State for linear discrete time-varying systems, with applications to dissipative systems

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Abstract

We study the notion of state for a class of linear discrete time-varying systems and provide a characterization of polynomial operator in the shift used to define state trajectory from both external and auxiliary trajectories. We then characterize lossless systems and show that a storage function for a dissipative system can be written as a quadratic function of the state.
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1. Introduction

We consider sets of solutions of systems of higher-order difference equations with time-varying coefficients, together with the following concept of state crucially related to *trajectory concatenability* (see [8]). Two trajectories $w_i : \mathbb{Z} \rightarrow \mathbb{R}^q$, $i = 1, 2$ produced by a linear, time-varying system have the *same state at time* k_0 if their *concatenation at* k_0, defined by

$$\left(w_1 \underset{k_0}{\wedge} w_2 \right)(k) := \begin{cases} w_1(k) & \text{if } k < k_0 \\ w_2(k) & \text{if } k \geq k_0 \ , \end{cases} \tag{1}$$

is also a solution of the same system of difference equations.

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We show that given a system trajectory, a corresponding state trajectory can be computed by applying a polynomial difference operator in the time-shift to it. We call such operator a *state map*, (see [3, 4]); it can be computed directly from the system equations. Equations of first order in the state and of zeroth order in the system variables can be computed directly from a state map.

We also discuss some consequences of our definition of state in dissipativity theory. We show that *path-sums* (the discrete-time analogous of path-integrals in continuous-time) of constant functionals of the variables of a linear, discrete time-varying system are quadratic functions of its state. In the special case of linear, time-invariant systems we conclude from such more general result that storage functions are a quadratic function of the state.

The paper is organised as follows. In Sect. 2 we introduce some of the main concepts used in this paper, i.e. the notion of the state, system representations and adjoint operator in the shift over the ring of Laurent polynomials with time-varying coefficients. In Sect. 3 we characterize concatenability of system trajectories for various system representations. We prove that storage functions are quadratic function of the state, compare our results with the literature and point to future directions of research in Sect. 4. We conclude the paper in Sect. 5.

2. Background

2.1. System representations

We consider linear discrete time-varying systems characterized by their *behavior*, i.e. the set of trajectories satisfying a system of linear difference equation with time-varying coefficient (see Section V.A of [5]). We first consider equations that describe the behavior using only the variables of interest, called *external variables*, denoted by w:

$$R_{L_1}(k)w(k+L_1)+R_{L_1+1}(k)w(k+L_1+1)+\ldots+R_{L_2}(k)w(k+L_2)=0\ ,\ k\in\mathbb{Z}\ , \tag{2}$$

where $L_1, L_2 \in \mathbb{Z}$, and R_j for $j = L_1, \ldots, L_2$: are time-dependent matrices.

Denote by $\mathcal{M}[\xi]$ the ring of polynomials in the indeterminate ξ with time-varying coefficients, and by $\mathcal{M}[\xi^{-1},\xi]$ the ring of Laurent polynomials in the indeterminate ξ with time-varying coefficients. The ring $\mathcal{M}[\xi]$ is a domain, that is $R_1(\xi)R_2(\xi) = 0 \Rightarrow R_1(\xi) = 0$ or $R_2(\xi) = 0$ and multiplication is not commutative (more details about multiplication are provided later in the paper.) It has a (right- and left) division-with-remainder operation and it admits a field of rational functions, consisting of elements of the form $\frac{n}{d}$ with $n, d \in \mathcal{M}\left[\xi^{-1},\xi\right]$ and $d \neq 0$.

We denote by σ and σ^{-1} the *left*, respectively *right shift*, operators acting on the space of real-valued, q-dimensional sequences $\left(\mathbb{R}^q\right)^{\mathbb{Z}} := \{w : \mathbb{Z} \to \mathbb{R}^q\}$. They are defined by

$$
\begin{aligned}
&\sigma, \sigma^{-1} : \left(\mathbb{R}^q\right)^{\mathbb{Z}} \to \left(\mathbb{R}^q\right)^{\mathbb{Z}} \\
&(\sigma w)(k) := w(k+1) \\
&(\sigma^{-1} w)(k) := w(k-1) \ .
\end{aligned}
$$

From (2), define $R(\xi^{-1},\xi) := R_{L_1}\xi^{L_1} + \ldots + R_{L_2}\xi^{L_2} \in \mathcal{M}[\xi^{-1},\xi]^{g\times q}$, and associate with it the polynomial difference operator $R(\sigma^{-1},\sigma)$ defined by

$$
\begin{aligned}
R(\sigma^{-1},\sigma) \quad &: \quad \left(\mathbb{R}^q\right)^{\mathbb{Z}} \to \left(\mathbb{R}^g\right)^{\mathbb{Z}} \\
R(\sigma^{-1},\sigma)w \quad &:= \quad R_{L_1}\sigma^{L_1}w + R_{L_1+1}\sigma^{L_1+1}w + \ldots + R_{L_2}\sigma^{L_2}w \ ,
\end{aligned}
$$

where $R(\sigma^{-1},\sigma)w(k) = \sum_{i=L_1}^{L_2} R_i(k)\left(\sigma^i w\right)(k)$. Hence, we write (2) equivalently as

$$
R(\sigma^{-1},\sigma)w = 0 \ . \tag{3}
$$

We call (2) (equivalently (3)) *kernel representations* of the linear discrete time-varying *behavior*

$$
\mathfrak{B} := \{w : \mathbb{Z} \to \mathbb{R}^q \mid w \ \text{ satisfies (3)}\} \ , \tag{4}
$$

since $\mathfrak{B} = \ker\ R(\sigma^{-1},\sigma)$.

Example 1. Consider a time-varying mass-spring system with $m(k)$ the varying mass hanging on a spring with a spring constant k_s. We denote by w_x the position of the mass and by w_F the force acting on it, see p.2501 of [5]. The system is described by the difference equation

$$\begin{aligned}\big(T_d^2 k_s + m(k)\big) w_x(k) - \big(m(k+1) + m(k)\big) w_x(k+1)\\ + m(k+1) w_x(k+2) - T_d^2 w_F(k) = 0.\end{aligned}$$

where $T_d > 0$ is the sampling rate. Let $w = [w_x, w_f]^\top$ then

$$\begin{aligned}\underbrace{\begin{bmatrix} T_d^2 k_s + m(k) & -T_d^2 \end{bmatrix}}_{=:R_0(k)} \begin{bmatrix} w_x(k) \\ w_F(k) \end{bmatrix} - \underbrace{\begin{bmatrix} (m(k+1) + m(k) & 0 \end{bmatrix}}_{=:R_1(k)} \begin{bmatrix} w_x(k+1) \\ w_F(k+1) \end{bmatrix}\\ + \underbrace{\begin{bmatrix} m(k+1) & 0 \end{bmatrix}}_{=:R_2(k)} \begin{bmatrix} w_x(k+2) \\ w_F(k+2) \end{bmatrix} = 0 \ .\end{aligned}$$

These equations correspond to the kernel representation

$$\begin{bmatrix} T_d^2 k_s + m(k) & -T_d^2 \\ -\big((m(k+1) + m(k)\big)\sigma & 0 \\ \big(m(k+1)\big)\sigma^2 & 0 \end{bmatrix} \begin{bmatrix} w_x(k) \\ w_F(k) \end{bmatrix} = 0 \ ,$$

with

$$\underbrace{\begin{bmatrix} T_d^2 k_s + m(k) & -T_d^2 \\ -\big((m(k+1) + m(k)\big)\xi & 0 \\ \big(m(k+1)\big)\xi^2 & 0 \end{bmatrix}}_{=:R(\xi)} \in \mathcal{M}[\xi^{-1}, \xi]^{3\times 2} \ ,$$

The concept of *rank* of a matrix in $\mathcal{M}\left[\xi^{-1}, \xi\right]^{g\times q}$ follows in a straightforward way from that of rank of the module generated by its columns or rows. A matrix $U \in \mathcal{M}\left[\xi\right]^{g\times g}$ is called *unimodular* if its inverse is also an element of $\mathcal{M}\left[\xi\right]^{g\times g}$. As already mentioned multiplication in $\mathcal{M}\left[\xi\right]$ is not commutative, that is $\xi R \neq R\xi$ and $\xi^{-1}R \neq R\xi^{-1}$; indeed, $\xi R(k) = R(k+1)\xi$ and $\xi^{-1}R(k) = R(k-1)\xi^{-1}$ for $R \in \mathcal{M}$. It follows that

$$\sigma R(\sigma^{-1}, \sigma) w = \sum_{i=L_1}^{L_2} R_i(k+1)\sigma^i w(k+1)$$

and that

$$\sigma^{-1} R(\sigma^{-1}, \sigma) w = \sum_{i=L_1}^{L_2} R_i(k-1)\sigma^i w(k-1).$$

