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The phenomenology of dynamical neutron star tides

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ABSTRACT

We introduce a phenomenological, physically motivated, model for the effective tidal deformability of a neutron star, adding the frequency dependence (associated with the star's fundamental mode of oscillation) that comes into play during the late stages of the binary inspiral. Testing the model against alternative descriptions, we demonstrate that it provides an accurate representation of the dynamical tide up to close to merger. The simplicity of the prescription makes it an attractive alternative for a gravitational-wave data analysis implementation, facilitating an inexpensive construction of a large number of templates covering the relevant parameter space.

Key words: gravitational waves – stars: neutron – neutron star mergers.

1 INTRODUCTION

The inspiral and merger of binary neutron stars have long been considered a bread-and-butter source for advanced gravitational-wave detectors. Hence, the release of pent-up excitement following the spectacular GW170817 event (Abbott et al. 2017a, 2019) came as no surprise. After all, these events have the potential to unlock scientific mysteries from astrophysics (confirming binary mergers as the origin of short gamma-ray bursts and explaining the cosmic generation of heavy elements) and cosmology (through inferred values of the Hubble constant) through to nuclear physics (as the imprint of matter on the gravitational-wave signal helps constrain the equation of state relevant for the extreme conditions represented by neutron stars). Remarkably, the GW170817 event led to progress in all these directions (Abbott et al. 2017b, c, 2018; Cowperthwaite 2017; Kasliwal et al. 2017) and the number of relevant analyses and discussions is already overwhelming.

The analysis of the GW170817 data has led to a – perhaps surprisingly tight – constraint on the neutron star tidal deformability, commonly expressed in terms of the dimensionless parameter (Flanagan & Hinderer 2008; Hinderer et al. 2010):

$$\Lambda_l = \frac{2}{(2l-1)!!}\frac{k_l}{\mathcal{C}^{2l+1}} , \qquad (1)$$

where k_l is the Love number, l is the relevant multipole (with the main contribution from the quadrupole, $l = 2$), and $\mathcal{C} = GM_\star/Rc^2$ is the compactness of the star ($M_\star$ is the mass and R is the radius, and in the following we will use geometric units with $c = G = 1$). Given that the tidal imprint enters, formally, at the fifth post-Newtonian order (Flanagan & Hinderer 2008), the mass of the two binary partners can be inferred from lower order post-Newtonian terms in the signal. A constraint on Λ_l can then be turned into a constraint on the neutron star radius (Abbott et al. 2018). For example, the analysis of Radice et al. (2018) suggests $400 \lesssim \tilde{\Lambda}_2 \lesssim 800$ (for a suitably averaged quadrupole tidal deformability $\tilde{\Lambda}_2$, depending on the mass ratio and with the lower limit somewhat model dependent). This then leads to the radius being constrained to (roughly) the range of 10–13 km, which overlaps with the first results for PSR J0030+0451 from NICER (Miller et al. 2019; Riley et al. 2019). The bottomline is that our current understanding of the neutron star radius comes with error bars at the 10 per cent level. This demonstrates how observations are beginning to constrain the theory and the neutron star equation of state, but we have not yet reached the (better than) 5 per cent error bars on the radius, the rough target for NICER and the point at which astrophysics observations would be doing 'better' than current and upcoming nuclear physics experiments. Relevant efforts to constrain the strong interaction in laboratory experiments include the PREX effort, which probes the neutron skin thickness in lead, where a thick neutron skin would correspond to a large value for the neutron star tidal deformability (Fattoyev, Piekarewicz & Horowitz 2018).

The error bars on the neutron star radius will tighten with future NICER data and precision measurements of Λ_2 for neutron star binaries. However, the general expectation is that – unless we get very lucky! – the latter may require third-generation gravitational-wave instruments (like the Einstein Telescope or the Cosmic Explorer, Hall & Evans 2019 or, perhaps, a dedicated high-frequency detector, Ackley et al. 2020), which might be able to constrain Λ_2 to the few per cent level (Martynov et al. 2019). This should then, assuming that the neutron star masses can be inferred to higher precision, lead to a neutron star radius estimate accurate to the few per cent level, as well. This possibility motivates the discussion – the main point of which is to reduce the physics ambiguities and explore possible parameter degeneracies – in this paper.

We outline a phenomenological (yet, physically motivated) model for the dynamical tide in a neutron star binary, adding frequency dependence to the tidal deformability that comes into play during the late stages of inspiral. This is a small effect, but it should be within

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the reach of precision measurements for particularly 'bright' future events. Moreover, the analysis is useful as it provides insight into the 'systematic errors' associated with the tidal deformability, in the same way that our recent work (Andersson & Pnigouras 2020) sheds light on the level at which the matter composition in the neutron star core comes into play. Moreover, the simplicity of the model makes it an attractive alternative for a data analysis implementation, which requires inexpensive construction of a large number of templates covering the relevant parameter space. Having said that, the aim of our discussion is manifestly not to develop such a model. Our focus is on the physics of the problem.

