On Composition of Bounded-Recall Plans

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Abstract

The article studies the ability of agents with bounded memory to execute consecutive composition of plans. It gives an upper limit on the amount of memory required to execute the composed plans and shows that the limit cannot be improved. Furthermore, the article shows that there are, essentially, no other universal properties of plans for bounded-recall agents expressible through the relation "there is a plan for an agent with a given memory size to navigate from one given set of states to another".

Keywords: working memory; plan; navigability; bounded-recall; Mealy machine; composition; logic; axiomatization

1. Introduction

In this article we study how working memory capacity affects ability of agents to execute a plan of actions. Although the capacity of the human brain is enormous, only a very limited amount of it, called "working memory" is used to make decisions. In fact, in some situations animals have a larger capacity of working memory than humans.

1.1. Inoue and Matsuzawa Experiment

In their experiments, Inoue and Matsuzawa [1] present chimpanzees with numbers 1, 2, ..., 9 randomly placed on a computer screen. Once a chimpanzee touches

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number 1, the number disappears and the rest of the numbers are replaced with white squares. The chimpanzee is then expected to touch the squares in the order they had been numbered. Figure 1 illustrates[1] this experiment. We use letters $a, \ldots, i$ to denote the various positions on the screen. In our example, originally these positions display numbers 5, 2, 6, 4, 9, 1, 7, 8, and 3 respectively. Once the chimpanzee touches position f of number 1, this position becomes dark and the rest of the positions become white. This transition is shown on the figure by a directed edge labeled with f. Next, the chimpanzee is expected to touch position b, where number 2 used to be, then position i, and so on. In their experiments, Inoue and Matsuzawa observed that one of the chimpanzees, named Ayumu, was able to correctly touch the positions of all numbers 1 through 9 [1, Supplementary Movie S1]. In tests with just 5 numbers, Ayumu outperformed human subjects on accuracy [1].

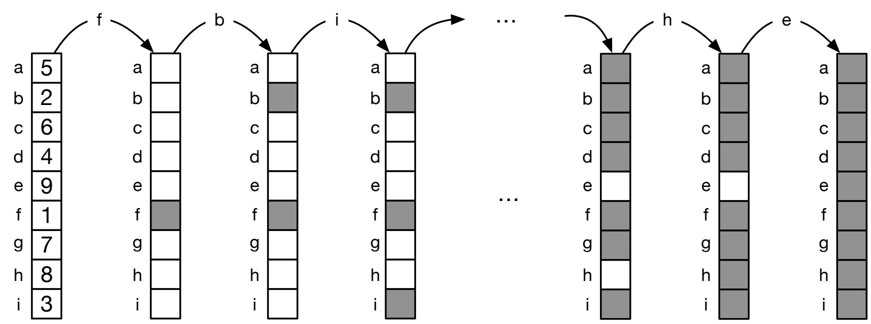


Figure 1: Inoue and Matsuzawa Experiment.

Inoue and Matsuzawa [1] refer to the chimpanzees' ability to remember the positions of numbers on the screen as working memory. At the same time, Carruthers [2] points to a distinction between short-term memory and working memory. Short-term memory requires no attention, but can be used to store information for only up to about two seconds. Working memory requires attention, but can be used to store information for longer periods of time. Since Inoue and Matsuzawa's experiment lasts for about two seconds, Carruthers [2] argues that there is no clear reason to use the term "working memory" rather than "short-term memory" while describing this experiment. However, following the original work of Inoue and

[1]Figure 1 depicts a successful run of the experiment in which the chimpanzee touches all squares correctly. If this illustration were to be treated more formally, a failure state and transitions to the failure state would need to be added to the diagram.

Matsuzawa [1] we use the term “working memory”.

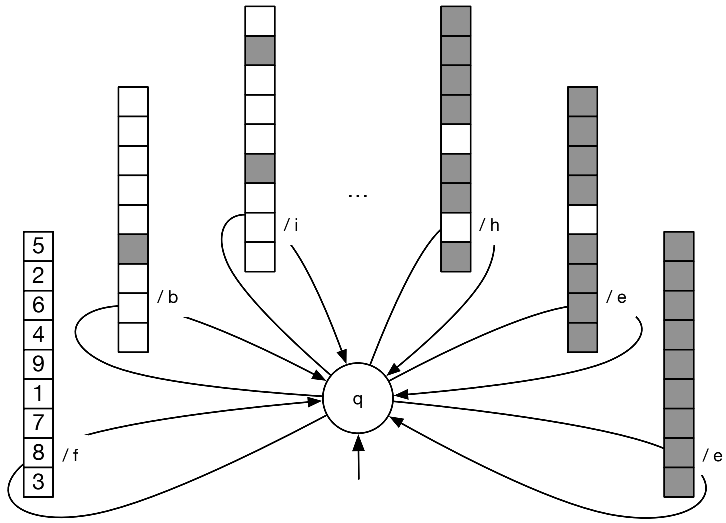


Figure 2: One-State Mealy Machine.

1.2. Mealy Machines and Worlds

In this article, to formally capture the abilities of agents with limited working memory, we represent agents as *Mealy machines* [3]. These machines determine an output (action) based on the current state and the current input (observation). Although other formalisms (for example, *Moore machines* [4] or random access memory machines) could be used instead, Mealy machine notations allow us to express properties of agents with limited working memory more elegantly. We use the size (number of states) of a Mealy machine required to accomplish a task as a measure of the size of the working memory needed. Note that to accomplish the task depicted in Figure 1, one actually needs only a single-state Mealy machine. An example of this machine is given in Figure 2. This machine contains a single state q, which is also the initial state of the machine. Possible transitions from state q back to the same state are denoted by the labeled directed edges. Each label has the form x/y where x is the current input (observation) of the machine and y is the output (action). For example, the left-most edge on this figure shows that if the machine observes the initial configuration of the numbers, then it touches position f on the screen.

Although the Mealy machine in Figure 2 has only one state, we do not claim that in Inoue and Matsuzawa's experiment the agent does not need any working

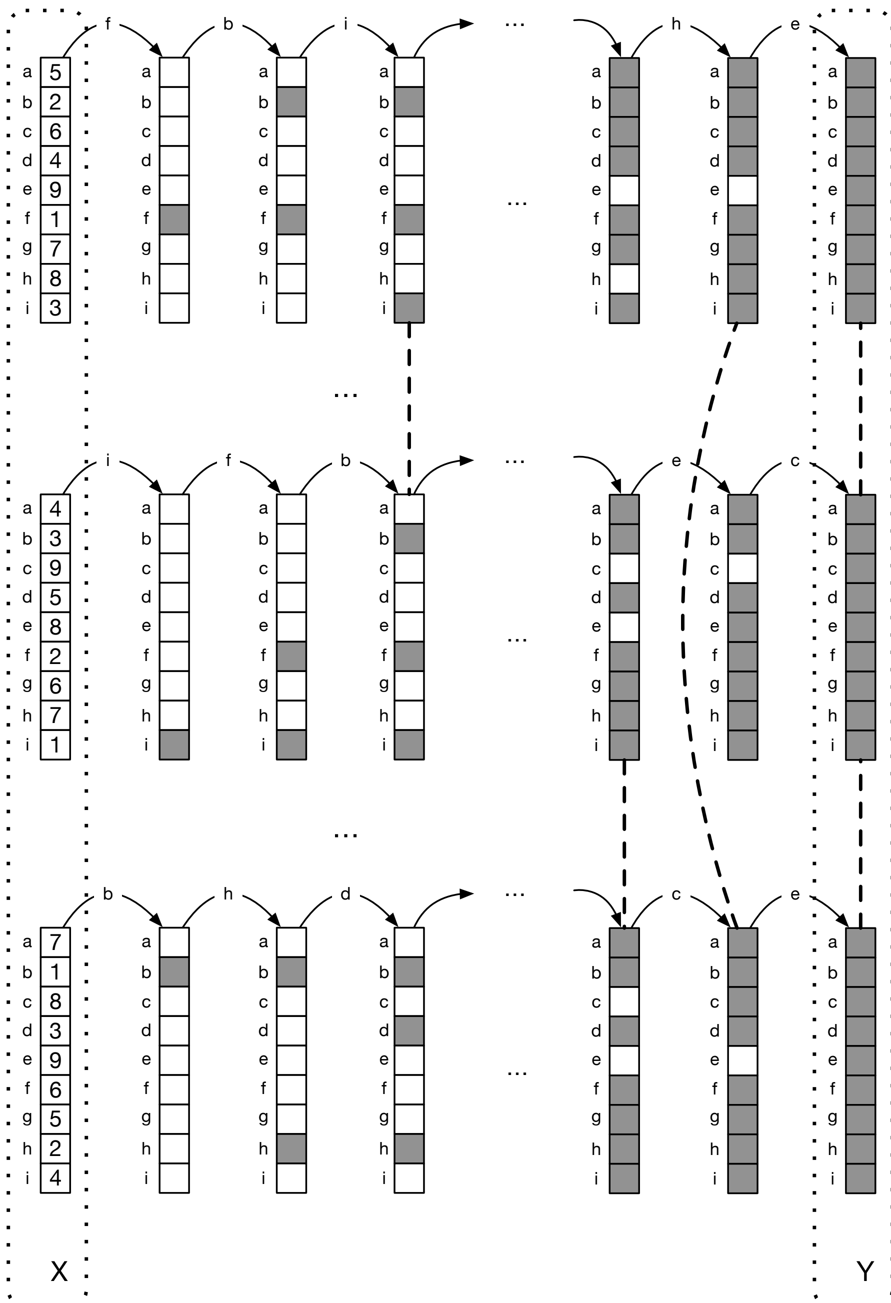


Figure 3: A fragment of the world from Inoue and Matsuzawa's experiment.

memory to correctly touch the positions in the given order. Figure 2 depicts the machine that can do this test for a *specific* configuration of the numbers on the screen. However, chimpanzee Ayumu can do it for *any* configuration. In other words, Ayumu can navigate from any of the states in set X to a state in set Y of the world partially depicted in Figure 3. This world has total of $10! + 1$ states, of which only 21 are shown in the figure. Set X consists of $9!$ states that are the possible initial states of the experiment. Only three of these states are shown on the diagram. These states represent the different initial configurations of the numbers on the screen in the beginning of the experiment. If the chimpanzee touches the correct position in one of the initial states, the world transitions to the next state depicted to the right of the initial state. If the chimpanzee touches a wrong position, the world transitions to a "failure" state from which there is no way out. The failure state, the transitions into the failure state, and the loop transitions from the failure state back into the failure state are *not* shown in Figure 3. Once the world transitions into one of the states in set Y, it will remain there no matter which position is touched. The corresponding loop transitions are also not shown in Figure 3. The dashed (but not the dotted) lines between states on the diagram represent states indistinguishable to the agent. For example, all initial states in set X are distinguishable, while all states in set Y are indistinguishable.

