

Portfolio Selection in Quantile Decision Models ^{*}

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Abstract

This paper develops a model for optimal portfolio allocation for an investor with quantile preferences, i.e., who maximizes the τ -quantile of the portfolio return, for $\tau \in (0, 1)$. Quantile preferences allow to study heterogeneity in individuals' portfolio choice by varying the quantiles, and have a solid axiomatic foundation. Their associated risk attitude is captured entirely by a single dimensional parameter (the quantile τ), instead of the utility function. We formally establish the properties of the quantile model. The presence of a risk-free asset in the portfolio produces an all-or-nothing optimal response to the risk-free asset that depends on investors' quantile preference. In addition, when both assets are risky, we derive conditions under which the optimal portfolio decision has an interior solution that guarantees diversification vis-à-vis fully investing in a single risky asset. We also derive conditions under which the optimal portfolio decision is characterized by two regions: full diversification for quantiles below the median and no diversification for upper quantiles. These results are illustrated in an exhaustive simulation study and an empirical application using a tactical portfolio of stocks, bonds and a risk-free asset. The results show heterogeneity in portfolio diversification across risk attitudes.

Keywords: Optimal Asset Allocation, Quantile Preferences, Portfolio Theory, Risk Attitude.

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1 Introduction

Portfolio selection is a fundamental topic in economics and finance and one of the leading applications of decision theory under uncertainty. Modern portfolio theory derives its main results on diversification and risk under the important paradigm of the expected utility (EU) theory; see, for instance, [Cochrane \(2005\)](#) and [Campbell \(2018\)](#). Nevertheless, the EU framework has been subjected to a number of criticisms, mostly arising from experimental evidence.¹

In this paper we depart from the EU framework and investigate the optimal portfolio allocation of individuals concerned with maximizing a specific quantile of the distribution of portfolio returns. This individual’s behavior is motivated from both practical and theoretical points of view. Regarding the former, quantiles have been used in practical decision making in banking and investment (in the form of Value-at-Risk² and goal-reaching problems) and in mining, oil and gas industries (in the form of “probabilities of exceeding”³ a certain level of production). On the latter motivation, there has been increasing theoretical, empirical and experimental interest in decision under uncertainty under quantile preferences (QP). This alternative preference has been characterized in early work by [Manski \(1988\)](#), who studied properties of a quantile model for individual’s behavior. More recently, QP have been formally axiomatized by [Chambers \(2009\)](#), [Rostek \(2010\)](#), and [de Castro and Galvao \(2021\)](#). [Mendelson \(1987\)](#) introduced the concept of quantile-preserving spread, which is a notion of risk aversion for the quantile model that establishes a parallelism with mean-preserving spreads in the standard EU framework. [Chambers \(2007\)](#) considered the problem of aggregating a profile of interpersonally comparable utilities into a social utility and provides a quantile representation of such social aggregation. [Bhattacharya \(2009\)](#) studies the problem of optimally dividing individuals into peer groups to maximize a quantile of social gains from heterogeneous peer effects. [Giovannetti \(2013\)](#) modeled a two-period economy with one risky and one risk-free asset, where the agent has QP instead of the standard EU. [de Castro and Galvao \(2019\)](#) developed a dynamic model of rational behavior under uncertainty, in which the agent maximizes a stream of the future quantile utilities. From an experimental point of view, [de Castro et al. \(2020\)](#) shown evidence that the behavior of between 30 and 50% of the individuals in their experiment can be better described with QP rather than EU.

Overall, QP have several attractive features.⁴ An individual’s decision is independent of

¹Some studies suggesting that individuals did not always employ objective probabilities resulted in, among others, Prospect Theory ([Kahneman and Tversky, 1979](#)), Rank-Dependent Expected Utility Theory ([Quiggin, 1982](#)), Cumulative Prospect Theory ([Tversky and Kahneman, 1992](#)), Regret ([Bell, 1982](#)), and Ambiguity Aversion ([Gilboa and Schmeidler, 1989](#)). [Rabin \(2000\)](#) criticized EU theory arguing that EU would require unreasonably large levels of risk aversion to explain the data from some small-stakes laboratory experiments. See also [Simon \(1979\)](#), [Tversky and Kahneman \(1981\)](#), [Payne et al. \(1992\)](#) and [Baltussen and Post \(2011\)](#) as examples providing experimental evidence on the failure of the EU paradigm.

²See, e.g., [Duffie and Pan \(1997\)](#) and [Jorion \(2007\)](#). The VaR measure is one of the main practical tools for reporting the exposure to risk by financial institutions.

³See, e.g., [Apiwatcharoenkul et al. \(2016\)](#) and [Fanchi and Christiansen \(2017\)](#).

⁴[Rostek \(2010\)](#) discusses other advantages of the QP, such as robustness, ability to deal with categorical

the form of her utility function and thus an optimal choice is relatively easy to compute.⁵ The measure of risk aversion is simple, intuitive, and determined by the quantile $\tau \in (0, 1)$. The increasing interest on this recent approach to modeling individuals' behavior under uncertainty suggests that it is important to understand portfolio choice in this context, and this paper fulfills this gap.

Our study of portfolio choice begins with the observation that the individuals' risk attitude under QP is captured by a single-dimensional parameter, the quantile $\tau \in (0, 1)$. The lower the τ , the more averse to risk the τ -quantile-maximizer decision maker (τ -DM) is. Next we formally establish properties of the quantile model. First, we focus on a simple portfolio given by a risk-free and a risky asset. In contrast to the capital market line characterizing the mutual fund separation theorem in a mean-variance setting (Tobin, 1958), the optimal portfolio allocation under QP is to fully invest on the risk-free asset for QP given by τ below the magnitude of the risk-free rate, or to fully invest on the risky asset, otherwise. The extension of the portfolio to considering a risk-free asset and two risky assets provides similar insights indicating an optimal binary response with respect to the risk-free asset. However, in this case, we find that diversification between the two risky assets may also be an optimal outcome for middle quantiles even if the allocation to the risk-free asset is null.

The optimal portfolio choice problem is then extended to consider a portfolio of two risky assets. We formally show that a τ -DM always diversifies (invest in both assets) if the distribution functions of the assets in the portfolio have same lower end (worst-case scenario) and τ is sufficiently low. In contrast, if τ is sufficiently high, we find that there is no diversification at all: the τ -DM only invests on the riskier asset. We illustrate these rich heterogeneous behaviors and theoretical insights with examples of two uniform random variables, a case in which we are able to obtain an analytical solution of the portfolio selection problem. We then provide further conditions under which the optimal portfolio decision has an interior solution with diversification vis-à-vis no diversification. The intuition behind the obtained characterization is that, in general, there will be diversification for investors concerned with low quantiles and downside risk. These diversification insights are illustrated in numerical simulation exercises that cover several canonical cases under different scenarios. For the particular case of two independent and identically distributed (iid) random variables, full diversification is optimal for $\tau \leq \tau_0$ but not for values of $\tau > \tau_0$. The optimal strategy in the upper part of the distribution is investing fully in one risky asset. Therefore, the quantile model is flexible and allows for the possibility of underdiversification in the sense that, in some scenarios, the optimal portfolio choice may be no diversification. This is consistent with evidence of underdiversification compared to standard mean-variance efficient allocations; see Campbell (2006), Mitton and

(instead of continuous) variables, and the flexibility of offering a family of preferences indexed by quantiles.

⁵Intuitively, the monotonicity of quantiles allows one to avoid modeling individuals' utility function. This is because the maximization problem is invariant to monotonic transformations of the distribution of portfolio returns.

Vorkink (2007) and references therein.

The insights of the QP model for optimal portfolio allocation are applied to an empirical example to illustrate the similarities and differences between the optimal portfolio choices of EU and QP individuals. To do this, we consider a simple portfolio selection of stocks, bonds and cash, and compare the optimal asset allocation of QP individuals with that of EU individuals with mean-variance and power utility functions. We consider monthly data collected from three assets: the risk-free one-month nominal yield on the U.S. Treasury bill rate, the S&P 500 and the G0Q0 Bond Index, for the period January 1980 to December 2016. We compute the optimal portfolio allocation for the full range of $\tau \in (0, 1)$. For low enough quantiles, the optimal strategy is to invest fully in the risk-free asset, and for high enough quantiles, the optimal solution is fully invest in S&P 500 index. There is a rich diversification pattern among the three assets for middle quantile indexes. Overall, the results show portfolio diversification heterogeneity across risk attitudes, with no diversification for very low and large quantiles. These empirical findings contrast with the results obtained from two standard EU cases, the mean-variance and CRRA utility cases that exhibit full diversification under risk aversion.

We remark that the initial reaction to the consideration of QP might be of doubt, since quantile maximization is different from the familiar and well known EU model. Quantile maximization implies, indeed, some choices that might seem unusual at first glance, but can be considered reasonable after we overcome the influence of the EU over our intuition. More than that, there are situations where the maximization of a quantile seems very natural and by varying τ they encompass the whole range of risk aversion attitudes encountered in EU. It is not our contention, nevertheless, that quantile maximization is a decision method to be prescribed in all cases and problems, but rather that it is a complement analysis to EU. As such, it is important to document its properties and implications on relevant economic settings, such as the portfolio selection problem that is the focus of this paper. Those properties could then be tested in laboratories or confronted with data.

The remainder of the paper is laid out as follows. Section 1.1 has a brief review of the literature. Section 2 discusses the risk attitudes in QP models. Section 3 contains the main results of the paper. This section derives conditions under which there is full or null diversification in the tails, and conditions that provide focal optimal portfolios for risk averse and risk loving individuals. In addition, this section presents a numerical simulation exercise that illustrates the theoretical insights of the paper. Section 4 presents a simple portfolio allocation exercise among stocks, bonds and a risk-free asset, and Section 5 concludes. All mathematical derivations and proofs are collected in the Appendix.

1.1 Literature review

This paper relates to a number of streams of literature in portfolio selection and economic theory. First, the paper relates to the extensive literature on optimal decision theory under

uncertainty and portfolio selection. Optimal portfolio decision based on the EU has been the basis of asset pricing equilibrium models such as the Sharpe-Lintner CAPM (Sharpe, 1964; Lintner, 1965) and more recent alternatives based on factor models. In this context, the investor’s optimal portfolio decision relies heavily on the specification of the utility function for modeling individuals’ preferences. Thus the theoretical optimal portfolio allocation of individuals with constant absolute risk aversion and constant relative risk aversion preferences may be very different although, in practice, it may be difficult to differentiate between both attitudes towards risk from real data. A robust approach within the EU paradigm is stochastic dominance. This theory allows one to rank risky alternatives without relying on specific forms of the individuals’ utility function. Early work by Porter (1974) and Fishburn (1977) characterize the optimal portfolio decisions of EU individuals using stochastic dominance criteria of different orders. However, the equivalence between EU maximization and stochastic dominance is only satisfied, under risk aversion, for well-behaved (increasing and concave) utility functions.

Second, this paper is related to a branch of the literature on models for portfolio selection with alternative preferences to the EU. Many alternative preference measures to the EU have been put forward in the portfolio choice literature. Most of these approaches replace the utility function, which is essentially a distortion in wealth, by a distortion in the probability distribution of wealth. This probability distortion function, as Yaari (1987) shows, represents the risk preference in a different way. Similar approaches involving subjective probability distributions include, most significantly, Kahneman and Tversky (1979)’s prospect theory. Garlappi et al. (2007) develop a model for an investor with multiple priors and aversion to ambiguity. We extend the last two literatures by replacing EU and its variations with QP.

Third, this paper is related to a several works on models using the QP. As mentioned previously, QP were first studied by Manski (1988) and subsequently axiomatized by Chambers (2009), Rostek (2010) and de Castro and Galvao (2021). The risk attitude under QP is based on the concept of quantile-preserving spreads introduced in Mendelson (1987) and reformulated in Manski (1988) in terms of a single-crossing criterion between distribution functions. More recently, QP models have been employed in modeling economic behavior in different frameworks, see e.g., Chambers (2007), Bhattacharya (2009), Giovannetti (2013) and de Castro and Galvao (2019). Also, as mentioned previously, de Castro et al. (2020) show experimental evidence on the use of QP. In the asset pricing literature the use of QP has been hardly explored though.

Fourth, there is a small literature on optimal portfolio allocation using a quantile target variable. Kulldorff (1993) and Föllmer and Leukert (1999) study the goal-reaching problem where the target variable is a specific quantile, and He and Zhou (2011) propose a portfolio choice model in continuous time, where the quantile function of the terminal cash flow is the decision variable. In a similar context, Brown and Sim (2009) provide a framework for measuring the quality of risky positions with respect to their ability to achieve some aspiration

level, that can be interpreted with a quantile probability. In the mutual fund industry, quantiles have been used as alternative performance measures, see, for example, [Kempf and Ruenzi \(2007\)](#). We contribute to these two last lines of research by taking the QP together with the quantile maximization to a portfolio selection model and deriving its properties.

We conclude this section by discussing a related literature that shares some of the insights of QP theory but is different in scope. This discussion may serve as further general motivation for use of QP. Quantile measures have been used as risk measures in optimal portfolio allocation. In particular, Value-at-Risk (VaR) and expected shortfall models are closely linked to a low quantile selection, see [Duffie and Pan \(1997\)](#) and [Jorion \(2007\)](#) for a comprehensive review of VaR models. In an optimal asset allocation context, the VaR quantiles act as constraints in the asset allocation optimization exercise rather than as target variables to be optimized. These mean-risk models discussed in [Fishburn \(1977\)](#) can be considered as an extension of standard mean-variance formulations, see [Markowitz \(1952\)](#), rather than as QP models for optimal portfolio allocation. The relevant literature includes [Basak and Shapiro \(2001\)](#), [Krokhmal et al. \(2001\)](#), [Campbell et al. \(2001\)](#), [Wu and Xiao \(2002\)](#), [Bassett et al. \(2004\)](#), [Engle and Manganelli \(2004\)](#) and [Ibragimov and Walden \(2007\)](#), among others, and sheds an interesting light on the properties of VaR-optimal portfolios while acknowledging considerable computational difficulties ([Gaivoronski and Pflug, 2005](#); [Rachev et al., 2007](#)).

2 Quantile preferences and the risk attitude

This section briefly reviews the definition of risk under quantile preferences (QP). We first introduce the notation and definition of QP. Given any random variable $X : \Omega \rightarrow \mathbb{R}$, we denote by $F_X : \mathbb{R} \rightarrow [0, 1]$ the cumulative distribution function (CDF) of X . Given $\tau \in (0, 1)$, the τ -quantile of X is defined as

$$Q_\tau[X] \equiv \inf \{x \in X : F_X(x) \geq \tau\}.$$

A well-known and important property of quantiles, used below, is its invariance with respect to monotonic transformations. More formally, if $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and strictly increasing, then

$$Q_\tau[\psi(X)] = \psi(Q_\tau[X]). \tag{1}$$

A preference \succeq over random variables is a τ -quantile preference for some fixed $\tau \in (0, 1)$ if

$$X \succeq Y \iff Q_\tau[u(X)] \geq Q_\tau[u(Y)], \tag{2}$$

where $u(\cdot)$ is the utility function over the possible outcomes of the random variables X and Y . Note that $u(X)$ and $u(Y)$ are also random variables. [Manski \(1988\)](#) was the first to study QP as in (2). [Chambers \(2009\)](#) shows that these preferences satisfy the properties of monotonicity,

ordinal covariance, and continuity. In contrast, [Rostek \(2010\)](#) axiomatized the QP in the context of [Savage \(1954\)](#)'s subjective framework. Recently, [de Castro and Galvao \(2021\)](#) provide an alternative axiomatization for the static case using an uncertainty setting and finite state space.

It is important to notice that the QP defined by (2) are in fact independent of the utility function. Indeed, for any continuous and strictly increasing $u : \mathbb{R} \rightarrow \mathbb{R}$, from (1),

$$X \succeq Y \iff Q_\tau[u(X)] \geq Q_\tau[u(Y)] \iff u(Q_\tau[X]) \geq u(Q_\tau[Y]) \iff Q_\tau[X] \geq Q_\tau[Y]. \quad (3)$$

This result shows that the utility function plays absolutely no role in defining the preference. We can use (1) to make any transformation of u ; therefore, we could transform a concave utility function into a convex one without changing the preference. In particular, this implies that the concavity of the utility function has absolutely no implication for the risk attitude (nor any property) of QP.

The risk attitude in QP models is captured by the parameter $\tau \in (0, 1)$. This is discussed in detail by [de Castro and Galvao \(2019\)](#) and [de Castro et al. \(2020\)](#). Here we just summarize the main idea. For comparison of two decision makers, consider the following definition by [Ghirardato and Marinacci \(2002, Definition 4, p. 263\)](#):

Definition 2.1 (Ghirardato-Marinacci). *A preference \succeq' is more uncertainty averse than preference \succeq if for any $q \in \mathbb{R}$, and random variable X , $q \succeq X \Rightarrow q \succeq' X$ and $q \succ X \Rightarrow q \succ' X$.*

[Ghirardato and Marinacci \(2002\)](#)'s definition is a generalization of the standard notion of risk aversion in the context of risk under expected utility. The idea is that if a DM with preference \succeq would rather have the certain outcome $q \in \mathbb{R}$ than the risky prospect X , then the more uncertainty averse \succeq' DM prefers it as well.

The following Proposition 2.2 shows that \succeq^τ is more risk averse than $\succeq^{\tau'}$ if and only if $\tau < \tau'$. This property implies that an agent with QP given by τ is more risk preferring than another agent with QP given by τ' if $\tau > \tau'$. Thus, a QP decision maker that maximizes a lower quantile is more risk averse than one who maximizes a higher quantile.

Proposition 2.2. *Consider quantile maximizing preferences \succeq_τ and $\succeq_{\tau'}$. The following statements are equivalent:*

- (1) $\tau \geq \tau'$;
- (2) $\succeq_{\tau'}$ is more uncertainty averse than \succeq_τ ;

Proof. See [de Castro et al. \(2020\)](#). □

In fact, [de Castro et al. \(2020\)](#) consider two more equivalent conditions, based on the notion of quantile-preserving spreads introduced by [Mendelson \(1987\)](#), that extend [Rothschild and Stiglitz \(1970\)](#)'s mean-preserving notion. See [de Castro and Galvao \(2019\)](#) for a more detailed discussion.

3 Optimal portfolio choice problem

In this section we first study the case of a portfolio given by a risk-free and a risky asset. This simple portfolio problem serves to establish the intuition about the optimal behavior of individuals with quantile preferences (QP) and motivate the problem of interest, which is the study of the optimal portfolio allocation between two or more risky assets. We focus on the analysis of two risky assets. Prior to this, we set the foundations of the optimal portfolio allocation problem.