2 A SIMPLE PHENOMENOLOGICAL MODEL

We take as our starting point the discussion of Andersson & Pnigouras (2020), where the tidal response of a star is expressed in terms of the star's normal modes of oscillation.[1] The original analysis aimed to provide an idea of the 'error' associated with the assumption that a deformed neutron star is described by a barotropic (beta equilibrium) matter model rather than a model in which the matter composition is frozen as the system spirals through the sensitivity band of a gravitational-wave detector (which is expected as the time-scale associated with nuclear reactions is much longer than that of the inspiral). The question arises as most work on the tidal problem draws on phenomenological equation-of-state models (like piecewise polytropes or parametrizations based on the speed of sound), which do not account for fine-print issues like the state and composition of matter. The results from Andersson & Pnigouras (2020) demonstrate that the dynamical contribution to the tide is dominated by the excitation of the fundamental mode (f mode) of the star. This is not surprising. The result was established a long time ago (Lai 1994; Reisenegger & Goldreich 1994; Kokkotas & Schaefer 1995) in work aimed at quantifying the impact of mode resonances on the gravitational-wave signal (see Andersson & Ho 2018, for a recent discussion of this problem), but the discussion from Andersson & Pnigouras (2020) adds a twist to the story. The results demonstrate that the sum over modes converges to the usual Love number in the static limit. Again, this result could have been anticipated. As long as the modes form a complete set, they can be used as a basis to describe any dynamical response of the star. This is well known, but the implications appear not to have been explored in previous work.

Let us make pragmatic use of the results from Andersson & Pnigouras (2020) in order build a simple model that accounts for the main aspects of the dynamical tide. The basic idea is to include only the f-mode contribution to the mode sum and accept the contribution from other modes as a 'systematic error'. Based on the stratified Newtonian models considered by Andersson & Pnigouras (2020), we expect this error to be below the 5 per cent level. This level of uncertainty is smaller than our ignorance of (say) the neutron star equation of state, so the relation we write down should be precise enough for a 'practical' construction of gravitational-wave templates.

In essence, we start from a parametrized version of the Newtonian result for the effective Love number (Andersson & Pnigouras 2020):

$$k_l^{\rm eff} = -\frac{1}{2} + \frac{A_{\rm f}}{\tilde{\omega}_{\rm f}^2 - \tilde{\omega}^2}\left(1 - \tilde{\omega}^2 B_{\rm f}\right)\left(1 - \tilde{\omega}_{\rm f}^2 B_{\rm f}\right)^{-1}, \quad (2)$$

where $A_{\rm f}$ depends on the overlap integral between the f mode and the tidal driving, while $B_{\rm f}$ involves the ratio of the horizontal and radial mode eigenfunctions at the star's surface. The f-mode frequency $\tilde{\omega}_{\rm f}$ and the frequency associated with the Fourier transform (see Andersson & Pnigouras 2020, for discussion) $\tilde{\omega}$ are both scaled to $\sqrt{GM/R^3}$. We now insist that the relation (2) returns the usual Love number, k_l, in the static, $\tilde{\omega} \to 0$, limit.[2] This means that we must have

$$A_{\rm f}\left(1 - \tilde{\omega}_{\rm f}^2 B_{\rm f}\right)^{-1} = \tilde{\omega}_{\rm f}^2\left(k_l + \frac{1}{2}\right). \quad (3)$$

We can use this relation to replace one of the unknown parameters, $A_{\rm f}$ or $B_{\rm f}$. Opting to replace the former, we write (2) as

$$k_l^{\rm eff} = \frac{\tilde{\omega}_{\rm f}^2 k_l}{\tilde{\omega}_{\rm f}^2 - \tilde{\omega}^2} + \frac{\tilde{\omega}^2}{\tilde{\omega}_{\rm f}^2 - \tilde{\omega}^2}\left[\frac{1}{2} - \tilde{\omega}_{\rm f}^2 B_{\rm f}\left(k_l + \frac{1}{2}\right)\right]. \quad (4)$$

This expression is instructive. First of all, let us make the connection with an inspiralling binary by adding the usual assumption of adiabaticity, which links the Fourier frequency of the tidal response to the orbital frequency Ω. Focusing on the quadrupole modes, which make the main contribution to the gravitational-wave signal, we then have $\tilde{\omega} = 2\tilde{\Omega}$. We can also connect to the post-Newtonian expansion, which is commonly expressed in terms of the dimensionless parameter $x = (\Omega M)^{2/3}$, where M is the total mass of the system ($x = v^2$, where v is the orbital velocity, and the static tide enters the post-Newtonian expansion at order $v^5 = x^{5/2}$, representing a fifth-order contribution in the usual counting). From (4), we then see that – in contrast to other models that aim to account for the dynamical tide by including the main mode resonance (essentially adding a harmonic oscillator term to the adiabatic inspiral Hamiltonian), e.g. Hinderer et al. (2016) and Steinhoff et al. (2016) – the effective Love number is not (overall) proportional to the static one. However, the difference appears at a higher post-Newtonian order. Specifically, the difference enters as we, formally, add an order $x^{11/2} = v^{11}$ term to the expansion. The fact that this is a very high order contribution, which one would normally safely neglect, accords with the recent results of Schmidt & Hinderer (2019) (which, in turn, draw on the formulation from Flanagan & Hinderer 2008). However, the argument is somewhat misleading because a formal low-frequency expansion fails to represent the f-mode resonance feature that (for typical neutron star parameters) will be prominent as the system approaches merger. This is, indeed, evident from the discussions of Hinderer et al. (2016) and Steinhoff et al. (2016). A reasonable model has to retain the resonance feature. With this in mind, it is useful to note the result for incompressible stars (Andersson & Pnigouras 2020). In this case, the relation

$$k_l^{\rm eff} = \frac{\tilde{\omega}_{\rm f}^2 k_l}{\tilde{\omega}_{\rm f}^2 - \tilde{\omega}^2} \quad (5)$$

is exact; that is, the leading term in the expression for the dynamical tide (4) takes the same form regardless of the matter compressibility.

