HOMOGENIZATION OF SANDWICH PANELS

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Abstract

Process of numerical modeling of plates with periodic corrugation requires some efforts to be put in careful and precise discretization of its complicated structure. This automatically generates very computationally expensive models to be computed. One of the most popular method of model simplification is analytical or numerical homogenization. The main goal of this paper is to present homogenization techniques that can be used to effectively model sandwich panels such as corrugated plates in elastic phase. Here, two methods of different complexity are described, namely: homogenization through application of classical laminated plate theory and Hill-Mandel conditions. Accuracy of those methods are compared with literature data and results of structural sample in two basic tests, i.e. four-point bending and uni-axial tensile tests. Results show that each method provides similar effective parameters which proves effectiveness of the methods presented.

1 Introduction

Sandwich panels or laminated plates are widely used as bearing structures in many industries including: packaging, structural engineering, shipbuilding, vehicle industry, aeronautics and many more.
Sandwich panels consists of lightweight core material inserted in between two or more stiff plates. The core depends on application of plate and can be built from some solid material such as Styrofoam, PUR foams, rock wool. The others constructions of core can be based on curved thin material shaped in triangles, hats or sine-waves, or even on 3D trusses usually shaped in pyramid-like units. Another important class of sandwich panels are panels where its core is composed of honeycomb structure. Such structures are characterized by great transverse load strength which is limited by stability of core's walls. The greatest advantage of sandwich plates is its ratio of bending stiffness to weight of element which leads to economical efficiency. Among important industrial applications of sandwich panels one can name: corrugated cardboard used to build paper boxes (one of most important container types in transport and storage), steel and aluminum sandwiches with corrugated core used in building ship hulls or airplanes bodies. Polymer sandwiches with honeycomb core are used in such fields as Formula 1 cars due to their load resistance and primarily light self-weight. Sandwich panels with thermal-insulating core are one of most important prefabricated materials in structural engineering which are used as walls and roofs and are typically applied on factory floors.

Complicated geometry of core is usually a factor that prevents modeling large structures consisting of sandwich panels as it enforces very fine mesh to be used, effectively making computations very expensive. In order to overcome this problem many homogenization techniques were developed. Homogenization is process in which some heterogeneous body is replaced with homogeneous material having stiffness equal to average stiffness of the heterogeneous one. This approach can lead to great reduction of problem size as standard shell elements are used instead, thus significantly less elements are used to mesh a domain.

Over the years many homogenization techniques were developed that are applicable to various problems including e.g. sandwich panels, continuum bodies with inclusions or solids which micro-structure is known and ordered. Within field of sandwich panels multiple methods of various complexity were presented. Each

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methods has its application range which depends both on geometry of the core and the constitutive model of layers that can be capture. Among easiest methods one can name Classical Laminated Plate Theory (CLPT) that can be used to predict effective stiffness of either panels consisting of flat layers such as carbon polymers of different orientations emerged in epoxy resins [18] or panels with corrugated core that is periodic in one plane direction - such as cardboards and steel corrugated plates [5, 2, 19]. These methods include standard Kirchhoff-Love's plate formulation where the stiffness of layered panel is obtained from integrating constitutive equations over the whole section of the plate. This approach limits application to elastic regime but due to its simplicity can be used without sophisticated FEM software. Another approach for homogenization was taken in [16, 4] where effective elastic parameters for corrugated boards are obtained using Hill-Mandel condition which is based on deformation energy-equivalence of structural and homogeneous samples. In this case FEM model of structure must be built in order to obtain global stiffness matrix of structure. In return some geometrical changes such as perforations or inclusions can be included in calculations to some extent. A comprehensive description of the techniques mentioned above with a particular application to corrugated paperboards can be found in [14, 8, 9, 10, 11].

In order to model panels with more complicated constitutive models (such as plasticity phase, delamination, perforations, etc.) or more complicated geometry (such as honeycomb or space trusses) other methods must be used. One of the most popular techniques is called Asymptotic Homogenization (AH) which is based on Asymptotic Expansion Theory [20, 13, 15, 6]. Although this method can be used for plates of an arbitrary geometry and constitutive formulation, it requires much effort to implement, such as modification of FEM code thus making it a non standard tool for engineering use. Another popular method that can include non-linear mechanics of sandwich panels is called multi-scale modeling in particular Arlequin Method [12, 17], which enables the modelling of macro and micro scales separately.