3.1 Quantile portfolio selection under quantile preferences

We start by formally describing the portfolio selection problem under QP. The portfolio manager has a budget $\mathbf{b} > 0$ to invest in n assets for a given fixed period of time. She will end up devoting $\mathbf{a}_i \in [0, \mathbf{b}]$ to asset i , satisfying $\sum_{i=1}^n \mathbf{a}_i = \mathbf{b}$. The initial price of asset i is $p_i > 0$, so that $\mathbf{a}_i = p_i q_i$, where q_i denotes the number of units of asset i that the portfolio manager buys. After the investment period, asset i 's price will be $\tilde{p}_i \geq 0$, which is random if asset i is not a risk-free asset. Therefore, the net return on asset i after the investment period is $\tilde{r}_i = \frac{\tilde{p}_i}{p_i} - 1$, which is a random variable. Consider the following portfolio

$$S_w = \sum_{i=1}^n w_i \tilde{r}_i,$$

where $w \equiv (w_1, \dots, w_n) \in [0, 1]^n$, with $\sum_{i=1}^n w_i = 1$. The weights $w_i = \frac{a_i}{\mathbf{b}} \geq 0$ denote the fraction of wealth invested on asset i . Implicitly, we are assuming that the portfolio manager does not short assets.⁶

To be consistent with the literature on optimal portfolio theory under EU preferences, we assume that individuals are endowed with a utility function $u(S_w)$, where $u : \mathbb{R} \rightarrow \mathbb{R}$, for describing individual's preferences on wealth. Then, for a given risk attitude $\tau \in (0, 1)$, the portfolio choice problem under QP is

$$\max_{w \in [0, 1]^n} Q_\tau [u(S_w)], \text{ s.t. } \sum_{i=1}^n w_i = 1. \quad (4)$$

As discussed in equation (3) above, the choice of utility function is irrelevant for portfolio selection under QP. Hence the quantile optimization problem (4) using a given utility is equivalent to maximizing the quantile obtained directly from the distribution of the random variable.

Lemma 3.1. *Let $u(\cdot)$ be a continuous and increasing utility function defined over the domain of the random variable S_w for $w \equiv (w_1, \dots, w_n) \in [0, 1]^n$. The maximization argument w^**

⁶Our model can deal with short sale, but we leave this to future work.

solves (4) if and only if it solves the following:

$$\max_{\mathbf{w} \in [0,1]^n} Q_\tau [S_{\mathbf{w}}], \text{ s.t. } \sum_{i=1}^n w_i = 1. \quad (5)$$

Given the result in equations (4) and (5), for the remaining of the paper, we focus the main analyses on the problem in (5).

It is worth noting that the above portfolio choice problem is different, and more general, than the goal-reaching problem proposed by [Kulldorff \(1993\)](#) in a quantile setting. This author maximizes the cumulative probability of $\sum_{i=1}^n w_i \tilde{r}_i$ subject to achieving some target return r_0 . More formally, the objective function is $\max_{(w_1, \dots, w_n)} P\{\sum_{i=1}^n w_i \tilde{r}_i \geq r_0\}$ subject to $\sum_{i=1}^n w_i = 1$.

The result in Lemma 3.1 is important in the context of portfolio allocation. Theoretically, it implies that the quantile choice rule is able to separate beliefs from tastes. The relevance of this separation criterion was put forward by [Ghirardato et al. \(2005\)](#) in the context of decision theory under uncertainty. These authors offered a result with this separation, but they did not insist on a complete separation of tastes and beliefs, because such a separation would rule out most of the choice rules commonly considered by decision theorists.⁷ In contrast, as shown in Lemma 3.1, the QP deliver a complete separation of tastes and beliefs. Empirically, this separation is very important as well. In particular, it allows portfolio managers to make choices on a particular portfolio without the knowledge of any specified utility function. For instance, a manager only needs to learn about the quantile τ of an agent to choose the portfolio weights from a given selection of returns \tilde{r} .

3.2 Optimal portfolio allocation when there is a risk-free asset

[Manski \(1988\)](#) derives the preferences of a quantile maximizer between two outcomes X and Y when one of the outcome measures is degenerate, and finds a complete separation in preferences between the degenerate and risky outcome. The deterministic choice is the preferred strategy for low quantiles. In contrast, for high quantiles, the risky outcome is the preferred strategy.

In this section, we provide further formality to the example in [Manski \(1988\)](#) and frame it in an optimal asset allocation context. We assume there is a riskless security that pays a rate of return equal to $R_f = \bar{r}$, and just one risky security that pays a stochastic rate of return equal to R with distribution function F_R . The portfolio return is defined by the convex combination

$$R_p = w\bar{r} + (1 - w)R = \bar{r} + (1 - w)(R - \bar{r}),$$

⁷For instance, if the preference is given by EU, the belief is captured by the probability while the tastes by the utility function over outcomes or consequences (such as monetary payoffs). Beliefs and tastes are not completely separated, however, because if we take a monotonic transformation of the utility function, which maintains the same tastes over consequences, we may end up with a *different* preference. That is, the pair beliefs and tastes come together and are stable only under affine transformations of the utility function. In other words, the EU preferences, as many other preferences, do not allow a separation of tastes and beliefs.

and the investor's maximization problem (5) for a specific quantile τ is $\arg \max_w Q_\tau[\mathbf{u}(\bar{r} + (1 - w)(R - \bar{r}))]$. Using the monotonicity of the quantile process, for a continuous and increasing utility function, the investor's problem simplifies to

$$\arg \max_w (1 - w)Q_\tau[R] + w\bar{r}.$$

Simple algebra shows that the individual portfolio choice w is then given by the following:

$$w^* = \begin{cases} 1 & \text{when } Q_\tau[R] < \bar{r} \\ 0 & \text{when } Q_\tau[R] > \bar{r} \\ \text{any } w \in [0, 1] & \text{when } Q_\tau[R] = \bar{r}. \end{cases}$$

The intuition of this solution is simple. For small values of τ the individual's optimal portfolio choice is $w^* = 1$ and corresponds to full investment on the risk-free asset. This is so because $\bar{r} > Q_\tau[R_p]$ for any combination R_p characterized by $0 < w < 1$. For larger values of τ , such that $Q_\tau[R] > \bar{r}$, the optimal portfolio decision reverses and yields $w^* = 0$. For $Q_\tau[R] = \bar{r}$, the QP maximizer is indifferent between the risk-free and the risky asset for any $w \in [0, 1]$ defining the portfolio return.

In contrast to the capital market line characterizing the mutual fund separation theorem in a mean-variance setting, [Tobin \(1958\)](#), the optimal portfolio allocation under QP specializes in the risk-free asset for individuals with τ below the magnitude of the risk-free rate and on the risky asset, otherwise. In [Appendix B.4](#), we extend the analysis of the risk-free asset by adding a second risky asset to the portfolio. We obtain the same findings indicating an optimal binary response to the risk-free asset. We notice that in this case, however, diversification between the two risky assets may be optimal for middle quantiles even if the allocation to the risk-free asset is null.

3.3 The case of two risky assets

This section considers the optimal portfolio allocation problem for an economy with two risky assets and a decision maker endowed with QP.⁸ Consider two risky assets represented, respectively, by the continuous random variables X and Y . Let the portfolio be defined as

$$S_w \equiv wX + (1 - w)Y, \tag{6}$$

with $0 \leq w \leq 1$ a scalar portfolio weight. The portfolio selection problem in this context will be:

$$\max_{w \in [0, 1]} Q_\tau[S_w]. \tag{7}$$

⁸See, e.g., [Damodaran \(2010\)](#) for an argument that every asset carries some risk.

Define the solution to this problem by $w^*(\tau) : (0, 1) \rightarrow [0, 1]$. Whenever there is no confusion we will use simply w^* . We will also assume that X and Y have joint distribution function given by a continuous probability density function (p.d.f.) f defined over intervals \mathcal{J}_X and \mathcal{J}_Y . More formally:

Assumption 1. X and Y have joint distribution function given by a continuous p.d.f. $f : \mathcal{J}_X \times \mathcal{J}_Y \rightarrow \mathbb{R}$, where $\mathcal{J}_X = [\underline{x}, \bar{x}]$, $\mathcal{J}_Y = [\underline{y}, \bar{y}]$, $-\infty \leq \underline{x} < \bar{x} \leq \infty$, $-\infty \leq \underline{y} < \bar{y} \leq \infty$ and $0 < f(x, y) < \infty$ for all $(x, y) \in \mathcal{J}_X \times \mathcal{J}_Y$.

It is important to emphasize that the above assumption does not exclude distributions with support in the whole real line. In particular, X and Y can be normal variables, for instance. In fact, almost all distributions studied in finance satisfy Assumption 1. It shall be understood that if $\bar{x} = \infty$ or $\bar{y} = \infty$, the intervals are, respectively, $[\underline{x}, \infty)$ and $[\underline{y}, \infty)$. An analogous observation holds when $\underline{x} = -\infty$ or $\underline{y} = -\infty$. We will maintain Assumption 1 in all results of this section and will not repeat it.

In the remaining of the section we establish the existence of the optimal portfolio choice under the QP theory as well as derive conditions that determine the existence of diversification.

Lemma 3.2. *The optimization problem (7) has at least one solution.*

Lemma 3.2 shows that the QP problem has at least one optimal vector, $w^*(\tau)$, that solves the problem for a given quantile τ . Deriving an explicit expression for w^* is in general a difficult task, but its value can be obtained through numerical calculations.

3.3.1 Diversification for low quantiles

Our first main result is that “in general” for τ sufficiently small, diversification is optimal, that is, there exists an interior solution $w^* \in (0, 1)$. As we will see in a moment, this requires some assumptions. Perhaps the most important setting is the one described in the following:

Theorem 1. *Assume that $\underline{x}, \underline{y} > -\infty$. If $\underline{x} = \underline{y}$ and $\tau \in (0, 1)$ is sufficiently small, then any optimal solution must be interior. More formally, there exists τ_0 such that if $\tau < \tau_0$ then $w \in \{0, 1\}$ does not solve (7) for that τ , or yet: if w^* solves (7), $w^* \in (0, 1)$.*

Intuitively, for a fixed small τ , the convex combination of the two assets is able to generate a larger quantile. Notice that this result requires no further assumption on the distributions other than the restriction of the same lower bound. Although this condition excludes normal distributions, it includes important distributions, as for example, the lognormal. Nevertheless, Theorem 1 can be extended to symmetric normal distributions, although we omit a formal statement for space considerations.

It is useful to illustrate Theorem 1 for the case of two standard uniform distributions, $X \sim \mathcal{U}(0, 1)$ and $Y \sim \mathcal{U}(0, 1)$; see Example 3.3. Figure 1 shows the CDFs of the random variable

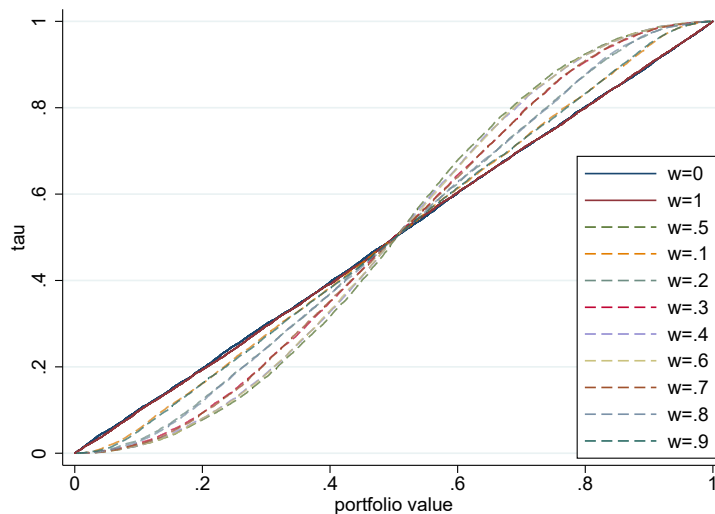


Figure 1: CDF of $S_w = wX + (1-w)Y$ indexed by $w \in [0, 1]$ when $X \sim \mathcal{U}(0, 1)$ and $Y \sim \mathcal{U}(0, 1)$.

S_w for different w . One can see that for low τ (in this case, $\tau \leq 0.5$), the CDF curve most to the right corresponds to $w^*(\tau) = 0.5$. If $\tau > 0.5$, the optimal $w^*(\tau)$ is either 0 or 1, that is, there is no diversification (both 0 and 1 are solutions because the two assets are identical in this case).

Example 3.3. Consider $X \sim \mathcal{U}(0, 1)$ and $Y \sim \mathcal{U}(0, 1)$, independent. With numerical calculations, in this case we have:

$$w^*(\tau) = \begin{cases} 0.5, & \text{if } \tau \in (0, \frac{1}{2}] \\ 1, & \text{if } \tau \in (\frac{1}{2}, 1). \end{cases}$$

Notice that there can be diversification even when one of the distribution functions stochastically dominates the other. See, for instance, Example 3.4 for the case of $X \sim \mathcal{U}(0, 2)$ and $Y \sim \mathcal{U}(0, 1)$. In this case, even though $\mathcal{U}(0, 2)$ stochastically dominates $\mathcal{U}(0, 1)$, there is diversification for $\tau \leq 1/4$: the optimal w is interior, $w^* = 0.5$.

Example 3.4. Consider $X \sim \mathcal{U}(0, 2)$ and $Y \sim \mathcal{U}(0, 1)$. In this case we have:

$$w^*(\tau) = \begin{cases} 0.5, & \text{if } \tau \in (0, \frac{1}{4}] \\ 1, & \text{if } \tau \in (\frac{1}{4}, 1). \end{cases}$$

This is an interesting result, because despite the fact that X first order stochastically dominates Y , there exists a convex combination S_w that dominates both random variables X and Y for low quantiles. Notice, however, that this feature is desirable, because the independence of X and Y makes a convex combination of the two less risky than any of them.

3.3.2 No diversification with different lower ends of the distributions

Given the result in Theorem 1, a natural question is what would happen if the assumption that $\underline{x} = \underline{y}$ of the theorem does not hold, that is, if $\underline{x} \neq \underline{y}$. The naïve intuition may be that for low quantiles diversification is always optimal. Theorem 1 shows that this occurs if τ is sufficiently small and there is no obvious difference in the lower limits given by \underline{x} and \underline{y} . However, when the tail behavior of the assets in the portfolio is very different, the next result shows that diversification is not optimal for low quantiles. More formally, Theorem 2 shows that for $\underline{x} - \underline{y} \geq \frac{M}{2m}$, with m and M suitable constants, the quantile of X is larger than the quantile of any convex combination of X and Y for low values of τ . A natural interpretation of this result in a risk management context is to say that the VaR of X is larger than the VaR of any diversified combination of the assets.⁹ The investor allocates all the portfolio weight in the variable that dominates the other in the left tail of the distribution. This is the message of the next result, where we denote X 's τ -quantile by x_τ .

Theorem 2. *Assume that $\underline{x} > \underline{y} > -\infty$. Fix $\bar{\tau} \in [0, 1]$. Let M and m be such that $m \leq f(x, y) \leq M$ for all $(x, y) \in [\underline{x}, x_{\bar{\tau}}] \times [\underline{y}, \bar{y}] \cup [\underline{x}, \bar{x}] \times [\underline{y}, x_{\bar{\tau}}]$.¹⁰ If $\underline{x} - \underline{y} \geq \frac{M}{2m}$, $w^* = 1$ is the unique solution to (7) for all $\tau \in (0, \bar{\tau})$.¹¹*

Theorem 2 shows that the optimal choice is $w^* = 1$ for all τ small, provided that the difference between the two distributions at the left end point is sufficiently large. Example 3.5 illustrates this result for the case of $X \sim \mathcal{U}(0.5, 1)$ and $Y \sim \mathcal{U}(0, 1)$. Since X and Y are uniform and $\underline{y} = 0$, we can take $m = M$ and the assumption of Theorem 2 simplifies to $\underline{x} \geq \frac{1}{2}$, which is precisely the condition satisfied in this example (with $\underline{x} = \frac{1}{2}$).

Example 3.5. *Consider $X \sim \mathcal{U}(0.5, 1)$ and $Y \sim \mathcal{U}(0, 1)$. Then, $w^* = 1$ for all $\tau \in (0, 1)$.*

Notice that in this example, we have the choice $w^* = 1$ for all $\tau \in (0, 1)$, which is stronger than the result stated in Theorem 2. We can in fact establish this stronger condition if the bounds m and M hold for the entire interval. More precisely, we can fix $\bar{\tau} = 1$ in Theorem 2 and conclude the following:

Corollary 3.6. *Assume that $\underline{x} > \underline{y} > -\infty$. Let M and m be such that $m \leq f(x, y) \leq M$ for all $(x, y) \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$. If $\underline{x} - \underline{y} \geq \frac{M}{2m}$, $w^* = 1$ is the unique solution to (7) for all $\tau \in (0, 1)$.*

The latter result shows the absence of diversification, across $\tau \in (0, 1)$, for individuals with QP under the conditions of the corollary. This result suggests that in many cases the efforts of portfolio managers may be futile under QP theory. A major implication of Corollary 3.6 is

⁹In the context of risk management, Theorem 2 provides theoretical support to the lack of subadditivity of VaR measures in general settings, see Artzner et al. (1999).

¹⁰The inferior limit m may be taken over a limited region, not over the whole support. That is, we can accommodate cases in which the support is infinite so that $f(x, y) \rightarrow 0$ when $y \rightarrow \infty$.

¹¹Of course, $w^* = 0$, when $\underline{y} > \underline{x} > -\infty$ and $\underline{y} - \underline{x} \geq \frac{M}{2m}$.

to show that, under QP, if there is no diversification in the lower left tail of the distribution it is very likely that there is no diversification for higher quantiles.

Nevertheless, the behavior with different lower end points can be complex. For instance, it may be the case that the optimal choice is $w^* \in \{0, 1\}$ for small τ , it becomes interior for intermediate values of τ and then becomes $w^* \in \{0, 1\}$ again for large τ 's. The following example illustrates a case when the conditions in the corollary are not satisfied. In this case we find the existence of diversification in the middle quantiles of the distribution despite the fact that there is no diversification in the lower left tail.

Example 3.7. Consider (X, Y) a bivariate random vector. Let $f_X(x) = M$ be the marginal density function of X if $0 < \underline{x} \leq x \leq \underline{d}$, $f_X(x) = m$ if $\underline{d} < x \leq \bar{x} < \infty$, and $f_X(x) = 0$ otherwise. The support of X is $[\frac{1}{2}, \frac{3}{2}]$, i.e. $\underline{x} = \frac{1}{2}$ and $\bar{x} = \frac{3}{2}$. Let $Y \sim \mathcal{U}(0, 1)$ such that $f_Y(y) = 1$ for $y \in [\underline{y}, \bar{y}]$, with $\underline{y} = 0$ and $\bar{y} = 1$, and $f_Y(y) = 0$, otherwise. Then, the following conditions of Corollary 3.6 are satisfied: (i) $\underline{x} > \underline{y} > -\infty$, (ii) M and m are such that $m \leq f(x, y) \leq M$ for all $(x, y) \in [\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$. However, for $M = 4$, $m = \frac{2}{3}$ and $\underline{d} = 0.6$, we have $\underline{x} - \underline{y} = 0.5 < 3 = \frac{M}{2m}$, such that the remaining condition of the corollary is not satisfied.

Figure 2 reports the CDFs for different combinations S_w indexed by $w \in [0, 1]$ for the density functions in Example 3.7. For small and large values of τ the optimal allocation is $w^* = 1$, however, there is a middle interval of τ for which diversification is optimal. This can be seen by noting that some CDF for $w \in (0, 1)$ crosses from below that of $w = 1$ that corresponds to X .

