Eigenvalue sensitivity minimisation for robust pole placement by the receptance method

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Abstract

The problem of robust pole placement in active structural vibration control by the method of receptance is considered in this paper. Expressions are derived for the eigenvalue sensitivities to parametric perturbations, which are subsequently minimised to improve performance robustness of the control of a dynamical system. The described approach has application to a vibrating system where variations are present due to manufacturing and material tolerances, damages and environment variabilities. The closed-loop eigenvalue sensitivities are expressed as a linear function of the velocity and displacement feedback gains, allowing their minimisation with carefully calculated feedback gains. The proposed algorithm involves curve fitting perturbed frequency response functions, FRFs, using the rational fraction polynomial method and implementation of a polynomial fit to the individual estimated rational fraction coefficients. This allows the eigenvalue sensitivity to be obtained entirely from structural FRFs, which is consistent with the receptance method. This avoids the need to evaluate the $\mathbf{M}, \mathbf{C}, \mathbf{K}$ matrices which are typically obtained through finite element modelling, that produces modelling uncertainty. It is also demonstrated that the sensitivity minimisation technique can work in conjunction with the pole placement and partial pole placement technique using the receptance method. To illustrate the working of the proposed algorithm, the controller is first implemented numerically and then experimentally.

Keywords — Active vibration control; Receptance method; Robust pole placement; Eigenvalue sensitivity

1 Introduction

The eigenvalues or poles of a system represent its dynamical behaviour and their assignment allows one to modify its' vibratory response [1]. It is well known that for a controllable system, if state feedback is employed, the eigenvalues or poles can be assigned [2]. It is often desirable to assign not only the eigenvalues, but the corresponding eigenvalue sensitivity. This is to ensure that any uncertainties such as manufacturing tolerances, environmental variabilities will not influence the controlled eigenvalues. The work presented in this paper is concerned with the improvement of performance robustness in an active controlled system by minimising the eigenvalue sensitivity.

An exact expression for the rate of change of eigenvalues with respect to any design parameter for undamped system were given by Fox and Kapoor [3], who also described two methods of evaluating the eigenvector

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sensitivities. Adhikari [4] later extended the method to non-classically damped system. The modal methods employed in Refs. [3, 4] are relatively straightforward, but the eigenvector sensitivity is expressed such that the full set of eigenvectors is required to calculate just a single eigenvector sensitivity, making it computationally expensive. Nelson [5] later developed an exact method for eigenvector sensitivity that only requires the eigenvalue and eigenvector under consideration, which is less computationally intensive compared to the modal method. Friswell [6] subsequently extended Nelson's method to enable the calculation of repeated eigenvalue sensitivities. A comparison of several methods for calculating the eigenvector sensitivities are provided by Sutter et al. [7]. The aforementioned literature were confined to requirements of full description of the $\mathbf{M}, \mathbf{C}, \mathbf{K}$ matrices, employing state-space analysis for the damped structural dynamics eigenvalue problem. Often, the system's matrices are acquired through modelling e.g. using the finite element method, that may introduce significant modelling errors compared with the real structure. A frequency-response-based methodology avoids the above difficulties, especially the spillover problems that usually originates from a discretised model of a continuous system [8]. A frequency-response-based methodology is superior than a state-space approach in that experimentally determined frequency response functions, FRFs, are used directly, without the need to know or evaluate the $\mathbf{M}, \mathbf{C}, \mathbf{K}$ matrices.

The earliest development of the frequency-response-based control may be traced back to the work by Jones [9] in 1985, where he expressed the dynamic behaviour of a modified structure through the original FRFs without relying on modal parameters of the original system. In 1989 and 1990 respectively, Sestieri and D'Ambrogio [10], and Özgüven [11] used the frequency response approach to deal with structural modification problems. Del Vescovo and D'Ambrogio [12] in 1995 described a new approach for active control using the measured FRFs to optimise the closed-loop frequency response. A decade later Ram and Mottershead [13] formalised a new, experimental-based pole-zero assignment using the measured receptance, termed the receptance method. The receptance method was studied extensively and tested experimentally in numerous settings as discussed in Refs. [14–21]. Mottershead et al. [18] applied the receptance method on the AgustaWestland W30 helicopter airframe to study the suppression of vibration level entering the cabin, thus demonstrating its practical merit. Fichera et al. [20] later applied the receptance method experimentally to extend the flutter envelope of a flexible wing, and similar work was done by Singh et al. [21]. Although the term receptance refers to force input to displacement output, the measured quantity may be of any frequency response, e.g. that from an actuator voltage input to a response sensor voltage output.

The receptance method was initially developed for a single-input linear system, but was later extended to multi-input-multi-output system [22]. Ghandchi Tehrani and Mottershead [17] extended the method to a class of non-linear system that is based on an iterative procedure. 4 years after the initial development of the receptnace method, Ghandchi Tehrani et al. [19] developed a partial pole-placement technique by the receptance method where a subset of poles are assigned to a specified values while leaving the other poles unchanged. This was tested experimentally on a lightweight glass-fibre beam with smart actuators and sensors, and on a heavy modular test structure. Recently, a partial eigenstructure assignment method by state feedback was proposed by Zhang and Ouyang [23]. It was shown that this new method has advantage over the original partial pole placement method when the eigenvector are hard to obtained or inaccurate. This technique was verified by means of numerical examples. A study on the effect of uncertainty on measured receptances was first studied by Ghandchi Tehrani et al. [24], these authors demonstrated the use of uncontrollability condition and sequential multi-input state feedback for poles' sensitivity minimisation. Mottershead et al. [25] considered the robustness of the closed-loop poles to the control gains based on the receptance method and Ghandchi Tehrani et al. [26] showed the assignment of the closed-loop poles as well as the assignment of eigenvalue sensitivities to control gains. A similar sensitivity-based approach was also considered by Liang et al. [27] where they considered how the uncertain physical parameters, such as contact damping and contact stiffness, affect the spread of the poles. Adamson et al. [28] considered a new method for robust control by means of globally optimising the placement of poles based on the variance of the real and imaginary part of each pole. Adamson et al. [29] in the subsequent year developed a new formulations for robust eigenstructure assignment using the receptance method. The variabilities arise from uncertainties in the open-loop poles, zeros and scaling factor due to curve fitting. For robust control, the sensitivity of the closed-loop poles with respect to the fitted parameter are derived analytically, which include the sensitivities with respect to the numerator and denominator coefficients; the authors only considered the model uncertainties. Wei et al. [30] considered the $\mathcal{H}_2$ optimal control using the receptance method where the poles were assigned whilst minimising the $\mathcal{H}_2$ norm. The proposed approach guaranteed the assignment of the system eigenvalues within a region with minimum energy. This technique however does not consider the robustness of the controller when the system is subjected to uncertainties.

Despite much published research in receptance-based eigenvalue sensitivity control, experimental application and validation of these methods is limited, mainly because of their complexity and the practical problems associated with the application of the algorithms. For example, the work in Ref. [24] requires a fully populated actuator distribution vector, $\mathbf{b}(s)$, which may limit the practicality as the number of sensors and actuators has to be the same for the application of the algorithm, making it expensive to implement experimentally. The focus of the present research is to apply the receptance-based eigenvalue sensitivity in an algorithm that is straightforward to implement.

In this paper, a new analytical expression for the eigenvalue sensitivities to physical uncertainties is derived that is based on the Sherman-Morrison formula of the receptance matrix. This expression is used to calculate the feedback gains required to assign a closed-loop eigenvalue with no sensitivity with respect to the model parameter, e.g. mass, stiffness, or damping. The method leads to linear sets of equation in the unknown gains that can be solved relatively easily without the high computation costs of other optimisation methods or Monte Carlo simulations. By assigning the poles and closed-loop eigenvalue sensitivities, the closed-loop poles' sensitivity is minimised under perturbation to the model parameters. The effect of uncertain structural parameters on a system will manifest itself in the rate of change of the numerator and denominator of the receptances. Therefore, no underlying model of the uncertain system is required for the implementation of this method. The method is demonstrated in a series of simple numerical examples, and validated on a three-degree-of-freedom DTOP structure [31]. Although a three DoF example is given in this article, this technique works well with as many DoFs as required provided that the number of conditions assigned is less than or equals to $2n$, n being the number of degrees of freedom. A least square method may be employed to assign any conditions upward of $2n$, as given by Eq. (25).

