HOMOTOPY DECOMPOSITIONS OF $PU(n)$-GAUGE GROUPS OVER RIEMANN SURFACES

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Abstract. We show that the gauge group of a principal $PU(n)$-bundle over a compact Riemann surface decomposes up to homotopy as the product of factors, one of which is a corresponding gauge group for S^2 and the others are immediately recognizable spaces. Further, when n is a prime p, the gauge group for S^2 decomposes as a product of immediately recognizable factors. These gauge groups have strong connections to moduli spaces of stable vector bundles.

1. Introduction

There is a deep connection between moduli spaces of stable vector bundles over a compact Riemann surface Σ_g and gauge groups of principal $U(n)$-bundles over Σ_g. This was recognized and exploited in spectacular fashion by Atiyah and Bott [1] to calculate the cohomology of the moduli spaces in many cases, leading to a whole new area of study that has attracted widespread interest. Daskalopolous and Uhlenbeck [5] made the connection more explicit by showing that the moduli space of rank n degree k stable vector bundles over Σ_g is homotopy equivalent through a dimensional range to the gauge group of the principal $U(n)$-bundle over Σ_g that is classified by having first Chern class k. Bradlow, Garcia-Prada and Gothen [4] went on to give an analogous homotopy equivalence in the case of moduli spaces of rank n degree k polystable Higgs bundles.

In [15] the author refined Daskalopolous and Uhlenbeck's homotopy equivalence in the case of principal $U(p)$-bundles when p is a prime by showing that the relevant gauge group decomposes up to homotopy as a product of recognizable factors. This allows for the calculation of the homotopy groups of the gauge group, or the moduli space, through a range based on known calculations for the factors. This approach was subsequently applied to non-orientable surfaces in [16] and real surfaces in [18]. The purpose of this paper is to return to compact, orientable surfaces but consider instead principal $PU(p)$-bundles.

The quotient map $U(n) \longrightarrow PU(n)$ induces mod-n reduction $\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ on π_1 and an isomorphism on π_m for $m \geq 2$. The goal, then, is to uncover the effect of this difference in the fundamental group on the corresponding gauge groups. This has its most complete description when $n = p$, in which case the effect is measured precisely by π_0 and π_1 of the gauge groups. The π_1 information is subtle and takes some work to tease out.

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To be more precise, first consider the case of principal $U(n)$-bundles over Σ_g. These are classified by the homotopy classes of maps $[\Sigma_g, BU(n)] \cong \mathbb{Z}$. Let $P_k \longrightarrow \Sigma_g$ be the principal $U(n)$-bundle classified by $k \in \mathbb{Z}$. The gauge group $\mathcal{G}_k(\Sigma_g, U(n))$ of P_k is the group of $U(n)$-equivariant automorphisms of P_k which fix Σ_g. In the special case when $\Sigma_g = S^2$, let $\mathcal{G}_k(U(n)) = \mathcal{G}_k(S^2, U(n))$. In [14] it was shown that there is an integral homotopy equivalence

$$\mathcal{G}_k(\Sigma_g, U(n)) \simeq \mathbb{Z}^{2g} \times \left(\prod_{i=1}^{2g} \Omega SU(n)\right) \times \mathcal{G}_k(U(n)) \tag{1}$$

where $\mathbb{Z}^{2g}$ is the product of $2g$ copies of $\mathbb{Z}$. This was refined in [15] when n is a prime p by decomposing $\mathcal{G}_k(U(n))$: if q is a prime different from p then there is a q-local homotopy equivalence

$$\mathcal{G}_k(U(p)) \simeq U(p) \times \Omega^2 U(p);$$

if $p \mid k$ there is a p-local homotopy equivalence

$$\mathcal{G}_k(U(p)) \simeq \prod_{i=0}^{p-1} S^{2i+1} \times \prod_{j=1}^{p-1} \Omega^2 S^{2j+1};$$

and if $p \nmid k$ there is a p-local homotopy equivalence

$$\mathcal{G}_k(U(p)) \simeq S^1 \times \prod_{i=0}^{p-2} S^{2i+1} \times \prod_{j=2}^{p} \Omega^2 S^{2j+1}.$$

In fact, the q-local homotopy equivalences for $q \neq p$ may be assembled to form a $\mathbb{Z}[\frac{1}{p}]$-local homotopy equivalence since the q-local equivalences are consequences of the fact that $\mathcal{G}_k(U(p))$ is the homotopy fibre of a map $U(p) \longrightarrow \Omega_0 U(p)$ of order p (here, $\Omega_0 U(p)$ is the connected component containing the basepoint). However, the p-local case is the more interesting and delicate one.

Principal $PU(n)$-bundles are classified by homotopy classes of maps $[\Sigma_g, BPU(n)] \cong \mathbb{Z}/n\mathbb{Z}$. If $P_k \longrightarrow \Sigma_g$ is the principal $PU(n)$-bundle classified by $k \in \mathbb{Z}/n\mathbb{Z}$ the gauge group $\mathcal{G}_k(\Sigma_g, PU(n))$ is the group of $PU(n)$-equivariant automorphisms of P_k which fix Σ_g. Write $\mathcal{G}_k(PU(n))$ for $\mathcal{G}_k(S^2, PU(n))$. Making use of the fibration $\mathbb{Z}/n\mathbb{Z} \longrightarrow SU(n) \longrightarrow PU(n)$, we prove the following.

Theorem 1.1. *For any $k \in \mathbb{Z}/n\mathbb{Z}$ there is a homotopy equivalence*

$$\mathcal{G}_k(\Sigma_g, PU(n)) \simeq (\mathbb{Z}/n\mathbb{Z})^{2g} \times \left(\prod_{i=1}^{2g} \Omega SU(n)\right) \times \mathcal{G}_k(PU(n)).$$

Theorem 1.2. *Fix a prime p and let $k \in \mathbb{Z}/p\mathbb{Z}$. The following hold:*

(a) *there is a $\mathbb{Z}[\frac{1}{p}]$-local homotopy equivalence*

$$\mathcal{G}_k(PU(p)) \simeq PU(p) \times \Omega^2 PU(p);$$

(b) *if $k = 0$ then there are p-local homotopy equivalences*

$$\mathcal{G}_k(PU(p)) \simeq PU(p) \times \Omega_0^2 PU(p) \simeq L \times \left(\prod_{i=1}^{p-2} S^{2i+1}\right) \times \left(\prod_{j=1}^{p-1} \Omega^2 S^{2j+1}\right)$$

where L is a retract of $PU(p)$ such that $\pi_1(L) \cong \pi_1(PU(p))$ and the universal cover of L is S^{2p-1};

(c) *if $k \neq 0$ then there is a p-local homotopy equivalence*

$$\mathcal{G}_k(PU(p)) \simeq \left(\prod_{i=0}^{p-2} S^{2i+1}\right) \times \left(\prod_{j=2}^{p} \Omega^2 S^{2j+1}\right).$$

In particular, when $k \neq 0$, $\mathcal{G}_k(PU(p))$ has one less factor of S^1 than the $p \nmid k$ case for $\mathcal{G}_k(U(p))$ and, interestingly, $\pi_1(\mathcal{G}_k(PU(p))) \cong \mathbb{Z}$. The torsion in $\pi_1(PU(p)) \cong \mathbb{Z}/p\mathbb{Z}$ is not directly reflected in a torsion property for the homotopy groups of $\mathcal{G}_k(PU(p))$. A notable special case is for $PU(2) \cong SO(3)$, when there is a 2-local homotopy equivalence $\mathcal{G}_k(PU(2)) \simeq S^1 \times \Omega^2 S^5$ if $k \neq 0$.

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2. $PU(n)$ gauge groups over Riemann surfaces

In this section we prove Theorem 1.1. The first step is to classify principal $PU(n)$-bundles. For a topological group G, let BG be its classifying space.