Newtonian results – like (4) – lead to useful intuition for the qualitative behaviour, but they do not provide quantitative solutions

[1]As a slight aside, it is worth noting the similarity between the mode-sum for the tidal response and the sum-rule for the electric and magnetic dipole polarizability in atoms and nuclei (Bernabéu, Gómez Dumm & Orlandini 1998; Mitroy, Safronova & Clark 2010), yet another connection between the astrophysics and laboratory experiments.

[2]This is the point where we ignore the formal contributions from other oscillations modes and, hence, the impact of the matter composition, which was the main focus of the discussion of Andersson & Pnigouras (2020).

to the actual problem (which obviously involves realistic neutron star models that require us to account for general relativity) we are interested in. However, the derivation of analogous results in the relativistic case poses a technical challenge and there has (at least so far) been very little work in this direction. This is unfortunate as it means that we do not yet have a precise model that can be meaningfully compared to, for example, the results of numerical simulations for the late stages of binary inspiral. However, if we assume that the form of the expression for the effective tide remains unchanged – and allow ourselves the freedom to introduce a couple of adjustable parameters – it is straightforward to write down a model that brings us closer to the result we need.

A logical first step in this direction involves the known 'universal relation' between the f-mode frequency and the tidal deformability (see Chan et al. 2014, as well as Andersson & Kokkotas 1998; Yagi & Yunes 2013). This relation allows us to replace the mode frequency $\tilde{\omega}_{\rm f}$ in (4) (say) with an expression involving k_l (or, equivalently, Λ_l). This step should be 'safe' given that the universal relation from Chan et al. (2014) is robust[3] (the evidence suggests that the errors involved are smaller than the error we introduce by assuming that the full mode-sum for the tide is replaced by the single contribution from the f mode in the first place). The relation we need takes the form

$$\bar{\omega}_{\rm f} = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 \ , \tag{6}$$

where

$$\bar{\omega}_{\rm f} \equiv M_\star \omega_{\rm f} = \tilde{\omega}_{\rm f} \mathcal{C}^{3/2} \tag{7}$$

and $y = \ln \Lambda_l$ (with the a_i parameters listed in table I of Chan et al. 2014). We see that we now need the stellar compactness $\mathcal{C}$. However, we can make use of another 'universal' relation, this time between the compactness and the (quadrupole) tidal deformability (Masselli et al. 2013)

$$\mathcal{C} \approx 3.71 \times 10^{-1} - 3.91 \times 10^{-2} y + 1.056 \times 10^{-3} y^2 \ , \tag{8}$$

to arrive at a one-parameter expression for the combination of the coefficients on the left-hand side of (3). Noting that $B_{\rm f} = 1/l$ for a homogeneous stellar model we represent the remaining parameter by ϵ, such that $B_{\rm f} = \epsilon/l$ (and it is worth noting that $\epsilon \approx 0.9$ for the polytropic models considered by Andersson & Pnigouras 2020).

We now have an explicit analytic formula for the effective Love number in terms of the result in the static limit, k_l, the (dimensionless) frequency $\bar{\omega} = \omega M_\star$ and the (to some extent) free parameter ϵ. Moreover, even though it was based on a Newtonian analysis, the result makes use of fully relativistic relations for the static Love number and the mode frequency. In order to make the connection with the gravitational-wave signal, we relate $\bar{\omega}$ to the (similarly scaled) orbital frequency $\bar{\Omega}$. Focusing on the $l = m = 2$ resonance, we then have (for an equal-mass binary)

$$\bar{\omega} = 2\bar{\Omega} = 2\Omega M_\star = \Omega M = x^{3/2}, \tag{9}$$

where M is the total mass of the system, as before, and x is the post-Newtonian expansion parameter.

Finally, we note that the mode frequency in (6) includes the gravitational redshift (i.e. refers to an observer at infinity), while the tidal interaction involves a binary partner at a finite distance. In order to account for this, we introduce a second free parameter, δ, such $\bar{\omega}_{\rm f}^2 \to \bar{\omega}_{\rm f}^2/\delta$. This then leads to the expression

$$k_l^{\rm eff} \approx \frac{\bar{\omega}_{\rm f}^2 k_l}{\bar{\omega}_{\rm f}^2 - \delta(2\bar{\Omega})^2} + \frac{(2\bar{\Omega})^2}{\bar{\omega}_{\rm f}^2 - \delta(2\bar{\Omega})^2}\left[\frac{\delta}{2} - \frac{\bar{\omega}_{\rm f}^2}{\mathcal{C}^3}\frac{\epsilon}{l}\left(k_l + \frac{1}{2}\right)\right] \ . \tag{10}$$