2 Theoretical development

2.1 Overview of pole placement by the receptance method

The principle of the receptance method may be formulated as follows. Consider a dynamic system with general mass, damping and stiffness matrices, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, and state feedback,

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{b}u(t) + \mathbf{p}(t) \tag{1}$$

with the single input control law defined as,

$$u(t) = -\mathbf{f}^{\mathrm{T}}\dot{\mathbf{x}}(t) - \mathbf{g}^{\mathrm{T}}\mathbf{x}(t) \tag{2}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K} \in \mathbb{R}^{n\times n}$; $\mathbf{M}$ is symmetric positive definite and $\mathbf{C}$ and $\mathbf{K}$ are symmetric positive semi-definite, $\mathbf{b} \in \mathbb{R}^n$ is the actuator distribution vector, u is the control force, $\mathbf{g}, \mathbf{f} \in \mathbb{R}^n$ are the displacement and velocity feedback gains respectively, and $\mathbf{p}$ is the external forces. In practice, each non-zero term in $\mathbf{b}$ implies the use of an actuator likewise each non-zero term in $\mathbf{f}$ and $\mathbf{g}$ implies the use of an actuator [13].

Combining Eq. (1) and Eq. (2) and expressing it in the s domain gives,

$$[\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K} + \mathbf{b}(\mathbf{g} + s\mathbf{f})^{\mathrm{T}}]\mathbf{x}(s) = \mathbf{p}(s) \tag{3}$$

which amounts to a rank-1 modification to the dynamic stiffness matrix as a consequence of the single input $u(t)$. The closed-loop transfer function is therefore given by,

$$\hat{\mathbf{H}}(s) = [\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K} + \mathbf{b}(\mathbf{g} + s\mathbf{f})^{\mathrm{T}}]^{-1} \tag{4}$$

The Sherman-Morrison formula [32] gives the inverse of a matrix with a rank-1 modification in terms of the inverse of the original matrix. So that the closed-loop receptance matrix, Eq. (4), can be rewritten as,

$$\hat{\mathbf{H}}(s) = \mathbf{H}(s) - \frac{\mathbf{H}(s)\mathbf{b}(\mathbf{g} + s\mathbf{f})^{\mathrm{T}}\mathbf{H}(s)}{1 + (\mathbf{g} + s\mathbf{f})^{\mathrm{T}}\mathbf{H}(s)\mathbf{b}} \tag{5}$$

where the open-loop receptance $\mathbf{H}(s) = [\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}]^{-1}$ in practice may be determined from frequency response measurements, as $\mathbf{H}(j\omega)$, which means there is no need to know the system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ as the receptances can be determined directly from vibration tests.

The closed-loop poles of the system, denoted by μ, can be obtained by solving the characteristic polynomial of the closed-loop system $1 + (\mathbf{g} + s\mathbf{f})^{\mathrm{T}}\mathbf{H}(s)\mathbf{b}$. With a known transfer matrix $\mathbf{H}(s)$, actuator distribution vector $\mathbf{b}$ and a complex set of desired closed-loop poles, $\{\mu_1, \mu_2, \cdots, \mu_{2n}\}$ closed under conjugation, the desired closed-loop poles, μ_k can be assigned using the feedback gains $\mathbf{f}$ and $\mathbf{g}$ such that it satisfy Eq. (25).

$$(\mathbf{g} + \mu_k\mathbf{f})^{\mathrm{T}}\mathbf{H}(\mu_k)\mathbf{b} = -1, \qquad k = 1, 2, \cdots, 2n \tag{6}$$

First denoting $\mathbf{r}_k = \mathbf{H}(\mu_k)\mathbf{b}$ then rewrite Eq. (6) in matrix form,

$$\mathbf{G}\begin{Bmatrix}\mathbf{g}\\ \mathbf{f}\end{Bmatrix} = \{-\mathbf{1}\}\ ,\ \mathbf{G}(\mu_k) = \begin{bmatrix}\mathbf{r}_1^{\mathrm{T}} & \mu_1\mathbf{r}_1^{\mathrm{T}}\\ \mathbf{r}_2^{\mathrm{T}} & \mu_2\mathbf{r}_2^{\mathrm{T}}\\ \vdots & \vdots\\ \mathbf{r}_{2n}^{\mathrm{T}} & \mu_{2n}\mathbf{r}_{2n}^{\mathrm{T}}\end{bmatrix} \tag{7}$$

The sets of $2n$ equations with $2n$ unknowns allows for the determination of $\mathbf{f}$ and $\mathbf{g}$, by the inversion of $\mathbf{G}(\mu_k)$. The two theorems that provide the conditions under which $\mathbf{G}(\mu_k)$ is invertible are given by Ram and Mottershead [13] as,

1. $\mathbf{G}(\mu_k)$ is invertible if the system is controllable and μ_k where $k = 1, 2, \cdots, 2n$ are distinct.
2. If $\mathbf{G}(\mu_k)$ is invertible and the set of $\{\mu_1, \mu_2, \cdots, \mu_{2n}\}$ is closed under conjugation, then $\mathbf{f}$ and $\mathbf{g}$ are strictly real.

Ghandchi Tehrani et al. [19] showed that $\{\mathbf{f}, \mathbf{g}\}$ may be obtained when a partial set of eigenvalues $\{\lambda_i, \lambda_i^*\}_{i=1}^{m}$ are assigned to a closed-loop set $\{\mu_i, \mu_i^*\}_{i=1}^{m}$ while all other eigenvalues are kept unchanged, that is $\{\lambda_i = \mu_i, \lambda_i^* = \mu_i^*\}_{i=m+1}^{n}$. Thus, this is termed partial pole placement. This method is achieved by implementing the uncontrollability condition, i.e. the control force distribution is orthogonal to the modal eigenvector, where it can be expressed as,

$$\mathbf{b}^{\mathrm{T}}\boldsymbol{\phi}_k = 0 \tag{8}$$

where $\boldsymbol{\phi}_k \in \{\boldsymbol{\phi}_i, \boldsymbol{\phi}_i^*\}_{i=1}^{n}$ are the eigenvectors of the system. Since $\mathbf{b} \in \mathbb{R}^n$ then Eq. (8) is true if [19],

$$\mathbf{b} = \text{null}\begin{bmatrix}\Re(\boldsymbol{\phi}_k)\\ \Im(\boldsymbol{\phi}_k)\end{bmatrix} \tag{9}$$

2.2 Eigenvalue sensitivity from the receptance matrix

We begin by expressing the open-loop receptances in terms of a numerator and denominator,

$$\mathbf{H}(s) = [\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}]^{-1} = \frac{\mathrm{adj}[\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}]}{\det[\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K}]} = \frac{\mathbf{N}(s)}{\mathrm{d}(s)} \tag{10}$$

The poles of the closed-loop system are obtained as the roots of the characteristic equation from the closed-loop transfer function Eq. (5). Denoting the characteristic equation as $\mathrm{p}(s)$,

$$\mathrm{p}(s) = 1 + (\mathbf{g} + s\mathbf{f})^{\mathrm{T}}\mathbf{H}(s)\mathbf{b} \tag{11}$$

Combining Eq. (10) and Eq. (11) and multiplying both sides by $\mathrm{d}(s)$,

$$\mathrm{d}(s)\mathrm{p}(s) = \mathrm{d}(s) + (\mathbf{g} + s\mathbf{f})^{\mathrm{T}}\mathbf{N}(s)\mathbf{b} \tag{12}$$

for the k-th closed-loop eigenvalue, Eq. (12) is written as,

$$\mathrm{d}(\mu_k) + (\mathbf{g} + \mu_k\mathbf{f})^{\mathrm{T}}\mathbf{N}(\mu_k)\mathbf{b} = 0 \tag{13}$$

