## UNIVERSITY OF SOUTHAMPTON

## Topics in the theory of soluble groups of finite rank


by

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A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy
in the
Faculty of Social, Human and Mathematical Sciences
Department of Mathematics

June 2018

## UNIVERSITY OF SOUTHAMPTON


#### Abstract

FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES Department of Mathematics

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This thesis contains a spectrum of different results all of which, broadly speaking, are motivated by the structure of soluble groups obeying various finiteness conditions. Chapter 1 contains introductory material required throughout the thesis. In chapters 2 and 3 , we study endomorphisms of nilpotent groups of finite rank and find criteria to guarantee that they are automorphisms, generalising (independent) work of Farkas [13] and Wehrfritz [32]. Chapter 4 exploits the Mal'cev correspondence for divisible nilpotent groups to characterise so-called powered nilpotent groups, and also contains refinements of results due to Segal [26]. Chapter 5 contains an explicit construction of the free Lie algebra on a module, along with an exposition of the theory of algebraic theories and functors. Finally, in chapter 6 we give an explicit characterisation of the socle series of certain modules over the class of commutative Von Neumann regular rings, confirming conjectures of Usher [29].


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## Southampton

## Academic thesis: Declaration of authorship

I, Hector Anthony Melville Durham, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

## Topics in the theory of soluble groups of finite rank

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1. This work was done wholly or mainly while in candidature for a research degree at this University;
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## Acknowledgements

First and foremost, I would like to thank my supervisor, Peter Kropholler, for his patient guidance and support, and for introducing me to a fascinating branch of mathematics. I would also express my gratitude to Bernhard Koeck for his continued advice throughout my time as a student. Furthermore I am indebted to Ian Leary for a careful reading of a draft version of this thesis, and for his time and advice towards the end of my PhD. The last four years have been an amazing experience, and I would like to express my thanks to all the postgraduate students and postdoctoral staff of the School of Mathematics at Southampton, past and present.

Finally and most especially I wish to thank my family for their unremitting support.

## Introduction

Let $G$ be a group. Recall that $G$ is said to be cohopfian if every injective endomorphism of $G$ is surjective. It is immediate that the integers are not cohopfian, and one might imagine that amongst groups belonging to the wider class of finitely generated torsion-free nilpotent groups (henceforth the class of $\mathfrak{T}$-groups) one cannot find cohopfian examples. That this is not the case was first observed by Smith [27], who manufactured examples by examining sufficiently pathological nilpotent Lie algebras and appealing to the Mal'cev correspondence. One has to look far to find these examples: any example necessarily has Hirsch length at least 7 and class at least 3. Further work in this area was undertaken by Belegradek [1], but a significant breakthrough was recently and almost simultaneously achieved by Cornulier [7] and Deré [10], who characterised the property of a $\mathfrak{T}$-group being cohopfian entirely in terms of the associated Lie algebra. A nontrivial consequence of this is that being cohopfian is a commensurability invariant for the class of $\mathfrak{T}$-groups.

We motivate our interest in this recent work by the following classical result concerning the structure of soluble constructable groups. In the elementary amenable case, there are many equivalent characterisations of this property in terms of various homological and homotopical finiteness conditions: we refer the reader to [19] and the references within for a detailed account of the various connections. A particular consequence of these equivalent structural constraints is that the groups are nilpotent-by-abelian-byfinite. The classical Bieri-Strebel invariants were defined in this setting in [2] and from this paper we highlight the following result, simplified for ease of exposition.

Theorem ( [2, 5.2]). Suppose that $G$ is a torsion-free finitely generated soluble group with nilpotent normal subgroup $N$ and abelian quotient $G / N$. Then the following are equivalent.

1. $G$ is constructable;
2. there exists some $g \in G$ and finitely generated $H \leqslant N$ such that $H^{g} \leqslant H$ and

$$
N=\bigcup_{i \geqslant 0} H^{g^{-i}}
$$

We remark here that a state of the art version of the above may also be found in [19]. In particular, if one is interested in the structure of these groups one method of attack is to begin by considering endomorphisms of $\mathfrak{T}$-groups and the structure of the resulting ascending unions.

Note in particular that outside of the polycyclic case, the nilpotent normal subgroup in question is not finitely generated. Inspired by this, we have been considering endomorphisms of nilpotent groups which are not finitely generated, but are nevertheless of finite rank. In particular, we obtain criteria for an individual endomorphism of a torsion-free nilpotent group to be an automorphism, in terms of the induced action on the torsionfree abelianisation and on the centre. Whilst the results we obtain are known in the finitely generated setting (see the chapter for further details and references), dropping this assumption adds several difficulties.

It is essentially elementary to show that the torsion-free abelianisation rather cleanly detects the surjectivity of an endomorphism, but one encounters significantly more difficulty by considering only the restriction to the centre: indeed the naive requirement that the restriction to the centre is an automorphism is far from sufficient. We refer the reader to chapter 3 for details, noting that chapter 2 contains preparatory material.

In chapter 4, we continue our focus on nilpotent groups, but this time heavily exploiting the Lie theoretic technology encoded in the Mal'cev correspondence. The first part is concerned with a characterisation of so-called powered nilpotent groups - essentially those which admit an endomorphism which acts as multiplication by a fixed nonzero rational on the abelianisation. By using the above cited recent work of Cornulier [7] we find that these correspond exactly to the class of nilpotent groups whose associated Lie algebra is 'Carnot'. Loosely speaking, this property may be thought of as being opposite to cohopfian in a strong sense.

We end chapter 4 with a section which generalises results of Segal [26] concerning how the Mal'cev correspondence behaves with respect to $\mathfrak{T}$-groups. In particular, we remove the need to consider a specific representation and also generalise certain statements to Lie rings.

Chapter 5 contains an explicit construction of the free $k$-Lie algebra on a $k$-module, where $k$ is a Dedekind domain. Our construction involves specific representations of the tensor and universal enveloping algebras. The first section of this chapter outlines a useful perspective from which one can easily deduce the existence of various adjoint functors: this is the language of algebraic functors and categories. In particular the existence of the aforementioned adjoint is easily obtained.

The thesis concludes with an explicit characterisation of the socle series of a semisimple module over a commutative Von Neumann regular ring, in terms of certain topological features of the spectrum. Originally this work was completed in the case of certain special semisimple modules over $\mathbb{C} C_{p \infty}$ - the motivation being to extend the novel techniques of Kropholler's theorem concerning soluble groups of finite cohomological dimension [18]. It was later found that the results hold in far more generality. The results in this chapter prove several conjectures of Usher's thesis [29] and fit rather cleanly with previous literature - see the final sections of the chapter for more details.

## Chapter 1

## Background group theory

In this first chapter we introduce some standard material to be used throughout the thesis. For the most part we will concern ourselves with infinite nilpotent groups of finite rank (we make this more precise later), and occasionally the more general class of infinite soluble groups of finite rank.

In the first section we detail the existence of certain canonical homomorphisms associated to any ascending or descending central series of any group, terminating or not. Next we specialise to the nilpotent case and establish some fundamental results concerning torsion and divisibility in this class, before introducing the crucial notion of an isolator. We then introduce two notions of rank: Hirsch length, for soluble groups in particular; and the Prüfer rank defined for any group. We note in particular that for the class of torsion-free nilpotent groups these coincide. We conclude the chapter by discussing classical results due to Mal'cev: the radicable hull, or Mal'cev completion of a nilpotent group, and the Mal'cev correspondence between certain nilpotent groups and Lie algebras.

Since the material here is well-known, we will occasionally only give the idea of the proof, or even just a direct reference. Nevertheless the proofs that we give here are our own, unless referenced otherwise.

Our convention for commutators in the following is:

$$
\begin{aligned}
{[x, y] } & :=x^{-1} y^{-1} x y \\
{\left[x_{1}, \ldots, x_{n}\right] } & :=\left[\left[x_{1}, \ldots, x_{n-1}\right], x_{n}\right] .
\end{aligned}
$$

### 1.1 The tensor power maps and their duals

In this section we document a tensor power map associated to any descending central series of a group, terminating or not. Dually, we discuss certain maps involving groups of homomorphisms associated to any ascending central series, again terminating or not. We will state the results in sufficient generality for later use.

This first lemma is presumably well known but we have been unable to find a reference for the general formulation we need. For the case when the descending series is precisely the lower central series, one reference is for example [21, 1.2.11].

Lemma 1.1.1. Let $G$ be a group and $\sigma$ an endomorphism of $G$. Suppose we are given a sequence of subgroups

$$
G=G_{1} \geqslant G_{2} \geqslant \cdots
$$

such that $\sigma\left(G_{i}\right) \leqslant G_{i}$ for each $i$, and such that for each $i, j \geqslant 1$ we have inclusions

$$
\left[G_{i}, G_{j}\right] \leqslant G_{i+j}
$$

Denote by $\mathbb{Z}[\sigma]$ the monoid ring on $\sigma$. Then for each $i \geqslant 1$ there are homomorphisms of $\mathbb{Z}[\sigma]$-modules

$$
\begin{aligned}
G_{i} / G_{i+1} \otimes_{\mathbb{Z}} G / G_{2} & \longrightarrow G_{i+1} / G_{i+2} \\
x G_{i+1} \otimes g G_{2} & \longmapsto[x, g] G_{i+2}
\end{aligned}
$$

and hence also module homomorphisms

$$
\left(G / G_{2}\right)^{\otimes i} \longrightarrow G_{i} / G_{i+1}
$$

where we equip tensor products with the diagonal action.

Proof. The details here are routine in view of the following standard commutator identities, holding for any $x, y, z$ in any group:

$$
\begin{aligned}
{[x, y z] } & =[x, z][x, y][x, y, z], \\
{[x y, z] } & =[x, z][x, z, y][y, z] .
\end{aligned}
$$

The equivariance follows immediately from the fact that endomorphisms commute with commutators.

The following lemma is perhaps less well-known but nevertheless elementary. We will apply it to the isolated central series, to be introduced in a later section.

Lemma 1.1.2. Let $G$ be a group and $\left(G_{i}\right)_{i \geqslant 1}$ a descending series as in lemma (1.1.1). Then the image of the tensor power map

$$
\left(G / G_{2}\right)^{\otimes i} \longrightarrow G_{i} / G_{i+1}
$$

is precisely $\gamma_{i}(G) G_{i+1} / G_{i+1}$.

Proof. We first note that for each $i \geqslant 1$, we have $\gamma_{i}(G) \leqslant G_{i}$ by the very definition of the series. Thus the statement makes sense, and we also obtain an epimorphism $G / \gamma_{2}(G) \rightarrow G / G_{2}$. We also note that the tensor power maps for the lower central series are certainly surjective, and furthermore that tensor powers of surjective maps are also surjective.

The following diagram is thus obtained, where the right hand vertical map is canonical, and the horizontal maps are as in lemma (1.1.1).


It is routine to check that it is commutative, and the result now follows, using the fact that the left hand vertical map is surjective.

We now concern ourselves with the dualized versions of the maps we have introduced above. In the sequel we will apply this only for the upper central series of a nilpotent group, but we give an analogous treatment to the above since this unifies the exposition.

It is important to note here that we will impose stronger conditions on the endomorphism in question: the definition of the action makes this transparent. In the sequel we will
need to deal with the situation where the endomorphism does not a priori satisfy this stronger requirement: this will occupy much of chapter 3 .

Once again this result is presumably well-known in this generality, but we are unaware of a reference. For the case when the ascending central series in question is precisely the upper central series, one reference is for example [21, 1.2.19].

Lemma 1.1.3. Let $G$ be a group and $\sigma$ an endomorphism of $G$ which induces an automorphism of the abelianisation of $G$, denoted by $\bar{\sigma}$. Suppose we are given a sequence of subgroups

$$
1=G^{0} \leqslant G^{1} \leqslant \cdots
$$

such that for each $i \geqslant 0$ we have $\sigma\left(G^{i}\right) \leqslant G^{i}$ and an inclusion $\left[G, G^{i+1}\right] \leqslant G^{i}$. Denoting again by $\mathbb{Z}[\sigma]$ the monoid ring on $\sigma$, we have for each $k \geqslant 0$ a homomorphism of $\mathbb{Z}[\sigma]$ modules

$$
\begin{aligned}
G^{k+2} / G^{k+1} & \longrightarrow \operatorname{Hom}\left(G / G^{\prime}, G^{k+1} / G^{k}\right), \\
x G^{k+1} & \longmapsto\left(g G^{\prime} \mapsto[x, g] G^{k}\right),
\end{aligned}
$$

where $\sigma$ acts on the right hand side by sending $\theta$ to the map $\theta^{\sigma}$, which for $g G^{\prime}$ in $G / G^{\prime}$ is defined by

$$
\theta^{\sigma}\left(g G^{\prime}\right)=\sigma \theta\left((\bar{\sigma})^{-1}\left(g G^{\prime}\right)\right)
$$

Proof. This is proved in a similar manner to lemma (1.1.1).

Note that we have a slightly weaker assumption on our central series compared to lemma (1.1.1): in that lemma we use $G / G_{2}$ there whilst only $G / G^{\prime}$ here.

We immediately deduce the following corollary for the upper central series. We state this separately since it will be used multiple times in the sequel.

Corollary 1.1.4. Let $G$ be a nilpotent group and denote the terms of the upper central series of $G$ by $Z^{i}$. Furthermore let $\alpha$ be an automorphism of $G$. Then for each $k \geqslant 0$ there is an $\alpha$-equivariant injective map

$$
\begin{aligned}
Z^{k+2} / Z^{k+1} & \longrightarrow \operatorname{Hom}\left(G / G^{\prime}, Z^{k+1} / Z^{k}\right) \\
x Z^{k+1} & \longmapsto\left(g G^{\prime} \mapsto[x, g] Z^{k}\right) .
\end{aligned}
$$

Proof. With lemma (1.1.3) in view, we need only check the injectivity of the maps in question, but this is elementary.

### 1.2 Cokernels of abelianisation maps

Let $G$ be a group and $H$ a subgroup of $G$. Then the inclusion of $H$ into $G$ induces a natural map of abelianisations $H / H^{\prime} \longrightarrow G / G^{\prime}$ with cokernel $G / H G^{\prime}$. Broadly speaking, the aim of this section is deduce information about $H$ by imposing information on this cokernel, especially in the nilpotent setting. Generalisations of the results in this section will be explored in chapter 2 , but the fundamental results in this section are needed earlier.

We begin with a pair of presumably well known elementary observations, whose statement only differs in the order of quantifiers. Therefore we prove only the first. Although in the sequel we will only require the result for $\mathbb{Z}$-modules, we state it in full generality here.

Lemma 1.2.1. Let $R$ be a commutative ring with 1 , and $\left(f_{i}: i=1, \ldots, n\right)$ a collection of $R$-module maps.

1. For each $i$, put $I_{i}:=\operatorname{Ann}\left(\operatorname{Coker} f_{i}\right)$. Then we have an inclusion

$$
I_{1} \cdots I_{n} \leqslant \operatorname{Ann}\left(\operatorname{Coker}\left(f_{1} \otimes \cdots \otimes f_{n}\right)\right)
$$

2. If each $f_{i}$ has torsion cokernel, then $f_{1} \otimes \cdots \otimes f_{n}$ also has torsion cokernel.

Proof. As noted above, we prove only the first part, the second part being essentially the same proof.

Note that for a module map $f: M \longrightarrow N$, the condition $r \in \operatorname{Ann}($ Coker $f)$ is equivalent so saying that for every $n \in N, r n \in \operatorname{Im} f$. To fix notation, suppose that $f_{i}: M_{i} \longrightarrow M_{i}^{\prime}$, and pick $r_{i} \in I_{i}$ so that $r:=r_{1} \cdots r_{n} \in I_{1} \cdots I_{n}$. Now let

$$
x=\sum_{j} m_{1 j}^{\prime} \otimes \cdots \otimes m_{n j}^{\prime} \in M_{1}^{\prime} \otimes \cdots \otimes M_{n}^{\prime}
$$

Then for each $m_{i j}^{\prime} \in M_{i}^{\prime}$ there is some $m_{i j} \in M_{i}$ with $f_{i}\left(m_{i j}\right)=r_{i} m_{i j}^{\prime}$, so that

$$
\begin{aligned}
r x & =r_{1} \cdots r_{n} \sum_{j} m_{1 j}^{\prime} \otimes \cdots \otimes m_{n j}^{\prime} \\
& =\sum_{j} r_{1} m_{1 j}^{\prime} \otimes \cdots \otimes r_{n} m_{n j}^{\prime} \\
& =\sum_{j} f_{1}\left(m_{1 j}\right) \otimes \cdots \otimes f_{n}\left(m_{n j}\right) \\
& \in \operatorname{Im} f_{1} \otimes \cdots \otimes f_{n} .
\end{aligned}
$$

It now follows that $I_{1} \cdots I_{n} \leqslant \operatorname{Ann}\left(\operatorname{Coker}\left(f_{1} \otimes \cdots \otimes f_{n}\right)\right)$, as desired.

As the following example shows, we cannot replace the product of annihilators in the above with the sum. Furthermore we see that the tensoring of maps does not commute with taking cokernels, even though the tensor product commutes with colimits.

Example 1.2.2. Consider the $\mathbb{Z}$-module inclusion $\iota: 2 \mathbb{Z} \hookrightarrow \mathbb{Z}$. The annihilator of the cokernel of this map is the ideal $I:=(2)$. The tensor square of this map is (up to isomorphism) the inclusion $4 \mathbb{Z} \longleftrightarrow \mathbb{Z}$. We thus have

$$
I+I=(2)+(2)=(2) \nless(4)=\operatorname{Ann}(\operatorname{Coker}(\iota \otimes \iota)) .
$$

Moreover we note that $\operatorname{Coker}(\iota \otimes \iota)=\mathbb{Z} / 4 \mathbb{Z} \not \not \mathbb{Z} / 2 \mathbb{Z}=\operatorname{Coker}(\iota) \otimes \operatorname{Coker}(\iota)$.

The next lemma is an immediate application of these results. We will immediately proceed to apply the first part, whilst an application of the second part will come in chapter 2.

Proposition 1.2.3. Let $G$ be a group and $H$ a subgroup of $G$. For $i \geqslant 1$ consider the natural maps

$$
\gamma_{i}(H) / \gamma_{i+1}(H) \xrightarrow{\beta_{i}} \gamma_{i}(G) / \gamma_{i+1}(G) .
$$

We consider two situations.

1. If $G / H G^{\prime}$ has exponent dividing $k$, then $\operatorname{Coker}\left(\beta_{i}\right)$ has exponent dividing $k^{i}$.
2. If $G / H G^{\prime}$ is torsion, then $\operatorname{Coker}\left(\beta_{i}\right)$ is also torsion.

Proof. Note that the cokernel of the abelianisation map $\beta_{1}$ is $G / H G^{\prime}$. Thus the first condition is equivalent to saying that $(k) \leqslant \operatorname{Ann}\left(\operatorname{Coker}\left(\beta_{1}\right)\right)$. We construct a commutative diagram as follows, where the first two vertical maps are as in lemma (1.1.1) applied
to the lower central series.


The result now follows upon applying the respective parts of lemma (1.2.1).

We may now show the following powerful result in the nilpotent setting, to be used multiple times in the sequel.

Proposition 1.2.4. Let $G$ be a nilpotent group of class $c, H \leqslant G$, and $\left|G: H G^{\prime}\right|=k$. Then for all $g \in G, g^{k^{1+\cdots+c}} \in H$.

Proof. The proof is by induction on the class $c$, the case $c=1$ being trivial.

Thus suppose $G$ is nilpotent of class $c$ and the result is true for nilpotent groups of class $c-1$. By induction we deduce that for all $g \in G, g^{k^{1+\cdots+c-1}} \in H \gamma_{c}(G)$, so in particular $g^{k^{1+\cdots+c-1}}=h z$ for $h \in H$ and $z \in \gamma_{c}(G)$. Lemma (1.2.3) now implies that the image of $\gamma_{c}(H)$ in $\gamma_{c}(G)$ has index dividing $k^{c}$, so that $z^{k^{c}} \in H$. Noting that $z$ is central we see then that

$$
g^{k^{1+\cdots+c}}=(h z)^{k^{c}}=h^{k^{c}} z^{k^{c}} \in H,
$$

as desired.

### 1.3 Torsion-free and divisible nilpotent groups

In this section we will see in particular that for the class of nilpotent groups, the properties of being torsion-free and divisible are detectable by the centre. The main tool here will be the upper central maps described in section (1.1).

We recall that for a prime $p$, a group $G$ is $p$-divisible if for all $x \in G$ there exists some $y \in G$ for which $x=y^{p}$. Then for a set $\pi$ of primes, we say that $G$ is said to be $\pi$-divisible if it is $p$-divisible for each $p \in \pi$. We begin with a pair of elementary observations.

Lemma 1.3.1. Let $A, B$ be abelian groups. Then if $B$ is torsion-free, so is $\operatorname{Hom}(A, B)$. If in addition $B$ is $\pi$-divisible for a set $\pi$ of primes, then again so is $\operatorname{Hom}(A, B)$.

Proof. These are both elementary. For the first part, suppose $n f=0$ for some $n \geqslant 1$ and homomorphism $f: A \rightarrow B$. Then if $a \in A$, we find $n f(a)=0$ so that $f(a)=0$ since $B$ is torsion-free.

For the second part, let $p \in \pi$ and $f$ as above. Then define $g: A \rightarrow B$ by setting $g(a)$ to be the $p$-th root of $f(a)$. This is well-defined and a homomorphism since $B$ torsion-free gives that $p$-th roots are unique.

This leads us to the following fundamental result concerning the extent to which the centre controls torsion in a nilpotent group.

Proposition 1.3.2. Let $G$ be a nilpotent group. Then $G$ is torsion-free if and only if $Z(G)$ is torsion-free.

Proof. Let $k \geqslant 0$. Denoting the terms of the upper central series by $Z^{k}$, we show that $Z^{k+1} / Z^{k}$ is torsion-free implies $Z^{k+2} / Z^{k+1}$ is torsion-free. Since extensions of torsionfree groups are torsion-free, this will immediately yield the result.

We recall from corollary (1.1.4) that there is an injective homomorphism

$$
Z^{k+2} / Z^{k+1} \Longleftrightarrow \operatorname{Hom}\left(G / G^{\prime}, Z^{k+1} / Z^{k}\right)
$$

induced by the bracket. The result now follows by lemma (1.3.1).

We have the following useful corollary:

Corollary 1.3.3. Suppose $G$ is torsion-free and nilpotent. Then $G / Z(G)$ is torsion-free.

Proof. This latter group has centre $Z^{2} / Z^{1}$, which is torsion-free by the proof of proposition (1.3.2). This proposition now yields the result.

Among the many useful consequences of the above result are the following two lemmas concerning divisibility in nilpotent groups; the first establishing that divisibility is unique in the torsion-free setting, and the second that the divisibility of a nilpotent group is detectable from its centre.

