of lattices are developable complexes of groups.

Proof. We first prove (i). The fundamental group Γ of $G(Z)$ acts on the universal cover Y and on X via the homomorphism $\psi : \Gamma \to H$. The action on the product space $X \times Y$ is properly discontinuous cocompact and by isometries. The kernel of the action is contained in the intersection $\bigcap_{\sigma \in Z'} \mathrm{Ker}(\psi|_{G_\sigma})$. But this acts faithfully on Y, thus, the action is faithful. It follows Γ is an $(H \times A)$-lattice.

We now prove the converse. Assume Γ is an $(H \times A)$-lattice, and note that the action of Γ on Y yields a developable complex of groups $G(Z) = (\Gamma_\sigma, \psi_a, g_{a,b})$ with spanning tree T and equipped with a homomorphism $\pi_H : \Gamma \to H$. It suffices to show the local groups corresponding to the vertices of Z are covirtually H-lattices. Indeed, for an edge $e \in EZ'$, if the index $|\Gamma_{t(e)} : \psi_e(\Gamma_{i(e)})|$ is infinite, then the universal cover of $G(Z)$ would not be locally finite. It follows that all of the local groups are commensurable and hence, commensurable in H. Consequently, the elements e^+ and e^- for all $e \in E^2Z'/T$ in Γ must commensurate the local groups.

Let $\sigma \in Y$ be a vertex and consider the stabiliser $\Gamma_\sigma < \Gamma$ for the action on $X \times Y$. Suppose Γ_σ does not act cocompactly on $X \times \sigma$, then there is no compact set whose Γ_σ translates cover $X \times \sigma$. Let D be a non-compact set whose Γ_σ-translates cover $X \times \sigma$, but there is a compact set C whose Γ translates cover $X \times Y$. We may arrange our subsets such that $C' = C \cap (X \times \sigma) \subseteq D$. In particular, there are elements $g_i \in \Gamma/\Gamma_\sigma$ whose translates of C' cover D. But some of these elements fix must $X \times \sigma$ yielding a contradiction. Hence, Γ_σ is cocompact.

It is clear that $\mathrm{Ker}(\Gamma_\sigma \to H)$ is finite. Otherwise Γ would act with infinite point stabilisers on $X \times Y$ contradicting the discreteness of Γ. It remains to show that the projection $\overline{\Gamma}_\sigma$ of Γ_σ to H is discrete. Assume that $\overline{\Gamma}_\sigma$ is not discrete, then there does not exists a neighbourhood N of $1 \in H$ such that $N \cap \overline{\Gamma}_\sigma = \{1\}$. But this immediately implies there does not exist a neighbourbood N' of $1 \in H \times A$ such that $N' \cap \Gamma = \{1\}$ which contradicts the discreteness of Γ. It follows Γ_σ is covirtually an H-lattice.

The final step is to show the elements e^+ and e^- for each $e \in EX'$ are mapped to elements of $\mathrm{Comm}_H(\pi_H(\Gamma_\sigma))$. But this is immediate since the local groups map to H with finite kernel, the elements e^+ and e^- commensurate the local groups, and so must still preserve the appropriate conjugation relations in the map to H. ◆

We now prove (ii). By (i), Γ acts properly discontinuously cocompactly on $X \times Y$. The result follows from the Švarc-Milnor Lemma [6, I.8.19]. ◆

The proof of (iii) is almost identical to (i) we will highlight the differences. Since X is a CAT(0) polyhedral complex, it follows that $X \times Y$ is. Now, we may apply Serre's Covolume Formula to Γ. Let Δ be a fundamental domain for Γ acting on $X \times Y$, then the covolume of Γ may be computed as

$$\sum_{\sigma\in\Delta^0}\frac{1}{|\Gamma_\sigma|} = \sum_{\sigma\in\pi_Y(\Delta^0)}\sum_{\tau\in\pi_Y^{-1}(\sigma)}\frac{1}{|\Gamma_\tau|} = \sum_{\sigma\in\pi_Y(\Delta^0)}\frac{1}{|K_\sigma|}\sum_{\tau\in\pi_Y^{-1}(\sigma)}\frac{|K_\sigma|}{|\Gamma_\tau|} = \sum_{\sigma\in\pi_Y(\Delta^0)}\frac{\mu(\pi_X(\Gamma_\sigma))}{|K_\sigma|}.$$

Since $\pi_Y(\Delta^0)$ can be identified with Z and the later sum converges by assumption, it follows as before that Γ acts faithfully properly discontinuously and isometrically with finite covolume on $X \times Y$. For the converse the only adjustment required is that the compact sets C and C' in the proof of (i) should be replaced with ones of finite covolume. The remainder of the proof is identical. ◆ □

4.6.3 Properties: L^2-cohomology and C^*-simplicity

In this section we will prove a result on L^2-cohomology in the spirit of Theorem 4.4.1 and a result on C^*-simplicity in the spirit of Theorem 4.4.7 for $(H \times A)$-lattices.

Theorem 4.6.3. *Let H be a unimodular locally compact group with lattices and X be a locally-finite* CAT(0) *polyhedral complex with cocompact minimal automorphism group A. Assume any two non-zero L^2-Betti numbers of an H-lattice are in dimensions separated by at least* $\dim(X)$ *and that A-lattices have at most one non-vanishing L^2-Betti number in dimension k. Let Γ be an $(H \times A)$-lattice and $\Delta^{(p)}$ be a representative set of p-cells for the action of Γ on X. We have*

$$b_n^{(2)}(\Gamma) = \sum_{p=0}^{\dim(X)}\sum_{\sigma\in\Delta^{(p)}}(-1)^p b_{n-k}^{(2)}(\Gamma_\sigma).$$

Proof. The proof is essentially identical to Theorem 4.4.1, except now we use a G-equivariant spectral sequence [17, Chapter VII.7] applied to the filtration of X by skeleta with $\mathcal{U}\Gamma$ coefficients. The assumption that any two non-zero L^2-Betti numbers of an H-lattice are in dimensions separated by at least $\dim(X)$ forces any higher differentials to be 0. In particular, the E^2-page equals the E^∞ page of the spectral sequence. Moreover, the E^2-page is computed by using the same measure equivalence argument as in Theorem 4.4.1. □

The proof of the following theorem is essentially the same measure equivalence and Künneth formula argument as in Theorem 4.4.7(iii).

Theorem 4.6.4. *Let $X = X_1 \times \cdots \times X_k$ be a product of proper minimal cocompact $\mathrm{CAT}(0)$-spaces each not isometric to $\mathbb{R}$ and let $H = \mathrm{Isom}(X_1) \times \cdots \times \mathrm{Isom}(X_k)$ act without fixed point at infinity. Let Y be a locally-finite $\mathrm{CAT}(0)$ polyhedral complex not quasi-isometric to $\mathbb{E}^n$ and let $A = \mathrm{Aut}(Y)$ act without fixed point at infinity. Let $\Gamma < H \times T$ be a finitely generated weakly irreducible lattice. If both H- and A-lattices have a non-zero L^2-Betti number and trivial amenable radical, then Γ is C^*-simple.*

4.7 Lattices with non-trivial Euclidean factor

In this section we will characterize irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices. We will also strengthen the virtual biautomaticity criterion for a Leary-Minasyan group [45, Theorem 8.5] to arbitrary CAT(0)-lattices. Along the way we will prove a number of results about $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices. To this end we will examine the projections $\pi_{\mathrm{Isom}(\mathbb{E}^n)}$ and $\pi_{\mathrm{O}(n)}$ more closely.

Lemma 4.7.1. *Let X be a proper $\mathrm{CAT}(0)$-space, let $H = \mathrm{Isom}(X)$, and let Γ be a finitely generated $(\mathrm{Isom}(\mathbb{E}^n) \times H)$-lattice. If the projection $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is not discrete, then $\pi_{\mathrm{O}(n)}(\Gamma)$ contains an element of infinite order.*

Proof. For $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ to be not discrete at least one of the following must be true:

(i) $\pi_{\mathrm{O}(n)}(\Gamma)$ is not discrete and thus, contains an element of infinite order.

(ii) There exists a sequence of elements $\mathbf{g}_i \in \mathbb{R}^n$ such that $\mathbf{g}_i \to \mathbf{0}$ as $i \to \infty$.

If the first case holds we are done, so assume it does not. After passing to a subsequence we may assume that each $\mathbf{g}_i$ is not some power or root of any other $\mathbf{g}_j$ and so $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma) \cap \mathbb{R}^n$ contains an infinitely generated abelian subgroup A. Since we have assumed the first case does not hold $\pi_{\mathrm{O}(n)}(\Gamma)$ is a finite group F and we have a short exact sequence

$$\{1\} \to A \to \pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma) \to F \to \{1\}.$$

But this implies $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is an infinitely generated quotient of the finitely generated group Γ, a contradiction. Hence, $\pi_{\mathrm{O}(n)}(\Gamma)$ contains an element of infinite order. □

The following propositions give criteria for irreducibility in terms of the action of $\pi_{\mathrm{O}(n)}(\Gamma)$ on $\mathbb{R}^n$.

Proposition 4.7.2. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice, then Γ is weakly and algebraically irreducible if and only if $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\mathrm{O}(n-1)$. In particular, if Γ is weakly irreducible, then no finite index subgroup of $\pi_{\mathrm{O}(n)}(\Gamma)$ fixes a 1-dimensional subspace of $\mathbb{R}^n$.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 4.7.3. *Let X be an irreducible locally finite $\mathrm{CAT}(0)$ polyhedral complex and let $A = \mathrm{Aut}(X)$ act cocompactly and minimally. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice, then Γ is weakly and algebraically irreducible if and only if $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\mathrm{O}(n-1)$. In particular, if Γ is weakly irreducible, then no finite index subgroup of $\pi_{\mathrm{O}(n)}(\Gamma)$ fixes a 1-dimensional subspace of $\mathbb{R}^n$.*

Proof of Proposition 4.7.2 and 4.7.3. Suppose Γ is reducible then Γ has a virtually normal $\mathbb{Z}$ subgroup. Clearly, $\pi_{O(n)}(\Gamma)$ virtually centralises this subgroup and so $\pi_{O(n)}(\Gamma)$ must be virtually contained in some $\mathrm{O}(n-1)$.

Conversely, suppose $\pi_{\mathrm{O}(n)}(\Gamma)$ is virtually contained in some $\mathrm{O}(n-1)$. Passing to the corresponding finite index subgroup Λ we see that the action of Λ preserves two subspaces of $\mathbb{R}^n$. One isomorphic to $\mathbb{R}^{n-1}$ and one isomorphic to $R \cong \mathbb{R}$. Now, Λ splits as a graph of lattices in which every vertex and edge group has an infinite order generator which acts freely cocompactly on R and stabilises the subspace $R^\perp$ setwise via $\pi_{\mathrm{Isom}(\mathbb{E}^n)}$. The infinite cyclic groups intersect in some infinite cyclic subgroup $Z < \Lambda$. The stable letters of Λ must virtually centralise Z since otherwise they would map R into $R^\perp$. Thus, Z is virtually normal in Λ and hence Γ. By [26, Theorem 2(ii)] Γ is reducible. □

The following corollary is immediate.

Corollary 4.7.4. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^2) \times T)$-lattice, then Γ is an irreducible lattice if and only if $\pi_{\mathrm{O}(2)}(\Gamma)$ contains an element of infinite order.*

The following propositions give criteria for irreducibility in terms of the action of Γ on $\mathcal{T}$.

Proposition 4.7.5. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Then Γ is weakly and algebraically irreducible if and only if Γ acts on $\mathcal{T}$ faithfully.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 4.7.6. *Let X be an irreducible locally finite $\mathrm{CAT}(0)$ polyhedral complex and let $A = \mathrm{Aut}(X)$ act cocompactly and minimally. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice, then Γ is weakly and algebraically irreducible if and only if Γ acts on X faithfully.*

Proof of Proposition 4.7.5 and 4.7.6. Assume Γ is irreducible. By [26, Corollary 3], Γ has finite amenable radical B. Such a non-trivial element $g \in B$ stabilises a vertex of the Bass-Serre tree $\mathcal{T}$ (resp. complex X). Now, either g has infinitely many conjugates which contradicts the finiteness of B, or g stabilises the whole of $\mathcal{T}$ (resp. X) and so is contained in $\Gamma \cap \mathrm{Isom}(\mathbb{E}^n)$. By Lemma 4.7.1 and Proposition 4.7.2 (Proposition 4.7.3) there is an infinite order element in $\pi_{O(n)}(\Gamma)$ and hence an infinite order element in $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ which does not commute with g. But now the normal closure of g in Γ must contained infinitely many conjugates of g. Hence, B is infinite, a contradiction. Thus, B must be trivial.

