Solving linear rational expectations models in the presence of structural change: Some extensions

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Abstract

Standard solution methods for linear rational expectations models assume a time-invariant structure. Recent work has gone beyond this by formulating solution methods for linear rational expectations models subject to structural changes, such as parameter shifts and policy reforms, that are announced in advance. This paper contributes to this literature by presenting solutions for some cases – imperfectly credible policy reforms; delayed announcement to some fraction of agents; and indeterminacy of the terminal solution (multiple equilibria) – that received little attention so far. These solutions are illustrated using several applications, including a New Keynesian model in which the Taylor principle is not satisfied by the terminal structure.

1 Introduction

Standard methods for solving linear rational expectations models, such as Blanchard and Kahn (1980), Anderson and Moore (1985), Binder and Pesaran (1997), King and Watson (1998), Uhlig (1999), Klein (2000) and Sims (2002), assume a time-invariant structure – that is, the parameters of the system are taken to be constant. As a result, such methods cannot be used to study occasional changes in structure, such as parameter shifts or policy reforms, which may be partially or fully anticipated.[1]

There are many reasons the structure of economic models might shift. A short list would include implementation of new policies like forward guidance or quantitative easing; reforms to current policies, such as inflation targets, pensions or taxes, which may be phased in gradually; and shifts in technology, competitiveness of product markets, or the reaction functions of policymakers. For example, Clarida et al. (2000) and Lubik and Schorfheide (2004) find

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[1] By contrast, Markov-switching linear rational expectations models (see Davig and Leeper (2007) or Farmer et al. (2009)) are used to study unanticipated recurrent events with a known probability distribution.

evidence of structural change in the Federal Reserve's interest rate rule, in the form of a more aggressive response to inflation in the post-Volcker period. From a policymaker perspective, the consequences of policy reforms may differ substantially depending on whether the reform in question is announced or unanticipated (see Mertens and Ravn (2012)). Hence, accurate policy evaluation requires solution methods that can deal with structural changes.

Recent work has moved in this direction by formulating solution methods for linear rational expectations models subject to structural change. This paper contributes to the literature by extending the solution of Kulish and Pagan (2017) to allow for some cases – imperfectly credible reforms; delayed announcement to some fraction of agents; indeterminacy of the terminal solution (i.e. multiple equilibria) – that received little attention so far. Thus, the paper is essentially a methodological contribution to the structural change literature.

The recent literature on structural change began with Cagliarini and Kulish (2013). Building on the time-invariant solution of Sims (2002), they set out a general method for solving linear rational expectations models subject to anticipated structural changes and establish conditions on existence and uniqueness of solutions. Kulish and Pagan (2017) present an alternative recursive solution based on the method of undetermined coeffficients. They show that since the model solution is given by a time-varying VAR, a likelihood function can be constructed to allow estimation of parameters and the dates of structural breaks.

Applications of these methods include Jones and Kulish (2013), who study the impact of unconventional monetary policies that involve announcements about the future path of short-term and long-term interest rates (forward guidance); Kulish et al. (2017), who estimate expected durations at the zero lower bound using U.S. data; and Gibbs and Kulish (2017), who study the output costs of disinflations using a model of 'unanchored' expectations due to adaptive learning by some agents. These works suggest that credible pre-announced policy changes may significantly improve economic outcomes relative to standard policies.

The present paper contributes to this literature by extending the recursive solution method in Kulish and Pagan (2017) to some additional cases that received little attention thus far: delayed announcement to a fraction of agents; imperfectly credible policy reforms; and structural changes involving indeterminacy (i.e. multiple equilibria) at the final stage.

Our delayed announcement approach allows some fraction of agents to be uninformed, for some time, about a future structural change, such as a policy reform. This can be motivated by situations such as pre-planned policies, which are known to agents in policy circles (private information) before they are made known (announced) to the general public. In particular, our approach has the informed fraction of agents forming model-consistent expectations, which take into account the ignorance of the uninformed agents. By contrast, previous work has considered uninformed agents in isolation (see Kulish and Pagan (2017)). Our framework also nests the special case where announcement dates (for the entire population) are controlled by a single parameter, namely, the number of periods in advance that the structural change is announced. We illustrate the latter using a pension reform example.

Our second extension relates to imperfect credibility, whereby some fraction of agents do not believe fully in the announced structure. A standard approach in the literature has been to model the expectations of disbelieving agents using simple rules of thumb such as previous policy targets or realized values of a subset of variables (see Goodfriend and King (2005), Nicolae and Nolan (2006), Ascari and Ropele (2013)). By comparison, we allow disbelieving agents form expectations that reflect their doubts about the announced future structure. In particular, following announced changes in policy regimes, such as the shift to a new policy or a lower inflation target, agents make forecasts based on an alternative sequence of structures for which policies may be expected to end prematurely, last longer than announced, or be permanently reversed in the future. To illustrate the distinction, we show that the standard approach is nested by our framework when some fraction of agents have expectations given by a VAR in which the matrices are specified by the researcher.

Finally, we show how the solution method of Kulish and Pagan (2017) can be extended to models with an indeterminate terminal solution. To do so, we use the approach in Farmer et al. (2015) which re-casts an indeterminate model, in which the solutions depend on sunspots, as one with fundamental shocks that can be solved using standard methods. Our extension using this method allows the terminal regime to have a particular sunspot solution and for agents to anticipate this in advance and coordinate their expectations on the sunspot, such that the transitional dynamics can be simulated using the method of Kulish and Pagan (2017).[2] As an application, we study a New Keynesian model in which agents anticipate that the Taylor principle is violated by the terminal structure.

We provide several other applications which illustrate our extensions by comparing with existing approaches or the benchmark of all agents with rational expectations. Since the extensions we consider are policy-relevant and have already been studied in fixed-structure models, they should be of interest to a wide audience, such as researchers at policy institutions who would like to take methods 'off the shelf' and use them in a variety of applications, including medium or large-scale DSGE models.

The paper proceeds as follows. Section 2 outlines rational expectations solutions in the absence of structural change. Section 3 sets out a benchmark model with structural change and derives a baseline result that we draw on in Section 4, where we present our extensions. Section 5 presents several numerical applications. Finally, Section 6 concludes.

[2]In independent work, Gibbs and McClung (2020) also present a method for solving models with indeterminate terminal solutions, but using the results in Bianchi and Nicolò (2021). Both their approach and ours are useful for solving models with anticipated structural change and an indeterminate terminal solution.

2 Solutions in the absence of structural change

Following Binder and Pesaran (1997), a linear rational expectations model of n equations and time-invariant structure may be written in the form:

$$B_1 x_t = B_2 E_t x_{t+1} + B_3 x_{t-1} + B_4 e_t + B_5, \quad \forall t \geq 0 \tag{1}$$

where x_t is an $n \times 1$ vector of endogenous state and jump variables, E_t is the conditional expectations operator, and e_t is an $m \times 1$ vector of white noise shocks with $E_t[e_{t+1}] = 0_{m\times 1}$. Without loss of generality, the covariance matrix of e_t is set equal to the identity matrix I_m. Note that serially correlated exogenous processes can be included in x_t.

Matrices B_i, $i \in [5]$, contain the model parameters. The B_i, $i \in \{1, 2, 3\}$, are $n \times n$ matrices, B_4 is an $n \times m$ matrix, and B_5 is an $n \times 1$ vector of intercepts. Time is discrete and starts at $t = 0$; hence $t \in \mathbb{N}$. As shown in Binder and Pesaran (1997), the formulation (1) is quite general as it can accommodate multiple leads and lags of the endogenous variables.

If a fundamental solution to system (1) exists, it will be a VAR of the form:

$$x_t = \Omega x_{t-1} + \Gamma e_t + \Psi \tag{2}$$

where Ω, Γ, Ψ are $n \times n$, $n \times m$ and $n \times 1$ matrices, respectively.

Given (2) and $E_t[e_{t+1}] = 0_{m\times 1}$, we have $E_t x_{t+1} = \Omega x_t + \Psi$. Substituting this expression into (1) determines the solution matrices in (2) as

$$\Omega = (B_1 - B_2\Omega)^{-1} B_3, \quad \Gamma = (B_1 - B_2\Omega)^{-1} B_4, \quad \Psi = (B_1 - B_2\Omega)^{-1}(B_2\Psi + B_5) \tag{3}$$

provided $\det[B_1 - B_2\Omega] \neq 0$.[3]

Note that (3) implies $B_2\Omega^2 - B_1\Omega + B_3 = 0_{n\times n}$ and so determines Ω. Once Ω is found, Γ and Ψ can be determined, providing a solution to the model.[4] It is now standard to solve the model using the methods in Blanchard and Kahn (1980), King and Watson (1998), Uhlig (1999), Klein (2000) and Sims (2002). There are also recursive methods that solve for the matrices Ω, Γ, Ψ, such as Binder and Pesaran (1997) and Cho and Moreno (2011). We assume readers have access to these standard solution methods.

[3]If $\det[B_1 - B_2\Omega] \neq 0$ and B_1 non-singular then $(B_1 - B_2\Omega)^{-1} = (I_n - B_1^{-1}B_2\Omega)^{-1}B_1^{-1}$. The solution matrices in (3) may then be written $\Omega = (I_n - A\Omega)^{-1}B$, $\Gamma = (I_n - A\Omega)^{-1}C$, $\Psi = (I_n - A\Omega)^{-1}(A\Psi + D)$, where $A = B_1^{-1}B_2$, $B = B_1^{-1}B_3$, $C = B_1^{-1}B_4$, and $D = B_1^{-1}B_5$ (see e.g. Kulish and Pagan (2017)).

[4]Note that Ψ is determined by Eq. (3) as $\Psi = (I_n - F)^{-1}(B_1 - B_2\Omega)^{-1}B_5$, where $F = (B_1 - B_2\Omega)^{-1}B_2$.

3 Solutions in the presence of structural change

3.1 Model

We now consider a multivariate rational expectations model as above, except that the structural matrices of parameters may change over time. To incorporate structural change, we assume two possible regimes, which we dub the "reference regime" and "alternative regime".

The reference reference is decribed by Eq. (4):

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5 \tag{4}$$

The alternative regime is described by Eq. (5):

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \tag{5}$$

The matrices $\overline{B}_i, \tilde{B}_i$, $i \in [5]$, have the same dimensions as in (1). For existence of a solution under structural change, we will require existence of a fixed-structure *terminal solution* that corresponds to the final structure. We assume, without loss of generality, that the final structure is the alternative regime, (5).[5]

In typical applications, one of the intercept matrices may be zero, as DSGE models are typically log-linearized around a non-stochastic steady state (see Uhlig (1999)).[6] At any given date t, either the reference regime applies or the alternative regime does. Given mutually exclusive regimes, we introduce an *indicator variable* $\mathbb{1}_t$ equal to 1 if the reference regime applies in period t and 0 if the alternative regime is in place. Our model is then:

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 0 \tag{6}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i \; \forall i \in [5]$ and $x_{-1} \in \mathbb{R}^n$ is given.

The information set at time t includes all current and future values of the indicator variable and all current and past values of the endogenous and exogenous variables. The indicator variable is an exogenous *predetermined* sequence $\{\mathbb{1}_t\}_{t=0}^{\infty}$; note that a change in its value indicates that a structural change takes place.

3.2 Solving the model

Definition 1 *A solution to the rational expectations model with structural change* (6) *is a function* $f : x_{t-1} \times e_t \to x_t$ *such that the system in* (6) *holds for all* t*, given the current evaluation of the indicator variable* $\mathbb{1}_t$ *and its known future values* $\mathbb{1}_{t+1}, \mathbb{1}_{t+2}, \ldots$*.*

[5] There is no loss of generality since either structure may be labelled as the alternative regime.

[6] If one of the regimes has a different steady state, intercepts will be present when the log-linearized model is written in terms of deviations x_t from a common steady state. See, e.g., Guerrieri and Iacoviello (2015).

An alternative way of characterizing a solution is in terms of a set of matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=0}^{\infty}$ that generalize the constant-coefficient decision rules of a linear rational expectations model:

$$x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t \tag{7}$$

where Ω_t is an $n \times n$ matrix, Γ_t is an $n \times m$ matrix and Ψ_t is an $n \times 1$ vector, and the t subscript indicates that the matrices are in general time-varying. Following Kulish and Pagan (2017), the matrices $\Omega_t, \Gamma_t, \Psi_t$ are determined recursively by simple formulas which are well-defined provided that a series of regularity conditions are met.

There are two key requirements for existence of a solution:

(i) *Existence of a rational expectations solution must hold at the terminal structure.* The corresponding solution matrices $\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$ (say) are used as inputs in the computation of the solution during the transition.

(ii) *A series of regularity conditions* $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ *must be met* for $t = 0, ..., \tilde{T}$, where $\tilde{T} + 1$ is the date at which the terminal structure is reached. If the regularity conditions are not met, then a solution will not exist even if requirement (i) is satisfied.

Requirement (i) is needed because the solution relies on 'backward induction' from a terminal solution. A terminal solution can be found using standard methods, such as Binder and Pesaran (1997), Sims (2002) or Dynare (Adjemian et al. (2011)). Conveniently, Jones (2016) shows that Dynare can be used to numerically log-linearize a nonlinear calibrated model and 'build' the corresponding structural matrices $\overline{B}_i, \tilde{B}_i$, $i \in [5]$. Existence of a terminal solution is necessary but not sufficient for existence of a solution because the regularity conditions in (ii) are needed to ensure that a solution exists during the transition.

Uniqueness is discussed further once the benchmark solution is presented. For the moment we note that Cagliarini and Kulish (2013) show that existence and uniqueness of a terminal solution are necessary conditions for the existence and uniqueness of the sequences $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=0}^{\infty}$ in (7), and they provide rank conditions that can be used to check for existence and uniqueness of the transition path $\{x_t\}_{t=0}^{\infty}$ under an arbitrary finite sequence of structural changes. Note that existence and uniqueness of a terminal solution can also be checked using standard methods such as Blanchard and Kahn (1980) and Sims (2002).