Lemma 1.3.4. Let $G$ be a torsion-free nilpotent group, and let $x, y \in G$ have $x^{n}=y^{n}$ for some $n \geqslant 1$. Then $x=y$.

Proof. We induct on the class $c$ of $G$, the case $c=1$ being clear. Suppose then that $c>1$ and let $Z$ denote the centre of $G$. By corollary (1.3.3) we know that $G / Z$ is torsion-free nilpotent of smaller class. In particular we deduce that $x=y z$ for some central $z$. But this implies that $z^{n}=1$ and so $G$ torsion-free yields that $z=1$, and we may conclude.

Next we have an analogue of proposition (1.3.2).

Lemma 1.3.5. Let $G$ be a torsion-free nilpotent group and $\pi$ a set of primes. Then $G$ is $\pi$-divisible if and only if $Z(G)$ is.

Proof. The if direction of this result has an identical proof to that of proposition (1.3.2), once we know that central extensions of $\pi$-divisible groups are again $\pi$-divisible: but this is elementary.

For the only if direction, we need to show that the centre is indeed a $\pi$-divisible subgroup. Thus choose $z$ central in $G$ and $p \in \pi$ so that there is some $y \in G$ with $y^{p}=z$. But then this $y$ represents a torsion element of $G / Z(G)$, and so $y \in Z(G)$ as desired, by lemma (1.3.3).

### 1.4 Isolators in nilpotent groups

A distinguishing feature of nilpotent groups amongst soluble ones is the extent to which torsion may be controlled. This section, containing various related results used throughout the sequel, may be viewed as an illustration of this fact. As a consequence, we will deduce in particular that the set of torsion elements in a nilpotent group forms a subgroup. The material here is standard, and our main reference is [21, 2.3].

Definition 1.4.1. Let $G$ be a nilpotent group and $H$ a subgroup of $G$. Then the isolator of $H$ in $G$, denoted either $\bar{H}$ or $I(H)$, is defined to be the subgroup

$$
\bar{H}=\left\{g \in G: g^{l} \in H \text { for some } l \geqslant 1\right\}
$$

In order for this definition to make sense it is required to show that such a set is indeed a subgroup. This is the content of the next result, an easy corollary of the results we have established in section (1.2).

Proposition 1.4.2. Let $G$ be a nilpotent group and $H$ a subgroup of $G$. Then the isolator of $H$ in $G$ is indeed a subgroup.

Proof. The only nontrivial axiom is closure under multiplication. Thus let $x, y \in \bar{H}$, so that there exist $s, t \geqslant 1$ with $x^{s}$ and $y^{t}$ both in $H$. Put $K:=\langle x, y\rangle$ and $L:=\left\langle x^{s}, y^{t}\right\rangle$, and note that $\left|K: L K^{\prime}\right|<\infty$ so that proposition (1.2.4) gives some $l \geqslant 1$ for which $(x y)^{l} \in L \leqslant H$. Thus $x y \in \bar{H}$ as required.

Note that in any nilpotent group, the isolator of the trivial subgroup is precisely the set of torsion elements - and we deduce in particular that this set is closed under multiplication. We proceed now to introduce and discuss the isolated central series.

Definition 1.4.3. For a group $G$, we define the $i$-th term of the isolated central series of $G$ to be $\Gamma_{i}(G):=\overline{\gamma_{i}(G)}$, denoted simply by $\Gamma_{i}$ if there is no ambiguity.

Note that we do not require that the group be nilpotent: one may view this group as the preimage of the torsion subgroup of the nilpotent quotient $G / \gamma_{i}(G)$, noting that the set of torsion elements in a nilpotent group form a subgroup, being precisely the isolator of the trivial subgroup.

We now show that it is indeed a central series.

Lemma 1.4.4. For any group $G$, the series defined by $\Gamma_{i}(G)$ is indeed a central series, and each term is fully invariant.

Proof. We show first that $\Gamma_{i}$ is fully invariant: if $\sigma$ is an endomorphism of $G$, and $x \in \Gamma_{i}$ so that $x^{l} \in \gamma_{i}$ for some $l \geqslant 1$, we find that $\sigma(x)^{l}=\sigma\left(x^{l}\right) \in \sigma\left(\gamma_{i}\right) \leqslant \gamma_{i}$, recalling that $\gamma_{i}$ is a fully invariant subgroup of $G$.

We need to show then that for each $i \geqslant 1, \Gamma_{i} / \Gamma_{i+1}$ is central in $G / \Gamma_{i+1}$. For simplicity suppose that $\Gamma_{i+1}$ is trivial, so that $G$ is torsion-free, and we need to show that $\Gamma_{i}$ is central in $G$. Thus $G$ is nilpotent of class $i$ so that $\gamma_{i}$ is central, and moreover by lemma (1.3.3) we have that $G / Z(G)$ is torsion-free. Consider the natural map $G / \gamma_{i} \longrightarrow G / Z(G)$. The subgroup $\Gamma_{i} / \gamma_{i}$ on the left hand side is torsion and hence lies in the kernel, which is $Z(G) / \gamma_{i}$. This is the desired inclusion.

The isolated descending central series may be characterised as the fastest descending central series with torsion-free factors, as we see next.

Lemma 1.4.5. For a group $G$, the series defined by $\Gamma_{i}(G)$ is the fastest descending central series such that the factor groups are torsion-free.

Proof. Suppose that $K_{i}$ is a descending central series with torsion-free factor groups. Since it is a central series, we deduce immediately that $\gamma_{i} \leqslant K_{i}$ for each $i$. Now consider the map $G / \gamma_{i} \longrightarrow G / K_{i}$, with kernel $K_{i} / \gamma_{i}$. The subgroup $\Gamma_{i} / \gamma_{i}$ is torsion, and hence lies in the kernel, since the right hand side is torsion-free. Thus $\Gamma_{i} \leqslant K_{i}$ as desired.

We now specialise to the torsion-free nilpotent setting and link the various central series.

Lemma 1.4.6. Let $G$ be a torsion-free nilpotent group of class $c$. Then for any $0 \leqslant i \leqslant c-1$ we have

$$
\gamma_{c-i} \leqslant \Gamma_{c-i} \leqslant Z_{i+1}
$$

Proof. We have that $\gamma_{c-i} \leqslant Z_{i+1}$ since the lower central series is the fastest descending series. Lemma (1.4.5) gives that $\Gamma_{c-i} \leqslant Z_{i+1}$, since in a torsion-free group the sections of the upper central series are torsion-free by proposition (1.3.2).

Remark 1.4.7. We might hope additionally that $Z_{i} \leqslant \gamma_{c-i}$ in the above. But, for example, we may take $\operatorname{Tr}_{1}(3, \mathbb{Z}) \times \mathbb{Z}$ of class 3 with $1 \times \mathbb{Z}$ central but not contained in the derived subgroup. Thus $Z_{1} \notin \gamma_{2}$ here.

We now state a useful corollary of a fundamental result due to Hall concerning the relationship between isolators and verbal subgroups.

Lemma 1.4.8 ( $[15,4.6])$. Let $G$ be a nilpotent group with subgroups $H_{1}, \ldots, H_{n}$. Then we have an inclusion $\left[\overline{H_{1}}, \ldots, \overline{H_{n}}\right] \leqslant \overline{\left[H_{1}, \ldots, H_{n}\right]}$.

Proof. This follows immediately from the corollary following the cited lemma, applied to the word $\theta\left(x_{1}, \ldots, x_{n}\right):=\left[x_{1}, \ldots, x_{n}\right]$.

This then yields the following general result.

Lemma 1.4.9. Let $G$ be a nilpotent group, and suppose that $H \leqslant G$ has $\bar{H}=G$. Then for each $i \geqslant 2$, we have that $\overline{\gamma_{i}(H)}=\Gamma_{i}(G)$.

Proof. It is an immediate consequence of the lemma above applied to $H_{j}=H$ for each $j$ that $\gamma_{i}(G)=\gamma_{i}(\bar{H}) \leqslant \overline{\gamma_{i}(H)}$. Taking isolators, this yields the inclusion $\Gamma_{i}(G) \leqslant \overline{\gamma_{i}(H)}$. The other inclusion follows by taking isolators on the inclusion $\gamma_{i}(H) \leqslant \gamma_{i}(G)$.

We now show that the isolated central series satisfies the conditions of lemma (1.1.1), so that we deduce the existence of certain tensor power maps. The details are contained in the next lemma.

Lemma 1.4.10. Let $G$ be a torsion-free nilpotent group and $\sigma$ an endomorphism of $G$. Then for any $i \geqslant 1$ we have that $\sigma\left(\Gamma_{i}\right) \leqslant \Gamma_{i}$. Furthermore given also $j \geqslant 1$, we find that $\left[\Gamma_{i}, \Gamma_{j}\right] \leqslant \Gamma_{i+j}$.

Proof. We have already seen in lemma (1.4.4) that the terms of the isolated central series are fully invariant. It thus suffices to prove the second part. Indeed for $i, j \geqslant 1$, we first recall the standard fact that $\left[\gamma_{i}, \gamma_{j}\right] \leqslant \gamma_{i+j}$. Applying now lemma (1.4.8) we deduce that $\left[\Gamma_{i}, \Gamma_{j}\right] \leqslant \overline{\left[\gamma_{i}, \gamma_{j}\right]} \leqslant \overline{\gamma_{i+j}}=\Gamma_{i+j}$.

We may now immediately deduce the following.
Lemma 1.4.11. Let $G$ be a torsion-free nilpotent group and $\sigma$ an endomorphism of $G$. Then there are $\mathbb{Z}[\sigma]$-module homomorphisms

$$
\alpha_{i}:\left(G / \Gamma_{2}\right)^{\otimes i} \longrightarrow \Gamma_{i} / \Gamma_{i+1}
$$

Furthermore the image of $\alpha_{i}$ is precisely $\gamma_{i} \Gamma_{i+1} / \Gamma_{i+1}$.

Proof. In view of lemmas (1.1.1) and (1.1.2), we see that the details are contained precisely in lemma (1.4.10).

### 1.5 Hirsch length

In this section we introduce the notion of Hirsch length, a generalisation of the torsionfree rank of abelian groups. For the most part, the material here is standard. For a reference we suggest for example [21, Section 5].

In this section, all groups will be assumed to be soluble, if we do not mention this. We recall firstly the definition of Hirsch length.

Definition 1.5.1. Let $G$ be a soluble group. The Hirsch length of $G$, denoted by $h(G)$, is defined to be

$$
h(G):=\sum_{i \geqslant 1} \operatorname{dim}_{\mathbb{Q}} \mathbb{Q} \otimes\left(G^{(i)} / G^{(i+1)}\right) .
$$

Note that this number could be infinite, even for finitely generated $G$ : for example the finitely generated metabelian group $\mathbb{Z} \imath \mathbb{Z}$ has infinite Hirsch length. Moreover, the following group $G$, due independently to Baumslag [3] and Remeslennikov [25], is finitely presented soluble, still metabelian, and with $G^{\prime}$ of infinite rank.

$$
G=\left\langle a, s, t \mid a^{t}=a a^{s},\left[a, a^{s}\right]=[s, t]=1\right\rangle
$$

Note also that if we take a different series exhibiting a group as soluble to define Hirsch length, we obtain the same number by the Schreier refinement theorem. We use this fact without mention in the sequel.

Hirsch length is a well behaved property, as the following lemma shows.

Proposition 1.5.2. The finiteness of Hirsch length is preserved by subgroups, quotients, and extensions. More precisely: if $G$ is a group of finite Hirsch length and is an extension of $N$ by $Q$, then $h(G)=h(N)+h(Q)$. Moreover, if $H$ is a (not necessarily normal) subgroup of $G$, then $h(H) \leqslant h(G)$.

Proof. These are all immediate consequences of the Schreier refinement theorem, combined with the fact that the torsion-free rank of abelian groups is a well-behaved invariant.

The only groups of Hirsch length zero are torsion groups, as we see next.

Lemma 1.5.3. A group $G$ has $h(G)=0$ if and only if it is torsion.

Proof. If $h(G)=0$, then $G$ is an extension of torsion groups, the result being clear for abelian groups. The converse is clear.

For the rest of this section, we will concern ourselves with subgroups of maximal Hirsch length. This property is well behaved with respect to subgroups and intersections:

Lemma 1.5.4. Suppose that $H \leqslant K \leqslant G$ with $h(H)=h(G)<\infty$. Then $h(K)=h(G)$. Furthermore if $N \geqq G$ then $h(H \cap N)=h(N)$.

Proof. By the monotonicity of Hirsch length given in proposition (1.5.2), we deduce that $h(H) \leqslant h(K) \leqslant h(G)$, so that we may immediately deduce $h(K)=h(G)$.

For the second part, we have by the second isomorphism theorem and proposition (1.5.2) that

$$
\begin{aligned}
h(N \cap H) & =h(H)-h(H / H \cap N) \\
& =h(H)-h(H N)+h(N) \\
& =h(H)-h(H)+h(N)=h(N)
\end{aligned}
$$

where $h(H N)=h(H)$ by the first part.

As a first example, finite index subgroups certainly have maximal Hirsch length:
Lemma 1.5.5. Suppose $H \leqslant_{f} G$ and $h(G)<\infty$. Then $h(H)=h(G)$.

Proof. Assume first that $H$ is normal. Then the result is a consequence of the previous two lemmas. For the general case, select $N$ normal of finite index in $G$, and contained in $H$. Then $h(H)=h(N)=h(G)$, as desired.

We will be interested in finding settings where the converse of lemma (1.5.5) holds. It fails in general even for minimax groups, for example $\mathbb{Z} \leqslant \mathbb{Z}[1 / 2]$. However we will see that for minimax groups, finite generation is sufficient. To ease the exposition we make a definition, for the purposes of the rest of the section.

Definition 1.5.6. Let $G$ be a soluble group of finite Hirsch length. If every subgroup of maximal Hirsch length has finite index, then we call G slender.

Note that there is the unrelated notion of slender groups in abelian group theory: irrelevant here. It is important to notice that slender groups are closed under extensions:

Proposition 1.5.7. Suppose that $N \geqq G$ and that both $N$ and $G / N$ are slender. Then $G$ is slender.

Proof. Suppose that $H \leqslant G$ has $h(H)=h(G)$. It follows from proposition (1.5.2) and lemma (1.5.4) that $h(H N / N)=h(G / N)$ so that $H N \leqslant_{f} G$ by hypothesis. Then $h(H \cap N)=h(N)$ by lemma (1.5.4) so that $H \cap N \leqslant_{f} N$ again by hypothesis. But $|N: N \cap H|=|H N: H|$, so that $H \leqslant_{f} H N \leqslant_{f} G$, giving the desired result.

We immediately deduce the following corollary.
Corollary 1.5.8. Polycyclic groups are slender.

Proof. The result is clear for finitely generated abelian groups. Apply proposition (1.5.7).

We now show that finitely generated minimax groups are slender. We thank Peter Kropholler here for pointing out the reduction step in lemma (1.5.10).

Proposition 1.5.9. Suppose $G$ is finitely generated minimax. Then $G$ is slender.

Proof. We induct on the derived length of $G$. If $G$ is abelian the result is clear. Thus, take $A=G^{(d-1)}>G^{(d)}=1$ and suppose that $H \leqslant G$ has $h(H)=h(G)$. Since $G / A$ is finitely generated of smaller derived length, and $h(H A / A)=h(G / A)$, we deduce by the inductive hypothesis that $H A \leqslant_{f} G$, and in particular that $H A$ is finitely generated. By lemma (1.5.10) below we may assume that $A$ is a finitely generated $\mathbb{Z} H A$ module. It now suffices to show that $H \leqslant_{f} H A$. Since $A$ is an abelian normal subgroup of $G$, we have that $H \cap A \preccurlyeq H A$, and thus we obtain a splitting

$$
\frac{H A}{H \cap A} \cong \frac{A}{H \cap A} \rtimes \frac{H}{H \cap A}
$$

It follows from our assumption that $h(A / H \cap A)=0$, so that it is a torsion abelian group. The key observation is that since $A$ is finitely generated as a module, we have that $A / H \cap A$ cannot have any $C_{p^{\infty}}$ sections, since $C_{p^{\infty}}$ cannot be finitely generated as a module over any ring. It follows that it is finite, and hence $|H A: H|=|A: A \cap H|<\infty$ as desired.

Lemma 1.5.10. Let $G=H A$ be a finitely generated group with $A$ an abelian normal subgroup of $G$. Then there is a subgroup $A_{0}$ of $G$ with the following properties:

1. $A_{0} \leqslant A$ and $A_{0} \preccurlyeq G$
2. $H A_{0}=H A$
3. $A_{0}$ is a finitely generated $\mathbb{Z} G$ module.

Proof. We write $G=H A=\left\langle h_{1} a_{1} \ldots, h_{n} a_{n}\right\rangle$, and claim that

$$
A_{0}:=\left\langle a_{1}, \ldots, a_{n}\right\rangle^{H}
$$

is a suitable subgroup. We verify the claims, which are routine. Since $A$ is normal in $G$, it follows that $A_{0} \leqslant A$. By construction it is normalised by $H$, and since $A$ is abelian
it is also normal in $A$. Thus it is normal in $H A=G$. Furthermore we also have that $H A_{0}=H A$, since it contains the generators $h_{i} a_{i}$. Finally $A_{0}$ is normally generated by the $a_{i}$, and so it is indeed a finitely generated $\mathbb{Z} G$ module.

We now have the following useful result concerning endomorphisms of certain soluble groups:

Proposition 1.5.11. Let $G$ be a finitely generated torsion-free minimax group, and $\sigma: G \longrightarrow G$ an endomorphism. Then $\sigma$ is injective if and only if $\sigma(G) \leqslant_{f} G$.

Proof. If $\sigma$ is injective, then certainly $h(\sigma(G))=h(G)$, since these groups are isomorphic. By proposition (1.5.9) we deduce immediately that $\sigma(G) \leqslant_{f} G$.

Conversely, set $K$ to be the kernel of $\sigma$. Combining lemma (1.5.5) and proposition (1.5.2), we deduce that $h(K)=0$. Thus $K$ is torsion by lemma (1.5.3). But now since $G$ is torsion-free we find that $K$ is trivial, so that $\sigma$ is injective as claimed.

We conclude this section with the following elementary diagram, which is used in the sequel without mention.

Remark 1.5.12. Let $G$ be a nilpotent group, and suppose that $H \leqslant G$. We then have the following implications.


### 1.6 Prüfer rank

In this section we introduce the Prüfer rank of a group, which does not require solubility to define. In the soluble setting the main difference between the Prüfer rank and Hirsch length is the detection of torsion. In the sequel we will tend to use Hirsch length, and thus the main aim of this section is to outline why for torsion-free nilpotent groups the Hirsch length and Prüfer rank coincide, modulo a technical lemma.

Definition 1.6.1. Let $G$ be a finitely generated group. Then we define $d(G)$ to be the cardinality of a minimal generating set for $G$. If now $G$ is arbitrary, we define the Prüfer rank of $G$ to be

$$
r(G):=\sup \{d(H): H \text { a finitely generated subgroup of } G\}
$$

or $\infty$, in case the supremum does not exist.

As with Hirsch length, the Prüfer rank is a well behaved invariant. This next result should be compared with proposition (1.5.2): it is a weaker result but holds in far greater generality: as mentioned before, we do not require that the group in question is soluble.

Proposition 1.6.2. Finite Prüfer rank is preserved by subgroups, quotients, and extensions. More precisely: if $G$ has finite Prüfer rank and is an extension of $N$ by $Q$, then $r(G) \leqslant r(N)+r(Q)$ and $r(Q) \leqslant r(G)$. Moreover, if $H$ is a (not necessarily normal) subgroup of $G$, then $r(H) \leqslant r(G)$.

Proof. The result for subgroups and quotients being clear, we show the desired inequality for extensions. Thus let $H \leqslant G$ be a finitely generated subgroup. Choose left coset representatives ( $x_{i}: i \in I$ ) of $H \cap N$ in $H$. Then since $H$ is finitely generated, there exist finitely many elements $x_{j_{i}} w_{i}$ with $w_{i} \in H \cap N$ for which $H=\left\langle x_{j_{i}} w_{i}\right\rangle$. Note also that we obtain a further equality $H=\left\langle x_{j_{i}}, w_{i}\right\rangle$. Working modulo $H \cap N$ and applying the second isomorphism theorem we may bound the number of $x_{j_{i}}$ by $r(Q)$ and moreover we may bound the number of $w_{i}$ by $r(N)$ since these all lie in $N$. In particular we may generate $H$ with no more than $r(N)+r(Q)$ elements and thus we have the desired bound on $d(H)$.

We now proceed to discuss the link between Prüfer rank and Hirsch length. Of course the main obstruction here is the presence of torsion. We first discuss an elementary link in the abelian setting.

Lemma 1.6.3. Let $A$ be a torsion-free abelian group and $l \in \mathbb{N}$. Then the following are equivalent.

1. $h(A)=l$;
2. $r(A)=l$;
3. A embeds into an l-dimensional rational vector space with torsion quotient.

Proof. We show that 1 and 2 are both equivalent to 3 .

Firstly then, suppose that $A$ is torsion-free abelian group of Hirsch length $l$. Modulo each abelian section in the derived series, we may select a maximal linearly independent set of elements. Lifting all of these back to $A$ we find a copy of $\mathbb{Z}^{l} \leqslant A$ with torsion quotient, a consequence of the results in the previous section. Then tensoring this inclusion with $\mathbb{Q}$ we find $A \leqslant \mathbb{Q}^{l}$, since $\mathbb{Q}$ is flat and kills torsion. Moreover the quotient is still torsion, being an image of $\mathbb{Q}^{l} / \mathbb{Z}^{l}$. Conversely, the fact that subgroups of finite dimensional rational vector spaces with torsion quotient have the same Hirsch length is immediate from proposition (1.5.2) and lemma (1.5.3).

Now suppose that $A$ has Prüfer rank $l$. Now $A$ certainly embeds into a rational vector space, and by considering the $\mathbb{Q}$-span of a maximal linearly independent (of cardinality $l$ by our hypothesis) set we see that it lies in a finite dimensional rational vector space. If the quotient were not torsion, we could lift back a suitable element to contradict maximality. For the converse, select a basis which lies in $A$. Then if we could find some finitely generated subgroup of $A$ of higher rank, we could extend a generating set of minimal cardinality to a basis of the vector space for a contradiction.

We may now easily deduce the following.
Proposition 1.6.4. Let $G$ be a torsion-free soluble group. Then if $G$ has finite Hirsch length it has finite Prüfer rank, and $r(G) \leqslant h(G)$. Conversely, if $G$ has finite Prüfer rank then its Hirsch length is also finite.

