IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received February 17, 2021, accepted March 1, 2021, date of publication March 12, 2021, date of current version March 29, 2021.

Digital Object Identifier 10.1109/ACCESS.2021.3065747

Constructing the Singular Roesser State-Space Model Description of 3D Spatio-Temporal Dynamics From the Polynomial System Matrix

MOHAMED S. BOUDELLIOUA [1], BARTŁOMIEJ SULIKOWSKI [2], KRZYSZTOF GAŁKOWSKI [3], (Member, IEEE), AND ERIC ROGERS [4], (Member, IEEE)

[1]Department of Mathematics, Sultan Qaboos University, Muscat, PC 123, Oman
[2]Institute of Control and Computation Engineering, University of Zielona Góra, 65-417 Zielona Góra, Poland
[3]Institute of Automation, Electronic and Electrical Engineering, University of Zielona Góra, 65-417 Zielona Góra, Poland
[4]School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, U.K.

Corresponding author: Mohamed S. Boudellioua (boudell@squ.edu.om)

This work was supported in part by the Sultan Qaboos University, Oman, under Grant IG-SCI-DOMS-18-11 and in part by the University of Zielona Góra Deputy Rector for Science and International Cooperation ''Small grant''.

ABSTRACT This paper considers systems theoretic properties of linear systems defined in terms of spatial and temporal indeterminates. These include physical applications where one of the indeterminates is of finite duration. In some cases, a singular Roesser state-space model representation of the dynamics has found use in characterizing systems theoretic properties. The representation of the dynamics of many linear systems is obtained in terms of transform variables and a polynomial system matrix representation. This paper develops a direct method for constructing the singular Roesser state-space realization from the system matrix description for 3D systems such that relevant properties are retained. Since this method developed relies on basic linear algebra operations, it may be highly effective from the computational standpoint. In particular, spatially interconnected systems of the form of the ladder circuits are considered as the example. This application confirms the usefulness and effectiveness of the proposed method independently of the system spatial order.

INDEX TERMS nD systems, polynomial system matrix, Roesser model, spatio-temporal dynamics.

I. INTRODUCTION

This paper considers linear systems described in terms of spatial and temporal indeterminates, which belong to the general class of nD, $n > 1$, linear systems. Often the dynamics of the system are described by algebraic equations obtained by applying transforms. The result is a system matrix description of the dynamics where the defining entries are polynomials in more than one indeterminate. Moreover, a transfer-function matrix description can be constructed if required.

In linear systems theory, transformations between model representations are standard in analysis and controller design. This general area is more involved and less well developed for nD linear systems, for which this paper gives new results.

The associate editor coordinating the review of this manuscript and approving it for publication was Bing Li.

These provide new techniques for onward use in systems theoretic analysis, stability analysis, and controller design.

The particular case of $n = 1$, sometimes termed 1D systems, there is one indeterminate (time), and primeness of polynomial matrix pairs is critical to the analysis of systems properties, e.g., controllability and observability, together with their preservation under equivalence transformations. Moreover, the construction of a state-space model from the system matrix or transfer-function matrix is well understood. The nD systems case is more complicated due to the underlying ring structure. Specifically, coprimeness is no longer a single property, and for $n \geq 3$ zero, factor and minor coprimeness are distinct properties (see e.g., [3], [14] and references therein).

A possible starting point for analyzing nD linear systems is system equivalence, defined in terms of the associated system matrix. Of significant importance, and especially for applications-related research, is constructing state-space models from the system matrix. Research on this problem can, e.g., be found in [2], [4], [5] for the Roesser model [19], and for the Fornassini-Marchessini model [10] in, e.g., [6], [7].

In most cases, the starting point is the so-called multivariate polynomial system matrix, which can be directly obtained for a given multidimensional system, e.g., the homogeneous, spatially distributed electrical system used to demonstrate the new results in this paper. However, for some problems, there is a need to construct a state-space model. This paper addresses the problem of transforming a system matrix of a 3D linear system to a singular (where in many cases, non-singular models do not exist), 3D Roesser [19] state-space model for examples where there are spatial and temporal indeterminates. It is established that this transformation can be completed by applying a sequence of elementary row and column operations together with an expansion of the original system matrix, and also the zero structure is preserved. Moreover, generalization to the nD case, $n > 3$, indeterminates is immediate.

The procedure developed relies on basic linear algebra operations and is therefore highly effective from the computational standpoint. Note that the overall computational effort required is determined by the number of states, inputs, and outputs in the system. Consider a system with r, l and m entries in, respectively, the state, input, and output vectors, respectively. This paper's new results will produce a singular 3D Roesser model of the dynamics that has order $8r + 15m + 22l$. Hence, the overall increase in the order is significant. However, the resulting model matrices are sparse, and this fact could be exploited by using known results for sparse matrices manipulations, see e.g. [8].

A physical circuit area is used to illustrate the application of the results. Note also that the use of elementary operations discussed in this paper is equivalent to applying multivariate polynomial matrix operations. Also, similar results and equivalent approaches can be found, e.g., in [1]. Further, these results could be the basis for the design of iterative learning control laws, see, e.g. [16].

System equivalence has been the subject of considerable attention in the literature for multidimensional systems see, e.g., [9], [11]–[13], [15], [17], but this was mostly for 2D systems. In contrast, this paper considers 3D case systems. Moreover, the example given to highlight this paper's new results is a physical example from the class of spatially distributed systems. As these results are the first from this approach to 3D systems, comparison with other approaches will have to await future research results.

In this paper, the notation $O_{e,h}$ and I_e denotes, respectively, the $e \times h$ null matrix and the $e \times e$ identity matrix. In cases where the dimensions are immediate, the subscripts will not be shown. Also, $\equiv$ denotes matrix equivalence.

II. PRELIMINARIES

Consider a general 3D discrete linear system described in the polynomial form as

$$P(z_1, z_2, z_3)\begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}, \tag{1}$$

where $x \in \mathbb{R}^r$ is the state vector, $u \in \mathbb{R}^l$ is the input vector, and $y \in \mathbb{R}^m$ is the output vector, and z_i, $1 \le i \le 3$, are the shift operators, which can unlike in the 1D systems case, be temporal or spatial. An example in the 2D case is linear repetitive process [20], where there is one spatial and one temporal variable and the latter is of the finite duration. The stability theory for such processes can be used to design iterative learning control laws, see, e.g., [16]. Moreover, other forms of iterative learning control dynamics can involve 3D (or nD with $n > 3$, in general) dynamics [21].

The associated system matrix for (1) is

$$P(z_1, z_2, z_3) = \begin{bmatrix} T(z_1, z_2, z_3) & U(z_1, z_2, z_3) \\ -V(z_1, z_2, z_3) & W(z_1, z_2, z_3) \end{bmatrix}, \tag{2}$$

where T, U, V and W, respectively, are polynomial matrices with entries in $\mathbb{R}[z_1, z_2, z_3]$ and of dimensions $r \times r$, $r \times l$, $m \times r$ and $m \times l$. This matrix characterizes the dynamics of the example considered. In many examples, no direct interaction between the system inputs and outputs of the system exists and, in any case, no loss of generality arises from setting $W(z_1, z_2, z_3) = 0$.

The following definitions and results are used in the development of the new results in this paper.