For convenience, we assume there is a unique stable terminal solution (Assumption 1). We maintain this assumption throughout the analysis unless otherwise specified.[7]

Assumption 1 *There exists a unique stable terminal solution* $x_t = \tilde{\Omega} x_{t-1} + \tilde{\Gamma} e_t + \tilde{\Psi}$, $t > \tilde{T}$. *We assume, without loss of generality, that the terminal structure is the alternative regime, and hence* $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_3$, $\tilde{\Gamma} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_4$, $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}(\tilde{B}_2\tilde{\Psi} + \tilde{B}_5)$.

[7] In particular, we relax this assumption when we consider indeterminate terminal solutions (Section 4.3).

3.3 Benchmark solution

We first study the benchmark solution in Kulish and Pagan (2017). Hence, at all dates $t \geq 0$ the future structure is known to agents.[8] We formalize this assumption in Definition 2.

Definition 2 *Structural change is said to be anticipated if all future values of the indicator variable, $\{\mathbb{1}_{t+j}\}_{j=1}^{\infty}$, are part of agents' information set for all $t \geq 0$. In this case, agents have full information about future structural changes and are said to be informed.*

By Definition 2, expectations under anticipated structural change satisfy

$$E_t\left[x_{t+j}\right] = E\left[x_{t+j}|I_t \cup \{\mathbb{1}_{t+1}, \mathbb{1}_{t+2}, ...\}\right], \quad \forall j > 0 \tag{8}$$

where $I_t = \{x_t, x_{t-1}, ..., e_t, e_{t-1}, ..., \mathbb{1}_t\}$ is the information set *excluding* the future structure.

Suppose that in periods $t \in [0, \tilde{T}]$ there is an arbitrary sequence of structures involving the reference and alternative regime.[9] The terminal structure is the alternative regime and is reached at $t = \tilde{T} + 1$ (see Assumption 1). The sequence of structures is reflected in the known path of the indicator variable, $\{\mathbb{1}_t\}_{t=0}^{\tilde{T}}$ and $\mathbb{1}_t = 0$, $\forall t > \tilde{T}$ (see Definition 2).

In the form of (6), the system to be solved is:

$$\begin{cases} B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, & 0 \leq t \leq \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T}. \end{cases} \tag{9}$$

For all $t > \tilde{T}$, the alternative regime is in place. By Assumption 1, the unique terminal solution is $x_t = \tilde{\Omega} x_{t-1} + \tilde{\Gamma} e_t + \tilde{\Psi}$, $\forall t > \tilde{T}$. Thus, the remaining system to be solved is:

$$\begin{aligned} &B_{1,0}x_0 = B_{2,0}E_0 x_1 + B_{3,0}x_{-1} + B_{4,0}e_0 + B_{5,0} \\ &\vdots \\ &B_{1,\tilde{T}}x_{\tilde{T}} = B_{2,\tilde{T}}E_{\tilde{T}}x_{\tilde{T}+1} + B_{3,\tilde{T}}x_{\tilde{T}-1} + B_{4,\tilde{T}}e_{\tilde{T}} + B_{5,\tilde{T}} \end{aligned} \tag{10}$$

where $E_{\tilde{T}}x_{\tilde{T}+1} = \tilde{\Omega} x_{\tilde{T}} + \tilde{\Psi}$. We can thus state the following result, which we draw on later.

Proposition 1 Consider the model (9)–(10). The solution is given by

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 0 \leq t \leq \tilde{T} \\ \tilde{\Omega} x_{t-1} + \tilde{\Gamma} e_t + \tilde{\Psi} & \text{for } t > \tilde{T} \end{cases} \tag{11}$$

[8] Note there is no loss of generality since if information about the future structure arrives at some arbitrary date $\tilde{t} > 0$ we may simply relabel the date 0 solution and apply the solution formulas from date $\tilde{t}$ onwards.

[9] To be both succinct and clear, we write $t \in [0, \tilde{T}]$ (with the understanding that t is discrete) rather than the more formal $t \in [[0, \tilde{T}]]$ or the longhand $t \in \{0, ..., \tilde{T}\}$.

where for $t = 0, ..., \tilde{T}$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t}, \tag{12}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}\Psi_{t+1} + B_{5,t}), \tag{13}$$

provided the following regularity condition holds: $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0 \; \forall t \in [0, \tilde{T}]$, and the terminal matrices $\Omega_{\tilde{T}+1} = \tilde{\Omega}$, $\Gamma_{\tilde{T}+1} = \tilde{\Gamma}$, $\Psi_{\tilde{T}+1} = \tilde{\Psi}$ satisfy Assumption 1.

Proof. See the Appendix. ■

Proposition 1 describes the benchmark solution (if it exists). The recursive solution matrices $\Omega_t, \Psi_t, \Gamma_t$ are equivalent to those given in Kulish and Pagan (2017); to see this, note that if $B_{1,t}$ is invertible then $\Omega_t = (I_n - A_t\Omega_{t+1})^{-1}B_t$, $\Gamma_t = (I_n - A_t\Omega_{t+1})^{-1}C_t$ and $\Psi_t = (I_n - A_t\Omega_{t+1})^{-1}(A_t\Psi_{t+1} + D_t)$, where $A_t := B_{1,t}^{-1}B_{2,t}$, $B_t := B_{1,t}^{-1}B_{3,t}$, $C_t := B_{1,t}^{-1}B_{4,t}$ and $D_t := B_{1,t}^{-1}B_{5,t}$. Note that if the structure were fixed at $B_{i,t} = \tilde{B}_i$ for all $i \in [5]$, $t \in [0, \tilde{T}]$ then the solution in Proposition 1 collapses to the fixed structure solution in (2)–(3).

Given our assumption of a unique terminal solution $(\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi})$, Proposition 1 shows that a unique solution $\{x_t\}_{t=0}^{\infty}$ exists provided the regularity (invertibility) conditions are satisfied; see Kulish and Pagan (2017) and Jones (2017). Note this result does not depend on the specifics of the regimes. For example, even if the structure for $t \in [0, \tilde{T}]$ implies indeterminacy in a fixed structure setting, the transition path is unique regardless of how large $\tilde{T}$ is.

A number of cases of interest are nested by Proposition 1. These include *permanent* anticipated structural change, such as entry into a monetary union ($B_{i,t} = \overline{B}_i$, $\forall t \in [0, \tilde{T}]$); *temporary* anticipated structural change implemented at some date $T+1$ and later reversed, such as forward guidance ($B_{i,t} = \tilde{B}_i \; \forall t \in [0, T]$, $B_{i,t} = \overline{B}_i \; \forall t \in [T+1, \tilde{T}]$); and the case of anticipated ('news') shocks.[10] The same formulas also apply for more than two regimes, for example, policy reforms phased in gradually in several steps.[11] In the next section we extend the benchmark model while preserving the recursive solution in Proposition 1.

Example 1 A Cagan model with a permanent anticipated monetary expansion:

$$p_t = \frac{1}{1+\eta}m_t + \frac{\eta}{1+\eta}E_t p_{t+1}, \qquad m_t = \begin{cases} \overline{m} & 0 \leq t \leq \tilde{T} \\ \overline{m}' & \forall t > \tilde{T} \end{cases}$$

where $\eta > 0$ and p_t, m_t are the (log) price level and money supply.

[10] To see this, suppose that $\forall t \in [0, \tilde{T}]$ the shock vector is given by $e_t = e_t^u + e_t^a$, where e_t^u is the unpredictable component of shocks ($E_t[e_{t+1}^u] = 0_{m\times 1}$) and e_t^a is the anticipated ('news') component ($E_t[e_{t+1}^a] = e_{t+1}^a$). Then the model $\forall t \in [0, \tilde{T}]$ is $B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t^u + \hat{B}_{5,t}$, where $\hat{B}_{5,t} := B_{4,t}e_t^a + B_{5,t}$.

[11] Suppose there are S different regimes. Then each regime $s \in [S]$ has its own indicator $\mathbb{1}_t^s$ and structure B_i^s ($i \in [5]$). The model for $t \in [0, \tilde{T}]$ is $B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}$ where $B_{i,t} = \sum_{s=1}^{S} \mathbb{1}_t^s B_i^s$.

Letting $x_t = [p_t]$, $B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}$, with $e_t = [0]$, $B_{1,t} = [1]$, $B_{2,t} = [\frac{\eta}{1+\eta}]$, $B_{i,t} = [0]$ for $i = 3, 4$ and $B_{5,t} = [\frac{\overline{m}}{1+\eta}]$ for $t \in [0, \tilde{T}]$, $B_{5,t} = [\frac{\overline{m}'}{1+\eta}]$ $\forall t > \tilde{T}$. Proposition 1 gives the same solution as in Rogoff and Obstfeld (1996) (Ch. 8):

$$p_t = \begin{cases} \Psi_t = \overline{m} + \left(\frac{\eta}{1+\eta}\right)^{\tilde{T}+1-t} (\overline{m}' - \overline{m}) & 0 \leq t \leq \tilde{T} \\ \tilde{\Psi} = \overline{m}' & \forall t > \tilde{T} \end{cases}$$

where $\Psi_{\tilde{T}} = \frac{\eta}{1+\eta}\tilde{\Psi} + \frac{\overline{m}}{1+\eta}$ and $\Psi_t = \frac{\eta}{1+\eta}\Psi_{t+1} + \frac{\overline{m}}{1+\eta}$, $\forall t < \tilde{T}$. The price level responds at $t = 0$ and accelerates toward its new value $\overline{m}'$ as implementation date $\tilde{T}+1$ approaches.

4 Extensions

We now consider three extensions of the baseline model: coexistence of informed and uninformed agents; imperfect credibility about announced changes in structure; and structural changes for which the terminal regime has multiple solutions that depend on sunspots.

4.1 Coexistence of informed and uninformed agents

As a first extension, we let some fraction of *uninformed* agents face an announcement lag. This is intended to capture the situation that some agents, such as those working in policy circles or government departments, may have advance information about future structural changes, such as policy reforms, not yet released to the general public. The present section thus builds on previous work that studied rational expectations and 'wrong beliefs' separately (Kulish and Pagan (2017): Sec. 3.2). As a special case, this approach also provides a formal framework for studying different announcement dates to the entire population.

Definition 3 *Consider a structural change that starts in period $T \in (0, \tilde{T}]$ and is complete by period $\tilde{T}+1$. Agents are said to be uninformed if the structural changes in periods T to $\tilde{T}+1$ are revealed to them in period $T-K$, where $K \in [0, T)$, and their expectations in periods $0 \leq t < T-K$ assume the current (original) structure will remain in place forever.*

By Definition 3, the expectations of uninformed agents satisfy

$$\hat{E}_t [x_{t+1}] = \begin{cases} E[x_{t+1}|I_t] & \text{for } 0 \leq t < T-K \\ E_t x_{t+1} & \text{for } t \geq T-K \end{cases} \tag{14}$$

where $I_t = \{x_t, x_{t-1}, ..., e_t, e_{t-1}, ..., \mathbb{1}_t\}$ as in (8).[12]

[12]Our choice of notation here reflects the fact that the uninformed initially have no information about future structural changes so that $\forall t < T-K$, $\hat{E}_t [x_{t+1}] = E[x_{t+1}|I_t \cup \emptyset] = E[x_{t+1}|I_t]$.

Note that uninformed agents have *no* knowledge of future structural changes in the first $T-K$ periods, but have full information from date $T-K$ onwards. In what follows, we denote the fraction of informed (i.e. rational) agents by $\lambda \in (0,1]$ (given and user-specified).

Without loss of generality, the initial structure in periods $t \in [0,T)$ is assumed to be the reference regime and, as in Proposition 1, the terminal structure for $t > \tilde{T}$ is the alternative regime. During the periods $t \in [T, \tilde{T}]$, any sequence of regimes is permitted, and thus generality of structural changes is preserved.[13] The informed agents ('insiders') are aware of all changes in structure from period $t=0$, whereas the uniformed agents ('outsiders') find out only in period $t = T-K$ and so receive the information with a $T-K$ period lag.

Our model for all $t \geq 0$ is

$$B_{1,t}x_t = B_{2,t}\tilde{E}_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \tag{15}$$

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)\hat{E}_t x_{t+1} \tag{16}$$

where $B_{i,t} = \overline{B}_i$ for $t \in [0,T)$ and $B_{i,t} = \tilde{B}_i$ for all $t > \tilde{T}$, $i \in [5]$.

The fraction λ of informed agents form their expectations with full knowledge of the future structure (see Def. 2) and take into account the expectations of the uninformed agents; hence their expectations are model-consistent in all periods, unlike those of the uninformed agents (cf. Def. 3). The specification in (15)–(16) nests the cases of all informed agents when $\lambda = 1$ (see Proposition 1) and all uninformed agents when $\lambda = 0$. In the special case $K = 0$, the structural reform is entirely unannounced to the uninformed agents; if $K = T-1$ the uninformed get the announcement with a one period lag relative to the 'insiders'.

In periods $0 \leq t < T-K$, the uninformed agents think the current (reference) regime will prevail in all future periods. Hence, their expectations are $\hat{E}_t x_{t+1} = \overline{\Omega}x_t + \overline{\Psi}$, $\forall t \in [0, T-K)$. Thus, by (15)–(16), and analogous to (9), our model is given by

$$\begin{cases} \hat{B}_1 x_t = \hat{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \hat{B}_5, & t \in [0, T-K) \\ B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, & t \in [T-K, \tilde{T}] \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T} \end{cases} \tag{17}$$

where $\hat{B}_1 := \overline{B}_1 - \overline{B}_2(1-\lambda)\overline{\Omega}$, $\hat{B}_2 := \overline{B}_2\lambda$ and $\hat{B}_5 := \overline{B}_2(1-\lambda)\overline{\Psi} + \overline{B}_5$.

Corollary 1 Consider the model in (17). The solution follows Proposition 1, except that $\forall t \in [0, T-K)$ the recursive formulas for $\Omega_t, \Gamma_t, \Psi_t$ have $B_{3,t} = \overline{B}_3$, $B_{4,t} = \overline{B}_4$, and the matrices $B_{1,t}, B_{2,t}, B_{5,t}$ must be replaced with the matrices $\hat{B}_1, \hat{B}_2, \hat{B}_5$.

Proof. It follows from Proposition 1 with an appropriate relabelling of matrices. ■

[13] Note that since period T is the first period of structural change, the structure in this period should be the alternative regime or else, by definition, the structural change would not have started in period T.