In this work different homogenization techniques (based on CLPT and Hill-Mandel condition) are presented that can be used to effectively model sandwich panels with corrugated core in elastic regime. Authors present algorithms associated with each method and then tests them against literature values of effective parameters of steel sandwich panel with corrugated core and compare response of homogeneous model to structural model for a given cardboard in a four-point bending tests and uni-axial tensile tests.

2 Mathematical formulation of plates

2.1 Shell Stiffness

Plate is structural element carrying transverse loads with one of the dimension being very low in comparison to the others. If this element is allowed to be loaded with in-plane force then it is referred as to shell element. In most of the applications sandwich structures can be modeled with shell elements.

Many plate theories were developed, though two most common ones are used to model sandwich panels, namely: Kirchhoff-Love's Plate Theory and Reissner-Mindlin's Plate Theory. The former is used for modeling thin plates (which ratio of thickness to smallest in-plane dimension is lower than 1/10), the latter is used for thick plates. According to Classical Plate Theory (Kirchhoff-Love's theory) strain over thickness of element can be disassembled to mid-plane straining and straining contributed by curvature which is linear over thickness:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_m + z\boldsymbol{\kappa}. \tag{1}$$

In case of linear elasticity it is assumed that shell elements undergo plane stress condition and stresses can be obtained using Hooke's law:

$$\boldsymbol{\sigma} = \mathbf{Q}(\boldsymbol{\varepsilon}_m + z\boldsymbol{\kappa}), \tag{2}$$

where $\mathbf{Q}$ is plane stress material stiffness matrix, formulated for orthotropic material in following way:

$$\mathbf{Q} = \mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix}^{-1}. \tag{3}$$

In order to reduce problem dimensionality from 3D to 2D, shell elements are usually formulated using internal forces instead of stresses [21]. Internal forces consists of in-plane normal and shearing forces N_x, N_y, N_{xy} and bending moments M_x, M_y, M_{xy} which can be obtained using their definitions:

$$
\begin{aligned}
\mathbf{N} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma dz = \int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{Q}(\varepsilon_m + z\kappa)dz, \\
\mathbf{M} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma z dz = \int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{Q}(\varepsilon_m + z\kappa)z dz.
\end{aligned} \tag{4}
$$

This system of equations can be written in alternative form:

$$
\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}, \tag{5}
$$

Where $\mathbf{A}, \mathbf{B}, \mathbf{D}$ can be obtained from (4):

$$
\begin{aligned}
\mathbf{A} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{Q} dz, \\
\mathbf{B} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{Q} z dz, \\
\mathbf{D} &= \int_{-\frac{t}{2}}^{\frac{t}{2}} \mathbf{Q} z^2 dz.
\end{aligned} \tag{6}
$$

Matrix

$$
\mathbf{ABD} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix},
$$

is called General Shell Stiffness Matrix and the primary focus of homogenization is identifying this matrix for a given sandwich panel.

Provided the **ABD** matrix is identified one must recover effective parameters as they are needed in most commercial software to define shells. Effective parameters can be obtained by using equations (6), which for a particular thickness of homogenized plate t reduces to:

$$
\begin{aligned}
\mathbf{A} &= \mathbf{Q}\ t_{hom}, \\
\mathbf{B} &= 0, \\
\mathbf{D} &= \frac{\mathbf{Q}\ t_{hom}^3}{12}.
\end{aligned} \tag{7}
$$