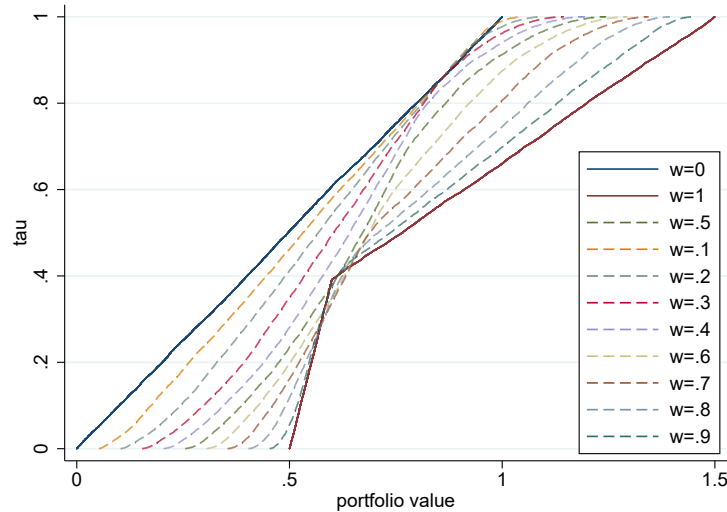


Figure 2: CDF of $S_w = wX + (1 - w)Y$ indexed by $w \in [0, 1]$ when (X, Y) are defined in the Example 3.7.

Another interesting case is posed by Theorem 2. One might think that the conclusion of this theorem could hold more generally, whenever $\underline{x} > \underline{y}$ and τ is sufficiently small. This is false, however: the difference between \underline{x} and \underline{y} must be bounded away from zero to ensure the conclusion. The following examples illustrate this observation.

Example 3.8. Consider $X \sim \mathcal{U}(0.25, 0.75)$ and $Y \sim \mathcal{U}(0, 1)$. We have $w^* \in (0, 1)$ for $\tau \in (0, \frac{1}{2})$ and $w^* = 0$ for $\tau \in (\frac{1}{2}, 1)$. See Figure 3. Its left panel plots the optimal allocation $w^*(\tau)$, while its right panel plots the τ -quantiles of X , Y and the optimal portfolio $S_{w^*(\tau)}$.

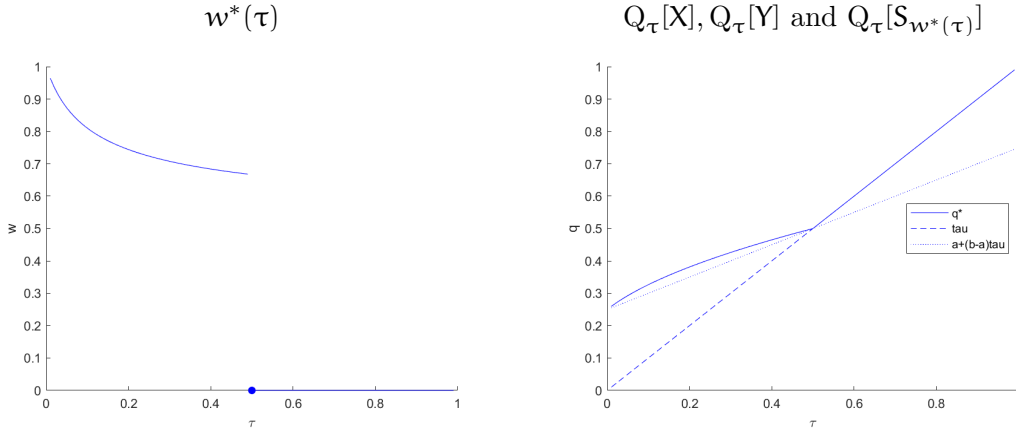


Figure 3: Illustration of Example 3.8—Optimal w^* for $X \sim \mathcal{U}(0.25, 0.75)$ and $Y \sim \mathcal{U}(0, 1)$.

Example 3.9. Consider $X \sim \mathcal{U}(0.25, 1.25)$ and $Y \sim \mathcal{U}(0, 1)$. We have $w^* \in (0, 1)$ for $\tau \in (0, 0.25)$ and $w^* = 1$ for $\tau > 0.25$. See Figure 4. As in the previous case, the left panel plots $w^*(\tau)$, while its right panel plots the τ -quantiles of X , Y and the optimal portfolio $S_{w^*(\tau)}$.

3.3.3 No diversification for large quantiles

The examples above suggest that for high τ , the optimal choice is $w^* \in \{0, 1\}$. This is indeed correct, as the following result establishes. It shows that the optimal portfolio choice for values of τ close to 1 is not interior.

Theorem 3. Assume that $\underline{x} = \underline{y} < \bar{y} = \bar{x} < \infty$. Fix $\bar{\tau} \in (0, 1)$. Let M and m be such that $m \leq f(x, y) \leq M$ for all $(x, y) \in [x_{\bar{\tau}}, \bar{x}] \times [\underline{y}, \bar{y}] \cup [\underline{x}, \bar{x}] \times [x_{\bar{\tau}}, \bar{y}]$.¹² If $x_{\bar{\tau}} - \underline{x} \geq \frac{M(\bar{x} - \underline{x})}{2m}$, then $w^*(\tau) \in \{0, 1\}$ for all $\tau \in [\bar{\tau}, 1)$.

Note that in this scenario we may have one or two solutions for τ sufficiently close to 1. In other words, there is no portfolio diversification in the upper tail of the distribution. These

¹²The inferior limit m may be taken over a limited region, not over the whole support. That is, we can accommodate cases in which the support is infinite so that $f(x, y) \rightarrow 0$ when $y \rightarrow \infty$.

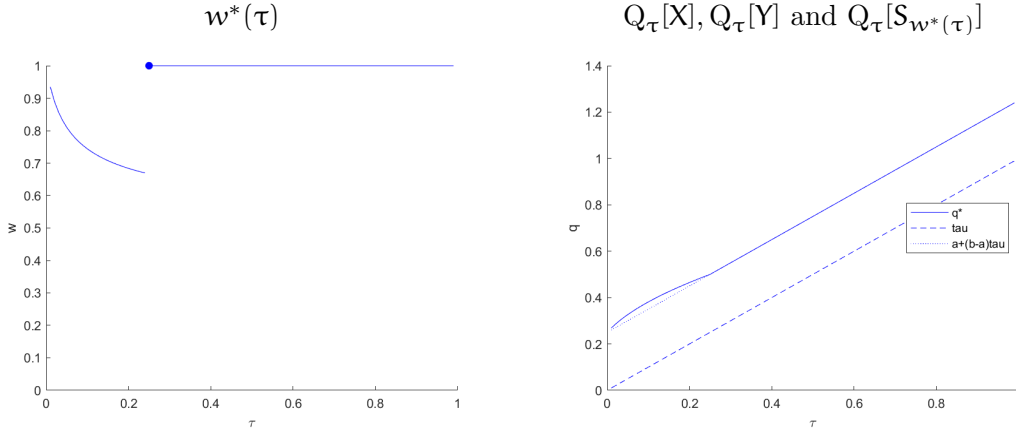


Figure 4: Illustration of Example 3.9—Optimal w^* for $X \sim \mathcal{U}(0.25, 1.25)$ and $Y \sim \mathcal{U}(0, 1)$.

findings are related to [Ibragimov and Walden \(2007\)](#). These authors find that for truncated versions of heavy-tailed distributions with unbounded support diversification may increase value at risk as long as the random variables are concentrated on a sufficiently large interval. The results in this section generalize [Ibragimov and Walden \(2007\)](#) by considering a wider class of distribution functions characterized by Assumption 1. In particular, we derive the conditions under which diversification in the tails may or may not be an optimal outcome for individuals endowed with QP.

In the following section we proceed to characterize the existence of interior solutions to the QP optimal portfolio choice problem.

3.3.4 Characterization of the interior solution in QP optimal portfolio choice

The preceding section has presented conditions on the support of the random variables X and Y that lead to either full diversification or null diversification. We have also extended the results to the tails of the distributions by showing that for lower bound supports far apart diversification is not an optimal outcome. In this section, we build upon these results and provide a characterization of the interior solutions to the optimal portfolio choice problem under QP theory. The section also establishes properties of the portfolio selection problem in a QP framework.

The proof of the results below will depend on some new definitions. First, let us denote $\Pr(S_w \leq q)$ by $h(w, q)$, that is,

$$h(w, q) \equiv \int_{J_X} \int_{J_Y \cap (-\infty, \frac{q-wx}{1-w}]} f(x, y) dy dx, \quad (8)$$

if $w \in (0, 1)$, $h(0, q) \equiv F_Y(q)$ and $h(1, q) \equiv F_X(q)$. Let us focus on the case $w \in (0, 1)$. From

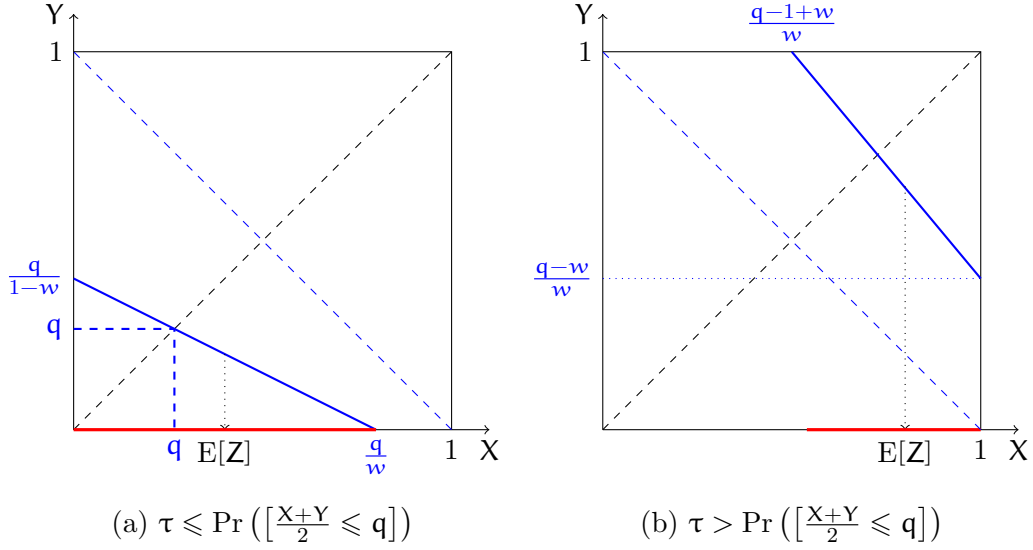


Figure 5: An illustration of Z for the case $J_X = J_Y = [0, 1]$.

Assumption 1, we know that as long as $\{(x, \frac{q-wx}{1-w}) : x \in \mathbb{R}\} \cap J_X \times J_Y$ contains more than one point, the equation $h(w, q) = \tau \in (0, 1)$ implicitly defines the quantile $q = q_{w, \tau}$ of S_w as a function of $w \in (0, 1)$ and $\tau \in (0, 1)$, that is,

$$h(w, q_{w, \tau}) = \tau \implies q(w, \tau) \equiv Q_\tau[S_w].$$

In what follows, we will omit τ from $q(w, \tau)$ such that $q(w, \tau) \equiv q(w)$.

Let us define $J_Z = J_Z^{w, q} = \{x \in \mathbb{R} : (x, \frac{q-wx}{1-w}) \in J_X \times J_Y\}$ and assume that w and q are such that $J_Z = J_Z^{w, q}$ is a proper interval. Now, let us define a random variable $Z = Z_{w, q}$ that is characterized by the following density function:

$$f_Z(x) \equiv \frac{f\left(x, \frac{q-wx}{1-w}\right)}{\int_{J_Z} f\left(t, \frac{q-tw}{1-w}\right) dt}, \quad (9)$$

for $x \in J_Z$. This density function is strictly positive on J_Z and zero otherwise. In order to understand what Z corresponds to, consider Figure 5. This figure plots the region $\{(x, \frac{q-wx}{1-w}) : x \in \mathbb{R}\} \cap [0, 1] \times [0, 1]$ in a XY plane for the case of two standard uniform distributions. The left plot in Figure 5 describes the problem for small quantiles ($\tau \leq \frac{1}{2}$) and $w = \frac{1}{2}$, and the right plot for large quantiles ($\tau > \frac{1}{2}$) and $w = \frac{1}{2}$. More generally, the random variable Z can be interpreted as the projection of $Y = \frac{q-wX}{1-w}$ onto X , and has support drawn in red and density given by the values in the probability distribution function along the blue line.¹³

¹³The two illustrations (a) and (b) correspond, respectively, to $\tau \leq \Pr\left(\left[\frac{X+Y}{2} \leq q\right]\right)$ and $\tau > \Pr\left(\left[\frac{X+Y}{2} \leq q\right]\right)$. In the first case (a), the line $wX + (1-w)Y = q$ is below the dashed blue line joining $(1, 0)$ to $(0, 1)$. In case (b) $\tau > \Pr\left(\left[\frac{X+Y}{2} \leq q\right]\right)$, the line $wX + (1-w)Y = q$ is above the dashed blue line.

The definition of the random variable Z through the characterization of its density function (9), allow us to state our first result in this section.

Proposition 3.10. *Let X and Y be random variables satisfying assumption 1, and let $Z = Z_{w,q}$ be a random variable characterized by the density function (9). Then, the function $q(w)$ is differentiable at $w \in (0, 1)$ and*

$$q'(w) = \frac{1}{1-w} (E[Z] - q(w)), \text{ for } \tau \in (0, 1), \quad (10)$$

provided that one of the following cases hold: (1) $\mathcal{J}_X = \mathcal{J}_Y = \mathbb{R}$; (2) $\mathcal{J}_X = \mathcal{J}_Y = \mathbb{R}_+$; (3) $\mathcal{J}_X = \mathcal{J}_Y = [0, c]$, with $c > 0$ a constant defining a compact support.

This result allows one to derive the optimality condition characterizing the optimal portfolio choice.

Corollary 3.11. *Let $w^* \in (0, 1)$ be an interior solution to the maximization problem (7) for a given $\tau \in (0, 1)$. Under the conditions of Proposition 3.10, the optimal w^* satisfies the condition $E[Z_{w^*,q}] = q(w^*)$, with $\frac{\partial}{\partial w} E[Z_{w^*,q}] < 0$.*

To understand the optimality condition in Corollary 3.11, we revisit Example 3.3 by considering again the particular case of two standard uniform random variables, $X, Y \sim \mathcal{U}(0, 1)$.

Example 3.12. *Let X and Y be iid $\mathcal{U}(0, 1)$ random variables. In this case, if $\tau \leq \frac{1}{2}$, we have $E[Z] = \frac{q}{2w}$. Then, $q(w)$ is increasing if*

$$E[Z] > q \iff \frac{q}{2w} > q \iff w < \frac{1}{2}.$$

The maximum is achieved at $w^ = \frac{1}{2}$ given that $E[Z] = q \iff \frac{q}{2w^*} = q \iff w^* = \frac{1}{2}$. To show that w^* is a maximum we also note that the function $q(w)$ is strictly decreasing for $w > \frac{1}{2}$ and $\tau \leq \frac{1}{2}$. This case is illustrated in Figure 5(a). On the other hand, if $\tau > \frac{1}{2}$, Figure 5(b) depicts Z . In this case, we can see that*

$$E[Z] = \frac{1}{2} \left(\frac{q-1+w}{w} + 1 \right) = 1 - \frac{1-q}{2w} > q \iff \frac{1-q}{2w} < 1-q \iff w > \frac{1}{2},$$

which implies that the function $q(w)$ is increasing for $w > \frac{1}{2}$ and achieves the maximum at $w^ = 1$, for all $\tau > \frac{1}{2}$.*

In this example there is a separation in the optimal portfolio allocation between risk-averse individuals and risk lovers. The optimal portfolio allocation of risk-averse individuals, characterized by $\tau \in (0, \frac{1}{2}]$, is full diversification given by $w^* = \frac{1}{2}$. Risk lovers, characterized by $\tau \in (\frac{1}{2}, 1)$, maximize the objective function (7) at $w^* = 1$. This important result separating

the optimal portfolio allocation into two regions broadly representing the risk preferences of risk-averse and risk-loving individuals can be generalized to other density functions under the following assumptions.

Assumption 2. *The joint density function $f(x, y)$ of the bivariate random variable (X, Y) is C^1 and satisfies that, for all fixed $\mu \in \mathbb{R}$, $f(\mu + \varepsilon, \mu - \frac{w}{1-w}\varepsilon)$ is unimodal on $\varepsilon \in \mathbb{R}$, with mode at $\varepsilon = 0$.*

Assumption 3. *The joint density function $f(x, y)$ of the bivariate random variable (X, Y) is such that there exists some $w^* \in (0, 1)$ satisfying the condition*

$$f\left(\mu + \varepsilon, \mu - \frac{w^*}{1-w^*}\varepsilon\right) = f\left(\mu - \varepsilon, \mu + \frac{w^*}{1-w^*}\varepsilon\right) \quad (11)$$

for all $\mu \in \mathbb{R}$ and $\varepsilon > 0$.

Assumption 2 evaluated at the optimal w^* guarantees the unimodality of the density function f_Z defined in (9). Similarly, Assumption 3 guarantees its symmetry. Note that condition (11) does not necessarily imply the symmetry of $f(x, y)$ in the sense $f(x, y) = f(y, x)$ unless $w^* = \frac{1}{2}$.

The following result constitutes one of the main results of this study, namely, we show the existence of two regimes in the optimal portfolio allocation for individuals characterized by QP and maximizing the objective function (7).

Proposition 3.13. *Let $f(x, y)$ be the joint density function of the pair (X, Y) . Under assumptions 1 to 3, the solution to the maximization problem (7), for all $\tau \in (0, \tau_0]$, is $w^* \in (0, 1)$ that satisfies expression (11). For $\tau > \tau_0$, the solution is $w^* \in \{0, 1\}$. More specifically, $w^* = 0$ for $\tau > \tau_0$ if $w^* \geq \frac{1}{2}$ in the interval $(0, \tau_0]$, and $w^* = 1$ for $\tau > \tau_0$ if $w^* \leq \frac{1}{2}$ in the interval $\tau \in (0, \tau_0]$. The threshold τ_0 is defined by the condition $\tau \leq P(Z \leq q)$ for $\tau \in (0, \tau_0]$, and $P(Z \leq q) < \tau$ for $\tau \in (\tau_0, 1)$.*

This result accommodates the iid case as a particular example.

Corollary 3.14. *Let X and Y be two iid random variables with unimodal density function $f_X(\cdot)$ that satisfies assumption 1. The solution to the maximization problem (7) is $w^* = \frac{1}{2}$ for $\tau \in (0, \tau_0]$, and $w^* \in \{0, 1\}$, indistinctively, for $\tau > \tau_0$.*

Proposition 3.13 and Corollary 3.14 offer very interesting insights about portfolio allocation and the importance of diversification. Under some general assumptions that accommodate, among other examples, very popular families of density functions such as the Normal distribution and the Student-t distribution, diversification is optimal for individuals with preferences characterized by risk aversion (low τ -quantiles). Individuals characterized by these preferences choose the same portfolio, regardless their specific τ , that is given by an interior solution. For iid random variables or for symmetric bivariate random variables the optimal allocation is full

diversification, interpreted as an equal contribution of each asset to the portfolio. In contrast, individuals with preferences characterized by high values of τ , do not diversify at all. The optimal investment strategy of these agents is full investment in the asset with more upside potential, that in these cases corresponds to the asset with higher downside risk too. In this case the specific value of τ characterizing the individual’s quantile preference is not relevant for determining the optimal portfolio allocation as long as this value is greater than the cut-off point τ_0 . For symmetric distributions, $\tau_0 = 0.5$.

