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USING THE VAN HIELE THEORY TO ANALYSE THE TEACHING OF GEOMETRICAL PROOF AT GRADE 8 IN SHANGHAI

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The data reported in this paper come from a study aimed at explaining how successful teachers teach proof in geometry. Through a careful analysis of a series of lessons taught in Grade 8 in Shanghai, China, the paper reports on the appropriateness of the van Hiele model of 'teaching phases' within the Chinese context. The analysis indicates that though the second and third van Hiele teaching phases could be identified in the Chinese lessons, the instructional complexity of, for example, the guided orientation phase means that more research is needed into the validity of the van Hiele model of teaching.

INTRODUCTION

The teaching of geometry, and, in particular, the teaching of geometrical proof, has received changing amounts of emphasis in recent curriculum reforms across many countries (compare, for example, the US NCTM *Standards*, 1989, 2000). For many, such as Wu (1996), plane geometry, taught well, is essential as it can give students at secondary school a first experience of the power and the economy of the basic axiom-theorem-deductive feature of mathematics. In China, the process and method of proof continues to be considered as an essential part of the school mathematics curriculum. For example, the *Shanghai Primary and Secondary School Curriculum Standard* (Shanghai Education Committee, 2004) specifies, for the lower secondary school level (Grade 6 to Grade 9; students' age 11-15 years), that the process of proving should be emphasized for the following reasons:

> ...to help students experience the developmental process from intuitive geometry to experimental geometry and then to deductive geometry; to establish the relationship and recognize the distinction between intuition and logical thinking; to perceive the meaning and the use of inductive reasoning, analogical reasoning, and deductive reasoning…; to experience the process of 'experiment-induction-conjecture-proof' (p35, translated by Ding).

Given the continuing debate across the world about the learning and teaching proof in geometry and the difficulties that many students encounter with this topic (see, for example, Jones 2000; Mammana and Villani, 1998), the research from which this paper is taken aims to contribute to understanding and interpreting, in depth, the teaching of geometrical proof by analysing classroom instruction at Grade 8 in Shanghai, China. The aim of this paper, following Whitman *et al* (1997), is to analyse the appropriateness of the van Hiele model of 'teaching phases' (see below) within the Chinese context, and, in particular, to see how well the model characterises the observed teaching in order to try to *explain* how a successful teacher teaches what is, by all accounts, an aspect of mathematics that is very difficult for many students at school.

RESEARCH VIEWS ON THE VAN HIELE THEORY

Based on their pedagogical experience and their teaching experiments, the van Hieles (husband and wife) proposed a psychological/pedagogical theory of thought levels in geometry (English version in Geddes *et al.*, 1984). For many researchers, such as Schoenfeld (1986), this model of thought levels provides a useful empirically-based description of what are likely to be relatively stable, qualitatively different, states or levels of understanding in learners. Accompanying this model of thought levels, the van Hieles proposed a model of teaching that specifies five sequential phases of instruction (see, for example, Clements & Battista, 1992, pp430-1) that, the van Hieles suggest, are a means of enhancing students' thinking from one thought level to the next. This model of teaching phases, as discussed below, is used as the main theoretical framework for this paper.

Originally, and in an attempt to understand the structure of geometry learning, Dina van Hiele-Geldof (see Geddes *et al*, 1984, pp217-223) focused on analyzing the relationship between student and subject matter in elementary geometry. As a result of her research, she suggested five teaching phases which, for the purposes of this paper, are termed as follows: 1) *Information*; 2) *Guided Orientation*; 3) *Explicitation*; 4) *Free Orientation*; 5) *Integration* (adapted from Clements & Battista, 1992, pp430-1; Geddes *et al*, 1984, p223; Hoffer, 1983).

At this point it is worth noting Hoffer's (1983) view that the third phase (*Explicitation*) was incorrectly given by Wirszup (1976, p83) as 'explanation', with Hoffer taking the view that, in this third phase, it is essential that "students make the observations explicitly rather than receive lectures (explanations) from the teacher" (*op cit*, p208). Furthermore, Clements and Battista (1992, pp430-1) call the second phase *Guided Orientation*, rather than use the Geddes *et al* term *Direct Orientation*.

