The pallet-loading vehicle routing problem with stability constraints
FIRST REVISION

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Abstract

This paper addresses an integrated routing and loading problem in which the pallets ordered by a set of customers have to be delivered by a set of trucks so that the total distance traveled is minimized. The problem has two main distinctive features. On the one hand, when assigning pallets to trucks, strict packing constraints, concerning axle weight, stability, and sequence loading constraints, must be considered. On the other hand, split delivery is allowed.

We have developed an integer linear model for the integrated problem, considering all routing and packing constraints. We have also designed a more efficient decomposition procedure in which some packing constraints are initially relaxed. Each time an integer solution is found in the search tree of the relaxed problem, it is checked whether it satisfies the remaining constraints and is therefore a feasible solution to the original problem. If it is not, a heuristic algorithm is first applied to rearrange the solution and, if it fails, an integer model is used, considering the packing problem for a single truck. If it also fails, a constraint is added to the relaxed problem to cut off the infeasible integer solution.

An extensive computational study shows how the integer linear model and the decomposition

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procedure work on a set of instances varying the number of customers, the number of pallets, and their weight distribution.

Keywords: Packing, 3D-VRP, pallet loading, stability, integer models

1. Introduction

When serving products from factories or distribution centres to customers, logistics companies have to solve complex problems arising from simultaneous planning of routing and packing. Excluding the especial case of point-to-point delivery, each route serving multiple customers has to be determined, resulting in well-known vehicle routing problems. Moreover, the demand of the customers served by any vehicle (route) defines a packing problem in which many constraints related to the dimensions and weight of the products, the characteristics of the vehicle, and the order in which the clients are visited have to be taken into account in order to obtain feasible solutions. In some cases this leads to complex packing problems. In the case of the company inspiring this study, customers are served using pallets, i.e., products are packed on pallets and only the pallets are loaded onto the trucks. An optimal solution of this problem involves designing the routes and determining a feasible packing of the pallets in each truck in such a way that the total distance travelled by all trucks is minimized and all the customer demands are met. Solving optimally the combined routing and packing problem has many economic and environmental benefits.

This problem is a combination of an optimisation problem and a decision problem. On the one hand we have the optimisation problem, which is related to the well-known Capacitated Vehicle Routing Problem (CVRP), and is NP-hard. On the other hand, we have to address the decision problem derived from the Container Loading Problem (CLP) and, therefore, combining these two problems into a single optimisation problem would end up with an NP-hard problem that could be challenging to solve in practice. It is therefore not surprising that in the past the two problems were solved separately by using both exact and heuristic methods, and afterwards the routing plan and the packing solutions were combined to obtain a complete solution, although sometimes quite far away from the overall optimal solution. More recently, there has been an increasing number of studies considering both problems simultaneously. Côté et al. (2017) have studied the value of addressing the routing and packing in the same optimisation problem, both theoretically and empirically, and concluded that the solution to the integrated problem was significantly better, keeping this advantage even if the integrated problem was solved heuristically.

Most of the existing studies on combined routing and packing problems consider only the basic constraints when defining the packing problem, concerning the orientation of the boxes, static stability, and multi-drop constraints in which the items of each customer have to be unloaded without moving the pallets of other customers that will be served later in the route. With the exception of the work by Pollaris et al. (2016), these studies do not consider weight distribution constraints, in which not only the total weight of the cargo is limited, but also the position of the centre of gravity and the weight on the axles of the truck. However, trucks with overloaded axles create serious problems on road networks, in terms of safety and road maintenance costs, and increasing controls are being enforced in many countries, using technologies such as Weight-In-Motion (WIM) systems (Jacob and Cottineau, 2016). These weight distribution constraints, which force the pallets loaded on the truck to be properly distributed, can in turn produce dynamic stability problems. Dynamic stability concerning the movements of the cargo when the truck is moving and subject to acceleration, braking, and turning, is implicitly satisfied if the truck is completely full and pallets are compactly packed, preventing each other from moving or falling. However, when handling heavy pallets, filling completely the truck is generally not possible and the pallets would have to be distributed along the length of the truck to satisfy the axle weight constraints. Therefore, stability constraints must be explicitly taken into account.

Moreover, concerning the features of the routing problem, all previous studies assume that each customer is visited just once, so that only one route visits each customer's location. This constraint is well considered in the literature of vehicle routing problems. Furthermore, it is true that in the VRP literature there are many papers allowing split delivery and showing its potential advantages, but this possibility has not yet been explored in the problem where routing and packing are integrated. In this paper we exploit this feature for several reasons. First, some customers may demand large quantities of pallets, not fitting in a single vehicle and therefore it would be necessary to split the delivery to find a feasible plan. Second, we could further minimise the total distance travelled by all vehicles. Note that if all customer pallets were homogeneous, this would not pose any especial difficulty: some trucks could be fully loaded and sent directly to the customer, while the remaining demand would be included in the combined routing and packing problem. However, in our problem, the pallets are heterogeneous with different weights, so assigning pallets to trucks satisfying the stability and weight distribution constraints is a difficult problem and, allowing multiple visits might be beneficial in finding suitable packing arrangements.

The problem considered in this paper studies these two characteristics that have not been sufficiently studied in the literature of combined routing and packing problems. On the one hand, we extend the packing constraints by including dynamic stability and weight distribution conditions. On the other hand, we allow for split delivery when designing the optimal set of routes to serve customers.

In this study we have first developed an integer linear model and then a decomposition approach for solving the problem optimally. We have also developed an extensive computational study on instances with different number of clients, number of pallets, and weight distributions. The results show that in most cases it is possible to obtain the optimal solution for small problems taking into account all packing constraints and that good feasible solutions are obtained for larger instances. The results also show that the weight distribution is an important factor in the combined routing-packing problem and that the possibility of split delivery is very often used as a part of optimal or near-optimal solutions.

The main contributions of the paper are: (1) a study of combined routing and packing problems in which split delivery is allowed, an aspect that has not yet been studied, proposing an integer linear model; (2) a decomposition procedure that solves much larger instances than the integer model; (3) a new benchmark in which the weight distribution is a controlled factor, allowing different influences of the packing part on the whole problem.

The remainder of the paper is organized as follows. Section 2 reviews related research. Section 3 describes the problem. Section 4 presents the integer linear model and Section 5 the decomposition approach developed to solve the problem. An upper bound, adapted from the well-known Clark and Wright algorithm is proposed in Section 6. The computational study is described in Section 7, while Section 8 contains conclusions and future research directions.

2. Related work

Single and Multi Container Loading Problems (SCLP, MCLP), as many other packing problems, have geometric constraints preventing items from overlapping and exceeding the dimensions of the container. An overview and classification of cutting and packing problems can be found in Wäscher et al. (2007). However, container loading problems have many other constraints arising from the fact that containers are moved by truck and, therefore, physical characteristics of both containers and pallets as well as their interaction have to be considered when providing solutions that can be applied in practice. Although these constraints have been discussed since the seminal paper by Bischoff and Ratcliff (1995), in which the authors listed 12 types of constraints appearing in practical situations, the survey by Bortfeldt and Wäscher (2013) concluded that many of them have been seldom considered and that very few of the published studies considered several of these constraints simultaneously. Nevertheless, the situation has improved in recent years. Surveys by Pollaris et al. (2015) and Zhao et al. (2016) show a growing interest in including constraints that are critical to ensure safe transportation of cargo, such as those concerning weight distribution and stability. Beyond the basic constraint limiting the total weight the container can carry, the critical question is the distribution of the weight, measured as the force exerted on the axles of the truck and the position of the centre of gravity. In terms of stability, two types can be distinguished, static stability of items when the vehicle transporting them is not moving, and dynamic stability, related to the possible displacement and damage of items when the vehicle is in motion and subject to acceleration, braking, and turns. Constraints limiting the weight on the axles have been included by Pollaris et al. (2016, 2017) and by Alonso et al. (2017, 2019), and constraints forcing the centre of gravity of the cargo to be on the geometric centre of the vehicle, with some tolerances, by Bortfeldt and Gehring (2001). More recently, Ramos et al. (2018), using data from truck manufacturers, have linked total weight to the centre of gravity, producing more realistic constraints. For many years, stability constraints were reduced to ensure full support, but, as Ramos et al. (2016) have shown, there are more efficient and robust ways of ensuring static stability. Dynamic stability has been considered by Ramos et al. (2015); Alonso et al. (2019, 2020).

