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Group Conformity in Social Networks

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Abstract Diffusion in social networks is a result of agents' natural desires to conform to the behavioral patterns of their peers. In this article we show that the recently proposed "propositional opinion diffusion model" could be used to model an agent's conformity to different social groups that the same agent might belong to, rather than conformity to the society as whole.

The main technical contribution of this article is a sound and complete logical system describing the properties of the influence relation in this model. The logical system is an extension of Armstrong's axioms from database theory by one new axiom that captures the topological structure of the network.

Keywords Social networks · Diffusion · Influence · Peer pressure · Conformity · Threshold model · Axiomatization · Completeness · Armstrong's axioms · Multiagent systems

1 Introduction

In this article we the study properties of a recently introduced model of diffusion in social networks, called the *propositional opinion diffusion model* [10]. This model generalizes the previously studied threshold model of diffusion. Diffusion models describe spread of a new idea, product, belief, or social norm, which from now on will be referred to as just "product". Models of diffusion can be divided into two classes: deterministic and probabilistic. In deterministic models the diffusion process is uniquely predetermined by the structure of the network and the initial distribution of the product. In probabilistic models

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only the probabilistic pattern of the diffusion is predetermined. The focus of this article is on deterministic models.

The article is organized as follows. In the next section we introduce the threshold model and discuss its limitations in capturing conformity to individual groups. Section 3 *informally* describes propositional diffusion model and explains how it can be used to model group conformity. Section 4 introduces notion of diffusion and states our main result – a sound and complete logical system that describes properties of diffusion specific to a given network structure. Section 5 places our result in the context of the existing literature.

The rest of the article is dedicated to the *formal* account of our logical system. Section 6 formally defines propositional diffusion model for monotonic function, that we call group conformity model. Section 7 introduces the syntax and the semantics of our system. Section 8 states the axioms of our system. Section 10 contains the proof of the soundness of these axioms and Section 11 establishes their completeness. Section 12 shows that our logical system is decidable. Section 13 concludes.

2 Threshold Model

The most commonly studied model of diffusion in social networks is the threshold model [11, 20]. The existing literature on threshold model includes the work that cover its algorithmic complexity [1] and its applications to externalities of interactions [9], selection of the most influential nodes [14, 15], and influence in strategic games [17].

The threshold model assigns threshold values to all nodes in the network and positive weights to all directed edges of the network. A directed edge represents possible influence of one agent on another. If the sum of weights on incoming edges from nodes that already adopted the product is at least as much as node's threshold value, then the node also adopts the product.

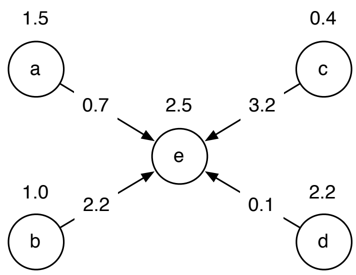

Fig. 1 Threshold Model.

An example of social network based on the threshold model is depicted in Figure 1. If node a in this example is given a free sample of the product and it starts using it, then the node puts peer pressure 0.7 on node e to adopt the product. Since 0.7 is less than threshold value 2.5 of node e, this pressure alone will not lead to node e adopting the product. However, if another free sample is given to node b and it also starts using it, then the total peer pressure on node e becomes $0.7 + 2.2 = 2.9$, which is greater than its threshold value 2.5. Thus, after nodes a and b adopt the product, they will influence node e to do the same. In a larger network, the process of the adoption might continue with more and more nodes putting peer pressure on others and, as a result, more and more nodes adopting the product. This process is called product diffusion in social networks.

A significant limitation of the threshold model comes from the fact that it treats peer pressure equally no matter what node or group of nodes it comes from. For example, consider a hypothetical person that has about the same number of two types of peers: co-workers and neighbors. If the person learns that 80% of co-workers at her new job drive a new model of a certain luxury car, then she would feel peer pressure to confirm to the group's norms and to buy this car even if most of her not-so-well-to-do neighbors drive a less expensive car. Similarly, if the same agent moves to a neighborhood and learns that 80% of her new neighbors drive this luxury car, then she would feel pressured into buying one even if most of her co-workers don't drive this car. At the same time, it is plausible that the agent might experience much less pressure to buy this car if 40% of her co-workers *and* 40% of her neighbors drive the car.

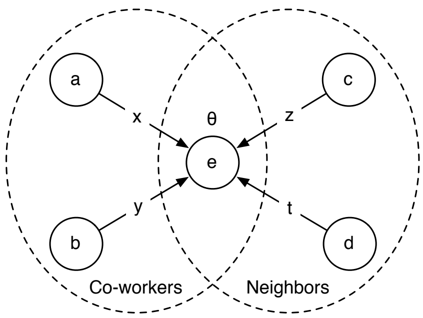

Fig. 2 Threshold Model.