The converse in the tree case follows from Proposition 4.3.4. If Γ acts on X faithfully, then the projection $\pi_A(\Gamma)$ is non-discrete. By Theorem 4.2.4 it suffices to show $P = \pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is non-discrete. Suppose P is discrete, then there is a finite index subgroup of P isomorphic to $Z = \mathbb{Z}^n$. But this is a virtually normal free abelian subgroup, so by [26, Theorem 2(ii)], Γ is reducible and so there is a finite index subgroup of Z which acts trivially on X, a contradiction. Thus, P is non-discrete and so Γ is weakly irreducible and by Theorem 4.2.4 algebraically irreducible. □

As an brief application we will construct (virtually) torsion-free irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T_{10})$-lattices.

Example 9. Recall the Leary-Minasyan group $\mathrm{LM}(A)$ where A is the matrix corresponding to the Pythagorean triple $(3,4,5)$ which acts on $\mathbb{E}^2 \times \mathcal{T}_{10}$. (Note that these groups were classified up to isomorphism by Valiunas [59].) By [45], this has presentation

$$\mathrm{LM}(A) = \langle a, b, t \mid [a,b],\ ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2 \rangle.$$

Using this group we will construct a virtually torsion-free irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice where T is the automorphism group of the $10n$-regular tree for all $n \geqslant 3$.

Let $\mathbb{Z}^n = \langle a_0, \dots, a_{n-1} \rangle$ and let $F = \langle f \rangle$ be a cyclic group of order n acting on L by cyclically permuting the a_i. Let $L = \mathbb{Z}^n \rtimes F$, this is a crystallographic group and so embeds into $\mathrm{Isom}(\mathbb{E}^n)$. Now, consider the $(n \times n)$-matrix B given by

$$B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & I_{n-2} \end{bmatrix}.$$

We define Γ_n to be the HNN extension of L by the matrix B, the Bass-Serre tree of this HNN extension will be regular of valence $10n$. This has generators $a_0, \dots, a_{n-1}, f, t$ and relations

$$f^n = 1,\ [a_i, a_j] = 1,\ f a_i f^{-1} = a_{i+1 \pmod n},\ [a_2, t] = 1,\ \dots,\ [a_{n-1}, t] = 1,$$

$$t a_0^2 a_1^{-1} t^{-1} = a_0^2 a_1,\ t a_0 a_1^2 t^{-1} = a_0^{-1} a_1^2,$$

where $i, j \in \{0, \dots, n-1\}$. Here the first three sets of relation come from L, the relations $[a_i, t] = 1$ for $i \geqslant 2$ come from the fact B fixes $\{a_2, \dots, a_{n-1}\}$ point-wise, and the last two relations arise from the action of B on $\langle a_0, a_1 \rangle$. Now, let $a := a_0$, then we may write Γ_n as

$$\Gamma_n = \langle a, f, t \mid f^n = 1,\ t a^2 a^{-f} t^{-1} = a^2 a^f,\ t(a^2)^f t^{-1} = a^{-1}(a^2)^f,\ [a^{f^i}, a^{f^j}] = 1 \rangle$$

for $i, j \in \{0, \dots, n-1\}$. Thus, Γ_n is a 3 generator, $\frac{1}{2}n(n-1) + 3$ relator group.

To see Γ_n is irreducible note that $\pi_{\mathrm{O}(n)}(\Gamma)$ is not virtually contained in some $\mathrm{O}(n-1) < \mathrm{O}(n)$. Indeed, consider the subgroup generated by the $\pi_{\mathrm{O}(n)}(f)$-orbit of $\pi_{\mathrm{O}(n)}(t)$. To show Γ_n is virtually torsion-free note that every torsion element of Γ_n has non-trivial image in $\pi_{\mathrm{O}(n)}(\Gamma_n)$. This is generated by the images of f and t and so is a finitely generated linear group and hence has a finite index torsion-free subgroup P_n. The preimage of P_n in Γ_n is torsion-free.

4.7.1 Biautomaticity

In this section we give a condition to determine the failure of biautomaticity for a CAT(0) group in the presence of a non-trivial Euclidean de Rham factor.

For the rest of this section we fix the following notation and terminology, the treatment roughly follows [45, Section 2] and [29, Section 2.3, 2.5]. Let $\mathcal{A}$ be a finite set and let Γ be a group with a map $\mu : \mathcal{A} \to \Gamma$. We say that Γ is *generated by* $\mathcal{A}$ if the unique extension of μ to the homomorphism from the free monoid $\mathcal{A}^*$ to Γ is surjective. We will call elements of $\mathcal{A}^*$ *words* and for any $w \in \mathcal{A}^*$, if $\mu(w) = g$ for some $g \in \Gamma$, we will say w *represents* g. We will always assume $\mathcal{A}$ is closed under inversion, that is, there is an involution $i : \mathcal{A} \to \mathcal{A}$ such that $\mu(i(a)) = \mu(a)^{-1}$, in this case we will denote $i(a)$ as a^{-1}. Any *subset* $\mathcal{L} \subseteq \mathcal{A}^*$ will be called a *language over* $\mathcal{A}$.

An *automatic structure* for a group Γ is a pair $(\mathcal{A}, \mathcal{L})$, where $\mathcal{A}$ is a finite generating set of Γ equipped with a map $\mu : \mathcal{A} \to \Gamma$ and closed under inversion, and $\mathcal{A} \subseteq \mathcal{A}^*$ is a language satisfying three conditions. Firstly, $\mu(\mathcal{L}) = \Gamma$, secondly $\mathcal{L}$ is a *regular language*, that is, it is accepted by some finite state automaton, and thirdly, it satisfies a fellow traveller property (which we will not make precise here). We say $(\mathcal{A}, \mathcal{L})$ is *biautomatic structure* if both $(\mathcal{A}, \mathcal{L})$ and $(\mathcal{A}, \mathcal{L}^{-1})$ are automatic structures. A group Γ is said to be *automatic (resp. biautomatic)* if it admits an automatic (resp. biautomatic) structure.

A (bi)automatic structure is *finite-to-one* if $|\mu^{-1}(g) \cap \mathcal{A}| < \infty$ for all $g \in \Gamma$. As noted in [45, Page 8] by [29, Theorem 2.5.1] it may be assumed that all (bi)automatic structures are finite-to-one. So without loss of generality we will make this assumption and we will also suppose that all the automata in this paper have no dead states.

A subgroup $H < \Gamma$ is $\mathcal{L}$*-quasiconvex* if there exists $\kappa \geqslant 0$ such that for any path p in the Cayley graph of Γ with respect to $\mathcal{A}$, starting at 1_Γ, ending at some $h \in H$, and labelled by a word $w \in \mathcal{L}$, then every vertex of p lies in the κ-neighbourhood of H. The main examples of $\mathcal{L}$-quasiconvex subgroups are centralisers of finite subsets as proved in [35, Proposition 4.3] and [29, Theorem 8.3.1 and Corollary 8.3.5].

Theorem 4.7.7. *Let $X = \prod_{i=1}^m X_i$ be a product of proper irreducible* $\mathrm{CAT}(0)$ *spaces each not isometric to $\mathbb{E}$ and $H < \mathrm{Isom}(X)$ be a closed subgroup acting minimally and cocompactly on X. Let $n \geqslant 2$ and let Γ be an $(\mathrm{Isom}(\mathbb{E}^n) \times H)$-lattice. If the projection $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is not discrete, then Γ is not virtually biautomatic.*

Proof. Assume $(\mathcal{B}, \mathcal{L})$ is a biautomatic structure on Γ. By [26, Theorem 2(i)] there exists a commensurated free abelian subgroup $A \leqslant \Gamma$ acting properly on $\mathbb{E}^n$ of rank n.

Claim: *There is a finite index subgroup of A that is $\mathcal{L}$-quasiconvex.*

By the Flat Torus Theorem the rank of a maximal abelian subgroup of Γ is bounded by the rank of a maximal flat in $X \times \mathbb{E}^n$. Let F be such a flat acted on by A. Fix a set of generators S_A for A and a set of generators S containing S_A for the maximal abelian subgroup containing A stabilising F.

We may split X into a product $Y_1 \times Y_2$ where A acts trivially on Y_1 and non-trivially on Y_2. For $j = 1, 2$ let $K_j = \mathrm{Isom}(Y_j) \cap H$. Since, A acts trivially on Y_1 it follows A and $\Gamma \cap K_1$ commute. Now, Γ splits as a complex of $(\mathrm{Isom}(\mathbb{E}^n) \times K_1)$-lattices. In particular, A is a subgroup of a vertex group Γ_v, which is covirtually virtually isomorphic to $A \times K_v$, where K_v is a lattice in K_1. Define S_K to be a set of generators for K_v and for each $s \in S_K$ let $s' \in K_v$ be some element which does no commute with s. Define a set $S_K' = \{s, s' \colon s \in S_K\}$ and note that it is finite.

Let $N = \mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$. For each irreducible factor Z_j for $j = 1, \dots, \ell$ of Y_2 choose some element $g_j \in N < \Gamma$ which acts non-trivially on Z_j. Note the kernel N is non-empty since otherwise Γ would be a finitely generated linear group and hence residually finite, contradicting [24, Theorem 2(iv)]. Now, we can choose such an element so that it centralises a finite index subgroup of A. Indeed, we may choose $g_j \in \langle\langle A \rangle\rangle \cap N$. Since A is commensurated g_j centralises $A^{g_j} \cap A$ a finite index subgroup of A. For each g_j pick another element g_j' which centralises a finite index subgroup of A and does not commute with g_j. Let $S_{Y_2} = \{g_j, g_j' \colon j = 1 \dots, \ell\}$ and note that it is finite. Let $A' = \left(\bigcap_{g \in S_{Y_2}} A^g\right) \cap A$, since this is the intersection of finitely many commensurable subgroups A' is a finite index subgroup of A. By construction A' is the centraliser of the finite set $S_K' \cup S_{Y_2} \cup S_A$. Thus, by [35, Proposition 4.3], A' is $\mathcal{L}$-quasiconvex. ◆

Now, by Lemma 4.7.1 there exists an element in $\bar{t} \in \pi_{O(n)}(\Gamma)$ with infinite order, let t denote a preimage of $\bar{t}$ in Γ. By [45, Corollary 5.4], there is a finite index subgroup $\Gamma^0 \leqslant \Gamma$ such that every finitely generated subgroup of Γ^0 centralises a finite index subgroup of A. After passing to a suitable power we may assume $t^k \in \Gamma^0$. But $\langle t^k \rangle$ does not centralise a finite index subgroup of A, a contradiction. Hence, there is no biautomatic structure on Γ. Since the hypotheses on Γ pass to finite index subgroups, it follows Γ is not virtually biautomatic. □

The following corollary characterises the biautomaticity of $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices.

Corollary 4.7.8. *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not quasi-isometric to $\mathbb{E}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let $n \geqslant 2$ and let Γ be a $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. Then, Γ is virtually biautomatic if and only if Γ is uniform and the projection $\pi_{\mathrm{O}(n)}(\Gamma)$ is finite.*

Proof. Note that a non-uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice is not finitely generated and hence, not virtually biautomatic. Indeed, it must split as a graph of groups with infinitely many vertices since $\mathrm{Isom}(\mathbb{E}^n)$ does not have any non-uniform lattices. Thus, we may assume Γ is uniform. Now, if Γ is virtually biautomatic then by Theorem 4.7.7 $\pi_{\mathrm{Isom}(\mathbb{E}^n)}(\Gamma)$ is discrete and hence $\pi_{\mathrm{O}(n)}(\Gamma)$ is finite. Conversely, if $\pi_{\mathrm{O}(n)}(\Gamma)$ is finite then Γ virtually splits as $\mathbb{Z}^n \times F_r$ which is biautomatic. □

Example 10. The group Γ_n for each $n \geqslant 2$ constructed in Example 9 is an irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T_{10n})$-lattice that is not virtually biautomatic.

Remark 4.7.9. In light of M. Valiunas' result [60, Theorem 1.2] Theorem 4.7.7 can be strengthened to state that Γ does not embed into any biautomatic group. It may also be possible to simplify the proof using their result.

4.7.2 Fibring

In this section we characterise irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices as those which do not virtually fibre. This result is new even for Leary-Minasyan groups.

Theorem 4.7.10. *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice, then Γ virtually algebraically fibres if and only if Γ is reducible.*

Proof. If Γ is reducible, then Γ virtually splits as $\mathbb{Z}\times\Gamma'$, in which case Γ virtually fibres.

We will now prove every irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice does not algebraically fibre, this will prove the theorem since a finite index subgroup of an irreducible lattice is an irreducible lattice. Now, suppose Γ is an irreducible uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice. By Theorem 4.3.3, the group Γ splits as a graph of $\mathrm{Isom}(\mathbb{E}^n)$-lattices, and so is the fundamental group of a graph of groups with vertex and edge stabilisers finite-by-$\mathrm{Isom}(\mathbb{E}^n)$-lattices. By the same argument as in the claim of the proof of Theorem 4.4.10 we may assume Γ is a reduced graph of groups which does not split as an ascending HNN-extension.