In recent years, problems that simultaneously address routing and packing have been extensively studied in many variants. These problems, where the routes are valid only if they satisfy certain loading constraints, can be classified as having 2D or 3D loading constraints, depending on whether items can be stacked on top of the others or not. A survey of all variants of routing and loading problems can be found in Pollaris et al. (2015), but here we will focus on 2D problems, which correspond to the case studied in this paper. Most of the approaches developed for these problems are metaheuristic: Tabu Search (Gendreau et al., 2008; Zachariadis et al., 2009), Ant Colony Optimization (Fuellerer et al., 2009; Zachariadis et al., 2013), VNS (Strodl et al., 2010), GRASP (Duhamel et al., 2011), Simulated Annealing (Leung et al., 2013; Wei et al., 2018), or column generation based primal heuristics (Pinto et al., 2018), and very few are exact algorithms: the branch and bound algorithm by Iori et al. (2007), the non-linear model for circular items by Martinez and Amaya (2013) and the integer linear model by Pollaris et al. (2016). Furthermore, there are some decomposition approaches that have been proposed in related packing problems. In Côté et al. (2014a) the authors presented a Bender's decomposition approach to solve the strip packing problem in which in the *master* problem the rectangles to be packed are decomposed into unit-width slices and in the *slave* problem the reconstruction of the rectangles is addressed. However, the decomposition presented in this paper differs from the traditional Bender's decomposition approach and is more aligned with a branch and cut as the one proposed in Côté et al. (2014b). The main difference between Côté et al. (2014a) and Côté et al. (2014b) is that in Côté et al. (2014b) the *master* problem is not optimally solved, and instead the subproblems induced by the decomposition are solved in specific nodes of the branching tree and may generate new valid cuts. One of the key contributions of Côté et al. (2014b) is to handle the sequential loading (LIFO) constraints efficiently, which we are aiming to extend in this paper by addressing a wider set of real world constraints.

To the best of our knowledge only a few studies on vehicle routing with loading constraints consider the case in which products are first put on pallets and these pallets are loaded onto vehicles. Doerner et al. (2007) study a problem in which products are put into pallets and pallets are stacked in a multi-pile vehicle routing problem that they solve by Tabu Search and Ant Colony Optimization. Zachariadis et al. (2012) introduce a pallet-packing vehicle routing problem in which boxes form pallets using a three-dimensional bin packing problem and then pallets are loaded onto vehicles. Routes are built first, using a local search algorithm, and then another heuristic procedure checks whether feasible loading patterns can be built. Pollaris et al. (2016) consider that pallets for a customer are homogeneous and can be placed in two rows in the truck but cannot be stacked, then producing a two-dimensional loading problem, including axle weight and sequence loading constraints, and propose a MIP model to minimize transportation costs. In Pollaris et al. (2017) the problem is solved heuristically using an Iterated Local Search algorithm. Moura and Bortfeldt (2017) solve their problem building first homogeneous pallets by adapting the Moura and Oliveira (2005) GRASP algorithm, and then loading them onto trucks using a tree search procedure.

It should be noted that only the papers by Pollaris et al. (2016, 2017) explicitly consider axle weight constraints and implicitly dynamic stability constraints not allowing empty spaces between pallets, while all previous studies only take into account simple orientation, non-overlapping, and sequence loading constraints. However, as all the other previous studies, they assume that customers can only be visited once, not allowing split delivery, and they also consider that pallets demanded by a customer are homogeneous. These two aspects are explored in this paper.

3. Problem description

The depot of the logistics centre and its set of clients can be seen as the nodes of a graph, $G = (V, A)$, where $V = V' \cup \{0\}$ is the set of nodes and A is the arc set. Nodes $V' = \{1, ..., n\}$ represent the clients and node 0 is the depot. Without loss of generality we can consider the graph to be complete, with a non-negative traveling cost c_{ij} associated with each arc $(i, j) \in A$ and these costs can be asymmetric.

The products demanded by each customer are packed on pallets of fixed dimensions (l, w, h). We assume that pallets are built in advance, so the demand of customer $i \in V'$ is given by a set of pallets D^i, where each pallet $t \in D^i$ has a known weight q_t^i, including the base of the pallet. The number of pallets demanded by each client i is denoted by $d^i = |D^i|$.

We assume that there is one type of truck of dimensions (L, W, H). The truck has two axles, the front axle located at a distance δ_1 from the cabin and the rear axle at a distance δ_2. The front and rear axles can support a maximum weight of Q_1 and Q_2 respectively, and Q_{max} is the maximum weight the truck can carry. The fleet consists of a set K of identical trucks, $|K|$ being large enough to supply the demand of all customers. The route of each truck starts and ends at the depot.

Pallets cannot be stacked on top of each other, so all of them must lie on the truck floor. Therefore, taking into account the dimensions of both the truck floor and the base of the pallets it is easy to work out the maximum number of pallets that can fit into one truck. It must be taken into account that, due to the dynamic stability constraints, pallets must be loaded in such a way that they are in contact with another pallet or the walls of the truck. Therefore, we can limit the placement of the pallets to a set of positions (P). The number of positions is calculated as: $|P| = \lfloor \frac{L}{w} \rfloor * \lfloor \frac{W}{l} \rfloor$. Figure 1 shows an example with $|P| = 22$. Two pallets can be placed widthwise, defining two rows, with 11 columns lengthwise. In what follows, we will assume that the truck has 2 rows, which is the more common situation, and $|P|/2$ columns. Columns are numbered from the cabin on the left to the door on the right. In the top view of the truck used in the paper, row 1 is the top row and row 2 is the bottom row.

The number of pallets demanded by a customer i, d^i, can be larger than the number of pallets fitting on a truck, $|P|$. In this case, more than one truck must be used to serve that customer. Some trucks may be loaded with pallets for a single customer, but there may be trucks containing pallets from different clients, which will be visited in the order defined by the truck route. In these cases, at each stop the pallets of the given client must be unloaded without reallocating any pallets from another customer that will be visited later by the same route. These constraints

Figure 1: Positions of pallets on the truck

are known as sequence-loading constraints (Pollaris et al., 2015) or LIFO constraints (Iori and Martello, 2010). They often limit the feasible packing solutions when combining the routing with the loading/unloading process.

In addition to the sequence-loading constraints, other load-related constraints must be taken into account. Concerning the weight, the maximum total weight and the maximum weights on the axles cannot be exceeded, and the centre of gravity must be as close as possible to the geometric centre of the truck and always between the axles. In addition, the cargo must be dynamically stable, so it must not be displaced when the truck is in motion and subject to acceleration, braking, and lateral forces. Therefore, no empty spaces between pallets are allowed. Figure 2 shows three examples of solutions. The first solution is feasible, satisfying both dynamic stability and sequence loading constraints. The second solution is not dynamically stable because it has empty spaces between pallets, and the third solution is neither stable nor satisfies the customers' unloading order. In the later solution, we can observe that the pallets to be unloaded on the first visit (customer 1) are placed at both ends of the container and, therefore, cannot be unloaded without reallocating (unloading and loading again) the pallets for customers 2 and 3.

4. An integer linear programming formulation

In this section we describe the integer linear model developed for the problem. As the model is complex, involving several sets of variables and constraints, its elements are described in several subsections.

4.1. Variables

The first decision variables of the model correspond to the use of trucks, their routes, and the location of the pallets.

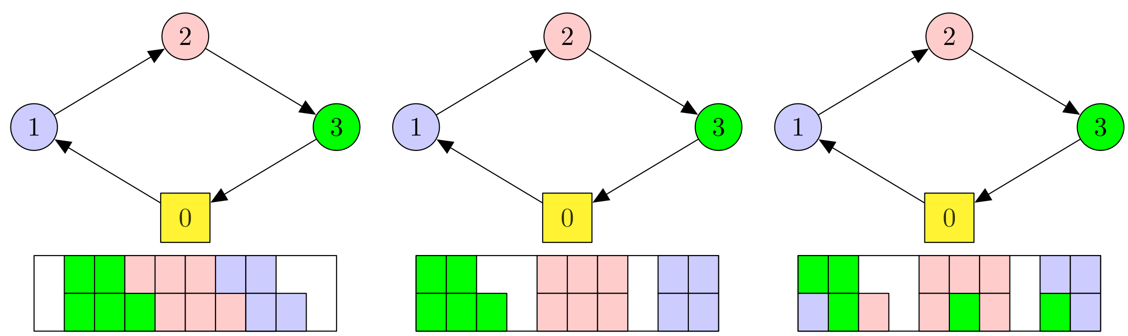

Figure 2: Solution types

$$
y_k = \begin{cases} 1, & \text{if truck } k \text{ is used}, \forall k \in K \\ 0, & \text{otherwise} \end{cases}
$$

$$
x_{ij}^k = \begin{cases} 1, & \text{if the route of truck } k \text{ goes through arc } (i,j), \forall k \in K, \forall (i,j) \in A \\ 0, & \text{otherwise} \end{cases}
$$

$$
z_{itp}^k = \begin{cases} 1, & \text{if pallet } t \text{ of client } i \text{ is located at position } p \text{ of truck } k, \forall k \in K, \forall p \in P, \forall i \in V', \forall t \in D^i \\ 0, & \text{otherwise} \end{cases}
$$

We need another set of variables to guarantee the dynamic stability of the cargo on each truck, which will be defined in Subsection 4.4. Also, to avoid invalid subtours in the routes, we use continuous variables to represent the flow of each vehicle in each arc. In Subsection 4.3 we introduce variables f_{ij}^k, which represent the flow of vehicle k through arc (i, j).