Definition 1: Let $\mathbb{P}(m, l)$ denote the class of $(r + m) \times (r + l)$ polynomial matrices corresponding to a system with l inputs and m outputs, which, by an obvious expansion, can be taken as having the same number of inputs and outputs. Two polynomial system matrices $P_1(z_1, z_2, z_3)$ and $P_2(z_1, z_2, z_3)$ are said to be zero coprime equivalent, if there exist polynomial matrices $S_1(z_1, z_2, z_3)$ and $S_2(z_1, z_2, z_3)$ of compatible dimensions such that

$$S_1(z_1, z_2, z_3)P_2(z_1, z_2, z_3) = P_1(z_1, z_2, z_3)S_2(z_1, z_2, z_3), \tag{3}$$

where P_1, S_1 are zero left coprime and P_2, S_2 are zero right coprime. In the case when P_1 and P_2 are of the same dimensions and S_1 and S_2 are unimodular, P_1 and P_2 are said to be unimodular equivalent.

In [18] it was shown that zero coprime equivalence characterizes fundamental algebraic properties of the system amongst its invariants.

A essential transformation developed for the nD system matrices study is zero coprime system equivalence, see, e.g., [11], [13]. This transformation is based on zero coprime equivalence, which is characterized by the following definition.

Definition 2: Two polynomial system matrices $P_1(z_1, z_2, z_3)$ and $P_2(z_1, z_2, z_3) \in \mathbb{P}(m, l)$ are said to be zero coprime system *equivalent (ZCSE), if they are related as*

$$\underbrace{\begin{bmatrix} M(z_1,z_2,z_3) & 0 \\ X(z_1,z_2,z_3) & I_m \end{bmatrix}}_{S_1(z_1,z_2,z_3)} P_2(z_1,z_2,z_3) = P_1(z_1,z_2,z_3) \underbrace{\begin{bmatrix} N(z_1,z_2,z_3) & Y(z_1,z_2,z_3) \\ 0 & I_n \end{bmatrix}}_{S_2(z_1,z_2,z_3)}, \quad (4)$$

where P_1, S_1 are zero left coprime, P_2, S_2 are zero right coprime and $M(z_1,z_2,z_3)$, $N(z_1,z_2,z_3)$, $X(z_1,z_2,z_3)$ and $Y(z_1,z_2,z_3)$ are polynomial matrices of compatible dimensions.

In the case when the system matrices have the same dimensions and $M(z_1,z_2,z_3)$ and $N(z_1,z_2,z_3)$ are unimodular, the transformation of ZCSE reduces to that of unimodular system equivalence, were two polynomial system matrices are ZCSE if, and only if, a trivial expansion or deflation of one is unimodular system equivalent to a trivial expansion or deflation of the other. The ZCSE property has a crucial role in several aspects of nD systems theory, see, e.g., [11], [13], [18], where the relevant property for the new developments in this paper is the following.

Lemma 1: [11] The transformation of ZCSE preserves the zero structure of the matrices

$$T(z_1,z_2,z_3), \quad P(z_1,z_2,z_3), \quad \begin{bmatrix} T(z_1,z_2,z_3) & U(z_1,z_2,z_3) \end{bmatrix}, \quad \begin{bmatrix} T(z_1,z_2,z_3) \\ -V(z_1,z_2,z_3) \end{bmatrix}.$$

In the 2D systems case, an equivalent singular Roesser state-space model description of the dynamics provided the route to the characterization of a controllability property that led to stabilizing controller's characterization and design. Therefore, it is conjectured that a similar role exists for 3D (or nD with $n > 3$ in general) linear systems, which is considered in the remainder of this paper.

Let $y(i,j,k)$, $x(i,j,k)$ and $u(i,j,k)$ be the output, state and input vectors for a linear system, where the indeterminates i, j and k correspond to the directions of information propagation. Then, with the same output, state and input dimensions as (1), the singular 3D Roesser state-space model has the form

$$E\begin{bmatrix} x_1(i+1,j,k) \\ x_2(i,j+1,k) \\ x_3(i,j,k+1) \end{bmatrix} = A\begin{bmatrix} x_1(i,j,k) \\ x_2(i,j,k) \\ x_3(i,j,k) \end{bmatrix} + Bu(i,j,k),$$
$$y(i,j,k) = C\begin{bmatrix} x_1(i,j,k) \\ x_2(i,j,k) \\ x_3(i,j,k) \end{bmatrix} + Du(i,j,k), \quad (5)$$

where the matrix E is singular. Also define z_1, z_2 and z_3, respectively, as the unit forward shift operator acting on i, j and k. This fact applied to (5) with assumed zero state initial conditions gives

$$P_{SR}(z_1,z_2,z_3)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \hline -u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline -y \end{bmatrix}, \quad (6)$$

with associated system matrix:

$$P_{SR}(z_1,z_2,z_3) = \left[\begin{array}{c|c} z_1E_1 + z_2E_2 + z_3E_3 - A & B \\ \hline -C & D \end{array}\right], \quad (7)$$

where E_1, E_2, E_3 are obtained from E by compatible partitioning. This last matrix is a particular case of (2) and the next section develops an algorithm for constructing the corresponding 3D singular Roesser state-space model for the case when $W(z_1,z_2,z_3) = 0$ and therefore $D = 0$ in (5).

III. CONSTRUCTING THE 3D SINGULAR ROESSER MODEL

Consider again (1) and (2). Then, the first stage in obtaining the singular 3D Roesser state-space model with a preserved zero structure (which is the essential requirement), for a system described by these equations is to construct the $[(r+m+l)+m] \times [(r+m+l)+l]$ normalized form, P_n, of P, i.e.,

$$P_n(z_1,z_2,z_3) = \left[\begin{array}{cc|c} P(z_1,z_2,z_3) & \mathcal{F}_{(r+m),m} & 0 \\ -\mathcal{F}^T_{(r+l),l} & 0_{l\times m} & I_l \\ \hline 0 & -I_m & 0 \end{array}\right], \quad (8)$$

where $\mathcal{F}^T_{\alpha,\beta} = \begin{bmatrix} 0_{\beta\times(\alpha-\beta)} & I_\beta \end{bmatrix}$. Also, it can be easily verified that the normalized system matrix P_n is ZCSE to the system matrix P.

Consider now P_n in the form

$$P_n = \left[\begin{array}{ccc|c} T & U & 0 & 0 \\ -V & 0 & I_m & 0 \\ 0 & -I_l & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right] \quad (9)$$

and apply, in turn, the following block elementary row and column operations, where R_e and C_f denote the eth block row and the fth column, respectively, of the underlying matrix. In these operations, $\leftarrow$ denotes the updating the variable on the left-hand side by the result of the operation on the right-hand side, and $\leftrightarrow$ denotes exchanging the entries on the left and right-hand sides.

$$\begin{aligned} &C_1 \leftarrow C_1 + C_3V, \quad R_1 \leftarrow R_1 + UR_3, \\ &R_4 \leftarrow R_4 + R_2, \quad C_4 \leftarrow C_4 + C_2, \\ &C_1 \leftrightarrow C_3, \quad R_1 \leftrightarrow R_2, \\ &R_2 \leftrightarrow R_3, \quad R_2 \leftarrow -I_lR_2. \end{aligned} \quad (10)$$

