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p-Hyperbolicity of homotopy groups via K-theory

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Abstract

We show that $S^n \vee S^m$ is $\mathbb{Z}/p^r$-hyperbolic for all primes p and all $r \in \mathbb{Z}^+$, provided $n, m \geq 2$, and consequently that various spaces containing $S^n \vee S^m$ as a p-local retract are $\mathbb{Z}/p^r$-hyperbolic. We then give a K-theory criterion for a suspension ΣX to be p-hyperbolic, and use it to deduce that the suspension of a complex Grassmannian $\Sigma Gr_{k,n}$ is p-hyperbolic for all odd primes p when $n \geq 3$ and $0 < k < n$. We obtain similar results for some related spaces.

Keywords Local hyperbolicity · K-theory

Mathematics Subject Classification 55Q52 · 55Q15 · 19L20

1 Introduction

A space X is called *rationally elliptic* if $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, and *rationally hyperbolic* if the dimension of $\bigoplus_{i \leq m} \pi_i(X) \otimes \mathbb{Q}$ grows exponentially in m. It was proved in [12, Chapter 33] that simply connected CW-complexes with rational homology of finite type and finite rational category are either rationally elliptic or rationally hyperbolic. In order to study the p-torsion analogue of this dichotomy, Huang and Wu [20] introduced the definitions of $\mathbb{Z}/p^r$- and p-hyperbolicity.

For p prime, by a *p-torsion summand* in an abelian group A, we mean a direct summand isomorphic to $\mathbb{Z}/p^r$ for some $r \geq 1$.

Definition 1 Let X be a space, and let p be a prime. We say that X is *p-hyperbolic* if the number of p-torsion summands in $\pi_*(X)$ grows exponentially, in the sense that

$$\liminf_m \frac{\ln(T_m)}{m} > 0,$$

where T_m is the number of p-torsion summands in $\bigoplus_{i \leq m} \pi_i(X)$.

The above definition counts $\mathbb{Z}/p^r$-summands for all values of r. It is also possible to consider only a single r, and by doing so we obtain the definition of $\mathbb{Z}/p^r$-hyperbolicity.

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Definition 2 Let X be a space, let p be a prime, and fix $r \in \mathbb{Z}^+$. We say that X is $\mathbb{Z}/p^r$-*hyperbolic* if the number of $\mathbb{Z}/p^r$-summands in $\pi_*(X)$ grows exponentially, in the sense that

$$\liminf_m \frac{\ln(t_m)}{m} > 0,$$

where t_m is the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \leq m} \pi_i(X)$.

Note that $\mathbb{Z}/p^r$-hyperbolicity for any r implies p-hyperbolicity. It follows immediately from a result of Henn [18, Corollary of Theorem 1] that the lim infs appearing in the above definitions must be finite if X is a simply connected finite CW-complex.

Huang and Wu show that for $n \geq 3$, $r \geq 1$ and p any prime, the Moore space $P^n(p^r)$ is $\mathbb{Z}/p^r$-hyperbolic and $\mathbb{Z}/p^{r+1}$-hyperbolic, and that $P^n(2)$ is also $\mathbb{Z}/8$-hyperbolic [20, Theorem 1.6]. More generally, they give criteria in terms of a functorial loop space decomposition due to Selick and Wu [29,30] for a suspension ΣX to be $\mathbb{Z}/p^r$-hyperbolic.

More recently, Zhu and Pan [35] use a classification of $(n-1)$-connected CW-complexes of dimension at most $n+2$, due to Chang [10], to show that, for $n \geq 4$, such a complex is $\mathbb{Z}/p$-hyperbolic, provided that it is not contractible or a sphere after p-localization. They also prove hyperbolicity results for several so-called elementary Chang complexes.

This paper studies p- and $\mathbb{Z}/p^r$-hyperbolicity of certain suspensions. Our first result is as follows.

Theorem 1 *Let $q_1, q_2 \geq 1$. Then $S^{q_1+1} \vee S^{q_2+1}$ is $\mathbb{Z}/p^r$-hyperbolic for all primes p and all $r \in \mathbb{Z}^+$.*

Let p be a prime. If a space X contains a wedge of two spheres as a p-local retract, then Theorem 1 implies that X is $\mathbb{Z}/p^r$-hyperbolic for all r. Various spaces have been shown to have such a wedge as a retract - examples of this sort are given in Section 2.1. A summary is as follows:

– for $n, k \geq 3$, the configuration space $\mathrm{Conf}_k(\mathbb{R}^n)$ is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Example 2);
– an $(n-1)$-connected $2n$-dimensional manifold M, where $H^n(M)$ is of rank at least 3, is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Example 3);
– a generalized moment-angle complex on a simplicial complex having two minimal missing faces which are not disjoint is $\mathbb{Z}/p^r$-hyperbolic for all p and r (Example 4);
– $\Sigma\mathbb{C}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all $p \neq 2$ and all r, and $\Sigma\mathbb{H}P^2$ is is $\mathbb{Z}/p^r$-hyperbolic for all $p \neq 2, 3$ and all r (Example 5).

Our other result is as follows.

Theorem 2 *Let p be an odd prime, and let X be a path connected space having the p-local homotopy type of a finite CW-complex. Suppose that there exists a map*

$$\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$$

with $q_i \geq 1$, such that the map

$$\widetilde{K}^*(\Sigma X) \otimes \mathbb{Z}/p \xrightarrow{(\mu_1 \vee \mu_2)^*} \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

is a surjection. Then ΣX is p-hyperbolic.

This criterion is quite different to that given by Huang and Wu [20, Theorem 1.5]. Their criterion is homotopical, using hypotheses on X to produce retracts of $\Omega\Sigma X$, whereas ours is cohomological, which makes it easier to check. On the other hand, their criterion is stronger, since it gives $\mathbb{Z}/p^r$-hyperbolicity, rather than just p-hyperbolicity. The examples they give, primarily various Moore spaces, differ from those we obtain, which are the suspensions of spaces related to complex projective space. More precisely, in Section 2.2, we show that the following spaces are p-hyperbolic for all $p \neq 2$:

- suspended complex projective space $\Sigma\mathbb{CP}^n$ for $n \geq 2$ (Example 6), and more generally;
- the suspended complex Grassmannian $\Sigma\mathrm{Gr}_{k,n}$ for $n \geq 3$ and $0 < k < n$ (Example 7);
- the suspended Milnor Hypersurface $\Sigma H_{m,n}$ for $m \geq 2$ and $n \geq 3$, (Example 8);
- the suspended unitary group $\Sigma U(n)$ for $n \geq 3$ (Example 9).

Both Theorems 1 and 2 will be proven by constructing an exponentially growing family of classes which generate summands in the relevant homotopy groups. We think of this family as 'witnessing' the hyperbolicity. For Theorem 1, one can proceed directly from the Hilton-Milnor decomposition of $S^n \vee S^m$ [19]. For Theorem 2, we employ K-theoretic methods originally used by Selick [28] to prove one direction of Moore's conjecture for suspensions having torsion-free homology.

If the map $\mu_1 \vee \mu_2$ of Theorem 2 induces a surjection on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$, then so does its suspension $\Sigma\mu_1 \vee \Sigma\mu_2$. The conclusion of Theorem 2 may therefore be strengthened in the following way.

Corollary 1 *With the hypothesis of Theorem 2, $\Sigma^n X$ is p-hyperbolic for all $n \geq 1$.* □

One might be motivated by this observation to ask whether, in the circumstances of Theorem 2, the stable homotopy groups of X satisfy the growth conditions of Definition 1 or 2. In the proofs of both Theorems 1 and 2, the classes that witness the hyperbolicity are composites involving Whitehead products. The suspension of a Whitehead product is always trivial [33, Theorem 3.11], so the classes we detect cannot be stable. Therefore, Corollary 1 does not suggest that the stable homotopy of ΣX should be p- or $\mathbb{Z}/p^r$-hyperbolic. On the other hand, it follows from our methods that, under the hypotheses of Theorem 2, the stable homotopy of $\Omega\Sigma X$ is p-hyperbolic.

By a result of Henn [17], any co-H space, and in particular any suspension, decomposes rationally as a wedge of spheres. It then follows from the Hilton-Milnor theorem [19] and the computation of the rational homotopy groups of spheres [31] that such a suspension is rationally hyperbolic precisely when there are at least two spheres (of dimension ≥ 2) in this decomposition.

If ΣX satisfies the hypotheses of Theorem 2 for any prime (including 2), then by Chern Character considerations the reduced rational homology of ΣX has dimension at least two, so ΣX is rationally a wedge of at least two spheres by the preceding discussion. This rational equivalence is a local equivalence at all but perhaps finitely many primes, so by Theorem 1, ΣX is $\mathbb{Z}/p^r$ hyperbolic for all r at all but finitely many primes p. One might therefore conjecture that the conclusion of Theorem 2 can be strengthened to give $\mathbb{Z}/p^r$-hyperbolicity for all r rather than p-hyperbolicity, but we do not know whether this is possible.

We now discuss situations in which it is adequate to consider ordinary cohomology, rather than K-theory. If ΣX has torsion-free integral (co)homology, or if its cohomology is concentrated in even degrees, then the Atiyah-Hirzebruch spectral sequence for $K^*(\Sigma X)$ collapses on the E^2 page [21]. It follows by naturality that the image of the map induced by $\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$ on K-theory is identified with the image of the induced map on cohomology. We may therefore replace K-theory with cohomology in Theorem 2, as follows.

Corollary 2 *Let X be a path connected space having the homotopy type of a finite CW-complex, such that the Atiyah-Hirzebruch spectral sequence for $K^*(\Sigma X)$ collapses on the E^2 page. Let p be an odd prime. Suppose that there exists a map $\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$ with $q_i \geq 1$, such that the map induced by $\mu_1 \vee \mu_2$ on $\widetilde{H}^*(\ ) \otimes \mathbb{Z}/p$ is a surjection. Then ΣX is p-hyperbolic.*

One advantage of ordinary cohomology is that it is connected to the homotopy groups integrally, via the universal coefficient theorem and Hurewicz map. We can exploit this as follows.

Example 1 Suppose that the Atiyah-Hirzebruch spectral sequence for $K^*(\Sigma X)$ collapses (for example, if ΣX has torsion-free homology) and that there exists $q \in \mathbb{Z}^+$ so that $\widetilde{H}_i(\Sigma X) = 0$ for $i \leq q$, and $\dim_{\mathbb{Q}}(\widetilde{H}_{q+1}(\Sigma X) \otimes \mathbb{Q}) \geq 2$. The Hurewicz map $\pi_{q+1}(\Sigma X) \to \widetilde{H}_{q+1}(\Sigma X)$ is an isomorphism, so there exists a map $\mu_1 \vee \mu_2 : S^{q+1} \vee S^{q+1} \to \Sigma X$ inducing the inclusion of a $\mathbb{Z}^2$-summand in $\widetilde{H}_{q+1}(\Sigma X)$. By the universal coefficient theorem relating ordinary homology and cohomology, $\mu_1 \vee \mu_2$ induces a surjection on integral cohomology, so by Corollary 2, ΣX is p-hyperbolic for all odd primes p.

The structure of this paper is as follows. In Sect. 2 we give applications of the main theorems and derive a simple lower bound for the growth of the number of $\mathbb{Z}/p^r$-summands in the homotopy groups of a wedge of two spheres (Corollary 3). The proofs of Theorems 1 and Theorem 2 may be read largely independently; Sect. 3 contains those preliminary results which are used in both cases. In Section 4, we give the proof of Theorem 1. The remainder of the paper is devoted to proving Theorem 2: Sects. 5 and 6 give the necessary background, and Sect. 7 contains the proof. An overview of the proof strategy can be found at the start of Sect. 7.

2 Applications

2.1 Spaces having a wedge of two spheres as a retract

Theorem 1 immediately implies that any space X which has $S^{q_1+1} \vee S^{q_2+1}$ as a retract after p-localization is $\mathbb{Z}/p^r$-hyperbolic for that p and all r. This implies that for all $n \geq 1$, $\Sigma^n X$ contains $S^{q_1+n+1} \vee S^{q_2+n+1}$ as a p-local retract, and so is $\mathbb{Z}/p^r$-hyperbolic for all r. We first consider examples of this form.

Example 2 It is known [23, Section 3.1] that $\mathrm{Conf}_k(\mathbb{R}^n)$, the ordered configuration space of k points in $\mathbb{R}^n$, contains $\bigvee_{k-1} S^{n-1}$ as a retract. It follows that, when $n, k \geq 3$, $\mathrm{Conf}_k(\mathbb{R}^n)$ is $\mathbb{Z}/p^r$-hyperbolic for all p and r.

Example 3 Let M be an $(n-1)$-connected $2n$-dimensional manifold. By the universal coefficient theorem, there can be no torsion in $H^n(M)$. Suppose that the rank of $H^n(M)$ is at least 3. By work of Beben and Theriault [9, Theorem 1.4], ΩM contains a wedge of two spheres as a retract after looping. Thus, M is again $\mathbb{Z}/p^r$-hyperbolic for all p and r.

Example 4 Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$, and let $(\underline{X}, \underline{A})$ be any sequence of pairs (D^{n_i}, S^{n_i-1}) with $n_i \geq 2$ for $1 \leq i \leq m$. If there exist two distinct minimal missing faces of K which are not disjoint, then by work of Hao, Sun and Theriault [16, Theorem 4.2] the polyhedral product $(\underline{X}, \underline{A})^K$ contains a wedge of two spheres as a retract after looping, and hence is $\mathbb{Z}/p^r$-hyperbolic for all p and all r.

Example 5 Localized away from 2, $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$. To see this, note that $\Sigma\mathbb{C}P^2$ has a CW-structure consisting of one 3-cell and one 5-cell, and that $\pi_4(S^3) \cong \mathbb{Z}/2$ [13]. This implies that the attaching map for the 5-cell is nullhomotopic after localization at an odd prime. Thus, $\Sigma\mathbb{C}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all r when $p \neq 2$.

Similarly, $\Sigma\mathbb{H}P^2$ admits a cell structure with one 5-cell and one 9-cell, and $\pi_8(S^5) \cong \mathbb{Z}/24$. Thus, $\Sigma\mathbb{H}P^2$ is $\mathbb{Z}/p^r$-hyperbolic for all r when $p \neq 2, 3$.

2.2 Suspensions of spaces related to $\mathbb{C}P^n$

Suppose that one has verified the hypotheses of Theorem 2 for a given space X and odd prime p, using a map $\mu_1 \vee \mu_2 : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$. If another space Y admits a map $\sigma : \Sigma X \to \Sigma Y$ which induces a surjection on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$, then it is immediate that $\sigma \circ (\mu_1 \vee \mu_2)$ satisfies the hypotheses of Theorem 2, and hence that ΣY is p-hyperbolic. The slogan is that K-theory surjections allow us to generate new examples from old ones.

In this section, we will apply this idea. We have seen in Example 5 that, localized away from 2, $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$, so certainly $\Sigma\mathbb{C}P^2$ satisfies the hypotheses of Theorem 2 at all odd primes p. We will now consider spaces X which are known to admit maps $\mathbb{C}P^2 \to X$ which induce surjections on integral K-theory, and hence on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$ for all odd p. It follows in each case that ΣX is p-hyperbolic, and further by Corollary 1, that $\Sigma^n X$ is p-hyperbolic for all $n \geq 1$.