Whatever the terms used, and the above illustrated some of the unresolved issues about the choice of terminology, the model is quite loose in that, as Schoenfeld (1986, p252) explains, and as Whitman *et al* (1997) found, the nature of the pedagogical sequence is far from clear. Not only that, but as the model is more a suggested process than a fixed formula, it is not at all obvious whether it is necessary for the teacher to go through each and every phase. Indeed, Hershkowitz (1998) is of the view that the van Hiele theory does not account well for the relationship between the context of the learning environment and the mathematical reasoning being developed. She suggests more context-specific research and this matches the call by Whitman *et al* (*ibid* p217) for more research to evaluate the use of the van Hiele theory with students of different cultural backgrounds. In general, the existing van Hiele-based research has yet to address systematically any of these issues concerning the nature and specification of the teaching phases.

In the little research that has directly examined the van Hiele teaching phases, Hoffer (1994) developed a way of codifying teacher behaviour in terms of the phases of instruction (which he characterised as "Familiarization", "Guided Orientation", "Free Orientation", "Verbalization", "Integration"). He then tested the coding procedure on a number of mathematics classes. Amongst his findings were that US mathematics teachers (not familiar with the van Hiele teaching phases) demonstrated a preponderance of phase 2 instruction (that is, "Guided Orientation") and, Hoffer claims, often interrupted student progress toward higher levels in order to return to phase 2 instruction. Taking up the Hoffer approach, Whitman *et al* (1997) applied Hoffer's instrument to the comparative study of geometry instruction in Japan and the US. What they found was that the US teacher, in general, taught using phase 2 instruction (that is, "Guided Orientation") but that "the class showed multiple phases ….within one module" (*ibid* p228) whereas in the case of the Japanese teacher "there was ambiguity in trying to identify the phase at which the teacher was teaching because it appeared that more than one interpretation was available [to the research team]" (*ibid* p229). In both these cases, while Hoffer studied a number of teachers, and while Whitman *et al* selected lessons on congruence of triangles from one Japanese and one US teacher, the actual subject matter being taught received little attention in their published papers.

To contribute to the research base for this aspect of the van Hiele theory, and following Whitman *et al* (1997), the data reported in this paper come from a study aimed at seeing how well the van Hiele model of the five teaching phases accounts for the pedagogical methods used in teaching deductive geometry in classrooms in China. The key research question being addressed is to what extent the van Hiele model of five teaching phases accounts for the teaching of geometric proof by successful teachers in Chinese classrooms.

METHODOLOGICAL CONSIDERATIONS

The data reported in the paper come from a study of geometry teaching at Grade 8 in Shanghai (for other details, see Ding & Jones, 2006). In the city there are four grades at the lower secondary school level, from Grade 6 (students' age, 11-12 years old) to Grade 9 (students' age, 14-15 years old). As the school geometry curriculum is divided into three stages, namely intuitive, experimental and deductive geometry, students at Grade 8 (13-14 years old) start to learn more formal deductive geometry and practice proof writing. Consequently, studying this Grade offers the opportunity to analyse how Chinese teachers lead students at this Grade level to learn proof in deductive geometry.

For the purposes of this paper, data, collected in 2006, is selected from the teaching of one teacher, referred to as Lily (pseudonym), in an ordinary public school in a typical suburb of the city. The teacher, selected because of very good reputation for student success, had over 20 years teaching experience of secondary school mathematics. At the time of the data collection, there were 39 students in the class and mathematics lessons, each 40 minutes long, took place six times each week. Every lesson with this teacher was observed over a three week period. During this time, 12 geometry lessons were observed with topics concerning parallelograms, rectangles, rhombi and squares. In total, four definitions and fifteen theorems were taught during the three-week observation period. Given the known expertise of the teacher, supporting evidence showed that the students were ready for this level of mathematics.

The data collected included classroom observations notes, audio-recordings of lessons (transcribed), and other field notes. During each lesson, photographs were taken to provide information which could not be recorded by audio-recorder or field notes (for example, recording work presented on the blackboard).

USING THE MODEL OF TEACHING PHASES TO ANALYSE LESSONS

In analysing the data, it was vital to understand, in depth, the nature of each phase in the van Hiele model. Pierre van Hiele (1986, p177) suggested that the teacher conducts the teaching process as follows: in the first phase, "by placing at the children's disposal (putting into discussion) material clarifying the context"; in the second phase, "by supplying the material by which the pupils learn the principal connections in the field of thinking"; in the third phase, "by leading class discussions that will end in a correct use of language"; in the fourth phase, "by supplying materials with various possibilities of use and giving instructions to permit various performances"; in the fifth phase, "by inviting the pupils to reflect on their actions, by having rules composed and memorized, and so on". This illustrates that, as a teacher moves through the teaching phases, there is a transition from forms of direct instruction towards the students' independence from the teacher.