To formally show that threshold model is not suitable to model the above situation, let us consider a simpler setting depicted in Figure 2. Here node e has just two co-workers: a and b and two neighbors: c and d. Suppose that if both co-workers buy the car, they put enough peer pressure on node e to do the same. In other words, $x + y \geq \theta$. Let us also assume that if both neighbors get the car, they influence the node to do the same: $z + t \geq \theta$. Note now that assumption $x + y \geq \theta$ implies that $(x \geq \theta/2) \vee (y \geq \theta/2)$ and assumption

$z + t \geq \theta$ implies that $(z \geq \theta/2) \vee (t \geq \theta/2)$. Hence,

$$[(x \geq \theta/2) \vee (y \geq \theta/2)] \wedge [(z \geq \theta/2) \vee (t \geq \theta/2)].$$

Thus, due to distributively of conjunction over disjunction,

$$[(x \geq \theta/2) \wedge (z \geq \theta/2)] \vee [(x \geq \theta/2) \wedge (t \geq \theta/2)] \vee$$
$$[(y \geq \theta/2) \wedge (z \geq \theta/2)] \vee [(y \geq \theta/2) \wedge (t \geq \theta/2)].$$

Then,

$$(x + z \geq \theta) \vee (x + t \geq \theta) \vee (y + z \geq \theta) \vee (y + t \geq \theta).$$

In other words, there is a co-worker and a neighbor such that if both of them buy the car, they put enough pressure on node e to do the same. This paradoxical conclusion comes from the fundamental assumption of the threshold model that influence is determined by the total of peer pressure no matter from which agents this pressure comes from. This limitation could be overcome using a diffusion model recently proposed by Grandi, Lorini, and Perrussel [10].

3 Group Conformity Model

Grandi, Lorini, and Perrussel [10] call their model of diffusion "propositional opinion diffusion". In case of a single product/opinion, they assign a single Boolean function to each node that computes the Boolean state of the node based on the Boolean states of its neighbors. This is a very general model that might lead to oscillation of node values. For example, if in a two-node network one node has a Boolean function which is the negation of the other node and the over node's Boolean function is the identity, then the nodes will oscillate between two Boolean values and never reach a stable configuration. However, if the Boolean function assigned to each node is monotonic [1], then the network will always achieve a stable configuration. They call such Boolean functions "ballot-monotonic".

Note that monotonic Boolean functions can be represented as disjunctions of conjunctions of atomic Boolean variables[2]. If such a Boolean function is used to model influence on a node in a social network, then each disjunct in the disjunction can be viewed as a *conformity group* of the node. For the above example with co-workers and neighbors, see Figure 2, the Boolean function of node e could be $(a \wedge b) \vee (c \wedge d)$. Disjuncts $a \wedge b$ and $c \wedge d$ correspond to conformity groups $\{a, b\}$ and $\{c, d\}$. If all nodes in at least one conformity group adopt the product, then node e also adopts the product.

Visually, we represent a conformity group by a point on the border of a node where one or several arrowheads reach the border. For example, node e in the social network depicted in Figure 3 has two such points: one on the left,

[1] Boolean function $f(x_1, \dots, x_n)$ is monotonic if $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$ for each all Boolean (0 or 1) values $x_1 \leq y_1, x_2 \leq y_2, \dots, x_n \leq y_n$.

[2] For the sake of completeness we prove this claim in Lemma 20 located in the appendix to this article.

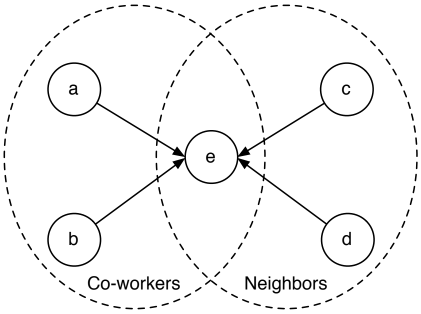

Fig. 3 Node e adopts the product if either both co-workers or both neighbors do it.

where incoming arrows from nodes a and b meet and another on the right, where incoming arrows from nodes c and d meet. In order for a node to adopt a product, all nodes connected to at least one of its points must adopt the product first. We refer to Grandi, Lorini, and Perrussel's propositional opinion diffusion model with monotonic Boolean functions as the *group conformity model* of diffusion in social networks.

4 Diffusion

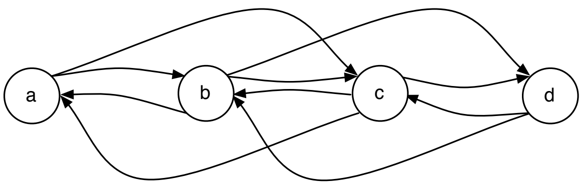

Fig. 4 $\{a\} \rhd \{c, d\}$.

The main technical contribution of this article is a complete axiomatization of the influence relation in the proposed model. We write $A \rhd B$ and say that a set of nodes A influences set of nodes B if after free samples of the product are given to all nodes in set A and they start using the product, all nodes in set B will *eventually* adopt the product. Word "eventually" here means that the influence might spread not directly from set A to set B, but through other nodes. For example, $\{a\} \rhd \{c, d\}$ in the social network depicted in Figure 4. Indeed, if a free sample of the product is given to node a, then node b adopts the product because the set $\{a\}$ is one of the conformity groups of node b. Then, node c adopts the product because the set $\{a, b\}$ is one of the conformity groups of node c. Finally, node d adopts the product because $\{b, c\}$ is a conformity group of node d. In this article we study universal properties of the influence relation that are true for all social networks based on the group conformity model. It is easy to see that the following three principles are examples of such properties:

1. Reflexivity: $A \rhd B$, where $A \subseteq B$,
2. Augmentation: $A \rhd B \to (A \cup C) \rhd (B \cup C)$,
3. Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$.

