Supercharged AdS$_3$ Holography

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ABSTRACT: Given an asymptotically Anti-de Sitter supergravity solution, one can obtain a microscopic interpretation by identifying the corresponding state in the holographically dual conformal field theory. This is of particular importance for heavy pure states that are candidate black hole microstates. Expectation values of light operators in such heavy CFT states are encoded in the asymptotic expansion of the dual bulk configuration. In the D1-D5 system, large families of heavy pure CFT states have been proposed to be holographically dual to smooth horizonless supergravity solutions. We derive the precision holographic dictionary in a new sector of light operators that are superdescendants of scalar chiral primaries of dimension (1,1). These operators involve the action of the supercharges of the chiral algebra, and they play a central role in the proposed holographic description of recently-constructed supergravity solutions known as "supercharged superstrata". We resolve the mixing of single-trace and multi-trace operators in the CFT to identify the combinations that are dual to single-particle states in the bulk. We identify the corresponding gauge-invariant combinations of supergravity fields. We use this expanded dictionary to probe the proposed holographic description of supercharged superstrata, finding precise agreement between gravity and CFT.

Contents

1 Introduction

Black holes are some of the most interesting objects in our universe. Despite many theoretical advances, we lack a detailed understanding of their internal structure at the quantum level, in particular the resolution of their singularities. String Theory is the leading framework within which to study the quantum properties of black holes and their microstates.

Holographic duality is a powerful tool in the study of black hole microstates. Given an asymptotically AdS supergravity solution, by identifying the corresponding state in the dual conformal field theory, we can gain valuable insight into its microscopic interpretation. Large families of pure CFT states are understood to be holographically dual to smooth horizonless supergravity solutions in the bulk, see e.g. [1–9]. We are primarily interested in pure CFT states that are 'heavy' in the sense that their conformal dimension is proportional to the central charge of the CFT. When the mass and charges of such states agree with those of a black hole solution, such states are interpreted as microstates of the corresponding black hole.

In this paper we work in the D1-D5 system in Type IIB compactified on $\mathcal{M} = T^4$ or K3, and consider heavy bound states carrying an additional momentum charge P along the spatial direction common to the D1 and D5 branes. The black holes we study are the BPS D1-D5-P black hole, either non-rotating [10] or rotating [11]. On the bulk side, in this paper we will work in the supergravity approximation, though there has been recent progress on describing black hole microstates in worldsheet models [12–16].

The state-of-the-art constructions of supersymmetric microstate solutions in supergravity involve a breaking of the isometries that are preserved by the corresponding black hole, and are known as 'superstrata'. For some early constructions and studies of superstrata, see e.g. [17–22]. The construction of these solutions involves solving a set of layered BPS equations in sequence. A solution to the first two layers typically has a number of unfixed parameters. Solving the final layer of equations and imposing smoothness gives rise to algebraic relations on the parameters. This procedure is in general known as 'coiffuring' [23–25].

There is a proposal for the dual CFT description of superstrata, which has been developed hand-in-hand with (and indeed has informed) the supergravity constructions, see e.g. [17–20, 26]. The parameters in the supergravity solutions have a specific holographic interpretation in the proposed dual CFT states. Precision holography, in particular protected three-point correlation functions that consist of the expectation value of a light operator in the heavy background, can be used to investigate such proposals [6, 27–31].

The first families of superstrata that were constructed had limitations in that smooth solutions were not available for all choices of parameters. However recently, more general families of D1-D5-P superstrata have been constructed, that have resolved these limitations, as follows.

In the corresponding proposed dual CFT states, some of the momentum-carrying excitations are left-moving supercharges of the small (4,4) superconformal algebra. This is a novel feature with respect to the first families of superstrata that were constructed. As a result, these more recently constructed solutions are known as 'supercharged super-

strata' [32–35]. Supercharged superstrata and their proposed holographic description are a primary motivation for this work.

Precision holography relates expectation values of light operators in heavy CFT states to gauge-invariant combinations of supergravity fields in an asymptotic expansion at large radial distance in AdS. Coefficients of successively higher terms in the radial expansion in supergravity correspond to expectation values of successive higher dimension operators [36, 37]. Recently it was observed that in order to carry out precision holographic tests of superstrata, it is necessary to derive the explicit holographic dictionary to higher order than was previously known, and the precision holographic dictionary for a set of dimension two operators was derived [31]. Specifically, a set of chiral primary operators (and affine descendants) of dimension (1,1) was considered, and the mixing between single-trace operators and multi-trace operators was resolved. This enabled precision holographic tests of the first families of superstrata constructed in [17–20] and their proposed dual CFT states. In addition, a simple preliminary test of the supercharged superstrata constructed in [32] was computed. The results of [31] support the proposed dictionary in all examples, and also give a CFT interpretation of the coiffuring relations of non-supercharged superstrata.

In this paper we will derive the precision holographic dictionary in a novel sector, and use it to make precision holographic tests of supercharged superstrata. This novel sector consists of superdescendants of the dimension (1,1) chiral primary operators studied in [31]. Of particular interest to us is a new type of coiffuring relation, proposed in [33, 34], that resolves the limitations mentioned above. A main goal of this paper is to test this new coiffuring relation holographically. By doing so, we will also test the proposed holographic dictionary for the supercharged superstrata.

Although our primary motivations are in black hole physics and the fuzzball proposal [38–41], our results are relevant to more general aspects of holography that have been discussed in recent (and less recent) literature, specifically the mixing of single and multi-trace operators in the CFT that are dual to supergravity states in the bulk. As is well-known, supergravity fluctuations around the global AdS vacuum are holographically dual to short multiplets of CFT operators whose top components are chiral primaries (or anti-chiral primaries). The remainder of the short multiplet is obtained by acting on the chiral primaries with the generators of the anomaly-free subalgebra of the superconformal algebra (see e.g. the reviews [42, 43]).

However the precise form of this dictionary is not yet fully understood. In particular we will be interested in mixings between single-trace and multi-trace operators on the gauge theory side. In the case of a bound state of n_1 D1 branes and n_5 D5 branes the holographic CFT is a permutation orbifold CFT, where the gauge group is the permutation group S_N, where $N = n_1 n_5$. In this theory, traces are sums over the individual N copies of the CFT.

In the large N limit, for many purposes one can use the approximate dictionary that single-particle supergravity states correspond single-trace CFT operators. However for some time it has been understood that this approximate identification is not correct for all observables, for example extremal correlators, even at leading order in the large N limit. (Extremal correlators are those for which the conformal dimension of one operator

equals the sum of the conformal dimensions of the other operators in the correlator.) The correct dictionary instead relates single-particle supergravity states to specific admixtures of single-trace and multi-trace operators. The coefficients of the multi-trace operators in the admixture involve powers of $1/N$, yet they can contribute to the value of extremal correlators at leading order in large N. The precise general form of this dictionary has until recently been elusive; for previous work, see e.g. [27, 28, 44–50].

Aside from extremal correlators, certain (non-extremal) mixed heavy-light correlators require the correct dictionary to be used, even when working at leading order in the large N limit. It is these types of correlators that we consider in the present work. We consider correlators in which the light CFT operator is dual to a single-particle supergravity state. In the correlators we consider, the multi-trace admixtures in these light operators can contribute to the value of the correlator at leading order in large N because they can have a Wick contraction with the heavy state that produces other factors of N. We will exhibit examples in detail.

Recently it has been proposed that single-particle supergravity fluctuations around global AdS$_5 \times$S^5 are holographically dual to $\mathcal{N} = 4$ SYM operators (in short multiplets) that are orthogonal to all multi-trace operators [51, 52]. This condition identifies specific admixtures of single-trace and multi-trace operators. For $\mathcal{N} = 4$ SYM, this condition is sufficient to uniquely define this set of CFT operators up to normalization, and has passed recent checks [52].

In this paper we will emphasize that in AdS$_3 \times$S$^3 \times \mathcal{M}$ the situation is more complicated than in AdS$_5 \times$S^5. In the AdS$_3$ case, it is not sufficient to consider the set of orbifold CFT operators that are orthogonal to all multi-trace operators; one must resolve additional operator mixing. Such mixing has been studied previously in [28]. We will resolve this point fully in the sector in which we work, by combining and improving upon the results of [31] and [28].

The method that we pursue is as follows. We first recast our recent work on AdS$_3$ holography [31] in the single-particle basis. This involves taking slightly different combinations of the single-trace and multi-trace operators of dimension (1,1) from those used in [31]. In doing so we resolve the additional operator mixing, combining [31] and [28]. We then act with the supercharges of the small $\mathcal{N} = 4$ superconformal algebra to generate the superdescendants within this supermultiplet that are our primary interest in this paper. By construction, all the resulting CFT operators are in the single-particle half-BPS basis.

We then derive the gauge-invariant combinations of supergravity fields that describe the single-particle excitations of interest. We do so to the required order in the asymptotic expansion, and identify the terms in this expansion that encode the expectation values of the superdescendant operators that we study.

We determine the normalization coefficients of the holographic dictionary by taking a set of test CFT states and proposed dual supergravity superstrata solutions. For consistency these normalization coefficients must depend at most on charges and moduli of the theory, and not on any property of the microstates chosen for this calibration computation, and this is indeed the case. Furthermore one expects that these coefficients respect the symmetries of the supergravity theory, and our results indeed do so.

Having normalized the dictionary, we make two non-trivial holographic tests of superstrata: first we test the coiffuring proposal of general non-supercharged superstrata of [33, 34]. Second, we test a 'hybrid' superstratum involving both supercharged and non-supercharged elements. These tests also represent further cross-checks on the holographic dictionary itself.

All our tests result in non-trivial and elegant agreement between supergravity and CFT. While the correlators we compute cannot prove that the proposed holographic description of superstrata is correct in all its details, the agreement we find provides state-of-the-art evidence that supports the proposed holographic description, and gives a CFT interpretation to the supercharged coiffuring relation of [33, 34].

The structure of this paper is as follows. In section 2 we review the D1-D5 CFT and perform a preliminary computation to be used later in the paper. Section 3 is a review of relevant aspects of the supergravity theory, superstrata, and precision holography. In section 4 we begin the construction of the dictionary in both gravity and CFT. In section 5 we fix the normalization coefficients in the dictionary and perform tests of two distinct families of superstrata. In section 6 we discuss our results; technical details are recorded in four appendices.

2 D1-D5 CFT

In this section we review the D1-D5 CFT, introduce the operators we shall consider in this work, and perform our first new calculation (in section 2.4) for use later in the paper.

2.1 D1-D5 CFT and structure of short multiplets

We begin in Type IIB string theory compactified on $\mathcal{M} \times S^1$ where $\mathcal{M}$ is T^4 or K3. We take $\mathcal{M}$ to be microscopic and the S^1 to be large. We coordinatize the S^1 by y, and we denote by R_y the asymptotic proper radius of the S^1. We consider bound states of n_1 D1-branes wrapped on the S^1 and n_5 D5-branes wrapped on $\mathcal{M} \times S^1$, with $g_s n_1 \gg 1$ and $g_s n_5 \gg 1$. The AdS$_3$ decoupling limit can be formulated as a large R_y scaling limit, see e.g. [13]. This decouples the original asymptotic region, resulting in an asymptotically AdS$_3 \times S^3 \times \mathcal{M}$ bulk [53], in which y is the angular direction in AdS$_3$.

At low energies, the worldvolume theory on this D1-D5 system flows to a two-dimensional $\mathcal{N} = (4, 4)$ SCFT with central charge $c = 6n_1 n_5 \equiv 6N$. There is considerable evidence that there is a locus in moduli space at which the theory is a sigma model with target space $\mathcal{M}^N/S_N$, where S_N is the symmetric group; for some early work, see e.g. [10, 54–58].[1] We will work with $\mathcal{M} = T^4$ for concreteness and ease of presentation, however our main results concern the universal sector of the compactification on $\mathcal{M}$ and so all of our main results carry over appropriately to the K3 compactification.

The symmetric orbifold CFT for $\mathcal{M} = T^4$ contains N copies of the $c = 6$ sigma model on T^4. This $c = 6$ model has a free field realization in terms of four free bosons plus four left-moving and four right-moving free fermions.

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[1]Recent work has explored a stringy version of this duality in the S-dual NS5-F1 system, for the case of a single NS5-brane; see [59, 60] and references within.

State	J^3	L_0	$\bar{J}^3$	$\bar{L}_0$	SU(2)$_C$
$|CP\rangle$	h	h	$\bar{h}$	$\bar{h}$	**1**
$G^2|CP\rangle$	$h-1$	$h+1$	$\bar{h}$	$\bar{h}$	**1**
$\bar{G}^2|CP\rangle$	h	h	$\bar{h}-1$	$\bar{h}+1$	**1**
$G\bar{G}|CP\rangle$	$h-1/2$	$h+1/2$	$\bar{h}-1/2$	$\bar{h}+1/2$	**1** $\oplus$ **3**
$G^2\bar{G}^2|CP\rangle$	$h-1$	$h+1$	$\bar{h}-1$	$\bar{h}+1$	**1**

Table 1. Bosonic structure of the short multiplets.

Let us introduce some notation that we will require. We label different copies of the $c = 6$ CFT by the index $r = 1, 2, \ldots, N$. The small $\mathcal{N} = (4, 4)$ superconformal algebra has left and right R-symmetry groups SU(2)$_L \times$ SU(2)$_R$; we use indices $\alpha, \dot{\alpha} = \pm$ for the fundamental representation of each respectively. There is an additional organizational SU(2)$_C \times$ SU(2)$_A \sim$ SO(4)$_I$ which is inherited from the rotation group on the tangent space of $\mathcal{M}$; we use indices $A, \dot{A} = 1, 2$ for the fundamental representation of each respectively.

The four bosons and four left-moving and four right-moving free fermions on copy r of the CFT are then denoted respectively by

$$X^{A\dot{A}}_{(r)}, \qquad \psi^{\alpha\dot{A}}_{(r)}, \qquad \bar{\psi}^{\dot{\alpha}\dot{A}}_{(r)}. \tag{2.1}$$

The orbifold theory also contains spin-twist operators, labelled by conjugacy classes of the permutation group, which modify the boundary conditions of the fields. The orbifold nature of the target spaces decomposes the Hilbert space of the theory into twisted sectors corresponding to these operators. A generic element $g \in S_N$ involves various permutation cycles; let us denote their lengths by k_i. An orbifold CFT state involving twist operator excitations is often described as involving a collection of 'strands' of length k_i occurring with degeneracy N_i, possibly carrying excitations and/or polarization indices, subject to the 'strand budget' constraint that the total number of copies of the full CFT is N,

$$\sum_i \mathsf{k}_i N_i = N. \tag{2.2}$$

We now introduce the chiral primary operators (CPOs) of the theory, which are the top components of the short multiplets of the SU(1, 1|2)$_L \times$ SU(1, 1|2)$_R$ symmetry. These operators, together with their descendants under the generators of the anomaly-free part of the small $\mathcal{N} = (4, 4)$ superconformal algebra, i.e. $\{J_0^-, L_{-1}, G_{-1/2}^{-;A}\}$ and $\{\bar{J}_0^-, \bar{L}_{-1}, \bar{G}_{-1/2}^{-;A}\}$, play a central role in the construction of the holographic dictionary, because of their relation to single-particle excitations in supergravity [55–57].

The bosonic structure of the SU(1, 1|2)$_L \times$ SU(1, 1|2)$_R$ short multiplets is sketched in table 1. In table 1, G^2 is a short hand for the combination

$$G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2h} J_0^+ L_{-1}, \tag{2.3}$$

where h is the eigenvalue of L_0 for the CPO we act upon; similarly for $\bar{G}^2$. By working with this linear combination, one obtains a state that is orthogonal to the other descendants of the CPO, which is then dual to an independent supergravity fluctuation [32, 61, 62].

Moreover, this combination gives a state that is an eigenstate of the Casimirs of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, as one can check making use of the anomaly-free part of the chiral algebra in the NS-NS sector, composed of L_0, $L_{\pm 1}$, J_0^a, $G_{\pm 1/2}^{\alpha A}$. Setting temporarily $m, n = 1, 0, -1$ and $r, s = \pm\frac{1}{2}$, the anomaly-free part of the chiral superconformal algebra is

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad [J_0^a, J_0^b] = i\epsilon^{abc}J_0^c, \qquad [L_n, J_0^a] = 0,$$

$$[J_0^a, G_s^{\alpha A}] = \frac{1}{2}G_s^{\beta A}(\sigma^a)_\beta^\alpha, \qquad [L_m, G_s^{\alpha A}] = \left(\frac{m}{2} - s\right)G_{m+s}^{\alpha A}, \tag{2.4}$$

$$\{G_r^{\alpha A}, G_s^{\beta B}\} = \epsilon^{\alpha\beta}\epsilon^{AB}L_{r+s} + (r-s)\epsilon^{AB}(\sigma^{aT})_\gamma^\alpha\epsilon^{\gamma\beta}J_{r+s}^a,$$

where $a, b, c = \{\pm, 3\}$ are indices in the adjoint of $\mathrm{SU}(2)_L$.

When discussing superstrata, it will be convenient for our conventions to work with anti-chiral primary operators (ACPOs): these are descendants of CPOs obtained acting the maximal number of times with the generators J_0^-, $\bar{J}_0^-$ and are characterized by $h = -j$, $\bar{h} = -\bar{j}$. Denoting an anti-chiral primary by O_{AC}, we are interested in its bosonic descendants under the left-moving anomaly-free chiral algebra. These can be written in the form

$$(J_0^+)^{\mathsf{m}-\mathsf{q}}L_{-1}^{\mathsf{n}-\mathsf{q}}\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2} + \frac{1}{2h}J_0^+L_{-1}\right)^{\mathsf{q}}|O_{\mathrm{AC}}\rangle. \tag{2.5}$$

States with $\mathsf{q} = 1$ are the building-blocks of the proposed holographic dual states of super-charged superstrata [32].

2.2 Low-dimension operators

We now introduce a particular set of chiral primaries and review the holographic dictionary for these operators, constructed in [6, 27, 29, 31]. It was shown in [63] that expectation values of CPOs in 1/4 or 1/8-BPS states are independent of the moduli of the theory, so that their value at the free orbifold point can be reliably compared with the boundary expansion of the supergravity solution. In paricular, we will focus on light CPOs, with dimension $\Delta = h + \bar{h} \leq 2$.

In the symmetric product orbifold CFT, by definition operators must be invariant under permutations of the copies. One can construct gauge-invariant operators by tracing (i.e. summing) over the N copies of the basic fields and/or their products. We begin with single-trace CPOs, obtained taking a single sum over the N copies.

For single-trace operators we will work with non-unit-normalized operators, for consistency with previous papers [29]. This choice is particularly convenient for the $\mathrm{SU}(2)$ currents, as it means that they satisfy the standard algebra (in the spin basis, i.e. where $[J^+, J^-] = 2J^3$). By contrast, for the multi-trace operators that we will introduce in the remainder of the paper, we will find it convenient to define them with unit normalization in the large N limit.

Among the CPOs with $\Delta = 1$, the only ones with non vanishing conformal spin $\mathsf{s} = h - \bar{h}$ are the R-symmetry currents (all sums over copy indices $r, s, \ldots$ run from 1 to N unless otherwise indicated):

$$J^+ = \sum_r J_{(r)}^+ = \sum_r \psi_{(r)}^{+1}\psi_{(r)}^{+2}, \qquad \bar{J}^+ = \sum_r \bar{J}_{(r)}^+ = \sum_r \bar\psi_{(r)}^{+1}\bar\psi_{(r)}^{+2}. \tag{2.6}$$

They are characterized respectively by $(h = j = 1, \bar{h} = \bar{j} = 0)$ and $(h = j = 0, \bar{h} = \bar{j} = 1)$.

Next, let us denote the dimension of the $(1,1)$ cohomology of $\mathcal{M}$ by $h^{1,1}(\mathcal{M})$. Then there are $n \equiv (h^{1,1}(\mathcal{M}) + 1)$ CPOs that have $\Delta = 1$ and $\mathsf{s} = 0$, i.e. $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$ (For T^4, $n = 5$, while for K3, $n = 21$). First we have a twist-two operator. To write this operator we first define the 'bare' twist-two operator $\sigma_{(rs)}$ to be the lowest-dimension (spin)-twist operator associated with the permutation (rs). We also define the spin fields $S^+, \bar{S}^+$ that map NS to R boundary conditions on the fermions, and vice versa. The operator Σ_2^{++} is given by

$$\Sigma_2^{++} = \sum_{r<s} S^+ \bar{S}^+ \sigma_{(rs)} = \sum_{r<s} \sigma_{(rs)}^{++} \,. \tag{2.7}$$

Our focus in this paper is on heavy pure microstates that are invariant on $\mathcal{M}$. Of the $n-1$ remaining operators with $\Delta = 1$ and $\mathsf{s} = 0$, only the following one can have a non-vanishing expectation value on this class of microstates:

$$O^{++} = \sum_r O_r^{++} = \sum_r \frac{\epsilon^{\dot{A}\dot{B}}}{\sqrt{2}} \psi_{(r)}^{+\dot{A}} \bar{\psi}_{(r)}^{+\dot{B}} \,. \tag{2.8}$$

Next, we discuss the set of CPOs of dimension $\Delta = 2$ that are conformal scalars, and so have $h = j = \bar{h} = \bar{j} = 1$. There are $n + 1$ operators with these quantum numbers. First, in the untwisted sector, we have the single-trace product of one left and one right current:

$$\Omega^{++} = \sum_r J_{(r)}^+ \bar{J}_{(r)}^+ = \sum_r \psi_{(r)}^{+1} \psi_{(r)}^{+2} \bar{\psi}_{(r)}^{+1} \bar{\psi}_{(r)}^{+2} \,. \tag{2.9}$$

Second, we have a twist-three operator which contains bare twist operators $\sigma_{(qrs)}, \sigma_{(qsr)}$ associated with the inequivalent permutations (qrs) and (qsr) respectively, dressed with current modes that add the required charge:

$$\Sigma_3^{++} = \sum_{q<r<s} \bar{J}_{-1/3}^+ J_{-1/3}^+ \left(\sigma_{(qrs)} + \sigma_{(qsr)} \right) \,. \tag{2.10}$$

Among the $n-1$ remaining operators, only the following one can have a non-vanishing expectation value on the class of $\mathcal{M}$-invariant microstates,

$$O_2^{++} = \sum_{r<s} (O_{(r)}^{++} + O_{(s)}^{++}) \sigma_{(rs)}^{++} = \sum_{r<s} O_{(rs)}^{++} \,. \tag{2.11}$$

We next consider double-trace operators, obtained taking the product of two single traces.[2] We focus on the dimension $\Delta = 2$ scalar double-trace operators, which are defined via

$$
\begin{aligned}
(\Sigma_2 \cdot \Sigma_2)^{++} &= \frac{2}{N^2} \sum_{(r<s),(p<q)} \sigma_{(rs)}^{++} \sigma_{(pq)}^{++}, & (J \cdot \bar{J})^{++} &= \frac{1}{N} \sum_{r,s} J_{(r)}^+ \bar{J}_{(s)}^+, \\
(\Sigma_2 \cdot O)^{++} &= \frac{\sqrt{2}}{N^{3/2}} \sum_{r<s,q} \sigma_{(rs)}^{++} O_{(q)}^{++}, & (O \cdot O)^{++} &= \frac{1}{N} \sum_{r,s} O_{(r)}^{++} O_{(s)}^{++} \,.
\end{aligned}
\tag{2.12}
$$

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[2]Our convention for the double-traces is different from that in eq. (4.5) of [31], where the double-traces were defined as the off-diagonal product of single-traces. The reason for this will be discussed in section 4.1.

