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A Second Order Dynamical System for Equilibrium Problems

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Abstract We consider a second order dynamical system for solving equilibrium problems in Hilbert spaces. Under mild conditions, we prove existence and uniqueness of strong global solution of the proposed dynamical system. We establish the exponential convergence of trajectories under strong pseudo monotonicity and Lipschitz-type conditions. We then investigate a discrete version of the second order dynamical system, which leads to a fixed point type algorithm with inertial effect and relaxation. The linear convergence of this algorithm is established under suitable conditions on parameters. Finally some numerical experiments are reported confirming the theoretical results.

Keywords Dynamic programming · Equilibrium problem · Monotonicity · Lipschitz continuity · Exponential stability · Linear convergence.

Mathematics Subject Classification (2000) 47J20 · 49J40 · 49M30

1 Introduction

Let H be a real Hilbert space endowed with an inner product and its induced norm denoted $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let f be a function from $H \times H$ to $\mathbb{R}$ satisfying for each $x \in H$, $f(x,x) = 0$ and such that the function $f(x,\cdot)$ is convex and lower semicontinuous. The equilibrium problem, also known as Ky Fan variational inequality, associated with f in the sense of [7], is denoted by $EP(f,C)$, and

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consists in finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0 \quad \text{for every } y \in C.$$

We denote the solution set of $EP(f,C)$ by $Sol(f,C)$.

EP is a general mathematical model which includes, as special cases, the optimization problem, the variational inequality, the saddle point problem, the Nash equilibrium problem in noncooperative games, the fixed point problem, and others; see, for instance, [7,23,24] and references quoted therein. For example, when $f(x,y) = \langle F(x), y-x \rangle$ for all $x,y \in H$, the equilibrium problem $EP(f,C)$ reduces to the classical variational inequality $VI(F,C)$ which consists in finding a point $x^* \in C$ such that

$$\langle F(x^*), y-x^* \rangle \geq 0 \quad \text{for every } y \in C,$$

where $F : H \to H$ is a continuous mapping.

Another important case of EP is the saddle point problem: Given two sets $C_1 \subset H_1$ and $C_2 \subset H_2$, where H_1 and H_2 are two real Hilbert spaces, a saddle point of a function $G : H_1 \times H_2 \to \mathbb{R}$ is any $x^* = (x_1^*, x_2^*) \in C_1 \times C_2$ such that

$$G(x_1^*, y_2) \leq G(x_1^*, x_2^*) \leq G(y_1, x_2^*)$$

holds for any $y = (y_1, y_2) \in C_1 \times C_2$. Finding a saddle point of G amounts to solving $EP(f,C)$ with $C = C_1 \times C_2$ and

$$f((x_1,x_2),(y_1,y_2)) = G(y_1,x_2) - G(x_1,y_2).$$

A saddle point of G is a Nash equilibrium in a two-person zero-sum game, that is a noncooperative game where the cost function of the first player is G and the cost function of the second player is $-G$ (see, e.g., [1,6]).

In recent years, a large number of applications has been described successfully via the concept of equilibrium solution and therefore many researchers devoted their efforts to study EPs. We recommend the readers the excellent monograph [6] for a comprehensive survey on existence of equilibrium points and solution methods for finding them.

For solving EPs, many solution methods have been proposed, and most of them are adapted from solution methods of a particular model of EPs, see for example [6,21,22, 19,23,29,31,32,33]. Fixed point-type methods are highly recommended because they are simple in form and useful in practice. Indeed, these methods are extended from the classical projection method for solving a variational inequality (VI for short) [15]. Mastroeni proved that fixed point methods can efficiently solve the class of strongly monotone EPs [23]. Later, Muu and Quoc established the linear convergence rate of these methods [25].

Recently, first and second order dynamical system approaches have been widely investigated for solving fixed point problems, variational inequalities and monotone inclusions [2,3,4,8,9,11,12,13,16,27,28,34,35]. As a natural extension, it is interesting to study EPs from a continuous time perspective. The first attempt was recently studied in [36], where a first order dynamical system for solving EPs was proposed and investigated. The key idea is to reformulate the EP as a fixed point problem of a suitable operator. Then, the solutions set of the EP is approached by considering a dynamical system associated with the fixed point map, which is similar to the strategy using in [11,12,34] for monotone inclusions and variational inequality. Under strong pseudo-monotonicity and Lipschizt continuity, it was proved that the trajectories generated by the first order dynamical system converges exponentially to the uniques solution. In addition, a discrete dynamical system was considered leading to a linear convergent fixed point algorithm for solving EPs.

In this paper, we continue this research direction by considering second order dynamical system associated with this fixed point reformulation (see e.g. [4,9,12,34]), for which we obtain the exponential stability. In addition, we consider a discrete version of the proposed dynamical system, which leads to a fixed point method with inertial effect and relaxation. We establish the linear convergence of the iterative sequence generated by this algorithm to the unique solution of the equilibrium problem

The remaining part of the paper is organized as follows. Section 2 consists of some preliminaries. In Section 3, we propose the second order dynamical system and establish the solution existence and uniqueness of trajectories. Section 4 describes the global exponential convergence of the proposed dynamical system. A discrete version dynamical system and its linear convergence are presented in Section 5. Finally, some numerical experiments are reported in Section 6 to illustrate the obtained theoretical results.

2 Preliminaries

In this section we recall some well known definitions useful in the sequel.

Let $g : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. We call g subdifferentiable at x if the set

$$\partial g(x) = \{u \in H : g(y) \geq g(x) + \langle u, y-x\rangle \ \forall y \in H\}$$

is nonempty. Then, $\partial g(x)$ is called the subdifferential of g at x and vector $w \in \partial g(x)$ is called a subgradient of g at x. The function g is subdifferentiable on C if it is subdifferentiable at each point of C. Note that if the function g is convex, lsc and has full domain, then it is continuous on the whole space [5, Corollary 8.30]. In this case, g is subdifferentiable on H [5, Proposition 16.14]. In addition, let f and g be proper, convex, lsc functions such that $\mathrm{dom} f \cap \mathrm{int}\, \mathrm{dom} g \neq \emptyset$ or $\mathrm{dom}(g) = H$, here $\mathrm{dom} f = \{x \in H, f(x) < +\infty\}$ denotes the domain of f, then $\partial(f+g) = \partial f + \partial g$ [5, Corollary 16.38].

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, x-y\rangle \geq 0, \forall y \in C\},$$

and $N_C(x) = \emptyset$ if $x \notin C$. The indicator function of C is defined as $i_C(x) = 0$ if $x \in C$ and $i_C(x) = +\infty$ otherwise. In addition, we have $\partial i_C(x) = N_C(x)$ for all $x \in H$.

For every $x \in H$, the metric projection $P_C(x)$ of x onto C is defined by

$$P_C(x) = \arg\min\{\|y-x\| : y \in C\}.$$

Since C is nonempty, closed and convex, $P_C(x)$ exists and is unique. For more details as well as for unexplained terminologies and notations we refer to [5].

Definition 1 A mapping $f : H \times H \to \mathbb{R}$ is said to be

(a) strongly monotone with modulus $\delta > 0$ on C if

$$f(x,y) + f(y,x) \leq -\delta\|x-y\|^2 \quad \forall x, y \in C;$$

(b) strongly pseudo-monotone with modulus $\delta > 0$ on C if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq -\delta \|x-y\|^2$$

for all $x, y \in C$;

(c) satisfying a Lipschitz-type condition on C if there exists a constant $L > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - L\|x-y\| \|y-z\| \quad \forall x,y,z \in C. \tag{1}$$

The following example provides a class of EPs where the function f is strongly pseudo-monotone and not strongly monotone on C.