An example of a Mealy machine that can navigate from any of the states in set X to a state in set Y is depicted in Figure 4. This machine has one initial state q and $9!$ "working" states. From the initial state the machine transitions to one of the working states, depending on the configurations of the numbers on the screen it observes. It can then accomplish its task in the working state, similarly to the machine from Figure 2. Although our formal definition of a Mealy machine, given later, requires the machine to specify its actions in each state of the world, for the sake of simplicity, the diagram in Figure 4 does not specify how the machine acts in the failure state.

1.3. Main Contribution

In this article we study the ternary relation $X \rhd_n Y$ that stands for "there is a Mealy machine of size at most n that can navigate from each of the states in set X to a state in set Y of a given world". Since we allow for some states to be indistinguishable, it will be convenient to assume that X and Y are sets of equivalence classes of states rather then sets of states. In other words, if set X (or set Y) contains a state q, then it also must contain any state that the agent cannot distinguish from state q.

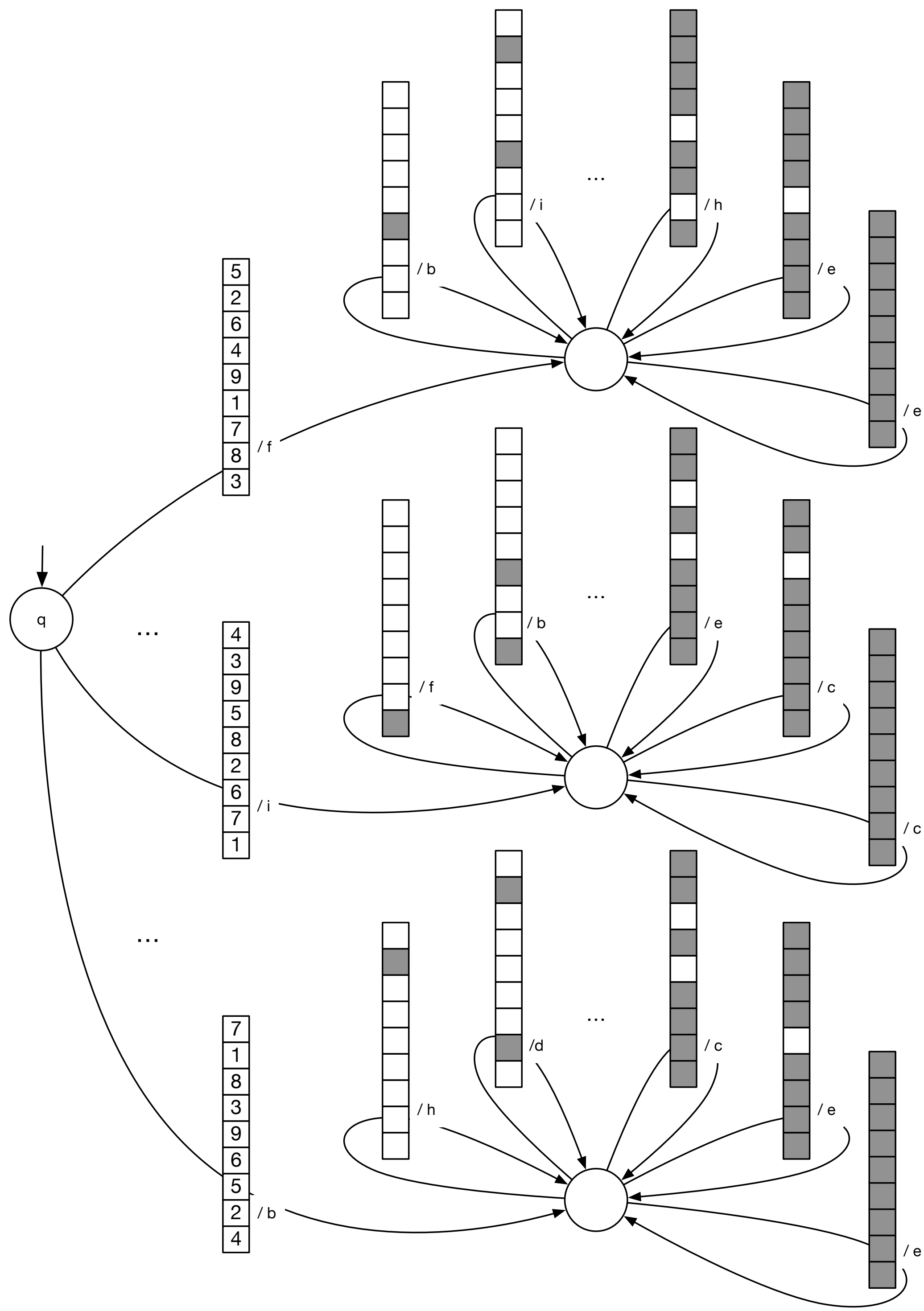


Figure 4: Mealy machine for Inoue and Matsuzawa experiment.

The most interesting property of this relation is captured by what we call the Composition axiom: $X \rhd_n Y \to (Y \rhd_k Z \to X \rhd_{n+k} Z)$. It states that if there is a Mealy machine of size n that can navigate from set X to set Y and a Mealy machine of size k that can navigate from set Y to set Z, then there is a Mealy machine of size $n + k$ that can navigate from set X to set Z. Recall that a Mealy machine is just one way to capture the agent's memory size. In practice people usually describe memory size in bits rather than number of states. A machine that has m bits of memory is a 2^m-state machine. An n-state machine needs $\lceil \log_2 n \rceil$ bits of memory. Thus, another way to state the Composition axiom is: *if one needs m_1 bits of memory to navigate from X to Y and m_2 bits of memory to navigate from Y to Z, then one can navigate from X to Z using $\lceil \log_2(2^{m_1} + 2^{m_2}) \rceil$ bits of memory.*

The Composition axiom states that one *can* navigate using that much memory, but does one actually *need* that much memory? In some cases an agent can navigate from X to Z using less than $\lceil \log_2(2^{m_1} + 2^{m_2}) \rceil$ bits of memory. For example, if X is a subset of Z, then no matter how hard it is to navigate from X to Y and from Y to Z, no memory is required to navigate from X to Z. We call this observation the Reflexivity axiom: $X \rhd_1 Z$, where $X \subseteq Z$. The subscript of this formula refers to a memoryless single-state Mealy machine.

In Theorem 1, we show that *in general* an agent requires all $\lceil \log_2(2^{m_1} + 2^{m_2}) \rceil$ bits of memory to navigate. In fact, we show that statement

$$X \rhd_n Y \to (Y \rhd_k Z \to X \rhd_{n+k-1} Z) \tag{1}$$

is not universally true even when sets X, Y, and Z are singletons. This, leaves a question if there are any other ways to generalize the Composition axiom that will keep it universally true. In this article we answer this question by giving a complete axiomatization of all universal properties of the relation $X \rhd_n Y$. This axiomatization includes the Composition axiom, the Reflexivity axiom, and two more axioms. A preliminary version of this work, without proofs, appeared in [5].

1.4. Literature Review

Finite state machines have been previously used in game theory to model players with *bounded* rationality in iterative strategic games [6, 7, 8, 9, 10], to model the evolution of rational players [11], to analyze two-armed Bernoulli bandit problems [12], to model AI agents behavior in video games [13], and to specify AI agents for border patrol [14]. Kanovich, Kirgin, Nigam, and Scedrov proved NP-completeness of a security problem in collaborative systems with bounded-recall agents [15]. Nikolaidis, Hsu, and Srinivasa proposed a bounded-memory model that captures human adaptive behaviors and used it to help robots to interact with humans in hybrid human-machine environments [16].

Most of the above papers represent finite state machines as Moore machines [4] whose output is determined solely by the current state. In the current article we focus on Mealy machines because they yield a more elegant logical system for reasoning about navigability. Mealy machines have been previously used in circuit design [17], in machine learning [18, 19, 20], and for software specification [21, 22].