The new parameter δ has to account for two relativistic effects. First, we have the gravitational redshift of the mode frequencies that we have already alluded to. Secondly, we need to consider the rotational frame-dragging induced by the orbital motion. The problem is not trivial, but an argument by Steinhoff et al. (2016) suggests that the two effects almost cancel so – as a first approximation – we are motivated to simply remove the gravitational redshift altogether. Simplistically, for a signal emitted at r_0 and observed at r_1 we have

$$\frac{\omega_1}{\omega_0} = \left(\frac{1 - 2M_\star/r_0}{1 - 2M_\star/r_1}\right)^{1/2} . \tag{11}$$

If ω_0 is the frequency 'emitted' at R and $\omega_{\rm f}$ is the frequency observed at infinity, then

$$\omega_{\rm f}^2 = (1 - 2\mathcal{C})\,\omega_0^2. \tag{12}$$

Keeping in mind that the mode frequencies we use include the gravitational redshift, we may simply take $\delta = 1 - 2\mathcal{C}$ to remove it. This then reduces the result to a one-parameter expression for the effective Love number. As we will now demonstrate, this expression performs surprisingly well.

As a suitable comparison (in fact, the only comparable results in the literature), we consider the results for the dynamical tide from Hinderer et al. (2016) and Steinhoff et al. (2016), which are similar in spirit as they introduce the notion of an effective tidal deformability. However, the main focus of Hinderer et al. (2016) and Steinhoff et al. (2016) was to extend the effective-one-body framework to account for the dynamical tide. In addition to this, Steinhoff et al. (2016) provide an approximate analytical formula from matched asymptotics. This result has been tested against numerical relativity simulations (see e.g. Foucart et al. 2019). As it appears to perform well in such comparisons (Dietrich & Hinderer 2017), it provides a natural benchmark against which to test our simple closed-form expression.

Focusing on the example illustrated by Steinhoff et al. (2016), i.e. an equal-mass neutron star binary with $M_\star = 1.350\ {\rm M}_\odot$ and $R = 13.5$ km, leading to $\mathcal{C} = 0.148$ and $\Lambda_2 = 1111$, it is easy to demonstrate that we obtain an accurate representation of the dynamical tide throughout the relevant frequency range (up to close to merger where our simple prescription obviously breaks as the expression for $k_l^{\rm eff}$ diverges[4]) by tuning the parameter ϵ. The effective Love number obtained from (10) is compared to the results from Steinhoff et al. (2016) in Fig. 1, for both $l = 2$ and 3. As our formula still leaves ϵ as a free parameter, we show results for the range $\epsilon = 0.85$–0.9 and, as is clear from the result in the figure, the

[3]While we are confident in this argument, one should be aware that the universal relation involves a conservative view of the equation of state, e.g. the absence of sharp phase transitions (Han & Steiner 2019). It would be useful to establish what happens for models that include phase transitions. However, if the relation breaks then so do related assumptions, like the I–Love–Q relations (Yagi & Yunes 2013) that are already used to break degeneracies in binary inspiral waveforms, but there is no indication that this is the case (Paschalidis et al. 2018). In that situation, one might still be able to make progress by separately constraining the tidal deformability and the f-mode frequency (Pratten, Schmidt & Hinderer 2020).

[4]In principle, it is fairly straightforward to regularize the resonance should one want to do so; see, for example, the stationary phase approximation used by Steinhoff et al. (2016). However, as it is not clear how this procedure changes when the problem is considered in general relativity, and the spirit of our strategy is to facilitate an intuitive extension in that direction, we will leave the singular behaviour in the phenomenological model.

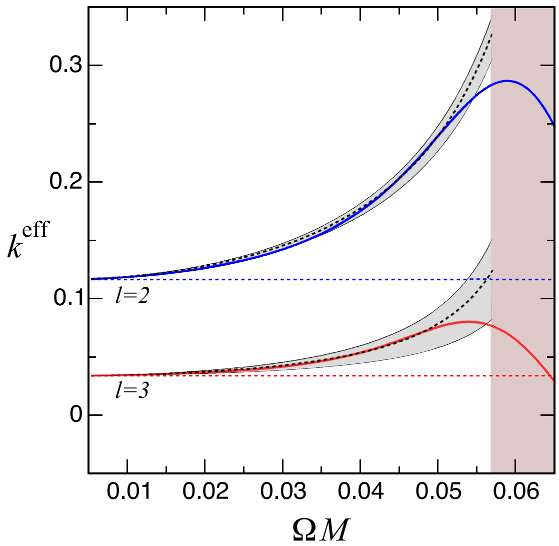

Figure 1. Comparing the effective Love number $k_l^{\rm eff}$ from (10) with $\delta = 1 - 2\mathcal{C}$ to the results from Steinhoff et al. (2016), for both $l = 2$ and 3. The dashed horizontal lines represent the static Love number ($k_l^{\rm eff}$ in the $\Omega \to 0$ limit). The results of Steinhoff et al. (2016) are shown as solid curves (blue for $l = 2$ and red for $l = 3$). Estimates from (10) are shown for the range $\epsilon = 0.85$–0.9 with the latter representing the lower edge of the filled band in each case. The particular choice $\epsilon = 0.875$ (dashed black curves) provides an excellent fit to the results from Steinhoff et al. (2016). Finally, we indicate the region beyond the (approximate) merger frequency, $\Omega M \gtrsim 0.057$ in this case, by the shaded area in the figure.