Example 1 Let $C \subset H$ and

$$f(x,y) = \alpha(x,y) g(x,y)$$

where

$$\alpha(x,y) \geq \alpha > 0 \quad \forall x,y \in C$$

and g is strongly monotone on C with modulus $\delta > 0$, for instance $g(x,y) := \langle x, y-x \rangle$ (hence $\delta = 1$). We can verify that f is strongly pseudo-monotone on C. Indeed, let $x, y \in C$ such that $f(x,y) \geq 0$, since $\alpha(x,y) \geq \alpha > 0$ we have $g(x,y) \geq 0$. By the δ-strong monotonicity of g on C we deduce $g(y,x) \leq -\delta \|x-y\|^2$ for every $x, y \in C$. Hence

$$f(y,x) = \alpha(y,x) g(y,x) \leq -\delta \alpha(y,x) \|x-y\|^2 \leq -\delta \alpha \|x-y\|^2,$$

i.e. f is strongly pseudo-monotone on C with modulus $\delta \alpha > 0$.
Note that, in general f is neither strongly monotone nor monotone. To see this, let $H = \mathbb{R}^n$, $\alpha(x,y) = R - \|x\|$, $g(x,y) = \langle x, y-x \rangle$ and $C = \{x \in H, \|x\| \leq r\}$ with $R > r > R/2$. Choosing $x = (R/2, 0, 0, ...)$, $y = (r, 0, 0...) \in C$ we have

$$f(x,y) + f(y,x) = (r - R/2)^3 > 0.$$

Remark 1 (i) The implications (a)$\Longrightarrow$(b) is evident. Note also that property (b) guarantees that $EP(f,C)$ cannot have more than one solution. Indeed, it was proved in [26] that if the function f is strongly pseudo-monotone and continuous then the equilibrium problem $EP(f,C)$ has a unique solution x^*. When EP reduces to VI, i.e., $f(x,y) = \langle F(x), y-x \rangle$ for all $x, y \in H$, the (generalized) monotonicity of function f defined above corresponds to the well known (generalized) monotonicity of mapping F (see [20]).

(ii) We note that (1) is weaker than the Lipschitz-type condition introduced by Antipin [1], which can be written as

$$|[f(x,y) - f(x,z)] - [f(u,y) - f(u,z)]| \leq L\|x-u\| \|y-z\| \quad \forall u,x,y,z \in C. \tag{2}$$

Indeed, taking $u = y$ in (2), since $f(y,y) = 0$, we can deduce

$$-f(x,y) + f(x,z) - f(y,z) \leq |f(x,y) - f(x,z) + f(y,z)| \leq L\|x-y\| \|y-z\| \quad \forall x,y,z \in C,$$

which implies (1). On the other hand, (1) implies the Lipschitz-type condition in the sense of Mastroeni [23]

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 \|x-y\|^2 - c_2 \|y-z\|^2 \quad \forall x,y,z \in C,$$

where c_1, c_2 are two given positive constants.

When $f(x,y) = \langle F(x), y-x\rangle$ for all $x,y \in H$, i.e., EP collapses to a VI problem, then f satisfies Antipin's Lipschitz condition (2) on C (and hence also (1)) if F is Lipschitz continuous on C with a Lipschitz constant $L > 0$. Indeed, since F is Lipschitz continuous, using the Cauchy-Schwarz inequality, we have for all $u,x,y,z \in C$ that

$$\begin{aligned} |[f(x,y)-f(x,z)]-[f(u,y)-f(u,z)]| &= |\langle F(x)-F(u), y-z\rangle| \\ &\leq \|F(x)-F(u)\|\|y-z\| \\ &\leq L\|x-u\|\|y-z\|. \end{aligned}$$

To establish the main results of the paper, we need to recall the stability concepts of an equilibrium point of the general dynamical system

$$\dot{x}(t) = T(x(t)), \quad t \geq 0 \tag{3}$$

where T is a continuous mapping from H to H and $x : [0,+\infty) \to H$.

Definition 2 [28]

(a) A point $x^* \in H$ is an equilibrium point for (3) if $T(x^*) = 0$;
(b) An equilibrium point x^* of (3) is stable if, for any $\varepsilon > 0$, there exists $r > 0$ such that, for every $x_0 \in B(x^*,r)$, the solution $x(t)$ of the dynamical system with $x(0) = x_0$ exists and is contained in $B(x^*,\varepsilon)$ for all $t > 0$, where $B(x^*,r)$ denotes the open ball with center x^* and radius r;
(c) A stable equilibrium point x^* of (3) is asymptotically stable if there exists $r > 0$ such that, for every solution $x(t)$ of (3) with $x(0) \in B(x^*,r)$, one has

$$\lim_{t\to+\infty} x(t) = x^*;$$

(d) An equilibrium point x^* of (3) is exponentially stable if there exist $r > 0$ and constants $\kappa > 0$ and $\theta > 0$ such that, for every solution $x(t)$ of (3) with $x(0) \in B(x^*,r)$, one has

$$\|x(t)-x^*\| \leq \kappa\|x(0)-x^*\|e^{-\theta t} \quad \forall t \geq 0. \tag{4}$$

Furthermore, x^* is globally exponentially stable if (4) holds true for all solutions $x(t)$ of (3).

3 A second order dynamical system

Let $\alpha,\beta : [0,+\infty) \to [0,+\infty)$ be Lebesgue measurable function, $\lambda > 0$ and $x_0, v_0 \in H$. In this section we will approach the solution set of $EP(f,C)$ from a continuous perspective. For every $\lambda > 0$, we define a function $\psi_\lambda : H \to H$ by

$$\psi_\lambda(x) = \operatorname{argmin}_{y\in C}\{\lambda f(x,y) + \frac{1}{2}\|y-x\|^2\} \quad \forall x \in H. \tag{5}$$

For every $x \in H$ and $\lambda > 0$, since $f(x,\cdot)$ is convex and C is a convex set, the problem

$$\min_{y\in C}\{\lambda f(x,y) + \frac{1}{2}\|y-x\|^2\} \tag{6}$$

is a strongly convex problem, hence it has a unique solution. Therefore, the function ψ_λ is well defined and has single values on H.

Following the works in [4, 8, 12, 34], we consider the following second order dynamical system:

$$\begin{cases} \ddot{x}(t) + \alpha(t)\dot{x}(t) + \beta(t)(x(t) - \psi_\lambda(x(t))) = 0, \\ x(0) = x_0, \quad \dot{x}(0) = v_0. \end{cases} \tag{7}$$

Definition 3 A function $x : [0,b] \to H$ (where $b > 0$) is said to be absolutely continuous if one of the following equivalent properties holds:

(i) there exists an integrable function $y : [0,b] \to H$ such that

$$x(t) = x(0) + \int_0^t y(s)ds \quad \forall t \in [0,b];$$

(ii) x is continuous and its distributional derivative $\dot{x}$ is Lebesgue integrable on $[0,b]$.

Before stating the existence and uniqueness of the trajectory of (7), we need to recall the definition of its strong global solution.

Definition 4 We say that $x : [0,+\infty) \to H$ is a strong global solution of dynamical system (7) if the following properties are satisfied:

(i) $x, \dot{x} : [0,+\infty) \to H$ are locally absolutely continuous, in other word, absolutely continuous on each interval $[0,b]$ for $0 < b < +\infty$;
(ii) $\ddot{x}(t) + \alpha(t)\dot{x}(t) + \beta(t)(x(t) - \psi_\lambda(x(t))) = 0$ for almost every $t \in [0,+\infty)$;
(iii) $x(0) = x_0$ and $\dot{x}(0) = v_0$.

The following Lemma will be useful in the sequel for convergence analysis [31, 36]. We provide a simple proof for completeness.

Lemma 1 *For any $\lambda > 0$ and $x \in H$, setting $z := \psi_\lambda(x) \in C$, then it holds*

$$\lambda\,(f(x,y) - f(x,z)) \geq \langle x - z, y - z\rangle \quad \forall y \in C. \tag{8}$$

Proof For every $x \in H$, z is the unique solution of the strongly convex minimization problem (6). The optimality condition associated with (6) implies that $0 \in \partial G(z)$, where $G(y) := \lambda f(x,y) + \frac{1}{2}\|y - x\|^2 + i_C(y)$. Since the first two terms of G have full domain, the sum rule of subdifferential can be applied. Hence, there exists $s \in \partial_2 f(x,z)$ such that

$$0 \in \lambda s + z - x + N_C(z),$$

where $N_C(z)$ denotes the normal cone to C at z. Hence, by definition of this cone, we have

$$\langle x - z - \lambda s, y - z\rangle \leq 0 \quad \forall y \in C. \tag{9}$$

On the other hand, since $s \in \partial_2 f(x,z)$, it follows

$$f(x,y) - f(x,z) \geq \langle s, y - z\rangle \quad \forall y \in C. \tag{10}$$

Combining (9) and (10), we obtain

$$\lambda\,(f(x,y) - f(x,z)) \geq \langle \lambda s, y - z\rangle \geq \langle x - z, y - z\rangle \quad \forall y \in C.$$

As a consequence of Lemma 1 we have the following result.