Proof. We induct on the derived length of $G$, the abelian case being lemma (1.6.3). Suppose then that $G$ has derived length greater than 1. Since $G / G^{\prime}$ need not be torsion free we consider the isolator $\Gamma_{2}(G)$. Then if $G$ has finite Hirsch length, the quotient $G / \Gamma_{2}(G)$ does also, and hence has finite Prüfer rank bounded by its Hirsch length by lemma (1.6.3). In particular

$$
r(G) \leqslant r\left(\Gamma_{2}(G)\right)+r\left(G / \Gamma_{2}(G)\right) \leqslant h\left(\Gamma_{2}(G)\right)+h\left(G / \Gamma_{2}(G)\right)=h(G)
$$

by propositions (1.6.2) and (1.5.2).
The converse is essentially an identical proof, although we cannot expect a similar bound since we only have an inequality in proposition (1.6.2).

For certain classes of soluble groups these invariants do coincide. One key result to establish this is the following, which holds in much greater generality - see the cited references for more details. We concern ourselves only with the nilpotent case in the sequel and thus we only state a special case here. Recall that a $\mathfrak{T}$-group is a finitely generated torsion-free nilpotent group.

Lemma 1.6.5 ( [21, 5.2.9] [36]). Let $G$ be a $\mathfrak{T}$-group of Hirsch length $r$. Then for any prime $p$, there exists an elementary abelian section of $G$ of order $p^{r}$.

We then have the following link between Prüfer rank and Hirsch length.
Proposition 1.6.6. Let $G$ be a torsion-free nilpotent group. Then $G$ has finite Hirsch length if and only if it has finite Prüfer rank, and moreover these invariants coincide.

Proof. By proposition (1.6.4) it suffices to show that if $G$ has Hirsch length $r$ then it has Prüfer rank at least $r$. Select a finitely generated subgroup $H \leqslant G$ of Hirsch length $r$. To see that this is possible one may for example take the subgroup generated by the preimages of a maximal linearly independent set in the abelianisation. Then one may see that such a subgroup has maximal Hirsch length by considering for example the tensor power maps (1.1.1). Now select a prime $p$ and apply lemma (1.6.5) to this $H$ to find subgroups $K \triangleleft L \leqslant H$ with $L / K$ elementary $p$-abelian of rank $r$. Note that $L$ is finitely generated since $H$ is polycyclic and requires at least $r$ generators, since the elementary abelian section does. Thus $d(L) \geqslant r$ which sets the Prüfer rank of $G$ to be at least $r$, as desired.

### 1.7 The Mal'cev Completion

In this section we introduce the Mal'cev completion, or radicable hull, of a torsion-free nilpotent group, the idea being to embed such a group into a divisible one in a minimal fashion. The analogy for torsion-free abelian groups is with the injective hull: obtained by tensoring with $\mathbb{Q}$. Our main references here are $[15,21,26,31]$. We first give the full statement in the categorical framework.

Mal'cev Completion ( $[31,8.5])$. The inclusion of the category of radicable torsion-free nilpotent groups into the category of torsion-free nilpotent groups admits a left adjoint $(-)^{\mathbb{Q}}$, called the Mal'cev Completion functor. In particular to any torsion-free nilpotent group $N$ there is an injective group homomorphism $\iota: N \longrightarrow N^{\mathbb{Q}}$, where $N^{\mathbb{Q}}$ is a radicable torsion-free nilpotent group, obeying the following universal property: given any
group homomorphism $f: N \longrightarrow R$ where $R$ is a radicable nilpotent group, there exists a unique group homomorphism $f^{\mathbb{Q}}: N^{\mathbb{Q}} \longrightarrow R$ such that the following diagram commutes.


As per Warfield [31], it is a standard argument to see that this adjunction exists. The work is in showing that for $N$ torsion-free nilpotent the canonical map $N \rightarrow N^{\mathbb{Q}}$ is injective. We discuss here one coherent way to realise the Mal'cev completion concretely and obtain the injectivity of the canonical map. Our discussion will roughly follow that of [21, 2.1]. One starts by considering the finitely generated case. Thus let $G$ be a $\mathfrak{T}$ group of Hirsch length $n$. Since the sections of the upper central series are torsion-free, one may refine it to be a series where the sections are all infinite cyclic. If one lifts cyclic generators back to $G$, say to $x_{1}, \ldots x_{n}$, then an easy induction shows for each element $a$ of $G$ there exist unique integers $r_{1}, \ldots, r_{n}$ such that $a=x_{1}^{r_{1}} \cdots x_{n}^{r_{n}}$. We notate this by $x^{\mathbf{r}}$ for short with $\mathbf{r}:=\left(r_{1}, \ldots, r_{n}\right)$. One says that the sequence of elements $x_{1}, \ldots, x_{n}$ forms a fundamental basis of $G$.

By this (set-theoretic) correspondence we may interpret the multiplication and powering maps in $G$ as vectors of functions as follows, where $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^{n}$ and $l \in \mathbb{Z}$.

$$
\begin{aligned}
x^{\mathbf{r}} \cdot x^{\mathbf{s}} & =x_{1}^{\alpha_{1}(\mathbf{r}, \mathbf{s})} \cdots x_{n}^{\alpha_{n}(\mathbf{r}, \mathbf{s})}, \\
\left(x^{\mathbf{r}}\right)^{l} & =x_{1}^{\beta_{1}(\mathbf{r}, l)} \cdots x_{n}^{\beta_{n}(\mathbf{r}, l)} .
\end{aligned}
$$

The bulk of the work is now in the following well-known result due to Hall.

Proposition 1.7.1 ([15, 6.5]). Let $G$ be a $\mathfrak{T}$-group with functions $\alpha_{i}: \mathbb{Z}^{2 n} \longrightarrow \mathbb{Z}$ and $\beta_{j}: \mathbb{Z}^{n+1} \longrightarrow \mathbb{Z}$ obtained as above from a choice of fundamental basis. Then each $\alpha_{i}$ and $\beta_{j}$ is a polynomial function with rational coefficients.

One may now proceed to define the Mal'cev completion $G^{\mathbb{Q}}$ of a $\mathfrak{T}$-group $G$ of Hirsch length $n$. As a set, this is defined to be $\mathbb{Q}^{n}$, and the group operations are defined by extending the polynomials $\alpha_{i}$ and $\beta_{j}$ uniquely - this is possible since knowing the values of a polynomial on the integers determines the polynomial. That this is a group is a
consequence of the fact that the group axioms for $G$ express themselves purely as equations involving the aforementioned polynomials: thus they hold in $G^{\mathbb{Q}}$. Furthermore, one may directly see that this construction is unique up to unique isomorphism.

Finally, one generalises the construction to a torsion-free nilpotent group $N$ by expressing it as a direct limit of its finitely generated subgroups and completing each subgroup in a compatible manner.

We mention another approach to the Mal'cev completion of a $\mathfrak{T}$-group, also based on a linearity result due to Hall. Recall that $\operatorname{Tr}_{1}(n, k)$ denotes the group of $n \times n$ upper triangular matrices with entries in a ring $k$ and with 1 s on the main diagonal. We then have

Proposition 1.7.2 ( $[15,7.5])$. Every $\mathfrak{T}$-group may be embedded as a subgroup of $\operatorname{Tr}_{1}(n, \mathbb{Z})$ for some $n$.

Thus for a $\mathfrak{T}$-group $G$ one may isolate the image of $G$ in $\operatorname{Tr}_{1}(n, \mathbb{Q})$ to obtain the Mal'cev completion, as this latter group is radicable. This is the approach taken by Segal in [26].

We consider as an illustration of these ideas the case of the Heisenberg group.

Example 1.7.3. The integral Heisenberg group $H$ is defined by

$$
H:=\left(\begin{array}{ccc}
1 & \mathbb{Z} & \mathbb{Z} \\
& 1 & \mathbb{Z} \\
& & 1
\end{array}\right)=\operatorname{Tr}_{1}(3, \mathbb{Z})
$$

We may give an explicit fundamental basis as $x_{1}, x_{2}, x_{3}$ defined by

$$
\begin{aligned}
& x_{1}:=\left(\begin{array}{lll}
1 & 1 & 0 \\
& 1 & 0 \\
& & 1
\end{array}\right), \\
& x_{2}:=\left(\begin{array}{lll}
1 & 0 & 0 \\
& 1 & 1 \\
& & 1
\end{array}\right), \\
& x_{3}:=\left(\begin{array}{lll}
1 & 0 & 1 \\
& 1 & 0 \\
& & 1
\end{array}\right) .
\end{aligned}
$$

One then verifies immediately the following, where each of $r_{i}, s_{j}$ and $l$ are integers:

$$
\begin{aligned}
x_{1}^{r_{1}} x_{2}^{r_{2}} x_{3}^{r_{3}} \cdot x_{1}^{s_{1}} x_{2}^{s_{2}} x_{3}^{s_{3}} & =x_{1}^{r_{1}+s_{1}} x_{2}^{r_{2}+s_{2}} x_{3}^{r_{3}+s_{1} s_{2}+r_{1} r_{2}+r_{1} s_{2}+s_{3}}, \\
\left(x_{1}^{r_{1}} x_{2}^{r_{2}} x_{3}^{r_{3}}\right)^{l} & \left.=x_{1}^{l r_{1}} x_{2}^{l r_{2}} x_{3}^{\left({ }^{(+1}+{ }_{2}^{2}\right.}\right) r_{1} r_{2}+l r_{3}
\end{aligned} .
$$

In particular, we see that $\beta_{3}$ does not have integer coefficients in this case.
Note that the Mal'cev completion of $H$ is precisely

$$
H^{\mathbb{Q}}:=\left(\begin{array}{ccc}
1 & \mathbb{Q} & \mathbb{Q} \\
& 1 & \mathbb{Q} \\
& & 1
\end{array}\right)=\operatorname{Tr}_{1}(3, \mathbb{Q})
$$

We will proceed to list various specific properties of the Mal'cev completion which we require later. Thus we give more details in this final section. Recall that two abstract groups are said to be commensurate if they have isomorphic subgroups of finite index.

Proposition 1.7.4. Two $\mathfrak{T}$-groups are commensurate if and only if they have isomorphic Mal'cev completions.

Proof. We first show that if $H \leqslant_{f} G$ are $\mathfrak{T}$-groups, then $H^{\mathbb{Q}} \cong G^{\mathbb{Q}}$. This is an easy consequence of our earlier discussion: one sees that one way to characterise $G^{\mathbb{Q}}$ amongst radicable torsion-free nilpotent groups is that for each $x \in G^{\mathbb{Q}}$ there is some $n \geqslant 1$ for which $x^{n} \in G$. To see this, note that this is inherent in the particular construction we outline above. Then the functorality is saying that this is true of any particular Mal'cev completion. Bearing this in mind it follows immediately from the fact that $H$ has finite index in $G$ that $H^{\mathbb{Q}} \cong G^{\mathbb{Q}}$.

Thus suppose $G$ and $H$ are commensurate $\mathfrak{T}$-groups, so that there are subgroups $K \leqslant_{f} G$ and $L \leqslant_{f} H$ with $K \cong L$. But then by our earlier remark and the functorality of the Mal'cev completion we deduce $G^{\mathbb{Q}} \cong K^{\mathbb{Q}} \cong L^{\mathbb{Q}} \cong H^{\mathbb{Q}}$ as desired.

Conversely, suppose that $G^{\mathbb{Q}} \cong H^{\mathbb{Q}}$. By considering the image of $H$ in $G^{\mathbb{Q}}$ we may suppose that both $G$ and $H$ are subgroups of $G^{\mathbb{Q}}$. Suppose $H$ is generated by $h_{1}, \ldots h_{r}$. Since $H \leqslant G^{\mathbb{Q}}$, there exists some $n \geqslant 1$ for which $h_{i}^{n} \in G$ for each $i$. But now the subgroup $K:=\left\langle h_{i}^{n}: 1 \leqslant i \leqslant r\right\rangle$ has finite index in $H$ : certainly it does modulo $H^{\prime}$ and we may then appeal to proposition (2.1.5). Thus since $K \leqslant H \cap G \leqslant H$ we deduce $H \cap G \leqslant_{f} H$.

By symmetry $H \cap G$ also has finite index in $G$, and we may conclude.

We will need to consider how the Mal'cev completion behaves with respect to endomorphisms of nilpotent groups of finite rank (with sections (1.5) and (1.6) in view, we see that we may take either finite Hirsch length or finite Prüfer rank here). We start with a general lemma with no finiteness of rank assumption, and denote isolators by bars.

Lemma 1.7.5. Let $f: M \rightarrow N$ be a homomorphism of nilpotent groups of finite rank, and let $f_{\times}$denote the induced homomorphism on the Mal'cev completion. Then $f_{\times}$ extends $f$, and furthermore we have $\operatorname{Ker} f_{\times}=\overline{\operatorname{Ker} f}$ and $\operatorname{Im} f_{\times}=\overline{\operatorname{Im} f}$.

Proof. That $f_{\times}$extends $f$ is a consequence of the injectivity of the natural map $N \rightarrow N^{\mathbb{Q}}$. The two stated equations follow immediately from the definitions and lemma (1.3.4).

We then have the following crucial corollary in the finite rank case: note in particular the surjectivity of the induced map.

Corollary 1.7.6. Let $N$ be a nilpotent group of finite rank and $\sigma$ an injective endomorphism of $N$. Then the induced endomorphism $\sigma_{\times}$is an automorphism of $N^{\mathbb{Q}}$ extending $\sigma$.

Before we conclude this section, we briefly discuss how the upper central series interacts with the Mal'cev completion.

Proposition 1.7.7. Let $N$ be torsion-free nilpotent and set $R:=N^{\mathbb{Q}}$. Then for each $i \geqslant 0$, we have that $Z^{i}(N)=Z^{i}(R) \cap N$.

Proof. We induct on $i$, the base case of $i=0$ being trivial. It will suffice to show that $Z^{i}(N) \leqslant Z^{i}(R)$, so assume this is true at the $(i-1)$ st step. Thus let $x \in Z^{i}(N)$ and $w \in R$, so that $w^{n} \in N$ for some $n \geqslant 1$. Then certainly $\left[x, w^{n}\right] \in Z^{i-1}(N) \leqslant Z^{i-1}(R)$. Set $H_{1}=\langle x\rangle$ and $H_{2}=\left\langle w^{n}\right\rangle$. Then note that, denoting isolators by bars,

$$
\begin{aligned}
{[x, w] } & \in\left[\bar{H}_{1}, \bar{H}_{2}\right] \\
& \leqslant \overline{\left[H_{1}, H_{2}\right]} \\
& \leqslant \overline{Z^{i-1}(N)} \\
& \leqslant \overline{Z^{i-1}(R)} \\
& =Z^{i}(R),
\end{aligned}
$$

as desired, where the first inclusion is lemma (1.4.8).

The following is an immediate and useful consequence of the above proposition.
Corollary 1.7.8. Let $N$ be torsion-free nilpotent of finite rank and $\sigma$ an injective endomorphism of $N$. Then the upper central series of $N$ is preserved by $\sigma$.

Proof. Fix $i \geqslant 0$ and set $R:=N^{\mathbb{Q}}$. Then, using proposition (1.7.7) and corollary (1.7.6), we find

$$
\sigma\left(Z^{i}(N)\right)=\sigma_{\times}\left(Z^{i}(R)\right) \cap \sigma(N)=Z^{i}(R) \cap \sigma(N) \leqslant Z^{i}(N)
$$

as desired.

### 1.8 The Mal'cev Correspondence

Following on naturally from the previous section, we end this chapter with a brief introduction to the Mal'cev correspondence. This is an equivalence of categories between rational nilpotent Lie algebras and torsion-free radicable nilpotent groups. The main reference here is [17].

We may state immediately
The Mal'cev Correspondence ( [17, p. 118]). The categories of rational nilpotent Lie algebras and of torsion-free radicable nilpotent groups are equivalent. To each torsionfree radicable nilpotent group $G$ we may associate its Lie algebra $\log (G)$, and conversely given a rational nilpotent Lie algebra $L$ the associated group is $\exp (L)$.

We state the correspondence here removed from its geometric context. One may view torsion-free radicable nilpotent groups as unipotent algebraic groups over $\mathbb{Q}$, and thereby realise the correspondence by associating to such a group its tangent space at the identity with its induced Lie algebra structure. It is then a theorem of algebraic groups that this correspondence does indeed induce an equivalence of categories. In this regard we refer the reader to [14]. The approach of Khukhro in [17], which we outline in some detail below, is to explicitly witness the correspondence for free objects and then to generalise this to all objects.

A further approach to the correspondence is outlined in [31]: one may view the correspondence explicitly inside a certain completed group algebra.

Before sketching a way to construct the correspondence, we state the main result we require in the sequel.

As per Segal in [26, p. 102], we define, given a tuple $\mathbf{e}=\left(e_{1}, \ldots, e_{r}\right)$ of positive integers,

$$
[x, y]_{\mathbf{e}}:=[x, \underbrace{y, \ldots, y}_{e_{1}}, \underbrace{x, \ldots, x}_{e_{2}}, y, \ldots] .
$$

We now cite certain formulae pertaining to the Mal'cev correspondence, including the celebrated Baker-Campbell-Hausdorff formula and its inverse, which exhibit the uniform way in which the lie algebra structure determines the group structure, and vice versa.

Theorem 1.8.1 ( [17, p. 118]). There exist constants $a_{e}, b_{e}, c_{e}, d_{\boldsymbol{e}} \in \mathbb{Q}$, for each vector of positive integers $\mathbf{e}$, such that $\log$ and $\exp$ formally obey the following relations for all $X, Y, g, h$ :

$$
\begin{align*}
\log (\exp (X) \exp (Y)) & =X+Y+\sum_{e} a_{e}[X, Y]_{e}  \tag{1.8.1}\\
\log [\exp (X), \exp (Y)] & =\sum_{e} b_{e}[X, Y]_{e}  \tag{1.8.2}\\
\exp (\log (g)+\log (h)) & =g h \prod_{e}[g, h]_{e}^{c_{e}}  \tag{1.8.3}\\
\exp [\log (g), \log (h)] & =\prod_{e}[g, h]_{e}^{d_{e}} \tag{1.8.4}
\end{align*}
$$

More precisely, given any pair $(L, G)$ consisting of a finite dimensional nilpotent Lie algebra with $\exp L=G$ its associated radicable nilpotent group, then the formulas above are valid for any $x, y \in L$ and $g, h \in G$, noting that they are finite by nilpotency.

We now proceed to outline an argument to establish the Mal'cev correspondence. Our brief exposition here closely follows that of Khukhro [17], although at the end we do differ slightly. The strategy is to first manufacture the correspondence in the free setting and deduce the Baker-Campbell-Hausdorff formula and its variants here. Next, one sets up the correspondence objectwise by artificially defining new operations using the aforementioned formulae. That this process works will follow from the freeness assumption.

Thus, let $X$ be a set and $c \geqslant 1$. We consider the free nilpotent of class $c$ associative $\mathbb{Q}$-algebra on $X$, denoted simply by $A$. Concretely, this has $\mathbb{Q}$-basis monomials $x_{1} \cdots x_{k}$ where $k \leqslant c$ and each $x_{j} \in X$, multiplication is concatenation, and any $(c+1)$-fold product of elements of $X$ vanishes. By setting $[x, y]:=x y-y x$, one endows $A$ with a rational Lie algebra structure. Now let $L$ denote the sub Lie ring of $A$ generated by $X$. Then we may realise the free nilpotent of class $c$ rational Lie algebra on $X$ as a subalgebra of $A$ as $\mathbb{Q} L=\{r l: l \in L\}$, as remarked by Khukhro [17, p. 102].

Adjoin now a formal identity 1 to $A$ by considering the algebra $\mathbb{Q} \oplus A$ and consider the subset $1+A=\{1+a: a \in A\}$. We formally define maps $\log : 1+A \rightarrow A$ and $\exp : A \rightarrow 1+A$ as follows:

$$
\begin{aligned}
\log (1+a) & :=a-\frac{a^{2}}{2}+\cdots+(-1)^{i-1} \frac{a^{i}}{i}+\cdots \\
\exp (a) & :=1+\frac{a}{1}+\cdots+\frac{a^{i}}{i!}+\cdots
\end{aligned}
$$

Note that these are well-defined by our nilpotency assumption. Furthermore, since an element of an associative algebra commutes with its powers, it is a straightforward verification that these maps are mutually inverse.

It is important to notice that $1+A$ is in fact a group under multiplication with identity 1 , the inverse of $1+a$ being $1-a+\cdots+(-1)^{c-1} a^{c}$. Inside this group one wishes to identify the free nilpotent group of class $c$ on $X$, and one may do so rather explicitly, as in the following.

Proposition 1.8.2 ( [17, Theorem 9.2]). The set $\left\{e^{x}: x \in X\right\} \leqslant 1+A$ freely generates a free nilpotent group $G$ of class $c$.

One may now properly introduce the Baker-Campbell-Hausdorff formula in this setting.
Theorem 1.8.3. For $x, y \in A$ free nilpotent of class $c$ we define the polynomial $H_{c}(x, y)$ to satisfy

$$
e^{H_{c}(x, y)}=e^{x} e^{y}
$$

Then if $x, y \in X$, we have that $H_{c}(x, y) \in \mathbb{Q} L$.

Note that by the freeness assumption we have that for any $c, i \geqslant 1$, the polynomial $H_{c}$ is a truncation of $H_{c+i}$, in the obvious sense. In particular we may consistently define the power series $H$ as in theorem (1.8.1) above. An important corollary of the fact that the finite polynomials lie in $\mathbb{Q} L$ is the following.

Corollary 1.8.4 ( [17, Theorem 9.2]). The set $e^{\mathbb{Q} L}$ is a group, and furthermore is the Mal'cev completion of $G$ defined in proposition (1.8.2).

Using the machinery we have outlined above, one may now proceed to establish the Mal'cev correspondence in general. Suppose that we wish to witness the correspondence for a particular finite dimensional nilpotent rational Lie algebra $M$ of class $c$. Formally, the corresponding group is the set $e^{M}=\left\{e^{x}: x \in M\right\}$, with group operation defined
by $e^{x} \cdot e^{y}:=e^{H_{c}(x, y)}$. The fact that this operation defines a group is a consequence of corollary (1.8.4). Take for example the case of the associative law. We require that for all $x_{1}, x_{2}, x_{3}$ in $M$ that $H_{c}\left(\left(H_{c}\left(x_{1}, x_{2}\right), x_{3}\right)=H_{c}\left(x_{1}, H_{c}\left(x_{2}, x_{3}\right)\right)\right.$. Denote $A$ as above to be the free nilpotent of class $c$ associative $\mathbb{Q}$-algebra on $\left\{a_{1}, a_{2}, a_{3}\right\}$ so that the Lie algebra $\mathbb{Q} L$ is free on this set, as remarked above. By corollary (1.8.4) we certainly have $H_{c}\left(\left(H_{c}\left(a_{1}, a_{2}\right), a_{3}\right)=H_{c}\left(a_{1}, H_{c}\left(a_{2}, a_{3}\right)\right)\right.$. Now using our freeness assumption map $\mathbb{Q} L$ onto $M$ via $a_{i} \mapsto x_{i}$ to achieve the desired result. In a similar manner one may establish the reverse direction of the correspondence.