Lemma 2.1. *There is an isomorphism of groups $[S^2, BPU(n)] \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. The homotopy fibration $\mathbb{Z}/n\mathbb{Z} \longrightarrow SU(n) \longrightarrow PU(n)$ classifies to give a homotopy fibration $B\,\mathbb{Z}/n\mathbb{Z} \longrightarrow BSU(n) \longrightarrow BPU(n)$. Taking homotopy groups and noting that $BSU(n)$ is 3-connected immediately gives $\pi_2(BPU(n)) \cong \pi_1(B\,\mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. □

Let Σ_g be a surface of genus g. If $g = 0$ then $\Sigma_g \simeq S^2$, and if $g \geq 1$ then it is well known that there is a homotopy cofibration sequence

$$S^1 \xrightarrow{f} \bigvee_{i=1}^{2g} S^1 \longrightarrow \Sigma_g \xrightarrow{q} S^2 \xrightarrow{\Sigma f} \bigvee_{i=1}^{2g} S^2 \tag{2}$$

where f is the attaching map for the top cell of Σ_g, q is the pinch map to the top cell, and Σf is null homotopic.

Lemma 2.2. *If Σ_g is a surface of genus $g \geq 1$ then the map $\Sigma_g \xrightarrow{q} S^2$ induces an isomorphism of sets $[S^2, BPU(n)] \xrightarrow{q^*} [\Sigma_g, BPU(n)]$. Consequently, $[\Sigma_g, BPU(n)] \cong \mathbb{Z}/n\mathbb{Z}$.*

Proof. From the homotopy cofibration (2) we obtain an exact sequence

$$[\bigvee_{i=1}^{2g} S^2, BPU(n)] \xrightarrow{(\Sigma f)^*} [S^2, BPU(n)] \xrightarrow{q^*} [\Sigma_g, BPU(n)] \longrightarrow [\bigvee_{i=1}^{2g} S^1, BPU(n)].$$

Observe that $(\Sigma f)^* = 0$ since Σf is null homotopic and $[\bigvee_{i=1}^{2g} S^1, BPU(n)] \cong 0$ since $BPU(n)$ is simply-connected. Therefore q^* is an isomorphism. That $[\Sigma_g, BPU(n)] \cong \mathbb{Z}/p\mathbb{Z}$ now follows from Lemma 2.1. □

Next, we describe a context in which homotopy theory can be used to study gauge groups. In general, let G be a topological group, X a pointed space, and P a principal G-bundle over X classified by a map $f\colon X \longrightarrow BG$. The gauge group $\mathcal{G}_f(P)$ of P is the group of G-equivariant automorphisms of P that fix X. In [1, 6] it was shown that there is a homotopy equivalence

$$B\mathcal{G}_f(P) \simeq \mathrm{Map}_f(X, BG)$$

where the right side is the component of the space of continuous maps from X to BG that contains the map f. This description is advantageous as there is an evaluation fibration sequence

$$G \xrightarrow{\partial_f} \mathrm{Map}^*_f(X, BG) \longrightarrow \mathrm{Map}_f(X, BG) \xrightarrow{ev} BG \tag{3}$$

where ev evaluates a map at the basepoint, $\mathrm{Map}^*_f(X, BG)$ is the component of the space of continuous pointed maps from X to BG that contains f, and ∂_f is the fibration connecting map. It is worth pointing out a subtlety in (3): $\mathrm{Map}_f(X, BG)$ consists of maps f' homotopic to f by unbased homotopies while $\mathrm{Map}^*_f(X, BG)$ consists of maps homotopic to f by based homotopies, so if f' is in the fibre of the evaluation map then an argument is needed to say that the homotopy between f' and f may be chosen to be pointed so that $f' \in \mathrm{Map}^*_f(X, BG)$; such an argument is given in [6, Lemma 5.5]. The salient point of the homotopy fibration (3) is that $\mathcal{G}_f(P)$ is the homotopy fibre of ∂_f.

In our case, let $\mathcal{G}_k(\Sigma_g, PU(n))$ be the gauge group of the principal $PU(n)$-bundle classified by $k \in [\Sigma_g, BPU(n)] \cong \mathbb{Z}/n\mathbb{Z}$. In the special case when $\Sigma_g = S^2$ write $\mathcal{G}_k(PU(n))$. The isomorphism $[S^2, BPU(n)] \xrightarrow{q^*} [\Sigma_g, BPU(n)]$ in Lemma 2.2 implies that the map $\Sigma_g \xrightarrow{q} S^2$ induces a one-to-one correspondence between the components of $\mathrm{Map}^*(\Sigma^g, BPU(n))$ and $\mathrm{Map}^*(S^2, BPU(n))$. Since $PU(n)$ is path-connected, taking π_0 in the homotopy fibration (3) implies that there is a matching one-to-one correspondence between the components of $\mathrm{Map}(\Sigma^g, BPU(n))$ and $\mathrm{Map}(S^2, BPU(n))$. Therefore there is a homotopy commutative diagram of evaluation fibrations

$$\begin{array}{ccccccc}
PU(n) & \xrightarrow{\partial_k} & \mathrm{Map}^*_k(S^2, BPU(n)) & \longrightarrow & \mathrm{Map}_k(S^2, BPU(n)) & \xrightarrow{ev} & BPU(n) \\
\| & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \| \\
PU(n) & \xrightarrow{\overline{\partial}_k} & \mathrm{Map}^*_k(\Sigma_g, BPU(n)) & \longrightarrow & \mathrm{Map}_k(\Sigma_g, BPU(n)) & \xrightarrow{ev} & BPU(n).
\end{array} \tag{4}$$

It is well known that the components of $\mathrm{Map}^*(S^2, BPU(n)) \simeq \Omega PU(n)$ are all homotopy equivalent. Sutherland [13] showed that the same is true for the components of $\mathrm{Map}^*(\Sigma_g, BPU(n))$, and the homotopy equivalences are compatible in the sense that there is a homotopy fibration diagram

$$\begin{array}{ccc}
\mathrm{Map}^*_k(S^2, BPU(n)) & \xrightarrow{\simeq} & \mathrm{Map}^*_0(S^2, BPU(n)) \\
\downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\
\mathrm{Map}^*_k(\Sigma_g, BPU(n)) & \xrightarrow{\simeq} & \mathrm{Map}^*_0(\Sigma_g, BPU(n)).
\end{array} \tag{5}$$

Write ∂_k also for the composite $PU(n) \xrightarrow{\partial_k} \mathrm{Map}^*_k(S^2, BPU(n)) \xrightarrow{\simeq} \mathrm{Map}^*_0(S^2, BPU(n))$, and do likewise for $\overline{\partial}_k$. Write $\mathrm{Map}^*_0(S^2, BPU(n))$ as $\Omega_0 PU(n)$, the component of $\Omega PU(n)$ containing the basepoint. We have $\mathrm{Map}_k(S^2, BPU(n)) \simeq B\mathcal{G}_k(PU(n))$ and $\mathrm{Map}_k(\Sigma_g, BPU(n)) \simeq B\mathcal{G}_k(\Sigma_g, PU(n))$. Then from (4) and (5) we obtain a homotopy fibration diagram

$$
(6)\qquad
\begin{array}{ccccccc}
PU(n) & \xrightarrow{\partial_k} & \Omega_0 PU(n) & \longrightarrow & B\mathcal{G}_k(PU(n)) & \xrightarrow{ev} & BPU(n) \\
\Vert & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \Vert \\
PU(n) & \xrightarrow{\overline{\partial}_k} & \mathrm{Map}^*_0(\Sigma_g, BPU(n)) & \longrightarrow & B\mathcal{G}_k(\Sigma_g, PU(n)) & \xrightarrow{ev} & BPU(n).
\end{array}
$$

Lemma 2.3. *There is a homotopy fibration* $\prod_{i=1}^{2g} \Omega PU(n) \xrightarrow{(\Sigma f)^*} \Omega_0 PU(n) \xrightarrow{q^*} Map^*_0(\Sigma_g, BPU(n))$.