These three principles known in the database theory as Armstrong's axioms [8, p. 81], where they give a sound and complete axiomatization of functional dependency [2]. The same three principles also give a sound and complete axiomatization of universal properties of influence relation in threshold model of social networks [3]. As we will see later in the article, these axioms are also sound and complete with respect to group conformity semantics. This, however, is just a special case of the main technical result of this article, which is a complete axiomatization of all universally true properties of influence relation for all social networks with a given topology.

By a network topology we mean a directed graph that shows connections in a social network. Informally, an edge from a node a to a node b in such a graph means that node b "knows" agent a and thus can be potentially influenced by it. Figure 5 depicts a topology of the social network from Figure 3. Although in this example each edge of the topology corresponds to at least one edge in the social network, generally speaking this does not have to be true. In other words, a node might "know" some nodes that do not belong to any of its conformity groups. By Adj_n we denote the set of all nodes that are starting points of edges of the network topology graph that end at node n. For example, $Adj_e = \{a, b, c, d\}$ for the network topology depicted in Figure 5. We use abbreviation $Adj_n(A)$ for the intersection $Adj_n \cap A$.

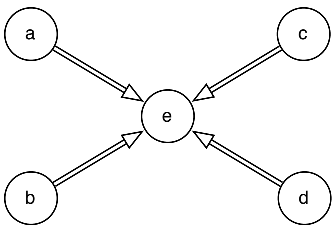

Fig. 5 Network topology of the social network from Figure 3.

In this article we give a sound and complete axiomatization of properties of influence relation $\rhd$ that are true for all social networks with a given network topology. This axiomatization consists of the listed above Armstrong axioms and the following additional principle:

4. Partition: $\bigvee_{b\in B}(A \rhd b) \to \bigvee_{b\in B}(Adj_b(A) \rhd b)$, where A, B is a partition of the set of all vertices.

Informally, the partition axiom expresses the fact that if one side of a partition can influence at least one element on the other side of the partition, then there must be an element on the other side of the partition that is influenced first. Formula $\bigvee_{b\in B}(A \rhd b)$ captures the assumption that side A of a partition A, B can influence at least one element $b \in B$ on the other side of the partition. Formula $\bigvee_{b\in B}(Adj_b(A) \rhd b)$ says that there is an element $b \in B$ on the other side of the partition that is influenced only by the elements of the set A "known" to node b. We will give a formal proof of the soundness of this axiom in Lemma 12.

5 Literature Review

Several logical frameworks for reasoning about diffusion in social networks have been studied before. Seligman, Liu, and Girard [21] proposed Facebook Logic for capturing properties of epistemic social networks in a modal language, but did not give any axiomatization for this logic. They further developed this approach in papers [22,16] where they introduced dynamic friendship relations. Christoff and Hansen [6] simplified Seligman, Liu, and Girard setting and gave a complete axiomatization of the logical system for this new setting. Christoff and Rendsvig proposed Minimal Threshold Influence Logic [7] that uses modal language to capture dynamic of diffusion in a threshold model and gave a complete axiomatization of this logic. Baltag, Christoff, Rendsvig, and Smets [4] discussed logics for informed update and prediction update. Xiong, Ågotnes, Seligman, Zhu introduced logic of tweeting that describes propagation of believes in a social network through public announcements [26].

Informally, the languages of the described above systems feel significantly richer than the more succinct language of our system. However, neither of these systems capture principles similar to our Partition axiom. Naumov and Tao [18,19] used Armstrong's axioms to describe influence in social networks. They considered relation $A \rhd_b B$ that stands for "given marketing budget b, group of agents A can influence group of agents B" and gave an Armstrong-like axioms for this relation. Since they do not assume a fixed topology of the network, their approach does not capture any properties similar to our partition axiom.

Azimipour and Naumov [3] studied properties of influence relation $A \rhd B$ for a given network topology in the setting of threshold model. They observed that in addition to Armstrong's axioms such properties include many other properties. For example, any threshold social network with network topology depicted in Figure 5 satisfies the following property:

$$\left(\bigvee_{x\in\{a,b,c,d\}} \varnothing \rhd x\right) \vee \left((a,b \rhd e) \wedge (c,d \rhd e) \to \bigvee_{x\in\{a,b\}} \bigvee_{y\in\{c,d\}} x \rhd y\right). \quad (1)$$

In fact we have essentially proven this property for an arbitrary threshold social network in Section 2. Indeed, note that formula $\varnothing \rhd A$ means that all nodes in set A will eventually adopt the product even if no node is given a free sample. Thus, the additional assumption $\bigvee_{x\in\{a,b,c,d\}} \varnothing \rhd x$ is required to guarantee that each node in set $\{a,b,c,d\}$ cannot adopt the product on its own without any external peer pressure, which is equivalent to the assumption that all four of these nodes have threshold values greater than zero.

Intuitively, principle (1) reflects the specifics of threshold model much more than it captures a property of real-world diffusion in social networks. To exclude properties like principle (1) from their logical system, Azimipour and Naumov modified the notion of a network topology by considering values assigned to the edges, but not values assigned to the nodes, to be a part of the network topology. In other words, they focused on the properties of social networks that are true no matter what the threshold values are. Their main result is a sound and complete axiomatization of such properties.