Now consider a small change in the system parameter $\theta + \delta\theta$, so that the new poles become $\mu_k + \delta\mu_k$ and the first order Taylor's expansion of denominator, $\mathrm{d}(\theta + \delta\theta, \mu_k + \delta\mu_k)$ and numerator, $\mathbf{N}(\theta + \delta\theta, \mu_k + \delta\mu_k)$ are

$$\begin{aligned} \mathrm{d}(\theta + \delta\theta, \mu_k + \delta\mu_k) &= \mathrm{d}(\theta, \mu_k) + \left.\frac{\partial \mathrm{d}}{\partial s}\right|_{s=\mu_k} \frac{\partial \mu_k}{\partial \theta}\delta\theta + \left.\frac{\partial \mathrm{d}}{\partial \theta}\right|_{s=\mu_k} \delta\theta \\ \mathbf{N}(\theta + \delta\theta, \mu_k + \delta\mu_k) &= \mathbf{N}(\theta, \mu_k) + \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\mu_k} \frac{\partial \mu_k}{\partial \theta}\delta\theta + \left.\frac{\partial \mathbf{N}}{\partial \theta}\right|_{s=\mu_k} \delta\theta \end{aligned} \tag{14}$$

The poles of the perturbed system are then given by solving,

$$\mathrm{d}(\theta + \delta\theta, \mu_k + \delta\mu_k) + (\mathbf{g} + (\mu_k + \delta\mu_k)\mathbf{f})^{\mathrm{T}}\mathbf{N}(\theta + \delta\theta, \mu_k + \delta\mu_k)\mathbf{b} = 0 \tag{15}$$

which can be expanded to give,

$$\begin{aligned} &\left[\mathrm{d}(\theta, \mu_k) + \left.\frac{\partial \mathrm{d}}{\partial s}\right|_{s=\mu_k} \frac{\partial \mu_k}{\partial \theta}\delta\theta + \left.\frac{\partial \mathrm{d}}{\partial \theta}\right|_{s=\mu_k} \delta\theta\right] + (\mathbf{g} + (\mu_k + \delta\mu_k)\mathbf{f})^{\mathrm{T}} \\ &\left[\mathbf{N}(\theta, \mu_k) + \left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\mu_k} \frac{\partial \mu_k}{\partial \theta}\delta\theta + \left.\frac{\partial \mathbf{N}}{\partial \theta}\right|_{s=\mu_k} \delta\theta\right]\mathbf{b} = 0 \end{aligned} \tag{16}$$

combining Eq. (13) and (16) and neglecting δ^2 or higher order terms,

$$\left.\frac{\partial \mathrm{d}}{\partial s}\right|_{s=\mu_k} \frac{\partial \mu_k}{\partial \theta} + \left.\frac{\partial \mathrm{d}}{\partial \theta}\right|_{s=\mu_k} + \frac{\partial \mu_k}{\partial \theta}\mathbf{f}^{\mathrm{T}}\mathbf{N}(\theta, \mu_k)\mathbf{b} + (\mathbf{g} + \mu_k\mathbf{f})^{\mathrm{T}}\left[\left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\mu_k} \frac{\partial \mu_k}{\partial \theta} + \left.\frac{\partial \mathbf{N}}{\partial \theta}\right|_{s=\mu_k}\right]\mathbf{b} = 0 \tag{17}$$

The sensitivity of the poles are then given by,

$$\frac{\partial \mu_k}{\partial \theta} = \frac{-\left.\frac{\partial \mathrm{d}}{\partial \theta}\right|_{s=\mu_k} - (\mathbf{g} + \mu_k\mathbf{f})^{\mathrm{T}}\left.\frac{\partial \mathbf{N}}{\partial \theta}\right|_{s=\mu_k}\mathbf{b}}{\left.\frac{\partial \mathrm{d}}{\partial s}\right|_{s=\mu_k} + (\mathbf{g} + \mu_k\mathbf{f})^{\mathrm{T}}\left.\frac{\partial \mathbf{N}}{\partial s}\right|_{s=\mu_k}\mathbf{b} + \mathbf{f}^{\mathrm{T}}\mathbf{N}(\theta, \mu_k)\mathbf{b}} \tag{18}$$

To minimise the eigenvalue of the closed-loop poles, one can solve for the feedback gains $\mathbf{f}$ and $\mathbf{g}$ such that the numerator of $\partial\mu_k/\partial\theta = 0$ and thus,

$$-\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_k} - (\mathbf{g}+\mu_k\mathbf{f})^{\mathrm{T}}\left.\frac{\partial \mathbf{N}}{\partial\theta}\right|_{s=\mu_k}\mathbf{b} = 0 \tag{19}$$

or

$$(\mathbf{g}+\mu_k\mathbf{f})^{\mathrm{T}}\mathbf{q}_k = -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_k} \tag{20}$$

where,

$$\mathbf{q}_k(\mu_k) = \left.\frac{\partial \mathbf{N}}{\partial\theta}\right|_{s=\mu_k}\mathbf{b} \tag{21}$$

Transpose both sides Eq. (20) and expanding them becomes

$$\mathbf{q}_k^{\mathrm{T}}\mathbf{g} + \mu_k\mathbf{q}_k^{\mathrm{T}}\mathbf{f} = -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_k} \tag{22}$$

The set of $2n$ equations with $2n$ unknowns may be written in the matrix form as,

$$\mathbf{S}\begin{Bmatrix}\mathbf{g}\\ \mathbf{f}\end{Bmatrix} = \begin{Bmatrix} -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_1} \\ -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_2} \\ \vdots \\ -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_{2n}} \end{Bmatrix} \tag{23}$$

with,

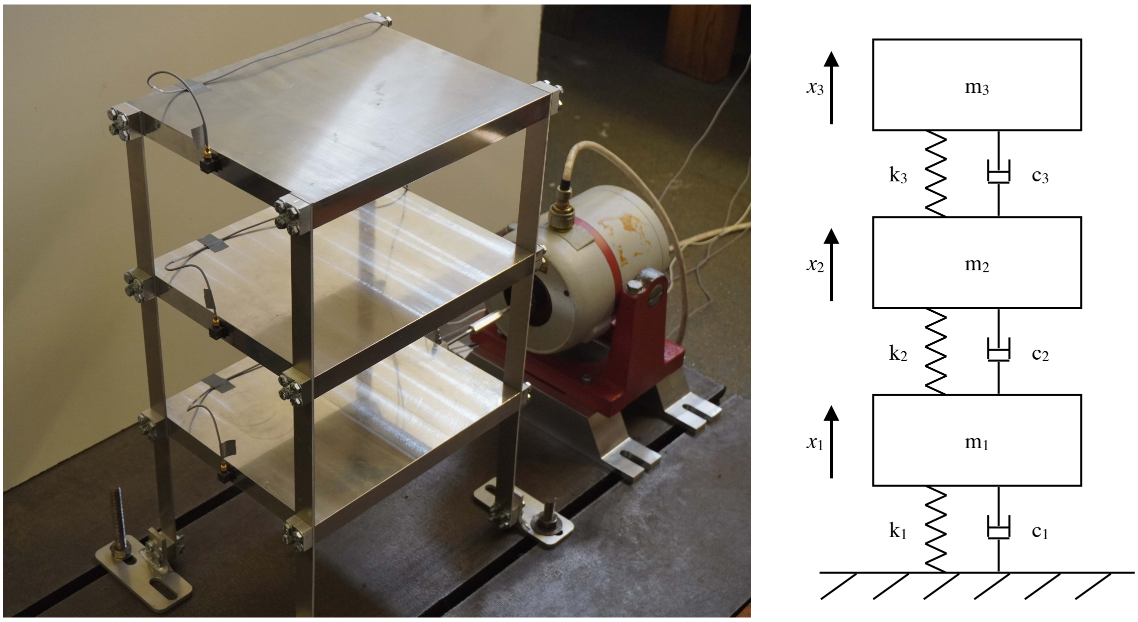
$$\mathbf{S} = \begin{bmatrix} \mathbf{q}_1^{\mathrm{T}} & \mu_1\mathbf{q}_1^{\mathrm{T}} \\ \mathbf{q}_2^{\mathrm{T}} & \mu_2\mathbf{q}_2^{\mathrm{T}} \\ \vdots & \vdots \\ \mathbf{q}_{2n}^{\mathrm{T}} & \mu_{2n}\mathbf{q}_{2n}^{\mathrm{T}} \end{bmatrix} \tag{24}$$

The following observation may be noted from Eq. (18): the sensitivity of the poles occur in complex conjugate pairs since the poles themselves are closed under conjugation and the feedback gains are strictly real for a real problem. In practice, the partial derivative of the numerator and denominator of the receptance can be obtained by collecting a set of FRFs with random parameter variations and fitting a function to the individually identified polynomial coefficients. This allows one to evaluate the slope of the numerator and denominator and hence the feedback gains may be obtained by solving Eq. (23). The sensitivity equation may also be combined with the pole placement linear equation Eq. (7) for a simultaneous assignment of the closed-loop eigenvalues and eigenvalue sensitivities.

