Abstract—Multi-carrier faster-than-Nyquist (MFTN) signaling constitutes a promising spectrally efficient non-orthogonal physical layer waveform. In this correspondence, we propose a pair of low-complexity generalized approximate message passing (GAMP)-based frequency-domain equalization (FDE) algorithms for MFTN systems operating in multipath channels. To mitigate the ill-condition of the resultant equivalent channel matrix, we construct block circulant interference matrices by inserting a few cyclic postfixes, followed by truncating the duration of the inherent two-dimensional interferences. Based on the decomposition of the block circulant matrices, we develop a novel frequency-domain received signal model using the two-dimensional fast Fourier transform for mitigating the colored noise imposed by the non-orthogonal matched filter. Moreover, we derive a GAMP-based FDE algorithm and its refined version, where the latter relies on approximations for circumventing the emergence of the ill-conditioned matrices. Our simulation results demonstrate that, for a fixed spectral efficiency, MFTN signaling can significantly improve the bit error rate (BER) performance by jointly optimizing the time- and frequency-domain packing factors. Compared to its Nyquist-signaling counterpart, our proposed MFTN systems employing the refined GAMP equalizer can achieve about 39% higher transmission rates at a negligible BER performance degradation.

Index Terms—Multi-carrier faster-than-Nyquist signaling, frequency-domain equalization, generalized approximate message passing, block circulant matrix with circulant blocks.

I. INTRODUCTION

Given the paucity of spectral resources, faster-than-Nyquist (FTN) signaling has attracted substantial attention as a benefit of its high capacity [1]–[4]. It constitutes a prominent member of the non-orthogonal multiple access (NOMA) family [5], [6] and it readily lends itself to amalgamation with multiple-input multiple-output (MIMO) [7] systems. To further increase its spectral efficiency (SE), multi-carrier FTN (MFTN) signaling was then proposed in [8]. By relaxing the constraints on the orthogonality of the signaling pulse and subcarriers with respect to the symbol interval and frequency spacing, MFTN signaling can provide a higher SE than one-dimensional FTN signaling. However, MFTN signaling inevitably incurs severe intersymbol interferences (ISIs) and intercarrier interferences (ICIIs), which result in a prohibitively high receiver complexity. Accordingly, conceiving low-complexity equalization is critical, but challenging for MFTN systems.

A popular class of MFTN signaling employs transmit precoding [9], [10]. Specifically in [9], a linear precoding scheme combined with receiver-side ICI cancellation was developed for MFTN systems. To circumvent the potential transmission contamination caused by small eigenvalues of the interference matrix, a Tomlison-Harashima precoding (THP) method was designed for one-dimensional frequency packed MFTN systems using high-order modulation in [10].

The other class of techniques employs advanced equalization at the receiver [11]–[16]. In [11], the optimal maximum a posteriori (MAP) method was derived by harnessing the Bahl-Cocke-Jelinek-Raviv (BCJR) algorithm, which however imposed an exponentially escalating complexity. In [12], a suboptimal reduced-state BCJR detector based on successive interference cancellation (SIC) was proposed. However, its complexity still increases exponentially with the length of the truncated interferences. As a further advance, a memoryless turbo receiver having linearly increasing complexity was designed for MFTN systems in [13]. Hence, this less complex, but also less powerful receiver results in a bit error rate (BER) performance loss in aggressive time-frequency packing scenarios. To strike a BER performance versus complexity trade-off, a pair of minimum mean-squared error (MMSE)-based equalizers were developed for MFTN signaling in [14]. Unfortunately, the two-dimensional (2D) MMSE-based equalizer suffers from a significant performance loss in severe ICI scenarios. By contrast, the one-dimensional version of [14] combined with SIC can asymptotically approach the MAP performance. However, the complexity of both methods is still dominated by the associated matrix inversion calculations. For longhaul optical transmission scenarios, turbo parallel interference cancellation (PIC) schemes combined with BCJR-based ISI cancellation were proposed for time-frequency packed (TFP) wavelength-division-multiplexed (WDM) systems [15], [16]. These systems separately eliminate the inherent 2D interferences of MFTN signaling, but a joint 2D interference cancellation scheme is expected to achieve better performance.