All these operators have unit norm in the large N limit. These operators are highest-weight with respect the $SU(2)_L \times SU(2)_R$ R-symmetry group; the rest of the R-symmetry multiplet can be constructed as usual by acting with the zero modes of $J^-, \bar{J}^-$.

For the scalar operators of dimension one, (2.7), (2.8), we denote the members of the $(j, \bar{j}) = (\frac{1}{2}, \frac{1}{2})$ R-symmetry multiplet as $\Sigma_2^{\alpha, \dot{\alpha}}$ and $O^{\alpha, \dot{\alpha}}$ with $\alpha, \dot{\alpha} = \pm$. For the scalar operators of dimension two, (2.9)–(2.12), we label the $(j, \bar{j}) = (1, 1)$ R-symmetry multiplet with indices $a, \dot{a} = \pm, 0$, and we choose to normalize the descendants such that they have the same norm as the highest-weight state: for instance, we define $\Sigma_3^{0,+} = \frac{1}{\sqrt{2}}[J_0^-, \Sigma_3^{++}]$. For the R-symmetry currents themselves, we denote the members of the multiplet by J^a, and we normalize the descendants such that the standard $SU(2)$ commutation relations $[J^+, J^-] = 2J^3$ hold (similarly for the right currents $\bar{J}^{\dot{a}}$).

2.3 Holography for D1-D5(-P) black hole microstates

We now review the holographic description of two-charge D1-D5 black hole microstates and three-charge D1-D5-P superstrata, including supercharged superstrata. For the two-charge states we follow in places the discussion in [29].

The relevant sector of the CFT to describe black hole microstates is the Ramond-Ramond (RR) sector. We begin with RR ground states. A strand of length k in a RR ground state is denoted by $|s\rangle_\mathsf{k} \equiv |m, \bar{m}\rangle_\mathsf{k}$ where[3] $(m, \bar{m})$ take values $(\pm, \pm)$ or $(0, 0)$. The labels $(m, \bar{m})$ denote the respective eigenvalues of the $SU(2)_L \times SU(2)_R$ currents $J_0^3, \bar{J}_0^3$ on the state $|s\rangle_\mathsf{k}$.

These ground states are generated by acting on the strand $|++\rangle_\mathsf{k}$ with the zero modes of the untwisted chiral primaries: when $\mathsf{k} = 1$, for example, one has

$$J_0^- |++\rangle_1 = |-+\rangle_1 , \quad \bar{J}_0^- |++\rangle_1 = |+-\rangle_1 , \quad O_0^{--}|++\rangle_1 = |00\rangle_1 , \quad \Omega_0^{--}|++\rangle_1 = |--\rangle_1 . \tag{2.13}$$

A basis for the RR ground states is given by taking tensor products of $N_\mathsf{k}^{(s)}$ strands of type $|s\rangle_\mathsf{k}$, yielding the following eigenstates of the R-symmetry currents:

$$\psi_{\{N_\mathsf{k}^{(s)}\}} \equiv \prod_{\mathsf{k},s} (|s\rangle_\mathsf{k})^{N_\mathsf{k}^{(s)}} , \tag{2.14}$$

subject to the "strand budget" constraint

$$\sum_{\mathsf{k},s} \mathsf{k} N_\mathsf{k}^{(s)} = N . \tag{2.15}$$

It is convenient to work with non-normalized states. Due to this choice, the norm of the state is required when computing 3-pt functions. The norm of the state (2.14) is defined as the number of inequivalent elements that belong to the conjugacy class of S_N defined by the partition (2.15); it was derived in [29] and is given by

$$\left| \psi_{\{N_\mathsf{k}^{(s)}\}} \right|^2 = \frac{N!}{\prod_{\mathsf{k},s} N_\mathsf{k}^{(s)}! \, \mathsf{k}^{N_\mathsf{k}^{(s)}}} . \tag{2.16}$$

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[3]These are only 5 of the 16 RR ground states at twist k: we restrict to these as we are interested in bosonic states which are invariant under rotations of the internal manifold $\mathcal{M}$.

The two-charge black hole microstates that are well-described in the supergravity limit are coherent states [27] that are linear combinations of the $\psi_{\{N_k^{(s)}\}}$, weighted by coefficients $A_k^{(s)}$

$$\psi(\{A_k^{(s)}\}) \equiv \sum_{\{N_k^{(s)}\}}{}' \prod_{k,s} (A_k^{(s)}|s\rangle_k)^{N_k^{(s)}}, \qquad (2.17)$$

such that the sum is peaked for large values of $N_k^{(s)}$; here the prime on the summation symbol indicates that the sum is subject to the constraint (2.15). The $A_k^{(s)}$ can in general be complex, however for our practical applications in this paper we will take them to be real. The average number $\overline{N}_k^{(s)}$ of strands of type $|s\rangle_k$ in the semi-classical limit is fixed by the coefficients $A_k^{(s)}$. This relation is determined by calculating the norm of (2.17) using (2.16), and then taking its variation with respect to the $N_k^{(s)}$: this leads to [29]

$$k\,\overline{N}_k^{(s)} = |A_k^{(s)}|^2. \qquad (2.18)$$

Comparison with the strand budget constraint (2.15) implies the relation

$$\sum_{k,s} |A_k^{(s)}|^2 = N. \qquad (2.19)$$

We now introduce the states that are proposed to be holographically dual to superstrata. To do so we will make use of the spectral flow transformation in the $\mathcal{N} = (4,4)$ superconformal algebra of the D1-D5 CFT (see e.g. [9, 62, 64] for more details and [65] for a very recent application), which connects the RR and the NS-NS sectors of the theory. In particular, it defines a map between RR ground states and anti-chiral primaries in the NS sector. Although the CFT states dual to superstrata are in the RR sector of the theory, it is convenient for ease of notation and of the computations to introduce their corresponding states in the NS-NS sector.

The momentum-carrying building blocks of the 1/8-BPS states dual to superstrata are the descendant states obtained by acting upon the anti-chiral primary $|O_k^{--}\rangle_{\mathrm{NS}}$, which corresponds to the RR ground state $|00\rangle_k$, with the holomorphic generators of the small $\mathcal{N} = 4$ superconformal algebra J_0^+, L_{-1} and $G_{-\frac{1}{2}}^{+A}$. We will denote these by $|k, m, n, q\rangle$, and one has [17–19, 21, 32, 33]

$$|k, m, n, q\rangle = \frac{1}{(m-q)!(n-q)!} (J_0^+)^{m-q} L_{-1}^{n-q} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{k} J_0^+ L_{-1} \right)^q |O_k^{--}\rangle_{\mathrm{NS}}, \qquad (2.20)$$

where the parameters can take values $q = 0$ or 1, $n \geq q$, $k > 0$, and $q \leq m \leq k - q$ [32] (our notation follows [33]). As discussed around eq. (2.5), we describe states with $q = 1$ as supercharged states.

We are interested in superstrata whose CFT dual state is made up of strands of type $|k, m, n, q\rangle$ and the NS-NS vacuum $|0\rangle_1$: a basis for these states is given by the following eigenstates of the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ currents

$$\psi_{\{N_0, N_i\}} \equiv |0\rangle_1^{N_0} \prod_i |k_i, m_i, n_i, q_i\rangle^{N_i} \qquad (2.21)$$

subject to the constraint that the total number of copies must saturate the strand budget:

$$N_0 + \sum_i \mathsf{k}_i N_i = N \,. \tag{2.22}$$

Again, in order to construct states whose holographic duals are well described in supergravity, one must consider coherent states obtained superposing the $\psi_{\{N_0, N_i\}}$, weighted by coefficients A, B_i

$$\psi(\{A, B_i\}) \equiv \sum_{\{N_0, N_i\}}' \left[\prod_i \left(A |0\rangle_1 \right)^{N_0} \left(B_i |\mathsf{k}_i, \mathsf{m}_i, \mathsf{n}_i, \mathsf{q}_i\rangle \right)^{N_i} \right] \,, \tag{2.23}$$

where the prime symbol denotes that the sum is subject to the constraint (2.22). Again the coefficients A, B_i can in principle be complex, however we shall take them to be real in all our examples.

In the remainder of the paper we will work with NS-NS sector states unless otherwise specified. As a result, for ease of notation in the following we will mostly suppress the subscript NS on the states.

2.4 Norm of supercharged superstrata CFT states

In this section we compute the norm of the states (2.23), for use later in the paper. This computation is new and generalizes the discussion in [29] and [21] to the states (2.23).

We do so by first computing the norm of the building-block states (2.21). We then determine the average numbers $\{\bar{N}_0, \bar{N}_i\}$ over which the sum in the full coherent state (2.23) is peaked. The norm of (2.21) is obtained combining the combinatorial contribution in (2.16) with the effects of the momentum carrying excitations on the norm of each strand. In order to determine the norm of a single strand $|\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{q}\rangle$, we will make use of the algebra eq. (2.4), together with the fact that an anti-chiral primary is annihilated by $G_{-1/2}^{-A}$ and J_0^- (in addition to all positive modes of the anomaly-free subalgebra), plus the following relations on hermitian conjugation:

$$(G_n^{\alpha A})^\dagger = -\epsilon_{\alpha\beta}\epsilon_{AB} G_{-n}^{\beta B} \,, \qquad (J_n^a)^\dagger = J_{-n}^{-a} \,, \qquad (L_n)^\dagger = L_{-n} \,. \tag{2.24}$$

We start by considering $\mathsf{m} = \mathsf{n} = \mathsf{q} = 1$, which gives:

$$\langle \mathsf{k}, 1, 1, 1 | \mathsf{k}, 1, 1, 1 \rangle = \mathsf{k}^2 - 1 \,. \tag{2.25}$$

In order to compute the contribution to the norm coming from the contractions of the J_0^+ insertions, it is useful to consider the following state, using the shorthand $\hat{\mathsf{m}} = \mathsf{m} - \mathsf{q}$:

$$\begin{aligned}
J_0^- &\left(J_0^+ \right)^{\hat{\mathsf{m}}} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{\mathsf{k}} J_0^+ L_{-1} \right)^{\mathsf{q}} |O_{\mathsf{k}}^{--}\rangle \\
&= \left(-2 \left(\hat{\mathsf{m}} - 1 \right) + \mathsf{k} - 2q + J_0^+ J_0^- \right) \left(J_0^+ \right)^{\hat{\mathsf{m}}-1} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{\mathsf{k}} J_0^+ L_{-1} \right)^{\mathsf{q}} |O_{\mathsf{k}}^{--}\rangle \\
&= \hat{\mathsf{m}} \left(\mathsf{k} - \hat{\mathsf{m}} + 1 - 2\mathsf{q} \right) \left(J_0^+ \right)^{\hat{\mathsf{m}}-1} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{\mathsf{k}} J_0^+ L_{-1} \right)^{\mathsf{q}} |O_{\mathsf{k}}^{--}\rangle \,. \tag{2.26}
\end{aligned}$$

Here we have used the relations $J_0^3 |\mathsf{k}, \mathsf{q}, \mathsf{q}, \mathsf{q}\rangle = \left(-\frac{\mathsf{k}}{2} + \mathsf{q} \right) |\mathsf{k}, \mathsf{q}, \mathsf{q}, \mathsf{q}\rangle$ and $J_0^- |\mathsf{k}, \mathsf{q}, \mathsf{q}, \mathsf{q}\rangle = 0$.

Iterating this procedure and using (2.25) one obtains

$$\langle \mathsf{k}, \mathsf{m}, \mathsf{q}, \mathsf{q} | \mathsf{k}, \mathsf{m}, \mathsf{q}, \mathsf{q} \rangle = \binom{\mathsf{k} - 2\mathsf{q}}{\mathsf{m} - \mathsf{q}} (\mathsf{k}^2 - 1)^{\mathsf{q}} . \qquad (2.27)$$

We proceed similarly for the Virasoro generators, using $\hat{\mathsf{n}} = \mathsf{n} - \mathsf{q}$:

$$L_1 (L_{-1})^{\hat{\mathsf{n}}} \left(G^{+1}_{-\frac{1}{2}} G^{+2}_{-\frac{1}{2}} + \frac{1}{\mathsf{k}} J^+_0 L_{-1} \right)^{\mathsf{q}} |O^{--}_{\mathsf{k}}\rangle$$
$$= \hat{\mathsf{n}} (\mathsf{k} + \hat{\mathsf{n}} - 1 + 2\mathsf{q}) \left(G^{+1}_{-\frac{1}{2}} G^{+2}_{-\frac{1}{2}} + \frac{1}{\mathsf{k}} J^+_0 L_{-1} \right)^{\mathsf{q}} |O^{--}_{\mathsf{k}}\rangle . \qquad (2.28)$$

Here we have used the relations $L_0|\mathsf{k}, \mathsf{q}, \mathsf{q}, \mathsf{q}\rangle = \left(\frac{\mathsf{k}}{2} + \mathsf{q} \right) |\mathsf{k}, \mathsf{q}, \mathsf{q}, \mathsf{q}\rangle$ and $L_1|\mathsf{k}, \mathsf{q}, \mathsf{q}, \mathsf{q}\rangle = 0$.

Iterating this procedure and using (2.25) one obtains

$$\langle \mathsf{k}, \mathsf{q}, \mathsf{n}, \mathsf{q} | \mathsf{k}, \mathsf{q}, \mathsf{n}, \mathsf{q} \rangle = \binom{\mathsf{k} + \mathsf{n} + \mathsf{q} - 1}{\mathsf{n} - \mathsf{q}} (\mathsf{k}^2 - 1)^{\mathsf{q}} . \qquad (2.29)$$

Since the Virasoro generators commute with $J^\pm_0$, we can directly combine the results in (2.25), (2.27) and (2.29) to obtain

$$\langle \mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{q} | \mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{q} \rangle = \binom{\mathsf{k} - 2\mathsf{q}}{\mathsf{m} - \mathsf{q}} \binom{\mathsf{k} + \mathsf{n} + \mathsf{q} - 1}{\mathsf{n} - \mathsf{q}} (\mathsf{k}^2 - 1)^{\mathsf{q}} . \qquad (2.30)$$

This result, along with (2.16), gives the norm of the building-block state (2.21),

$$|\psi_{\{N_0, N_i\}}|^2 = \frac{N!}{N_0!} \prod_i \frac{1}{N_i!} \left[\frac{(\mathsf{k}_i^2 - 1)^{\mathsf{q}_i}}{\mathsf{k}_i} \binom{\mathsf{k}_i - 2\mathsf{q}_i}{\mathsf{m}_i - \mathsf{q}_i} \binom{\mathsf{k}_i + \mathsf{n}_i + \mathsf{q}_i - 1}{\mathsf{n}_i - \mathsf{q}_i} \right]^{N_i} . \qquad (2.31)$$

This implies that the norm of the full coherent state (2.23) is

$$|\psi(\{A, B_i\})|^2 \equiv \sideset{}{'}\sum_{\{N_0, N_i\}} \frac{N!}{N_0!} |A|^{2N_0} \prod_i \frac{1}{N_i!} \left[|B_i^2| \frac{(\mathsf{k}_i^2 - 1)^{\mathsf{q}_i}}{\mathsf{k}_i} \binom{\mathsf{k}_i - 2\mathsf{q}_i}{\mathsf{m}_i - \mathsf{q}_i} \binom{\mathsf{k}_i + \mathsf{n}_i + \mathsf{q}_i - 1}{\mathsf{n}_i - \mathsf{q}_i} \right]^{N_i} . \qquad (2.32)$$

We now determine the average number of strands in the coherent state by requiring that the variation of the summand of (2.32) with respect to N_0, N_i vanishes [29], obtaining

$$\bar{N}_0 = |A|^2 \qquad \mathsf{k}_i \bar{N}_i = (\mathsf{k}_i^2 - 1)^{\mathsf{q}_i} \binom{\mathsf{k}_i - 2\mathsf{q}_i}{\mathsf{m}_i - \mathsf{q}_i} \binom{\mathsf{k}_i + \mathsf{n}_i + \mathsf{q}_i - 1}{\mathsf{n}_i - \mathsf{q}_i} |B_i|^2 . \qquad (2.33)$$

This equation, combined with the strand budget constraint (2.22), implies

$$|A|^2 + \sum_i (\mathsf{k}_i^2 - 1)^{\mathsf{q}_i} \binom{\mathsf{k}_i - 2\mathsf{q}_i}{\mathsf{m}_i - \mathsf{q}_i} \binom{\mathsf{k}_i + \mathsf{n}_i + \mathsf{q}_i - 1}{\mathsf{n}_i - \mathsf{q}_i} |B_i|^2 = N . \qquad (2.34)$$

3 Supergravity and superstrata

In this section we briefly review the supergravity theory in which we work, the superstratum solutions we study, and the Kaluza-Klein spectrum of the six-dimensional supergravity theory reduced on S^3.

3.1 Six-dimensional supergravity fields

The D1-D5 system admits an AdS_3 decoupling limit leading to configurations that have $AdS_3 \times S^3 \times \mathcal{M}$ asymptotics [53]. In the limit in which the internal manifold $\mathcal{M}$ is microscopic, dimensional reduction of the 10-dimensional theory on $\mathcal{M}$ gives a $D = 6$ supergravity theory with n tensor multiplets,[4] whose equations of motion were first derived by Romans in [66]. The bosonic field content of the theory is as follows: a graviton g_{MN} ($M, N = 0, \ldots, 5$ are curved 6D indices), 5 2-forms whose field strengths H^m are self-dual ($m = 1, \ldots, 5$ is a vector index of $SO(5)$), n 2-forms whose field strengths H^r are anti-selfdual ($r = 6, \ldots, n + 5$ is a vector index of $SO(n)$) and $5n$ scalars ϕ^{mr}. Dimensional reduction on the T^4 also gives rise to 16 vectors: their CFT duals, however, belong to the short multiplets of fermionic CPOs and we shall not consider them further in the present work. The scalars live in the coset space $SO(5, n)/(SO(5) \times SO(n))$, which can be parametrized by vielbeins (V_I^m, V_I^r) where $I = (m, r)$ is an $SO(5, n)$ vector index. We also introduce field strengths G^I which are related to the selfdual and anti-selfdual field strengths through the vielbeins via $H^m = G^I V_I^m$ and $H^r = G^I V_I^r$. In order to support the global $AdS_3 \times S^3$ vacuum, one must turn on one of the fluxes, which we will take to be $H^{m=5}$. We parametrize fluctuations of the six-dimensional supergravity fields around the $AdS_3 \times S^3$ background as follows:

$$g_{MN} = g_{MN}^0 + h_{MN}, \quad G^{(A)} = g^{0(A)} + g^{(A)}, \quad V_I^m = \delta_I^m + \phi^{(mr)}\delta_I^r, \quad V_I^r = \delta_I^r + \phi^{(mr)}\delta_I^m.$$
(3.1)

The explicit expression for the background fields g_{MN}^0 and $g^{0(A)}$ is given in eq. (3.11) below.

3.2 Superstrata

In this subsection we briefly review certain aspects of superstrata that will be relevant to the remainder of the paper.

Superstrata are supersymmetric supergravity solutions in which the isometries preserved by the corresponding black hole solution are broken by momentum-carrying waves, see e.g. [17–22, 26, 32–35]. These include the first families of smooth horizonless solutions with large BTZ-like AdS_2 throats, general angular momentum, and identified holographic duals in the AdS_3 limit [19, 21]. In the D1-D5-P frame, these are typically constructed in the context of the general 1/8-BPS ansatz of Type IIB supergravity that carries D1, D5, P charges and is invariant on $\mathcal{M}$. This was derived in [67] and is reproduced in appendix C for completeness.