4.2. Objective function

The objective is to minimize the transportation cost as follows:

$$
\min \quad \sum_{k \in K} \sum_{i \in V} \sum_{j \in V \setminus \{i\}} c_{ij} x_{ij}^k \tag{1}
$$

Other aspects could be considered in the objective function, such as minimizing the number of trucks if a fixed cost is associated with their use or reducing the pollutant emissions. However, in this paper we focus on the total distance travelled by all vehicles.

4.3. Routing constraints

- Each truck starts from the depot:

$$\sum_{i \in V'} x_{0i}^k = y_k \qquad \forall k \in K \tag{2}$$

- If a truck arrives at a node, it has to leave it:

$$\sum_{j \in V, j \neq i} x_{ji}^k = \sum_{j \in V, j \neq i} x_{ij}^k \qquad \forall k \in K, \forall i \in V' \tag{3}$$

- If truck k travels from i to j, then it is used in the solution:

$$y_k \geq x_{ij}^k \qquad \forall i, j \in V, \forall k \in K \tag{4}$$

- Each customer is visited at most once by each route:

$$\sum_{j \in V, j \neq i} x_{ij}^k \leq 1 \qquad \forall i \in V', \forall k \in K \tag{5}$$

- Subtour elimination constraints

  We use a formulation based on the single-commodity flow, proposed first by Gavish and Graves (1978), where different goods are assigned to different vehicles. We use a variable f_{ij}^k to represent the flow k through arc (i, j).

  - The flow from a customer to the depot must be 0:

  $$f_{i0}^k = 0 \qquad \forall k \in K, \forall i \in V' \tag{6}$$

  - The maximum flow coming from the depot has to be less than or equal to the number of customers on the route of truck k:

  $$f_{0i}^k \leq |V'| \qquad \forall k \in K, \forall i \in V' \tag{7}$$

  - The flow arriving at a node i should be equal to the flow leaving i plus one if truck k visits node i:

  $$\sum_{j \in V, j \neq i} f_{ji}^k = \sum_{j \in V, j \neq i} f_{ij}^k + \sum_{j \in V, j \neq i} x_{ij}^k \qquad \forall k \in K, \forall i \in V' \tag{8}$$

  - No flow goes through an arc if the route does not use the arc:

  $$|V'| \, x_{ij}^k \geq f_{ij}^k \qquad \forall k \in K, \forall i, j \in V, j \neq i \tag{9}$$

Figure 4.3 is an example of a flow demand for three customers. According to equation (6) the flow going from the last customer to the depot must be 0. The remaining values are determined by equation (8), where the flow that arrives at each node must be equal to the flow leaving plus one. For each route the value of the f_{ij}^k decreases when a customer is visited until it takes value 0, when the truck arrives at the depot.

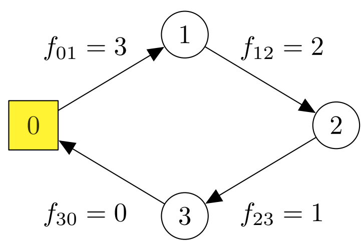

Figure 3: Flow circulating through a route with three customers

4.4. Packing constraints

- At each position of each truck, at most one pallet can be placed:

$$\sum_{i \in V'} \sum_{t \in D^i} z_{itp}^k \leq 1 \qquad \forall k \in K, \forall p \in P \tag{10}$$

- Each pallet must be placed once:

$$\sum_{k \in K} \sum_{p \in P} z_{itp}^k = 1 \qquad \forall i \in V', \forall t \in D^i \tag{11}$$

- The weight of each truck k must not exceed the maximum weight the truck can carry:

$$\sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} q_t^i z_{itp}^k \leq Q \qquad \forall k \in K \tag{12}$$

- Axle weight constraints

  The force exerted by each pallet on each axle depends on its weight and also on its position in the truck, and can be calculated according to the law of levers (Alonso et al., 2017). The sum of all the forces exerted by the pallets should be lower than of equal to the maximum force each axle can support:

$$\sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} (q_t^i z_{itp}^k)(\delta_2 - o_p^x) \leq Q_1(\delta_2 - \delta_1) \qquad \forall k \in K \tag{13}$$

$$\sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} (q_t^i z_{itp}^k)(o_p^x - \delta_1) \leq Q_2(\delta_2 - \delta_1) \qquad \forall k \in K \tag{14}$$

where o_p^x is the x-coordinate of the centre of position p on the truck.

- The geometric centre of the truck is the ideal position for the center of gravity $(G_x, G_y) = (L/2, W/2)$, but deviations from it are allowed, within given lengthwise, τ_1^x, τ_2^x, and widthwise, τ_1^y, τ_2^y, tolerances, to be set by the user. Figure 4 shows the region in which the centre of gravity must lie. In our tests, the lateral deviations, $\tau_1^y = \tau_2^y = W/8$, allow only small deviations from the truck centre. With respect to longitudinal deviations, $\tau_1^x = \delta_2 - L/2$, $\tau_2^x = L/2$, just imposing the condition that the deviation in the door direction, τ_1^x, cannot exceed the position of the rear axle, while no condition is set for the deviation in the cabin direction, τ_2^x. The weight of the empty truck, Q_e, is added to the equations, to prevent small displacements of the load from producing large displacements of the position of the centre of gravity. These conditions are guaranteed only when the truck is fully loaded, i.e., when the truck leaves the depot.

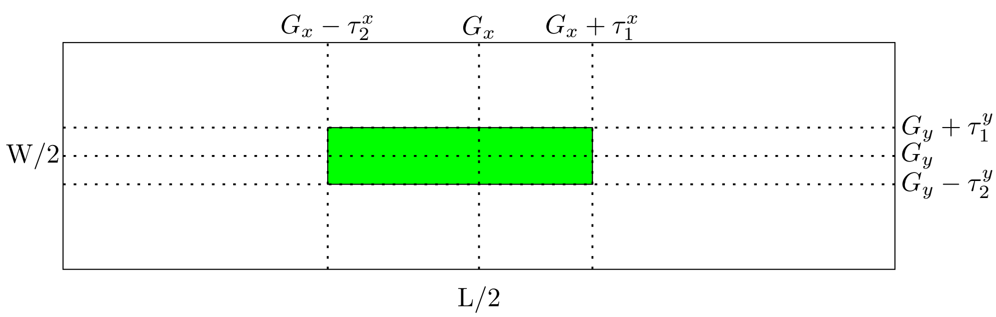

Figure 4: Feasibility area for the centre of gravity

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$$Q_e G_x + \sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} (o_p^x q_t^i z_{itp}^k) \leq (\sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} q_t^i z_{itp}^k + Q_e)(G_x + \tau_1^x) \qquad \forall k \in K \tag{15}$$

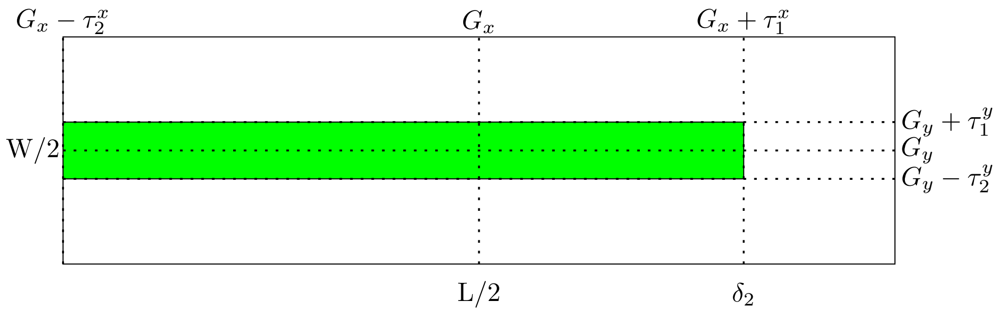

Figure 5: Feasibility area for the centre of gravity

$$Q_e G_x + \sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} (o_p^x q_t^i z_{itp}^k) \geq (\sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} q_t^i z_{itp}^k + Q_e)(G_x - \tau_2^x) \qquad \forall k \in K \qquad (16)$$

$$Q_e G_y + \sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} (o_p^y q_t^i z_{itp}^k) \leq (\sum_{j \in V'} \sum_{t \in D^i} \sum_{p \in P} q_t^i z_{itp}^k + Q_e)(G_y + \tau_1^y) \qquad \forall k \in K \qquad (17)$$

$$Q_e G_y + \sum_{i \in V'} \sum_{t \in D^i} \sum_{p \in P} (o_p^y q_t^i z_{itp}^k) \geq (\sum_{j \in V'} \sum_{t \in D^i} \sum_{p \in P} q_t^i z_{itp}^k + Q_e)(G_y - \tau_2^y) \qquad \forall k \in K \qquad (18)$$