## Chapter 2

## Supplement results for nilpotent groups

In a group $G$, a supplement of a subgroup $H \leqslant G$ is a subgroup $K$ such that $G=H K$. By an almost supplement of $H$ we will mean a subgroup $K$ such that $H K$ is close to $G$ in some sense - for example it has the same rank, or is of finite index. In this section we discuss various results, some of which are well known, concerning the role of supplements and their generalisations in nilpotent groups. In the first section we discuss subgroups for which the derived subgroup is a supplement, or almost one. We then move to discuss the isolated derived subgroup and its almost supplements in various contexts.

### 2.1 Almost supplements of the derived subgroup

Here we will see in particular that the derived subgroup is only a supplement for the whole group. Interestingly for finitely generated soluble groups we will even see that this property of the derived subgroup characterises nilpotent groups. More generally, we may think of the results in this section as an illustration of the extent to which the abelianisation controls the subgroup structure of a nilpotent group.

We begin with a useful and elementary illustration of this principle.

Proposition 2.1.1. Let $G$ be a nilpotent group, $H \leqslant G$ such that $G=H G^{\prime}$. Then $G=H$.

Proof. We induct on the class $c$ of $G$, the case $c=1$ a triviality. If $c>1$ we deduce by induction that $G=H \gamma_{c}(G)$. But then

$$
G=H G^{\prime}=H\left(H \gamma_{c}(G)\right)^{\prime}=H H^{\prime}=H,
$$

as desired.
Alternatively, one may take $k=1$ in proposition (1.2.4).

Interestingly we find that proposition (2.1.1) admits a partial converse. We first require a lemma, which is a trivial consequence of two lemmas in [21], generalising in particular a classical theorem due to Wielandt [35]. Recall for a group $G$, we denote by $\operatorname{Frat}(G)$ the Frattini subgroup of $G$, defined to be the intersection of all maximal subgroups.

Lemma 2.1.2. Let $G$ be a finitely generated soluble group. Then $G^{\prime} \leqslant \operatorname{Frat}(G)$ if and only if $G$ is nilpotent.

Proof. If $G^{\prime} \leqslant \operatorname{Frat}(G)$, we deduce immediately that $G / \operatorname{Frat}(G)$ is polycyclic, so that [21, 7.4.10] yields that $G$ is polycyclic. But now [21, 1.3.20] yields that $G$ is nilpotent, as desired. The converse is also supplied by [21, 1.3.20].

We now have
Proposition 2.1.3. Let $G$ be a finitely generated soluble group such that for all subgroups $H \leqslant G, H G^{\prime}=G \Longrightarrow H=G$. Then $G$ is nilpotent.

Proof. By lemma (2.1.2), it suffices to show that $G^{\prime} \leqslant \operatorname{Frat}(G)$, where $\operatorname{Frat}(G)$ is the Frattini subgroup of $G$. To this end, suppose that $H$ is a maximal subgroup. The hypothesis immediately yields that $H=H G^{\prime}$, so that $G^{\prime} \leqslant H$, as desired.

We turn to studying almost supplement results for nilpotent groups, beginning by thinking of 'almost' as being 'has finite index in'. In the case of a normal subgroup we have a general result:

Proposition 2.1.4. Let $G$ be a nilpotent group, $H \preccurlyeq G$, and $\left|G: H G^{\prime}\right|<\infty$. Then $|G: H|<\infty$.

Proof. Select a transversal to $H G^{\prime}$ in $G$ and let $S$ be the (finitely generated) subgroup it generates, so that $G=S H G^{\prime}$. Since $H$ is normal, $S H$ is a subgroup and the corollary
above implies that $G=S H$. But now $G / H$ is finitely generated and torsion. Since it is nilpotent, it is also finite, as desired.

In the finite Prüfer rank situation we may show more.

Proposition 2.1.5. Let $G$ be a nilpotent group of finite Prüfer rank, $H \leqslant G$, and $\left|G: H G^{\prime}\right|<\infty$. Then $|G: H|<\infty$.

Proof. We induct on the length of a subnormal series for $H$, the base case being the theorem above. Thus suppose $H \preccurlyeq L \leqslant_{f} G$. Then $L / H$ is of finite Prüfer rank and is of finite exponent. Thus it is finite, again since it is nilpotent.

We move next to the most general almost supplement result we will prove, where $I$ denotes the isolator operator.

Proposition 2.1.6. Let $G$ be nilpotent and $H$ a subgroup of $G$ satisfying $I\left(H G^{\prime}\right)=G$. Then $I(H)=G$.

Proof. The proof is by induction on class, the case $c=1$ being trivial. Assume that $G$ has class $c>1$ and write $\gamma_{c}:=\gamma_{c}(G)$. One verifies immediately that

$$
I\left(\frac{H \gamma_{c}}{\gamma_{c}} \frac{G^{\prime}}{\gamma_{c}}\right)=\frac{I\left(H G^{\prime}\right)}{\gamma_{c}}=\frac{G}{\gamma_{c}}
$$

Thus by induction we deduce that $I\left(H \gamma_{c}\right)=G$. To conclude we claim that $I\left(H \gamma_{c}\right) \leqslant I(H)$. Suppose then that $g \in G$ has some $n \geqslant 1$ with $g^{n}=h z$ for $h \in H$ and $z$ central. By the second part of lemma (1.2.3) we see that $\gamma_{c}(G) / \gamma_{c}(H)$ is torsion, so that there is some $m \geqslant 1$ with $z^{m} \in H$. Applying the centrality of $z$ we deduce that $g^{n m}=h^{m} z^{m} \in H$, as desired.

We have the following useful corollary of the above.

Corollary 2.1.7. Let $G$ be a nilpotent group of finite Hirsch length, and $H$ a subgroup of $G$. Then $h\left(H G^{\prime}\right)=h(G) \Longrightarrow h(H)=h(G)$.

Proof. It suffices to show that for a nilpotent group of finite Hirsch length, the condition $h(H)=h(G)$ is equivalent to $I(H)=G$. For $H$ normal this is clear. For general $H$ we induct on subnormality length to find $K$ with $H \triangleleft K \leqslant G$ and $h(K)=h(G) \Longleftrightarrow I(K)=G$. If now $h(H)=h(G)$, then certainly $h(H)=h(K)$ so that $I(H)=K$ and $I(K)=G$. Since $I$ is a closure operator we deduce that $I(H)=G$.

Conversely if $I(H)=G$, then $I(K)=G$ so that $h(K)=h(G)$. But $K / H$ is torsion and hence $h(H)=h(K)=h(G)$ as desired.

Among the various applications of this general result we mention the following.
Corollary 2.1.8. Let $G$ be nilpotent of finite Hirsch length and of class c. Suppose $h\left(G / G^{\prime}\right)=n$. Denote the free nilpotent group on $n$ generators of class $c$ by $F_{n, c}$. Then we have that $h(G) \leqslant h\left(F_{n, c}\right)$.

Proof. Select $x_{1}, \ldots, x_{n} \in G$ which map to linearly independent elements mod $\Gamma_{2}(G)$. Let the subgroup they generate be $H$. Then the obvious epimorphism $F_{n, c} \longrightarrow H$ shows that $h(H) \leqslant h\left(F_{n, c}\right)$. But by construction $G / H \Gamma_{2}(G)$ is torsion, from which we easily deduce that $G / H G^{\prime}$ is torsion. In particular we deduce that $h\left(H G^{\prime}\right)=h(G)$, and the above result yields that $h(H)=h(G)$, as desired.

### 2.2 Almost supplements of the isolated derived subgroup

One might hope that an adjusted version of proposition (2.1.1) holds, where we replace $G^{\prime}$ by $\Gamma_{2}(G)$ but insist on $G$ torsion-free. Although this holds for abelian groups, this cannot hold for nilpotent groups in general. We will identify precisely the class of counterexamples in the finitely generated case, and then finally see that in an important case the result does hold, namely in the case where the subgroup is isomorphic to the whole group.

We would specifically like to express our gratitude to Dr Koeck for his careful reading of a previous version of this section, where a crucial error was found.

We begin with a result in the positive direction.

Lemma 2.2.1. Let $G$ be a nilpotent group, and $H \leqslant G$ with $H \gamma_{2}(G) \geqslant \Gamma_{2}(G)$. Then $G=H \Gamma_{2}(G)$ only if $G=H$.

Proof. This follows immediately from proposition (2.1.1): note that $G=H \Gamma_{2}(G) \leqslant H \gamma_{2}(G)$, so that $H=G$.

The following lemma now identifies precisely the class of counterexamples, in the case $G$ is finitely generated and torsion free.

Lemma 2.2.2. Let $G$ be a $\mathfrak{T}$-group. Then there exists a proper subgroup $H<_{f} G$ with $G=H \Gamma_{2}(G)$ if and only if $\gamma_{2}(G) \neq \Gamma_{2}(G)$.

Proof. If such a subgroup exists, then by lemma (2.2.1) we find $H \gamma_{2}(G) \nsupseteq \Gamma_{2}(G)$, so that $\gamma_{2}(G) \neq \Gamma_{2}(G)$. Conversely since $G$ is finitely generated, the torsion subgroup $\Gamma_{2}(G) / \gamma_{2}(G)$ of the abelianisation admits a complement $H / \gamma_{2}(G)$. Then $G=H \Gamma_{2}(G)$ and we have that $\gamma_{2}(G)<\Gamma_{2}(G)$ yields $H<G$.

We devote the rest of this section by showing that every commensurability class of nonabelian $\mathfrak{T}$-group admits a counterexample. By lemma (2.2.2) we see that it suffices to find a commensurate $\mathfrak{T}$-group which has torsion in its abelianisation. It thus suffices to prove the following proposition.

Proposition 2.2.3. Let $G$ be a non-abelian $\mathfrak{T}$-group. Then there is a finite index subgroup $H \leqslant G$ such that $H / H^{\prime}$ is not torsion-free.

We remark that the fact that there is always a finite index subgroup of a $\mathfrak{T}$-group with torsion-free abelianisation is a (deservedly) far better known and more useful lemma. Arguably then, the preceding proposition may be regarded as lying on a tangent. Nevertheless we feel it is of note, the proof being of independent interest and the result curious in and of itself. To prove it we require three lemmas.

Lemma 2.2.4. Let $G$ be a $\mathfrak{T}$-group with $\Gamma_{2}(G)=\gamma_{2}(G)$. Then for any $H \leqslant_{f} G$, we have that $H / H^{\prime}$ is torsion-free if and only if $H^{\prime}=H \cap G^{\prime}$.

Proof. Consider the natural map $H / H^{\prime} \longrightarrow G / G^{\prime}$. This has kernel $H \cap G^{\prime} / H^{\prime}$, which is a finite group since $H^{\prime} \leqslant{ }_{f} G^{\prime}$, as may be seen directly, or by appealing to lemma (1.4.9). Thus if $H / H^{\prime}$ is torsion free this subgroup is trivial, so that $H^{\prime}=H \cap G^{\prime}$. Conversely if this subgroup is trivial then the natural map is injective and $H / H^{\prime}$ is a subgroup of $G / G^{\prime}$, torsion-free by hypothesis.

The following lemma may be found in the literature, and is proved by examining the collection process for nilpotent groups.

Lemma 2.2.5 ([21, 2.2.5]). If $G$ is a nilpotent group of class $c$ and $p$ a prime exceeding $c$, then $G^{p}:=\left\langle g^{p}: g \in G\right\rangle$ consists of $p$-powers, so that in fact $G^{p}=\left\{g^{p}: g \in G\right\}$.

Finally we require the following.

Lemma 2.2.6. Let $G$ be a $\mathfrak{T}$-group of class $c$. Then for almost all primes $p$, we have that $\gamma_{c}(G) \nless G^{p}$.

Proof. We prove the lemma for any prime $p>c$ which does not divide the order of the finite abelian group $\Gamma_{c}(G) / \gamma_{c}(G)$. Now as a finitely generated torsion-free abelian group, $\gamma_{c}(G)$ cannot be $p$-divisible. Thus select $z \in \gamma_{c}(G) \backslash \gamma_{c}(G)^{p}$. We claim that $z$ is not an element of $G^{p}$. By lemma (2.2.5) it suffices to show that it is not a $p$-th power of another element, noting that we assumed that $p>c$. If it is, we find some $w \in G$ with $w^{p}=z$. But then by definition $w$ represents a torsion element in $\Gamma_{c}(G) / \gamma_{c}(G)$, and so by the coprimality hypothesis on $p$ we have $w \in \gamma_{c}(G)$. But $z$ was chosen to have no $p$-th root in $\gamma_{c}(G)$, a contradiction.

We may now prove the main proposition.

Proof (of proposition 2.2.3). If $G / G^{\prime}$ already has torsion, then we are done. Thus suppose that $\Gamma_{2}(G)=\gamma_{2}(G)$ so that lemma (2.2.4) applies. Select $p$ prime to satisfy lemma (2.2.6), and hence pick some $z \in \gamma_{c}(G) \backslash G^{p}$. We claim that $H:=G^{p}\langle z\rangle$ has torsion abelianisation. With lemma (2.2.4) in view, it will suffice to show that $z \in H \cap G^{\prime} \backslash H^{\prime}$. Indeed $z \in H$ by construction, and $z \in G^{\prime}$ since $c \geqslant 2$ by hypothesis (note: it is here that we see where our proof fails for the abelian case). Finally $H^{\prime}=\left[G^{p}\langle z\rangle, G^{p}\langle z\rangle\right]=\left[G^{p}, G^{p}\right] \leqslant G^{p}$, by the centrality of $z$. Since $z \notin G^{p}$ by construction, in particular $z \notin H^{\prime}$, as desired.

The desired corollary is then

Corollary 2.2.7. In any non-abelian $\mathfrak{T}$-group, there exists a finite index subgroup $G$ with a proper subgroup $H<_{f} G$ with $G=H \Gamma_{2}(G)$.

Proof. This is an immediate corollary of proposition (2.2.3) and lemma (2.2.2).

Nevertheless, there is an important special case where this result does hold, namely when the subgroup is a homomorphic image of the group. This holds for torsion-free nilpotent groups of finite rank, as we will see in the next section. See in particular theorem (3.1.3).

## Chapter 3

## Endomorphisms of nilpotent groups of finite rank

In this chapter, we concern ourselves with criteria which guarantee the bijectivity of an endomorphism of a nilpotent group of finite rank. By finite rank we mean either finite Hirsch length or finite Prüfer rank: by proposition (1.6.6) these coincide. We prove two results here. Stated briefly, we will show that both the the torsion-free abelianisation and centre detect whether an endomorphism is an automorphism. In certain special cases, these results are known.

In particular, suppose an endomorphism $\sigma$ of a $\mathfrak{T}$-group induces an isomorphism on the centre. Then (independent) work of Farkas [13] and Wehrfritz [32] shows that $\sigma$ is necessarily an automorphism. Meanwhile Wehrfritz [32] demonstrates that in the finite rank setting, still torsion-free and nilpotent, the result does not always hold.

For $\pi$ a set of rational primes, we obtain sufficient criteria in the $\pi$-divisible case by considering a generalisation of integer-like endomorphisms to so-called $\pi$-like endomorphisms. Invertible integer-like endomorphisms (the case $\pi=\emptyset$ ) are discussed in the context of nilpotent groups in [9]. Briefly, integer-like endomorphisms are those that preserve a maximal rank torsion-free finitely generated abelian group in the Lie algebra of the Mal'cev completion. As a special case of theorem (3.2.5), we then have the following.

Theorem. Let $N$ be torsion-free nilpotent of finite rank, and $\sigma$ an integer-like endomorphism with $\left.\operatorname{det} \sigma\right|_{Z(N)}= \pm 1$. Then $\sigma$ is an automorphism of $N$.

We give counterexamples (see 3.2.6) to show that one must have a condition on the determinant and furthermore that some version of integer-like must be considered. Meanwhile Farkas and Wehrfritz, again independently, show that an endomorphism of a polycyclic-by-finite group which induces an isomorphism on the Zaleskii subgroup is an automorphism - see also the later paper of Wehrfritz [33] in this context. We give an example (see 3.2.7) to show that this cannot be generalised to finitely generated minimax groups.

On the other hand, we show (theorem (3.1.3)) that the torsion-free abelianisation always detects the surjectivity of an endomorphism, a more straightforward result. This is the content of the first section of this chapter, since certain results are required later.

### 3.1 Detecting surjectivity on the torsion-free abelianisation

We begin with a certain rigidity result for torsion-free abelian groups of finite rank.
Recall that for an abelian group $A$ and natural number $l \geqslant 0$, the subgroup $A[l]$ of $A$ is by definition the subgroup consisting of elements of order dividing $l$.

Recall also that given a group $G$ with subgroup $H$ and an endomorphism $\sigma$ of $G, H$ is said to be invariant under $\sigma$, or preserved by $\sigma$, if $\sigma(H) \leqslant H$, and stabilised by $\sigma$ if $\sigma(H)=H$.

We then have the following.
Lemma 3.1.1. Let $p$ be a prime, and $T$ a p-torsion abelian group with $T[p]$ finite. Then every injective endomorphism of $T$ is surjective.

Proof. Let $\sigma$ be an injective endomorphism of $T$. Since $T=\bigcup_{i \geqslant 0} T\left[p^{i}\right]$ and these are fully invariant subgroups, it suffices to show that the restriction of $\sigma$ to $T\left[p^{i}\right]$ is surjective for each $i$. Since $\sigma$ is injective, it will suffice to show that these subgroups are all finite. The case $i=1$ is our hypothesis, so assume that $T\left[p^{i}\right]$ is finite and note that $T\left[p^{i+1}\right]$ is an extension of $T[p]$ by $p T\left[p^{i+1}\right] \leqslant T\left[p^{i}\right]$. Conclude.

The following useful proposition will be used in both parts of this chapter.
Proposition 3.1.2. Let $A$ be a torsion-free abelian group of finite rank and $B$ a subgroup of the same rank. Suppose $\sigma$ is an endomorphism of $A$ which stabilises $B$. Then $\sigma$ is an automorphism of $A$.

Proof. By the five lemma, it will suffice to show that the induced endomorphism on $A / B$ is surjective. One verifies immediately from the necessary injectivity of $\sigma$ on $A$ that is also injective on this quotient. Moreover, decomposing the torsion group $A / B$ into its primary components, we may assume that $A / B$ is $p$-torsion for some prime $p$. Lemma (3.1.1) now applies due to our finiteness of rank assumption.

We are now in a position to prove the following result. As mentioned before, one way of interpreting this result is as a supplement result in the sense of section 2.2. In particular we see that the presence of the endomorphism is key.

Theorem 3.1.3. Let $N$ be torsion-free nilpotent of finite rank, and $\sigma$ an endomorphism of $N$ which induces an automorphism on the torsion-free abelianisation, so that $\sigma(N) \Gamma_{2}(N)=N$. Then $\sigma$ is an automorphism of $N$.

Proof. Note firstly that $\sigma$ is necessarily injective by corollary (2.1.7). We induct on the class $c$ of $N$, the case $c=1$ being clear. For $N$ of class $c>1$, a further Hirsch length argument shows that the induced map $\bar{\sigma}: N / \Gamma_{c}(N) \rightarrow N / \Gamma_{c}(N)$ is injective. By induction we deduce that $N=\sigma(N) \Gamma_{c}(N)$.

Consider now the $\mathbb{Z}[\sigma]$-module homomorphism $\alpha_{c}$ as in lemma (1.4.11). Since by hypothesis $\sigma$ acts as an automorphism on $N / \Gamma_{2}(N)$, the image $\gamma_{c}(N)$ of $\alpha_{c}$ is stabilised by $\sigma$. Proposition (3.1.2) applies and we deduce that $\sigma\left(\Gamma_{c}(N)\right)=\Gamma_{c}(N)$. Finally $N=\sigma(N) \Gamma_{c}(N)=\sigma(N) \sigma\left(\Gamma_{c}(N)\right)=\sigma(N)$, as desired.

### 3.2 Detecting surjectivity via the centre

Let $N$ be a torsion-free nilpotent group of finite rank and $\sigma$ an injective endomorphism of $N$. Furthermore denote by $R$ the Mal'cev completion of $N$ and $\sigma_{\times}$the induced automorphism of $R$. Note that the upper central series of $N$ is indeed preserved by $\sigma$ : this is corollary (1.7.8). Together with the map $\beta_{1}$ obtained from corollary (1.1.4) by taking $k=0$, the natural maps $Z(N) \rightarrow Z(R)$ and $N / N^{\prime} \rightarrow R / R^{\prime}$ induce maps which
fit together as below.


The required properties of this sequence are detailed in the following proposition.

Proposition 3.2.1. Let $N, R, \sigma, \sigma_{\times}$be as above and consider diagram (3.2.1). We claim the following.

1. $\gamma$ is injective.
2. $\delta$ is an isomorphism.
3. $\delta^{-1} \circ \gamma \circ \beta_{1}$ is a $\mathbb{Z}\left[\sigma_{\times}\right]$-module map.
4. $\delta^{-1} \circ \gamma$ is a $\mathbb{Z}\left[\sigma_{\times}^{-1}\right]$-module map, provided that $\sigma(Z(N))=Z(N)$.

Proof. The first claim is an immediate consequence of the injectivity of $Z(N) \rightarrow Z(R)$. That $\delta$ is an isomorphism follows from the fact that $Z(R)$ is a $\mathbb{Q}$-vector space and the $\operatorname{map} N / N^{\prime} \rightarrow R / R^{\prime}$ is naturally isomorphic to tensoring with $\mathbb{Q}$.