Corollary 1 *For any $\lambda > 0$, $x \in Sol(f,C)$ if and only if $x = \psi_\lambda(x)$.*

Proof If $x = z$ then since $\lambda > 0$ we have from (8) and $f(x,x) = 0$ that

$$f(x,y) \geq 0 \quad \forall y \in C,$$

i.e., $x \in Sol(f,C)$. Conversely, if $x \in Sol(f,C)$, then substituting $y = x$ in (8), since $z \in C$ we obtain

$$\|x-z\|^2 \leq \lambda\,(f(x,x) - f(x,z)) = -\lambda f(x,z) \leq 0,$$

which implies $x = z$.

Remark 2 In the case $f(x,y) = \langle F(x), y-x\rangle$ for all $x,y \in H$, Corollary 1 reduces to the well-known characterization of the solution of $VI(F,C)$: For any $\lambda > 0$, x is a solution of $VI(F,C)$ if and only if $x = P_C(x - \lambda F(x))$, see e.g. [15]. Indeed,

$$\begin{aligned}\psi_\lambda(x) &= \operatorname{argmin}_{y\in C}\{\lambda f(x,y) + \frac{1}{2}\|y-x\|^2\}\\ &= \operatorname{argmin}_{y\in C}\{\lambda\,\langle F(x), y-x\rangle + \frac{1}{2}\|y-x\|^2\}\\ &= \operatorname{argmin}_{y\in C}\{\|y-(x-\lambda F(x))\|^2\} = P_C(x-\lambda F(x)),\end{aligned}$$

where P_C denotes the projection operator onto C.

The second order dynamical system (7) for $VI(F,C)$ is written as follows

$$\begin{cases}\ddot{x}(t) + \alpha(t)\dot{x}(t) + \beta(t)(x(t) - P_C(x(t) - \lambda F(x(t)))) = 0,\\ x(0) = x_0, \quad \dot{x}(0) = v_0,\end{cases} \tag{11}$$

whose the global exponential convergence has been recently established in [34].

The existence and uniqueness of the trajectory of (12) is stated in the following result, where we employ the technique used in [8].

Theorem 1 *Let $\alpha,\beta : [0,+\infty) \to [0,+\infty)$ be Lebesgue measurable functions such that $\alpha,\beta \in L^1_{loc}([0,+\infty))$ (that is $\alpha,\beta \in L^1_{loc}([0,b])$ for every $0 < b < +\infty$). Assume that ψ_λ is Lipschitz continuous for all $\lambda > 0$. Then for each $x_0, v_0 \in H$, there exists a unique strong global solution of the dynamical system* (7)*.*

Proof For all $x \in H$, defining $S : H \to H$ by

$$Sx := x - \psi_\lambda(x),$$

we can re-write dynamical system (7) as

$$\begin{cases}\ddot{x}(t) + \alpha(t)\dot{x}(t) + \beta(t)Sx(t) = 0,\\ x(0) = x_0, \quad \dot{x}(0) = v_0.\end{cases} \tag{12}$$

Moreover, Lemma 1 yields $Zeros(S) = Sol(f,C)$. By the Lipschitz continuity of $\psi_\lambda(\cdot)$ and the Cauchy-Schwartz inequality, it is clear that S is Lipschitz continuous. Hence, the conclusion is obtained following the proof of [8, Theorem 4].

We provide in the following some sufficient conditions for the Lipschitz continuity of ψ_λ.

Proposition 1 *If f satisfies Lipschitz condition* (2) *then ψ_λ is Lipschitz continuous with constant $1 + \lambda L$.*

Proof Let $x_1, x_2 \in H$ and $z_1 = \psi_\lambda(x_1), z_2 = \psi_\lambda(x_2)$ we have $z_1, z_2 \in C$ and from Lemma 1 it holds

$$\lambda\left(f(x_1,y) - f(x_1,z_1)\right) \geq \langle x_1 - z_1, y - z_1\rangle \quad \forall y \in C.$$

Substituting $y := z_2 \in C$ we obtain

$$\lambda\left(f(x_1,z_2) - f(x_1,z_1)\right) \geq \langle x_1 - z_1, z_2 - z_1\rangle.$$

Similarly, we can deduce

$$\lambda\left(f(x_2,z_1) - f(x_2,z_2)\right) \geq \langle x_2 - z_2, z_1 - z_2\rangle.$$

Adding the last two inequalities and using the Lipschitz condition (2) we have

$$\begin{aligned}\langle z_1 - z_2, z_1 - x_1 + x_2 - z_2\rangle &\leq \lambda\left([f(x_1,z_2) - f(x_1,z_1)] - [f(x_2,z_2) - f(x_2,z_1)]\right)\\ &\leq \lambda L\|x_1 - x_2\|\|z_1 - z_2\|,\end{aligned}$$

which implies

$$\|z_1 - z_2\|^2 \leq \langle z_1 - z_2, x_1 - x_2\rangle + \lambda L\|x_1 - x_2\|\|z_1 - z_2\| \leq (1+\lambda L)\|x_1 - x_2\|\|z_1 - z_2\|.$$

Hence

$$\|z_1 - z_2\| \leq (1+\lambda L)\|x_1 - x_2\|.$$

4 Global exponential convergence

In this section, we will investigate the exponential convergence of the trajectories $x(t)$ generated by dynamical system (7). We use a similar technique developed in [10, 12] for solving monotone inclusions. From now on, we assume that f is δ-strongly pseudo-monotone on C and satisfies the Lipschitz-type condition (1) with modulus $L > 0$ on H.
The following result will play an important role in our convergence analysis.

Proposition 2 *Let x^* be the unique solution of $EP(f,C)$ and $\lambda > 0$. Then, for any $x \in H$, we have*

$$\langle x - \psi_\lambda(x), x - x^*\rangle \geq \left(1 - \frac{\lambda L^2}{4\delta}\right)\|x - \psi_\lambda(x)\|^2 \tag{13}$$

and

$$\|x - x^*\| \leq \frac{1+\lambda\delta+\lambda L}{\lambda\delta}\|x - \psi_\lambda(x)\|. \tag{14}$$

Proof Setting $z := \psi_\lambda(x) \in C$ and substituting $y = x^* \in C$ into (8) we have

$$\lambda\left(f(x,x^*) - f(x,z)\right) \geq \langle x - z, x^* - z\rangle.$$

Combinning this inequality with the Lipschitz-type condition of f we obtain

$$\begin{aligned}\langle x - z, z - x^*\rangle &\geq \lambda\left(f(x,z) - f(x,x^*)\right)\\ &\geq -\lambda f(z,x^*) - \lambda L\|x - z\|\|z - x^*\|.\end{aligned} \tag{15}$$

Since $x^* \in Sol(f,C)$ and $z \in C$, it holds that $f(x^*,z) \geq 0$. Then by the δ-strong pseudomonoticity of f we have $f(z,x^*) \leq -\delta\|z - x^*\|^2$. It follows from (15) that

$$\langle x - z, z - x^*\rangle \geq \lambda\delta\|z - x^*\|^2 - \lambda L\|x - z\|\|z - x^*\|. \tag{16}$$

Therefore,

$$
\begin{aligned}
\langle x-\psi_\lambda(x), x-x^*\rangle &= \langle x-z, x-z+z-x^*\rangle \\
&= \|x-z\|^2 + \langle x-z, z-x^*\rangle \\
&\geq \|x-z\|^2 + \lambda\delta\|z-x^*\|^2 - \lambda L\|x-z\|\|z-x^*\| \\
&= \left(1-\frac{\lambda L^2}{4\delta}\right)\|x-z\|^2 \\
&\quad + \frac{\lambda L^2}{4\delta}\|x-z\|^2 + \lambda\delta\|z-x^*\|^2 - \lambda L\|x-z\|\|z-x^*\| \\
&\geq \left(1-\frac{\lambda L^2}{4\delta}\right)\|x-z\|^2,
\end{aligned}
$$

where we have used the Cauchy-Schwarz inequality in the last estimation. This implies (13). Again, from (16) and the Cauchy-Schwarz inequality we have

$$\lambda\delta\|z-x^*\|^2 \leq \|x-z\|\|z-x^*\| + \lambda L\|x-z\|\|z-x^*\|,$$

which implies

$$\|z-x^*\| \leq \frac{1+\lambda L}{\lambda\delta}\|x-z\|.$$

Hence

$$\|x-x^*\| \leq \|x-z\| + \|z-x^*\| \leq \frac{1+\lambda\delta+\lambda L}{\lambda\delta}\|x-z\|.$$

We are now in the position to establish the main result of this section where we employ the similar tools and techniques used in [34, 10, 1].