Upon reduction to $6D$, this ansatz gives rise to minimal 6D supergravity coupled to $n = 2$ tensor multiplets, which we will take to be labelled by $r = 6, 7$. The main interest of this work will be the holographic dictionary involving the field strength G^6 and G^7. Let us therefore discuss the relation between these fields and those given in appendix C. First,

---

[4]Some details of the theory change depending on whether the internal manifold is T^4 or K3. In the first case the number of supersymmetries is $\mathcal{N} = (2, 2)$, while in the other one has $\mathcal{N} = (2, 0)$. Moreover, $n = 5$ when the internal manifold is T^4 while $n = 21$ when $\mathcal{M} = K3$.

the 6D metric takes the form

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}}(dv + \boldsymbol{\beta})(du + \omega + \frac{\mathcal{F}}{2}(dv + \boldsymbol{\beta})) + \sqrt{\mathcal{P}}ds_4^2 . \tag{3.2}$$

We first define the following three-form field strengths (here and until the end of the subsection we use $a, b = 1, 2, 4$)

$$G^a = d\left[-\frac{1}{2}\frac{\eta^{ab}Z_b}{P}(du + \omega) \wedge (dv + \boldsymbol{\beta}) \right] + \frac{1}{2}\eta^{ab} \star_4 DZ_b + \frac{1}{2}(dv + \boldsymbol{\beta}) \wedge \Theta^a \tag{3.3}$$

where

$$\eta_{12} = \eta_{21} = -\eta_{44} = 1 , \qquad P = Z_1 Z_2 - Z_4^2 . \tag{3.4}$$

The field strengths G^a respect the self-duality condition:

$$\star_6 G^a = M^a{}_b G^b, \qquad M_{ab} = \frac{Z_a Z_b}{P} - \eta_{ab}. \tag{3.5}$$

The G^a arise from the dimensional reduction of the type IIB field strengths of C_2, C_6 and B, see appendix C. The relation between these 3-forms and the field strengths G^5, G^6, G^7 introduced above was derived in [6] and in our conventions is given by

$$G^5 = \frac{Q_1 G^1 + Q_5 G^2}{2Q_1 Q_5} , \qquad G^6 = -\frac{Q_1 G^1 - Q_5 G^2}{2Q_1 Q_5} , \qquad G^7 = \frac{1}{\sqrt{Q_1 Q_5}}G^4 . \tag{3.6}$$

The scalar fields $\phi^{(56)}$ and $\phi^{(57)}$ arise from the dilaton, C_0, and the component of C_4 with all legs on $\mathcal{M}$. These can be obtained from the vielbein matrix in [6, eqs. (3.33), (B.20)], along with eq. (3.1). In our conventions they take the form

$$\phi^{(56)} = \frac{1}{2\sqrt{Q_1 Q_5}}\left(\frac{Q_5 Z_1 - Q_1 Z_2}{\sqrt{Z_1 Z_2}} \right) , \qquad \phi^{(57)} = \frac{Z_4}{\sqrt{Z_1 Z_2}} . \tag{3.7}$$

The general structure of superstratum solutions is as follows. The construction begins with a seed solution which is usually taken to be a circular supertube [68, 69] with characteristic length-scale a. Momentum-carrying waves are added by a linear superposition of terms within the linear system of BPS equations, specifically at the level of the "first layer" recorded in eq. (C.8). These momentum-carrying waves come with a set of dimensionful Fourier coefficients b_i. This construction is designed to correspond to the structure of the CFT states in eq. (2.23). For further details, see e.g. [21, 33].

Smoothness of the supergravity solutions imposes the relation [33, eq. (4.13)–(4.14)]

$$\frac{Q_1 Q_5}{R^2} = a^2 + \sum_i \frac{b_i^2}{2}\hat{x}_i , \qquad \hat{x}_i = \left(\binom{k_i - 2q_i}{m_i - q_i}\binom{k_i + n_i + q_i - 1}{n_i - q_i}(k_i^2 - 1) \right)^{-1} \tag{3.8}$$

which has the same form of the CFT strand budget constraints (2.22), (2.34), and indeed this is no accident. By comparing the two, the proposed superstratum holographic dictionary involves the following map between the CFT coefficients A, B_i and the supergravity

coefficients $a\ b_i$:

$$
\begin{aligned}
\frac{A}{\sqrt{N}} &= R\sqrt{\frac{1}{Q_1 Q_5}}\, a \equiv \mathsf{a}\,, \\
\frac{B_i}{\sqrt{N}} &= R\sqrt{\frac{1}{2Q_1 Q_5}}\Big(\binom{k_i - 2q_i}{m_i - q_i}\binom{k_i + n_i + q_i - 1}{n_i - q_i}(k^2 - 1)^q\Big)^{-1} b_i \\
&= \sqrt{\frac{1}{2}}\Big(\binom{k_i - 2q_i}{m_i - q_i}\binom{k_i + n_i + q_i - 1}{n_i - q_i}(k^2 - 1)^q\Big)^{-1} \mathsf{b}_i\,,
\end{aligned}
\tag{3.9}
$$

where we have also defined the quantities a, b which will be used later in the paper.

3.3 Kaluza-Klein spectrum

In order to discuss the Kaluza-Klein spectrum of the 6D theory compactified on the S^3 [55, 57] (see also e.g. [27, 32]), we must expand the six-dimensional fluctuations in harmonics of S^3. Before doing this, however, it is convenient to perform the following rescalings:

$$
r \to a_0\,\tilde{r}\,, \quad t \to R_y\,\tilde{t}\,, \quad y \to R_y\,\tilde{y}\,, \quad \beta \to R_y\,\tilde{\beta}\,, \quad \omega \to R_y\,\tilde{\omega}\,, \quad Z_i \to \frac{\tilde{Z}_i}{a_0^2}\,,
\tag{3.10}
$$

with $a_0^2 \equiv \frac{Q_1 Q_5}{R_y^2}$. We then reabsorb the overall scale factor $\sqrt{Q_1 Q_5}$ in the metric[5] to obtain

$$
\begin{aligned}
g_{MN}^{(0)} &= \frac{d\tilde{r}^2}{\tilde{r}^2 + 1} - (\tilde{r}^2 + 1)d\tilde{t}^2 + \tilde{r}^2 d\tilde{y}^2 + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2\,, \\
g^{0(A=5)} &= \cos\theta\sin\theta d\phi \wedge d\psi \wedge d\theta - \tilde{r}d\tilde{r} \wedge d\tilde{t} \wedge d\tilde{y}\,, \qquad g^{0(A\neq 5)} = 0\,.
\end{aligned}
\tag{3.11}
$$

We now introduce a multi-index I for the S^3 harmonic degree k and $(j_3, \bar{j}_3)$ quantum numbers, $I = (k, m, \bar{m})$. In some places we will write k explicitly, and continue to use I for the remaining quantum numbers $(m, \bar{m})$. Harmonics on S^3 and AdS$_3$ are reviewed in appendix A. We split the 6D curved indices into AdS$_3$ indices $\mu, \nu = 0, 1, 2$ and S^3 indices $a, b = 1, 2, 3$. The subscript (ab) denotes the symmetric traceless component of the field. Then expanding the six-dimensional fluctuations in harmonics of S^3, one obtains [27, 55]

$$
\begin{aligned}
h_{\mu\nu} &= \sum h_{\mu\nu}^I Y^I \\
h_{\mu a} &= \sum h_{(v)\mu}^I Y_a^I + h_{(s)\mu}^I D_a Y^I \\
h_{(ab)} &= \sum \rho^I Y_{(ab)}^I + \rho_{(v)}^I D_a Y_b^I + \rho_{(s)}^I D_{(a} D_{b)} Y^I \\
h_a^a &= \sum \pi^I Y^I \\
g_{\mu\nu\rho}^A &= \sum 3 D_{[\mu} b_{\nu\rho]}^{(A)I} Y^I \\
g_{\mu\nu a}^A &= \sum b_{\mu\nu}^{(A)I} D_a Y^I + 2 D_{[\mu} Z_{\nu]}^{(A)I} Y_a^I \\
g_{\mu ab}^A &= \sum D_\mu U^{(A)I} \epsilon_{abc} D^c Y^I + 2 Z_\mu^{(A)I} D_{[b} Y_{a]}^I \\
g_{abc}^A &= -\sum \epsilon_{abc} \Lambda^I U^{(A)I} Y^I \\
\phi^{mr} &= \sum \phi^{(mr)I} Y^I\,.
\end{aligned}
\tag{3.12}
$$

In what follows, the main focus will be on $Z^{(6)}$ and $Z^{(7)}$.

---

[5]This step allows us to use eq. (4.12) in the form given.

4 Constructing the supercharged holographic dictionary

In this section we review the single-particle basis and use it derive the correspondence between the operators of interest and their dual bulk fields. In section 4.1 we discuss how the single-particle basis can be used to determine most, but not all, of the mixing between single and multi-trace operators from CFT arguments alone. In section 4.2 we compute the gauge-invariant fluctuations of supergravity fields that are dual to the CFT operators of interest. In section 4.3 we derive the holographic map in the sector we study, in the single-particle basis. This involves resolving the operator mixing by combining and refining the results of [31] and [28]. In section 4.4 we then generate the superdescendants within this supermultiplet that we use in the remainder of the paper. In section 4.5 we record the explicit holographic dictionary in the single-particle basis for convenient reference.

4.1 Single-particle operator basis

AdS/CFT duality relates AdS fields and boundary CFT operators through a matching of the observables of the theories; we shall focus on protected correlation functions, which can be compared between supergravity and the orbifold CFT.

On the bulk side, correlation functions can be computed by dimensionally reducing the 6D Lagrangian on S^3: the AdS$_3$ action takes the schematic form

$$S_{\text{AdS}_3} \sim \int_{\text{AdS}_3} \left(\mathcal{L}_2 + \mathcal{L}_3 + \cdots \right) \tag{4.1}$$

where $\mathcal{L}_n$ contains the interactions between n KK modes and is relevant for computing n-point functions and higher.

The first ingredient in the holographic dictionary is the identification of the quantum numbers of the fields and the dual operators. With linear field redefinitions one can diagonalize the quadratic term $\mathcal{L}_2$, and thus identify the quantum numbers that the CFT operators dual to each supergravity field must carry.

On the CFT side, however, there are degeneracies: in general there are single and multi-trace operators with the same quantum numbers. A further complication in the AdS$_3$ case (which is absent, for example, in the long-studied case of AdS$_5$) comes from the fact that there are degeneracies also between the single-traces: as discussed in section 2.2, the chiral primaries Σ_3 and Ω cannot be distinguished by their quantum numbers.

In order to identify the mixing matrix, one must analyze the three-point functions on both sides of the duality [70–73]. On the gravity side, these are generated by considering the cubic Lagrangian $\mathcal{L}_3$. The cubic Lagrangian was derived in [73] and a priori involves derivative couplings. It was shown, however, that the derivative couplings can be reabsorbed via a non-linear redefinition of the fields. While this transformation does not change non-extremal 3-point functions, it has been emphasized in [45, 46, 50] that extremal 3-point functions require special attention. An important fact is that non-derivative extremal cubic couplings vanish. This is no coincidence: in the extremal case, the spacetime integral that occurs in the Witten diagram diverges, so the extremal coupling must vanish in order to avoid a divergence in the value of the correlator. The extremal correlator can

still gain contributions from the derivative couplings in the Lagrangian, since they give rise to boundary terms upon partial integration [45]. The field redefinition, however, removes the derivative terms and thus all extremal three-point functions vanish.

The bulk field redefinition is interpreted on the CFT side as a change of basis [46], since it amounts to forming an admixture between the operator dual to the original field and certain multi-trace operators. By AdS/CFT, in this basis all CFT extremal three-point functions vanish. Let us denote by Φ^i the AdS field with respect to which the Lagrangian contains derivative terms, and let us denote its CFT dual by $\mathcal{O}^{\Delta_i}$. Then the operator $\tilde{\mathcal{O}}^{\Delta_i}$ dual to the redefined field $\tilde{\Phi}^i$ will take the form: $\tilde{\mathcal{O}}^{\Delta_i} = \mathcal{O}^{\Delta_i} + \frac{1}{\sqrt{N}} \sum_k c_{ik} \mathcal{O}^{\Delta_i - \Delta_k} \mathcal{O}^{\Delta_k} + \dots$ where the ellipses denote other double-traces or higher multi-trace operators, the only constraint being that they must have the same quantum numbers as the operator $\mathcal{O}^{\Delta_i}$. Note that the operators $\mathcal{O}_i^\Delta$ and $\tilde{\mathcal{O}}_i^\Delta$ coincide if $\Delta_i = 1$, as at dimension 1 the spectrum of the theory consists only of single-trace operators.

In generic correlators, the contribution of the double-trace operators to the correlator is subleading in the $1/N$ expansion (see e.g. the discussions in [28, 45]). This is the CFT version of the bulk statement that the field redefinition $\Phi^i \to \tilde{\Phi}^i$ leaves non-extremal correlators unchanged. However for certain correlators, the double-traces contribute at leading order in large N. This happens in extremal correlators and also in certain (non-extremal) mixed heavy-light correlators. In this paper we are interested in precisely such mixed heavy-light correlators.

In recent work it was proposed that single-particle supergravity excitations around global AdS$_5 \times$ S^5 are dual to CFT operators (in short multiplets) that are orthogonal to all multi-trace operators [51]. This was then extensively used in [52] to discuss the properties of the single-particle operator basis in free $\mathcal{N} = 4$ SYM.[6]

We now argue that the redefinition that removes cubic couplings gives rise to precisely the same set of single-particle CFT operators defined as those that are orthogonal to all multi-trace operators. We follow in part a discussion in [45]. First of all, we recall that conformal symmetry implies that a two-point function can be non-zero only if the two operators have the same dimension. To show that an operator with dimension k_1 is a single-particle operator, we must therefore show that it has vanishing two point function with all multi-traces $(\mathcal{O}_{k_2}\mathcal{O}_{k_3})(z)$ such that $k_1 = k_2 + k_3$. The first non-singular term in the OPE $\mathcal{O}_{k_2}(z_2)\mathcal{O}_{k_3}(z_3)$, with coefficient one, is the multi-trace $(\mathcal{O}_{k_2}\mathcal{O}_{k_3})(z_2)$. Now consider the extremal ($k_1 = k_2 + k_3$) three-point function (the coefficient c is zero in this basis however we keep it for convenience)

$$\langle \mathcal{O}_{k_1}(z_1)\mathcal{O}_{k_2}(z_2)\mathcal{O}_{k_3}(z_3)\rangle = \frac{c}{(z_1 - z_2)^{2k_2}(z_1 - z_3)^{2k_3}}. \tag{4.2}$$

By taking the $z_2 \to z_3$ limit (which is smooth at extremality) one obtains

$$\langle \mathcal{O}_{k_1}(z_1)(\mathcal{O}_{k_2}\mathcal{O}_{k_3})(z_2)\rangle = \frac{c}{(z_1 - z_2)^{2k_1}}. \tag{4.3}$$

This shows the equivalence of the single-particle CFT basis of [51, 52] and the CFT basis dual to supergravity fields with derivative cubic couplings removed. The vanishing of the

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[6]See also [74] for further discussion.

three-point coefficient c in (4.2) implies the orthogonality between the operator $\mathcal{O}_{k_1}$ and all multi-particle operators in eq. (4.3), and vice versa. This discussion motivates the change in the definition of double-trace operators with respect to the convention in [31] (see footnote 2).

Before proceeding to our analysis let us make a few comments on the derivation of the holographic dictionary for scalar operators of dimension two in [31]. This work did not use the single-particle basis, however we shall use the single-particle basis in the present work, so let us describe the difference in the two approaches.

The method used in [31] was as follows. First, the most general linear combination of single and double-trace operators allowed by the quantum numbers was worked out (higher multi-trace operators are trivially absent at dimension two). Then a set of different backgrounds were considered, for which there was already a well-established holographic description (the two-charge Lunin-Mathur solutions [1]). CFT expectation values of these light fields in a selection of these heavy states were then matched to the expansion of the dual bulk fields identified in [27] and [6]. By considering an exhaustive set of examples, the combinations of single and multi-trace operators in the CFT dual to certain supergravity fluctuations were fixed, as were the overall normalization coefficients of the holographic dictionary in this sector.

In the present paper we work in the single-particle basis. In this basis, the identification of the single-particle operators partially reduces to the identification of the operators that have the property that all their extremal three point functions vanish.[7] This is purely a CFT computation, which works as an input in constructing the holographic dictionary. Importantly, this does not resolve all the mixing, as we shall discuss shortly.

4.2 Gauge-invariant combinations of supergravity fields

We now determine the gauge-invariant combinations of supergravity fluctuations that are dual to the operators we consider.

Not all the fluctuations in the KK harmonic expansion (3.12) are independent: some of them are connected to the background fields or to other fluctuations through coordinate transformations that tend to zero at infinity. For instance, consider the vacuum state, which corresponds to empty global $AdS_3 \times S^3$. If one performs such a change of coordinates, one can turn on some of the AdS_3 fields in the KK harmonic expansion (3.12). Of course these are not physical excitations and there are no boundary operators that source them.

In the study of the KK spectrum in [55], the authors dealt with this redundancy by fixing the (de Donder) gauge. Here we follow instead the gauge-invariant KK reduction method developed in [37]. The strategy is to organize the AdS_3 fields in combinations that have the correct transformation properties under a gauge transformation.

The coordinate transformation

$$x^M \to x'^M = x^M - \xi^M \tag{4.4}$$

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[7] The authors thank Stefano Giusto and Rodolfo Russo for a discussion on this point.

generates a perturbation of the metric and 3-form which, up to linear order in the gauge parameter, reads

$$\delta h_{MN} = D_M \xi_N + D_N \xi_M + D_M \xi^R h_{RN} + D_N \xi^R h_{RM} + \xi^R D_R h_{MN} \,,$$
$$\delta g^A_{MNP} = 3D_{[M}\xi^R g^{0A}_{NP]R} + 3D_{[M}\xi^R g^A_{NP]R} + \xi^R D_R g^A_{MNP} \,,$$

(4.5)

where the background value g^{0A} was introduced in eq. (3.1). In order to deal with the non-linear terms, one must project onto the basis of S^3 harmonics.

As we will discuss in section 4.5, to study supercharged superstrata we will need to construct the holographic dictionary for the AdS$_3$ vector fields $Z^{(6)k=1}$ and $Z^{(7)k=1}$. The operators dual to $Z^{(6)k=1}$ and $Z^{(7)k=1}$ have dimension 3: in principle, we would need the transformation up to second order in the gauge parameter [75, 76], namely the terms quadratic in ξ and linear in $g^{0(A)}$. However, since $g^{0(A)} = 0$ for $A \neq 5$, these terms do not contribute in the analysis of the vector fields in the tensor multiplets and so the gauge-invariant combinations up to the order we are interested in can be obtained using eq. (4.5). Using the KK spectrum (3.12) together with the decomposition of ξ^M in harmonics,

$$\xi_\mu = \sum_I \xi_\mu^I Y^I \,, \qquad \xi_a = \sum_{I,J} \xi_v^I Y_a^I + \xi_s^J D_a Y^J \,,$$

(4.6)

one obtains that under such a diffeomorphism, the second order transformation of $Z^{(6)k=1}$ and $Z^{(7)k=1}$ reads (here $A \neq 5$, so for us $A = 6, 7$):

$$\delta Z_\mu^{(A)K} = D_\mu U^{(A)J} \left[\xi_s^I \left(n_{IJK}^v + c_{IJK}^v \right) + \xi_v^I \left(p_{IJK}^v + g_{IJK}^v \right) \right]$$
$$- \frac{E_{JIK}}{\lambda_k} \left[D_\mu \xi^{I\nu} D_\nu U^{(A)J} + \xi^{I\nu} D_\nu D_\mu U^{(A)J} \right] - \frac{\Lambda_k E_{KIJ}}{\lambda_k^2} \epsilon_{\mu\nu\rho} \xi^{I\nu} D_\rho U^{(A)J}$$
$$- \frac{\Lambda_j U^{(A)J}}{\lambda_k} \left[D_\mu \xi_v^I f_{IJK} + D_\mu \xi_s^I E_{IJK} \right] \,,$$

(4.7)

where the degree k associated with the multi-index K must be equal to 1 in order for the equality to hold. The triple overlap coefficients $n_{IJK}^v, c_{IJK}^v, p_{IJK}^v, g_{IJK}^v, E_{IJK}$ are defined in appendix A.1.3.

The gauge-invariant combination associated with the ($A \neq 5$) field $Z_\mu^{(A),k=1}$ will take the form $\boldsymbol{Z}_\mu^{(A),k=1} = Z_\mu^{(A),k=1} + \ldots$, where the ellipses represent fields and product of fields such that their transformation properties compensate those on the right-hand side of (4.7), such that the redefined field has the correct transformation properties. With this aim, we consider linear-order variations of

$$h_{(ab)} = \sum_{I,J,K} \rho_t^I Y_{ab}^I + \rho_v^J D_{(a} Y_{b)}^J + \rho_s^K D_{(a} D_{b)} Y^K \,,$$
$$h_{\mu a} = \sum_{I,J} h_\mu^{v,I} Y_a^I + h_\mu^{s,J} D_a Y^J \,,$$

(4.8)

where (ab) denotes symmetric traceless. The transformations read:

$$\delta h_{(ab)} = D_a \xi_b + D_b \xi_a = 2\xi_v^I D_{(a} Y_{b)}^I + 2\xi_s^J D_{(a} D_{b)} Y^J \,,$$
$$\delta h_{\mu a} = D_\mu \xi_a + D_a \xi_\mu = D_\mu \xi_v^I Y_a^I + (D_\mu \xi_s^J + \xi_\mu^J) D_a Y^J \,.$$

(4.9)

We thus obtain

$$\delta\rho_v^I = 2\xi_v^I, \qquad \delta\rho_s^I = 2\xi_s^I, \qquad \delta\hat{h}_\mu^{s,I} = \xi_\mu^I, \qquad \delta h_\mu^{v,I} = D_\mu\xi_v^I, \qquad (4.10)$$

where we have defined $\hat{h}_\mu^{s,I} = h_\mu^{s,I} - \frac{1}{2}D_\mu\rho_s^I$. One can further check that the fields $U^{(A\neq5)k=1}$ are gauge invariant, so one has (again for $A \neq 5$)

$$
\begin{aligned}
\mathbf{Z}_\mu^{(A)K} = Z_\mu^{(A)K} &+ \frac{E^{JIK}}{\lambda_k}\left(\hat{h}^{s,I,\nu}D_\nu D_\mu U^{(A)J} + D_\mu\hat{h}^{s,I,\nu}D_\nu U^{(A)J}\right) \\
&+ \frac{E^{KIJ}}{\lambda_k^2}\epsilon_{\mu\nu\rho}\hat{h}^{s,I,\nu}D_\rho U^{(A)J} + \frac{\Lambda_j U^{(A)J}}{\lambda_k}\left(\frac{1}{2}D_\mu\rho_v^I f^{IJK} + \frac{1}{2}D_\mu\rho_s^I E^{IJK}\right) \qquad (4.11) \\
&- D_\mu U^{(A)J}\left(\frac{1}{2}\rho_s^I n_{IJK}^v + \frac{1}{2}\rho_v^I p_{IJK}^v + \frac{1}{2}\rho_s^I c_{IJK}^v + \frac{1}{2}\rho_v^I g_{IJK}^v\right).
\end{aligned}
$$

In the following sections, we will also review the holographic dictionary for CPOs of dimension one and scalar chiral primaries of dimension two. Discussing this dictionary in an explicit gauge-invariant fashion requires studying the gauge-invariant combinations associated to other fields. In practise however, it is often convenient to partially fix the gauge and use the holographic dictionary in a preferred system of coordinates, and we will do this explicitly in appendix D.