Remark 1. It follows from the injective cogenerator property (see Appendix A p. 2511 of [5]) and the same argument of the proof of Th. 9 p. 125 of [10]) that a behavior $\mathfrak{B} = \ker\ R(\sigma^{-1},\sigma)$ is also associated with any representation of the form $U(\sigma^{-1},\sigma)R(\sigma^{-1},\sigma)$, with $U \in \mathcal{M}[\xi^{-1},\xi]^{g\times g}$ a unimodular matrix. In the following we will use such property, where expedient, to work with representations involving only the left-shift. Indeed, given $R(\xi^{-1},\xi) := R_{L_1}\xi^{L_1} + \ldots + R_{L_2}\xi^{L_2} \in \mathcal{M}[\xi^{-1},\xi]^{g\times q}$ with $L_1 < 0$, the behavior $\ker R(\sigma^{-1},\sigma)$ is also represented by $R'(\xi) := \xi^{-L_1}\left(R_{L_1}\xi^{L_1} + \ldots + R_{L_2}\xi^{L_2}\right) \in \mathcal{M}[\xi]^{g\times q}$.

Example 2. It is straightforward to check that the system in Ex. 1 also admits a kernel representation induced by

$$\underbrace{\begin{bmatrix} \big(T_d^2 k_s + m(k-1)\big)\xi^{-1} & -T_d^2\xi^{-1} \\ -\big((m(k) + m(k-1)\big) & 0 \\ \big(m(k)\big)\xi & 0 \end{bmatrix}}_{=:R(\xi^{-1},\xi)} \in \mathcal{M}[\xi^{-1},\xi]^{3\times 2},$$

with $\xi R(\xi^{-1},\xi) = R(\xi)$.

We now turn our attention to representations where besides the external variables w, also *auxiliary variables* ℓ are used, i.e.

$$M_{L_0}\ell(k+L_0) + \ldots + M_{L_1}\ell(k+L_1) = R_{L_2}w(k+L_2) + \ldots + R_{L_3}w(k+L_3)\ , \quad (5)$$

where $L_i \in \mathbb{Z}$, $i = 0,\ldots,3$; $M_j \in \mathcal{M}[\xi^{-1},\xi]^{g\times m}$ and $R_j \in \mathcal{M}[\xi^{-1},\xi]^{g\times q}$, $j = 0,\ldots,L_i$. Defining

$$\begin{aligned} M(\sigma^{-1},\sigma) &:= M_{L_0}\sigma^{L_0} + M_{L_0+1}\sigma^{L_0+1} + \ldots + M_{L_1}\sigma^{L_1} \\ R(\sigma^{-1},\sigma) &:= R_{L_2}\sigma^{L_2} + R_{L_2+1}\sigma^{L_2+1} + \ldots + R_{L_3}\sigma^{L_3}\ , \end{aligned}$$

equation (5) can be rewritten as

$$M(\sigma^{-1},\sigma)\ell = R(\sigma^{-1},\sigma)w\ . \quad (6)$$

We call (5) (equivalently (6)) an *hybrid representation* of the linear discrete time-varying *external behavior*

$$\mathfrak{B} := \{w : \mathbb{Z} \to \mathbb{R}^q \mid \ \text{there exists } \ell : \mathbb{Z} \to \mathbb{R}^m \ \text{ such that}(w,\ell) \ \text{ satisfy } (5)\}\ . \quad (7)$$

We observe for future use that (5) and (6) also describe the *full behavior*

$$\mathfrak{B}_f := \left\{(w,\ell) : \mathbb{Z} \to \mathbb{R}^{q+m} \mid \quad (5) \text{ is satisfied}\right\} \ . \tag{8}$$

If the matrix $R(\xi^{-1},\xi)$ in (5) and (6) is the q-dimensional identity, i.e.

$$M(\sigma^{-1},\sigma)\ell = w \ , \tag{9}$$

we talk of an *image representation* of the external behavior, since in that case $\mathfrak{B} = \text{im}\left(M(\sigma^{-1},\sigma)\right)$. Note that in an image representation, the latent variable ℓ is *free*, i.e. for every choice of trajectory $\bar{\ell} : \mathbb{Z} \to \mathbb{R}^m$, there exists a trajectory $(w,\ell) \in \mathfrak{B}_f$ such that $\ell = \bar{\ell}$. A system admits an image representation if and only if it is *controllable* (see also Prop. 4.3 of [8] and Th. 6 of [10]).

Finally, we recall the following definition of *observability*.

Definition 1. *Let $\mathfrak{B}_f$ be a linear, discrete time-varying behavior with external variable w and auxiliary variable ℓ. Then ℓ is* observable *from w if*

$$(w,\ell_1),(w,\ell_2) \in \mathfrak{B}_f \Longrightarrow \ell_1 = \ell_2 \ . \tag{10}$$

In an observable system each external variable trajectory w uniquely identifies the auxiliary variable ℓ trajectory such that (5) holds. Consequently, (10) is equivalent with $\{\ell : \mathbb{Z} \to \mathbb{R}^m \mid M(\sigma)\ell = 0 \text{ in } (6)\} = \{0\}$. The following result can be proved using the injective co-generator property.

Proposition 1. *Define $\mathfrak{B}_f$ by (8). The following statements are equivalent:*

1. *ℓ is observable from w;*
2. *$M \in \mathcal{M}[\xi^{-1},\xi]^{g\times m}$ is left invertible;*
3. *there exists $F \in \mathcal{M}[\xi^{-1},\xi]^{m\times q}$ such that $\ell = F(\sigma^{-1},\sigma)w$.*

See Th. 9 p. 125 of [10] for the proof.

2.2. The state property

The following statements are a straightforward consequence of linearity.

Proposition 2. *Let $\mathfrak{B}$ be a linear discrete time-varying behavior. Let $w_1, w_2 \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$. The following statements are equivalent:*

1. $w_1 \underset{k_0}{\wedge} w_2 \in \mathfrak{B}$;
2. $0 \underset{k_0}{\wedge} w_2 - w_1 \in \mathfrak{B}$;
3. $w_2 - w_1 \underset{k_0}{\wedge} 0 \in \mathfrak{B}$.

A trajectory-based definition of state is as follows (see also [8] and Def. 13 p. 2505 of [5]).

Definition 2. *Let $\mathfrak{B}_f$ be a system with external variable w and auxiliary variable ℓ. ℓ has the* property of state *if*

$$(w_1, \ell_1), (w_2, \ell_2) \in \mathfrak{B}_f \text{ and } \ell_1(k_0) = \ell_2(k_0) \Longrightarrow (w_1, \ell_1) \underset{k_0}{\wedge} (w_2, \ell_2) \in \mathfrak{B}_f \ . \quad (11)$$

Thus a state variable is a special kind of auxiliary variable, that characterizes the property of *concatenability* between *full* (internal and external) trajectories.

Using the result of Prop. 2, one can prove in a straightforward way the following characterizations of state variable.

Proposition 3. *Let $\mathfrak{B}_f$ be a system with external variable w and auxiliary variable ℓ. Let $k_0 \in \mathbb{Z}$, then the following statements are equivalent:*

1. *ℓ has the property of state;*
2. *If $(w_1, \ell_1), (w_2, \ell_2) \in \mathfrak{B}_f$ and $\ell_1(k_0) = \ell_2(k_0)$, then $(0, 0) \underset{k_0}{\wedge} (w_2 - w_1, \ell_2 - \ell_1) \in \mathfrak{B}_f$;*
3. *If $(w_1, \ell_1), (w_2, \ell_2) \in \mathfrak{B}_f$ and $\ell_1(k_0) = \ell_2(k_0)$, then $(w_2 - w_1, \ell_2 - \ell_1) \underset{k_0}{\wedge} (0, 0) \in \mathfrak{B}_f$*

2.3. Adjoint operators in the shift

Denote by $\mathcal{L}_2(\mathbb{Z}, \mathbb{R}^q)$ the space of square-summable sequences equipped with the standard inner product, and let $P(\xi) = \sum_{j=0}^{L} P_j(k)\xi^j \in \mathcal{M}[\xi]^{q_1 \times q_2}$. The polynomial operator in the shift $P(\sigma)$ associated with P is defined by

$$\begin{aligned} P(\sigma) \quad &: \quad \mathcal{L}_2(\mathbb{Z}, \mathbb{R}^{q_2}) \to \mathcal{L}_2(\mathbb{Z}, \mathbb{R}^{q_1}) \\ (P(\sigma)w)(k) \quad &:= \quad P_0(k)w(k) + \ldots + P_L(k)w(k+L) \ , \end{aligned} \quad (12)$$

and it has a formal adjoint, defined in the natural way, which is also a polynomial operator in the shift with time-varying coefficients. The following result characterizes its representation as an element of $\mathcal{M}[\xi]^{q_2 \times q_1}$.