There have also been many works on logical systems for reasoning about strategies. In our paper [23] we considered relations $X \rhd_1 Y$ and $X \rhd_\infty Y$ for navigability by no recall and perfect recall strategies. Additionally, in [24] we gave a complete axiomatization of relation $X \rhd_Z Y$, which stands for "there is a no recall strategy to navigate from set X to set Y using only intermediate states in set Z". Others captured properties of navigability using modal language instead of relations. Non-epistemic logics of coalition power were developed by Pauly [25], who also proved the completeness of the basic logic of coalition power. His approach has been widely studied in the literature [26, 27, 28, 29, 30, 31, 32, 33]. More and Naumov proposed a logical system for coalition power based on a binary modality [34]. Alechina, Logan, Nga, and Rakib proposed a version of coalition logic with bounded resources [35]. Cao and Naumov developed a logic of strategic power with bounded cost and profit [36]. Alur, Henzinger, and Kupferman introduced Alternating-Time Temporal Logic (ATL) that combines temporal and coalition modalities [37]. Goranko and van Drimmelen gave a complete axiomatization of ATL [38]. Van der Hoek and Wooldridge proposed to combine ATL with epistemic modality to form Alternating-Time Temporal Epistemic Logic [39]. Aminof et al. studied model-checking problems of an extension of ATL with epistemic and "prompt eventually" modal operators [40]. Jamroga, Malvone, and Murano proposed a framework of natural strategies that can be used to express complexity of strategies [41, 42]. An alternative approach to expressing the power to achieve a goal in a temporal setting is the STIT logic [43, 44, 45, 46, 47]. Broersen, Herzig, and Troquard have shown that coalition logic can be embedded into a variation of STIT logic [48]. Strategy Logic introduces explicit quantifiers over strategies [49, 50, 51, 52, 53].

Since in the current work we assume that the agent might not be able to distinguish some of the states of the world, our setting is one with imperfect information. Strategies in such settings has been studied before under different names. Jamroga and Ågotnes talk about "knowledge to identify and execute a strategy" [54], Jamroga and van der Hoek discuss the "difference between an agent knowing that he has a suitable strategy and knowing the strategy itself [55]. Van Benthem calls such strategies “uniform” [56]. Naumov and Tao use the term “executable strategy” [57]. Recently, several modal logics that describe interplay between such strategies and knowledge have been proposed for no recall [58, 59, 60, 57, 61, 62] and perfect recall [63] settings.

Our system is closely related to Yanjing Wang’s Logic of Knowing How [64, 65] that describes properties of navigation by a single agent using a *linear* plan. Such a plan is a linear sequence of instructions to be executed in a given order. Navigation strategies based on linear plans are different from the uniform/know-how strategies that we consider in this article. However, we believe that navigation by linear strategies with *bounded length* is likely to yield a logical system similar to the one proposed in this article. Li and Wang extended the linear planning approach to navigability with intermediate constraints [66].

Navigability is a special case of planning. Planning with imperfect information has been studied before in various contexts [67]. De Giacomo, Murano, et al discuss the connection between such planning and two-player games [68]. They also devised a general technique to reduce planning under partial observability to two-player games with perfect information. Ferguson and Stentz suggest an algorithm for navigation planning using “maps plagued with uncertainty” [69]. Mentioned earlier work by Wang [65] develops a modal logic for linear plans. Unlike the navigation strategies that we consider in the current article, a linear plan is a fixed sequence of instructions to be executed without taking into account the agent’s observations.

Navigability is also closely related to reachability of a post-condition by a program. The classical example of a logical system for reasoning about such reachability is Hoare Logic [70] and a more recent one is Reachability Logic [71].

None of the works mentioned above develop complete logical systems for strategies of finite state machines or for strategies with bounded-recall in general. Ågotnes and Walther introduced Alternating-time Temporal Logic with Bounded Memory and discussed its expressive power [72]. Unlike us, they do not use a Mealy machine to model bounded recall. Instead, they define bounded recall as the memory of the last n states of the system. They list some properties of ATL with bounded recall, but do not prove completeness. Among the properties they list, none are similar to our Composition axiom. Complexity and model checking of ATL with bounded memory has been analyzed in [73, 74, 75]. In [75], bounded recall is defined as remembering the last n states, similarly to [72]. In [73], the bounded memory consists of several “cells” (similar to bits in our terminology). The authors define the size of memory as the number of cells, not as the number of states.

In [74], the bounded memory is represented by Mealy machines, just like in the current article. The author shows that the model checking problem for ATL is Δ_2^p-complete and it is PSPACE-complete for an ATL extension called ATL*. He also shows that ATL and ATL* model-checking is undecidable for finite-memory semantics in incomplete information games with at least three players. The model checking algorithms from this work potentially might be adopted to model checking of formulae in our language, but this would not imply the decidability and axiomatization results in the current article.

Unlike the current article, none of the above works on the bounded recall propose a sound and complete logical system for reasoning about bounded recall strategies or prove decidability of such a system.

1.5. Outline

In this article we propose a complete logical system for the bounded navigability relation $X \rhd_n Y$. The article is structured as follows. In the next section, we formally define worlds and Mealy machines. In Section 3 we give an counterexample for statement (1). The formal syntax and semantics of our logical system are defined in Section 4. In Section 5 we introduce and discuss the axioms of our system. In Section 6 we give examples of formal proofs in our system. Then, in Section 7 we prove the soundness of these axioms. Next, in Section 8 we prove the completeness of the system. Section 9 sketches the proof of the decidability of our logical system. Section 10 concludes.

2. Worlds and Mealy Machines

Although in the introduction we have talked about $X \rhd_n Y$ as a relation between sets of indistinguishability classes X and Y, in the formal account, we assume X and Y to be sets of *names* of indistinguishability classes. We make this change so that the logical system could be used to reason about any world, as long as its indistinguishability classes are named using some fixed set of names. For the rest of the article we fix a *finite* set of names N.

Next we formally define a world. All actions in the world from Inoue and Matsuzawa's experiment, see Figure 3, are deterministic. In general we allow actions, specified via relation Δ below, to have several possible outcomes. We require that each action has at least one outcome. Although the set of actions A in the definition below represents possible actions of a single agent, we allow for a possibility of other agents being present and acting in the system. The nondeterministic nature of the agent actions reflects possible actions of the other agents or the environment.

Definition 1. *A world is a tuple $(R, \sim, *, A, \Delta)$ where*

1. *R is a non-empty set of "states",*
2. *$\sim$ is an indistinguishability equivalence relation on R,*
3. *$*$ is an "enumeration" function from set of names N to set $R/\sim$,*
4. *A is a nonempty set of "actions",*
5. *$\Delta \subseteq R \times A \times R$ is a "transition" relation such that for each state $r \in R$ and each action $a \in A$ there is at least one $r' \in R$ where $(r, a, r') \in \Delta$.*

For example, in the world partially depicted in Figure 3, the set R consists of $10! + 1$ states. The states are represented by the vertical bars in the diagram. Only 21 of them are shown in the figure. The equivalence relation $\sim$ is captured by the dashed lines between the states. The set of actions A is $\{a, b, c, d, e, f, g, h, i\}$. The transition relation Δ is depicted by the direct edges between states labeled with the actions.

Recall that set N is already assumed to be finite. We say that a world is *finite* if sets R and A are finite.

Note that we do *not* assume that enumeration function $*$ is onto. That is, not all classes of states must have names. For every set $X \subseteq N$, let X^* be the image of set X with respect to the function $*$.

Definition 2. *A Mealy machine for a world $(R, \sim, *, A, \Delta)$ is a tuple (Q, s, α, δ) where*

1. *Q is an arbitrary finite set of states,*
2. *$s \in Q$ is a starting state,*
3. *$\alpha : (R/\sim) \times Q \to A$ is an "action function",*
4. *$\delta : (R/\sim) \times Q \to Q$ is a "transition function".*

For the Mealy machine depicted in Figure 4, the set Q includes state q and the 9! working states. The starting state q is labeled with an arrow pointing towards it. The action functions and transition functions are represented by the labeled directed edges between states.

Definition 3. *The size of a Mealy machine is the number of its states.*

In Figure 2 and in Figure 4, the sizes of Mealy machines are 1 and $9! + 1$ respectively. By Definition 2, the size of any Mealy machine is positive, because the machine must have a starting state.

In the definition below by $[r]$ we mean the equivalence class of state $r \in R$ with respect to the equivalence relation $\sim$.

Definition 4. *A path of a Mealy machine (Q, s, α, δ) in a world $(R, \sim, *, A, \Delta)$ is such an infinite sequence $r_0, q_0, a_0, r_1, q_1, a_1, r_2, \dots$, that*

1. *$r_0, r_1, r_2, \dots$ are states from set R,*
2. *$q_0, q_1, q_2 \dots$ are states from set Q,*
3. *$a_0, a_1, a_2, \dots$ are actions from set A,*
4. *$q_0 = s$,*
5. *$a_k = \alpha([r_k], q_k)$, for $k \geq 0$,*
6. *$(r_k, a_k, r_{k+1}) \in \Delta$, for $k \geq 0$,*
7. *$q_{k+1} = \delta([r_k], q_k)$, for $k \geq 0$.*

Figure 1 partially depicts a path of a Mealy machine from Figure 4 in the world of Figure 3. Namely, Figure 1 shows the states of the world the machine passes and the actions that it takes, but it does not show the states of the machine itself. Initially, the machine starts in state q, see Figure 4, from which the machine transitions into the top-most state and remains there for the rest of the path. By the above definition, the path is an infinite sequence. In our example, once the path reaches the right-most state in Figure 1 it loops in this state using action e.