3.4 Numerical results

In this section we study optimal asset allocation under QP for portfolios given by the mixture of two continuous random variables X and Y with payoff realizations defined as $S_w = wX + (1 - w)Y$. We also compare the results with the optimal asset allocation obtained under EU. Here, for simplicity and brevity, we concentrate on the simple case of two Gaussian independent and identically distributed random variables. Nevertheless, Appendix B presents a more comprehensive examination across different scenarios of the problem allowing for dependence and different configurations of the Gaussian and Chi-squared distributions.

In order to compute the optimal portfolio $w^*(\tau) : (0, 1) \rightarrow [0, 1]$ we are required to compute the cumulative distribution function F_{S_w} and quantile function $Q_\tau[S_w]$, for which in many cases we do not have an analytical expression to work with. We approximate the optimal portfolio allocation w^* and quantile function through simulation. The following example aims to illustrate the practical selection and diversification of portfolios, as well as the theoretical findings of the previous sections.

3.4.1 Design

In this numerical study we seek to compute the optimal portfolio weights $w^*(\tau)$ for the QP model in equation (5) for $\tau \in (0, 1)$. To simulate a portfolio, we draw $n = 10,000$ realizations of the random variables (X, Y) , which are chosen from different distributions, say Gaussian and Chi-squared. We investigate several alternative scenarios using independence and dependence of the variables X and Y . In addition, we vary the mean and variance of the distribution functions.

For each case below, we report two figures. The figures on the left panel plot the cumulative distribution functions generated by the portfolio value S_w , that is, we plot F_{S_w} for $w \in \{0.1, \dots, 0.9\}$. These figures intuitively characterize the solutions because for any τ we can draw a horizontal line and find the optimal QP solution by searching for the largest CDF. Second, figures on the right panel plot the graph of the portfolio selection $\{w^*(\tau), \tau\}$ for different values of τ .

3.4.2 Computation of portfolio

To implement the optimization, we use a simple grid search. Let $T_j = \{0 < \tau_1 < \dots < \tau_j < 1\}$ be a grid of values for the quantile index τ with j values, and let $\Lambda_l = \{0 \leq w_1 < \dots < w_l \leq 1\}$ be a grid of values for w with l values. For each fixed $\tau \in T_j$ we solve the following portfolio problem numerically

$$\hat{w}_n^*(\tau) = \arg \max_{\{w \in \Lambda_m\}} \hat{Q}_\tau[S_w], \quad (12)$$

where \hat{Q}_τ is the sample counterpart of the quantile function Q_τ .

We consider T_j with grid spacing of 0.01 ($\#T_j = 99$, with $\tau_1 = 0.01$ and $\tau_{99} = 0.99$) and Λ_l with grid spacing of 0.01 ($\#\Lambda_j = 101$, with $w_1 = 0$ and $w_{101} = 1$).

Consider a simulation exercise given by a very simple case where we consider a weighted combination of two independent standard Gaussian random variables, $X, Y \sim \text{iid } N(0, 1)$. It is well known that $S_w = wX + (1 - w)Y$ also follows a Gaussian distribution with $S_w \sim N(0, w^2 + (1 - w)^2)$. In this scenario, under risk aversion and for very general forms of the utility function, e.g. mean-variance, CRRA and CARA utility functions embedded in the optimal portfolio decision, risk-averse EU individuals choose the investment portfolio that minimizes the variance of the random variable S_w . In this setting this is given by $w^* = 0.5$.

The results for the QP model presented on the right panel of Figure 6 confirm these findings for low quantiles of the distribution but show more heterogeneity in the optimal portfolio decision for the upper quantiles. The findings of this simulation exercise provide empirical evidence showing that the optimal QP choice depends on the quantile $\tau \in (0, 1)$. In particular, for this specification of the random variables X and Y the optimal weight w^* is a function of τ such that $\hat{w}_n^*(\tau) = 0.5$ for $\tau \leq 0.5$, and either $\hat{w}_n^*(\tau) = 0$ or $\hat{w}_n^*(\tau) = 1$ for $\tau > 0.5$. This result is a consequence of the fact that the distributions of X and Y are identical and confirm the predictions in Corollary 3.14 about a lack of diversification in the upper tail of the distribution of portfolio returns. The left panel of the figure plots all combinations of distributions \hat{F}_{S_w} as a function of $w \in [0, 1]$. These distributions reveal a unique single-crossing point at $\tau_0 = 0.5$ for all w .

This simple exercise sheds light on portfolio selection under QP. As mentioned above, the QP do not rely on any functional form of the utility function. In addition, QP models have the advantage of allowing for heterogeneity through the quantiles because it offers a family of preferences indexed by τ , with the risk attitude under the QP being captured by the quantile. The results in Figure 6 show that individuals with QP encompass both risk-averse and risk-loving behavior. Those economic agents concerned with downside risk (low quantiles) diversify by investing equally in each asset, whereas individuals with preferences driven by the upper quantiles of the distribution of portfolio returns do not diversify at all. The critical point that determines whether an individual diversifies or not is $\tau_0 = 0.5$.¹⁴

¹⁴The simulation algorithm reflects this issue by selecting either $w = 0$ or $w = 1$ for $\tau_0 = 0.5$.

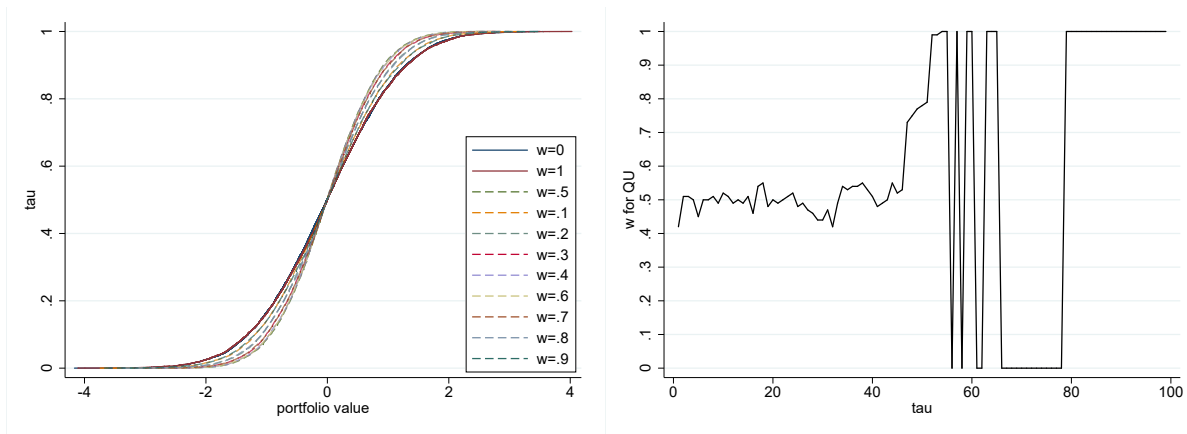


Figure 6: $X, Y \sim N(0, 1)$. Left panel plots the CDF of S_w . Right panel plots QP portfolio selection.

This neat separation between risk-aversion and risk-loving behaviors characterized by the quantile τ_0 is also observed, more generally, for distribution functions satisfying the conditions of Proposition 3.13. The result in the proposition accommodates departures from the iid case, otherwise, if the conditions of the proposition are not satisfied then the optimal asset allocation under QP is τ -dependent and shifts smoothly from a risk-averse optimal allocation to a risk-loving optimal allocation. Appendix B illustrates this case and collects several additional results across different scenarios, including the analysis of portfolios with a risk-free asset. The results provide strong support to the theoretical findings in this paper.

4 Empirical application to a portfolio of stocks, bonds and cash

This section provides an empirical application illustrating the methods developed in the paper. We investigate the optimal portfolio choice of an investor with quantile preferences (QP) that can allocate wealth among three assets: a risk-free asset (one-month Treasury bill rate), a bond index (G0Q0 Bond Index), and a stock index (S&P 500). We consider monthly data collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The G0Q0 Bond Index is a Bank of America and Merrill Lynch U.S. Treasury Index that tracks the performance of U.S. dollar denominated sovereign debt publicly issued by the U.S. government in its domestic market. The nominal yield on the U.S. one-month risk-free rate is obtained from Kenneth French website.

Prior to computing the portfolio weights, we report nonparametric kernel estimates of the unconditional density functions of the returns on the three assets. The results are given in Figure 7. Visual inspection of the densities shows that the three density functions are unimodal and exhibit similar mean returns but very different standard deviations. A formal statistical analysis rejects, however, the null hypothesis of equality of means for all pairwise combinations

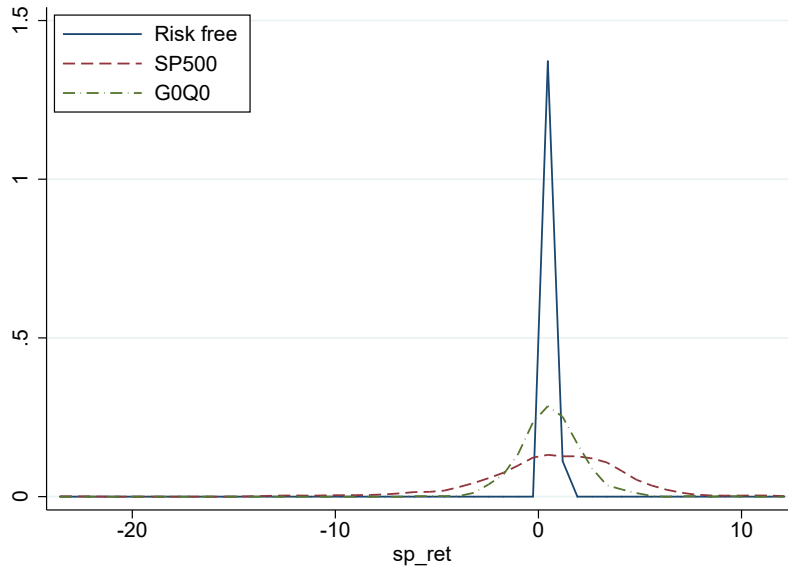


Figure 7: Nonparametric kernel estimates of the unconditional densities of monthly log-returns on the U.S. one-month Treasury bill, the G0Q0 bond index and the S&P 500 index. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

of the U.S. Treasury bill, the G0Q0 index and the S&P 500 index, and the null hypothesis of symmetry of the three density functions. More specifically, the mean return and standard deviation for the U.S. one-month Treasury bill are 0.363 and 0.296, respectively; the mean return and standard deviation for the G0Q0 bond index are 0.608 and 1.584, respectively, and 0.680 and 3.635 for the S&P 500 index. These summary statistics suggest that the equity index has the highest expected return and variance, and is followed by the bond index with regards to expected return and risk. In contrast, the U.S. Treasury bill has the lowest mean and variance.

We compute the QP portfolio optimal weights by solving the maximization problem (5) numerically, as in equation (12). We divide the analysis in two. First, we study the optimal portfolio allocation between the U.S. one-month Treasury bill and the G0Q0 index, and between the U.S. one-month Treasury bill and the S&P 500 index. Figure 8 clearly confirms the predictions of previous sections. The optimal portfolio allocation of a QP investor can be divided into two regions indexed by $\tau \in (0, 1)$. For low values of τ , individuals are risk averse and choose to minimize risk by allocating all the wealth on the risk-free asset. However, for high values of τ , individuals become risk lovers and choose the riskiest strategy that brings the highest upside potential. Interestingly, the results in Figure 8 highlight the role of the U.S. one-month Treasury bill as a risk-free investment and are consistent with the insights discussed in Section 3.2 on the mutual fund separation theorem for the QP case. For low quantiles, the

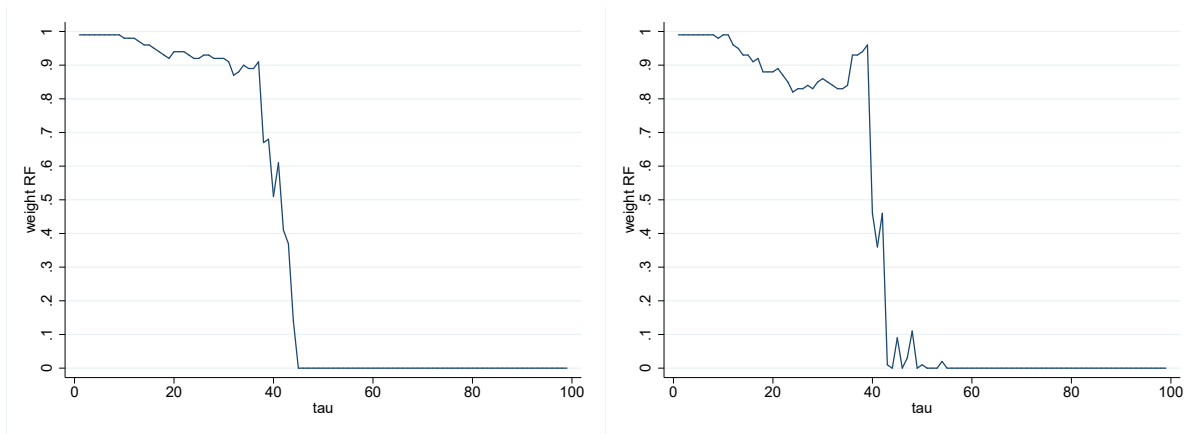


Figure 8: Left panel reports the optimal portfolio allocation between the U.S. risk-free asset and G0Q0 bond index. Right panel reports the optimal portfolio allocation between the U.S. risk-free asset and the S&P 500 index. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

optimal choice is to fully invest on the risk-free asset, whereas for high quantiles, the optimal choice is to invest fully on the risky alternative.

Second, we study the allocation problem where the investment universe comprises the three assets. In this case we report the weights of all the three assets. The results are given in Figure 9 and provide similar insights about the optimal portfolio allocation exercise. We observe a separation between the risk-free and riskiest asset for low and high quantiles. In particular, for very low quantiles the optimal portfolio allocation is given by only investing in the risk-free asset. In contrast, for values of τ beyond the mode, the optimal asset allocation is given by fully investing on the S&P 500 index. As τ increases, we find that the optimal portfolio allocation is given by a combination of the three assets, with a large share of investment on the risk-free asset and a small share of investment distributed equally between the G0Q0 bond index and the equity index. The allocation to the S&P 500 index with respect to the other two assets increases as the tolerance of the individual towards risk grows.

We conclude the section by comparing these results with the optimal portfolio allocation of a EU investor with preferences characterized by two different types of utility function: mean-variance and power utility functions. The mean-variance utility function is defined as

$$U(\mu, C) = w' \mu - \frac{\alpha}{2} w' C w, \quad (13)$$

with μ the vector of mean returns and C the covariance matrix of returns; the vector w denotes the optimal portfolio weights. This is performed for different degrees of risk aversion α . The left panel of Figure 10 presents the optimal portfolio allocations for values of α between 0 and

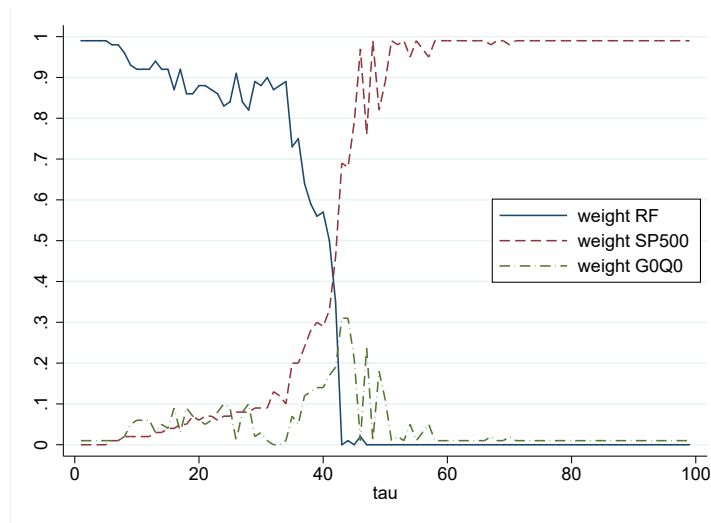


Figure 9: The optimal portfolio allocation between the U.S. risk-free asset, G0Q0 bond index and the S&P 500 index. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

0.5. In this range, the optimal portfolio allocation leads to a diversified portfolio that contains non-zero combinations of the three assets. For values of α close to zero, corresponding to risk neutrality, the optimal portfolio allocation is mainly driven by investment in the S&P 500 index. However, as the tolerance to risk decreases, investment in the bond index and the US Treasury bill gains importance. Investment in the risk-free asset is monotonically increasing on α and dominates the portfolio for values greater than 0.3. Comparison of both sets of results suggests interesting similarities and differences across types of individuals. Thus, there is a mapping between the optimal portfolio choice of the QP investor with risk preferences characterized by τ in the interval $(0.40, 0.60)$ and the optimal portfolio choice of the mean-variance investor with values of α in the range 0.05 to 0.45. The QP optimal allocation for values of τ close to zero is also similar to the mean-variance allocation for values of α greater than 0.5, signaling risk aversion. The main differences are, however, for behaviors related to risk-loving attitudes. Thus, for values of τ greater than 0.5 we find that the allocation of the QP individual is concentrated on the equity index. This result is only found for mean-variance investors for values of α very close to zero that reflect no risk penalty in the utility function (13).

For completeness, the right panel of Figure 10 reports the optimal asset allocation problem for a EU individual with a power utility function characterized by different degrees of relative risk aversion. Interestingly, the results are very similar to those obtained for the mean-variance case. In this example, risk-loving attitudes for EU individuals take place for values of γ that converge to zero.

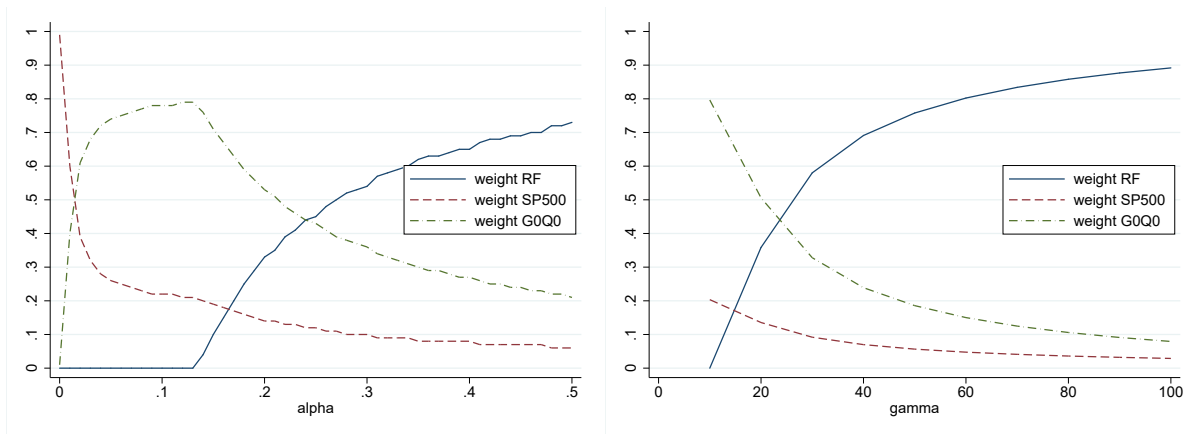


Figure 10: Left panel reports the optimal portfolio allocation for a mean-variance investor with risk preferences modeled by $\alpha \in [0, 0.5]$. Right panel reports the optimal portfolio allocation for an investor with a CRRA utility function with risk aversion coefficient given by $\gamma \in [10, 100]$. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index for the period January 1980 to December 2016. The risk-free rate is obtained from Kenneth French website.