The inclusion of $\mathbb{C}P^n$ into $\mathbb{C}P^{n+1}$ induces a surjection on K-theory, so it must still induce a surjection after suspending. Composing these inclusions with the local equivalence $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$ gives, for each $n \geq 2$, a map $S^3 \vee S^5 \to \Sigma\mathbb{C}P^n$ which still induces a surjection on $\widetilde{K}^*(\ ) \otimes \mathbb{Z}/p$ for all odd primes p. Applying Theorem 2 to this map gives the following.

Example 6 For $n \geq 2$, $\Sigma\mathbb{C}P^n$ is p-hyperbolic for all $p \neq 2$.

Now let $\mathrm{Gr}_{k,n}$ be the Grassmannian of k-dimensional complex subspaces of $\mathbb{C}^n$. First note that orthogonal complement gives a homeomorphism $\mathrm{Gr}_{k,n} \cong \mathrm{Gr}_{n-k,n}$. In particular $\mathrm{Gr}_{n-1,n} \cong \mathrm{Gr}_{1,n} \cong \mathbb{C}P^{n-1}$, so $\Sigma\mathrm{Gr}_{n-1,n}$ is p-hyperbolic. Other Grassmannians can be treated more uniformly, as follows.

Let $\gamma_{k,n}$ denote the tautological bundle over $\mathrm{Gr}_{k,n}$. Consider the inclusion

$$\begin{aligned} \iota_n &: \mathbb{C}^n \to \mathbb{C}^{n+1} \\ (x_1, x_2, \ldots, x_n) &\mapsto (x_1, x_2, \ldots, x_n, 0). \end{aligned}$$

This inclusion induces a map $i_{k,n} : \mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k,n+1}$, defined on $V \in \mathrm{Gr}_{k,n}$ by $V \mapsto \iota_n(V)$. It follows from this definition that $i^*_{k,n}(\gamma_{k,n+1}) = \gamma_{k,n}$. Letting e_i denote the i-th standard basis vector in $\mathbb{C}^n$, we also have a map $j_{k,n} : \mathrm{Gr}_{k,n} \to \mathrm{Gr}_{k+1,n+1}$, defined on $V = \mathrm{Span}(v_1, v_2, \ldots, v_k) \in \mathrm{Gr}_{k,n}$ by $V \mapsto \mathrm{Span}(\iota(v_1), \iota(v_2), \ldots, \iota(v_k), e_{n+1})$. It follows from this definition that $j^*_{k,n}(\gamma_{k+1,n+1}) = \gamma_{k,n} \oplus \underline{\mathbb{C}^1}$, where $\underline{\mathbb{C}^1}$ is the 1-dimensional trivial bundle.

Since $K^*(\mathbb{C}P^n)$ is generated by the class of the tautological bundle, composing the maps $i_{k,n}$ and $j_{k,n}$ for different values of k and n will give maps $\mathbb{C}P^2 = \mathrm{Gr}_{1,3} \to \mathrm{Gr}_{k,n}$ for all $1 \leq k \leq n-2$ and $n \geq 3$ which induce surjections in integral K-theory. As in Example 6, this implies the following (the case $k = n-1$ is $\mathrm{Gr}_{n-1,n}$, which was treated first).

Example 7 For $n \geq 3$ and $0 < k < n$, the suspended complex Grassmannian $\Sigma \mathrm{Gr}_{k,n}$ is p-hyperbolic for all $p \neq 2$.

For $m \leq n$, the *Milnor Hypersurface* $H_{m,n}$ is defined by

$$H_{m,n} = \{([z],[w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^{m} z_i w_i = 0\}.$$

Suppose that $m \geq 2$ and $n \geq 3$. Then there is an inclusion $\iota : \mathbb{C}P^2 \to H_{m,n}$, defined by

$$\iota([z_0 : z_1 : z_2]) = ([z_0 : z_1 : z_2 : 0 : \cdots : 0], [0 : \cdots : 0 : 1]).$$

Write π_1 for the projection $H_{m,n} \to \mathbb{C}P^m$. Then the inclusion $\mathbb{C}P^2 \to \mathbb{C}P^m$ factors as

$$\begin{tikzcd} \mathbb{C}P^2 \arrow[r, "\iota"] \arrow[dr] & H_{m,n} \arrow[d, "\pi_1"] \\ & \mathbb{C}P^m. \end{tikzcd}$$

This implies that ι induces a surjection on integral K-theory, so we obtain the following.

Example 8 For $m \geq 2$ and $n \geq 3$, the suspended Milnor Hypersurface $\Sigma H_{m,n}$ is p-hyperbolic for all $p \neq 2$.

Let $U(n)$ denote the unitary group. There is a well-known map $r : \Sigma \mathbb{C}P^{n-1} \to U(n)$ (see for example [34]) which induces a surjection on K-theory. From this we obtain

Example 9 For $n \geq 3$, the suspended unitary group $\Sigma U(n)$ is p-hyperbolic for all $p \neq 2$.

2.3 Quantitative lower bounds on growth

In Sect. 4, we will derive the following simple lower bound for the lim inf in the definition of $\mathbb{Z}/p^r$-hyperbolicity, for the space $S^{q_1+1} \vee S^{q_2+1}$.

Corollary 3 *Let p be a prime and $r \in \mathbb{Z}^+$. Let t_m be the constants of Definition 2 for $X = S^{q_1+1} \vee S^{q_1+1}$. Then*

$$\liminf_m \frac{\ln(t_m)}{m} \geq \frac{\ln(2)}{\max(q_1, q_2)}.$$

This implies that t_m eventually exceeds $((1-\varepsilon)2)^{\frac{m}{\max(q_1,q_2)}}$ for any $\varepsilon > 0$. The constant 2 reflects the number of wedge summands. Note that this bound is independent of p and r.

Example 10 Taking $\varepsilon = \frac{1}{4}$, we find that for all $r \in \mathbb{Z}^+$ and all primes p the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \leq m} \pi_i(S^2 \vee S^2)$ eventually exceeds $(\frac{3}{2})^m$.

One can produce an analogous quantitative bound on the lim inf in the case of Theorem 2, but this bound is very weak. In particular, it depends on knowledge of the Adams operations on $K^*(X)$, and is at best $\frac{\ln(2)}{2(p-1)}$.

3 Preliminary results

Both Theorems 1 and 2 will be proven by means of Lemma 2. Our first goal is to establish this lemma.

Let L be the free Lie algebra over $\mathbb{Q}$ on basis elements $x_1, \ldots, x_n$. Write $\mathscr{L}_k$ for the subset of L consisting of the *basic products* of the x_i of *weight* k, in the sense of [19], where the basic products of weight 1 are taken to be the x_i, ordered by $x_1 < x_2 < \cdots < x_n$. The union $\mathscr{L} = \bigcup_{k=1}^{\infty} \mathscr{L}_k$ is a vector space basis for L (see for example [32, Theorem 5.3], but note that Serre uses the name *Hall basis* for the set of basic products).

Let $\mu : \mathbb{Z}^+ \longrightarrow \{-1, 0, 1\}$ be the Möbius inversion function, defined by

$$\mu(s) = \begin{cases} 1 & s = 1 \\ 0 & s > 1 \text{ is not square free} \\ (-1)^{\ell} & s > 1 \text{ is a product of } \ell \text{ distinct primes.} \end{cases}$$

The Witt Formula $W_n(k)$ is then defined by

$$W_n(k) = \frac{1}{k} \sum_{d|k} \mu(d) n^{\frac{k}{d}}.$$

Theorem 3 [19, Theorem 3.3] *Let L be the free Lie algebra over $\mathbb{Q}$ on basis elements $x_1, \ldots, x_n$. Then $|\mathscr{L}_k| = W_n(k)$.* □

Lemma 1 [7, Introduction] *The ratio*

$$\frac{W_n(k)}{\frac{1}{k} n^k}$$

tends to 1 as k tends to ∞. □

In particular, this implies that for $n \geq 2$, the Witt function $W_n(k)$ grows exponentially in k. It should follow that if the number of p-torsion summands in $\bigoplus_{i \leq k} \pi_i(Y)$ exceeds $W_2(k)$, then Y is p-hyperbolic. The following lemma makes a slightly generalised form of this idea precise.

Lemma 2 *Let Y be a space. Suppose that there exist $a, b \in \mathbb{Z}^+$ such that the number of p-torsion summands (respectively, $\mathbb{Z}/p^r$-summands) in $\bigoplus_{i \leq ak+b} \pi_i(Y)$ exceeds $W_2(k)$, for all k large enough. Then Y is p-hyperbolic (respectively, $\mathbb{Z}/p^r$-hyperbolic).*

Proof The proofs for p- and $\mathbb{Z}/p^r$-hyperbolicity are identical, so we give only the former. Reframing the hypothesis in terms of the sequence $\{T_m\}_m$ of Definition 1, we are assuming precisely that $T_{ak+b} > W_2(k)$ for sufficiently large k. We then have that

$$\liminf_m \frac{\ln(T_m)}{m} = \liminf_k \frac{\ln(T_{ak+b})}{ak+b} \geq \liminf_k \frac{\ln(W_2(k))}{ak+b}.$$

It then follows from Lemma 1 that if $1 > \varepsilon > 0$, once k is large enough, we have

$$W_2(k) > (1-\varepsilon)\frac{1}{k}2^k.$$

This implies that

$$\liminf_k \frac{\ln(W_2(k))}{ak+b} \geq \liminf_k \frac{\ln((1-\varepsilon)\frac{1}{k}2^k)}{ak+b},$$

and since this holds for all $\varepsilon > 0$,

$$\liminf_{m} \frac{\ln(T_m)}{m} \geq \liminf_{k} \frac{\ln(W_2(k))}{ak+b} \geq \liminf_{k} \frac{\ln(\frac{1}{k}2^k)}{ak+b} = \liminf_{k} \frac{\ln(\frac{1}{k}) + k\ln(2)}{ak+b} = \frac{\ln(2)}{a},$$

which is greater than zero, as required. □

3.1 Existence of summands in the stable stems

We write π_j^S for the j-th stable stem in the homotopy groups of spheres, that is

$$\pi_j^S := \lim_{n\to\infty} \pi_{n+j}(S^n).$$

The proof of Theorem 1, depends on having, for each p and r, some j such that π_j^S contains a $\mathbb{Z}/p^r$-summand. The purpose of this subsection is to show that the existence of such a j follows from existing work of Adams and others.

Lemma 3 *For any prime p and any $r \in \mathbb{Z}^+$, there exists j such that $\mathbb{Z}/p^r$ is a direct summand in π_j^S. That is, for a fixed choice of such a j, $\mathbb{Z}/p^r$ is a direct summand in $\pi_{n+j}(S^n)$ for all $n \geq j+2$.*

Proof We write $\nu_p(s)$ for the largest power of p dividing the integer s.

CASE 1 (p odd): Set $t{:=}p^{r-1}(p-1)$, and notice that, since $(p-1)$ is even, $j{:=}4t-1$ is congruent to 7 mod 8. Theorem 1.6 of [2], and the discussion immediately following it, then tells us that π_j^S contains a direct summand isomorphic to $\mathbb{Z}/m(2t)$, for a function m which Adams defines. By decomposing this subgroup into direct summands of prime power order, it suffices to show that $\nu_p(m(2t)) = r$.

The discussion after Theorem 2.5 in [1] gives that since $t \equiv 0 \bmod (p-1)$,

$$\nu_p(m(2t)) = 1 + \nu_p(2t).$$

Now, $\nu_p(2t)$ is equal to $(r-1)$, by definition of t, so $\nu_p(m(2t)) = r$, as required.

CASE 2 ($p = 2$, $r \geq 3$): Set $t{:=}2^{r-3}$, and set $j{:=}4t-1$. From Theorem 1.5, and the discussion following Theorem 1.6 in [2], π_j^S has a direct summand isomorphic to $\mathbb{Z}/m(2t)$, regardless of whether j is congruent to 3 or 7 mod 8. Again, referring to the discussion after Theorem 2.5 of [1], we see that

$$\nu_2(m(2t)) = 2 + \nu_2(2t) = 3 + \nu_2(t) = r,$$

as required.

CASE 3 ($p^r = 2$ and $p^r = 4$): It is known from [13] that $\pi_1^S \cong \mathbb{Z}/2$, and from [8] that $\pi_{34}^S \cong \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^3$. □

4 Proof of Theorem 1

In this section we prove Theorem 1, which says that the wedge of two spheres is $\mathbb{Z}/p^r$-hyperbolic for all p and r. We also prove Corollary 3, which extracts some simple quantitative information from the proof of Theorem 1. We first record the following simple observation.

Remark 1 Let $k_1, \ldots, k_n$ and $q_1, \ldots q_n$ be non-negative integers. Suppose that $q_1 \leq q_2 \leq \cdots \leq q_n$, and let $k = \sum_{i=1}^n k_i$. Then

$$kq_1 \leq \sum_{i=1}^n k_i q_i \leq kq_n.$$

Proof of Theorem 1 Assume without loss of generality that $q_1 \leq q_2$. By Lemma 2 it suffices to show that there exist constants a and b such that the number of $\mathbb{Z}/p^r$-summands in

$$\bigoplus_{i \leq ak+b} \pi_i(S^{q_1+1} \vee S^{q_2+1})$$

exceeds $W_2(k)$, for k large enough.

We first apply the Hilton-Milnor Theorem. Since we are dealing with spheres, we need only the original form, due to Hilton in [19]:

$$\Omega(S^{q_1+1} \vee S^{q_2+1}) \simeq \Omega\Sigma(S^{q_1} \vee S^{q_2}) \simeq \Omega \prod_{B \in \mathscr{L}} S^{k_1 q_1 + k_2 q_2 + 1},$$

where, as in Sect. 3, $\mathscr{L} = \bigcup_{k=1}^{\infty} \mathscr{L}_k$ is Hilton's 'basic product' basis for L, the free Lie Algebra over $\mathbb{Q}$ on two generators x_1 and x_2, and k_i is the number of occurrences of the generator x_i in the bracket B. Recall also from Section 3 that the weight k of a bracket B is equal to $k_1 + k_2$, and that the cardinality of $\mathscr{L}_k$ is given by the Witt formula $W_2(k)$ by Theorem 3.

For fixed $k \in \mathbb{Z}^+$, consider the factor in the above product corresponding to $\mathscr{L}_k \subset \mathscr{L}$:

$$F_k := \Omega \prod_{B \in \mathscr{L}_k} S^{k_1 q_1 + k_2 q_2 + 1}.$$

The associated subgroup of $\pi_*(S^{q_1+1} \vee S^{q_2+1})$,

$$\bigoplus_{B \in \mathscr{L}_k} \pi_*(S^{k_1 q_1 + k_2 q_2 + 1}),$$

is a direct summand.

We will first find a $\mathbb{Z}/p^r$-summand in the homotopy groups of each of the spheres appearing in F_k. Since $q_1 \leq q_2$, Remark 1 applies, and we may lower bound the dimensions of spheres appearing in F_k by $k_1 q_1 + k_2 q_2 + 1 \geq kq_1 + 1$. By Lemma 3, there exists $j \in \mathbb{Z}^+$ such that $\pi_{j+\ell}(S^\ell)$ has a direct summand $\mathbb{Z}/p^r$ for $\ell \geq j + 2$. Therefore, if k is large enough that $kq_1 \geq j + 1$, then $k_1 q_1 + k_2 q_2 + 1 \geq j + 2$ - that is, the j-th stem is stable on all of the spheres occurring in F_k. Thus, for k large enough, there is a $\mathbb{Z}/p^r$ summand in $\pi_{j+k_1 q_1 + k_2 q_2 + 1}(S^{k_1 q_1 + k_2 q_2 + 1})$ whenever $k_1 + k_2 = k$.