Most of the above-mentioned equalizers were proposed for additive white Gaussian noise (AWGN) channels, and they cannot deal with the colored noise imposed by non-orthogonal matched filtering. It has been demonstrated that mitigating the effect of colored noise improves the BER performance [3]. Hence, it is necessary to tackle the inherent colored noise problem in order to design an efficient MFTN receiver. In [17],
the authors proposed a Gaussian message passing (GMP)-based time-domain equalizer (TDE) relying on a vector-form trellis interference structure for MFTN systems operating in multipath channels. The truncated non-diagonal covariance matrix of the colored noise was exploited for improving the accuracy of message updating. To develop a parametric message passing receiver, the discrete a priori probabilities of the transmitted symbols are approximated by Gaussian distributions in [17], which results in an additional BER performance degradation. Moreover, MFTN signaling tends to suffer from a severe ill-conditioning problem, when the packing factors are significantly reduced, which may lead to the divergence of the equalization algorithms. However, there is a paucity of studies on tackling the ill-conditioning problem of MFTN signaling.

In this correspondence, we propose a pair of low-complexity equalization algorithms for MFTN systems operating in multipath channels. By inserting several cyclic postfixes and ignoring the insignificant interferences, we construct a block circulant interference matrix constituted by circulant blocks for reducing the condition number and mitigating the ill-conditioning problem of MFTN signaling. To diagonalize the covariance matrix of the colored noise, we reformulate the frequency-domain (FD) received signal model with the aid of the 2D fast Fourier transform (FFT). Building on this model, a parametric generalized approximated message passing (GAMP)-based FD equalization (FDE) algorithm is derived. To avoid the potential noise enhancement caused by the ill-conditioned matrices, we also design a refined GAMP-based FDE algorithm by invoking certain approximations. As a benefit, the complexity will only grow logarithmically with the number of transmitted symbols.

Notations: \(0_N\) and \(1_N\) represent all-zero and all-one column-vectors, respectively, while \(0_{N \times N}\) and \(1_N\) denote all-zero and identity matrices of size \(N \times N\). Each element of the normalized discrete Fourier transform (DFT) matrix \(F_N\) is \(F_{m,n} = N^{-1/2} \exp(-j2\pi(m-1)(n-1)/N)\). The operation \(\mathcal{D}(x)\) or \(\mathcal{D}(X)\) construct a diagonal matrix from the vector \(x\) or from the main diagonal vector of the square matrix \(X\). The operators \((\cdot)^T\) and \((\cdot)^H\) calculate the transpose and conjugate transpose of a matrix. The operators \(\circ\), \(\otimes\), and \(\otimes\) denote element-wise product, element-wise division, Kronecker product, and equality up to a constant normalization factor. The notation \(|\cdot|\) is used to compute the absolute value element-by-element. The complex Gaussian distribution of \(x\) with the mean vector \(\mathbf{m}_{x}\) and covariance matrix \(\mathbf{V}_{x}\) is expressed as \(CN(x; \mathbf{m}_{x}, \mathbf{V}_{x})\). Moreover, \(\mathbb{E}\{\cdot\}\) and \(\mathbb{V}\{\cdot\}\) denote the calculations of expectation and variance.