corresponding curves for $\epsilon = 0.875$ provide an excellent match to the results from Steinhoff et al. (2016). It is also worth noting that $\epsilon \approx 0.935$ removes the second term in (10), leaving an expression of the form expected from incompressible models. We note from the figure that this would tend to underestimate the value for $k_l^{\rm eff}$ (the corresponding result would lie beneath the grey regions shown in the figure), which indicates the level at which the parameter ϵ impacts the result. The essence of the comparison is that our formula (10) performs (perhaps surprisingly) well. It may be phenomenological in origin, but there can be little doubt that our simple expression provides an effective representation of the required behaviour, and hence reflects the underlying physics.

It is worth noting that, while the two sets of results diverge for large values of ΩM in Fig. 1, the corresponding frequencies are close to (or indeed beyond) the merger frequency. As the picture of two separate, tidally deformed, bodies breaks down, there is no reason to expect the model to make sense beyond this point. The post-merger region is indicated by the shaded area in Fig. 1. In order to estimate the merger frequency, we have simply taken the corresponding orbital separation to be the sum of the neutron star radii, which leads to $\Omega M \approx \mathcal{C}^{3/2}$. For the model used in Fig. 1, the estimated merger frequency would then be $\Omega M \approx 0.057$ (corresponding to a gravitational-wave frequency of about 1400 Hz for this specific model). A more precise estimate could be obtained from the results of Read et al. (2013), but this would not change the conclusions.

3 THE STATE OF PLAY

The favourable comparison to the results from Steinhoff et al. (2016) suggests that the simple relation for the effective tidal deformability (10) remains useful up to close to the final merger. The main lesson from this is that the phenomenology of the problem is clear. The discussion admittedly does not add much to the well-established logic developed for the Newtonian tidal problem (Lai 1994; Reisenegger & Goldreich 1994; Kokkotas & Schaefer 1995; Andersson & Pnigouras 2020), but the steps we have taken towards a relativistic model provide us with a better handle on the involved systematics. Moreover, building on this, the evaluation of the large set of templates required to span the parameter space relevant for gravitational-wave searches should not be computationally costly. In particular, it would be straightforward to combine (10) with any current waveform model that implements the static tide. However, the emphasis of this paper is not on the development of an alternative waveform model – there are many such efforts in the literature. Rather, we are interested in the physics of the tidal problem, to what extent we can develop a simple representation for the dynamical tide and – now that this has been demonstrated – to what extent such a model provides useful insight. One particular issue, which will inevitably come to the fore as the discussion of third-generation ground-based detectors gathers pace, relates to the precision with which the tidal information can be inferred from observations and turned into constraints on the neutron star radius and the matter equation of state. With this in mind, it is evident from Fig. 1 that the dynamical tide dominates during the late stages of inspiral (as the system approaches resonance). The simple fact that it strengthens the tidal imprint should be good for observations and, combined with the simple link to the static tide provided by (say) (10), one might hope to be able to facilitate a more precise extraction of the Love number k_2 and hence more secure radius constraints. A related aspect concerns the measurement uncertainties. Assuming that the observational errors of the tidal deformability Λ_2 may reach the per cent level, how do we make sure that the parameter inference is not limited by the theory? Could it, for example, be the case that this level of precision requires information beyond bulk properties like mass and radius (e.g. the internal composition discussion by Andersson & Pnigouras 2020 or the presence of superfluidity in the neutron star core considered by Yu & Weinberg 2017)?

As we reflect on the different options, we need to consider the all-important benchmarking of phenomenological (computationally efficient) models against (computationally costly) non-linear numerical simulations. This is relevant for many reasons. Perhaps most importantly, while an absolute requirement for a description of the complex merger dynamics, numerical simulations are unlikely to be able to track the many thousand binary orbits required to model a system that evolves through the detector sensitivity band. This will always require an approximate description. For well-separated binaries, this need is satisfied by post-Newtonian (essentially point particle) results but this description becomes less reliable during the late stages of inspiral – largely due to the emergence of finite size effects, like the tidal deformability. The importance of the problem, both for signal detection and the extraction of parameters from an observation, has driven the development of reliable alternatives involving dynamical aspects of the tide, like the effective-one-body framework that forms the basis for the work of Hinderer et al. (2016) and Steinhoff et al. (2016).

An important point, which appears to be commonly overlooked but is evident from the discussion leading to (10), involves the natural parameters to use in a phenomenological model. The typical approach connects with post-Newtonian logic by expressing the results in terms of the dimensionless parameter $x = (\Omega M)^{2/3}$. This is an obvious choice for the main contribution to the gravitational-wave signal, which depends on the orbital motion, but it is less clear that this parameter makes sense during the late stages of inspiral, for which numerical simulations are viable (recall that it is rare that binary neutron star simulations are carried out for more than the last 10–20 orbits and it is only very recently that such simulations have been carried out with sub-radian precision in the accumulated gravitational-wave phase; Dietrich, Bernuzzi & Tichy 2017; Kiuchi et al. 2017; Kawaguchi et al. 2018; Foucart et al. 2019). There is no reason why the matter effects would be naturally expressed in terms of a parameter based on the orbital dynamics in this regime. In fact, this would be counter-intuitive. This point is neatly illustrated by (10) that encodes the matter dynamics (for each binary companion) in terms of the f-mode oscillation frequency, a behaviour that would be obscured by a formal expansion in x.