Proof. If $g = 0$ then q^* is the identity map and the product $\prod_{i=1}^{2g} \Omega PU(n)$ is a point, so the assertion holds. If $g \geq 1$, the homotopy cofibration sequence $\bigvee_{i=1}^{2g} S^1 \longrightarrow \Sigma_g \xrightarrow{q} S^2 \xrightarrow{\Sigma f} \bigvee_{i=1}^{2g} S^2$ implies that there is a homotopy fibration sequence

$$\mathrm{Map}^*(\bigvee_{i=1}^{2g} S^2, BPU(n)) \xrightarrow{(\Sigma f)^*} \mathrm{Map}^*(S^2, BPU(n)) \xrightarrow{q^*} \mathrm{Map}^*(\Sigma_g, BPU(n)) \longrightarrow \mathrm{Map}^*(\bigvee_{i=1}^{2g} S^1, BPU(n)).$$

Observe that there is a homotopy equivalence $\mathrm{Map}^*(\bigvee_{i=1}^{2g} S^1, BPU(n)) \cong \prod_{i=1}^{2g} PU(n)$; in particular, this space is connected. Restricting the map $\mathrm{Map}^*(\Sigma_g, BPU(n)) \longrightarrow \prod_{i=1}^{2g} PU(n)$ to the 0-component of $\mathrm{Map}^*(\Sigma_g, BPU(n))$ therefore gives a homotopy fibration sequence

$$\prod_{i=1}^{2g} \Omega PU(n) \xrightarrow{(\Sigma f^*)} \mathrm{Map}^*_0(S^2, BPU(n)) \xrightarrow{q^*} \mathrm{Map}^*_0(\Sigma_g, BPU(n)) \longrightarrow \prod_{i=1}^{2g} PU(n).$$

Rewriting $\mathrm{Map}^*_0(S^2, BPU(n))$ as $\Omega_0 PU(n)$ gives the asserted homotopy fibration. □

From the left square of (6) and Lemma 2.3 we obtain a homotopy fibration diagram

$$
(7)\qquad
\begin{array}{ccccc}
 & & \Omega\mathrm{Map}^*_0(\Sigma_g, BPU(n)) & = & \Omega\mathrm{Map}^*_0(\Sigma_g, BPU(n)) \\
 & & \downarrow & & \downarrow{\scriptstyle a} \\
\mathcal{G}_k(PU(n)) & \longrightarrow & \mathcal{G}_k(\Sigma_g, PU(n)) & \xrightarrow{b} & \prod_{i=1}^{2g} \Omega PU(n) \\
\Vert & & \downarrow & & \downarrow{\scriptstyle (\Sigma f)^*} \\
\mathcal{G}_k(PU(n)) & \longrightarrow & PU(n) & \xrightarrow{\partial_k} & \Omega_0 PU(n) \\
 & & \downarrow{\scriptstyle \overline{\partial}_k} & & \downarrow{\scriptstyle q^*} \\
 & & \mathrm{Map}^*_0(\Sigma_g, BPU(n)) & = & \mathrm{Map}^*_0(\Sigma_g, BPU(n))
\end{array}
$$

which defines the maps a and b.

Proof of Theorem 1.1. Since Σf is null homotopic, so is $(\Sigma f)^*$. Therefore the map a in (7) has a right homotopy inverse. The top square in (7) then implies that the map b has a right homotopy inverse as well. Since $\mathcal{G}_k(\Sigma_g, PU(n))$ is an H-space, the right homotopy inverse for b can be multiplied with the map from the fibre of b to obtain a homotopy equivalence

$$\mathcal{G}_k(\Sigma_g, PU(n)) \simeq \left(\prod_{i=1}^{2g} \Omega PU(n)\right) \times \mathcal{G}_k(PU(n)).$$

The statement of the theorem is now obtained by substituting in $\Omega PU(n) \simeq \mathbb{Z}/n\mathbb{Z} \times \Omega SU(n)$, which is obtained from the homotopy fibration $\mathbb{Z}/n\mathbb{Z} \longrightarrow SU(n) \longrightarrow PU(n)$. □

3. Properties of ∂_k when $n = p$

Fix a prime p. In this section we specialize to $\mathcal{G}_k(PU(p))$ and factor the connecting map $PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ for the evaluation fibration. To start, we need some known results about $PU(p)$. In what follows, homology and cohomology will be taken with mod-p coefficients.

The mod-p cohomology of $PU(p)$ was calculated by Baum and Browder [2].

Lemma 3.1. *There is an algebra isomorphism*

$$H^*(PU(p)) \cong \mathbb{Z}/p\mathbb{Z}[y]/(y^p) \otimes \Lambda(z_1, z_3, \cdots, z_{2p-3})$$

where $|y| = 2$, $|z_i| = i$, *and* $\beta(z_1) = y$. □

A p-local homotopy decomposition for $PU(p)$ was proved by Kishimoto and Kono [9].

Proposition 3.2. *Localized at an odd prime p, there is a homotopy equivalence*

$$PU(p) \simeq \left(\prod_{i=1}^{p-2} S^{2i+1}\right) \times L$$

where L is a space with $\pi_1(L) \cong \pi_1(PU(p))$ and the universal cover of L is S^{2p-1}. □

Remark 3.3. When $p = 2$, $PU(p) \cong SO(3)$ so the statement of Proposition 3.2 still holds.

Observe that by comparing the cohomology decomposition of $H^*(PU(p))$ with that induced by the homotopy decomposition in Proposition 3.2, we obtain

$$H^*(L) \cong \mathbb{Z}/p\mathbb{Z}[y]/(y^p) \otimes \Lambda(z_1).$$

Remark 3.4. The space L is not identified in [9] as the lens space $S^{2p-1}/(\mathbb{Z}/p\mathbb{Z})$ obtained as the $(2p-1)$-skeleton of $B\mathbb{Z}/p\mathbb{Z}$, but they are homotopy equivalent. For the generator $z_1 \in H^1(L)$ is represented by a map $f\colon L \longrightarrow K(\mathbb{Z}/p\mathbb{Z}, 1) \simeq B\mathbb{Z}/p\mathbb{Z}$. It is well known that $H^*(B\mathbb{Z}/p\mathbb{Z}) \cong \Lambda(u) \otimes \mathbb{Z}/p\mathbb{Z}[v]$ where $|u| = 1$, $|v| = 2$ and $\beta(u) = v$. Now $f^*(u) = z_1$ and the Bockstein implies that $f^*(v) = y$, so as f^* is an algebra map it is a surjection. In fact, the restriction of f^* to the $(2p-1)$-skeleton S of $B\mathbb{Z}/p\mathbb{Z}$ is an isomorphism. As a CW-complex, L is $(2p-1)$-dimensional, so by cellular approximation the map f factors through the $(2p-1)$-skeleton S of $B\mathbb{Z}/p\mathbb{Z}$. Thus the resulting map $L \longrightarrow S$ induces an isomorphism in cohomology and on the fundamental group, and so is a homotopy equivalence by Whitehead's Theorem.

In Lemma 3.6 we will show that ∂_k factors through L. This begins with a general definition.

Definition 3.5. Suppose that A is an H-space with multiplication m. A map $f\colon A\longrightarrow F$ is a *homotopy action* if there is a map $\theta\colon A\times F\longrightarrow F$ extending $A\vee F\xrightarrow{f\vee 1}F$ and satisfying a homotopy commutative diagram

$$\begin{array}{ccc} A\times A & \xrightarrow{m} & A \\ \downarrow{\scriptstyle 1\times f} & & \downarrow{\scriptstyle f} \\ A\times F & \xrightarrow{\theta} & F. \end{array}$$

The canonical example of a homotopy action is the connecting map $\Omega B\longrightarrow F$ of a homotopy fibration $F\longrightarrow E\longrightarrow B$. In our case, the homotopy fibration sequence $PU(p)\xrightarrow{\partial_k}\Omega_0 PU(p)\longrightarrow B\mathcal{G}_k(PU(p))\longrightarrow BPU(p)$ from (6) gives a homotopy action $\theta\colon PU(p)\times\Omega_0PU(p)\longrightarrow\Omega_0PU(p)$.