We now upper bound the dimension of the homotopy groups in which these summands appear. Since $q_1 \leq q_2$ we have by Remark 1 that $j + k_1 q_1 + k_2 q_2 + 1 \leq kq_2 + 1 + j$, so each of the $\mathbb{Z}/p^r$-summands we have identified is a distinct direct summand in

$$\bigoplus_{i \leq kq_2+1+j} \bigoplus_{B \in \mathscr{L}_k} \pi_i(S^{k_1 q_1 + k_2 q_2 + 1}),$$

hence in

$$\bigoplus_{i \leq kq_2+1+j} \pi_i(S^{q_1+1} \vee S^{q_2+1}).$$

We have identified one such summand for each $B \in \mathscr{L}_k$, so the number of $\mathbb{Z}/p^r$-summands in $\bigoplus_{i \le kq_2+1+j} \pi_i(S^{q_1+1} \vee S^{q_2+1})$ is at least $|\mathscr{L}_k| = W_2(k)$. Thus, taking $a = q_2$ and $b = 1+j$ in Lemma 2 suffices. □

Proof of Corollary 3 The last line of the proof of Lemma 2 shows that $\liminf_m \frac{\ln t_m}{m} > \frac{\ln 2}{a}$. The last line of the proof of Theorem 1 implies that a may be taken to be q_2, under the assumption that $q_1 \le q_2$, which implies the result. □

5 *K*-theory and *K*-homology of $\Omega\Sigma X$

The remainder of this paper proves Theorem 2. Sections 5 and 6 assemble necessary background, which we will use in Sect. 7 to prove the result.

When studying the homotopy groups of a suspension ΣX, as in Theorem 2, the following approach is natural. Since $\pi_*(\Sigma X) \cong \pi_{*-1}(\Omega\Sigma X)$, we may instead study $\Omega\Sigma X$. This is useful because $\Omega\Sigma X$ is well understood homologically via the Bott-Samelson theorem, which decomposes its homology as the tensor algebra on $\widetilde{H}_*(X)$. Because we will need to use Adams' e-invariant, which is defined in terms of K-theory, we wish to replace ordinary homology with K-homology.

The purpose of Sect. 5 is to record the version of the Bott-Samelson theorem which applies to (torsion-free) K-homology, along with a universal coefficient theorem for passing between K-theory and K-homology. All of the material here is already known (in particular much of it is in [28]) so its summary here is for convenience and clarity.

Our conventions on definition of $\widetilde{K}^*(X)$ are those of [6]. In particular, we define $\widetilde{K}^{-1}(X) := \widetilde{K}^0(\Sigma X)$, and set $\widetilde{K}^*(X) := \widetilde{K}^0(X) \oplus \widetilde{K}^{-1}(X)$. We regard $\widetilde{K}^*(X)$ and $\widetilde{K}_*(X)$ as being $\mathbb{Z}/2$-graded. It is shown in [6] that $\widetilde{K}^*(X)$ is a $\mathbb{Z}/2$-graded ring.

We will wish to work with K-theory and K-homology modulo the torsion subgroup. For a space X, write $\widetilde{K}_*^{\mathrm{TF}}(X)$ and $\widetilde{K}_{\mathrm{TF}}^*(X)$ for the quotients of the reduced K-homology and K-theory of X by their torsion subgroups. The same convention applies in the unreduced case.

5.1 Künneth and universal coefficient theorems

The universal coefficient theorem for K-theory first appears in some unpublished lecture notes of Anderson, and is first published by Yosimura.

Theorem 4 (Universal coefficient theorem). *For any CW-complex X and each integer n there is a short exact sequence*

$$0 \to \mathrm{Ext}(K_{n-1}(X), \mathbb{Z}) \to K^n(X) \to \mathrm{Hom}(K_n(X), \mathbb{Z}) \to 0.$$

□

In the torsion-free case, the universal coefficient theorem is as follows, where, unsurprisingly, we write $\mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})^{\mathrm{TF}}$ for the quotient of $\mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})$ by its torsion subgroup.

Corollary 4 1. *For any CW-complex X and each integer n there is a short exact sequence*

$$0 \to \mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})^{\mathrm{TF}} \to \widetilde{K}^n_{\mathrm{TF}}(X) \to \mathrm{Hom}(\widetilde{K}_n^{\mathrm{TF}}(X), \mathbb{Z}) \to 0.$$

2. *If X is a finite CW-complex, then* $\mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})^{\mathrm{TF}} = 0$*, and we obtain an isomorphism* $\widetilde{K}^n_{\mathrm{TF}}(Y) \xrightarrow{\cong} \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_n(Y), \mathbb{Z})$.

Proof Let T_1 denote the torsion subgroup of $\mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})$, and let T_2 be the torsion subgroup of $\widetilde{K}^n(X)$. The Universal Coefficient Sequence of Theorem 4 gives an injection $T_1 \to \widetilde{K}^n(X)$, which must have image contained in T_2, thus lift to an injection $T_1 \to T_2$. For any group G, $\mathrm{Hom}(G, \mathbb{Z})$ is torsion-free, so the composite $T_2 \to \widetilde{K}^n(X) \to \mathrm{Hom}(K_n(X), \mathbb{Z})$ is trivial, and by exactness we obtain a lift $T_2 \to \mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})$. The image of this map must be torsion, which is to say that it must be contained in T_1, so the aforementioned map $T_1 \to T_2$ is a surjection. That is, the Universal Coefficient Sequence of Theorem 4 has last term torsion-free and first map restricting to an isomorphism of torsion subgroups. This implies the first statement.

For the second statement, we need only note that if X is finite, then $\widetilde{K}^{\mathrm{TF}}_{n-1}(X)$ is finitely generated, so $\mathrm{Ext}(\widetilde{K}_{n-1}(X), \mathbb{Z})$ is torsion, as required. □

Selick [28] deduces the following from work of Atiyah [5], Mislin [24] and Adams [3].

Theorem 5 (Künneth theorem for K-homology). *Let X and Y be of the homotopy type of finite complexes. Then there is an isomorphism of $\mathbb{Z}/2$-graded $\mathbb{Z}$-modules:*

$$\widetilde{K}^{\mathrm{TF}}_*(X \wedge Y) \cong \widetilde{K}^{\mathrm{TF}}_*(X) \otimes \widetilde{K}^{\mathrm{TF}}_*(Y).$$

□

Remark 2 It follows immediately from Corollary 4 (and knowledge of $\widetilde{K}^*(S^q)$) that $\widetilde{K}^{\mathrm{TF}}_*(S^q) \cong \mathbb{Z}$. We write ξ_q for the generator of $\widetilde{K}^{\mathrm{TF}}_*(S^q)$. By the Künneth Theorem (Theorem 5), we may choose the ξ_q so that $\xi_n \otimes \xi_m$ is identified with ξ_{n+m} under the homeomorphism $S^n \wedge S^m \cong S^{n+m}$.

In the case of K-theory, the analogous result follows directly from [3].

Theorem 6 (Künneth theorem for K-theory). *Let X and Y be of the homotopy type of finite complexes. Then the external product on K-theory defines an isomorphism of $\mathbb{Z}/2$-graded rings:*

$$\widetilde{K}^*_{\mathrm{TF}}(X) \otimes \widetilde{K}^*_{\mathrm{TF}}(Y) \xrightarrow{\cong} \widetilde{K}^*_{\mathrm{TF}}(X \wedge Y).$$

5.2 The James construction

For a space X, let X^s denote the product of s copies of X. Let $\sim$ be the relation on X^s defined by

$$(x_1, \ldots, x_{i-1}, *, x_{i+1}, x_{i+2}, \ldots x_s) \sim (x_1, \ldots, x_{i-1}, x_{i+1}, *, x_{i+2}, \ldots x_s).$$

Let $J_s(X)$ be the space $X^s/\sim$. There is a natural inclusion

$$\begin{aligned} J_s(X) &\overset{J}{\hookrightarrow}_{s+1}(X) \\ (x_1, \ldots, x_s) &\mapsto (x_1, \ldots, x_s, *). \end{aligned}$$

The *James construction* JX is defined to be the colimit of the diagram consisting of the spaces $J_s(X)$ and the above inclusions. Write $i_s : J_s(X) \to JX$ for the map associated to the colimit. Notice that JX carries a product given by concatenation, which makes it into the free topological monoid on X, and that a topological monoid is in particular an H-space.

Let $X^{\wedge i}$ denote the smash product of i copies of X, and let $\eta : X \to \Omega\Sigma X$ be the unit of the adjunction $\Sigma \dashv \Omega$. Explicitly, $\eta(x) = (t \mapsto \langle x, t\rangle \in \Sigma X)$.

Theorem 7 [22]

1. *There is a homotopy equivalence $JX \xrightarrow{\simeq} \Omega\Sigma X$ which respects the H-space structures and identifies i_1 with η.*
2. *There is a homotopy equivalence $\bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \xrightarrow{\simeq} \Sigma JX$ which restricts to a homotopy equivalence $\bigvee_{i=1}^{s} \Sigma X^{\wedge i} \xrightarrow{\simeq} \Sigma J_s(X)$ for each $s \in \mathbb{Z}^+$.* □

Lemma 4 [28, Lemma 7] *Let X have the homotopy type of an $(r-1)$-connected CW-complex.*

1. $(i_s)_* : \pi_N(J_s(X)) \to \pi_N(JX)$ *is an isomorphism for* $N < r(s+1) - 1$.
2. *Let $x \in \pi_N(J_s(X))$ for any N. If Σx is nontrivial then $(i_s)_*(x)$ is also nontrivial.*

Proof The first part follows by cellular approximation from the observation that $J_s(X)$ contains the $(r(s+1)-1)$-skeleton of JX. The second part follows from the observation that Σi_s has a retraction by Theorem 7. □

For spaces X and Y, let $X * Y$ denote the *join*, which we define to be the homotopy pushout of the projections $X \times Y \to X$ and $X \times Y \to Y$. The join is naturally a quotient of $X \times I \times Y$, where I denotes the unit interval. Following the treatment in [4], let C_1 denote the subspace of $X * Y$ consisting of points of the form $(x, t, *)$, for $t \in I$ and $x \in X$, and let C_2 be the subspace consisting of points of the form $(*, t, y)$. The subspace $C_1 \cup C_2 \cong CX \cup CY$ is contractible, so the quotient map $q : X * Y \to X * Y/C_1 \cup C_2$ is a homotopy equivalence. The quotient $X * Y/C_1 \cup C_2$ is homeomorphic to $\Sigma X \wedge Y$. The suspended product $\Sigma(X \times Y)$ is also a quotient of $X \times I \times Y$, and this quotient lies between $X * Y$ and $X * Y/C_1 \cup C_2$.

This gives a factorization of q as $X * Y \to \Sigma(X \times Y) \to \Sigma(X \wedge Y)$. Let q^{-1} denote any choice of homotopy inverse to q; all possible choices are homotopic. We may form a new map $\delta_{X,Y}$ as the composite $\Sigma(X \wedge Y) \xrightarrow{q^{-1}} X * Y \to \Sigma(X \times Y)$. It is automatic that $\delta_{X,Y}$ splits the quotient map $\pi : \Sigma(X \times Y) \to \Sigma X \wedge Y$. The homotopy class of $\delta_{X,Y}$ is well-defined, and we will call $\delta_{X,Y}$ the *canonical splitting* of π. Note that $\delta_{X,Y}$ is natural in maps of spaces in the sense that given $f : A \to X$ and $g : B \to Y$ we obtain a commutative diagram

$$\begin{tikzcd} \Sigma A \wedge B \arrow[r, "\delta_{A,B}"] \arrow[d, "\Sigma(f\wedge g)"] & \Sigma(A \times B) \arrow[d, "\Sigma(f\times g)"] \\ \Sigma X \wedge Y \arrow[r, "\delta_{X,Y}"] & \Sigma(X \times Y). \end{tikzcd}$$

For $s \geq 3$, consider the quotient map $\Sigma X^s \to \Sigma X^{\wedge s}$. We define the *canonical splitting* of this quotient to be the composite of canonical splittings

$$\Sigma X^{\wedge s} \to \Sigma(X \times X) \wedge X^{\wedge(s-2)} \to \Sigma((X \times X) \times X) \wedge X^{\wedge(s-3)} \to \cdots \to \Sigma X^s.$$

Of course, we chose an order of multiplication here. This canonical splitting is natural as before.

Definition 3 For a $\mathbb{Z}$-graded (respectively $\mathbb{Z}/2$-graded) module M, let $T(M) = \bigoplus_{k=1}^{\infty} M^{\otimes k}$ denote the *tensor algebra* on M. The product is given by concatenation. We refer to $M^{\otimes k}$ as the *weight k component* of the tensor algebra $T(M)$. We define a $\mathbb{Z}$-grading (respectively $\mathbb{Z}/2$-grading) on $T(M)$ by setting $|x_1 \otimes x_2 \otimes \cdots \otimes x_k| = \sum_{i=1}^{k} |x_i|$.

Definition 4 For a space Y, let $\sigma : \widetilde{K}_*^{\mathrm{TF}}(Y) \xrightarrow{\cong} \widetilde{K}_{*+1}^{\mathrm{TF}}(\Sigma Y)$ be the suspension isomorphism. Let $\varphi : \widetilde{K}_*^{\mathrm{TF}}(\Sigma Y) \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma Y)$ be a homomorphism of graded groups, not necessarily induced by a map of spaces. We call the composite $\sigma^{-1} \circ \varphi \circ \sigma$ the *desuspension* of θ, denoting it by $S^{-1}\varphi$.

Write $m_s : (\Omega\Sigma X)^s \to \Omega\Sigma X$ for the map given by iteratively performing the standard loop multiplication on $\Omega\Sigma X$ in any choice of order. Up to homotopy, m_s is independent of this choice of order, since $\Omega\Sigma X$ is homotopy associative.

Theorem 7 gives the existence of a homotopy equivalence $\Gamma : \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \to \Sigma\Omega\Sigma X$. There are many choices of Γ, up to homotopy. The next lemma asserts that Γ can be chosen in a way which suits our purpose. Selick [28] describes the composite $\bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \xrightarrow{\Gamma} \Sigma\Omega\Sigma X \xrightarrow{\simeq} \Sigma J X$ of Γ with the homotopy equivalence of Theorem 7 (1). This immediately implies the following description of Γ.

Lemma 5 [28] *Let X be a finite CW-complex. The homotopy equivalence $\Gamma : \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \to \Sigma\Omega\Sigma X$ may be chosen such that:*

1. $S^{-1}(\Gamma_*) : T(\widetilde{K}_*^{\mathrm{TF}}(X)) \xrightarrow{\cong} K_*^{\mathrm{TF}}(\Omega\Sigma X)$ *is an isomorphism of algebras;*
2. *the restriction of Γ to $\Sigma X^{\wedge s}$ is homotopic to the composite*

$$\Sigma X^{\wedge s} \to \Sigma X^s \xrightarrow{\Sigma(\eta)^s} \Sigma(\Omega\Sigma X)^s \xrightarrow{\Sigma m_s} \Sigma\Omega\Sigma X,$$

where the unlabelled arrow is the canonical splitting.