$$\begin{bmatrix} \mathbf{r}_1^{\mathrm{T}} & \mu_1\mathbf{r}_1^{\mathrm{T}} \\ \mathbf{r}_2^{\mathrm{T}} & \mu_2\mathbf{r}_2^{\mathrm{T}} \\ \vdots & \vdots \\ \mathbf{r}_m^{\mathrm{T}} & \mu_m\mathbf{r}_m^{\mathrm{T}} \\ \mathbf{q}_1^{\mathrm{T}} & \mu_1\mathbf{q}_1^{\mathrm{T}} \\ \vdots & \vdots \\ \mathbf{q}_{2n-m}^{\mathrm{T}} & \mu_{2n-m}\mathbf{q}_{2n-m}^{\mathrm{T}} \end{bmatrix}\begin{Bmatrix}\mathbf{g}\\ \mathbf{f}\end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_1} \\ \vdots \\ -\left.\frac{\partial \mathrm{d}}{\partial\theta}\right|_{s=\mu_{2n-m}} \end{Bmatrix} \tag{25}$$

For an exact assignment of poles and sensitivity, the total number of equations must not exceed $2n$ as shown by Eq. (25). A least squares solution is required otherwise, which, however, will not weight the solution by differences in the poles and the sensitivities, which might well have different magnitudes and scales. One must therefore decide between optimising the placement of poles and minimisation of the eigenvalues by assigning a weight to the pole placement and sensitivity equations respectively.

3 Numerical examples



(a) (b)

Figure 1: (a) DTOP structure [31]. (b) Three degree of freedom representation of the DTOP structure. Note that the physical structure in (a) moves from side to side, but this is represented as equivalent vertical motion in (b).

In the numerical examples, a three-degree-of-freedom system is modelled from the DigiTwin Operating Platform (DTOP) extracted from [33], as shown in Figure 1a. This is also the system used for the experimental validation discussed below. The actuator dynamics is not considered in the numerical examples, for simplicity, but these are included in the experimental validation in the following section. The $\mathbf{M}$ [kg], $\mathbf{C}$ [kg s^{-1}] and $\mathbf{K}$ [kg s^{-2}] matrices of the system are assumed to be,

$$\mathbf{M} = \begin{bmatrix} m_3 & 0 & 0 \\ 0 & 5.144 & 0 \\ 0 & 0 & 5.362 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1.340 & -1.340 & 0 \\ -1.340 & 2.355 & -1.016 \\ 0 & -1.016 & 2.714 \end{bmatrix}, \quad \mathbf{K} = 10^4 \begin{bmatrix} 4.589 & -4.589 & 0 \\ -4.589 & 9.053 & -4.464 \\ 0 & -4.464 & 8.310 \end{bmatrix}$$

The uncertain parameter is chosen as m_3 with its nominal mass taken as $m_3 = 5.142$ kg. A perturbation of $m_3 \pm 20\%$ is introduced, or uniformly distributed between 4.114 kg and 6.170 kg.

$$m_3 \sim \mathcal{U}(4.114,\ 6.170)$$

It is assumed that the sensors are available on all three masses but the actuator only acts on one of the masses. In practice, a shaker was connected to the structure at the bottom mass, it is through this shaker that the control force was applied, as shown in Figure 1a. The actuator distribution vector is thus,

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

The open-loop poles of the nominal system were found to be,

$$\lambda_{1,2} = -0.028405 \pm 39.696j$$
$$\lambda_{3,4} = -0.21006 \pm 112.74j$$
$$\lambda_{5,6} = -0.37382 \pm 166.54j$$

3.1 Example 1

It is required to assign the first and third pairs of poles and minimise the eigenvalue sensitivity of the third mode when subjected to one random parameter, m_3. The desired closed-loop poles for assignment are hence chosen as the following and the sensitivity, S of the third pole (both conjugate pair) is set to zero,

$$\mu_{1,2} = -0.14203 \pm 43.66j$$
$$\mu_{5,6} = -1.8691 \pm 183.19j$$
$$S_{5,6} = 0 \pm 0j$$

solving for the control gains,

$$\begin{bmatrix} \mathbf{r}_1^{\mathrm{T}} & \mu_1\mathbf{r}_1^{\mathrm{T}} \\ \mathbf{r}_2^{\mathrm{T}} & \mu_2\mathbf{r}_2^{\mathrm{T}} \\ \mathbf{r}_5^{\mathrm{T}} & \mu_5\mathbf{r}_5^{\mathrm{T}} \\ \mathbf{r}_6^{\mathrm{T}} & \mu_6\mathbf{r}_6^{\mathrm{T}} \\ \mathbf{q}_5^{\mathrm{T}} & \mu_5\mathbf{q}_5^{\mathrm{T}} \\ \mathbf{q}_6^{\mathrm{T}} & \mu_6\mathbf{q}_6^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{g} \\ \mathbf{f} \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ y_5 \\ y_6 \end{Bmatrix} \tag{26}$$

with

$$\mathbf{r}_k = \mathbf{H}(\mu_k)\mathbf{b}\ , \quad \mathbf{q}_k = \left.\frac{\partial \mathbf{N}}{\partial \theta}\right|_{s=\mu_k} \mathbf{b}\ , \quad y_k = -\left.\frac{\partial \mathrm{d}}{\partial \theta}\right|_{s=\mu_k}$$

leads to,

$$\mathbf{g} = \begin{bmatrix} 73564 & 1737 & -23955 \end{bmatrix}^{\mathrm{T}} \qquad \mathbf{f} = \begin{bmatrix} 20.155 & -1.6247 & 2.7489 \end{bmatrix}^{\mathrm{T}}$$

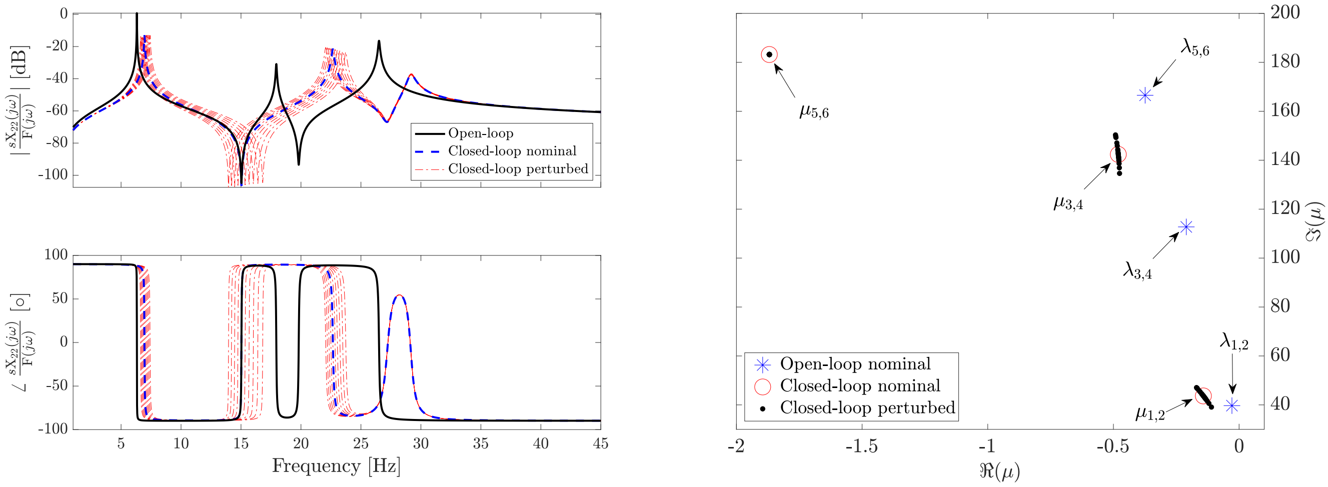
The closed-loop poles are found to be,

$$\mu_{1,2} = -0.1420 \pm 43.66j$$
$$\mu_{3,4} = -0.4805 \pm 142.37j$$
$$\mu_{5,6} = -1.8691 \pm 183.19j$$

and the closed-loop eigenvalue sensitivity are found as,

$$S_{1,2} = 0.0199 \pm 2.6232j$$
$$S_{3,4} = 0.0054 \pm 5.2908j$$
$$S_{5,6} = 1.2621 \times 10^{-14} \pm 5.0779 \times 10^{-14}j$$

The first and third pairs of the closed-loop poles have shifted exactly to the desired locations with the second unassigned pair remained stable and moved to a new location as a result of re-assigning the other two eigenvalues. The numerical values of the eigenvalue sensitivity indicated that the third eigenvalue sensitivity pair is much smaller in the absolute sense when compared with the other unassigned sensitivity pairs. Thus, indicating the successful minimisation of third eigenvalue sensitivity. The insensitivity of the mode may also be observed in Figure 2a where the third peak is unperturbed under $\pm 20\%$ m_3 perturbation. The variability of the poles in Figure 2b also demonstrates a tight cluster of the perturbed poles in the upper left quadrant signifying that the controlled mode is unchanged under the influence of the uncertain mass.



