International Game Theory Review

The Polymatrix Gap Conjecture

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The article proposes a novel way to compare classes of strategic games based on their sets of pure Nash equilibria. This approach is then used to relate the classes of zero-sum games, polymatrix, and k-polymatrix games. The article concludes with a conjecture that k-polymatrix games form an increasing chain of classes.

Keywords: Zero-sum games; polymatrix games; pure Nash equilibria, relation on classes of games

Subject Classification: 91A06, 91A43, 91A99.

1. Introduction

Strategic games could be viewed as abstract mathematical structures. As such, they can be compared using homomorphisms, as it is common in mathematics. One can also compare classes of games using game-specific properties such as Nash equilibria. Gottlob *et al.* [2005] suggested to compare classes of games based on the complexity of their sets of equilibria. In this article, we propose to compare classes of games by using their sets of pure Nash equilibria. For example, although any set of two-player strategy profiles is a Nash equilibrium of some game, not each such set is a set of Nash equilibria of a zero-sum game. Being able to compare different classes of games would have important implications for understanding the strengths or limitations of mechanism design, and more broadly, for discovering connections between seemingly unrelated classes of games.

In this work we consider strategic games with a finite number of players, each player with a finite number of strategies. We say that the class of zero-sum games is *coarser* than the class of all games. In general, we say that a class of games C is *coarser* than a class of games D if for any game $c \in C$ there is a game $d \in D$ such that games c and d have the same set of Nash equilibria. We say that class C is *strictly coarser* than class D if class C is *coarser* than class D, but not the other way around. If class C is *coarser* than class D, and class D is *coarser* than class C, then we say that these two classes are *Nash-equivalent*. In other words, two classes of games are *Nash-equivalent* if they have the same sets of Nash equilibria.

Next we use this general framework to investigate how the class of zero-sum games relates to several classes of games on graphs. In particular, we prove that if the number of players is greater than 4, polymatrix games are *strictly coarser* than zero-sum games. Then we extend the definition of polymatrix game, where each player simultaneously plays multiple 2-person games, to include the more general situation where each player simultaneously plays multiple k-person games, which we call k-polymatrix games. We prove that for every $2 < k \leq n-1$, the class of k-polymatrix games is *coarser* than the class of zero-sum games, and that $(n-1)$-polymatrix games are *Nash-equivalent* to zero-sum games. We conclude the article with a conjecture that k-polymatrix games form an increasing chain of classes.

Although k-polymatrix game is a new notion proposed in this article, polymatrix games have been widely studied. Howson Jr [1072] and Audet *et al.* [2006] studied existence and algorithms for finding mixed equilibria in such games. Govindan and Wilson [2004] computed equilibria of an arbitrary n-person game by approximating it with a sequence of polymatrix games. Quintas [1989] gave a characterization of the set of all mixed Nash equilibria in polymatrix games. Cai *et al.* [2016] have shown how linear programming could be used to find mixed Nash equilibria in zero-sum polymatrix games. Irfan and Ortiz [2014], Proposition 3.8 have shown that the class of linear-influence games is Nash-equivalent to 2-action polymatrix games. The polymatrix and k-polymatrix games are also related to graphical games, in which payoff function of each player (node in a graph) depends only on the actions of the adjacent nodes Elkind *et al.* [2006]; Elkind *et al.* [2007]; Dilkina*et al.* [2007]; Daskalakis and Papadimitriou [2006]; Kearns *et al.* [2001].

2. Preliminaries

In this section we introduce basic notions used throughout the article, illustrate them with examples, and prove their basic properties.

Definition 1. *A frame is a pair* $(N, \{A_i\}_{i\in N})$*, where*

(1) $N = \{1, 2, \dots, n\}$ *is a finite list of "players" for some integer* $n > 0$*, and*
(2) A_i *is a finite nonempty set of "strategies" for each* $i \in N$*.*

Integer n *is called the size of the frame.*

Definition 2. *A game* $\mathbf{u}$ *over a frame* $\mathcal{F} = (N, \{A_i\}_{i\in N})$ *is a set of "utility" functions* $\mathbf{u} = \{u_i\}_{i\in N}$ *such that* $u_i : A_1 \times \cdots \times A_n \to \mathbb{R}$ *for each* $i \in N$*. The set of all games over frame* $\mathcal{F}$ *is denoted by* $G(\mathcal{F})$*.*

We often write $A_1 \times \cdots \times A_n$ as $\prod_{i\in N} A_i$. The elements of this set are called strategy profiles. If $s = \{s_i\}_{i\in N}$ is a strategy profile and $e = \{v_1, \dots, v_k\}$ is a set of players, then by projection $s|_e$ we mean tuple $\{s_{v_i}\}_{i\in e} \in \prod_{i\le k} A_{v_i}$.

Definition 3. *A strategy profile $s \in \prod_{i\in N} A_i$ of a game $\{u_i\}_{i\in N} \in G(N, \{A_i\}_{i\in N})$ is a Nash equilibrium if $u_i(s_{-i}, a) \le u_i(s)$ for each player $i \in N$ and each strategy $a \in A_i$.*

The set of all Nash equilibria of a game $\mathbf{u}$ is denoted by $NE(\mathbf{u})$.

Lemma 1. *If game $\{u_i\}_{i\in N} \in G(N, \{A_i\}_{i\in N})$ has no Nash equilibria, then there are at least two sets in family $\{A_i\}_{i\in N}$ both of which have at least two elements.* ⊠

Definition 4. *For any frame $\mathcal{F} = (N, \{A_i\}_{i\in N})$, a game $\{u_i\}_{i\in N} \in G(\mathcal{F})$ is called a zero-sum game if*

$$\sum_{i\in N} u_i(s) = 0$$

for each strategy profile $s \in \prod_{i\in N} A_i$. The set of all zero-sum games over frame $\mathcal{F}$ is denoted by $Z(\mathcal{F})$.