Now, $H^1(\Gamma;\mathbb{Z})\otimes\mathbb{R}\cong H^1(\Gamma;\mathbb{Z})$ and by Proposition 4.7.11, for every character $\phi\in H^1(\Gamma;\mathbb{R})$ we see that ϕ restricted to a vertex or edge group is zero. Since Γ is the fundamental group of a reduced graph of groups, is not an ascending HNN extension, and ϕ vanishes on every edge group, we may apply [21, Proposition 2.5] to deduce that $\phi\notin\Sigma(\Gamma)$. Hence, Γ does not algebraically fibre. □

Proposition 4.7.11. *Let $\mathcal{T}$ be a locally-finite leafless unimodular tree, not isometric to $\mathbb{R}$, let $T=\mathrm{Aut}(\mathcal{T})$, and let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice. If Γ is irreducible, then $H^1(\Gamma;\mathbb{Z})=H^1(\mathcal{T}/\Gamma;\mathbb{Z})$.*

The analogous result for $(\mathrm{Isom}(\mathbb{E}^n)\times A)$-lattices is as follows. We will prove both results simultaneously.

Proposition 4.7.12. *Let X be an irreducible locally finite $\mathrm{CAT}(0)$ polyhedral complex and let $A=\mathrm{Aut}(X)$ act cocompactly and minimally, and let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times A)$-lattice. If Γ is algebraically irreducible, $H^1(\Gamma;\mathbb{Z})=H^1(X/\Gamma;\mathbb{Z})$.*

Proof of Proposition 4.7.11 and 4.7.12. Let $\phi\in H^1(\Gamma;\mathbb{Z})=\mathrm{Hom}(\Gamma,\mathbb{Z})$. Suppose ϕ is non-zero on some local group L, then after passing to a finite index subgroup the restriction of ϕ is non-zero on some subgroup isomorphic to $\mathbb{Z}^n$. In particular, ϕ defines a codimension 1 subgroup of $\mathbb{Z}^n$ contained in $\mathrm{Ker}(\phi)$. Moreover, after passing to a further finite index subgroup $L'\cong\mathbb{Z}^n$, by commensurability of the local groups, there is codimension 1 subgroup $K\cong\mathbb{Z}^{n-1}$ of L' which is contained in every local group. Now, the flat $\mathbb{R}\otimes K$ is an $(n-1)$-dimensional flat stabilised by $P=\pi_{\mathrm{O}(n)}(\Gamma)$, contradicting Proposition 4.7.2 (Proposition 4.7.3). Thus, every local group is contained in $\mathrm{Ker}(\phi)$.

The isomorphism now follows from applying the equivariant spectral sequence to the filtration of $\mathcal{T}$ or X by skeleta (see [17, Chapter VII.7]). The previous paragraph shows that $E_2^{0,1}=0$, thus $H^1(\Gamma;\mathbb{Z})=E_2^{1,0}=E_\infty^{1,0}=H^1(X/\Gamma;\mathbb{Z})$. □

4.7.3 A characterisation

We are now ready to prove the characterisation of irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices (Theorem 4.B) from the introduction.

Theorem 4.7.13 (Theorem 4.B)**.** *Let $\mathcal{T}$ be a locally finite unimodular leafless tree not isometric to $\mathbb{R}$ and let $T = \mathrm{Aut}(\mathcal{T})$. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice. The following are equivalent:*

(i) *Γ is a weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice;*

(ii) *Γ is irreducible as an abstract group;*

(iii) *Γ acts on $\mathcal{T}$ faithfully;*

(iv) *Γ does not virtually fibre;*

(v) *Γ is C^*-simple;*

(vi) *and if $n = 2$, Γ is non-residually finite and not virtually biautomatic.*

Proof. The equivalence of (i) and (ii) is given by Theorem 4.2.4. The equivalence of (i) and (iii) is given by Proposition 4.7.5. The equivalence of (i) and (iv) is given by Theorem 4.7.10.

To see (i) and (iii) imply (v), observe that by [26, Theorem 2(iv)] Γ is non-residually finite and so $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$ is infinite. Now, Γ satisfies the conditions of Theorem 4.4.7(ii) and so Γ is C^*-simple. If Γ is reducible, then Γ virtually splits as $\Lambda = \mathbb{Z}^k \times \Gamma'$ for some $1 \leqslant k \leqslant n$. In particular, Λ is not C^*-simple since Λ has non-trivial amenable radical. It follows that Γ is not C^*-simple. Thus, (v) is equivalent to (i).

Assume $n = 2$ and note, by Corollary 4.7.4, Γ is irreducible if and only if $\pi_{O(n)}(\Gamma)$ contains an infinite order element. It follows from [26, Theorem 2(iv)] that Γ is reducible if and only if Γ is residually finite. The equivalence of (i) and (vi) is given by Corollary 4.7.8. □

4.8 Products with Salvetti complexes

In this section we will adapt a construction of Horbez and Huang [38, Proposition 4.5] to extend actions from trees to Salvetti complexes. Horbez–Huang constructed an example of a non-uniform lattice acting on the universal cover of the Salvetti complex $\widetilde{S}_L$ provided L is not a complete graph. We generalise this to construct a tower of uniform lattices in $\mathrm{Aut}(\widetilde{S}_L)$ and with an additional hypothesis on L non-biautomatic lattices in $\mathrm{Isom}(\mathbb{E}^n) \times \mathrm{Aut}(\widetilde{S}_L)$.

4.8.1 Graph and polyhedral products

Let K be a simplicial complex on the vertex set $[m] := \{1, \dots, m\}$. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i) \mid i \in [m]\}$ be a collection of CW-pairs. The *polyhedral product* of $(\underline{X}, \underline{A})$ and K, is the space

$$(\underline{X}, \underline{A})^K := \bigcup_{\sigma \in K} \prod_{i=1}^{m} Y_i^\sigma \subseteq \prod_{i=1}^{m} X_i \quad \text{where} \quad Y_i^\sigma = \begin{cases} X_i & \text{if } i \in \sigma, \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

Let K be a simplicial complex on $[m]$ vertices. Let $\underline{\Gamma} = \{\Gamma_1, \dots, \Gamma_m\}$ be a set of discrete groups. The *graph product* of $\underline{\Gamma}$ and K, denoted $\underline{\Gamma}^K$ is quotient of the free product $*_{i \in [m]} \Gamma_i$ by the relations $[\gamma_i, \gamma_j] = 1$ for all $\gamma_i \in \Gamma_i$ and $\gamma_j \in \Gamma_j$ if i and j are connected by an edge in K. Let $B\underline{\Gamma} = \{B\Gamma_1, \dots, B\Gamma_n\}$. The graph product $\underline{\Gamma}^K$ is the fundamental group of the polyhedral product $X = (B\underline{\Gamma}, *)^K$. Moreover, if K is a *flag complex*, i.e. every nonempty set of vertices which are pairwise connected by edges spans a simplex, then X is a $K(\underline{\Gamma}^K, 1)$ [57, Theorem 1.1].

If every vertex group in a graph product $\underline{\Gamma}^L$ is $\mathbb{Z}$ then we call the group a *right-angled Artin group (RAAG)* and denote $\underline{\Gamma}^L$ by A_L. In this case we will identify the generating set of A_L with the vertex set VL of L. The polyhedral product $(S^1, *)^L$ is a classifying space for A_L, is referred to as the *Salvetti complex* for A_L and denoted by S_L. We denote the universal cover by $\widetilde{S}_L$.

4.8.2 Extending actions over the Salvetti complex

We will now adapt the construction of Horbez and Huang [38, Proposition 4.5] to extend actions from trees to Salvetti complexes and present some applications.

Construction 4.8.1. *Let L be a finite simplicial graph on vertices $\{v_1, \dots, v_m\}$ and suppose $\langle v_1, \dots, v_k \rangle = F_k < A_L$ is a free subgroup. Let Γ be a group acting on $\mathcal{T}_{2k}$ by isometries such that the action is label-preserving, then the action of Γ on $\mathcal{T}$ extends to an action of $\widetilde{\Gamma}$ on $\widetilde{S}_L$ by isometries. Moreover, if Γ is a T_{2k}-lattice then $\widetilde{\Gamma}$ is an $\mathrm{Aut}(\widetilde{S}_L)$-lattice.*

Proof. There is an isometric embedding $\mathcal{T}_{2k} \rightarrowtail \widetilde{S}_L$ with edges labelled by $\mathcal{V} = \{v_1, \dots, v_k\} \subseteq VL$. Define $\phi : A_L \twoheadrightarrow F_k$ by $v \mapsto 1$ unless $v \in \mathcal{V}$ and let $\pi : \widetilde{S}_L \to X$ be the covering space corresponding to $\mathrm{Ker}(\phi)$. Let Γ be a group acting on $\mathcal{T}_{2k}$ preserving the labelling, we want to extend the action of Γ on $\mathcal{T}_{2k}$ to an action on $\widetilde{S}_L$.

We may identify the vertex set of $\mathcal{T}_{2k}$ with the vertex set of X via the embedding of $\mathcal{T}_{2k} \rightarrowtail \widetilde{S}_L$. We orient each edge of $\widetilde{S}_L$ and endow X with the induced labelling and orientation. The 1-skeleton $X^{(1)}$ of X is obtained from $\mathcal{T}_{2k}$ by attaching to each vertex of $\mathcal{T}_{2k}$ a circle for each $v \in VL \backslash \mathcal{V}$.

Since Γ acts by isometries on $\mathcal{T}_{2k}$ label preservingly, it follows Γ acts by isometries on $X^{(1)}$ label preservingly and preserves the orientation of edges in $VL \backslash \mathcal{V}$. It follows the action extends to X. Let $\widetilde{\Gamma}$ be the group of lifts of all automorphisms in Γ, we have a short exact sequence

$$1 \longrightarrow \mathrm{Aut}(\pi) \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 1.$$

We have $\widetilde{S}_L/\widetilde{\Gamma} = X/\Gamma$ so there is a bijection between the $\widetilde{\Gamma}$-orbits of $\widetilde{S}_L^{(0)}$ and the Γ-orbits of $\mathcal{T}_{2k}^{(0)}$. For a vertex $v \in X$, each lift of $g \in \mathrm{Stab}_\Gamma(\mathrm{v})$ fixes a unique vertex $\tilde{v} \in \widetilde{S}_L$. In particular, the cardinality of the vertex stabilisers is preserved. It follows from Serre's covolume formula that if Γ was a T_{2k}-lattice, then $\widetilde{\Gamma}$ is an $\mathrm{Aut}(\widetilde{S}_L)$-lattice. □

Proposition 4.8.2. *There is an ascending tower of lattices in $T_4 = \mathrm{Aut}(\mathcal{T}_4)$ with label preserving action.*

Proof. The groups will be index two subgroups of the HNN extensions constructed in [10, Example 7.4]. We describe them here for the convenience of the reader. Let $V_r = \{f : \mathbb{Z}_r \to \mathbb{Z}_2 \ : \ f \text{ a function}\} \cong \mathbb{Z}_2^r$ and $\alpha_r \in \mathrm{Aut}(V_r)$ by $\alpha_r(f)(i) = f(i+1)$. Let $W_r = \{f \in V_r \ : \ f(0) = 1\} \cong \mathbb{Z}_2^{r-1}$ and define Γ_r to be the HNN extension

$$\langle V_r, t \mid f^t = \alpha_r(f) \ \forall f \in W_r \rangle.$$

By [10, Proposition 7.6] the group Γ_r acts faithfully on $\mathcal{T}_4$ with quotient a loop (one vertex and one edge) and covolume $1/m^r$. Moreover, if $r|r'$ then $\Gamma_r \leqslant \Gamma_{r'}$ with index $m^{r'-r}$ and so for $r \geqslant 2$, the sequence $(\Gamma_{r^s})_{s\geqslant 1}$ is an infinite ascending chain in $\mathrm{Lat}_{\mathrm{u}}(\mathcal{T}_4)$.