Proposition 4. *The adjoint of $P(\sigma)$ defined by* (12)*, denoted by $P(\sigma)^*$, is*

$$\begin{aligned} P(\sigma)^* &: \quad \mathcal{L}_2(\mathbb{Z},\mathbb{R}^{q_1}) \to \mathcal{L}_2(\mathbb{Z},\mathbb{R}^{q_2}) \\ \left(P(\sigma)^* w'\right)(k) &:= \quad P_L(k-L)^\top w'(k-L) + \ldots + P_0(k)^\top w'(k) \; . \end{aligned} \tag{13}$$

Proof. To prove the claim, let $w' \in \mathcal{L}_2(\mathbb{Z},\mathbb{R}^{q_2})$. The formal adjoint of $P(\sigma)$ is defined by the condition

$$\begin{aligned} \langle w', P(\sigma)w \rangle = \sum_{k=-\infty}^{+\infty} w'(k)^\top \left(P(\sigma)w\right)(k) &= \langle P(\sigma)^* w', w \rangle \\ &= \sum_{k=-\infty}^{+\infty} \left(P(\sigma)^* w'\right)(k)^\top w(k) \; . \end{aligned}$$

Consider $0 \leq j \leq L$, and rewrite the expression

$$\sum_{k=-\infty}^{+\infty} w'(k)^\top \left(P(\sigma)w\right)(k) = \sum_{k=-\infty}^{+\infty} w'(k)^\top \left(\sum_{j=0}^{L} P_j(k) w(k+j) \right) ,$$

by gathering the terms associated with $w(k+j)$:

$$\begin{aligned} \sum_{k=-\infty}^{+\infty} w'(k)^\top \left(P(\sigma)w\right)(k) = & \sum_{k=-\infty}^{+\infty} \big(w'(k+j-L)^\top P_L(k+j-L) \\ & + w'(k+j-L+1)^\top P_{L-1}(k+j-L+1) \\ & + \ldots + w'(k+j)^\top P_0(k+j) \big) \, w(k+j) \; . \end{aligned}$$

Defining $k' := k+j$, this expression is rewritten as

$$\sum_{k=-\infty}^{+\infty} w'(k)^\top \left(P(\sigma)w\right)(k) = \sum_{k'=-\infty}^{+\infty} \left(P(\sigma)^* w'\right)(k')^\top w(k') \; ,$$

where $P(\sigma)^*(k') := P_L(k'-L)^\top \sigma^{-L} + \ldots + P_0(k')^\top$. This proves the claim. □

Note that the adjoint of $P(\sigma)$ induced by a polynomial matrix $P \in \mathcal{M}[\xi]^{q_1 \times q_2}$ is represented by an element of the Laurent polynomial matrix $P \in \mathcal{M}[\xi^{-1}, \xi]^{q_1 \times q_2}$.

3. Concatenability conditions and state maps

3.1. Kernel representations

The main results in this subsection are equivalent to some in Section V.B of [5], whose approach is more akin to the classical realization problem, i.e. that of devising auxiliary variables with respect to which first-order representations can be computed. Our only claim to originality is that we explicitly show the connection of the state property with the computation of state variables from the system equations.

Without loss of generality (see Rem. 1) we consider kernel representations (2) induced by matrices $R \in \mathcal{M}[\xi]^{g\times q}$, i.e.

$$R(\xi) = R_0 + \ldots + R_L\xi^L \ , \ L \in \mathbb{N} \ . \tag{14}$$

We define the following two Laurent polynomial matrices with time-varying coefficients, whose value at k is defined by:

$$X^+(\xi) \quad = \quad \begin{bmatrix} R_1(k-1) + \ldots + R_L(k-1)\xi^{L-1} \\ R_2(k-2) + \ldots + R_L(k-2)\xi^{L-2} \\ \vdots \\ R_L(k-L) \end{bmatrix} \tag{15}$$

and

$$X^-(\xi^{-1}) \quad = \quad \begin{bmatrix} R_0(k-1)\xi^{-1} \\ R_0(k-2)\xi^{-2} + R_1(k-2)\xi^{-1} \\ \vdots \\ R_0(k-L)\xi^{-L} + \ldots + R_{L-1}(k-L)\xi^{-1} \end{bmatrix} \tag{16}$$

We characterize the concatenability of a trajectory in $\mathfrak{B}$ with zero.

Theorem 1. *Consider a kernel representation* (14). *Let* $w, w_1, w_2 \in \ker\ R(\sigma)$ *and* $k_0 \in \mathbb{Z}$. *Define* $X^+(\xi)$ *by (15), and* $X^-(\xi)$ *by (16). Then*

1. $0 \underset{k_0}{\wedge} w \in \mathfrak{B} \Longleftrightarrow X^+(\sigma)w(k_0) = 0$;
2. $w \underset{k_0}{\wedge} 0 \in \mathfrak{B} \Longleftrightarrow X^-(\sigma)w(k_0-1) = 0$;
3. $w_1 \underset{k_0}{\wedge} w_2 \in \mathfrak{B} \Longleftrightarrow X^-(\sigma)w_1(k_0-1) = -X^+(\sigma)w_2(k_0)$.

Proof. To prove (1), observe first that since $w \in \mathfrak{B}$, $(R(\sigma)w)(k) = 0$ for all $k \geq k_0$. Moreover, since $\left(0 \underset{k_0}{\wedge} w\right)(k) = 0$ for all $k \leq k_0 - L - 1$, it follows that

$R(\sigma)\left(0 \underset{k_0}{\wedge} w\right)(k) = 0$ for $k \leq k_0 - L - 1$. Conclude that $0 \underset{k_0}{\wedge} w \in \mathfrak{B}$ if and only if $R(\sigma)\left(0 \underset{k_0}{\wedge} w\right)(k) = 0$ for $k = k_0 - 1, \ldots, k_0 - L$, equivalently if and only if

$$\begin{aligned}
R_1(-1+k_0)w(k_0) + \cdots + R_L(-1+k_0)w(k_0+L-1) &= 0 \\
R_2(-2+k_0)w(k_0) + \cdots + R_L(-2+k_0)w(k_0+L-2) &= 0 \\
\vdots \qquad\qquad\qquad & \vdots \\
R_{L-1}(1-L+k_0)w(k_0) + R_L(1-L+k_0)w(k_0+1) &= 0 \\
R_L(-L+k_0)w(k_0) &= 0\,.
\end{aligned}$$

It is straightforward to check that these are equivalent with $\left(X^+(\sigma)w\right)(k_0) = 0$.

To prove (2), observe that $0 \in \mathfrak{B}$ hence $\left(R(\sigma)w\right)(k) = 0$ for all $k \geq k_0$. Now $R(\sigma)\left(w \underset{k_0}{\wedge} 0\right)(k) = 0$ for $k = k_0 - 1, \ldots, k_0 - L$ if and only if the following equations hold:

$$\begin{aligned}
R_0(-1+k_0)w(k_0-1) &= 0 \\
R_0(-2+k_0)w(k_0-2) + R_1(-2+k_0)w(k_0-1) &= 0 \\
\vdots \qquad\qquad\qquad & \vdots \\
R_0(-L+k_0)w(k_0-L) + \ldots R_{L-1}(-L+k_0)w(k_0-1) &= 0\,.
\end{aligned}$$

It is a matter of straightforward verification to check that these equations are equivalent with $(X^-(\sigma)w)(k_0-1) = 0$.