- To ensure a dynamically stable cargo, we have to place the pallets in consecutive positions widthwise and lengthwise. To include dynamic stability constraints we consider new variables:

$$l_{kp} = \begin{cases} 1, & \text{if there is an empty space to the left of position } p \text{ of truck } k \\ 0, & \text{otherwise} \end{cases}$$

$$r_{kp} = \begin{cases} 1, & \text{if there is an empty space to the right of position } p \text{ of truck } k \\ 0, & \text{otherwise} \end{cases}$$

$$s_{kc} = \begin{cases} 1, & \text{if there is only one pallet in the column } c \text{ in truck } k \\ & c \in \{1, ..., |P|/2\} \\ 0, & \text{otherwise} \end{cases}$$

Variable l_{kp} takes value 1 if in truck k there is a pallet in position p but not in position $(p-1)$. In the opposite direction, variable r_{kp} takes value 1 if there is a pallet in position p but not in position $(p+1)$. Variable s_{kc} takes value 1 if in column c of truck k there is one pallet in one of the rows and an empty space in the other row. Using these variables, we define the following dynamic stability constraints:

$$l_{kp} \geq \sum_{i \in V'} \sum_{t \in D^i} z^k_{itp} - \sum_{i \in V'} \sum_{t \in t_i} z^k_{itp-1} \qquad k \in K, p \in \{1, ..., |P| - 1\} \setminus \{|P|/2\} \tag{19}$$

$$l_{kp} \geq \sum_{i \in V'} \sum_{t \in D^i} z^k_{itp} \qquad k \in K, p \in \{0, |P|/2\}, \tag{20}$$

$$r_{kp} \geq \sum_{i \in V'} \sum_{t \in D^i} z^k_{itp} - \sum_{i \in V'} \sum_{t \in t_i} z^k_{itp+1} \qquad k \in K, p \in \{0, ..., |P| - 1\} \setminus \{|P|/2 - 1\} \tag{21}$$

$$r_{kp} \geq \sum_{i \in V'} \sum_{t \in D^i} z^k_{itp} \qquad k \in K, p \in \{|P|/2 - 1, |P|\} \tag{22}$$

$$s_{kc} \geq \sum_{i \in V'} \sum_{t \in D^i} z^k_{itc} - \sum_{i \in V'} \sum_{t \in D^i} z^k_{itc+|P|/2} \qquad k \in K, c \in \{0, ..., |P|/2 - 1\} \tag{23}$$

$$s_{kc} \geq \sum_{i \in V'} \sum_{t \in D^i} z^k_{itc+|P|/2} - \sum_{i \in V'} \sum_{t \in D^i} z^k_{itc} \qquad k \in K, c \in \{0, ..., |P|/2 - 1\} \tag{24}$$

$$\sum_{1 \leq p \leq |P|/2} l_{kp} \leq 1 \qquad k \in K \tag{25}$$

$$\sum_{|P|/2+1 \leq p \leq |P|} l_{kp} \leq 1 \qquad k \in K \tag{26}$$

$$\sum_{1 \leq p \leq |P|/2} r_{kp} \leq 1 \qquad k \in K \tag{27}$$

$$\sum_{|P|/2+1 \leq p \leq |P|} r_{kp} \leq 1 \qquad k \in K \tag{28}$$

$$\sum_{1 \leq p \leq |P|/2} s_{kc} \leq 1 \qquad k \in K \tag{29}$$

$$l_{kp}, r_{kp} \in \{0, 1\} \qquad k \in K, p \in P \tag{30}$$

$$s_{kc} \in \{0, 1\} \qquad k \in K, p \in \{1, ..., |P|/2\} \tag{31}$$

Constraints (19) define the value of variables l_{kp} according to the values of variables z^k_{itp}, which indicate the positions of pallets in the truck. Constraints (20) consider the special cases of the positions in the front column. In these cases, there cannot be pallets to the left of these positions and, to properly count the number of empty spaces in a row, variables l_{kp} take their value from variables z^k_{itp}. Constraints (21) play the same role for the r_{kp} variables. The special cases are now in the back column and are considered in constraints (22). Constraints (23) make variables $s_{kc} = 1$ if there is a pallet in a given position c in the first row ($c \in \{0, ..., |P|/2\}$) and an empty space in the same column of the second row, position $c + |P|/2$. Constraints (24) play the same role changing the order of the rows. Constraints (25) to (29) limit to one the empty spaces to the left, to the right, or in the other row of a given column.

Figure 6 shows a bird's eye view of a truck with 22 positions, including the variables l_{kp}, r_{kp}, and s_{kc} taking value 1. This pallet configuration does not satisfy the longitudinal and lateral stability constraints. Figure 7 shows a configuration in which these stability constraints are satisfied.

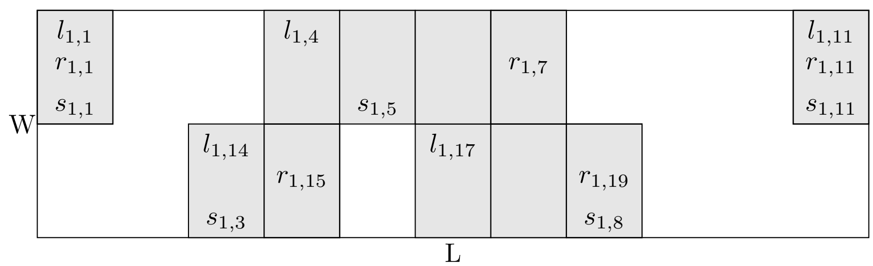

Figure 6: A configuration not satisfying stability constraints

- Symmetry constraints

As the trucks are all identical, there can be many equivalent solutions with the same assignment of pallets to trucks by just exchanging the indices of the trucks. To avoid this, we impose a partial order in the allocation of pallets to trucks. Specifically, the first pallet of each customer i ($i \leq K$) can only be placed on trucks $1, ..., i$

$$\sum_{k \in K, k \leq i} \sum_{p \in P} z_{i1p}^{k} = 1 \qquad \forall i \in V', i \leq K \tag{32}$$

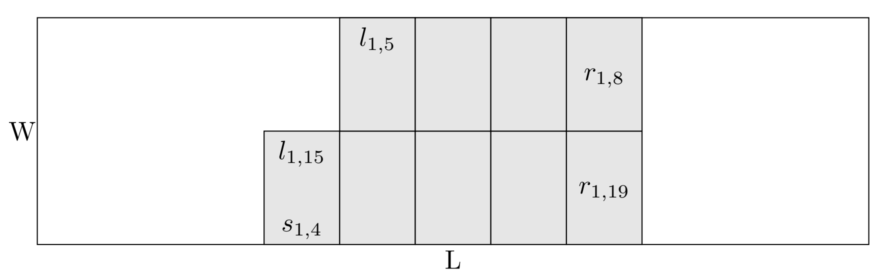

Figure 7: A configuration satisfying stability constraints

4.5. Relations routing - packing

- If a truck travels from i to j, it has to deliver at least one pallet of customer j:

$$\sum_{i \in V} x_{ij}^{k} \leq \sum_{t \in D^{j}} \sum_{p \in P} z_{jtp}^{k} \qquad \forall j \in V', k \in K \tag{33}$$

- If some pallets of customer j are loaded on truck k, the route of truck k visits customer j:

$$\sum_{t \in D^{j}} \sum_{p \in P} z_{jtp}^{k} \leq \sum_{i \in V} d^{j} x_{ij}^{k} \qquad \forall j \in V', k \in K \tag{34}$$

- Sequence loading constraints

  Pallets of different customers must be placed in the truck in such a way that the pallets of a customer who is being visited first on the route have to be unloaded without moving the pallets of other customers who are being visited later on the route. The relationship between the positions in the defined grid composed of rows and columns can be defined in several ways. Following Pollaris et al. (2016), we consider that, if $p_{a,b}$ and $p'_{a',b'}$ are pallet positions in the truck, where a, a' represent the rows, $a, a' \in \{0, 1\}$ and b, b' the columns, $b, b' \in \{0, ..., |P|/2 - 1\}$, then $p'_{a',b'} \succ p_{a,b}$ if and only if $b' > b$, that is, in a column b' larger than b. If the positions are in the same column, there is no precedence between them. The sequence loading constraints are, then:

$$\sum_{t \in D^{i}} z_{itp}^{k} + \sum_{t \in D^{j}} z_{jtp'}^{k} \leq 2 - x_{ij}^{k} \qquad \forall k \in K, \forall p, p' \in P | p' \succ p, \forall i, j \in V' \tag{35}$$

  If a truck k visits first customer i and then customer j ($x_{ij}^{k} = 1$), and a pallet of customer i is placed at a column b, no pallet of customer j being visited later can be placed at a column greater than b (although it can be in the same column).