5 Conclusion

This paper studies the optimal asset allocation problem for individuals with quantile preferences (QP). The proposed QP model has several attractive features: (i) the portfolio choice is independent of the utility function and related to the risk attitude τ ; (ii) the ability to capture heterogeneity by varying the quantiles; (iii) robustness; and (iv) it has a solid axiomatic foundation.

We divide the portfolio allocation problem in two scenarios: with and without a risk-free asset. In the former case, we find that an investor with quantiles preferences with risk aversion characterized by a small τ reacts to the presence of a risk-free asset by fully investing on it. Otherwise, for higher values of τ the optimal strategy is to fully invest on the risky portfolio. This result is in stark contrast with the standard mutual fund separation theorem that shows that the optimal combination between the risk-free asset and a risky portfolio is convex and determined by the investor's risk aversion profile. Under quantile preferences, we observe an all-or-nothing behavior, instead. For the case of two risky assets, we derive theoretically conditions on the support of the random variables under which the optimal portfolio decision has an interior solution. This result provides the setup under which diversification strategies are optimal for investors endowed with quantile preferences. These insights are in clear contrast to the EU paradigm that claims that diversification is always an optimal strategy under very general forms of risk aversion.

The paper has also explored the optimality of diversification strategies in the tails of the distribution of portfolio returns. In particular, we have derived conditions under which diver-

sification in the tails is outperformed by fully investing in one risky asset. This result is of interest for the characterization of underdiversification strategies in finance.

Additionally, we have characterized the optimal portfolio allocation under QP when an interior solution exists. Under unimodality and a symmetry condition on the bivariate distribution of the random lotteries, the individual's optimal portfolio decision under QP is characterized by two regions: for quantiles below the median full diversification is optimal, for quantiles above the median diversification is dominated by fully investing on the asset with highest upside potential. This strategy characterizes the optimal portfolio decision of risk-loving individuals.

Appendix

A Proofs

This appendix collects the proofs for the results in the main text.

Proof of Lemma 3.1:

Proof. The objective function of the primal problem in (4) is

$$\max_{\mathbf{w} \in [0,1]^n} Q_\tau [\mathbf{u}(\mathbf{S}_w)].$$

Noticing that $\mathbf{u}(\cdot)$ is continuous and increasing, and that the quantile is invariant with respect to monotone transformations, then the above maximization argument is given by

$$\begin{aligned} \arg \max_w Q_\tau [\mathbf{u}(\mathbf{S}_w)] &= \arg \max_w \mathbf{u} (Q_\tau [\mathbf{S}_w]) \\ &= \arg \max_w Q_\tau [\mathbf{S}_w]. \end{aligned}$$

□

Proof of Lemma 3.2:

Proof. It is sufficient to show that $w \mapsto Q_\tau [wX + (1 - w)Y] = Q_\tau [\mathbf{S}_w]$ is continuous. But this follows from Assumption 1, which implies that the CDF of $\mathbf{S}_w = wX + (1 - w)Y$ is strictly increasing, thus making its quantile continuous. □

Proof of Theorem 1:

Proof. (1) By appealing to translations if necessary, we may assume without loss of generality that $\underline{x} = \underline{y} = 0$, that is, if this is not the case, we may consider the random variables $X' = X - \underline{x}$ and $Y' = Y - \underline{y}$. We may also assume without loss of generality that $\bar{x}, \bar{y} < \infty$, for if $\bar{x} = \infty$ or $\bar{y} = \infty$, we may truncate the distribution at large \hat{x}, \hat{y} so that $\Pr(\{(x, y) : x > \hat{x} \text{ or } y > \hat{y}\}) < \epsilon$ for some small $\epsilon > 0$. Thus, without loss of generality, we may assume that there exists $m, M \in \mathbb{R}_+$ such that $0 < m < f(x, y) < M < \infty$ for all $(x, y) \in [0, \bar{x}] \times [0, \bar{y}]$.

We will prove that we can choose τ small enough so that $w = 1$ is dominated by $w = \frac{1}{2}$. The argument that $w = 0$ is also dominated by $w = \frac{1}{2}$ for sufficiently small τ is analogous. For this, we will assume that $w = 1$ is better than $w = \frac{1}{2}$ and find a contradiction for τ small enough. Let x_τ denote the τ -quantile of X and assume that $q = q(\frac{1}{2}, \tau) \leq x_\tau$. See Figure 11 where $q < x_\tau$. From the definition, the probability under the red line $\frac{X+Y}{2} \leq 2$ is τ , which is the same of the probability of the vertical band from 0 to x_τ . Therefore, the blue area in the

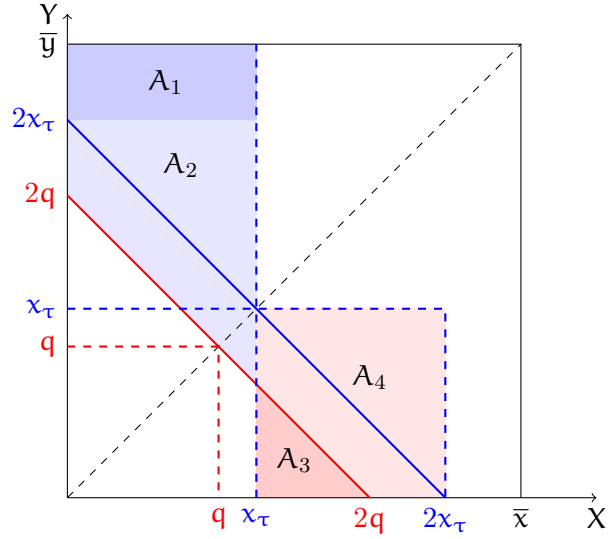


Figure 11: Illustration of the proof.

graph (both light and dark blue together) have to be the same as the dark red that corresponds to the small triangle from x_τ to $2q$ in the horizontal axis and from 0 to $2q - x_\tau$. To better follow the argument, let A_1 correspond to the probability in the dark blue area and A_2 , the probability of the light blue area. Let A_3 correspond to the probability of the dark red area and A_4 , the probability of the light red area. Therefore, as observed above, by definition,

$$A_1 + A_2 = A_3. \quad (14)$$

We will show, however, that we can choose τ sufficiently small so that

$$A_1 > A_3 + A_4. \quad (15)$$

Observe that (15) implies

$$A_1 + A_2 > A_1 > A_3 + A_4 > A_3,$$

which contradicts (14). In order to obtain (15), observe that

$$A_1 = \int_0^{x_\tau} \left[\int_{2x_\tau}^{\bar{y}} f(x, y) dy \right] dx$$

$$A_3 + A_4 = \int_{x_\tau}^{2x_\tau} \left[\int_0^{x_\tau} f(x, y) dy \right] dx.$$

Since $m < f(x, y) < M$ for $(x, y) \in [0, \bar{x}]^2$, we have

$$A_1 > \int_0^{x_\tau} \left[\int_{2x_\tau}^{\bar{y}} m dy \right] dx = mx_\tau (\bar{y} - 2x_\tau)$$

$$A_3 + A_4 < \int_{x_\tau}^{2x_\tau} \left[\int_0^{x_\tau} M dy \right] dx = Mx_\tau^2.$$

Therefore, we obtain (15) if

$$mx_\tau (\bar{y} - 2x_\tau) > Mx_\tau^2$$

$$\iff x_\tau [m\bar{y} - (2m + M)x_\tau] > 0$$

$$\iff x_\tau < \frac{\bar{y}}{\left(\frac{M}{m} + 2\right)}.$$

Since \bar{y} , M and m are given constants, the above inequality is satisfied if x_τ is small enough, that is, if τ is sufficiently close to 0. □

Proof of Theorem 2:

Proof. Note that by translating the distribution (along the 45° line) if necessary, we may assume without loss of generality that $\underline{y} = 0$.

Let us fix any $\tau \in (0, \bar{\tau}]$ and let x_τ, y_τ denote the τ -quantiles of X and Y , respectively. Of course, $x_\tau > \underline{x}$. Figure 12 below will be useful to illustrate the reasoning in this proof.

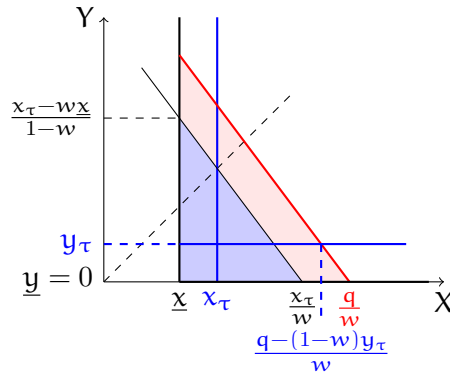


Figure 12: Illustration of the proof.

Let \bar{M} be the average $f_X(x)$ density on the region $x \in [\underline{x}, x_\tau]$, that is, $\bar{M} \equiv \frac{\tau}{x_\tau - \underline{x}}$. Therefore, $m \leq \bar{M} \leq M$ and $m \leq f(x, y) \leq M$ for all $(x, y) \in [\underline{x}, x_\tau] \times [\underline{y}, \bar{y}]$. We want to show that $q(w, \tau) < x_\tau$ for all $w \in [0, 1)$. To show this, it is enough to show that for any $q \geq x_\tau$ and

$w \in [0, 1]$, the probability under the line $wx + (1 - w)y \leq q$ is larger than τ , that is,

$$h(w, q) = \int_{\underline{x}}^{\frac{x_\tau}{w}} \left[\int_0^{\frac{x_\tau - wx}{1-w}} f(x, y) dy \right] dx > \tau.$$

For this, it is sufficient to show that the area in blue in Figure 12 is larger than τ already for $q = x_\tau$, that is,

$$q \geq x_\tau \implies h(w, q) \geq h(w, x_\tau) = \int_{\underline{x}}^{\frac{x_\tau}{w}} \left[\int_0^{\frac{x_\tau - wx}{1-w}} f(x, y) dy \right] dx > \tau. \quad (16)$$

Since $f(x, y) > m$ for all $(x, y) \in [\underline{x}, x_\tau] \times [\underline{y}, \bar{y}] \cup [\underline{x}, \bar{x}] \times [\underline{y}, x_\tau]$, we have

$$h(w, x_\tau) > \frac{m}{2} \left(\frac{x_\tau}{w} - \underline{x} \right) \left(\frac{x_\tau - w\underline{x}}{1-w} \right) = \frac{m(x_\tau - w\underline{x})^2}{2w(1-w)}.$$

Thus, to establish (16), it is sufficient to show that

$$\begin{aligned} \frac{m(x_\tau - w\underline{x})^2}{2w(1-w)} > \tau &\iff m(x_\tau - w\underline{x})^2 > 2\tau w(1-w) \\ &\iff m[(x_\tau)^2 - 2w\underline{x}x_\tau + w^2\underline{x}^2] - 2\tau w + 2w^2\tau > 0 \\ &\iff w^2(2\tau + m\underline{x}^2) - 2w(m\underline{x}x_\tau + \tau) + mx_\tau^2 > 0. \end{aligned}$$

Let us define the quadratic polynomial:

$$p(w) \equiv w^2(2\tau + m\underline{x}^2) - 2w(m\underline{x}x_\tau + \tau) + mx_\tau^2. \quad (17)$$

Notice that $p(0) = mx_\tau^2 > 0$ and for $w = 1$:

$$p(1) = (2\tau + m\underline{x}^2) - 2(m\underline{x}x_\tau + \tau) + mx_\tau^2 = m(\underline{x}^2 - 2\underline{x}x_\tau + x_\tau^2) = m(x_\tau - \underline{x})^2 > 0.$$

Let w_V denote the vertex of the quadratic $p(w)$, which is given by

$$w_V = \frac{m\underline{x}x_\tau + \tau}{2\tau + m\underline{x}^2}.$$

Note that w_V is always positive.

We can conclude that the quadratic form $p(w)$ in (17) is positive if $p(w_V) > 0$. To verify

this, we can substitute w_V in the quadratic form to obtain:

$$\begin{aligned} p(w_V) &= \left[\frac{m x_\tau \underline{x} + \tau}{2\tau + m \underline{x}^2} \right]^2 (2\tau + m \underline{x}^2) - 2 \left[\frac{m x_\tau \underline{x} + \tau}{2\tau + m \underline{x}^2} \right] (m \underline{x} x_\tau + \tau) + m x_\tau^2 \\ &= \frac{-(m x_\tau \underline{x} + \tau)^2}{2\tau + m \underline{x}^2} + m x_\tau^2 = \frac{m x_\tau^2 (2\tau + m \underline{x}^2) - (m x_\tau \underline{x} + \tau)^2}{2\tau + m \underline{x}^2}, \end{aligned}$$

which is positive as long as

$$\begin{aligned} 2\tau m x_\tau^2 + m^2 x_\tau^2 \underline{x}^2 &> (m x_\tau \underline{x} + \tau)^2 = m^2 x_\tau^2 \underline{x}^2 + 2\tau m x_\tau \underline{x} + \tau^2 \\ \iff 2\tau m x_\tau (x_\tau - \underline{x}) &> \tau^2 \\ \iff 2m x_\tau (x_\tau - \underline{x}) &> \tau. \end{aligned}$$

Since $\tau = \bar{M}(x_\tau - \underline{x})$, the above condition is equivalent to

$$2m x_\tau > \bar{M} \iff x_\tau > \frac{\bar{M}}{2m}.$$

Since $\bar{M} \leq M$ and $x_\tau > \underline{x} \geq \frac{M}{2m}$, we have $x_\tau > \frac{\bar{M}}{2m}$, thus proving the result. \square

Remark A.1. *One can also conclude that $p(w) > 0$ for all $w \in [0, 1]$ if $w_V > 1$. Unfortunately, the condition for this is stronger than the one given in Theorem 2. Indeed, $w_V > 1$ if*

$$w_V = \frac{m x_\tau \underline{x} + \tau}{2\tau + m \underline{x}^2} \geq 1,$$

which is equivalent to

$$\begin{aligned} m x_\tau \underline{x} + \tau &\geq 2\tau + m \underline{x}^2 \\ \iff m \underline{x} (x_\tau - \underline{x}) &\geq \tau. \end{aligned}$$

Since $\tau = \bar{M}(x_\tau - \underline{x})$, the above condition simplifies to

$$m \underline{x} \geq \bar{M} \iff \underline{x} \geq \frac{\bar{M}}{m}.$$

Since $\bar{M} \leq M$, if we have $\underline{x} \geq \frac{M}{m}$ then $\underline{x} \geq \frac{\bar{M}}{m}$. However, this assumption $\underline{x} \geq \frac{M}{m}$ is stronger than the condition in Theorem 2.

Proof of Theorem 3:

Proof. Fix $\tau \geq \bar{\tau}$ and assume that $\underline{x} = \underline{y}$ and $\bar{x} = \bar{y}$. We want to show that $w^*(\tau) = 1$ for sufficiently high τ , which is equivalent to show that $q(w, \tau) < x_\tau$ for all $w \in (0, 1)$ and

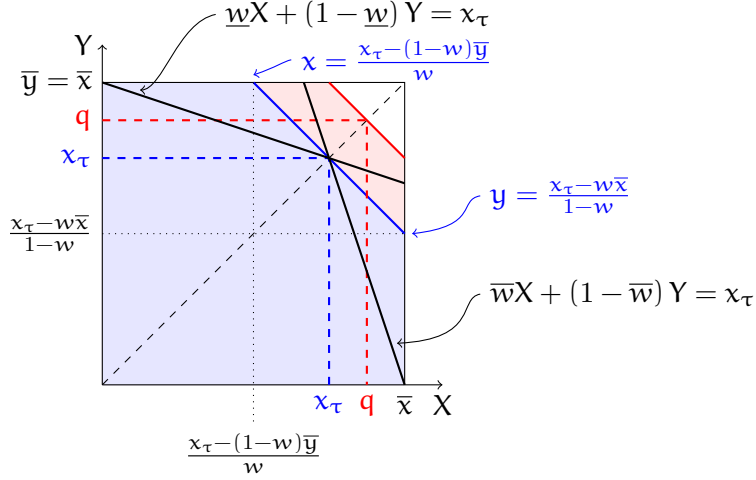


Figure 13: Illustration for the proof of Theorem 3: the case of τ close to 1.

$\tau \geq \bar{\tau}$. To show this, it is enough to show that for any $q > x_\tau$ the probability under the line $wx + (1-w)y \leq q$ is larger than τ , that is,

$$q > x_\tau \implies h(w, q) > h(w, x_\tau) = \int_{A(w, x_\tau)} f(x, y) dy dx \geq \tau,$$

where $A(w, q) \equiv \{(x, y) \in [\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}] : wx + (1-w)y \leq q\}$. This is equivalent to show that

$$1 - h(w, x_\tau) = \int_{A^c(w, x_\tau)} f(x, y) dy dx \leq 1 - \tau.$$

For estimating this integral, we have to consider three intervals for w : $(0, \underline{w})$, $[\underline{w}, \bar{w}]$ and $(\bar{w}, 1)$, where \underline{w} and \bar{w} correspond to the lines that pass by (\underline{x}, \bar{x}) and (\bar{x}, \underline{x}) respectively, that is,

$$\begin{aligned} \underline{w} \underline{x} + (1 - \underline{w}) \bar{x} = x_\tau &\iff \underline{w} = \frac{\bar{x} - x_\tau}{\bar{x} - \underline{x}}; \\ \bar{w} \bar{x} + (1 - \bar{w}) \underline{x} = x_\tau &\iff \bar{w} = \frac{x_\tau - \underline{x}}{\bar{x} - \underline{x}}. \end{aligned}$$

Note that $\underline{w} = 1 - \bar{w} < \bar{w}$ for large τ .

Figure 13 shows the set $A(w, x_\tau)$ in blue, for $w \in [\underline{w}, \bar{w}]$. Thus, if $w \in [\underline{w}, \bar{w}]$,

$$1 - h(w, x_\tau) = \int_{\frac{x_\tau - (1-w)\bar{x}}{w}}^{\bar{x}} \left[\int_{\frac{x_\tau - wx}{1-w}}^{\bar{x}} f(x, y) dy \right] dx.$$

For $w \in (0, \underline{w})$,

$$1 - h(w, x_\tau) = \int_{A^c(w, x_\tau)} f(x, y) dy dx = \int_{\underline{x}}^{\bar{x}} \left[\int_{\frac{x_\tau - wx}{1-w}}^{\bar{x}} f(x, y) dy \right] dx,$$

and, for $w \in (\bar{w}, 1)$,

$$1 - h(w, x_\tau) = \int_{A^c(w, x_\tau)} f(x, y) dy dx = \int_{\underline{x}}^{\bar{x}} \left[\int_{\frac{x_\tau - (1-w)y}{w}}^{\bar{x}} f(x, y) dx \right] dy.$$

We will consider each of the above cases.

Case 1: $w \in [\underline{w}, \bar{w}]$.