These operations applied to (9) result in the trivially expanded system matrix, denoted by P_e, of P as

$$P_e = \left[\begin{array}{cc|c} I_{m+l} & 0 & 0 \\ 0 & T & U \\ \hline 0 & -V & 0 \end{array}\right] \equiv \begin{bmatrix} I_{m+l} & 0 \\ 0 & P \end{bmatrix} \quad (11)$$

and it follows that P_n and P are related by the following ZCSE transformation:

$$\underbrace{\left[\begin{array}{c|c} I_r & 0_{r,m} \\ 0_{m,r} & 0_{m,m} \\ 0_{l,r} & 0_{l,m} \\ \hline 0_{m,r} & I_m \end{array}\right]}_{S_1}\underbrace{\left[\begin{array}{c|c} T & U \\ \hline -V & 0 \end{array}\right]}_{P} = \left[\begin{array}{c|c} T & U \\ 0 & 0 \\ 0 & 0 \\ \hline -V & 0 \end{array}\right]. \tag{12}$$

Let $r_1 = r + m + l$, $s_1 = r_1 + m$ and $q_1 = r_1 + l$ and write P_n as

$$\begin{aligned} P_n &\equiv \left[\begin{array}{c|c} T_n & U_n \\ \hline -V_n & 0 \end{array}\right] = P_{n,0} + z_1 P_{n,1} \\ &\equiv \begin{bmatrix} T_{n,0} & U_{n,0} \\ -V_{n,0} & 0 \end{bmatrix} + z_1 \begin{bmatrix} T_{n,1} & U_{n,1} \\ -V_{n,1} & 0 \end{bmatrix}, \end{aligned} \tag{13}$$

where $P_{n,0}$ and $P_{n,1}$ are $s_1 \times q_1$ polynomial matrices over $\mathbb{R}[z_2, z_3]$. Now let

$$\begin{aligned} F_1 &= \left[\begin{array}{ccc|c} I_{q_1} & 0 & 0 & 0 \\ P_{n,1} & I_{s_1} & 0 & 0 \\ 0 & 0 & I_l & 0 \\ \hline 0 & 0 & 0 & I_m \end{array}\right], \\ H_1 &= \left[\begin{array}{ccc|c} I_{q_1} & -z_1 I_{q_1} & 0 & 0 \\ 0 & I_{q_1} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ \hline 0 & 0 & 0 & I_l \end{array}\right], \end{aligned} \tag{14}$$

where it is routine to verify that F_1 and H_1 are unimodular. Also

$$\begin{aligned} &F_1 P_{nne} H_1 \\ &\quad = Q \\ &\quad \equiv \left[\begin{array}{ccccc|c} I_{r_1} & 0 & -z_1 I_{r_1} & 0 & 0 & 0 \\ 0 & I_l & 0 & -z_1 I_l & 0 & 0 \\ T_{n,1} & U_{n,1} & T_{n,0} & U_{n,0} & 0 & 0 \\ -V_{n,1} & 0 & -V_{n,0} & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_l & 0 & I_l \\ \hline 0 & 0 & 0 & 0 & -I_m & 0 \end{array}\right], \end{aligned} \tag{15}$$

where P_{nne} is given by:

$$P_{nne} = \left[\begin{array}{ccc|c} I_{q_1} & 0 & 0 & 0 \\ 0 & P_n & \mathcal{F}_{s_1,m} & 0 \\ 0 & -\mathcal{F}_{q_1,l}^T & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right]. \tag{16}$$

To show that P_{nne} is the unimodular system equivalent trivial expansion of P_n, apply the following elementary row and column operations (similar to (10)) to P_{nne}

$$\begin{aligned} &R_4 \leftrightarrow R_2, \quad C_2 \leftrightarrow C_4, \\ &C_2 \leftrightarrow C_3, \end{aligned} \tag{17}$$

to obtain

$$\left[\begin{array}{ccc|c} I_{q_1} & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 \\ 0 & 0 & I_l & -\mathcal{F}_{q_1,l}^T \\ \hline 0 & \mathcal{F}_{s_1,m} & 0 & P_n \end{array}\right]. \tag{18}$$

Expand, next $\mathcal{F}_{s_1,m}$ and $\mathcal{F}_{q_1,l}$ to obtain

$$\left[\begin{array}{ccc|cc} I_{q_1} & 0 & 0 & 0 & 0 \\ 0 & -I_m & 0 & 0 & 0 \\ 0 & 0 & I_l & 0 & -I_l \\ \hline 0 & 0 & 0 & T_n & U_n \\ 0 & I_m & 0 & -V_n & 0 \end{array}\right] \tag{19}$$

and then apply the elementary operations

$$\begin{aligned} &R_5 \leftarrow R_5 + R_2, \quad C_5 \leftarrow C_5 + C_3, \\ &R_2 \leftarrow -I_m R_2. \end{aligned} \tag{20}$$

These steps result in

$$P_{ne} = \begin{bmatrix} I_{q_1+m+l} & 0 \\ 0 & P_n \end{bmatrix} \tag{21}$$

and it follows in that Q and P_n are related by the ZCSE transformation

$$\underbrace{\left[\begin{array}{c|c} 0_{r_1,r_1} & 0_{r_1,m} \\ 0_{l,r_1} & 0_{l,m} \\ I_{r_1} & 0_{r_1,m} \\ 0_{m,r_1} & 0_{m,m} \\ 0_{l,r_1} & 0_{l,m} \\ \hline 0_{m,r_1} & I_m \end{array}\right]}_{S_3}\underbrace{\left[\begin{array}{c|c} T_n & U_n \\ \hline -V_n & 0 \end{array}\right]}_{P_n} = Q\underbrace{\left[\begin{array}{c|c} z_1 I_{r_1} & 0 \\ 0 & z_1 I_l \\ I_{r_1} & 0 \\ 0 & I_l \\ V_n & 0 \\ \hline 0 & I_l \end{array}\right]}_{S_4}. \tag{22}$$

Moreover

$$QS_4 = \begin{bmatrix} 0_{r_1,r_1} & 0_{r_1,l} \\ 0_{l,r_1} & 0_{l,l} \\ T_n & U_n \\ 0_{m,r_1} & 0_{r,l} \\ 0_{l,r_1} & 0_{l,l} \\ -V_n & 0_{m,l} \end{bmatrix} \tag{23}$$

and therefore Q is ZCSE to P_n and P. Also

$$\begin{aligned} Q &\equiv \begin{bmatrix} T_Q & U_Q \\ -V_Q & 0 \end{bmatrix} = Q_0 + z_2 Q_1 \\ &\equiv \begin{bmatrix} T_{Q,0} & U_{Q,0} \\ -V_{Q,0} & 0 \end{bmatrix} + z_2 \begin{bmatrix} T_{Q,1} & U_{Q,1} \\ -V_{Q,1} & 0 \end{bmatrix}, \end{aligned} \tag{24}$$

where Q_0 and Q_1 are $s_2 \times q_2$ polynomial matrices over $\mathbb{R}[z_1, z_3]$ and $s_2 = 2r_1 + 2m + 2l$ and $q_2 = 2r_1 + m + 3l$.