4.3 Refining the existing holographic dictionary

The action for the AdS$_3$ fields is obtained by substituting the KK harmonic expansion (3.12) into the six-dimensional Lagrangian. This procedure leads to a three-dimensional Lagrangian which has a non-diagonal mass matrix. The duality between 3D fields and operators prescribes that the dimension of the operator corresponds to the energy of the bulk excitation. Thus in order to identify the AdS fields dual to the operator of the D1-D5 CFT, one must perform the linear field redefinition that diagonalizes the mass matrix.

Moreover, as discussed in section 4.1, by performing quadratic field redefinitions it possible to recast the cubic Lagrangian into a form with no derivative couplings: this corresponds to the basis of single-particle excitations. For a general discussion we refer to [55, 73]; here we just discuss the AdS fields that will enter the holographic dictionary we are going to construct. Recall that in [55, 73] the KK spectrum was been studied in de Donder gauge, while we are interested in a gauge independent discussion. It follows from [37] that this can be obtained by simply replacing the fields with the corresponding gauge-invariant combination: in the following, unless explicitly stated, this replacement will be understood.

The field redefinitions that diagonalize the linearized field equations are ($r = 6, 7$)

$$
\begin{aligned}
s_I^{(r)k} &= \frac{\sqrt{k}}{\sqrt{k+1}}\left(\phi_I^{(5r)k} + 2(k+2)U_I^{(r)k}\right), \\
\sigma_I^k &= \frac{\sqrt{k(k-1)}}{\sqrt{k+1}}\left(6(k+2)U_I^{(5)k} - \pi_I^k\right), \qquad\qquad (4.12) \\
A_{I\mu}^{(\pm)k} &= \pm 2Z_{I\mu}^{(5)(\pm)k} - h_{I\mu}^{(\pm)k}, \qquad Z_{I\mu}^{(r)k} \to 4\sqrt{k+1}Z_{I\mu}^{(r)k},
\end{aligned}
$$

where the superscripts $(\pm)$ are used to distinguish the fields that couple to left $(+)$ and right $(-)$ SU(2) vector harmonics. The overall k-dependent factors are needed to canonically normalize the quadratic Lagrangian [73]. These fields have masses:

$$m^2_{s^{(r)k}} = m^2_{\sigma^k} = k(k-2)\,, \qquad m_{A^{(\pm)k}} = k-1\,, \qquad m_{Z^{(r)k}} = k+1\,. \qquad (4.13)$$

Restriction to fields with low k. We focus on the low-order fields, in particular we shall restrict to $s^{(r)k}$ with $k = 1, 2$;
σ^k with $k = 2$; $A^{(\pm)k}$ with $k = 1$; and $Z^{(r)k}$ with $k = 1$. Among these fields, only $\sigma^{k=2}$ has a cubic coupling involving derivatives: the equation of motion reads[8]

$$\Box \sigma_I^{(k=2)} = \frac{11}{3} \sum_{r=6,7} \left(s_i^{r(k=1)} s_j^{r(k=1)} - D_\mu s_i^{r(k=1)} D^\mu s_j^{r(k=1)} \right) a_{Iij}\,, \qquad (4.14)$$

where a_{Iij} is defined as the following triple overlap in eq. (A.9).

Following the discussion in section 4.1, we wish to remove the derivative coupling, which can be done with the following field redefinition:

$$\sigma_I^{(k=2)} \;\to\; \tilde{\sigma}_I^{(k=2)} = \sigma_I^{(k=2)} + \frac{11}{6} \sum_{r=6,7} s_i^{r(k=1)} s_j^{r(k=1)} a_{Iij}\,. \qquad (4.15)$$

We now identify the single-particle operators of the D1-D5 CFT that are dual to the fields in (4.12), (4.15). At dimension one there are no multi-trace operators, nor there are degeneracies among the single traces, so the basis of single-trace and single-particle operators coincide. The explicit dictionary for these fields was derived in [29] and is recorded in table 2 below.

For dimension two operators the situation is more complicated, as follows. The spectrum of single-trace CPOs discussed in section 2.2 splits into two subsectors, according to the quantum numbers. The supergravity theory has an SO(n) symmetry that acts on the tensor multiplets. However, in the full String Theory, only an SO($n-1$) subgroup is preserved [28]. This means that the dimension two operator O_2 can only mix with the double trace $(\Sigma_2 \cdot O)$. By contrast, since Σ_3 and Ω are scalars under this SO($n-1$), they mix with each other and with the multi-traces $(\Sigma_2 \cdot \Sigma_2)$, $(J \cdot \bar{J})$ and $(O \cdot O)$. This has been explicitly verified in [31].

The single-particle CPO in the first of these subsectors is [31]

$$\tilde{O}_2^{++} = \left(\frac{\sqrt{2} \, O_2^{++}}{N} - \frac{1}{\sqrt{N}} (\Sigma_2 \cdot O)^{++} \right). \qquad (4.16)$$

The coefficient in front of O_2^{++} is chosen such that this first term on the right-hand side is unit-normalized in the large N limit. Since $(\Sigma_2 \cdot O)$ is unit-normalized at large N, the full operator $\tilde{O}_2^{++}$ also has unit norm at large N. We see that the mixing coefficient between the unit-normalized operators scales as $1/\sqrt{N}$. This is a general feature of all the

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[8]This is [27, eq. (5.8)] with implemented SO($h^{1,1}(\mathcal{M}) + 1$) invariance.

examples we study.[9] As noted above, in generic correlators the multi-trace contribution is subleading at large N, however in extremal or certain heavy-light correlators it contributes at leading order in large N. Note that for this operator, all coefficients are fixed from CFT considerations.

In the second subsector, Σ_3 and Ω mix among themselves, and also with the multi-traces $(\Sigma_2 \cdot \Sigma_2)$, $(J \cdot \bar{J})$ and $(O \cdot O)$. CFT considerations alone are not sufficient to identify the two individual single-particle CPOs: if one imposes orthonormality and orthogonality with all multi-traces, one is left with a one-parameter family of possible pairs of candidate single-particle operators. We discuss this and the following steps in more detail in appendix B. To proceed, we fix the mixing among the single-traces Σ_3 and Ω using additional information from comparison with supergravity, using the mixing matrix derived in [28]. Once we incorporate this single-trace mixing, imposing orthonormality and orthogonality with all multi-traces determines all the remaining admixture coefficients, resulting in the single-particle operators:

$$
\begin{aligned}
\tilde{\Sigma}_3^{++} &\equiv \frac{3}{2} \left[\left(\frac{\Sigma_3^{++}}{N^{\frac{3}{2}}} - \frac{\Omega^{++}}{3N^{\frac{1}{2}}} \right) + \frac{1}{N^{\frac{1}{2}}} \left(-\frac{2}{3}(\Sigma_2 \cdot \Sigma_2)^{++} + \frac{1}{6}(O \cdot O)^{++} + \frac{1}{3}(J \cdot \bar{J})^{++} \right) \right], \\
\tilde{\Omega}^{++} &\equiv \frac{\sqrt{3}}{2} \left[\left(\frac{\Sigma_3^{++}}{N^{\frac{3}{2}}} + \frac{\Omega^{++}}{N^{\frac{1}{2}}} \right) + \frac{1}{N^{\frac{1}{2}}} \left(-(\Sigma_2 \cdot \Sigma_2)^{++} - \frac{1}{2}(O \cdot O)^{++} - (J \cdot \bar{J})^{++} \right) \right].
\end{aligned}
\tag{4.17}
$$

Let us make a similar comment on the factors of N and numerical coefficients in eq. (4.17). The linear combinations of the single-particle operators inside Σ_3^{++} and Ω^{++} ensure that these single-trace combinations (and thus the full single-particle operators) are orthonormal in the large N limit, as we discuss in more detail in appendix B. Again the admixture coefficients between the unit-normalized single-traces and multi-traces are of order $1/\sqrt{N}$; in generic correlators the contributions from the multi-traces are subleading; but in extremal or certain heavy-light correlators, the multi-traces contribute at leading order in large N.

We now make two observations. We note that in this paper, the coefficients of the multi-traces have been derived from a purely CFT calculation of orthogonality with all multi-traces. The only direct supergravity input here is the mixing between Ω and Σ_3 derived in [28]. This contrasts with the method of [31] which fixed the multi-trace coefficients holographically. The fact that these two methods agree is non-trivial, and is explored further in appendix D.

Secondly, let us emphasize that the non-trivial mixing between Ω and Σ_3 demonstrates that there is no one-to-one correspondence between k-cycles in the CFT and single-particle supergravity excitations, even at the level of single traces: single-particle states in the bulk are dual to a linear combination of single cycles of the symmetric group (cf. [28, 32]).

---

[9]The scaling of $1/\sqrt{N}$ per additional trace for the mixing coefficients of multi-trace operators has appeared before in discussions of extremal correlators [28]. Note that in the case of a bound state of N_3 D3 branes giving rise to SU(N_3) $\mathcal{N} = 4$ SYM, the analogous scaling of such admixture coefficients is $1/N_3$ per additional trace (see e.g. [52] and references within).

AdS$_3$ field	Dual operator	$(j_{\mathrm{sl}}, \bar{j}_{\mathrm{sl}})$	$(j_{\mathrm{su}}, \bar{j}_{\mathrm{su}})$
$s^{(6)k=1}$	Σ_2	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$s^{(6)k=2}$	$\tilde{\Sigma}_3$	$(1, 1)$	$(1, 1)$
$s^{(7)k=1}$	O	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$s^{(7)k=2}$	$\tilde{O}_2$	$(1, 1)$	$(1, 1)$
$\tilde{\sigma}^{k=2}$	$\tilde{\Omega}$	$(1, 1)$	$(1, 1)$
$A_\mu^{(+)k=1}$	J	$(1, 0)$	$(1, 0)$
$A_\mu^{(-)k=1}$	$\bar{J}$	$(0, 1)$	$(0, 1)$
$Z_\mu^{6(-)k=1}$	$GG\tilde{\Sigma}_3$	$(2, 1)$	$(0, 1)$
$Z_\mu^{7(-)k=1}$	$GG\tilde{O}_2$	$(2, 1)$	$(0, 1)$

Table 2. This table shows the duality between AdS$_3$ fields and the CFT operator. We denote with $(j_{\mathrm{sl}}, \bar{j}_{\mathrm{sl}})$ the quantum numbers associated with the Casimirs of the two copies of SL$(2, \mathbb{R})$ and with $(j_{\mathrm{su}}, \bar{j}_{\mathrm{su}})$ those associated to the Casimir of the two copies of SU(2).

4.4 Supercharged CFT operators

In the following, we will construct the holographic dictionary for the bosonic 1/8-BPS supercharged descendants of the single-particle operators $\tilde{O}_2$ and $\tilde{\Sigma}_3$. We denote these by $GG\tilde{O}_2$ and $GG\tilde{\Sigma}_3$ respectively, and we obtain:

$$
\begin{aligned}
\left(GG\tilde{O}_2\right)^{(0,a)} &\equiv \frac{1}{\sqrt{3}} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) \tilde{O}_2^{-,a} \\
&= \frac{1}{\sqrt{3}} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) \left[\frac{\sqrt{2} O_2}{N} - \frac{1}{N^{1/2}} \left(\Sigma_2 \cdot O \right) \right]^{-,a}, \quad (4.18) \\
\left(GG\tilde{\Sigma}_3\right)^{(0,a)} &\equiv \frac{1}{\sqrt{3}} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) \tilde{\Sigma}_3^{-,a} \\
&= \frac{\sqrt{3}}{2} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) \\
&\quad \times \left[\left(\frac{\Sigma_3}{N^{\frac{3}{2}}} - \frac{\Omega}{3N^{\frac{1}{2}}} \right) + \frac{1}{N^{\frac{1}{2}}} \left(-\frac{2}{3} \left(\Sigma_2 \cdot \Sigma_2 \right) + \frac{1}{6} \left(O \cdot O \right) + \frac{1}{3} \left(J \cdot \bar{J} \right) \right) \right]^{-,a}.
\end{aligned}
$$

The overall numerical coefficients follow from eqs. (2.25), (4.16) and (4.17) and are required to normalize the operators to one at large N. Being descendants of single-particle operators, they are orthogonal to all multi-trace operators. They carry quantum numbers $(j_{\mathrm{sl}}, \bar{j}_{\mathrm{sl}}) = (2, 1)$ associated with the Casimirs of SL$(2, \mathbb{R})$ and quantum numbers $(j_{\mathrm{su}}, \bar{j}_{\mathrm{su}}) = (0, 1)$ associated to the Casimirs of SU$(2)_L \times$ SU$(2)_R$ (we refer the reader to appendix A for explicit definitions).

Using the relation between the mass m of a field in AdS$_{d+1}$ and the dimension of the dual operator (see for example [42]) one can identify the map between single-particle operators and AdS$_3$ fields. The residual degeneracy between $s^{(6)k=2}$ and $\tilde{\sigma}^{k=2}$ can be fixed

by comparing non-extremal three-point functions [28]: $s^{(6)k=2}$ is dual to $\tilde{\Sigma}_3$ and $\tilde{\sigma}^{k=2}$ is dual to $\tilde{\Omega}$. These results are recorded in table 2.

4.5 Refined holographic dictionary at dimension one and two

For convenient reference we now record the holographic dictionary for CPOs of dimension one and scalar CPOs of dimension two derived in [6, 27, 29, 31], after having recast it in the single-particle basis. The dictionary relates the asymptotic expansion of the AdS$_3$ fields in a non-trivial background with the expectation value of the dual operators $\mathcal{O}$ in the dual heavy CFT state $|H\rangle$,

$$\langle \mathcal{O} \rangle \equiv \langle H|\mathcal{O}(\tilde{t}, \tilde{y})|H \rangle \,, \tag{4.19}$$

where $(\tilde{t}, \tilde{y})$ is a generic insertion point on the CFT cylinder.

For scalars, the holographic prescription relates the expectation value of a scalar operator of dimension Δ with the coefficient of $\tilde{r}^{-\Delta}$ of the large $\tilde{r}$ expansion of the dual scalar field. The mass of the scalar fields $s^{(r)k}$ and σ^k in eq. (4.13) implies that their dual operators have dimension $\Delta = k$. This motivates introducing the following asymptotic expansion of scalar fields: we denote with $\left[\Phi_k\right]$ the first non-vanishing term of the expansion of the scalar field Φ_k,

$$\Phi_k = \frac{\left[\Phi_k\right]}{\tilde{r}^k} + O(\tilde{r}^{-(k+1)}) \,. \tag{4.20}$$

Similarly, for the one-forms $A_{k=1}^{a(\pm)}$ we expand as [27, 77]

$$A_{k=1}^{a(\pm)} = \left[A_{k=1}^{a(\pm)}\right](d\tilde{t} \pm d\tilde{y}) + O(\tilde{r}^{-1}) \,. \tag{4.21}$$

We then have the dictionary

$$
\begin{aligned}
\frac{1}{\sqrt{N}}\langle J^\pm \rangle &= -\frac{\sqrt{N}}{\sqrt{2}}\left[A_{k=1}^{\mp(+)}\right], & \frac{1}{\sqrt{N}}\langle \bar{J}^\pm \rangle &= -\frac{\sqrt{N}}{\sqrt{2}}\left[A_{k=1}^{\mp(-)}\right], \\
\frac{1}{\sqrt{N}}\langle J^3 \rangle &= -\frac{\sqrt{N}}{2}\left[A_{k=1}^{0(+)}\right], & \frac{1}{\sqrt{N}}\langle \bar{J}^3 \rangle &= -\frac{\sqrt{N}}{2}\left[A_{k=1}^{0(-)}\right], \\
\frac{\sqrt{2}}{N}\langle \Sigma_2^{\alpha\dot{\alpha}} \rangle &= (-1)^{\alpha\dot{\alpha}}\frac{\sqrt{N}}{\sqrt{2}}\left[s_{k=1}^{(6)(-\alpha,-\dot{\alpha})}\right], & \frac{1}{\sqrt{N}}\langle O^{\alpha\dot{\alpha}} \rangle &= (-1)^{\alpha\dot{\alpha}}\frac{\sqrt{N}}{\sqrt{2}}\left[s_{k=1}^{(7)(-\alpha,-\dot{\alpha})}\right], \\
\langle \tilde{\Omega}^{a,\dot{a}} \rangle &= -(-1)^{a+\dot{a}}\frac{\sqrt{N}}{\sqrt{2}}\left[\tilde{\sigma}_{k=2}^{(-a,-\dot{a})}\right], & & \tag{4.22} \\
\langle \tilde{\Sigma}_3^{a,\dot{a}} \rangle &= (-1)^{a+\dot{a}}\frac{\sqrt{N}}{\sqrt{2}}\left[s_{k=2}^{(6)(-a,-\dot{a})}\right], & \langle \tilde{O}_2^{a,\dot{a}} \rangle &= (-1)^{a+\dot{a}}\frac{\sqrt{N}}{\sqrt{2}}\left[s_{k=2}^{(7)(-a,-\dot{a})}\right].
\end{aligned}
$$

The numerical coefficients and factors of N on the left hand side of each equality are such that the operators are unit-normalized at large N. With this choice, we note that the coefficients in the third and fifth lines of the dictionary respect the SO(n) symmetry between the n tensor multiplets of the supergravity theory.

5 Supercharged holographic dictionary

In this section we construct the holographic dictionary for the single-particle operators $GG\tilde{O}_2^{(0,a)}$ and $GG\tilde{\Sigma}_3^{(0,a)}$ defined in (4.18). We saw in table 2 that the expectation value of these operators corresponds to the bulk asymptotic expansion of the vector fields $Z_\mu^{7(-)k=1}$ and $Z_\mu^{6(-)k=1}$. The linearized equation of motion for these fields is [73]

$$\star \, dZ_k^{(A)(-)} = -(k+1)Z_k^{(A)(-)} \,, \tag{5.1}$$

for $A = 6, 7$. We note that this is the equation obeyed by the left AdS$_3$ harmonics $B_{\mathrm{L},l}^{(\pm)l,\bar{l}}$ in eq. (A.28), with the identification $l = k + 3$. Left vector harmonics on AdS$_3$ are discussed in appendix A.2.2: their large $\tilde{r}$ expansion reads

$$B_{\mathrm{L},l}^{(\pm)l,\bar{l}} \sim \frac{d\tilde{t} + d\tilde{y}}{\tilde{r}^{(l-2)}} + O\left(\frac{1}{r^{l-1}}\right) \,. \tag{5.2}$$

This motivates, for $k = 1$, the following asymptotic expansion of the bulk fields, where we use the same square bracket notation introduced in the previous subsection for the leading term:

$$Z_{k=1}^{7(a,-)} = \left[Z_{k=1}^{7(a,-)}\right]\frac{d\tilde{t} + d\tilde{y}}{\tilde{r}^2} + O\left(\frac{1}{\tilde{r}^3}\right) \,, \qquad Z_{k=1}^{6(a,-)} = \left[Z_{k=1}^{6(a,-)}\right]\frac{d\tilde{t} + d\tilde{y}}{\tilde{r}^2} + O\left(\frac{1}{\tilde{r}^3}\right) \,. \tag{5.3}$$

We consider the following ansatz for the dictionary involving supercharged operators dual to vector fields in the tensor multiplet:

$$\begin{aligned}
\langle GG\tilde{O}_2^{(0,a)}\rangle &= \alpha \left[Z_{k=1}^{7(a,-)}\right] \,, \\
\langle GG\tilde{\Sigma}_3^{(0,a)}\rangle &= \beta \left[Z_{k=1}^{6(a,-)}\right] \,.
\end{aligned} \tag{5.4}$$

where α and β are unknown coefficients that we will determine by evaluating the ansatz (5.4) on some reference heavy states. Consistency of the dictionary requires that α and β depend neither on the SU(2)$_R$ quantum number a nor on the heavy state considered.

Moreover, based on the SO(n) symmetry between the tensor multiplets of the supergravity theory, one expects to find $\alpha = \beta$, and we shall verify explicitly that this is the case.

5.1 Normalizing the supercharged holographic dictionary

In this section we fix the coefficient α in the supercharged holographic dictionary in eq. (5.4) by looking at one of the simplest examples of supercharged superstrata: the one sourced by the mode $(\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{q}) = (2, 1, \mathsf{n}, 1)$. This supergravity solution was constructed in [32]. Since we work in the conventions of [33], a slightly more convenient reference for the explicit form of the supergravity quantities we need in the following[10] is [33, eqs. (2.4), (4.1), (4.4), (4.13), (4.14)].

---

[10]In what follows we will label with b the supergravity coefficient that is denoted by c_4 in [33].

The CFT state that is proposed to be dual to this bulk solution is

$$\sum_p (A|0\rangle_1)^{N-2p} \left(B \frac{L_{-1}^{n-1}}{(n-1)!} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) |O_2^{--}\rangle \right)^p. \tag{5.5}$$

The single-particle operator $G\tilde{G}\tilde{O}_2$ has a non-vanishing expectation value on this state, sourced by its single-trace constituent. In order to compute the correlator, we first consider the basic process described by the correlator

$$\langle O_2^{++}| \left(-G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2} + \frac{1}{2} J_0^- L_{+1} \right) \frac{L_1^{n-1}}{(n-1)!} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) O_2^{--} (\tilde{t}, \tilde{y}) |0\rangle_1^{\otimes 2} \tag{5.6}$$

$$= z^2 \bar{z} \langle O_2^{++}| \left(-G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2} + \frac{1}{2} J_0^- L_{+1} \right) \frac{L_1^{n-1}}{(n-1)!} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) O_2^{--} (z, \bar{z}) |0\rangle_1^{\otimes 2},$$

where we have mapped the one-point function from the cylinder to the plane and inserted the appropriate conformal factor.