Now consider the special case $\lambda = 0$, nested by the above. This corresponds to all agents having delayed information about a future structural change. To make this concrete, suppose the government plans a structural reform starting at $t = T$ and has the option to make this information public (at some date $T - K$) or to keep it private until date T. If $K = 0$, the reform is unannounced, whereas if $K \in (0, T-1]$ agents find out about the reform in advance, but with a delay of $T - K$ periods. Note that $K = T$ corresponds to full information as in Proposition 1 (i.e. immediate disclosure).

One potential application of the above framework is *timing of announcements*. Since different announcement dates will generally imply different paths for the endogenous variables, policymakers face the question: what is the optimal reform announcement date? We provide an application of optimal announcement dates (pension reform) in Section 5.2.

Henceforth, rather than presenting several further corollaries, we just describe how the solutions presented relate to Proposition 1.

4.2 Imperfect credibility

A reform is said to be *imperfectly credible* if some agents think structural changes will *not* be implemented as announced. Imperfect credibility concerns were an important consideration in the Bank of Canada's decision to renew its inflation target in 2011, rather than switch to a price-level target (see Ambler (2014)). We first show that past approaches to imperfect credibility are nested by the structural change solution before considering two new cases.

4.2.1 Past approaches to imperfect credibility

One strand of literature considers settings where the structure *is* implemented as announced, but imperfect credibility arises because some fraction of agents doubt whether the actual structure will match the announced one. Intuitively, this approach isolates the specific impact of policymakers not being believed. We focus here on this approach to imperfect credibility; hence our analysis does not relate to some alternative approaches.[14]

We assume any changes in the structure $B_{i,t}$ ($i \in [5]$) are complete by date $T+1$, but that imperfect credibility may persist beyond this to some date $\tilde{T} \geq T$. A fraction λ of agents form rational expectations, while the fraction $1 - \lambda$ who doubt the announced structure base their expectations on previous targets or realized outcomes. For example, Goodfriend and King (2005) and Ascari and Ropele (2013) model disinflations using some inflation expectations of the form $\tilde{E}_t \pi_{t+1} = \lambda E_t \pi_{t+1} + (1 - \lambda)\pi^H$, where π^H is the old inflation target, whereas

[14] One alternative strand of literature considers agents who solve optimal filtering problems to disentangle temporary and permanent (unanticipated) policy shifts (see Erceg and Levin (2003)). Following Ball (1995), a seperate literature allows policy promises to be reneged on with some exogenous probability (see Schaumburg and Tambalotti (2007)) or endogenously in response to welfare considerations (Haberis et al. (2019)). Here, by contrast, we rule out reneging on policy promises – i.e. structures are implemented as announced.

Nicolae and Nolan (2006) assume that money supply expectations depend partly on realized money supply. In the above papers, λ is exogenous and, in some cases, time varying.

These expectations specifications can easily be nested using our method. Suppose a fraction of agents $\lambda_t \in (0,1]$ (given) have rational expectations and the remaining fraction, $1-\lambda_t$, have expectations $E_t^{IC}x_{t+1}$ that may depend on realized values of the endogenous and exogenous variables or on target values (i.e. intercepts). Our model for all $t \geq 0$ is:

$$B_{1,t}x_t = B_{2,t}\tilde{E}_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \tag{18}$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1-\lambda_t)E_t^{IC}x_{t+1} \tag{19}$$

where $E_t^{IC}x_{t+1} = F_0 x_t + F_1 x_{t-1} + F_2 e_t + F_3$ (for user-specified matrices F_0, F_1, F_2, F_3).

Note that the above specification allows expectations to depend on the current shocks e_t. This is useful because some authors (e.g. Yetman (2005)) allow expectations to differ from announced target values by random errors, in which case the vector e_t can be enlarged.

We assume $\lambda_t = 1$ for all $t \geq \tilde{T}+1$. This implies, as standard in the literature, that imperfect credibility eventually 'dies out' (e.g. Nicolae and Nolan (2006)). For $t \in [0,\tilde{T}]$, credibility is imperfect and determined by the known sequence $\{\lambda_t\}_{t=0}^{\tilde{T}}$; for example, if we allow credibility to improve over time (as is common in the literature) then λ_t will be increasing in each period up to date $\tilde{T}$.

Using Eq. (19) in (18) we have:

$$\hat{B}_{1,t}x_t = \hat{B}_{2,t}E_t x_{t+1} + \hat{B}_{3,t}x_{t-1} + \hat{B}_{4,t}e_t + \hat{B}_{5,t}, \quad \forall t \geq 0 \tag{20}$$

where $\hat{B}_{1,t} := B_{1,t} - B_{2,t}(1-\lambda_t)F_0$, $\hat{B}_{2,t} := B_{2,t}\lambda_t$, $\hat{B}_{j,t} := B_{j,t} + B_{2,t}(1-\lambda_t)F_{j-2}$, $j \in \{3,4,5\}$.

Note that $\forall t \geq \tilde{T}+1$, we have $\lambda_t = 1$ and $\hat{B}_{i,t} = \tilde{B}_i$, $i \in [5]$. Hence, analogous to (9):

$$\begin{cases} \hat{B}_{1,t}x_t = \hat{B}_{2,t}E_t x_{t+1} + \hat{B}_{3,t}x_{t-1} + \hat{B}_{4,t}e_t + \hat{B}_{5,t}, & 0 \leq t \leq \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T}. \end{cases} \tag{21}$$

It follows that the solution is given by Proposition 1, except that $\forall t \in [0,\tilde{T}]$ the matrices $B_{i,t}$, $i \in [5]$, must be replaced with the matrices $\hat{B}_{i,t}$ defined in (20). Note that the above approach generalizes easily to cases where imperfect credibility is specific to particular variables, since the scalars λ_t and $1-\lambda_t$ may be replaced with a diagonal matrix Λ_t and $I_n - \Lambda_t$.[15]

The above approach to imperfect credibility is intuitive and easy to implement, but the expectations $E_t^{IC}x_{t+1}$ are clearly ad hoc. The imperfect credibility solutions we set out below are instead based on an alternative sequence of structures while retaining the assumption that some fraction of agents form rational (i.e. model-consistent) expectations in all periods.

[15] In particular, Λ_t is an $n \times n$ diagonal matrix with $\lambda_{1,t}, \lambda_{2,t}, ..., \lambda_{n,t}$ on the main diagonal. To model imperfect credibility for the first variable (say inflation) we can let $\lambda_{1,t} \in (0,1]$ and set $\lambda_{2,t}, ..., \lambda_{n,t} = 1$, $\forall t$.

4.2.2 Imperfect credibility: alternative solutions

We now present two alternative solutions. The key distinction is that instead of using user-specified 'rules of thumb' such as past targets or realized values of variables (as in the previous section), agents who doubt the announced structure form expectations based on the alternative sequence of structures that they expect to materalize.

There are some papers in the literature in which agents take expectations based on alternative structures. Kryvtsov et al. (2008) consider imperfect credibility following a permanent (unanticipated) switch from inflation to price-level targeting, whereas Gibbs and Kulish (2017) study disinflations when some fraction of agents are adaptive learners.[16] The solutions presented below differ from these approaches because some fraction of agents have model-consistent expectations *in all periods*. As a result, we stay in the framework of rational expectations, and the expectations of the rational agents take into account the presence of the doubting agents who disbelieve the announced structure.

Type 1 Credibility

We first consider the simple case where the doubting agents do not believe *at all* in the announced sequence of structures. We refer to this as Type 1 credibility. As above, we let $\lambda_t \in (0, 1]$ (given and user-specified) denote the fraction of agents with rational expectations; the fraction of sceptical agents is therefore $1 - \lambda_t$. Our model for all $t \geq 0$ is:

$$B_{1,t}x_t = B_{2,t}\tilde{E}_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \tag{22}$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t)E_t^{IC} x_{t+1} \tag{23}$$

where $E_t^{IC} x_{t+1}$ is the conditional expectation of the sceptical agents.

We assume sceptical agents know the properties of e_t, the current structure, and observe x_t. They forecast using the correct functional form (that includes x_t); however, they do not know the future structure and think it will differ from the announced one in one or more periods. As in the last section, we assume $\lambda_t = 1$ for all $t \geq \tilde{T} + 1$ (imperfect credibility 'dies out'); in the earlier periods $t \in [0, \tilde{T}]$ the λ_t may be time-varying or (as a special case) be fixed at some value between 0 and 1. Any other structural change (captured by $B_{i,t}$, $i \in [5]$) is complete by date $T + 1$, where $T \leq \tilde{T}$.

The expectations of the sceptical agents are based on a different sequence of structures $\{\hat{B}_{1,s}, \hat{B}_{2,s}, \hat{B}_{3,s}, \hat{B}_{4,s}, \hat{B}_{5,s}\}_{s=t+1}^{\infty}$ that they expect to materialize.[17] We assume the matrices $\hat{B}_{i,t}$ are fixed for all $t \geq \tilde{T} + 1$; note this allows the possibility that sceptical agents expect

[16] In Gibbs and Kulish (2017), non-learners form conditional expectations but do not take into account the presence of the learners – i.e. their expectations are rational only if learners are absent (see p. 161).

[17] Analogous to (6), we may think of $\{\hat{B}_{1,s}, \hat{B}_{2,s}, \hat{B}_{3,s}, \hat{B}_{4,s}, \hat{B}_{5,s}\}_{s=t+1}^{\infty}$ as given by $\hat{B}_{i,s} = \mathbb{1}_s^{IC}\overline{B}_i + (1 - \mathbb{1}_s^{IC})\tilde{B}_i$, $i \in [5]$, where $\{\mathbb{1}_s^{IC}\}_{s=t+1}^{\infty}$ is a given (user-specified) indicator variable describing beliefs.

a structural change (say forward guidance) to last longer than announced or to end prematurely.[18] The terminal $\hat{B}_{i,t}$ matrices may potentially differ from the actual terminal structure $\{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4, \tilde{B}_5\}$ – i.e. it could be $\{\overline{B}_1, \overline{B}_2, \overline{B}_3, \overline{B}_4, \overline{B}_5\}$ instead.

Analogous to Proposition 1, the matrix recursions under the 'hat' structure are

$$\Omega_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}\hat{B}_{3,t}, \quad \Gamma_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}\hat{B}_{4,t}, \tag{24}$$

$$\Psi_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}(\hat{B}_{2,t}\Psi_{t+1}^{IC} + \hat{B}_{5,t}) \tag{25}$$

provided $\det[\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC}] \neq 0$ for all $t \in [0, \tilde{T}]$, where

$$\Omega_{\tilde{T}+1}^{IC} = \overline{\Omega} := (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3, \quad \Gamma_{\tilde{T}+1}^{IC} = \overline{\Gamma} := (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_4,$$

$$\Psi_{\tilde{T}+1}^{IC} = \overline{\Psi} := (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$$

and we assume, for concreteness, that the (perceived) terminal structure of the sceptical agents is $\{\overline{B}_1, \overline{B}_2, \overline{B}_3, \overline{B}_4, \overline{B}_5\}$ and $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$ are unique and well-defined.[19]

Our sceptical agents know the correct functional form of the solution x_t, but not the correct solution matrices. Similar to the structural forecasters in Gibbs and Kulish (2017), they make forecasts by using the (would-be) solution matrices under the structure they anticipate (see (24)–(25)) in conjunction with the observed data:

$$E_t^{IC} x_{t+1} = \Omega_{t+1}^{IC} x_t + \Psi_{t+1}^{IC} \tag{26}$$

where we use the zero-mean property of e_{t+1}. Note that the expectation in (26) would be rational if the matrices $\Omega_{t+1}^{IC}, \Psi_{t+1}^{IC}$ coincided with the model-consistent ones Ω_{t+1}, Ψ_{t+1}.

Using (26) in (22)–(23), our model is:

$$B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC}, \quad \forall t \geq 0 \tag{27}$$

where $B_{1,t}^{IC} := B_{1,t} - B_{2,t}(1-\lambda_t)\Omega_{t+1}^{IC}$, $B_{2,t}^{IC} := B_{2,t}\lambda_t$ and $B_{5,t}^{IC} := B_{2,t}(1-\lambda_t)\Psi_{t+1}^{IC} + B_{5,t}$.

Note that for all $t \geq \tilde{T}+1$, $\lambda_t = 1$ and $B_{i,t}^{IC} = B_{i,t} = \tilde{B}_i$, $i \in [5]$. Hence, we have:

$$\begin{cases} B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC}, & 0 \leq t \leq \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T}. \end{cases} \tag{28}$$

Since this is in the same form as (9), the solution to (28) follows Proposition 1, except that $\forall t \in [0, \tilde{T}]$ the matrices $B_{i,t}$, $i \in \{1, 2, 5\}$, in the recursions for $\Omega_t, \Gamma_t, \Psi_t$ must be replaced

[18] In the former case, we must set $\tilde{T} > T$. In the opposite case where the structural change is expected to end sooner than announced, the matrices $\hat{B}_{1,t}, \hat{B}_{2,t}, \hat{B}_{3,t}, \hat{B}_{4,t}, \hat{B}_{5,t}$ will be fixed from some date $t' < T$. The recursions (24)–(25) (see below) will then give fixed matrices $\Omega^{IC}, \Gamma^{IC}, \Psi^{IC}$ for all $t \geq t'$.

[19] Note that either the reference regime or the alternative regime may be used in the final step.

by the matrices $B_{i,t}^{IC}$ in (27) (where $\Omega_t^{IC}, \Gamma_t^{IC}, \Psi_t^{IC}$ are given by (24)–(25) and we assume the invertibility condition is met). To make this concrete, two applications are described below.

- *Permanent structural change.* Consider a permanent one-off change in structure (such as in Example 1). Suppose the change in structure occurs at date $T = 8$ and that some fixed fraction $\alpha \in (0, 1)$ of agents are sceptical about the announced change in structure, believing fully in the original structure (the reference regime). They are sceptical in periods $t \in [0, \tilde{T}]$, where $\tilde{T} = 12$, but believe fully in the announced structure thereafter. This is a special case of the above where $\lambda_t = 1 - \alpha$ for $t \in [0, 12]$, $\lambda_t = 1$ for all $t > 12$ and $\hat{B}_{i,t} = \overline{B}_i$ for all $t \geq 1$. Accordingly, the expectations $E_t^{IC} x_{t+1} = \overline{\Omega} x_t + \overline{\Psi}$ enter the solution in periods $t \in [0, 12]$ but not thereafter.