Let

$$\begin{aligned} F_2 &= \left[\begin{array}{ccc|c} I_{q_2} & 0 & 0 & 0 \\ Q_1 & I_{s_2} & 0 & 0 \\ 0 & 0 & I_l & 0 \\ \hline 0 & 0 & 0 & I_m \end{array}\right], \\ H_2 &= \left[\begin{array}{ccc|c} I_{q_2} & -z_2 I_{q_2} & 0 & 0 \\ 0 & I_{q_2} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ \hline 0 & 0 & 0 & I_l \end{array}\right] \end{aligned} \tag{25}$$

and introduce $r_2 = 2r_1 + m + 2l$. Also the matrices F_2 and H_2 are unimodular and hence

$$F_2 Q_{ne} H_2$$

$$= R \equiv \left[\begin{array}{ccc|c} I_{q_2} & -z_2 I_{q_2} & 0 & 0 \\ Q_1 & Q_0 & \mathcal{F}_{s_2,m} & 0 \\ 0 & -\mathcal{F}_{q_2,l}^T & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right]$$

$$\equiv \left[\begin{array}{ccccc|c} I_{r_2} & 0 & -z_2 I_{r_2} & 0 & 0 & 0 \\ 0 & I_l & 0 & -z_1 I_l & 0 & 0 \\ T_{Q,1} & U_{Q,1} & T_{Q,0} & U_{Q,0} & 0 & 0 \\ -V_{Q,1} & 0 & -V_{Q,0} & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_l & 0 & I_l \\ \hline 0 & 0 & 0 & 0 & -I_m & 0 \end{array}\right], \tag{26}$$

where

$$Q_{ne} = \left[\begin{array}{ccc|c} I_{q_2} & 0 & 0 & 0 \\ 0 & Q & \mathcal{F}_{s_2,m} & 0 \\ 0 & -\mathcal{F}_{q_2,l}^T & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right]. \tag{27}$$

Applying block elementary and column operations that mirror those for (17)-(20) to Q_{ne} gives the unimodular system equivalent trivial expansion of Q as

$$Q_e = \begin{bmatrix} I_{q_2+m+l} & 0 \\ 0 & Q \end{bmatrix} \tag{28}$$

and it follows in that Q and R are related by the following ZCSE transformation

$$\underbrace{\left[\begin{array}{c|c} 0_{r_2,r_2} & 0_{r_2,m} \\ 0_{l,r_2} & 0_{l,m} \\ I_{r_2} & 0_{r_2,m} \\ 0_{m,r_2} & 0_{m,m} \\ 0_{l,r_2} & 0_{l,m} \\ \hline 0_{m,r_2} & I_m \end{array}\right]}_{S_5} \underbrace{\left[\begin{array}{c|c} T_Q & U_Q \\ \hline -V_Q & 0 \end{array}\right]}_{Q} = RS_6, \tag{29}$$

where R is given by (26) and

$$S_6 = \left[\begin{array}{c|c} z_2 I_{r_2} & 0 \\ 0 & z_2 I_l \\ I_{r_2} & 0 \\ 0 & I_l \\ V_Q & 0 \\ \hline 0 & I_l \end{array}\right]. \tag{30}$$

Also

$$RS_6 = \begin{bmatrix} 0_{r_2,r_2} & 0_{r_2,l} \\ 0_{l,r_2} & 0_{l,l} \\ T_Q & U_Q \\ 0_{m,r_2} & 0_{m,l} \\ 0_{l,r_2} & 0_{l,l} \\ -V_Q & 0_{m,l} \end{bmatrix}. \tag{31}$$

Therefore R is ZCSE to Q and P_n and P.

Let $r_3 = q_2 + s_2 + l$, $s_3 = r_3 + m$ and $q_3 = r_3 + l$ and write R as

$$R \equiv \begin{bmatrix} T_R & U_R \\ -V_R & 0 \end{bmatrix} = R_0 + z_3 R_1$$
$$\equiv \begin{bmatrix} T_{R,0} & U_{R,0} \\ -V_{R,0} & 0 \end{bmatrix} + z_3 \begin{bmatrix} T_{R,1} & U_{R,1} \\ -V_{R,1} & 0 \end{bmatrix}, \tag{32}$$

where R_0 and R_1 are $s_3 \times q_3$ polynomial matrices over $\mathbb{R}[z_1, z_2]$. Also introduce

$$F_3 = \left[\begin{array}{ccc|c} I_{q_3} & 0 & 0 & 0 \\ R_1 & I_{s_3} & 0 & 0 \\ 0 & 0 & I_l & 0 \\ \hline 0 & 0 & 0 & I_m \end{array}\right],$$
$$H_3 = \left[\begin{array}{ccc|c} I_{q_3} & -z_3 I_{q_3} & 0 & 0 \\ 0 & I_{q_3} & 0 & 0 \\ 0 & 0 & I_m & 0 \\ \hline 0 & 0 & 0 & I_l \end{array}\right], \tag{33}$$

which are unimodular and

$$F_3 R_{ne} H_3 = \tilde{P} = \left[\begin{array}{ccccc|c} I_{r_3} & 0 & -z_3 I_{r_3} & 0 & 0 & 0 \\ 0 & I_l & 0 & -z_3 I_l & 0 & 0 \\ T_{R,1} & U_{R,1} & T_{R,0} & U_{R,0} & 0 & 0 \\ -V_{R,1} & 0 & -V_{R,0} & 0 & I_m & 0 \\ 0 & 0 & 0 & -I_l & 0 & I_l \\ \hline 0 & 0 & 0 & 0 & -I_m & 0 \end{array}\right], \tag{34}$$

where

$$R_{ne} = \left[\begin{array}{ccc|c} I_{q_3} & 0 & 0 & 0 \\ 0 & R & \mathcal{F}_{s_3,m} & 0 \\ 0 & -\mathcal{F}_{q_3,l}^T & 0 & I_l \\ \hline 0 & 0 & -I_m & 0 \end{array}\right]. \tag{35}$$

Applying the block elementary row and column operations defined by (17) to R_{ne} gives the unimodular system equivalent trivial expansion of R as

$$R_e = \begin{bmatrix} I_{q_3+m+l} & 0 \\ 0 & R \end{bmatrix} \tag{36}$$

and it follows that $\tilde{P}$ and R are related by the ZCSE transformation

$$\underbrace{\left[\begin{array}{c|c} 0_{r_3,r_3} & 0_{r_3,m} \\ 0_{l,r_3} & 0_{l,m} \\ I_{r_3} & 0_{r_3,m} \\ 0_{m,r_3} & 0_{m,m} \\ 0_{r_3,r_3} & 0_{r_3,m} \\ \hline 0_{m,r_3} & I_m \end{array}\right]}_{S_7} \underbrace{\left[\begin{array}{c|c} T_R & U_R \\ \hline -V_R & 0 \end{array}\right]}_{R} = \tilde{P}S_8, \tag{37}$$

where $\tilde{P}$ given in (34) is the $s_4 \times q_4$ polynomial matrix over $\mathbb{R}[z_1, z_2, z_3]$, ($s_4 = s_3 + q_3 + l + m$, $s_4 = q_3 + m + l$) and

$$S_8 = \left[\begin{array}{c|c} z_3 I_{r_3} & 0 \\ 0 & z_3 I_l \\ I_{r_3} & 0 \\ 0 & I_l \\ V_R & 0 \\ \hline 0 & I_l \end{array}\right]. \tag{38}$$

Moreover,

$$\tilde{P}S_8 = \begin{bmatrix} 0_{r_3,r_3} & 0_{r_3,l} \\ 0_{l,r_3} & 0_{l,l} \\ T_R & U_R \\ 0_{r,r_3} & 0_{r,l} \\ 0_{r_3,r_3} & 0_{r_3,l} \\ -V_R & 0 \end{bmatrix} \tag{39}$$

and it follows that $\tilde{P}$ is ZCSE to R and P_n and P and the polynomial matrix $\tilde{P}$ corresponds to the 3D singular Roesser state-space model given by (5). Hence, the following theorem has therefore been constructively established.