  Figures 10 and 11 show two solutions in which three customers share the same truck and the order of visits is 1-2-3. The sequence loading constraints are met in Figure 11 but not in Figure 10.

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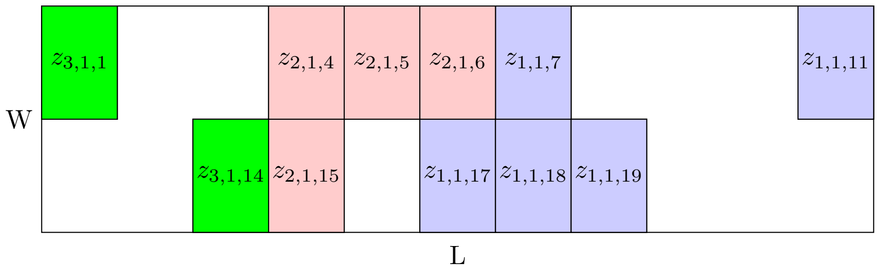


Figure 8: A configuration satisfying multidrop constraints. Route 1-2-3

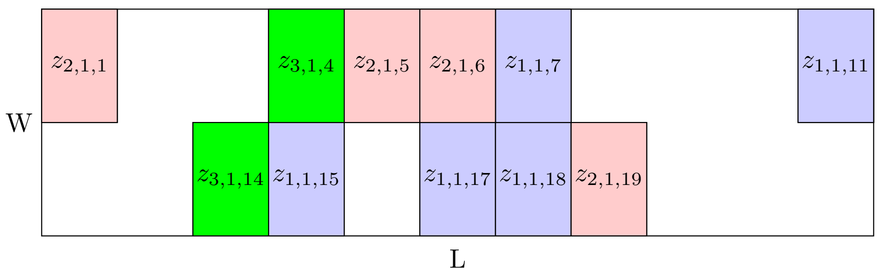


Figure 9: A configuration not satisfying multidrop constraints. Route 1-2-3

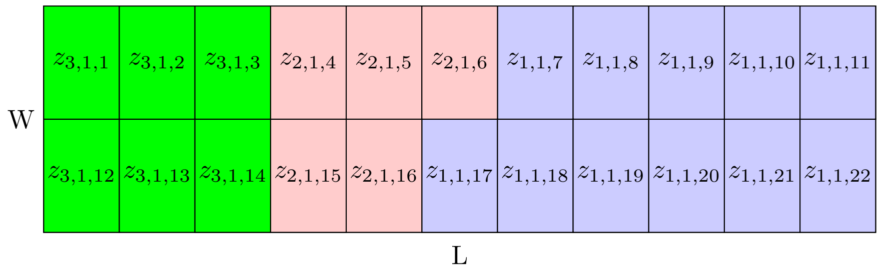


Figure 10: A configuration satisfying sequential loading constraints. Route 1-2-3

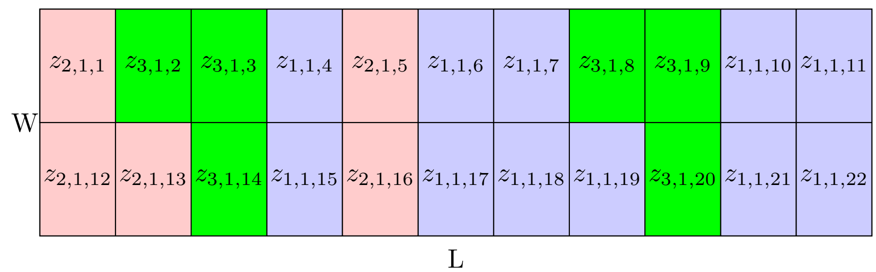


Figure 11: A configuration not satisfying sequential loading constraints. Route 1-2-3

5. A decomposition approach

Ensuring dynamic stability comes at a high cost in terms of the number of variables l_{kp}, r_{kp}, s_{kc} and constraints (19) to (29). Sequence loading conditions (35) also add a large number of constraints to an already complex formulation. As will be shown later in the computational study, this prevents the model from solving medium and large instances. Therefore, we have developed a decomposition approach in which these stability and sequence loading constraints are initially relaxed. Every time the branch and bound developed by the CPLEX solver finds a solution in which variables y_k, x_{ij}^k, and z_{itp}^k are integer, the stability and sequence loading constraints are checked for every truck k. If for a given truck k they are not satisfied, a simple construction heuristic is used that tries to load all the pallets assigned to this truck in positions producing a dynamically stable configuration that satisfies the sequence loading constraints. The outline of the constructive heuristic applied to each truck k in any integer solutions is:

- *Step 1. Initialization*

  Let $\mathcal{C}$ be the list of clients whose products are loaded onto truck k, from the first customer visited on the route to the last.

- *Step 2. Loading the customer's pallets*

  Taking each customer in $\mathcal{C}$ on turn, load their pallets from the back of the truck to the front, with no gaps, always covering both positions in each column before moving on to the next.

- *Step 3. Checking the weight constraints*

  All the pallets involved fit in the truck because they come from an integer solution of the relaxed model. Moreover, stability and sequence loading constraints are satisfied by the way in which the pallets are placed at Step 2. Concerning the weight constraints, although the limits on the weight that axles can support were satisfied when the pallets were placed in the positions given by the solution of the relaxed model, they have to be checked for the new positions in which they have been placed by the loading procedure in Step 2.

  If the axle weight constraints are satisfied for both axles, the procedure ends up with a feasible solution for the packing problem on truck k. Otherwise, if the rear axle limit is exceeded and there is some empty space at the front of the truck, all pallets are shifted forward to the adjacent column until the limit is satisfied or there are no more empty positions. If after one shift both axle weight limits are satisfied, a feasible solution has been found. Otherwise, the procedure ends without providing a feasible solution.

If the heuristic algorithm finds solutions packing all pallets for all the trucks, the solution is feasible for the original problem. If for some truck k the heuristic algorithm fails to pack all the assigned pallets, an integer linear model is called. The model is adapted from Alonso et al. (2019), considering now that we have an ordered set of clients $I \subseteq V'$, with a subset of their pallets T^i assigned to the truck k and we want to check whether is possible to pack all pallets in $T = \bigcup_i T^i$ in one truck satisfying the weight and stability and sequence loading constraints or not. The single-truck model can also be seen as a simplification of the model in Section 4, eliminating variables and constraints concerning routing, and eliminating the reference to truck k in the definition of variables and the construction of constraints related to packing. More precisely, the single-truck model will be defined using variables

$$u_{itp} = \begin{cases} 1, & \text{if pallet } t \text{ of client } i \text{ is packed in position } p \text{ of the truck} \\ 0, & \text{otherwise} \end{cases}$$

and variables l_{kp}, r_{kp}, s_{kc}, required for the stability constraints, but removing index k.

No objective function is needed, as we have a feasibility problem. The constraints are adapted from constraints (10)-(31) of the model in Section 4. The only constraints that must be changed are the sequence loading constraint, because now the order of clients is fixed by the solution of the relaxed problem. Here, the constraint is:

$$\sum_{t \in T^i} u_{itp} + \sum_{t \in T^j} u_{jtp'} \leq 1 \qquad \forall p, p' \in P | p' \succ p, \forall i, j \in I | i \succ j \tag{37}$$

Clients i a truck are ordered from cabin to door, so $i \succ j$ indicates that client i is visited before client j and must be placed nearer the door.

If the problem has a feasible solution, all pallets initially assigned to the truck can be loaded in a dynamically stable way and satisfying the customers' unloading order and, therefore, the solution is feasible for the original problem. Otherwise, this set of pallets cannot go on the same truck and a constraint forbidding this possibility is added into the model. The constraint forbids the set of pallets assigned to truck k, given by variables z^k_{itp}, to be together in a truck following the route specified by variables x^k_{ij}.

$$\sum_{i\in I}\sum_{t\in T^i}\sum_{p\in P} z^k_{itp} + \sum_{i\in I}\sum_{j\in I} x^k_{ij} \leq |T| + |I| - 1 \tag{38}$$

6. Upper bounds

In this section we propose an adaptation of the well-known *Clark and Wright* algorithm (Clarke and Wright, 1963). This algorithm was first proposed for the Capacitated Vehicle Routing Problem, where the customer demand and the capacity of the vehicles are computed as the number of items, i.e, there is only one type of items.