[3] is based on threshold models of diffusion. In this article we consider the more general class of group conformity models. In this class, threshold-specific principles like (1) are no longer universally true and all topology-specific properties of diffusion can be captured by the Partition axiom.

In sociology, conformity to multiple groups has been usually discussed in terms of multiple *roles* played by an agent and distinct *reference groups* corresponding to these roles [5] (or to social identities [24]). As Turner [25] notes, the reference groups might be "a group with which one compares himself to in making a self-judgment" [12], "a source of an individual's values" [13], or "a group whose acceptance one seeks" [23].

6 Diffusion in Social Networks: Group Conformity Model

This section formally describes the group conformity model of diffusion in social networks, gives the related notions of A^n and A^* closures that will be used later in this article, and proves basic properties of these notions.

Definition 1 A directed graph is (V,E), where V is a finite set of vertices and $E \subseteq V \times V$ is a set of edges such that $(v,v) \notin E$ for each $v \in V$.

Informally, vertices in set V represent agents in a social network and edges in set E capture connections between agents along which peer-pressure could potentially exist. An example of such graph is depicted in Figure 5.

Definition 2 Let $Adj_v(A) = \{u \in A \mid (u,v) \in E\}$, for any directed graph (V,E), set $A \subseteq V$, and any vertex $v \in V$.

In other words, $Adj_v(A)$ is the set of all vertices in set A that are connected by a directed edge to vertex v.

The next definition is the key definition of this section. It specifies the group conformity model of social networks. In this definition we use the notion of upward closed family of subsets. Family $\mathcal{F}$ of subsets of U is upward closed if for each $X, Y \subseteq U$ if $X \in \mathcal{F}$ and $X \subseteq Y$, then $Y \in \mathcal{F}$.

Definition 3 A social network is a triple $(V, E, \{C_v\}_{v\in V})$ where (V, E) is a directed graph and $C_v \subseteq \mathcal{P}(Adj_v(V))$ is an upward closed family of conformity groups. The topology of social network $(V, E, \{C_v\}_{v\in V})$ is directed graph (V, E).

The next definition describes how the influence spreads through the social network once a set of agents A adopts the product. The set A^n is the set of all agents who adopts the product after n steps of the diffusion.

Definition 4

$$A^n = \begin{cases} A, & \text{if } n = 0, \\ A^{n-1} \cup \{v \in V \mid Adj_v(A^{n-1}) \in C_v\} & \text{if } n > 0. \end{cases}$$

Note that $A^{n-1} \subseteq A^n$ by the above definition. In other words, once an agent is influenced, she remains influenced. For example, for the social network depicted in Figure 4, if $A = \{a\}$, then $A_0 = A = \{a\}$, $A_1 = \{a, b\}$, $A_2 = \{a, b, c\}$, and $A_3 = \{a, b, c, d\}$, see Figure 6.

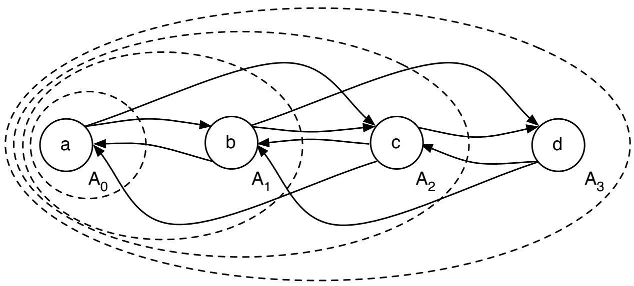

Fig. 6 Steps of Diffusion.

The next two lemmas capture basic properties of diffusion. The first of them immediately follows from Definition 4.

Lemma 1 *If $A^1 = A$, then $A^n = A$.* □

Lemma 2 *If $A \subseteq B$, then $A^n \subseteq B^n$ for each $n \geq 0$.*

Proof We prove this statement by induction on n. If $n = 0$, then $A^0 = A \subseteq B = B^0$ by the assumption $A \subseteq B$ and Definition 4.

Suppose that $n > 0$ and $v \in A^n$. Then, by Definition 4, either $v \in A^{n-1}$ or $Adj_v(A^{n-1}) \in C_v$. In the first case, $v \in B^{n-1}$ by the induction hypothesis. Thus, $v \in B^n$ by Definition 4. Let us next consider the case $Adj_v(A^{n-1}) \in C_v$.

Note that $A^{n-1} \subseteq B^{n-1}$ by the induction hypothesis. Thus, $Adj_v(A^{n-1}) \subseteq Adj_v(B^{n-1})$. Hence, $Adj_v(B^{n-1}) \in C_v$ because C_v is an upward closed family of sets (see Definition 3) and $Adj_v(A^{n-1}) \in C_v$. Therefore, $v \in B^n$ by Definition 4. $\square$

Definition 4 specified the result of n-th step of the diffusion. The next definition specifies the final result A^* of diffusion after potentially infinitely many steps. Later, in Lemma 5, we observe that A^* is reached already after finitely many steps.

Definition 5

$$A^* = \bigcup_{n\geq 0} A^n.$$

For example, for the social network depicted in Figure 4, if $A = \{a\}$, then $A^* = \{a, b, c, d\}$. We conclude this section with several properties of A^* that will be used later.