The description of the map Γ in Lemma 5 has the following consequence. For a space Y, let $\mathrm{ev} : \Sigma\Omega Y \to Y$ be the evaluation map, which may be described explicitly by $\mathrm{ev}(\langle\gamma, t\rangle) = \gamma(t)$ for $\gamma \in \Omega Y$.

Lemma 6 *Let $\Gamma : \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i} \to \Sigma\Omega\Sigma X$ be the homotopy equivalence of Lemma 5. The composite $\mathrm{ev} \circ \Gamma$ is homotopic to the projection onto the first wedge summand.*

Proof Let $\iota_s : \Sigma X^{\wedge s} \to \bigvee_{i=1}^{\infty} \Sigma X^{\wedge i}$ be the inclusion of the s-th wedge summand. We must show that

$$\mathrm{ev} \circ \Gamma \circ \iota_s \simeq \begin{cases} 1_{\Sigma X} & \text{if } s = 1, \text{ and} \\ * & \text{otherwise.} \end{cases}$$

The following diagram commutes up to homotopy

$$\begin{tikzcd} \Sigma X \arrow[r, "\Sigma\eta"] \arrow[dr, "1_{\Sigma X}"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\ & \Sigma X. \end{tikzcd}$$

By Lemma 5, $\Sigma\eta = \Gamma \circ \iota_1$, which implies the $s = 1$ statement.

Now let $s \geq 2$. Ganea [14, Theorems 1.1 and 1.4] shows that the homotopy fibre of ev is given by

$$\Sigma(\Omega\Sigma X \wedge \Omega\Sigma X) \xrightarrow{v} \Sigma\Omega\Sigma X \xrightarrow{\mathrm{ev}} \Sigma X,$$

where the map v is equal to the composite

$$\Sigma(\Omega\Sigma X \wedge \Omega\Sigma X) \to \Sigma(\Omega\Sigma X \times \Omega\Sigma X) \xrightarrow{\Sigma m_2} \Sigma\Omega\Sigma X$$

of Σm_2 with the canonical splitting. We will show that $\Gamma \circ \iota_s$ factors through v, and hence composes trivially with ev. Consider the following diagram, where the unlabelled arrows are all canonical splittings:

$$\begin{tikzcd}
 & \Sigma(\Omega\Sigma X \wedge \Omega\Sigma X) \arrow[r] \arrow[rr, bend left, "v"] & \Sigma(\Omega\Sigma X \times \Omega\Sigma X) \arrow[r, "\Sigma m_2"'] & \Sigma\Omega\Sigma X \\
\Sigma(\Omega\Sigma X)^{\wedge s} \arrow[r] & \Sigma((\Omega\Sigma X)^{(s-1)} \wedge \Omega\Sigma X) \arrow[r] \arrow[u, "\Sigma(m_{(s-1)} \wedge 1)"] & \Sigma(\Omega\Sigma X)^s \arrow[r, "\Sigma m_s"'] \arrow[u, "\Sigma(m_{(s-1)} \times 1)"] & \Sigma\Omega\Sigma X \arrow[u, equal] \\
\Sigma X^{\wedge s} \arrow[r] \arrow[u, "\Sigma\eta^{\wedge s}"] & \Sigma(X^{(s-1)} \wedge X) \arrow[r] \arrow[u, "\Sigma(\eta^{(s-1)} \wedge \eta)"] & \Sigma X^s. \arrow[u, "\Sigma\eta^s"] &
\end{tikzcd}$$

The composite along the bottom of the diagram is $\Gamma \circ \iota_s$, so to obtain the desired factorization of $\Gamma \circ \iota_s$ through v, it suffices to show that the diagram commutes up to homotopy.

The top right square commutes because $m_2 \circ m_{(s-1)} \simeq m_s$, by homotopy associativity of the H-space $\Omega\Sigma X$. The remaining three squares commute by naturality of our canonical splitting. This completes the proof. $\square$

Let ρ_k be the projection $T(\widetilde{K}_*^{\mathrm{TF}}(X)) \to \widetilde{K}_*^{\mathrm{TF}}(X)^{\otimes k}$. The next corollary is immediate from Lemma 6.

Corollary 5 $S^{-1}(\mathrm{ev}_* \circ \Gamma_*) = \rho_1 : T(\widetilde{K}_*^{\mathrm{TF}}(X)) \to \widetilde{K}_*^{\mathrm{TF}}(X)$. $\square$

5.3 Primitives and commutators

It follows from the Künneth Theorem (Theorem 5), and the fact that $\Sigma(Y \times Y) \simeq \Sigma Y \vee \Sigma Y \vee \Sigma(Y \wedge Y)$, that $K_*^{\mathrm{TF}}(Y \times Y) \cong K_*^{\mathrm{TF}}(Y) \otimes K_*^{\mathrm{TF}}(Y)$. We may therefore make the following definition. A class $y \in \widetilde{K}_*^{\mathrm{TF}}(Y)$ is called *primitive* if $\Delta_*(y) = y \otimes 1 + 1 \otimes y$, where $\Delta : Y \to Y \times Y$ is the diagonal, defined by $\Delta(y) = (y, y)$.

The comultiplication $Y \to Y \vee Y$ on a co-H-space Y is a factorization of Δ via the inclusion $Y \vee Y \hookrightarrow Y \times Y$. From this point of view, the following lemma is immediate.

Lemma 7 *If Y is a co-H-space, then all elements in $\widetilde{K}_*^{\mathrm{TF}}(Y)$ are primitive.* $\square$

If Y is an H-group, then the multiplication $m : Y \times Y \to Y$ induces a map $\widetilde{K}_*^{\mathrm{TF}}(Y) \otimes \widetilde{K}_*^{\mathrm{TF}}(Y) \to \widetilde{K}_*^{\mathrm{TF}}(Y)$. We will denote this map by juxtaposition, so that $m_*(y_1 \otimes y_2) = y_1 y_2$. Furthermore, the commutator $Y \times Y \to Y$ descends to a map $c : Y \wedge Y \to Y$. Expanding the definition of the commutator in terms of the K-homology Künneth Theorem (Theorem 5) gives the following lemma.

Lemma 8 *Let Y be an H-group, and let $c : Y \wedge Y \to Y$ be the commutator. If y_1 and $y_2 \in \widetilde{K}_*^{\mathrm{TF}}(Y)$ are primitive, then $c_*(y_1 \otimes y_2) = y_1 y_2 - (-1)^{|y_1||y_2|} y_2 y_1$.* $\square$

6 The category of ψ-modules

In [2], Adams defines an abelian category which we will follow Selick [28] in calling ψ-modules. The e-invariant, which is our central tool, is defined by Adams in terms of ψ-modules. The purpose of this section is to record results about ψ-modules for later use.

A ψ-module consists of an abelian group M, with homomorphisms

$$\psi^\ell : M \to M$$

for each $\ell \in \mathbb{Z}$, satisfying the axioms of [2, Section 6]. If X is a space then the group $\widetilde{K}^0(X)$, together with its Adams operations, is a ψ-module. Since we defined $\widetilde{K}^{-1}(X)$ by setting $\widetilde{K}^{-1}(X) = \widetilde{K}^0(\Sigma X)$, it too has the structure of a ψ-module. Maps of spaces induce maps of ψ-modules. The Adams operation ψ^ℓ on $\widetilde{K}^0(S^{2n})$ is multiplication by ℓ^n, so in particular Adams operations do not commute with the Bott isomorphism.

For graded ψ-modules M and N we will write $\mathrm{Hom}_{\psi-\mathrm{Mod}}(M, N)$ for the abelian group consisting of graded ψ-module homomorphisms. The unadorned notation $\mathrm{Hom}(M, N)$ will mean homomorphisms of the underlying graded abelian groups.

Lemma 9 *Let M and N be ψ-modules, with N torsion-free. The inclusion of $\mathbb{Z}$-modules* $\mathrm{Hom}_{\psi-\mathrm{Mod}}(M, N) \hookrightarrow \mathrm{Hom}(M, N)$ *is an injection onto a summand.*

Proof Let $\varphi : M \to N$ be a homomorphism of underlying $\mathbb{Z}$-modules. If, for some $k \in \mathbb{Z}\backslash\{0\}$, $k \cdot \varphi$ is a ψ-module homomorphism, then, since N is torsion-free, φ is also a ψ-module homomorphism. This implies that Coker $(\mathrm{Hom}_{\psi-\mathrm{Mod}}(M, N) \hookrightarrow \mathrm{Hom}(M, N))$ is torsion-free, which implies the result. □

For the avoidance of doubt, by the e-invariant we will always mean what Adams calls the complex e-invariant e_C [1,2].

Definition 5 (Adams' e-invariant). Suppose that $f : X \to Y$ induces the trivial map on $\widetilde{K}^*$. Then the cofibre sequence of f gives a short exact sequence of ψ-modules

$$0 \leftarrow \widetilde{K}^0(Y) \leftarrow \widetilde{K}^0(C_f) \leftarrow \widetilde{K}^0(\Sigma X) \leftarrow 0.$$

The *e-invariant* of f is the element of $\mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Y), \widetilde{K}^0(\Sigma X))$ represented by this exact sequence.

The e-invariant does not commute with the Bott isomorphism, but the interaction between the Bott isomorphism and the Adams operations is easy to describe, as follows. Let ψ^ℓ_Y be the homomorphism $\psi^\ell : \widetilde{K}^0(Y) \to \widetilde{K}^0(Y)$. Then, modulo the Bott isomorphism, we have $\psi^\ell_{\Sigma^2 X} = \ell \cdot \psi^\ell_X$. That is 'upon double suspending, the Adams operations gain a factor ℓ'. In terms of the e-invariant, all we need to know is the following.

Lemma 10 [2, Proposition 3.4b)]. *There is a homomorphism*

$$T : \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Y), \widetilde{K}^0(\Sigma X)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(\Sigma^2 Y), \widetilde{K}^0(\Sigma^3 X)),$$

such that $T(e(f)) = e(\Sigma^2 f)$. □

We will be concerned only with the e-invariants of maps whose domain is a sphere. One of the two K-groups of a sphere vanishes, in the dimension matching the parity of the sphere, but the e-invariant, as defined above, lives only in K^0. In order to detect maps regardless of the parity of the sphere on which they are defined, we will need to keep track of the e-invariants of f and Σf, so we will use the following modified e-invariant.

Definition 6 (Double e-invariant). Let

$$\mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Y),\widetilde{K}^*(\Sigma X)) := \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Y),\widetilde{K}^0(\Sigma X)) \oplus \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^{-1}(Y),\widetilde{K}^{-1}(\Sigma X)).$$

Suppose that $f : X \to Y$ induces the trivial map on $\widetilde{K}^*$. Then the *double e-invariant of* f is $\overline{e}(f) = (e(f), e(\Sigma f)) \in \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Y),\widetilde{K}^*(\Sigma X))$.

Pullback of an extension along a homomorphism defines a map

$$\mathrm{Hom}_{\psi-\mathrm{Mod}}(M, B) \otimes \mathrm{Ext}_{\psi-\mathrm{Mod}}(B, A) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(M, A).$$

If $g : Y \to Z$ then $e(g \circ f)$ is represented by the pullback of $e(f)$ and $g^* : \widetilde{K}^0(Z) \to \widetilde{K}^0(Y)$ [2, Proposition 3.2 b)]. To describe $\overline{e}(g \circ f)$ we need only apply this result degree-wise, as follows. For convenience, we write $g^* \cdot e(f)$ for the pullback of g^* and $e(f)$. Define the map

$$\begin{aligned}&\theta_0(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Z),\widetilde{K}^0(Y)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^0(Z),\widetilde{K}^0(\Sigma X))\\ &\theta_0(f)(x) = x \cdot e(f).\end{aligned}$$

Likewise, define

$$\begin{aligned}&\theta_{-1}(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^{-1}(Z),\widetilde{K}^{-1}(Y)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^{-1}(Z),\widetilde{K}^{-1}(\Sigma X))\\ &\theta_{-1}(f)(x) = x \cdot e(\Sigma f).\end{aligned}$$

Combining these, let

$$\theta(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z),\widetilde{K}^*(Y)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z),\widetilde{K}^*(\Sigma X))$$

be the direct sum $\theta_0(f) \oplus \theta_{-1}(f)$. These definitions, together with Adams' above result, give the following lemma.

Lemma 11 *For maps $f : X \to Y$ and $g : Y \to Z$, the following diagram commutes:*

$$\begin{tikzcd}
{[Y, Z]} \arrow[r, "f^*"] \arrow[d, "\deg"'] & {[X, Z]} \arrow[d, "\overline{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z),\widetilde{K}^*(Y)) \arrow[r, "\theta(f)"] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(Z),\widetilde{K}^*(\Sigma X)).
\end{tikzcd}$$

Following [28], write $\mathbb{Z}(n)$ for the ψ-module $\widetilde{K}^0(S^{2n})$. Explicitly, $\mathbb{Z}(n)$ has underlying abelian group $\mathbb{Z}$, and ψ^ℓ acts by multiplication by ℓ^n. It follows that $\widetilde{K}^{-1}(S^{2n+1}) := \widetilde{K}^0(S^{2n+2}) \cong \mathbb{Z}(n+1)$.

Lemma 12 [2, Proposition 7.8, 7.9] *If $n < m$ then $\mathrm{Ext}_{\psi-\mathrm{Mod}}(\mathbb{Z}(n),\mathbb{Z}(m))$ injects into $\mathbb{Q}/\mathbb{Z}$. The e-invariant of a map $f : S^{2m-1} \to S^{2n}$ may therefore be regarded as an element of $\mathbb{Q}/\mathbb{Z}$. Furthermore, the value $e(f)$ in $\mathbb{Q}/\mathbb{Z}$ satisfies $e(\Sigma^2 f) = e(f)$, so in particular, when f is a map between spheres, $e(f)$ depends only on the stable homotopy class of f.* □

The following theorem is the main technical component of Selick's paper [28].

Theorem 8 [28, Theorem 6] *Let $f' : S^{2m-1} \to S^{2n}$ be such that $p^{t-1}e(f') \neq 0$ in $\mathbb{Q}/\mathbb{Z}$, for p prime and some $t \in \mathbb{Z}^+$. Let Y have the homotopy type of a finite CW-complex and let $g : S^{2n} \to Y$ be such that $\mathrm{Im}(g^* : \widetilde{K}^0(Y) \to \widetilde{K}^0(S^{2n}))$ contains $up^s\widetilde{K}^0(S^{2n})$, for $s \in \mathbb{Z}^+$ and u prime to p. If $s < t$, and there exists some $\ell \in \mathbb{Z}^+$ for which*

$$\psi^\ell \otimes \mathbb{Q} : \widetilde{K}^0(Y) \otimes \mathbb{Q} \to \widetilde{K}^0(Y) \otimes \mathbb{Q}$$

does not have ℓ^m as an eigenvalue, then $e(g \circ f') \neq 0$. □

The following theorem of Gray [15] will provide the map f' for Theorem 8. Specifically, this theorem provides a linearly spaced family of stems, each of which has a stable p-torsion class which is born on S^3 and detected by the e-invariant.