Now, define $\phi : \Gamma_r \to \mathbb{Z}_2$ by $\phi(V_r) = 0$ and $\phi(t) = 1$. The kernel Λ_r is an index two subgroup which satisfies the same properties as Γ_r except now the quotient has fundamental domain the first barycentric subdivision of a loop (two vertices and two edges) and covolume $2/m^r$. □

Corollary 4.8.3. *Let L be a finite flag complex which is not a full simplex, then the automorphism group $\mathrm{Aut}(\widetilde{S}_L)$ of the universal cover of the Salvetti complex contains a tower of uniform lattices.*

Proof. Fix $r \geqslant 2$. We apply Construction 4.8.1 to the lattices Λ_{r^s} for $s \geqslant 1$ in the preceding proposition and obtain a sequence of lattices $\widetilde{\Lambda}_{r^s}$ in $\mathrm{Aut}(\widetilde{S}_L)$. The group $\widetilde{\Lambda}_{r^s}$ has two orbits of vertices, each stabilised by a group of order m^{r^s}, it follows from Serre's Covolume Formula that $\widetilde{\Lambda}_{r^s}$ has covolume equal to $2/m^{r^s}$. It remains to show that the inclusions $\Lambda_{r^s} \rightarrowtail \Lambda_{r^{s'}}$ induce inclusions $\widetilde{\Lambda}_{r^s} \rightarrowtail \widetilde{\Lambda}_{r^{s'}}$ for $s' < s$. Consider the covering space $\pi : \widetilde{S}_L \to X$ where X is as in Construction 4.8.1. Note that X and hence $\mathrm{Aut}(\pi)$ does not depend on r or s since each group acts with the same fundamental domain. In particular, as $\Lambda_{r^s} < \Lambda_{r^{s'}}$ we have $\widetilde{\Lambda}_{r^s} < \widetilde{\Lambda}_{r^{s'}}$ for $s < s'$. □

Theorem 4.8.4. *Let L be a finite simplicial graph on vertices $\mathcal{V} = \{v_1, \dots, v_m\}$ and suppose $\langle v_1, \dots, v_k\rangle = F_k < A_L$ is a free subgroup and that $\{v_1, \dots, v_k\} \subseteq \mathrm{Aut}(L)\cdot v_1$. Let X be a proper $\mathrm{CAT}(0)$ space and assume $H < \mathrm{Isom}(X)$ acts cocompactly and minimally.*

(i) *Let Γ be a group acting on $\mathcal{T}_{2k}$ by isometries, then the action of Γ on $\mathcal{T}$ extends to an action of $\widetilde{\Gamma}$ on $\widetilde{S}_L$ by isometries.*

(ii) *If Γ is a uniform lattice in $H \times T_{2k}$, then $\widetilde{\Gamma}$ is a uniform lattice in $H \times \mathrm{Aut}(\widetilde{S}_L)$.*

(iii) *If in addition X is a $\mathrm{CAT}(0)$ polyhedral complex and Γ is an $(H \times T_{2k})$-lattice, then $\widetilde{\Gamma}$ is an $(H \times \mathrm{Aut}(\widetilde{S}_L))$-lattice.*

(iv) *If the projection of Γ to H (resp. T_{2k}) is non-discrete, then so is the projection of $\widetilde{\Gamma}$ to H (resp. $\mathrm{Aut}(\widetilde{S}_L)$).*

Proof. The proof of (i) is identical to Construction 4.8.1 except now we do not require the action to be label preserving on $\mathcal{T}_{2k}$. Indeed, the assumption that $\{v_1, \dots, v_k\} \subseteq \mathrm{Aut}(L)\cdot v_1$ implies there is an isometry of $\widetilde{S}_L$ that permutes the edges around any vertex of $\mathcal{T}_{2k}$ and so we can extend any action on $\mathcal{T}_{2k}$ to $\widetilde{S}_L$. ◆

The proof of (ii) follows from taking the diagonal embedding $\widetilde{\Gamma} \rightarrowtail H \times \mathrm{Aut}(\widetilde{S}_L)$ and then noting that the quotient $(\widetilde{S}_L \times X)/\widetilde{\Gamma}$ is compact and that cardinality of each of the vertex stabilisers is finite. ◆

We prove (iii) in the same manner, noting the covolume on the product space is finite by Serre's Covolume Formula. ◆

The images of the projections of Γ and $\widetilde{\Gamma}$ to H coincide. Since any element of Γ which acts non-trivially on $\mathcal{T}_{2k}$ lifts to an element acting non-trivially on $\widetilde{S}_L$, the non-discreteness of $\pi_{T_{2k}}(\Gamma)$ implies the non-discreteness of $\pi_{\mathrm{Aut}(\widetilde{S}_L)}(\widetilde{\Gamma})$. This proves (iv). ◆ □

Example 11. Applying the previous theorem to the Leary-Minasyan group $\mathrm{LM}(A)$ which acts irreducibly on the product of a 10-regular tree and $\mathbb{E}^2$ we obtain a lattice Γ in $\mathrm{Isom}(\mathbb{E}^2) \times \mathrm{Aut}(\widetilde{S}_L)$. Moreover, the projection to either factor is non-discrete. Thus, if $\widetilde{S}_L$ is irreducible, then Γ is algebraically irreducible by [23]. By Theorem 4.7.7 the group Γ is not virtually biautomatic.

Recall that a group Γ is *quasi-isometrically rigid* if every group quasi-isometric to Γ is virtually isomorphic to Γ. The quasi-isometric rigidity of right angled Artin groups has received a lot of attention recently (see for instance [41] and the references therein). The following corollary is immediate and appears to be new if L has no induced 4-cycle [41] and A_L is not a free group [45].

Corollary 4.8.5 (Corollary 4.C)**.** *Let L be a finite simplicial graph on vertices $V = \{v_1, \dots, v_m\}$ and let $W = \{v_1, \dots, v_5\}$. Suppose $A_W < A_L$ is a free subgroup and that $\mathrm{Sym}(W) \leqslant \mathrm{Aut}(L)$. If A_L is irreducible, then there exists a weakly irreducible uniform lattice in $\mathrm{Aut}(\widetilde{S}_L) \times \mathrm{Isom}(\mathbb{E}^n)$ which is not virtually biautomatic nor residually finite. In particular, $A_L \times \mathbb{Z}^2$ is not quasi-isometrically rigid.*

Proof. The group Γ constructed in Example 11 is algebraically irreducible, non-residually finite, and quasi-isometric to $A_L \times \mathbb{Z}^2$. Both properties are virtual isomorphism invariants but $A_L \times \mathbb{Z}^2$ is algebraically reducible and residually finite. In particular, $A_L \times \mathbb{Z}^2$ is quasi-isometric to Γ but not virtually isomorphic to Γ and so cannot be quasi-isometrically rigid. □

Remark 4.8.6. It seems likely that one could take a polyhedral product of locally CAT(0) cube complexes over a flag complex and then repeat the above constructions to obtain towers of lattices in the automorphism group of the universal cover and more weakly irreducible lattices in mixed products.

4.9 From trees to right-angled buildings

In this section will show that the functors introduced by A. Thomas in [58] take graphs of H-lattices with a fixed Bass-Serre tree to complexes of H-lattices whose development is a “sufficiently symmetric" right-angled building (we will make this precise later). Finally, we will combine these tools to construct a number of examples. In particular, non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattices where A is the automorphism group of a sufficiently symmetric right-angled building, and non-residually finite algebraically irreducible lattices in products of arbitrarily many isometric and non-isometric sufficiently symmetric right-angled buildings.

4.9.1 Right angled buildings

Let (W, I) be a right-angled Coxeter system. Let N be the finite nerve of (W, I) and P' be the simplicial cone on N' with vertex x_0. A *right-angled building* of type (W, I) is a polyhedral complex X equipped with a maximal family of subcomplexes called *apartments*. Such an apartment is isometric to the Davis complex for (W, I) and the copies of P' in X are called *chambers*. Moreover, the apartments and chambers satisfy the axioms for a Bruhat–Tits building.

Let $\mathcal{S}$ denote the set of $J \subseteq I$ such that $W_J \leqslant W$ is finite. Note that $W_\varnothing = \{1\}$ so $\varnothing \in \mathcal{S}$. For each $i \in I$, the vertex P' of *type* $\{i\}$ will be called an i*-vertex*, and the union of the simplices of P' which contains the i-vertex but not x_0 will be called the i*-face* There is a one-to-one correspondence between the vertices of P' and the types $J \in \mathcal{S}$.

Let X be a right-angled building. A vertex of X has a type $J \in \mathcal{S}$ induced by the types of P'. For $i \in I$ an $\{i\}$*-residue* of X is the connected subcomplex consisting of all chambers which meet in a given i-face. The cardinality of the $\{i\}$-residue is the number of copies of P' in it.

Theorem 4.9.1 ([40])**.** *Let (W, I) be a right-angled Coxeter system and $\{q_i : i \in I\}$ a set of integers such that $q_i \geqslant 2$, then up to isometry there exists a unique building X of type (W, I) such that for each $i \in I$ the $\{i\}$-residue of X has cardinality q_i.*

If (W, I) is generated by reflections in an n-dimensional right-angled hyperbolic polygon P, then P' is the barycentric subdivision of P. Moreover, the apartments of X are isometric to $\mathbb{R}\mathbf{H}^n$. In this case we call X a *hyperbolic building.* We remark that a right-angled building can be expressed as the universal cover of a polyhedral product, however, we will not use this observation elsewhere.

Remark 4.9.2. Let (W, I) be a right-angled Coxeter system with parameters $\{q_i\}$ and nerve N. Let E_i be a set of size q_i and let CE_i denote the simplicial cone on E_i, denote the collections of these by $\underline{E}$ and $C\underline{E}$ respectively. The right-angled building of type (W, I) with parameters $\{q_i\}$ is the universal cover of the polyhedral product $(C\underline{E}, \underline{E})^N$.

4.9.2 A functor theorem

In this section we will recap a functorial construction of A. Thomas which takes graphs of groups with a given universal covering tree to complexes of groups with development a right-angled building. We will then show that this functor takes graphs of lattices to complexes of lattices and deduce some consequences.

Let X be a right-angled building of type (W, I) and parameters $\{q_i\}$ with chamber P'. Suppose $m_{i_1,i_2} = \infty$ and define the following two symmetry conditions due to Thomas [58]:

(T1) There exists a bijection g on I such that $m_{i,j} = m_{g(i),g(j)}$ for all $i, j \in I$, and $g(i_1) = i_2$.

(T2) There exists a bijection $h : \{i \in I : m_{i_1,i} < \infty\} \to \{i \in I : m_{i_2,i} < \infty\}$ such that $m_{i,j} = m_{h(i),h(j)}$ for all i, j in the domain, $h(i_1) = i_2$, and for all i in the domain $q_i = q_{h(i)}$.

We include the construction adapted from [58] for completeness and for utility in the proofs of the new results which will follow. An example of the construction for a graph of groups consisting of a single edge is given in Figure 4.2

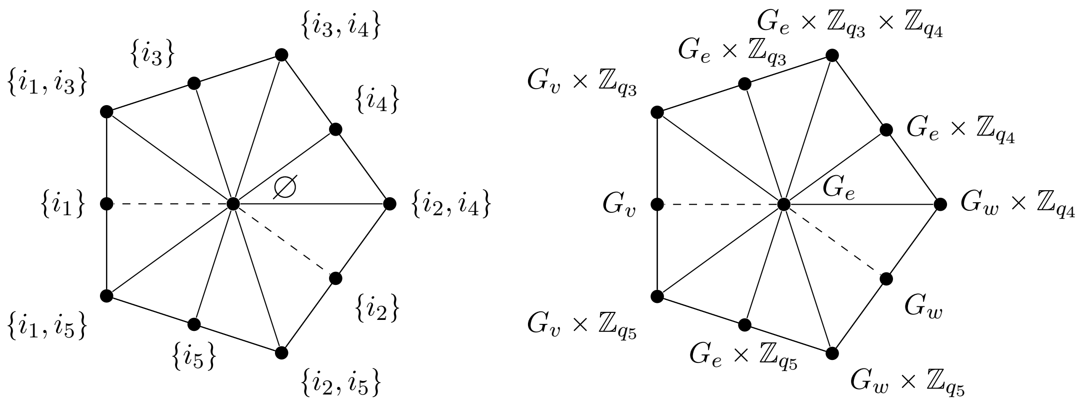

FIGURE 4.2: The left pentagon shows a labelling of the types $J \in \mathcal{S}$. The right pentagon shows the local groups after applying Thomas' functor to a graph of groups with a single edge. In both pentagons the dashed line shows the embedding of the graph. If the graph of groups has a single vertex, then $G_v = G_w$, $q_1 = q_2$, $q_3 = q_4$, the edge $(\{i_1, i_5\}, \{i_1\})$ is glued to $(\{i_2, i_5\}, \{i_2\})$, and the edge $(\{i_1, i_3\}, \{i_1\})$ is glued to $(\{i_2, i_4\}, \{i_2\})$.

Construction 4.9.3 (Thomas' Functor [58]). *Let X be a right-angled building of type (W, I) and parameters $\{q_i\}$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h. Then there is functor $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(X)$ preserving faithfulness and coverings.*

We will construct F as a composite $F_2 \circ F_1$. We first define $F_1 : \mathcal{G} \to \mathcal{C}_1$. Let $(A, \mathcal{A})$ be a graph of groups and $|A|$ the geometric realisation of A. We will construct a complex of groups $F_1(A)$ over $|A|$. For the objects we have:

- The local groups at the vertices of $|A|$ are the vertex groups of $\mathcal{A}$.
- For all $e \in EA$ let $\sigma_e = \sigma_{\overline{e}}$ be the vertex of the barycentric subdivision $|A|'$ at the midpoint of e.
- The local group at σ_e in $F_1(A)$ is $A_e = A_{\overline{e}}$.
- A monomorphism $\alpha_e : A_e \to A_{i(e)}$ in A induces the same monomorphism in $F_1(A)$.

Let $\phi : A \to B$ be a morphism of graphs of groups over a map of graphs f, note that by [58, Proposition 2.1] F_1 is not injective on morphisms. We define $F_1(\phi)$ as follows:

- The map f induces a polyhedral map $f' : |A|' \to |B|'$ so we will define $F_1(\phi) : F_1(A) \to F_1(B)$ over f.

- Now take the morphisms on the local groups to be the same as for ϕ.