Theorem 1: The matrix $\tilde{P}$ is unimodular system equivalent to the trivial expansion matrix of P

$$P_{ee} = \begin{bmatrix} I_{s_4-r-m} & 0 \\ 0 & P \end{bmatrix}. \tag{40}$$

Also the matrices $\tilde{P}$ and P are related by the ZCSE transformation:

$$\tilde{S}_1\, P = \tilde{P}\tilde{S}_2 = \left[\begin{array}{c|c} T & U \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline -V & 0 \end{array}\right], \tag{41}$$

where

$$\tilde{S}_1 = S_7S_5S_3S_1 = \left[\begin{array}{c|c} I_r & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \hline 0 & I_m \end{array}\right],$$

$$\tilde{S}_2 = S_8S_6S_4S_2 = \left[\begin{array}{c|c} I_r & 0 \\ 0 & I_l \\ V & 0 \\ 0 & I_l \\ V & 0 \\ 0 & I_l \\ V & 0 \\ 0 & I_l \\ V & 0 \\ \hline 0 & I_l \end{array}\right]. \tag{42}$$

IV. CASE STUDY

Consider a particular case of spatially interconnected systems in the form of the ladder circuit of Fig. 1. To obtain the 3D model take the state vector as the inductor current and capacitor voltage in each internal node p, f, $0 < p < N$, $0 < q < M$, i.e.,

$$x(p, q, t) = \begin{bmatrix} U_C(p, q, t) \\ i_L(p, q, t) \end{bmatrix}. \tag{43}$$

Also, as one option, take the system output to be the state vector and the sources to be controlled as

$$i(p, q, t) := \gamma\, i_c(p+1, q, t) = \gamma\, \mathsf{C}\dot{U}_c(p+1, q, t). \tag{44}$$

By Kirchoff laws, the dynamics of this circuit are described by the state-space model given as (45) below where

$$\hat{\mathcal{A}}_{111} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{L} \end{bmatrix}, \hat{\mathcal{A}}_{101} = \begin{bmatrix} 0 & 0 \\ 0 & \mathsf{L} \end{bmatrix}, \hat{\mathcal{A}}_{011} = \begin{bmatrix} -\mathsf{C} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\hat{\mathcal{A}}_{110} = \begin{bmatrix} 0 & 0 \\ -1 & -\mathsf{R}_1 \end{bmatrix}, \quad \hat{\mathcal{A}}_{001} = \begin{bmatrix} -\mathsf{C} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\hat{\mathcal{A}}_{010} = \begin{bmatrix} -\dfrac{1}{\mathsf{R}_2} & 1+\gamma \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathcal{A}}_{100} = \begin{bmatrix} 0 & 1 \\ 0 & \mathsf{R}_1 \end{bmatrix},$$
$$\hat{\mathcal{A}}_{000} = \begin{bmatrix} \dfrac{1}{\mathsf{R}_2} & -1-\gamma \\ 0 & 0 \end{bmatrix}, \quad \hat{\mathcal{B}}_{110} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \hat{\mathcal{B}}_{100} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \tag{46}$$

In this case both state variables, i.e., $U_C(p, q, t)$ and $i_L(p, q, t)$, are selected as outputs and hence $\hat{\mathcal{C}} = I_2$.

To obtain a discrete representation of the dynamics of (45), as shown at the bottom of the next page, the forward Euler algorithm with discretion period h is employed, i.e.,

$$\dot{x}(p, q, t) \to \frac{x(p, q, k+1) - x(p, q, k)}{h}$$
$$x(p, q, t) \to x(p, q, k).$$

The resulting discrete state-space model is given in (47), as shown at the bottom of the next page, below with

$$\mathcal{A}_{111} = \frac{\hat{\mathcal{A}}_{111}}{h}, \ \mathcal{A}_{101} = \frac{\hat{\mathcal{A}}_{101}}{h}, \ \mathcal{A}_{011} = \frac{\hat{\mathcal{A}}_{011}}{h}$$
$$\mathcal{A}_{110} = \hat{\mathcal{A}}_{111} + \frac{\hat{\mathcal{A}}_{110}}{h}, \ \mathcal{A}_{001} = \frac{\hat{\mathcal{A}}_{001}}{h},$$
$$\mathcal{A}_{010} = \hat{\mathcal{A}}_{010} - \frac{\hat{\mathcal{A}}_{011}}{h}, \ \mathcal{A}_{100} = \hat{\mathcal{A}}_{100} - \frac{\hat{\mathcal{A}}_{101}}{h},$$
$$\mathcal{A}_{000} = \hat{\mathcal{A}}_{000} - \frac{\hat{\mathcal{A}}_{001}}{h}, \ \mathcal{B}_{110} = \hat{\mathcal{B}}_{110}, \mathcal{B}_{100} = \hat{\mathcal{B}}_{100}.$$

The related polynomial system matrix (1) is obtained by applying the shift operators

$$z_1g(p, q, k) = g(p+1, q, k), \ z_2g(p, q, k) = g(p, q+1, k),$$
$$z_3g(p, q, k) = g(p, q, k+1),$$

where z_1 is a temporal shift operator and z_2 and z_3 spatial. In the resulting matrix

$$T(z_1, z_2, z_3) = z_1z_2z_3\mathcal{A}_{111} - z_1z_2\mathcal{A}_{110}$$
$$-z_1z_3\mathcal{A}_{101} - z_2z_3\mathcal{A}_{011} - z_1\mathcal{A}_{100}$$

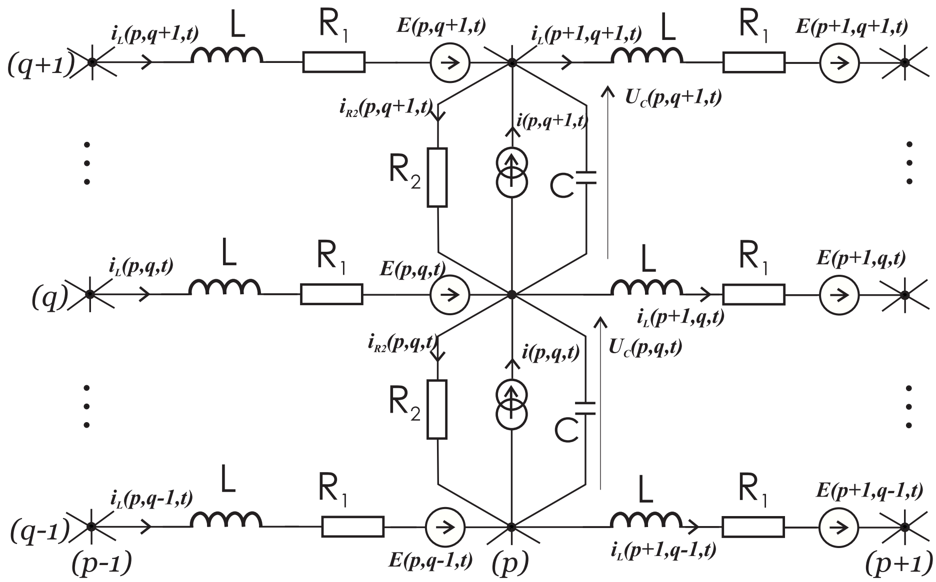

FIGURE 1. Structure of a spatially interconnected RLC circuit.