Theorem 2 *Let x^* be the unique solution of $EP(f,C)$, let $0<\lambda<\frac{4\delta}{L^2}$ and $\xi = 1-\frac{\lambda L^2}{4\delta} > 0$. Let $\alpha,\beta : [0,+\infty) \to [0,+\infty)$ be a locally absolutely continuous functions fulfilling for every $t\in[0,+\infty)$*

(i) $1<\alpha\leq\alpha(t)\leq\frac{\xi\lambda^2\delta^2}{(1+\lambda\delta+\lambda L)^2}\beta(t)+1$;
(ii) $\dot\alpha(t)\leq 0$ *and* $\frac{d}{dt}\left(\frac{\alpha(t)}{\beta(t)}\right)\leq 0$;
(iii) $\alpha^2(t)-\alpha(t)-\frac{2\beta(t)}{\xi}\geq 0$.

Then any strong global solution $x(t)$ to the dynamical system (7) *converges exponentially to x^* as $t\to\infty$, i.e., there exist positive numbers κ,θ such that*

$$\|x(t)-x^*\| \leq \kappa\|x(0)-x^*\|e^{-\theta t} \quad \forall t\geq 0.$$

Proof Consider for every $t\in[0,+\infty)$ the Lyapunov function $\mathscr{E}(t)=\frac{1}{2}\|x(t)-x^*\|^2$. Then

$$\dot{\mathscr{E}}(t) = \langle x(t)-x^*, \dot x(t)\rangle, \quad \ddot{\mathscr{E}}(t) = \|\dot x(t)\|^2 + \langle x(t)-x^*, \ddot x(t)\rangle.$$

Setting $z(t) := \psi_\lambda(x(t))$ and taking to account of (7) we obtain for every $t\in[0,+\infty)$ that

$$\ddot{\mathscr{E}}(t) + \alpha(t)\dot{\mathscr{E}}(t) + \beta(t)\langle x(t)-x^*, x(t)-z(t)\rangle = \|\dot x(t)\|^2,$$

which, together with (13) implies

$$\ddot{\mathscr{E}}(t) + \alpha(t)\dot{\mathscr{E}}(t) + \xi\beta(t)\|x(t)-z(t)\|^2 \leq \|\dot x(t)\|^2,$$

where $\xi = 1 - \frac{\lambda L^2}{4\delta} > 0$. Again it follows from (7) that

$$\ddot{\mathscr{E}}(t) + \alpha(t)\dot{\mathscr{E}}(t) + \frac{\xi}{2}\beta(t)\|x(t) - z(t)\|^2 + \frac{\xi}{2\beta(t)}\|\ddot{x}(t) + \alpha(t)\dot{x}(t)\|^2 \leq \|\dot{x}(t)\|^2.$$

Applying (14), we obtain from the last inequality

$$\begin{aligned} &\ddot{\mathscr{E}}(t) + \alpha(t)\dot{\mathscr{E}}(t) + \xi_1\beta(t)\mathscr{E}(t) + \frac{\xi}{2\beta(t)}\|\ddot{x}(t)\|^2 \\ &+ \left(\frac{\xi\alpha^2(t)}{2\beta(t)} - 1\right)\|\dot{x}(t)\|^2 + \frac{\xi\alpha(t)}{\beta(t)}\langle\ddot{x}(t), \dot{x}(t)\rangle \leq 0, \end{aligned} \tag{17}$$

where $\xi_1 = \frac{\xi\lambda^2\delta^2}{(1+\lambda\delta+\lambda L)^2}$. Since $\frac{d}{dt}\|\dot{x}(t)\|^2 = 2\langle\ddot{x}(t), \dot{x}(t)\rangle$, setting for every $t \in [0, +\infty)$

$$a(t) := \xi_1\beta(t), \quad b(t) := \frac{\xi\alpha(t)}{2\beta(t)}, \quad c(t) := \frac{\xi\alpha^2(t)}{2\beta(t)} - 1, \quad u(t) := \|\dot{x}(t)\|^2$$

and eliminating a non-negative term $\frac{\xi}{2\beta(t)}\|\ddot{x}(t)\|^2$ in (17) we obtain

$$\ddot{\mathscr{E}}(t) + \alpha(t)\dot{\mathscr{E}}(t) + a(t)\mathscr{E}(t) + b(t)\dot{u}(t) + c(t)u(t) \leq 0. \tag{18}$$

Multiplying both side of (18) with $e^t > 0$, and using the identities

$$\begin{aligned} e^t\ddot{\mathscr{E}}(t) &= \frac{d}{dt}\left(e^t\dot{\mathscr{E}}(t)\right) - e^t\dot{\mathscr{E}}(t) \\ e^t\dot{\mathscr{E}}(t) &= \frac{d}{dt}\left(e^t\mathscr{E}(t)\right) - e^t\mathscr{E}(t) \\ e^t\dot{u}(t) &= \frac{d}{dt}\left(e^t u(t)\right) - e^t u(t) \end{aligned}$$

we obtain

$$\begin{aligned} &\frac{d}{dt}\left(e^t\dot{\mathscr{E}}(t)\right) + (\alpha(t) - 1)\frac{d}{dt}\left(e^t\mathscr{E}(t)\right) + (a(t) + 1 - \alpha(t))\, e^t\mathscr{E}(t) \\ &+ b(t)\frac{d}{dt}\left(e^t u(t)\right) + (c(t) - b(t))e^t u(t) \leq 0. \end{aligned} \tag{19}$$

From assumptions (i) and (iii) we have

$$a(t) + 1 - \alpha(t) \geq 0, \quad c(t) - b(t) \geq 0 \quad \forall t \in [0, +\infty).$$

Hence from (19) we can write

$$\frac{d}{dt}\left(e^t\dot{\mathscr{E}}(t)\right) + (\alpha(t) - 1)\frac{d}{dt}\left(e^t\mathscr{E}(t)\right) + b(t)\frac{d}{dt}\left(e^t u(t)\right) \leq 0. \tag{20}$$

Since

$$\begin{aligned} (\alpha(t) - 1)\frac{d}{dt}\left(e^t\mathscr{E}(t)\right) &= \frac{d}{dt}\left[(\alpha(t) - 1)\, e^t\mathscr{E}(t)\right] - \dot{\alpha}(t)e^t\mathscr{E}(t) \\ b(t)\frac{d}{dt}\left(e^t u(t)\right) &= \frac{d}{dt}\left(b(t)e^t u(t)\right) - \dot{b}(t)e^t u(t), \end{aligned}$$

we have from (20)

$$\frac{d}{dt}\left(e^t\dot{\mathscr{E}}(t)\right)+\frac{d}{dt}\left[(\alpha(t)-1)\,e^t\mathscr{E}(t)\right]-\dot{\alpha}(t)e^t\mathscr{E}(t)+\frac{d}{dt}\left(b(t)e^tu(t)\right)-\dot{b}(t)e^tu(t)\le 0. \quad (21)$$

By assumption (ii), $\dot{\alpha}(t)\le 0$ and $\dot{b}(t)\le 0$ for all $t\in[0,+\infty)$. Therefore, we have from (21)

$$\frac{d}{dt}\left[e^t\dot{\mathscr{E}}(t)+(\alpha(t)-1)\,e^t\mathscr{E}(t)+b(t)e^tu(t)\right]\le 0.$$

This implies that the function

$$t\to e^t\dot{\mathscr{E}}(t)+(\alpha(t)-1)\,e^t\mathscr{E}(t)+b(t)e^tu(t)$$

is monotonically decreasing, hence there exists $M>0$ such that

$$e^t\dot{\mathscr{E}}(t)+(\alpha(t)-1)\,e^t\mathscr{E}(t)+b(t)e^tu(t)\le M.$$

Since $b(t),u(t)\ge 0$, we get

$$\dot{\mathscr{E}}(t)+(\alpha(t)-1)\,\mathscr{E}(t)\le Me^{-t},$$

hence

$$\dot{\mathscr{E}}(t)+(\alpha-1)\,\mathscr{E}(t)\le Me^{-t}$$

for every $t\in[0,\infty)$. This implies that

$$\frac{d}{dt}\left[e^{(\alpha-1)t}\mathscr{E}(t)\right]\le Me^{(\alpha-2)t}$$

for every $t\in[0,\infty)$. By integration, we have

(i) if $1<\alpha<2$ then

$$0\le\mathscr{E}(t)\le\left(\mathscr{E}(0)+\frac{M}{\alpha-2}\right)e^{-(\alpha-1)t};$$

(ii) if $2<\alpha$ then

$$0\le\mathscr{E}(t)\le\mathscr{E}(0)e^{-(\alpha-1)t}+\frac{M}{2-\alpha}e^{-t}\le\left(\mathscr{E}(0)+\frac{M}{2-\alpha}\right)e^{-t};$$

(ii) if $\alpha=2$ then

$$0\le\mathscr{E}(t)\le(\mathscr{E}(0)+Mt)\,e^{-t}.$$

This implies that $x(t)$ converges exponentially to x^*.