Let $\mathcal{C}(\mathcal{T}) = \mathrm{Im}(F_1(\mathcal{G}(\mathcal{T})))$ and $G(Y) \in \mathcal{C}(\mathcal{T})$. Now, we will define $F_2 : \mathcal{C}(\mathcal{T}) \to \mathcal{C}(X)$ as follows:

- We first embed Y' into a canonically constructed polyhedral complex $F_2(Y)$. For each $e \in EY$ let P'_e be a copy of P' and identify the midpoint of e with the cone vertex x_0 of P'_e.
- If Y is 2-colourable with colours i_1 and i_2 (from the valences of the Bass-Serre tree if $q_{i_1} \neq q_{i_2}$), then we identify the vertex of e of type i_j with the i_j-vertex of P'_e.
- Suppose Y is not 2-colourable. If $e \in EY$ is not a loop in Y then identify one vertex of e with the i_1-vertex of P'_e and the other with the i_2-vertex. If e forms a loop then we attach P'_e/h (where h is the isometry from the assumption) and identify the vertex of e to the image of the i_1- and i_2-vertices of in P'_e/h.
- Glue together, either by preserving type on the i_1- and i_2-faces or by the isometry h, the faces of the the P'_e and P'_e/h whose centres correspond to the same vertex of Y. Let $F_2(Y)$ denote the resulting polyhedral complex.
- Note that $Y' \rightarrowtail F_2(Y)$ and that each vertex of $F_2(Y)$ has a unique type $J \in \mathcal{S}$ or two types J and $h(J)$ where $i_1 \in J \in \mathcal{S}$ and h is the isometry from the assumption.
- Fix the local groups and structure maps induced by the embedding of Y' in $F(Y)$. For each $i \in I$ let $G_i = \mathbb{Z}_{q_i}$ and for $J \subseteq I$ let $G_J = \prod_{j \in J} G_j$. For each $e \in EY$ let G_e be the local group at the midpoint of e.
- Let $J \in \mathcal{S}$ such that neither i_1 or i_2 are in J. The local group at a vertex of type J is $G_e \times G_J$. The structure maps between such local groups are the natural inclusions.
- Let $J \in \mathcal{S}$ and suppose $i_k \in J$ for one of $k = 1$ or $k = 2$. Since $m_{i_1,i_2} = \infty$ both i_1 and i_2 cannot be in J. Let F_e be the i_k-face of P'_e or the glued face of P'_e/h. The vertex of type J in P'_e or P'_e/h is contained in F_e. Let v be the vertex of Y identified with the centre of F_e and let G_v be the local group at v in $G(Y)$
- The local group at the vertex of type J is $G_v \times G_{J \setminus \{i_k\}}$. For each $J' \subset J$ with $i_k \in J'$ the structure map $G_v \times G_{J' \setminus \{i_k\}} \rightarrowtail G_v \times G_{J \setminus \{i_k\}}$ is the natural inclusion. For each $J' \subset J$ with $i_k \notin J'$ the structure map $G_e \times G_{J'} \rightarrowtail G_v \times G_{J \setminus \{i_k\}}$ is the product of the structure map $G_e \rightarrowtail G_v$ in $G(Y)$ and the natural inclusion.

Now, let $\phi : G(Y) \to H(Z)$ be a morphism in $\mathcal{C}(\mathcal{T})$ over a non-degenerate polyhedral map $f : Y \to Z$. We will define $F_2(\phi)$ as follows:

- If Y and Z are two colourable f extends to a polyhedral map $F_2(f) : F_2(Y) \to F_2(Z)$. Otherwise we use (T1) to construct $F_2(f)$.
- If $\tau \in VF(Y)$ then $G_\tau = G_\sigma \times G_J$ where σ is a vertex of Y'. The homomorphism of local groups $G_\sigma \times G_J \to H_{f(\sigma)} \times G_J$ is ϕ_σ on the first factor and the identity on the other factors.
- Let $a \in EF(Y)$. If ψ_a, the structure map along $a \in F_2(G(Y))$, has a structure map ψ_b from $G(Y)$ as its first factor, put $F_2(\phi)(b) = \phi(a)$. Otherwise set $F_2(\phi)(b) = 1$.

We will now show the functor takes graphs of lattices to complexes of lattices and deduce a number of consequences. Recall for a locally compact group H that $\mathrm{Lat(H)}$ denotes the (po)set of H-lattices and $\mathrm{Lat_u(H)}$ denotes the (po)set of uniform H-lattices.

Theorem 4.9.4. *Let Y be a right-angled building of type (W, I) and parameters $\{q_i\}$ and let $A = \mathrm{Aut}(Y)$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h, and let $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(Y)$ be Thomas' functor. Let X be a finite dimensional proper $\mathrm{CAT}(0)$ space and assume $H = \mathrm{Isom}(X)$ contains a cocompact lattice. The following conclusions hold:*

(i) *If $G(\mathcal{T})$ is a graph of H-lattices, then $F(G(\mathcal{T}))$ is a complex of H-lattices.*

(ii) *F induces an inclusion of sets $\mathrm{Lat_u(H \times T)} \rightarrowtail \mathrm{Lat_u(H \times A)}$.*

(iii) *If Y is a $\mathrm{CAT}(0)$ polyhedral complex then F induces an inclusion of sets $\mathrm{Lat(H \times T)} \rightarrowtail \mathrm{Lat(H \times A)}$.*

Let Γ be a uniform $(H \times T)$-lattice and let $F\Gamma$ be the corresponding $(H \times A)$-lattice.

(iv) *$\pi_T(\Gamma)$ is discrete if and only if $\pi_A(F\Gamma)$ is discrete. Moreover, $\pi_H(\Gamma) = \pi_H(F\Gamma)$.*

(v) *If Γ satisfies any of {algebraically irreducible, non-residually finite, not virtually torsion free}, then so does $F\Gamma$.*

Proof. We first prove (i). We will first verify the conditions on the local groups and then construct a morphism to H. Let $(B, \mathcal{B}, \psi)$ be a graph of H-lattices and consider the image $L(Z)$ of $\mathcal{B}$ under F. Here $Z = F(B)$. Each local group in $L(Z)$ is of the form $G_\sigma \times G_J$ where G_σ is a local group in $\mathcal{B}$ and G_J is a finite product of finite cyclic groups. We have a morphism $\psi : \mathcal{B} \to H$ such that the image of each local group G_σ is an H-lattice and the restriction to G_σ has finite kernel. Thus, by construction the local groups in $L(Z)$ are commensurable in $\pi_1(L(Z))$. We define $F(\psi_\sigma)$ to be the composite $\psi|_{G_\sigma} \circ \pi_\sigma : G_\sigma \times G_J \twoheadrightarrow G_\sigma \to \psi(G_\sigma)$, thus commensurability of the images in H is immediate.

We will now deal with the edges. Note the twisting elements in $L(Z)$ are all trivial and the complex of groups H has all structure maps the identity. Let the structure maps in $L(Z)$ be denoted by λ_a for $a \in EZ'$ and the structure maps in $\mathcal{B}$ by α_e for $e \in EB$. The family of elements $(t_e)_{e \in EB}$ in the path group $\pi(\mathcal{B})$ are mapped under ψ to elements of $\mathrm{Comm}_H(\psi(G_\sigma))$ where G_σ is some local group. Now, let $a \in EZ'$, then by construction a either corresponds to a subdivision of an edge a in EB in which case we define $(F\psi)(a) = \psi(a)$. Or, a corresponds to a inclusion of local groups $G_\sigma \times G_{J'} \to G_\sigma \times G_J$, in which case we define $(F\psi)(a) = 1_H$.

It remains to verify the two edge axioms for a morphism. For each $a \in EZ'$ corresponding to the subdivision of an edge a in EB we have

$$\mathrm{Ad}((F\psi)(a)) \circ F(\psi_{i(a)}) = \mathrm{Ad}(\psi(a)) \circ \psi_{i(a)} \circ \pi_a = \psi_{t(a)} \circ \alpha_a \circ \pi_a = F(\psi_{t(a)}) \circ F(\alpha_a),$$

where π_a is the surjection $G_a \times G_J \twoheadrightarrow G_a$. For any other edge $a \in EZ'$ we have

$$\mathrm{Ad}((F\psi)(a)) \circ F(\psi_{i(a)}) = F(\psi_{i(a)}) \text{ and } F(\psi_{t(a)}) \circ \lambda_a = F(\psi_{i(a)}).$$

Finally, the other condition that $(F\psi)(ab) = (F\psi)(a)(F\psi)(b)$ for $(a,b) \in E^2Z'$ is verified trivially. Thus, $F(\mathcal{B}) = L(Z)$ is a complex of H-lattices. ◆

We will next prove (ii). Let Γ be an $(H \times T)$-lattice. By Theorem 4.3.3, Γ splits as graph of H-lattices $\mathcal{B}$. Thus, by (i) we obtain a complex of H-lattices $F(\mathcal{B})$ with fundamental group Λ. By Theorem 4.6.2(i) it suffices to show that for each local group G_σ in $F(\mathcal{B})$ the kernel $K_\sigma = \mathrm{Ker}(\pi_H|_{FG_\sigma})$ acts faithfully on X. Now, K_σ is a direct product of $L_\sigma = \mathrm{Ker}(\pi_H|_{G_\sigma})$ with a direct product of cyclic groups G_J, where G_σ is a local group in $\mathcal{B}$. By construction G_J acts faithfully on X and by Theorem 4.3.3, K_σ acts faithfully on $\mathcal{T}$ whose automorphism group embeds into A. In particular, K_σ acts faithfully on X. ◆

We will next prove (iii). We construct a complex of lattices as in the previous case. The proof for (iii) is now identical once we have verified that covolume condition in Theorem 4.6.2(iii). Let c denote the covolume of an $(H \times T)$-lattice Γ with associated graph of lattices $(B, \mathcal{B})$, this is given by the formula $c = \sum_{\sigma \in VA} \mu(\Gamma_\sigma) < \infty$. Now, every vertex of the complex $Z = F(B)$ has local group isomorphic to a finite extension of some Γ_σ. In particular we may bound $\sum_{\sigma \in Z} \mu(\Gamma_\sigma)$ by $\ell \times c$ where ℓ is the number of vertices in the finite Coxeter nerve of X. ◆

The proof of (iv) follows from the proof of (i). ◆

The proof of (v) follows from either applying Theorem 4.2.4 to (iv) (algebraically irreducible) or the fact $\Gamma \rightarrowtail F\Gamma$ and the properties of residual finiteness and virtual torsion-freeness are subgroup closed. ◆ □

4.9.3 Examples and applications

In this section we will detail some sample examples and applications of the functor theorem.

We can obtain a number of examples by applying Thomas' functor to any irreducible $(\mathrm{Isom}(\mathbb{E}^n)\times T)$-lattice. This will give a non-biautomatic group acting properly discontinuously cocompactly on $\mathbb{E}^n\times X$ where X is a sufficiently symmetric right-angled building. More precisely, we have the following corollary:

Corollary 4.9.5 (General version of Corollary 4.D). *Let Y be a right-angled building of type (W, I) and parameters $\{q_i\}$ and let $A = \mathrm{Aut}(Y)$. For each $i_1, i_2 \in I$ such that $m_{i_1,i_2} = \infty$ let $\mathcal{T}$ be the (q_{i_1}, q_{i_2})-biregular tree and let $T = \mathrm{Aut}(\mathcal{T})$. Suppose (T1) holds and if $q_{i_1} = q_{i_2}$ then (T2) holds with g an extension of h and let $F : \mathcal{G}(\mathcal{T}) \to \mathcal{C}(Y)$ be Thomas' functor. Let Γ be a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice and suppose $\pi_{\mathrm{O}(n)}(\Gamma)$ is infinite, then $F\Gamma$ is a uniform $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice which is not virtually biautomatic nor residually finite. In particular, if Y is irreducible, then the direct product of a uniform A-lattice with $\mathbb{Z}^2$ is not quasi-isometrically rigid.*

Proof. By Theorem 4.9.4 $F\Gamma$ is a uniform $(\mathrm{Isom}(\mathbb{E}^n)\times A)$-lattice with a non-discrete projection to $\mathrm{O}(n)$. That $F\Gamma$ is not virtually biautomatic then follows from Theorem 4.7.7. The failure of quasi-isometric rigidity follows from the fact that the direct product of a uniform A lattice with $\mathbb{Z}^2$ is reducible, whereas, the weakly irreducible lattice is algebraically irreducible by Theorem 4.2.4 and so does not virtually split as a direct product of two infinite groups. In particular, the groups cannot by virtually isomorphic. □

Example 12. Let $\Gamma = \mathrm{LM}(A)$ where A is the matrix corresponding to the Pythagorean triple $(3, 4, 5)$. Recall the group acts on $\mathbb{E}^2 \times \mathcal{T}_{10}$. Let X be the right angled building whose Coxeter nerve is the regular pentagon and whose parameters are given by $q_1 = q_2 = 10$, $q_3 = q_4 = k$, and $q_5 = \ell$. Let A be the automorphism group of X and consider $F\Gamma$ the image of Γ under Thomas' functor F as in Figure 4.2. By Theorem 4.9.4, the group $F\Gamma$ is a non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with non-discrete projections to both factors and is irreducible as an abstract group. Moreover, by the previous corollary, $F\Gamma$ is not virtually biautomatic.