Since $m \leq f(x, y) \leq M$ for all $(x, y) \in [x_\tau, \bar{x}] \times [\underline{x}, \bar{x}]$, we have

$$1 - h(w, x_\tau) \leq \frac{M}{2} \left[\bar{x} - \frac{x_\tau - (1-w)\bar{x}}{w} \right] \left(\bar{x} - \frac{x_\tau - w\bar{x}}{1-w} \right).$$

Using $\bar{x} = \bar{x}$, the left hand side becomes $\frac{M(\bar{x} - x_\tau)^2}{2w(1-w)}$. Thus, it is enough to prove that

$$\frac{M(\bar{x} - x_\tau)^2}{2w(1-w)} \leq 1 - \tau \iff w(1-w) \geq \frac{M(\bar{x} - x_\tau)^2}{2(1-\tau)}.$$

Since $w \in [\underline{w}, \bar{w}]$, we have

$$w(1-w) \geq \min\{\underline{w}(1-\underline{w}), \bar{w}(1-\bar{w})\} = \underline{w}\bar{w} = \frac{x_\tau - \underline{x}}{\bar{x} - \underline{x}} \cdot \frac{\bar{x} - x_\tau}{\bar{x} - \underline{x}}.$$

Thus, it is sufficient to show that:

$$\begin{aligned} \frac{(x_\tau - \underline{x})(\bar{x} - x_\tau)}{(\bar{x} - \underline{x})^2} &\geq \frac{M(\bar{x} - x_\tau)^2}{2(1-\tau)} \\ \iff 2(1-\tau)(x_\tau - \underline{x}) &\geq M(\bar{x} - \underline{x})^2 (\bar{x} - x_\tau). \end{aligned} \quad (18)$$

We now define $\bar{m} = \frac{(\bar{x} - x_\tau)(\bar{x} - \underline{x})}{1-\tau}$ so that $1 - \tau = \bar{m}(\bar{x} - x_\tau)(\bar{x} - \underline{x}) = \bar{m}(\bar{x} - x_\tau)(\bar{x} - \underline{x})$.

Since

$$1 - \tau = \int_{x_\tau}^{\bar{x}} \left[\int_{\underline{x}}^{\bar{x}} f(x, y) dy \right] dx \geq m(\bar{x} - x_\tau)(\bar{x} - \underline{x}),$$

we have $\bar{m} \geq m \implies \frac{M}{\bar{m}} \leq \frac{M}{m}$. Therefore, the assumption gives

$$x_\tau - \underline{x} \geq x_\tau - \underline{x} \geq \frac{M(\bar{x} - \underline{x})}{2m} \geq \frac{M(\bar{x} - \underline{x})}{2\bar{m}},$$

as we wanted to show.

Case 2: $w \in (0, \underline{w})$.

Since $m \leq f(x, y) \leq M$ for all $(x, y) \in [x_{\bar{\tau}}, \bar{x}] \times [\underline{x}, \bar{x}]$, we have

$$1 - h(w, x_{\tau}) \leq \frac{M}{2} (\bar{x} - \underline{x}) \left(\bar{x} - \frac{x_{\tau} - w\bar{x}}{1 - w} \right) = \frac{M (\bar{x} - \underline{x}) (\bar{x} - x_{\tau})}{2(1 - w)}.$$

Thus, it is enough to prove

$$\frac{M (\bar{x} - \underline{x}) (\bar{x} - x_{\tau})}{2(1 - w)} \leq 1 - \tau \iff 1 - w \geq \frac{M (\bar{x} - \underline{x}) (\bar{x} - x_{\tau})}{2(1 - \tau)}.$$

Since $w \in (0, \underline{w})$, we have $1 - w > 1 - \underline{w} = \frac{x_{\tau} - \underline{x}}{\bar{x} - \underline{x}}$. Thus, it is sufficient to show that:

$$2(1 - \tau)(x_{\tau} - \underline{x}) \geq M(\bar{x} - \underline{x})^2 (\bar{x} - x_{\tau}).$$

This is exactly condition (18) above, which we have shown to be implied by the assumption.

Case 3: $w \in (\bar{w}, 1)$.

Since $m \leq f(x, y) \leq M$ for all $(x, y) \in [x_{\bar{\tau}}, \bar{x}] \times [\underline{x}, \bar{x}]$, we have

$$1 - h(w, x_{\tau}) \leq \frac{M}{2} (\bar{x} - \underline{x}) \left[\bar{x} - \frac{x_{\tau} - (1 - w)\bar{x}}{w} \right] = \frac{M (\bar{x} - \underline{x}) (\bar{x} - x_{\tau})}{2w}.$$

Thus, it is enough to prove

$$\frac{M (\bar{x} - \underline{x}) (\bar{x} - x_{\tau})}{2w} \leq 1 - \tau \iff 2(1 - \tau)w \geq M (\bar{x} - \underline{x}) (\bar{x} - x_{\tau}).$$

Since $w \in (\bar{w}, 1)$, $w > \bar{w} = \frac{x_{\tau} - \underline{x}}{\bar{x} - \underline{x}}$, it is sufficient to show again condition (18) above, which we have shown to be implied by the assumption of Theorem 3. \square

Proof of Proposition 3.10:

Proof. Assume first that $q(w)$ is differentiable at $w \in (0, 1)$. Taking the total derivative with respect to w on the equation $h(w, q) = \tau$, we obtain

$$\partial_w h(w, q) + \partial_q h(w, q) \cdot q'(w) = 0 \implies q'(w) = -\frac{\partial_w h(w, q)}{\partial_q h(w, q)}. \quad (19)$$

From (8), it is clear that h is differentiable and that $\partial_q h(w, q) > 0$. Therefore, by the Implicit Function Theorem, q is differentiable and given by (19). Now, we will calculate $\partial_w h(w, q)$

and $\partial_q h(w, q)$. Define the function $y(w, q) \equiv \frac{q-wx}{1-w}$. Then,

$$\begin{aligned}\partial_q y(w, q) &= \frac{1}{1-w}, \text{ and} \\ \partial_w y(w, q) &= \frac{(-x)(1-w) - (q-wx)(-1)}{(1-w)^2} = \frac{q-x}{(1-w)^2}.\end{aligned}$$

Support on \mathbb{R}

Here we consider first the case in which $\mathcal{J}_X = \mathcal{J}_Y = \mathbb{R}$. The other two cases are similar and considered below.¹⁵ It is clear from (8) and Assumption 1 that h is C^1 and:

$$\partial_w h(w, q) = \int_{-\infty}^{\infty} f\left(x, \frac{q-wx}{1-w}\right) \left[\frac{q-x}{(1-w)^2}\right] dx \quad (20)$$

and

$$\partial_q h(w, q) = \int_{-\infty}^{\infty} f\left(x, \frac{q-wx}{1-w}\right) \frac{1}{1-w} dx. \quad (21)$$

From this,

$$\begin{aligned}\partial_w h(w, q) &= \frac{q}{(1-w)^2} \int_{-\infty}^{\infty} f\left(x, \frac{q-wx}{1-w}\right) dx - \frac{1}{(1-w)^2} \int_{-\infty}^{\infty} xf\left(x, \frac{q-wx}{1-w}\right) dx \\ &= \frac{q}{1-w} \partial_q h(w, q) - \frac{1}{(1-w)} E[Z] \cdot \partial_q h(w, q) \\ &= \frac{1}{1-w} \cdot (q - E[Z]) \cdot \partial_q h(w, q).\end{aligned}$$

Therefore,

$$q'(w) = -\frac{\partial_w h(w, q)}{\partial_q h(w, q)} = \frac{1}{1-w} (E[Z] - q),$$

as we wanted to show. □

Support on $[0, \infty)$

Now, we consider the case $\mathcal{J}_X = \mathcal{J}_Y = \mathbb{R}_+ = [0, \infty)$. In this case, we have

$$h(w, q) = \int_0^{\frac{q}{w}} \int_0^{\frac{q-wx}{1-w}} f(x, y) dy dx. \quad (22)$$

Let us define:

$$g(w, q, x) \equiv \int_0^{\frac{q-wx}{1-w}} f(x, y) dy,$$

¹⁵When \mathcal{J}_X or \mathcal{J}_Y are not \mathbb{R} , then we have to consider limits that make the derivatives more complex.

for $x < \frac{q}{w}$ and $g(w, q, \frac{q}{w}) = 0$ so that g is continuous and differentiable. Also,

$$h(w, q) = \int_0^{\frac{q}{w}} g(w, q, x) dx.$$

Then,

$$\begin{aligned} \partial_w h(w, q) &= g\left(w, q, \frac{q}{w}\right) \left(\frac{-q}{w^2}\right) + \int_0^{\frac{q}{w}} \partial_w g(w, q, x) dx \\ &= \frac{1}{(1-w)^2} \int_0^{\frac{q}{w}} (q-x) f\left(x, \frac{q-wx}{1-w}\right) dx, \end{aligned}$$

and

$$\begin{aligned} \partial_q h(w, q) &= g\left(w, q, \frac{q}{w}\right) \left(\frac{1}{w}\right) + \int_0^{\frac{q}{w}} \partial_q g(w, q, x) dx \\ &= \frac{1}{1-w} \int_0^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx. \end{aligned}$$

From this, we observe that the same expressions remain valid:

$$\begin{aligned} \partial_w h(w, q) &= \frac{q}{(1-w)^2} \int_0^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx - \frac{1}{(1-w)^2} \int_0^{\frac{q}{w}} xf\left(x, \frac{q-wx}{1-w}\right) dx \\ &= \frac{1}{1-w} \cdot (q - E[Z]) \cdot \partial_q h(w, q) \end{aligned}$$

and

$$q'(w) = -\frac{\partial_w h(w, q)}{\partial_q h(w, q)} = \frac{1}{1-w} (E[Z] - q), \quad (23)$$

as before.

Support on $[0, c]$

We now consider the case $\mathcal{J}_X = \mathcal{J}_Y = [0, c]$. There are two subcases to consider, as show in Figure 14.

Case (a)

In this case,

$$h(w, q) = \int_0^{\frac{q-(1-w)c}{w}} \int_0^c f(x, y) dy dx + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \int_0^{\frac{q-wx}{1-w}} f(x, y) dy dx.$$

Let

$$g(\alpha, \beta, x) \equiv \int_{\alpha}^{\beta} f(x, y) dy.$$

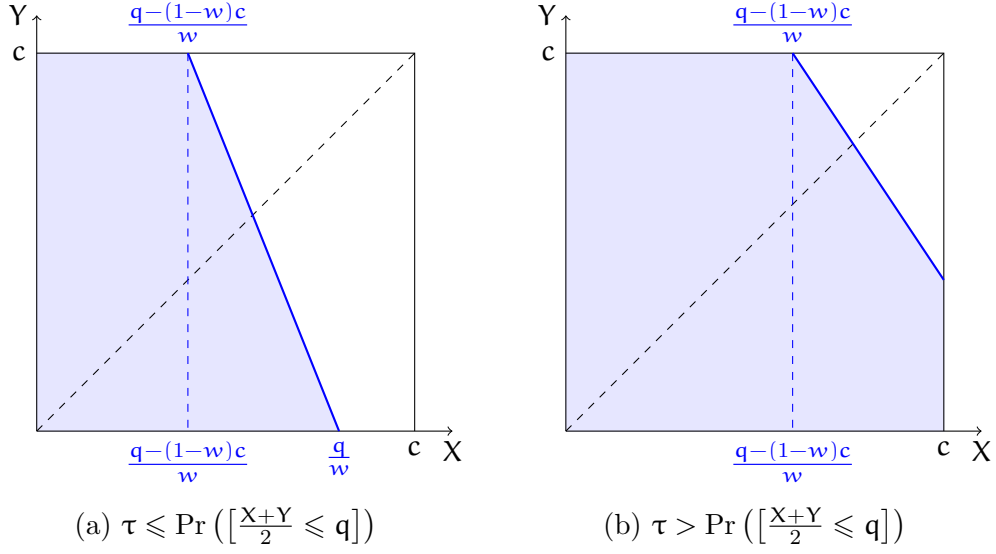


Figure 14: An illustration of Z for the case $\mathcal{J}_X = \mathcal{J}_Y = [0, c]$. The blue area has probability τ .

Therefore,

$$h(w, q) = \int_0^{\frac{q-(1-w)c}{w}} g(0, c, x) dx + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} g\left(0, \frac{q-wx}{1-w}, x\right) dx.$$

Thus,

$$\begin{aligned} \partial_w h(w, q) &= g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right] \\ &\quad + \int_0^{\frac{q-(1-w)c}{w}} \partial_w [g(0, c, x)] dx \\ &\quad + g\left(0, 0, \frac{q}{w}\right) \left(\frac{-q}{w^2}\right) - g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right] \\ &\quad + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_w \left[g\left(0, \frac{q-wx}{1-w}, x\right)\right] dx. \end{aligned}$$

Since $g(0, 0, \cdot) = 0$ and $\partial_w g(0, c, x) = 0$, and $g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right]$ appear with

+ and - signs, we have

$$\begin{aligned}
\partial_w h(w, q) &= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_w \left[g\left(0, \frac{q-wx}{1-w}, x\right) \right] dx \\
&= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) \partial_w \left[\frac{q-wx}{1-w} \right] dx \\
&= \frac{q-x}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx.
\end{aligned}$$

We have:

$$\begin{aligned}
\partial_q h(w, q) &= g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_q \left[\frac{q-(1-w)c}{1-w} \right] \\
&\quad + \int_0^{\frac{q-(1-w)c}{w}} \partial_q [g(0, c, x)] dx \\
&\quad + g\left(0, 0, \frac{q}{w}\right) \left(\frac{1}{w}\right) - g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_q \left[\frac{q-(1-w)c}{1-w} \right] \\
&\quad + \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_q \left[g\left(0, \frac{q-wx}{1-w}, x\right) \right] dx.
\end{aligned}$$

Analogously to the previous case,

$$\begin{aligned}
\partial_q h(w, q) &= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} \partial_q \left[g\left(0, \frac{q-wx}{1-w}, x\right) \right] dx \\
&= \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) \partial_q \left[\frac{q-wx}{1-w} \right] dx \\
&= \frac{1}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} (q-x) f\left(x, \frac{q-wx}{1-w}\right) dx.
\end{aligned}$$

Note that in this case,

$$f_Z(z) = \frac{f\left(z, \frac{q-wz}{1-w}\right)}{\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx},$$

so that

$$E[Z] = \frac{\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} xf\left(x, \frac{q-wx}{1-w}\right) dx}{\int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx}.$$

From this, we observe that the same expressions remain valid:

$$\begin{aligned}
\partial_w h(w, q) &= \frac{q}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} f\left(x, \frac{q-wx}{1-w}\right) dx - \frac{1}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^{\frac{q}{w}} xf\left(x, \frac{q-wx}{1-w}\right) dx \\
&= \frac{q}{1-w} \partial_q h(w, q) - \frac{1}{(1-w)} E[Z] \cdot \partial_q h(w, q) \\
&= \frac{1}{1-w} \cdot (q - E[Z]) \partial_q h(w, q),
\end{aligned}$$

and

$$q'(w) = -\frac{\partial_w h(w, q)}{\partial_q h(w, q)} = \frac{1}{1-w} (E[Z] - q),$$

as before.

Case (b)

In this case,

$$h(w, q) = \int_0^{\frac{q-(1-w)c}{w}} \int_0^c f(x, y) dy dx + \int_{\frac{q-(1-w)c}{w}}^c \int_0^{\frac{q-wx}{1-w}} f(x, y) dy dx,$$

and

$$h(w, q) = \int_0^{\frac{q-(1-w)c}{w}} g(0, c, x) dx + \int_{\frac{q-(1-w)c}{w}}^c g\left(0, \frac{q-wx}{1-w}, x\right) dx.$$

Thus,

$$\begin{aligned}
\partial_w h(w, q) &= g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right] \\
&\quad + \int_0^{\frac{q-(1-w)c}{w}} \partial_w [g(0, c, x)] dx \\
&\quad - g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right] \\
&\quad + \int_{\frac{q-(1-w)c}{w}}^c \partial_w \left[g\left(0, \frac{q-wx}{1-w}, x\right)\right] dx.
\end{aligned}$$

Since $\partial_w g(0, c, x) = 0$, and $g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_w \left[\frac{q-(1-w)c}{1-w}\right]$ appear with + and - signs,

$$\begin{aligned}\partial_w h(w, q) &= \int_{\frac{q-(1-w)c}{w}}^c \partial_w \left[g\left(0, \frac{q-wx}{1-w}, x\right) \right] dx \\ &= \int_{\frac{q-(1-w)c}{w}}^c f\left(x, \frac{q-wx}{1-w}\right) \partial_w \left[\frac{q-wx}{1-w} \right] dx \\ &= \frac{q-x}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^c f\left(x, \frac{q-wx}{1-w}\right) dx.\end{aligned}$$

Also,

$$\begin{aligned}\partial_q h(w, q) &= g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_q \left[\frac{q-(1-w)c}{1-w} \right] \\ &\quad + \int_0^{\frac{q-(1-w)c}{w}} \partial_q [g(0, c, x)] dx \\ &\quad - g\left(0, c, \frac{q-(1-w)c}{w}\right) \partial_q \left[\frac{q-(1-w)c}{1-w} \right] \\ &\quad + \int_{\frac{q-(1-w)c}{w}}^c \partial_q \left[g\left(0, \frac{q-wx}{1-w}, x\right) \right] dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\partial_q h(w, q) &= \int_{\frac{q-(1-w)c}{w}}^c \partial_q \left[g\left(0, \frac{q-wx}{1-w}, x\right) \right] dx \\ &= \int_{\frac{q-(1-w)c}{w}}^c f\left(x, \frac{q-wx}{1-w}\right) \partial_q \left[\frac{q-wx}{1-w} \right] dx \\ &= \frac{1}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^c (q-x) f\left(x, \frac{q-wx}{1-w}\right) dx.\end{aligned}$$

Note that in this case,

$$f_Z(z) = \frac{f\left(z, \frac{q-wz}{1-w}\right)}{\int_{\frac{q-(1-w)c}{w}}^c f\left(x, \frac{q-wx}{1-w}\right) dx},$$

so that

$$E[Z] = \frac{\int_{\frac{q-(1-w)c}{w}}^c xf\left(x, \frac{q-wx}{1-w}\right) dx}{\int_{\frac{q-(1-w)c}{w}}^c f\left(x, \frac{q-wx}{1-w}\right) dx}.$$

From this, we observe that the same expressions remain valid:

$$\begin{aligned}
\partial_w h(w, q) &= \frac{q}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^c f\left(x, \frac{q-wx}{1-w}\right) dx - \frac{1}{(1-w)^2} \int_{\frac{q-(1-w)c}{w}}^c xf\left(x, \frac{q-wx}{1-w}\right) dx \\
&= \frac{q}{1-w} \partial_q h(w, q) - \frac{1}{(1-w)} E[Z] \cdot \partial_q h(w, q) \\
&= \frac{1}{1-w} \cdot (q - E[Z]) \partial_q h(w, q)
\end{aligned}$$

and

$$q'(w) = -\frac{\partial_w h(w, q)}{\partial_q h(w, q)} = \frac{1}{1-w} (E[Z] - q),$$

as before. □

Proof of Corollary 3.11:

Proof. The proof of this result is immediate from imposing $q'(w^*) = 0$ in Proposition 3.10, and noting that $q''(w) = \frac{\partial}{\partial w} E[Z_{w,q}]$, with $q''(\cdot)$ the second derivative with respect to w . To show this, we take the first derivative of $q'(w)$ in (10). Then,

$$q''(w) = \frac{\left(\frac{\partial}{\partial w} E[Z] - q'(w)\right) (1-w) + E[Z] - q(w)}{(1-w)^2}. \quad (24)$$

Then, noting that $E[Z] = q(w^*)$ and $q'(w^*) = 0$, we have

$$q''(w) = \frac{1}{(1-w)^2} \frac{\partial}{\partial w} E[Z]. \quad (25)$$

□

Proof of Proposition 3.13:

Proof. The proof of this result is shown in three different stages. First, we note from Corollary 3.11 that the first order condition characterizing an extremum of the optimization problem (7) is $E[Z] = q(w^*)$, with $w^* \in (0, 1)$. Now, given that the density function $f_Z(z)$ is evaluated over the line $y = \frac{q(w^*) - w^*x}{1-w^*}$, it is not difficult to see that the projection of this line on the y -axis is $y^* = E[Z]$ for $x^* = E[Z]$. This result implies that for different quantile values $q(w^*) \equiv q(w^*, \tau)$ indexed by $\tau \in (0, 1)$, the mean value of the random variable Z characterized by the density function $f_Z(z)$ in (9) is in the 45% degree line such that $(x^*, y^*) = (E[Z], E[Z])$.