- *Temporary structural change.* Consider a temporary change in structure that starts in period 1 and is reversed after period $T = 8$. A fixed fraction $\alpha \in (0, 1)$ of agents are sceptical about the announced change in structure in periods 0 to 8 (so $\tilde{T} = T = 8$ in this example). In particular, they think the reform will be abandoned *sooner* than announced, say in period 6 (the policy could be, e.g., forward guidance). We thus have $\lambda_t = 1 - \alpha$ for $t \in [0, 8]$ and $\lambda_t = 1$ for all $t \geq 9$, along with $\hat{B}_{i,t} = \overline{B}_i$ for $t \in [1, 6]$ and $\hat{B}_{i,t} = \tilde{B}_i$ for all $t \geq 7$. The sceptics' expectations are thus $E_t^{IC} x_{t+1} = \tilde{\Omega} x_t + \tilde{\Psi}$ for all $t \geq 7$ and are given by the recursions (24)–(25) for $t \in [0, 6]$.

Type 2 Credibility

We now present a simple generalization of the previous solution in which 'sceptics' weight both the announced structure and a different structure. We refer to this as Type 2 credibility because the agents are more sophisticated in their thinking: they *doubt* the announced (i.e. actual) structure rather than rejecting it completely. Our model for all $t \geq 0$ is

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t} \tag{29}$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t) E_t^{IC} x_{t+1} \tag{30}$$

where, as before, $\lambda_t \in (0, 1]$ for $t \in [0, \tilde{T}]$ and $\lambda_t = 1$ for all $t \geq \tilde{T} + 1$.

The expectation $E_t^{IC} x_{t+1}$ of the sceptical agents is given by

$$E_t^{IC} x_{t+1} = p_t E_t^* x_{t+1} + (1 - p_t) \hat{E}_t x_{t+1} \tag{31}$$

where $p_t \in (0, 1]$ (given) denotes the weight on the announced structure and $1 - p_t$ is the weight on the alternative structure. Type 1 credibility is the special case $p_t = 0$. Note that because $\lambda_t = 1$ for all $t \geq \tilde{T} + 1$, the value of p_t is irrelevant after period $\tilde{T}$. Therefore, for concreteness, we assume $p_t = p_{\tilde{T}}$ for all $t \geq \tilde{T} + 1$. In the earlier periods $t \in [0, \tilde{T}]$, p_t follows a known sequence specified by the researcher in which p_t may be time-varying or constant.

The expectation $E_t^* x_{t+1}$ is based on the announced structure, whereas $\hat{E}_t x_{t+1}$ is based on the alternative structure. We have $\hat{E}_t x_{t+1} = \Omega_{t+1}^{IC} x_t + \Psi_{t+1}^{IC}$ as in (24)–(26) and $E_t^* x_{t+1} = \Omega_{t+1}^* x_t + \Psi_{t+1}^*$, where $\Omega_{t+1}^*, \Psi_{t+1}^*$ come from Proposition 1 (with matrices $B_{1,t}, ..., B_{5,t}$).[20] The $E_t^* x_{t+1}$ is like the structural forecast in Gibbs and Kulish (2017): it would be a rational expectation if non-rational agents were absent (i.e. if $\lambda_t = 1$ for all t).

Equation (31) has the intuitive intepretation that, in a given period $t \in [0, \tilde{T}]$, doubting agents attach a subjective probability p_t to the announced structure and a probability $1 - p_t$ to implementation of the different structure $\{\hat{B}_{1,s}, \hat{B}_{2,s}, \hat{B}_{3,s}, \hat{B}_{4,s}, \hat{B}_{5,s}\}_{s=t+1}^{\infty}$.

Using the above expressions in (30), the economy-wide expectation is given by

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t)[\tilde{\Omega}_{t+1}^{IC} x_t + \tilde{\Psi}_{t+1}^{IC}] \tag{32}$$

where $\tilde{\Omega}_{t+1}^{IC} := p_t \Omega_{t+1}^* + (1 - p_t)\Omega_{t+1}^{IC}$ and $\tilde{\Psi}_{t+1}^{IC} := p_t \Psi_{t+1}^* + (1 - p_t)\Psi_{t+1}^{IC}$.

Using (32) in (29)–(30), we have $\forall t \geq 0$, $B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC}$, where $B_{1,t}^{IC} := B_{1,t} - B_{2,t}(1 - \lambda_t)\tilde{\Omega}_{t+1}^{IC}$, $B_{2,t}^{IC} := B_{2,t}\lambda_t$ and $B_{5,t}^{IC} := B_{2,t}(1 - \lambda_t)\tilde{\Psi}_{t+1}^{IC} + B_{5,t}$. Note that $\forall t \geq \tilde{T} + 1$, we have $\lambda_t = 1$ and $B_{i,t}^{IC} = B_{i,t} = \tilde{B}_i$, $i \in [5]$. Hence, our system is:

$$\begin{cases} B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC}, & 0 \leq t \leq \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T}. \end{cases} \tag{33}$$

The solution again follows Proposition 1, except that for $t \in [0, \tilde{T}]$, the matrices $B_{1,t}, B_{2,t}, B_{5,t}$ must be replaced by the matrices $B_{1,t}^{IC}, B_{2,t}^{IC}, B_{5,t}^{IC}$, which depend on the sequences $\{\lambda_t, p_t\}_{t=0}^{\tilde{T}}$ and the recursions $\{\Omega_{t+1}^*, \Psi_{t+1}^*\}_{t=0}^{\tilde{T}}$ (see (12)–(13)) and $\{\Omega_{t+1}^{IC}, \Psi_{t+1}^{IC}\}_{t=0}^{\tilde{T}}$ (see (24)–(26)).

4.3 Indeterminacy

In this section we show how to deal with non-fundamental (sunspot) solutions. For concreteness we assume the terminal structure implies indeterminacy, such that there are multiple stable terminal solutions that depends on sunspots (see Assumption 2).

Assumption 2 *The terminal structure $\{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4, \tilde{B}_5\}$ is indeterminate, i.e. there are many stable terminal solutions.*

Several methods have been put forward in the literature for solving indeterminate models, including Lubik and Schorfheide (2003), Farmer et al. (2015) and Bianchi and Nicolò (2021). All three approaches are equivalent in terms of finding equilibria, but the simplest method is that of Farmer et al. (2015); the analysis that follows thus utilizes their method.

The method of Farmer et al. (2015) involves moving non-fundamental expectational errors to the vector of fundamental shocks. Since this reclassification resolves the indeterminacy

[20] Again, we use the zero-mean property of e_{t+1}. In general, neither $E_t^* x_{t+1}$ or $\hat{E}_t x_{t+1}$ are rational.

problem, the resulting model can be solved using standard methods. In this way, the Farmer et al. (2015) approach allows us to solve for a specific sunspot solution while sticking in the same framework used thus far. Our starting model is:

$$B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, \;\; \forall t \geq 0 \tag{34}$$

where $B_{i,t} = \mathbb{1}_t B_i + (1 - \mathbb{1}_t)\tilde{B}_i$ ($\mathbb{1}_t = 0, \forall t > \tilde{T}$) and Assumption 2 holds.

Suppose there is indeterminacy of degree $k \in [n]$. The degree of indeterminacy can be determined using standard methods, like Sims (2002).[21] If there are *more than* k expectational variables, the user must choose k of them to be hit by sunspots (Farmer et al. (2015)).

Consider first the terminal periods, $t > \tilde{T}$. Following the approach in Farmer et al. (2015), we define an augmented terminal structure based on (34) and k new equations:

$$\begin{cases} \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \\ x_t^k = s_{t-1} + v_t \end{cases}, \quad \forall t > \tilde{T} \tag{35}$$

where $s_t := E_t x_{t+1}^k$ and x_t^k is a vector of the k forward-looking endogenous variables whose expectations are hit by zero-mean (sunspot) shocks $v_{i,t}$, $i = 1,..,k$, drawn from some *assumed* distribution and potentially correlated with the fundamental shocks e_t.[22]

The intuition for the augmented system, (35), is quite simple. Degree k indeterminacy requires adding k extra equations in which the expectations $E_{t-1}x_t^k$ $(= s_{t-1})$ are state variables and each of these variables $x_{i,t}^k$, $i \in [k]$, is subject to self-fulfilling expectations shocks $v_{i,t}$ drawn from some assumed distribution. For convenience, we assume these 'sunspot variables' are ordered first in x_t.

Partitioning the top line of (35) into sunspot and non-sunspot variables leads to

$$\begin{bmatrix} \tilde{B}_{11} & \tilde{B}_{12} \end{bmatrix} \begin{bmatrix} x_t^k \\ x_t^{n-k} \end{bmatrix} = \begin{bmatrix} \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix} \begin{bmatrix} E_t x_{t+1}^k \\ E_t x_{t+1}^{n-k} \end{bmatrix} + \begin{bmatrix} \tilde{B}_{31} & \tilde{B}_{32} \end{bmatrix} \begin{bmatrix} x_{t-1}^k \\ x_{t-1}^{n-k} \end{bmatrix} + \tilde{B}_4 e_t + \tilde{B}_5 \tag{36}$$

where $\tilde{B}_{i1}, \tilde{B}_{i2}$, $i \in \{1, 2, 3\}$, are $n \times k$ and $n \times (n - k)$ matrices, respectively.

Following Farmer et al. (2015), $s_t := E_t x_{t+1}^k$ is used to substitute s_t for $E_t x_{t+1}^k$ in (36).[23]

[21] Formally, if there are n_1 unstable generalized eigenvalues and p non-fundamental errors then (under some regularity assumptions) there are $k = p - n_1$ degrees of indeterminacy.

[22] On the covariance matrix of the shocks, see Eq. (21) in Farmer et al. (2015) and the related discussion.

[23] Note that substituting s_t in place of $E_t x_{t+1}^k$ is not strictly necessary (see Gibbs and McClung (2020)) but reduces the dimension of the vector $\tilde{x}_t$ and hence also dimensionality of the matrices $\tilde{\mathcal{B}}_i$; see (37).

The terminal system in (35) can then be written in the form:

$$\tilde{\mathcal{B}}_1 \begin{bmatrix} x_t^k \\ x_t^{n-k} \\ s_t \end{bmatrix} = \tilde{\mathcal{B}}_2 \begin{bmatrix} E_t x_{t+1}^k \\ E_t x_{t+1}^{n-k} \\ E_t s_{t+1} \end{bmatrix} + \tilde{\mathcal{B}}_3 \begin{bmatrix} x_{t-1}^k \\ x_{t-1}^{n-k} \\ s_{t-1} \end{bmatrix} + \tilde{\mathcal{B}}_4 \begin{bmatrix} v_t \\ e_t \end{bmatrix} + \tilde{\mathcal{B}}_5$$

i.e.

$$\tilde{\mathcal{B}}_1 \tilde{x}_t = \tilde{\mathcal{B}}_2 E_t \tilde{x}_{t+1} + \tilde{\mathcal{B}}_3 \tilde{x}_{t-1} + \tilde{\mathcal{B}}_4 \tilde{e}_t + \tilde{\mathcal{B}}_5, \quad \forall t > \tilde{T}, \tag{37}$$

where

$$\tilde{\mathcal{B}}_1 = \begin{bmatrix} I_k & 0_{k\times n-k} & 0_{k\times k} \\ \tilde{B}_{11} & \tilde{B}_{12} & -\tilde{B}_{21} \end{bmatrix}, \quad \tilde{\mathcal{B}}_2 = \begin{bmatrix} 0_{k\times k} & 0_{k\times n-k} & 0_{k\times k} \\ 0_{n\times k} & \tilde{B}_{22} & 0_{n\times k} \end{bmatrix},$$

$$\tilde{\mathcal{B}}_3 = \begin{bmatrix} 0_{k\times k} & 0_{k\times n-k} & I_k \\ \tilde{B}_{31} & \tilde{B}_{32} & 0_{n\times k} \end{bmatrix}, \quad \tilde{\mathcal{B}}_4 = \begin{bmatrix} I_k & 0_{k\times m} \\ 0_{n\times k} & \tilde{B}_4 \end{bmatrix}, \quad \tilde{\mathcal{B}}_5 = \begin{bmatrix} 0_{k\times 1} \\ \tilde{B}_5 \end{bmatrix}.$$

Equation (37) defines the model in the terminal periods, $t > \tilde{T}$. Note that since the realized sunspot shocks are included in the vector of fundamental shocks, the system in (37) can be solved using standard methods. Thus, analogous to Proposition 1, there is now a unique terminal solution of the form $\tilde{x}_t = \tilde{\Omega}\tilde{x}_{t-1} + \tilde{\Gamma}\tilde{e}_t + \tilde{\Psi}$, $\forall t > \tilde{T}$, where $\tilde{\Omega} = (\tilde{\mathcal{B}}_1 - \tilde{\mathcal{B}}_2\tilde{\Omega})^{-1}\tilde{\mathcal{B}}_3$, $\tilde{\Gamma} = (\tilde{\mathcal{B}}_1 - \tilde{\mathcal{B}}_2\tilde{\Omega})^{-1}\tilde{\mathcal{B}}_4$, $\tilde{\Psi} = (\tilde{\mathcal{B}}_1 - \tilde{\mathcal{B}}_2\tilde{\Omega})^{-1}(\tilde{\mathcal{B}}_2\tilde{\Psi} + \tilde{\mathcal{B}}_5)$.

We assume that agents' expectations coordinate on the sunspot solution in the terminal regime, as in Gibbs and McClung (2020). With this assumption, a solution can be recovered using the same method as in the determinate case (see below). Specifically, given a particular terminal sunspot solution $\tilde{x}_t = \tilde{\Omega}\tilde{x}_{t-1} + \tilde{\Gamma}\tilde{e}_t + \tilde{\Psi}$, $\forall t > \tilde{T}$, a solution for $t \in [0, \tilde{T}]$ can be determined using the recursive formulas in Proposition 1 (provided the invertibility conditions are satisfied). Note that (if it exists) the solution $\{\tilde{x}_t\}_{t=0}^{\infty}$ will be unique given our assumption of coordination on a particular sunspot process v_t.