Theorem 9 [15, Corollary of Theorem 6.2] *Let p be an odd prime and let $j \in \mathbb{Z}^+$. Then there exists a class $f_{p,j} \in \pi_{2j(p-1)+2}(S^3)$ with $e(f_{p,j}) = \frac{-1}{p} \in \mathbb{Q}/\mathbb{Z}$.* □

The corresponding 2-primary result is as follows. Adams [2, Theorem 1.5 and Proposition 7.14] shows that, for $j > 0$, the $(8j+3)$-rd stem contains a direct summand whose 2-primary component has order 8, and that on this component the e-invariant is a surjection onto $\mathbb{Z}/4$. The sphere of origin of the classes in this component was deduced by Curtis in [11].

Theorem 10 [2,11] *Let $j \in \mathbb{Z}^+$. Then there exists a class $f_{2,j} \in \pi_{8j+6}(S^3)$ of order 4, with $e(f_{2,j}) = \frac{-1}{2} \in \mathbb{Q}/\mathbb{Z}$.* □

7 Main construction

Having assembled preliminaries in Sects. 5 and 6, we can begin to work towards the proof of Theorem 2. Our approach is as follows. From the data of Theorem 2, we will construct a commutative diagram of (roughly) the following form, where $\mathscr{B}$ is a set and the other objects are $\mathbb{Z}$-modules.

$$\begin{tikzcd}
\mathscr{B}^k \arrow[r] \arrow[d] & \pi_*(\Omega\Sigma X) \arrow[d] \\
I^k \arrow[r] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*(\Omega\Sigma X), \widetilde{K}^*(S^*)).
\end{tikzcd}$$

We will argue that

- The image of the top map consists of classes of order dividing p.
- The image of the left vertical map generates a submodule isomorphic to the weight k component of the free graded Lie algebra over $\mathbb{Z}/p$ on two generators.
- The bottom map is injective.

Together, these facts imply that there is a submodule of $\pi_*(\Omega\Sigma X) \cong \pi_{*+1}(\Sigma X)$, consisting of classes of order dividing p, and surjecting onto a module isomorphic to the weight k component of the free graded Lie algebra over $\mathbb{Z}/p$ on two generators. This submodule (which is necessarily a $\mathbb{Z}/p$-vector space) must therefore have dimension at least $W_2(k)$ (Theorem 11), which will imply that ΣX is p-hyperbolic (Lemma 2).

The diagram will be obtained by juxtaposing three squares. Sections 7.1–7.3 each construct one of these squares. In Sect. 7.4 we put them together and prove Theorem 2. Roughly speaking, the top map of the diagram should be thought of as first taking a family of Samelson products and then pulling them back along some suitable map f coming from Gray's work (Theorem 9). The vertical maps should be thought of as passing from maps of spaces to K-theoretic invariants, and the bottom map (therefore) should be thought of as tracking the effect of the top map on these invariants.

Because of the need to work with a finite CW-complex in Selick's Theorem (Theorem 8) we will restrict the right hand side of the diagram to instead refer to some finite skeleton $J_s(X)$ of the James construction.

7.1 Samelson products and their Hurewicz images in *K*-homology

Let R be a commutative ring with unit. We take a *graded Lie algebra* over R to be defined as in [26]. For a non-negatively graded R-module V, let $L(V)$ denote the *free graded Lie algebra* [26, Section 8.5]. Write $L^k(V)$ for the submodule of $L(V)$ generated by the brackets of length k in the elements of V. We will call $L^k(V)$ the *weight k component* of $L(V)$. Note that this convention differs from Neisendorfer's - he writes $L(V)_k$ for the weight k component.

Definition 7 Let Y be an H-group, and let $c : Y \wedge Y \to Y$ be the commutator of Lemma 8. Let $\alpha \in \pi_N(Y)$, and let $\beta \in \pi_M(Y)$. The *Samelson product* of α and β, written $\langle \alpha, \beta \rangle \in \pi_{N+M}(Y)$, is the composite

$$\langle \alpha, \beta \rangle : S^{N+M} \cong S^N \wedge S^M \xrightarrow{\alpha \wedge \beta} Y \wedge Y \xrightarrow{c} Y.$$

Samelson products are bilinear, graded anticommutative, and satisfy the graded Jacobi identity. They fail, however, to make $\pi_*(Y)$ into a graded Lie algebra over $\mathbb{Z}$ in Neisendorfer's sense [27, Section 7]. This is important because we want the natural map from the free Lie algebra to the corresponding tensor algebra to be an injection onto a summand (Lemma 14). One could define an appropriate notion of 'free graded pseudo-Lie algebra', and proceed as follows with that in place of the set $\mathscr{B}(\pi_*(A))$, which we use in what follows, but we prefer to avoid making the extra definition.

For a graded R-module V, let $U(V)$ denote the graded set of homogeneous elements in V. Let $\mathscr{B}(V)$ be the set of formal iterated brackets of the elements of $U(V)$. Bracket gives a natural operation on $\mathscr{B}(V)$, which we write as $[x, y]$. Elements of $\mathscr{B}(V)$ are nonassociative words in the elements of $U(V)$, so we may define a grading on $\mathscr{B}(V)$ which extends the grading on $U(V)$ via $|[x, y]| = |x| + |y|$. The *weight* of an element of $\mathscr{B}(V)$ is its word length. Write $\mathscr{B}_N(V)$ for the subset of elements in degree N, $\mathscr{B}^k(V)$ for the subset of elements of weight k, and set $\mathscr{B}^k_N(V) = \mathscr{B}^k(V) \cap \mathscr{B}_N(V)$.

Let $\nu : A \to \Omega\Sigma X$ be a map. By definition of $\mathscr{B}(\pi_*(A))$, there exists a map

$$\Phi^\pi_\nu : \mathscr{B}(\pi_*(A)) \to \pi_*(\Omega\Sigma X)$$

which extends ν_* and satisfies $\Phi^\pi_\nu([x, y]) = \langle \Phi^\pi_\nu(x), \Phi^\pi_\nu(y) \rangle$ for all $x, y \in \mathscr{B}(\pi_*(A))$.

YFor a $\mathbb{Z}/2$-graded $\mathbb{Z}$-module V, we define a non-negatively graded $\mathbb{Z}$-module $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), V)$, by setting

$$\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), V)_N = \begin{cases} \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), V) & \text{if } N > 0, \text{ and} \\ 0 & \text{if } N \leq 0, \end{cases}$$

where the homomorphisms are understood to respect the $\mathbb{Z}/2$-grading on $\widetilde{K}_*$ and V.

In the case that $V = L$ is a $\mathbb{Z}/2$-graded Lie algebra over $\mathbb{Z}$, $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L)$ inherits a non-negatively graded Lie algebra structure as follows. Let the generators ξ_N of $\widetilde{K}^{\mathrm{TF}}_*(S^N)$ be as in Remark 2. Then the bracket $[f, g]$ of $f \in \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), L)$ and $g \in \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^M), L)$ is the homomorphism $\widetilde{K}^{\mathrm{TF}}_*(S^M) \to L$ carrying ξ_{N+M} to $[f(\xi_N), g(\xi_M)] \in L$. The squaring operation is defined in the same way. Likewise, if V is a $\mathbb{Z}/2$-graded associative algebra over $\mathbb{Z}$, then $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), V)$ inherits the structure of a non-negatively graded associative algebra.

Let $\nu : A \to \Omega\Sigma X$ be a map. There is a composition

$$L(\widetilde{K}^{\mathrm{TF}}_*(A)) \to T(\widetilde{K}^{\mathrm{TF}}_*(A)) \to \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X),$$

where the first map is the natural map which is the identity on $\widetilde{K}^{\mathrm{TF}}_*(A)$ and satisfies $[x, y] \mapsto xy - (-1)^{|x||y|}yx$, and the second map is obtained by applying the universal property of the tensor algebra to ν_*. Let

$$\Phi^K_\nu : \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X))$$

be the pushforward along the above composite. It is then automatic that Φ^K_ν is a map of non-negatively graded Lie algebras over $\mathbb{Z}$, where the structures are defined as above.

We write $\deg : \pi_N(Y) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(Y))$ for the map $f \mapsto f_*$. Let $\deg' : \mathscr{B}(\pi_*(A)) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A)))$ be the unique map which restricts to $\deg : \pi_*(A) \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(A)) \subset \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A)))$ and carries brackets to brackets. The above maps are related as follows.

Lemma 13 *Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes. The following diagram commutes:*

$$\begin{tikzcd}
\mathscr{B}(\pi_*(A)) \arrow[r, "\Phi^\pi_\nu"] \arrow[d, "\deg'"] & \pi_*(\Omega\Sigma X) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \arrow[r, "\Phi^K_\nu"] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)).
\end{tikzcd}$$

Proof By construction of $\mathscr{B}(\pi_*(A))$, it suffices to show that the restriction of the diagram to the weight 1 component $\mathscr{B}^1(\pi_*(A)) = \pi_*(A)$ commutes, and that all maps respect the bracket operations. By definition, $L^1(\widetilde{K}^{\mathrm{TF}}_*(A)) = \widetilde{K}^{\mathrm{TF}}_*(A)$. It then follows immediately from the definitions of Φ^π_ν and Φ^K_ν that restricting the left hand side of the diagram to weight 1 components gives the diagram

$$\begin{tikzcd}
\pi_*(A) \arrow[r, "\nu_*"] \arrow[d, "\deg"] & \pi_*(\Omega\Sigma X) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(A)) \arrow[r, "\nu_*"] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)),
\end{tikzcd}$$

which commutes, since it just expresses naturality of deg.

It remains to show that all maps respect bracket operations. The maps Φ^π_ν and $\deg'$ respect the bracket operations by definition, and Φ^K_ν respects bracket operations by construction. We therefore only need show that deg respects brackets. Let $f \in \pi_N(\Omega\Sigma X)$, and let $g \in \pi_M(\Omega\Sigma X)$. We must show that $\deg(\langle f, g\rangle)$ is the commutator $\deg(f)\deg(g) - (-1)^{NM}\deg(g)\deg(f)$ with respect to the algebra operation on $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X))$.

Since $\widetilde{K}^{\mathrm{TF}}_*(S^{N+M}) \cong \mathbb{Z}$, it suffices to show that the two homomorphisms agree on the generator ξ_{N+M} (Remark 2). By Definition 7 and the Künneth Theorem (Theorem 5),

$$\deg(\langle f, g\rangle)(\xi_{N+M}) = c_* \circ (f_* \otimes g_*)(\xi_N \otimes \xi_M) = c_* \circ (f_*(\xi_N) \otimes g_*(\xi_M)).$$

Spheres of dimension at least 1 are co-H spaces, so by Lemma 7, ξ_N and ξ_M are primitive. By naturality of the diagonal $f_*(\xi_N)$ and $g_*(\xi_M)$ are still primitive, so by Lemma 8,

$$c_* \circ (f_*(\xi_N) \otimes g_*(\xi_M)) = f_*(\xi_N)g_*(\xi_M) - (-1)^{NM}g_*(\xi_M)f_*(\xi_N),$$

which by definition of the multiplication on $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X))$ is the result of evaluating $\deg(f)\deg(g) - (-1)^{NM}\deg(g)\deg(f)$ on ξ_{N+M}, as required. □

We now lift the previous result to $J_s(X)$, thereby producing the first square of the diagram promised at the start of this section. Recall that we write $i_s : J_s(X) \to JX$ for the inclusion, and that by Theorem 7 we have a homotopy equivalence $JX \xrightarrow{\simeq} \Omega\Sigma X$. We will abuse notation and also write i_s for the composite $J_s(X) \to JX \xrightarrow{\simeq} \Omega\Sigma X$.

Corollary 6 *Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. If $N, s \in \mathbb{Z}^+$ satisfy $N < r(s+1) - 1$, then $(i_s)_* : \pi_N(J_s X) \to \pi_N(\Omega\Sigma X)$ is an isomorphism and for each $k \leq s$ there exists a commutative diagram:*

$$\begin{tikzcd}
\mathscr{B}^k_N(\pi_*(A)) \arrow[r, "\widetilde{\Phi^\pi_\nu}"] \arrow[d, "\deg'"] & \pi_N(J_s X) \arrow[d, "\deg"] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), L^k(\widetilde{K}^{\mathrm{TF}}_*(A))) \arrow[r, "\widetilde{\Phi^K_\nu}"] & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(J_s X)),
\end{tikzcd}$$

with $(i_s)_* \circ \widetilde{\Phi^\pi_\nu} = \Phi^\pi_\nu$ and $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), (i_s)_*) \circ \widetilde{\Phi^K_\nu} = \Phi^K_\nu$.

Proof Consider the diagram of Lemma 13. Lemma 4 shows that $(i_s)_*$ is an isomorphism on π_N, so let $\widetilde{\Phi^\pi_\nu}$ be the unique map such that the condition $(i_s)_* \circ \widetilde{\Phi^\pi_\nu} = \Phi^\pi_\nu$ holds. By Theorem 7 (2) and Lemma 5, the map $(i_s)_* : \widetilde{K}^{\mathrm{TF}}_N(J_s(X)) \to \widetilde{K}^{\mathrm{TF}}_N(\Omega\Sigma X)$ is the inclusion of the tensors of length at most s. Since $k \leq s$, we may therefore define $\widetilde{\Phi^K_\nu}$ to be the unique map such that the condition $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), (i_s)_*) \circ \widetilde{\Phi^K_\nu} = \Phi^K_\nu$ holds. Commutativity then follows from Lemma 13 by naturality of deg, since $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), (i_s)_*)$ is injective. □

Lemma 14 *Let V be a non-negatively- or $\mathbb{Z}/2$-graded $\mathbb{Z}$-module which is free and finitely generated in each dimension. Then*

- *$L(V)$ and $T(V)$ are free $\mathbb{Z}$-modules in every dimension.*
- *The natural map $L(V) \to T(V)$, $[x, y] \mapsto xy - (-1)^{|x||y|}yx$ is an injection onto a summand.*

Proof The non-negatively graded case is immediate from [26], Proposition 8.3.1 and p282. For the $\mathbb{Z}/2$-graded case, first observe that there is a forgetful functor U from $\mathbb{Z}$-graded modules to $\mathbb{Z}/2$-modules which carries $\mathbb{Z}$-graded (Lie) algebras to $\mathbb{Z}/2$-graded (Lie) algebras, and a functor C from $\mathbb{Z}/2$-graded modules to $\mathbb{Z}$-modules which puts V_0 in any even dimension and V_1 in any odd dimension. Both C and U respect freeness and split injections, and there are natural isomorphisms $UT(CV) \cong T(V)$ and $UL(CV) \cong L(V)$. This implies the $\mathbb{Z}/2$-graded result. □

The graded version of Theorem 3 now follows immediately from Hilton's paper:

Theorem 11 [19, Theorem 3.2, 3.3] *Let V be a torsion-free $\mathbb{Z}$- or $\mathbb{Z}/2$-graded $\mathbb{Z}$-module of total dimension n. Then the total dimension of $L^k(V)$ is $W_n(k)$.* □

Let R be a commutative ring with unit. Let M be an R-module, and as usual let $T(M)$ denote the tensor algebra on M. Let $\iota_k : M^{\otimes k} \to T(M)$ be the inclusion, and let $\rho_k : T(M) \to M^{\otimes k}$ be the projection. Let $\tau : T(M) \to T(M)$ be the composite $\iota_1 \circ \rho_1$. Given an R-algebra A, and a map $\varphi : M \to A$, we write $\widetilde{\varphi}$ for the induced map $T(M) \to A$, that is, the unique map of algebras such that $\widetilde{\varphi} \circ \iota_1 = \varphi$.