Second, we prove that condition (11) evaluated at w^* guarantees the symmetry around $E[Z]$ of the density function f_Z for $x \in \mathcal{J}_Z = \mathcal{J}_Z^{w^*, q} = \{x \in \mathbb{R} : \left(x, \frac{q-w^*x}{1-w^*}\right) \in \mathcal{J}_X \times \mathcal{J}_Y\}$. More formally, condition (11) implies $f_Z(E[Z] + \varepsilon) = f_Z(E[Z] - \varepsilon)$ for all $\varepsilon > 0$. This is so by noting

$$\text{that } f_Z(E[Z] + \varepsilon) = \frac{f(E[Z] + \varepsilon, E[Z] - \frac{w^*}{1-w^*} \varepsilon)}{\int_{J_Z} f\left(t, \frac{E[Z] - tw^*}{1-w^*}\right) dt} \text{ and } f_Z(E[Z] - \varepsilon) = \frac{f(E[Z] - \varepsilon, E[Z] + \frac{w^*}{1-w^*} \varepsilon)}{\int_{J_Z} f\left(t, \frac{E[Z] - tw^*}{1-w^*}\right) dt}.$$

Now, we show that under assumption 2, the density function f_Z is unimodal. To show this, we note that $f_Z(z) = \frac{f\left(x, \frac{E[Z] - w^*x}{1-w^*}\right)}{\int_{J_Z} f\left(t, \frac{E[Z] - tw^*}{1-w^*}\right) dt}$, such that under the change of variable $x = E[Z] + \varepsilon$, we obtain $f_Z(z) = \frac{f(E[Z] + \varepsilon, E[Z] - \frac{w^*}{1-w^*} \varepsilon)}{\int_{J_Z} f\left(t, \frac{E[Z] - tw^*}{1-w^*}\right) dt}$. Assumption 2 evaluated at $\mu = E[Z]$ implies that the numerator of this expression increases with ε up to $\varepsilon = 0$ and then decreases. This condition is sufficient to show that $f_Z(z)$ is unimodal with mode at $E[Z]$.

These findings (unimodality and symmetry of $f_Z(\cdot)$) apply to every $\tau \in (0, 1)$ and imply that the point $(x^*, y^*) = (E[Z], E[Z])$ divides the line $\frac{q(w^*) - w^*x}{1-w^*}$ in two equal segments for all quantile values $q(w^*) \equiv q(w^*, \tau)$ indexed by $\tau \in (0, 1)$. This property also implies that $E[Z]$ is the median of the distribution of Z with support the projection of the line on the x-axis. In this scenario no other combination \tilde{w} , with $\tilde{w} \neq w^*$, and such that $q(\tilde{w}, \tau)$ and $q(w^*, \tau)$ defines two different lines for the same $\tau \in (0, 1)$, yields a line $y = \frac{q(\tilde{w}) - \tilde{w}x}{1-\tilde{w}}$ that intersects $y = x$ at $(x^*, y^*) = (E[Z], E[Z])$. Let (\tilde{x}, \tilde{x}) denote such intersection, and let $\tau_0 \in (0, 1)$ be defined by the condition $\tau \leq P(Z \leq q)$ for $\tau \leq \tau_0$, and $P(Z \leq q) < \tau$ for $\tau \in (\tau_0, 1]$. Then, the condition $\tau \leq P(Z \leq q)$, for all $\tau \in (0, \tau_0]$, implies that the projection of the crossing point (\tilde{x}, \tilde{x}) on the x-axis, given by \tilde{x} , is smaller than the corresponding projection for w^* , that is $x^* = E[Z]$. Then, for all $\tilde{w} \in (0, 1)$ with $\tilde{w} \neq w^*$, it follows that $\tilde{x} < E[Z]$, that is equivalent to the condition $q(\tilde{w}) < q(w^*)$ since $q(\tilde{w}) = \tilde{w}x + (1 - \tilde{w})y$. Then, for $x = y = \tilde{x}$, we have $q(\tilde{w}) = \tilde{x}$ and for $x = y = E[Z]$, we have $q(w^*) = E[Z]$. Hence, the quantity w^* maximizes the quantile function for all $\tau \in (0, \tau_0]$.

It remains to see that the condition $q(w^*) = E[Z]$ characterizes a minimum of the optimization problem (7) for $\tau > \tau_0$. In this scenario, the solution to the optimization problem (7) is also in the 45° degree line, however, for $\tau \in (\tau_0, 1]$, it follows that $P(Z \leq q) < \tau$ (see Figure 5(b)). Then, for any $\tilde{w} = w^* \pm \varepsilon$, with $\varepsilon > 0$, the projection of the crossing point (\tilde{x}, \tilde{x}) on the x-axis is larger than the projection of the crossing point $(E[Z], E[Z])$ associated to w^* . Thus, for any $\tilde{w} = w^* \pm \varepsilon$ with $\varepsilon > 0$, it follows that $\tilde{x} > E[Z]$ and, using the above arguments, $q(\tilde{w}) > q(w^*)$ for all $\tilde{w} \neq w^*$, with $w^*, \tilde{w} \in (0, 1)$.

The continuity of $q(w)$ with respect to w implies that the solution to (7) is a corner solution. For $w^* = \frac{1}{2}$ in the region $\tau \in (0, \tau_0]$, the solution for $\tau > \tau_0$ is indistinctively zero or one due to the symmetry of condition (11). Otherwise, if $w^* \neq \frac{1}{2}$ in the region $\tau \in (0, \tau_0]$, the solution to the maximization problem for $\tau > \tau_0$ is given by the random variable with largest upside potential. More formally, it will be one if $F_X(z) < F_Y(z)$ uniformly over z , for z sufficiently large. In contrast, the solution will be zero for $\tau > \tau_0$ if $F_X(z) > F_Y(z)$ uniformly over z , for z sufficiently large. This property is determined by whether $w^* < \frac{1}{2}$ in condition (11) or not. \square

Proof of Corollary 3.14:

Proof. To prove the result in the corollary, it is sufficient to show that the iid assumption implies assumptions 2 and 3 with $w^* = \frac{1}{2}$. Then, applying Proposition 3.13 the result follows.

To prove assumption 2, we note that under the iid condition, for any $\mu \in \mathbb{R}$, it holds that $f(\mu + \varepsilon, \mu - \varepsilon) = f_X(\mu + \varepsilon)f_X(\mu - \varepsilon)$. Now, taking the first derivative with respect to ε , we obtain

$$\frac{\partial f(\mu + \varepsilon, \mu - \varepsilon)}{\partial \varepsilon} = \frac{\partial f_X(\mu + \varepsilon)}{\partial \varepsilon} f_X(\mu - \varepsilon) - \frac{\partial f_X(\mu - \varepsilon)}{\partial \varepsilon} f_X(\mu + \varepsilon). \quad (26)$$

Let m_X denote the mode of the random variable X . For $\mu = m_X$, the unimodality of $f_X(\cdot)$ implies that $\frac{\partial f_X(\mu + \varepsilon)}{\partial \varepsilon} > 0$ for $\varepsilon < 0$, $\frac{\partial f_X(\mu + \varepsilon)}{\partial \varepsilon} = 0$ at $\varepsilon = 0$ and $\frac{\partial f_X(\mu + \varepsilon)}{\partial \varepsilon} < 0$ for $\varepsilon > 0$. Under these conditions, expression (26) yields $\frac{\partial f(\mu + \varepsilon, \mu - \varepsilon)}{\partial \varepsilon} > 0$ for $\varepsilon < 0$; $\frac{\partial f(\mu + \varepsilon, \mu - \varepsilon)}{\partial \varepsilon} = 0$ at $\varepsilon = 0$ and $\frac{\partial f(\mu + \varepsilon, \mu - \varepsilon)}{\partial \varepsilon} < 0$ for $\varepsilon > 0$, as stated in assumption 2.

The proof is a bit more complex for $\mu \neq m_X$. In this case, for $\mu < m_X$ there exists an interval $|\varepsilon| < m_X - \mu$ that needs to be carefully evaluated. More specifically, for values of ε inside the interval, we note that expression (26) is positive if $\varepsilon < 0$ and negative if $\varepsilon > 0$. This follows from noting that for $\varepsilon < 0$ and $\mu - m_X < \varepsilon$, the following condition is satisfied:

$$\frac{\partial f_X(\mu + \varepsilon)/\partial \varepsilon}{\partial f_X(\mu - \varepsilon)/\partial \varepsilon} > \frac{f_X(\mu + \varepsilon)}{f_X(\mu - \varepsilon)}. \quad (27)$$

To show this condition it is sufficient to show that $\frac{\partial f_X(\mu + \varepsilon)}{\partial \varepsilon} > \frac{\partial f_X(\mu - \varepsilon)}{\partial \varepsilon}$ given that $f_X(\mu + \varepsilon) < f_X(\mu - \varepsilon)$ for $\varepsilon < 0$. This condition is, however, fulfilled for $x \in (-\infty, m_X)$ if the density function f_X is unimodal. Similarly, for $\varepsilon > 0$ and $\varepsilon < m_X - \mu$, the following condition is satisfied:

$$\frac{\partial f_X(\mu + \varepsilon)/\partial \varepsilon}{\partial f_X(\mu - \varepsilon)/\partial \varepsilon} < \frac{f_X(\mu + \varepsilon)}{f_X(\mu - \varepsilon)}. \quad (28)$$

This condition can be shown using the above arguments and noting that $\varepsilon > 0$.

For values of ε outside the interval $(0, m_X - \mu)$ the above proof follows straightforwardly.

For $\mu > m_X$, there exists an interval $|\varepsilon| < \mu - m_X$ that needs to be carefully evaluated. Nevertheless, the proof in this case follows as in the previous case. Also, for values of ε outside the interval the above proof follows similarly.

To verify Assumption 3, it is sufficient to show that for $w^* = \frac{1}{2}$ and $\varepsilon > 0$, equation (11) becomes

$$f(\mu + \varepsilon, \mu - \varepsilon) = f(\mu - \varepsilon, \mu + \varepsilon), \quad (29)$$

for all $\mu \in \mathbb{R}$, with $f(\cdot, \cdot)$ the joint density function of the random variables X and Y . This condition is, however, satisfied by construction under the iid assumption. This is so because

$$f(\mu + \varepsilon, \mu - \varepsilon) = f_X(\mu + \varepsilon)f_X(\mu - \varepsilon) = f(\mu - \varepsilon, \mu + \varepsilon), \quad (30)$$

for all $\mu \in \mathbb{R}$.

□

B Numerical simulation study

This Appendix presents additional results on the optimal asset allocation under quantile preferences (QP) for a portfolio $S_w = wX + (1 - w)Y$ given by the mixture of two continuous random variables X and Y , which are chosen from different distribution functions. For illustrative purposes, we consider pairs of Gaussian and Chi-squared random variables, under mutual independence and also with dependence. These results are also compared with the optimal asset allocation obtained under EU. Numerical computation of the portfolios are as described in Section 3.4.

B.1 Independent and identically distributed random variables

Two Chi-Squared independent and identically distributed random variables

To illustrate the optimal portfolio allocation problem for asymmetric distributions, we investigate the case of two independent and identically distributed standard Chi-squared distributions, X and Y . Under risk aversion, the iid assumption is sufficient to guarantee that the optimal portfolio choice of a EU individual will be a fully diversified portfolio given by $w^* = 0.5$. For QP, the same result holds true for values of τ identified with risk aversion. The left panel of Figure 15 shows that the family of distribution functions F_{S_w} satisfies the single-crossing condition defined by Manski (1988); see also de Castro and Galvao (2021, Section 5). The right panel of Figure 15 shows the optimal portfolio allocation is divided into two regions: a first region given by $\hat{w}_n^*(\tau) = 0.5$, for $\tau \leq \tau_0 \approx 0.80$, and a second region given by $\hat{w}_n^*(\tau) = \{0, 1\}$, for $\tau > \tau_0$. The shift in the cut-off point τ_0 compared to $\tau_0 = 0.5$ is due to the asymmetry of the Chi-squared distribution.

B.2 Independently distributed random variables

Two Gaussian independent random variables with different variances

We extend the Gaussian case discussed above and consider the case of independent random variables X and Y with different variances. Let $X \sim N(0, 1)$ and $Y \sim N(0, 2)$, independent of each other. In this case Y is a mean-preserving spread of the random variable X . This observation implies that X second order stochastically dominates Y and, using Fishburn (1977), X is preferred to Y for EU individuals endowed with an increasing and concave utility function. We can extend this result to derive the optimal portfolio allocation of a EU risk averse individual. In this case, the combination S_{w^*} with $w^* = 2/3$ has zero mean and minimizes the variance of the family of random variables S_w . This combination second order stochastically dominates

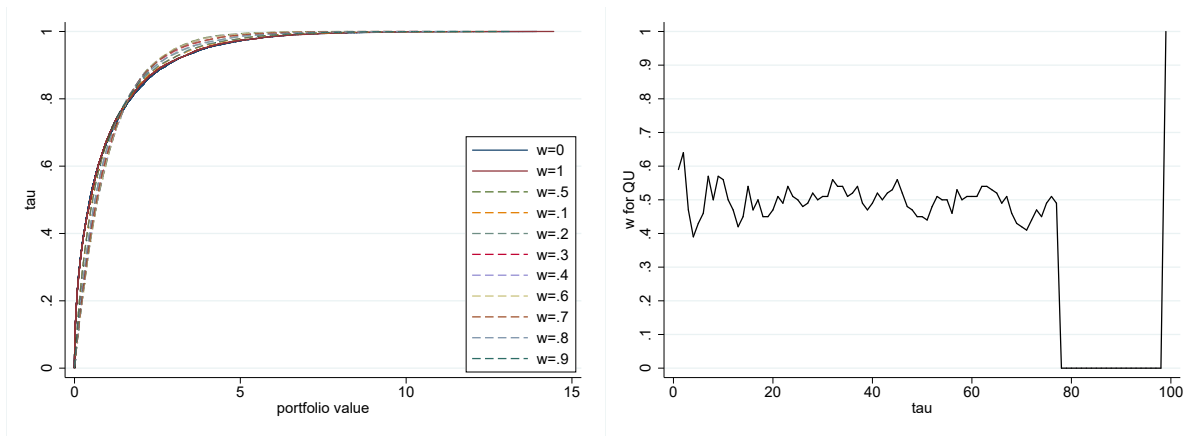


Figure 15: $X, Y \sim \chi_1^2$. Left box plots the CDF of S_w . Right box plots QP portfolio selection.

any other convex combination including X and Y . Then, using [Fishburn \(1977\)](#), S_{w^*} is the optimal strategy for EU individuals endowed with an increasing and concave utility function.

For individuals endowed with QP, we note that condition (11) is satisfied for $w^* = 2/3$. Then, Proposition 3.13 implies that the optimal combination of a QP individual is $w^* = \frac{2}{3}$ for $\tau \leq \tau_0$, with $\tau_0 = \frac{1}{2}$. This result is illustrated in the numerical exercise on the right panel of Figure 16. Diversification takes place for $\tau \leq \frac{1}{2}$ and no diversification is the optimal result for $\tau > \frac{1}{2}$.

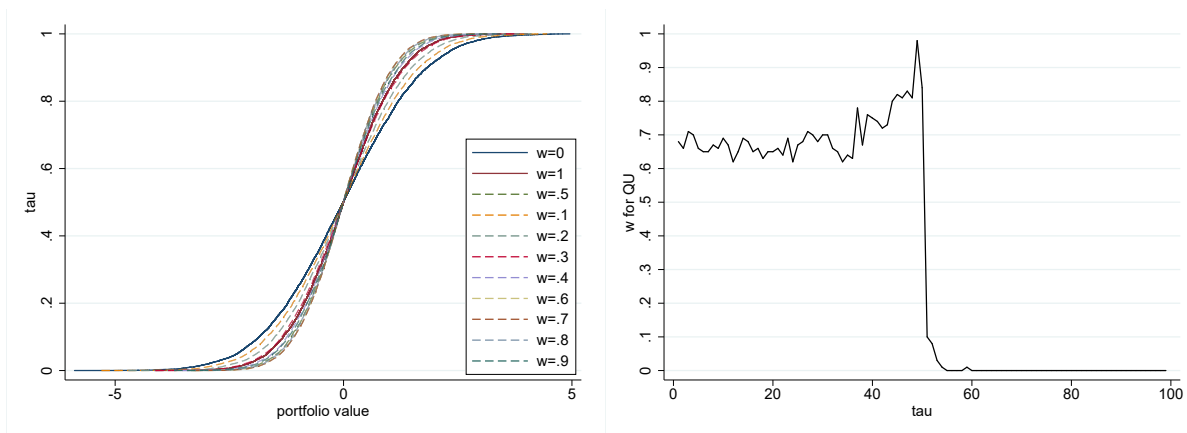


Figure 16: $X \sim N(0,1)$ and $Y \sim N(0,2)$. Left panel plots the CDF of S_w . Right panel plots QP portfolio selection.

Two Gaussian independent random variables with different means

Consider now the case of two independent Gaussian random variables with different means. Let $X \sim N(1,1)$ and $Y \sim N(1.5,1)$, independent of each other. In this case the random variable

Y first order stochastically dominates the random variable X . In this scenario the optimal portfolio decision of an EU individual has to be calculated for each utility function separately because it depends on how the individual faces the trade-off between risk and return.

The results for the QP case are provided in Figure 17. The results show the presence of an interior solution for $\tau \leq \tau_0$, with τ_0 around 0.30, and a corner solution $\hat{w}_n^*(\tau) = 0$ for $\tau > 0.30$. The solution in the right tail of the distribution is rationalized by the first order stochastic dominance of Y over any other convex combination of X and Y for τ sufficiently large.

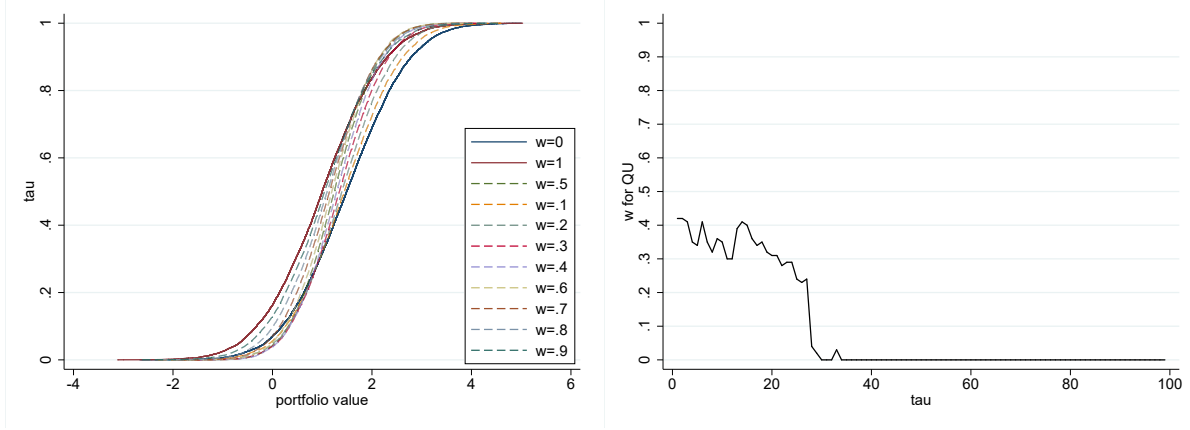


Figure 17: $X \sim N(1, 1)$ and $Y \sim N(1.5, 1)$. Left box plots the CDF of S_w . Right box plots QP portfolio selection.