We will now construct a presentation for $\Lambda_{k,\ell} := F\Gamma$. The group has generators a, b, x_3, x_4, x_5, t and relations

$$x_3^k = x_4^k = x_5^\ell = 1,\ [a, b],\ [a, x_3],\ [a, x_4],\ [a, x_5],\ [b, x_3],\ [b, x_4],\ [b, x_5],\ [x_3, x_4],$$

$$ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2,\ tx_3t^{-1} = x_4, [t, x_5].$$

The following proposition shows the group is virtually torsion-free.

Proposition 4.9.6. *The group $\Lambda_{2,2}$ in Example 12 is virtually torsionfree. This is witnessed by the index 16 subgroup*

$$\Delta := \langle a,\ b,\ x_3tx_4t^{-1},\ x_3x_4t^{-2},\ (x_5x_3)^2,\ (x_5x_4)^2,\ t^{-1}x_3x_4t^{-1},\ (tx_5x_4t^{-1})^2\rangle.$$

Proof. The quotient $\Lambda_{2,2}/\Delta$ is isomorphic to $D_4\times\mathbb{Z}_2$ which has order 16. By construction every torsion element of $\Lambda_{2,2}$ is conjugate to some power of x_3, x_4, x_5 or x_3x_4. Indeed, every torsion element is contained in a vertex or edge stabiliser of the action on the pentagonal building and acts trivially on $\mathbb{E}^2$. Each of these elements is mapped to a non-trivial element of $D_4\times\mathbb{Z}_2$. In particular, the kernel Δ is torsion-free. □

Corollary 4.9.7. *The group Δ admits a presentation with 8 generators*

$$a,\ b,\ y_1,\ y_2,\ y_3,\ y_4,\ y_5,\ y_6$$

and 20 relations

$$[a,b],\ [a,y_4],\ [a,y_3],\ [b,y_3],\ [b,y_4],$$
$$a^{-2}b^{-1}y_6a^2by_6^{-1},$$
$$a^{-1}y_1ba^2y_1^{-1}b^{-1}a^{-1},$$
$$ba^{-1}by_1^{-1}b^{-2}ay_1,$$
$$y_6y_2^{-1}b^{-1}y_2y_6^{-1}y_2^{-1}by_2,$$
$$y_2y_6^{-1}y_2^{-1}a^{-1}y_2y_6y_2^{-1}a,$$
$$y_2^{-1}ab^{-1}ay_2y_5^{-1}a^{-1}ba^{-1}y_5,$$
$$y_5a^{-2}b^{-1}y_5^{-1}y_3y_5a^2by_5^{-1}y_3^{-1},$$
$$y_4y_5^{-1}y_3^{-1}y_5y_1^{-1}y_2y_6y_2^{-1}y_4^{-1}y_3y_1y_6^{-1},$$
$$y_5^{-1}ab^{-3}y_5y_6y_5^{-1}b^2a^{-1}by_5y_6^{-1},$$
$$y_5^{-1}ba^{-1}b^2y_5y_4y_5^{-1}b^{-1}ab^{-2}y_5y_4^{-1},$$
$$y_5^{-1}ab^{-3}y_5b^{-1}a^{-2}y_5^{-1}b^2a^{-1}by_5a^2b,$$
$$b^{-1}a^{-3}b^{-1}a^{-2}y_5^{-1}b^{-1}ab^{-3}a^2y_5ba^2b^{-3},$$
$$y_2^{-1}baby_2y_5^{-1}b^{-1}ab^{-2}y_5b^{-1}a^{-2}b^{-2}ay_5^{-1}b^{-1}ab^{-2}y_5b^{-1}a^{-2},$$
$$ay_5a^4ba^2b^2y_5^{-1}b^2a^{-1}b^3a^{-1}b^3a^{-1}by_5a^4b^2y_5^{-1}b^2,$$
$$y_3y_5y_4^{-1}y_6aba^3by_5^{-1}b^3a^{-1}y_5a^2by_5^{-1}b^2a^{-1}by_5y_6^{-1}y_4y_5^{-1}y_3^{-1}b^2ay_5ba^4by_5^{-1}b^2a^{-1}b$$

and the abelianization of Δ is isomorphic to $\mathbb{Z}_8^2\oplus\mathbb{Z}^6$.

Remark 4.9.8. It follows immediately from the presentation of Δ that it and hence $\Lambda_{2,2}$ contain a subgroup isomorphic to $\mathbb{Z}^3$. For example $\langle a,b,y_3\rangle$ or $\langle a,b,y_4\rangle$. Note that this coincides with the dimension of a maximal flat in $X \times \mathbb{E}^2$. Since both groups have a commensurated abelian subgroup their L^2-cohomology vanishes (see Proposition 4.4.3).

Example 13. Let $n \geqslant 2$ and let Γ_n be the irreducible lattice constructed in Example 9 acting on $\mathbb{E}^n \times \mathcal{T}_{10n}$. Let X be a right angled building satisfying (T1) and (T2) with automorphism group A and parameters $\{q_i\}$ all equal to $10n$. Applying Thomas' functor and Theorem 4.9.4 to Γ_n we obtain a non-residually finite $(\mathrm{Isom}(\mathbb{E}^n) \times A)$-lattice with non-discrete projections to both factors. Moreover by Corollary 4.9.5, Γ_n is not virtually biautomatic.

We will now show the existence of non-residually finite lattices in arbitrary products of sufficiently symmetric isometric and non-isometric right-angled buildings. We note that Bourdon's "hyperbolization of Euclidean buildings" [16, Section 1.5.2] can be used to construct weakly irreducible uniform lattices in products of hyperbolic buildings. We will provide a number of examples to show that the groups we construct here are distinct.

Corollary 4.9.9. *Let Γ be a weakly irreducible lattice in product of trees $\mathcal{T}_1 \times \cdots \times \mathcal{T}_n$ such that $\mathcal{T}_k$ is (t_{k_1}, t_{k_2})-biregular. Let $X_1 \times \cdots \times X_n$ be a product of irreducible right angled buildings satisfying (T1) and (T2). Suppose X_k is of type (W_k, I_k), has parameters $\{t_{k_1}, t_{k_2}, q_{k_3}, \ldots, q_{k_{n_k}}\}$ where $m_{k_{i_1}, k_{i_2}} = \infty$ and $A_k = \mathrm{Aut}(X_k)$. The lattice $\Lambda = F^n\Gamma$ obtained by applying Thomas' functor n times (once for each tree $\mathcal{T}_k$ corresponding to the building X_k) is a lattice in $A_1 \times \cdots \times A_n$, is weakly and algebraically irreducible, and is non-residually finite.*

Proof. Let $T_k = \mathrm{Aut}(\mathcal{T}_k)$. The result follows from applying Theorem 4.9.4 n times as follows. Consider Γ as a graph of $(T_2 \times \cdots \times T_n)$-lattices and apply F to obtain a $(A_1 \times T_2 \times \cdots \times T_n)$-lattice with the desired properties (non-residual finiteness follows from the fact that the projection to $T_2 \times \cdots \times T_n$ has a non-trivial kernel). Now, consider $F\Gamma$ as a graph of $(A_1 \times T_3 \times \ldots T_n)$-lattices and proceed by induction on the index k. □

Examples 14. We will detail three examples:

(i) In [53, Theorem 2.27, Theorem 3.15] the authors construct infinite series of explicit examples of irreducible S-arithmetic quaternionic lattices acting simply transitively on the vertices of products of $n \geqslant 1$ trees of constant valency, in each case we may apply Theorem 4.9.9 to obtain algebraically and weakly irreducible non-residually finite uniform lattices acting on a product of n buildings. It is unclear whether these groups are related to the groups constructed by Bourdon's hyperbolization.

(ii) In [14] [12] Burger and Mozes construct for each pair of sufficiently large even integers (m, n) a finitely presented simple group as a uniform lattices in a product of trees $\mathcal{T}_m \times \mathcal{T}_n$ (for more examples see [52] [51] [50]). Applying Theorem 4.9.9, we obtain uniform non-residually finite algebraically and weakly irreducible lattices acting on a product of buildings $X_1 \times X_2$ each satisfying (T1) and (T2) with X_1 having some parameters equal to m and X_2 having some parameters equal to n.

(iii) Applying Theorem 4.9.9 to the non-uniform lattices in products of arbitrarily many trees constructed in Example 7 yields weakly irreducible non-uniform lattices in products of arbitrarily many sufficiently symmetric right-angled buildings.

4.10 Some questions

In this section we will raise a conjecture and some questions left open by this paper. In light of the results in Section 4.4.4 showing that many CAT(0) groups are autostackable (in particular the Leary-Minasyan groups) we raise the following conjecture:

Conjecture 4.10.1. *Every* CAT(0) *group is autostackable.*

In every example of an $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice known to the author, the lattice is virtually torsion-free. Note that if there was a non-virtually torsion-free $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice Γ, then any inseparable torsion element must be contained in $\mathrm{Ker}(\pi_{\mathrm{Isom}(\mathbb{E}^n)})$.

Question 4.10.2. Are there non-virtually torsion-free $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices?

Since it is possible to characterise $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattice in terms of C^*-simplicity and virtual fibring, it would be interesting to recover the characterisation for complexes of $\mathrm{Isom}(\mathbb{E}^n)$-lattices.

Question 4.10.3. Are the weakly irreducible non-biautomatic groups constructed in Section 4.8 and Section 4.9 C^*-simple? Do they virtually fibre?

More generally we ask:

Question 4.10.4. When is a CAT(0) lattice C^*-simple?

The characterisation of weakly irreducible $(\mathrm{Isom}(\mathbb{E}^n) \times T)$-lattices (Theorem 4.B) suggests the following question:

Question 4.10.5. Can C^*-simplicity and virtual fibring of a Leary-Minasyan group $\mathrm{LM}(A)$ be determined by properties of the matrix A?

Finally, we remark that in [42] the constructions in this paper were used by the author to construct an example of a hierarchically hyperbolic group which is not virtually torsion-free.

References

[1] Peter Abramenko and Bertrand Rémy. Commensurators of some non-uniform tree lattices and Moufang twin trees. In *Essays in geometric group theory*, volume 9 of *Ramanujan Math. Soc. Lect. Notes Ser.*, pages 79–104. Ramanujan Math. Soc., Mysore, 2009.

[2] Hyman Bass. Covering theory for graphs of groups. *J. Pure Appl. Algebra*, 89(1-2):3–47, 1993.

[3] Uri Bader, Alex Furman, and Roman Sauer. Weak notions of normality and vanishing up to rank in L^2-cohomology. *Int. Math. Res. Not. IMRN*, (12):3177–3189, 2014.

[4] Uri Bader, Alex Furman, and Roman Sauer. An adelic arithmeticity theorem for lattices in products. *Math. Z.*, 293(3-4):1181–1199, 2019.

[5] Nadia Benakli and Yair Glasner. Automorphism groups of trees acting locally with affine permutations. *Geom. Dedicata*, 89:1–24, 2002.

[6] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[7] Mark Brittenham, Susan Hermiller, and Derek Holt. Algorithms and topology of Cayley graphs for groups. *J. Algebra*, 415:112–136, 2014.

[8] Mark Brittenham, Susan Hermiller, and Ashley Johnson. Homology and closure properties of autostackable groups. *J. Algebra*, 452:596–617, 2016.

[9] Mark Brittenham, Susan Hermiller, and Tim Susse. Geometry of the word problem for 3-manifold groups. *J. Algebra*, 499:111–150, 2018.

[10] Hyman Bass and Ravi Kulkarni. Uniform tree lattices. *J. Amer. Math. Soc.*, 3(4):843–902, 1990.

[11] Emmanuel Breuillard, Mehrdad Kalantar, Matthew Kennedy, and Narutaka Ozawa. C^*-simplicity and the unique trace property for discrete groups. *Publ. Math. Inst. Hautes Études Sci.*, 126:35–71, 2017.

[12] Marc Burger and Shahar Mozes. Finitely presented simple groups and products of trees. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(7):747–752, 1997.

[13] Marc Burger and Shahar Mozes. Groups acting on trees: from local to global structure. *Inst. Hautes Études Sci. Publ. Math.*, 92:113–150 (2001), 2000.

[14] Marc Burger and Shahar Mozes. Lattices in product of trees. *Inst. Hautes Études Sci. Publ. Math.*, 92:151–194 (2001), 2000.

[15] Robert Bieri, Walter D. Neumann, and Ralph Strebel. A geometric invariant of discrete groups. *Invent. Math.*, 90(3):451–477, 1987.

[16] Marc Bourdon. Sur les immeubles fuchsiens et leur type de quasi-isométrie. *Ergodic Theory Dynam. Systems*, 20(2):343–364, 2000.

[17] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[18] Uri Bader and Yehuda Shalom. Factor and normal subgroup theorems for lattices in products of groups. *Invent. Math.*, 163(2):415–454, 2006.