Lemma 3 *If $A^1 = A$, then $A^* = A$.*

Proof The statement of the lemma follows from Lemma 1 and Definition 5. $\square$

Lemma 4 *If $A \subseteq B$, then $A^* \subseteq B^*$.*

Proof The statement of the lemma follows from Lemma 2 and Definition 5. $\square$

Lemma 5 *$A^* = A^n$ for some $n \geq 0$.*

Proof By Definition 4, we have $A^k \subseteq A^{k+1}$ for each $k \geq 0$. Thus, $A^0 \subseteq A^1 \subseteq A^2 \cdots \subseteq V$. At the same time, set V is finite by Definition 1. Thus, there is $n \geq 0$ such that $A^n = A^k$ for each $k > n$. Therefore, $A^* = A^n$ by Definition 5. $\square$

Lemma 6 *$(A^*)^* \subseteq A^*$.*

Proof By Lemma 5 there are $n, m \geq 0$ such that $A^* = A^n$ and $(A^n)^* = (A^n)^m$. Therefore, $(A^*)^* = (A^n)^* = (A^n)^m = A^{n+m} \subseteq A^*$ by Definition 4 and Definition 5. $\square$

7 Syntax and Semantics

In this section we define formal syntax and formal semantics of our logical system. The axioms of the system are stated in Section 8.

Definition 6 For any finite set V, let $\Phi(V)$ be the minimal set of formulae such that

1. $A \rhd B \in \Phi(V)$ for all sets $A, B \subseteq V$,

2. $\varphi \to \psi, \neg\varphi \in \Phi(V)$ for each $\varphi, \psi \in \Phi(V)$.

We assume that disjunction $\vee$ is defined through implication $\to$ and negation $\neg$ in the standard way. The next definition is a key definition of this article. It specifies the meaning of the statement $A \rhd B$.

Definition 7 For any social network $N = (V, E, \{C_v\}_{v \in V})$ and any formula φ, satisfiability relation $N \vDash \varphi$ is defined as follows:

1. $N \vDash A \rhd B$ if $B \subseteq A^*$, where A, B are subsets of V,
2. $N \vDash \psi \to \chi$ if $N \nvDash \psi$ or $N \vDash \chi$,
3. $N \vDash \neg\psi$ if $N \nvDash \psi$.

For example, if N is the social network depicted in Figure 4, then $N \vDash a \rhd d$ because $\{d\} \subseteq \{a\}^* = \{a, b, c, d\}$.

8 Axioms

In this section for any given directed graph $G = (V, E)$ we specify the axioms of the logical system that captures the influence properties of all social networks with topology G. In addition to substitution instances of propositional tautologies in the language $\Phi(V)$, this system contains the following axioms for all subsets A, B, C of set V:

1. Reflexivity: $A \rhd B$, where $B \subseteq A$,
2. Augmentation: $A \rhd B \to A, C \rhd B, C$,
3. Transitivity: $A \rhd B \to (B \rhd C \to A \rhd C)$,
4. Partition: $\bigvee_{b \in B}(A \rhd b) \to \bigvee_{b \in B}(Adj_b(A) \rhd b)$, where A, B is a partition of the set of all vertices of graph G,

where here and in the rest of the article by X, Y we mean the union of sets X and Y. We write $\vdash_G \varphi$ if formula $\varphi \in \Phi(V)$ is provable from the above axioms using Modus Ponens inference rule. We write $X \vdash_G \varphi$ if formula φ is provable using an additional set of axioms X. We often omit subscript G when its value is clear from the context.

9 Examples of Derivations

In this section we give two examples of formal derivations in our logical system.

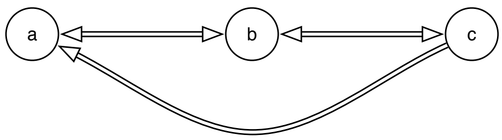

Fig. 7 Network Topology T_1.

Lemma 7 *$\vdash a \rhd c \to (a \rhd b \vee \varnothing \rhd c)$ for the network topology T_1 depicted in Figure 7.*

Proof Consider partition $\{a\}, \{b, c\}$ of the network. Note that $Adj_b(\{a\}) = \{a\}$ and $Adj_c(\{a\}) = \varnothing$. Thus, $\vdash a \rhd b \vee a \rhd c \to a \rhd b \vee \varnothing \rhd c$ by the Partition axiom. Therefore, $a \rhd c \to a \rhd b \vee \varnothing \rhd c$ by propositional reasoning. □

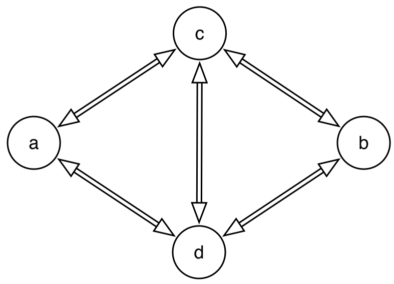

Fig. 8 Network Topology T_2

Lemma 8 *$\vdash a \rhd b \to (b \rhd a \to c, d \rhd a, b)$ for the network topology T_2 depicted in Figure 8.*