Localize at p. By Proposition 3.2, there is a homotopy equivalence $PU(p)\simeq\left(\prod_{i=1}^{p-2}S^{2i+1}\right)\times L$. Let δ_k be the composite

$$\delta_k\colon L\hookrightarrow PU(p)\xrightarrow{\partial_k}\Omega_0PU(p).$$

Lemma 3.6. *Localize at p. For any $k\in\mathbb{Z}/p\mathbb{Z}$ there is a homotopy commutative diagram*

$$\begin{array}{ccc} PU(p) & \xrightarrow{\partial_k} & \Omega_0PU(p) \\ \downarrow{\scriptstyle\varpi} & & \| \\ L & \xrightarrow{\delta_k} & \Omega_0PU(p) \end{array}$$

where ϖ has a right homotopy inverse.

Proof. The argument proceeds in several steps.

Step 1: Alter the homotopy equivalence $PU(p)\simeq\left(\prod_{i=1}^{p-2}S^{2i+1}\right)\times L$ so it is written as a product of maps. The decomposition implies that in mod-p cohomology there is an algebra isomorphism $H^*(PU(p))\cong\otimes_{i=1}^{p-2}H^*(S^{2i+1})\otimes H^*(L)$. For $1\leq i\leq p-2$, let $g_i\colon S^{2i+1}\longrightarrow PU(p)$ be the inclusion and let $\mathfrak{g}\colon L\longrightarrow PU(p)$ be the inclusion. Since each g_i for $1\leq i\leq p-2$ and $\mathfrak{g}$ have left homotopy inverses, the maps g_i^* and $\mathfrak{g}^*$ are surjections. Consider the composite

$$e\colon\left(\prod_{i=1}^{p-2}S^{2i+1}\right)\times L\xrightarrow{(\prod_{i=1}^{p-2}g_i)\times\mathfrak{g}}\prod_{i=1}^{p-1}PU(p)\xrightarrow{\quad m\quad}PU(p)$$

where m is the iterated multiplication on $PU(p)$. For $1\leq i\leq p-2$, the restriction of e to S^{2i+1} is g_i, so e^* surjects onto $H^*(S^{2i+1})$. The restriction of e to L is $\mathfrak{g}$, so e^* surjects onto $H^*(L)$. Therefore e^* is an algebra map surjecting onto the generating set of $\left(\otimes_{i=1}^{p-2}H^*(S^{2i+1})\right)\otimes H^*(L)$, implying that e^* is a surjection. The homotopy equivalence $PU(p)\simeq\left(\prod_{i=1}^{p-2}S^{2i+1}\right)\times L$ implies that $H^*(PU(p))$ and $\left(\otimes_{i=1}^{p-2}H^*(S^{2i+1})\right)\otimes H^*(L)$ have the same Euler-Poincaré series, so e^* being a surjection implies it must be an isomorphism. Therefore, e is a homotopy equivalence by Whitehead's Theorem.

Step 2: For $1 \leq i \leq p-2$ the composite $\partial_k \circ g_i$ is null homotopic. Consider the composite

$$S^{2i+1} \xrightarrow{g_i} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p).$$

Since $SU(p)$ is the universal cover of $PU(p)$ and the even dimensional homotopy groups of $SU(p)$ are trivial in dimensions $\leq 2p-2$, we have $\pi_{2i+1}(\Omega_0 PU(p)) \cong \pi_{2i+1}(\Omega SU(p)) \cong 0$. Thus $\partial_k \circ g_i$ is null homotopic.

Step 3: A null homotopy for ∂_k applied to the product of the maps g_i. For $2 \leq t \leq p-2$, let f_t be the composite

$$f_t\colon \prod_{i=1}^{t} S^{2i+1} \xrightarrow{\prod_{i=1}^{t} g_i} \prod_{i=1}^{t} PU(p) \xrightarrow{\ m\ } PU(p).$$

We now claim that the composite $\prod_{i=1}^{p-2} S^{2i+1} \xrightarrow{f_{p-2}} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is null homotopic. Since ∂_k is the connecting map in a homotopy fibration, there is a homotopy action $\theta\colon PU(p) \times \Omega_0 PU(p) \longrightarrow \Omega_0 PU(p)$. Consider the diagram

$$\begin{array}{ccccc}
\left(\prod_{i=1}^{t-1} S^{2i+1}\right) \times S^{2t+1} & \xrightarrow{f_{t-1}\times g_t} & PU(p) \times PU(p) & \xrightarrow{\ m\ } & PU(p) \\
& \searrow^{f_{t-1}\times *} & \downarrow{\scriptstyle 1\times\partial_k} & & \downarrow{\scriptstyle \partial_k} \\
& & PU(p) \times \Omega_0 PU(p) & \xrightarrow{\ \theta\ } & \Omega_0 PU(p).
\end{array}$$

The left triangle homotopy commutes since $\partial_k \circ g_t$ is null homotopic and the right square homotopy commutes since θ is a homotopy action. The top row is homotopic to f_t. As the restriction of θ to $PU(p)$ is ∂_k, the lower direction around the diagram is homotopic to $\partial_k \circ f_{t-1} \circ \pi$, where π is the projection onto $\prod_{i=1}^{t-1} S^{2i+1}$. Thus the homotopy commutativity of the diagram implies that $\partial_k \circ f_t \simeq \partial_k \circ f_{t-1} \circ \pi$. Consequently, a null homotopy for $\partial_k \circ f_{t-1}$ implies that $\partial_k \circ f_t$ is also null homotopic. When $t = 2$ we have $f_{t-1} = g_1$ and $\partial_k \circ g_1$ is null homotopic. Thus, iteratively, f_t is null homotopic for all $2 \leq t \leq p-2$. In particular, f_{p-2} is null homotopic, as claimed.

Step 4: The factorization of ∂_k through δ_k. Consider the diagram

$$\begin{array}{ccccccc}
PU(p) & \xrightarrow{e^{-1}} & L \times \left(\prod_{i=1}^{p-2} S^{2i+1}\right) & \xrightarrow{\mathfrak{g}\times f_{p-2}} & PU(p) \times PU(p) & \xrightarrow{\ m\ } & PU(p) \\
& & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle 1\times\partial_k} & & \downarrow{\scriptstyle \partial_k} \\
& & L & \xrightarrow{i_1\circ\mathfrak{g}} & PU(p) \times \Omega_0 PU(p) & \xrightarrow{\ \theta\ } & \Omega_0 PU(p)
\end{array}$$

where π is the projection and i_1 is the inclusion of the first factor. The left square homotopy commutes since $\partial_k \circ f_{p-2}$ is null homotopic and the right square homotopy commutes since θ is a homotopy action. Along the top row, $m \circ (\mathfrak{g} \times f_{p-2})$ is homotopic to e so the top row is homotopic to the identity map. Since the restriction of θ to $PU(p)$ is ∂_k, the bottom row is homotopic to $\partial_k \circ \mathfrak{g}$, which by definition, is δ_k. Thus the homotopy commutativity of the diagram implies that

$\partial_k \simeq \delta_k \circ \pi \circ e^{-1}$. Finally, let $\varpi = \pi \circ e^{-1}$. Then ϖ has a right homotopy inverse since π does and since e^{-1} is a homotopy equivalence, and $\partial_k \simeq \delta_k \circ \varpi$, as asserted. □

Next, we consider δ_k more closely. An important step is to determine a p-local homotopy decomposition for ΣL_{2p-2} when p is odd, where L_{2p-2} is the $(2p-2)$-skeleton of L. We will freely use the cohomology of $PU(p)$ and L described earlier.