As a measure of the current level of uncertainty, let us compare different suggested models for the tidal contribution. This kind of comparison is straightforward, as several alternatives have been given in closed form. However, one has to be careful because the associated assumptions impact the result. We need to compare apples with apples. As will soon become clear, this turns out to be less straightforward. A relevant comparison, with immediate implication for gravitational-wave astronomy, involves the accumulated phase associated with the tidal contribution. At the very least, one would expect to be able to distinguish between models that differ by at least half a cycle (1 rad) in the inspiral signal (although the large signal-to-noise detections expected from third-generation detectors should allow much better precision than this). Effectively, for the quadrupole contribution to an equal-mass binary signal we need to integrate an expression of the form (Hinderer et al. 2010; Andersson & Ho 2018)

$$\frac{\mathrm{d}\Phi_T}{\mathrm{d}x} = -\frac{65}{2^5}\frac{k_2}{C^5}x^{3/2}f(x) \tag{13}$$

for the tidal contribution to the phase, Φ_T. The Newtonian pre-factor is the same for all models, but the factor $f(x)$ differs, e.g. depending on which point-particle inspiral model we consider. Since the models we consider can be described analytically, it is easy to obtain an idea of the difference between the tidal prescriptions.

Fig. 2 provides a summary of the current state of the art (for the same stellar model as before). The figure shows (all for $l = 2$): the post-Newtonian model from Damour et al. (2012), the fit to numerical data from Dietrich et al. (2017), a similar fit from Kawaguchi et al. (2018), and the recent expression from Dietrich et al. (2019b). The last three of these models are based on (different) sets of non-linear simulations for the late stages of inspiral, matched to a chosen post-Newtonian model for low frequencies. The results in Fig. 2 show[5] that there is significant variation between the models, suggesting that the discussion of the problem has not yet 'converged'. In principle, the discussion should implement as much 'known' post-Newtonian information as possible. In practice, this involves deciding which post-Newtonian (or, indeed, effective-one-body) model one should take as the baseline. The problem is that one would expect post-Newtonian estimates to lead to larger neutron stars and hence an enhanced tidal effect (Samajdar & Dietrich 2018). This expectation is illustrated by the results for the post-Newtonian model from Damour et al. (2012) (see their equation 31), which differs dramatically from fits based on numerical simulations for $x \gtrsim 0.1$ (see Fig. 2). Another key take-home message from the data concerns the low-frequency

[5]We have treated the various expressions for the accumulated phase Φ_T as 'exact' and taken the derivative. As the expressions we consider involve order-by-order post-Newtonian expansion or Padé approximants to improve the models, this derivative is not particularly consistent, but the results should be considered as illustrative so this should not be too much of a concern.

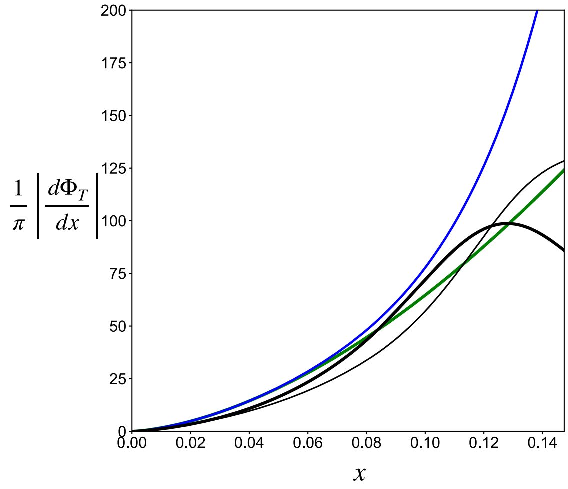

Figure 2. Comparing the impact of the tidal deformability for different closed-form expressions based on high-order post-Newtonian calculations or non-linear inspiral simulations (all for $l = 2$). We show the derivative (with respect to x) of the tidal contribution to the gravitational-wave phase. From top to bottom on the right edge, the curves represent: the post-Newtonian model from Damour, Nagar & Villain (2012) (solid blue curve), the fit to numerical data from Dietrich et al. (2017) (thin black curve), a fit to the numerical data from Kawaguchi et al. (2018) (solid green curve), and the recent fit to numerical data from Dietrich et al. (2019b) (thick black curve). The results illustrate the level of 'uncertainty' in current state-of-the-art models.

behaviour. We note that the approximation from Kawaguchi et al. (2018) asymptotes to the result from Damour et al. (2012) as $x \to 0$, while the low-frequency slope of the two models from Dietrich et al. (2017, 2019b) (see also Samajdar & Dietrich 2018; Dietrich et al. 2019a) is different. The latter approaches the integrated version of (13) with a fixed k_2 and an $f(x)$ such that the relevant factor in the phase is $1 + c_1 x$ with $c_1 = 3115/624$ (in agreement with the post-Newtonian correction from Vines & Flanagan 2013). This observation relates directly to the fact that the dynamical tide arises at a high post-Newtonian order (see the analysis of Schmidt & Hinderer 2019). In effect, we need to account for this before we compare an expression like (10) to the existing models. Effectively, noting that the function $f(x)$ represents non-dynamical aspects of the tide, we need to decide which model to use as our benchmark for comparison. For natural reasons, as it is the most recent discussion of the problem, we will compare to the result from Dietrich et al. (2019b) in the following.