Proof By the Reflexivity axiom, $a, c, d \rhd a$. Then, the assumption $a \rhd b$ implies $a, c, d \rhd b$ by the Transitivity axiom. Consider now partition $\{a, c, d\}$, $\{b\}$ of the network. Note that $Adj_n(\{a, c, d\}) = \{c, d\}$. Thus, $a, c, d \rhd b \to c, d \rhd b$ by the Partition axiom. Hence, $c, d \rhd b$ by the Modus Ponens inference rule. Similarly, one can show $c, d \rhd a$. Statements $c, d \rhd b$ and $c, d \rhd a$, by the Augmentation axiom, imply $c, d \rhd c, d, b$ and $c, d, b \rhd a, b$ respectively. Therefore, $c, d \rhd a, b$ by the Transitivity axiom. □

10 Soundness

In this section we prove the soundness of our logical system. First, we show soundness of each axiom as a separate lemma and later state the soundness for the whole system as Theorem 1.

Lemma 9 *$N \vDash A \rhd B$, for each $B \subseteq A$.*

Proof By Definition 4 and Definition 5, $B \subseteq A = A^0 \subseteq A^*$. Therefore, $N \vDash A \rhd B$ by Definition 7. □

Lemma 10 *If $N \vDash A \rhd B$, then $N \vDash A, C \rhd B, C$.*

Proof By Definition 7, assumption $N \vDash A \rhd B$ implies that $B \subseteq A^*$. Thus, $B \subseteq (A \cup C)^*$ by Lemma 4. At the same time, $C = C^0$ by Definition 4. Hence, $C = C^0 \subseteq C^*$ by Definition 5. Then, $C \subseteq (A \cup C)^*$ by Lemma 4.

Finally, statements $B \subseteq (A \cup C)^*$ and $C \subseteq (A \cup C)^*$ imply that $B \cup C \subseteq (A \cup C)^*$. Therefore, $N \vDash A, C \rhd B, C$ by Definition 7. □

Lemma 11 *If $N \vDash A \rhd B$ and $N \vDash B \rhd C$, then $N \vDash A \rhd C$.*

Proof By Definition 7, assumption $N \vDash A \rhd B$ implies that $B \subseteq A^*$. Thus, $B^* \subseteq (A^*)^*$ by Lemma 4. Hence, $B^* \subseteq A^*$ by Lemma 6. At the same time, $C \subseteq B^*$ by the assumption $N \vDash B \rhd C$ and Definition 7. Then, $C \subseteq B^* \subseteq A^*$. Therefore, $N \vDash A \rhd C$ by Definition 7. □

Lemma 12 *For any partition A, B of the set of all vertices V, if $N \vDash A \rhd b_0$ for some $b_0 \in B$, then there is $b \in B$ such that $N \vDash Adj_b(A) \rhd b$.*

Proof Assumption $N \vDash A \rhd b_0$ by Definition 7 implies that $b_0 \in A^*$. Thus, $b_0 \in A^* \setminus A$ because $b_0 \in B$ and A, B is a partition of set V. Hence, $A^* \neq A$. Then, $A^1 \neq A$ by Lemma 3. Thus, by Definition 4, there is $b \in (V \setminus A) = B$ such that $Adj_b(A) \in C_b$. Then, $Adj_b(Adj_b(A)) = Adj_b(A) \in C_b$. Thus, $b \in (Adj_b(A))^1$ by Definition 4. Hence, $b \in (Adj_b(A))^*$ by Definition 5. Therefore, $N \vDash Adj_b(A) \rhd b$ by Definition 7. □

We are now ready to state the soundness theorem for our logical system, which follows from the lemmas above.

Theorem 1 *If $\vdash_G \varphi$, then $N \vDash \varphi$ for each social network N with topology G.* □

11 Completeness

In this section we prove the completeness of our logical system, which is stated later as Theorem 2. We start, however, by defining a *canonical* social network $N(X) = (V, E, \{C_v\}_{v \in V})$ for each given directed graph (V, E) and each maximal consistent subset X of $\Phi(V)$.

Definition 8 Let $C_v = \{A \subseteq Adj_v(V) \mid A \rhd v \in X\}$.

By Definition 3, to show that $(V, E, \{C_v\}_{v \in V})$ is a social network, it suffices to prove that family of sets C_v is upward closed for each $v \in V$. We do this in the following lemma.

Lemma 13 *If $A \in C_v$ and $A \subseteq B$, then $B \in C_v$ for each $v \in V$ and all $A, B \subseteq Adj_v(V)$.*

Proof By Definition 8, assumption $A \in C_v$ implies that $A \rhd v \in X$. At the same time, $B \rhd A$ is an instance of the Reflexivity axiom due to the assumption $A \subseteq B$. Hence, $X \vdash B \rhd v$ by the Transitivity axiom. Thus, $B \rhd v \in X$ because X is a maximal consistent set of formulas. Therefore, $B \in C_v$ by Definition 8. □

This concludes the definition of the canonical social network $N(X)$. In the rest of this section we use this network to prove the completeness of our logical system stated as Theorem 2. The theorem follows from Lemma 19 in the standard way. Lemma 19 is proven by induction on the structural complexity of a formula. The base case of this induction is established in Lemma 14 and Lemma 18.