Definition 5. *For any frame $\mathcal{F}$ and any two sets of games $C, D \subseteq G(\mathcal{F})$, let $C \preceq_{\mathcal{F}} D$ if for each game $\mathbf{u} \in C$ there is a game $\mathbf{u}' \in D$ such that $NE(\mathbf{u}) = NE(\mathbf{u}')$.*

If $C \preceq_{\mathcal{F}} D$, then we say that C is a *Nash-coarser* and D is a *Nash-finer* class of games. Note that $\preceq_{\mathcal{F}}$ is a reflexive and transitive relation on subsets of $G(\mathcal{F})$. We write $C \equiv_{\mathcal{F}} D$ if $C \preceq_{\mathcal{F}} D$ and $D \preceq_{\mathcal{F}} C$, in which case we say that classes C and D are *Nash-equivalent*.

Example 1.

Let frame $\mathcal{F} = (\{1, 2\}, \{A_i\}_{i\in\{1,2\}})$ be a two-player frame such that sets A_1 and A_2 each have at least two elements. It is easy to see that for any set $X \subseteq A_1 \times A_2$ there is a game $\{u_i\}_{i\in\{1,2\}} \in G(\mathcal{F})$ such that $NE(\{u_i\}_{i\in\{1,2\}}) = X$. At the same time, due to the interchangeability theorem for two-player zero-sum games Osborne and Rubinstein [1994], p.23, there is a set $X \subseteq A_1 \times A_2$ for which there is no zero-sum game $\{u_i\}_{i\in\{1,2\}} \in Z(\mathcal{F})$ such that $NE(\{u_i\}_{i\in\{1,2\}}) = X$. Hence, $Z(\mathcal{F}) \preceq_{\mathcal{F}} G(\mathcal{F})$, but $G(\mathcal{F}) \not\preceq_{\mathcal{F}} Z(\mathcal{F})$.

Definition 6. *Let $k \geq 1$ be an integer. Game $\{u_i\}_{i\in N} \in G(N, \{A_i\}_{i\in N})$ is called a k-polymatrix game, if for each player $i \in N$ and each k-element set $e \subseteq N$ containing player i, there is such function $f_i^e : \prod_{j\in e} A_j \to \mathbb{R}$ that*

$$u_i(s) = \sum_{e \ni i} f_i^e(s|_e),$$

for each strategy profile $s \in \prod_{\ell\in N} A_\ell$.

Definition 7. *For any frame $\mathcal{F}$ and any $k \geq 1$, let $H_k(\mathcal{F})$ be the set of all k-polymatrix games over frame $\mathcal{F}$. The games in the set $H_2(\mathcal{F})$ are also known as polymatrix games.*

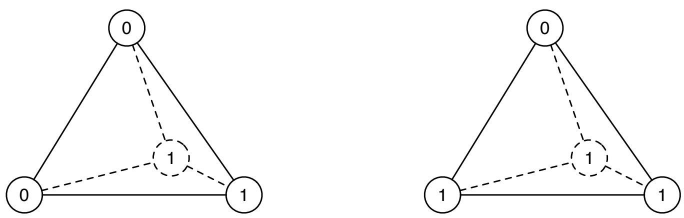

Fig. 1. Two strategy profiles of the game from Example 2.

Example 2. Let N be $\{1, 2, 3, 4\}$ and frame $\mathcal{F} = (N, \{A_i\}_{i\in N})$ be such that $A_i = \{0, 1\}$ for each $i \in N$. We can think about players in this game being vertices of a tetrahedron and a strategy profile being an assignment of either 0 or 1 to each vertex of the tetrahedron, see Figure 1. Consider a game in class $H_3(\mathcal{F})$ such that utility function for each vertex is the sum of three functions corresponding to the three faces of the tetrahedron containing this vertex. A function f corresponding to a face is defined in terms of the sum x of all values assigned to the vertices on the face:

(1) if x is even, then pay-off of each node on the face is 0.

(2) if $x = 1$, then each node on the face is penalized by 100.

(3) if $x = 3$, then each node on the face is penalized by 1.

Strategy profile $(0, 0, 0, 0)$ is a Nash equilibrium of this game because under this strategy profile each player gets pay-off zero, which is the largest possible pay-off in the game. Strategy profile $(0, 0, 1, 1)$, depicted in Figure 1, left, is *not* a Nash equilibrium of this game. Indeed, pay-off of each player using strategy 0 under this strategy profile is $-100 + 0 - 100 = -200$. The pay-off will increase to $0 - 1 + 0 = -1$ for a player who switches the strategy from 0 to 1. After such a switch by a single player, the strategy profile might become, for example, $(1, 0, 1, 1)$, depicted in Figure 1, right. It is easy to see that this strategy profile is a Nash equilibrium of the game. In general, the Nash equilibria of this game are $(0, 0, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 0, 1)$, $(1, 0, 1, 1)$, and $(0, 1, 1, 1)$. In other words, the set of Nash equilibria is $\{s \in \{0, 1\}^4 \ :\ |s| \equiv 0 \pmod 3\}$, where $|s|$ is the number of 1s in profile s.

The game described in this example belongs to class $H_3(\mathcal{F})$. At the same this game does not belong to class $H_2(\mathcal{F})$. In other words, this game is not a polymatrix game. This follows from Lemma 5 that we prove in Section 4.

We conclude this section with a simple technical observation about Nash equilibria of k-polymatrix games that will be used later. The observation is true because the matching pennies game only requires two strategies.

Lemma 2. *For any frame $(N, \{A_i\}_{i \in N})$ of size $n \geq 3$ such that there are at least two sets in family $\{A_i\}_{i \in N}$ with at least two elements, there is $(n - 1)$-polymatrix game in $G(N, \{A_i\}_{i \in N})$ that has no Nash equilibria.* ⊠

3. First Result

In this section we prove our first result relating k-polymatrix and zero-sum games. Namely, we prove that the class of k-polymatrix games is coarser than the class of zero-sum games for each $k \geq 2$.

Theorem 1. *$H_k(\mathcal{F}) \preceq_{\mathcal{F}} Z(\mathcal{F})$ for each frame $\mathcal{F} = (N, \{A_i\}_{i \in N})$ of size $n \geq 2$ and each $k \leq n-1$.*

Proof. Consider any game $\{u_i\}_{i \in N} \in H_k(\mathcal{F})$. By Definition 5, it suffices to show that there is a game $\{u'_i\}_{i \in N} \in Z(\mathcal{F})$ such that $NE(\{u_i\}_{i \in N}) = NE(\{u'_i\}_{i \in N})$.