Definition 5. *For any world and an arbitrary set of names $X \subseteq N$, let $Path_m(X)$ be the set of all paths $r_0, q_0, a_0, r_1, \dots$ of a Mealy machine m such that $[r_0] \in X^*$.*

Definition 6. *For any world and an arbitrary set of names $Y \subseteq N$, let $Visit_m(Y)$ be the set of all paths $r_0, q_0, a_0, r_1, \dots$ of a Mealy machine m such that $[r_i] \in Y^*$ for at least one integer $i \geq 0$.*

In other words, $Path_m(X)$ and $Visit_m(Y)$ are the sets of all paths of machine m that originate in X^* and pass through Y^* respectively.

3. Limits of Bounded-Recall Navigability

As has been stated in Section 1.3, one of the key contributions of this article is the Composition axiom for navigability with bounded-recall. This axiom states that if there is a Mealy machine of size n that can navigate from set X to set Y and a Mealy machine of size k that can navigate from set Y to set Z, then there is a Mealy machine of size $n+k$ that can navigate from set X to set Z. We formally state this axiom in Section 5 and prove its soundness in Section 7. The next theorem shows that, in general, the $n + k$ bound cannot be improved.

Theorem 1. *If the set of name N contains at least three names $\overline{x}$, $\overline{y}$, and $\overline{z}$, then for any positive integers n and k, there is a world $(R, \sim, *, A, \Delta)$, and Mealy machines m_1 and m_2 of sizes n and k, respectively, such that*

1. $Path_{m_1}(\{\overline{x}\}) \subseteq Visit_{m_1}(\{\overline{y}\})$,
2. $Path_{m_2}(\{\overline{y}\}) \subseteq Visit_{m_2}(\{\overline{z}\})$,
3. $Path_{m}(\{\overline{x}\}) \nsubseteq Visit_{m}(\{\overline{z}\})$ *for each Mealy machine m of size less than $n+k$.*

Proof. Consider the "wormhole" world $(R, \sim, *, A, \Delta)$ partially depicted in Figure 5. It consists of $n + k + 4$ states: x, $w_1, \ldots, w_n$, y, $w_{n+1}, \ldots, w_{n+k}$, z, and $\odot$. We refer to state $\odot$, which is not shown on the diagram, as the "black hole" state. States $w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+k}$ are indistinguishable. This is shown in the diagram by dashed lines. The set of actions A in this world is the set $\{0, \ldots, n+k\}$. As shown in the diagram, action 0 transitions the world from state x to state w_1 (and from state y to state w_{n+1}), action 1 from state w_1 to state w_2, etc. Furthermore, we assume that the execution of any action not shown in the diagram transitions the system into the black hole state $\odot$. In particular, any action executed in state z transitions the world into $\odot$. Let $(\overline{x})^* = [x] = \{x\}$, $(\overline{y})^* = [y] = \{y\}$, and $(\overline{z})^* = [z] = \{z\}$.

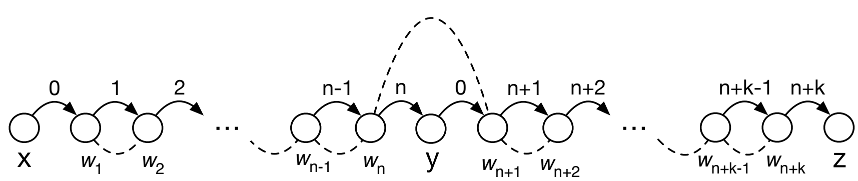


Figure 5: "Wormhole" world $(R, \sim, *, A, \Delta)$.

Figures 6 and 7 depict Mealy machines m_1 and m_2 respectively. Machine m_1 can navigate from state x to state y and machine m_2 can navigate from state y to state z. In other words, $Path_{m_1}(\{\overline{x}\}) \subseteq Visit_{m_1}(\{\overline{y}\})$ and $Path_{m_2}(\{\overline{y}\}) \subseteq Visit_{m_2}(\{\overline{z}\})$.

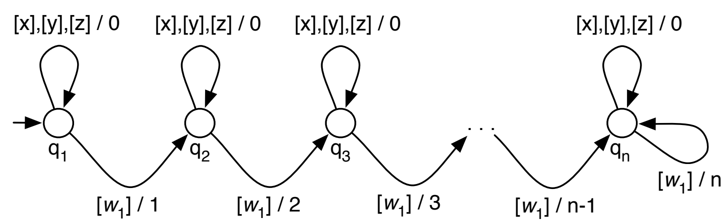


Figure 6: Mealy machine m_1.

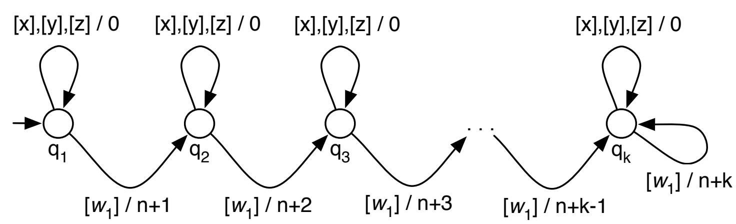


Figure 7: Mealy machine m_2.

To conclude the proof, we need to show that there is no Mealy machine m of size less that $n + k$ that can navigate from state x to state z. Suppose that such a machine $m = (Q, s, \alpha, \delta)$ exists. Consider any path

$$\begin{aligned}\pi \;\; = \;\; & x, q_0, a_0, w_1, q_1, a_1, w_2, q_2, a_2, \dots, w_n, q_n, a_n, y, \hat{q}, 0, w_{n+1}, q_{n+1}, a_{n+1}, \\ & w_{n+2}, q_{n+2}, a_{n+2}, \dots, w_{n+k}, q_{n+k}, a_{n+k}, z, \dots \in Path_m(\{\overline{x}\}).\end{aligned}$$

Since machine m has less than $n+k$ states, by the pigeonhole principle, at least two of the states $q_1, q_2, \dots, q_n, q_{n+1}, \dots, q_{n+k}$ must be the same. Let $q_i = q_j$ for some $i \neq j$. Hence, $a_i = \alpha([w_i], q_i) = \alpha([w_j], q_j) = a_j$ by item 5 of Definition 4 and because states w_i and w_j are indistinguishable. Note however, that by the design of the world $(R, \sim, *, A, \Delta)$, see Figure 5, in order for the path to pass through the wormhole, the machine must use action i in state w_i and action j in state w_j. Since $a_i = a_j$, machine m is using a wrong instruction in at least one of these two states along path π. Therefore, $\pi \notin Visit_m(\{\overline{z}\})$. $\boxtimes$

4. Syntax and Semantics

In this section we formally define the syntax and semantics of our logical system.

Definition 7. *Let Φ be the minimal set of formulae such that*

1. $X \rhd_n Y \in \Phi$ *for all nonempty sets* $X, Y \subseteq N$ *and each integer* $n \geq 1$,
2. $\varphi \to \psi, \neg\varphi \in \Phi$ *for all formulae* $\varphi, \psi \in \Phi$.

In other words, language Φ is defined by the following grammar:

$$\varphi := \neg\varphi \mid \varphi \to \varphi \mid X \rhd_n Y.$$

The next definition is one of the most important definitions of this section. It formally specifies the meaning of the relation $X \rhd_n Y$.

Definition 8. *For any world M, let $M \vDash X \rhd_n Y$ if $Path_m(X) \subseteq Visit_m(Y)$ for some Mealy machine m of size at most n.*

For example, consider sets of states X and Y shown on Figure 3. Note that all states in these sets are distinguishable, so each of them forms its own class. Let $\mathbf{X}$ and $\mathbf{Y}$ be the set of names of classes $\{[r] \mid r \in X\}$ and $\{[r] \mid r \in Y\}$ respectively. Then, $M \vDash \mathbf{X} \rhd_{9!+1} \mathbf{Y}$ because there is a Mealy machine of size $9! + 1$, namely the machine from Figure 4, such that any path of this machine originating in a class from set $\mathbf{X}^*$ will eventually pass through a class from set $\mathbf{Y}^*$.

5. Axioms

In this article we give a sound and complete axiomatization of the relation $X \rhd_n Y$. It consists of the following four axioms for all sets $X, Y, Z \subseteq N$,

1. Reflexivity: $X \rhd_n X$,
2. Augmentation: $X \rhd_n Y \to X \cup Z \rhd_n Y \cup Z$,
3. Composition: $X \rhd_n Y \to (Y \rhd_k Z \to X \rhd_{n+k} Z)$,
4. Monotonicity: $X' \rhd_n Y \to X \rhd_n Y$, if $X \subseteq X'$.

The Reflexivity axiom states that for any positive integer n and any set $X \subseteq N$ there is a Mealy machine of size at most n that can navigate from X to X. In fact, *any* machine can do this because such navigation does not require any transition at all. The Augmentation axiom says that if there is a Mealy machine that can navigate from X to Y, then there is a Mealy machine of the same size that can navigate from $X \cup Z$ to $Y \cup Z$. Indeed, the same machine that can navigate from X to Y can also navigate from $X \cup Z$ to $Y \cup Z$: if the machine starts in Z, then it is already in $Z \subseteq Y \cup Z$; otherwise, it will eventually arrive to $Y \subseteq$

$Y \cup Z$. The Composition axiom states that a machine that can navigate from X to Y and a machine that can navigate from Y to Z can be combined into a machine that can navigate from X to Z. The size of the combined machine is the sum of the sizes of the original machines. We prove the soundness of this axiom in Lemma 6. The Reflexivity, the Augmentation, and the Composition axioms, without the subscript, are known in the database theory [76, p. 81] as Armstrong's axioms [77]. The Monotonicity axiom states that if a Mealy machine can navigate from set X' to set Y, then a machine of the same size (in fact, the very same machine) can navigate from any subset $X \subseteq X'$ to Y.