Now, let M and N be R-modules, and let $\varphi : M \to T(N)$ be a map. In the proof of Theorem 12, we will wish to make a 'leading terms' style argument. This is made precise in the next Lemma, which compares $\widetilde{\varphi}$ with $\widetilde{\tau \circ \varphi}$. Informally, we think of $\widetilde{\tau \circ \varphi}$ as the 'leading terms part' of $\widetilde{\varphi}$.

Lemma 15 *Let R be a commutative ring with unit. Let M and N be $\mathbb{Z}$- or $\mathbb{Z}/2$-graded R-modules. Let $\iota_k : M^{\otimes k} \to T(M)$ be the inclusion, let $\rho_k : T(N) \to N^{\otimes k}$ be the projection, and let $\tau : T(N) \to T(N)$ be as above. Let $\varphi : M \to T(N)$ be a map. Then $\rho_k \circ \widetilde{\varphi} \circ \iota_k = \rho_k \circ \widetilde{\tau \circ \varphi} \circ \iota_k$.*

Proof It suffices to check equality on basic tensors. Let $v \in M^{\otimes k}$ be a basic tensor, so that $v = v_1 \otimes v_2 \otimes \cdots \otimes v_k$, for $v_i \in M$. Then

$$\begin{aligned}\widetilde{\varphi} \circ \iota_k(v) &= \widetilde{\varphi}(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \varphi(v_1) \otimes \varphi(v_2) \otimes \cdots \otimes \varphi(v_k)\\ &= \tau(\varphi(v_1)) \otimes \tau(\varphi(v_2)) \otimes \cdots \otimes \tau(\varphi(v_k)) + \text{terms of weight } > k.\end{aligned}$$

Applying ρ_k to both sides yields the result. □

Theorem 12 *Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/p$ for p prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes. Let $\overline{\nu} : \Sigma A \to \Sigma X$ be the adjoint of ν. If*

$$\overline{\nu}_* \otimes \mathbb{F} : \widetilde{K}^{\mathrm{TF}}_*(\Sigma A) \otimes \mathbb{F} \to \widetilde{K}^{\mathrm{TF}}_*(\Sigma X) \otimes \mathbb{F}$$

is an injection, then

$$\Phi^K_\nu \otimes \mathbb{F} : \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \otimes \mathbb{F} \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)) \otimes \mathbb{F}$$

is also an injection.

Remark 3 In the case where $\overline{\nu}$ is a suspension $\Sigma\zeta$, we have a diagram

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "\Omega\overline{\nu}"] & \Omega\Sigma X \\ A \arrow[u, "\eta"] \arrow[ur, "\nu"] \arrow[r, "\zeta"] & X, \arrow[u, "\eta"'] \end{tikzcd}$$

so in particular ν_* factors through the weight 1 component $\widetilde{K}^{\mathrm{TF}}_*(X)$ of the tensor algebra decomposition of $\widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)$. This dramatically simplifies the proof, removing the need for Lemma 15. In practice this is not a reasonable assumption - for example, the map $\mu : S^3 \vee S^5 \to \Sigma\mathbb{C}P^2$ of Example 6 (which plays the role of $\overline{\nu}$) does not desuspend.

Proof In this proof, for a space Y, we will identify the algebras $T(\widetilde{K}^{\mathrm{TF}}_*(Y))$ and $\widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma Y)$, omitting the isomorphism $S^{-1}\Gamma_*$ of Lemma 5. We defined Φ^K_ν to be the pushforward along a certain map $L(\widetilde{K}^{\mathrm{TF}}_*(A)) \to \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X)$. Call this map $\Phi^{K'}_\nu$. It suffices to prove that $\Phi^{K'}_\nu \otimes \mathbb{F}$ is an injection.

The triangle identities for the adjunction $\Sigma \dashv \Omega$ give a commutative diagram

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "\Omega\overline{\nu}"] & \Omega\Sigma X \\ A. \arrow[u, "\eta"] \arrow[ur, "\nu"] & \end{tikzcd}$$

Since $\Phi^{K'}_\nu$ is the unique map of Lie algebras extending ν, we have a commuting diagram

$$\begin{tikzcd} T(\widetilde{K}^{\mathrm{TF}}_*(A)) \cong \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma A) \arrow[r, "(\Omega\overline{\nu})_*"] & \widetilde{K}^{\mathrm{TF}}_*(\Omega\Sigma X) \\ L(\widetilde{K}^{\mathrm{TF}}_*(A)), \arrow[u, hook] \arrow[ur, "\Phi^{K'}_\nu"'] & \end{tikzcd}$$

where we note that that by Lemma 14, the natural map $L(\widetilde{K}^{\mathrm{TF}}_*(A)) \to T(\widetilde{K}^{\mathrm{TF}}_*(A))$ is an injection onto a summand. It therefore suffices to show that $(\Omega\overline{\nu})_* \otimes \mathbb{F}$ is an injection.

Let $\widetilde{(\nu_*)}$ denote the extension of ν_* to $T(\widetilde{K}^{\mathrm{TF}}_*(A))$, so that $\widetilde{(\nu_*)} = (\Omega\overline{\nu})_*$ (modulo the isomorphism $S^{-1}\Gamma_*$, as above). Since $(\rho_i \circ \widetilde{(\nu_*)} \circ \iota_k) = 0$ for $i < k$, it further suffices to show that $(\rho_k \circ \widetilde{(\nu_*)} \circ \iota_k) \otimes \mathbb{F}$ is an injection for each k. By Lemma 15, with $M = \widetilde{K}^{\mathrm{TF}}_*(A)$ and $N' = \widetilde{K}^{\mathrm{TF}}_*(X)$, we have that $\rho_k \circ \widetilde{(\nu_*)} \circ \iota_k = \rho_k \circ \widetilde{(\tau \circ \nu_*)} \circ \iota_k$.

As previously, let $\mathrm{ev} : \Sigma\Omega Y \to Y$ denote the evaluation map. The following diagram commutes:

$$\begin{tikzcd} \Sigma A \arrow[r, "\Sigma\nu"] \arrow[dr, "\overline{\nu}"'] & \Sigma\Omega\Sigma X \arrow[d, "\mathrm{ev}"] \\ & \Sigma X. \end{tikzcd}$$

The hypothesis therefore implies that the composite $(\mathrm{ev}_* \circ \Sigma\nu_*) \otimes \mathbb{F}$ is an injection. Desuspending, we have that $(S^{-1}\mathrm{ev}_* \circ \nu_*) \otimes \mathbb{F}$ is an injection. By Lemma 5,

$$(\rho_1 \circ \nu_*) \otimes \mathbb{F} : \widetilde{K}^{\mathrm{TF}}_*(A) \otimes \mathbb{F} \to \widetilde{K}^{\mathrm{TF}}_*(X) \otimes \mathbb{F}$$

is an injection of $\mathbb{F}$-vector spaces. Thus, the image $(\rho_1 \circ \nu_*)(\widetilde{K}^{\mathrm{TF}}_*(A)) \otimes \mathbb{F} \subset \widetilde{K}^{\mathrm{TF}}_*(X) \otimes \mathbb{F}$ is a direct summand. Thus, the extension $\widetilde{(\tau \circ \nu_*)} \otimes \mathbb{F}$ is an injection, and $\widetilde{(\tau \circ \nu_*)}\widetilde{K}^{\mathrm{TF}}_*(A)^{\otimes k} \subset \widetilde{K}^{\mathrm{TF}}_*(X)^{\otimes k}$ for each k. This implies that $\rho_k \circ \widetilde{(\tau \circ \nu_*)} \circ \iota_k$ is an injection for each k, as required. □

The following corollary, which lifts the injectivity back to $J_s(X)$, is immediate from Theorem 12 and Lemma 6.

Corollary 7 *Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{Z}/p$ for p prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. Suppose that $N, s, k \in \mathbb{Z}^+$ satisfy $k \leq s$, so that the map $\widetilde{\Phi^K_\nu}$ is as in Corollary 6. If*

$$\overline{\nu}_* \otimes \mathbb{F} : \widetilde{K}^{\mathrm{TF}}_*(\Sigma A) \otimes \mathbb{F} \to \widetilde{K}^{\mathrm{TF}}_*(\Sigma X) \otimes \mathbb{F}$$

is an injection, then

$$\widetilde{\Phi^K_\nu} \otimes \mathbb{F} : \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), L^k(\widetilde{K}^{\mathrm{TF}}_*(A))) \otimes \mathbb{F} \to \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(J_s X)) \otimes \mathbb{F}$$

is also an injection. □

We have now established all that we will need to know about this 'first square'.

7.2 Maps derived from the universal coefficient isomorphism

In this subsection we will build the second square of our diagram. This square is really just the Universal Coefficient Theorem (Corollary 4) in a different form. We will write deg for both K-homological and K-theoretic degree.

Lemma 16 *Let Y be a space having the homotopy type of a finite CW-complex. There exists an isomorphism $\mathscr{U}$ making the following diagram commute.*

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, equal] \arrow[d, "\deg"] & \pi_N(Y) \arrow[d, "\deg"] \\
\operatorname{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(Y)) \arrow[r, "\mathscr{U}"] & \operatorname{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)).
\end{tikzcd}$$

Proof For $\beta : \widetilde{K}_*^{\mathrm{TF}}(S^N) \to \widetilde{K}_*^{\mathrm{TF}}(Y)$, let $\mathscr{U}(\beta)$ be the unique map making the following diagram commute

$$\begin{tikzcd}
\widetilde{K}^*_{\mathrm{TF}}(Y) \arrow[r, "\cong"] \arrow[d, "\mathscr{U}(\beta)"] & \operatorname{Hom}(\widetilde{K}_*^{\mathrm{TF}}(Y), \mathbb{Z}) \arrow[d, "\operatorname{Hom}(\beta, \mathbb{Z})"] \\
\widetilde{K}^*_{\mathrm{TF}}(S^N) \arrow[r, "\cong"] & \operatorname{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \mathbb{Z})
\end{tikzcd}$$

where the isomorphisms are those of Corollary 4. Since $\widetilde{K}_*^{\mathrm{TF}}(Y)$ is a finitely generated free $\mathbb{Z}$-module, $\beta \mapsto \operatorname{Hom}(\beta, \mathbb{Z})$ is an isomorphism, so $\mathscr{U}$ is also an isomorphism. Commutativity of the diagram from the statement of this lemma is by naturality of Lemma 4. □

Corollary 8 *Let Y be a space having the homotopy type of a finite CW-complex. For a $\mathbb{Z}$-module M, let $\tau_p : M \to M \otimes \mathbb{Z}/p$ be the natural map. There exists an injection $\mathscr{U}'$ making the following diagram commute.*

$$\begin{tikzcd}
\pi_N(Y) \arrow[rrr, equal] \arrow[d, "\deg"] \arrow[ddr] & & & \pi_N(Y) \arrow[d, "\deg"] \arrow[ddr] & \\
\operatorname{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(Y)) \arrow[dd, "\tau_p"] & & & \operatorname{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[dd, "\tau_p"] & \\
& \operatorname{Im}(\tau_p \circ \deg) \arrow[dl, hook] \arrow[rrr, "\mathscr{U}'"] & & & \operatorname{Im}(\tau_p \circ \deg) \arrow[dl, hook] \\
\operatorname{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(Y)) \otimes \mathbb{Z}/p & & & \operatorname{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. &
\end{tikzcd}$$

Proof By Lemma 16, we have a commutative diagram

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, equal] \arrow[d, "\deg"] & \pi_N(Y) \arrow[d, "\deg"] \\
\operatorname{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(Y)) \arrow[r, "\mathscr{U}"] & \operatorname{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)).
\end{tikzcd}$$

with $\mathscr{U}$ an isomorphism, so $\mathscr{U} \otimes \mathbb{Z}/p$ is also an isomorphism. By Lemma 9, the map

$$\operatorname{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p \to \operatorname{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p$$

is an injection. Maps of spaces induce maps of ψ-modules on K-theory, so the image of $\mathscr{U} \circ \deg$ is contained in $\operatorname{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N))$, and hence there exists a map $\mathscr{U}'$ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Im}(\tau_p \circ \deg) \arrow[r, dotted, "\mathscr{U}'"] \arrow[d, hook] & \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p \arrow[d, hook] \\
\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), \widetilde{K}_*^{\mathrm{TF}}(Y)) \otimes \mathbb{Z}/p \arrow[r, "\mathscr{U} \otimes \mathbb{Z}/p \cong"] & \mathrm{Hom}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p.
\end{tikzcd}$$

Both vertical maps are injections, so $\mathscr{U}'$ has the required properties. □

7.3 Pulling back along classes defined on S^3

Let $f \in \pi_j(S^3)$, and let $N \geq 3$. Then, for $\omega \in \pi_N(Y)$, the composite

$$S^{N+j-3} \xrightarrow{\Sigma^{N-3} f} S^N \xrightarrow{\omega} Y$$

is defined. The class $\omega \circ \Sigma^{N-3} f$ lies in $\pi_{M-1}(Y)$, where $M - 1 = N + j - 3$.

Thus motivated, we define the map $f^*_\Sigma : \pi_*(Y) \to \pi_*(Y)$ on $\omega \in \pi_N(Y)$ by setting $f^*_\Sigma(\omega) = (\Sigma^{N-3} f)^*\omega = \omega \circ \Sigma^{N-3} f$. In words, f^*_Σ pulls classes back along the appropriate suspension of f. Strictly speaking, f^*_Σ is only a partial map, because it is undefined on π_N for $N \leq 2$, but this will be unimportant.

Recall the definition of the double e-invariant $\overline{e}$ (Definition 6). On $\pi_N(Y)$, we have by definition that $f^*_\Sigma = (\Sigma^{N-3} f)^*$. By Lemma 11 we have a commuting square:

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, "f^*_\Sigma"] \arrow[d, "\deg"'] & \pi_{N+j-3}(Y) \arrow[d, "\overline{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta(\Sigma^{N-3} f)"] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^{N+j-2})).
\end{tikzcd}$$

Mimicking the convention for f^*_Σ, let

$$\theta_\Sigma(f) : \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^*)) \to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^{*+j-2}))$$

be the map which is defined to be equal to $\theta(\Sigma^{N-3} f)$ on the degree N component $\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N))$ of $\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^*))$.