By Definition 6 and the assumption $\{u_i\}_{i \in N} \in H_k(\mathcal{F})$, for each player $i \in N$ and each edge e of size k, where $i \in e$, there is a function $f_i^e : \prod_{j \in e} A_j \to \mathbb{R}$ such that

$$u_i(s) = \sum_{e \ni i} f_i^e(s|_e). \tag{1}$$

Define

$$u'_i(s) = u_i(s) - \frac{1}{|N| - k} \sum_{e \not\ni i} \sum_{j \in e} f_j^e(s|_e).$$

First, we show that $\{u'_i\}_{i \in N}$ is a zero-sum game. Indeed, for any strategy profile

$s \in \prod_{i\in N} A_i$, due to equation (1),

$$\begin{aligned}
\sum_{i\in N} u'_i(s) &= \sum_{i\in N}\left(u_i(s) - \frac{1}{|N|-k}\sum_{e\not\ni i}\sum_{j\in e} f^e_j(s|_e)\right)\\
&= \sum_{i\in N} u_i(s) - \sum_{i\in N}\frac{1}{|N|-k}\sum_{e\not\ni i}\sum_{j\in e} f^e_j(s|_e)\\
&= \sum_{i\in N} u_i(s) - \frac{1}{|N|-k}\sum_{e}\sum_{i\notin e}\sum_{j\in e} f^e_j(s|_e)\\
&= \sum_{i\in N} u_i(s) - \frac{1}{|N|-k}\sum_{e}(|N|-k)\sum_{j\in e} f^e_j(s|_e)\\
&= \sum_{i\in N} u_i(s) - \sum_{e}\sum_{j\in e} f^e_j(s|_e)\\
&= \sum_{i\in N} u_i(s) - \sum_{j\in N}\sum_{e\ni j} f^e_j(s|_e)\\
&= \sum_{i\in N} u_i(s) - \sum_{j\in N} u_j(s) = 0.
\end{aligned}$$

Next we show that $NE(\{u_i\}_{i\in N}) = NE(\{u'_i\}_{i\in N})$. Indeed, for any strategy profile s, any player $i \in N$, and any strategy $a \in A_i$,

$$\begin{aligned}
u'_i(s) - u'_i(s_{-i},a) &= \left(u_i(s) - \frac{1}{|N|-k}\sum_{e\not\ni i}\sum_{j\in e} f^e_j(s|_e)\right)\\
&\quad - \left(u_i(s_{-i},a) - \frac{1}{|N|-k}\sum_{e\not\ni i}\sum_{j\in e} f^e_j((s_{-i},a)|_e)\right)\\
&= \left(u_i(s) - \frac{1}{|N|-k}\sum_{e\not\ni i}\sum_{j\in e} f^e_j(s|_e)\right)\\
&\quad - \left(u_i(s_{-i},a) - \frac{1}{|N|-k}\sum_{e\not\ni i}\sum_{j\in e} f^e_j(s|_e)\right)\\
&= u_i(s) - u_i(s_{-i},a).
\end{aligned}$$

Hence, $u'_i(s) \le u'_i(s_{-i},a)$ if and only if $u_i(s) \le u_i(s_{-i},a)$ for each strategy profile s, each player $i \in N$, and each strategy $a \in A_i$. Therefore, by Definition 3, games $\{u_i\}_{i\in N}$ and $\{u'_i\}_{i\in N}$ have the same set of Nash equilibria. □

4. Second Result

In this section we show that is a frame $\mathcal{F}$ for which the set of zero-sum games is *not* Nash-coarser than the set of polymatrix games. Together with Theorem 1 this shows that the class of polymatrix games over frame $\mathcal{F}$ is strictly coarser than the class of zero-sum games.

Theorem 2. *For each $n \geq 4$ there is a frame $\mathcal{F}$ of size n such that*

$$Z(\mathcal{F}) \not\preceq_{\mathcal{F}} H_2(\mathcal{F}).$$

To prove the theorem, for any given $n \geq 4$, we define frame $\mathcal{F}$ to be $(N, \{A_i\}_{i\in N})$, where $N = \{1, 2, 3, 4, \ldots, n\}$ and $A_i = \{0, 1\}$ for each $i \in N$. Next we define game $\{u'_i\}_{i\in N} \in G(\mathcal{F})$. For any strategy profile $s = \{s_i\}_{i\in N}$, let

$$u'_i(s) = \begin{cases} 2 - n, & \text{if } s_i = 0 \text{ and } |s| = n - 2, \\ 2, & \text{if } s_i = 1 \text{ and } |s| = n - 2, \\ |s|, & \text{if } s_i = 0 \text{ and } |s| \neq n - 2, \\ |s| - n, & \text{if } s_i = 1 \text{ and } |s| \neq n - 2, \end{cases} \tag{2}$$

where $|s|$ is the number of 1s in the strategy profile s. Intuitively, game $\{u'_i\}_{i\in N}$ could be described as follows:

(1) if $|s| = n-2$, then each player who picked strategy 0 pays 1 unit to each player who picked strategy 1,

(2) if $|s| \neq n-2$, then each player who picked strategy 1 pays 1 unit to each player who has chosen strategy 0.

Since the pay-offs in the game $\{u'_i\}_{i\in N}$ consist in payments between players, the following lemma holds.

Lemma 3. $\{u'_i\}_{i\in N} \in Z(\mathcal{F})$. ⊠

The next lemma describes the set of Nash equilibria of the game $\{u'_i\}_{i\in N}$.

Lemma 4. $NE(\{u'_i\}_{i\in N}) = \{s \in \{0,1\}^n \mid |s| \equiv 0 \pmod{(n-1)}\}$.