[19] Nathan Corwin, Gili Golan, Susan Hermiller, Ashley Johnson, and Zoran Šunić. Autostackability of Thompson's group F. *J. Algebra*, 545:111–134, 2020.

[20] Pierre-Emmanuel Caprace, Peter H. Kropholler, Colin D. Reid, and Phillip Wesolek. On the residual and profinite closures of commensurated subgroups. *Mathematical Proceedings of the Cambridge Philosophical Society*, page 1–22, 2019.

[21] Christopher H. Cashen and Gilbert Levitt. Mapping tori of free group automorphisms, and the Bieri-Neumann-Strebel invariant of graphs of groups. *J. Group Theory*, 19(2):191–216, 2016.

[22] Pierre-Emmanuel Caprace and Adrien Le Boudec. Bounding the covolume of lattices in products. *Compos. Math.*, 155(12):2296–2333, 2019.

[23] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: discrete subgroups. *J. Topol.*, 2(4):701–746, 2009.

[24] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: structure theory. *J. Topol.*, 2(4):661–700, 2009.

[25] Pierre-Emmanuel Caprace and Nicolas Monod. A lattice in more than two Kac-Moody groups is arithmetic. *Israel J. Math.*, 190:413–444, 2012.

[26] Pierre-Emmanuel Caprace and Nicolas Monod. Erratum and addenda to "isometry groups of non-positively curved spaces: discrete subgroups", 2019. arXiv:1908.10216 [math.GR].

[27] Pierre de la Harpe. On simplicity of reduced C^*-algebras of groups. *Bull. Lond. Math. Soc.*, 39(1):1–26, 2007.

[28] Pierre de la Harpe and Jean-Philippe Préaux. C^*-simple groups: amalgamated free products, HNN extensions, and fundamental groups of 3-manifolds. *J. Topol. Anal.*, 3(4):451–489, 2011.

[29] David B. A. Epstein, James W. Cannon, Derek F. Holt, Silvio V. F. Levy, Michael S. Paterson, and William P. Thurston. *Word processing in groups.* Jones and Bartlett Publishers, Boston, MA, 1992.

[30] Damien Gaboriau. Invariants l^2 de relations d'équivalence et de groupes. *Publ. Math. Inst. Hautes Études Sci.*, 95:93–150, 2002.

[31] Giovanni Gandini. Bounding the homological finiteness length. *Bull. Lond. Math. Soc.*, 44(6):1209–1214, 2012.

[32] H. Garland and M. S. Raghunathan. Fundamental domains for lattices in rank one semisimple Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 62:309–313, 1969.

[33] H. Garland and M. S. Raghunathan. Fundamental domains for lattices in (R-)rank 1 semisimple Lie groups. *Ann. of Math. (2)*, 92:279–326, 1970.

[34] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[35] S. M. Gersten and H. B. Short. Rational subgroups of biautomatic groups. *Ann. of Math. (2)*, 134(1):125–158, 1991.

[36] André Haefliger. Complexes of groups and orbihedra. In *Group theory from a geometrical viewpoint (Trieste, 1990)*, pages 504–540. World Sci. Publ., River Edge, NJ, 1991.

[37] André Haefliger. Extension of complexes of groups. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):275–311, 1992.

[38] Camille Horbez and Jingyin Huang. Measure equivalence classification of transvection-free right-angled artin groups, 2020. arXiv:2010.03613 [math.GR].

[39] Susan Hermiller and Conchita Martínez-Pérez. HNN extensions and stackable groups. *Groups Geom. Dyn.*, 12(3):1123–1158, 2018.

[40] Frédéric Haglund and Frédéric Paulin. Constructions arborescentes d'immeubles. *Math. Ann.*, 325(1):137–164, 2003.

[41] Jingyin Huang. Commensurability of groups quasi-isometric to RAAGs. *Invent. Math.*, 213(3):1179–1247, 2018.

[42] Sam Hughes. Hierarchically hyperbolic groups, products of CAT(-1) spaces, and virtual torsion-freeness, 2021. arXiv:2105.02847 [math.GR].

[43] Mehrdad Kalantar and Matthew Kennedy. Boundaries of reduced C^*-algebras of discrete groups. *J. Reine Angew. Math.*, 727:247–267, 2017.

[44] Aditi Kar and Michah Sageev. Ping pong on CAT(0) cube complexes. *Comment. Math. Helv.*, 91(3):543–561, 2016.

[45] Ian J. Leary and Ashot Minasyan. Commensurating HNN-extensions: non-positive curvature and biautomaticity, 2019. arXiv:1907.03515 [math.GR].

[46] Wolfgang Lück. *L^2-invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.

[47] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[48] Robert T. Powers. Simplicity of the C^*-algebra associated with the free group on two generators. *Duke Math. J.*, 42:151–156, 1975.

[49] Henrik Densing Petersen, Roman Sauer, and Andreas Thom. L^2-Betti numbers of totally disconnected groups and their approximation by Betti numbers of lattices. *J. Topol.*, 11(1):257–282, 2018.

[50] Nicolas Radu. New simple lattices in products of trees and their projections. *Canad. J. Math.*, 72(6):1624–1690, 2020. With an appendix by Pierre-Emmanuel Caprace.

[51] Diego Rattaggi. A finitely presented torsion-free simple group. *J. Group Theory*, 10(3):363–371, 2007.

[52] Diego Rattaggi. Three amalgams with remarkable normal subgroup structures. *J. Pure Appl. Algebra*, 210(2):537–541, 2007.

[53] Nithi Rungtanapirom, Jakob Stix, and Alina Vdovina. Infinite series of quaternionic 1-vertex cube complexes, the doubling construction, and explicit cubical Ramanujan complexes. *Internat. J. Algebra Comput.*, 29(6):951–1007, 2019.

[54] Roman Sauer. Homological Invariants and Quasi-Isometry *Geom. funct. anal.* 16:476—515, 2006.

[55] Jean-Pierre Serre. Cohomologie des groupes discrets. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 77–169. Ann. of Math. Studies, No. 70, 1971.

[56] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell, Corrected 2nd printing of the 1980 English translation.

[57] Mentor Stafa. On the fundamental group of certain polyhedral products. *J. Pure Appl. Algebra*, 219(6):2279–2299, 2015.

[58] Anne Thomas. Lattices acting on right-angled buildings. *Algebr. Geom. Topol.*, 6:1215–1238, 2006.

[59] Motiejus Valiunas. Isomorphism classification of leary-minasyan groups, 2020. arXiv:2011.08143 [math.GR].

[60] Motiejus Valiunas. Leary-minasyan subgroups of biautomatic groups, 2021. arXiv:2104.13688 [math.GR].

Paper 5

HIERARCHICALLY HYPERBOLIC GROUPS AND VIRTUAL TORSION-FREENESS

SAM HUGHES

ABSTRACT. We construct non-virtually torsion-free hierarchically hyperbolic groups.

5.1 Introduction

Hierarchically hyperbolic groups (HHGs) and spaces (HHSs) were introduced by Behrstock, Hagen and Sisto in [5]. Hierarchically hyperbolic groups are known to satisfy a number of properties such as having finite asymptotic dimension [4, Theorem A], having a uniform bound on the conjugator length of Morse elements [1], and for virtually torsion-free HHGs, their uniform exponential growth is well understood [2]. HHGs belong to the class of semihyperbolic groups [13, Corollary F] (see also [11]). In particular, they have undistorted abelian subgroups, solvable conjugacy problem, finitely many conjugacy classes of finite subgroups, and are of type FP_∞.

That HHGs have only many finitely many conjugacy classes of finite subgroups implies that every residually finite HHG is in fact virtually torsion-free. This motivates the question of whether there exist any non-virtually torsion-free HHGs. The question is of considerable interest to specialists since, for example, a number of theorems about HHGs require the assumption of virtual torsion-freeness (see for instance [2, Theorem 1.1] and [22, Theorem 1.2(3')]).

In this paper we construct a CAT(0) lattice Γ acting faithfully and geometrically on a product of trees. We then prove that Γ is a hierarchically hyperbolic group and has no finite index torsion-free subgroups.

Theorem 5.A (Theorem 5.4.3)**.** *There exist hierarchically hyperbolic groups which are not virtually torsion-free.*

To the author's knowledge this is the first explicit example of an HHG which is not virtually torsion-free. The author suspects that it is possible to apply the results of Hagen–Susse [14] to Wise's examples in [24] to obtain an HHG which is not virtually torsion-free, however, the construction presented here is much more elementary and gives an explicit HHG structure.

Acknowledgements

The author would like to thank his PhD supervisor Ian Leary for his guidance and support. Additionally, the author would like to thank Mark Hagen, Ashot Minasyan, Harry Petyt, and Motiejus Valiunas for helpful correspondence and conversations. Finally, the author would like to thank Yves de Cornulier for the idea inspiring the examples in Section 5.4 .

5.2 Definitions

In this section we will give the relevant background on HHSs and HHGs for our endeavours. The definitions are rather technical so we will only focus on what we need, for a full account the reader should consult [3, Definition 1.1, 1.2.1]. We will follow the treatment in [18, Section 2]. To this end, a *hierarchically hyperbolic space (HHS)* is pair $(X, \mathfrak{S})$ where X is an ϵ-quasigeodesic space and $\mathfrak{S}$ is a set with some extra data which essentially functions as a coordinate system on X where each coordinate entry is a hyperbolic space. The relevant parts of the axiomatic formalisation are described as follows:

- For each *domain* $U \in \mathfrak{S}$, there is a hyperbolic space $\mathcal{C}U$ and *projection* $\pi_U : X \to \mathcal{C}U$ that is coarsely Lipschitz and coarsely onto [3, Remark 1.3].
- $\mathfrak{S}$ has a partial order $\sqsubseteq$, called *nesting*. Nesting chains are uniformly finite, and the length of the longest such chain is called the *complexity* of $(X, \mathfrak{S})$.
- $\mathfrak{S}$ has a symmetric relation $\perp$, called *orthogonality*. The complexity bounds pairwise orthogonal sets of domains.
- The relations $\sqsubseteq$ and $\perp$ are mutually exclusive. The complement of $\sqsubseteq$, $\perp$ and $=$ is called *transversality* and denoted $\pitchfork$.
- If $U \in \mathfrak{S}$ and there is some domain orthogonal to U, then there is some $W \in \mathfrak{S}$ such that $V \sqsubset W$ whenever $V \perp U$. We call W an *orthogonal container*.
- Whenever $U \pitchfork V$ or $U \sqsubset V$ there is a bounded set $\rho_V^U \subset \mathcal{C}V$. These sets, and projections of elements $x \in X$, are *consistent* in the following sense:
  - *ρ-consistency:* Let $U, V, W \in \mathfrak{S}$ such that $U \sqsubset V$ and ρ_W^V is defined, then ρ_W^U coarsely agrees with ρ_W^V;
  - If $U \pitchfork V$ then $\min\{d_{\mathcal{C}U}(\pi_U(x), \rho_U^V), d_{\mathcal{C}V}(\pi_V(x), \rho_V^U)\}$ is bounded.

All coarseness may taken to be uniform so we can and will fix a uniform constant ϵ [3, Remark 1.6].

We remind the reader that these axioms for an HHS are not a complete set but only recall the structure we will need. For the full definition the reader should consult [3, Definition 1.1, 1.2.1]. The following definition of an HHG is however complete.

Let X be the Cayley graph of a group Γ and suppose $(X, \mathfrak{S})$ is an HHS, then $(\Gamma, \mathfrak{S})$ is a *hierarchically hyperbolic group structure (HHG)* if it also satisfies the following:

(i) Γ acts cofinitely on $\mathfrak{S}$ and the action preserves the three relations. For each $g \in G$ and each $U \in \mathfrak{S}$, there is an isometry $g : \mathcal{C}U \to \mathcal{C}gU$ and these isometries satisfy $g \cdot h = gh$;

(ii) for all $U, V \in \mathfrak{S}$ with $U \pitchfork V$ or $V \sqsubset U$ and all $g, x \in \Gamma$ there is equivariance of the form $g\pi_U(gx) = \pi_{gU}(gx)$ and $g\rho^V_U = \rho^{gV}_{gU}$.

Note that this is not the original definition of a HHG as given in [3]. Instead, we have adopted the simpler axioms from [18], the axioms we have given imply the original axioms, however, by [12, Section 2.1] they are in fact equivalent.

5.3 Hierarchical hyperbolicity and products

In this section we provide a proof of the folklore result that a group acting geometrically on a product of δ-hyperbolic spaces with equivariant projections is an HHG. Let X be a proper metric space and let $H = \mathrm{Isom}(X)$, then H is a locally compact group with the topology given by uniform convergence on compacta. Let Γ be a discrete subgroup of H. We say Γ is a *uniform lattice* if X/Γ is compact. Recall that $\mathrm{Isom}(\mathbb{E}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$, we denote by $\pi_{\mathrm{O}(n)}$ the projection to $\mathrm{O}(n)$.