For $t \geq 2$, the mod-p Moore space $P^t(p)$ is the homotopy cofibre of the degree p map on S^{t-1}. Note that $\Sigma P^t(p) \simeq P^{t+1}(p)$. The Moore space is characterized by the fact that $H^*(P^t(p)) \cong \mathbb{Z}/p\mathbb{Z}\{u_{t-1}, v_t\}$ where $|u_{t-1}| = t-1$, $|v_t| = t$ and $\beta(u) = v$. In particular, this implies that the 2-skeleton of $PU(p)$ is $P^2(p)$. Let $j\colon P^2(p) \longrightarrow PU(p)$ be the skeletal inclusion. Writing u, v for u_2, v_2, we have $j^*(z_1) = u$ and $j^*(y) = v$. For $1 \leq m \leq p-1$, let $\bar{j}_m$ be the composite

$$\bar{j}_m\colon \prod_{i=1}^m P^2(p) \xrightarrow{\prod_{i=1}^m j} \prod_{i=1}^m PU(p) \xrightarrow{\ m\ } PU(p)$$

where m is the multiplication on $PU(p)$. In cohomology, m^* induces the comultiplication in the Hopf algebra structure on $H^*(PU(p))$, and this Hopf algebra structure implies that $m^*(y^m) = \otimes_{i=1}^m y + A$ where A is a sum of tensor products involving y^k for $k \geq 2$. As $j^*(y) = v$ and $j^*(y^k) = 0$ for $k \geq 2$, we obtain $(\bar{j}_m)^*(y^m) = \otimes_{i=1}^m v$. After suspending, we obtain a map

$$\bar{\bar{j}}_m\colon \Sigma \bigwedge_{i=1}^m P^2(p) \longrightarrow \Sigma\left(\prod_{i=1}^m P^2(p)\right) \xrightarrow{\Sigma \bar{j}_m} \Sigma PU(p)$$

with $(\bar{\bar{j}}_m)^*(\sigma y^m) = \sigma(\otimes_{i=1}^m v)$. By [11], if p is odd there is a homotopy equivalence $P^s(p) \wedge P^t(p) \simeq P^{s+t}(p) \vee P^{s+t-1}(p)$. Iterating this, we obtain a map $P^{2m+1}(p) \longrightarrow \Sigma \bigwedge_{i=1}^m P^2(p)$ which induces an isomorphism on $H_{2m+1}(\ \ )$. Therefore we obtain a composite

$$j_m\colon P^{2m+1}(p) \longrightarrow \Sigma \bigwedge_{i=1}^m P^2(p) \xrightarrow{\bar{\bar{j}}_m} \Sigma PU(p)$$

with the property that $(j_m)^*(\sigma y^m) = v_{2m+1}$, where v_{2m+1} is a generator of $H^{2m+1}(P^{2m+1}(p))$. Let $u_{2m} \in H^{2m}(P^{2m+1}(p))$ be a generator with the property that $\beta(u_{2m}) = v_{2m+1}$. Since $\beta(z_1 \otimes y^{m-1}) = y^m$ in $H^*(PU(p))$, the naturality of the Bockstein implies that $(j_m)^*(\sigma(z_1 \otimes y^{m-1})) = u_{2m}$. Therefore $(j_m)^*$ is an epimorphism. Let $\overline{\psi}$ be the wedge sum of the maps j_m for $1 \leq m \leq p-1$,

$$\overline{\psi}\colon \bigvee_{m=1}^{p-1} P^{2m+1}(p) \longrightarrow \Sigma PU(p).$$

Localize at p. Using the map ϖ in Lemma 3.6 gives a composite

$$\psi\colon \bigvee_{m=1}^{p-1} P^{2m+1}(p) \xrightarrow{\overline{\psi}} \Sigma PU(p) \xrightarrow{\Sigma\varpi} \Sigma L.$$

Notice that the domain of ψ has dimension $2p-1$, so ψ factors as a composite

$$\bigvee_{m=1}^{p-1} P^{2m+1}(p) \xrightarrow{\psi'} \Sigma L_{2p-2} \longrightarrow \Sigma L$$

for some map ψ'.

Lemma 3.7. *Localized at an odd prime p, the map $\bigvee_{m=1}^{p-1} P^{2m+1}(p) \xrightarrow{\psi'} \Sigma L_{2p-2}$ is a homotopy equivalence.*

Proof. Since $PU(p) \xrightarrow{\varpi} L$ has a right homotopy inverse, it induces an inclusion onto the subalgebra $\mathbb{Z}/p\mathbb{Z}[y]/(y^p) \otimes \Lambda(z_1)$ in cohomology. Therefore the definition of ψ as the wedge sum of the maps j_m for $1 \leq m \leq p-1$ implies that ψ^* induces an isomorphism in cohomology in dimensions $\leq 2p-1$. If $\Sigma L_{2p-2} \longrightarrow \Sigma L$ also induces an isomorphism in cohomology in dimensions $\leq 2p-1$, then $(\psi')^*$ induces an isomorphism in cohomology in all dimensions and so is a homotopy equivalence by Whitehead's Theorem.

To finish, it suffices to show that the skeletal inclusion $L_{2p-2} \longrightarrow L$ induces an isomorphism in cohomology in dimensions $\leq 2p-2$. This is true for skeletal reasons in dimensions $\leq 2p-3$. In dimension $2p-2$, the homotopy cofibration $L_{2p-2} \xrightarrow{s} L \xrightarrow{t} S^{2p-1}$, where s is the skeletal inclusion and t is the pinch map to the top cell, induces a long exact sequence

$$0 \longrightarrow H^{2p-1}(\Sigma L) \xrightarrow{(\Sigma s)^*} H^{2p-1}(\Sigma L_{2p-2}) \longrightarrow H^{2p-1}(S^{2p-1}) \xrightarrow{t^*} H^{2p-1}(L) \longrightarrow 0.$$

Since $H^{2p-1}(S^{2p-1})$ and $H^{2p-1}(L)$ have a single $\mathbb{Z}/p\mathbb{Z}$ generator and t^* is a surjection by exactness, t^* must be an isomorphism. Thus, by exactness, $(\Sigma s)^*$ is also an isomorphism. Hence $L_{2p-2} \xrightarrow{s} L$ induces an isomorphism in cohomology in dimensions $\leq 2p-2$. □

Remark 3.8. When $p = 2$, $L = PU(2)$, the analogue of ψ is the map $P^3(2) \xrightarrow{\Sigma j} \Sigma PU(2)$, and the analogue of ψ' is the identity map. Thus the statement of Lemma 3.7 also holds for $p = 2$.

To use Lemma 3.7 to further analyze the map δ_k in Lemma 3.6, we need a general lemma. For a space X, let $X^{\times m}$ be the m-fold Cartesian product of X with itself.

Lemma 3.9. *Suppose that there is a homotopy fibration sequence $\Omega B \xrightarrow{\partial} F \longrightarrow E \longrightarrow B$ and a map $a\colon A \longrightarrow \Omega B$ such that $\partial \circ a$ is null homotopic. For $m \geq 1$, let a_m be the composite*

$$a_m\colon A^{\times m} \xrightarrow{a^{\times m}} \Omega B^{\times m} \xrightarrow{\mu} \Omega B$$

where the right map is loop multiplication on ΩB. Then $\partial \circ a_m$ is null homotopic for all $m \geq 1$.