Proof. Consider an arbitrary strategy profile $s = \{s_i\}_{i\in N} \in \{0,1\}^n$. We consider the following five cases separately:

(1) $|s| = 0$. Consider any $i_0 \le n$. Note that $|(s_{-i_0}, 1)| = 1 \ne n-2$ and $|s| = 0 \ne n-2$ because $n \ge 4$. To show that profile s is a Nash equilibrium, note that, due to equation (2),

$$u'_{i_0}(s_{-i_0}, 1) = |(s_{-i_0}, 1)| - n = 1 - n \le 1 - 4 < 0 = |s| = u'_{i_0}(s).$$

(2) $1 \le |s| \le n-3$. Consider any $i_1 \le n$ such that $s_{i_1} = 1$. Note that $|(s_{-i_1}, 0)| \le n-4$. To show that profile s is not a Nash equilibrium, note that, due to equation (2) and because $|(s_{-i_1}, 0)| \ne n-2$ and $|s| \ne n-2$,

$$u'_{i_1}(s_{-i_1}, 0) = |(s_{-i_1}, 0)| \ge 0 > -3 = (n-3) - n \ge |s| - n = u'_{i_1}(s).$$

(3) $|s| = n-2$. Consider any $i_0 \le n$ such that $s_{i_0} = 0$. Then, $|(s_{-i_0}, 1)| = n-1$. To show that profile s is not a Nash equilibrium, note that, due to equation (2) and because $n \ge 4$,

$$u'_{i_0}(s_{-i_0}, 1) = |(s_{-i_0}, 1)| - n = (n-1) - n = -1 > 2 - n = u'_{i_0}(s).$$

(4) $|s| = n-1$. Consider any $i_0, i_1 \le n$ such that $s_{i_0} = 0$ and $s_{i_1} = 1$. Then, $|(s_{-i_0}, 1)| = n$ and $|(s_{-i_1}, 0)| = n-2$. To show that profile s is a Nash equilibrium, note that, due to equation (2) and because $n \ge 4$,

$$u'_{i_0}(s_{-i_0}, 1) = |(s_{-i_0}, 1)| - n = n - n = 0 < n - 1 = |s| = u'_{i_0}(s),$$
$$u'_{i_1}(s_{-i_1}, 0) = 2 - n < (n-1) - n = |s| - n = u'_{i_1}(s).$$

(5) $|s| = n$. Consider any $i_1 \le n$. Note that $s_{i_1} = 1$ because $|s| = n$. Also,

$|(s_{-i_1}, 0)| = n - 1$. To show that profile s is not a Nash equilibrium, note that, due to equation (2) and because $n \geq 4$,

$$u'_{i_1}(s_{-i_1}, 0) = |(s_{-i_1}, 0)| = n - 1 > n - n = |s| - n = u'_{i_1}(s). \qquad \square$$

The next lemma captures a technical but important property of polymatrix games. Namely, it shows that the set described in the statement of Lemma 4 cannot be the set of Nash equilibria of a polymatrix game.

Lemma 5. *For any game* $\{u_i\}_{i\in N} \in H_2(\mathcal{F})$*, if*

$$\{s \in \{0,1\}^n \mid |s| \equiv 0 \pmod{(n-1)}\} \subseteq NE(\{u_i\}_{i\in N}),$$

then $(1, 1, \ldots, 1) \in NE(\{u_i\}_{i\in N})$.

Proof. Suppose that $(1, 1, \ldots, 1) \notin NE(\{u_i\}_{i\in N})$. Thus, by Definition 3, there is a player $i \in N$ such that $u_i((1, 1, \ldots, 1)_{-i}, 0) > u_i(1, 1, \ldots, 1)$. Without loss of generality, assume that $i = 1$. Hence,

$$u_1(0, 1, 1, \ldots, 1) > u_1(1, 1, 1, \ldots, 1). \tag{3}$$

Note that $n + 1$ strategy profiles

$$(0, 0, 0, \ldots, 0), (0, 1, 1, \ldots, 1), (1, 0, 1, \ldots, 1), (1, 1, 0, \ldots, 1), \ldots, (1, 1, 1, \ldots, 0)$$

are Nash equilibria of game $\{u_i\}_{i\in N}$ by the assumption of the lemma. Thus, by

Definition 3,

$$
\begin{aligned}
u_1(0,0,0,\dots,0) &\geq u_1((0,0,0,\dots,0)_{-1},1),\\
u_1(0,1,1,\dots,1) &\geq u_1((0,1,1,\dots,1)_{-1},1),\\
u_1(1,0,1,\dots,1) &\geq u_1((1,0,1,\dots,1)_{-1},0),\\
u_1(1,1,0,\dots,1) &\geq u_1((1,1,0,\dots,1)_{-1},0),\\
&\dots\\
u_1(1,1,1,\dots,0) &\geq u_1((1,1,1,\dots,0)_{-1},0).
\end{aligned}
$$

In other words,

$$
\begin{aligned}
u_1(0,0,0,\dots,0) &\geq u_1(1,0,0,\dots,0),\\
u_1(0,1,1,\dots,1) &\geq u_1(1,1,1,\dots,1),\\
u_1(1,0,1,\dots,1) &\geq u_1(0,0,1,\dots,1),\\
u_1(1,1,0,\dots,1) &\geq u_1(0,1,0,\dots,1),\\
&\dots\\
u_1(1,1,1,\dots,0) &\geq u_1(0,1,1,\dots,0).
\end{aligned}
\tag{4}
$$

Recall that $\{u_i\}_{i\in N} \in H_2(\mathcal{F})$ by the assumption of the lemma. Hence, by Definition 6, for each edge incident to node i, utility function u_i can be written as a sum of pay-off functions. In particular, there are functions $f_i : \{0,1\} \times \{0,1\} \to \mathbb{R}$ such that

$$u_1(s) = \sum_{i=2}^{n} f_i(s_1, s_i). \tag{5}$$

Note that here, in the case of polymatrix games, we use slightly simplified notations for functions f_i than in Definition 6, that deals with a more general setting of k-polymatrix games for an arbitrary k.