(a) (b)

Figure 2: Example 1, (a) Open-loop and perturbed closed-loop FRFs. (b) Open-loop and closed-loop poles variability.

3.2 Example 2

In the second example, it is required to assign the second and third pole pairs and minimise the sensitivity of the second mode. The objectives are therefore set as,

$$\mu_{3,4} = -1.0503 \pm 124.01j$$
$$\mu_{5,6} = -1.8691 \pm 183.19j$$
$$S_{3,4} = 0 \pm 0j$$

solving for the control gains,

$$\begin{bmatrix} \mathbf{r}_3^{\mathrm{T}} & \mu_3\mathbf{r}_3^{\mathrm{T}} \\ \mathbf{r}_4^{\mathrm{T}} & \mu_4\mathbf{r}_4^{\mathrm{T}} \\ \mathbf{r}_5^{\mathrm{T}} & \mu_5\mathbf{r}_5^{\mathrm{T}} \\ \mathbf{r}_6^{\mathrm{T}} & \mu_6\mathbf{r}_6^{\mathrm{T}} \\ \mathbf{q}_3^{\mathrm{T}} & \mu_3\mathbf{q}_3^{\mathrm{T}} \\ \mathbf{q}_4^{\mathrm{T}} & \mu_4\mathbf{q}_4^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \mathbf{g} \\ \mathbf{f} \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ y_3 \\ y_4 \end{Bmatrix} \tag{27}$$

leads to,

$$\mathbf{g} = \begin{bmatrix} 67680 & 27125 & 70220 \end{bmatrix} \qquad \mathbf{f} = \begin{bmatrix} 27.572 & 9.4753 & 20.008 \end{bmatrix}$$

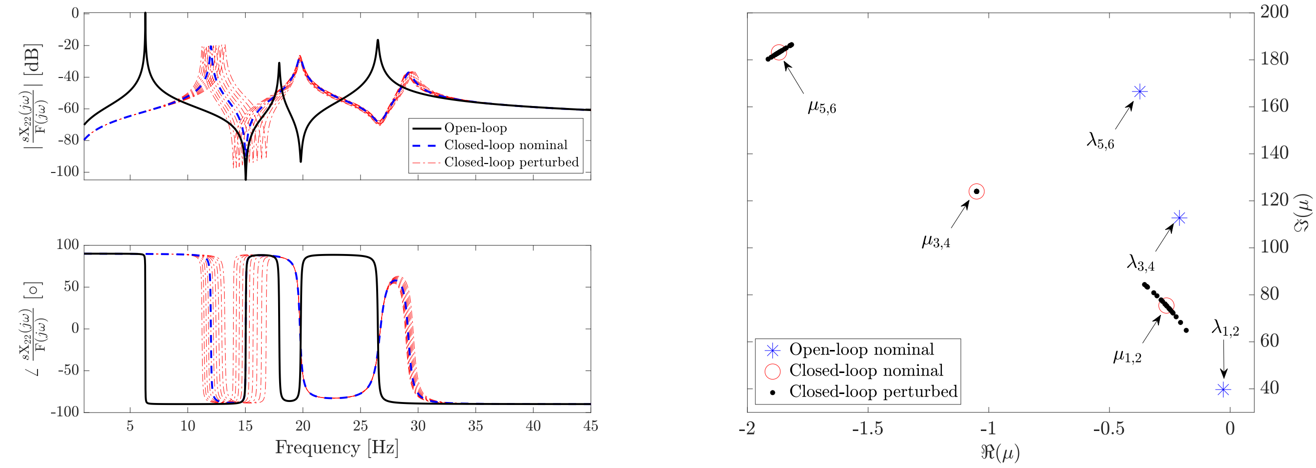
The closed-loop poles are found to be,

$$\mu_{1,2} = -0.2640 \pm 75.455j$$
$$\mu_{3,4} = -1.0503 \pm 124.01j$$
$$\mu_{5,6} = -1.8691 \pm 183.19j$$

and the closed-loop eigenvalue sensitivity are found as,

$$S_{1,2} = 0.0581 \pm 6.4866j$$
$$S_{3,4} = -9.2149 \times 10^{-15} \pm 1.2904 \times 10^{-15}j$$
$$S_{5,6} = -0.0327 \pm 2.0651j$$

as expected, the assigned closed-loop poles had shifted exactly to the desired locations and the closed-loop eigenvalue sensitivities of the minimised second mode had numerically reduced to a negligible level. It can



(a) (b)

Figure 3: Example 2, (a) Open-loop and perturbed closed-loop FRFs. (b) Open-loop and closed-loop poles variability.

again be observed from the FRFs plot and the poles' variability in Figure 3a and 3b respectively, where the second peak in the FRFs that correspond to the second mode is not distributed and the perturbed eigenvalues are clustered tightly in a single point around $-1.05 + 124i$ — the desired closed-loop poles. Here the eigenvalue of the first mode is not assigned due to the desire to assign the eigenvalues and sensitivities exactly as explained by Eq. (25), it is therefore allowed to move freely as a result of the feedback gains. For this particular example, the first mode remained stable as it lies on the left-half of the complex plane, this however may not always be the case for a generic system. A solution to this may be assigning a condition to the problem for which all of the real part of the eigenvalues must be negative, i.e. lying on the left-half complex plane. Another solution might be to adopt the partial pole placement technique, developed by Ghandchi Tehrani et al. [19]. The latter method is achieved by implementing the uncontrollability condition, whereby the force actuated is orthogonal to the mode that is required to remain unchanged under the effect of feedback. A numerical example for the partial pole placement technique is given in Section 3.3.

3.3 Example 3

Consider again the objective in Example 2, but now with the uncontrollability condition implemented onto the first mode, the actuator distribution vector is obtained by solving Eq. (9) and found to be,

$$\mathbf{b} = \begin{bmatrix} 0.571 & -0.810 & 0.179 \end{bmatrix}^{\mathrm{T}}$$

Solving Eq. (27) for this case the feedback gains are found as,

$$\mathbf{g} = \begin{bmatrix} -90426 & -142110 & -95432 \end{bmatrix}^{\mathrm{T}} \qquad \mathbf{f} = \begin{bmatrix} -38.985 & -44.358 & -8.399 \end{bmatrix}^{\mathrm{T}}$$

Figure 4 shows that the first closed-loop poles successfully remained in the open-loop location, with a numerical value of $\mu_{1,2} = -0.028405 \pm 39.696j$ thus showing that the mode is unchanged under the effect of feedback gains, i.e. $\mu_{1,2} = \lambda_{1,2}$. This is especially useful to render a mode uncontrollable if they are susceptible to becoming unstable under perturbation. Hence, increasing the robustness of the system as a whole. The second mode remained insensitive as required, with a numerical value of $S_{3,4} = -1.2964 \times 10^{-15} \pm 1.7666 \times 10^{-14}j$, thus proving the developed eigenvalue sensitivity equation works well with the existing partial pole placement method. There is however a slight trade off in terms of the robustness of the third mode when Figure 4 is compared to Figure 3b, where the distribution of the