For the third part, note that $\sigma_{\times}$has a well defined action (as specified in lemma (1.1.3)) on $\operatorname{Hom}\left(R / R^{\prime}, Z(R)\right)$ since it is an automorphism of $R$. In order to show it is equivariant, let $w Z$ be an element of $Z^{2}(N) / Z^{1}(N)$. It is required to show that we have an equality of maps

$$
\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(\sigma(w) Z)=\left(\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(w Z)\right)^{\sigma_{\times}}: R / R^{\prime} \longrightarrow Z(R)
$$

Since $\delta$ is an isomorphism it suffices to check that these are equal after precomposing with the natural map $N / N^{\prime} \rightarrow R / R^{\prime}$. It thus suffices to show that if $x \in N$ that

$$
\left(\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(\sigma(w) Z)\right)\left(x R^{\prime}\right)=\sigma_{\times}\left(\left(\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(w Z)\right)\left(\sigma_{\times}^{-1}(x) R^{\prime}\right)\right)
$$

The left hand side now immediately reduces to $[\sigma(w), x]$, so that this equality holds if and only if

$$
\begin{equation*}
\sigma_{\times}^{-1}[\sigma(w), x]=\left(\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(w Z)\right)\left(\sigma_{\times}^{-1}(x) R^{\prime}\right) \tag{3.2.2}
\end{equation*}
$$

Note that $\sigma_{\times}^{-1}(x)$ may not lie in $N$, but there certainly exists some $l \geqslant 1$ for which $\sigma_{\times}^{-1}(x)^{l} \in N$. Taking the $l$-th multiple in $Z(R)$ of the right hand side of equation (3.2.2) we thus see that

$$
\begin{aligned}
l \cdot\left(\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(w Z)\right)\left(\sigma_{\times}^{-1}(x) R^{\prime}\right) & =\left(\left(\delta^{-1} \circ \gamma \circ \beta_{1}\right)(w Z)\right)\left(\sigma_{\times}^{-1}(x)^{l} R^{\prime}\right) \\
& =\left[w, \sigma_{\times}^{-1}(x)^{l}\right] \\
& =l \cdot\left[w, \sigma_{\times}^{-1}(x)\right] .
\end{aligned}
$$

Appealing to the unique divisibility of $Z(R)$ we deduce that the right hand side of equation (3.2.2) is precisely $\left[w, \sigma_{\times}^{-1}(x)\right]=\sigma_{\times}^{-1}[\sigma(w), x]$, as desired.

We now show the final part, and note here that the very fact that the abelian group $\operatorname{Hom}\left(N / N^{\prime}, Z(N)\right)$ is a $\mathbb{Z}\left[\sigma_{\times}^{-1}\right]$-module is because $\sigma$ restricted to the centre is an isomorphism. Thus it is exactly here that use our hypothesis. Indeed, given $\theta: N / N^{\prime} \longrightarrow Z(N)$ and $g \in N$ we find that $\theta^{\sigma^{-1}}\left(g N^{\prime}\right)$ is precisely

$$
\sigma_{\times}^{-1} \theta\left(\sigma(g) N^{\prime}\right) \in \sigma_{\times}^{-1}(Z(N))=Z(N) .
$$

We now show that $\delta^{-1} \circ \gamma$ is indeed $\sigma_{\times}^{-1}$-equivariant. Thus let $\theta: N / N^{\prime} \longrightarrow Z(N)$. We need to show that

$$
\delta^{-1} \circ \gamma\left(\theta^{\sigma_{\times}^{-1}}\right)=\left(\delta^{-1} \circ \gamma(\theta)\right)^{\sigma_{\times}^{-1}}
$$

as maps $R / R^{\prime} \longrightarrow Z(R)$. As above it suffices to check for $g \in N$ that we have

$$
\delta^{-1} \circ \gamma\left(\theta^{\sigma_{\times}^{-1}}\right)\left(g R^{\prime}\right)=\sigma_{\times}^{-1}\left(\left(\delta^{-1} \circ \gamma(\theta)\right)\left(\sigma(g) R^{\prime}\right)\right) .
$$

The left hand side is just $\theta^{\sigma_{\times}^{-1}}\left(g N^{\prime}\right)=\sigma_{\times}^{-1} \theta\left(\sigma(g) N^{\prime}\right)$ by our calculation above. Thus the equation holds precisely when $\left(\delta^{-1} \circ \gamma(\theta)\right)\left(\sigma(g) R^{\prime}\right)=\theta\left(\sigma(g) N^{\prime}\right)$, but this is again true by definition of the maps under consideration.

For the rest of this section, $\pi$ will denote a (possibly empty) set of prime numbers. We begin by recalling the following standard notions. A $\pi$-number is a rational integer with
all prime divisors contained in $\pi$, and we notate $\mathbb{Z}[1 / \pi]:=\left\{\frac{m}{n}: m \in \mathbb{Z}, n\right.$ a non-zero $\pi$-number $\}$. Finally a $\pi$-unit is a unit in this ring. If $\pi=\{p\}$, we notate $\mathbb{Z}[1 / \pi]=\mathbb{Z}[1 / p]$ as usual. We now introduce the notion of $\pi$-like morphisms.

Definition 3.2.2. Let $V$ be a rational vector space of dimension $n<\infty$, and let $\nu$ be an automorphism of $V$. We say that $\nu$ is $\pi$-like if one of the following equivalent conditions hold.

- The coefficients of the characteristic polynomial of $\nu$ lie in $\mathbb{Z}[1 / \pi]$.
- There exists some $W \leqslant V$ preserved by $\nu$ with $W \cong \mathbb{Z}[1 / \pi]^{n}$.

Now let $\sigma$ be an injective endomorphism of a torsion-free nilpotent group $N$ of finite rank. We say that $\sigma$ is $\pi$-like if the induced automorphism of the associated rational Lie algebra of $N$ is $\pi$-like.

The equivalence of these two conditions is standard linear algebra. In case $\pi=\emptyset$, this is the notion of integer-like automorphisms, as considered in [?]. This is a particularly well-behaved class of endomorphisms, as we see next.

Proposition 3.2.3. Let $N$ be torsion-free nilpotent of finite rank and $\sigma$ an injective endomorphism of $N$. Then the following are equivalent.

1. $\sigma$ is $\pi$-like.
2. For any central series of $N$ preserved by $\sigma$ with torsion-free sections, the action of $\sigma$ on each section is $\pi$-like.
3. The induced map on $N / \Gamma_{2}(N)$ is $\pi$-like.

Proof. $1 \Longrightarrow$ 2: Mal'cev complete at the central series to obtain a decomposition of the associated rational Lie algebra $V$ as $0=V_{0} \leqslant V_{1} \leqslant \cdots \leqslant V_{r}=V$ with $\sigma_{\times}\left(V_{i}\right)=V_{i}$ for each $i$, where $\sigma_{\times}$is the induced automorphism. Let the characteristic polynomial of the induced automorphism on the section $V_{i} / V_{i-1}$ be $f_{i}$ and the characteristic polynomial of $\sigma_{\times}$be $f$. We obtain a factorisation $f=f_{1} \cdots f_{n}$. By hypothesis $f$ has coefficients in $\mathbb{Z}[1 / \pi]$, and Gauss' Lemma implies that each $f_{i}$ has coefficients in this ring too.
$2 \Longrightarrow$ 3: Trivial.
$3 \Longrightarrow 1$ : Select a subgroup isomorphic to a free $\mathbb{Z}[1 / \pi]$-module of maximal rank in $N / \Gamma_{2}(N)$ preserved by $\sigma$. The image of this subgroup under the tensor power maps
described in lemma (1.4.11) show that the action on each section of the isolated central series is $\pi$-like. Upon Mal'cev completing, this factorises the characteristic polynomial of the induced automorphism $\sigma_{\times}$into polynomials with coefficients in $\mathbb{Z}[1 / \pi]$. Conclude.

A trivial consequence of the above proposition is the following, which we state separately for later clarity.

Corollary 3.2.4. Let $V$ be a finite dimensional $\mathbb{Q}$-vector space and $\omega$ a $\pi$-like automorphism of $V$. Suppose that $U$ is a subspace of $V$ stabilised by $\omega$. Then $\left.\omega\right|_{U}$ is $\pi$-like.

We may now show the following.
Theorem 3.2.5. Let $N$ be a $\pi$-divisible torsion-free nilpotent group of finite rank, and let $\sigma$ be a $\pi$-like endomorphism of $N$ such that $\left.\operatorname{det} \sigma\right|_{Z(N)}$ is a $\pi$-unit. Then $\sigma \in \operatorname{Aut}(N)$.

Proof. The proof is by induction on the class $c$ of $N$. If $c=1$ and $N$ has rank $n$, select $W \cong \mathbb{Z}[1 / \pi]^{n}$ preserved by $\sigma$ and $W \leqslant N$. Then the hypothesis on the determinant implies that $\sigma(W)=W$, and we may conclude with proposition (3.1.2).

Thus suppose $c>1$. By considering $N / Z(N)$ of smaller class and centre $Z^{2}(N) / Z^{1}(N)$, it will suffice to show that the determinant of the induced map on $Z^{2}(N) / Z^{1}(N)$ is also a $\pi$-unit, noting also that $N / Z(N)$ is still $\pi$-divisible by the results of section (1.3). In order to proceed, we will show that the action of $\sigma_{\times}^{-1}$ on $\operatorname{Hom}\left(R / R^{\prime}, Z(R)\right)$ is $\pi$-like. This will follow from the final part of proposition (3.2.1) once we know that the action of $\sigma_{\times}^{-1}$ on $\operatorname{Hom}\left(N / N^{\prime}, Z(N)\right)$ is $\pi$-like. Since $Z(N)$ is torsion-free, it is equivalent to show this for $\operatorname{Hom}\left(N / \Gamma_{2}(N), Z(N)\right)$. Our hypothesis implies, by proposition (3.2.3), that the action of $\sigma$ on $N / \Gamma_{2}(N)$ is $\pi$-like. Moreover by our determinant hypothesis we may deduce that the action of $\sigma^{-1}$ on $Z(N)$ is $\pi$-like. Thus there are subgroups $S, T$ of $N / \Gamma_{2}(N)$ and $Z(N)$ respectively, both of maximal Hirsch length and isomorphic to direct sums of copies of $\mathbb{Z}[1 / \pi]$, with $\sigma(S) \leqslant S$ and $\sigma^{-1}(T) \leqslant T$. Consider the subgroup $H_{S, T} \leqslant \operatorname{Hom}\left(N / \Gamma_{2}(N), Z(N)\right)$, consisting by definition of those $f$ for which $f(S) \leqslant T$. One may conclude by observing that $H_{S, T}$ is preserved by $\sigma^{-1}$, and that $H_{S, T}$ is of maximal Hirsch length and also isomorphic to a direct sum of copies of $\mathbb{Z}[1 / \pi]$.

Now let $U$ be the $\mathbb{Q}$-span of the image of $Z^{2}(N) / Z^{1}(N)$ in $\operatorname{Hom}\left(R / R^{\prime}, Z(R)\right)$ under $\delta^{-1} \circ \gamma \circ \beta_{1}$, in the notation of proposition (3.2.1). Then in particular $\sigma_{\times}^{-1}(U)=U$ and corollary (3.2.4) applies with $\omega=\sigma_{\times}^{-1}$ : denoting the map $\sigma_{\times}$induces on $U$ by $\bar{\sigma}_{\times}$, we
deduce that $\left(\operatorname{det} \bar{\sigma}_{\times}\right)^{-1}$ is a $\pi$-number. Moreover $\operatorname{det} \bar{\sigma}_{\times}$is a $\pi$-number by proposition (3.2.3). Thus it is a $\pi$-unit, as desired.

We now justify why we prove theorem (3.2.5) under the stated hypotheses.
Example 3.2.6. We consider the nilpotent group

$$
N:=\left(\begin{array}{ccc}
1 & \mathbb{Z} & \mathbb{Z}[1 / 2] \\
& 1 & \mathbb{Z}[1 / 2] \\
& & 1
\end{array}\right) \leqslant \mathrm{GL}_{3}(\mathbb{Q}), \quad Z(N)=\left(\begin{array}{ccc}
1 & 0 & \mathbb{Z}[1 / 2] \\
& 1 & 0 \\
& & 1
\end{array}\right)
$$

with endomorphisms

$$
\varphi_{1}\left(\begin{array}{ccc}
1 & a & c \\
& 1 & b \\
& & 1
\end{array}\right)=\left(\begin{array}{ccc}
1 & 2 a & c \\
& 1 & b / 2 \\
& & 1
\end{array}\right), \quad \varphi_{2}\left(\begin{array}{ccc}
1 & a & c \\
& 1 & b \\
& & 1
\end{array}\right)=\left(\begin{array}{ccc}
1 & 2 a & 2 c \\
& 1 & b \\
& & 1
\end{array}\right)
$$

Both $\varphi_{1}$ and $\varphi_{2}$ are injective endomorphisms of $N$ which are not surjective, but induce isomorphisms on the centre.

The first example demonstrates that we must assume that the induced map on the torsion-free abelianisation is $\pi$-like. (Note that $\pi=\emptyset$ here.) However proposition (3.2.3) shows that this already implies that the whole endomorphism is $\pi$-like, whence this hypothesis. Meanwhile, the second endomorphism is $\pi$-like but the determinant on the centre is not a $\pi$-unit, whence our second hypothesis.

It is shown independently in [13] and [32] that an endomorphism of a polycyclic-by-finite group which restricts to an isomorphism of the Zaleskii subgroup is an automorphism. (Recall that the Zaleskii subgroup is the centre of the Fitting subgroup of the group modulo its largest normal periodic subgroup). We now show that this cannot hold in the finitely generated minimax setting.

Example 3.2.7. Let $N$ be as above and set

$$
t:=\left(\begin{array}{ccc}
1 / 2 & & \\
& 1 & \\
& & 1
\end{array}\right), \quad x:=\left(\begin{array}{lll}
1 & & \\
& 1 & \\
& & 2
\end{array}\right)
$$

Then $G:=\langle N, x\rangle$ is finitely generated minimax with Fitting subgroup $N$. Conjugating by $t$ gives a proper inclusion $G^{t}<G$ and moreover induces $\varphi_{2}$ on $N$ above. The Zaleskii
subgroup here is precisely the centre of the Fitting subgroup, where our endomorphism induces an isomorphism.

## Chapter 4

## Applications of the Mal'cev correspondence

In this chapter we concern ourselves with deducing structural information about nilpotent groups using the associated Mal'cev Lie algebra. The first section will deal with gradings on Lie algebras and is a minor extension of work by de Cornulier in [7], and the second section will extend various results due to Segal in [26].

### 4.1 Carnot gradings and powered nilpotent groups

In this section we concern ourselves with a strong negation of the cohopfian property, that of being powered. Loosely speaking that there is an injective endomorphism of the group acting as powering on the abelianisation. The notion of a Carnot grading is discussed in some detail in the work of de Cornulier in [7] and it is from here that we have the idea for the main result. In particular, we aim to show the following.

Proposition 4.1.1. Let $\Gamma$ be a $\mathfrak{T}$-group. Then $\Gamma$ is powered if and only if the Lie algebra associated to the Mal'cev completion of $\Gamma$ is Carnot.

We first introduce the relevant notions.

Definition 4.1.2. Let $G$ be a radicable torsion-free nilpotent group of finite rank. Given $\lambda \in \mathbb{Q}^{\times}$, we say that an automorphism $\varphi$ of $G$ is a $\lambda$-powering automorphism if the following diagram commutes, where the vertical maps are the canonical projections, and $\times \lambda$ denotes the map given by multiplication by $\lambda$.


We say that $G$ is powered if there exists a $\lambda$-powering automorphism of $G$ for some $\lambda \in \mathbb{Q}^{\times} \backslash\{ \pm 1\}$. We say that a $\mathfrak{T}$-group is powered if its Mal'cev completion is powered.

One defines in an identical fashion the notion of a $\lambda$-powering automorphism of a finite dimensional rational Lie algebra, and thus also the notion of a powered Lie algebra.

Note that this is well defined here since the abelianisation of a Mal'cev complete nilpotent group is torsion-free. As one would hope, one has that a radicable nilpotent group is powered if and only if its corresponding Lie algebra is:

Lemma 4.1.3. Let $G$ be a torsion-free radicable nilpotent group of finite rank with associated Lie algebra $\mathfrak{g}$. Then $G$ is powered as a group if and only if $\mathfrak{g}$ is powered as a Lie algebra. More precisely, for any $\lambda$, a $\lambda$-powering automorphism $\varphi$ of $G$ induces a $\lambda$-powering automorphism of $\mathfrak{g}$, and vice versa.

Proof. Note firstly that an automorphism $\varphi$ of $G$ is $\lambda$-powering if and only if for each $g \in G$ one has that $\varphi(g) g^{-\lambda} \in \gamma_{2}(G)$. Analogously for $\mathfrak{g}$, an automorphism $\psi$ is $\lambda$ powering if and only if for each $x \in \mathfrak{g}$ one has that $\psi(x)-\lambda x \in \gamma_{2}(\mathfrak{g})$.

With this in mind, suppose that $\varphi$ is $\lambda$-powering for $G$, and let $\hat{\varphi}$ be the induced automorphism of $\mathfrak{g}$. Given $x \in \mathfrak{g}$, we then have that

$$
\begin{aligned}
\hat{\varphi}(x)-\lambda x & =\log \circ \varphi \circ \exp (x)+\log \left(\exp (x)^{-\lambda}\right) \\
& =\log \left(\varphi(\exp (x)) \exp (x)^{-\lambda} w\right)
\end{aligned}
$$

for some $w \in \gamma_{2}(G)$, by an application of the inverse Baker-Campbell-Hausdorff formula (1.8.3). But now by hypothesis on $\varphi$, we see that this final element lies in $\log \left(\gamma_{2}(G)\right)=\gamma_{2}(\mathfrak{g})$, as desired.

Conversely, suppose that $\psi$ is a $\lambda$-powering automorphism for $\mathfrak{g}$ with induced automorphism $\hat{\psi}$ on $G$. Then for any $g \in G$, we have

$$
\begin{aligned}
\hat{\psi}(g) g^{-\lambda} & =\exp \circ \psi \circ \log (g) \exp (-\lambda \log (g)) \\
& =\exp \left(\psi(\log (g))-\lambda \log (g)+w^{\prime}\right)
\end{aligned}
$$

for some $w^{\prime} \in \gamma_{2}(\mathfrak{g})$, in this case by the Baker-Campbell-Hausdorff formula (1.8.1). Again by hypothesis on $\psi$ this yields that this last element lies in $\exp \left(\gamma_{2}(\mathfrak{g})\right)=\gamma_{2}(G)$.

The following lemma is also true in the group setting, but is not required in the sequel.
Lemma 4.1.4. Let $\mathfrak{g}$ be a powered finite dimensional nilpotent rational Lie algebra, and let $\psi$ be a $\lambda$-powering automorphism for $\mathfrak{g}$. Then for each $i \geqslant 1, \psi$ acts as multiplication by $\lambda^{i}$ on $\gamma_{i}(\mathfrak{g}) / \gamma_{i+1}(\mathfrak{g})$.

Proof. The proof is by induction on $i$, the case $i=1$ true by hypothesis. Consider then $x \in \gamma_{i}(\mathfrak{g})$ for some $i>1$. Since $\psi$ is linear, we may assume that $x=[y, z]$ for some $y \in \mathfrak{g}$ and some $z \in \gamma_{i-1}(\mathfrak{g})$. Since $\psi$ is powering, we find some $y^{\prime} \in \gamma_{2}(\mathfrak{g})$ such that $\psi(y)-\lambda y=y^{\prime}$, and by the inductive hypothesis we find some $z^{\prime} \in \gamma_{i}(\mathfrak{g})$ for which $\psi(z)-\lambda^{i-1} z=z^{\prime}$. We then compute

$$
\begin{aligned}
\psi(x) & =[\psi(y), \psi(z)] \\
& =\left[\lambda y+y^{\prime}, \lambda^{i-1} z+z^{\prime}\right] \\
& =\lambda^{i}[y, z]+\left[\lambda y, z^{\prime}\right]+\left[y^{\prime}, \lambda^{i-1} z\right]+\left[y^{\prime}, z^{\prime}\right] \\
& \in \lambda^{i} x+\left[\mathfrak{g}, \gamma_{i}(\mathfrak{g})\right]+\left[\gamma_{2}(\mathfrak{g}), \gamma_{i-1}(\mathfrak{g})\right]+\left[\gamma_{2}(\mathfrak{g}), \gamma_{i}(\mathfrak{g})\right] \\
& \subseteq \lambda^{i} x+\gamma_{i+1}(\mathfrak{g}),
\end{aligned}
$$

where the final inclusion holds due to the Jacobi identity. This completes the proof.

We now turn to the notion of a graded Lie algebra. The next few definitions and the following proposition are adapted from de Cornulier in [7].

Definition 4.1.5 ( [7, p. 16]). Let $\mathfrak{g}$ be a Lie algebra over a ring $R$, and let $(A,+)$ be a magma (that is, a set with a single associative binary operation, denoted + ). We say that $\mathfrak{g}$ is graded in $A$ if there exists a decomposition

$$
\mathfrak{g}=\bigoplus_{\alpha \in A} \mathfrak{g}_{\alpha},
$$

of $\mathfrak{g}$ as an $R$-module, with the property that $\left[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}\right] \subseteq \mathfrak{g}_{\alpha+\beta}$ for each $\alpha, \beta \in A$.

We introduce now the notion of a Carnot graded Lie algebra.
Definition 4.1.6 ([7, Definition 3.3]). A grading on a Lie algebra $\mathfrak{g}$ is said to be Carnot if the grading is in the positive integers and the Lie algebra is generated by $\mathfrak{g}_{1}$. A graded Lie algebra is Carnot-graded if the grading is Carnot. Finally we say that a Lie algebra is Carnot if it admits a Carnot grading.

To any Lie algebra there is associated a canonical Carnot-graded Lie algebra (not necessarily isomorphic):

Definition 4.1.7 ([7, Definition 3.4]). Let $\mathfrak{g}$ be a Lie algebra with lower central series $\left(\gamma_{i}(\mathfrak{g})\right)_{i \geqslant 1}$. The associated Carnot-graded Lie algebra of $\mathfrak{g}$ is defined to be

$$
\operatorname{Car}(\mathfrak{g}):=\bigoplus_{i \geqslant 1} \gamma_{i}(\mathfrak{g}) / \gamma_{i+1}(\mathfrak{g})
$$

with Lie bracket induced from the bracket on $\mathfrak{g}$.

For any Lie algebra $\mathfrak{g}, \operatorname{Car}(\mathfrak{g})$ is indeed Carnot graded, and up to isomorphism all Carnot gradings look like this:

Lemma 4.1.8 ( [7, Proposition 3.5]). A Lie algebra $\mathfrak{g}$ is Carnot if and only if it is isomorphic to $\operatorname{Car}(\mathfrak{g})$. Furthermore for any Carnot grading on $\mathfrak{g}$, there is an isomorphism of graded Lie algebras $\mathfrak{g} \cong \operatorname{Car}(\mathfrak{g})$.

The key step to proving proposition (4.1.1) is now the following lemma, one direction of which is essentially contained in [7].

Lemma 4.1.9. Let $\mathfrak{g}$ be a finite dimensional nilpotent rational Lie algebra. Then $\mathfrak{g}$ is powered if and only if it is Carnot.