Proof. The proof is by induction on m. Since $a_1 = a$, the statement is true for $m = 1$ by hypothesis. Suppose that $\partial \circ a_{m-1}$ is null homotopic. Consider the diagram

$$\begin{array}{ccccc} A \times A^{\times(m-1)} & \xrightarrow{a \times a_{m-1}} & \Omega B \times \Omega B & \xrightarrow{\mu} & \Omega B \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle 1 \times \partial} & & \downarrow{\scriptstyle \partial} \\ A & \xrightarrow{i_1 \circ a} & \Omega B \times F & \xrightarrow{\theta} & F \end{array}$$

where π_1 is the projection onto the first factor, i_1 is the inclusion of the first factor, and θ is the homotopy action associated to the given homotopy fibration sequence. The left square homotopy commutes by the inductive hypothesis and the right square homotopy commutes by the homotopy action. Notice that the top row is homotopic to a_m, while the bottom row is homotopic to $\partial \circ a$, which is null homotopic. Thus the homotopy commutativity of the diagram as a whole implies that $\partial \circ a_m$ is null homotopic. □

Lemma 3.10. *Let p be a prime and suppose that the composite $P^2(p) \xrightarrow{j} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is null homotopic. Then, localized at p, the composite $L_{2p-2} \longrightarrow L \xrightarrow{\delta_k} \Omega_0 PU(p)$ is null homotopic.*

Proof. Let $a = j$ and for each $m \geq 1$, let a_m be the composite

$$a_m \colon P^2(p)^{\times m} \xrightarrow{a^{\times m}} PU(p)^{\times m} \xrightarrow{\mu} PU(p).$$

Since ∂_k is the connecting map for the homotopy fibration $\Omega_0 PU(p) \longrightarrow B\mathcal{G}_k(PU(p)) \xrightarrow{ev} BPU(p)$ and $\partial_k \circ a$ is null homotopic, Lemma 3.9 implies that $\partial_k \circ a_m$ is null homotopic for all $m \geq 1$. Taking adjoints, the composite

$$\Sigma P^2(p)^{\times m} \xrightarrow{\Sigma a_m} \Sigma PU(p) \xrightarrow{\overline{\partial}_k} PU(p)$$

is null homotopic, where $\overline{\partial}_k$ is the adjoint of ∂_k. Localize at p and consider the diagram

$$(8) \qquad \begin{array}{ccccccc} P^{2m+1}(p) & \longrightarrow & \Sigma(P^2(p)^{\times m}) & \xrightarrow{\Sigma a_m} & \Sigma PU(p) & \xrightarrow{\Sigma\varpi} & \Sigma L \\ & & \downarrow & & \downarrow{\scriptstyle \overline{\partial}_k} & & \downarrow{\scriptstyle \overline{\delta}_k} \\ & & * & \longrightarrow & PU(p) & = & PU(p). \end{array}$$

where $\overline{\delta}_k$ is the adjoint of δ_k. The left square homotopy commutes since $\overline{\partial}_1 \circ \Sigma a_k$ is null homotopic and the right square homotopy commutes by taking the adjoint of the homotopy commutative diagram in the statement of Lemma 3.6. Recall the maps that were defined leading to the homotopy decomposition of ΣL_{2p-2} in Lemma 3.7. The composite $P^{2m+1}(p) \longrightarrow \Sigma(P^2(p)^{\times m}) \xrightarrow{\Sigma a_m} \Sigma PU(p)$ along the top row in (8) is the definition of the map j_m. Thus the diagram shows that $\overline{\partial}_k \circ j_m$ is null homotopic. The map $\bigvee_{m=1}^{p-1} P^{2m+1}(p) \xrightarrow{\overline{\psi}} \Sigma PU(p)$ is the wedge sum of the maps j_m for $1 \leq m \leq p-1$, so $\overline{\partial}_k \circ \overline{\psi}$ is null homotopic. The map $\bigvee_{m=1}^{p-1} P^{2m+1}(p) \xrightarrow{\psi} \Sigma L$ is the composite of $\overline{\psi}$ with $\Sigma\varpi$, so the homotopy commutativity of the right square in (8) implies that $\overline{\delta}_k \circ \psi$ is null homotopic. The map $\bigvee_{m=1}^{p-1} P^{2m+1}(p) \xrightarrow{\psi'} \Sigma L_{2p-2}$ is the factorization of ψ through the $(2p-1)$-skeleton of ΣL, so $\overline{\delta}_k \circ \psi'$ is null homotopic. But by Lemma 3.7, ψ' is a homotopy equivalence, so the composite $\Sigma L_{2p-2} \longrightarrow \Sigma L \xrightarrow{\overline{\delta}_k} PU(p)$ is null homotopic. Hence, taking adjoints, the composite $L_{2p-2} \longrightarrow L \xrightarrow{\delta_k} \Omega_0 PU(p)$ is null homotopic. □

4. Samelson products on $PU(p)$ and the fundamental group of $\mathcal{G}_k(PU(p))$.

In this section we use the information obtained about the boundary map ∂_k to show the existence of a non-trivial Samelson product on $PU(p)$ and use this to determine $\pi_1(\mathcal{G}_k(PU(p)))$.

We need to compare $PU(p)$ gauge groups with those for $U(p)$. In general, for any $n \geq 2$, the fibration $S^1 \longrightarrow U(n) \longrightarrow PU(n)$ induces an epimorphism $\pi_1(U(n)) \cong \mathbb{Z} \longrightarrow \pi_1(PU(n)) \cong \mathbb{Z}/n\mathbb{Z}$. Equivalently, there is an epimorphism $[S^2, BU(n)] \cong \mathbb{Z} \longrightarrow [S^2, BPU(n)] \cong \mathbb{Z}/n\mathbb{Z}$. For $\bar{k} \in \mathbb{Z}$, let $\mathcal{G}_{\bar{k}}(U(n))$ be the gauge group of the principal $U(n)$-bundle over S^2 classified by the degree $\bar{k}$ map in $\mathbb{Z} \cong [S^2, BU(n)]$. As was the case for $PU(n)$, there is a homotopy equivalence $B\mathcal{G}_{\bar{k}}(U(n)) \simeq \mathrm{Map}_{\bar{k}}(S^2, BU(n))$ and the components of $\mathrm{Map}^*(S^2, BU(n))$ are all homotopy equivalent to $\mathrm{Map}^*_0(S^2, BU(n)) = \Omega_0 U(n)$. If the mod-$n$ reduction of $\bar{k}$ is k then the epimorphism $[S^2, BU(n)] \longrightarrow [S^2, BPU(n)]$ and the naturality of the evaluation fibration implies that there is a homotopy commutative diagram of fibration sequences

$$\begin{array}{ccccccc} U(n) & \xrightarrow{\partial_{\bar{k}}} & \Omega_0 U(n) & \longrightarrow & B\mathcal{G}_{\bar{k}}(U(n)) & \longrightarrow & BU(n) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ PU(n) & \xrightarrow{\partial_k} & \Omega_0 PU(n) & \longrightarrow & B\mathcal{G}_k(PU(n)) & \longrightarrow & BPU(n). \end{array} \tag{9}$$

Note that $\Omega_0 U(n) \simeq \Omega SU(n) \simeq \Omega_0 PU(n)$ and the map $\Omega_0 U(n) \longrightarrow \Omega_0 PU(n)$ in (9) is a homotopy equivalence.

Let $\epsilon_1 \colon S^1 \longrightarrow U(n)$ represent a generator of $\pi_1(U(n)) \cong \mathbb{Z}$ and let $i \colon S^1 \longrightarrow PU(n)$ be the composite $S^1 \xrightarrow{\epsilon_1} U(n) \longrightarrow PU(n)$, so i represents a generator of $\pi_1(PU(n)) \cong \mathbb{Z}/n\mathbb{Z}$. Write 1 for the identity map of $U(n)$ or $PU(n)$, the context making clear which is intended. The following was proved in [10].