II. System Model

We consider the universal FDE-based low-density parity-check (LDPC)-coded MFTN system of Fig. 1. The \(N_c\) input bits \(b = [b_0, \cdots, b_{N_c-1}]^T\) are encoded into \(N_c\) LDPC-coded bits \(c = [c_0, \cdots, c_{N_c-1}]^T\), which are then mapped onto an \(M\)-ary constellation to generate \(N_c\) modulated symbols \(x = [x_0, \cdots, x_{N_c-1}]^T\). Assume that the coded MFTN system contains \(K\) subcarriers and each subcarrier conveys \(N = N_c/K\) symbols. Through a serial-to-parallel (S/P) converter, we obtain \(K\) parallel modulated symbol vectors \(x_0, \cdots, x_{K-1}\) associated with \(x_k = [x_{k,0}, \cdots, x_{k,N-1}]^T, k = 0, \cdots, K-1\). Then, the first \(2N_p\) modulated symbols of each subcarrier are concatenated to \(x_k\) as cyclic postfixes. The extended \(x_k\) is then passed through a pulse shaping filter \(p(t)\). Each output sequence is then assigned to a subcarrier and the first \(2K_p\) subcarriers are also inserted as cyclic postfixes in the FD. For MFTN signaling, we employ pulse shaping filters associated with the time interval \(\tau T\) and the overlapped subcarriers with frequency spacing of \(\nu F\) for improving the SE, where \(\tau, \nu \in (0, 1)\) are the time and frequency packing factors, respectively. \(T\) is the Nyquist time interval, and \(F\) is the minimum orthogonal frequency spacing. Then non-orthogonal multi-carrier modulation can be efficiently carried out by inverse FFT (IFFT) modules [18]. After that, through a parallel-to-serial (P/S) converter, the baseband transmitted MFTN signal is expressed as

\[
s(t) = \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} x_{k,n} p(t - n\tau T)e^{j2\pi\nu F t},
\]

where \(n\) and \(k\) are the time and subcarrier indexes, \(K = K + 2K_p\), and \(N = N + 2N_p\). In this correspondence, we employ a root-raised cosine (RRC) shaping pulse \(p(t)\) with a roll-off factor \(\beta\). For other non-orthogonal signaling schemes [19]–[21], the shaping pulse \(p(t)\) can be adjusted and the following derivations are still valid.

After transmission over a multipath channel, the received MFTN signal is fed into a non-orthogonal matched filter, which yields

\[
r_{k,n} = \sum_{l=0}^{L-1} h_l \int_{-\infty}^{\infty} s(t - l\tau T)p(t - n\tau T)e^{-j2\pi\nu F \tau} dt + \omega_{k,n},
\]

\[
= \sum_{l=0}^{L-1} h_l \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} x_{k,n} e^{-j2\pi\nu F \tau} \int_{-\infty}^{\infty} p(t - n\tau T)
\]

\[
\times p(t - n\tau T - l\tau T)e^{j2\pi(k - k_p)\nu F \tau} dt + \omega_{k,n},
\]

\[
= \sum_{l=0}^{L-1} h_l \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} A_p(\nu F, k, \nu F) \psi_{k,n} x_{k,n} + \omega_{k,n},
\]

where \(h_l\) is the \(l\)-th channel coefficient, \(L\) is the channel’s impulse-response duration, \(A_p(\tau T, \nu F) = \int_{-\infty}^{+\infty} p(t)p(t - \tau T)\nu F dt + \omega_{k,n}.

Fig. 1: FDE-based transceiver structure of MFTN signaling.
\[ \tau | e^{j2\pi \nu \tau} d\tau \] is the ambiguity function, \( \psi_{k_\Delta,n}^l \) is the equivalent channel matrix with \( G_l = \sum_{k_\Delta \in Z} \Theta_{k_\Delta} \Xi_{k_\Delta,\nu} \Lambda_{\mu} \xi + \omega = G x + \omega, \quad \text{where} \quad G = \sum_{k_\Delta \in Z} h_i G_i \]

where \( \Xi_{k_\Delta,\nu} \) is the equivalent channel matrix with \( \Theta_{k_\Delta} = 1_k \otimes D^{(k_\Delta)} \) is an \( N_s \times N_s \) diagonal matrix with \( \theta_{k_\Delta} = [\theta_{k_\Delta}^0, \ldots, \theta_{k_\Delta}^{N_s-1}]^T \), \( \Xi_{k_\Delta,\nu} \) is the colored noise having the covariance matrix \( R_\omega = \sigma_\omega^2 G_0 \).