Thus, from inequalities (4) and equation (5),

$$
\begin{aligned}
f_2(0,0) + f_3(0,0) + \cdots + f_n(0,0) &\geq f_2(1,0) + f_3(1,0) + \cdots + f_n(1,0),\\
f_2(0,1) + f_3(0,1) + \cdots + f_n(0,1) &\geq f_2(1,1) + f_3(1,1) + \cdots + f_n(1,1),\\
f_2(1,0) + f_3(1,1) + \cdots + f_n(1,1) &\geq f_2(0,0) + f_3(0,1) + \cdots + f_n(0,1),\\
f_2(1,1) + f_3(1,0) + \cdots + f_n(1,1) &\geq f_2(0,1) + f_3(0,0) + \cdots + f_n(0,1),\\
&\cdots\\
f_2(1,1) + f_3(1,1) + \cdots + f_n(1,0) &\geq f_2(0,1) + f_3(0,1) + \cdots + f_n(0,0).
\end{aligned}
$$

Summing the left and the right hand sides of the above $n+1$ inequalities and combining terms one gets

$$
\begin{aligned}
&\sum_{j=2}^{n} \left(f_j(0,0) + f_j(0,1) + f_j(1,0) + (n-2) f_j(1,1)\right)\\
&\geq \sum_{j=2}^{n} \left(f_j(1,0) + f_j(1,1) + f_j(0,0) + (n-2) f_j(0,1)\right).
\end{aligned}
$$

Hence, by cancelling equal terms on both sides and factoring out $n-3$,

$$(n-3)\sum_{j=2}^{n} f_j(1,1) \geq (n-3)\sum_{j=2}^{n} f_j(0,1).$$

Recall that $n \geq 4$ by the assumption of the theorem. Hence,

$$\sum_{j=2}^{n} f_j(1,1) \geq \sum_{j=2}^{n} f_j(0,1).$$

Then, equality (5) implies that $u_1(1,1,1,\ldots,1) \geq u_1(0,1,1,\ldots,1)$, which is a contradiction with inequality (3). □

To finish the proof of Theorem 2, note that $\{u'_i\}_{i\in N} \in Z(\mathcal{F})$ by Lemma 3. At the same time, by Lemma 4 and Lemma 5 there is no game $\{u_i\}_{i\in N} \in H_2(\mathcal{F})$ such that $NE(\{u'_i\}_{i\in N}) = NE(\{u_i\}_{i\in N})$. Therefore, $Z(\mathcal{F}) \not\preceq H_2(\mathcal{F})$ by Definition 5.

5. Third Result

In Theorem 1 we have shown that for any frame of size n, the class of k-polymatrix games is coarser than the class of zero-sum games for each $k \leq n-1$. In Theorem 2 we have shown that, in the case $k = 2$, zero-sum games are *not* coarser than polymatrix games. In other words, classes of polymatrix and zero-sum games are not Nash-equivalent. Our last, and probably the most interesting observation is that classes of $(n-1)$-polymatrix and zero-sum games are Nash-equivalent. Due to Theorem 1, we only need to prove the following result.

Theorem 3. *$Z(\mathcal{F}) \preceq_{\mathcal{F}} H_{n-1}(\mathcal{F})$ for each frame $\mathcal{F}$ of size $n \geq 3$.*

The rest of this section is dedicated to the proof of Theorem 3. Let $\mathcal{F} = (N, \{A_i\}_{i\in N})$, where $N = \{1, \dots, n\}$. Before starting the proof, we introduce two technical notions and prove an auxiliary proposition.

Definition 8. *For each strategy profile $s \in \prod_{i\in N} A_i$ and each $j \in N$, let $(s_{-j}, *)$ be the cylinder set $\{(s_{-j}, x) \mid x \in A_j\}$.*

Definition 9. *For any strategy profiles $s, s' \in \prod_{i\in N} A_i$ let $\mathsf{h}(s, s')$ be the Hamming distance between tuples s and s'. The Hamming distance $\mathsf{H}(X, Y)$ between nonempty sets $X, Y \subseteq \prod_{i\in N} A_i$ is defined as usual:*

$$\mathsf{H}(X, Y) = \min\{\mathsf{h}(s, s') \mid s \in X, s' \in Y\}.$$

$\mathsf{H}(X, Y)$ is well-defined because set $\prod_{i\in N} A_i$ is finite by Definition 2.

Proposition 1. *For any profile $s = \{s_i\}_{i\in N} \in \prod_{i\in N} A_i$ and any nonempty set $X \subseteq \prod_{i\in N} A_i$, if $\mathsf{H}(\{s\}, X) > 0$, then $\mathsf{H}((s_{-j}, *), X) \geq \mathsf{H}(\{s\}, X) - 1$ for every $j \in N$.* ⊠

Proposition 2. *For any strategy profile* $s = \{s_i\}_{i \in N} \in \prod_{i \in N} A_i$ *and any nonempty set* $X \subseteq \prod_{i \in N} A_i$ *if* $\mathsf{H}(\{s\}, X) > 0$*, then there is a player* $j \in N$ *and strategy* $q \in A_j$ *such that* $\mathsf{H}((s_{-j}, q), X) = \mathsf{H}(\{s\}, X) - 1$.

Proof. By Definition 9, there exists a profile $x = \{x_i\}_{i \in N} \in X$ such that $\mathsf{H}(\{s\}, X) = \mathsf{h}(s, x)$. Then, assumption $\mathsf{H}(\{s\}, X) > 0$ implies that strategy profiles s and x differ for at least one player $j \in N$. Let $q = x_j$. Then, $\mathsf{h}((s_{-j}, q), x) = \mathsf{h}(s, x) - 1$. Thus,

$$\mathsf{H}(\{(s_{-j}, q)\}, X) \leq \mathsf{h}((s_{-j}, q), x) = \mathsf{h}(s, x) - 1 = \mathsf{H}(\{s\}, X) - 1.$$

Next, we show that $\mathsf{H}(\{(s_{-j}, q)\}, X) = \mathsf{H}(\{s\}, X) - 1$. Suppose that

$$\mathsf{H}(\{(s_{-j}, q)\}, X) < \mathsf{H}(\{s\}, X) - 1. \tag{6}$$

By Definition 9, there exists a strategy profile $y \in X$ such that $\mathsf{h}((s_{-j}, q), y) = \mathsf{H}(\{(s_{-j}, q)\}, X)$. Hence, by the triangle inequality and inequality (6),

$$\begin{aligned}\mathsf{H}(\{s\}, X) &\leq \mathsf{h}(s, y) \leq \mathsf{h}(s, (s_{-j}, q)) + \mathsf{h}((s_{-j}, q), y) = 1 + \mathsf{h}((s_{-j}, q), y) \\ &= 1 + \mathsf{H}(\{(s_{-j}, q)\}, X) < 1 + \mathsf{H}(\{s\}, X) - 1 = \mathsf{H}(\{s\}, X),\end{aligned}$$

which is a contradiction. □

We are now ready to prove Theorem 3.