We write $X \vdash \varphi$ if formula φ is derivable from the propositional tautologies, the above four axioms, and an additional set of hypotheses X using the Modus Ponens inference rule. We gave similar axiomatizations for no recall and perfect recall navigability in [23] and no recall navigability with intermediate constraints in [24].

6. Examples of Derivations

In this section we give three examples of formal proofs in our logical system. The Monotonicity axiom states that if the left argument of the relation $\rhd_n$ is replaced with a subset, then the relation remains true. Intuitively, the right argument of this relation could be replaced with a superset, but such an axiom is missing from our list. In our first example we show that this principle is derivable from the rest of the axioms of our system.

Lemma 1. *$X \rhd_n Y \to X \rhd_n Y'$, if $Y \subseteq Y'$.*

Proof. By the Augmentation axiom,

$$\vdash X \rhd_n Y \to (X \cup (Y' \setminus Y)) \rhd_n (Y \cup (Y' \setminus Y)).$$

Hence, due to the assumption $Y \subseteq Y'$,

$$\vdash X \rhd_n Y \to (X \cup (Y' \setminus Y)) \rhd_n Y'.$$

At the same time, by the Monotonicity axiom,

$$\vdash (X \cup (Y' \setminus Y)) \rhd_n Y' \to X \rhd_n Y'.$$

Therefore, $\vdash X \rhd_n Y \to X \rhd_n Y'$ by the laws of propositional reasoning. $\boxtimes$

The next two results are used later in the proof of the completeness.

Lemma 2. $\vdash X \rhd_n Y \to X \rhd_{n'} Y$, *if* $n \le n'$.

Proof. If $n = n'$, then $X \rhd_n Y \to X \rhd_{n'} Y$ is a propositional tautology. Let us now assume that $n < n'$. Thus, $\vdash Y \rhd_{n'-n} Y$ by the Reflexivity axiom. At the same time, by the Composition axiom,

$$\vdash X \rhd_n Y \to (Y \rhd_{n'-n} Y \to X \rhd_{n+(n'-n)} Y).$$

Thus, by the laws of propositional reasoning,

$$\vdash X \rhd_n Y \to X \rhd_{n+(n'-n)} Y.$$

Therefore, $\vdash X \rhd_n Y \to X \rhd_{n'} Y$. $\boxtimes$

Lemma 3. $\vdash X_1 \rhd_{n_1} Y_1 \to (X_2 \rhd_{n_2} Y_2 \to X_1 \cup X_2 \rhd_{n_1+n_2} Y_1 \cup Y_2)$.

Proof. By the Augmentation axiom, both

$$\vdash \quad X_1 \rhd_{n_1} Y_1 \to X_1 \cup X_2 \rhd_{n_1} Y_1 \cup X_2, \tag{2}$$
$$\vdash \quad X_2 \rhd_{n_2} Y_2 \to Y_1 \cup X_2 \rhd_{n_2} Y_1 \cup Y_2. \tag{3}$$

At the same time, by the Composition axiom,

$$\begin{aligned} \vdash \quad & X_1 \cup X_2 \rhd_{n_1} Y_1 \cup X_2 \\ & \to (Y_1 \cup X_2 \rhd_{n_2} Y_1 \cup Y_2 \to X_1 \cup X_2 \rhd_{n_1+n_2} Y_1 \cup Y_2). \end{aligned} \tag{4}$$

Statements (2), (3), and (4) imply

$$\vdash X_1 \rhd_{n_1} Y_1 \to (X_2 \rhd_{n_2} Y_2 \to X_1 \cup X_2 \rhd_{n_1+n_2} Y_1 \cup Y_2)$$

by the laws of propositional reasoning. $\boxtimes$

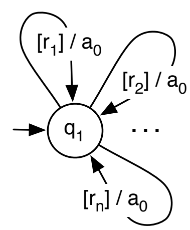
7. Soundness

Theorem 2 (soundness). *If* $\vdash \varphi$, *then* $M \vDash \varphi$ *for any world* M.

We prove the soundness of each of the four axioms of our system as four separate lemmas.

Lemma 4 (Reflexivity). *$M \vDash X \rhd_n X$, for any set $X \subseteq N$, any integer $n \geq 1$, and any world M.*

Proof. By Definition 1, set A contains at least one action a_0. Let $[r_1], [r_2], \dots, [r_n]$ be the equivalence classes of states of world M. Consider the following single-state Mealy machine m that uses action a_0 in all states of the world:



By Definition 8, it suffices to show that $Path_m(X) \subseteq Visit_m(X)$. Indeed, consider a path $\pi = r'_0, q_0, a_0, r'_1, \cdots \in Path_m(X)$. Note that $[r'_0] \in X^*$ by Definition 5. Hence, $\pi \in Visit_m(X)$ by Definition 6. $\boxtimes$

Lemma 5 (Augmentation). *If $M \vDash X \rhd_n Y$, then $M \vDash X \cup Z \rhd_n Y \cup Z$, for all sets $X, Y, Z \subseteq N$, any integer $n \geq 1$, and any world M.*

Proof. By Definition 8, it suffices to show that if $Path_m(X) \subseteq Visit_m(Y)$, then

$$Path_m(X \cup Z) \subseteq Visit_m(Y \cup Z).$$

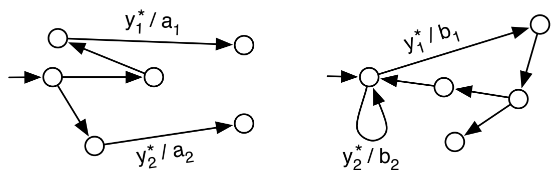
Consider a path $\pi = r_0, q_0, a_0, r_1, \cdots \in Path_m(X \cup Z)$. If $r_0 \in Z$, then $\pi \in Visit_m(Z) \subseteq Visit_m(Y \cup Z)$ by Definition 6.

Suppose $r_0 \notin Z$. Thus, the assumption $\pi \in Path_m(X \cup Z)$ implies that $\pi \in Path_m(X)$. Hence, $\pi \in Visit_m(Y)$ due to the assumption $Path_m(X) \subseteq Visit_m(Y)$. Therefore, $\pi \in Visit_m(Y \cup Z)$ by Definition 6. $\boxtimes$

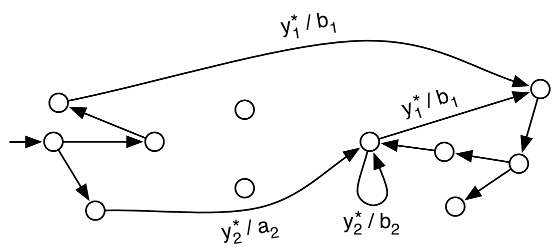
Lemma 6 (Composition). *If $M \vDash X \rhd_n Y$ and $M \vDash Y \rhd_k Z$, then $M \vDash X \rhd_{n+k} Z$, for all sets $X, Y, Z \subseteq N$, all integers $n, k \geq 1$, and any world M.*

Proof. By Definition 8, assumptions $M \vDash X \rhd_n Y$ and $M \vDash Y \rhd_k Z$ imply that there are Mealy machines m_1 and m_2 of sizes at most n and k with initial states q_1 and q_2 respectively, such that $Path_{m_1}(X) \subseteq Visit_{m_1}(Y)$ and $Path_{m_2}(Y) \subseteq$

$Visit_{m_2}(Z)$. By Definition 8, it suffices to construct a Mealy machine m with at most $n + k$ states such that $Path_m(X) \subseteq Visit_m(Z)$. We construct machine m by building a Mealy machine that simulates machine m_1 (from its initial state q_1) till the moment it reaches a state in set Y^* for the first time, and then simulates machine m_2 (from its initial state q_2) indefinitely. For any two given Mealy machines m_1 and m_2:



Mealy machine m could be constructed using steps 1 through 5 as following:



1. choose a transition of Mealy machine m_1 labeled by y^*/a where $y \in Y$ and $a \in A$,
2. identify a transition of Mealy machine m_2 from its *initial state* q_2, labeled by y^*/b for the same y as above and for some action $b \in A$,
3. redirect the transition of machine m_1, as identified in step 1, to lead to the ending state of the transition of machine m_2, as identified in step 2,
4. repeat steps 1 through 3 for each transition of machine m_1 labeled by y^*/a where $y \in Y$ and $a \in A$,
5. make state q_1 to be the initial state of machine m.

Therefore, $M \vDash X \rhd_{n+k} Z$. $\boxtimes$

Lemma 7 (Monotonicity). *If $M \vDash X' \rhd_n Y$ and $X \subseteq X'$, then $M \vDash X \rhd_n Y$, for all sets $X, X', Y \subseteq N$, any integer $n \geq 1$, and any world M.*

Proof. By Definition 8, assumption $M \vDash X' \rhd_n Y$ implies that there is a Mealy machine m of size at most n such that $Path_m(X') \subseteq Visit_m(Y)$. At the same time, by Definition 5, assumption $X \subseteq X'$ implies that $Path_m(X) \subseteq Path_m(X')$. Thus, $Path_m(X) \subseteq Path_m(X') \subseteq Visit_m(Y)$. Therefore, $M \vDash X \rhd_n Y$ by Definition 8. ⊠

8. Completeness

In this section we prove the strong completeness of our logical system. This proof is a non-trivial generalization of the proof of Theorem 1. While the proof of that theorem constructed a counterexample for formula

$$\{\bar{x}\} \rhd_n \{\bar{y}\} \to (\{\bar{y}\} \rhd_k \{\bar{z}\} \to \{\bar{x}\} \rhd_{n+k-1} \{\bar{z}\}),$$

in this section we construct a counterexample for an arbitrary formula not provable in our logical system.