The pseudo \( a \) priori messages \( \nu^p(i) \) and \( \hat{p}(i) \) updated in Line 3 and 4 of Algorithm 1 are the approximations of the messages propagated from the factor nodes \( p(\tilde{z}) \) to the variable nodes \( z \). The messages \( \nu^1(i) \) and \( \gamma(i) \) obtained in Line 7 and 8 update the outgoing messages from the factor nodes \( p(\tilde{z}) \) to the variable nodes \( z \). By taking the product of both incoming messages at nodes \( x \), we express the \( a \) posteriori probability of the transmitted symbols as

\[ p(x|\hat{r}) \approx \frac{p(x)N(x;\gamma(i), D(\nu^1(i)))}{\int_x p(x)N(x;\gamma(i), D(\nu^1(i)))}, \]

where \( p(x) \) is the discrete \( a \) priori probability density function of \( x \), which depends both on the constellation mapping rules and on the \( a \) priori probabilities of the coded bits. In the turbo receiver, the latter is updated as \( p(c_n) = \frac{1}{2} [1 + (-1)^{c_n} \tanh (L_s^{c_n}(c_n))] \), where \( L_s^{c_n}(c_n) \) denotes the soft extrinsic information gleaned from the channel decoder. Assuming that each element of \( x \) belongs to the constellation set \( S = \{S_1, \ldots, S_M\} \), the \( a \) posteriori expectation and variance
of the transmitted symbols in Line 9 and 10 are given by
\[
\hat{x}_{k,n}(i+1) = \sum_{m} \mathcal{S}_m p(x_{k,n} = \mathcal{S}_m) \xi_{k,n}^m(i), \tag{6}
\]
\[
\nu_{k,n}^m(i+1) = \frac{1}{\mathcal{S}_m} |\mathcal{S}_m - \hat{\gamma}_{k,n}(i)|^2 / \nu_{k,n}^\gamma(i), \tag{7}
\]
where \( \xi_{k,n}^m(i) = \exp(-|\mathcal{S}_m - \hat{\gamma}_{k,n}(i)|^2 / \nu_{k,n}^\gamma(i)). \)

B. Refined GAMP-Based FDE Algorithm

Since the squared modulus operations increase the condition number of the equivalent channel matrix, the message updates of the proposed GAMP-based FDE algorithm are sensitive to small perturbations. In this section, we introduce average approximations for tackling this problem and then develop a refined GAMP-based FDE algorithm.

We rewrite pseudo a priori variance vector \( \nu^p(i) \) of the noiseless measurements in Line 3 of Algorithm 1 as
\[
\nu^p(i) = \sum_{k \in \mathcal{I}} \mathcal{D}(\Phi_{k\Delta}) \nu(i)) \Phi_{k\Delta}^H 1_N,
\]
\[
\approx \mathcal{D}\left[ \sum_{l,l'} h_l h_{l'}^\ast \tilde{v}_{l,l'}(i) I_{N_1} A_{0,l'}^H \right] 1_N,
\]
\[
+ \sum_{k \in \mathcal{I}} \mathcal{D}\left[ \sum_{l,l'} h_l h_{l'}^\ast \tilde{v}_{l,l'}(i) (F_K \otimes F_N) \Theta_{k\Delta} \right] \xi_{k\Delta}(i) 1_N,
\]
\[
= \frac{1}{N_1} \sum_{l,l'} h_l h_{l'}^\ast \tilde{v}_{l,l'}(i) \left( A_{0,l'} A_{0,l'}^H + \sum_{k \in \mathcal{I}} \Theta_{k\Delta} \right) 1_N, \tag{8}
\]
where \( \tilde{v}_{l,l'}(i) \) is a pseudo a priori variance vector of the noiseless measurements in Line 7 of Algorithm 1 is approximated as
\[
\nu(i) \approx 1 / \sum_{l,l'} h_l h_{l'}^\ast \left( \tilde{v}_{l,l'}(i) + \sum_{k \in \mathcal{I}} \Theta_{k\Delta} \right) 1_N,
\]
\[
= 1 / \xi(i) 1_N, \tag{9}
\]
where \( \tilde{v}_{l,l'}(i) \) is the refined version only requires \( N_1 N_c \) transmitted symbols. The MMSE-based TDE of [24] has a complexity order of \( O(N_c^2) \) due to the matrix inversion. The complexity of the MMSE-SIC TDE of [14] is then simplified as
\[
\tilde{\gamma}(i) = \hat{x}(i) + \xi(i) \sum_{k \Delta \in \mathcal{I}} \Phi_{k\Delta}^H \hat{s}(i). \tag{10}
\]