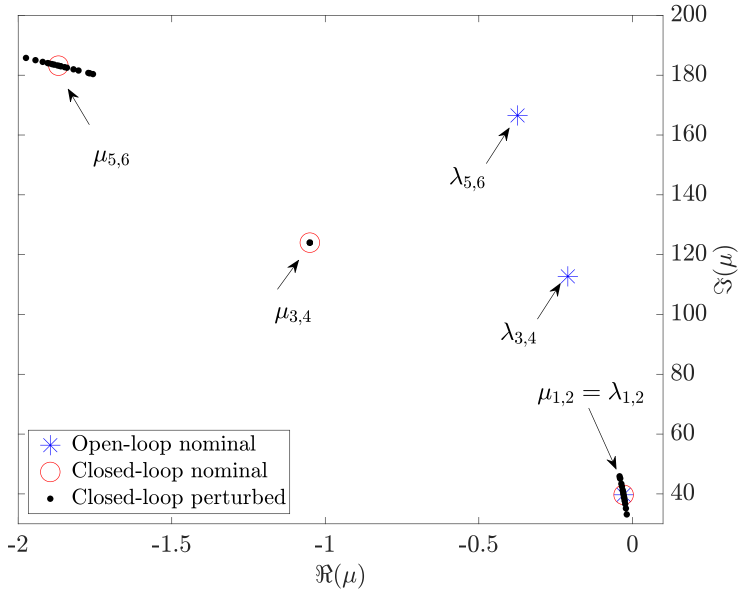


Figure 4: Open-loop and closed-loop poles variabilities for Example 3.

third eigenvalue is seen to be slightly greater when perturbed in this case, and the control effort is also observed to have increased. This is likely due to the fact that extra effort is required to cause a mode to be unchanged under the influence of feedback in addition to minimising the eigenvalue sensitivity of the second mode.

4 Experimental Implementation

4.1 Set-up

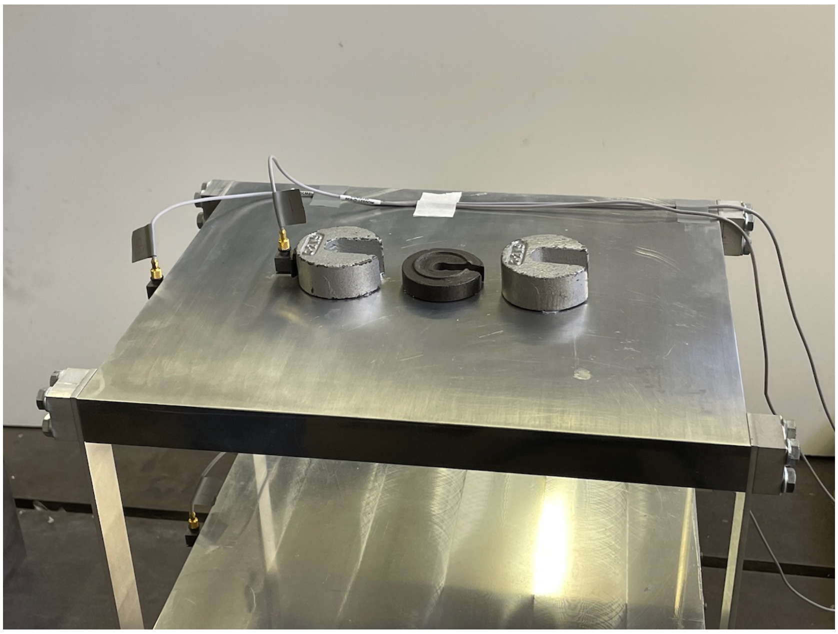
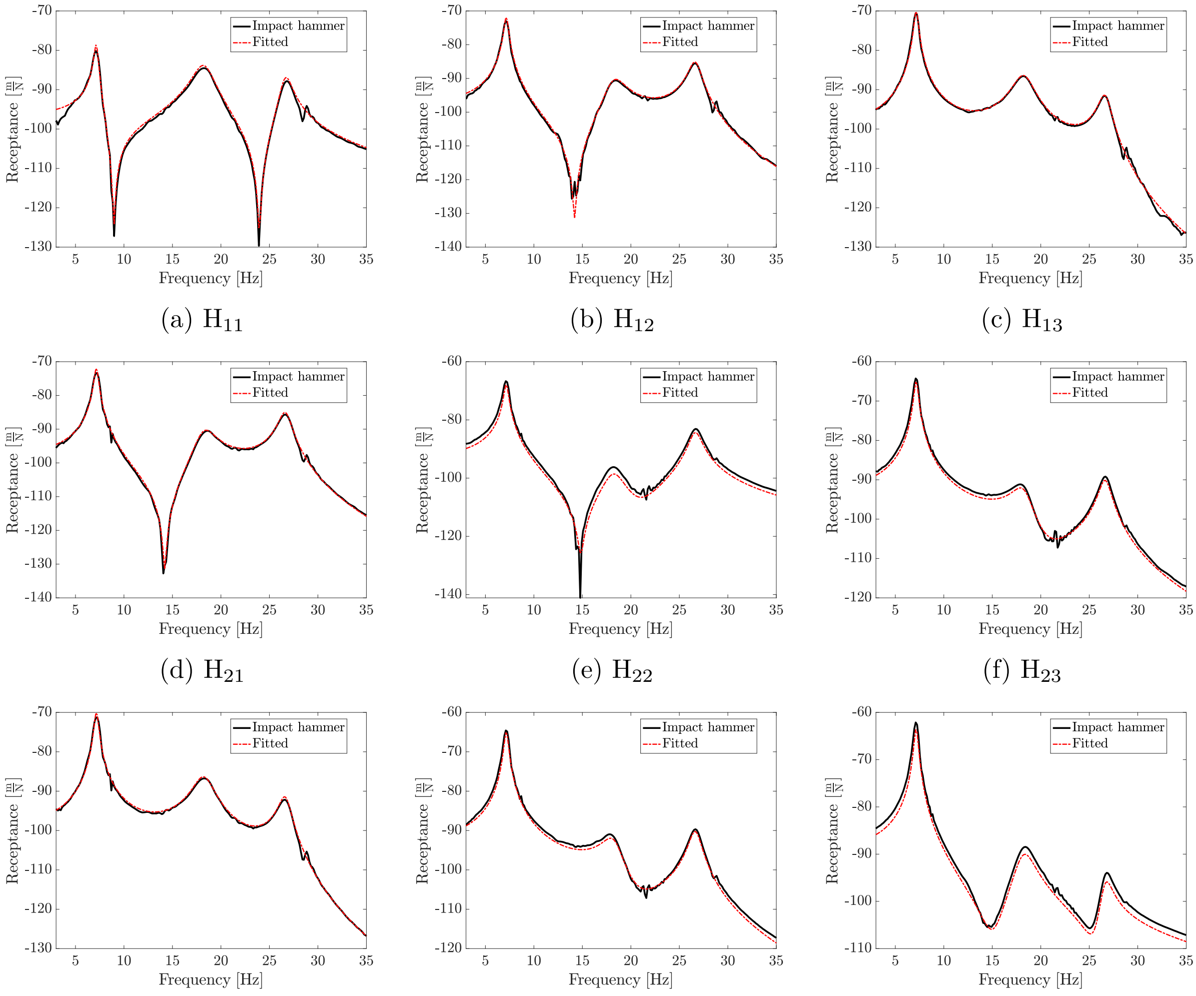


Figure 5: Added mass to simulate a perturbed system.



(a) H_{11} (b) H_{12} (c) H_{13}

(d) H_{21} (e) H_{22} (f) H_{23}

(g) H_{31} (h) H_{32} (i) H_{33}

Figure 6: Open-loop measured and curve fitted FRFs.

Experiments were conducted on a three-degree-of-freedom spring-mass system which is shown in Figure 1a. The structure was developed as part of the EPSRC DigiTwin project [31] and comprises three 5 kg masses connected by a series of parallel beams that served as spring elements. The lower end of the beams were connected to a very rigid test base by four L-brackets. The upper mass was considered uncertain and was varied by adding or removing up to ten 0.1 kg mass as shown in Figure 5. The nominal system is defined as the 5 kg mass plus five 0.1 kg mass, i.e. the upper mass varies from 5 kg to 6 kg, a perturbation of ± 0.5 kg around the nominal mass of 5.5 kg. This is to ensure that the effect of adding and subtracting mass are both considered equally.