We start the proof by fixing an arbitrary subset Φ_0 of the set of all formulae Φ and a maximal consistent subset Ω of set Φ_0. Later, in the proof of the strong completeness we will choose Φ_0 to be the whole set Φ and in the proof of the completeness with respect to finite models we will choose Φ_0 to be a specific finite subset of Φ. In the proof of Theorem 1 a rough equivalent of set Ω is the singleton set of formulae $\{\{\bar{x}\} \rhd_n \{\bar{y}\} \to (\{\bar{y}\} \rhd_k \{\bar{z}\} \to \{\bar{x}\} \rhd_{n+k-1} \{\bar{z}\})\}$.

Next, we define the canonical world $M(\Phi_0, \Omega) = (R, \sim, *, A, \Delta)$.

8.1. Canonical World

The world on Figure 5 can be viewed as consisting of three types of states: main states x, y, and z, a "black hole" state (not shown on the figure), and auxiliary states $w_1, \dots, w_n, w_{n+1}, \dots, w_{n+k}$ that form two wormholes. Informally, the world in the canonical model also consists of the same three types of states: a distinct "main" state for each element of set N, a "black hole" state $\odot$ that has no way out, and a set of auxiliary states whose sequences form passages ("wormholes") between states of the first type. If set Ω contains a formula $X \rhd_n Y$, then the canonical world has a wormhole (see Figure 8) of size n that can be used to transition from *each* main state in set $X \subseteq N$ into *each* main state in set $Y \subseteq N$. By size $|W|$ of a wormhole W we mean the number of auxiliary states in this wormhole. The wormholes that we introduce here generalize the wormholes from Section 3. Namely, Figure 5 depicts wormholes between single states: one wormhole between states x and y

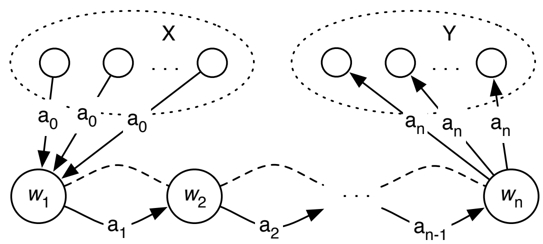


Figure 8: Wormhole W consists of the auxiliary states $w_1, \dots, w_n$. In this diagram, a_i stands for $a_i(X, n, Y)$.

and another between states y and z. Here we consider wormholes between *sets* of states X and Y. In the formal definition below, we choose *distinct* elements $w_i(X, n, Y)$ for each $X, Y \subseteq N$ and each integer i.

Definition 9. *The set of states R of the canonical world $M(\Omega)$ is the disjoint union*

$$N \cup \{\odot\} \cup \{w_i(X, n, Y) \mid X \rhd_n Y \in \Omega, 1 \leq i \leq n\}.$$

Note that set R might be infinite. Just like in Figure 5, we assume that all states in the world are distinguishable except for the auxiliary states that form wormholes. Two auxiliary states are indistinguishable no matter if they come from the same or different wormholes.

Definition 10. *For any two states $x, y \in R$, let $x \sim y$ if either $x = y$ or $x, y \notin N \cup \{\odot\}$.*

Recall that, in general, set N is the set of names of indistinguishability classes. In the case of the canonical model, set N also serves as the set of "main" states in the world. Since each main state is distinguishable from the other states of the world, any such state r forms its own indistinguishability class $[r]$. We connect these two meanings of the elements of set N by defining the name $r \in N$ to be used as a name of class $[r]$:

Definition 11. $*(r) = [r]$ *for each $r \in N$.*

In Figure 5, to advance from state x to state z a machine needs to use different actions in each state (even if the states are located in different wormholes) with the exception of the same action 0 used to advance from state x to state w_1 and from state y to state w_{n+1}. It is important for the proof of part 3 of Theorem 1 that the machine has to use different actions in wormhole states. At the same time, we have chosen to use the same action 0 for states x and y for the sake of simplicity. In the case of the canonical world the construction is very similar except that we now require all actions to be different without any exceptions. This change is important because there can be multiple wormholes starting from the same "main" state of the world and we need a way for the machine to specify which wormhole it wants to enter.

Consider the wormhole of the world that corresponds to a formula $X \rhd_n Y \in \Omega$. The canonical world has a distinct action $a_0(X, n, Y)$ to get from X to the first auxiliary state in the wormhole, a distinctive action $a_i(X, n, Y)$ to advance from i-th state of the wormhole to the next one, and a distinctive action $a_n(X, n, Y)$ that leads out of the wormhole into a randomly selected state of set Y. These actions are distinct not only for passages within the given wormhole, but they also differ from one wormhole to another.

Definition 12. $A = \{a_i(X, n, Y) \mid 0 \leq i \leq n, \Omega \vdash X \rhd_n Y\}$.

Definition 13. *For any wormhole W corresponding to a formula $X \rhd_n Y \in \Omega$, actions $a_i(X, n, Y)$ advance the machine through the wormhole as specified above. If this action is executed in a state which is not the i-th state of wormhole W, then the system transitions into the black hole state $\odot$. Any action executed in state $\odot$ transitions the system back into the same state $\odot$.*

This concludes the definition of the canonical world $M(\Phi_0, \Omega)$.

Lemma 8. *World $M(\Phi_0, \Omega)$ is finite for any finite set Φ_0.* $\boxtimes$

8.2. Properties of Wormholes

If wormhole W corresponds to formula $X \rhd_n Y \in \Omega$, then we refer to sets X and Y as $In(W)$ and $Out(W)$ respectively.

Lemma 9. $\Omega \vdash In(W) \rhd_{|W|} Out(W)$.

Proof. Let wormhole W correspond to a formula $X \rhd_n Y \in \Omega$. Thus, $In(W) = X$, $Out(W) = Y$, and $|W| = n$. Therefore, $\Omega \vdash In(W) \rhd_{|W|} Out(W)$ by the assumption $X \rhd_n Y \in \Omega$. $\boxtimes$

Lemma 10. *Sets $In(W)$ and $Out(W)$ are nonempty.*

Proof. Let wormhole W correspond to formula $X \rhd_n Y \in \Phi$. Sets X and Y are nonempty by Definition 7. Therefore, sets $In(W)$ and $Out(W)$ are nonempty. $\boxtimes$

As usual, the key step in the proof of the completeness is the "truth" or the induction lemma that establishes the connection between the maximal consistent set of formulae Ω and the canonical world $M(\Phi_0, \Omega)$. In our case, this is Lemma 22. Since language Φ of our logical system consists of atomic propositions $X \rhd_n Y$ and Boolean connectives only, the hardest step in proving Lemma 22 is the base case. We divided the base case into two separate statements: Lemma 11 and Lemma 21.

The proof of Lemma 11 is significantly simpler than the proof of Lemma 21 because we constructed the canonical world in such a way that if $X \rhd_n Y \in \Omega$, then the world has a wormhole of size n from set X to set Y. Thus, the proof of Lemma 11 comes down to constructing a Mealy machine of size n that can navigate through this wormhole. We call this machine "wormhole navigator".

The proof of Lemma 11 also closely resembles the proof of parts 1 and 2 of Lemma 1. The wormhole navigator, depicted in Figure 9, is a straightforward modification of Mealy machines m_1 and m_2 from Figures 6 and 7 respectively.

Lemma 11. *If $X \rhd_n Y \in \Omega$, then $M(\Phi_0, \Omega) \vDash X \rhd_n Y$.*

Proof. Suppose that $X \rhd_n Y \in \Omega$. By Definition 8, it suffices to show that there is a Mealy machine m with n states such that $Path_m(X) \subseteq Visit(Y)$. Let m be the Mealy machine depicted in Figure 9, where N_- is the set $N^* \setminus \{[w_1]\}$, $a_i =$

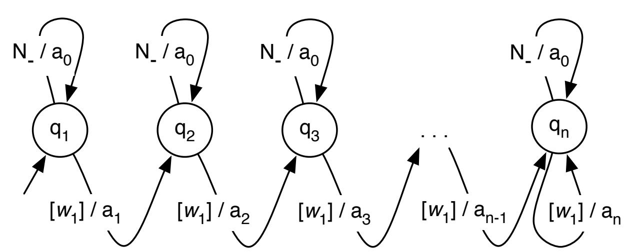


Figure 9: Wormhole Navigator m.

$a_i(X, n, Y)$ for each i such that $0 \le i \le n$, and $w_1 = w_1(X, n, Y)$ (recall that $[w_1] = [w_i(X, n, Y)]$ for each i such that $2 \le i \le n$). In other words, machine m has states $q_1, \dots, q_n$, where q_1 is the starting state. If the machine is outside of the "wormhole" corresponding to the formula $X \rhd_n Y$, it applies the "default" action a_0 and remains in the same state. If the machine is in the "wormhole", then in state q_i it applies action a_i and transitions into state q_{i+1}, unless $i = n$, in which case it still applies action a_i but remains in state q_i.