We now prove statement (3), observe that since $w_1, w_2 \in \mathfrak{B}$, then $\left(R(\sigma)w_1\right)(k) = 0$ for all $k \leq k_0$ and $\left(R(\sigma)w_2\right)(k) = 0$ for all $k \geq k_0$. Now $w_1 \underset{k_0}{\wedge} w_2 \in \mathfrak{B}$ if and only if $R(\sigma)\left(w_1 \underset{k_0}{\wedge} w_2\right)(k) = 0$ for $k = k_0 - 1, \ldots, k_0 - L$, equivalently if and only if

$$\begin{aligned}
R_0(k_0-1)w_1(k_0-1) + R_1(k_0-1)w_2(k_0) + \cdots + R_L(k_0-1)w_2(k_0+L-1) &= 0 \\
R_0(k_0-2)w_1(k_0-2) + R_1(k_0-2)w_1(k_0-1) + \cdots + R_L(k_0-2)w_2(k_0+L-2) &= 0 \\
\vdots \qquad\qquad\qquad & \vdots \\
R_0(k_0-L)w_1(k_0-L) + \cdots + R_{L-1}(k_0-L)w_1(k_0-1) + R_L(k_0-L)w_2(k_0) &= 0.
\end{aligned}$$

Rewrite gathering terms involving w_1 and w_2 .

$$R_0(k_0-1)w_1(k_0-1) = -R_1(k_0-1)w_2(k_0) - \cdots - R_L(k_0-1)w_2(k_0+L-1)$$
$$R_0(k_0-2)w_1(k_0-2) + R_1(k_0-2)w_1(k_0-1) = - \cdots - R_L(k_0-2)w_2(k_0+L-2)$$
$$\vdots$$
$$R_0(k_0-L)w_1(k_0-L) + \cdots + R_{L-1}(k_0-L)w_1(k_0-1) = -R_L(k_0-L)w_2(k_0).$$

The equations on the left are equivalent with $(X^-(\sigma)w_1)(k_0-1)$ and those of the right with $\left(X^+(\sigma)w_2\right)(k_0)$, hence $(X^-(\sigma)w_1)(k_0-1) = -\left(X^+(\sigma)w_2\right)(k_0)$

□

Since the value of $X^+(\sigma)w$ at k depends on the values of w at $k, k+1, \ldots$, and since that of $X^-(\sigma)w$ depends on the values of w at $k-1, k-2, \ldots$, we call them respectively the *future-* and the *past-induced state maps*.

Example 3. From Ex. 1 the future- and past-induced state maps are

$$\begin{aligned} X^+(\xi) &= \begin{bmatrix} -\big(m(k)+m(k-1)\big)+m(k)\xi & 0 \\ m(k-1) & 0 \end{bmatrix} \\ X^-(\xi^{-1}) &= \begin{bmatrix} \big(T_d^2 k_s + m(k-1)\big)\xi^{-1} & -T_d^2\xi^{-1} \\ \big(T_d^2 k_s + m(k-2)\big)\xi^{-2} - \big(m(k-1)+m(k-2)\big)\xi^{-1} & -T_d^2\xi^{-2} \end{bmatrix}. \end{aligned}$$

The following is a straightforward consequence of Th. 1.

Corollary 1. *Consider a representation (2); define $X^+(\xi)$ by (15) and $X^-(\xi)$ by (16) . Let $w \in \ker\ R(\sigma)$, and define $x := X^+(\sigma)w$ and $x' := X^-(\sigma)w$. Then x and x' are state variables for the system with hybrid representation respectively*

$$\begin{aligned} R(\sigma)w &= 0 \\ X^+(\sigma)w &= x\ , \end{aligned} \tag{17}$$

and

$$\begin{aligned} R(\sigma)w &= 0 \\ X^-(\sigma)w &= x'\ . \end{aligned} \tag{18}$$

Remark 2. In Th. 4 in Section IV.C of [5] it is shown that the property of state is equivalent with the existence of first-order representations in the state variable, and zeroth order in the external one. Moreover, issues such as minimality and the computation of input-state-output representations of $\mathfrak{B}$ given $X^+(\xi)$ and an input-output partition of the variables w are also discussed, see p. 2510 *ibid.*

Remark 3. To prove Th. 1 one can also use an argument based on the concept of adjoint polynomial operator in the shift (see Prop. 4); we sketch it here. It is straightforward to see that $R(\sigma)w = 0$ if and only if $\sum_{k=-\infty}^{+\infty} f(k)^\top \left(R(\sigma)w\right)(k) = 0$ for all $f : \mathbb{Z} \to \mathbb{R}^g$ of compact support. Using duality, the last condition is equivalent with $\sum_{k=-\infty}^{+\infty} \left(R(\sigma)^* f(k)\right)^\top w(k) = 0$ for all f of compact support. Now assume that $0 \underset{k_0}{\wedge} w \in \mathfrak{B}$; then $\sum_{k=k_0}^{+\infty} \left(R(\sigma)^* f(k)\right)^\top w(k) = 0$ for all f of compact support. Write

$$
\begin{aligned}
&\sum_{k=k_0}^{+\infty} \left(R(\sigma)^* f(k)\right)^\top w(k) \\
&= \left(f(k_0)^\top R_0(k_0) + \ldots + f(k_0 - L)^\top R_L(k_0 - L)\right) w(k_0) \\
&+ \left(f(k_0+1)^\top R_0(k_0+1) + \ldots + f(k_0 - L + 1)^\top R_L(k_0 - L + 1)\right) w(k_0+1) \\
&\vdots \\
&+ \left(f(k_0+L-1)^\top R_0(k_0+L-1) + \ldots + f(k_0 - 1)^\top R_L(k_0 - 1)\right) w(k_0+L-1) \\
&+ \left(f(k_0+L)^\top R_0(k_0+L) + \ldots + f(k_0)^\top R_L(k_0)\right) w(k_0+L) + \ldots
\end{aligned}
$$

Now rewrite gathering the terms involving $f(j)^\top$ for $j = k_0 - L, \ldots, k_0 + L$:

$$
\begin{aligned}
&f(k_0 - L)^\top R_L(k_0 - L) w(k_0) \\
&+ f(k_0 - L + 1)^\top \left(R_{L-1}(k_0 - L + 1) w(k_0) + R_L(k_0 - L + 1) w(k_0 + 1)\right) + \ldots + \\
&f(k_0 - 2)^\top \left(R_2(k_0 - 2) w(k_0) + \ldots + R_L(k_0 - 2) w(k_0 + L - 2)\right) \\
&+ f(k_0 - 1)^\top \left(R_1(k_0 - 1) w(k_0) + \ldots + R_L(k_0 - 1) w(k_0 + L - 1)\right) \\
&+ f(k_0)^\top \left(R_0(k_0) w(k_0) + R_1(k_0) w(k_0 + 1) + \ldots + R_L(k_0) w(k_0 + L)\right) + \ldots
\end{aligned}
$$

It follows from $R(\sigma)w = 0$ that all but the first L of these terms are zero, and that $0 \underset{k_0}{\wedge} w \in \ker\ R(\sigma)$ if and only if

$$\begin{aligned}
&f(k_0-L)^\top R_L(k_0-L)w(k_0)\\
&+f(k_0-L+1)^\top \left(R_{L-1}(k_0-L+1)w(k_0)+R_L(k_0-L+1)w(k_0+1)\right)+\ldots+\\
&f(k_0-2)^\top \left(R_2(k_0-2)w(k_0)+\ldots+R_L(k_0-2)w(k_0+L-2)\right)\\
&+f(k_0-1)^\top \left(R_1(k_0-1)w(k_0)+\ldots+R_L(k_0-1)w(k_0+L-1)\right)=0\ ,
\end{aligned}$$

for all sequences f of compact support. Since f is an arbitrary compact support trajectory, it follows that $0 \underset{k_0}{\wedge} w \in \ker\ R(\sigma)$ if and only if

$$\begin{aligned}
R_L(k_0-L)w(k_0) &= 0\\
R_{L-1}(k_0-L+1)w(k_0)+R_L(k_0-L+1)w(k_0+1) &= 0\\
&\vdots\\
R_1(k_0-1)w(k_0)+R_2(k_0-1)w(k_0+1)+\ldots+R_L(k_0-1)w(k_0+L-1) &= 0\ ,
\end{aligned}$$

i.e. if and only if $\left(X^+(\sigma)w\right)(k_0)=0$.

A similar argument based on duality and the adjoint polynomial operator in the shift can be used to prove that $w \underset{k_0}{\wedge} 0 \in \mathfrak{B}$ if and only if the conditions in statement 2) of Th. 1 hold. We will not enter into such details here.