$$
\begin{aligned}
&-z_3\mathcal{A}_{001}-z_2\mathcal{A}_{010}-\mathcal{A}_{000},\\
U(z_1,z_2,z_3)&=z_1\,z_2\mathcal{B}_{110}+z_1\mathcal{B}_{100},\\
V(z_1,z_2,z_2)&=I_m,\quad W(z_1,z_2,z_3)=0. \qquad (48)
\end{aligned}
$$

As a numerical example, the data $\mathsf{L}=0.1\,H$, $\mathsf{C}=20\times10^{-6}\,F$, $\mathsf{R}_1=2000\,\Omega$, $\mathsf{R}_2=1000\,\Omega$ and $h=0.01$ secs is used. For ease of presentation the following notation is used: $X(a,b)$ denotes the entry in row a and column b of the matrix X and $X(c:d,e:f)$ denotes the matrix formed by rows c through to d and columns e through to f.

For this example, P_n defined in (13) is given by (49), as shown at the bottom of the next page. Also, $T_{n,0}$ and $T_{n,1}$ are given in (50), as shown at the bottom of the next page, and

$$
\begin{aligned}
U_{n,0}&=\begin{bmatrix}0&0&0&0&1\end{bmatrix}^T,\\
U_{n,1}&=\begin{bmatrix}0&0&0&0&0\end{bmatrix}^T,\\
V_{n,0}&=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\end{bmatrix},
\end{aligned}
$$

$$
V_{n,1}=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\end{bmatrix}.
$$

In what follows, $Q\in\mathbb{R}(z_1,z_2,z_3)^{16\times15}$ of (15) has zero entries except for

$$
\begin{aligned}
Q(9:16,7:14)&=-I_8\\
Q(7,2)&=-1\\
Q(1:6,1:6)&=I_6\\
Q(9:14,10:15)&=I_6\\
Q(1:6,7:12)&=-z_1I_6\\
Q(8,1)&=100z_2\\
Q(8,3)&=z_2-1\\
Q(7,8)&=1.01-1.01z_2\\
Q(7,7)&=0.002z_3-0.001\,z_2+0.002z_2z_3\\
&\quad-0.003\\
Q(8,2)&=200000z_2-10*z3+10*z2*z3\\
&\quad-1990.
\end{aligned}
$$

$$
\begin{aligned}
\hat{\mathcal{A}}_{111}\dot{x}(p+1,q+1,t)&=\hat{\mathcal{A}}_{101}\dot{x}(p+1,q,t)+\hat{\mathcal{A}}_{011}\dot{x}(p,q+1,t)+\hat{\mathcal{A}}_{110}x(p+1,q+1,t)+\hat{\mathcal{A}}_{001}\dot{x}(p,q,t)\\
&\quad+\hat{\mathcal{A}}_{010}x(p,q+1,t)+\hat{\mathcal{A}}_{100}x(p+1,q,t)+\hat{\mathcal{A}}_{000}x(p,q,t)+\hat{\mathcal{B}}_{110}E(p+1,q+1,t)\\
&\quad+\hat{\mathcal{B}}_{100}E(p+1,q,t),\\
y(p,q,t)&=\hat{\mathcal{C}}x(p,q,t) \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_{111}x(p+1,q+1,k+1)&=\mathcal{A}_{101}x(p+1,q,k+1)+\mathcal{A}_{011}x(p,q+1,k+1)+\mathcal{A}_{110}x(p+1,q+1,k)\\
&\quad+\mathcal{A}_{001}x(p,q,k+1)+\mathcal{A}_{010}x(p,q+1,k)+\mathcal{A}_{100}x(p+1,q,k)+\mathcal{A}_{000}x(p,q,k)\\
&\quad+\mathcal{B}_{110}E(p+1,q+1,k)+\mathcal{B}_{100}E(p+1,q,k),\\
y(p,q,k)&=\mathcal{C}x(p,q,k) \qquad (47)
\end{aligned}
$$

Hence, the entries in $T_{Q,0}$ and $T_{Q,1}$ are zero except for

$$
\begin{aligned}
T_{Q,0}(9:14,7:12) &= -I_6\\
T_{Q,0}(7:8,2:3) &= -I_2\\
T_{Q,0}(1:6,1:6) &= I_6\\
T_{Q,0}(9:13,10:14) &= I_5\\
T_{Q,0}(1:6,7:12) &= -z_1 I_6\\
T_{Q,0}(7,8) &= 1.01\\
T_{Q,0}(8,2) &= -10z_3-1990\\
T_{Q,0}(7,7) &= 0.002z_3-0.003\\
T_{Q,1}(8,1) &= 100\\
T_{Q,1}(8,2) &= 10z_3+200000\\
T_{Q,1}(8,3) &= 1\\
T_{Q,1}(7,7) &= 0.002z_3-0.001\\
T_{Q,1}(7,7) &= -1.01
\end{aligned}
$$

and

$$
\begin{aligned}
U_{Q,0} &= \begin{bmatrix} 0_{13\times 1}\\ 1 \end{bmatrix},\\
U_{Q,1} &= \begin{bmatrix} 0_{14\times 1} \end{bmatrix},\\
V_{Q,0} &= \begin{bmatrix} 0_{2\times 12} & I_2 \end{bmatrix},\\
V_{Q,1} &= \begin{bmatrix} 0_{2\times 14} \end{bmatrix}.
\end{aligned}
$$

Also, $R \in \mathbb{R}(z_1,z_2,z_3)^{34\times 33}$ of (26) has zero entries except for

$$
\begin{aligned}
R(22:23,17:18) &= -I_2\\
R(24:34,22:32) &= -I_{11}\\
R(1:21,1:21) &= I_{21}\\
R(24:32,25:33) &= I_9,\\
R(23,3) &= 1\\
R(23,1) &= 100\\
R(16:21,22:27) &= -z_1 I_6\\
R(1:15,16:30) &= -z_2 I_{15}\\
R(22,23) &= 1.01\\
R(22,8) &= -1.01\\
R(23,17) &= -10z_3-1990\\
R(23,2) &= 10z_3+200000\\
R(22,7) &= 0.002\,z_3-0.001\\
R(22,22) &= 0.002\,z_3-0.003
\end{aligned}
$$

and hence $T_{R,0}$ has zero entries except for

$$
\begin{aligned}
T_{R,0}(23,17) &= -1990\\
T_{R,0}(22:23,17:18) &= -I_2\\
T_{R,0}(24:32,22:30) &= -I_9\\
T_{R,0}(1:21,1:21) &= I_{21}\\
T_{R,0}(24:31,25:32) &= I_8,\ T_{R,0}(23,3)=1\\
T_{R,0}(23,1) &= 100\\
T_{R,0}(23,2) &= 200000\\
T_{R,0}(16:21,22:27) &= -z_1 I_6\\
T_{R,0}(1:15,16:30) &= -z_2 I_{15}\\
T_{R,0}(22,23) &= 1.01\\
T_{R,0}(22,8) &= -1.01\\
T_{R,0}(22,7) &= -0.001\\
T_{R,0}(22,22) &= -0.003.
\end{aligned}
$$