B.3 Dependent random variables

We study now the case of two dependent random variables. We differentiate between symmetric and asymmetric random variables.

Two Gaussian dependent random variables with same mean

In this exercise, we consider a bivariate Normal random variable (X, Y) with covariance matrix $\Sigma = [1 \ \rho; \rho \ 1]$, where $\rho = 0.5$ is the correlation parameter. This scenario is rationalized by Proposition 3.13. More specifically, condition (11) is satisfied for $\hat{w}_n^*(\tau) = 0.5$, such that an interior solution is obtained for $\tau \leq \tau_0 = 0.5$. This result is observed in Figure 18. For values of τ greater than 0.5 the optimal solution is, indistinctively, $\hat{w}_n^*(\tau) = \{0, 1\}$.

Chi-squared dependent random variables

We complete the case of dependent assets with asymmetric distribution functions. Suppose that $X \sim \chi_1^2 + 1$ and $Y \sim \chi_2^2 + X - 2$ two dependent assets with expected value equal to two in both cases. For QP individuals, the optimal portfolio allocation has an interior solution for

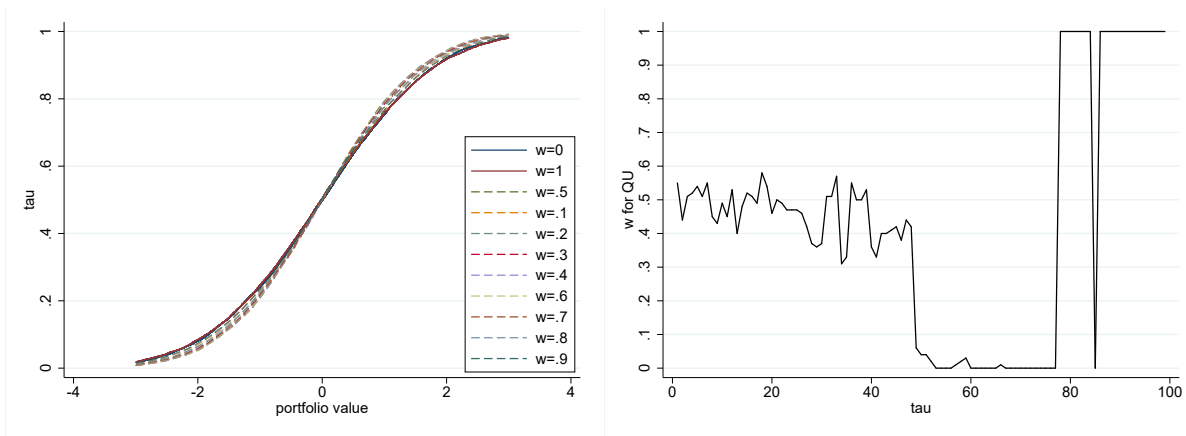


Figure 18: $(X, Y) \sim N(0, \Sigma)$ with $\Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$. Left box plots CDF of S_w . Right box plots QP portfolio selection.

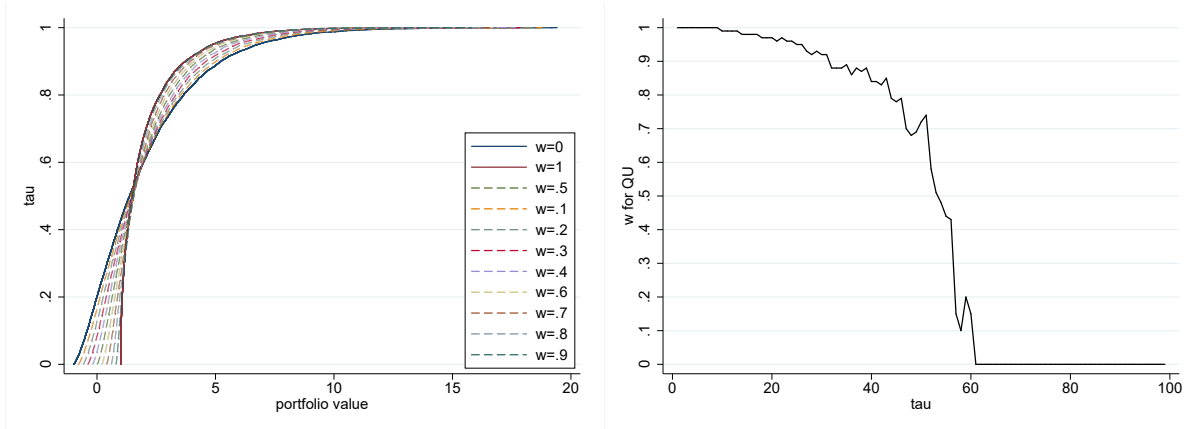


Figure 19: $X \sim \chi_1^2 + 1$ and $Y \sim \chi_2^2 + X - 2$. Left box plots the CDF of S_w . Right box plots QP portfolio selection.

$\tau \leq \tau_0$, with $\tau_0 = 0.60$. The presence of an interior solution for small values of τ is rationalized by Theorem 1. However, in contrast to previous examples, assumptions 3 is not satisfied implying that the optimal portfolio allocation varies over τ and there is no separation between risk aversion and risk loving behavior. Figure 19 reports the optimal portfolio allocation in this case.

B.4 Optimal portfolio allocation when there is a risk-free asset

Building upon previous insights of the QP theory in Section 3.2, we can extend the mutual fund separation theorem to the case of one risk free asset with returns \bar{r} and two risky assets R_1, R_2 with distribution functions F_{R_1}, F_{R_2} , respectively, and such that F_{R_1} crosses F_{R_2} from below at point x_{12} . This assumption implies that R_2 is riskier than R_1 . The portfolio return

is defined by the convex combination $R_p = w_0\bar{r} + w_1R_1 + w_2R_2$, with $w_0 + w_1 + w_2 = 1$. The investor's optimization problem is

$$\arg \max_{\{w_0, w_1, w_2\}} Q_\tau[w_0\bar{r} + w_1R_1 + w_2R_2]. \quad (31)$$

In this case the individual's optimal portfolio choice is given by the risk-free rate when $Q_\tau[R_p(w)] < \bar{r}$, for any combination of weights $w = \{w_0, w_1, w_2\}$ that is different from $w^* = \{w_0^* = 1, w_1^* = w_2^* = 0\}$. For higher quantiles, the solution to the maximization problem (31) is, in principle, quantile-specific. We illustrate this scenario by simulating the returns on three assets with returns \bar{r} , $R_1 \sim N(\mu_1, 1)$ and $R_2 \sim N(\mu_2, 1)$. The solution for $0 \leq \tau \leq 1$ is obtained by simulating $n = 10,000$ realizations of the random variables.

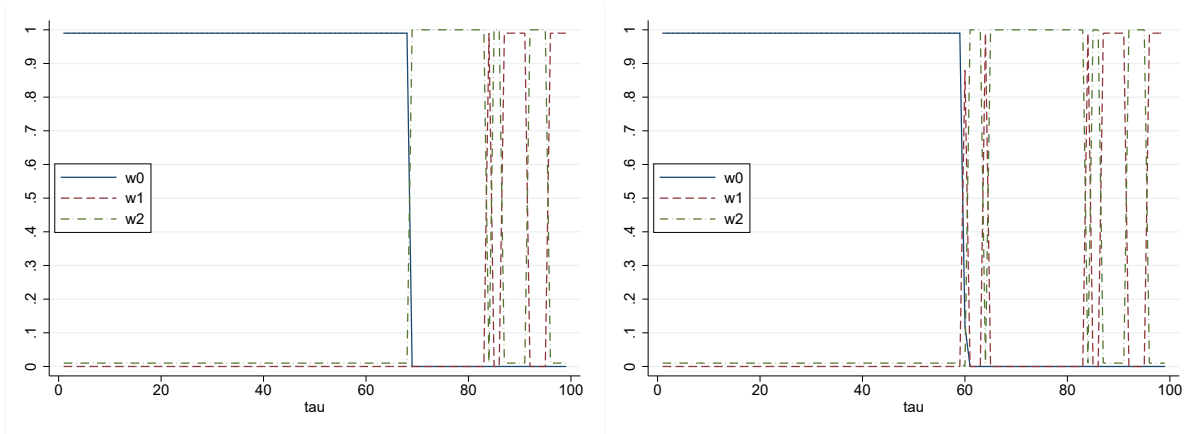


Figure 20: $R_p = w_0\bar{r} + w_1R_1 + w_2R_2$, with \bar{r} constant, $R_1 \sim N(\mu_1, 1)$ and $R_2 \sim N(\mu_2, 1)$. Left box plots $\bar{r} = 0.5$, $\mu_1 = \mu_2 = 0$. Right box plots $\bar{r} = 0.25$, $\mu_1 = \mu_2 = 0$.

The left panel of Figure 20 reports the case $\bar{r} = 0.5$, $\mu_1 = \mu_2 = 0$. The right panel considers the case $\bar{r} = 0.25$, $\mu_1 = \mu_2 = 0$. As discussed above, for $Q_\tau[R_p(w)] < \bar{r}$, the optimal allocation to the portfolio is given by investing on the risk-free asset. For the parameterization $\bar{r} = 0.5$, $\mu_1 = \mu_2 = 0$ - left panel - the value of τ that yields the condition $Q_\tau[R_p(w^*)] = \bar{r}$ is $\tau_0 \approx 0.7$. For values of $\tau > 0.7$, the optimal allocation to the risk-free asset is zero, and the QP individual is indifferent between assets 1 and 2. The right panel characterized by a smaller risk-free return ($\bar{r} = 0.25$) presents a similar outcome. In this scenario the value of τ that satisfies the condition $Q_\tau[R_p(w^*)] = \bar{r}$ is $\tau_0 \approx 0.6$. For higher quantiles, the optimal portfolio allocation is the same as for the left panel and entails a zero allocation to the risk-free rate.¹⁶

¹⁶The case of two risky assets with different means is available from the authors upon request.

References

- APIWATCHAROENKUL, W., L. LAKE, AND J. JENSEN (2016): “Uncertainty in Proved Reserves Estimates by Decline Curve Analysis,” in *SPE/IAEE Hydrocarbon Economics and Evaluation Symposium*, Society of Petroleum Engineers.
- ARTZNER, P., F. DELBAEN, J. EBER, AND D. HEATH (1999): “Coherent Measures of Risk,” *Mathematical Finance*, 9, 203–228.
- BALTUSSEN, G. AND G. T. POST (2011): “Irrational Diversification: An Examination of Individual Portfolio Choice,” *Journal of Financial and Quantitative Analysis*, 46, 1463–1491.
- BASAK, S. AND A. SHAPIRO (2001): “Value-at-Risk Based Risk Management: Optimal Policies and Asset Prices,” *Review of Financial Studies*, 14, 371–405.
- BASSETT, G. W., R. KOENKER, AND G. KORDAS (2004): “Pessimistic Portfolio Allocation and Choquet Expected Utility,” *Journal of Financial Econometrics*, 2, 477–492.
- BELL, D. E. (1982): “Regret in Decision Making Under Uncertainty,” *Operations Research*, 30, 961–981.
- BHATTACHARYA, D. (2009): “Inferring Optimal Peer Assignment From Experimental Data,” *Journal of the American Statistical Association*, 104, 486–500.
- BROWN, D. B. AND M. SIM (2009): “Satisficing Measures for Analysis of Risky Positions,” *Management Science*, 55, 71–84.
- CAMPBELL, J. (2006): “Household Finance,” *Journal of Financial Economics*, 61, 1553–1604.
- CAMPBELL, J. Y. (2018): *Financial decisions and markets: a course in asset pricing*, Princeton University Press.
- CAMPBELL, R., R. HUISMAN, AND K. KOEDIJK (2001): “Optimal Portfolio Selection in a Value-at-Risk Framework,” *Journal of Banking and Finance*, 25, 1789–1804.
- CHAMBERS, C. P. (2007): “Ordinal Aggregation and Quantiles,” *Journal of Economic Theory*, 137, 416–431.
- (2009): “An Axiomatization of Quantiles on the Domain of Distribution Functions,” *Mathematical Finance*, 19, 335–342.
- COCHRANE, J. H. (2005): *Asset Pricing, Revised Edition*, Princeton University Press.
- DAMODARAN, A. (2010): “Into the Abyss: What If Nothing is Risk Free?” Available at SSRN: <https://ssrn.com/abstract=1648164> or <http://dx.doi.org/10.2139/ssrn.1648164>.

- DE CASTRO, L. AND A. F. GALVAO (2019): “Dynamic Quantile Models of Rational Behavior,” *Econometrica*, 87, 1893–1939.
- (2021): “Static and Dynamic Quantile Preferences,” Forthcoming, *Economic Theory*.
- DE CASTRO, L., A. F. GALVAO, C. N. NOUSSAIR, AND L. QIAO (2020): “Do People Maximize Quantiles?” Working paper, University of Arizona.
- DUFFIE, D. AND J. PAN (1997): “An Overview of Value at Risk,” *Journal of Derivatives*, 4, 7–49.
- ENGLE, R. F. AND S. MANGANELLI (2004): “CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles,” *Journal of Business and Economic Statistics*, 22, 367–381.
- FANCHI, J. R. AND R. L. CHRISTIANSEN (2017): *Introduction to Petroleum Engineering*, Wiley Online Library.
- FISHBURN, P. C. (1977): “Mean-Risk Analysis with Risk Associated with Below-Target Returns,” *American Economic Review*, 67, 116–126.
- FÖLLMER, H. AND P. LEUKERT (1999): “Quantile Hedging,” *Finance and Stochastics*, 3, 251–273.
- GAIVORONSKI, A. AND G. PFLUG (2005): “Value-at-Risk in Portfolio Optimization: Properties and Computational Approach,” *Journal of Risk*, 7(2), 1–31.
- GARLAPPI, L., R. UPPAL, AND T. WANG (2007): “Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach,” *Review of Financial Studies*, 20, 41–81.
- GHIRARDATO, P., F. MACCHERONI, AND M. MARINACCI (2005): “Certainty Independence and the Separation of Utility and Beliefs,” *Journal of Economic Theory*, 120, 129–136.
- GHIRARDATO, P. AND M. MARINACCI (2002): “Ambiguity Made Precise: A Comparative Foundation,” *Journal of Economic Theory*, 102, 251–289.
- GILBOA, I. AND D. SCHMEIDLER (1989): “Maxmin Expected Utility with a Non-unique Prior,” *Journal of Mathematical Economics*, 18, 141–153.
- GIOVANNETTI, B. C. (2013): “Asset Pricing under Quantile Utility Maximization,” *Review of Financial Economics*, 22, 169–179.
- HE, X. D. AND X. Y. ZHOU (2011): “Portfolio Choice Via Quantiles,” *Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics*, 21, 203–231.
- IBRAGIMOV, R. AND D. WALDEN (2007): “The Limits of Diversification When Losses May Be Large,” *Journal of Banking and Finance*, 31, 2551–2569.

- JORION, P. (2007): *Value at Risk. The New Benchmark for Managing Financial Risk*, vol. 81, McGraw-Hill. 3rd Edition.
- KAHNEMAN, D. AND A. TVERSKY (1979): "Prospect Theory: An Analysis of Decision under Risk," *Econometrica*, 47, 263–292.
- KEMPF, A. AND S. RUENZI (2007): "Tournaments in Mutual-Fund Families," *The Review of Financial Studies*, 21, 1013–1036.
- KROKHMAL, P., S. URYASEV, AND J. PALMQUIST (2001): "Portfolio Optimization with Conditional Value-at-Risk Objective and Constraints," *Journal of Risk*, 4(2), 43–68.
- KULLDORFF, M. (1993): "Optimal Control of Favorable Games with a Time Limit," *SIAM Journal on Control and Optimization*, 31, 52–69.
- LINTNER, J. (1965): "The Valuation of Risky Assets and the Selection of Risky Investments in the Portfolios and Capital Budgets," *The Review of Economics and Statistics*, 47, 13–37.
- MANSKI, C. (1988): "Ordinal Utility Models of Decision Making under Uncertainty," *Theory and Decision*, 25, 79–104.
- MARKOWITZ, H. (1952): "Portfolio Selection," *Journal of Finance*, 7, 77–91.
- MENDELSON, H. (1987): "Quantile-Preserving Spread," *Journal of Economic Theory*, 42, 334–351.
- MITTON, T. AND K. VORKINK (2007): "Equilibrium underdiversification and the preference for skewness," *The Review of Financial Studies*, 20, 1255–1288.
- PAYNE, J. W., J. R. BETTMAN, AND E. J. JOHNSON (1992): "Behavioral Decision Research: A Constructive Processing Perspective," *Annual Review of Psychology*, 43, 87–131.
- PORTER, R. B. (1974): "Semivariance and Stochastic Dominance: A Comparison," *American Economic Review*, 64, 200–204.
- QUIGGIN, J. (1982): "A Theory of Anticipated Utility," *Journal of Economic Behavior & Organization*, 3, 323–343.
- RABIN, M. (2000): "Risk Aversion and Expected-Utility Theory: A Calibration Theorem," *Econometrica*, 68, 1281–1292.
- RACHEV, S. T., S. STOYANOV, AND F. J. FABOZZI (2007): *Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization: The Ideal Risk, Uncertainty, and Performance Measures*, John Wiley, Finance.

- ROSTEK, M. (2010): “Quantile Maximization in Decision Theory,” *Review of Economic Studies*, 77, 339–371.
- ROTHSCHILD, M. AND J. E. STIGLITZ (1970): “Increasing Risk: I. A Definition,” *Journal of Economic Theory*, 2, 225 – 243.
- SAVAGE, L. (1954): *The Foundations of Statistics*, New York: Wiley.
- SHARPE, W. F. (1964): “Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk,” *The Journal of Finance*, 19, 425–462.
- SIMON, H. A. (1979): “Rational Decision Making in Business Organizations,” *American Economic Review*, 69, 493–513.
- TOBIN, J. (1958): “Liquidity Preference as Behavior Towards Risk,” *Review of Economic Studies*, 67, 65–86.
- TVERSKY, A. AND D. KAHNEMAN (1981): “The Framing of Decisions and the Psychology of Choice,” *Science*, 211, 453–458.
- (1992): “Advances in Prospect Theory: Cumulative Representation of Uncertainty,” *Journal of Risk and Uncertainty*, 5, 297–323.
- WU, G. AND Z. XIAO (2002): “An Analysis of Risk Measures,” *Journal of Risk*, 4(4), 53–75.
- YAARI, M. E. (1987): “The Dual Theory of Choice Under Risk,” *Econometrica*, 95–115.