Lemma 14 *If $C \rhd D \in X$, then $N(X) \vDash C \rhd D$.*

Proof Suppose that $N(X) \nvDash C \rhd D$. Thus, $D \nsubseteq C^*$ by Definition 7. Hence, there is $d \in D$ such that $d \notin C^*$. Then $D \rhd d$ is an instance of the Reflexivity axiom. Thus, $X \vdash C \rhd d$ by Transitivity axiom and the assumption $C \rhd D \in X$ of the lemma. Note that $C = C^0 \subseteq C^*$ by Definition 4 and Definition 5. Thus, $C^* \rhd C$ is an instance of the Reflexivity axiom. Then, $X \vdash C^* \rhd d$ by the Transitivity axiom. At the same time

$$\bigvee_{v \in V \setminus C^*} (C^* \rhd v) \to \bigvee_{v \in V \setminus C^*} (Adj_v(C^*) \rhd v)$$

is an instance of Partition axiom. Thus, since $d \in V \setminus C^*$ and $X \vdash C^* \rhd d$,

$$X \vdash \bigvee_{v \in V \setminus C^*} (Adj_v(C^*) \rhd v).$$

Since set X is a maximal consistent set of formulas, there must exist a vertex $v \in V \setminus C^*$ such that $X \vdash Adj_v(C^*) \rhd v$. Thus, $Adj_v(C^*) \rhd v \in X$ due to X being a maximal consistent set of formulas. Hence, $Adj_v(C^*) \in C_v$ by Definition 8. By Lemma 5, there is $n \geq 0$ such that $C^* = C^n$. Then, $Adj_v(C^n) \in C_v$. Thus, $v \in C^{n+1}$ by Definition 4. Therefore, $v \in C^*$ by Definition 5, which is a contradiction with the choice of vertex v. □

Before proving Lemma 18, we need to establish three technical results that are used in the proof of this lemma.

Lemma 15 *If $B = \{b_1, \dots, b_n\}$ are such that $A \rhd b_i \in X$ for each $i \geq 1$, then $X \vdash A \rhd B$.*

Proof We prove this statement by induction on integer n. If $n = 0$, then $B = \varnothing \subseteq A$. Thus, $\vdash A \rhd B$ by the Reflexivity axiom.

If $n > 0$, then suppose that $X \vdash A \rhd \{b_1, \dots, b_{n-1}\}$. Thus, $X \vdash A \cup \{b_n\} \rhd \{b_1, \dots, b_{n-1}, b_n\}$ by the Augmentation axiom. At the same time, assumption $A \rhd b_n \in X$ implies $X \vdash A \rhd A \cup \{b_n\}$ also by the Augmentation axiom. Therefore, $X \vdash A \rhd \{b_1, \dots, b_{n-1}, b_n\}$ by the Transitivity axiom. □

Lemma 16 *$X \vdash C^{n-1} \rhd C^n$ for each $n \geq 1$.*

Proof Let vertices $d_1, \dots, d_n$ be all such vertices $v \in V$ that $Adj_v(C^{n-1}) \in C_v$.

Consider an arbitrary integer i such that $1 \leq i \leq n$. Then, $Adj_{d_i}(C^{n-1}) \in C_{d_i}$ by Definition 8. Thus, $Adj_{d_i}(C^{n-1}) \rhd d_i \in X$ for each $i \leq n$. At the same time, $Adj_{d_i}(C^{n-1}) \subseteq C^{n-1}$ by Definition 2. Hence, $\vdash C^{n-1} \rhd Adj_{d_i}(C^{n-1})$ by the Reflexivity axiom. Thus, $X \vdash C^{n-1} \rhd d_i$ by the Transitivity axiom for all i such that $1 \le i \le n$.

Thus, $X \vdash C^{n-1} \rhd \{d_1, \dots, d_n\}$ by Lemma 15. Hence, by the choice of vertices $d_1, \dots, d_n$,

$$X \vdash C^{n-1} \rhd \{v \in V \mid Adj_v(C^{n-1}) \in C_v\}.$$

Thus, by the Augmentation axiom,

$$X \vdash C^{n-1} \rhd C^{n-1} \cup \{v \in V \mid Adj_v(C^{n-1}) \in C_v\}.$$

Therefore, $X \vdash C^{n-1} \rhd C^n$ by Definition 4. □

Lemma 17 *$X \vdash C \rhd C^n$.*

Proof We prove this statement by induction on n. If $n = 0$, then $C^n = C^0 = C$ by Definition 4. Thus, $\vdash C \rhd C^n$ by the Reflexivity axiom. If $n > 0$, the suppose that $X \vdash C \rhd C^{n-1}$. Thus, $X \vdash C \rhd C^n$ by Lemma 16 and the Transitivity axiom. □

Lemma 18 *If $N(X) \vDash C \rhd D$, then $C \rhd D \in X$.*

Proof By Definition 7, assumption $N(X) \vDash C \rhd D$ implies that $D \subseteq C^*$. Thus, by Lemma 5, there is $n \ge 0$ such that $D \subseteq C^n$. Then, $\vdash C^n \rhd D$ by the Reflexivity axiom. Hence, $X \vdash C \rhd D$ by Lemma 17 and the Transitivity axiom. Therefore, $C \rhd D \in X$ due to the maximality of set X. □

Lemma 19 *$N(X) \vDash \varphi$ iff $\varphi \in X$, for any $\varphi \in \Phi(V)$.*

Proof We prove the lemma by induction on structural complexity of formula φ. The base case follows from Lemma 14 and Lemma 18. The induction step follows from maximality and consistency of the set X in the standard way. □

We are now ready to state and to prove the completeness of our logical system.