Remark 3 As in [34, 10], we notice that it is easy to find functions α,β satisfying assumptions (i)-(iii) in Theorem 2. For example, if we choose $\alpha(t)=\alpha+\frac{1}{t+1}$ and $\beta(t)=\beta-\frac{1}{t+1}$ for all $t\in[0,+\infty)$ and $\alpha>1$, then (ii) is fulfilled. Assumption (iii) is equivalent to

$$\alpha^2-\alpha-\frac{2}{\xi}\beta+\frac{1}{(t+1)^2}+\frac{1}{t+1}+\frac{2}{\xi(t+1)}\ge 0,$$

which can be guaranteed if

$$\alpha^2-\alpha-\frac{2}{\xi}\beta\ge 0,$$

or

$$\alpha \geq \frac{1}{2}\left(1+\sqrt{1+\frac{8\beta}{\xi}}\right).$$

Assumption (i) reads as

$$\alpha + \frac{1}{t+1} \leq \xi_1\left(\beta - \frac{1}{t+1}\right) + 1,$$

i.e.,

$$\alpha \leq \xi_1\beta - \frac{\xi_1+1}{t+1} + 1,$$

which is guaranteed whenever

$$\alpha \leq \xi_1\beta - (\xi_1+1) + 1 = (\beta-1)\xi_1.$$

Therefore, to fulfill assumptions (i)-(iii), it is sufficient to choose β (large enough) and α satisfying

$$\frac{1}{2}\left(1+\sqrt{1+\frac{8\beta}{\xi}}\right) \leq \alpha \leq (\beta-1)\xi_1. \tag{22}$$

Also any constant functions $\alpha(t) = \alpha$ and $\beta(t) = \beta$ for all $t \in [0,+\infty)$ with α and β satisfying

$$\frac{1}{2}\left(1+\sqrt{1+\frac{8\beta}{\xi}}\right) \leq \alpha \leq \beta\xi_1 + 1,$$

fulfill assumptions (i)-(iii).

5 Linear convergence of a discrete system

Explicit discretization: A finite-difference scheme for (7) with respect to the time variable t, with stepsize $h_k > 0$, relaxation variable $\beta_k > 0$, damping variable $\alpha_k > 0$, and initial points x_0 and x_1 yields the following iterative scheme:

$$\frac{1}{h_k^2}(x_{k+1} - 2x_k + x_{k-1}) + \frac{\alpha_k}{h_k}(x_k - x_{k-1}) + \beta_k(y_k - \psi_\lambda(y_k)) = 0, \tag{23}$$

where y_k is an extrapolated point from x_k and x_{k-1} that will be chosen later (because $S := I - \psi_\lambda(\cdot)$ is Lipschitz continuous, there is some flexibility in this choice). We can write (23) as

$$x_{k+1} = x_k + (1-\alpha_k h_k)(x_k - x_{k-1}) - h_k^2\beta_k(y_k - \psi_\lambda(y_k)).$$

Setting $\rho_k = 1 - \alpha_k h_k, \eta_k = h_k^2\tau_k$ and choose $y_k := x_k + \rho_k(x_k - x_{k-1})$ we can write the above scheme as

$$\begin{cases} y_k = x_k + \rho_k(x_k - x_{k-1}) \\ x_{k+1} = (1-\eta_k)y_k + \eta_k\psi_\lambda(y_k), \end{cases}$$

which is a relaxed version of fixed point algorithm with additional inertial effects. In this section, we will investigate the convergence properties of (24). For the sake of simplicity, we only consider the case where all parameters are constants, i.e., $\rho_k = \rho$ and $\eta_k = \eta$ for all k.

$$\begin{cases} y_k = x_k + \rho(x_k - x_{k-1}) \\ x_{k+1} = (1-\eta)y_k + \eta\psi_\lambda(y_k). \end{cases} \tag{24}$$

We also make the following assumptions on the parameters:
(A1): $\theta > 0$ and $0 < \lambda < \frac{2\delta}{\theta L^2}$;
(A2):

$$0 < \eta < \min\left\{\frac{1}{1-q}, 1 + q\left(1 - \frac{1}{\theta}\right)\right\}$$

where $q := \frac{1}{1+\lambda(2\delta - \lambda\theta L^2)}$.
(A3):

$$0 \le \rho \le \min\left\{\frac{1-\eta+\eta q}{3}, \frac{(1-\frac{1}{\theta})q + 1 - \eta}{(1-\frac{1}{\theta})q + 1 + \eta}\right\}.$$

Remark 4 Note that (A2) allows over relaxation, i.e. $\eta > 1$, which can accelerates the convergence speed in certain examples. If $\eta = 1$ and (A1) is fulfilled, then (A2) holds for any $\theta > 1$ and (A3) becomes

$$0 \le \rho \le \min\left\{\frac{q}{3}, \frac{(1-\frac{1}{\theta})q}{(1-\frac{1}{\theta})q + 2}\right\}.$$

In this case, (24) reduces to fixed point method with inertial effect

$$\begin{cases} y_k = x_k + \rho(x_k - x_{k-1}) \\ x_{k+1} = \psi_\lambda(y_k). \end{cases}$$

Algorithm (24) is very general, in the sence that it includes many others algorithm as a special case. For example, when $\rho = 0$ and $\eta = 1$, it reduces to the fixed point algorithm studied in [23,**?**]; when $\rho = 0$ it reduces to the relaxed fixed point algorithm considered in [36]; when $\eta = 1$ it is the fixed point algorithm with inertial effect proposed in [17].

As in Section 4, we assume that f is δ-strongly pseudo-monotone on C and satisfies the Lipschitz-type condition (1) with modulus $L > 0$ on H. Using similar tools and techniques as in [34, 10], the linear convergence of scheme (24) is stated as follows.

Theorem 3 *Let the parameters λ, η, ρ be such that assumptions (A1), (A2) and (A3) are fulfilled. Then the sequence $\{x_k\}$ generated by* (24) *converges linearly to the unique solution x^* of $EP(f,C)$.*

Proof Setting $z_k := \psi_\lambda(y_k) \in C$ and substituting $y = x^* \in C$ into (8) we have

$$\lambda\left(f(y_k, x^*) - f(y_k, z_k)\right) \ge \langle y_k - z_k, x^* - z_k\rangle.$$

Combinning this inequality with the Lipschitz-type condition of f we obtain

$$\begin{aligned} \langle y_k - z_k, z_k - x^*\rangle &\ge \lambda\left(f(y_k, z_k) - f(y_k, x^*)\right) \\ &\ge -\lambda f(z_k, x^*) - \lambda L\|y_k - z_k\|\|z_k - x^*\|. \end{aligned} \tag{25}$$

Since $x^* \in Sol(f,C)$ and $z_k \in C$, it holds that $f(x^*,z_k) \geq 0$. Then by the δ-strong pseudomonoticity of f we have $f(z_k,x^*) \leq -\delta\|z_k - x^*\|^2$. It follows from (25) that

$$\langle y_k - z_k, z_k - x^* \rangle \geq \lambda\delta\|z_k - x^*\|^2 - \lambda L\|y_k - z_k\|\|z_k - x^*\|.$$