The results in Fig. 3 provide a comparison of the model based on (10) and the result from Dietrich et al. (2019b). First of all, we can compare the impact of the effective tidal deformability from (10) to the static tide. Fig. 3 then indicates that the contribution from the dynamical tide brings us closer to the numerical relativity fit than the static tide. This is not a surprise – the numerical simulations of Foucart et al. (2019) suggest an agreement with the effective tidal formulation from Steinhoff et al. (2016) (and by implication from Fig. 1, our expression). However, in the case we consider here the match is not great beyond something like $x \approx 0.06$ (roughly corresponding to a gravitational-wave frequency of 350 Hz). It turns out that we can improve the match by making use of the free parameter (ϵ). Somewhat surprisingly, the phenomenological model performs well up to significantly higher frequencies if we replace the 'best-fitting' value $\epsilon \approx 0.875$ suggested by Fig. 1 with $\epsilon = 0$. It is not clear, at least not at this point, why this should be the case – or, indeed, if this choice (which represents the horizontal component of

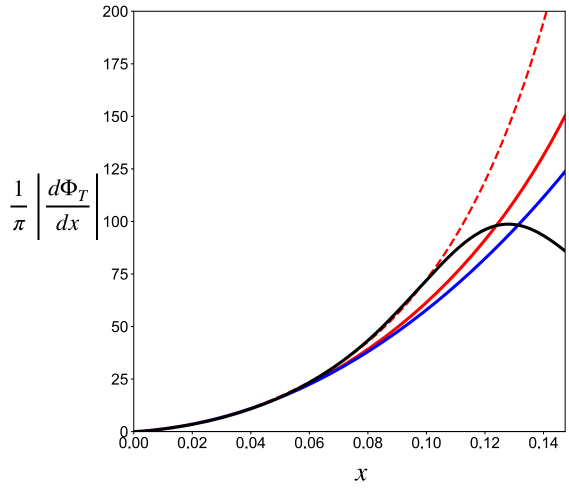

Figure 3. Different approximate results for the tidal contribution, taking the recent results from Dietrich et al. (2019b) (solid black curve, based on a selection of numerical relativity simulations) as the benchmark for comparison. In turn, we show the static contribution to the tide (as a blue curve), our approximate expression for the dynamical tide from (10) [solid red curve, using the 'best-fitting' value $\epsilon \approx 0.875$, which we know (from Fig. 1) is close to the result from Hinderer et al. (2016) for much of the inspiral]. We also show that simply taking $\epsilon = 0$ (dashed red curve) in (10) leads to a surprisingly good approximation to the numerical relativity result for much of the frequency range.

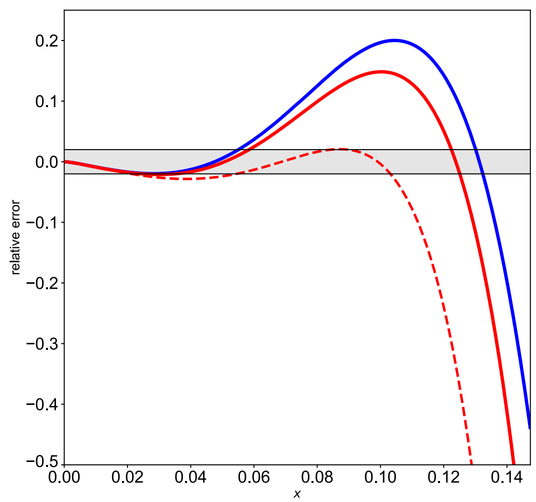

Figure 4. Same as Fig. 3, but showing the relative difference between the different approximations and the numerical relativity-based model from Dietrich et al. (2019b). As a rough indication of the accuracy of the different models for the tidal contribution, we show (as a grey horizontal band) the ± 2 per cent error band.

the displacement vanishing at the surface) makes physical sense – but the model now works well up to about $x \approx 0.1$ (roughly 760 Hz). The result is emphasized by Fig. 4, which shows the relative difference between the model from (10) and the expression from Dietrich et al. (2019b). In this case, we also provide a 2 per cent error band as an indication of the level of precision that may be required when we consider third-generation detectors. The theory has clearly not reached this level yet.

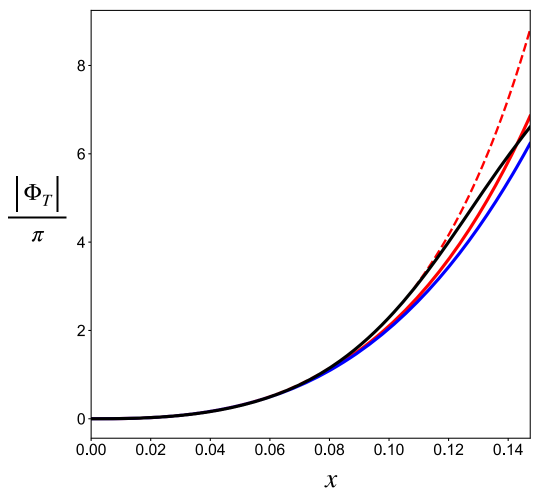

Figure 5. The accumulated gravitational-wave phase Φ_T for the models considered in Fig. 3, highlighting the expected result that our simple expression (10) extends the frequency range across which the approximation remains faithful to the results inferred from numerical relativity (in this case, the model from Dietrich et al. 2019b), but also that the result diverges for the very late stages of inspiral (as expected).