Theorem 2 *For any directed graph G, if $N \vDash \varphi$ for any social network N with topology G, then $\vdash_G \varphi$.*

Proof Suppose that $\nvdash \varphi$. Let X be a maximal consistent set of formulae containing $\neg\varphi$. Then, $N(X) \vDash \neg\varphi$ by Lemma 19. Therefore, $N(X) \nvDash \varphi$. □

Note that if $G = (V, E)$ is a complete graph, then $Adj_b(A) = A$ for each set of vertices $A \subseteq V$ and each vertex $b \in B$. Thus, in case of a complete graph the Partition axiom is a propositional tautology. Hence, all influence properties of a network with a complete graph topology follow from the three original Armstrong's axioms. By Definition 3, any social network with a set of vertices V could be considered to be a network over the complete graph. Therefore, by Theorem 2, all properties which are true for all social networks with the set of vertices C are provable from the three original Armstrong's axioms.

12 Decidability

The logical system introduced in this article is decidable. Namely, for any graph $G = (V, E)$, the set $\{\varphi \in \Phi(V) \mid\ \vdash_G \varphi\}$ is decidable. Indeed, by Definition 3, there are only finitely many social networks based on graph G. Thus, decidability of the logical system follows from the soundness and the completeness.

13 Conclusion

In this article we proposed a group conformity interpretation of the propositional opinion diffusion model of social networks. It has been shown in the introduction that the this model can simulate social behaviors that the existing threshold model cannot. It is also easy to see that any threshold model can be represented as a group conformity model in which conformity groups of a node are all sets of its neighbors whose total pressure on the node is at least as high as the node's threshold value. The same argument shows that a hypothetical "hybrid" conformity-threshold model that uses an individual threshold for each conformity group also can be simulated by the group conformity model proposed in this article.

Our main technical contribution is a sound and complete axiomatic system that describes all properties of influence common to all group conformity models with the same topological structure of the social network.

A Monotonic Boolean Functions

In this appendix we prove the property of monotonic Boolean functions that we referred to in the introduction.

Lemma 20 *Any monotonic Boolean formula can be written as a disjunction of several disjuncts where each disjunct is a conjunction of several propositional variables.*

Proof Consider any monotonic Boolean formula $\varphi(x_1, \dots, x_n)$. If $b_1, \dots, b_n$ are Boolean values, then by $\varphi\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{smallmatrix}\right)$ we denote the Boolean value of formula φ on Boolean arguments $b_1, \dots, b_n$. Consider Boolean expression

$$\psi = \bigvee \left\{ x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k} \mid \varphi\left(\begin{smallmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ 0 & \dots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right) = 1 \right\}. \quad (2)$$

Next we show that formulae φ and ψ are equivalent. Consider any Boolean values $b_1, \dots, b_n$. We will show that $\varphi\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{smallmatrix}\right) = 1$ if and only if $\psi\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{smallmatrix}\right) = 1$
($\Rightarrow$) Let $i_1, \dots, i_n$ be all indices i such that $b_i = 1$. Then,

$$\varphi\left(\begin{smallmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ 0 & \dots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right) = 1.$$

Hence, conjunction $x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k}$ is one of disjuncts in formula ψ. Note that

$$(x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k})\left(\begin{smallmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ 0 & \dots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right) = 1 \wedge \dots \wedge 1 = 1.$$

Therefore,

$$\psi\left(\begin{smallmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ 0 & \dots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right) = 1.$$

because conjunction $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$ is one of disjuncts in formula ψ.
($\Leftarrow$) In order for a disjunction to have value 1 at least one disjunct must have value 1. Thus, assumption $\psi\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{smallmatrix}\right) = 1$ and equation (2) imply that there are indices $i_1, \dots, i_k$ such that

$$(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k})\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{smallmatrix}\right) = 1 \tag{3}$$

and

$$\varphi\left(\begin{smallmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ 0 & \dots & 0 & 1 & 0 & \cdots & 0 & 1 & 0 & \dots & 0 \end{smallmatrix}\right) = 1. \tag{4}$$

Note that $(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k})\left(\begin{smallmatrix} 1 & 2 & 3 & \cdots & n \\ b_1 & b_2 & b_3 & \cdots & b_n \end{smallmatrix}\right) = b_{i_1} \wedge \cdots \wedge b_{i_k}$. Thus, equality (3) implies that $b_{i_1} = \cdots = b_{i_k} = 1$. Hence, it follows from equation (4) that

$$\varphi\begin{pmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ 0 & \dots & 0 & b_i & 0 & \cdots & 0 & b_k & 0 & \dots & 0 \end{pmatrix} = 1.$$

Therefore,

$$\varphi\begin{pmatrix} 1 & \dots & i_1-1 & i_1 & i_1+1 & \dots & i_k-1 & i_k & i_k+1 & \dots & n \\ b_1 & \dots & b_{i-1} & b_i & b_{i+1} & \cdots & b_{k-1} & b_k & b_{k+1} & \dots & b_n \end{pmatrix} = 1.$$

due to the assumption that formula φ is monotonic. $\square$

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