The idea of the *Clark and Wright* algorithm is to initially consider a solution in which each route visits only one customer. If any customer requires two or more trucks to deliver all the pallets demanded, then the initial solution might use more trucks than the number of customers, which differs from the standard CVRP. In successive steps, routes are merged whenever the combined route is feasible. We represent the initial solution as $s = \{T^s_1, \dots, T^s_{n_t}, R^s_1, \dots, R^s_{n_t}\}$, where n_t is the number of trucks used in the solution, T^s_i is a vector containing the pallets loaded on truck and R^s_i is a vector containing the list of customers in the order in which they are visited on the route, $i \in \{1, \dots, n_t\}$. The following conditions apply in the solution s_1.

- All pallets are loaded, i.e., for a given pallet $j^i \in \{1^i, \dots, d^l\}$ of customer $l \in V'$ there is exactly one truck $i \in \{1, \dots, n_t\}$ such that $j^i \in T^s_i$.
- Each truck only visits one customer, i.e., for each $i \in \{1, \dots, n_t\}$, $R^s_i = \{0, l, 0\}$, being $l \in V'$.

Then, the adapted *Clark and Wright* algorithm is described in Algorithm 1.

In lines 8 and 12 all packing constraints are checked, so the combined routes will be valid, but by design the algorithm does not allow splitting a customer's load between trucks, apart from the initial split when a customer's load does not fit in a single truck.

7. Testing the models

We performed an extensive computational study for testing models and algorithms. The computer used was a Linux OS machine, Linux version 3.10.0-693.5.2.el7.x86-64, gcc version 4.8.5 20150623, Red Hat 4.8.5-16 (GCC), with 4 CPU, 8 Threads, 2.40GHz, and 4GB of RAM. The model and the algorithm were coded in $C++$ and the model solved using CPLEX 12.8.0.0, with a time limit of 3600 seconds for the model or the complete algorithm on each instance.

Algorithm 1 Adapted *Clark and Wright* algorithm

Compute s^1

$List = \emptyset$

Set $s = s_1$ and $Improvements = True$

for each pair of trucks $i_1, i_2 \in \{1, \ldots, n_t\}$ in solutions s **do**

  Create combined route $R^* = R^s_{i_1} \cup R^s_{i_2}$ (removing the depot after the last customer in $R^s_{i_1}$)

  Create combined route $\overline{R}^* = R^s_{i_2} \cup R^s_{i_1}$ (removing the depot after the last customer in $R^s_{i_2}$)

  Set $T^* = T^s_{i_1} \cup T^s_{i_2}$.

  Check the packing constraints, using the heuristic described in Section 5 for (R^*, T^*)

  if Valid solution found **then**

    Compute the savings and set $List = List \cup (R^*, T^*)$

  end if

  Check the packing constraints, using the heuristic described in Section 5 for $(\overline{R}^*, T^*)$

  if Valid solution found **then**

    Compute savings and set $List = List \cup (\overline{R}^*, T^*)$

  end if

end for

if $List \neq \emptyset$ **then**

  Sort $List$ by non-decreasing savings

  Let (R^0, T^0) be the first route in $List$

  Update s. Include route (R^0, T^0) and remove the two combined routes to obtain (R^0, T^0)

  $Improvements = True$

end if

7.1. Test instances

The closest reference to our study is the paper by Pollaris et al. (2016). In their computational study, they generated instances with networks of 10, 15, 20 and 25 customers, with demands subject to low or high variability (4-7 or 1-15 pallets per customer) and with pallets of two weight classes, heavy (1000-1500 kg.) and light (100-500 kg.). However, in their instances, the pallets demanded by a client always fit in a truck, so clients were visited only once, and all pallets demanded by a customer had the same weight so their positions on the truck were interchangeable.

We have generated new test instances in a similar way, but to cover a wider range of cases, from almost pure routing problems to almost pure packing problems, we have combined three factors, the weight distribution of the pallets, the number of customers and the type of demand:

1. Weight distribution of pallets
   (a) **P1**: pallets with weights taken from a uniform distribution $(1700, 2300)$ kg. (only heavy pallets)
   (b) **P2**: pallets with weights taken from a uniform distribution $(700, 1300)$ kg. (medium weight pallets)
   (c) **P3**: pallets with weights taken from a uniform distribution $(300, 1000)$ kg. (light pallets)
   (d) **P4**: pallets with weights taken from a uniform distribution $(100, 1900)$ kg. (mixed weights)

   Using these weight distributions, 4 classes of instances were generated:

   Set A: (a) **P1**: 15 pallets; (b) **P2**: 30 pallets; (c) **P3**: 60 pallets; (d) **P4**: 30 pallets

   Set B: (a) **P1**: 20 pallets; (b) **P2**: 40 pallets; (c) **P3**: 60 pallets; (d) **P4**: 40 pallets

   Set C: (a) **P1**: 30 pallets; (b) **P2**: 60 pallets; (c) **P3**: 90 pallets: (d) **P4**: 60 pallets

   Set D: (a) **P1**: 40 pallets; (b) **P2**: 80 pallets; (c) **P3**: 120 pallets; (d) **P4**: 80 pallets
2. Number of clients: 5, 10, 15
3. Demands
   (a) Uniform: All clients have the same demand ($K = \left\lfloor \frac{\text{Number of pallets being sent}}{\text{Number of clients}} \right\rfloor$)
   (b) Variable: The demands are taken from a uniform distribution in $(\frac{1}{3}K, \frac{5}{3}K)$

Five instances were generated for each combination of factors, resulting in a benchmark set of 480 instances.

The clients were chosen at random from the Berlin52.tsp instance (available in `http://comopt.ifi.uni-heidelberg.de/software/TSPLIB95/tsp/`), so that the set of 15 clients included the set of 10 clients and this set included the set of 5 clients.

Figure 14 shows the distribution of the set of 15 clients. The depot is placed at the bottom-left corner.

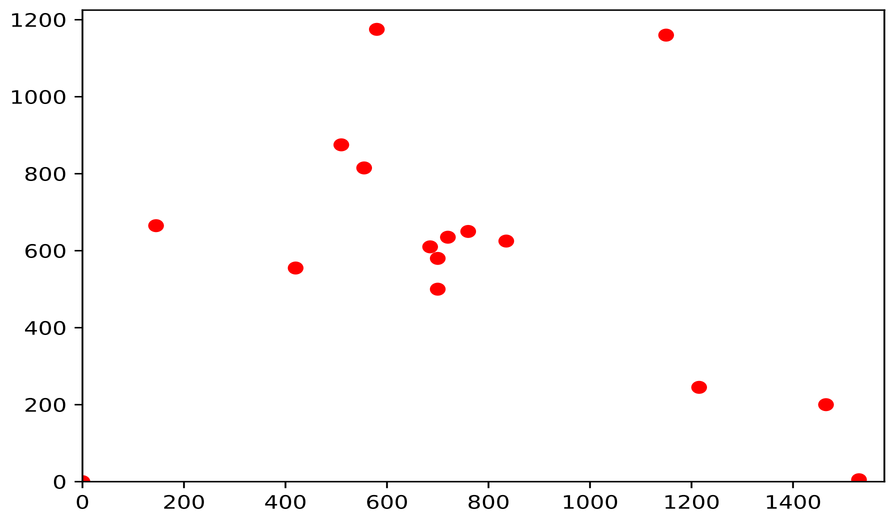


Figure 12: Spatial distribution of the 15 clients

There is only one type of truck in which 32 pallets can be loaded, in 2 rows and 16 columns. The maximum weight they can carry is 20411 kgr. and the maximum weights on the front and rear axles are 10432 and 9979 kgr, respectively.

The instances of sets B, C, and D have been designed in such a way that their pallets almost completely fill 2, 3, and 4 trucks, respectively. The average weight of pallets in instances P1 and P2 are 40000, 60000, and 80000 kgr., very close to the total weight limit in each case. Therefore, the packing problems are hard, with few but heavy pallets, making it difficult to meet the axle weight and stability conditions. Instances in P3 have more pallets but they are lighter, although they also fill the trucks by number of pallets. The packing problems are easier, more similar to pure capacitated routing problems. P4 instances combine light and heavy pallets, and although the total weight is near the total weight limit, the packing problems tend to be easier than those with only heavy pallets. Using these instances, it would be possible to assess the impact of the packing problem comparing the performance of the integer models across the four types of weight distribution. The characteristics of the instances in set A are very similar to those in set B, also requiring 2 trucks, but in classes P1, P2, and P4 there are fewer pallets, making the packing problem easier.

7.2. Performance of the complete integer model

We tested the integer linear model described in Section 4 and also the decomposition approach in Section 5 in which the initial solution is provided by the adapted *Clark and Wright* heuristic algorithm described above. The comparison of these two approaches is shown in the tables. We also tested an intermediate strategy in which the stability and sequence loading constraints were treated as lazy constraints, but its results did not improve on those of the other strategies and are not reported here.