Lemma 4.1. *The adjoint of $\partial_{\bar{k}}$ is $\bar{k}$ times the Samelson product $\langle \epsilon_1, 1 \rangle \colon S^1 \wedge U(n) \longrightarrow U(n)$, and the adjoint of ∂_k is k times the Samelson product $\langle i, 1 \rangle \colon S^1 \wedge PU(n) \longrightarrow PU(n)$.* □

Let $\epsilon_n \colon S^{2n-1} \longrightarrow U(n)$ represent the generator of $\pi_{2n-1}(U(n)) \cong \mathbb{Z}$. By Lemma 4.1 and the naturality of the Samelson product, the adjoint of the composite $S^{2n-1} \xrightarrow{\epsilon_n} U(n) \xrightarrow{\partial_{\bar{k}}} \Omega_0 U(n)$ is $\bar{k}$ times the Samelson product $\langle \epsilon_1, \epsilon_n \rangle \colon S^1 \wedge S^{2n-1} \longrightarrow U(n)$. By [3], $\langle \epsilon_1, \epsilon_n \rangle$ has order n. By Remark 3.4, we may regard L up to homotopy equivalence as the lens space $S^{2p-1}/(\mathbb{Z}/p\mathbb{Z})$.

Lemma 4.2. *Localize at a prime p. There is a homotopy commutative diagram*

$$\begin{array}{ccc} S^{2p-1} & \xrightarrow{\epsilon_p} & U(p) \\ \downarrow{\scriptstyle i_p} & & \downarrow \\ L & \longrightarrow & PU(p) \end{array}$$

where i_p is homotopic, up to multiplication by a unit in $\mathbb{Z}_{(p)}$, to the quotient map for the lens space L.

Proof. If $p = 2$ then $L = PU(2)$ and $U(2) \longrightarrow PU(2)$ induces an isomorphism on π_3, so we can choose ϵ_2 to be a lift of the quotient map $S^3 \longrightarrow PU(2)$, in which case i_2 is the quotient map.

If p is odd then Proposition 3.2 states that there is a p-local homotopy equivalence $PU(p) \simeq \left(\prod_{i=1}^{p-2} S^{2i+1}\right) \times L$. Taking homotopy groups p-locally, by [17] $\pi_{2p-1}(S^{2i+1}) \cong 0$ for $1 \le i \le p-2$. Therefore the composite $S^{2p-1} \xrightarrow{\epsilon_p} U(p) \longrightarrow PU(p)$ factors through L, as claimed. Observe that the map $L \longrightarrow PU(p)$ is an isomorphism on π_{2p-1}, so i_p represents the generator of $\pi_{2p-1}(L) \cong \mathbb{Z}_{(p)}$. Therefore i_p is homotopic to the lens space quotient map, up to multiplication by a unit in $\mathbb{Z}_{(p)}$. $\square$

Proposition 4.3. *Let p be a prime and suppose that $(k, p) = 1$. The following hold:*

(a) *the composite $P^2(p) \xrightarrow{j} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is nontrivial;*
(b) *the Samelson product $\langle i, j\rangle \colon S^1 \wedge P^2(p) \longrightarrow PU(p)$ is nontrivial;*
(c) $\pi_1(\mathcal{G}_k(PU(p))) \simeq \mathbb{Z}$.

Proof. For part (a), consider the cofibration $L_{2p-2} \longrightarrow L \longrightarrow S^{2p-1}$. The standard cell structure of the lens space $L = S^{2p-1}/(\mathbb{Z}/p\mathbb{Z})$ implies that the composite $S^{2p-1} \longrightarrow L \longrightarrow S^{2p-1}$ of the quotient map on the left and the pinch map to the top cell on the right is the degree p map. Now suppose that the composite $P^2(p) \xrightarrow{j} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is null homotopic. We will show that this leads to a contradiction. Localize at p. Then by Lemma 3.10 the composite $L_{2p-2} \longrightarrow L \xrightarrow{\delta_k} \Omega_0 PU(p)$ is null homotopic. Therefore δ_k extends to a map $\delta'_k \colon S^{2p-1} \longrightarrow \Omega_0 PU(p)$. Consider the diagram

$$\begin{array}{ccccc} S^{2p-1} & \xrightarrow{\epsilon_p} & U(p) & \xrightarrow{\partial_{\bar{k}}} & \Omega_0 U(p) \\ \downarrow{\scriptstyle i_p} & & \downarrow & & \downarrow{\scriptstyle \simeq} \\ L & \longrightarrow & PU(p) & \xrightarrow{\partial_k} & \Omega_0 PU(p) \\ \downarrow & & & \nearrow{\scriptstyle \delta'_k} & \\ S^{2p-1} & & & & \end{array} \tag{10}$$

The upper left square homotopy commutes by Lemma 4.2, the upper right square homotopy commutes by (9), and the lower triangle homotopy commutes by the definition of δ'_k.

The adjoint of $\partial_{\bar{k}} \circ \epsilon_p$ is $\bar{k}$ times the Samelson product $\langle \epsilon_1, \epsilon_p\rangle$, and $\langle \epsilon_1, \epsilon_p\rangle$ has order p. Note that $\pi_{2p-1}(\Omega_0 U(p)) \cong \pi_{2p}(U(p)) \cong \mathbb{Z}/p!\mathbb{Z}$ ($\cong \mathbb{Z}/p\mathbb{Z}$ as we are localized at p). Therefore if $(\bar{k}, p) = 1$ then $\partial_{\bar{k}} \circ \epsilon_p$ represents a generator of $\pi_{2p-1}(\Omega_0 U(p))$. On the other hand, by Lemma 4.2, i_p is homotopic to the quotient map defining the lens space, up to multiplication by a unit in $\mathbb{Z}_{(p)}$. Therefore the left column in (10) is $u \cdot p$ for some unit $u \in \mathbb{Z}_{(p)}$. The homotopy commutativity of (10) then implies that $\partial_{\bar{k}} \circ \epsilon_p \simeq u \cdot p \cdot \delta'_k$. This is a contradiction since $\partial_{\bar{k}} \circ \epsilon_p$ generates $\pi_{2p-1}(\Omega_0 U(p)) \cong \mathbb{Z}/p\mathbb{Z}$. Hence the composite $P^2(p) \xrightarrow{j} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is nontrivial, proving part (a).

For part (b), by Lemma 4.1 and the naturality of the Samelson product, the adjoint of the composite $P^2(p) \xrightarrow{j} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is k times the Samelson product $\langle i, j\rangle \colon S^1 \wedge P^2(p) \longrightarrow PU(p)$. Therefore part (b) follows immediately from part (a).

For part (c), consider the homotopy fibration sequence $\Omega PU(p) \longrightarrow \Omega_0^2 PU(p) \longrightarrow \mathcal{G}_k(PU(p)) \longrightarrow PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$. Applying π_1 gives an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(\mathcal{G}_k(PU(p))) \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow 0. \tag{11}$$

There are two options: either (11) splits or $\pi_1(\mathcal{G}_k(PU(p))) \cong \mathbb{Z}$. If (11) splits then there is a map $\delta\colon S^1 \longrightarrow \mathcal{G}_k(PU(p))$ representing the $\mathbb{Z}/p\mathbb{Z}$ generator of $\pi_1(\mathcal{G}_k(PU(p)))$ that extends to a map $\delta'\colon P^2(p) \longrightarrow \mathcal{G}_k(PU(p))$. The composite $P^2(p) \xrightarrow{\delta'} \mathcal{G}_k(PU(p) \longrightarrow PU(p)$ therefore induces an isomorphism on π_1 and hence an isomorphism on $H_1(\ ;\mathbb{Z})$ by the Hurewicz Isomorphism. This implies that the inclusion $P^2(p) \xrightarrow{j} PU(p)$ lifts through $\mathcal{G}_k(PU(p)) \longrightarrow PU(p)$, and hence the composite $P^2(p) \xrightarrow{j} PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$ is null homotopic. If $(k,p)=1$ this contradicts part (a). Thus, if $(k,p)=1$ we must have $\pi_1(\mathcal{G}_k(PU(p))) \cong \mathbb{Z}$, proving part (c). □

Remark 4.4. Rounding off the story on the fundamental group of $\mathcal{G}_k(PU(p))$, when $k=0$ part (a) of Theorem 1.2 will show that $\pi_1(\mathcal{G}_k(PU(p))) \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$.