Hence

$$\begin{aligned}-2\lambda\delta\|z_k - x^*\|^2 + 2\lambda L\|y_k - z_k\|\|z_k - x^*\| &\geq 2\langle y_k - z_k, x^* - z_k\rangle \\ &= \|z_k - x^*\|^2 - \|y_k - x^*\|^2 + \|y_k - z_k\|^2,\end{aligned}$$

which implies

$$\begin{aligned}(1+2\lambda\delta)\|z_k - x^*\|^2 &\leq \|y_k - x^*\|^2 - \|y_k - z_k\|^2 + 2\lambda L\|y_k - z_k\|\|z_k - x^*\| \\ &\leq \|y_k - x^*\|^2 - \|y_k - z_k\|^2 + \frac{1}{2}\|y_k - z_k\|^2 + 2\lambda^2 L^2\|z_k - x^*\|^2 \\ &\leq \|y_k - x^*\|^2 - \frac{1}{2}\|y_k - z_k\|^2 + 2\lambda^2 L^2\|z_k - x^*\|^2,\end{aligned}$$

or equivalently,

$$\left[1 + 2\lambda(\delta - \lambda L^2)\right]\|z_k - x^*\|^2 \leq \|y_k - x^*\|^2 - \frac{1}{2}\|y_k - z_k\|^2.$$

Therefore,

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|(1-\eta)y_k + \eta z_k - x^*\|^2 \\ &= \|(1-\eta)(y_k - x^*) + \eta(z_k - x^*)\|^2 \\ &= (1-\eta)\|y_k - x^*\|^2 + \eta\|z_k - x^*\|^2 - \eta(1-\eta)\|y_k - z_k\|^2 \\ &\leq (1-\eta)\|y_k - x^*\|^2 + \frac{\eta}{1+2\lambda(\delta - \lambda L^2)}\|y_k - x^*\|^2 \\ &\quad - \frac{1}{2(1+2\lambda(\delta - \lambda L^2))}\|y_k - z_k\|^2 - \eta(1-\eta)\|y_k - z_k\|^2 \\ &= \kappa_1\|y_k - x^*\|^2 - \kappa_2\|y_k - z_k\|^2, \end{aligned} \tag{26}$$

where

$$\kappa_1 := 1 - \eta + \frac{\eta}{1+2\lambda(\delta - \lambda L^2)} = 1 - \eta + \eta q > 0$$

and

$$\kappa_2 := \frac{1}{2(1+2\lambda(\delta - \lambda L^2))} + \eta(1-\eta) = \frac{q}{2} + \eta(1-\eta) > 0$$

by assumption (A2).

Let us estimate the right hand side of (26). We have

$$\begin{aligned}\eta^2\|y_k - z_k\|^2 &= \|x_{k+1} - y_k\|^2 \\ &= \|x_{k+1} - x_k - \rho(x_k - x_{k-1})\|^2 \\ &= \|x_{k+1} - x_k\|^2 + \rho^2\|x_k - x_{k-1}\|^2 - 2\rho\langle x_{k+1} - x_k, x_k - x_{k-1}\rangle \\ &\geq \|x_{k+1} - x_k\|^2 + \rho^2\|x_k - x_{k-1}\|^2 - 2\rho\|x_{k+1} - x_k\|\|x_k - x_{k-1}\| \\ &\geq \|x_{k+1} - x_k\|^2 + \rho^2\|x_k - x_{k-1}\|^2 - \rho\|x_{k+1} - x_k\|^2 - \rho\|x_k - x_{k-1}\|^2 \\ &= (1-\rho)\|x_{k+1} - x_k\|^2 - \rho(1-\rho)\|x_k - x_{k-1}\|^2. \end{aligned} \tag{27}$$

In addition

$$\begin{aligned}\|y_k - x^*\|^2 &= \|(1+\rho)(x_k - x^*) - \rho(x_{k-1} - x^*)\|^2 \\ &= (1+\rho)\|x_k - x^*\|^2 - \rho\|x_{k-1} - x^*\|^2 + \rho(1+\rho)\|x_k - x_{k-1}\|^2.\end{aligned}$$

Combining (26), (27) and (28), we obtain

$$\begin{aligned}\|x_{k+1} - x^*\|^2 \le\ & \kappa_1(1+\rho)\|x_k - x^*\|^2 - \kappa_1\rho\|x_{k-1} - x^*\|^2 + \kappa_1\rho(1+\rho)\|x_k - x_{k-1}\|^2 \\ & + \frac{\kappa_2\rho(1-\rho)}{\eta}\|x_k - x_{k-1}\|^2 - \frac{\kappa_2(1-\rho)}{\eta}\|x_{k+1} - x_k\|^2.\end{aligned}$$

Since $\kappa_1 \in (0,1)$, the last inequality implies

$$\begin{aligned}& \|x_{k+1} - x^*\|^2 - \rho\|x_k - x^*\|^2 + \frac{\kappa_2(1-\rho)}{\eta}\|x_{k+1} - x_k\|^2 \\ \le\ & \|x_{k+1} - x^*\|^2 - \kappa_1\rho\|x_k - x^*\|^2 + \frac{\kappa_2(1-\rho)}{\eta}\|x_{k+1} - x_k\|^2 \\ \le\ & \kappa_1\left[\|x_k - x^*\|^2 - \rho\|x_{k-1} - x^*\|^2 + \frac{\kappa_2(1-\rho)}{\eta}\|x_k - x_{k-1}\|^2\right] \\ & - \left(\frac{\kappa_1\kappa_2(1-\rho)}{\eta} - \frac{\kappa_2\rho(1-\rho)}{\eta} - \kappa_1\rho(1+\rho)\right)\|x_k - x_{k-1}\|^2. \qquad (28)\end{aligned}$$

Setting

$$A_k := \|x_k - x^*\|^2 - \rho\|x_{k-1} - x^*\|^2 + \frac{\kappa_2(1-\rho)}{\eta}\|x_k - x_{k-1}\|^2,$$

we prove that $A_k \ge 0$ for all $k \ge 0$. Indeed, applying the following inequality with $\xi > 0$

$$a^2 + \xi b^2 \ge \min\{1,\xi\}(a^2 + b^2) \ge \min\{1/2, \xi/2\}(a-b)^2,$$

we obtain

$$\|x_k - x^*\|^2 + \frac{\kappa_2(1-\rho)}{\eta}\|x_k - x_{k-1}\|^2 \ge \min\left\{\frac{1}{2}, \frac{\kappa_2(1-\rho)}{2\eta}\right\}\|x_{k-1} - x^*\|^2$$

From assumption (A3) we have $\rho \le \frac{\kappa_2}{\kappa_2 + 2\eta}$, i.e. $\rho \le \frac{\kappa_2(1-\rho)}{2\eta}$ and

$$0 \le \rho \le \min\left\{\frac{\kappa_1}{3}, \frac{\kappa_2}{\kappa_2 + 2\eta}\right\} < \frac{1}{3}. \qquad (29)$$

Hence

$$A_k \ge \min\left\{\frac{1}{2}, \frac{\kappa_2(1-\rho)}{2\eta}\right\}\|x_{k-1} - x^*\|^2 - \rho\|x_{k-1} - x^*\|^2 \ge 0.$$

In addition, from (29) we have

$$\begin{aligned}\frac{\kappa_2\rho(1-\rho)}{\eta} &\le \frac{\kappa_1\kappa_2\rho(1-\rho)}{3\eta} \\ \kappa_1\rho(1+\rho) &< \kappa_1\frac{\kappa_2(1-\rho)}{2\eta}\frac{4}{3} = \frac{2\kappa_1\kappa_2(1-\rho)}{3\eta},\end{aligned}$$

which implies

$$\frac{\kappa_1\kappa_2(1-\rho)}{\eta} - \frac{\kappa_2\rho(1-\rho)}{\eta} - \kappa_1\rho(1+\rho) \ge \sigma > 0,$$

where $\sigma := \frac{2\kappa_1\kappa_2(1-\rho)}{3\eta} - \kappa_1\rho(1+\rho) > 0$. Therefore, (28) implies that

$$A_{k+1} \leq \kappa_1 A_k - \sigma\|x_k - x_{k-1}\|^2 \leq \kappa_1 A_k,$$

for all $k \geq 0$, from which we deduce

$$A_k \leq A_0\kappa_1^k \quad \text{and} \quad \sigma\|x_k - x_{k-1}\|^2 \leq \kappa_1 A_k \leq A_0\kappa_1^{k+1}$$

i.e., $\{A_k\}$ and $\{\|x_k - x_{k-1}\|\}$ converge linearly to 0 and this immediately implies that the sequence $\{x_k\}$ converges linearly to the unique solution x^*.

6 Numerical examples

In this section, we consider some numerical results to illustrate the global exponential stability of the unique equilibrium point of dynamical system (7) and the linear convergence of its discretization. Codes are implemented in MATLAB 2019b running on a Macbook Pro laptop with an Intel core CPU i7 at 2.6 GHz and 16 GB memory. The stopping condition is $\|x(t) - \psi_\lambda(x(t))\| \leq \varepsilon$ for all test problems, where $\varepsilon = 10^{-4}$. In our codes, the subproblem (6) is solved using the "fmincon" function from Matlab whenever there is no explicit solution.