Proof. Consider any zero-sum game $\mathbf{u}' = \{u'_i\}_{i \in N} \in Z(\mathcal{F})$. By Definition 5, it suffices to show that there is a game $\mathbf{u} \in H_{n-1}(\mathcal{F})$ such that $NE(\mathbf{u}') = NE(\mathbf{u})$.

Indeed, if set $NE(\mathbf{u}')$ is empty, then the family of sets $\{A_i\}_{i \in N}$ has at least two sets with at least two elements, by Lemma 1. Hence, there is a game $\mathbf{u} \in H_{n-1}(\mathcal{F})$ such that $NE(\mathbf{u}') = \varnothing = NE(\mathbf{u})$ by Lemma 2. In the rest of the proof we assume that set $NE(\mathbf{u}')$ is nonempty.

By Definition 2, set A_i is finite for each $i \in N$. Thus, there is a real number $C > 0$ such that

$$\forall i \in N \; \forall s \in \prod_k A_k, \quad |u'_i(s)| < C. \tag{7}$$

For any player $j \in N$ and each strategy profile $s \in \prod_{k\in N} A_k$, let

$$f^j(s) = -\max_{q\in A_j} u'_j(s_{-j}, q) - 2C(|N|-1) \cdot H\left((s_{-j}, *), NE(\mathbf{u}')\right). \tag{8}$$

Function $f^j(s)$ is well-defined because set A_j is nonempty by Definition 2 and set $NE(\mathbf{u}')$ is nonempty due to an assumption earlier in this proof.

For each player $i \in N$ and each strategy profile $s \in \prod_{k\in N} A_k$, let

$$u_i(s) = \sum_{j\neq i} f^j(s). \tag{9}$$

Let $\mathbf{u} = \{u_i\}_{i\in N}$.

Claim 1. $\mathbf{u} \in H_{n-1}(\mathcal{F})$.

Proof. By Definition 6 and equation (9), it suffices to show that function $f^j(s)$ does not depend on the j-th component of strategy profile s. In other words, that $f^j(s) = f^j(s_{-j}, a)$ for each player $j \in N$, each strategy profile $s \in \prod_{k\in N} A_k$, and each strategy $a \in A_j$. The last statement follows from equation (8). □

Claim 2. *$u'_i(s) = u_i(s)$ for any strategy profile $s \in NE(\mathbf{u}')$ and any $i \in N$.*

Proof. For any $s \in NE(\mathbf{u}')$, equations (9) and (8) imply that

$$\begin{aligned} u_i(s) &= \sum_{j\neq i} f^j(s) \\ &= \sum_{j\neq i} \left(-\max_{q\in A_j} u'_j(s_{-j}, q) - 2C(|N|-1) \cdot H\left((s_{-j}, *), NE(\mathbf{u}')\right) \right). \end{aligned}$$

Assumption $s \in NE(\mathbf{u}')$ implies that $H\left((s_{-j}, *), NE(\mathbf{u}')\right) = 0$. Thus,

$$u_i(s) = \sum_{j\neq i} -\max_{q\in A_j} u'_j(s_{-j}, q).$$

Therefore,

$$u_i(s) = \sum_{j \neq i} - \max_{q \in A_j} u'_j(s_{-j}, q) = \sum_{j \neq i} -u'_j(s) = u'_i(s),$$

because $s \in NE(\mathbf{u}')$ and $\mathbf{u}'$ is a zero-sum game. □

We are now ready to show that $NE(\mathbf{u}') = NE(\mathbf{u})$.

$(\subseteq)$: Suppose that $s \in NE(\mathbf{u}')$. We will show that $s \in NE(\mathbf{u})$. Consider any player $i \in N$ and any strategy $q \in A_i$. It suffices to show that $u_i(s) \geq u_i(s_{-i}, q)$. We denote (s_{-i}, q) by s'. Then, by Claim 2 and the assumption $s \in NE(\mathbf{u}')$,

$$u_i(s) = u'_i(s) \geq u'_i(s_{-i}, q) = u'_i(s'). \tag{10}$$

At the same time, because $\mathbf{u}'$ is a zero-sum game,

$$\begin{aligned} u'_i(s') &= \sum_{j \neq i} -u'_j(s') \geq \sum_{j \neq i} - \max_{r \in A_j} u'_j(s'_{-j}, r) \\ &\geq \sum_{j \neq i} \left(- \max_{r \in A_j} u'_j(s'_{-j}, r) - 2C(|N| - 1) \cdot H\left((s'_{-j}, *), NE(\mathbf{u}')\right) \right). \end{aligned}$$

Then, taking into account equations (8) and (9) and because $s' = (s_{-i}, q)$,

$$u'_i(s') \geq \sum_{j \neq i} f^j(s') = u_i(s') = u_i(s_{-i}, q).$$

Therefore, $u_i(s) \geq u_i(s_{-i}, q)$ by equation (10).

$(\supseteq)$: Consider any strategy profile s such that $s \notin NE(\mathbf{u}')$, it suffices to show that $s \notin NE(\mathbf{u})$. First, we consider the special case when for each $i \in N$,

$$\mathsf{H}\left((s_{-i}, *), NE(\mathbf{u}')\right) = 0. \tag{11}$$

Note that the left-hand-side of the above equation is well-defined due to the assumption in the beginning of the proof of the theorem that set $NE(\mathbf{u}')$ is not empty.