Moreover, the 32×32 matrix $T_{R,1}$ has zero entries except for

$$T_{R,1}(2,23) = 10$$

$$
P_n = \begin{bmatrix}
0.002\,z_3-0.001\,z_2+0.002\,z_2\,z_3-0.003 & 1.01-1.01z_2-z_1 & 0 & 0 & 0 & 0\\
100z_1\,z_2 & 200000\,z_1\,z_2-1990\,z_1-10\,z_1\,z_3+10\,z_1\,z_2\,z_3 & z_1\,z_2-z_1 & 0 & 0 & 0\\
-1 & 0 & 0 & 1 & 0 & 0\\
0 & -1 & 0 & 0 & 1 & 0\\
0 & 0 & -1 & 0 & 0 & 1\\
0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix} \quad (49)
$$

$$
T_{n,0} = \begin{bmatrix}
0.002z_3-0.001\,z_2+0.002\,z_2\,z_3-0.003 & 1.01-1.01\,z_2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 1 & 0\\
0 & -1 & 0 & 0 & 1\\
0 & 0 & -1 & 0 & 0
\end{bmatrix} \quad (50)
$$

$$
T_{n,1} = \begin{bmatrix}
0 & -1 & 0 & 0 & 0\\
100\,z_2 & 200000\,z_2-10\,z_3+10\,z_2\,z_3-1990 & z_2-1 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad (51)
$$

$$T_{R,1}(7,22) = 0.002$$
$$T_{R,1}(17,23) = -10$$
$$T_{R,1}(22,22) = 0.002$$

and

$$U_{R,0} = \begin{bmatrix} 0_{31\times 1} \\ 1 \end{bmatrix}$$
$$U_{R,1} = \begin{bmatrix} 0_{32\times 1} \end{bmatrix}$$
$$V_{R,0} = \begin{bmatrix} 0_{2\times 30} & I_2 \end{bmatrix}$$
$$V_{R,1} = \begin{bmatrix} 0_{2\times 32} \end{bmatrix}.$$

In the next step $\tilde{P} \in \mathbb{R}(z_1, z_2, z_3)^{70\times 69}$ of (38) is obtained. Again, this matrix has zero entries except for

$$\tilde{P}(55:56, 50:51) = -I_2$$
$$\tilde{P}(57:70, 55:68) = -I_{14}$$
$$\tilde{P}(56,17) = -10$$
$$\tilde{P}(56,50) = -1990$$
$$\tilde{P}(1:54, 1:54) = I_{54}$$
$$\tilde{P}(57:68, 58:69) = I_{12},$$
$$\tilde{P}(56,36) = 1$$
$$\tilde{P}(56,2) = 10$$
$$\tilde{P}(56,34) = 100$$
$$\tilde{P}(56,35) = 200000$$
$$\tilde{P}(49:54, 55:60) = -z_1 I_6$$
$$\tilde{P}(34:48, 49:63) = -z_2\, I_{15}$$
$$\tilde{P}(1:33, 34:66) = -z_3\, I_{33}$$
$$\tilde{P}(55,56) = 1.01$$
$$\tilde{P}(55,41) = -1.01$$
$$\tilde{P}(55,40) = -0.001$$
$$\tilde{P}(55,55) = -0.003$$
$$\tilde{P}(55,7) = 0.002$$
$$\tilde{P}(55,22) = -0.002.$$

Moreover, $\tilde{P}$ retains the structure defined in (34) and hence corresponds to (6) with

$$E_1 = \begin{bmatrix} 0_{48\times 54} & 0_{48\times 6} & 0_{48\times 8} \\ 0_{6\times 54} & -I_6 & 0_{6\times 6} \\ 0_{14\times 54} & 0_{14\times 6} & 0_{14\times 8} \end{bmatrix}$$
$$E_2 = \begin{bmatrix} 0_{33\times 48} & 0_{33\times 15} & 0_{33\times 5} \\ 0_{15\times 48} & -I_{15} & 0_{15\times 5} \\ 0_{20\times 48} & 0_{20\times 15} & 0_{20\times 5} \end{bmatrix}$$
$$E_3 = \begin{bmatrix} 0_{33\times 33} & -I_{33} & 0_{33\times 2} \\ 0_{35\times 33} & 0_{35\times 33} & 0_{35\times 2} \end{bmatrix},$$

Also, $A \in \mathbb{R}^{68\times 68}$ has zero entries except for

$$A(1:54, 1:54) = -I_{54}$$
$$A(57:67, 58:68) = -I_{11}$$
$$A(57:68, 55:66) = I_{12}$$
$$A(55:56, 50:51) = I_2$$
$$A(56,36) = -1$$
$$A(56,50) = 1990$$
$$A(56,17) = 10$$
$$A(56,2) = -10$$
$$A(56,34) = -100$$
$$A(56,35) = -200000$$
$$A(55,7) = -0.002$$
$$A(55,22) = -0.002$$
$$A(55,40) = 0.001$$
$$A(55,56) = -1.01$$
$$A(55,55) = 0.003$$
$$A(55,41) = 1.01$$
$$C = \begin{bmatrix} 0_{2\times 66} & I_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0_{1\times 67} \\ 1 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and these matrices form the singular 3D Roesser state-space model of (5) in this case.

To summarize the case study presented, the model of 3D spatially interconnected system with respectively: 2 states, 1 input, and 2 outputs, has resulted in a singular 3D Roesser model of 68th order. A significant increase in the order will eventually appear for more complicated systems but, due to the sparse model matrices, this should not be computationally prohibitive.

V. CONCLUSION

A constructive method has been developed that constructs a singular Roesser state-space model of a 3D linear system with spatial and temporal indeterminates from its system matrix. The form of transformation used in the reduction procedure is generated by a finite sequence of elementary row and column operations and a trivial expansion of the original polynomial system. An advantage of the developed method is the possible application to the analysis and synthesis of 3D spatially interconnected systems. Since there are no results referring directly applying the original models, the transformation into the known 3D Roesser model and the theory developed for this class of systems is promising for future research. Similar structure *n*D models appear in modeling other classes of interconnected systems, e.g., mechanical (connections of masses and dampers) heat exchangers and electromechanical systems. The results in this paper are also applicable.

REFERENCES

[1] E. Antoniou and S. Vologiannidis, "On the reduction of 2-D polynomial systems into first order equivalent models," *Multidimensional Syst. Signal Process.*, vol. 31, no. 1, pp. 249–268, Jan. 2020.
[2] J. A. Ball, "Multidimensional circuit synthesis and multivariable dilation theory," *Multidimensional Syst. Signal Process.*, vol. 22, nos. 1–3, pp. 27–44, Mar. 2011.
[3] N. K. Bose, B. Buchberger, and J. Guiver, *Multidimensional Systems Theory and Applications*. Dordrecht, The Netherlands: Kluwer, 2003.

[4] M. S. Boudellioua, K. Gakowski, and E. Rogers, "Characterization of a class of spatially interconnected systems (ladder circuits) using two-dimensional systems theory," *Multidimensional Syst. Signal Process.*, vol. 30, no. 4, pp. 2185–2197, Oct. 2019.

[5] M. S. Boudellioua, K. Galkowski, and E. Rogers, "Equivalence of wave linear repetitive processes and the singular 2-D Roesser state-space model," *Multidimensional Syst. Signal Process.*, vol. 31, no. 1, pp. 103–116, Jan. 2020.