A LDS V406 electrodynamic shaker powered by a LDS PA25E voltage driven voltage amplifier was connected to the bottom mass through a stinger as shown in Figure 1a. It is through this shaker that the control force was applied. Hence, the actuator distribution matrix is defined as $\mathbf{b} = [0 \quad 0 \quad 1]^{\mathrm{T}}$. Each mass was instrumented with a B&K Type 4507 accelerometer. All the input and output signals were acquired by the analogue-to-digital and digital-to-analogue converter (ADC and DAC) dSPACE DS1103 PPC Controller Board sampled at 1024Hz. A PC with MATLAB and Simulink was first used to calculate the receptance gain using the pole placement and sensitivity algorithm, the required feedback gains were then uploaded onto dSPACE controller board via Simulink.

The open-loop receptances were acquired by performing an instrumented hammer test on the structure with the shaker attached and the feedback gains set to zero. This is to ensure that the dynamics of the shaker and all other instruments were included in the measurements, and the only difference between the closed-loop and open-loop systems are the feedback gains. The acceleration sensors signal were digitally integrated using MATLAB/Simulink to enable velocity and displacement feedback. The open-loop receptance $\mathbf{H}(j\omega)$ between the sensors' voltage / hammers' voltage was obtained by taking the FFT of the respective time-domain signals. The transfer function $\mathbf{H}(s)$ was approximated by a curve fit to the measured receptance $\mathbf{H}(j\omega)$ using the Rational Fraction Polynomial (RFP) method [34], with a 3 degree-of-freedom fit, i.e. 6 poles. Good agreement between the curve fitted FRFs and the measured FRFs are obtained as shown in Figure 6. The open-loop poles of the nominal system were $\lambda_{1,2} = -1.28 \pm 44.4j$ (first mode, 7.07Hz), $\lambda_{3,4} = -6.51 \pm 114j$ (second mode, 18.2Hz) and $\lambda_{5,6} = -4.23 \pm 167j$ (third mode, 26.5Hz)

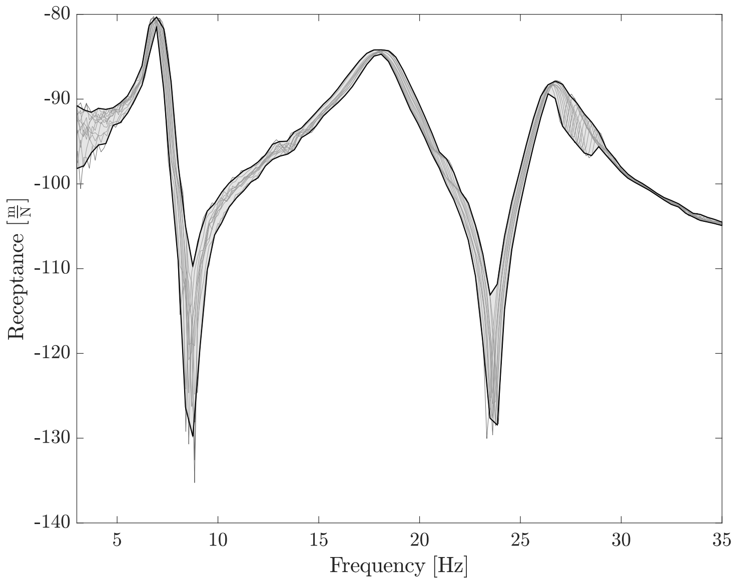


Figure 7: Measured perturbed open-loop receptance at H_{11} with 10 perturbations of 0.1 kg increments in the upper mass, m_3.

Figure 7 illustrate the measured perturbed FRFs at H_{11}. A total of 10 perturbations around the nominal mass m_3 in 0.1 kg increments were added and removed from the upper mass, giving a variation of approximately $\pm 9\%$ m_3. Each of the perturbed FRFs were fitted and the individual transfer function polynomial coefficients are plotted against the perturbed parameter m_3. In theory, this could be performed for any perturbed parameter. The numerical value of the open-loop eigenvalue sensitivity were found as, $S_{1,2} = -0.23561 \pm 2.723j$ (first mode), $S_{3,4} = -0.51035 \pm 3.2196j$ (second mode) and $S_{5,6} = -2.0516 \pm 2.8219j$ (third mode). A polynomial curve fit is performed on the individual estimated rational fraction coefficients to obtain a function that describes the rate of change of coefficients with respect to the perturbed parameter. Thus, the slope of the numerator and denominator of the transfer function in Eq. (19) can be obtained. To demonstrate, assume a fitted transfer function H_{11} as,

$$\mathrm{H}_{11}(s, \theta_z) = \frac{\mathrm{N}(s, \theta_z)}{\mathrm{d}(s, \theta_z)} = \frac{a_{6,z}s^5 + a_{5,z}s^4 + \cdots + a_{1,z}}{b_{7,z}s^6 + b_{6,z}s^5 + \cdots + b_{1,z}} \tag{28}$$

with z being the different perturbation samples to the uncertain model parameter θ and the fitted polynomial coefficients $(a_{i,z})_{i=1}^{6}$ and $(b_{i,z})_{i=1}^{7}$ are also a function of θ_z. With enough samples, one could express the fitted coefficients as,

$$a_i(\theta) = \sum_{j=1}^{\eta} \alpha_j \theta^{j-1} \tag{29}$$

and similarly,

$$b_i(\theta) = \sum_{j=1}^{\eta} \beta_j \theta^{j-1} \tag{30}$$

where Eqs.(29) and (30) are functions of order η that describes the fitted transfer function polynomial coefficients to the perturbed parameter θ. Consequently, the derivative of the numerator and denominator of the measured frequency response function with respect to θ can be evaluated by the virtue of Eqs. (29) and (30), as follows,

$$\frac{\partial \mathrm{N}(s,\theta)}{\partial \theta} = \frac{\partial \mathrm{a}_6(\theta)}{\partial \theta} s^5 + \frac{\partial \mathrm{a}_5(\theta)}{\partial \theta} s^4 + \cdots + \frac{\partial \mathrm{a}_1(\theta)}{\partial \theta} = \sum_{i=1}^{6} a_{i,\theta} s^{i-1} \tag{31}$$

$$\frac{\partial \mathrm{d}(s,\theta)}{\partial \theta} = \frac{\partial \mathrm{b}_6(\theta)}{\partial \theta} s^5 + \frac{\partial \mathrm{b}_5(\theta)}{\partial \theta} s^4 + \cdots + \frac{\partial \mathrm{b}_1(\theta)}{\partial \theta} = \sum_{i=1}^{7} b_{i,\theta} s^{i-1} \tag{32}$$

where $a_{i,\theta}$, and $b_{i,\theta}$ stands for $\partial \mathrm{a}_i(\theta)/\partial\theta$ and $\partial \mathrm{b}_i(\theta)/\partial\theta$ respectively. Thus, Eq. (25) may subsequently be evaluated with a desired objective that yields the required feedback gains for active control. In the proceeding experiments, the eigenvalue sensitivity of the second and the third modes are minimised. First, simulation was performed to predict the closed-loop response of the experiments prior to implementing active control in real time. The simulated results were obtained via the closed-loop transfer matrix given in Eq. (5), where the measured open-loop frequency response and the required feedback gains were used to predict the closed-loop responses. The predicted results were obtained prior to the implementation of active control in real-time. The results showed very good agreement between the predicted and the measured closed-loop response as shown in the following examples.

4.2 Experiment 1

In the first experiment, the objective was to shift the first and the third modes and make the third mode insensitive to m_3 perturbation. The desired closed-loop poles were chosen to be,

$$\mu_{1,2} = -1.41 \pm 44.4j$$
$$\mu_{5,6} = -4.66 \pm 200j$$
$$S_{5,6} = 0 \pm 0j$$

and the sensitivity of the third mode is to be minimised. Solving Eq. (25) to obtain the control gains

$$\mathbf{g} = \begin{bmatrix} 3526 & -2497 & 462.8 \end{bmatrix}^{\mathrm{T}} \qquad \mathbf{f} = \begin{bmatrix} 6.311 & 3.750 & 3.450 \end{bmatrix}^{\mathrm{T}}$$

The desired closed-loop poles and the experimentally measured closed-loop poles are summarised in Table 1.