Remark 4. The result of Th. 1 corroborates the statements on pp. 202, 211 and 212 of [7] that the dimension of the state-space of linear time-varying systems is in general not constant. Indeed, from Th. 1 it follows that the number of independent linear equations expressing the concatenability conditions at k_0 equals the rank of the matrices $X^+(k_0)$ and $X^-(k_0)$, which may vary with k_0.

3.2. Image representations

Proposition 5. *Denote by $\mathfrak{B}_f$ the full behavior associated with an image representation (9). Assume that ℓ is observable from w. Denote by $\mathfrak{B}$ the manifest behaviour of (9), and let $k_0 \in \mathbb{Z}$. The following statements are equivalent:*

1. $(0,0) \underset{k_0}{\wedge} (w,\ell) \in \mathfrak{B}_f$;
2. $0 \underset{k_0}{\wedge} w \in \mathfrak{B}$.

Proof. The implication (1) $\Longrightarrow$ (2) follows from the definition of external behavior. To prove (2) $\Longrightarrow$ (1), denote by $\ell_- \underset{k_0}{\wedge} \ell_+$ the latent variable trajectory corresponding to the external one $0 \underset{k_0}{\wedge} w$, that is $(\ell_- \underset{k_0}{\wedge} \ell_+, 0 \underset{k_0}{\wedge} w) \in \mathfrak{B}_f$; we now show that $\ell_- = 0$.

Recall that the variable ℓ in an image representation is free. Since $(\ell_- \underset{k_0}{\wedge} \ell_+, 0 \underset{k_0}{\wedge} w) \in \mathfrak{B}_f$, then the external trajectory corresponding to the latent variable one ℓ_- at k_0 is identically zero on all $\mathbb{Z}$: $(\ell_-, 0) \in \mathfrak{B}_f$. However, the only latent variable trajectory corresponding to the identically zero external one is zero on all $\mathbb{Z}$; it follows that $\ell_- = 0$ and the claim is proved. $\square$

We assume without loss of generality (see Rem. 1) that in (9) only the left shift appears, i.e. that $M \in \mathcal{M}[\xi]^{q \times m}$:

$$M(\sigma)\ell = w \; . \tag{19}$$

We rewrite the equations (19) as a kernel representation of the full behavior:

$$\begin{bmatrix} -I_q & M(\sigma) \end{bmatrix} \begin{bmatrix} w \\ \ell \end{bmatrix} = 0 \; . \tag{20}$$

The following characterization of concatenability conditions for observable image representations follows in a straightforward way from (20) and Th. (1).

Theorem 2. *Let $M(\xi)$ in (19) induce an image representation of $\mathfrak{B}_f$, $(w, \ell) \in \mathfrak{B}$ and $k_0 \in \mathbb{Z}$. In (21) and (22) define past and future state maps associated with $M(\xi)$.*

$$X^+(\xi) \;\; := \;\; \begin{bmatrix} M_1(k-1) + \ldots + M_L(k-1)\xi^{L-1} \\ M_2(k-2) + \ldots + M_L(k-2)\xi^{L-2} \\ \vdots \\ M_L(k-L) \end{bmatrix} \tag{21}$$

$$X^-(\xi^{-1}) \;\; := \;\; \begin{bmatrix} M_0(k-1)\xi^{-1} \\ M_0(k-2)\xi^{-2} + M_1(k-2)\xi^{-1} \\ \vdots \\ M_0(k-L)\xi^{-L} + \ldots + M_{L-1}(k-L)\xi^{-1} \end{bmatrix} \tag{22}$$

Assume that ℓ is observable from w, then

1. $(0,0) \underset{k_0}{\wedge} (w,\ell) \in \mathfrak{B}_f$ *if and only if* $\left(X^+(\sigma)\ell\right)(k_0) = 0$.
2. $(w,\ell) \underset{k_0}{\wedge} (0,0) \in \mathfrak{B}_f$ *if and only if* $\left(X^-(\sigma)\ell\right)(k_0 - 1) = 0$.

Proof. Follows in a straightforward way from Prop. 5 and Th. 1, since $\sigma^j I_q = 0_{q\times q}$ for all $j \geq 1$. □

In the following we show, using the results of Th. 2, that storage functions are quadratic functions of the state induced by the polynomial operator in the shift associated with $X^+(\xi)$.

4. Dissipative systems and storage functions

A two-variable polynomial framework to discuss quadratic functionals of system variables for the linear, time-invariant case has been introduced in [9] for the continuous-time case, and in [1] for the discrete-time case. In Th. 5.5 p. 1721 of [9] it is proved that every storage function of a continuous-time system dissipative with respect to a constant supply rate is a quadratic function of the state. In this section we show that the same holds true for discrete-time-varying systems. The linear, time-invariant case is a special case of our result.

We first consider the *lossless* case: we prove that such storage function is a quadratic function of the state of the system. We then show that the results for lossless systems also apply to *dissipative systems*.

4.1. Lossless systems

Let $\mathfrak{B}$ be a linear, discrete time-varying behavior, and let $S = S^\top \in \mathbb{R}^{q\times q}$. In the following we consider the quadratic functional Q_S of the system variables whose value at k is defined by

$$\begin{aligned} Q_S \quad &: \quad \mathfrak{B} \to \left(\mathbb{R}^q\right)^{\mathbb{Z}} \\ Q_S(w)(k) \quad &:= \quad w(k)^\top S w(k)\ . \end{aligned} \tag{23}$$

$\mathfrak{B}$ is called *dissipative* with respect to Q_S (for brevity, S-*dissipative*) if there exists a quadratic functional Q_Ψ of the system variables such that for every $w \in \mathfrak{B}$ and every $k \in \mathbb{Z}$ it holds that

$$w(k)^\top S w(k) \geq Q_\Psi(w)(k+1) - Q_\Psi(w)(k) \; . \tag{24}$$

In this case, Q_S is called a *supply rate* and Q_Ψ is called a *storage function* for Q_S. $\mathfrak{B}$ is *lossless* with respect to Q_S (for brevity, S-*lossless*) if (24) holds with equality.

In Prop. 3.2 p. 35 of [1] a characterization is given of losslessness for linear, time-invariant behaviors, and for the existence of storage functions; the argument used in the proof is based on Fourier analysis and is not immediately adaptable to the time-varying case. We now provide self-contained proofs for the latter situation, based on the notion of dual polynomial shift operator introduced in Section 2.3 of this paper.

We begin with the following auxiliary result.

Lemma 1. *Let $M \in \mathcal{M}[\xi]^{q\times m}$, and $S = S^\top \in \mathbb{R}^q$. Define $X^+(\xi)$ as in Th. 2. Then for every $\ell_i \in \mathcal{L}_2(\mathbb{Z}, \mathbb{R}^m)$, $i = 1, 2$ and every $k_0, k_1 \in \mathbb{Z}$ it holds that*

$$\begin{aligned}
&\sum_{k=k_0}^{k_1} \left(M(\sigma)\ell_1\right)(k)^\top S \left(M(\sigma)\ell_2\right)(k) \\
&= \sum_{k=k_0}^{k_1} \ell_1(k)^\top \left(M(\sigma)^* S (M(\sigma)\ell_2\right)(k) \\
&- \left(X^+(\sigma)\ell_1\right)(k_1+1) \mathrm{col}\left(SM(\sigma)\ell_2(k_1-j)\right)_{j=0,\dots,L-1} \\
&+ \left(X^+(\sigma)\ell_1\right)(k_0) \mathrm{col}\left(SM(\sigma)\ell_2(k_0-j)\right)_{j=1,\dots,L}
\end{aligned} \tag{25}$$

Proof. It is a matter of straightforward verification to check that by gathering the terms associated with $\ell_1(k)^\top$, the expression

$$\sum_{k=-\infty}^{+\infty} \left(M(\sigma)\ell_1\right)(k)^\top S \left(M(\sigma)\ell_2\right)(k)$$

can be rewritten as

$$\begin{aligned}
\sum_{k=-\infty}^{+\infty} \ell_1(k)^\top \quad & \big(M_0(k)^\top S \left(M(\sigma)\ell_2\right)(k) + M_1(k-1)^\top S \left(M(\sigma)\ell_2\right)(k-1) \\
& + \ldots + M_L(k-L)^\top S \left(M(\sigma)\ell_2\right)(k-L)\big) \; .
\end{aligned} \tag{26}$$