Proposition 5.3.1. *Let $n \geqslant 0$ and let $H \leqslant \mathrm{Isom}(\mathbb{E}^n) \times \prod_{i=1}^m \mathrm{Isom}(X_i)$ be a closed subgroup, where each X_i is a proper non-elementary δ-hyperbolic space. Assume H acts minimally and cocompactly on $X = \mathbb{E}^n \times \prod_{i=1}^m X_i$ and let Γ be a uniform H-lattice. If $\pi_{\mathrm{O}(n)}(\Gamma)$ is trivial, then Γ is a hierarchically hyperbolic group.*

Proof. Let q be a Γ-equivariant quasi-isometry $\mathrm{Cay}(\Gamma, \mathrm{A}) \to \mathrm{X}$ given by the Švarc-Milnor Lemma [6, I.8.19]. If $n > 0$, then for $j \in \{1-n, \dots, 0\}$ let $X_j = \mathbb{R}$ and $H_j = \mathrm{Isom}(\mathbb{E})$. If $n > 0$, then let $i \in \{1-n, \dots, m\}$, otherwise let $i \in \{1, \dots, m\}$. Now, products of HHSs are HHSs so $(X, \mathfrak{S})$ is an HHS [3, Proposition 8.27]. Moreover, by the description given in the proof of [3, Proposition 8.27] every domain of $\mathfrak{S}$ is either bounded (in fact a point) or some X_i.

Note that $\mathfrak{S}$ is finite and the action on $\mathfrak{S}$ is trivial. Every domain of the structure is either bounded (in fact a point) or one of the X_i. In the first case the Γ action is trivial and in the second case Γ acts via π_{H_i}. This immediately yields the first axiom.

For the second axiom consider the following diagram where the vertical arrows are given by applying the obvious group action:

$$\begin{array}{ccc} \Gamma \times \mathrm{Cay}(\Gamma, \mathrm{A}) & \xrightarrow{(\pi_{H_i}, \pi_{X_i} \circ q)} & \pi_{H_i}(\Gamma) \times X_i \\ \downarrow & & \downarrow \\ \mathrm{Cay}(\Gamma, \mathrm{A}) & \xrightarrow[\pi_{X_i} \circ q]{} & X_i. \end{array}$$

We will verify the diagram commutes. Let $x \in \mathrm{Cay}(\Gamma, \mathrm{A})$ and $g \in \Gamma$. First, we evaluate the composite map going down then across, we have

$$(g, x) \mapsto gx \mapsto \pi_{X_i}(q(gx)).$$

Going the other way we have

$$(g, x) \mapsto (\pi_{H_i}(g), \pi_{X_i}(q(x))) \mapsto (\pi_{H_i}(g), \pi_{X_i}(q(x)) = \pi_{X_i}(gq(x)) = \pi_{X_i}(q(gx))$$

where the last equality is given by the Γ-equivariance of q. In particular, $g\pi_{X_i}(x) = \pi_{gX_i}(gx) = \pi_{X_i}(gx)$. The other condition for equivariance is established immediately since any two domains that are not points are orthogonal to each other. □

We restate this result in terms of groups acting geometrically on products of CAT(-1) spaces. For an introduction to CAT(κ) groups and spaces see [6]. We will assume some non-degeneracy conditions on the CAT(0) spaces to avoid many technical difficulties associated with the CAT(0) condition (see [10, Section 1.B] for a thorough explanation). A group H acting on a CAT(0) space X is *minimal* if there is no H-invariant closed convex subset $X' \subset X$. If $\mathrm{Isom}(X)$ is minimal, then we say X is minimal.

Corollary 5.3.2. *Let Γ be a group acting properly cocompactly by isometries on a finite product of proper irreducible minimal* CAT(-1)*-spaces without permuting isometric factors, then Γ is a hierarchically hyperbolic group.*

Proof. The group Γ splits as a short exact sequence

$$\{1\} \rightarrowtail F \rightarrowtail \Gamma \twoheadrightarrow \Lambda \twoheadrightarrow \{1\},$$

where Λ satisfies the conditions of the previous theorem and F is the kernel of the action onto the product space. Since F acts trivially on the product space, it acts trivially on the HHG structure for Λ. The epimorphism $\varphi : \Gamma \twoheadrightarrow \Lambda$ induces an equivariant quasi-isometry ψ on the associated Cayley graphs. Thus, we may precompose every map in the previous theorem with φ or ψ to endow Γ with the structure of a HHG. □

To prove a converse to this corollary one may need to investigate the commensurators of maximal abelian subgroups of a hierarchically hyperbolic group Γ. Indeed, the CAT(0) not biautomatic groups introduced by Leary–Minasyan [17] and the groups constructed by the author in [15] have maximal abelian subgroups which have infinite index in their commensurator and are not virtually normal. All of these groups have a non-discrete projection to $\mathrm{O}(n) \leqslant \mathrm{Isom}(\mathbb{E}^n)$.

Question 5.3.3. Suppose a hierarchically hyperbolic group Γ has the property that every abelian subgroup is contained in a maximal abelian subgroup. Then, is a maximal abelian subgroup A of Γ either finite index in its commensurator $\mathrm{Comm}_\Gamma(A)$ or virtually normal?

5.4 Non-virtually torsion-free HHGs

In this section we will construct a hierarchically hyperbolic group which is not virtually torsion-free.

Let Λ be a Burger-Mozes simple group [7] [8] [9] acting on $\mathcal{T}_1 \times \mathcal{T}_2$ splitting as an amalgamated free product $F_n *_{F_m} F_n$ with embeddings $i, j : F_m \to F_n$. This defines a groups Λ which embeds discretely into the product of $T_1 = \mathrm{Aut}(\mathcal{T}_1)$ and $T_2 = \mathrm{Aut}(\mathcal{T}_2)$ with compact quotient. For instance one may take Rattaggi's example of a lattice in the product of an 8-regular and 12-regular tree which splits as $F_7 *_{F_{73}} F_7$ [21] (see also [20] or one of Radu's examples [19]).

Define $A = \mathbb{Z}_p \rtimes F_n$ for p prime such that the F_n-action is non-trivial. Consider the embeddings $\widetilde{i}, \widetilde{j} : F_m \rightarrowtail F_n \rightarrowtail A$ given by the composition of i or j with the obvious inclusion. Now, we build a group Γ as an amalgamated free product $A *_{F_m} A$, note that Γ surjects onto the original Burger-Mozes group Λ with kernel the normal closure of the torsion elements. Let $\mathcal{T}_3$ denote the Bass-Serre tree of Γ and let T_3 denote the corresponding automorphism group.

Proposition 5.4.1. Γ *is a uniform* $(T_1 \times T_3)$*-lattice which does not permute the factors.*

This can be easily deduced by endowing Γ with a graph of lattices structure in the sense of [15, Defintion 3.1] and then applying [15, Theorem A]. Instead we will provide a direct proof.

Proof. The group Γ acts on Bass-Serre tree $\mathcal{T}_3$ and also on $\mathcal{T}_1$ via the homomorphism $\psi : \Gamma \to T_1$ defined by taking the surjection $\Gamma \twoheadrightarrow \Lambda$. The diagonal action on the product space $\mathcal{T}_1 \times \mathcal{T}_3$ is properly discontinuous cocompact and by isometries. The kernel of the action is trivial, since the only elements which could act trivially are the torsion elements. However, these all clearly act non-trivially on $\mathcal{T}_3$. Thus, the action is faithful. We conclude that Γ is a uniform $(T_1 \times T_3)$-lattice. □

It remains to show Γ is not virtually torsion-free.

Proposition 5.4.2. Γ *is not virtually torsion-free.*

The author thanks Yves de Cornulier for the following argument.

Proof. Note that F_n normally generates A. Indeed, let a be a generator of $\mathbb{Z}_p$ and $f \in F_n$ act by $fa^if^{-1} \mapsto a^2i$. The elements $a^{-1}fa$ and $af^{-1}a^{-1}$ are in $\langle\langle F_n \rangle\rangle$ by definition, thus, $a^{-1}faaf^{-1}a^{-1} = a^{-1}a^4a^{-1} = a^2 \in \langle\langle F_n \rangle\rangle$. It immediately follows $a \in \langle\langle F_n \rangle\rangle$ and so F_n normally generates A.

Now, the finite residual of Γ, that is the intersection of all finite-index subgroups of Γ, denoted $\Gamma^{(\infty)}$, clearly contains the Burger-Mozes simple group Λ. Thus, both copies of F_n are contained in $\Gamma^{(\infty)}$ since these are subgroups of Λ. As F_n normally generates A, it follows $\Gamma^{(\infty)} = \Gamma$. Since, A is not torsion-free, we conclude Γ is not virtually torsion-free. □

To summarise we have the following theorem.

Theorem 5.4.3 (Theorem 5.A)**.** Γ *is a hierarchically hyperbolic group which is not virtually torsion-free.*

Proof. By Proposition 5.4.1 and Corollary 5.3.2 we see Γ is a hierarchically hyperbolic group. By Proposition 5.4.2 we see Γ is not virtually torsion-free. □

Remark 5.4.4. In [15, Corollary 9.9] the author gave a way to use A. Thomas's construction in [23] to promote lattices in products of trees to lattices in products of "sufficiently symmetric" right-angled buildings. Applying [15, Corollary 9.9] to one of the non-virtually torsion-free lattices Γ we obtain a non-virtually torsion-free lattice Λ acting on a product of "sufficiently symmetric" right-angled hyperbolic buildings each not quasi-isometric to a tree. Moreover, by Corollary 5.3.2 Λ is hierarchically hyperbolic.

References

[1] Carolyn Abbott and Jason Behrstock. Conjugator lengths in hierarchically hyperbolic groups, 2019. arXiv:1808.09604 [math.GR].

[2] Carolyn Abbott, Thomas Ng, and Davide Spriano. Hierarchically hyperbolic groups and uniform exponential growth, 2019. arXiv:1909.00439 [math.GR].

[3] Jason Behrstock, Mark Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces II: Combination theorems and the distance formula. *Pacific J. Math.*, 299(2):257–338, 2019.

[4] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups. *Proc. Lond. Math. Soc. (3)*, 114(5):890–926, 2017.

[5] Jason Behrstock, Mark F. Hagen, and Alessandro Sisto. Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups. *Geom. Topol.*, 21(3):1731–1804, 2017.

[6] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[7] Marc Burger and Shahar Mozes. Finitely presented simple groups and products of trees. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(7):747–752, 1997.

[8] Marc Burger and Shahar Mozes. Groups acting on trees: from local to global structure. *Inst. Hautes Études Sci. Publ. Math.*, 92:113–150 (2001), 2000.

[9] Marc Burger and Shahar Mozes. Lattices in product of trees. *Inst. Hautes Études Sci. Publ. Math.*, 92:151–194 (2001), 2000.

[10] Pierre-Emmanuel Caprace and Nicolas Monod. Isometry groups of non-positively curved spaces: structure theory. *J. Topol.*, 2(4):661–700, 2009.

[11] Matthew G. Durham, Yair N. Minsky, and Alessandro Sisto. Stable cubulations, bicombings, and barycenters, 2020. arXiv:2009.13647 [math.GR].

[12] Matthew Gentry Durham, Mark F. Hagen, and Alessandro Sisto. Correction to the article Boundaries and automorphisms of hierarchically hyperbolic spaces. *Geom. Topol.*, 24(2):1051–1073, 2020.

[13] Thomas Haettel, Nima Hoda, and Harry Petyt. The coarse helly property, hierarchical hyperbolicity, and semihyperbolicity, 2020. arXiv:2009.14053 [math.GR]

[14] Mark F. Hagen and Tim Susse. On hierarchical hyperbolicity of cubical groups. *Israel J. Math.*, 236(1):45–89, 2020.

[15] Sam Hughes. Graphs and complexes of lattices, 2021. arXiv:2104.13728 [math.GR].

[16] Michael Kapovich and Bernhard Leeb. Actions of discrete groups on nonpositively curved spaces. *Math. Ann.*, 306(2):341–352, 1996.

[17] Ian J. Leary and Ashot Minasyan. Commensurating HNN-extensions: non-positive curvature and biautomaticity, 2019. arXiv:1907.03515 [math.GR].

[18] Harry Petyt and Davide Spriano. Unbounded domains in hierarchically hyperbolic groups, 2020. arXiv:2007.12535 [math.GR].

[19] Nicolas Radu. New simple lattices in products of trees and their projections. *Canad. J. Math.*, 72(6):1624–1690, 2020. With an appendix by Pierre-Emmanuel Caprace.

[20] Diego Rattaggi. A finitely presented torsion-free simple group. *J. Group Theory*, 10(3):363–371, 2007.

[21] Diego Rattaggi. Three amalgams with remarkable normal subgroup structures. *J. Pure Appl. Algebra*, 210(2):537–541, 2007.

[22] Bruno Robbio and Davide Spriano. Hierarchical hyperbolicity of hyperbolic-2-decomposable groups, 2020. arXiv:2007.13383 [math.GR].

[23] Anne Thomas. Lattices acting on right-angled buildings. *Algebr. Geom. Topol.*, 6:1215–1238, 2006.

[24] Daniel T. Wise. Complete square complexes. *Comment. Math. Helv.*, 82(4):683–724, 2007.