The proposed refined GAMP-based FDE algorithm is summarized in Algorithm 2.

C. Complexity Analysis

The complexity comparisons of the proposed GAMP-based FDE and the existing equalization algorithms are summarized in TABLE I. The complexity of GMP-based TDE algorithm of [17] is dominated by the inversion of the truncated interference matrix, having a complexity order of \( O(N_s(2N_1 + 1)^2(2K_1 + 1)^3) \). The GAMP-based TDE algorithm of [23] has to evaluate \( N^2 \) complex multiplications for \( N \) transmitted symbols. The MMSE-based TDE of [24] has a complexity order of \( O(N_c^2) \) due to the matrix inversion. The complexity of the MMSE-SIC TDE of [14] is then simplified as
\[
\hat{\gamma}(i) = \hat{x}(i) + \xi(i) \sum_{k \Delta \in \mathcal{I}} \Phi_{k\Delta}^H \hat{s}(i). \tag{10}
\]

The proposed refined GAMP-based FDE algorithm is summarized in Algorithm 2.

IV. SIMULATION RESULTS

In this section, we evaluate the BER performance of the proposed FDE algorithms and analyze their convergence properties using extrinsic information transfer (EXIT) charts [25], [26]. We consider a rate-3/4 LDPC code having a length of \( N_c = 4032 \) and QPSK modulation. The number of subcarriers...
### TABLE I: Complexity Analysis

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Complexity of the equalizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>GMP-TDE</td>
<td>$O(N_c(2N_t + 1)^2(2K_t + 1)^3)$</td>
</tr>
<tr>
<td>GAMP-TDE</td>
<td>$O(N_c^2)$</td>
</tr>
<tr>
<td>MMSE-TDE</td>
<td>$O(N_c^2)$</td>
</tr>
<tr>
<td>MMSE-SIC-TDE</td>
<td>$O((K-1)N_t) + O(L_c^2)$</td>
</tr>
<tr>
<td>GAMP-FDE</td>
<td>$O(N_c^2) + O(N_tN_c \log N_c)$</td>
</tr>
<tr>
<td>Refined GAMP-FDE</td>
<td>$O(N_tN_c \log N_c)$</td>
</tr>
</tbody>
</table>

Fig. 3: BER performance of different equalization algorithms for MFTN systems with $\tau = 0.9$, $\nu = 0.8$.

is $K = 32$ and the number of symbols per subcarrier is $N = 256$. The roll-off factor of the RRC shaping pulse is $\beta = 0.3$. The number of cyclic postfixes is $K_p = 1$ and $N_p = 12$, where $K_t$ and $N_t$ equal to them, respectively. The fixed truncated lengths can cover the dominant 2D interferences in the following simulations. A multipath channel having $L = 8$ taps is considered with the power delay profile $\sigma_{h_t}^2 = \exp(-l)/(\sum_s \sigma_{h_s}^2)$. The number of iterations between the equalizer and channel decoder is $I_c = 50$ and the maximum number of LDPC decoding iterations is $I_e = 15$.