Recall the definition (13) of adjoint polynomial operator in the shift, and note that (26) is equivalent with

$$\sum_{k=-\infty}^{+\infty} \left(M(\sigma)\ell_1\right)(k)^\top S \left(M(\sigma)\ell_2\right)(k) = \sum_{k=-\infty}^{+\infty} \ell_1(k)^\top \left(M(\sigma)^* S M(\sigma)\ell_2\right)(k-L) \, .$$

Now consider

$$\sum_{k=\mathrm{k}_0}^{k_1} \left(M(\sigma)\ell_1\right)(k)^\top S \left(M(\sigma)\ell_2\right)(k) \, ; \tag{27}$$

we want to rewrite this expression using terms

$$\begin{aligned} \ell_1(k)^\top \qquad & \big(M_0(k)^\top S \left(M(\sigma)\ell_2\right)(k) + M_1(k-1)^\top S \left(M(\sigma)\ell_2\right)(k-1) \\ & + \ldots + M_L(k-L)^\top S \left(M(\sigma)\ell_2\right)(k-L)\big) \ , \end{aligned} \tag{28}$$

like those appearing in (26). Note that if $k-j<k_0$ in (28), terms appear in (28) that are absent from (27) at k_0. Also, if $k+j>k_1$ in (28), terms are missing in (28) that are present in (27) at k_1.

Consequently, to rewrite the left-hand side of (25) using terms as in (28), we need to consider the values of (28) at $k-j<k_0$ and at $k+j>k_1$, and add or subtract appropriate expressions. In the first case, at $k=k_0$ we need to take into account the terms (multiplied by $\ell_1(k_0)^\top$ on the left)

$$M_1(k_0-1)^\top S \left(M(\sigma)\ell_2\right)(k_0-1) + \ldots + M_L(k_0-L)^\top S \left(M(\sigma)\ell_2\right)(k_0-L) \, ;$$

at $k=k_0+1$, the terms (multiplied by $\ell_1(k_0+1)^\top$ on the left)

$$M_2(k_0-1)^\top S \left(M(\sigma)\ell_2\right)(k_0-1) + \ldots + M_L(k_0-L+1)^\top S \left(M(\sigma)\ell_2\right)(k_0-L+1) \, ;$$

and so on until $k=k_0+L-2$ we take into account the terms (multiplied by $\ell_1(k_0+L-2)^\top$ on the left)

$$M_{L-1}(k_0-1)^\top S \left(M(\sigma)\ell_2\right)(k_0-1) + M_L(k_0-2)^\top S \left(M(\sigma)\ell_2\right)(k_0-2) \, ,$$

and finally at $k=k_0+L-1$, the term (multiplied by $\ell_1(k_0+L-1)^\top$ on the left) $M_L(k_0-1)^\top S \left(M(\sigma)\ell_2\right)(k_0-1)$. We now rewrite the sum of these terms by separating those multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_0-1)$:

$$\begin{aligned}&\ell_1(k_0)^\top M_1(k_0-1)^\top S+\ell_1(k_0+1)^\top M_2(k_0-1)^\top S\\&+\ldots+\ell_1(k_0+L-2)^\top M_{L-1}(k_0-1)^\top S+\ell_1(k_0+L-1)^\top M_L(k_0-1)^\top S\ ;\end{aligned}$$

those multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_0-2)$:

$$\begin{aligned}&\ell_1(k_0)^\top M_2(k_0-2)^\top S+\ell_1(k_0+1)^\top M_3(k_0-2)^\top S+\ldots\\&\ldots+\ell_1(k_0+L-2)^\top M_L(k_0-2)^\top S\ ;\end{aligned}$$

and so on, until those that are multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_0-L+1)$:

$$\ell_1(k_0)^\top M_{L-1}(k_0-L+1)^\top S+\ell_1(k_0+1)^\top M_L(k_0-L+1)^\top S\ ;$$

and finally, that multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_0-L)$, namely:

$$\ell_1(k_0)^\top M_L(k_0-L)^\top S\ .$$

It is straightforward to verify that the sum of these expression equals

$$\left(X^+(\sigma)\ell_1\right)(k_0)^\top \mathrm{col}\left(S\left(M(\sigma)\ell_2\right)(k_0-j)\right)_{j=1,\ldots,L}\ .$$

We now consider the values of (28) at $k+j>k_1$, i.e. the terms involving $\ell_1(k_1+j)$, $j=1,\ldots,L$, namely

$$\left(M_1(k_1)\ell_1(k_1+1)+\ldots+M_L(k_1)\ell_1(k_1+L)\right)^\top S\ ,$$

which is multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_1)$;

$$\left(M_2(k_1-1)\ell_1(k_1+1)+\ldots+M_L(k_1-1)\ell_1(k_1+L-1)\right)^\top S\ ,$$

which is multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_1-1)$; and so on, until

$$\left(M_{L-1}(k_1-L+2)\ell_1(k_1+1)+M_L(k_1-L+2)\ell_1(k_1+2)\right)^\top S\ ,$$

which is multiplied on the right by $\left(M(\sigma)\ell_2\right)(k_1-L+2)$; and finally,

$$\left(M_L(k_1-L+1)\ell_1(k_1+1)\right)^\top S\ ,$$

which is multiplied on the right by $(M(\sigma)\ell_2)(k_1-L+1)$. It is straightforward to verify that the sum of these expression equals

$$\left(X^{+}(\sigma)\ell_1\right)(k_1+1)^{\top}\mathrm{col}\left(S\left(M(\sigma)\ell_2\right)(k_1-j)\right)_{j=0,\ldots,L-1}\ .$$

Conclude from this argument that

$$\begin{aligned}&\sum_{k=k_0}^{k_1}\ell_1(k)^{\top}\left(M(\sigma)^{*}S(M(\sigma)\ell_2\right)(k)\\&=\sum_{k=k_0}^{k_1}\left(M(\sigma)\ell_1\right)(k)^{\top}S\left(M(\sigma)\ell_2\right)(k)\\&+\left(X^{+}(\sigma)\ell_1\right)(k_1+1)\mathrm{col}\left(SM(\sigma)\ell_2(k_1-j)\right)_{j=0,\ldots,L-1}\\&-\left(X^{+}(\sigma)\ell_1\right)(k_0)\mathrm{col}\left(SM(\sigma)\ell_2(k_0-j)\right)_{j=1,\ldots,L}\end{aligned}$$

This concludes the proof of the claim. □

The following result can be proved with an argument symmetric to that used to prove Lemma 1.

Corollary 2. *Let $M\in\mathcal{M}[\xi]^{q\times m}$, and $S=S^{\top}\in\mathbb{R}^{q}$. Define $X^{+}(\xi)$ as in Th. 2. Then for every $\ell_i\in\mathcal{L}_2(\mathbb{Z},\mathbb{R}^m)$, $i=1,2$ and every $k_0,k_1\in\mathbb{Z}$ it holds that*

$$\begin{aligned}&\sum_{k=k_0}^{k_1}\left(M(\sigma)\ell_1\right)(k)^{\top}S\left(M(\sigma)\ell_2\right)(k)\\&=\sum_{k=k_0}^{k_1}\ell_1(k)^{\top}\left(M(\sigma)^{*}S(M(\sigma)\ell_2\right)(k)\\&+\mathrm{col}\left(SM(\sigma)\ell_1(k_1-j)\right)_{j=0,\ldots,L-1}^{\top}\left(X^{+}(\sigma)\ell_2\right)(k_1+1)\\&+\mathrm{col}\left(SM(\sigma)\ell_1(k_0-j)\right)_{j=1,\ldots,L}^{\top}\left(X^{+}(\sigma)\ell_2\right)(k_0)\end{aligned}$$

We now give some characterizations of losslessness for controllable systems.