Upon expanding, we obtain four terms. The one with four supercharge modes evaluates to

$$-\langle O_2^{++}|(G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2}) \frac{L_1^{n-1}}{(n-1)!} (G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2}) O_2^{--}(z, \bar{z})|0\rangle_1^{\otimes 2}$$

$$= -\frac{1}{(n-1)!} \langle O_2^{++}|(G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2}) \Big[(n^2 - 3n) G_{\frac{1}{2}}^{+2} G_{\frac{1}{2}}^{+1} L_1^{n-3} + (n-1) G_{\frac{1}{2}}^{+2} G_{-\frac{1}{2}}^{+1} L_1^{n-2}$$

$$+ (n-1) G_{-\frac{1}{2}}^{+2} G_{\frac{1}{2}}^{+1} L_1^{n-2} + G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} L_1^{n-1} \Big] O_2^{--}(z, \bar{z})|0\rangle_1^{\otimes 2}$$

$$= \frac{(n^2 + 2n + 1)}{(n-1)!} \langle O_2^{++}|L_1^{n-1} O_2^{--}(z, \bar{z})|0\rangle_1^{\otimes 2}, \tag{5.7}$$

where we have used the anomaly-free algebra (2.4). By similar standard manipulations the other three terms evaluate to

$$-\frac{1}{2(n-1)!} \langle O_2^{++}|L_1^n L_{-1} O_2^{--}(z, \bar{z})|0\rangle = \frac{n(n+1)}{(n-1)!} \langle O_2^{++}|L_1^{n-1} O_2^{--}(z, \bar{z})|0\rangle_1^{\otimes 2}. \tag{5.8}$$

We note that the commutation relation between L_1^n and a primary with left dimension h is

$$[L_1^n, O_h] = \sum_{m=0}^{n} \frac{n!}{(n-m)!m!} w^{n+m} \frac{(2h+n-1)!}{(2h+m-1)!} \partial^m O_h. \tag{5.9}$$

Collecting all terms and using (5.6) and (5.9) we obtain

$$\langle O_2^{++}| \left(-G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2} + \frac{1}{2} J_0^- L_{+1} \right) \frac{L_1^{n-1}}{n-1!} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) O_2^{--} (\tilde{t}, \tilde{y}) |0\rangle_1^{\otimes 2}$$

$$= \frac{n(n+1)(n+2)}{2} e^{i((n+2)\tilde{t} + n\tilde{y})}. \tag{5.10}$$

This basic process describes the contribution to the expectation value of the single-particle operator when it acts on two copies of the CFT vacuum. We now use this to

– 25 –

compute the effect of the single-particle operator acting on the full state (5.5). To do so, we must compute a combinatorial factor, as we shall describe momentarily. Combining the amplitude in (5.10) with this combinatorial factor, the relevant contribution is represented by

$$GG\tilde{O}_2^{(0,-)}(\tilde{t},\tilde{y})\left[\left(|0\rangle_1^{N-2p}\right)\left(\frac{L_{-1}^{\mathsf{n}-1}}{(\mathsf{n}-1)!}\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2}+\frac{1}{2}J_0^+L_{-1}\right)|O_2^{--}\rangle\right)^p\right] \tag{5.11}$$

$$=\frac{\sqrt{2}\,(p+1)}{\sqrt{3}N}e^{i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})}\left[\left(|0\rangle_1^{N-2p-2}\right)\left(\frac{L_{-1}^{\mathsf{n}-1}}{(\mathsf{n}-1)!}\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2}+\frac{1}{2}J_0^+L_{-1}\right)|O_2^{--}\rangle\right)^{p+1}\right].$$

The factors in (5.11) arise as follows (cf. [29]). The norm of the states on each side of the equality must match. The factor of $\sqrt{2}/(\sqrt{3}N)$ comes from the normalization of $\tilde{O}_2$, (4.18). The factor of $(p+1)$ is a combination of a combinatorial factor and the n-dependent factor in (5.10). The norm of the state on the left-hand side (l.h.s.) of the equation is given by the norm of the state in square brackets (on the l.h.s.) multiplied by the number of ways in which the single-particle operator can act on any two of the $N-2p$ vacua, namely $\binom{N-2p}{2}$. The norm of the states in square brackets on both the l.h.s. and r.h.s. are given in eq. (2.31). These factors all combine with the n-dependent prefactor in (5.10) to give (5.11).

When we compute the amplitude with the full coherent state (5.5), we obtain an additional factor of A^2/B due to the fact that the process annihilates two A-type strands and creates one B-type strand (cf. [29, eq. (4.19)]). Furthermore, we work at large N and with coherent states in which the average $\bar{p}$ is of order N, so we approximate $(\bar{p}+1)\simeq\bar{p}$. We then obtain the amplitude

$$\langle GG\tilde{O}_2^{(0,-)}\rangle=\frac{\sqrt{2}\bar{p}}{\sqrt{3}N}\frac{A^2}{B}e^{i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})}. \tag{5.12}$$

From eq. (2.33), $\bar{p}$ is of order B^2. Using eqs. (2.33) and (3.9), we obtain the final result for the correlator,

$$\langle GG\tilde{O}_2^{(0,-)}\rangle=\sqrt{N}\frac{\mathsf{a}^2\mathsf{b}}{2\sqrt{3}}e^{i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})}. \tag{5.13}$$

Since $GG\tilde{O}_2^{(0,+)}=(GG\tilde{O}_2^{(0,-)})^\dagger$, the expectation value of the operator $GG\tilde{O}_2^{(0,+)}$ must also be non-vanishing. We thus obtain

$$\langle GG\tilde{O}_2^{(0,+)}\rangle=\langle GG\tilde{O}_2^{(0,-)}\rangle^*=\sqrt{N}\frac{\mathsf{a}^2\mathsf{b}}{2\sqrt{3}}e^{-i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})}. \tag{5.14}$$

In order to evaluate the coefficient α in eq. (5.4), we must perform an asymptotic expansion of the gauge-invariant combination $\boldsymbol{Z}_{k=1}^7$, given in eq. (4.11). The supergravity solution with modes $(\mathsf{k},\mathsf{m},\mathsf{n},\mathsf{q})=(2,1,\mathsf{n},1)$ is characterized by $U_{k=1}^7=0$, which implies that the field $Z_{k=1}^7$ that appears in the Kaluza-Klein reduction (3.12) coincides with the

gauge-invariant combination $\mathbf{Z}_{k=1}^7$. We thus obtain (recall $\mathsf{n} \geq 1$)

$$\mathbf{Z}_{k=1}^{7(--)} = Z_{k=1}^{7(--)} = -\mathsf{a}^2\mathsf{b}\, e^{-i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})} \frac{\tilde{r}^{\mathsf{n}-1}}{(\tilde{r}^2+1)^{\mathsf{n}/2+2}} \Big(+ id\tilde{r} + \tilde{r}(\tilde{r}^2+1)(d\tilde{t}+d\tilde{y}) \Big),$$

$$\mathbf{Z}_{k=1}^{7(+-)} = Z_{k=1}^{7(+-)} = -\mathsf{a}^2\mathsf{b}\, e^{i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})} \frac{\tilde{r}^{\mathsf{n}-1}}{(\tilde{r}^2+1)^{\mathsf{n}/2+2}} \Big(- id\tilde{r} + \tilde{r}(\tilde{r}^2+1)(d\tilde{t}+d\tilde{y}) \Big),$$

(5.15)

where we recall that the rescaling in eq. (4.12) has been performed, and where the superscript $(\pm-)$ indicates that the field couples to the $Y^{(\pm-)}$ harmonic respectively. The large $\tilde{r}$ expansion of (5.15) gives

$$\left[Z_{k=1}^{7(--)} \right] = -\mathsf{a}^2\mathsf{b}\, e^{-i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})}, \qquad \left[Z_{k=1}^{7(+-)} \right] = -\mathsf{a}^2\mathsf{b}\, e^{i((\mathsf{n}+2)\tilde{t}+\mathsf{n}\tilde{y})}. \tag{5.16}$$

Using the ansatz for the holographic dictionary in eq. (5.4), along with eqs. (5.13), (5.14) and (5.16), we obtain

$$\alpha = -\frac{N^{1/2}}{2\sqrt{3}}. \tag{5.17}$$

We observe that the value of α is independent of the quantum numbers that specify the state, as required.

5.2 Holographic test of general non-supercharged superstrata

We now compute the coefficient β defined in eq. (5.4) and find that $\alpha = \beta$, verifying the expectation discussed below eq. (5.4). In doing so, we will also test the coiffuring proposal for general multi-mode superstrata developed in [33, 34]. In the present subsection we test the non-supercharged part of this proposal, and in the following subsection we shall test the hybrid supercharged plus non-supercharged part.

As described in the Introduction, "coiffuring" refers to imposing a set of algebraic relations on the parameters in the supergravity solution, required for smoothness [23–25]. A significant achievement of [33] was a proposal for particular families of multi-mode superstrata that had proven impossible to construct with previous methods, as we describe in more detail below. This came as part of a more general proposal for the coiffuring of multi-mode superstrata, where the modes could be either of supercharged or non-supercharged type. This was expressed in a more general holomorphic formalism in [34].

From the supergravity point of view, coiffuring relations are often not easy to give an interpretation to, beyond being a consequence of requiring solutions to be smooth. By contrast, holographic calculations can give a microscopic interpretation to coiffuring relations, as has been done in [29, 31]. We will now test the new type of coiffuring relation proposed in [33, 34], and we will find perfect agreement.

We focus on the family of $(1, \mathsf{m}, \mathsf{n})$ multimode non-supercharged superstrata constructed in [35, appendix, D]. This family of solutions is given in terms of two holomorphic functions F_0, F_1 of a complex variable ξ which is related to the standard six-dimensional coordinates in section 3.3 via

$$\xi = \frac{\tilde{r}}{\sqrt{\tilde{r}+1}} e^{i(\tilde{t}+\tilde{y})}. \tag{5.18}$$

We consider the two-mode solution in which both F_0 and F_1 consist of a single mode,

$$F_0(\xi) = b\xi^{\mathsf{n}_b}, \qquad F_1(\xi) = d\xi^{\mathsf{n}_d}. \tag{5.19}$$

The explicit supergravity solution is given in full detail in [35, eqs. (D.15)–(D.29)] and we shall not reproduce it here.

The proposed family of dual CFT states is

$$\sum_{p,q} \left(A|0\rangle_1\right)^{N-p-q} \left(B \frac{1}{\mathsf{n}_b!} L_{-1}^{\mathsf{n}_b} |O^{--}\rangle\right)^p \left(D \frac{1}{\mathsf{n}_d!} J_0^+ L_{-1}^{\mathsf{n}_d} |O^{--}\rangle\right)^q. \tag{5.20}$$

The supergravity mode parameters (b,d), the CFT parameters (B,D), and the convenient parameters (b,d) are related as before by eq. (3.9), which in this specific case takes the form

$$\frac{B}{\sqrt{N}} = \frac{R}{\sqrt{2\,Q_1 Q_5}} b = \frac{\mathsf{b}}{\sqrt{2}}, \qquad \frac{D}{\sqrt{N}} = \frac{R}{\sqrt{2\,Q_1 Q_5}} d = \frac{\mathsf{d}}{\sqrt{2}}. \tag{5.21}$$

When $\mathsf{n}_b \neq \mathsf{n}_d$, this is a class of multimode superstrata where $(\mathsf{k}_b \mathsf{m}_d - \mathsf{k}_d \mathsf{m}_b)(\mathsf{k}_b \mathsf{n}_d - \mathsf{k}_d \mathsf{n}_b) \neq 0$. This is the class of superstrata that had evaded construction since [21] until the proposal of [33]. We have taken $k_b = k_d = 1$, since higher values of k_b and/or k_d would require extending the holographic dictionary even further beyond the sector of conformal dimensions that we consider in this work. With $k_b = k_d = 1$, this family of states essentially contains the full class of states in which $(\mathsf{k}_b \mathsf{m}_d - \mathsf{k}_d \mathsf{m}_b)(\mathsf{k}_b \mathsf{n}_d - \mathsf{k}_d \mathsf{n}_b) \neq 0$: since $\mathsf{m} \leq \mathsf{k}$, we must have one excited strand with $\mathsf{m} = 0$ and one with $\mathsf{m} = 1$. Furthermore, once one has control over the general two-mode family (5.19)–(5.20), adding further modes of the same type is a straightforward generalization in both supergravity and CFT.

We start from the CFT side. We shall see that when $\mathsf{n}_b \neq \mathsf{n}_d$, the operator $GG\tilde{\Sigma}_3^{(0,a)}$ has a non-vanishing expectation value in this state. Expanding the definition of $GG\tilde{\Sigma}_3^{(0,a)}$, we have

$$\langle GG\tilde{\Sigma}_3^{(0,a)} \rangle \equiv \left\langle \frac{1}{\sqrt{3}} (G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1}) \tilde{\Sigma}_3^{-,a} \right\rangle = \tag{5.22}$$

$$\left\langle \frac{\sqrt{3}}{2} (G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1}) \left[\left(\frac{\Sigma_3}{N^{\frac{3}{2}}} - \frac{\Omega}{3N^{\frac{1}{2}}} \right) + \frac{1}{N^{\frac{1}{2}}} \left(-\frac{2}{3} (\Sigma_2 \cdot \Sigma_2) + \frac{1}{6} (O \cdot O) + \frac{1}{3} (J \cdot \bar{J}) \right) \right]^{-,a} \right\rangle.$$

This expectation value is sourced only by the double-trace term (we suppress overall factors and restore them at the end)

$$GG(O \cdot O) = \frac{1}{N} \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) \sum_{r,s} O_r^{--} O_s^{--}. \tag{5.23}$$

Expanding this product, the GG combination of modes can act either on strand r or strand s, giving rise to eight terms. Two of these terms give rise to fermionic strands, and so do not contribute to the correlator. The remaining terms can be written as

$$\frac{1}{N} \sum_{r,s} \left[-\frac{1}{2} O_r^{--} (J_0^+ L_{-1} O_s^{--}) + \frac{1}{2} (L_{-1} O_r^{--}) (J_0^+ O_s^{--}) \right], \tag{5.24}$$

where we have used the relation $(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + J_0^+ L_{-1}) O^{--} = 0$, which holds because O^{--} is a scalar operator of dimension one.

The operator in eq. (5.24) transforms two copies of the vacuum into one strand of type $|1, 0, \mathsf{n}_b, 0\rangle$ and another of type $|1, 1, \mathsf{n}_d, 0\rangle$. We now compute the contribution of this fundamental process. After doing so, we will again dress it with the appropriate combinatorial factor. The initial state is given by two copies of the vacuum $|0\rangle_{r=1}|0\rangle_{r=2}$, which can be transformed into the state $|1, 0, \mathsf{n}_b, 0\rangle_{r=1}|1, 1, \mathsf{n}_d, 0\rangle_{r=2}$ or $|1, 0, \mathsf{n}_d, 0\rangle_{r=1}|1, 1, \mathsf{n}_b, 0\rangle_{r=2}$: the two processes contribute with equal amplitudes, so we compute only one of them, and multiply the result by two. For ease of notation (and to avoid confusion with the twist-two operator O_2), we shall abbreviate the subscripts $r = 1, 2$ to $(1), (2)$.

We proceed to compute

$$
{}_{(1)}\langle O^{++}|\frac{L_1^{\mathsf{n}_b}}{\mathsf{n}_b!} \, {}_{(2)}\langle O^{++}|\frac{L_1^{\mathsf{n}_d}}{\mathsf{n}_d!} J_0^- \left[\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) O_{(1)}^{--} O_{(2)}^{--} \right] (\tilde{t}, \tilde{y}) \, |0\rangle_{(1)}|0\rangle_{(2)} \quad (5.25)
$$

$$
= z^2 \bar{z} \Bigg[-\frac{1}{2} \, {}_{(1)}\langle O^{++}|\frac{L_1^{\mathsf{n}_b}}{\mathsf{n}_b!} O_{(1)}^{--} (z, \bar{z}) \, |0\rangle_{(1)} \, {}_{(2)}\langle O^{++}|\frac{L_1^{\mathsf{n}_d}}{\mathsf{n}_d!} J_0^- \left(J_0^+ L_{-1} O_{(2)}^{--} \right) (z, \bar{z}) \, |0\rangle_{(2)}
$$

$$
+ \frac{1}{2} \, {}_{(1)}\langle O^{++}|\frac{L_1^{\mathsf{n}_b}}{\mathsf{n}_b!} \left(L_{-1} O_{(1)}^{--} \right) (z, \bar{z}) \, |0\rangle_{(1)} \, {}_{(2)}\langle O^{++}|\frac{L_1^{\mathsf{n}_d}}{\mathsf{n}_d!} J_0^- \left(J_0^+ O_{(2)}^{--} \right) (z, \bar{z})|0\rangle_{(2)} \Bigg].
$$

Using standard manipulations, together with the commutation relation in eq. (5.9) and the anomaly-free algebra (2.4), one can rewrite this amplitude as

$$
{}_{(1)}\langle O^{++}|\frac{L_1^{\mathsf{n}_b}}{\mathsf{n}_b!} \, {}_{(2)}\langle O^{++}|\frac{L_1^{\mathsf{n}_d}}{\mathsf{n}_d!} J_0^- \left[\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1} \right) O_{(1)}^{--} O_{(2)}^{--} \right] (\tilde{t}, \tilde{y}) \, |0\rangle_{(1)}|0\rangle_{(2)}
$$

$$
= \frac{1}{2} \left[-(\mathsf{n}_d + 1) + (\mathsf{n}_b + 1) \right] e^{i(\mathsf{n}_b + \mathsf{n}_d + 2)\tilde{t} + i(\mathsf{n}_b + \mathsf{n}_d)\tilde{y}} \quad (5.26)
$$

$$
= \frac{1}{2} (\mathsf{n}_b - \mathsf{n}_d) \, e^{i(\mathsf{n}_b + \mathsf{n}_d + 2)\tilde{t} + i(\mathsf{n}_b + \mathsf{n}_d)\tilde{y}} .
$$

Already we see that the expectation value is non-zero only when $\mathsf{n}_b \neq \mathsf{n}_d$.

We now dress this with the combinatorial factor as before. The relevant contribution can be represented as

$$
(GG (O \cdot O))^{0-} (\tilde{t}, \tilde{y}) \left[\left(|0\rangle_1^{N-p-q} \right) \left(\frac{1}{\mathsf{n}_b!} L_{-1}^{\mathsf{n}_b} |O^{--}\rangle \right)^p \left(\frac{1}{\mathsf{n}_d!} J_0^+ L_{-1}^{\mathsf{n}_d} |O^{--}\rangle \right)^q \right]
$$

$$
= \left((\mathsf{n}_b - \mathsf{n}_d) \, e^{i(\mathsf{n}_b + \mathsf{n}_d + 2)\tilde{t} + i(\mathsf{n}_b + \mathsf{n}_d)\tilde{y}} \right) \frac{(p+1)(q+1)}{N} \quad (5.27)
$$

$$
\times \left[\left(|0\rangle_1^{N-p-q-2} \right) \left(\frac{1}{\mathsf{n}_b!} L_{-1}^{\mathsf{n}_b} |O^{--}\rangle \right)^{p+1} \left(\frac{1}{\mathsf{n}_d!} J_0^+ L_{-1}^{\mathsf{n}_d} |O^{--}\rangle \right)^{q+1} \right] .
$$

The first term on the r.h.s. is the contribution of the fundamental process, given by twice the result in eq. (5.26). The second term comes requiring that the normalization of the two sides of the equality are the same. The expectation value of the single-particle operator is then obtained by combining eq. (5.27) with the normalization factors we suppressed from eq. (5.22), along with the relation between the CFT and the supergravity coefficients in

eq. (5.21). This gives

$$
\begin{aligned}
\langle GG\tilde{\Sigma}_3^{0-}\rangle &= \frac{1}{8\sqrt{3}N^{1/2}}\langle\,(GG(O\cdot O))^{0-}\,\rangle \\
&= \frac{1}{4\sqrt{3}}\frac{\bar{p}\,\bar{q}}{N^{3/2}}\frac{A^2}{BD}(\mathsf{n}_b-\mathsf{n}_d)e^{i(\mathsf{n}_b+\mathsf{n}_d+2)\tilde{t}+i(\mathsf{n}_b+\mathsf{n}_d)\tilde{y}} \\
&= \frac{\sqrt{N}}{8\sqrt{3}}\mathsf{a}^2\,\mathsf{bd}(\mathsf{n}_b-\mathsf{n}_d)e^{i(\mathsf{n}_b+\mathsf{n}_d+2)\tilde{t}+i(\mathsf{n}_b+\mathsf{n}_d)\tilde{y}}\,, \\
\langle GG\tilde{\Sigma}_3^{0+}\rangle &= (\langle GG\tilde{\Sigma}_3^{0-}\rangle)^* \\
&= \frac{\sqrt{N}}{8\sqrt{3}}\mathsf{a}^2\,\mathsf{bd}(\mathsf{n}_b-\mathsf{n}_d)e^{-i(\mathsf{n}_b+\mathsf{n}_d+2)\tilde{t}-i(\mathsf{n}_b+\mathsf{n}_d)\tilde{y}}\,.
\end{aligned}
\tag{5.28}
$$

These CFT expectation values are holographically encoded in the expansion of the gauge-invariant vector field $\boldsymbol{Z}^6_{k=1}$. The supergravity solution is obtained combining [35, eqs. (D.15)-(D.29)] with the holomorphic functions F_0 and F_1 in eq. (5.19). One can check that, as in the example studied in section 5.1, the AdS$_3$ scalar field $U^6_{k=1}$ vanishes on this background. Eq. (4.11) then implies that $\boldsymbol{Z}^6_{k=1} = Z^6_{k=1}$. The vector fields are given by

$$
Z^{6(+-)}_{k=1} = -\frac{\mathsf{a}^2\,\mathsf{bd}}{4}(\mathsf{n}_b-\mathsf{n}_d)e^{i((\mathsf{n}_b+\mathsf{n}_d+2)\tilde{t}+(\mathsf{n}_b+\mathsf{n}_d)\tilde{y})}\frac{\tilde{r}^{\mathsf{n}_b+\mathsf{n}_d-1}}{(1+\tilde{r}^2)^{\frac{(\mathsf{n}_b+\mathsf{n}_d+4)}{2}}}\left(-id\tilde{r}+\tilde{r}(\tilde{r}^2+1)(d\tilde{t}+d\tilde{y})\right),
$$

$$
Z^{6(--)}_{k=1} = -\frac{\mathsf{a}^2\,\mathsf{bd}}{4}(\mathsf{n}_b-\mathsf{n}_d)e^{-i((\mathsf{n}_b+\mathsf{n}_d+2)\tilde{t}+(\mathsf{n}_b+\mathsf{n}_d)\tilde{y})}\frac{\tilde{r}^{\mathsf{n}_b+\mathsf{n}_d-1}}{(1+\tilde{r}^2)^{\frac{(\mathsf{n}_b+\mathsf{n}_d+4)}{2}}}\left(id\tilde{r}+\tilde{r}(\tilde{r}^2+1)(d\tilde{t}+d\tilde{y})\right),
$$

$$
\tag{5.29}
$$

where we have used the normalization in eq. (4.12) and the relation between the CFT and the supergravity modes in eq. (5.21).