After a very careful study of the van Hieles' original work, together with van Hiele-based research on the teaching phases, we seek to formulate an operational characterisation of the teaching phases in geometrical proof teaching and use this to analyse data collected in the Chinese classroom. The characteristics and terms of each phase described by the van Hieles (see Geddes *et al*., 1984), Hoffer (1983, 1994) and Clements and Battista (1992) were utilised. In what follows, an analysis of the teaching of proof in two geometry lessons (lesson Z2 and lesson Z3 - designations for identification purposes only) given by the case-study teacher, Lily (pseudonym), is presented in which each of the van Hiele phases is practically characterised. In these two lessons, there were two types of proof teaching: 1) teaching new geometrical theorems (Proof 1 and 2, involving theorems verifying a parallelogram by its opposite sides); 2) teaching proof of problem solving, namely, exercises consisting of two relatively simple problems (Exercises 4-5) and three complex problems (Exercises 6-8).

Characterizing the *Information* phase of teaching

The *Information* phase can be characterised when the teacher provides inquiry-based learning activities in which students carry out 'experiments' and make inductive reasoning and conjectures relating to a geometrical proof. In the analysis of the observed lessons, this phase was not found in either lesson, perhaps because the observed lessons were not at the start of the teaching of geometrical proof to these particular students.

Characterizing the *Guided Orientation* phase of teaching

In the analysed lessons, the phase of *Guided Orientation* was characterised by the teacher guiding students to uncover the links that form relationships of a proof problem, as exemplified by the following extract related to Proof 3 of lesson Z2 (see Figure 2)

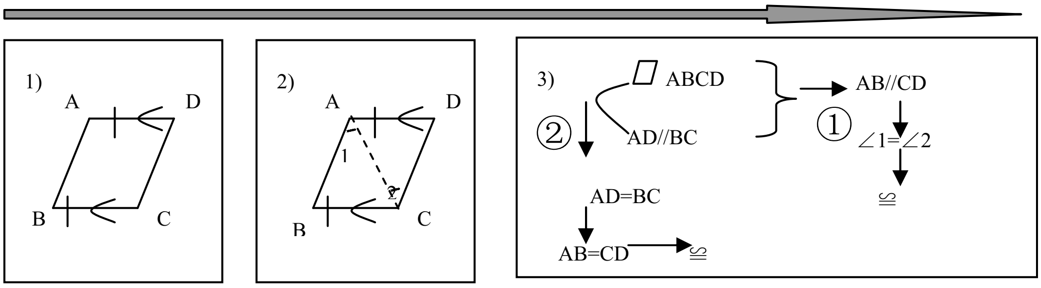


Figure 1: Proof 3, lesson Z2

The teacher briefly presented the ‘given’ for the problem (AD//=BC) and the statement to be proved (ABCD is a parallelogram), putting marks for the ‘given’ on the figure on the blackboard (see figure 1-1).

91 Lily: So far, ***how many methods did we learn to verify a parallelogram?*** (Some students answered the definition (AB//CD, AD//BC), and some answered Proof2 (from the previous proof, students know that AB=CD, AD=BC); detailed student dialogue omitted)

101 Lily: OK. Now, if I need to prove that this is a parallelogram, ***what is given?*** (Some students suggested AD//BC, some talked about AD=BC; detailed student dialogue omitted)

107 Lily: ***How do you make a decision?*** (Some students suggested the definition (AB//CD, AD//BC), while others suggested AB=CD, AD=BC; the teacher highlighted the given AD//BC, students discussed the use of the definition; detailed student dialogue omitted)

115 Lily: If I use the definition to prove, ***what should I prove first?***

116 Linlin (Boy): Parallel sides.

120 Lily: ***How to prove the parallel lines?*** (AB//CD). (Students suggested linking AC; the teacher used a board ruler to link AC - see figure 1-2; student dialogue omitted)

126 Lily: To prove AB//CD, ***what should I turn to prove first***? (Some students discussed equal angles, some answered alternate interior angles; student dialogue omitted)

129 Lily: ***Which pair of angles?*** (Using the students’ answers, the teacher highlight angles BAC and ACD; see figure 1-2; detailed student dialogue omitted.)

132 Lily: To prove ∠1=∠2, ***what should we turn to prove first…?*** (The class then discussed the idea of proving congruent triangles; dialogue omitted)

(While the teacher asked students these questions, she gradually wrote down an analytic structure of the proof on the blackboard - see figure 1-3 . She then used a similar sequence of questions to organize the analytic structure of another proof; see figure 1-3②)

Characterizing the *Explicitation* phase of teaching

The *Explicitation* phase of teaching was determined when students had knowledge, and were able to use mathematical language, to present the general structure of a proof. For instance, the extract from Exercise 4 of lesson Z2 (see figure 2) is characteristic of the explicitation phase. The extract follows the teacher explaining that ABCD (figure 2-1) is a parallelogram and that points E and F are 'dynamic' points that can move such that BE is always equal to DF (figure 2-2). The problem to prove what shape is quadrilateral BEDF (figure 2-3).