By (34) and our coordination assumption, the system during the transition is

$$\begin{cases} B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \\ x_t^k = s_{t-1} \end{cases}, \quad \forall t \in [0, \tilde{T}]. \tag{38}$$

Note that this system $\forall t \in [0, \tilde{T}]$ is analogous to (35), except that the time-indexed structural matrices $B_{i,t}$ appear and the sunspot process v_t does not enter, as multiple equilibria are absent in the transition periods when agents use backward induction from the terminal solution. Note that this result arises because the (future) sunspots are mean-zero and hence do not influence agents' expectations during the transition periods $t \in [0, \tilde{T}]$.[24]

Since (38) is analogous to (35), an appropriate partition of the $B_{i,t}$ matrices (as in (36))

[24]See also the worked example in the Appendix of Gibbs and McClung (2020).

and the substitution of s_t for $E_t x^k_{t+1}$ (as in (37)) gives us the following system:

$$\mathcal{B}_{1,t}\tilde{x}_t = \mathcal{B}_{2,t}E_t\tilde{x}_{t+1} + \mathcal{B}_{3,t}\tilde{x}_{t-1} + \mathcal{B}_{4,t}\tilde{e}_t + \mathcal{B}_{5,t}, \quad \forall t \in [0, \tilde{T}], \tag{39}$$

where $\tilde{x}_t$, $\tilde{e}_t$ are defined in (37) and

$$\mathcal{B}_{1,t} = \begin{bmatrix} I_k & 0_{k\times n-k} & 0_{k\times k} \\ B_{11,t} & B_{12,t} & -B_{21,t} \end{bmatrix}, \quad \mathcal{B}_{2,t} = \begin{bmatrix} 0_{k\times k} & 0_{k\times n-k} & 0_{k\times k} \\ 0_{n\times k} & B_{22,t} & 0_{n\times k} \end{bmatrix},$$

$$\mathcal{B}_{3,t} = \begin{bmatrix} 0_{k\times k} & 0_{k\times n-k} & I_k \\ B_{31,t} & B_{32,t} & 0_{n\times k} \end{bmatrix}, \quad \mathcal{B}_{4,t} = \begin{bmatrix} 0_{k\times k} & 0_{k\times m} \\ 0_{n\times k} & B_{4,t} \end{bmatrix}, \quad \mathcal{B}_{5.t} = \begin{bmatrix} 0_{k\times 1} \\ B_{5,t} \end{bmatrix}.$$

In summary, our augmented model is

$$\mathcal{B}_{1,t}\tilde{x}_t = \mathcal{B}_{2,t}E_t\tilde{x}_{t+1} + \mathcal{B}_{3,t}\tilde{x}_{t-1} + \mathcal{B}_{4,t}\tilde{e}_t + \mathcal{B}_{5,t}, \quad \forall t \geq 0 \tag{40}$$

where $\mathcal{B}_{i,t} = \tilde{\mathcal{B}}_i$, $\forall t > \tilde{T}$ (see (37)) and $\tilde{e}_t$ zero mean with covariance matrix $\Sigma = \begin{bmatrix} \Sigma_v & \Sigma_{ve} \\ \Sigma_{ve} & I_m \end{bmatrix}$.

Since the augmented model (40) is in the usual form, the solution (if it exists) will be given by Proposition 1, except that matrices $B_{i,t}, \tilde{B}_i$ are replaced by $\mathcal{B}_{i,t}, \tilde{\mathcal{B}}_i$, $\forall i \in [5]$.

Example 2 Consider the following New Keynesian model:

$$\pi_t = \beta E_t\pi_{t+1} + \kappa y_t + e_{\pi,t}, \quad y_t = E_t y_{t+1} - \sigma(R_t - E_t\pi_{t+1}) + e_{y,t} \tag{41}$$

where $\beta \in (0,1)$, $\kappa > 0$; $e_{\pi,t}$, $e_{y,t}$ are exogenous, zero-mean IID disturbances and

$$R_t = \theta_{\pi,t}\pi_t, \qquad \theta_{\pi,t} = \begin{cases} \overline{\theta}_\pi = 0 & \text{for } t \in [0,4] \\ \tilde{\theta}_\pi \in (0,1) & \text{for } t \geq 5. \end{cases} \tag{42}$$

For convenience, we substitute the Taylor rule (42) into the IS curve in (41) to get:

$$\pi_t = \beta E_t\pi_{t+1} + \kappa y_t + e_{\pi,t}, \quad y_t = E_t y_{t+1} - \sigma(\theta_{\pi,t}\pi_t - E_t\pi_{t+1}) + e_{y,t}. \tag{43}$$

For $\theta_{\pi,t} = \theta_\pi$ $\forall t$, the determinacy properties are well known (Bullard and Mitra (2002)): if $\theta_\pi > 1$, Taylor principle holds and there is a unique stable equilibrium, whereas if $\theta_\pi \in (0,1)$ there is one degree of indeterminacy ($k = 1$) and thus many stable solutions. By (42), $\theta_{\pi,t} = \tilde{\theta}_\pi \in (0,1)$ for $\forall t \geq 5$. Hence, the terminal structure does not satisfy the Taylor principle and there is indeterminacy of degree $k = 1$. As there is one degree of indeterminacy and two forward-looking variables, we may pick either to be hit by sunspots. Here we pick inflation and hence $x^k_t = [\pi_t]$ and $x^{n-k}_t = [y_t]$.

To write the terminal structure as in (37), we must introduce a 'sunspot equation' (see (35)) for inflation:

$$\pi_t = s_{t-1} + v_{\pi,t}, \quad \forall t \geq 5, \tag{44}$$

where $s_t := E_t\pi_{t+1}$ and $v_{\pi,t}$ is an exogenous sunspot shock with $E_{t-1}v_{\pi,t} = 0$. Using the substitution $E_t\pi_{t+1} = s_t$ in (43) and adding (44) to the system, we have $\forall t \geq 5$: $\tilde{\mathcal{B}}_1\tilde{x}_t = \tilde{\mathcal{B}}_2 E_t\tilde{x}_{t+1} + \tilde{\mathcal{B}}_3\tilde{x}_{t-1} + \tilde{\mathcal{B}}_4\tilde{e}_t$, which is a special case of (37):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\kappa & -\beta \\ \sigma\tilde{\theta}_\pi & 1 & -\sigma \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \\ s_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t y_{t+1} \\ E_t s_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ s_{t-1} \end{bmatrix} + \begin{bmatrix} v_{\pi,t} \\ e_{\pi,t} \\ e_{y,t} \end{bmatrix} \tag{45}$$

For $t \in [0,4]$, the structure is given by (43), with $\theta_{\pi,t} = \overline{\theta}_\pi = 0$, along with $\pi_t = s_{t-1}$. As in (39), we may write the system as $\mathcal{B}_1\tilde{x}_t = \mathcal{B}_2 E_t\tilde{x}_{t+1} + \mathcal{B}_3\tilde{x}_{t-1} + \mathcal{B}_4\tilde{e}_t$, $\forall t \in [0,4]$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\kappa & -\beta \\ 0 & 1 & -\sigma \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \\ s_t \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t y_{t+1} \\ E_t s_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ s_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{\pi,t} \\ e_{\pi,t} \\ e_{y,t} \end{bmatrix} \tag{46}$$

By Proposition 1, the solution is given by

$$\tilde{x}_t = \begin{cases} \Omega_t\tilde{x}_{t-1} + \Gamma_t\tilde{e}_t & \text{for } 0 \leq t \leq 4 \\ \tilde{\Omega}\tilde{x}_{t-1} + \tilde{\Gamma}\tilde{e}_t & \text{for } t \geq 5 \end{cases} \tag{47}$$

where $\tilde{\Omega} = (\tilde{\mathcal{B}}_1 - \tilde{\mathcal{B}}_2\tilde{\Omega})^{-1}\tilde{\mathcal{B}}_3$, $\tilde{\Gamma} = (\tilde{\mathcal{B}}_1 - \tilde{\mathcal{B}}_2\tilde{\Omega})^{-1}\tilde{\mathcal{B}}_4$ and, for $t \in [0,4]$,

$$\Omega_t = (\mathcal{B}_1 - \mathcal{B}_2\Omega_{t+1})^{-1}\mathcal{B}_3, \quad \Gamma_t = (\mathcal{B}_1 - \mathcal{B}_2\Omega_{t+1})^{-1}\mathcal{B}_4, \quad \text{with } \Omega_5 = \tilde{\Omega}.$$

Eq. (47) determines $\tilde{x}_t$ given draws $v_{\pi,t}, e_{\pi,t}, e_{y,t}$ from assumed distributions. Note that the sunspots $v_{\pi,t}$ enter the solution in the terminal periods $t \geq 5$ (see (45)), but not in the earlier periods $t \in [0,4]$ (cf. (46)).

5 Applications

We now consider some numerical applications. The first application uses a New Keynesian model to study several monetary policy announcements, including an example where agents anticipate that the terminal structure does not satisfy the Taylor principle. The second application is a pension reform in the Diamond (1965) model that is used to study the optimal announcement date. Further details of the applications and the codes are given in the *Supplementary Appendix*.

5.1 A New Keynesian model

Following Cagliarini and Kulish (2013), we consider a simplified version of the New Keynesian model in Ireland (2007) in which the inflation target π^* is non-stochastic; the deviation of technology from its steady state, a_t, is stationary; and there are no habits in consumption. Under these assumptions, the model is given by the following set of log-linear equations:

$$\pi_t = \frac{1}{(1+\beta\alpha)} \left(\beta E_t \pi_{t+1} + (1+\beta\alpha-\alpha-\beta)\pi^* + \alpha\pi_{t-1} + \psi\sigma y_t - \psi a_t - \mu_t\right) \tag{48}$$

$$y_t = E_t y_{t+1} - \sigma^{-1}(R_t - E_t\pi_{t+1}) + \frac{(1-\rho_g)}{\sigma} g_t - \sigma^{-1}\ln(\beta) \tag{49}$$

$$R_t = (1-\rho_R)\overline{R} + \rho_R R_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) \tag{50}$$

where the shocks to technology, demand and the mark-up follow AR(1) processes:

$$u_t = \rho_u u_{t-1} + \sigma_u \epsilon_{u,t}, \qquad u \in \{a, g, \mu\}, \qquad \epsilon_{u,t} \sim N(0,1).$$

Eqs. (48)–(50) are the Phillips curve, the IS curve and the interest rate rule. In these equations, π_t is the log inflation rate; y_t is output expressed in log deviations from steady state; and R_t is the log nominal interest rate. The steady-state nominal interest rate is $\overline{R} = \pi^* - \ln(\beta)$. We use the same parameter values as in Cagliarini and Kulish (2013): $\pi^* = 0.0125$ (quarterly), $\beta = 0.9925$, $\sigma = 1$, $\alpha = 0.25$, $\psi = 0.1$, $\rho_R = 0.65$, $\theta_\pi = 0.5$, $\theta_y = 0.1$, $\theta_{dy} = 0.2$; $\rho_g, \rho_a, \rho_\mu = 0.9$, $\sigma_g = 0.02$, $\sigma_a = 0.007$ and $\sigma_\mu = 0.001$. The above model can easily be cast in matrix form by letting $x_t = \begin{bmatrix}\pi_t & y_t & R_t & a_t & g_t & \mu_t\end{bmatrix}'$ and $e_t = \begin{bmatrix}\epsilon_{a,t} & \epsilon_{g,t} & \epsilon_{\mu,t}\end{bmatrix}'$.

5.1.1 A change in the inflation target

As a first exercise, suppose the central bank announces a lower inflation target as in Cagliarini and Kulish (2013). The announcement refers to the future value of π^*, which is to be reduced permanently from 5% to 2.5% per annum. To motivate a disinflation policy, the economy starts at steady state but is hit in period 1 with an unanticipated demand shock, $\epsilon_{g,1} = 0.02$. The announcement of a lower inflation target is made in period 4 and is implemented in period 8 (see Figure 1). This is an application of the benchmark solution in Proposition 1, where the two regimes differ only in the intercept matrices $\overline{B}_5$ (initial structure) and $\tilde{B}_5$ (terminal structure) due to the shift in the inflation target (see (4)–(6)).

The impulse responses in Figure 1 replicate those in Cagliarini and Kulish (2013). Inflation and output increase in response to the demand shock and remain elevated up to period 3; however, both inflation and output fall sharply when the lower inflation target is announced in period 4. Because the monetary policy rule responds to the deviation of inflation from the current target rather than the announced target, a substantial negative

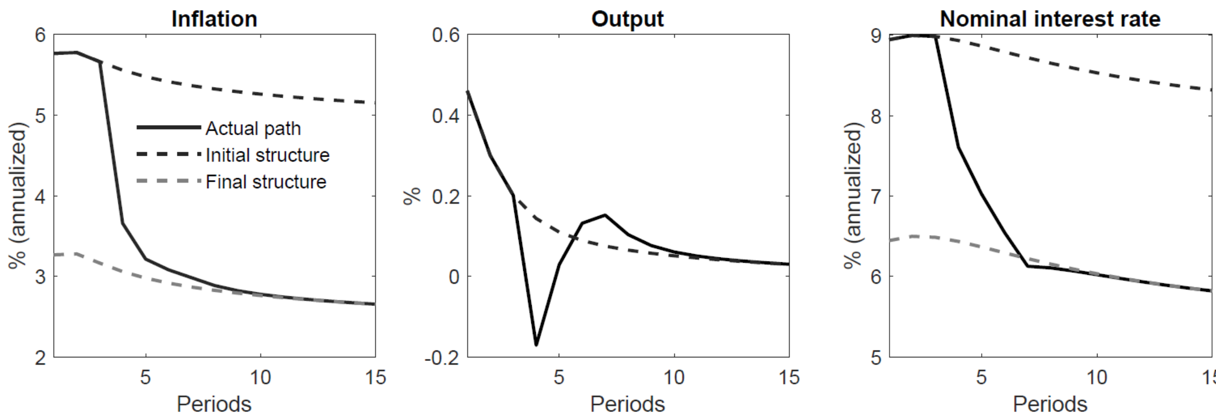

Figure 1: **IRFs with anticipated reduction in π^*: announced $t = 4$, implemented $t = 8$.** The figure shows impulse responses to a one standard deviation demand shock $\epsilon_{g,0} = 0.02$ followed by an announcement (in period 4) that the inflation target will be permanently reduced from 5% p.a. to 2.5% p.a. in period 8.

inflation gap opens up in period 4 and the nominal interest rate is cut sharply. However, as expected inflation also falls sharply, the real interest rate does not change much, and hence output falls below steady state despite the cut in nominal interest rates. By the time the inflation target is actually reduced in period 8, output is above its steady state value and most of the disinflation has already happened via expectations.