Finally, turning to the accumulated phase, illustrated (for the same set of models as in Figs 3 and 4) in Fig. 5, we see that the effective dynamical tide leads to a slight (sub-radian) change in the gravitational-wave phasing. This is a small effect, but it should be distinguishable by the high-signal-to-noise detections expected by third-generation detectors (Hall & Evans 2019; Martynov et al. 2019) or dedicated high-frequency instruments (Ackley et al. 2020). In this case, it is worth noting that the 'best-fitting' value of $\epsilon \approx 0.875$ from Fig. 1 performs much better than the ad hoc $\epsilon = 0$ above $x \approx 0.12$, i.e. just before merger. This is simply a reflection of the fact that the phenomenological expression from (10) diverges as we approach the mode resonance (see footnote 4). It is also worth noting that the same phenomenological model stays close to the numerical fit from Dietrich et al. (2017) provided we change $c_1 \to c_1/2$. In fact, in that case the models agree well up to $x \approx 0.12$ (a gravitational-wave frequency close to 1200 Hz).

In summary, the phenomenological model from (10) performs well in comparisons with models drawing on numerical simulations, and hence provides useful understanding of the main physics that need to be accounted for in a model for the dynamical tide. Of course, the issue of the 'correct' form for $f(x)$ requires further thinking. The comparison with numerical relativity also requires some care as one would anticipate numerical dissipation to enhance the energy loss in the system, leading to a faster inspiral in non-linear simulations and this may (to some extent) mimic the tide. It could be that the simulations do not yet have the level of precision we need for a true comparison. The results from Foucart et al. (2019) would seem to support this. However, one has to be careful. It is notable that, even though the three numerical-relativity-inspired descriptions in Fig. 2 agree to sub-radian precision in the overall gravitational-wave phase, the formulas from Kawaguchi et al. (2018) and Dietrich et al. (2017, 2019b) match different models in the low-frequency limit. Keeping in mind that the numerical simulations involve only the final 10–20 orbits, one should really focus on the region above $x \approx 0.1$ in the different figures. This brings us to another key point, where further deliberation is needed. A given numerical simulation does not automatically represent the past history of a binary inspiral. The initial data does not have the required 'memory' (it may, for example, involve some level of unwanted eccentricity; Bernuzzi et al. 2015; Dietrich & Hinderer 2017). As simulations become more precise (with differences at the sub-radian level required for the results to be reliable enough for gravitational-wave data analysis), the matching to the low-frequency part of the inspiral signal inevitably comes to the fore. A better understanding of the physics associated with the tidal response should be valuable for this effort.

4 FINAL REMARKS

We have introduced a phenomenological, physically motivated, model for the effective tidal deformability of a neutron star binary, adding frequency dependence that comes into play during the late stages of inspiral. A comparison against alternative descriptions (like the results from Steinhoff et al. 2016) suggests that we have at hand a simple, yet accurate, description of the tidal imprint. This should make the model an attractive alternative for an implementation of the matter effects in gravitational-wave data analysis algorithms. The remaining free parameter of the model (ϵ) tends not to have much impact on the gravitational-wave phasing, unless we push it outside the range expected from the results of Andersson & Pnigouras (2020). Having said that, we have noted that simply setting this parameter to zero (leaving us with no free parameters, at all!) leads to a model that agrees very well with expressions for the dynamical tide based on fits to numerical simulations (see, in particular, Fig. 4). It is not clear at this point why this should be the case; the issue requires further exploration.

Our results suggest interesting avenues for future work. For example, it ought to be straightforward to extend the logic to rotating systems by making use of the phenomenological relations from Doneva et al. (2013) that encode the effect that spin has on the fundamental mode (although it is worth keeping in mind that merging neutron star binaries are likely to be old enough that the stars will have had plenty of time to spin-down, an assumption of slow spin may well be adequate). At a more formal (and challenging) level, we need to extend the mode-sum approach from Andersson & Pnigouras (2020) to general relativity. This is an essential step if we want to base the analysis on the use of realistic matter equations of state. Efforts in this direction should allow an actual derivation of a result along the lines of (10) (rather than the present argument, which was based on analogy with the Newtonian analysis). However, the relativistic problem is technically difficult because the stellar oscillation modes are quasi-normal (with inevitable damping due to gravitational-wave emission) and known not to be complete (as the scattering of waves by the space–time curvature leads to a late-time power-law tail; Price 1972). Nevertheless, this should be a priority issue, as one would also arrive at a precise description of mode resonances (Andersson & Ho 2018). The importance of such a development is clear, but the technical challenge should not be underestimated.

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DATA AVAILABILITY

All relevant data required to reproduce the results are incorporated into the article. Additional material is available on request.

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