Problem 1. The bifunction f of the equilibrium problem comes from the Cournot-Nash oligopolistic equilibrium model with the price and fee-fax functions being affine considered in [14, 29, 30]. The test problem is described as follows: assume that there are m companies that produce a commodity. Let x denote the vector whose entry x_j stands for the quantity of the commodity produced by company j. We suppose that the price $p_j(s)$ is a decreasing affine function of s with $s = \sum_{j=1}^m x_j$, i.e. $p_j(s) = \alpha_j - \beta_j s$, where $\alpha_j > 0, \beta_j > 0$. The profit made by company j is given by $f_j(x) = p_j(s)x_j - c_j(x_j)$, where $c_j(x_j)$ is the tax and fee for generating x_j. Suppose that $C_j = [x_j^{min}, x_j^{max}]$ is the strategy set of company j , then the strategy set of the model is $C := C_1 \times C_2 ... \times C_m$. Each company seeks to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies is a parametric input. A commonly used approach to this model is based upon the famous Nash equilibrium concept. We recall that a point $x^* \in C = C_1 \times C_2 ... \times C_m$ is an equilibrium point of the model if

$$f_j(x^*) \geq f_j(x^*[x_j]) \quad \forall x_j \in C_j, \forall j = 1, 2, ..., m,$$

where the vector $x^*[x_j]$ stands for the vector obtained from x^* by replacing x_j^* with x_j. By taking $f(x,y) := \phi(x,y) - \phi(x,x)$ with $\phi(x,y) := -\sum_{j=1}^m f_j(x[y_j])$, the problem of finding a Nash equilibrium point of the model can be formulated as follows: Find $x^* \in C$ such that

$$f(x^*, x) \geq 0, \forall x \in C.$$

Now, assume that the tax-fee function $c_j(x_j)$ is increasing and affine for every j . This assumption means that both of the tax and fee for producing a unit are increasing as the quantity of the production gets larger. In that case, the bifunction f can be formulated in the following form [29, 30]

$$f(x,y) = \langle Px + Qy + r, y - x\rangle \tag{30}$$

where $r \in \mathbb{R}^m$, and P and Q are two square matrices of order m . It was proved in [29] that the function f is strongly pseudo-monotone with modulus $\delta = \lambda_{min}(P - Q)$, the smallest eigenvalue of $P-Q$ and f satisfies the Lipschitz-type condition with modulus $L=\|P-Q\|$, f is convex in the second variable whenever Q is positive semidefinite. As in [29,31], in our test, the vector r and the matrices P and Q are chosen as follows:

$$r=\begin{bmatrix}1\\-2\\-1\\2\\-1\end{bmatrix};\quad P=\begin{bmatrix}3.1&2&0&0&0\\2&3.6&0&0&0\\0&0&3.5&2&0\\0&0&2&3.3&0\\0&0&0&0&3\end{bmatrix};\quad Q=\begin{bmatrix}1.6&1&0&0&0\\1&1.6&0&0&0\\0&0&1.5&1&0\\0&0&1&1.5&0\\0&0&0&0&2\end{bmatrix}.$$

The constraint set of this problem is defined by

$$C=\{x\in\mathbb{R}^5\,|\,\sum_{i=1}^{5}x_i\geq 0,\quad -5\leq x_i\leq 5,\quad i=1,2,3,4,5\},$$

and its solution x^* is given by

$$x^*=(-0.725388,0.803109,0.72000,-0.866667,0.200000)^T.$$

For this problem, $\delta=\lambda_{min}(P-Q)=0.7192$ and $L=\|P-Q\|=2.905$, and we choose

$$\theta=2;\quad \lambda=\frac{1.9*\delta}{\theta L^2}=0.0811;\quad \eta=0.9*\min\left\{\frac{1}{1-q},1+q\left(1-\frac{1}{\theta}\right)\right\}=1.3474;$$

$$\rho=\min\left\{\frac{1-\eta+\eta q}{3},\frac{\left(1-\frac{1}{\theta}\right)q+1-\eta}{\left(1-\frac{1}{\theta}\right)q+1+\eta}\right\}=0.0526.$$

Hence Assumptions (A1), (A2) and (A3) are fulfilled. Figure 1 displays the trajectories generated by the dynamical system (7). It is clear that $x(t)$ converges exponentially to the unique equilibrium point x^*.

Figure 2 compares the behavior of algorithm (24) in four cases when $x_0=(2,1,4,-1,-2)^T$: the relaxed-inertial algorithm (RI-FixedPoint: $\eta_k=1.3474,\rho_k=0.0526$) , the inertial algorithm (I-FixedPoint: $\eta_k=1,\rho_k=0.0526$), the relaxed algorithm (RI-FixedPoint: $\eta_k=1.3474,\rho_k=0$), and the fixed point algorithm (FixedPoint: $\eta_k=1,\rho_k=0$) for all k. Note that the inertial algorithm, the relaxed algorithm and the fixed point algorithm have been proposed in [17], [36] and [23,25], respectively. We can see that while all algorithms converge linearly, the relaxed-inertial method takes advantage in this example.

Problem 2. We consider the well-known Rosen-Suzuki optimization problem [18, Problem 43], where the cost function ϕ defined for $x=(x_1,x_2,x_3,x_4)$ by

$$\phi(x)=x_1^2+x_2^2+2x_3^2+x_4^2-5x_1-5x_2-21x_3+7x_4.$$

The constraint set is given by $C=\{x\in\mathbb{R}^4\,|\,g_i(x)\leq 0,i=1,2,3\}$, with

$$\begin{aligned}g_1(x)&=x_1^2+x_2^2+x_3^2+x_4^2+x_1-x_2+x_3-x_4-8,\\g_2(x)&=x_1^2+2x_2^2+x_3^2+2x_4^2-x_1-x_4-10,\\g_3(x)&=2x_1^2+x_2^2+x_3^2+2x_1-x_2-x_4-5.\end{aligned}$$

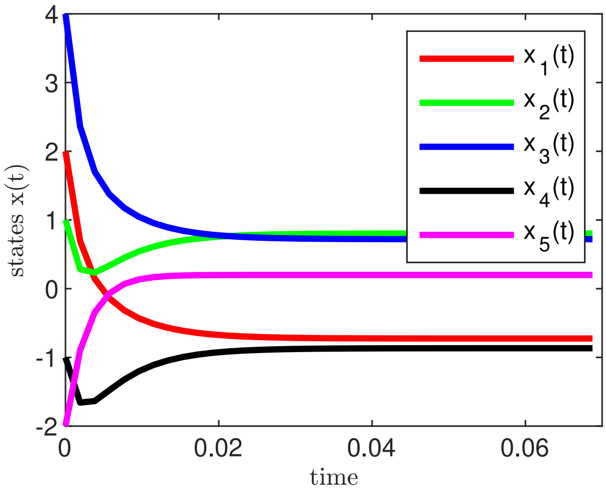

Fig. 1: The trajectory of dynamical system (7) for Problem 1 when $x_0 = (2,1,4,-1,-2)^T$, where $x_i(t)$ stands for the quantity of the commodity produced by company i at time t.

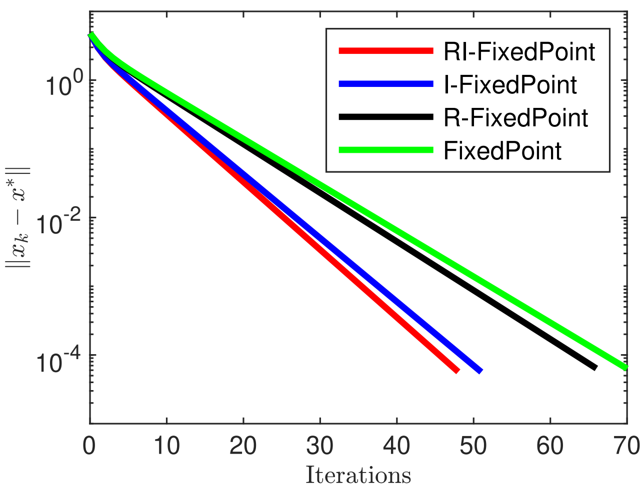

Fig. 2: Comparison of the relaxed inertial algorithm with the inertial algorithm, the relaxed algorithm and the fixed point algorithm when $x_0 = (2,1,4,-1,-2)^T$ for Problem 1.