Now suppose the above policy is *imperfectly credible*, as in Section 4.2. In particular, suppose the reduction in the inflation target is imperfectly credible in the announcement period ($t = 4, ..., 7$) and that these doubts persist for some time after implementation in period $t = 8$, namely, until period $\tilde{T} = 10$. The 'sceptics' thus continue to doubt the policy for three quarters once it has been implemented. The fraction of rational agents, who believe the announced policy, is fixed at $\lambda_t = \lambda$ for $t \in [4, 10]$; our assumption that imperfect credibility ends in period 10 means $\lambda_t = 1$ for $t \geq 11$. In what follows, we set $\lambda = 0.85$ or $\lambda = 0.70$; these values imply, respectively, that 15% or 30% of agents are 'doubters' in periods 4 to 10. Note that $\lambda_t = 1$, $\forall t$, corresponds to perfect credibility as in Figure 1.

We consider two different imperfect credibility specifications. The first specification (or 'standard approach', Section 4.2.1) has the doubting agents using rule-of-thumb inflation expectations $E_t^{IC}\pi_{t+1} = \pi^*_{orig}$, as in Goodfriend and King (2005) and Ascari and Ropele (2013). The economy-wide inflation expectation is thus $\tilde{E}_t\pi_{t+1} = \lambda E_t\pi_{t+1} + (1 - \lambda)\pi^*_{orig}$ for $t \in [4, 10]$ and $\tilde{E}_t\pi_{t+1} = E_t\pi_{t+1}$ for all $t \geq 11$. The only difference to model (48)–(50) is that the expectation $E_t\pi_{t+1}$ is replaced by $\tilde{E}_t\pi_{t+1}$ in periods 4 to 10. The second specification is Type 1 credibility (Section 4.2.2). We assume that in periods $t \in [4, 10]$ the doubting agents expect the *original structure* to be in place in all *future* periods, i.e. they have expectations

$E_t^{IC} x_{t+1} = \overline{\Omega} x_t + \overline{\Psi}$, where x_t is the vector of endogenous variables (defined above) and $\overline{\Omega}, \overline{\Psi}$ are the fixed solution matrices under the original structure. Hence, they think a permanent reversion to the original structure (with the 5% inflation target) will occur next period.

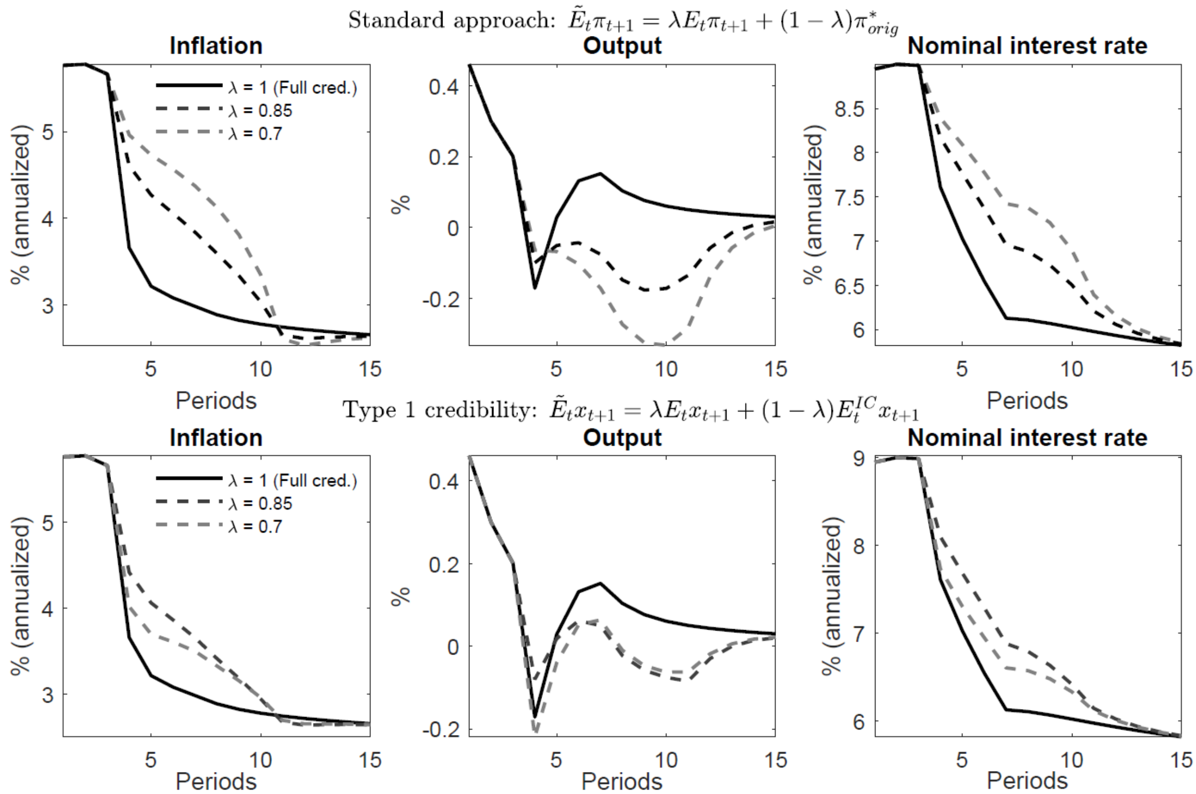

Figure 2: **IRFs with anticipated reduction in π^*: impact of imperfect credibility.** The figure shows impulse responses to a one standard deviation demand shock $\epsilon_{g,0} = 0.02$ followed by an announcement (in period 4) that the inflation target will be permanently reduced from 5% p.a. to 2.5% p.a. in period 8. The announcement is imperfectly credible to a fraction $1 - \lambda$ of agents in periods $t = 4, ..., 10$.

Figure 2 reports impulse responses. The solid line shows the case of full credibility as in Figure 1. There is a marked difference in the imperfect credibility cases (dashed lines). For both types of imperfect credibility, inflation is higher during the announcement period, and this continues until imperfect credibility 'dies out' in period 11. Intuitively, inflation increases because the rational fraction of agents is 'diluted' and replaced by agents with 'sticky' expectations either due to a rule of thumb (top panel) or reliance on the past structure (lower panel); hence the population as a whole is less responsive to the announcement that the inflation target will be reduced and expectations remain high for some periods.

Notably, there are substantive differences in the impulse responses for the two types of imperfect credibility. Under the standard approach (top panel), inflation expectations are sharply elevated relative to full credibility and inflation remains stubbornly high despite the announcement. As a result, the real interest rate is lowered relative to full credibility, such that output initially increases (top panel, middle). Only once inflation expectations and nominal interest rates fall substantially do we see output fall well below its steady state value. By comparison, inflation is less elevated under Type 1 credibility (lower panel, left) because agents use realized current values x_t as a basis for their expectations and hence there is feedback from the expectations of the rational agents, which lower actual inflation, to less stubborn expectations of the doubting agents.[25] As a result, the reponse of the real interest rate is smaller and output is much closer to its simulated path under full credibility.

These results are interesting because several papers have studied the real effects of disinflations under imperfect credibility, typically finding non-trivial output and welfare implications when some fraction of agents are rational as here (Goodfriend and King (2005), Nicolae and Nolan (2006), Ascari and Ropele (2013)). By comparison, the results for Type 1 credibility suggest that these 'rule of thumb' approaches might overstate the impact of imperfect credibility; it would be interesting to investigate this further in future research.

5.1.2 Forward guidance

We now consider forward guidance in the above model. In particular, we compare 'plain vanilla' forward guidance to versions of forward guidance with imperfect credibility and delayed announcement for some fraction of the population. We assume that the period of low interest rates is announced in period 0 and implemented in periods 2 to 5. Given the presence of forward guidance, the interest rate rule (50) must be amended to

$$R_t = \begin{cases} \underline{R} := 0 & \text{for } t \in [2,5] \\ (1-\rho_R)\overline{R} + \rho_R R_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) & \text{otherwise.} \end{cases} \tag{51}$$

To motivate the use of forward guidance, we hit the economy with a large negative demand shock in period 0 ($\epsilon_{g,0} = -0.08$). The model parameters are unchanged and we use the smaller inflation target of 2.5% p.a. from the previous section ($\pi^* = 0.00625$, quarterly). The three cases of forward guidance we consider are: a 'plain vanilla' baseline case that is fully anticipated; forward guidance that remains unannounced to a fraction $1-\lambda$ of agents ($K = 0$, Section 4.1); and Type 2 credibility when a fraction $1-\lambda$ expect forward guidance to end 2 periods 'early' and permanently revert to the Taylor-type rule. In the latter cases, the fraction of agents with rational expectations is set at $\lambda = 0.7$ and for Type 2 credibility we give doubting agents a subjective probability $p = 0.5$ that places equal weight on the announced structure and that under early reversion.[26] The results are reported in Figure 3.

[25] Since the rational fraction of agents form expectations that take into account the beliefs of the 'doubters', their inflation expectations do not fall as much as under full credibility, though there is still a big impact.

[26] It is assumed that imperfect credibility dies out when the forward guidance periods ends.

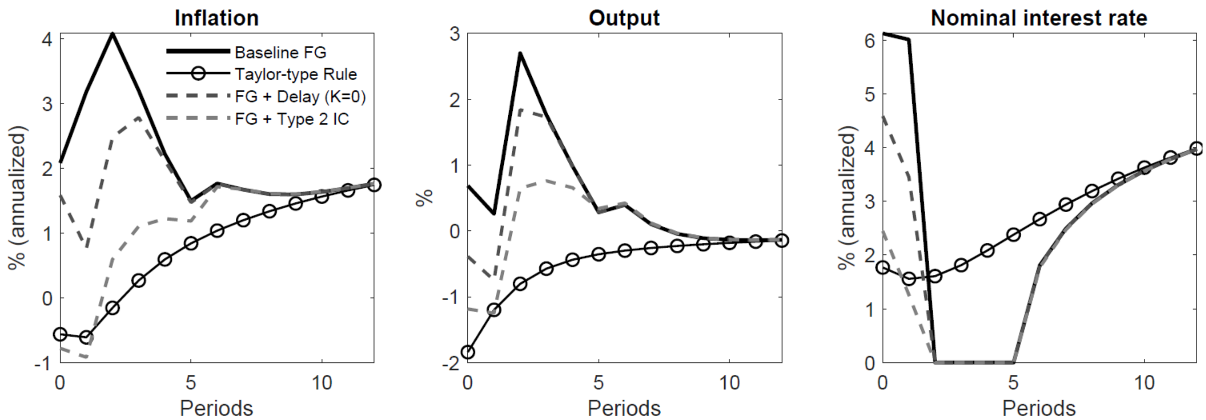

Figure 3: **IRFs to large negative demand shock + forward guidance (various cases).** The figure shows impulse responses to a four-s.d. demand shock $\epsilon_{g,0} = -0.08$ followed by forward guidance (FG) that is announced in period 0 and implemented from periods 2 to 5. FG + Type 2 IC refers to guidance that is imperfectly credible ($p = 0.5$) to 30% of agents, whereas FG + Delay has 30% of agents not receiving the announcement.

Compared to the Taylor-type rule (Fig. 3, circles), forward guidance is less aggressive in cutting interest rates on impact following the negative demand shock (right panel). The reason is that in all three forward guidance cases, the expansionary future monetary policy is fully anticipated by some agents and thus, in the absence of 'tight' monetary policy today, inflation and output would be somewhat above their target values. The highly expansionary impact of forward guidance through the expectations channel is known in the literature as the 'forward guidance puzzle' (Del Negro et al. (2012)). We see this most clearly in the baseline case, where interest rates initially *increase* relative to their steady state and inflation and output remain close to their target values; i.e. monetary policy *restrains* the initial stimulus.

The expansionary impact of forward guidance is attenuated somewhat when some agents do not receive advance information about the policy (delayed announcement, black dash) or doubt the announcement by placing some probability on the event that forward guidance ends 2 periods prematurely (Type 2 credibility, grey dash). The impact of imperfect credibility is quite strong, with the initial interest rate response and the responses of inflation and output being quite close to those under the Taylor-type rule (see left and middle panel); hence, imperfect credibility may help explain why, in practice, inflation and output do not respond as strongly to forward guidance as standard models predict (see also Haberis et al. (2019)). After the announcement period is over and interest rates are reduced to zero, both inflation and output have 'hump-shaped' dynamics, with the main difference relative to the Taylor-type rule being that output is above steady state most of the time (middle panel).

5.1.3 Forward guidance + indeterminacy

We now consider a case where the terminal solution is indeterminate because the Taylor principle is violated. Consider a simple forward guidance policy that is announced and implemented in period 0. The policy holds nominal interest rates at a lower bound $\underline{R}$ until period 7. From period 8 onwards, interest rates are determined by a Taylor-type rule. For ease of interpretation, we write the monetary policy rule with re-scaled coefficients as

$$R_t = \begin{cases} \underline{R} := 0 & t \in [0,7] \\ \rho_R R_{t-1} + (1-\rho_R)\left[\overline{R} + \delta_\pi(\pi_t - \pi^*) + \delta_y y_t + \delta_{dy}(y_t - y_{t-1})\right] & \text{otherwise} \end{cases} \tag{52}$$

where $\delta_z = \frac{\theta_z}{1-\rho_R}$ for $z \in \{\pi, y, dy\}$.

We consider two different values for the parameter δ_π. We set the first at the baseline value used above (i.e. $\delta_\pi = 0.5/0.35 \approx 1.429$); we think of this as a benchmark case where there is a determinate (i.e unique) terminal solution. Our second choice is $\delta_\pi = 0.9$, which is weaker than the more than one-for-one response suggested by the Taylor principle. In this case there is degree 1 indeterminacy – i.e. there are multiple stable terminal solutions.[27] Accordingly, we use the approach in Section 4.3 that redefines sunspots as fundamental shocks so that determinacy is restored and we can proceed using standard methods.