By the definition of the world $M(\Phi_0, \Omega)$ and the choice of Mealy machine m, any path in set $Path_m(X)$ has the form: $x, q_1, a_0, w_1, q_1, a_1, w_2, q_2, a_2, \dots, a_{n-1}, w_n, q_n, a_n, y, q_n, \dots$ where $x \in X^*$, $y \in Y^*$. Thus, $Path_m(X) \subseteq Visit_m(Y)$. $\boxtimes$

8.3. Outline of the Proof of Lemma 21

Statement of Lemma 21 resembles part 3 of Theorem 1. However, the proof of Lemma 21 is significantly more complicated than the proof of Theorem 1. In fact, the proof of this lemma occupies most of the remaining part of this article. The complexity of the proof rises from the fact that world $M(\Phi_0, \Omega)$ has a much larger size and a much more complicated structure than the linear world consisting of just two wormholes depicted in Figure 5.

Just like in the case of part 3 of Theorem 1, the proof of Lemma 21 is based on the pigeonhole principle. Note that in the proof of Theorem 1 we applied this principle to the set of all wormhole states in Figure 5. However, this would not be useful in our case because a Mealy machine does not have to pass through all states in the canonical world $M(\Phi_0, \Omega)$ in order to navigate from set X to set Y. Instead, in this proof we use the pigeonhole principle in Lemma 12 to put the limit on the total size of all wormholes that are passable (can be navigated through) by a given Mealy machine.

Next, we define a chain of sets of states $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ for a given set Y and a given Mealy machine. Informally, set Y_i consists of all states of the world from which the Mealy machine is guaranteed to navigate to set Y using at most i passable wormholes. Note that if the Mealy machine can navigate from set X to set Y, then it must be true that $X \subseteq \bigcup_i Y_i$.

The introduction of the chain $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ allows us to use induction to combine the statements of Lemma 9 for each individual passable wormhole into the claim that $\Omega \vdash \bigcup_i Y_i \rhd_n Y$. Therefore, $\Omega \vdash X \rhd_n Y$ by the Monotonicity axiom, just like Lemma 21 states.

The detailed proof of Lemma 21 is given in the next two sections. The first of them introduces passable wormholes and the chain $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ for an arbitrary Mealy machine and an arbitrary set Y of states in the canonical world. It also proves key facts about these notions. The proof of Lemma 21 and the remainder of the proof of the completeness is given in Section 8.5.

8.4. Passable Wormholes and Sets Y_i

In this section we fix a nonempty set Y of states in a canonical world $M(\Phi_0, \Omega)$ and a Mealy machine m of size at most n. We define passable wormholes, the rank of a passable wormhole, and prove their basic properties.

Definition 14. *Wormhole $W = (w_1, \dots, w_k)$ is "passable" if there is a state q of Mealy machine m such that m, starting in machine state q, can navigate from world state w_1 into the set of world states $Out(W)$. The set of all passable wormholes is denoted by P.*

For any set of wormholes X, let $\|X\| = \sum_{W \in X} |W|$.

Lemma 12. *$\|P\| \le n$.*

Proof. By Definition 13, to navigate through a wormhole W from a state in set $In(W)$ to a state in set $Out(W)$, machine m must pass through all states in the wormholes and leave each state of the wormhole using a state-specific action $a_i(In(W), n, Out(W))$. By Definition 4, the action of a machine is determined (via function δ) by the state of the machine and the state of the world the machine is in. By Definition 10, the states in all wormholes are indistinguishable. Thus, the action in a wormhole state is determined only by the state of the machine. Hence, by the pigeonhole principle, the total number of states in all passable wormholes cannot be more than the number of states of machine m. Therefore, $\|P\| \le n$. $\boxtimes$

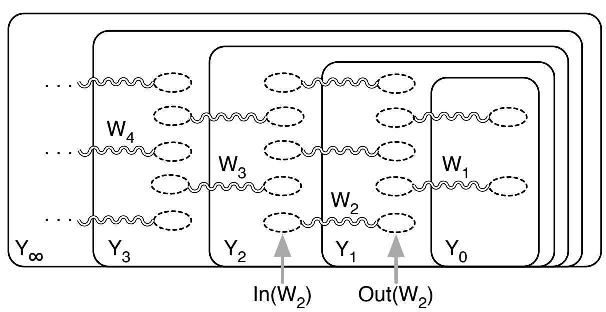


Figure 10: Sets $Y = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots Y_\infty$ and function $rank(W)$. Wormholes W_1, W_2, W_3, and W_4 have rank 1, 2, 3, and 4 respectively.

We now define the chain of sets of states $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$, see Figure 10. Informally, set Y_i consists of all wormholes W such that set Y_0 is reachable from $In(W)$ through at most i wormholes.

Definition 15. *For any* $i \geq 0$*, let*

$$Y_i = \bigcup \left\{ In(W) \,\middle|\, W \in P, Out(W) \subseteq \bigcup_{j<i} Y_j \right\} \cup Y.$$

Lemma 13. $Y_0 = Y$.

Proof. The statement of the lemma follows from Definition 15 because by Lemma 10 set $Out(W)$ is not empty for each wormhole W. $\boxtimes$

Definition 16. $Y_\infty = \bigcup_{i \geq 0} Y_i$.

We define the rank of an arbitrary passable wormhole W, see Figure 10, to be the smallest $r \geq 1$ such that set $Out(W)$ is a subset of $Y_0 \cup Y_1 \cup \cdots \cup Y_{r-1}$. If such an integer r does not exist, then the rank is not defined.

Definition 17. *For any passable wormhole* W*,*

$$rank(W) = \min \left\{ r \geq 1 \mid Out(W) \subseteq Y_{r-1} \right\}.$$

Definition 18. *For any integer* $r \geq 1$*, set* P_r *is the set of all passable wormholes of rank* r*.*

Lemma 14. $Y_i = Y_{i-1} \cup \left(\bigcup_{W \in P_i} In(W) \right)$ *for each* $i \geq 1$.

Proof. The statement of the lemma follows from Definition 15, Definition 17, and Definition 18. $\boxtimes$

Lemma 15. *There is an integer* $k \geq 0$ *such that* $Y_k = Y_\infty$.

Proof. By Definition 15, sets $Y_0, Y_1, \cdots \subseteq N$ form an increasing chain $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$. Note that set N is finite by the assumption in the Section 4. Thus, there is an integer $k \geq 0$ such that $Y_k = Y_{k+1} = Y_{k+2} = \ldots$. Therefore, $Y_\infty = Y_k$. $\boxtimes$

Let $\ell \geq 0$ be the minimal integer such that $Y_\ell = Y_\infty$. Such an integer exists by Lemma 15.

Lemma 16. $\|P_i\| > 0$ *for each i such that* $1 \leq i \leq \ell$.

Proof. Suppose that $\|P_i\| = 0$ for some i such that $1 \leq i \leq \ell$. Note that, by Definition 9, the length of each wormhole is at least 1. Thus, set P_i is empty. Then, $Y_i = Y_{i-1}$ by Lemma 14. Hence, $Y_{i-1} = Y_i = Y_{i+1} = ...$ by Definition 15. Thus, $Y_\infty = Y_{i-1}$. At the same time $i - 1 < i \leq \ell$, which contradicts the choice of ℓ as the minimal integer such that $Y_\ell = Y_\infty$. $\boxtimes$

We are now ready to prove that $\Omega \vdash \bigcup_i Y_i \rhd_n Y$. As we discussed in Section 8.3, this is a core fact in the proof of Lemma 21. We show this fact by induction. The next lemma is an auxiliary lemma for the induction. Lemma 18 carries out the induction itself. Lemma 19 rephrases the result of Lemma 18 into the required form.

Lemma 17. $\Omega \vdash Y_i \rhd_{\|P_i\|} Y_{i-1}$ *for each i such that* $1 \leq i \leq \ell$.

Proof. Lemma 16 implies that $P_i = \{W_1, W_2, \dots, W_k\}$ for some $k \geq 1$. Then $\Omega \vdash In(W_j) \rhd_{|W_j|} Out(W_j)$ by Lemma 9, for each $j \leq k$. Thus, by Lemma 3 applied $k - 1$ times,

$$\Omega \vdash \bigcup_{j\leq k} In(W_j) \rhd_{|W_1|+|W_2|+\cdots+|W_k|} \bigcup_{j\leq k} Out(W_j).$$

Hence, by the choice of $W_1, W_2, \dots, W_k$,

$$\Omega \vdash \bigcup_{j\leq k} In(W_j) \rhd_{\|P_i\|} \bigcup_{j\leq k} Out(W_j).$$

Then, by the Augmentation axiom,

$$\Omega \vdash Y_{i-1} \cup \left(\bigcup_{j\leq k} In(W_j)\right) \rhd_{\|P_i\|} Y_{i-1} \cup \left(\bigcup_{j\leq k} Out(W_j)\right).$$

Thus, by Lemma 14,

$$\Omega \vdash Y_i \rhd_{\|P_i\|} Y_{i-1} \cup \left(\bigcup_{j\leq k} Out(W_j)\right).$$

At the same time, $Out(W) \subseteq Y_{i-1}$ for each wormhole $W \in P_i$ by Definition 17. Therefore, $\Omega \vdash Y_i \rhd_{\|P_i\|} Y_{i-1}$ by the choice of wormholes $W_1, W_2, \dots, W_k \in P_i$. $\boxtimes$

Lemma 18. $\Omega \vdash Y_i \rhd_{\|P_1\|+\dots+\|P_i\|} Y$ *for each integer i such that* $1 \le i \le \ell$.