The coefficient β in eq. (5.4) is obtained performing the asymptotic expansion (5.3) of the three-dimensional vectors (5.29) and comparing it with the CFT result in eq. (5.28). We find that

$$
\beta = -\frac{N^{1/2}}{2\sqrt{3}}\,,
\tag{5.30}
$$

thus explicitly verifying that $\alpha = \beta$.

We emphasize again that this amplitude is non-vanishing only when $n_b \neq n_d$. This can be interpreted as the CFT telling us that for the set of states (5.20), when $n_b \neq n_d$ the bulk solution must involve an extra field that is not turned on when $n_b = n_d$. This is precisely the field that was introduced in the more general coiffuring proposal of [33]. So we have seen that a very non-trivial holographic test of this more general coiffuring is passed.

5.3 Holographic test of hybrid supercharged superstrata

At this point we have fixed all coefficients of the holographic dictionary in the sector in which we work. We now use the dictionary to make a precision holographic test of a "hybrid" superstratum solution that combines non-supercharged and supercharged elements. Our computation tests both the proposed holographic dictionary for hybrid superstrata and the supergravity coiffuring procedure for combining non-supercharged and supercharged modes of [33, 34].

This test also serves as an additional non-trivial cross-check of the operator mixing in the dictionary. Note that this operator mixing has already passed thorough holographic tests [31]. The tests in [31] were performed in a different basis to the single-particle basis, however our results are equivalent, as demonstrated in appendix D. The test that follows involves a non-trivial and delicate cancellation between a set of terms.

We consider the multi-mode hybrid superstratum composed of modes $(\mathsf{k}_1, \mathsf{m}_1, \mathsf{n}_1, \mathsf{q}_1) = (2, 1, 0, 0)$, and $(\mathsf{k}_2, \mathsf{m}_2, \mathsf{n}_2, \mathsf{q}_2) = (2, 1, 1, 1)$, constructed in [34, appendix (B.1)]. There, the solution was given in terms of two holomorphic functions F and S of the complex variable ξ defined in eq. (5.18). In our conventions, we have

$$F(\xi) = b, \qquad S(\xi) = d\frac{\xi}{6}. \tag{5.31}$$

The supergravity solution is given explicitly in [34, eqs. (6.8), (6.9), (B.1)–(B.12)] and so we shall not reproduce it here. Performing the Kaluza-Klein reduction of this background, one obtains that the gauge-invariant field $\boldsymbol{Z}^6_{k=1}$ in eq. (4.11) vanishes. The holographic dictionary in eq. (5.4) then predicts that the expectation values of the single-particle operator $(GG\tilde{\Sigma}_3)^{0a}$ on the dual CFT state must vanish. We will now explicitly check that this is indeed the case.

The proposed dual CFT state is

$$\sum_{p,q} (A|0\rangle_1)^{N-2p-2q} \left(B J_0^+ |O_2^{--}\rangle\right)^p \left(D \left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1}\right) |O_2^{--}\rangle\right)^q. \tag{5.32}$$

The CFT coefficients (B, D) and the supergravity Fourier coefficients (b, d) in (5.31) are again related via eq. (3.9), which in the present example becomes

$$\frac{B}{\sqrt{N}} = \frac{R}{2\sqrt{2\,Q_1Q_5}}\,b, \qquad \frac{D}{\sqrt{N}} = \frac{R}{3\sqrt{2\,Q_1Q_5}}\,d. \tag{5.33}$$

The only $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ component of the single-particle operator $(GG\tilde{\Sigma}_3)^{0a}$ that can have a non-vanishing expectation value is $(GG\tilde{\Sigma}_3)^{00}$. Indeed, this single-particle operator contains three operators that have non-zero expectation values in the state (5.32): the single-trace operators $(GG\Sigma_3)^{00}$, $(GG\Omega)^{00}$ and the double-trace operator $(GGJ\bar{J})^{00}$.

First, we compute the expectation value of $(GG\Sigma_3)^{00}$, which arises from the basic process in which one B-type strand plus one A-type vacuum strand are converted into one D-type strand plus one A-type vacuum strand:

$$\left(_1\langle 0|\langle O_2^{++}| \left(-G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2} + \frac{1}{2} J_0^- L_{+1}\right)\right) \left(\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2} J_0^+ L_{-1}\right) \Sigma_3^{-0}\right) \left(J_0^+ |O_2^{--}\rangle|0\rangle_1\right)$$

$$= 3\,_1\langle 0|\langle O_2^{++}|\Sigma_3^{-0} J_0^+ |O_2^{--}\rangle|0\rangle_1$$

$$= -3 \left(\sqrt{2}\,_1\langle 0|\langle O_2^{++}|\Sigma_3^{00}|O_2^{--}\rangle|0\rangle - \,_1\langle 0|\langle O_2^{++}|J_0^+ \Sigma_3^{-,0}|O_2^{--}\rangle|0\rangle_1\right)$$

$$= -3\sqrt{2}\,_1\langle 0|\langle O_2^{++}|\Sigma_3^{00}|O_2^{--}\rangle|0\rangle_1$$

$$= -\frac{1}{\sqrt{2}}. \tag{5.34}$$

In the equation above, following the prescription after eq. (2.12)), the descendant is normalized so that it has the same norm as the highest weight state, that is $\Sigma_3^{00} = \frac{1}{\sqrt{2}}[J_0^+, \Sigma_3^{-0}]$. This three-point function is independent of the insertion point of the light operator on the cylinder. The last equality follows from

$$_1\langle 0|\langle O_2^{++}|\Sigma_3^{00}|O_2^{--}\rangle|0\rangle_1 = \frac{1}{6}\,, \tag{5.35}$$

which can be derived using the covering-space method of Lunin-Mathur [71, 78] analogously to the computation in [31, appendix (B.2)].

We now use the basic amplitude in eq. (5.34) to compute the expectation value of the single-trace operator $(GG\Sigma_3)^{00}$ on the full coherent state (5.32). The relevant contribution can be represented by

$$(GG\Sigma_3)^{00} \left[\left(|0\rangle_1^{N-2p-2q}\right) \left(J_0^+|O_2^{--}\rangle\right)^p \left(\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+L_{-1}\right)|O_2^{--}\rangle\right)^q \right]$$

$$= -\frac{2\sqrt{2}}{3}\left(N - 2\left(p+q\right)\right)\left(q+1\right) \tag{5.36}$$

$$\times \left[\left(|0\rangle_1^{N-2(p+q)}\right) \left(J_0^+|O_2^{--}\rangle\right)^{p-1} \left(\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+L_{-1}\right)|O_2^{--}\rangle\right)^{q+1} \right].$$

The overall factor on the r.h.s. is obtained as before by requiring that the norms on the two sides of the equality sign are the same. In doing so, we combined eq. (5.34) with a combinatorial factor that represents the number of ways that the operator $(GG\Sigma_3)^{00}$ can act on the state on the l.h.s. In particular, it can act on any of the $(N - 2p - 2q)p$ pairs of strands $|0\rangle_1(J_0^+|O^{--}\rangle_2)$ and can cut-and-join these in two inequivalent ways.

We now use eq. (2.33) to express the average number of strands in a coherent state with the CFT coefficients A, B, D. We obtain that, in the large N-limit,

$$\langle(GG\Sigma_3)^{00}\rangle = -\frac{2\sqrt{2}}{3}(N - 2\bar{p} - 2\bar{q})\,\bar{q}\,\frac{B}{D} = -\sqrt{2}A^2BD\,. \tag{5.37}$$

Second, we compute the expectation value of the single-trace operator $GG\Omega$, which arises from the basic process in which a B-type strand is converted into a D-type strand:

$$\left(\langle O_2^{++}|\left(-G_{\frac{1}{2}}^{-1}G_{\frac{1}{2}}^{-2} + \frac{1}{2}J_0^-L_{+1}\right)\right)\left(\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+L_{-1}\right)\Omega^{-0}\right)\left(J_0^+|O_2^{--}\rangle\right)$$

$$= 3\langle O_2^{++}|\Omega_3^{-0}J_0^+|O_2^{--}\rangle \tag{5.38}$$

$$= -3\sqrt{2}\langle O_2^{++}|\Omega^{00}|O_2^{--}\rangle$$

$$= -3\sqrt{2}\,.$$

The last equality follows from [31, eq. (5.40)]. To compute the expectation value of the operator on the full coherent state we again combine the above result with a combinatorial factor. Doing so, we obtain

$$(GG\Omega)^{00} \left[\left(|0\rangle_1^{N-2p-2q}\right) \left(J_0^+|O_2^{--}\rangle\right)^p \left(\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+L_{-1}\right)|O_2^{--}\rangle\right)^q \right] \tag{5.39}$$

$$= -2\sqrt{2}\,(q+1) \left[\left(|0\rangle_1^{N-2p-2q}\right) \left(J_0^+|O_2^{--}\rangle\right)^{p-1} \left(\left(G_{-\frac{1}{2}}^{+1}G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+L_{-1}\right)|O_2^{--}\rangle\right)^{q+1} \right].$$

Here we have used the fact that the operator $(GG\Omega)$ can act on any of the p strands of type $J_0^+|O^{--}\rangle_2$, and we have matched the norms on the two sides of the equation. The expectation value on the full coherent state then follows from eq. (2.33). We obtain

$$\langle (GG\Omega)^{00} \rangle = -\frac{2\sqrt{2}\,\bar{q}\,B}{D} = -\frac{3\sqrt{2}}{N}BD(A^2 + 3D^2 + 2B^2)\,. \tag{5.40}$$

For later convenience, in the last equality we have used the strand budget constraint (2.34).

The final operator that contributes to the expectation value of $(GG\tilde{\Sigma}_3)^{00}$ is

$$\left(GGJ\bar{J} \right)^{00} = \sqrt{2}\left(\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+ L_{-1} \right) J^- \right) \bar{J}^3\,, \tag{5.41}$$

where the factor of $\sqrt{2}$ follows from the normalization of the $SU(2)_R$ descendant.

This operator contributes through the basic process in which a B-type strand is converted into a D-type strand. This process is mediated only by the holomorphic part of the operator $(GGJ\bar{J})^{00}$. Both B-type and D-type strands are eigenstates of $\bar{J}_0^3$, so we treat this contribution separately below. Focusing for now on the holomorphic part, we have the amplitude

$$\begin{aligned}
&\left(\langle O_2^{++}| \left(-G_{\frac{1}{2}}^{-1} G_{\frac{1}{2}}^{-2} + \frac{1}{2}J_0^- L_{+1} \right) \right) \left(\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+ L_{-1} \right) J^- \right) \left(J_0^+|O_2^{--}\rangle \right) \\
&= -6\,\langle O_2^{++}|J^3|O_2^{--}\rangle \\
&= 6\,.
\end{aligned} \tag{5.42}$$

We now compute the expectation value of the multi-trace operator in the full coherent state (5.32). We include the antiholomorphic part at this point. The relevant contribution can be represented by

$$\begin{aligned}
&\left(GGJ\bar{J} \right)^{00} \left[\left(|0\rangle_1^{N-2p-2q} \right) \left(J_0^+|O_2^{--}\rangle \right)^p \left(\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+ L_{-1} \right) |O_2^{--}\rangle \right)^q \right] \\
&= -4\sqrt{2}\,(p+q)\,(q+1) \left[\left(|0\rangle_1^{N-2p-2q} \right) \left(J_0^+|O_2^{--}\rangle \right)^{p-1} \left(\left(G_{-\frac{1}{2}}^{+1} G_{-\frac{1}{2}}^{+2} + \frac{1}{2}J_0^+ L_{-1} \right) |O_2^{--}\rangle \right)^{q+1} \right]\,.
\end{aligned} \tag{5.43}$$

The prefactor on the r.h.s. is a combination of the basic amplitude (5.42), the action of $\bar{J}$, and a combinatorical factor. The combinatorics has two parts: first, the operator $\bar{J}$ can act either on one of the p strands of type $J_0^+|O_2^{--}\rangle$ or on one of the q strands of type $GG|O_2^{--}\rangle$. Second, the operator GGJ can only act upon strands of type $GG|O_2^{--}\rangle$. Using eqs. (2.33) and (2.34) we obtain

$$\left\langle \left(GGJ\bar{J} \right)^{00} \right\rangle = -\frac{4\sqrt{2}\bar{q}\,(\bar{p}+\bar{q})\,B}{D} = -6\sqrt{2}BD\left(\frac{3}{2}D^2 + B^2 \right)\,. \tag{5.44}$$

Finally, we combine the three contributions in eqs. (5.37), (5.40) and (5.44) using the definition of the single-particle operator $GG\tilde{\Sigma}$ in eq. (4.18) to obtain the anticipated cancellation:

$$\langle GG\tilde{\Sigma}_3^{00} \rangle = 0\,. \tag{5.45}$$

This result agrees with the vanishing of the dual AdS$_3$ field $\boldsymbol{Z}^6_{k=1}$ in the proposed dual supergravity solution. Thus we see that the proposed holographic dictionary for hybrid superstrata has passed a non-trivial test. This computation also represents a non-trivial cross-check of the operator mixing involved in the single-particle operator dual to the AdS$_3$ vector field $\boldsymbol{Z}^6_{k=1}$.

5.4 Summary of precision holographic dictionary

For convenient reference we record here a summary of the precision holographic dictionary for single-particle scalar operators of dimension one and two, together with our new entries for the superdescendant operators $GG\tilde{\Sigma}_3$, $GG\tilde{O}_2$.

$$
\begin{aligned}
\frac{1}{\sqrt{N}}\langle J^{\pm}\rangle &= -\frac{\sqrt{N}}{\sqrt{2}}\Big[A^{\mp(+)}_{k=1}\Big], & \frac{1}{\sqrt{N}}\langle \bar{J}^{\pm}\rangle &= -\frac{\sqrt{N}}{\sqrt{2}}\Big[A^{\mp(-)}_{k=1}\Big], \\
\frac{1}{\sqrt{N}}\langle J^3\rangle &= -\frac{\sqrt{N}}{2}\Big[A^{0(+)}_{k=1}\Big], & \frac{1}{\sqrt{N}}\langle \bar{J}^3\rangle &= -\frac{\sqrt{N}}{2}\Big[A^{0(-)}_{k=1}\Big], \\
\frac{\sqrt{2}}{N}\langle \Sigma_2^{\alpha\dot\alpha}\rangle &= (-1)^{\alpha\dot\alpha}\frac{\sqrt{N}}{\sqrt{2}}\Big[s^{(6)(-\alpha,-\dot\alpha)}_{k=1}\Big], & \frac{1}{\sqrt{N}}\langle O^{\alpha\dot\alpha}\rangle &= (-1)^{\alpha\dot\alpha}\frac{\sqrt{N}}{\sqrt{2}}\Big[s^{(7)(-\alpha,-\dot\alpha)}_{k=1}\Big], \\
\langle \tilde{\Omega}^{a,\dot a}\rangle &= -(-1)^{a+\dot a}\frac{\sqrt{N}}{\sqrt{2}}\Big[\tilde{\sigma}^{(-a,-\dot a)}_{k=2}\Big], & & \text{(5.46)} \\
\langle \tilde{\Sigma}_3^{a,\dot a}\rangle &= (-1)^{a+\dot a}\frac{\sqrt{N}}{\sqrt{2}}\Big[s^{(6)(-a,-\dot a)}_{k=2}\Big], & \langle \tilde{O}_2^{a,\dot a}\rangle &= (-1)^{a+\dot a}\frac{\sqrt{N}}{\sqrt{2}}\Big[s^{(7)(-a,-\dot a)}_{k=2}\Big], \\
\langle GG\tilde{\Sigma}_3^{(0,a)}\rangle &= -\frac{\sqrt{N}}{2\sqrt{3}}\Big[Z^{6(-a,-)}_{k=1}\Big], & \langle GG\tilde{O}_2^{(0,a)}\rangle &= -\frac{\sqrt{N}}{2\sqrt{3}}\Big[Z^{7(-a,-)}_{k=1}\Big].
\end{aligned}
$$

6 Discussion

In this paper we have derived the precision holographic dictionary in a new sector of superdescendants of scalar operators of dimension two. In doing so we also expressed the existing precision dictionary in the single-particle basis. Our results for the dictionary in these sectors are summarized in eq. (5.46).

We considered the recent proposal that single-particle supergravity fluctuations around global AdS are holographically dual to half-BPS operators that are orthogonal to all multi-trace operators [51, 52]. We emphasized that this proposal is not sufficient to determine the single-particle basis in the D1-D5 CFT. We thus combined this proposal with the mixing between single-trace operators worked out in [28] to obtain the refined dictionary for operators of dimension one, and scalar operators of dimension two summarized in the first five lines of eq. (5.46). We then derived the new part of the dictionary in the last line of eq. (5.46).

We note that the dictionary in eq. (5.46) has a lot of structure: (i) the normalization coefficients in the various sectors respect the SO(n) symmetry of the supergravity theory; (ii) single-particle states are orthogonal to all multi-particle states; (iii) the mixing between single-trace operators of dimension two is fixed as described above. Furthermore,

the overall normalization of each subsector of the dictionary has been calibrated successively on the well-established holographic description of two-charge microstates with $\mathbb{R}^4$ base polarizations, and non-supercharged superstrata, in [31]. Our computation of the coefficient β in section 5.2 can also be regarded as a calibration on a non-supercharged superstratum, and is consistent with the value of α that we obtained by comparison with a supercharged superstratum.

Having derived the new sector of the dictionary for superdescendant operators, we used this to perform tests of a set of non-supercharged superstrata of particular interest, and a set of 'hybrid' superstrata that involve both supercharged and non-supercharged elements. We found precise agreement between gravity and CFT.

The agreement we find in this work does not prove that the proposal for the dual CFT states of supercharged superstrata is precisely correct. The reason is simply the usual limitation of precision holographic studies: for a given superstratum solution in supergravity, and a given precision involving a finite set of expectation values of light operators, there can be other CFT states that have the same values of those correlators. However, our results provide evidence in support of the proposed holographic description of both non-supercharged and supercharged superstrata, and demonstrate that the proposal for the dual CFT states passes all available state-of-the-art tests.

Our results open up various possibilities for future work. The most obvious ones are to use our dictionary (5.46) to perform precision tests of other families of solutions, and to generalize the dictionary both to primaries of higher dimension and to other superdescendants. Of course, the higher one goes in dimension, the more complex the task of constructing the explicit precision dictionary. Our present extension of the dictionary has allowed us to test the main features of the supercharged and hybrid superstrata that have been constructed to date. Future constructions of more general classes of superstrata, or the desire for input into the construction of such new solutions, may provide particular motivation to expand the dictionary into further new sectors.

Looking beyond three-point functions, there has been much progress on holographic four-point functions in the D1-D5 system in recent years. Heavy-heavy-light-light (HHLL) four-point functions were computed in [79–81], and generalized to correlators in which the heavy operators are simple non-supercharged and supercharged superstrata [82]. This enabled new results on LLLL four-point functions to be derived [83–87]. More recently, HHLL correlators in the Regge limit have been computed [88, 89], including those in which the light operators are multi-trace operators [90]. Our results on resolving the operator mixing among light operators should enable new classes of four-point functions to be studied.

Holography has been an invaluable tool in the development of the fuzzball description of black hole microstates, and in this work we have further enlarged the precision holographic dictionary. We expect that our results should prove useful to further our understanding of the heavy bound states that comprise the quantum description of black holes in String Theory.

Acknowledgments

A Harmonics on S³ and AdS₃

A.1 Harmonics on S₃

A.1.1 Spherical harmonics

The spherical harmonics on S³ are a representation of the isometry group of the three-sphere $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. We will use spherical coordinates in the $\mathbb{R}^4$ base space that are related to the Cartesian coordinates via

$$
\begin{aligned}
x^1 &= r \sin\theta \cos\phi\,, & x^2 &= r \sin\theta \sin\phi\,, \\
x^3 &= r \cos\theta \cos\psi\,, & x^4 &= r \cos\theta \sin\psi\,,
\end{aligned}
\tag{A.1}
$$

where $\theta \in [0, \frac{\pi}{2}]$ and $\psi, \phi \in [0, 2\pi)$. With this coordinate choice, the S³ line element ds_3^2 is given by $ds_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2$. We use conventions in which $\epsilon_{\theta\phi\psi}=1$. The generators of the isometry group of S³, written in terms of the standard SU(2) generators, are

$$
\begin{aligned}
J^{\pm} &= \frac{1}{2}e^{\pm i(\phi+\psi)}\big(\pm\partial_\theta + i\cot\theta\partial_\phi - i\tan\theta\partial_\psi\big)\,, & J^3 &= -\frac{i}{2}(\partial_\phi + \partial_\psi)\,, \\
\bar{J}^{\pm} &= \frac{1}{2}e^{\pm i(\phi-\psi)}\big(\mp\partial_\theta - i\cot\theta\partial_\phi - i\tan\theta\partial_\psi\big)\,, & \bar{J}^3 &= -\frac{i}{2}(\partial_\phi - \partial_\psi)\,,
\end{aligned}
\tag{A.2}
$$

which satisfy the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ algebra:

$$
\begin{aligned}
[J^+, J^-] &= 2J^3\,, & [J^3, J^+] &= J^+\,, & [J^3, J^-] &= -J^-\,, \\
[\bar{J}^+, \bar{J}^-] &= 2\bar{J}^3\,, & [\bar{J}^3, \bar{J}^+] &= \bar{J}^+\,, & [\bar{J}^3, \bar{J}^-] &= -\bar{J}^-\,.
\end{aligned}
\tag{A.3}
$$

The left quadratic Casimir operator is $J^2 = \frac{1}{2}\big(J^+J^- + J^-J^+\big) + \big(J^3\big)^2$, and likewise for $\bar{J}^2$. A state in a representation with principal quantum number j_{su} has J^2 eigenvalue $j_{\mathrm{su}}(j_{\mathrm{su}}+1)$.