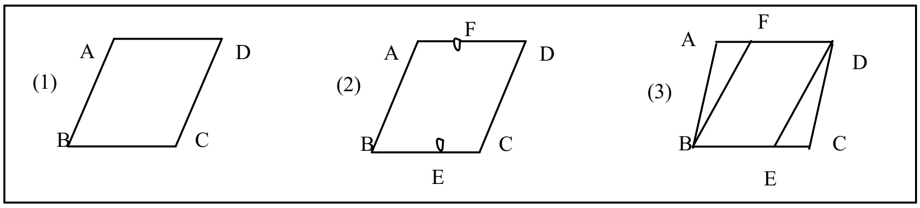


Figure 2: Exercise 4, lesson Z2

210 Lily: ***What does quadrilateral BEDF look like?*** (Students answer a parallelogram, dialogue omitted; the teacher asks the student to discuss why this might be the case)

206.1 Beibei: If a pair of opposite sides is equal and parallel, then….

209 Liuliu: (responded to Beibei) Yes, parallel and equal…???

215 Liuliu: Opposite sides are equal; I could use this to prove this problem. (this statement is taken to mean FD=BE, BF=DE).

221.1 Beibei: (Responding on Liuliu) Why?

221.2 Liuliu: You could see here. First, to calculate that ABF and ECD are congruent. Next, BF and DE are congruent. Oh, equal. BE and FD are already known.

221.3 Liuliu: This is to prove quadrilateral BEDF is a parallelogram.

221.4 Beibei: It is already given that a pair of opposite sides is equal.

221.5 Liuliu: You need to calculate that its opposite sides are equal. One pair of sides is given, yet you need to know another pair of sides.

221.6 Beibei: It is already given that BE=FD.

221.7 Liuliu: BE=DF. But you need to prove that BF=DE.

221.8 Beibei: If a pair of opposite sides of a quadrilateral is not only equal, but also parallel, then it is a parallelogram. (Liuliu does not reply to Beibei at this point; both listen to another student's presentation of the proof invited by the teacher.)

Characterizing the *Free orientation* phase of teaching

The *Free Orientation* phase of teaching, according to the van Hiele model and in the context of teaching geometrical proof, is when students learn their own ways to prove multi-step proof problems. This phase was not found in Lily's lesson 2 and 3, perhaps because the sampled lessons were in the *Guided Orientation* phase of teaching.

Characterizing the *Integration* phase of teaching

The *Integration* phase of teaching, according to the van Hiele model and in the context of teaching geometrical proof, is when students review and reflect the methods used in a set of proofs. This phase was not found in Lily's lesson 2 and 3, perhaps because the sampled lessons were in the *Guided Orientation* phase of teaching.

DEVELOPING AN OPERATIONAL MODEL OF THE VAN HIELE PHASES

An operational model of the van Hiele phases for the *process of teaching proof in geometry* is proposed as one outcome of this analysis. Descriptors of the *Guided Orientation* phase of this framework were drawn from a detailed analysis, exemplified above, of the case study lessons. The operational model is arranged in terms of the van Hiele phases of teaching:

1. <u>Information</u>: The teacher provides students inquiry-based learning activities in which students do experiments and make inductive reasoning and conjecture for a proof.

2. <u>Guided Orientation</u>: The teacher guides students to uncover the links that form a proof.

--a) The teacher demonstrates the 'Given' and the 'To Prove' statement or a problem; draws a figure and put marks on the figure on the blackboard; asks a set of questions and corrects students' answers to help them understand the requirement of a problem; provides students time to read the problem and to draw the figure on their own.

--b) The teacher encourages students to outline the different known theorems of a figure; helps students review the nature of the known definition/theorem and uncover their relationship; guides students to use deductive method to obtain new theorem from other known definition/theorems; shows how to write a formal proof; helps students evaluate the nature of the new theorem; guides students to use words and mathematical language to precisely present the new theorem.

--c) The teacher encourages students to outline the different ways to prove a problem; guides students to present the general structure of a proof and correct errors and emphasizes the rigor in proving; demonstrates the use of a new theorem in solving a set of problems.