To motivate forward guidance, the economy is hit in period 0 with the same negative demand shock $\epsilon_{g,0} = -0.08$ as in our previous exercise. Since there is degree 1 indeterminacy when $\delta_\pi = 0.9$ and two forward-looking variables (π_t and y_t), we must decide which one is hit with sunspot fluctuations. Here we take inflation and hence $\pi_t = s_{t-1} + v_{\pi,t}$, where $v_{\pi,t}$ is a zero-mean sunspot and s_t $(= E_t\pi_{t+1})$ is a new state variable. The augmented vectors are thus $\tilde{x}_t = \begin{bmatrix}\pi_t & y_t & R_t & a_t & g_t & \mu_t & s_t\end{bmatrix}'$ and $\tilde{e}_t = \begin{bmatrix}v_{\pi,t} & \epsilon_{a,t} & \epsilon_{g,t} & \epsilon_{\mu,t}\end{bmatrix}'$ (see (36)–(37)).

As in Gibbs and McClung (2020), we assume that the policy announcement coordinates agents' expectations on the sunspot. We assume, for simplicity, that the sunspots $v_{\pi,t}$ are uncorrelated with the structural shocks and have an $N(0, 0.0005^2)$ distribution; the latter makes multiple solution paths clearly visible in simulations.[28]

Figure 4 plots the deterministic solutions (i.e. impulse responses) following the negative demand shock in period 0. In the *determinate* case, we see the standard result that forward guidance has very strong effects on inflation and output, with both substantially above steady state despite the deflationary shock (solid line, left and middle). We also see 'hump-shaped' dynamics in inflation and output as in our earlier forward guidance exercise (Fig. 3), with

[27] In the model presented here (with predetermined endogenous variables) there is no analytically tractable determinacy condition in terms of $\delta_\pi, \delta_y, \delta_{dy}$; hence we check for determinacy numerically.

[28] We check the Blanchard and Kahn (1980) determinacy condition in the original model, given a particular calibration, and then write system in terms of the vectors $\tilde{x}_t$ and $\tilde{e}_t$, with the structural matrices $\mathcal{B}_{i,t}$ adjusted according to whether there is a sunspot solution or a unique terminal solution.

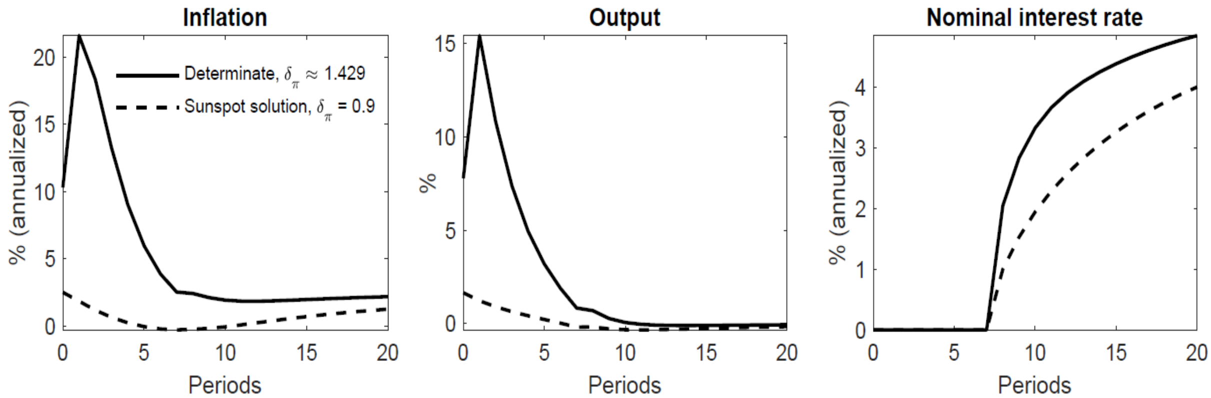

Figure 4: **Deterministic simulation: determinate vs sunspot solution.** The figure shows impulse responses to a four-s.d. demand shock $\epsilon_{g,0} = -0.08$ followed by forward guidance that is announced in period 0 and implemented in periods 0 to 7.

both variables reaching a peak in period 1. Thereafter, inflation, output and interest rates move smoothly – and fairly rapidly – toward the steady state, with a small 'bump' in period 8 when policy switches back to the Taylor-type rule.

In the indeterminate case, the impulse responses of inflation and output are tame by comparison, and both variables have a slight U-shape response (dashed line, left and middle panels). The possibility that sunspot solutions might not display the usual forward guidance puzzle is highlighted by Gibbs and McClung (2020), who argue that iterative expectational stability determines whether a particular model has bounded impulse responses as the forward guidance horizon is increased. For the sunspot solution here, the responses of endogenous variables are bounded as the horizon is increased and hence the puzzle is absent; the same result is found by Gibbs and McClung (2020) (Fig. 3) in numerical analysis of the Smets-Wouters model.[29] Finally, note that the impulse responses of the sunspot solution are more persistent, which is intuitive since inflation expectations are a state variable.

Figure 5 plots stochastic simulation paths following the same initial demand shock as in Figure 4; note that in this case the subsequent structural shocks to technology, mark-ups and demand are drawn from normal distributions with the standard deviations in Section 5.1. In the determinate case (solid line), the solution follows a typical 'jagged' pattern with mild persistence, and, given the draws of shocks, the initial boom in inflation and output is followed immediately by deflation and a large drop in output. In the indeterminate case, there are multiple terminal solutions for $t > 7$ that depend on the sunspots $v_{\pi,t}$; two such solutions are plotted in Figure 5 (dashed lines).[30] These simulation paths are based on the

[29] Our code allows the forward guidance horizon to be arbitrarily long, and hence it is simple to check this.

[30] In the deterministic simulation, we set the sunspots equal to zero. Note that in both the deterministic and stochastic cases the simulation paths coincide for $t \leq 6$ (see (39)).

same sequences of structural shocks $\epsilon_{a,t}, \epsilon_{g,t}, \epsilon_{\mu,t}$ but with two different sequences of sunspot shocks $v_{\pi,t}$ drawn from the $N(0, 0.0005^2)$ distribution.

Non-trivial differences in the simulation paths for inflation and nominal interest rates emerge under the two sunspot solutions. By comparison to the determinate solution, the simulation paths for inflation and the nominal interest rate display long-lived cycles, indicating strong persistence. It is well-known that these differences between sunspot and determinate solutions may arise (e.g. Benhabib and Farmer (1999)).

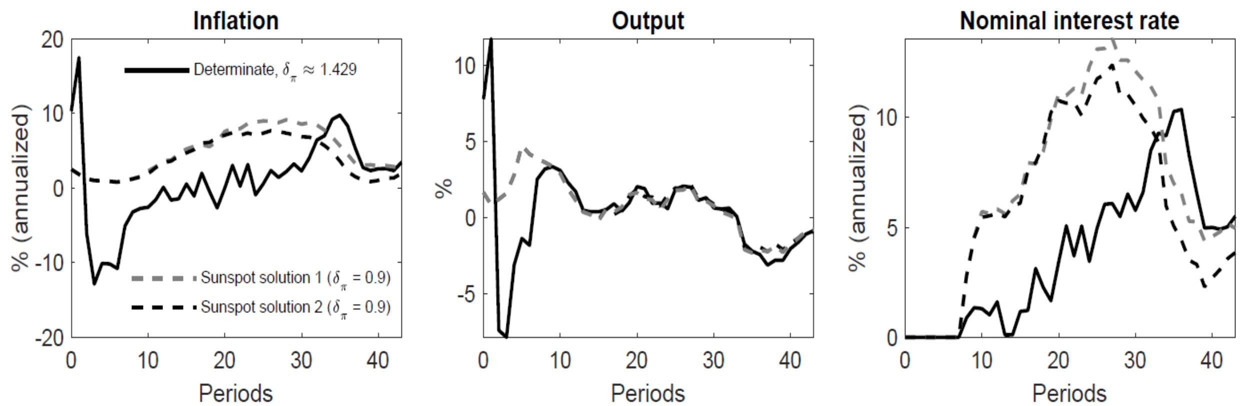

Figure 5: **Stochastic simulation: determinate vs sunspot solutions.** The figure shows stochastic simulation paths after a four-s.d. demand shock $\epsilon_{g,0} = -0.08$ followed by forward guidance that is announced in period 0 and implemented in periods 0 to 7.

5.2 Pension reform in the Diamond model

As a second application, we consider a social security reform. In particular, we consider a pension reform that reduces the pension benefit (and contribution) rate in the Diamond (1965) model. This example is useful because it illustrates the study of optimal announcement dates (see Section 4.1), as well as providing a case where the solution method is applied to an underlying non-linear model that is log-linearized whilst taking into account the change in steady state triggered by the reform.[31] Since such pension reforms have been studied previously, we also keep contact with known results in the literature.

Pension reform in the log-utility case is studied by Fedotenkov (2016). We use the same perfect foresight model, except for the addition of CES utility U_t with elasticity σ^{-1}. There are two generations alive at any given date t, the young (y) and the old (o). Households have two-period lives and care only about consumption; hence $U_t = \frac{c_{t,y}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{t+1,o}^{1-\sigma}}{1-\sigma}$, where $\beta > 0$. The pension contribution rate is τ and the pension system is balanced budget. Population

[31] We proceed by first expressing all variables in log deviations from steady state and then re-writing both regimes in terms of a common steady-state deviation x_t (see Fn. 6, Fn. 33 and Supplementary Appendix).

N_t grows at rate n and individual labour supply is 1. Output is produced by a representative firm with a Cobb-Douglas production function, $Y_t = K_t^{\alpha} N_t^{1-\alpha}$. The factor prices of labour and capital are w_t and R_t, respectively. Capital depreciates fully in a generation.

The central equation of the model is the capital accumulation equation:

$$k_{t+1} = \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}}}{(1 + \frac{(1-\alpha)\tau}{\alpha}) R_{t+1} + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}}} \left(\frac{(1-\tau)(1-\alpha)}{1+n} \right) k_t^{\alpha}$$

where $k_{t+1} := K_{t+1}/N_{t+1}$ and $R_t = \alpha k_t^{\alpha-1}$.

The above equation makes clear that capital accumulation depends on the pension contribution rate, τ. Except for the case of log utility ($\sigma = 1$), there is no analytical solution. Accordingly, we log-linearize the non-linear model to get the following system:

$$\hat{c}_{t,y} = -\frac{1}{\sigma}\hat{R}_{t+1} + \hat{c}_{t+1,o}, \qquad \hat{R}_t = (\alpha - 1)\hat{k}_t, \qquad \hat{w}_t = \alpha \hat{k}_t$$
$$\hat{c}_{t,y} = \frac{(1-\tau)(1-\alpha)k^{\alpha}}{c_y}\hat{w}_t - \frac{(1+n)k}{c_y}\hat{k}_{t+1}, \qquad \hat{c}_{t,o} = \alpha \hat{k}_t$$

where $c_y = (1-\tau)(1-\alpha)k^{\alpha} - (1+n)k$, k is steady state capital and 'hats' are log deviations from steady state.[32] These equations are, respectively, the Euler equation, the equilibrium factor prices, and the budget identities of young and old after market clearing is imposed.

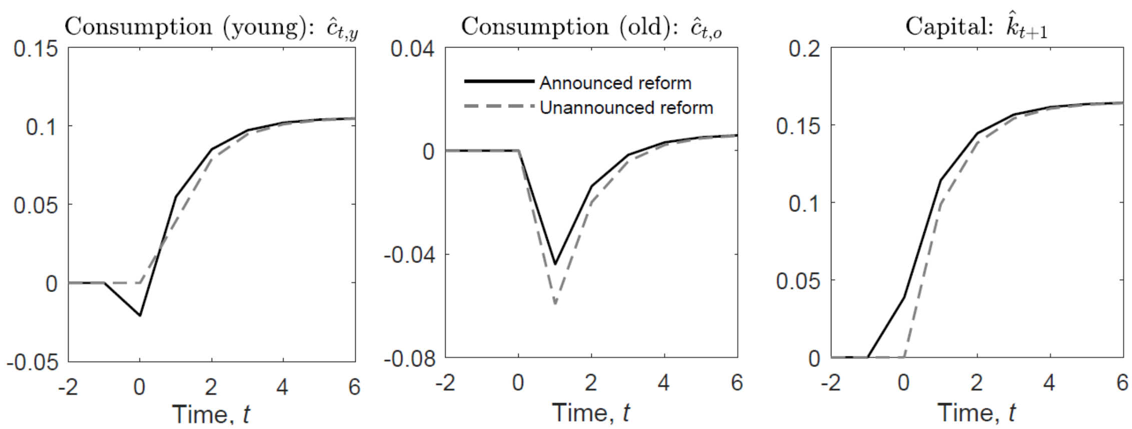

Figure 6: Impact of a pension reform at date 1. Announced vs unannounced (log utility).

As in Fedotenkov (2016), we consider a pension reform at date $t = 1$ that may be announced (in period 0) or unannounced. The pension contribution rate is reduced from $\tau = 0.20$ to $\tau' = 0.15$. The other parameters are $\alpha = 0.4$, $\beta = \frac{1}{(1+0.01)^{35}} = 0.760$, $n = 0$, and we initially set $\sigma = 1$ (log utility) to keep contact with known results. The transition paths

[32] The steady state cannot be found analytically if $\sigma \neq 1$, so we rely on numerical methods.

in Figure 6 are consistent with the results in Fedotenkov (2016) and the reply by Hatcher (2019). Note that whereas these authors solved the fully non-linear model under log utility, the results here are based on the linearized system above while allowing for the change in the contribution rate and the new steady state to which the model converges.[33]

The announced reform initially lowers consumption by the young, as they respond to the future reduction in pension benefits by saving more, so that capital accumulation rises (left and right panels). Consumption of the old falls when the reform is implemented at $t = 1$, but the drop in consumption is lower if the reform is announced, as pensioners can consume out of their extra savings when young. Since both consumption of the young and expected old-age consumption fall on announcement in period $t = 0$, lifetime utility goes down (Fedotenkov (2016)) – and so may social welfare. Hatcher (2019) studies social welfare for a range of social discount factors $\gamma \in (0, 1)$, again working with log utility. We now reconsider the impact on social welfare for the case of CES utility.[34]

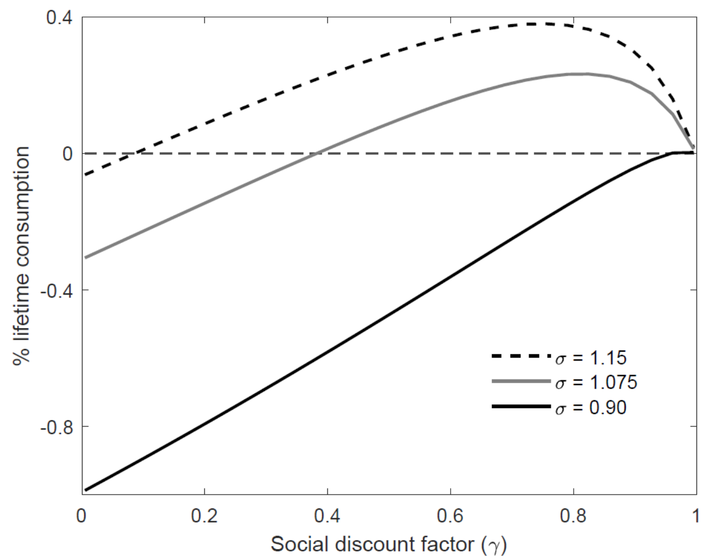

Figure 7: Is it optimal to announce pension reform? Welfare gain (loss) of announced reform.