This problem is reformulated as an equilibrium problem with function f defined for each $x, y \in \mathbb{R}^4$ by

$$f(x,y) = \langle \nabla \phi(x), y - x \rangle,$$

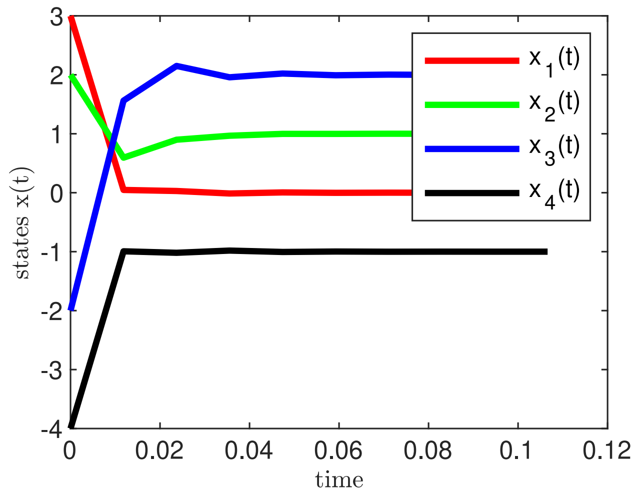

Fig. 3: The trajectory of dynamical system (7) for Problem 1 when $x_0 = (3, 2-2, -4)^T$.

here $\nabla\phi$ denotes the gradient of ϕ. For this problem, we have $\delta = 2$ and $L = 4$, and we choose

$$\theta = 2; \quad \lambda = \frac{1.9 * \delta}{\theta L^2} = 0.1188; \quad \eta = 0.9 * \min\left\{\frac{1}{1-q}, 1+q\left(1-\frac{1}{\theta}\right)\right\} = 1.3396;$$

$$\rho = \min\left\{\frac{1-\eta+\eta q}{3}, \frac{\left(1-\frac{1}{\theta}\right)q+1-\eta}{\left(1-\frac{1}{\theta}\right)q+1+\eta}\right\} = 0.0526.$$

The optimal solution of this problem is $x^* = (0, 1, 2, -1)^T$.

Figure 3 displays the trajectories generated by the dynamical system (7). It is clear that $x(t)$ converges exponentially to the unique equilibrium point x^*. Figure 4 presents the behavior of algorithm (24) in four cases when $x_0 = (1, -1, 2, -3)^T$: the relaxed-inertial algorithm (RI-FixedPoint: $\eta_k = 1.3396, \rho_k = 0.0526$), the inertial algorithm (I-FixedPoint: $\eta_k = 1, \rho_k = 0.0526$), the relaxed algorithm (RI-FixedPoint: $\eta_k = 1.3396, \rho_k = 0$), and the fixed point algorithm (FixedPoint: $\eta_k = 1, \rho_k = 0$) for all k. One can see that all algorithms converge linearly and the relaxed-inertial method outperforms the others.

Problem 3. Finally, we consider the strongly pseudo-monotone EP as in Example 1 with

$$f(x,y) = (R - \|x\|)\langle x, y-x\rangle \quad \text{and} \quad C = \{x \in H, \|x\| \leq r\}$$

with $R > r > R/2 > 0$. Note that the function f is neither monotone nor strongly monotone, but it is strongly pseudo-monotone with modulus $\delta = R - r$. We verify that f satisfies

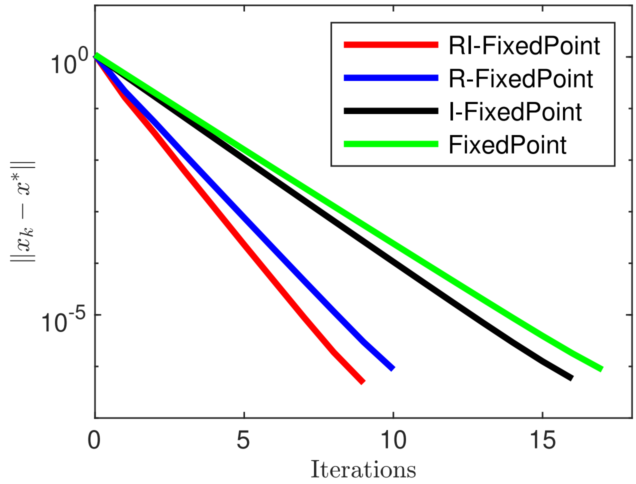

Fig. 4: Comparison of the relaxed inertial algorithm with the inertial algorithm, the relaxed algorithm and the fixed point algorithm for Problem 2 when $x_0 = (1,-1,2,-3)^T$.

Lipschitz-type condition (1) on C with $L = R + 2r$. Indeed let $x, y, z \in C$ we have

$$\begin{aligned} f(x,y) + f(y,z) - f(x,z) &= (R - \|x\|)\langle x, y - x\rangle + (R - \|y\|)\langle y, z - y\rangle - (R - \|x\|)\langle x, z - x\rangle \\ &= (R - \|x\|)\langle x, y - z\rangle - (R - \|y\|)\langle y, y - z\rangle \\ &= \langle Rx - Ry, y - z\rangle + \langle \|y\|y - \|x\|x, y - z\rangle \\ &\geq -R\|x - y\|\|y - z\| - \|\|y\|y - \|x\|x\|\|y - z\| \\ &\geq -R\|x - y\|\|y - z\| - 2r\|y - x\|\|y - z\| \\ &= -L\|x - y\|\|y - z\|. \end{aligned}$$

In our experiment we choose $H = \mathbb{R}^{100}$, $R = 8, r = 5$ and

$$\theta = 2; \quad \lambda = \frac{1.9 * \delta}{\theta L^2} = 0.0088; \quad \eta = 0.9 * \min\left\{\frac{1}{1-q}, 1 + q\left(1 - \frac{1}{\theta}\right)\right\} = 1.3488;$$

$$\rho = \min\left\{\frac{1 - \eta + \eta q}{3}, \frac{\left(1 - \frac{1}{\theta}\right)q + 1 - \eta}{\left(1 - \frac{1}{\theta}\right)q + 1 + \eta}\right\} = 0.0526.$$

The optimal solution of this problem is $x^* = 0 \in \mathbb{R}^{100}$. We do not display the trajectories generated by the dynamical system (7) as it is cumbersome. Figure 5 presents the behavior of algorithm (24) in four cases when x_0 is chosen randomly in $[-10, 10]^{100}$: the relaxed-inertial algorithm (RI-FixedPoint: $\eta_k = 1.3488, \rho_k = 0.0526$), the inertial algorithm (I-FixedPoint: $\eta_k = 1, \rho_k = 0.0526$), the relaxed algorithm (RI-FixedPoint: $\eta_k = 1.3488, \rho_k = 0$), and the fixed point algorithm (FixedPoint: $\eta_k = 1, \rho_k = 0$) for all k. It is clear that all algorithms converge linearly and the relaxed fixed point method is comparable with relaxed-inertial fixed point method and both outperform the others.

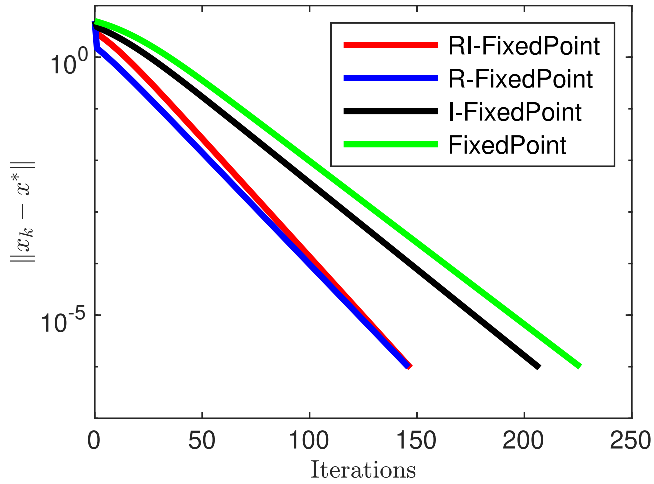

Fig. 5: Comparison of the relaxed inertial algorithm with the inertial algorithm, the relaxed algorithm and the fixed point algorithm for Problem 3 with a random starting point.

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Conflict of Interest

The authors declare that they have no conflict of interest.

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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