[6] D. Zhao, K. Galkowski, B. Sulikowski, and L. Xu, "Derivation and reduction of the singular Fornasini–Marchesini state-space model for a class of multidimensional systems," *IET Control Theory Appl.*, vol. 14, no. 4, pp. 634–645, Mar. 2020.

[7] D. Zhao, S. Yan, S. Matsushita, and L. Xu, "An approach to multidimensional Fornasini–Marchesini state-space model realization w.r.t. columns of transfer matrices," *Syst. Control Lett.*, vol. 123, pp. 116–123, Jan. 2019.

[8] I. S. Duff, "A survey of sparse matrix research," *Proc. IEEE*, vol. 65, no. 4, pp. 500–535, Apr. 1977.

[9] E. M. O. El-Nabrawy, "Algebraic issues in linear multi-dimensional system theory," Ph.D. dissertation, Dept. Math. Sci., Loughborough Univ., Loughborough, U.K., 2006.

[10] E. Fornasini and G. Marchesini, "State-space realization theory of two-dimensional filters," *IEEE Trans. Autom. Control*, vol. AC-21, no. 4, pp. 484–492, Aug. 1976.

[11] D. S. Johnson, "Coprimeness in multidimensional system theory and symbolic computation," Ph.D. dissertation, Dept. Math. Sci., Loughborough Univ. Technol., Loughborough, U.K., 1993.

[12] D. S. Johnson, A. C. Pugh, E. Rogers, G. E. Hayton, and D. H. Owens, "A polynomial matrix theory for a certain class of two-dimensional linear systems," *Linear Algebra Appl.*, vols. 241–243, pp. 669–703, Jul. 1996.

[13] B. C. Levy, "2-D polynomial and rational matrices and their applications for the modelling of 2-D dynamical systems," Ph.D. dissertation, Dept. Elect. Eng., Stanford Univ., USA, Stanford, CA, USA, 1981.

[14] Z. L. Lin Xu and H. Fan, "On minor prime factorizations for n-D polynomial matrices," *IEEE Trans. Circuits Systems II, Exp. Briefs*, vol. 52, no. 9, pp. 568–571, Sep. 2005.

[15] S. J. McInerney, A. C. Pugh, M. S. Boudellioua, and G. E. Hayton, "On the reduction of a general 2-D polynomial system matrix to GSS form," in *Proc. Amer. Control Conf. (ACC)*, Philadelphia, PA, USA, 1998, pp. 2522–2526.

[16] W. Paszke, E. Rogers, and K. Gałkowski, "Experimentally verified generalized KYP lemma based iterative learning control design," *Control Eng. Pract.*, vol. 53, pp. 57–67, Aug. 2016.

[17] A. C. Pugh, S. J. McInerney, and E. M. O. El-Nabrawy, "Equivalence and reduction of 2-D systems," *IEEE Trans. Circuits Syst.*, vol. 52, no. 5, pp. 275–371, Jul. 2005.

[18] A. C. Pugh, S. J. McInerney, M. Hou, and G. E. Hayton, "A transformation for 2-D systems and its invariants," in *Proc. 35th IEEE Conf. Decis. Control*, Kobe, Japan, 1996, pp. 2157–2158.

[19] R. Roesser, "A discrete state-space model for linear image processing," *IEEE Trans. Autom. Control*, vol. AC-20, no. 1, pp. 1–10, Feb. 1975.

[20] E. Rogers, K. Galkowski, and D. H. Owens, *Control Systems Theory and Applications for Linear Repetitive Processes* (Control and Information Sciences). Berlin, Germany: Springer-Verlag, 2007.

[21] B. Sulikowski, K. Galkowski, E. Rogers, and A. Kummert, "Robust iterative learning control for ladder circuits with mixed uncertainties," in *Proc. 16th Int. Conf. Control, Autom., Robot. Vis. (ICARCV)*, Shenzhen, China, Dec. 2020, pp. 932–937.

MOHAMED S. BOUDELLIOUA received the B.Eng. degree in mechanical engineering, the M.Sc. degree in control engineering from the University of Bradford, in 1981 and 1982, respectively, and the Ph.D. degree in applied mathematics, in 1986, for a thesis on multidimensional systems theory. He worked with Nottingham Trent Polytechnic (now Nottingham Trent University), as a Research Assistant and a Demonstrator, from 1983 to 1986. From 1987 to 1994, he was an Associate Professor with Skikda University, Algeria. From 1994 to 2001, he was with the Al-Ahsa College of Technology, Saudi Arabia. Since 2001, he has been with Sultan Qaboos University, Oman. His main research interest includes algebraic methods in multidimensional systems theory.

BARTŁOMIEJ SULIKOWSKI received the Ph.D. and D.Sc. degrees from the University of Zielona Gora, Poland, in 2006 and 2019, respectively. His publication record counts over 50 papers in the world-known journals and conference proceedings, such as *Automatica*, *Multidimensional Systems*, *Signals Processing*, and the *International Journal of Control*. His research interests include multidimensional (nD) systems, linear repetitive processes, classes of spatially interconnected systems modeled as 2D/nD systems (recent years), and the application of numerical methods to control problems for nD systems. For many years, he has been a member of teams carrying out the national grants funded by the National Centre of Science, Poland (previously National Committee for Scientific Research), and international ones, such as Polonium and DAAD.

KRZYSZTOF GAŁKOWSKI (Member, IEEE) is currently a Professor of technical sciences with the University of Zielona Gora and also with Nicolaus Copernicus University in Torun, Poland. His major research interests lie in the areas of multidimensional (nD) systems and control theory and applications for repetitive processes, including application to iterative learning control with experimental validation. He has authored over 100 articles in leading peer reviewed journals and over 200 papers in the proceedings of major international conferences. He is also an Associate Editor of *IET Control Theory and Applications* and a member of the editorial boards of the *International Journal of Control and Multidimensional Systems and Signal Processing*. He is also a coauthor/editor of four research monographs/books and also a Guest Editor of special issues in leading international journals, including the *International Journal of Control, Multidimensional Systems and Signal Processing* and the *Asian Journal of Control*. He is also a Co-Founder of the international workshops on multidimensional systems (NDS). He is also a Visiting Professor with the School of Electronics and Computer Science, University of Southampton, U.K., and was three times a Gerhard Mercator Guest Professor, funded by DFG, in The University of Wuppertal, Germany, (September 2004–August 2005), (October 2006–September 2007), and (March 2011–February 2012). During the period 2011–2014, he was a Chair Professor with the East China University of Science and Technology in Shanghai, China.

ERIC ROGERS (Member, IEEE) was born in Northern, Ireland. He received the bachelor's degree in mechanical engineering from Queen's University Belfast and the master's and Ph.D. degrees in control systems from The University of Sheffield, U.K. He is currently a Professor of control systems theory and design with the School of Electronics and Computer Science, University of Southampton, U.K. His major current research interests include multidimensional systems theory and applications, iterative learning control and applications, including application to robotic-assisted stroke 123 Multidimensional Systems and Signal Processing (2019) (30:2185–2197 2197) rehabilitation with supporting clinical trials and smart rotor control for wind turbines and estimation and control of autonomous underwater vehicles with particular interests in long range autonomy and close sea bed following. He is also the Editor-in-Chief of the *International Journal of Control* and *Multidimensional Systems and Signal Processing*.