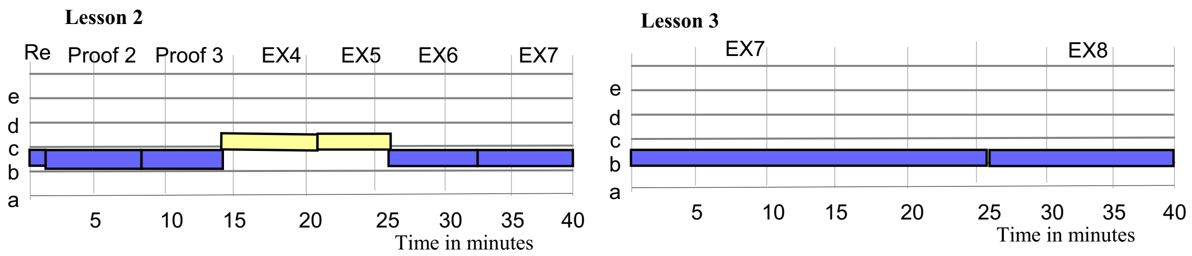
--d) The teacher provides multi-step problems that help students understand the network of definition/theorems; encourages students discover the hidden property by a set of questions and by uncovering a basic figure from the complicated figure; guides student to evaluate an appreciate method of a proof; helps students recognize the nature of different theorems of a figure;

3. Explicitation: The teacher ensures that students have the knowledge to present ideas and the general structure of a proof before the teacher's guidance. S/he begins to accurately use mathematical language in presenting a proof. In this phase, the teacher gets to understand what students have learned of the proof topic.

4. Free Orientation: The teacher ensures that students learn their own way to prove multi-step problems, often in a variety of ways.

5. Integration: The teacher ensures that students review, and reflect on, the methods used in a set of proofs.

Using this operational model, the teaching of proof in Lily's lesson 2 and 3 is shown in Figure 3.



a- Information; b- Guided Orientation; c- Explicitation; d- Free Orientation; e- Integration. EX-exercises

[colours not significant]

Figure 3: Proof teaching phases in Lily's lesson Z2 and Z3

DISCUSSION

The analysis presented in this paper indicates that the van Hiele theory can be a way of characterising the teaching phases in geometrical proof. In studying the relevant research, and in carrying out the analysis presented in this paper, it is clear that many questions about the teaching phases remain unanswered. As Clements and Battista (1992, p434) note, overall, and primarily because of a lack of research, many issues remains unclear, including how the phases of teaching relate to the subject matter and the students' prior attainment, whether the phases are followed in a linear fashion or iteratively within topic or even within individual lessons, whether one or more mathematical concepts can be included within one sequence of teaching phases, whether a different emphasis on particular phases depends on what is being taught (such as concepts, or skills, or problem-solving), and so on.

In terms of how long a teaching phase may last, Hoffer (1994), in his study, broken down lessons into discernible activities lasting 3-20 minutes and codified these in terms of the van Hiele teaching phases. In analysing the geometry lessons observed in Shanghai, the second and third of the van Hiele teaching phases were found across the range of lessons observed for this project (beyond the two lessons reported in this paper). Even so, the study indicates that the instructional complexity of the 'guided orientation' phase means that far more research is needed in the van Hiele teaching phases. For example, in lesson Z2 and Z3 (as analysed in this paper), the teacher's intention was carefully to lead students to experience the systematic network of theorems in constructing a proof through a sequence of well-designed, though demanding, multi-steps exercises. Moreover, the analysis of the instructional structure of the individual problem in the lesson suggests that the teacher was likely to develop students' abstract thinking and extend the structure of thinking through the model 'new theorem - simple problems - complicated problems'. According to interviews conducted with the teacher, she considered mathematical problems as a means of helping students practice the use of new theorems in further proofs. In terms of her instructional view, there were two types of problems in proof teaching: 1) simple problem, by which she meant one-step problems which directly use the new theorem; 2) complicated problem, which, for her, consist of both 'latitudinal' and 'longitudinal' problems – with a latitudinal problem containing a system of knowledge, (for instance, theorems of a parallelogram may link to those of a triangle or a circle, a parallelogram may link to function or equation) and a longitudinal problem entailing using a theorem in depth in a proof (for instance, using a theorem twice in a proof, with the second use probably requiring the use of an auxiliary line).

All these considerations means that further study is essential if *explanations* of how teachers, in China or elsewhere, effectively support students to extend their geometric thinking and proving. Given the aim of this study is interpreting, in depth, the teaching of geometrical proof in classroom, the intention is that the operational model of the van Hiele phases proposed in this paper (based primarily on two case study lessons) is to be further refined through additional analysis of all observed data.

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