Proof. We prove this statement by induction on integer i. Suppose that $i = 1$. Thus, $\Omega \vdash Y_i \rhd_{\|P_i\|} Y_0$ by Lemma 17. Therefore, $\Omega \vdash Y_i \rhd_{\|P_i\|} Y$ by Lemma 13.

By the induction hypothesis, $\Omega \vdash Y_{i-1} \rhd_{\|P_1\|+\dots+\|P_{i-1}\|} Y$. At the same time $\Omega \vdash Y_i \rhd_{\|P_i\|} Y_{i-1}$ by Lemma 17. Therefore, $\Omega \vdash Y_i \rhd_{\|P_1\|+\dots+\|P_i\|} Y$, by the Composition axiom. ⊠

Lemma 19. $\Omega \vdash Y_\infty \rhd_n Y$.

Proof. If $\ell = 0$, then $Y_\infty = Y_\ell = Y_0 = Y$ by Lemma 13 and the choice of ℓ. Thus, $\Omega \vdash Y_\infty \rhd_n Y$ by the Reflexivity axiom.

Suppose now that $\ell > 0$. By Lemma 12 and because sets $P_1, \dots, P_\ell$ are disjoint, $\|P_1\| + \dots + \|P_\ell\| \le \|P\| \le n$. Therefore, $\Omega \vdash Y_\infty \rhd_n Y$ by Lemma 18 and Lemma 2. ⊠

Informally, the next lemma states there is no way to enter set Y_∞ from outside of this set using a passable wormhole.

Lemma 20. *If $Out(W) \subseteq Y_\infty$, then $In(W) \subseteq Y_\infty$, for any passable wormhole W.*

Proof. Recall that $Y_\infty = Y_\ell$ by the choice of integer ℓ. Hence, $Out(W) \subseteq Y_\ell$. Thus, $rank(W) \le \ell$ by Definition 17. Then, $In(W) \subseteq Y_{\ell+1}$ by Lemma 14. Therefore, $In(W) \subseteq Y_\infty$ by Definition 16. ⊠

8.5. Completeness: Final Steps

Recall that the previous section was the preparation for the proof of the following lemma. This lemma can be viewed as a very general form of the contrapositive of part 3 of Theorem 1. We use this lemma as one part of the base case in the induction proof of Lemma 22.

Lemma 21. *If $X \rhd_n Y \in \Phi_0$ and $M(\Phi_0, \Omega) \vDash X \rhd_n Y$, then $X \rhd_n Y \in \Omega$.*

Proof. By Definition 8, the assumption $M(\Phi_0, \Omega) \vDash X \rhd_n Y$ implies that there is a Mealy machine m of size at most n such that $Path_m(X) \subseteq Visit_m(Y)$. Let sets $\{Y_i\}_i$, Y_∞, P, and $\{P_i\}_i$ be defined for set Y and Mealy machine m as specified in Section 8.4. We consider the following two cases separately:

Case I: $X \subseteq Y_\infty$. Thus, $\Omega \vdash X \rhd_n Y$ by Lemma 19 and the Monotonicity axiom. Recall that $X \rhd_n Y \in \Phi_0$ by the assumption of the lemma. Therefore, $X \rhd_n Y \in \Omega$ because Ω is a maximal consistent subset of Φ_0.

Case II: $X \nsubseteq Y_\infty$. Consider any $x_0 \in X \setminus Y_\infty$. If Mealy machine m starts in state x_0 and in the initial state q_1 of the machine, then, by Definition 13, only one of the following three cases take place, see Figure 11:

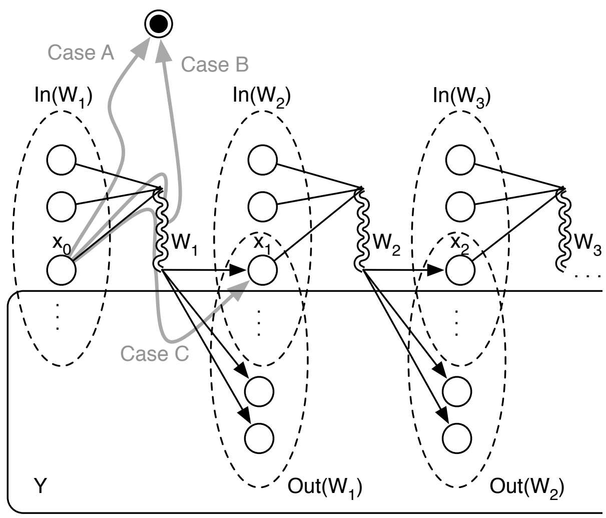


Figure 11: Towards proof of Lemma 21.

Case A: machine m transitions from state x_0 and initial state q_1 into black hole state $\odot$. By line 5 of Definition 1, the machine will be able to continue making transitions indefinitely. However, by Definition 13, the machine will never leave state $\odot \notin Y$. Thus, $Path_m(X) \nsubseteq Visit_m(Y)$, because the states of the wormholes are not in set N and thus these states do not belong to set Y. Statement $Path_m(X) \nsubseteq Visit_m(Y)$ contradicts the choice of machine m.

Case B: machine m transitions from state x_0 and initial state q_1 into a wormhole W_1, but exits the wormhole before reaching set $Out(W_1)$. In this case, by Definition 13, machine m must exit the wormhole into black hole state $\odot$. Just like in case A, this means that $Path_m(X) \nsubseteq Visit_m(Y)$, which again contradicts the choice of machine m.

Case C: machine m transitions from state x_0 and initial state q_1 into a wormhole W_1, navigates through the wormhole, and exits it into set $Out(W_1)$. Lemma 20 implies that $Out(W_1) \nsubseteq Y_\infty$ because $x_0 \in In(W_1) \setminus Y_\infty$. Consider any state $x_1 \in$

$Out(W_1) \setminus Y_\infty$. Thus, wormhole W is passable. By Definition 13, machine *m might* exit from wormhole W_1 into state x_1. Repeat the steps above ad infinitum to either (a) construct a path that goes through finitely many states $x_0, x_1, \dots, x_m \notin Y_\infty$ and finitely many wormholes $W_1, W_2, \dots, W_m$ and then reaches black hole state $\circledcirc$ to remain there forever, or (b) construct a path that goes through infinitely many states $x_0, x_1, \dots, \notin Y_\infty$ and infinitely many wormholes $W_1, W_2, \dots$ never reaching set Y_0. In either case, $Path_m(X) \nsubseteq Visit_m(Y)$. ⊠

Lemma 22. *$M(\Phi_0, \Omega) \vDash \varphi$ iff $\varphi \in \Omega$, when set Φ_0 is closed with respect to subformulae and $\varphi \in \Phi_0$.*

Proof. We prove the statement of the lemma by structural induction on the complexity of formula φ. The base case follows from Lemma 11 and Lemma 21. The induction step follows from Definition 8, the maximality, and the consistency of set Ω in the standard way. ⊠

Next, we state and prove a strong completeness theorem for our logical system.

Theorem 3. *If $X \nvdash \varphi$, then there is a world M such that $M \vDash \chi$ for each $\chi \in X$ and $M \nvDash \varphi$.*

Proof. Suppose $\nvdash \varphi$. Thus, set $X \cup \{\neg\varphi\}$ is consistent. By Lindenbaum's lemma for propositional logic [78, Proposition 2.14], this set has a maximal consistent extension $\Omega \subseteq \Phi$. Then, $\varphi \notin \Omega$ due to the consistency of set Ω. Let $\Phi_0 = \Phi$. Therefore, $M(\Phi_0, \Omega) \vDash \chi$ for each $\chi \in X$ and $M(\Phi_0, \Omega) \nvDash \varphi$ by Lemma 22. ⊠

9. Finite Completeness and Decidability

Recall that a world $(R, \sim, *, A, \Delta)$ is *finite* if sets R and A are finite. In this section we prove (weak) completeness of our logical system with respect to finite worlds and decidability of the set of all theorems of our system.

Theorem 4. *If $\nvdash \varphi$, then there is a finite world M such that $M \nvDash \varphi$.*

Proof. Suppose that $\nvdash \varphi$. Let Φ_0 be the finite set of all subformulae of formula $\neg\varphi$ and set Ω be a maximal consistent subset for Φ_0 such that $\neg\varphi \in \Omega$. Then, $\varphi \notin \Omega$ because set Ω is consistent. Thus, $M(\Phi_0, \Omega) \nvDash \varphi$ by Lemma 22. World $M(\Phi_0, \Omega)$ is finite by Lemma 8. ⊠

Theorem 5. *Set $\{\varphi \in \Phi \mid\ \vdash \varphi\}$ is decidable.*

Proof. This set is recursively enumerable because it is axiomatizable. The complement of the set is recursively enumerable by Theorem 4. Therefore, the set is decidable. $\boxtimes$

10. Conclusion

The contribution of this article is three-fold. First, we observe that if one needs m_1 bits of memory to navigate from X to Y and m_2 bits of memory to navigate from Y to Z, then one can navigate from X to Z using $\lceil \log_2(2^{m_1} + 2^{m_2}) \rceil$ bits of memory. Second, we show that this result cannot be improved. Third, we describe all properties of bounded-recall plans. We do this by giving a sound and complete logical system that captures all properties of the navigability relation. This work is a non-trivial extension of our previous papers [23] and [24], where we considered no recall and perfect recall strategies.

In the future, we plan to study the properties of bounded-recall navigability in multi-agent settings. One can consider coalitions of agents in which each agent has a limited working memory, but perhaps it is even more interesting to study the properties of coalitions with bounded shared memory.

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