Degree k scalar harmonics live in the $(j_{\mathrm{su}}, \bar{j}_{\mathrm{su}}) = (k/2, k/2)$ representation of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$. We denote these by $Y_k^{m,\bar{m}}$, and $(m, \bar{m})$ are the spin charges under $(J^3, \bar{J}^3)$. They satisfy the following Laplace equation:

$$
\Box_{S_3} Y_k^{m_1,\bar{m}_2} = -k(k+2) Y_k^{m_1,\bar{m}_2}\,.
\tag{A.4}
$$

Denoting the volume of S³ by $\Omega_3 = 2\pi^2$, we use normalized spherical harmonics

$$
\int Y_{k_1}^{*m_1,\bar{m}_1} Y_{k_2}^{m_2,\bar{m}_2} = \Omega_3\, \delta_{k_1,k_2} \delta^{m_1,m_1} \delta^{\bar{m}_1,\bar{m}_2}\,.
\tag{A.5}
$$

One can generate the degree k scalar spherical harmonic wavefunctions acting with the lowering operators in (A.2) on the highest-weight wavefunctions, which are

$$
Y_k^{\pm\frac{k}{2},\pm\frac{k}{2}} = \sqrt{k+1}\,\sin^k\theta\, e^{\pm ik\phi}\,.
\tag{A.6}
$$

The degree $k = 1, 2$ normalized scalar spherical harmonics are given by:

$$Y_1^{+\frac{1}{2},+\frac{1}{2}} = \sqrt{2}\sin\theta\, e^{i\phi} \qquad , Y_1^{+\frac{1}{2},-\frac{1}{2}} = \sqrt{2}\cos\theta\, e^{i\psi}\,,$$
$$Y_1^{-\frac{1}{2},+\frac{1}{2}} = -\sqrt{2}\cos\theta\, e^{-i\psi} \quad , \quad Y_1^{-\frac{1}{2},-\frac{1}{2}} = \sqrt{2}\sin\theta\, e^{-i\phi}\,; \tag{A.7}$$

$$\begin{aligned}
Y_2^{+1,+1} &= \sqrt{3}\sin^2\theta\, e^{2i\phi}\,, & Y_2^{+1,0} &= \sqrt{6}\sin\theta\cos\theta\, e^{i(\phi+\psi)}\,, \\
Y_2^{+1,-1} &= \sqrt{3}\cos^2\theta e^{2i\psi}\,, & Y_2^{0,+1} &= -\sqrt{6}\sin\theta\cos\theta\, e^{i(\phi-\psi)}\,, \\
Y_2^{0,0} &= -\sqrt{3}\cos 2\theta\,, & Y_2^{0,-1} &= \sqrt{6}\sin\theta\cos\theta\, e^{-i(\phi-\psi)} \\
Y_2^{-1,+1} &= \sqrt{3}\cos^2\theta\, e^{-2i\psi}\,, & Y_2^{-1,0} &= -\sqrt{6}\sin\theta\cos\theta\, e^{-i(\phi+\psi)}\,, \\
Y_2^{-1,-1} &= \sqrt{3}\sin^2\theta\, e^{-2i\phi}\,.
\end{aligned} \tag{A.8}$$

We also define the following triple overlap,

$$\int Y_k^{M_k,\bar{M}_k} Y_{k_1}^{m_1,\bar{m}_1} \left(Y_{k_2}^{m_2,\bar{m}_2}\right)^* = \Omega_3\; a_{(m_1,\bar{m}_1)(m_2,\bar{m}_2)}^{M_k,\bar{M}_k}\,. \tag{A.9}$$

A.1.2 Vector harmonics

Degree k left vector harmonics live in the $(j_{\text{su}}, \bar{j}_{\text{su}}) = (\frac{k+1}{2}, \frac{k-1}{2})$ representation of $\text{SU}(2)_L \times \text{SU}(2)_R$. We denote these by $Y_{\text{L},k}^{m,\bar{m}}$ where L stands for left; we shall suppress the label L when we write explicit S^3 vector indices (which will be denoted by a, b). Similarly, degree k right vector harmonics have $(j_{\text{su}}, \bar{j}_{\text{su}}) = (\frac{k-1}{2}, \frac{k+1}{2})$ and we denote them by $Y_{\text{R},k}^{m,\bar{m}}$. We shall use $Y_{v,k}^{m,\bar{m}}$ to denote a vector harmonic which can be either right or left. Vector harmonics satisfy

$$\nabla_{S^3}^2 Y_{v,k}^{m,\bar{m}} = -(k^2 + 2k - 1)Y_{v,k}^{m,\bar{m}}\,, \qquad D^a(Y_{v,k}^{m,\bar{m}})_a = 0\,, \tag{A.10}$$

where $\nabla_{S^3}^2 = g^{ab} D_a D_b$ and where D_a is the covariant derivative with Levi-Civita connection.

One could generate the $(1,0)$ and $(0,1)$ vector harmonics by dualizing (A.2), i.e. through $V_a = g_{ab} V^b$. We choose a different normalization for these harmonics by imposing

$$\int (Y_1^{\hat{a}A})_a^* (Y_1^{\hat{b}B})^a = \Omega_3\, \delta^{\hat{a},\hat{b}} \delta^{A,B}\,, \tag{A.11}$$

where $\hat{a}, \hat{b} = \pm, 0$ denote the range of m and $A, B = \pm$ denotes left/right vector harmonics. With this choice, the degree 1 vector spherical harmonics expressed as one-forms are

$$\begin{aligned}
Y_1^{++} &= \frac{1}{\sqrt{2}} e^{i(\phi+\psi)} \left[-i\, d\theta + \sin\theta\cos\theta\, d(\phi - \psi)\right]\,, \\
Y_1^{-+} &= \frac{1}{\sqrt{2}} e^{-i(\phi+\psi)} \left[i\, d\theta + \sin\theta\cos\theta\, d(\phi - \psi)\right]\,, \\
Y_1^{0+} &= -\cos^2\theta\, d\psi - \sin^2\theta\, d\phi\,, \\
Y_1^{+-} &= \frac{1}{\sqrt{2}} e^{i(\phi-\psi)} \left[i\, d\theta - \sin\theta\cos\theta\, d(\phi + \psi)\right]\,, \\
Y_1^{--} &= -\frac{1}{\sqrt{2}} e^{-i(\phi-\psi)} \left[i\, d\theta + \sin\theta\cos\theta\, d(\phi + \psi)\right]\,, \\
Y_1^{0-} &= \cos^2\theta\, d\psi - \sin^2\theta\, d\phi\,.
\end{aligned} \tag{A.12}$$

One can generate a degree k vector harmonic using the SU(2) tensor product decomposition

$$\left(\frac{k}{2}, \frac{k}{2}\right) \otimes (1,0) = \left(\frac{k}{2}+1, \frac{k}{2}\right) \oplus \left(\frac{k}{2}, \frac{k}{2}\right) \oplus \left(\frac{k}{2}-1, \frac{k}{2}\right). \tag{A.13}$$

The highest weight state of $(\frac{k}{2}+1, \frac{k}{2})$ is obtained by simply multiplying the highest weight states of $(\frac{k}{2}, \frac{k}{2})$ and that of $(1,0)$ (the Clebsch-Gordan coefficient in this case is always one). One can then generate all the descendants by acting with the lowering operator J^-. In order to satisfy the generalized normalization condition

$$\int (Y_{v,k_1}^{m_1, \bar{m}_1})_a^* (Y_{v,k_2}^{m_2, \bar{m}_2})^a = \Omega_3 \, \delta^{k_1, k_2} \delta^{m_1, m_2} \, \delta^{\bar{m}_1, \bar{m}_2}, \tag{A.14}$$

one can use the SU(2) algebra to write

$$Y_{\mathrm{L},k}^{\frac{k+1}{2}-m, \frac{k-1}{2}-\bar{m}} = \sqrt{\frac{(k+1-m)!}{(k+1)!m!}} \sqrt{\frac{(k-1-\bar{m})!}{(k-1)!\bar{m}!}} \left[(J^-)^m (\bar{J}^-)^{\bar{m}}, Y_{\mathrm{L},k}^{\frac{k+1}{2}, \frac{k-1}{2}} \right]. \tag{A.15}$$

All the previous discussion proceeds analogously for right vector harmonics. We define the following triple integral:

$$\int Y_k^{(m_k, \bar{m}_k)} (Y_1^{a-})_\mu (Y_1^{b+})^\mu = \Omega_3 f_{(m_k, \bar{m}_k)ab}^{(k)}. \tag{A.16}$$

The explicit value of the components of $f_{(m_k, \bar{m}_k)ab}^{(k)}$, defined in (A.16), that have been used in this paper are

$$f_{(0,0)00}^{(2)} = \frac{1}{\sqrt{3}}, \qquad f_{(1,1)--}^{(2)} = \frac{1}{\sqrt{3}}, \qquad f_{(\pm 1, \pm 1)00}^{(2)} = 0. \tag{A.17}$$

One also has $\epsilon_{abc} D^b (Y_{\mathrm{L},k}^{m, \bar{m}})^c = (k-1)(Y_{\mathrm{L},k}^{m, \bar{m}})_a$ and $\epsilon_{abc} D^b (Y_{\mathrm{R},k}^{m, \bar{m}})^c = -(k-1)(Y_{\mathrm{R},k}^{m, \bar{m}})_a$.

A.1.3 Useful definitions

Due to the non-linearity of the gauge-invariant combinations at higher order, one must project products of harmonics into harmonics of higher order. The following definitions are useful:

(I) $(Y^I)^\mu Y^J = \frac{E^{IJK}}{\Lambda_K} D^\mu Y^K + f^{IJK}(Y^K)^\mu$

(II) $\epsilon_{\mu\nu\rho} D^\sigma Y^I D_\sigma D_\rho Y^J = c_{IJK}^s \epsilon_{\mu\nu\rho} D^\rho Y^K + c_{IJK}^v D_{[\nu} Y_{\mu]}^K$

(III) $Y^{\sigma,I} \epsilon_{\mu\nu\rho} D_\sigma D^\rho Y^J = g_{IJK}^s \epsilon_{\mu\nu\rho} D^\rho Y^K + g_{IJK}^v D_{[\nu} Y_{\mu]}^K$

(IV) $2 D_{[\mu} D^\rho Y^I \epsilon_{\nu]\rho\sigma} D^\sigma Y^J = n_{IJK}^s \epsilon_{\mu\nu\rho} D^\rho Y^K + n_{IJK}^v D_{[\nu} Y_{\mu]}^K$

(V) $2 D_{[\mu} (Y^I)^\rho \epsilon_{\nu]\rho\sigma} D^\sigma Y^J = p_{IJK}^s \epsilon_{\mu\nu\rho} D^\rho Y^K + p_{IJK}^v D_{[\nu} Y_{\mu]}^K$

where I, J, K are the multi-indices defined below eq. (3.11), and where on the right-hand side the index K is summed over.

A.2 Harmonics on AdS$_3$

A.2.1 Scalar harmonics

The spherical harmonics are a representation of the AdS$_3$ isometry group $\mathrm{SO}(2,2) \simeq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We use coordinates where the line element reads:

$$ds^2_{\mathrm{AdS}_3} = -(\tilde{r}^2 + 1)d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 + 1} + \tilde{r}^2 d\tilde{y}^2\,. \tag{A.18}$$

In our conventions $\epsilon_{\tilde{t}\tilde{r}\tilde{y}}=1$. The generators of the isometry group are given by:

$$
\begin{aligned}
L_{\pm 1} &= ie^{\pm i(\tilde{t}+\tilde{y})}\left(-\frac{1}{2}\frac{\tilde{r}}{\sqrt{\tilde{r}^2+1}}\partial_{\tilde{t}} - \frac{1}{2}\frac{\sqrt{\tilde{r}^2+1}}{r}\partial_{\tilde{y}} \pm \frac{i}{2}\sqrt{r^2+1}\,\partial_{\tilde{r}}\right)\,, & L_0 &= \frac{i}{2}\left(\partial_{\tilde{t}}+\partial_{\tilde{y}}\right)\,, \\
\bar{L}_{\pm 1} &= ie^{\pm i(\tilde{t}-\tilde{y})}\left(-\frac{1}{2}\frac{\tilde{r}}{\sqrt{\tilde{r}^2+1}}\partial_{\tilde{t}} + \frac{1}{2}\frac{\sqrt{\tilde{r}^2+1}}{\tilde{r}}\partial_{\tilde{y}} \pm \frac{i}{2}\sqrt{\tilde{r}^2+1}\,\partial_{\tilde{r}}\right)\,, & \bar{L}_0 &= \frac{i}{2}\left(\partial_{\tilde{t}}-\partial_{\tilde{y}}\right)
\end{aligned}
\tag{A.19}
$$

and they respect the algebra

$$[L_0, L_\pm] = \mp L_\pm \qquad [L_1, L_{-1}] = 2L_0 \qquad [\bar{L}_0, \bar{L}_\pm] = \mp \bar{L}_\pm \qquad [\bar{L}_1, \bar{L}_{-1}] = 2\bar{L}_0\,. \tag{A.20}$$

The quadratic Casimirs are $L^2 = \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) - (L_0)^2$ and the corresponding antiholomorphic operator $\bar{L}^2$. For a state of L^2 quantum number j_{sl}, we have $L^2|j_{\mathrm{sl}}\rangle = -j_{\mathrm{sl}}(j_{\mathrm{sl}}-1)|j_{\mathrm{sl}}\rangle$.

Scalar harmonics have $j_{\mathrm{sl}} = \bar{j}_{\mathrm{sl}}$. We introduce for convenience $l = 2j_{\mathrm{sl}}$. We introduce scalar harmonics $B_l^{(\pm)}$, where the superscript $\pm$ denotes the positive and negative frequency modes. $B_l^{(+)00}$ is the lowest-weight state in the discrete series representation D^+, and $B_l^{(-)00}$ is the highest-weight state in the discrete series representation D^-. For ease of language we refer to these both as highest-weight states. These harmonics solve the following Laplace equation:

$$\Box_{\mathrm{AdS}_3} B_l^{(\pm)} = l(l-2)B_l^{(\pm)} \tag{A.21}$$

We then have

$$B_l^{(\pm)} = \frac{e^{\mp il\tilde{t}}}{\sqrt{\tilde{r}^2+1}^{\,l}}\,, \qquad L_0 B_l^{(\pm)} = \bar{L}_0 B_l^{(\pm)} = \pm\frac{l}{2}B_l^{(\pm)}\,. \tag{A.22}$$

The fact that these are highest-weight states can be seen as follows:

$$\left[L_1, B_l^{(+)}\right] = \left[\bar{L}_1, B_l^{(+)}\right] = 0 \qquad \left[L_{-1}, B_l^{(-)}\right] = \left[\bar{L}_{-1}, B_l^{(-)}\right] = 0\,. \tag{A.23}$$

A.2.2 Vector harmonics

We now introduce vector harmonics on AdS$_3$. Those that live in the representation labeled by L^2 quantum numbers $(j_{\mathrm{sl}}, \bar{j}_{\mathrm{sl}}) = (\frac{l-2}{2}, \frac{l}{2})$ will be denoted by $B_{\mathrm{R},l}^{(\pm)}$, where R stands for right. Analogously, there is also the left representation with quantum numbers $(j_{\mathrm{sl}}, \bar{j}_{\mathrm{sl}}) = (\frac{l}{2}, \frac{l-2}{2})$, and we will denote it by $B_{\mathrm{L},l}^{(\pm)}$. The vector harmonics satisfy the following Laplace equation

$$(d\delta + \delta d)B_{v,l}^{(\pm)} = (l-2)^2 B_{v,l}^{(\pm)}\,. \tag{A.24}$$

Here the subscript v stands for vector, meaning that the formula applies to both left and right vector harmonics. The $l = 0$ vector harmonics are the Killing one forms, which can be obtained can be obtained dualizing eq. (A.19),

$$
\begin{aligned}
L_0 &= -\frac{i}{2}\left(\left(\tilde{r}^2 + 1\right)d\tilde{t} - \tilde{r}^2 d\tilde{y}\right), \\
L_\pm &= e^{\pm i(\tilde{t}+\tilde{y})}\frac{\tilde{r}}{2\sqrt{\tilde{r}^2+1}}\left(\mp\frac{d\tilde{r}}{\tilde{r}} + i\left(\tilde{r}^2+1\right)(d\tilde{t} - d\tilde{y})\right), \\
\bar{L}_0 &= -\frac{i}{2}\left(\left(\tilde{r}^2 + 1\right)d\tilde{t} + \tilde{r}^2 d\tilde{y}\right), \\
\bar{L}_\pm &= e^{\pm i(\tilde{t}-\tilde{y})}\frac{\tilde{r}}{2\sqrt{\tilde{r}^2+1}}\left(\mp\frac{d\tilde{r}}{\tilde{r}} + i\left(\tilde{r}^2+1\right)(d\tilde{t} + d\tilde{y})\right).
\end{aligned}
\tag{A.25}
$$

One can generate degree l vector harmonics, as in the S^3 case, by multiplying scalar harmonics with the one-forms in eq. (A.25) and exploiting the $\mathrm{SL}(2,\mathbb{R})$ tensor product decomposition. For concreteness, let us consider right vector harmonics, which live in the representation in the first term of the following direct sum:

$$
\left(\frac{l}{2}, \frac{l}{2}\right) \otimes (-1, 0) = \left(\frac{l-2}{2}, \frac{l}{2}\right) \oplus \left(\frac{l}{2}, \frac{l}{2}\right) \oplus \left(\frac{l+2}{2}, \frac{l}{2}\right).
\tag{A.26}
$$

Let us focus on the $B^{(+)}$ modes. It is important to note that the scalar harmonic $B_l^{(+)}$ is a lowest weight state; in order to obtain the lowest weight of the vectorial representation we need to take the product with L_1, and one has:

$$
\begin{aligned}
B_{\mathrm{R},l}^{(+)} = B_l^{(+)} \otimes L_1, &\qquad \left[L_0, B_{\mathrm{R},l}^{(+)}\right] = \frac{l-2}{2}B_{\mathrm{R},l}^{(+)}, \\
\left[\bar{L}_0, B_{\mathrm{R},l}^{(+)}\right] = \frac{l}{2}B_{\mathrm{R},l}^{(+)}, &\qquad \left[L_1, B_{\mathrm{R},l}^{(+)}\right] = \left[\bar{L}_1, B_{\mathrm{R},l}^{(+)}\right] = 0.
\end{aligned}
\tag{A.27}
$$

Analogous relations hold for the left vector harmonics. Vector harmonics on AdS$_3$ are also eigenstates of the following operator, where $\star$ is the Hodge star on global AdS$_3$, (A.18):

$$
\star dB_{\mathrm{R},l}^{(\pm)} = (l-2)B_{\mathrm{R},l}^{(\pm)}, \qquad \star dB_{\mathrm{L},l}^{(\pm)} = -(l-2)B_{\mathrm{L},l}^{(\pm)}.
\tag{A.28}
$$

B Extremal 3-point functions

In this appendix we will derive the scalar chiral primary operators of dimension two in the single-particle basis. The vanishing of extremal three-point functions built with the operator $\tilde{O}_2$ in (4.16) was discussed in section (4.1) of [31]; here we shall focus on the operators $\tilde{\Omega}$ and $\tilde{\Sigma}_3$ defined in eq. (4.17). These operators involve a mixing between the single-trace operators Σ_3, Ω and the double-trace operators $(\Sigma_2 \cdot \Sigma_2)$, $(J \cdot \bar{J})$ and $(O \cdot O)$. Let us first discuss the mixing matrix between the single-traces [28]. The single-particle operators take the form

$$
\tilde{\Omega} \sim a_1\frac{\Sigma_3}{N^{\frac{3}{2}}} + a_2\frac{\Omega}{N^{\frac{1}{2}}}, \qquad \tilde{\Sigma}_3 \sim b_1\frac{\Sigma_3}{N^{\frac{3}{2}}} + b_2\frac{\Omega}{N^{\frac{1}{2}}},
\tag{B.1}
$$

where $\sim$ means that we are retaining only the single-trace contributions, and where we have included factors of N such that each term contributes at order N^0 to the norm of the respective single-particle operator at large N. As we shall discuss below, at large N, the multi-trace contribution to the norm of the single-particle operator is subleading. Thus, the orthonormality conditions

$$\langle \tilde{\Sigma}_3^{++} \tilde{\Sigma}_3^{--} \rangle = \langle \tilde{\Omega}^{++} \tilde{\Omega}^{--} \rangle = 1 \,, \qquad \langle \tilde{\Sigma}_3^{++} \tilde{\Omega}^{--} \rangle = 0 \,, \tag{B.2}$$

give three constraints on the four coefficients a_1, a_2, b_1 and b_2. In order to completely fix the mixing matrix in eq. (B.1) we need a fourth constraint, which was derived in [28] from comparison with non-extremal supergravity correlators. The result is

$$\tilde{\Omega} \sim \frac{\sqrt{3}}{2} \left(\frac{\Sigma_3}{N^{\frac{3}{2}}} + \frac{\Omega}{N^{\frac{1}{2}}} \right) \,, \qquad \tilde{\Sigma}_3 \sim \frac{3}{2} \left(\frac{\Sigma_3}{N^{\frac{3}{2}}} - \frac{\Omega}{3N^{\frac{1}{2}}} \right) \,. \tag{B.3}$$

We now turn to the mixing of multi-trace operators in the single-particle basis. As we shall see, having fixed the single-trace mixing in (B.3), the multi-trace admixtures can then be completely fixed by CFT computations. By contrast, if we had not fixed the single-trace mixing in (B.3) and performed the following steps with the general admixture (B.1), we would not have enough constraints to determine the mixing.