The FRFs in Figure 8 demonstrated that the algorithm had successfully shifted the first and third mode, while the third mode had also become insensitive to the perturbation of $\pm 10\%$ m_3. The three perturbed closed-loop FRFs represent the $-0.5,\ 0,\ +0.5$, kg perturbations. It can be clearly seen that the peak in the third mode is shifted and is not influenced by the perturbation, this is due to the assignment of eigenvalue sensitivity minimisation. Good agreement can be seen between the simulated and experimental results in Figure 8a and 8b. The numerical value of the third mode sensitivity was reduced from the open-loop value of $S_{5,6} = -2.0516 \pm 2.8219j$ to a new closed-loop value of $\hat{S}_{5,6} = -0.22068 \pm 1.4098j$ with a reduction of 89.2% and 50.0% to the the real and imaginary components of the sensitivity from the open-loop case. Technically speaking, the numerical value of the assigned closed-loop eigenvalue sensitivity should be zero, as it was prescribed. However, this is difficult to achieved experimentally on account of measurement and curve fitting errors.

Table 1: Experiment 1: The desired and measured closed-loop poles.

Closed-loop poles	Desired	Obtained
$\mu_{1,2}$	$-1.41 \pm 44.4j$	$-1.92 \pm 43.3j$
$\mu_{3,4}$	Unconstrained	$-9.10 \pm 124j$
$\mu_{5,6}$	$-4.66 \pm 200j$	$-18.7 \pm 201j$

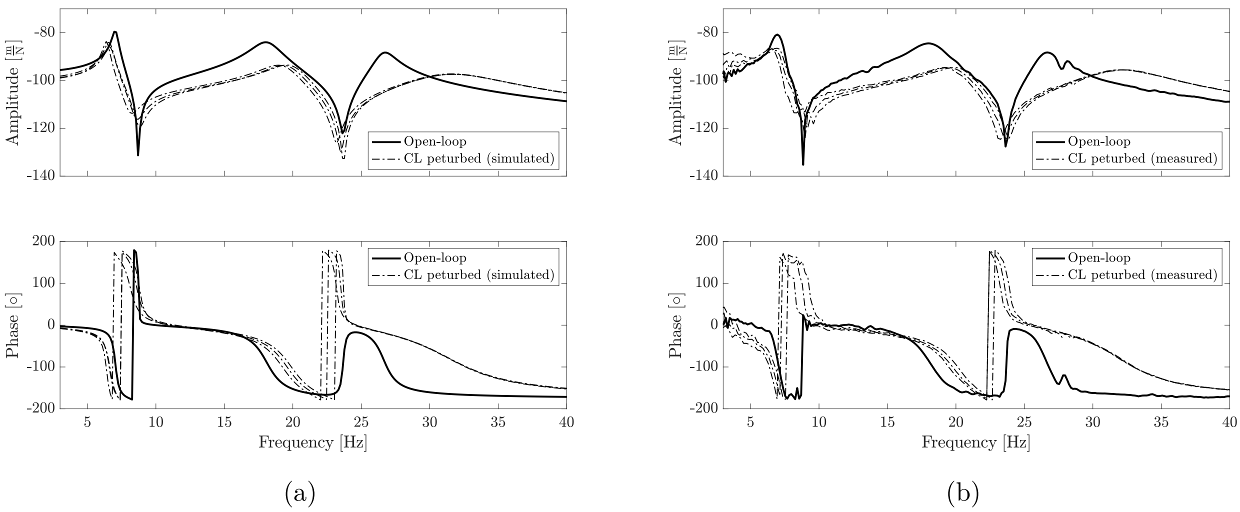


Figure 8: H_{11} receptance FRFs (Experiment 1). (a) Simulation (b) Experiment

4.3 Experiment 2

In the second experiment, it was desired to shift the poles of the first and the second mode and make the second mode insensitive to m_3 perturbation. The objective was therefore chosen to be,

$$\mu_{1,2} = -1.92 \pm 44.4j$$
$$\mu_{3,4} = -9.76 \pm 120j$$
$$S_{3,4} = 0 \pm 0j$$

which require the control gains of

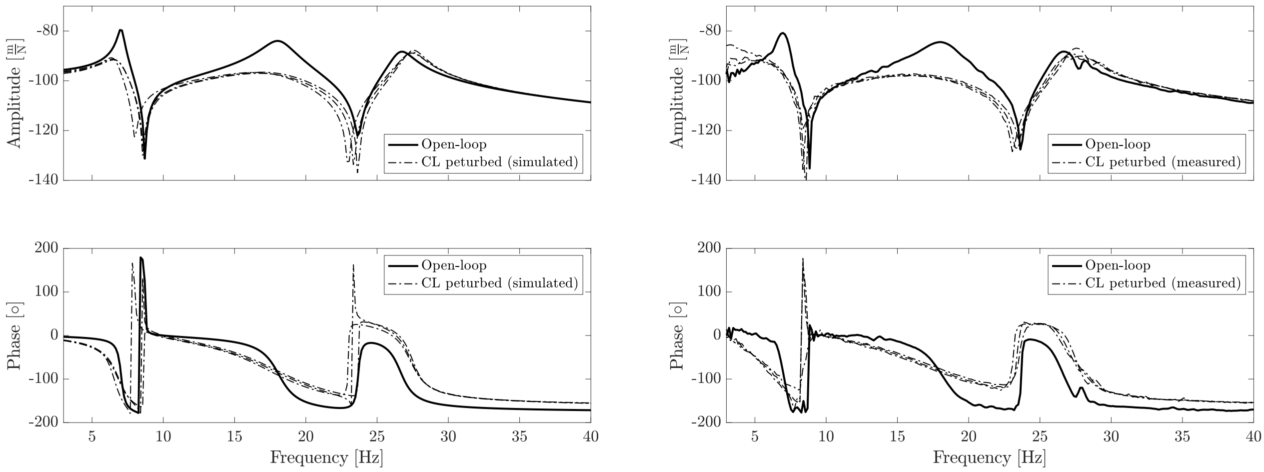
$$\mathbf{g} = \begin{bmatrix}1035 & -372.5 & 174.1\end{bmatrix}^{\mathrm{T}} \qquad \mathbf{f} = \begin{bmatrix}12.47 & 14.64 & -2.226\end{bmatrix}^{\mathrm{T}}$$

The desired closed-loop poles and the experimentally measured closed-loop poles are summarised in Table 2.

Table 2: Experiment 2: The desired and measured closed-loop poles.

Closed-loop poles	Desired	Obtained
$\mu_{1,2}$	$-1.92 \pm 44.4j$	$-3.3 \pm 42.8j$
$\mu_{3,4}$	$-9.76 \pm 120j$	$-23.4 \pm 119j$
$\mu_{5,6}$	Unconstrained	$-6.03 \pm 173j$

It was found that in order to achieve the objective of making the second mode insensitive, the algorithm attempts to apply a high level of damping to the second mode, as can be observed from the FRFs in



(a) (b)

Figure 9: H_{11} receptance FRFs (Experiment 2). (a) Simulation (b) Experiment

Figure 9. Physical insight into this can be explained by the fact that a heavily damped mode is naturally insensitive to perturbation. It is also worth noting that no constraint is being imposed on the closed-loop poles, which allows the algorithm to place the new closed-loop poles far from its open-loop position to a location where it is least sensitive, as demanded. The numerical value of the assigned closed-loop eigenvalue sensitivity in this case were found as $\hat{S}_{3,4} = -0.46484 \pm 0.44628j$, that gives a 9.80% and 86.1% reduction to the real and imaginary parts of the eigenvalue sensitivity from the open-loop case.

5 Conclusions

In this research, a new method of minimising the eigenvalue sensitivity using the Sherman-Morrison formula description of the receptance method is presented. This method is based on a first order Taylor's approximation to the closed-loop receptance when subjected to uncertain parameters. The rate of change of the closed-loop eigenvalues to an uncertain parameter are then subsequently expressed in a linear equation in the unknown receptance gains. Eigenvalue sensitivity minimisation can then be achieved by solving for the unknown gains that assign the closed-loop poles with no sensitivities. The minimisation of the eigenvalue sensitivity ensures that a small perturbation to the system results in a small effect on the placement of the closed-loop poles. It was shown that implementing the sensitivity equation in the pole placement formulation, eigenvalue sensitivity can be significantly reduced. This method has been tested both numerically and experimentally on a three-degree-of-freedom structure and it was shown in both cases that the algorithm assigned the closed-loop poles and renders it insensitive to ±20% and ±9% perturbation in one of the masses for the numerical and experimental examples respectively. In the considered experiment, it was found that the algorithm reduced the eigenvalue sensitivity by 89.2% and 86.1% to the real and imaginary part across both experiments.

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