Proof. If we assume that $\mathfrak{g}$ is Carnot, then we may equip $\mathfrak{g}$ with a Carnot grading $\mathfrak{g}=\bigoplus_{i \geqslant 1} \mathfrak{g}_{i}$. Given now $\lambda \in \mathbb{Q}^{\times} \backslash\{ \pm 1\}$ arbitrary, we may define an automorphism $\psi$ of $\mathfrak{g}$ which acts on $\mathfrak{g}_{i}$ as multiplication by $\lambda^{i}$. This is a Lie algebra homomorphism since if $x \in \mathfrak{g}_{i}$ and $y \in \mathfrak{g}_{j}$, we have that $\psi[x, y]=\lambda^{i+j}[x, y]=\left[\lambda^{i} x, \lambda^{j} y\right]=[\psi(x), \psi(y)]$. To see that it is a powering automorphism, choose an isomorphism of graded Lie algebras $\mathfrak{g} \cong \operatorname{Car}(\mathfrak{g})$ as in lemma (4.1.8) to see that $\gamma_{2}(\mathfrak{g})=\bigoplus_{i \geqslant 2} \mathfrak{g}_{i}$, which yields that $\bmod \gamma_{2}(\mathfrak{g})$, $\psi$ acts as multiplication by $\lambda$, as desired.

Conversely, suppose that we are given a $\lambda$-powering automorphism $\psi$ of $\mathfrak{g}$ for some $\lambda \in \mathbb{Q}^{\times} \backslash\{ \pm 1\}$. We claim that this map is diagonalisable. Indeed, we show that the minimum polynomial of $\psi$ divides the polynomial $\prod_{i=1}^{c}\left(X-\lambda^{i}\right)$, where $c$ denotes the nilpotency class of $\mathfrak{g}$. To see this it suffices to see that the endomorphism $\prod_{i=1}^{c}\left(\psi-\lambda^{i}\right)$ of $\mathfrak{g}$ is the zero map.

More precisely, we claim by induction on $j$ that $\prod_{i=1}^{j}\left(\psi-\lambda^{i}\right)(\mathfrak{g}) \subseteq \gamma_{j+1}(\mathfrak{g})$. When $j=1$ this is the definition of a $\lambda$-powering automorphism. For the inductive step, we see that lemma (4.1.4) yields an inclusion $\left(\psi-\lambda^{j+1}\right)\left(\gamma_{j+1}(\mathfrak{g})\right) \subseteq \gamma_{j+2}(\mathfrak{g})$. Combining this with the inductive hypothesis then gives the result.

One thus obtains an eigenspace decomposition $\mathfrak{g}=\bigoplus_{i \geqslant 1} E_{\lambda^{i}}(\psi)$. It remains to check that this is a Carnot grading. Indeed, given $x \in E_{\lambda^{i}}(\psi)$ and $y \in E_{\lambda^{j}}(\psi)$, one has $\psi[x, y]=[\psi(x), \psi(y)]=\left[\lambda^{i} x, \lambda^{j} y\right]=\lambda^{i+j}[x, y]$, so that $[x, y] \in E_{\lambda^{i+j}}(\psi)$, as desired. Finally since $E_{\lambda^{1}}(\psi) \cong \mathfrak{g} / \gamma_{2}(\mathfrak{g})$, we see that the algebra is indeed generated by the degree 1 component, as desired.

In the presence of lemma (4.1.3), we see that the previous lemma proves proposition (4.1.1).

### 4.2 Uniformly sandwiching $\mathfrak{T}$-groups

The Mal'cev correspondence is a categorical equivalence for radicable nilpotent groups, but one often works with specific $\mathfrak{T}$-groups. It is of interest then to see to what extent the correspondence can be made to work in the finitely generated setting. Immediately one can find examples of $\mathfrak{T}$-groups for which their logarithm is not even closed under addition: it is well known that one only needs to pass to a finite index subgroup or overgroup to arrange for the image under the logarithm to be a subgroup. Our aim here will be to generalise a result in this direction due to Segal in [26]. We are interested principally in removing the dependence of the constants on a specific representation, and also to generalise a specific result to closure under the Lie bracket. In detail, our aim will be to show the following.

Proposition 4.2.1. Let $G$ be a torsion-free radicable nilpotent group of finite rank and of class at most $c$. There exists a constant $\alpha$, which depends on $c$ only, such that for any subgroup $\Gamma \leqslant G$, we have that the images of $\Gamma^{\alpha}$ and $\left(\Gamma^{1 / \alpha}\right)^{\alpha}$ are Lie subrings of $\log G$.

Before embarking on the proof, we require a specific result concerning log. This is an immediate generalisation of corollaries 2 and 3 to be found in [26, pp. 102-103].

Lemma 4.2.2. Let $G$ be a torsion-free radicable nilpotent group of finite rank and of class at most $c$. Given elements $x_{1}, \ldots, x_{s} \in G$, we have a relation of the form

$$
\begin{equation*}
\left[\log \left(x_{1}\right), \ldots, \log \left(x_{s}\right)\right]=\log \left[x_{1}, \ldots, x_{s}\right]+\sum_{i} s_{i} \log v_{i} \tag{4.2.1}
\end{equation*}
$$

where each of the $v_{i}$ is an iterated commutator of length at least $s+1$ in $x_{1}, \ldots, x_{s}$, and the constants $s_{i} \in \mathbb{Q}$ depend only on $c$.

Proof. This follows from the cited results, with two modifications. One first notes that the eventual vanishing of commutators is a consequence of bounded nilpotency class. To see that the coefficients are suitably universal in this setting, one first performs the proof in the Mal'cev completion of the free nilpotent group of class $c$ on $x_{1}, \ldots, x_{s}$, and then specialises to $G$.

The following is an adapted and generalised version of [26, Lemma 1, p. 105].
Lemma 4.2.3. Suppose $\Gamma$ is a $\mathfrak{T}$-group of class at most $c, \Gamma^{\mathbb{Q}}$ its Mal'cev completion, and $H \geqq \Gamma$. Denote by $\gamma_{j}$ the $j$ th term of the lower central series of $\Gamma$. Then there exists a constant $r$, depending only on $c$, such that for any $0 \leqslant t \leqslant c-1$,

$$
\begin{equation*}
r^{2^{t}-1} \mathbb{Z} \log \left(\gamma_{c-t} \cap H\right) \subseteq \log \left(\gamma_{c-t} \cap H\right) . \tag{4.2.2}
\end{equation*}
$$

We have the following immediate corollary of the above lemma.
Corollary 4.2.4. Suppose $\Gamma$ is a $\mathfrak{T}$-group of class at most $c$, and let $\Gamma^{\mathbb{Q}}$ be its Mal'cev completion. There exists a constant m, depending only on $c$, such that

$$
m \mathbb{Z} \log \Gamma \subseteq \log \Gamma
$$

Proof. It suffices to take $H=\Gamma$ and $t=c-1$ in lemma 4.2.3, so that $m=r^{2^{c-1}-1}$ works.

Proof (of lemma 4.2.3). Let $F_{c}=F_{c}(x, y)$ denote the free nilpotent group of class $c$ on $x, y$. Now choose $r \in \mathbb{N}$ so that for each vector $\mathbf{e}$ of positive integers corresponding to a
commutator $[x, y]_{\mathbf{e}}$ of total length at most $c$, we have that

$$
\begin{equation*}
a_{\mathbf{e}}[\log (x), \log (y)]_{\mathbf{e}} \in \frac{1}{r} \sum_{\mathbf{f}} \mathbb{Z} \log [x, y]_{\mathbf{f}} \tag{4.2.3}
\end{equation*}
$$

That this constant exists depends on lemma 4.2.2, and that it depends only on $c$ is a consequence of the universality of the $a_{\mathbf{e}}$ and the freeness of $F_{c}$.

Now let $j \geqslant 1$. Then for any $g_{1}, \ldots, g_{s} \in \gamma_{j} \cap H$, we have

$$
\begin{equation*}
\log \left(g_{1}\right)+\cdots+\log \left(g_{s}\right)-\log \left(g_{1} \cdots g_{s}\right) \in \frac{1}{r} \sum_{\mathbf{f}} \mathbb{Z} \log \left(\gamma_{j+1} \cap H\right) \tag{4.2.4}
\end{equation*}
$$

This follows by induction on $s$ from the following more general observation to be used later: given also $i \geqslant 1$ and any pair $x \in \gamma_{i} \cap H, y \in \gamma_{j} \cap H$, we have that

$$
\begin{align*}
\log (x)+\log (y)-\log (x y) & =-\sum_{\mathbf{e}} a_{\mathbf{e}}[\log (x), \log (y)]_{\mathbf{e}} \\
& \in \frac{1}{r} \sum_{\mathbf{f}} \mathbb{Z} \log [x, y]_{\mathbf{f}} \\
& \subseteq \frac{1}{r} \sum_{\mathbf{f}} \mathbb{Z} \log \left(\gamma_{i+j} \cap H\right) \tag{4.2.5}
\end{align*}
$$

where the final line follows since for any vector $\mathbf{e}$ of positive integers we have the inclusion

$$
\left[\gamma_{i} \cap H, \gamma_{j} \cap H\right]_{\mathbf{e}} \subseteq \gamma_{i+j} \cap H
$$

We now show that 4.2.2 holds when $t=0$. Since $\gamma_{c} \cap H$ is central, we see immediately that

$$
r^{2^{0}-1} \mathbb{Z} \log \left(\gamma_{c-0} \cap H\right)=\mathbb{Z} \log \left(\gamma_{c} \cap H\right)=\log \left(\gamma_{c} \cap H\right)
$$

Thus suppose $t>0$ and that (4.2.2) holds for all $0 \leqslant q<t$. Take $\lambda \in \mathbb{Z} \log \left(\gamma_{c-t} \cap H\right)$, so that we have

$$
\lambda=\log \left(g_{1}\right)+\cdots+\log \left(g_{s}\right)
$$

with each $g_{i} \in \gamma_{c-t} \cap H$. Our observation in line (4.2.4) yields that

$$
r^{2^{t-1}} \lambda-\log (w) \in r^{2^{t-1}-1} \mathbb{Z} \log \left(\gamma_{c-(t-1)} \cap H\right)
$$

where $w=\left(g_{1} \cdots g_{s}\right)^{r^{2^{t-1}}}$. By induction we have some $h_{t-1} \in \gamma_{c-(t-1)} \cap H$ such that $r^{2^{t-1}} \lambda=\log (w)+\log \left(h_{t-1}\right)$, and applying our observation in line (4.2.5) we see that

$$
r^{2^{t-1}+1} \lambda=\log \left(w^{\prime}\right)+\mu,
$$

for $w^{\prime}=\left(w h_{t-1}\right)^{r}$ and $\mu \in \mathbb{Z} \log \left(\gamma_{c-(t-2)} \cap H\right)$ at least (in fact $\mu \in \mathbb{Z} \log \left(\gamma_{2(c-t)+1)} \cap H\right)$ ). Now by induction again we have that $r^{2^{t-2}-1} \mu \in \log \gamma_{c-(t-2)} \cap H$, so that there is some $h_{t-2} \in \gamma_{c-(t-2)} \cap H$ with $r^{2^{t-2}-1} \mu=\log \left(h_{t-2}\right)$, yielding

$$
r^{2^{t-1}+2^{t-2}} \lambda=\log \left(w^{\prime \prime}\right)+\log \left(h_{t-2}\right) .
$$

Continue this process to find $z \in \gamma_{c-t} \cap H$ and $h_{0} \in \gamma_{c} \cap H$.

$$
\begin{aligned}
r^{2^{t-1}+2^{t-2}+\cdots+2^{0}} \lambda & =\log (z)+\log \left(h_{0}\right) \\
& =\log \left(z h_{0}\right),
\end{aligned}
$$

the last line following by the centrality of $h_{0}$. Since $2^{t-1}+2^{t-2}+\cdots+2^{0}=2^{t}-1$, we've shown

$$
r^{2^{2}-1} \lambda \in \log \left(\gamma_{c-t} \cap H\right),
$$

as desired.

We next have the following key lemma. This is a generalised and adapted version of [26, Lemma 3, p. 112].

Lemma 4.2.5. Suppose $\Gamma$ is a $\mathfrak{T}$-group of class at most $c$, let $\Gamma^{\mathbb{Q}}$ be its Mal'cev completion, and suppose that $K \geqq G \leqslant \Gamma^{\mathbb{Q}}$. There exists a constant $l \in \mathbb{N}$, depending on $c$ only, such that for all $g \in G$ and $k \in K$ we have

$$
\begin{align*}
\log (g)+l \log (k) & \in \log K\langle g\rangle  \tag{4.2.6}\\
l \log (g)+\log (k) & \in \log K\left\langle g^{l}\right\rangle  \tag{4.2.7}\\
{[l \log (g), \log (k)] } & \in \log K\left\langle g^{l}\right\rangle \tag{4.2.8}
\end{align*}
$$

Proof. We first of all refine the definition of $r$ in lemma (4.2.3) as follows. Let again $F_{c}=F_{c}(x, y)$ denote the free nilpotent group of class $c$ on $x, y$. For each vector $\mathbf{e}$ of positive integers corresponding to a commutator $[x, y]_{\mathbf{e}}$ of total length at most $c$, ensure
that

$$
a_{\mathbf{e}}[\log (x), \log (y)]_{\mathbf{e}} \in \frac{1}{r} \sum_{\mathbf{f}} \mathbb{Z} \log [x, y]_{\mathbf{f}}
$$

as before, and that furthermore

$$
b_{\mathbf{e}}[\log (x), \log (y)]_{\mathbf{e}} \in \frac{1}{r} \sum_{\mathbf{f}} \mathbb{Z} \log [x, y]_{\mathbf{f}}
$$

We note that the proof of lemma 4.2 .3 still goes through with this $r$.

The strategy of the proof is now as follows. We will show by induction on $i$ that if $K \leqslant \gamma_{c-i}$, then equations (4.2.6), (4.2.7) and (4.2.8) hold with $l=r^{2^{i}}$. Then since $K \leqslant \gamma_{c-(c-1)}=G$, we take $l=r^{2^{c-1}}$ to complete the proof.

If $i=0$ then $K$ is central, and we immediately see

$$
\begin{aligned}
\log (g)+r \log (k) & =\log \left(g k^{r}\right) \in \log K\langle g\rangle, \\
r \log (g)+\log (k) & =\log \left(g^{r} k\right) \in \log K\left\langle g^{r}\right\rangle, \\
{[r \log (g), \log (k)] } & =0 \in \log K\left\langle g^{r}\right\rangle,
\end{aligned}
$$

as desired.

Now we suppose that $i \geqslant 1$ and that equations (4.2.6), (4.2.7) and (4.2.8) hold with $l=r^{2^{i-1}}$ for any $K \leqslant \gamma_{c-(i-1)}$. Furthermore suppose that $K \leqslant \gamma_{c-i}$, and set $H=[K, G]$. Then $H \leqslant\left[\gamma_{c-i}, G\right]=\gamma_{c-(i-1)}$, and moreover $H \preccurlyeq G$.

Now let $g \in G$ and $k \in K$, and note that for any $l \in \mathbb{N}$,

$$
\begin{aligned}
\log (g)+l \log (k)-\log \left(g k^{l}\right) & =-\sum a_{\mathbf{e}}[\log (g), l \log (k)]_{\mathbf{e}} \\
& =-l \sum a_{\mathbf{e}} l^{n_{\mathbf{e}}}[\log (g), \log (k)]_{\mathbf{e}} \\
& \in \frac{l}{r} \sum_{\mathbf{f}} \mathbb{Z} \log [g, k]_{\mathbf{f}}
\end{aligned}
$$

where $n_{\mathbf{e}}+1$ is the number of occurrences of $Y$ in $[X, Y]_{\mathbf{e}}$. Taking now $l=r^{2^{i}}$ and noting that $[g, k]_{\mathbf{f}} \in H$ for all $\mathbf{f}$, we have that

$$
\log (g)+l \log (k)-\log \left(g k^{l}\right) \in r^{2^{i}-1} \mathbb{Z} \log H
$$

But now applying lemma (4.2.3) with $t=i-1$ and $\Gamma=K$, we see that

$$
r^{2^{i-1}-1} \mathbb{Z} \log H \subseteq \log H
$$

Combining now the previous two lines and noting that $2^{i}-1=2^{i-1}-1+2^{i-1}$ we find some $h \in H$ for which

$$
\log (g)+l \log (k)=\log \left(g k^{l}\right)+r^{2^{i-1}} \log (h) .
$$

Applying now by induction line (4.2.6) for $K=H$ and $g=g k^{l}$ yields

$$
\begin{aligned}
\log \left(g k^{l}\right)+r^{2^{i-1}} \log (h) & \in \log H\left\langle g k^{l}\right\rangle \\
& \subseteq \log K\langle g\rangle,
\end{aligned}
$$

as desired. To verify line (4.2.7), we interchange the roles of $g$ and $k$ in the above to deduce that for some $h \in H$,

$$
l \log (g)+\log (k)=\log \left(g^{l} k\right)+r^{2^{i-1}} \log (h)
$$

Applying again by induction line (4.2.6) yields

$$
\begin{aligned}
\log \left(g^{l} k\right)+r^{2^{i-1}} \log (h) & \in \log H\left\langle g^{l} k\right\rangle \\
& \subseteq \log K\left\langle g^{l}\right\rangle
\end{aligned}
$$

as desired.

To verify line (4.2.8), we begin by noting from (1.8.2) that for any $l \in \mathbb{N}$,

$$
\begin{aligned}
{[l \log (g), \log (k)]-\log \left[g^{l}, k\right] } & =-\sum_{\mathbf{e} \neq(1)} b_{\mathbf{e}}[l \log (g), \log (k)]_{\mathbf{e}} \\
& =-l \sum_{\mathbf{e} \neq(1)} b_{\mathbf{e}} \mathbf{l}^{l_{\mathbf{e}}}[\log (g), \log (k)]_{\mathbf{e}} \\
& \in \frac{l}{r} \sum_{\mathbf{f}} \mathbb{Z} \log [g, k]_{\mathbf{f}},
\end{aligned}
$$

where here $n_{\mathbf{e}}$ denotes the number of occurrences of $X$ in $[X, Y]_{\mathbf{e}}$. Now proceeding exactly as above we find some $h \in H$ for which

$$
[l \log (g), \log (k)]=\log \left[g^{l}, k\right]+r^{2^{i-1}} \log (h)
$$

A final inductive application of line (4.2.6) yields

$$
\begin{aligned}
\log \left[g^{l}, k\right]+r^{2^{i-1}} \log (h) & \in \log H\left\langle\left[g^{l}, k\right]\right\rangle \\
& \subseteq \log K\left\langle g^{l}\right\rangle
\end{aligned}
$$

as desired.

The main application of this lemma is as follows, a generalisation of $[26$, Theorem $4, \mathrm{p}$. 113].

Theorem 4.2.6. Suppose $\Gamma$ is a $\mathfrak{T}$-group of class at most $c$, let $\Gamma^{\mathbb{Q}}$ be its Mal'cev completion, and let $l=l(c)$ be as in lemma (4.2.5). If $\Gamma \geqq \Gamma^{1 / l}=\left\langle g^{1 / l}: g \in \Gamma\right\rangle$, then $\log \Gamma$ is a Lie subring of $\log \Gamma^{\mathbb{Q}}$.

Proof. Let $x, y \in \Gamma$. Then $x=g^{l}$ for some $g \in \Gamma^{1 / l}$. By lemma (4.2.5) with $K=\Gamma$ and $G=\Gamma^{1 / l}$ we have

$$
\log (x)+\log (y)=l \log (g)+\log (y) \in \log \Gamma\left\langle g^{l}\right\rangle=\log \Gamma
$$

so that $\log \Gamma$ is a $\mathbb{Z}$-submodule, and furthermore by lemma (4.2.5) again

$$
[\log (x), \log (y)]=[l \log (g), \log (y)] \in \log \Gamma\left\langle g^{l}\right\rangle=\log \Gamma
$$

So that $\log \Gamma$ is closed under the bracket, as desired.

We now have the following corollary of this theorem, the deduction of which is identical to the deduction of $[26$, Theorem 5, p. 114]. This corollary is itself a generalisation of this theorem.

Corollary 4.2.7. Let $G$ be a Mal'cev complete nilpotent torsion-free group of finite rank and of class at most $c$. There exists a constant $\alpha$, which depends on $c$ only, such that for any subgroup $\Gamma \leqslant G$, the images of $\Gamma^{\alpha}$ and $\left(\Gamma^{1 / \alpha}\right)^{\alpha}$ are Lie subrings of $\log G$.

## Chapter 5

## Free constructions in algebra

Let $k$ be a commutative ring with unity. In this section we will discuss the left adjoint to the forgetful functor

$$
U: k \text {-Mod } \longrightarrow k \text {-Lie, }
$$

where by $k$-Mod we mean the category of $k$-modules and by $k$-Lie we mean the category of Lie algebras over $k$. We will begin by outlining a very general and useful theory which shows why this adjunction (and many others) exist, before proceeding to give an explicit construction of this object in case $k$ is a Dedekind domain.

### 5.1 Categorical universal algebra

Here we outline a particular perspective which allows us to immediately deduce the existence of various adjoint functors. Our main reference throughout is Borceux [4, Chapter 3]. Nothing in this section is new, but we feel that this perspective is underrepresented in the group theoretic literature, and so we give a brief exposition here.

We begin by rethinking the notion of group. We claim in particular that the axioms of a group may be interpreted solely in terms of morphisms and commutative diagrams in the category of sets. In particular, if $G$ is a group there is inherent the following data:

- a binary operation $m: G^{2} \rightarrow G$,
- an inverse function $\iota: G \rightarrow G$ and
- an identity, which may be thought of as a function $G^{0} \rightarrow G$.

Now we claim that each axiom which specifies how these functions interact may be expressed as the commutativity of a certain diagram. For example, one sees immediately that the commutativity of the following diagram expresses exactly the associative axiom in $G$.


It is an instructive exercise to write the correct diagrams encoding the other axioms.

Generalising, one says that the theory of groups consists in three operations, of arity varying from 0 to 2 , obeying certain commutative diagrams. One immediate issue is that the above describes only the fact that this $G$ is a group: a set-theoretical remedy to this problem is contained in [4, p. 123]. We will omit the potentially delicate issues here and outline the categorical framework only. Thus we now define what we mean by an algebraic theory in full generality.

Definition 5.1.1 ( [4, 3.3.1]). An algebraic theory is a category $\mathcal{T}$ with

$$
\operatorname{ob} \mathcal{T}=\left\{x^{i}: 0 \leqslant i<\omega\right\}
$$

consisting of distinct objects, where $x^{i}$ is the categorical product of $i$ copies of $x^{1}$.

A model of $\mathcal{T}$ is a functor $\mathcal{T} \longrightarrow$ Set which preserves products, and a homomorphism between two models is a natural transformation of functors. The category of all models of $\mathcal{T}$ is denoted by $\operatorname{Mod}_{\mathcal{T}}$.

Since this section is for the most part expository, we illustrate these ideas through several examples.

Example 5.1.2. We consider the following examples.