Tables 1 and 2 show the results of the integer model in Section 4 on sets A and B, respectively. For each group of 10 instances defined by the pallet weight distributions, columns 2-4 show the number of optimal solutions, the number of feasible solutions when optimality is not proven, and the number of times a feasible solution is not even found. The remaining columns show averages: gap, when it exists, value of the best solution found, value of the initial solution provided by the heuristic algorithm, and CPU time in seconds. It can be observed in Table 1 that only the instances with 5 clients are successfully solved. For almost half of the 10-client instances not even a feasible solution was found and for 15-client instances only in 9 out of 40 a feasible solution was found. The results in Table 2 are even worse, reflecting the fact that pallets considered in instances P1, P2, and P4 are heavier and, therefore, the packing problem is more challenging. Only for the instances in P3, with light pallets, the results are similar to those in the previous table.

For the larger instances in sets C and D, only for some of the 5-customer instances a feasible solution was found. It can be concluded that the complete integer model is not really useful to solve the problem.

7.3. Performance of the decomposition strategy

Tables 3, 4, and 5 show the results of the decomposition procedure on sets B, C, and D, and a more detailed analysis is provided in Tables 6, 7, 8, and 9. Set A is a simpler version of set B and its results are no reported. Each row shows a summary of the results of the 10 instances in each group of instances. Columns 2-7 show the number of optimal solutions, the average gap if an upper bound is found, the average value of the initial solution provided by the heuristic algorithm, the average value of the best solution found, the average running time in seconds, and the number of routes in which at least one customer on the route is not totally served, so its demand is completed by other routes. The last five columns show details of the decomposition procedure. *Call_Heur* indicates the average number of times in which a node in the search tree has an integer solution

Table 1: Performance of the integer linear model on set A instances

	Optimal	Feasible	No solution	Gap (%)	Best	Initial	Time
5 clients							
P1	10	0	0		6016	6016	12
P2	10	0	0		6001	6167	76
P3	10	0	0		6273	7525	382
P4	10	0	0		6045	6060	109
10 clients							
P1	9	1	0	3	6383	6676	1416
P2	0	1	9			6896	3600
P3	0	10	0	25	8348	7794	3600
P4	0	0	10			6844	3600
15 clients							
P1	5	4	1			7501	2977
P2	0	0	10			7518	3600
P3	0	0	10			8354	3600
P4	0	0	10			7499	3600

Table 2: Performance of the integer linear model on set B instances

	Optimal	Feasible	No solution	Gap (%)	Best	Initial	Time
5 clients							
P1	5	5	0	14	6780	7497	2653
P2	3	7	0	20	7423	7663	3123
P3	10	0	0	0	6306	7690	386
P4	9	1	0	14	6416	7690	1475
10 clients							
P1	0	3	7			8033	3600
P2	0	1	9			8081	3600
P3	0	9	1			7706	3600
P4	0	1	9			8044	3600
15 clients							
P1	0	2	8			8537	3600
P2	0	0	10			8681	3600
P3	0	0	10			8421	3600
P4	0	0	10			8549	3600

in the variables of the relaxed model and the heuristic is called to check for each truck whether its pallets can be arranged to satisfy the axle weight and sequence loading constraints. *Sol_Heur* is the average number of times in which the heuristic finds a feasible solution for one truck. For trucks in which the heuristic does not find a feasible solution, the one-truck integer model is called and the number of times this happens appears in column *Call_Model*, and from those the model finds a feasible solution the number of times indicated by *Sol_Model*. Finally, if also the model fails, column *Cuts* show how many times a constraint is added to cut off the infeasible solution.

Table 3: Performance of the decomposition approach on set B instances

	Optimal	Gap	Best	Initial	Time	Splits	Call Heur	Sol Heur	Call Model	Sol Model	Cuts
5 clients											
P1	8	5	6359	7497	825	10	334.9	160.4	174.5	3.2	171.3
P2	8	12	6451	7663	1172	11	36.7	7.1	29.6	9.6	20.0
P3	10		6305	7690	6	9	7.4	6.0	1.4	1.4	0.0
P4	10		6319	7690	15	9	8.2	4.0	4.2	4.2	0.0
10 clients											
P1	8	2	6512	8033	1387	1	285.0	74.5	210.5	2.3	208.2
P2	3	13	6911	8081	2838	21	64.6	9.1	55.5	6.7	48.8
P3	10	0	6494	7706	74	3	18.0	13.0	5.0	4.2	0.8
P4	9	6	6508	8044	894	8	42.8	14.7	28.1	15.8	12.3
15 clients											
P1	5	5	7396	8537	2759	5	606.7	222.9	383.8	3.6	380.2
P2	0	13	7797	8681	3600	13	91.9	10.5	81.4	9.5	71.9
P3	10	0	7239	8421	409	3	33.8	23.4	10.4	5.2	5.2
P4	7	5	7355	8549	1857	6	64.3	18.5	45.8	17.2	28.6

Figures 7.3 and 14 show two examples of solutions. Figure 7.3 corresponds to an instance in class C, with 5 customers requiring 6 heavy pallets of class P1 each. The solution requires 3 trucks. The left-hand side of the figure shows the routes, while the right-hand side shows the position of the pallets in the trucks, with the door on the right. Each truck can only accommodate 10 pallets due to the total weight limit. The solution shows that the demands from customers 3 and 4 are split between different trucks. Had split delivery not been allowed, the solution would have required 5 trucks. As pointed out by Archetti et al. (2008), the clearest case of advantage of split delivery arises when each customer's demand is slightly more than half the capacity of the truck, as in this case. Figure 14 corresponds to an instance in class B, with 15 customers whose demand varies between 1 and 4 mixed-weight pallets of class P4. In this case, the customers can be served by 2 trucks. Only

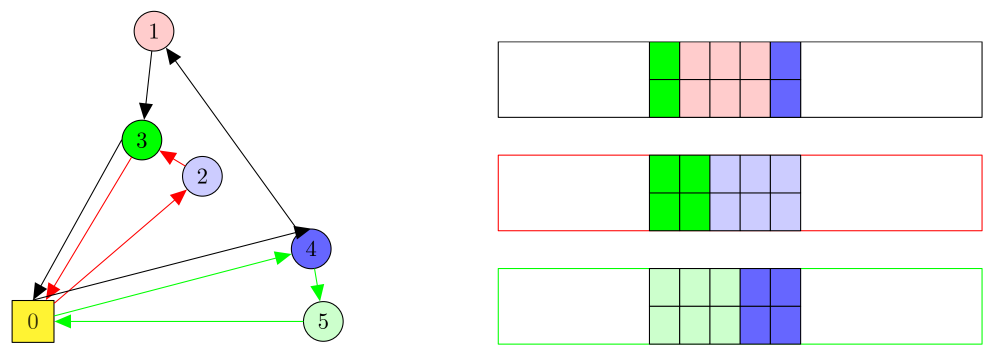
Table 4: Performance of the decomposition approach on set C instances

	Optimal	Gap	Best	Initial	Time	Splits	Call Heur	Sol Heur	Call Model	Sol Model	Cuts
5 clients											
P1	7	2	8833	10946	1170	20	289.8	150.6	139.2	17.5	121.7
P2	9	18	8941	11211	950	20	43.7	10.7	33.0	17.6	15.4
P3	10	0	8782	10878	70	15	18.5	13.0	5.5	5.1	0.4
P4	10	0	8724	11111	53	17	27.9	11.2	16.7	16.6	0.1
10 clients											
P1	4	4	8715	9783	2703	17	654.6	175.8	478.8	107.2	371.6
P2	1	26	9229	10010	3411	13	104.9	18.3	86.6	28.9	57.7
P3	9	4	8303	9706	1194	10	84.6	36.8	47.8	17.2	30.6
P4	10	0	8359	10037	687	13	42.6	16.8	25.8	21.8	4.0
15 clients											
P1	0	8	9441	10890	3600	8	498.7	208.7	290.0	14.4	275.6
P2	0	24	10301	10820	3600	19	103.1	22.5	80.6	21.2	59.4
P3	1	5	9260	10642	3350	4	57.2	35.1	22.1	15.8	6.3
P4	0	11	9338	10778	3600	14	95.8	32.2	63.6	45.2	18.4

Table 5: Performance of the decomposition approach on set D instances

	Optimal	Gap	Best	Initial	Time	Splits	Call Heur	Sol Heur	Call Model	Sol Model	Cuts
5 clients											
P1	8	0,4	11174	12438	929	31	135.0	34.5	100.5	33.1	67.4
P2	9	3	11080	12720	1478	31	78.8	20.2	58.6	37.2	21.4
P3	10	0	11002	12423	370	20	27.0	16.6	10.4	10.1	0.3
P4	10	0	10939	12361	208	25	34.2	12.7	21.5	21.5	0.0
10 clients											
P1	1	5	10949	12719	3600	24	671.7	280.6	391.1	112.0	279.1
P2	0	32	12170	12800	3600	36	123.2	22.3	100.9	42.8	58.1
P3	0	12	10893	12931	3600	20	77.2	51.6	25.6	23.9	1.7
P4	2	7	10982	13061	3182	26	61.8	21.6	40.2	38.9	1.3
15 clients											
P1	0	16	11596	12562	3600	18	223.2	122.1	101.1	23.4	77.7
P2	0	24	12099	12896	3600	33	120.1	37.2	82.9	37.5	45.4
P3	0	18	11307	12674	3600	17	98.6	50.6	48.0	42.9	5.1
P4	1	14	10969	12936	3525	24	102.9	37.5	65.4	55.8	9.6

the pallets for customer 15 are split into two trucks.