Proposition 6. *Let $\mathfrak{B}=\mathrm{im}\ M(\sigma)$ be a linear discrete time-varying behavior, and let $S=S^{\top}\in\mathbb{R}^{q}$. The following statements are equivalent:*

1. *$\mathfrak{B}$ is S-lossless;*
2. *$M(\sigma)^{*}SM(\sigma)$ is the zero operator, i.e.*

$$M(\sigma)^{*}SM(\sigma)\ell=0 \text{ for every } \ell:\mathbb{Z}\rightarrow\mathbb{R}^{m}\ .$$

3. *For all $k \in \mathbb{Z}$, it holds that*

$$\begin{bmatrix} M_L(k)^\top \\ \vdots \\ M_0(k+L)^\top \end{bmatrix} S \begin{bmatrix} M_0(k) & \dots & M_L(k) \end{bmatrix} = 0 \; ;$$

Proof. To prove the equivalence of the first two statements, let $\ell_1, \ell_2 : \mathbb{Z} \to \mathbb{R}^m$ be two arbitrary square-integrable trajectories. Since $\lim_{k\to\pm\infty} \ell_i(k) = 0$, $i = 2, 1$, it follows that $\lim_{k\to\pm\infty} \left(X^+(\sigma)\ell_1\right)(k) = 0$. Consequently, $\mathfrak{B}$ is lossless if and only if

$$0 = \sum_{k=-\infty}^{+\infty} \left(M(\sigma)\ell_1\right)(k)^\top S \left(M(\sigma)\ell_2\right)(k) = \sum_{k=-\infty}^{+\infty} \ell_1(k)^\top \left(M(\sigma)^* S (M(\sigma)\ell_2\right)(k) \; .$$

Use the arbitrariness of ℓ_1, ℓ_2 in $\mathcal{L}_2(\mathbb{Z}, \mathbb{R}^m)$ to conclude that $\left(M(\sigma)^* S (M(\sigma)\ell\right)(k) = 0$ for all $\ell : \mathbb{Z} \to \mathbb{R}^m$ and all $k \in \mathbb{Z}$. The equivalence (1) $\iff$ (2) is proved.

To prove the equivalence of (2) and (3), consider that at every $k \in \mathbb{Z}$ the following equality holds

$$0 = \left(M(\sigma)^* S (M(\sigma)\ell\right)(k) = \begin{bmatrix} M_L(k)^\top \\ \vdots \\ M_0(k+L)^\top \end{bmatrix} S \begin{bmatrix} M_0(k) & \dots & M_L(k) \end{bmatrix} \begin{bmatrix} \ell(k) \\ \vdots \\ \ell(k+L) \end{bmatrix} \; ,$$

and use the arbitrariness of ℓ. $\square$

We now prove the main result of this section: if a system is lossless, then the storage function is a quadratic function of the state of $\mathfrak{B}$. In our proof we use the notion of *coefficient matrix* associated with $P \in \mathcal{M}[\xi]^{q_1 \times q_2}$. The value of the coefficient matrix $\widetilde{P}$ at $k \in \mathbb{Z}$ is defined by

$$\widetilde{P}(k) := \begin{bmatrix} P_0(k) & P_1(k) & \dots & P_L(k) & 0_{q_1 \times q_2} & \dots \end{bmatrix} \; .$$

Note that $\widetilde{P}$ has an infinite number of columns, but only a finite number of nonzero entries.

Theorem 3. *Let $\mathfrak{B}$ be a linear, time-varying behavior described in image form by (9), and let X^+ be the future state map for $\mathfrak{B}$ defined in Th. 2. Let $S =$*

$S^\top \in \mathbb{R}^{q\times q}$, and assume that $\mathfrak{B}$ is lossless with respect to the supply rate Q_S. There exists $P : \mathbb{Z} \to \mathbb{R}^{qL_1\times qL_1}$ such that for all $(w,\ell) \in \mathfrak{B}_f$ and all $k \in \mathbb{Z}$ it holds that

$$Q_S(w)k) = \left(X^+(\sigma)\ell\right)^\top P(k{+}1)\left(X^+(\sigma)\ell\right)(k{+}1) - \left(X^+(\sigma)\ell\right)^\top P(k)\left(X^+(\sigma)\ell\right)(k) \; .$$

Proof. We prove the claim by first showing that there exists a bilinear form B_Ψ such that for every $(w_i,\ell_i) \in \mathfrak{B}_f$, $i = 1,2$, and every $k \in \mathbb{Z}$ the following equality holds:

$$w_1(k)^\top S w_2(k) = B_\Psi(\ell_1,\ell_2)(k+1) - B_\Psi(\ell_1,\ell_2)(k) \; . \tag{29}$$

We then prove that B_Ψ is a quadratic function of the state, i.e. that there exists K such that $B_\Psi(\ell_1,\ell_2)(k) = \left(X^+(\sigma)\ell_1\right)^\top K\left(X^+(\sigma)\ell_2\right)(k)$.

Use Lemma 1 and the definition of losslessness, to conclude that for all ℓ_i, $i = 1,2$ of compact support contained in $[k_0,k_1]$ it holds that

$$\begin{aligned}
0 &= \sum_{k=k_0}^{k_1} \ell_1(k)^\top \left(M(\sigma)^* S(M(\sigma)\ell_2\right)(k) \\
&= \sum_{k=k_0}^{k_1} \left(M(\sigma)\ell_1\right)(k)^\top S\left(M(\sigma)\ell_2\right)(k) \\
&\quad + \left(X^+(\sigma)\ell_1\right)(k_1+1)\mathrm{col}\left(SM(\sigma)\ell_2(k_1-j)\right)_{j=0,\dots,L-1} \\
&\quad - \left(X^+(\sigma)\ell_1\right)(k_0)\mathrm{col}\left(SM(\sigma)\ell_2(k_0-j)\right)_{j=1,\dots,L} \; .
\end{aligned}$$

From this and the statement 2) of Prop. 6 conclude that

$$\begin{aligned}
&\sum_{k=k_0}^{k_1} \left(M(\sigma)\ell_1\right)(k)^\top S\left(M(\sigma)\ell_2\right)(k) \\
&= \left(X^+(\sigma)\ell_1\right)(k_1+1)\mathrm{col}\left(SM(\sigma)\ell_2(k_1-j)\right)_{j=0,\dots,L-1} \\
&- \left(X^+(\sigma)\ell_1\right)(k_0)\mathrm{col}\left(SM(\sigma)\ell_2(k_0-j)\right)_{j=1,\dots,L} \; .
\end{aligned}$$

Define B_Ψ to be the bilinear form acting on trajectories from $\mathbb{Z}$ to $\mathbb{R}^m$ defined by

$$B_\Psi(\ell_1,\ell_2)(k) := \left(X^+(\sigma)\ell_1\right)(k)\mathrm{col}\left(SM(\sigma)\ell_2(k-j)\right)_{j=1,\dots,L} \; , \tag{30}$$

and let $k_0 = k_1 := k$. Equation (29) is proved. We now prove that B_Ψ is a quadratic function of the state.

In the following, we associate to $X^+(\xi) = X_0 + \ldots + X_{L-1}\xi^{L-1}$ its coefficient matrix $\widetilde{X^+}$, and with the expression $\operatorname{col}\left(SM(\sigma)\ell_2(k-j)\right)_{j=1,\ldots,L}$ the polynomial operator in the shift defined by

$$\begin{bmatrix} S\left(M_0(k-1) + M_1(k-1)\xi + \ldots + M_L(k-1)\xi^L\right) \\ S\left(M_0(k-2) + M_1(k-2)\xi + \ldots + M_L(k-2)\xi^L\right) \\ \vdots \\ S\left(M_0(k-L) + M_1(k-L)\xi + \ldots + M_L(k-L)\xi^L\right) \end{bmatrix},$$

and we denote by $\widetilde{F}$ its coefficient matrix. Then

$$\begin{aligned} &\left(X^+(\sigma)\ell_1\right)(k)\operatorname{col}\left(SM(\sigma)\ell_2(k-j)\right)_{j=1,\ldots,L} \\ &= \begin{bmatrix} \ell_1(k)^\top & \ldots & \ell_1(k+L-1)^\top & \ell_1(k+L) & \ldots \end{bmatrix} \widetilde{X^+}\widetilde{F} \begin{bmatrix} \ell_2(k) \\ \vdots \\ \ell_2(k+L) \\ \vdots \end{bmatrix} \end{aligned}$$

Using Cor. 2, and the definition of $\mathfrak{B}_\Psi$ in (30), conclude that for every $k \in \mathbb{Z}$ and every $\ell_i \in \mathcal{L}_2(\mathbb{Z}, \mathbb{R}^m)$, $i = 1, 2$ it holds that

$$\begin{aligned} &\left(X^+(\sigma)\ell_1\right)(k)\operatorname{col}\left(SM(\sigma)\ell_2(k-j)\right)_{j=1,\ldots,L} \\ &= \operatorname{col}\left(SM(\sigma)\ell_1(k-j)\right)_{j=1,\ldots,L}\left(X^+(\sigma)\ell_2\right)(k) \,. \end{aligned}$$

Since ℓ_1 and ℓ_2 are arbitrary, we conclude that at every $k \in \mathbb{Z}$ it holds that $\widetilde{X}(k)^\top \widetilde{F}(k) = \widetilde{F}(k)^\top \widetilde{X}(k)$, and consequently $\widetilde{X}^\top \widetilde{F} = \widetilde{F}^\top \widetilde{X}$; note that this is an equality between matrices with time-varying coefficients.