Assumption $s \notin NE(\mathbf{u}')$ implies that there is player $j \in N$ and strategy $p \in A_j$ such that $u'_j(s) < u'_j(s_{-j}, p)$. Note that equality (11) implies that there is strategy

$q \in A_j$ such that $(s_{-j}, q) \in NE(\mathbf{u}')$. Hence,

$$u'_j(s) < u'_j(s_{-j}, p) \leq u'_j(s_{-j}, q). \tag{12}$$

Thus, due to equality (9), equality (8), equality (11), the assumption that $\mathbf{u}'$ is a zero-sum game, inequality (12), and Claim 2, and the assumption $(s_{-j}, q) \in NE(\mathbf{u}')$,

$$\begin{aligned}
u_j(s) &\overset{(9)}{=} \sum_{i \neq j} f^i(s) \\
&\overset{(8)}{=} \sum_{i \neq j} \left(-\max_{r \in A_j} u'_j(s_{-j}, r) - 2C(|N| - 1) \cdot \mathsf{H}\left((s_{-j}, *), NE(\mathbf{u}')\right) \right) \\
&\overset{(11)}{=} \sum_{i \neq j} -\max_{r \in A_i} u'_i(s_{-i}, r) \\
&\leq \sum_{i \neq j} -u'_i(s) = u'_j(s) \overset{(12)}{<} u'_j(s_{-j}, q) = u_j(s_{-j}, q).
\end{aligned}$$

Therefore, $s \notin NE(\mathbf{u})$. This concludes the proof in the special case when equality (11) holds for each player $i \in N$.

We now can assume that there is at least one player $i \in N$ for which equality (11) does not hold. We denote such player by j. Thus,

$$\mathsf{H}\left(\{s\}, NE(\mathbf{u}')\right) \geq \mathsf{H}\left((s_{-j}, *), NE(\mathbf{u}')\right) > 0. \tag{13}$$

Thus, by Proposition 2, there exist a player $m \in N$ and a strategy $q \in A_m$ such that

$$\mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) = \mathsf{H}\left(\{s\}, NE(\mathbf{u}')\right) - 1. \tag{14}$$

Hence, for each player $i \in N$, due to the choice of m, by Proposition 1,

$$\begin{aligned}
\mathsf{H}((s_{-i}, *), NE(\mathbf{u}')) &\geq \mathsf{H}\left(\{s\}, NE(\mathbf{u}')\right) - 1 \\
&= \mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) \\
&\geq \mathsf{H}(((s_{-m}, q)_{-i}, *), NE(\mathbf{u}')).
\end{aligned} \tag{15}$$

Claim 3. *There is a player $i \in N$ such that*

$$\mathsf{H}((s_{-i}, *), NE(\mathbf{u}')) > \mathsf{H}(((s_{-m}, q)_{-i}, *), NE(\mathbf{u}')).$$

Proof. By inequality (13), we have $\mathsf{H}(\{s\}, NE(\mathbf{u}')) > 0$. We consider the following two cases separately:

Case 1: $\mathsf{H}(\{s\}, NE(\mathbf{u}')) = 1$. Thus, $\mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) = 0$ by inequality (14). Hence, $\mathsf{H}(((s_{-m}, q)_{-i}, *), NE(\mathbf{u}')) = 0$ for each player $i \in N$. Recall that we have previously chosen j that satisfies inequality (13). Consider $i = j$. Thus, by inequality (13),

$$\mathsf{H}((s_{-j}, *), NE(\mathbf{u}')) > 0 = \mathsf{H}(((s_{-m}, q)_{-j}, *), NE(\mathbf{u}')).$$

Case 2: $\mathsf{H}(\{s\}, NE(\mathbf{u}')) \geq 2$. Hence, $\mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) \geq 1$ by equality (14). Thus, by Proposition 2, there exist $m' \in N$ and $q' \in A_{m'}$ such that

$$\mathsf{H}(\{((s_{-m}, q)_{-m'}, q')\}, NE(\mathbf{u}')) = \mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) - 1.$$

Hence, by Proposition 1 and equation (14),

$$\begin{aligned}
\mathsf{H}((s_{-m'}, *), NE(\mathbf{u}')) &\geq \mathsf{H}(\{s\}, NE(\mathbf{u}')) - 1 \\
&= \mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) \\
&> \mathsf{H}(\{(s_{-m}, q)\}, NE(\mathbf{u}')) - 1 \\
&= \mathsf{H}(\{((s_{-m}, q)_{-m'}, q')\}, NE(\mathbf{u}')).
\end{aligned}$$

Let i be player m'. This concludes the proof of the claim. □

Let j_1 denote $i \in N$ whose existence is proven in Claim 3. Note that follows from the statement of Claim 3 that $j_1 \neq m$. Then,

$$\mathsf{H}((s_{-j_1}, *), NE(\mathbf{u}')) \geq \mathsf{H}(((s_{-m}, q)_{-j_1}, *), NE(\mathbf{u}')) + 1. \tag{16}$$

Hence, by equation (9) and equation (8),

$$\begin{aligned}
u_m(s) &= \sum_{j\neq m} f^j(s) \\
&= \sum_{j\neq m}\left(-\max_{p\in A_j} u'_j(s_{-j},p) - 2C(|N|-1)\cdot \mathsf{H}\left((s_{-j},*),NE(\mathbf{u}')\right)\right) \\
&= \sum_{j\neq m,j_1}\left(-\max_{p\in A_j} u'_j(s_{-j},p) - 2C(|N|-1)\cdot \mathsf{H}\left((s_{-j},*),NE(\mathbf{u}')\right)\right) \\
&\quad - \max_{p\in A_{j_1}} u'_{j_1}(s_{-j_1},p) - 2C(|N|-1)\cdot \mathsf{H}\left((s_{-j_1},*),NE(\mathbf{u}')\right). \\
&\overset{(15)}{\le} \sum_{j\neq m,j_1}\left(-\max_{p\in A_j} u'_j(s_{-j},p) - 2C(|N|-1)\cdot \mathsf{H}(((s_{-m},q)_{-j},*),NE(\mathbf{u}'))\right) \\
&\quad - \max_{p\in A_{j_1}} u'_{j_1}(s_{-j_1},p) - 2C(|N|-1)\cdot \mathsf{H}((s_{-j_1},*),NE(\mathbf{u}')) \\
&\overset{(16)}{\le} \sum_{j\neq m,j_1}\left(-\max_{p\in A_j} u'_j(s_{-j},p) - 2C(|N|-1)\cdot \mathsf{H}(((s_{-m},q)_{-j},*),NE(\mathbf{u}')))\right) \\
&\quad - \max_{p\in A_{j_1}} u'_{j_1}(s_{-j_1},p) - 2C(|N|-1)\cdot \left(\mathsf{H}\left(((s_{-m},q)_{-j_1},*),NE(\mathbf{u}')\right)+1\right) \\
&= \sum_{j\neq m}\left(-\max_{p\in A_j} u'_j(s_{-j},p) - 2C(|N|-1)\cdot \mathsf{H}\left(((s_{-m},q)_{-j},*),NE(\mathbf{u}')\right)\right) \\
&\quad -2C(|N|-1). \qquad (17)
\end{aligned}$$