Graph

Trucks

Figure 13: Solution of a class C instance with 5 clients, uniform demand, and weight distribution P1

The results in Tables 3, 4, and 5 can be analyzed from two perspectives, the size of the problem, measured in number of trucks and customers, and the weight distribution of the pallets. Tables 6 and 7 summarize the results from the point of view of the size. Rows correspond to instance sets, B, C, and D, and columns to the number of clients. Table 6 contains the number of optimal solutions, while Table 7 shows the average gaps. Looking at the tables from the columns, it can be observed that 5-customer instances are well solved, even when the problem size increases. The 10-customer instances are quite well solved in set B, but the quality of solutions decreases in set C and even more sharply in set D. For the 15-customer instances, even the instances in set C cannot be solved efficiently. When viewed from the rows, the instances in set B are solved quite well, with a slow decrease in quality with the number of customers. However, the quality of solutions decreases very fast with the number of customers in set C and even more steeply in set D.

Table 6: Optimal solutions by type of instance and number of clients

Set	5 clients	10 clients	15 clients	Total
B	36	30	22	88
C	36	24	1	61
D	37	3	1	41
Total	109	57	24	190

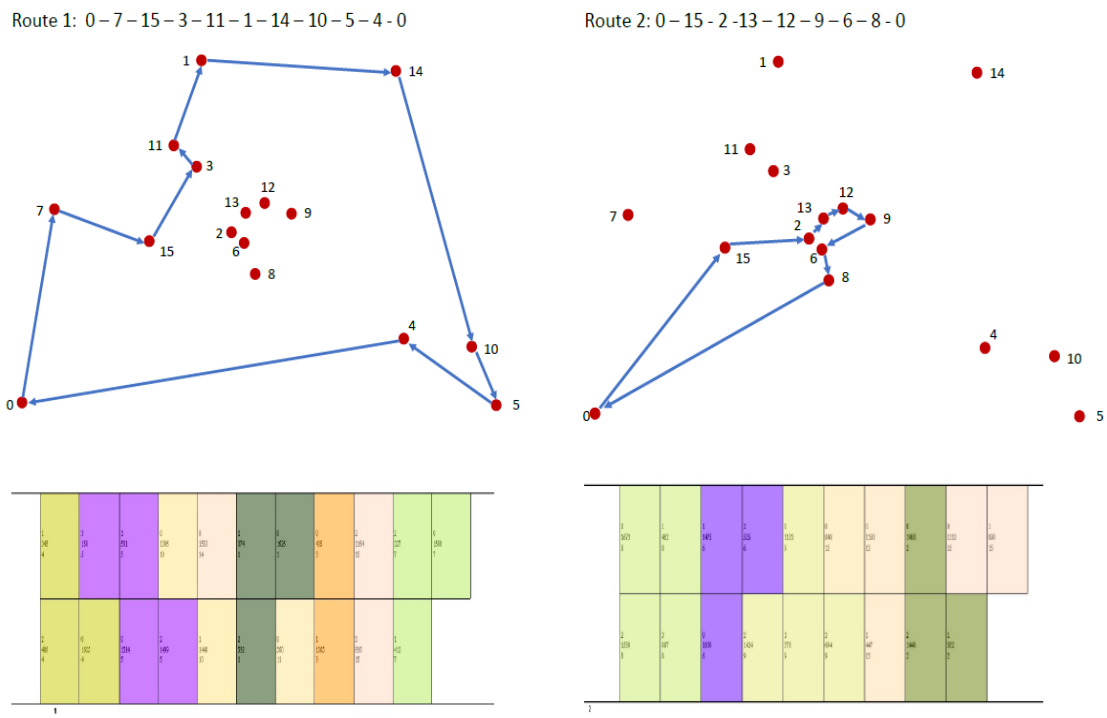


Figure 14: Solution of an instance of set B with 15 clients, variable demand, and weight distribution P4

The results can also be analyzed from the perspective of the weight distribution. Table 8 shows a first summary of the influence of the weight distribution on the number of optimal solutions, average gap, and percentage of routes in which there is split delivery.

It can be seen that, among the four classes of instances according to weight distribution, P2 is clearly the most difficult class, with the fewest number of optimal solutions and the largest average gap. Each truck has to accommodate many medium weight pallets and packing them to satisfy the axle weight and multi-drop constraints is difficult. In contrast, the P3 and P4 classes, consisting of lightweight or mixed pallets seem much easier. The P1 class occupies an intermediate position. The pallets are heavier, but there are fewer. The relatively high number of split routes is an appealing feature of the solutions obtained. There has been an interesting debate on the cases in which split delivery actually improves the solution of routing problems. Here, it seems that for the cases in

Table 7: Average gaps by type of instance and number of clients

Set	5 clients	10 clients	15 clients	Average
B	4.2	5.2	5.7	5.6
C	5.0	8.5	12.0	8.5
D	0.8	11.0	18.0	9.9
Average	3.3	8.2	11.9	7.8

Table 8: Comparison of results by weight distribution

Weight class	Optimal sol.	Average gap	Split delivery
P1	41	5.3%	46%
P2	30	18.3%	72%
P3	60	3.0%	35%
P4	59	4.8%	50%

which weight conditions are difficult to satisfy, many solutions split the customer demand into more than one route.

Table 9 shows how the decomposition approach works for the four classes of instances. Column 2 shows the average number of times in which an integer solution of the relaxed problem is found and the distribution of pallets on the trucks is checked to see whether it satisfies the axle weight and sequence loading constraints by calling the heuristic algorithm. Columns 3, 4, 5 show the percentages of times a feasible solution is obtained by the heuristic, obtained by the one-truck model, or not found and then a cut is added to the model so that the current solution is no longer feasible for the original problem.

The results in Table 9 confirm that packing problems are much more difficult in the P1 and P2 classes. More than half of the integer solutions found when solving the relaxed model cannot be converted into feasible solutions that satisfy axle weight and sequence loading constraints. In classes P3 and P4 the packing constraints are almost reduced to a one-dimensional characteristic, the number of pallets, making them more similar to the capacitated routing problem, and most of the integer solutions of the relaxed problem are converted into feasible solutions. Between P1 and P2 the difference in the number of integer solutions found when solving the relaxed problem is quite

Table 9: Comparison of results of the decomposition approach by weight distribution

	Trucks checked	Heur. solved	Model solved	Cut added
P1	1233	39%	9%	52%
P2	257	20%	28%	52%
P3	141	64%	28%	8%
P4	160	37%	50%	13%

remarkable. As in Table 8, P2 appears as the hardest class to solve.

8. Conclusions

We have studied a combined routing and packing problem, inspired by a problem in practice with hard packing constraints concerning axle weight, stability, and sequence loading constraints, and allowing the possibility of split delivery, not only when it is necessary because the demand of a client exceeds the truck capacity but also whenever it is convenient to obtain a better solution in terms of the number of trucks and the total distance travelled.

Aiming at solving the problem optimally, we have first developed a complex integer linear model, including all routing and packing constraints. As this model is only able to optimally solve very small instances, we have designed a decomposition approach in which some packing constraints are initially relaxed and, whenever an integer solution of the relaxed problem is obtained, the relaxed constraints are checked, the solution is rearranged to satisfy them using a heuristic and an auxiliary integer model and, failing that, an inequality is added to cut off the unfeasible solution.

We have designed an extensive computational experiment to study the influence of the number of clients and pallets and also the influence of the pallet weight distribution. As expected, the number of optimal solutions decreases and the gaps increase when the size of the problem increases but, more interestingly, the study shows how the weight distribution of the pallets is a major factor, differentiating for cases where the packing is less challenging, leading to pure routing problems, and cases in which packing is the most influential factor.

In future work, with the aim of solving larger instances, we will use this model in a matheuristic approach and we will also use the knowledge of the problem to develop new metaheuristic approaches that will not rely on solving mathematical models.

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