Now use unimodular operations on $\widetilde{X}$ and $\widetilde{F}$, respectively, to bring them to the form $U_1\widetilde{X} = \begin{bmatrix} \widetilde{X}' \\ 0 \end{bmatrix}$ and $U_2\widetilde{F} = \begin{bmatrix} \widetilde{F}' \\ 0 \end{bmatrix}$, with $\widetilde{X}'$, $\widetilde{F}'$ of full row rank. Use such factorizations to write $\widetilde{X}^\top \widetilde{F} = \widetilde{X}'^\top \widetilde{G}\widetilde{F}'$, with $\widetilde{G}$ a nonsingular matrix. From the equality $\widetilde{X}'^\top \widetilde{G}\widetilde{F}' = \widetilde{F}'^\top \widetilde{G}^\top \widetilde{X}'$ conclude that the row space of $\widetilde{X}'$ contains that of $\widetilde{F}'$. Since the former is contained in the row space of $\widetilde{X}$, conclude that row space $\left(\widetilde{F}'\right) \subseteq$ row space $\left(\widetilde{X}\right)$. It follows that $P : \mathbb{Z} \to \mathbb{R}^{qL_1 \times qL_1}$ exists such that $\widetilde{X}^\top \widetilde{F} = \widetilde{X}^\top P \widetilde{X}$. This concludes the proof of the second part of the claim. $\square$

Remark 5. If $\mathfrak{B}$ is time-invariant, then the matrix $M(\xi)$ associated to an image representation has constant coefficients. The argument used in the proof of Th. 3 can be used *verbatim* to prove the existence of a *constant* matrix P such that $B_\Psi(\ell) = (X(\sigma)\ell)^\top P\,(X(\sigma)\ell)$. It follows that in the lossless case storage functions in discrete-time are *always* quadratic functions of the state, and Th. 3 is the counterpart of Th. 5.2 p. 1381 in [2].

The result of Th. 3 is instrumental in proving that under some technical assumptions, also for dissipative (non-lossless) systems the storage function is a quadratic function of the state.

4.2. Dissipative systems

Consider the functional Q_Δ of the system variables defined by difference between the supply rate and the storage function, defined by

$$Q_\Phi(w)(k) - (Q_\Psi(w)(k+1) - Q_\Psi(w)(k)) = Q_\Delta(w)(k)\ , \qquad (31)$$

is called a *dissipation rate*. It follows from (24) that Q_Δ is nonnegative. It follows from the *dissipation equality* (31) that a system is dissipative with respect to the supply rate Q_Φ if and only if it is lossless with respect to the new supply rate $Q_\Phi - Q_\Delta$. The following result is a straightforward consequence of this observation.

Corollary 3. *Let $\mathfrak{B} = \operatorname{im} M(\sigma)$ and $S = S^\top \in \mathbb{R}^{q\times q}$. Define $X^+(\xi)$ by (21). Assume that $\mathfrak{B}$ is S-dissipative, with a dissipation rate Q_Δ that is a quadratic function of $X^+(\sigma)\ell$ and $w = M(\sigma)\ell$. Then any storage function Q_Ψ is a quadratic function of the state, i.e. there exists $P \in \mathcal{M}^{qL\times qL}$ such that*

$$Q_\Psi(\ell)(k) = \left(X^+(\sigma)\ell\right)(k)^\top P\left(X^+(\sigma)\ell\right)(k)\ .$$

Proof. To prove the claim we define a new system from $\mathfrak{B}$ and the dissipation rate. By assumption, there exists a time-varying matrix $D = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^\top & D_{22}\end{bmatrix}$ of suitable dimensions, such that

$$Q_\Delta(\ell) = \begin{bmatrix} X^+(\sigma)\ell^\top & M(\sigma)\ell^\top \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^\top & D_{22}\end{bmatrix} \begin{bmatrix} X^+(\sigma)\ell \\ M(\sigma)\ell \end{bmatrix}\ .$$

Now consider the system $\mathfrak{B}'$ represented in image form by

$$\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} X^{+}(\sigma) \\ M(\sigma) \end{bmatrix} \ell \, . \tag{32}$$

It is straightforward to verify that since $x = X^{+}(\sigma)\ell$ is a state for $\mathfrak{B}$, it is also a state for $\mathfrak{B}'$. Moreover, $\mathfrak{B}'$ is lossless with respect to the new supply rate

$$\begin{aligned} w^{\top} S w - \begin{bmatrix} x^{\top} & w^{\top} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^{\top} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} &= \begin{bmatrix} x^{\top} & w^{\top} \end{bmatrix} \begin{bmatrix} -D_{11} & -D_{12} \\ -D_{12}^{\top} & S - D_{22} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \\ &= Q_{\Psi}(\ell) \, . \end{aligned}$$

Now apply Th. 3 to $\mathfrak{B}'$. $\square$

Remark 6. We continue the discussion of Rem. 5 on how our results are related to those in [2]. We analyze the case of the system with image representation

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \sigma + 1 \\ \sigma \end{bmatrix} \ell \, ,$$

considered in Ex. 4.1 p. 1380 of [2]. Straightforward manipulations yield that this system is also described in kernel form by $\begin{bmatrix} \sigma & \sigma - 1 \end{bmatrix} \begin{bmatrix} w_2 \\ w_2 \end{bmatrix} = 0$. For future reference note that defining $x := w_1 - w_2$, $y := w_1$ and $u := w_2$, the system has the state representation

$$\begin{aligned} \sigma x &= u \\ y &= x + u \, . \end{aligned} \tag{33}$$

The system is dissipative with respect to the supply rate $w_1^2 + w_2^2$, equivalently expressed by $2(\sigma\ell)^2 + \ell^2 + 2\ell\sigma(\ell)$ as a function of the latent variable ℓ. In terms of the variables x, y and u defined above, the supply rate is written as

$$w_1^2 + w_2^2 = (w_1 - w_2)^2 + 2w_1w_2 = x^2 + 2xu + 2u^2 \, .$$

In [2] the authors take the following quadratic functional of ℓ:

$$\ell^2 + 2\ell(\sigma\ell) + (\sigma\ell)^2 + (\sigma^3\ell)^2 \, , \tag{34}$$

as dissipation rate. Such functional is nonnegative; straightforward manipulations can be used to show that the difference between supply and dissipation rate is the increment of $(\sigma\ell)^2 + (\sigma^2\ell)^2$. This expression can be rewritten as a function of x and u, resulting in

$$x^2 + 2xu + 2u^2 - \left(x^2 + 2xu + u^2 + (\sigma^2 u)^2\right) = u^2 - (\sigma^2 u)^2 = u^2 - (\sigma^3 x)^2 ,$$

which is not a quadratic function of the state.

This example highlights the importance of the assumption that the dissipation rate is a quadratic function of the state and the input of the system. The dissipation rate in (34) is a *dynamic* functional of x and u: using the easily established equality $\ell = w_1 - w_2 = x$ and (33), (34) can be rewritten as

$$x^2 + 2xu + u^2 + (\sigma^3 x)^2 = x^2 + 2xu + u^2 + (\sigma^2 u)^2 ,$$

a functional of x, u, and $\sigma^2 x = \sigma u$. On this issue, see also Section 6 of [6].

Remark 7. The assumption on the dissipation rate being a quadratic function of the state and the system variables is straightforward to verify in the time-invariant case, where a *canonical factorization* (see [9]) of its coefficient matrix can be computed using standard Linear Algebra techniques and used to verify the condition. Such procedures are not available for the case of time-varying matrices; the development of computationally effective tests for verifying the assumption of Cor. 3 is a matter for further research.

5. Conclusions

Starting from an intrinsic, trajectory-based definition of state, we provided a procedure to compute a state variable for systems described by higher-order difference equations with time-varying coefficients. We also showed that the storage function of a dissipative system is a quadratic function of the state.

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