Let us consider the most general linear combination allowed by the quantum numbers that can give rise to the single-particle operators:

$$\alpha \frac{\Sigma_3}{N^{3/2}} + \beta \frac{\Omega}{N^{1/2}} + \gamma(\Sigma_2 \cdot \Sigma_2) + \delta(J \cdot \bar{J}) + \epsilon(O \cdot O)^{++} \,. \tag{B.4}$$

The two sets of values for the coefficients α and β are given by eq. (B.3). We now fix the two sets of coefficients γ, δ and ϵ by imposing that all extremal three-point functions containing the operator (B.4) and operators of lower dimension vanish. Equivalently, this imposes that the operator (B.4) is orthogonal to all multi-trace operators, as discussed around eq. (4.3).

There are three non-trivial such correlators, corresponding respectively to the last three terms in (B.4). We work at leading order in large N. For ease of notation we will suppress the standard space-time dependence in the following correlators.

- The first constraint follows from imposing that the three-point function containing the operator (B.4) and two O^{--} operators vanishes. Since O^{--} belongs to the untwisted sector of the theory, the contributions coming from the twist operators in (B.4) are trivially zero; moreover the contribution of the multi-trace $(J\bar{J})$ is subleading at large N. We obtain

$$\left\langle \left(\beta \frac{\Omega^{++}}{N^{1/2}} + \epsilon(O \cdot O)^{++} \right) O^{--} O^{--} \right\rangle = \beta N^{1/2} + 2\epsilon N = 0 \,, \tag{B.5}$$

where we have used the definitions of O, Ω and $(O \cdot O)$ in eqs. (2.8), (2.9) and (2.12).

- Let us now consider the current insertions: again, since they carry no twist, the twist operators in eq. (B.4) do not contribute to the correlator; the contribution of the double-trace operator $(O \cdot O)^{++}$ is subleading at large N. The computation leads to

$$\left\langle \left(\beta \frac{\Omega^{++}}{N^{1/2}} + \delta \left(J \cdot \bar{J} \right)^{++} \right) J^- \bar{J}^- \right\rangle = \beta N^{1/2} + \delta N = 0 \,, \tag{B.6}$$

where we have used the definitions of J, $\bar{J}$, Ω and $(J \cdot \bar{J})$ in eqs. (2.6), (2.9) and (2.12).

- The final constraint comes from inserting two twist operators Σ_2^{--}. There are three leading-order contributions to the extremal three-point function, from the operators Σ_3^{++}, $(\Sigma_2 \cdot \Sigma_2)^{++}$ and Ω^{++} in (B.4). First, the contraction with Σ_3^{++} gives at large N

$$\left\langle \sum_{r<s<t} (\sigma_{(rst)}^{++} + \sigma_{(rts)}^{++}) \sum_{a<b} \sigma_{(ab)}^{--} \sum_{c<d} \sigma_{(cd)}^{--} \right\rangle = \frac{3N^{3/2}}{4} \,. \tag{B.7}$$

The combinatorics works as follows. The choice of the copies r, s, t gives $\binom{N}{3}$ and, for each choice, there are two inequivalent cycles. Thus the twist-three operator can glue three strands out of N in $2\binom{N}{3}$ ways. For concreteness let us suppose that the orientation of the 3 copies r, s, t is chosen to be (123). Then there remains the freedom to take the first σ_2 to be the 2-cycle (12), (23) or (13) (this fixes the second σ_2 to be (23), (13) and (12) respectively): this gives an additional factor of three. Eq. (B.7) follows from combining this combinatorial factor with the building block

$$\langle \sigma_{(123)}^{++} \sigma_{(12)}^{--} \sigma_{(23)}^{--} \rangle = \frac{3}{4} \,, \tag{B.8}$$

derived in [71, eq. (5.25)].

Second, the contribution from the double-trace $(\Sigma_2 \cdot \Sigma_2)^{++}$ in (B.4) is

$$\left\langle \frac{2}{N^2} \sum_{(r<s) \neq (p<q)} \sigma_{(rs)}^{++} \sigma_{(pq)}^{++} \sum_{a<b} \sigma_{(ab)}^{--} \sum_{c<d} \sigma_{(cd)}^{--} \right\rangle = N^2 \,. \tag{B.9}$$

The computation goes as follows. The choice of the copies r, s and p, q gives the combinatorial factors $N(N-1)/2$ and $(N-2)(N-3)/2$ respectively; moreover, one can perform two inequivalent Wick contractions, which give an extra factor of two. Taking into account the normalization of the multi-trace operator, one obtains eq. (B.9).

The third contribution comes from the operator Ω^{++} in (B.4). One could evaluate this amplitude either by using the techniques developed in [71, 78], or by exploiting Ward identities that relate different n-point functions in the same R-symmetry multiplet together with some results obtained in [31]; we shall use the latter method.

We first use the permutation property of the correlator (see e.g. [91, eq. (2.2.48)]) to reorder the operators as

$$\langle \Omega^{++} \Sigma_2^{--} \Sigma_2^{--} \rangle = \langle \Sigma_2^{--} \Omega^{++} \Sigma_2^{--} \rangle \,. \tag{B.10}$$

We next write the operator Σ_2^{--} as $[J_0^- \bar{J}_0^-, \Sigma_2^{++}]$ and move the current modes onto the other operators, to obtain

$$\langle \Sigma_2^{--} \, \Omega^{++} \, \Sigma_2^{--} \rangle = \langle \Sigma_2^{++} \, (2\Omega^{00}) \, \Sigma_2^{--} \rangle = \frac{N^2}{4} \, . \tag{B.11}$$

The last equality is obtained by spectral flowing the correlator to the RR sector. On doing so we notice that Ω carries no twist so that the twist operators need to act on the same copies, which introduces a combinatorial factor of $\binom{N}{2}$. We then use [31, eq. (5.40)].

Combining eqs. (B.5)–(B.7), (B.9), and (B.11), we find that the required vanishing of the three-point functions considered imposes the three constraints

$$\gamma = -\frac{3\alpha}{4N^{1/2}} - \frac{\beta}{4N^{1/2}} \, , \qquad \delta = -\frac{\beta}{N^{1/2}} \, , \qquad \epsilon = -\frac{\beta}{2N^{1/2}} \, . \tag{B.12}$$

Combining these constraints with the two sets of values of α and β in eq. (B.3), we obtain

$$\tilde{\Sigma}_3^{++} = \frac{3}{2} \left[\left(\frac{\Sigma_3^{++}}{N^{\frac{3}{2}}} - \frac{\Omega^{++}}{3N^{\frac{1}{2}}} \right) + \frac{1}{N^{\frac{1}{2}}} \left(-\frac{2}{3}(\Sigma_2\cdot\Sigma_2)^{++} + \frac{1}{6}(O\cdot O)^{++} + \frac{1}{3}(J\cdot\bar{J})^{++} \right) \right] ,$$

$$\tilde{\Omega}^{++} = \frac{\sqrt{3}}{2} \left[\left(\frac{\Sigma_3^{++}}{N^{\frac{3}{2}}} + \frac{\Omega^{++}}{N^{\frac{1}{2}}} \right) + \frac{1}{N^{\frac{1}{2}}} \left(-(\Sigma_2\cdot\Sigma_2)^{++} - \frac{1}{2}(O\cdot O)^{++} - (J\cdot\bar{J})^{++} \right) \right] , \tag{B.13}$$

as written in the main text in eq. (4.17). Recall that the factors of N multiplying the single-trace operators are such that these terms all contribute at leading order (order N^0) to the norm of the operators. By contrast, since the double-trace operators are already unit normalized, the admixture factors of $1/\sqrt{N}$ imply for instance that the double-trace contribution to the norm of the single-particle operators is subleading at large N.

C Type IIB supergravity ansatz and BPS equations

In this appendix we record for completeness the general solution to Type IIB supergravity compactified on T^4 that is 1/8-BPS, carries D1-D5-P charges, and is invariant on T^4 [67, (E.7)]:

$$ds_{10}^2 = \sqrt{\alpha} \, ds_6^2 + \sqrt{\frac{Z_1}{Z_2}} \, d\hat{s}_4^2 \, ,$$

$$ds_6^2 = -\frac{2}{\sqrt{\mathcal{P}}} \, (dv + \boldsymbol{\beta}) \left[du + \omega + \frac{\mathcal{F}}{2} \, (dv + \boldsymbol{\beta}) \right] + \sqrt{\mathcal{P}} \, ds_4^2 \, ,$$

$$e^{2\Phi} = \frac{Z_1^2}{\mathcal{P}} \, ,$$

$$B = -\frac{Z_4}{\mathcal{P}} \, (du + \omega) \wedge (dv + \boldsymbol{\beta}) + a_4 \wedge (dv + \boldsymbol{\beta}) + \gamma_4 \, ,$$

$$C_0 = \frac{Z_4}{Z_1} \, ,$$

$$C_2 = -\frac{Z_2}{\mathcal{P}} \, (du + \omega) \wedge (dv + \boldsymbol{\beta}) + a^1 \wedge (dv + \boldsymbol{\beta}) + \gamma_2 \, ,$$

$$C_4 = \frac{Z_4}{Z_2} \, \widehat{\text{vol}}_4 - \frac{Z_4}{\mathcal{P}} \, \gamma_2 \wedge (du + \omega) \wedge (dv + \boldsymbol{\beta}) + x_3 \wedge (dv + \boldsymbol{\beta}) \, ,$$

$$C_6 = \widehat{\text{vol}}_4 \wedge \left[-\frac{Z_1}{\mathcal{P}} \, (du + \omega) \wedge (dv + \boldsymbol{\beta}) + a^2 \wedge (dv + \boldsymbol{\beta}) + \gamma_1 \right] , \tag{C.1}$$

where

$$\alpha = \frac{Z_1 Z_2}{Z_1 Z_2 - Z_4^2}\,, \qquad \mathcal{P} = Z_1 Z_2 - Z_4^2\,. \tag{C.2}$$

Here $d\hat{s}_4^2$ denotes the flat metric on T^4, and $\widehat{\mathrm{vol}}_4$ denotes the corresponding volume form. This ansatz contains all fields that are known to arise from world-sheet calculations of the backreaction of D1-D5-P bound states invariant on $\mathcal{M}$ [92, 93].

The BPS equations for this ansatz have the following structure. The base metric, ds_4^2, and the one-form $\boldsymbol{\beta}$ satisfy non-linear equations known as the "zeroth layer". Having solved these initial equations, the remaining BPS equations are organized into two further layers of linear equations [67, 94].

We denote the exterior differential on the spatial base $\mathcal{B}$ by $\tilde{d}$, and following [95] we introduce

$$\mathcal{D} \equiv \tilde{d} - \boldsymbol{\beta} \wedge \frac{\partial}{\partial v}\,. \tag{C.3}$$

In the current work we consider only solutions in which the four-dimensional base space is flat $\mathbb{R}^4$, and in which $\boldsymbol{\beta}$ is independent of v. The BPS equation for $\boldsymbol{\beta}$ is then

$$d\boldsymbol{\beta} = *_4 d\boldsymbol{\beta}\,, \tag{C.4}$$

where $*_4$ stands for the flat $\mathbb{R}^4$ Hodge dual.

To write the remaining layers of BPS equations in a covariant form, we make the rescaling $(Z_4, a_4, \gamma_4) \to (Z_4, a_4, \gamma_4)/\sqrt{2}$ for the remainder of this appendix (and only here). We introduce the SO$(1,2)$ Minkowski metric η_{ab} $(a = 1, 2, 4)$ in the null form

$$\eta_{12} = \eta_{21} = 1\,, \qquad \eta_{44} = -1\,. \tag{C.5}$$

This metric is used to raise and lower a, b indices. We introduce the two-forms Θ^1, Θ^2, Θ^4 as follows:[11]

$$\Theta^b \equiv \mathcal{D}a^b + \eta^{bc}\dot{\gamma}_c\,. \tag{C.6}$$

We then have

$$\mathcal{P} \equiv \tfrac{1}{2}\eta^{ab}Z_a Z_b = Z_1 Z_2 - \tfrac{1}{2}Z_4^2\,. \tag{C.7}$$

The "first layer" of the BPS equations is then

$$*_4\, D\dot{Z}_a = \eta_{ab}D\Theta^b\,, \qquad D *_4 DZ_a = -\eta_{ab}\Theta^b \wedge d\boldsymbol{\beta}\,, \qquad \Theta^a = *_4\Theta^a\,, \tag{C.8}$$

while the "second layer" is given by

$$\begin{aligned}
D\omega + *_4 D\omega + \mathcal{F}\, d\boldsymbol{\beta} &= Z_a\Theta^a\,, \\
*_4 D *_4 \left(\dot{\omega} - \tfrac{1}{2}\, D\mathcal{F}\right) &= \ddot{\mathcal{P}} - \tfrac{1}{2}\eta^{ab}\dot{Z}_a\dot{Z}_b - \tfrac{1}{4}\eta_{ab} *_4 \Theta^a \wedge \Theta^b\,.
\end{aligned} \tag{C.9}$$

---

[11]The relation to the notation in [21] is that $\Theta_{\mathrm{here}}^1 = \Theta_1^{\mathrm{there}}$, $\Theta_{\mathrm{here}}^2 = \Theta_2^{\mathrm{there}}$, $(1/\sqrt{2})\Theta_4^{\mathrm{here}} = \Theta_4^{\mathrm{there}}$.

D Gauge-fixed holographic dictionary

In section 4.5 we discussed the holographic dictionary that relates the expectation value of single-particle operators at dimension one and scalar single-particle operators at dimension two in term of gauge-invariant combination of KK fields. This formulation has the advantage of being general and manifestly gauge invariant: it can be used to probe any microstate solution in any coordinate system. From a practical point of view, however, it is useful to formulate the dictionary in a preferred gauge, as this makes it more manageable to use. In this appendix we study the dictionary in such a gauge.

In this appendix we consider the general class of solutions for which the four-dimensional base metric ds_4^2 in (3.2), (C.1) is flat $\mathbb{R}^4$. This class includes all solutions discussed in this paper. For solutions in which the base metric is not flat, one should instead use the gauge-invariant formulation of the dictionary in eq. (4.22).

We choose coordinates such that the flat $\mathbb{R}^4$ base metric takes the form:

$$ds_4^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2).\tag{D.1}$$

In general, the functions Z_1, Z_2 and Z_4 obey a Poisson-type equation, where the Θ_i play the role of sources; see eq. (C.8). For 1/4-BPS solutions, one has $\Theta_1 = \Theta_2 = \Theta_4 = 0$, and so the Z_a are harmonic. For all the (1/8-BPS) superstrata we deal with in the present work, the first term in the large r expansion of the source $\Theta_a \wedge d\beta$ is of order $1/r^k$ where $k \geq 7$.

For convenience we perform the transformation in eq. (3.10) and reabsorb the overall factor $\sqrt{Q_1 Q_5}$ in the metric. We thus define the following expansion[12] [6, 27]:

$$\tilde{Z}_1 = \frac{Q_1}{\tilde{r}^2}\left(1 + \sum_{k=1}^{2}\sum_{m_k,\bar{m}_k=-k/2}^{k/2} f_{1k}^{(m_k,\bar{m}_k)}\frac{Y_k^{m_k,\bar{m}_k}}{\tilde{r}^k} + O(\tilde{r}^{-3})\right),$$

$$\tilde{Z}_2 = \frac{Q_5}{\tilde{r}^2}\left(1 + \sum_{k=1}^{2}\sum_{m_k,\bar{m}_k=-k/2}^{k/2} f_{5k}^{k\,(m_k,\bar{m}_k)}\frac{Y_k^{m_k,\bar{m}_k}}{\tilde{r}^k} + O(\tilde{r}^{-3})\right),$$

$$\tilde{Z}_4 = \frac{\sqrt{Q_1 Q_5}}{\tilde{r}^2}\left(\sum_{k=1}^{2}\sum_{m_k,\bar{m}_k=-k/2}^{k/2} \mathcal{A}_k^{(m_k,\bar{m}_k)}\frac{Y_k^{m_k,\bar{m}_k}}{\tilde{r}^k} + O(\tilde{r}^{-3})\right),\tag{D.2}$$

$$\tilde{A} = \frac{1}{\tilde{r}^2}\sum_{a=1}^{3}(a_{a+}Y_1^{a+} + a_{a-}Y_1^{a-}) + O(\tilde{r}^{-3}),\qquad \mathcal{F} = -\frac{2Q_p}{a_0\tilde{r}^2} + O(r^{-3}),$$

where we have denoted with $Y_k^{m_k,\bar{m}_k}$ the scalar harmonics on S^3 of degree k and with $Y_1^{a\pm}$ the vector harmonics of degree 1 (see appendix A), and where a_0 was defined below (3.10). We are still left with the freedom of choosing the origin of the coordinate system on the flat base. We fix this redundancy by requiring

$$f_{11}^{(m_1,\bar{m}_1)} + f_{51}^{(m_1,\bar{m}_1)} = 0,\tag{D.3}$$

which corresponds to placing the origin at the center of mass of the D1-D5 system. The choices in eqs. (D.2) and (D.3) imply that, at order $k = 1$, all fields in (3.12) that can be

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[12]Note that this equation is more general than [31, eq. (3.6)], which applies only to solutions in which the Z_a are harmonic.

set to zero by a gauge transformation vanish. With these choices, the expansion of the bulk quantities in the dictionary (4.22) in terms of (D.2) yields the gauge-fixed single-particle fields in supergravity:

$$
\begin{aligned}
\left[s_{k=1}^{(6)(\alpha,\dot\alpha)}\right] &= -2\sqrt{2}f_{51}^{(\alpha,\dot\alpha)}\,, \qquad \left[s_{k=2}^{(6)(a,\dot a)}\right] = \sqrt{\frac{3}{2}}\left(f_{12}^{(a,\dot a)}-f_{52}^{(a,\dot a)}\right)\,, \\
\left[s_{k=1}^{(7)(\alpha,\dot\alpha)}\right] &= 2\sqrt{2}\mathcal{A}_1^{(\alpha,\dot\alpha)}\,, \qquad \left[s_{k=2}^{(7)(a,\dot a)}\right] = \sqrt{6}\left(\mathcal{A}_2^{(a,\dot a)}\right)\,, \\
\left[\tilde\sigma_{k=2}^{(a,\dot a)}\right] &= -\frac{1}{\sqrt{2}}\left(f_{12}^{(a,\dot a)}+f_{52}^{(a,\dot a)}\right)+2\sqrt{2}\left(f_{51}^{(\alpha,\dot\alpha)}f_{51}^{(\beta,\dot\beta)}+\mathcal{A}_1^{(\alpha,\dot\alpha)}\mathcal{A}_1^{(\beta,\dot\beta)}\right)a_{(\alpha,\dot\alpha)(\beta,\dot\beta)}^{(a,\dot a)} \\
&\quad + 4\sqrt{2}a^{c+}a^{d-}f_{(a,\dot a)cd}^1\,, \\
\left[A_{k=1}^{a(\pm)}\right] &= -2a^{-a,\pm}\,,
\end{aligned}
\tag{D.4}
$$

where $a_{(\alpha,\dot\alpha)(\beta,\dot\beta)}^{(a,\dot a)}$ and $f_{(a,\dot a)cd}^1$ are the triple overlap coefficients defined in (A.9) and (A.16).

Let us compare the dictionary for operators of dimension two given in eq. (4.22) with the one given in [31]. First, we note that the dictionary for the operator $\tilde{O}_2$ given in [31, eqs. (5.35)–(5.36)] is already formulated in the single-particle basis.

Next, we take [31, eqs. (5.33)–(5.34)] and rotate them to express them in terms of the geometric quantities $g_{a,\dot a}$, $\tilde{g}_{a,\dot a}$. We note that the operator dual to $g_{a,\dot a}$ coincides with $\tilde{\Sigma}_3$, thus in this rotated basis the holographic dictionary for $g_{a,\dot a}$ is given in terms of a single-particle operator.

By contrast, the operator dual to $\tilde{g}_{a,\dot a}$ is not yet $\tilde{\Omega}$. The dictionary for $\tilde{g}_{a,\dot a}$ at this point reads:

$$
\begin{aligned}
\tilde{g}_{a,\dot a} &= \sqrt{2}\left(-\left(f_{12}^{(a,\dot a)}+f_{52}^{(a,\dot a)}\right)+8a^{c+}a^{d-}f_{(a,\dot a)cd}^1\right) \\
&= -(-1)^{a+\dot a}\sqrt{6}\left[\frac{1}{N^{3/2}}\langle\Sigma_3^{a\dot a}\rangle+\frac{1}{N^{3/2}}\left(\langle\Omega^{a\dot a}\rangle-\frac{1}{3}\langle(\Sigma_2\cdot\Sigma_2)^{a\dot a}\rangle-\langle\left(J\cdot\bar{J}\right)^{a\dot a}\rangle+\frac{1}{6}\langle(O\cdot O)^{a\dot a}\rangle\right)\right].
\end{aligned}
\tag{D.5}
$$

We now observe that the right-hand side of the first line of (D.5) differs from the second-last line of eq. (D.4) through the terms that involve f_{51} and $\mathcal{A}_1$. These are gauge-fixed versions of gauge-invariant quantities. The holographic dictionary for f_{51} and $\mathcal{A}_1$ is obtained by combining eqs. (D.4) and (4.22). We can thus improve on the dictionary (D.5) by adding the terms in f_{51} and $\mathcal{A}_1$ to both sides of the equation. Doing so results precisely in the dictionary given in eq. (D.4). This demonstrates the consistency of the results of [31] and the present work, and is a non-trivial check of the independent methods used in the two works.

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