Lemma 17 *Let p be a prime, and let $f \in \pi_j(S^3)$ with $e(f)$ defined. If $pf = 0$, then there exists a map $\theta^p_\Sigma(f)$ making the following diagram commute for all N:*

$$\begin{tikzcd}
\pi_N(Y) \arrow[r, "f^*_\Sigma"] \arrow[d, "\deg"'] & \pi_{N+j-3}(Y) \arrow[d, "\overline{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta_\Sigma(f)"] \arrow[d, "\tau_p"'] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^{N+j-2})) \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(Y), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. \arrow[ur, dotted, "\theta^p_\Sigma(f)"] &
\end{tikzcd}$$

Proof Since $pf = 0$, we have that $p\overline{e}(\Sigma^{N-3} f) = 0$ for all N, which implies that $\theta_\Sigma(f)$ vanishes on p-divisible elements, so there exists a unique map $\theta^p_\Sigma(f)$ making the diagram commute, as required. □

Lemma 18 *Let X be a finite CW-complex. Let λ_ℓ^X be the largest eigenvalue of the rational Adams operation*

$$\psi^\ell \otimes \mathbb{Q} : \widetilde{K}^0(X) \otimes \mathbb{Q} \to \widetilde{K}^0(X) \otimes \mathbb{Q}.$$

Then, for $i \geq 0$

- *the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma^{2i} J_s(X)) \otimes \mathbb{Q}$ is $\ell^i (\lambda_\ell^X)^s$, and*
- *the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma^{2i+1} J_s(X)) \otimes \mathbb{Q}$ is $\ell^i \lambda_\ell^{\Sigma X} (\lambda_\ell^X)^{s-1}$.*

Proof When $i \geq 1$, Theorem 7 gives that $\Sigma J_s(X) \simeq \Sigma \bigvee_{t=1}^s X^{\wedge t}$, so $\Sigma^{2i} J_s(X) \simeq S^{2i} \wedge \bigvee_{t=1}^s X^{\wedge t}$, and $\Sigma^{2i+1} J_s(X) \simeq S^{2i} \wedge \Sigma X \wedge \bigvee_{t=1}^{s-1} X^{\wedge t}$. By the Künneth theorem (Theorem 6), this implies isomorphisms of rings

$$\widetilde{K}^0_{\mathrm{TF}}(\Sigma^{2i} J_s(X)) \cong \bigoplus_{t=1}^{s} \widetilde{K}^0_{\mathrm{TF}}(S^{2i}) \otimes \widetilde{K}^0_{\mathrm{TF}}(X)^{\otimes t}, \text{ for } i \geq 1,$$

and

$$\widetilde{K}^0_{\mathrm{TF}}(\Sigma^{2i+1} J_s(X)) \cong \bigoplus_{t=0}^{s-1} \widetilde{K}^0_{\mathrm{TF}}(S^{2i}) \otimes \widetilde{K}^0_{\mathrm{TF}}(\Sigma X) \otimes \widetilde{K}^0_{\mathrm{TF}}(X)^{\otimes t} \text{ for } i \geq 0.$$

The Künneth isomorphism of Theorem 6 is given by the external product of K-theory. Since the Adams operations are ring homomorphisms, the above isomorphisms are also isomorphisms of ψ-modules. In particular, the Adams operations on the left are the tensor product of the corresponding operations on the right.

These decompositions hold for $\widetilde{K}^0_{\mathrm{TF}}$, so they also hold for $\mathbb{Q} \otimes \widetilde{K}^0$, and the remaining problem is to determine the largest eigenvalue of the relevant tensor products of Adams operations. The eigenvalues of a tensor product of linear endomorphisms are precisely the products of the eigenvalues. The operation ψ^ℓ acts on S^{2i} by multiplication by ℓ^i. Together, these observations imply the result. □

Lemma 19 *Let p be an odd prime. Let X be an $(r-1)$-connected finite CW-complex. Let $N, s \in \mathbb{Z}^+$. Consider the diagram of Lemma 17 for $Y = J_s(X)$ and $f = f_{p,j} \in \pi_{2j(p-1)+2}(S^3)$, the map of Theorem 9:*

$$\begin{tikzcd}
\pi_N(J_s(X)) \arrow[r, "f^*_\Sigma"] \arrow[d, "\deg"'] & \pi_{N+2j(p-1)-1}(J_s(X)) \arrow[d, "\overline{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta_\Sigma(f)"] \arrow[d, "\tau_p"'] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+2j(p-1)})) \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \otimes \mathbb{Z}/p. \arrow[ur, "\theta^p_\Sigma(f)"'] &
\end{tikzcd}$$

For $\ell \in \mathbb{Z}^+$, let λ_ℓ^Y be the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(Y) \otimes \mathbb{Q}$, and let $\lambda_\ell =$ $\max(\lambda_\ell^X, \lambda_\ell^{\Sigma X})$. *If there exists $\ell \in \mathbb{Z}^+$ such that $\ell^{j(p-1)+\frac{N-1}{2}} > \lambda_\ell^s$ then* $\mathrm{Ker}(\overline{e} \circ f^*_\Sigma) \subset \mathrm{Ker}(\tau_p \circ \deg)$*, and hence the restriction of $\theta^p_\Sigma(f)$ to* $\mathrm{Im}(\tau_p \circ \deg)$ *is an injection.*

Proof First note that $pf = 0$ by Theorem 9, so $\theta^p_\Sigma(f)$ is well-defined by Lemma 17. Let $\omega \in \pi_N(J_s(X))$. Suppose that $\omega \in \mathrm{Ker}(\overline{e} \circ f^*_\Sigma)$, that is, that the $\overline{e}$-invariant of the composite

$$S^{N+2j(p-1)-1} \xrightarrow{\Sigma^{N-3} f} S^N \xrightarrow{\omega} J_s(X)$$

is trivial. By Lemma 10, this implies that $\Sigma^i(\omega \circ \Sigma^{N-3} f)$ has trivial e-invariant for all i. In particular, $e(\Sigma\omega \circ \Sigma^{N-2} f)$ and $e(\omega \circ \Sigma^{N-3} f)$ are both 0.

By Lemma 18, the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(J_s(X)) \otimes \mathbb{Q}$ is at most λ^s_ℓ, and the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma J_s(X)) \otimes \mathbb{Q}$ is also at most λ^s_ℓ. We now divide into cases, based on the parity of N.

CASE 1 (N even): Write $N = 2n$. Let $f' = \Sigma^{N-3} f$ and $g = \omega$ in Theorem 8. The domain of $\omega \circ \Sigma^{N-3} f$ is S^{M-1}, where $M - 1 = N + 2j(p-1) - 1$, so M is even, as is required. To check the eigenvalue hypothesis of Theorem 8, write $M = 2m$. By Lemma 18, the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(J_s(X)) \otimes \mathbb{Q}$ is at most λ^s_ℓ, and $\ell^m = \ell^{j(p-1)+n} > \ell^{j(p-1)+\frac{N-1}{2}}$, which we assumed was greater than λ^s_ℓ. This means that ℓ^m cannot be an eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(J_s(X)) \otimes \mathbb{Q}$. Now, $e(f) \neq 0$ by construction (Theorem 9), so $e(\Sigma^{N-3} f) \neq 0$ by stability (Lemma 12). Since $e(\omega \circ \Sigma^{N-3} f) = 0$, the contrapositive of Theorem 8 gives that ω^* has p-divisible image in $\widetilde{K}^0(S^N)$. Since N is even, this implies that $\tau_p \circ \deg(\omega) = 0$, as required.

CASE 2 (N odd): Write $n = 2n + 1$. Let $f' = \Sigma^{N-2} f$ and $g = \Sigma\omega$ in Theorem 8, and proceed similarly to case 1. The domain of $\Sigma\omega \circ \Sigma^{N-2} f$ is S^{M-1}, where $M - 1 = N + 2j(p-1)$, so M is even, as is required. To check the eigenvalue hypothesis of Theorem 8, write $M = 2m$. By Lemma 18, the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma J_s(X)) \otimes \mathbb{Q}$ is at most λ^s_ℓ, and $\ell^m = \ell^{j(p-1)+n} = \ell^{j(p-1)+\frac{N-1}{2}}$, which we assumed was greater than λ^s_ℓ. This means that ℓ^m cannot be an eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(\Sigma J_s(X)) \otimes \mathbb{Q}$. As in the previous case, $e(\Sigma^{N-2} f) \neq 0$. Since $e(\Sigma\omega \circ \Sigma^{N-2} f) = 0$, the contrapositive of Theorem 8 gives that $(\Sigma\omega)^*$ has p-divisible image in $\widetilde{K}^0(S^{N+1})$. Since N is odd, this implies that $\tau_p \circ \deg(\omega) = 0$, as required. This completes the case, and hence the proof. □

7.4 Proof of Theorem 2

Construction 13 *Let p be an odd prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. Let $f \in \pi_i(S^3)$ with $\overline{e}(f)$ defined. Suppose that $N, k, s \in \mathbb{Z}^+$ satisfy $N < r(s+1) - 1$ and $k \leq s$. The diagrams of the preceding subsections may be combined as follows.*

Recall the definition of $\deg'$ *from the preamble to Lemma 13. Let $I(A)$ be the submodule of* $\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^*), L(\widetilde{K}^{\mathrm{TF}}_*(A))) \otimes \mathbb{Z}/p$ *generated by* $\mathrm{Im}(\tau_p \circ \deg')$. *The same grading conventions as usual apply: we write $I^k(A)$ for the weight k part, we write $I_N(A)$ for the degree N part, and let $I^k_N(A) = I^k(A) \cap I_N(A)$.*

From Corollary 6, using the assumptions that $N < r(s+1) - 1$ and $k \leq s$ (which make $\widetilde{\Phi^\pi_\nu}$ and $\widetilde{\Phi^K_\nu}$ well-defined) we obtain the following diagram, where the images of the vertical maps have been 'popped out' to their right.

$$
\begin{tikzcd}
\mathscr{B}^k_N(\pi_*(A)) \arrow[rr, "\widetilde{\Phi^\pi_\nu}"] \arrow[dr] \arrow[dd, "\tau_p\circ\deg'"'] & & \pi_N(J_s(X)) \arrow[dr] \arrow[dd, "\tau_p\circ\deg" near start] & \\
& I^k_N(A) \arrow[rr] \arrow[dl, hook] & & \mathrm{Im}(\tau_p\circ\deg) \arrow[dl, hook] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), L^k(\widetilde{K}^{\mathrm{TF}}_*(A)))\otimes\mathbb{Z}/p \arrow[rr, "\widetilde{\Phi^K_\nu\otimes\mathbb{Z}/p}"] & & \mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(J_s(X)))\otimes\mathbb{Z}/p. &
\end{tikzcd}
$$

Next, from Corollary 8 (with $Y = J_s(X)$) we have a diagram

$$
\begin{tikzcd}
\pi_N(J_s(X)) \arrow[rr, equal] \arrow[dr] \arrow[dd, "\tau_p\circ\deg"'] & & \pi_N(J_s(X)) \arrow[dr] \arrow[dd, "\tau_p\circ\deg"' near start] & \\
& \mathrm{Im}(\tau_p\circ\deg) \arrow[rr, "\mathscr{U}'"] \arrow[dl, hook] & & \mathrm{Im}(\tau_p\circ\deg) \arrow[dl, hook] \\
\mathrm{Hom}(\widetilde{K}^{\mathrm{TF}}_*(S^N), \widetilde{K}^{\mathrm{TF}}_*(J_s(X)))\otimes\mathbb{Z}/p & & \mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N))\otimes\mathbb{Z}/p. &
\end{tikzcd}
$$

Lastly, we obtain the following diagram from Lemma 17:

$$
\begin{tikzcd}
\pi_N(J_s(X)) \arrow[r, "f^*_\Sigma"] \arrow[d, "\deg"'] & \pi_{N+i-3}(J_s(X)) \arrow[d, "\bar{e}"] \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N)) \arrow[r, "\theta_\Sigma(f)"] \arrow[d, "\tau_p"'] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+i-2})) \\
\mathrm{Hom}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^N))\otimes\mathbb{Z}/p. \arrow[ur, "\theta^p_\Sigma(f)"'] &
\end{tikzcd}
$$

Concatenating these diagrams gives a diagram as follows:

$$
\begin{tikzcd}
\mathscr{B}^k_N(\pi_*(A)) \arrow[r, "f^*_\Sigma\circ\widetilde{\Phi^\pi_\nu}"] \arrow[d, "\tau_p\circ\deg'"'] & \pi_{N+i-3}(J_s(X)) \arrow[d, "\bar{e}"] \\
I^k_N(A) \arrow[r, "\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu\otimes\mathbb{Z}/p})"] & \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)), \widetilde{K}^*_{\mathrm{TF}}(S^{N+i-2})).
\end{tikzcd}
$$

In this subsection, we combine the results of the previous subsections to produce results about this diagram.

Theorem 14 *Let p be an odd prime. Let $\nu : A \to \Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X $(r-1)$-connected for $r \geq 1$. Let $N, k, s \in \mathbb{Z}^+$ with $N < r(s+1) - 1$ and $k \leq s$. Let $f = f_{p,j} \in \pi_{2j(p-1)+2}(S^3)$, the map of Theorem 9.*

For $\ell \in \mathbb{Z}^+$, let λ^Y_ℓ be the largest eigenvalue of $\psi^\ell \otimes \mathbb{Q}$ on $\widetilde{K}^0(Y)\otimes\mathbb{Q}$, and let $\lambda_\ell = \max(\lambda^X_\ell, \lambda^{\Sigma X}_\ell)$. If

- *$\overline{\nu}_* \otimes \mathbb{Z}/p : \widetilde{K}^{\mathrm{TF}}_*(\Sigma A)\otimes\mathbb{Z}/p \to \widetilde{K}^{\mathrm{TF}}_*(\Sigma X)\otimes\mathbb{Z}/p$ is an injection, and*
- *there exists $\ell \in \mathbb{Z}^+$ such that $\ell^{j(p-1)+\frac{N-1}{2}} > \lambda^s_\ell$,*

then $\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p): I^k_N(A)\to \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)),\widetilde{K}^*_{\mathrm{TF}}(S^{N+2j(p-1)}))$ *is an injection.*

Proof By Corollary 7, since $\overline{\nu}_*\otimes\mathbb{Z}/p$ is an injection, $\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p$ is also an injection. By Corollary 8 $\mathscr{U}'$ is an injection. By Lemma 19 the hypothesis on ℓ implies that the restriction of $\theta^p_\Sigma(f)$ to $\mathrm{Im}(\tau_p\circ\deg)$ is an injection. The map $\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p)$ is thus a composite of injections, hence an injection, as required. □

In the proof of Theorem 2, we will wish to restrict attention to those elements of $\mathscr{B}(\pi_*(A))$ who are brackets of classes in $\pi_*(A)$ in some dimensional range $q_{\min}\le n\le q_{\max}$. All such classes lie in dimensions $kq_{\min}\le N\le kq_{\max}$. Said more precisely, we have an inclusion $\mathscr{B}^k(\bigoplus_{n=q_{\min}}^{q_{\max}}\pi_n(A))\subset\bigcup_{N=kq_{\min}}^{kq_{\max}}\mathscr{B}^k_N(\pi_*(A))$. We will now study the diagram of Construction 13 in this dimensional range.