The results in Figure 7 show that as the elasticity of substitution is reduced (increasing σ), an announced reform is more likely to raise social welfare. This result is quite intuitive: if households are not as willing to substitute current for future consumption then the drop in young-age consumption on announcement is attenuated relative to Figure 6, and hence the smaller utility losses of the young at $t = 0$ may be more than offset by the (discounted)

[33]The system is log-linearized around the original steady state for $t = 0$ and around the new steady state thereafter. The model is then written in terms of log deviations from the original steady state, with any constants collected in the matrix $\tilde{B}_5$. Further details are provided in the *Supplementary Appendix*.

[34]Social welfare is $W_0 = \sum_{t=-1}^{\infty} \gamma^t \hat{U}_t$, where $\hat{U}_t$ is a second-order approximation of lifetime utility around steady state. Following Walsh (2017), $\hat{U}_t = U_{SS} + c_y^{1-\sigma}[\hat{c}_{t,y} - \frac{(\sigma-1)}{2}\hat{c}_{t,y}^2] + \beta c_o^{1-\sigma}[\hat{c}_{t+1,o} - \frac{(\sigma-1)}{2}\hat{c}_{t+1,o}^2]$.

utility gains of future generations. The above result clearly speaks to the issue of optimal announcement dates, i.e. the question of *when* to announce a future reform in order to maximize social welfare. We leave further study as a topic for future research.

6 Conclusion

This paper builds on the recursive method in Kulish and Pagan (2017) for solving linear rational models subject to anticipated structural changes. We presented three extensions to the benchmark solution that allow for a fraction of uninformed agents; imperfect credibility of structural changes due to some fraction of agents basing their expectations on an alternative sequence of structures; and structural changes that involve indeterminacy of the terminal solution (i.e. multiple equilibria). These extensions should interest a wide audience, especially those who want to take methods 'off the shelf' and use them in a variety of applications, including solving medium and large-scale DSGE models.

We illustrated our extensions using several numerical applications. For example, we saw that indeterminacy of the terminal solution can be dealt with using standard solution methods and may have important implications for the impact of policy announcements. We also saw that our extension with uninformed agents may be used to study announced versus unannounced reforms and optimal announcement dates, whereas our imperfect credibility approach can have quite different implications to conventional approaches that warrant further investigation.

There are several promising avenues for future research. First, it would be of interest to estimate models incorporating the extensions presented in this paper, thus building on works by Kulish and Pagan (2017) and Kulish et al. (2017). Second, further research on optimal announcement dates would be of interest given the widespread use of policies like forward guidance that rely on announcement or signalling effects due to the release of new information. Finally, since a linearized model may not provide a good approximation to an underlying nonlinear model, extending solutions to non-linear models would be of value. One method that could improve accuracy is the dynamic perturbation approach of Mennuni and Stepanchuk (2018), which approximates a model at many points along the transition path. We leave a formal investigation of these issues for future research.

References

Adjemian, S., Bastani, H., Juillard, M., Mihoubi, F., Perendia, G., Ratto, M., and Villemot, S. (2011). Dynare: Reference manual, version 4.

Ambler, S. (2014). Price-level targeting: A post-mortem? *CD Howe Institute Commentary*, (400).

Anderson, G. and Moore, G. (1985). A linear algebraic procedure for solving linear perfect foresight models. *Economics Letters*, 17(3):247–252.

Ascari, G. and Ropele, T. (2013). Disinflation effects in a medium-scale new keynesian model: money supply rule versus interest rate rule. *European Economic Review*, 61:77–100.

Ball, L. (1995). Disinflation with imperfect credibility. *Journal of Monetary Economics*, 35(1):5–23.

Benhabib, J. and Farmer, R. E. (1999). Indeterminacy and sunspots in macroeconomics. *Handbook of Macroeconomics*, 1:387–448.

Bianchi, F. and Nicolò, G. (2021). A generalized approach to indeterminacy in linear rational expectations models. *Quantitative Economics*, 12(3):843–868.

Binder, M. and Pesaran, M. H. (1997). Multivariate linear rational expectations models: characterization of the nature of the solutions and their fully recursive computation. *Econometric Theory*, pages 877–888.

Blanchard, O. J. and Kahn, C. M. (1980). The solution of linear difference models under rational expectations. *Econometrica*, pages 1305–1311.

Bullard, J. and Mitra, K. (2002). Learning about monetary policy rules. *Journal of monetary economics*, 49(6):1105–1129.

Cagliarini, A. and Kulish, M. (2013). Solving linear rational expectations models with predictable structural changes. *Review of Economics and Statistics*, 95(1):328–336.

Cho, S. and Moreno, A. (2011). The forward method as a solution refinement in rational expectations models. *Journal of Economic Dynamics and Control*, 35(3):257–272.

Clarida, R., Gali, J., and Gertler, M. (2000). Monetary policy rules and macroeconomic stability: evidence and some theory. *The Quarterly Journal of Economics*, 115(1):147–180.

Davig, T. and Leeper, E. M. (2007). Generalizing the taylor principle. *American Economic Review*, 97(3):607–635.

Del Negro, M., Giannoni, M. P., and Patterson, C. (2012). The forward guidance puzzle. *FRB of New York Staff Report*, (574).

Diamond, P. A. (1965). National debt in a neoclassical growth model. *The American Economic Review*, 55(5):1126–1150.

Erceg, C. J. and Levin, A. T. (2003). Imperfect credibility and inflation persistence. *Journal of Monetary Economics*, 50(4):915–944.

Farmer, R. E., Khramov, V., and Nicolò, G. (2015). Solving and estimating indeterminate dsge models. *Journal of Economic Dynamics and Control*, 54:17–36.

Farmer, R. E., Waggoner, D. F., and Zha, T. (2009). Understanding markov-switching rational expectations models. *Journal of Economic Theory*, 144(5):1849–1867.

Fedotenkov, I. (2016). Ignorance is bliss: Should a pension reform be announced? *Economics Letters*, 147:135–137.

Gibbs, C. and McClung, N. (2020). Does my model predict a forward guidance puzzle? *Available at SSRN 3662025*.

Gibbs, C. G. and Kulish, M. (2017). Disinflations in a model of imperfectly anchored expectations. *European Economic Review*, 100:157–174.

Goodfriend, M. and King, R. G. (2005). The incredible volcker disinflation. *Journal of Monetary Economics*, 52(5):981–1015.

Guerrieri, L. and Iacoviello, M. (2015). Occbin: A toolkit for solving dynamic models with occasionally binding constraints easily. *Journal of Monetary Economics*, 70:22–38.

Haberis, A., Harrison, R., and Waldron, M. (2019). Uncertain policy promises. *European Economic Review*, 111:459–474.

Hatcher, M. (2019). Should a pension reform be announced? a reply. *Economics Letters*, 183:108583.

Ireland, P. N. (2007). Changes in the federal reserve's inflation target: Causes and consequences. *Journal of Money, credit and Banking*, 39(8):1851–1882.

Jones, C. (2016). Extracting rational expectations model structural matrices from dynare.

Jones, C. (2017). Unanticipated shocks and forward guidance at the zero lower bound. Technical report, NYU Working Paper.

Jones, C. and Kulish, M. (2013). Long-term interest rates, risk premia and unconventional monetary policy. *Journal of Economic Dynamics and Control*, 37(12):2547–2561.

King, R. G. and Watson, M. W. (1998). The solution of singular linear difference systems under rational expectations. *International Economic Review*, pages 1015–1026.

Klein, P. (2000). Using the generalized schur form to solve a multivariate linear rational expectations model. *Journal of Economic Dynamics and Control*, 24(10):1405–1423.

Kryvtsov, O., Shukayev, M., and Ueberfeldt, A. (2008). Adopting price-level targeting under imperfect credibility. Technical report, Bank of Canada Working Paper.

Kulish, M., Morley, J., and Robinson, T. (2017). Estimating dsge models with zero interest rate policy. *Journal of Monetary Economics*, 88:35–49.

Kulish, M. and Pagan, A. (2017). Estimation and solution of models with expectations and structural changes. *Journal of Applied Econometrics*, 32(2):255–274.

Lubik, T. A. and Schorfheide, F. (2003). Computing sunspot equilibria in linear rational expectations models. *Journal of Economic Dynamics and Control*, 28(2):273–285.

Lubik, T. A. and Schorfheide, F. (2004). Testing for indeterminacy: An application to us monetary policy. *American Economic Review*, 94(1):190–217.

Mennuni, A. and Stepanchuk, S. (2018). Dynamic perturbation. Technical report, Working Paper.

Mertens, K. and Ravn, M. O. (2012). Empirical evidence on the aggregate effects of anticipated and unanticipated us tax policy shocks. *American Economic Journal: Economic Policy*, 4(2):145–81.

Nicolae, A. and Nolan, C. (2006). The impact of imperfect credibility in a transition to price stability. *Journal of Money, Credit and Banking*, pages 47–66.

Rogoff, K. and Obstfeld, M. (1996). Foundations of international macroeconomics, mit press.

Schaumburg, E. and Tambalotti, A. (2007). An investigation of the gains from commitment in monetary policy. *Journal of Monetary Economics*, 54(2):302–324.

Sims, C. A. (2002). Solving linear rational expectations models. *Computational Economics*, 20(1-2):1.

Uhlig, H. (1999). A toolkit for analyzing nonlinear dynamic stochastic models easily,. *In (Marimon, Ramon and Scott, Andrew eds.) Computational Methods for the Study of Dynamic Economies*, Oxford University Press.

Walsh, C. E. (2017). *Monetary Theory and Policy*. MIT press.

Yetman, J. (2005). The credibility of the monetary policy “free lunch”. *Journal of Macroeconomics*, 27(3):434–451.

Appendix

Proof of Proposition 1

The model is given by

$$\begin{cases} B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, & 0 \le t \le \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & t > \tilde{T} \end{cases} \tag{A1}$$

where $E_t[e_{t+1}] = 0_{m\times 1}$.

Consider first the periods $0 \le t \le \tilde{T}$. Suppose there exist a set of well-defined matrices $\{\Omega_t, \Gamma_t, \Psi_t\}$ (with non-stochastic entries) such that for all $t \in \{0, ..., \tilde{T}\}$,

$$x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t. \tag{A2}$$

Shifting (A2) forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = \Omega_{t+1} x_t + \Psi_{t+1}, \quad 0 \le t \le \tilde{T} - 1. \tag{A3}$$

Substituting (A3) into the first line of (A1) and rearranging gives

$$(B_{1,t} - B_{2,t}\Omega_{t+1})x_t = B_{3,t}x_{t-1} + B_{4,t}e_t + B_{2,t}\Psi_{t+1} + B_{5,t}, \quad 0 \le t \le \tilde{T} - 1. \tag{A4}$$

Provided $\Omega_{\tilde{T}}, \Gamma_{\tilde{T}}, \Psi_{\tilde{T}}$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$, the set $\{\Omega_t, \Gamma_t, \Psi_t\}$ is well defined for t where these matrices are given by Proposition 1. Therefore, if $\Omega_{\tilde{T}}, \Gamma_{\tilde{T}}, \Psi_{\tilde{T}}$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ for $t = 0, ..., \tilde{T} - 1$, the sequences of $\{\Omega_t, \Gamma_t, \Psi_t\}$ are well-defined for $t = 0, ..., \tilde{T} - 1$.

For $t > \tilde{T}$, we have by Assumption 1, $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Gamma}e_t + \tilde{\Psi}$ where $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_3$, $\tilde{\Gamma} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_4$, $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}(\tilde{B}_2\tilde{\Psi} + \tilde{B}_5)$ are unique and well-defined. Hence,

$$E_t x_{t+1} = \tilde{\Omega}x_t + \tilde{\Psi}, \quad \forall t \ge \tilde{T}. \tag{A5}$$

The matrices $\Omega_{\tilde{T}}, \Gamma_{\tilde{T}}, \Psi_{\tilde{T}}$ are determined by the first line of (A1) and (A5) at $t = \tilde{T}$:

$$B_{1,\tilde{T}}x_{\tilde{T}} = B_{2,\tilde{T}}E_{\tilde{T}}x_{\tilde{T}+1} + B_{3,\tilde{T}}x_{\tilde{T}-1} + B_{4,\tilde{T}}e_{\tilde{T}} + B_{5,\tilde{T}}, \qquad E_{\tilde{T}}x_{\tilde{T}+1} = \tilde{\Omega}x_{\tilde{T}} + \tilde{\Psi}$$

or $(B_{1,\tilde{T}} - B_{2,\tilde{T}}\tilde{\Omega})x_{\tilde{T}} = B_{3,\tilde{T}}x_{\tilde{T}-1} + B_{4,\tilde{T}}e_{\tilde{T}} + B_{2,\tilde{T}}\tilde{\Psi} + B_{5,\tilde{T}}$. Provided $\det[B_{1,\tilde{T}} - B_{2,\tilde{T}}\tilde{\Omega}] \neq 0$, the matrices $\Omega_{\tilde{T}}, \Gamma_{\tilde{T}}, \Psi_{\tilde{T}}$ are given by the expressions in Proposition 1. ■