1. Let us examine exactly how groups form an algebraic theory in this precise sense. The theory $\mathcal{T}$ of groups will have at the very least objects $\left\{x^{i}: 0 \leqslant i<\omega\right\}$, with $x^{i}$ the $i$-fold product of $x^{1}$. Note that as part of this definition there is an abundance of morphisms enforced by the fact that the category has products.

To this set of morphisms we add the following, as per our discussion above: a morphism $m: x^{2} \rightarrow x^{1}$, a morphism $\iota: x^{1} \rightarrow x^{1}$ and finally a morphism $e: x^{0} \rightarrow x^{1}$ :


Finally we insist that these morphisms obey various relations, those which encode the axioms as commutative diagrams.

For example, we insist that $m \circ(\mathrm{id} \times m)=m \circ(m \times \mathrm{id})$ as per diagram (5.1.1) above. This clearly satisfies the requirements of a category.

We claim that the models of this theory may be thought of as groups in an obvious way. Indeed a model is precisely a functor $F: \mathcal{T} \rightarrow$ Set which preserves products. Thus the image of $x^{i}$ is $G^{i}$ for one particular set $G:=F\left(x^{1}\right)$. But now the functorality of $F$ and our construction of $\mathcal{T}$ means exactly that $G$ is a group as per our discussion above. (Note that $G$ must be nonempty by the presence of the 0 -ary axiom.) Furthermore we clearly see that any particular group fits into this framework.

Finally, it is an easy check to see that the notion of homomorphism and natural transformation of functor coincide under this correspondence. Thus we see that the category of models over this theory is equivalent to the category of groups.
2. The category of modules over a ring $k$ is also an algebraic category. There is a subtlety in encoding the notion of a $k$-module as we did for groups above: a priori, a $k$-module $M$ is equipped with a map

$$
k \times M \longrightarrow M
$$

encoding the scalar multiplication. The way to remedy this, as per [4, 3.3.5.g] is to add, for each $r \in k$, a unary operation $M \rightarrow M$ representing scalar multiplication by this particular $r$. Now the module axioms are easily expressed as commutative diagrams.
3. It is now easy to deduce that both the category of $k$-Lie algebras and associative $k$-algebras are algebraic in this sense.
4. The theory of sets is trivially algebraic - we add no extra morphisms other than those required by the definition.
5. For a nonexample, we may take the category of fields. Intuitively it seems reasonable that this cannot be algebraic: we need to insist that $0 \neq 1$, and furthermore that the inverse map is partially defined. This seems difficult to prescribe in terms of a commutative diagram, and indeed we will prove in example (5.1.6) that this category cannot be algebraic.

We now introduce the relevant notion of morphism of theories.
Definition 5.1.3 ( $[4,3.7 .1])$. Let $\mathcal{T}_{1}$ and $\mathcal{T}_{2}$ denote algebraic theories, with

$$
\mathrm{ob} \mathcal{T}_{j}=\left\{x_{j}^{i}: 0 \leqslant i<\omega\right\} .
$$

A morphism of theories $\mathcal{T}_{1} \longrightarrow \mathcal{T}_{2}$ is a functor which preserves finite products and sends $x_{1}^{1}$ to $x_{2}^{1}$. Such a morphism of theories induces a functor

$$
\operatorname{Mod}_{\mathcal{T}_{2}} \longrightarrow \operatorname{Mod}_{\mathcal{T}_{1}} .
$$

The induced functor is by definition an algebraic functor.

The main theorem we wish to state is the following.
Theorem 5.1.4 ( [4, 3.7.7]). Algebraic functors have left adjoints.
Example 5.1.5. The most elementary example of an algebraic functor is one induced by an inclusion of algebraic theories. For example, the algebraic theory of monoids includes into the algebraic theory of groups, and the induced algebraic functor, up to equivalence, is the forgetful functor from the category of groups to the category of monoids. The left adjoint of this functor is the Grothendieck completion functor.

The theory of sets includes into any algebraic theory, and one obtains an algebraic functor sending any model of a theory to its underlying set: this is the traditional 'forgetful functor'. That this functor will always have a left adjoint is the existence of free objects on sets: for example, free groups, free monoids, free semigroups and free commutative rings (=polynomial rings) all arise in this context.

We can now also show the following.

Example 5.1.6. The category of fields is not algebraic. To see this suppose it is algebraic, so that there exists the notion of a free field on a set. Denote this functor by $F$, and let the characteristic of $F(\emptyset)$ be $p$. Select a field $F_{q}$ of characteristic $q \neq p$ and note that

$$
\emptyset=\operatorname{Hom}_{\text {Field }}\left(F(\emptyset), F_{q}\right)=\operatorname{Hom}_{\text {Set }}\left(\emptyset, F_{q}\right)=\{\emptyset\},
$$

for a contradiction.

Of particular use in the next section is the following corollary.

Corollary 5.1.7. The forgetful functor

$$
k \text {-Lie } \longrightarrow k \text {-Mod }
$$

admits a left adjoint.

Proof. The categories $k$-Mod and $k$-Lie are both model categories of the corresponding algebraic theories of $k$-modules and $k$-Lie algebras, respectively. We deduce that the forgetful functor is algebraic, since it is induced by an inclusion of theories. Thus it has a left adjoint by theorem (5.1.4).

### 5.2 The free Lie algebra on a module

The aim of this section will be to establish a concrete realisation of the free lie algebra on a module - that is, the adjoint whose existence is guaranteed by corollary (5.1.7). This construction is mentioned in the literature for $k=\mathbb{Z}[22, \mathrm{p} .3]$ but not proven. Before we state the result, we introduce several other related functors which will play a role. Note that in the sequel when we refer to a commutative diagram of functors we only insist that it is commutative up to a natural isomorphism.

Proposition 5.2.1. Let $k$ be a commutative ring with unity and consider the following commutative diagram of functors, where $k$-Assoc denotes the category of unital associative $k$-algebras, and the functor ()$_{L}$ sends an algebra $A$ to the Lie algebra with the same underlying module structure and equipped with the commutator bracket defined for
$a, b \in A$ by $[a, b]:=a b-b a$.


Each of these functors has a left adjoint, and we obtain a commutative diagram of left adjoint functors:


Here, $\boldsymbol{T}$ denotes the tensor algebra functor, and $\boldsymbol{U}$ denotes the universal enveloping algebra functor. Given a $k$-Lie algebra $\boldsymbol{L}$ with Lie bracket $[-,-]$, we may take $\boldsymbol{U}(L)$ to be the quotient of $\boldsymbol{T}(L)$ by the ideal $\langle x \otimes y-y \otimes x-[x, y]: x, y \in L\rangle$.

Proof. That $\mathcal{L}$ exists is corollary (5.1.7). The remaining details here are standard, see for example [8].

Our construction of $\mathcal{L}$ will depend on the following result concerning the universal enveloping algebra due to Higgins [16], a corollary of the Poincaré-Birkhoff-Witt Theorem.

Theorem 5.2.2. [16] Let $k$ be a Dedekind domain, and let $M$ be a $k$-Lie algebra. Then the canonical map $M \longrightarrow \boldsymbol{U}(M)$ is injective.

We now state our construction of $\mathcal{L}$. We thank Peter Kropholler here for the diagrammatic representation of this proof.

Theorem 5.2.3. Suppose that $k$ is a Dedekind domain and $M$ a $k$-module. We may then take $\mathcal{L}(M)$ to be the Lie subalgebra of $(\boldsymbol{T}(M))_{L}$ generated by (the canonical image of) $M$.

Proof. We will show in particular that $M$, together with the natural map $M \rightarrow \mathcal{L}(M)$, satisfies the required universal property. Precisely, we will show that given any $k$-Lie algebra $L$ and $k$-module homomorphism $f: M \rightarrow L$, there is a unique $k$-Lie algebra
homomorphism $\bar{f}: \mathcal{L}(M) \rightarrow L$ such that the following diagram commutes.


Since by definition the image of $M$ in $\mathcal{L}(M)$ generates the Lie algebra, the uniqueness of any extension is clear. Thus is suffices to show existence.

The first step is to consider the composition $\pi \circ \boldsymbol{T}(f): \boldsymbol{T}(M) \rightarrow \boldsymbol{U}(L)$, where $\pi$ denotes the canonical projection, recalling our definition of the universal enveloping algebra as in proposition (5.2.1). Applying the functor ()$_{L}$ we obtain a commutative square, where the vertical maps are canonical.


The left hand map certainly factorises through $\mathcal{L}(M)$, and now the composition

$$
\mathcal{L}(M) \rightarrow(\boldsymbol{T}(M))_{L} \rightarrow(\pi \circ \boldsymbol{T}(f))_{L}
$$

has image contained in $\iota L$, since $L$ is a Lie algebra. But the injectivity of $\iota$ as per theorem (5.2.2) yields the desired factorisation of this latter map into $\bar{f}$. That this choice of $\bar{f}$ makes diagram (5.2.1) commute requires a final application of the injectivity of $\iota$.

This categorical perspective allows us to obtain easily the following. Note that we do not require $k$ to be a Dedekind domain here.

Proposition 5.2.4. Let ( $)_{a b}$ denote the abelianisation functor $k$-Lie $\longrightarrow k$-Mod defined by quotienting out by the commutator ideal. Then ( $)_{a b} \circ \mathcal{L}$ is naturally isomorphic to the identity functor. That is, the abelianisation of the free Lie algebra on a $k$-module $M$ is isomorphic to $M$.

Proof. Define a functor $F: k$-Mod $\longrightarrow k$-Lie defined by turning a module into a Lie algebra with the same underlying module and trivial bracket. One checks immediately that this is right adjoint to our abelianisation functor. There is a commutative diagram of functors

and hence a commutative diagram of left adjoints

whence our result.

## Chapter 6

## Semisimple modules over Von Neumann regular rings

In this chapter, we establish a direct link between the derived sets of a certain topological space and the socle series of a semisimple module over a Von Neumann regular ring. In particular, the work fits nicely with earlier work of Tiwary [28] and Usher [29].

### 6.1 Internal and external derived sets

In this section, we introduce the necessary topological lemmas for the following sections. We begin with the following fundamental definition, where we recall that for a subset $Y$ of a topological space $X$, the set $Y^{\prime}$ is by definition the set of limit points of $Y$ in $X$.

Definition 6.1.1. Let $X$ be a topological space and $Y \subseteq X$. The internal derived sets of $Y$, denoted $Y^{(\alpha)}$, are defined for any ordinal $\alpha$ as follows:

$$
Y^{(\alpha)}:= \begin{cases}Y & \text { if } \alpha=0 \\ \left(Y^{(\alpha-1)}\right)^{\prime} \cap Y^{(\alpha-1)} & \text { if } \alpha \text { is a successor ordinal } \\ \bigcap_{\beta<\alpha} Y^{(\beta)} & \text { if } \alpha \text { a limit ordinal }\end{cases}
$$

Similarly the external, or relative derived sets of $Y$, denoted $Y^{\alpha}$, are defined for any ordinal $\alpha \geqslant 1$ as follows:

$$
Y^{\alpha}:= \begin{cases}Y^{\prime} & \text { if } \alpha=1 \\ \left(Y^{\alpha-1}\right)^{\prime} & \text { if } \alpha \text { is a successor ordinal }>1 \\ \bigcap_{\beta<\alpha} Y^{\beta} & \text { if } \alpha \text { a limit ordinal }\end{cases}
$$

We remark briefly that the reason we define the relative version only for non-zero ordinals is that we do not wish to insist that $Y^{\alpha} \subseteq Y$ for limit ordinals $\alpha$.

These notions of derived set are certainly different. For example, consider the set $X:=\{1 / n: n \geqslant 1\} \subset \mathbb{R}$. Then $\emptyset=X^{(1)} \subsetneq X^{1}=\{0\}$. For closed sets, however, it is easy to see that these notions coincide, and more generally it is not difficult to relate these two notions of derived set, as follows.

Lemma 6.1.2. Let $X$ be a topological space and $Y \subseteq X$. Then for any ordinal $\alpha \geqslant 1$, $Y^{\alpha}=\bar{Y}^{(\alpha)}$.

Proof. We first note that the result is certainly true if $Y$ is closed in $X$, since closed sets contain their limit points. Therefore to show the result for a general $Y$, it suffices to show for all ordinals $\alpha \geqslant 1$ that $Y^{\alpha}=\bar{Y}^{\alpha}$. For $\alpha=1$ this is the elementary fact that $Y^{\prime}=\bar{Y}^{\prime}$, and an induction completes the result.

We now introduce the following property of a topological space.
Definition 6.1.3. A topological space $X$ is said to be scattered if every nonempty subset has an isolated point.

We may immediately characterise this property in terms of the interior derivatives of a set, as follows. Note that we could also use the relative derived sets in the next result since there is no subspace involved. Nevertheless we state it in this way for clarity in the sequel.

Lemma 6.1.4. A space $X$ is scattered if and only if there exists an ordinal $\alpha$ for which $X^{(\alpha)}$ is empty.

Proof. Suppose firstly that $X$ is not scattered, so that there exists some nonempty subset $Y$ of $X$ which has no isolated point. It follows that $Y^{\prime}=\bar{Y}$. Taking limit points again
yields that $Y^{\prime \prime}=Y^{\prime}$. In particular we have that $Y \subseteq \bigcap_{\alpha} X^{(\alpha)}$, so that no derivative of $X$ can ever be empty.

If conversely we have that $X^{(\alpha)}$ is nonempty for each $\alpha$, it follows that their intersection $Y:=\bigcap_{\alpha} X^{(\alpha)}$ satisfies $Y^{\prime}=Y$ and is nonempty: in particular it has no isolated points.

For the remainder of this section we concern ourselves with the relative derivatives of a set, discussing various results needed in the sequel. As we progress, we will require further separation properties of the ambient space. It will become clear later why we consider the internal derived sets: these arise naturally in the work of Tiwary. We begin with the following pair of elementary observations.

Lemma 6.1.5. Let $X$ be a topological space and $Y \subseteq X$. Let $x \in X$ and suppose that for some ordinal $\alpha$ we have $x \in Y^{\alpha}$, and suppose that $U$ is an open set containing $x$. Then firstly $U \cap(Y \backslash U)^{\alpha}=\emptyset$, so that in particular $x \notin(Y \backslash U)^{\alpha}$, and secondly $x \in(U \cap Y)^{\alpha}$.

Proof. We prove these each by induction. The first statement is clear when $\alpha=1$. Thus suppose that $U \cap(Y \backslash U)^{\alpha}=\emptyset$ for some ordinal $\alpha$, and suppose that

$$
y \in U \cap(Y \backslash U)^{\alpha+1}=U \cap\left((Y \backslash U)^{\alpha}\right)^{\prime}
$$

But then $U$ is an open set containing $y$, and $y$ is a limit point of $(Y \backslash U)^{\alpha}$, so that $U \cap(Y \backslash U)^{\alpha}$ cannot be nonempty, a contradiction.

Assume now that $\alpha$ is a limit ordinal and that the result is true for all smaller ordinals. Then

$$
U \cap(Y \backslash U)^{\alpha}=U \cap \bigcap_{\beta<\alpha}(Y \backslash U)^{\beta}=\bigcap_{\beta<\alpha} U \cap(Y \backslash U)^{\beta}=\emptyset
$$

as desired.

We move to the second claim. This is also clear when $\alpha=1$, so assume the result for an ordinal $\alpha$ and suppose that $x \in Y^{\alpha+1}$. Then there is some $y \in Y^{\alpha} \cap(U \backslash\{x\})$, so that by induction $y \in(U \cap Y)^{\alpha}$. In particular $y \in(U \cap Y)^{\alpha} \cap(U \backslash\{x\})$, so that $x \in(U \cap Y)^{\alpha+1}$, as desired.

Thus assume that $\alpha$ is a limit ordinal and that the result is true for all smaller ordinals. Then if we suppose that $x \in Y^{\alpha}$, we have that $x \in(U \cap Y)^{\beta}$ for any $\beta<\alpha$ by induction. In particular $x \in \bigcap_{\beta<\alpha}(U \cap Y)^{\beta}=(U \cap Y)^{\alpha}$, as desired.

It is clear that the internal derived sets of a subset always form a descending sequence of sets. If we impose a mild separation axiom on our space we may also deduce this for the relative derived sets.

Lemma 6.1.6. Let $X$ be a $T_{1}$ space, or equivalently a space with closed singletons [12, p.37], and suppose that $Y \subseteq X$. Then the relative derived sets of $Y$ in $X$ form $a$ descending series, so that if $1 \leqslant \beta \leqslant \alpha$ are ordinals, there is an inclusion $Y^{\beta} \supseteq Y^{\alpha}$.

Proof. The key point is that in a space with closed singletons, the set $Y^{\prime}$ is closed in $X$, and hence contains its limit points. Thus an easy induction gives the result.

In particular we have the following useful corollary of the above.

Corollary 6.1.7. Let $X$ be a $T_{1}$ space, and $A_{1}, \ldots, A_{n}$ a finite collection of subsets of $X$. Then for all ordinals $\alpha \geqslant 1$, we have that $A_{1}^{\alpha} \cup \cdots \cup A_{n}^{\alpha}=\left(A_{1} \cup \cdots \cup A_{n}\right)^{\alpha}$.

Proof. By induction on the number of sets it suffices to consider the case $n=2$. The case $\alpha=1$ is the elementary fact that $A_{1}^{\prime} \cup A_{2}^{\prime}=\left(A_{1} \cup A_{2}\right)^{\prime}$ and indeed this settles the case of a successor ordinal. Thus assume that $\alpha$ is a limit ordinal and that the result is true for all ordinals $\beta<\alpha$. We need to show that

$$
\bigcap_{\beta<\alpha} A_{1}^{\beta} \cup A_{2}^{\beta}=\left(\bigcap_{\beta<\alpha} A_{1}^{\beta}\right) \cup\left(\bigcap_{\beta<\alpha} A_{2}^{\beta}\right)
$$

The inclusion of the right hand side into the left is trivial, so pick an element $x$ in the left hand side and suppose it does not belong to $\bigcap_{\beta<\alpha} A_{2}^{\beta}$, so that there is some ordinal $\beta_{0}$ for which $x \notin A_{2}^{\beta_{0}}$ By lemma (6.1.6) we see that there is some ordinal $\beta_{0}$ such that for all ordinals $\beta_{0} \leqslant \beta<\alpha$, we have that $x \notin A_{2}^{\beta}$. We deduce by our hypothesis that for each of these ordinals $x \in A_{1}^{\beta}$. A further application of lemma (6.1.6) yields that $x \in \bigcap_{\beta<\alpha} A_{1}^{\beta}$, as desired.

We will also require the following separation lemma, where we require that the space is Hausdorff.

Lemma 6.1.8. Let $X$ be a Hausdorff space, and $Y \subseteq X$. Suppose that $\alpha$ is an ordinal and $Y^{\alpha}=\left\{x_{1}, \ldots, x_{n}\right\}$ for some finite $n$. Then there exists a partition of $Y$ into subsets $Y_{i}$ such that $Y_{i}^{\alpha}=\left\{x_{i}\right\}$ for each $1 \leqslant i \leqslant n$.

Proof. Since $X$ is Hausdorff we may find pairwise disjoint open sets $U_{i} \ni x_{i}$ for each $1 \leqslant i \leqslant n$. Now set

$$
Y_{i}:= \begin{cases}Y \cap U_{2}^{c} \cap \cdots \cap U_{n}^{c} & i=1 \\ Y \cap U_{i} & \text { otherwise. }\end{cases}
$$

These sets clearly form a partition of $Y$, and we immediately deduce that $Y_{i}^{\alpha} \subseteq\left\{x_{1}, \ldots, x_{n}\right\}$ for each $i$. We show firstly that $x_{j} \notin Y_{i}^{\alpha}$ for $j \neq i$. An application of lemma (6.1.5) gives that $U_{j} \cap\left(Y \backslash U_{j}\right)^{\alpha}=\emptyset$, but by construction we have that $Y_{i} \subseteq Y \backslash U_{j}$. Thus $U_{j} \cap Y_{i}^{\alpha}=\emptyset$, so that $x_{j} \notin Y_{i}^{\alpha}$. It thus suffices to show that $x_{i} \in Y_{i}^{\alpha}$, but another application of lemma (6.1.5) gives $x_{i} \in\left(U_{i} \cap Y\right)^{\alpha} \subseteq Y_{i}^{\alpha}$, as desired.

We conclude this section with an elementary fact concerning compact $T_{1}$ spaces.
Lemma 6.1.9. Let $X$ be a compact $T_{1}$ space and $Y \subseteq X$. Then $Y^{\prime}$ is empty if and only if $Y$ is finite.

Proof. Suppose firstly that $Y$ is finite and let $x \in X$. Then $x$ cannot be a limit point of $Y$. Indeed, let $y_{1}, \ldots, y_{n}$ be the elements of $Y$ which are distinct to $x$. Then $U:=\left\{y_{1}\right\}^{c} \cap \cdots \cap\left\{y_{n}\right\}^{c}$ is an open neighbourhood of $x$ since $X$ is $T_{1}$, and for this set $(U \backslash\{x\}) \cap Y=\emptyset$, as desired.

Conversely suppose that $Y$ is infinite and that $Y$ has no limit points in $X$. Then for each $x \in X$ there is an open $U_{x} \ni x$ for which $U_{x} \cap Y \subseteq\{x\}$. But then the open cover $\left(U_{x}: x \in X\right)$ of $X$ cannot have any finite subcover, since we would need infinitely many to even cover $Y$.

### 6.2 Von Neumann regular rings

In this section we briefly introduce the class of rings under consideration. There are of course other notions of regularity for rings, which we do not concern ourselves with here.

Definition 6.2.1. $A$ ring $R$ is said to be (Von Neumann) regular if for all $a \in R$ there exists $x \in R$ for which $a=$ axa.

As a first example, recall that a ring $R$ is boolean if for all $x \in R, x^{2}=x$. Thus certainly a boolean ring is regular. Rather more nontrivially, we have the following motivating example:

Proposition 6.2.2 ([6], [30]). Let $k$ be a field and $G$ a group. Then $k G$ is regular if and only if $G$ is locally finite and the characteristic of $k$ does not divide the order of any finite subgroup of $G$.

Interestingly enough, the condition of a ring being regular is seen rather cleanly by the module category:

Proposition 6.2.3 ( $[20,4.21])$. Let $R$ be a ring. Then the following are equivalent:

1. $R$ is regular;
2. every right $R$-module is flat;
3. every cyclic right $R$-module is flat.

Furthermore in the commutative setting we have the following:
Proposition 6.2.4 ( $[20,3.73])$. A commutative ring $R$ is regular if and only if every simple $R$-module is injective.

A well-known property of regular rings is that their spectrum is Hausdorff and totally disconnected. This we show now.