Construction 15 *Let p be an odd prime, $\nu: A\to\Omega\Sigma X$ for finite CW-complexes A and X with X $(r-1)$-connected for $r\ge 1$, and $f\in\pi_i(S^3)$ with $\overline{e}(f)$ defined. Let $q_{\max}>q_{\min}$ be natural numbers. Fix $k\in\mathbb{Z}^+$, and let $s=kq_{\max}+1$. For $N\in\mathbb{Z}^+$ with $kq_{\min}\le N\le kq_{\max}$, we have that $N<r(s+1)-1$ and $k\le s$. Combining the diagrams obtained from Construction 13 for this range of values of N gives the following diagram:*

$$\begin{tikzcd}
\bigcup_{N=kq_{\min}}^{kq_{\max}}\mathscr{B}^k_N(\pi_*(A)) \arrow[r, "f^*_\Sigma\circ\widetilde{\Phi^\pi_\nu}"] \arrow[d, "\tau_p\circ\deg'"'] & \bigoplus_{N=kq_{\min}}^{kq_{\max}}\pi_{N+i-3}(J_s(X)) \arrow[d, "\overline{e}"] \\
\bigoplus_{N=kq_{\min}}^{kq_{\max}} I^k_N(A) \arrow[r, "\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p)"] & \bigoplus_{N=kq_{\min}}^{kq_{\max}}\mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)),\widetilde{K}^*_{\mathrm{TF}}(S^{N+i-2})).
\end{tikzcd}$$

We now show that by choosing a large enough $c\in\mathbb{Z}^+$, and setting $f=f_{p,ck}$, the eigenvalue hypothesis of Theorem 14 may be satisfied across the dimensional range of Construction 15 for all sufficiently large k.

Corollary 9 *Let p be an odd prime. Let $\nu: A\to\Omega\Sigma X$, for spaces A and X having the homotopy type of finite CW-complexes, with X path-connected. Let $q_{\max}>q_{\min}$ be natural numbers. Let $c,k\in\mathbb{Z}^+$. Let $f=f_{p,ck}\in\pi_{2ck(p-1)+2}(S^3)$ be the map of Theorem 9. If*

$$\overline{\nu}_*\otimes\mathbb{Z}/p:\widetilde{K}^{\mathrm{TF}}_*(\Sigma A)\otimes\mathbb{Z}/p\to\widetilde{K}^{\mathrm{TF}}_*(\Sigma X)\otimes\mathbb{Z}/p$$

is an injection then there exists $c\in\mathbb{Z}^+$ such that for large enough $k\in\mathbb{Z}^+$,

$$\theta^p_\Sigma(f)\circ\mathscr{U}'\circ(\widetilde{\Phi^K_\nu}\otimes\mathbb{Z}/p):\bigoplus_{N=kq_{\min}}^{kq_{\max}} I^k_N(A)\to\bigoplus_{N=kq_{\min}}^{kq_{\max}}\mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}^*_{\mathrm{TF}}(J_s(X)),\widetilde{K}^*_{\mathrm{TF}}(S^{N+2ck(p-1)}))$$

is an injection.

Proof By Theorem 14, it suffices to show that for each N with $kq_{\min}\le N\le kq_{\max}$ there exists $\ell\in\mathbb{Z}^+$ such that $\ell^{ck(p-1)+\frac{N-1}{2}}>\lambda^s_\ell=\lambda^{kq_{\max}+1}_\ell$. Take any $\ell\ge 2$. Since $N\ge kq_{\min}$, it suffices to find c such that for large enough k we have $\ell^{ck(p-1)+\frac{kq_{\min}-1}{2}}>\lambda^{kq_{\max}}_\ell$. Taking logs on both sides, this is equivalent to

$$\left(ck(p-1)+\frac{kq_{\min}-1}{2}\right)\log(\ell)>kq_{\max}\log(\lambda_\ell).$$

It is now clear that we may choose c large enough that this equation holds for large enough k, in particular, any $c\ge\frac{1}{p-1}(q_{\max}\frac{\log(\lambda_\ell)}{\log(\ell)}-\frac{q_{\min}}{2})$ will do. □

7.5 Proof of Theorem 2

In this subsection we prove the main theorem. We first prove three lemmas. The first converts the K-theoretic hypothesis of Theorem 2 into the K-homological input our construction requires.

Lemma 20 *Let X be a space, and let p be prime. Let $\mu : S^{q_1+1} \vee S^{q_2+1} \to \Sigma X$ be a map with $q_i \geq 1$, such that the map*

$$\widetilde{K}^*_{\mathrm{TF}}(\Sigma X) \otimes \mathbb{Z}/p \xrightarrow{\mu^* \otimes \mathbb{Z}/p} \widetilde{K}^*_{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$$

is a surjection. Then $\mu_* \otimes \mathbb{Z}/p : \widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma X) \otimes \mathbb{Z}/p$ is an injection.

Proof Naturality of the Universal Coefficient Theorem (Corollary 4) relative to the map μ gives a diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \widetilde{K}^*_{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \arrow[r] & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}), \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u] & \widetilde{K}^*_{\mathrm{TF}}(\Sigma X) \arrow[r] \arrow[u, "\mu^*"] & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(\Sigma X), \mathbb{Z}) \arrow[r] \arrow[u, "\mathrm{Hom}(\mu_*,\mathbb{Z})"'] & 0,
\end{tikzcd}$$

where E denotes the quotient of $\mathrm{Ext}(\widetilde{K}_{*-1}(\Sigma X), \mathbb{Z})$ by its torsion subgroup. We will argue by contrapositive. Suppose that $\mu_* \otimes \mathbb{Z}/p$ is not injective. The $\mathbb{Z}/p$-vector space $\widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p$ has dimension 2, so there exists a basis x, y where $(\mu_* \otimes \mathbb{Z}/p)(y) = 0$. Prior to tensoring with $\mathbb{Z}/p$, this means that there exists a non-p-divisible element $\widetilde{y}$ of $\widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1})$ such that p divides $\mu_*(\widetilde{y})$. This implies that any element φ of $\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}), \mathbb{Z})$ with $\varphi(\widetilde{y})$ not p-divisible is not contained in the image of $\mathrm{Hom}(\mu_*, \mathbb{Z})$, hence, by the diagram, that $\mu^* \otimes \mathbb{Z}/p$ is not surjective, as required. $\square$

Some preamble to the second lemma is necessary. Let $h : \pi_*(A) \to \widetilde{K}_*^{\mathrm{TF}}(A)$ be the K-homological Hurewicz map, which sends $f \in \pi_N(A)$ to $f_*(\xi_N) \in \widetilde{K}_*^{\mathrm{TF}}(A)$. As with $\deg'$, let $h' : \mathscr{B}(\pi_*(A)) \to L(\widetilde{K}_*^{\mathrm{TF}}(A))$ be the unique map which restricts to $h : \pi_*(A) \to \widetilde{K}_*^{\mathrm{TF}}(A) \subset L(\widetilde{K}_*^{\mathrm{TF}}(A))$ and respects brackets.

Let M be a $\mathbb{Z}/2$-graded $\mathbb{Z}$-module. Let $\chi : \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), M) \to M$ be the map which carries $\varphi \in \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^N), M)$ to $\varphi(\xi_N) \in M$ (Remark 2). If $M = L$ is a $\mathbb{Z}/2$-graded Lie algebra, then it follows immediately from the definition of the bracket in $\mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), L)$ that χ is a map of Lie algebras.

Lemma 21 *For any space A, there is a commuting diagram*

$$\begin{tikzcd}
\mathscr{B}(\pi_*(A)) \arrow[r, "\deg'"] \arrow[d, "h'"'] & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), L(\widetilde{K}_*^{\mathrm{TF}}(A))) \arrow[dl, "\chi"] \\
L(\widetilde{K}_*^{\mathrm{TF}}(A)). &
\end{tikzcd}$$

Proof Commutativity of the diagram

$$\begin{tikzcd}
\pi_*(A) \arrow[r, "\deg"] \arrow[d, "h"'] & \mathrm{Hom}(\widetilde{K}_*^{\mathrm{TF}}(S^*), \widetilde{K}_*^{\mathrm{TF}}(A)) \arrow[dl, "\chi"] \\
\widetilde{K}_*^{\mathrm{TF}}(A). &
\end{tikzcd}$$

follows from the definitions. Commutativity of the diagram from the lemma statement then follows from the definition of $\mathscr{B}(\pi_*(A))$, since χ respects brackets. □

The third and fourth lemmas allow us to make the statement of Theorem 2 an entirely p-local one.

Lemma 22 *Let p be a prime, let X be a simply connected CW-complex, and let $f : S^{q_1+1} \vee S^{q_2+1} \longrightarrow X_{(p)}$ be any map. There exists a map $\widetilde{f} : S^{q_1+1} \vee S^{q_2+1} \longrightarrow X$, and a map $\varphi : S^{q_1+1} \vee S^{q_2+1} \longrightarrow S^{q_1+1} \vee S^{q_2+1}$ which is a homotopy equivalence after p-localization, making the diagram*

$$\begin{tikzcd}
S^{q_1+1} \vee S^{q_2+1} \arrow[r, "\widetilde{f}"] \arrow[d, "\varphi"'] & X \arrow[d] \\
S^{q_1+1} \vee S^{q_2+1} \arrow[r, "f"] & X_{(p)}
\end{tikzcd}$$

commute, where the vertical arrow is the localization.

Proof Write $f = f_1 \vee f_2$, for maps $f_i : S^{q_i+1} \longrightarrow X$. On homotopy groups, the localizing map may be identified with the tensor map

$$\pi_*(X) \longrightarrow \pi_*(X) \otimes \mathbb{Z}_{(p)} \cong \pi_*(X_{(p)}).$$

This implies that there exist integers u_1 and u_2, not divisible by p, such that $u_i f_i$ lifts to a map $\widetilde{f_i} : S^{q_i+1} \longrightarrow X$. Setting $\widetilde{f} = \widetilde{f_1} \vee \widetilde{f_2}$, and letting φ be the degree u_i map on each wedge summand, we obtain the desired diagram. □

Lemma 23 *Let p be prime, and let X and Y be connected CW-complexes. If $\mu : S^{q_1+1} \vee S^{q_2+1} \longrightarrow \Sigma X$ induces an injection on $\widetilde{K}_*() \otimes \mathbb{Z}/p$, and there is a homotopy equivalence of p-localizations $\Sigma Y_{(p)} \simeq \Sigma X_{(p)}$, then there exists a map $\mu' : S^{q_1+1} \vee S^{q_2+1} \longrightarrow \Sigma Y$ which also induces an injection on $\widetilde{K}_*() \otimes \mathbb{Z}/p$.*

Proof For any space A, the localizing map $A \longrightarrow A_{(p)}$ induces an isomorphism on $K_*() \otimes \mathbb{Z}/p$ [25]. It therefore suffices to show that there exists maps μ' and φ making the following diagram commute, with the localization $\varphi_{(p)}$ being a homotopy equivalence.

$$\begin{tikzcd}
S^{q_1+1} \vee S^{q_2+1} \arrow[r, dotted, "\mu'"] \arrow[d, dotted, "\varphi"'] & \Sigma Y \arrow[r] & \Sigma Y_{(p)} \\
S^{q_1+1} \vee S^{q_2+1} \arrow[r, "\mu"] & \Sigma X \arrow[r] & \Sigma X_{(p)}. \arrow[u, "\simeq"']
\end{tikzcd}$$

Such maps exist by Lemma 22. □

We are now ready to prove Theorem 2.

Proof of Theorem 2 Let $\mu = \mu_1 \vee \mu_2$, with adjoint $\overline{\mu} : S^{q_1} \vee S^{q_2} \to \Omega\Sigma X$. Let $f = f_{p,ck} \in \pi_{2ck(p-1)+2}(S^3)$. Consider the diagram of Construction 15, with $A = S^{q_1} \vee S^{q_2}$, $q_{\max} = \max(q_1, q_2)$, $q_{\min} = \min(q_1, q_2)$, and $\nu = \overline{\mu}$. We have such a diagram for each $k \in \mathbb{Z}^+$:

$$\begin{tikzcd}
\bigcup_{N=kq_{\min}}^{kq_{\max}} \mathscr{B}_N^k(\pi_*(S^{q_1} \vee S^{q_2})) \arrow[r, "f_\Sigma^* \circ \Phi_{\overline{\mu}}^{\widetilde{\pi}}"] \arrow[d, "\tau_p \circ \deg'"'] & \bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(J_s(X)) \arrow[d, "\overline{e}"] \\
\bigoplus_{N=kq_{\min}}^{kq_{\max}} I_N^k(S^{q_1} \vee S^{q_2}) \arrow[r, "\theta_\Sigma^p(f) \circ \mathscr{U}' \circ (\widetilde{\Phi_{\mu}^K} \otimes \mathbb{Z}/p)"] & \bigoplus_{N=kq_{\min}}^{kq_{\max}} \mathrm{Ext}_{\psi-\mathrm{Mod}}(\widetilde{K}_{\mathrm{TF}}^*(J_s(X)), \widetilde{K}_{\mathrm{TF}}^*(S^{N+2ck(p-1)})).
\end{tikzcd}$$

By assumption, $\mu^* \otimes \mathbb{Z}/p : \widetilde{K}^*(\Sigma X) \otimes \mathbb{Z}/p \to \widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p$ is a surjection. Since $\widetilde{K}^*(S^{q_1+1} \vee S^{q_2+1})$ is torsion-free, Lemma 20 then implies that

$$\mu_* \otimes \mathbb{Z}/p : \widetilde{K}_*^{\mathrm{TF}}(S^{q_1+1} \vee S^{q_2+1}) \otimes \mathbb{Z}/p \to \widetilde{K}_*^{\mathrm{TF}}(\Sigma X) \otimes \mathbb{Z}/p$$

is an injection. By Lemma 23, we may assume without loss of generality that X has the integral homotopy type of a finite CW-complex. Thus, by Corollary 9, we may fix c such that for large enough k, $\theta_p(f) \circ \mathscr{U}' \circ (\widetilde{\Phi_{\mu}^K} \otimes \mathbb{Z}/p)$ is an injection.

The Hurewicz map h is a surjection $\pi_*(S^{q_1} \vee S^{q_2}) \to \widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2})$, so the submodule generated by the image of the map $h' : \mathscr{B}(\pi_*(S^{q_1} \vee S^{q_2})) \to L(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2}))$ of Lemma 21 contains the submodule generated by $\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2})$ under the bracket operation. In particular, it contains the weight k component $L^k(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2}))$ for each k. By Theorem 3, $\dim_{\mathbb{Z}}(L^k(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2}))) = W_2(k)$. Note that $L^k(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2})) = \bigoplus_{N=kq_{\min}}^{kq_{\max}} L^k(\widetilde{K}_*^{\mathrm{TF}}(S^{q_1} \vee S^{q_2}))$.

It then follows from Lemma 21 that $\dim_{\mathbb{Z}/p}(\bigoplus_{N=kq_{\min}}^{kq_{\max}} I_N^k(S^{q_1} \vee S^{q_2})) \geq W_2(k)$. Since $\theta_\Sigma^p(f) \circ \mathscr{U}' \circ (\widetilde{\Phi_{\mu}^K} \otimes \mathbb{Z}/p)$ is an injection for large enough k, it follows that the dimension of $\overline{e}(\bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(J_s(X)))$ is at least $W_2(k)$. By Corollary 6 $(i_s)_*$ is an injection, so the dimension of $(i_s)_*(\bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(J_s(X))) \subset \bigoplus_{N=kq_{\min}}^{kq_{\max}} \pi_{N+2ck(p-1)-1}(\Omega\Sigma X)$ is also at least $W_2(k)$. Thus, ΣX satisfies the hypotheses of Lemma 2 with $a = 2c(p-1) + q_{\max} = 2c(p-1) + \max(q_1, q_2)$ and $b = 0$, and hence is p-hyperbolic. □

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