Next, note that by the triangle inequality and the choice of C, for any player $j\in N$,

$$\begin{aligned}
&\max_{p\in A_j} u'_j((s_{-m},q)_{-j},p) - \max_{p\in A_{j_1}} u'_j(s_{-j_1},p) \\
&\le \left|\max_{p\in A_j} u'_j((s_{-m},q)_{-j},p) - \max_{p\in A_{j_1}} u'_j(s_{-j_1},p)\right| \\
&\le \left|\max_{p\in A_j} u'_j((s_{-m},q)_{-j},p)\right| + \left|\max_{p\in A_{j_1}} u'_j(s_{-j_1},p)\right| < C+C=2C. \qquad (18)
\end{aligned}$$

Thus, by equation (17),

$$
\begin{aligned}
& u_m(s) \\
\leq\ & \sum_{j \neq m} \left(-\max_{p \in A_j} u'_j(s_{-j}, p) - 2C(|N|-1) \cdot \mathsf{H}\left(((s_{-m}, q)_{-j}, *), NE(\mathbf{u}')\right) \right) \\
& -2C(|N|-1) \\
=\ & \sum_{j \neq m} \left(-\max_{p \in A_j} u'_j((s_{-m}, q)_{-j}, p) - 2C(|N|-1) \cdot \mathsf{H}\left(((s_{-m}, q)_{-j}, *), NE(\mathbf{u}')\right) \right) \\
& + \sum_{j \neq m} \left(\max_{p \in A_j} u'_j((s_{-m}, q)_{-j}, p) - \max_{p \in A_j} u'_j(s_{-j}, p) \right) - 2C(|N|-1) \\
\overset{(18)}{<}\ & \sum_{j \neq m} \left(-\max_{p \in A_j} u'_j((s_{-m}, q)_{-j}, p) - 2C(|N|-1) \cdot \mathsf{H}\left(((s_{-m}, q)_{-j}, *), NE(\mathbf{u}')\right) \right) \\
& +2C(|N|-1) - 2C(|N|-1) \\
=\ & \sum_{j \neq m} \left(-\max_{p \in A_j} u'_j((s_{-m}, q)_{-j}, p) - 2C(|N|-1) \cdot \mathsf{H}\left(((s_{-m}, q)_{-j}, *), NE(\mathbf{u}')\right) \right) \\
\overset{(8)}{=}\ & \sum_{j \neq m} f^j(s_{-m}, q) \overset{(9)}{=} u_m(s_{-m}, q).
\end{aligned}
$$

Therefore, $s \notin NE(\mathbf{u})$.

6. The Conjecture

By Theorem 1, Theorem 2, and Theorem 3, for any $n \geq 4$ there is a frame $\mathcal{F}$ such that $Z(\mathcal{F}) \not\preceq_{\mathcal{F}} H_2(\mathcal{F})$ and $Z(\mathcal{F}) \equiv_{\mathcal{F}} H_{n-1}(\mathcal{F})$. Thus, for any $n \geq 4$ there is a frame $\mathcal{F}$ such that

$$H_2(\mathcal{F}) \not\equiv_{\mathcal{F}} H_{n-1}(\mathcal{F}).$$

Moreover, for any frame $\mathcal{F}$,

$$H_2(\mathcal{F}) \preceq_{\mathcal{F}} H_3(\mathcal{F}) \preceq_{\mathcal{F}} \cdots \preceq_{\mathcal{F}} H_{n-1}(\mathcal{F})$$

because any game in class $H_{k-1}(\mathcal{F})$ can be made into a game in class $H_k(\mathcal{F})$ by adding "dummy" arguments to functions f_i^e from Definition 6.

We make a conjecture that for each $n \geq 4$ there is a frame $\mathcal{F}$ such that

$$H_2(\mathcal{F}) \not\equiv_{\mathcal{F}} H_3(\mathcal{F}) \not\equiv_{\mathcal{F}} \cdots \not\equiv_{\mathcal{F}} H_{n-1}(\mathcal{F}).$$

7. Conclusion

In this article we introduced a new way to compare classes of games based on the richness of the sets of Nash equilibria of these classes. We generalized the class of polymatrix games to the class of k-polymatrix games. The original polymatrix games are 2-polymatrix games. Also, we proved that for games with n players, class $(n-1)$-polymatrix games is equivalent, in our sense, to zero-sum games, while the class of 2-polymatrix games is not. Although the results in this article are stated in terms of Nash equilibria, they hold for strict Nash equilibria too. These results highlight the limitations on the possible sets of Nash equilibria in different classes of games, and therefore they are relevant to mechanism design. Finally, we made a conjecture that the classes of k-polymatrix games are not equivalent for all $2 \leq k \leq n-1$.

Although in this article we defined the notions of coarser, strictly coarser, and Nash-equivalent classes based on sets of pure Nash equilibria, similar relations can be defined using mixed equilibria. The first and the second results in this article holds for mixed-equilibria-based coarser relation, which can be shown with through a straightforward modification of the proofs of Theorem 1 and Theorem 2. Whether the third result holds for mixed equilibria remains an open question.

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