Lemma 6.2.5. Let $R$ be a commutative regular ring. Then

$$
\operatorname{Spec}(R):=\{P \leqslant R: P \text { a prime ideal of } R\}
$$

is Hausdorff and totally disconnected.

Proof. Recall that a basis for the Zariski topology on $\operatorname{Spec}(R)$ consists of the sets

$$
D_{f}:=\{P: f \notin P\}
$$

where $f$ ranges over elements of $R$. Thus suppose now that $P$ and $Q$ are distinct prime ideals of $R$. Select $a \in P \backslash Q$, and choose $x$ for which $a=a x a$. This gives an equality of principal ideals $(a)=(a x)$, so that these elements are associate and hence we also have that $a x \in P \backslash Q$. Consider now the open sets $D_{a x}$ and $D_{1-a x}$ of $\operatorname{Spec}(R)$. Certainly $P \in D_{1-a x}$ and $Q \in D_{a x}$, and we claim moreover that there is a partition

$$
\operatorname{Spec}(R)=D_{a x} \sqcup D_{1-a x},
$$

from which the result will follow. That the union is the whole space is clear: no proper ideal can contain both $a x$ and $1-a x$. Finally noting that $a x(1-a x)=0$, we see that any prime ideal must contain one of these elements, so that the intersection is empty.

Remark 6.2.6. If $R$ is commutative regular, then every prime ideal is maximal. To see this, select a prime ideal $P$. Then the quotient ring is a regular domain, which is clearly a field. Thus $P$ is maximal as desired.

In particular $\operatorname{Spec}(R)$ is precisely the space of maximal ideals in $R$.

We proceed to specialise to our setting, beginning with a definition. The terminology here is due to Tiwary [28].

Definition 6.2.7. Let $R$ be a commutative regular ring, $P$ a maximal ideal of $R$, and $M$ a semisimple $R$-module. Then the $R / P$-monotypic component of $M$, denoted $M_{P}$, is by definition the direct sum of all submodules of $M$ isomorphic to $R / P$. If $m \in M$, write $m_{P}$ for the component of $m$ in $M_{P}$.

Furthermore we define the following subsets of the (maximal) spectrum of $R$ :

$$
\begin{aligned}
\operatorname{Supp}(M) & :=\left\{P \in \operatorname{Spec}(R): M_{P} \neq 0\right\}, \\
\operatorname{supp}(m) & :=\left\{P \in \operatorname{Spec}(R): m_{P} \neq 0\right\} .
\end{aligned}
$$

The monotypic components of a semisimple module are injective, as we see next. Tiwary does prove this, but our proof is different.

Lemma 6.2.8. Let $R$ be commutative regular. Then for any maximal ideal $P$, any direct sum of copies of $R / P$ is injective over $R$.

Proof. Let $M=\bigoplus_{i \in I} R / P$ be any such direct sum. Since $R$ is regular, each $R / P$ is injective by proposition (6.2.4), and as such the module $\prod_{i \in I} R / P$ is injective over $R$. But $M$ is an $R / P$-module summand of this larger module, and in particular an $R$-summand, so that it is injective over $R$ as desired.

Now let $M$ be semisimple over $R$, commutative and regular. Set $\mathfrak{X}:=\operatorname{Supp}(M)$. Then there is a canonical decomposition and inclusion

$$
\begin{equation*}
M=\bigoplus_{P \in \mathfrak{X}} M_{P} \hookrightarrow \prod_{P \in \mathfrak{X}} M_{P}, \tag{6.2.1}
\end{equation*}
$$

the right hand side of which is injective over $R$ by the previous lemma. Thus in particular the injective hull of $M$ lies in between these two modules. This was identified explicitly by Tiwary, whose work we will describe in section (6.4).

### 6.3 Limit points and the socle series

In this section we establish a precise link between the external derived sets and the socle series of a particular module, combining the results of the previous two sections.

We recall firstly the following.
Definition 6.3.1. Let $M$ be a module over a ring. Then the socle of $M$, denoted $\operatorname{soc}(M)$, is defined to be the join of the simple submodules of $M$. More generally the socle series $\left(S_{\alpha}(M)\right)_{\alpha \geqslant 0}$ of $M$ (with each term denoted $S_{\alpha}$ if the context is clear), is an increasing sequence of submodules indexed by ordinals, where $S_{0}(M)=0$ and we define $S_{\alpha+1}(M) / S_{\alpha}(M):=\operatorname{soc}\left(M / S_{\alpha}(M)\right)$, and for a limit ordinal $\alpha$ we define

$$
S_{\alpha}(M):=\bigcup_{\beta<\alpha} S_{\beta}(M)
$$

We may now state
Theorem 6.3.2. Let $R$ be a commutative Von Neumann regular ring and $M$ a semisimple $R$-module. Set $\mathfrak{X}:=\operatorname{Supp}(M)$. Then for any ordinal $\alpha \geqslant 1$ we have an equality

$$
\begin{equation*}
S_{\alpha}\left(\prod_{P \in \mathfrak{X}} M_{P}\right)=\left\{m \in \prod_{P \in \mathfrak{X}} M_{P}:(\operatorname{supp} m)^{\alpha}=\emptyset\right\} \tag{6.3.1}
\end{equation*}
$$

Proof. The proof will be by transfinite induction on $\alpha$.
Consider first the case $\alpha=1$. In view of lemma (6.1.9), we see that equation (6.3.1) holds for $\alpha=1$ precisely when

$$
\operatorname{soc}\left(\prod_{P \in \mathfrak{X}} M_{P}\right)=\bigoplus_{P \in \mathfrak{X}} M_{P}
$$

The sum is semisimple and hence certainly contained in the socle. Thus it will suffice to show that if $\langle m\rangle$ is a simple submodule of $\prod_{P} M_{P}$, then $\operatorname{supp}(m)$ is a singleton.

Suppose not, so that there are distinct maximal ideals $P, Q \in \operatorname{supp}(m)$. Choose $r \in P \backslash Q$ and consider the element $n:=r m$. It follows that $n_{P}=r m_{P}=0$ and $n_{Q}=r m_{Q} \neq 0$,
since for a maximal ideal $I$ the annihilator of any nonzero element of $M_{I}$ is precisely $I$. But then $n$ is a nonzero element of the simple module $\langle m\rangle$, so that in fact $\langle m\rangle=\langle n\rangle$. We deduce that there is some $s \in R$ for which $m=s n$, and in particular $\operatorname{supp}(m)=\operatorname{supp}(s n) \not \supset P$, a contradiction.

Now suppose that we know the result for some ordinal $\alpha$, and let $m \in \prod_{P \in \mathfrak{X}} M_{P}$. We claim that in order to show the result for $\alpha+1$, it suffices to show the following:

$$
\frac{\langle m\rangle+S_{\alpha}}{S_{\alpha}} \text { is simple } \Longleftrightarrow(\operatorname{supp} m)^{\alpha} \text { a singleton. }
$$

We show first that this does indeed suffice. Thus let $m \in S_{\alpha+1}$. Then $m=m_{1}+\cdots+m_{n}$ for some $m_{i}$, each generating a simple module modulo $S_{\alpha}$. Then by hypothesis this gives that each $\left(\operatorname{supp} m_{i}\right)^{\alpha}$ is a singleton. Furthermore one has an inclusion

$$
\operatorname{supp} m \subseteq \bigcup_{i=1}^{n} \operatorname{supp} m_{i}
$$

whence corollary (6.1.7) yields that

$$
(\operatorname{supp} m)^{\alpha} \subseteq \bigcup_{i=1}^{n}\left(\operatorname{supp} m_{i}\right)^{\alpha}
$$

is finite. In particular by lemma (6.1.9) we see that $(\operatorname{supp} m)^{\alpha+1}$ is empty as required.

Conversely, suppose that $(\operatorname{supp} m)^{\alpha+1}$ is empty, so that $(\operatorname{supp} m)^{\alpha}$ is finite, again by lemma (6.1.9). Now by an application of lemma (6.1.8) we may find finitely many elements $m_{1}, \ldots, m_{n}$ such that $m=m_{1}+\cdots+m_{n}$ and each $\left(\operatorname{supp} m_{i}\right)^{\alpha}$ is a singleton. Our hypothesis implies that each $m_{i}$ generates a simple module modulo $S_{\alpha}$, so that $m$ lies in the socle of this quotient, as desired.

We now prove that the claim does indeed hold. Recall that we assume that equation (6.3.1) holds at $\alpha$. Firstly then, suppose that $m$ generates a simple module in the quotient by $S_{\alpha}$. We want to show that $(\operatorname{supp} m)^{\alpha}$ is a singleton. Certainly it is not empty: $m$ is not an element of $S_{\alpha}$ and we may apply our induction hypothesis. Thus we assume for a contradiction that it contains two distinct elements $P, Q$. Recalling the basic open sets $D_{f}=\{I: f \notin I\}$ of the spectrum, select disjoint open neighbourhoods $D_{r} \ni Q$ and $D_{s} \ni P$, so that in particular $r \in P \backslash Q$. As above, consider the element $n:=r m$. There are now two cases to consider: either $n \in S_{\alpha}$ or not.

Suppose firstly that $n \in S_{\alpha}$, so that $(\operatorname{supp} n)^{\alpha}$ is empty by induction. We claim that there is an inclusion

$$
D_{r} \cap \operatorname{supp} m \subseteq \operatorname{supp} n
$$

Indeed if $I$ is in the left hand side then $r \notin I$ and $m_{I} \neq 0$, so that $n_{I}=r m_{I} \neq 0$, again using the elementary remark concerning annihilators. This yields the contradictory

$$
Q \in\left(D_{r} \cap \operatorname{supp} m\right)^{\alpha} \subseteq(\operatorname{supp} n)^{\alpha}=\emptyset
$$

where the first containment is by lemma (6.1.5).

Thus now suppose that $n \notin S_{\alpha}$. Then by our simplicity assumption, there exists some $t \in R$ for which $m-t n \in S_{\alpha}$. Applying now the induction hypothesis to this element we deduce that $(\operatorname{supp}(m-t n))^{\alpha}$ is empty. We now claim that there is an inclusion

$$
D_{s} \cap \operatorname{supp} m \subseteq \operatorname{supp}(m-t n)
$$

Again let $I$ be in the left hand side. Then since $D_{r}$ and $D_{s}$ are disjoint, it follows that $I \notin D_{r}$, so that $r \in I$. In particular $1-t r$ cannot be in $I$ since it is a proper ideal. Thus $m_{I} \neq 0$ gives $(m-t n)_{I}=(1-t r) m_{I} \neq 0$, as needed. Applying now again lemma (6.1.5) we obtain the contradictory

$$
P \in\left(D_{s} \cap \operatorname{supp} m\right)^{\alpha} \subseteq(\operatorname{supp}(m-t n))^{\alpha}=\emptyset
$$

as desired.

Conversely, we need to show that if $(\operatorname{supp} m)^{\alpha}=\{P\}$, then the module it generates modulo $S_{\alpha}$ is simple. It suffices to show that it is annihilated by $P$, so suppose that for some $a \in P$ we have that $a m \notin S_{\alpha}$. Select some $x$ for which $a=a x a$. We claim that there is an inclusion

$$
\operatorname{supp}(a m) \subseteq \operatorname{supp}(m) \backslash D_{1-a x}
$$

Indeed suppose that $I$ is in the left hand side, so that in particular $a \notin I$. Then $I \notin D_{1-a x}$ if $1-a x \in I$. But $0=a-a x a=a(1-a x) \in I$ and $I$ prime imply that $1-a x \in I$ as desired. Now taking derived sets and applying again lemma (6.1.5) we deduce that $(\operatorname{supp} a m)^{\alpha}$ is empty, which by induction gives that $a m \in S_{\alpha}$, a final contradiction.

It remains to show the case of a limit ordinal $\alpha$. Thus suppose the result for all smaller nonzero ordinals. Recall that by definition

$$
S_{\alpha}=\bigcup_{\beta<\alpha} S_{\beta} .
$$

Thus if $m \in S_{\alpha}$ then $m \in S_{\beta}$ for some $\beta<\alpha$. By induction we have that ( $\left.\operatorname{supp} m\right)^{\beta}$ is empty, so that

$$
\begin{equation*}
(\operatorname{supp} m)^{\alpha}=\bigcap_{\beta<\alpha}(\operatorname{supp} m)^{\beta} \tag{6.3.2}
\end{equation*}
$$

is also empty.
Conversely suppose that $(\operatorname{supp} m)^{\alpha}$ is empty. By line (6.3.2) we see that (supp $\left.m\right)^{\alpha}$ is an intersection of closed (and hence compact) sets, so that they cannot all be nonempty by Cantor's intersection theorem. Thus there is some $\beta<\alpha$ with (supp $m)^{\beta}$ empty, so that by induction $m \in S_{\beta} \subseteq S_{\alpha}$. The proof is now complete.

### 6.4 Injective hulls and the work of Tiwary

The following result identifies the injective hull of a semisimple module over a regular ring as a submodule of the product of monotypic components, as in equation (6.2.1).

Theorem 6.4.1 ( [28, Theorem 2]). Let $R$ be a commutative Von Neumann regular ring and $M$ a semisimple $R$-module, and set $\mathfrak{X}:=\operatorname{Supp}(M)$. Then the injective hull of $M$ in $\prod_{P \in \mathcal{X}} M_{P}$ is precisely

$$
H(M):=\left\{m \in \prod_{P \in \mathfrak{X}} M_{P}: \operatorname{supp}(m) \text { is scattered }\right\} .
$$

We proceed to link this with the socle series by making the following definition.
Definition 6.4.2. Let $M$ and $\mathfrak{X}$ be as in theorem (6.4.1). For any ordinal $\alpha$, define

$$
\begin{equation*}
H_{\alpha}\left(\prod_{P \in \mathfrak{X}} M_{P}\right):=\left\{m \in \prod_{P \in \mathfrak{X}} M_{P}:(\operatorname{supp} m)^{(\alpha)}=\emptyset\right\}, \tag{6.4.1}
\end{equation*}
$$

noting in particular the use of internal derived sets.

Applying this we obtain the following.

Corollary 6.4.3. Let $M$ and $\mathfrak{X}$ be as in theorem (6.4.1). Consider the socle series $\left(S_{\alpha}\right)_{\alpha \geqslant 1}$ and the series $\left(H_{\alpha}\right)_{\alpha \geqslant 1}$ introduced above of $\prod_{P \in \mathfrak{X}} M_{P}$. Then for each ordinal $\alpha \geqslant 1$ there is an inclusion $S_{\alpha} \leqslant H_{\alpha}$, and furthermore there is an equation

$$
H(M)=\bigcup_{\alpha} H_{\alpha}
$$

Proof. For an ordinal $\alpha \geqslant 1$, the inclusion $S_{\alpha} \leqslant H_{\alpha}$ is precisely the fact that for any space $X$ and subset $Y \subseteq X$ one has an inclusion $Y^{(\alpha)} \subseteq Y^{\alpha}$.

The fact that the union of the $H_{\alpha}$ 's is the injective hull is theorem (6.4.1) and lemma (6.1.4).

### 6.5 Usher's conjectures

We move now to the work of Usher in his thesis [29]. The following definition is given for modules over $k G$, for some field $k$ and group $G$. We note that this holds in more generality and give this definition here.

Definition 6.5.1 ( [29, 4.2.1]). Let $k$ be a field and $A$ a (unital) $k$-algebra. For an A-module $M$, we proceed to define an ordinal sequence $\mathcal{L}_{\alpha}(M)$ as follows. Firstly we set $\mathcal{L}_{0}(M):=0$ and for a successor ordinal $\alpha$, we define $\mathcal{L}_{\alpha}(M) / \mathcal{L}_{\alpha-1}(M)$ to be the largest locally finite-dimensional submodule of $M / \mathcal{L}_{\alpha-1}(M)$. For a limit ordinal $\alpha$ we put $\mathcal{L}_{\alpha}(M):=\bigcup_{\beta<\alpha} \mathcal{L}_{\beta}(M)$.

This agrees with the socle series in the commutative setting, as we show next. We require the following lemma.

Lemma 6.5.2. Let $k$ be a field and $A$ a commutative unital $k$-algebra. Then simple modules over $A$ are finite-dimensional.

Proof. A simple module over $A$ carries a field structure and is certainly finitely generated as a $k$-algebra. Now recall Zariski's lemma: the fact that finitely generated $k$-algebras which are fields form finite extensions over $k$. Thus we may conclude.

Lemma 6.5.3. Let $k$ be a field, $A$ a commutative unital $k$-algebra and $M$ a semisimple A-module. Then for all ordinals $\alpha$, we have that $\mathcal{L}_{\alpha}(M)=S_{\alpha}(M)$.

Proof. Plainly, it suffices to check the case of a successor ordinal $\alpha$. Furthermore since $\mathcal{L}_{\alpha}(M) / \mathcal{L}_{\alpha-1}(M)=\mathcal{L}_{1}\left(M / \mathcal{L}_{\alpha-1}(M)\right)$, it even suffices to consider the case $\alpha=1$. The inclusion $\mathcal{L}_{1}(M) \leqslant \operatorname{soc}(M)=S_{1}(M)$ is clear, and the reverse inclusion is supplied by lemma (6.5.2).

Now let $k$ be an algebraically closed field, $A$ a locally finite abelian group, and $M$ a semisimple module over $k A$ such that $M_{P}$ is one dimensional for each $P \in \operatorname{Spec}(M)$. In this special setting Usher conjectures in [29, 7.7.6] the exact statement of our theorem (6.3.2), albeit using the ordinal sequence $\mathcal{L}_{\alpha}$ instead of the socle series (equal here by our lemma above), which we show more generally for the class of commutative Von Neumann regular rings and all semisimple modules. Progress was made towards the successor ordinal case, see [29, 7.7.1, 7.7.4].

Furthermore, Usher conjectures a connection between the union of the sequence $\mathcal{L}_{\alpha}$ and the injective hull. This is supplied by Tiwary's result [28, Theorem 2] which we give above as theorem (6.4.1).

## Bibliography

[1] I. Belegradek. On co-Hopfian nilpotent groups. Bull. London Math. Soc. 35(6), 805-811, 2003.
[2] R. Bieri and R. Strebel. A geometric invariant for nilpotent-by-abelian-by-finite groups. J. Pure Appl. Algebra, 25(1):1-20, 1982.
[3] G. Baumslag. A finitely presented metabelian group with a free abelian derived group of infinite rank. Proc. Amer. Math. Soc., 35, 61-2, 1972.
[4] F. Borceux. Handbook of Categorical Algebra, volume II. Cambridge University Press, 1994.
[5] N. Bourbaki. Commutative Algebra, Addison-Wesley, 1972.
[6] I. G. Connell, On the group ring, Can. J. Math. 15, 650-685, 1963.
[7] Y. de Cornulier. Gradings on Lie algebras, systolic growth, and cohopfian properties of nilpotent groups. Bull. Soc. Math. France 144(4): 693-744, 2016.
[8] J. Dixmier. Enveloping algebras, Graduate Studies in Mathematics, 11, Providence, R.I.: American Mathematical Society, 1996.
[9] Dekimpe, K. and Deré J. Existence of Anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent Lie groups. Topol. Methods Nonlinear Anal. 46(1), 165189, 2015.
[10] Deré J. Gradings on Lie algebras with applications to infra-nilmanifolds. Groups Geom. Dyn. 11(1), 105-120, 2017.
[11] M. Dugas. Torsion-free abelian groups defined by an integral matrix. Int. J. Algebra, 6(1-4): 85-99, 2012.
[12] Engelking. General Topology. Heldermann, Berlin, 1989.
[13] Farkas, D.R.: Endomorphisms of polycyclic groups. Math. Zeit. 181, 567-574, 1982.
[14] F. Grunewald and D. Segal. Reflections on the classification of torsion-free nilpotent groups, Group theory, essays for Philip Hall, edited by K.W. Gruenberg and J.E. Roseblade, Academic Press, 121-158, 1984.
[15] P. Hall. The Edmonton Notes on Nilpotent Groups. Queen Mary College Mathematics Notes. Mathematics Department, Queen Mary College, London, 1969.
[16] P. J. Higgins. Baer invariants and the Birkhoff-Witt theorem, J. Algebra (11), 469482, 1956.
[17] E. Khukhro. p-Automorphisms of Finite p-Groups. London Math. Soc. Lecture Note Series, 246, Cambridge Univ. Press, 1998.
[18] P. H. Kropholler. Cohomological dimension of soluble groups. J. pure Appl. Algebra, 43(3), 281-7, 1986.
[19] P. H. Kropholler, C. Martinez-Pérez, and B. E. A. Nucinkis. Cohomological finiteness conditions for elementary amenable groups, J. Reine Angew. Math. 637, 49-62, 2009.
[20] T. Y. Lam. Lectures on Modules and Rings. Springer, Berlin, 1999.
[21] J. C. Lennox and D. J. S. Robinson. The Theory of Infinite Soluble Groups, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2004.
[22] R. Mikhailov. Homotopy theory of Lie functors. Draft. Available at http://www. mi.ras.ru/~romanvm/lie_derived13.pdf, 2012.
[23] M. Osborne. Basic Homological Algebra. New York: Springer, 2000.
[24] D. Passman. The algebraic structure of group rings. Dover ed., 2011.
[25] V. N. Remeslennikov. On finitely presented groups. Proceedings of the Fourth AllUnion Symposium on the Theory of Groups (Russian), 164-9, 1973.
[26] D. Segal, Polycyclic Groups. Cambridge, 1983.
[27] G. C. Smith, Compressibility in nilpotent groups, Bull. London Math. Soc. 17(5), 453-457, 1985.
[28] A. K. Tiwary, Injective hulls of semi-simple modules over regular rings, Pacific J. Math. 31, 247-254, 1969.
[29] A. Usher, Cluster Points and Cohomology for Abelian Groups, doctoral thesis, Queen Mary University of London, 2003.
[30] O. E. Villamayor, On weak dimension of algebras, Pac. J. Math. 9, 941-951, 1959.
[31] R. B. Warfield, Jr., Nilpotent Groups. Springer Lecture Notes in Mathematics No. 5 13, Springer-Verlag, Berlin, 1976.
[32] Wehrfritz, B.A.F.: Endomorphisms of polycyclic groups. Math. Zeit. 184, 97-99 1983.
[33] Wehrfritz, B.A.F.: Endomorphisms of polycyclic-by-finite groups. Math. Zeit. 264, 629-632 2010.
[34] C. Weibel. An introduction to homological algebra. Cambridge Studies in Advanced Mathematics. (No. 38). Cambridge: Cambridge University Press, 1997.
[35] H. Wielandt. Eine Kennzeichnung der direkten Produkte von p-Gruppen. Math. Zeit., 41, 281-2, 1937.
[36] D. I. Zaicev. Solvable groups of finite rank. (Russian). In Groups with Restricted Subgroups pp. 115-30. "Naukova Dumka", Kiev, 1971.