5. The proof of Theorem 1.2

Two more preliminary lemmas are needed.

Lemma 5.1. *Let p be a prime. If $(k,p)=1$ then there is an isomorphism $\pi_2(\mathcal{G}_k(PU(p))) \cong 0$.*

Proof. The homotopy fibration $\Omega_0^2 PU(p) \longrightarrow \mathcal{G}_k(PU(p)) \longrightarrow PU(p)$ induces an exact sequence

$$\pi_3(PU(p)) \xrightarrow{(\partial_k)_*} \pi_2(\Omega_0^2 PU(p)) \longrightarrow \pi_2(\mathcal{G}_k(PU(p))) \longrightarrow \pi_2(PU(p)). \tag{12}$$

Using the universal cover $SU(p)$ of $PU(p)$, we have $\pi_2(PU(p)) \cong \pi_2(SU(p)) \cong 0$ while $\pi_2(\Omega_0^2 PU(p)) \cong \pi_4(SU(p))$ is 0 if $p>2$ and is $\mathbb{Z}/2\mathbb{Z}$ if $p=2$. Thus if $p>2$ then $\pi_2(\mathcal{G}_k(PU(p))) \cong 0$. Note that this works for all $k \in \mathbb{Z}/p\mathbb{Z}$. If $p=2$ then $\pi_3(PU(2)) \cong \mathbb{Z}$ and we claim that $(\partial_k)_*$ is an epimorphism if $(2,k)=1$. By (9) and the fact that the quotient map $U(2) \longrightarrow PU(2)$ induces an isomorphism on π_m for $m \geq 2$, it is equivalent to show that $\pi_3(U(2)) \xrightarrow{(\partial_{\bar{k}})_*} \pi_2(\Omega_0^2 U(2))$ is an epimorphism. But ϵ_2 represents a generator of $\pi_3(U(2))$ and $(\partial_{\bar{k}})_*(\epsilon_2) = \partial_{\bar{k}} \circ \epsilon_2$ has order 2 if $(2,k)=1$, as noted following Lemma 4.1. Hence $(\partial_k)_*$ is onto. Therefore, in this case, exactness in (12) implies that $\pi_2(\mathcal{G}_k(PU(p))) \cong 0$. □

For a space X, let $\widetilde{X}$ be its universal cover.

Lemma 5.2. *Let p be a prime. If $(k,p)=1$ then there is a homotopy equivalence of universal covers $\widetilde{\mathcal{G}}_{\bar{k}}(U(p)) \simeq \widetilde{\mathcal{G}}_k(PU(p))$.*

Proof. Consider the homotopy fibration diagram

$$\begin{array}{ccccc}
S^1 & = & S^1 & & \\
\downarrow & & \downarrow & & \\
\mathcal{G}_{\bar{k}}(U(p)) & \longrightarrow & U(p) & \xrightarrow{\partial_{\bar{k}}} & \Omega_0 U(p) \\
\downarrow & & \downarrow & & \downarrow \simeq \\
\mathcal{G}_k(PU(p)) & \longrightarrow & PU(p) & \xrightarrow{\partial_k} & \Omega_0 PU(p)
\end{array}$$

induced by taking homotopy fibres in (9), where we assume that the mod-p reduction of $\bar{k} \in \mathbb{Z}$ is k. Taking homotopy groups for the homotopy fibration in the left column gives an exact sequence

$$\pi_2(\mathcal{G}_k(PU(p))) \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\mathcal{G}_{\bar{k}}(U(p))) \longrightarrow \pi_1(\mathcal{G}_k(PU(p))) \longrightarrow \pi_0(S^1).$$

Since $(k,p) = 1$, by Lemma 5.1 we have $\pi_2(\mathcal{G}_k(PU(p))) \cong 0$, and clearly $\pi_0(S^1) \cong 0$. Thus we in fact have a short exact sequence

$$0 \longrightarrow \pi_1(S^1) \longrightarrow \pi_1(\mathcal{G}_{\bar{k}}(U(p))) \longrightarrow \pi_1(\mathcal{G}_k(PU(p))) \longrightarrow 0.$$

The fact that the homotopy fibration $S^1 \longrightarrow \mathcal{G}_{\bar{k}}(U(p)) \longrightarrow \mathcal{G}_k(PU(p))$ induces a short exact sequence on π_1 implies that, upon taking universal covers, there is a homotopy fibration $* \longrightarrow \widetilde{\mathcal{G}}_{\bar{k}}(U(p)) \longrightarrow \widetilde{\mathcal{G}}_k(PU(p))$. Hence $\widetilde{\mathcal{G}}_{\bar{k}}(U(p)) \simeq \widetilde{\mathcal{G}}_k(PU(p))$ □

Finally we prove Theorem 1.2.

Proof of Theorem 1.2. Consider the homotopy fibration $\mathcal{G}_k(PU(p)) \longrightarrow PU(p) \xrightarrow{\partial_k} \Omega_0 PU(p)$. By Lemma 4.1, the adjoint of ∂_k is k times the Samelson product $S^1 \wedge PU(p) \xrightarrow{\langle i,1 \rangle} PU(p)$. Consequently, $\partial_k \simeq k \circ \partial_1$. By [12, Theorem 1], ∂_1 has order p. Therefore, if we either localize away from p or if $k = 0$ then ∂_k is null homotopic, implying that there is a homotopy equivalence

$$\mathcal{G}_k(PU(p)) \simeq PU(p) \times \Omega^2 PU(p).$$

(Here, note that $\Omega(\Omega_0 PU(p)) \simeq \Omega^2 PU(p)$ since $\pi_2(PU(p)) \cong \pi_2(U(p)) \cong 0$ so the space $\Omega^2 PU(p)$ has a single component.) This proves part (a) since it assumes localization away from p. The $k = 0$ case in part (b) follows by substituting in the homotopy equivalence $PU(p) \simeq L \times \left(\prod_{i=1}^{p-2} S^{2i+1} \right)$ from Proposition 3.2 and noting that $\Omega L \simeq \Omega S^{2p-1}$ because the universal cover of L is S^{2p-1}.

Next, suppose that $k \neq 0$, or equivalently, that $(k,p) = 1$. Consider the homotopy fibration defining the universal cover,

$$\widetilde{\mathcal{G}}_k(PU(p)) \longrightarrow \mathcal{G}_k(PU(p)) \longrightarrow K(\pi_1(\mathcal{G}_k(PU(p))), 1).$$

By Proposition 4.3 (c), $\pi_1(\mathcal{G}_k(PU(p))) \cong \mathbb{Z}$, implying (i) $K(\pi_1(\mathcal{G}_k(PU(p))), 1) \simeq S^1$ and (ii) the map $\mathcal{G}_k(PU(p)) \longrightarrow S^1$ has a right homotopy inverse. Therefore, as $\mathcal{G}_k(PU(p))$ is an H-space, there is a homotopy equivalence $\mathcal{G}_k(PU(p)) \simeq S^1 \times \widetilde{\mathcal{G}}_k(PU(p))$. By Lemma 5.2, $\widetilde{\mathcal{G}}_k(PU(p)) \simeq \widetilde{\mathcal{G}}_{\bar{k}}(U(p))$, so we obtain

$$\mathcal{G}_k(PU(p)) \simeq S^1 \times \widetilde{\mathcal{G}}_{\bar{k}}(U(p)).$$

The asserted homotopy decompositions in part (c) now follows from the analogous decomposition for $\mathcal{G}_{\bar{k}}(U(p))$ stated in the Introduction. □

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