In Fig. 3, the BER performance of the proposed GAMP-based FDE algorithms is compared to those of the existing GAMP-TDE of [23] and the MMSE-based TDEs of [14], [24]. The label ‘Gaussian’ and ‘Discrete’ refer to the algorithm employing the approximated Gaussian distributions and the exact discrete a priori distributions of the transmitted symbols, respectively. Due to the ill-conditioning problem of MFTN signaling, GAMP-TDE fails to converge, hence exhibiting an error floor at BER $= 10^{-5}$. Since the MMSE-SIC-TDE employs a truncated interference model for reducing the computational complexity, it inevitably leads to a performance degradation, compared to the MMSE-TDE algorithm. The BER performance of the GAMP-FDE approaches that of the MMSE-TDE, where the former has a significantly lower complexity. Observe that the refined GAMP-FDE algorithm outperforms the MMSE-TDE and an additional $E_b/N_0$ gain can be obtained when we employ the discrete a priori probability. This is because the refined GAMP-FDE introduces the average variance vectors for circumventing the problem of having the ill-conditioned matrices in (8) and (9).

The BER performance of the proposed refined GAMP-FDE algorithm for MFTN systems using different packing factors are shown in Fig. 4. Assuming that $\zeta$ is the cyclic postfix overhead, the SE is calculated as $\eta = \frac{R_e(1-\zeta) \log_2 M}{\nu r(1+\beta)}$ bits/s/Hz [2]. Compared to its Nyquist-signaling counterpart, our MFTN system using $\tau = 0.9$, $\nu = 0.8$ attains about 39% higher SE at a negligible BER loss. When we further reduce the packing factors, MFTN signaling can further improve the SE by up to 56% and 79%, respectively, at the cost of 1.2 dB and 1.4 dB $E_b/N_0$ losses at BER $= 10^{-5}$. In Fig. 4, we evaluate the interference energy $E_1 = \sum_n \sum_k |A_p(n\tau T, kvF)|^2 - |A_p(0,0)|^2$ of MFTN signaling, where we have $A_p(0,0) = \int_{-\infty}^{\infty} |p(t)|^2 dt = 1$ for a unit-energy shaping pulse. Observe that $E_1$ depends both on the packing factor combinations and on the pulse shaping filter. For a fixed SE, the effects of $\tau$ and $\nu$ on $E_1$ may be quite different. Here we only discuss MFTN systems employing the classic RRC pulse with $\beta = 0.3$. It is seen that MFTN signaling suffers from different $E_1$ for a fixed $\eta$. Hence, by jointly optimizing the time and frequency packing factors, we are able to reduce $E_1$ and accordingly improve the BER performance. As shown in Fig. 5, for $\eta = 1.38$ bits/s/Hz, MFTN signaling with $\tau = 0.9$, $\nu = 0.8$ suffers from a lower $E_1$, compared to the case of $\tau = 1.0$, $\nu = 0.72$. Accordingly,
The EXIT curves of the proposed GAMP-based FDE algorithms and that of the rate-3/4 LDPC decoder are shown in Fig. 6. There is no open tunnel between the equalizer and the decoder curve at $E_b/N_0 = 0$ dB, while an open tunnel emerges at $E_b/N_0 = 4$ dB. The stair-case-shaped decoding trajectories between the equalizer and the channel decoder at $E_b/N_0 = 4$ dB are also included for characterizing the exchange of extrinsic information. It is observed that the proposed GAMP-FDE equalizers require at least 3 iterations between the equalizer and channel decoder for reaching the maximum mutual information point. Moreover, the proposed refined GAMP-FDE equalizer converges faster than the GAMP-FDE, which demonstrates the power of the average approximation employed.

V. CONCLUSIONS

In this correspondence, we proposed low-complexity GAMP-based FDE algorithms for MFTN systems operating in multipath channels. To mitigate the ill-conditioning problem of MFTN signaling, we reformulated the received signal model with the aid of a block circular interference matrix via inserting a few cyclic postfixes. Then, the received signal was transformed to the FD by a 2D FFT to obtain the equivalent white Gaussian noise. Exploiting the GAMP rules, we derived a parametric GAMP-FDE algorithm, which was then further refined based on the average approximations of the variance vectors to circumvent the problems caused by the ill-conditioned matrices. Simulation results showed that MFTN signaling employing the refined GAMP-FDE algorithm significantly improves the transmission rates at a negligible BER degradation, compared to its Nyquist-signaling counterpart.

REFERENCES