Material Stiffness Matrix can be obtained with either **A** or **D** matrices. Effective parameters are then recovered by inverting **Q** matrix and reading appropriate terms of material compliance matrix. As sections of structural and homogenized panels are different, distribution of stresses along thickness is not conserved in process of homogenization. This leads to conclusion that for arbitrary thickness of homogenized panel only one of two sub-matrices will be recovered correctly. In order to overcome this problem, effective thickness must be firstly approximated. Such thickness has property that it allows to formulate **ABD** matrix with one set of effective elastic properties, in other words both matrices $\mathbf{A}, \mathbf{D}$ will be properly formulated. Effective thickness can be approximated using equation [1]:

$$
t_{hom} = \sqrt{12 \frac{D_{11} + D_{22} + D_{33}}{A_{11} + A_{22} + A_{33}}}, \tag{8}
$$

by making use of this thickness one can recover two sets of parameters through definitions of **A** and **D**, and those sets will not differ significantly. One can decide, based on application of sandwich panel, to use either set from in-plane stiffness, bending stiffness or average of those two.

3 Homogenization techniques

3.1 Homogenization based on Classical Laminated Plate Theory

Classical Laminated Plate Theory is usually used in reference to sandwich panels consisting of several flat layers with different material orientations. Such structure is commonly created with polymer composites with each layer differently oriented in plane (orientation of polymer chains on matrix provides orthotropy in each layer). Usually each layer is rotated against the neighboring ones by either 30, 45 or 90 degrees.

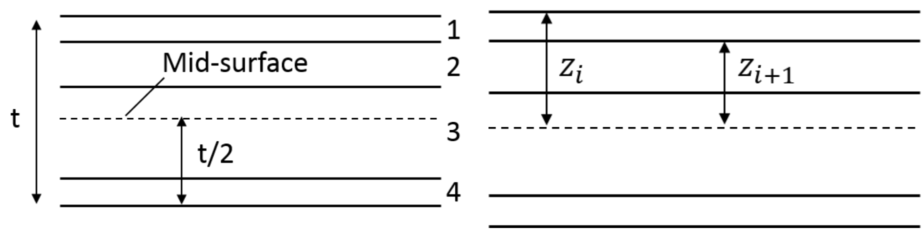

Figure 1: Example of 4-layered laminate panel. Definition of mid-surface and convention of layer's boundary coordinate.

As shown in Figure 1 laminate panel consist of few layers of different thickness, material type and material orientation. In order to calculate the General Shell Stiffness Matrix, one must define the mid-surface which lies in geometric middle of laminate. Each *i-th* layer contributes to the stiffness of the laminate and distance from it's top boundary to mid-surface is defined with z_i while distance from bottom boundary is z_{i+1}. General Shell Stiffness is then obtained by using definition (6).

$$\begin{aligned}
\mathbf{A} &= \int_{-t/2}^{t/2} \mathbf{Q}_i^* dz = \sum_{i=1}^{n} \mathbf{Q}_i^* (z_i - z_{i+1}), \\
\mathbf{B} &= \int_{-t/2}^{t/2} \mathbf{Q}_i^* z dz = \frac{1}{2} \sum_{i=1}^{n} \mathbf{Q}_i^* (z_i^2 - z_{i+1}^2), \\
\mathbf{D} &= \int_{-t/2}^{t/2} \mathbf{Q}_i^* z^2 dz = \frac{1}{3} \sum_{i=1}^{n} \mathbf{Q}_i^* (z_i^3 - z_{i+1}^3).
\end{aligned} \tag{9}$$

One must mind that $\boldsymbol{Q}_i^*$ must be built with material parameters' values accordingly to each layer orientation. Material stiffness matrix in plane can be rotated by using transformation matrices:

$$\boldsymbol{Q}^* = \boldsymbol{J}_{z,\sigma} \boldsymbol{Q}_{xyz} \boldsymbol{J}_{z,\varepsilon}^{-1}$$

$$\begin{aligned}
\boldsymbol{J}_{z,\sigma} &= \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \\
\boldsymbol{J}_{z,\varepsilon} &= \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ -\frac{1}{2}cs & \frac{1}{2}cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix},
\end{aligned} \tag{10}$$

where c, s are cosine and sine of the orientation angle (in-plane) respectively.

This method is very straight-forward and widely used in composite modeling, also due to fact that many commercial FEM systems allow to define shell with composite section, this means that process of homogenization is automated. It was extended to panels in which not all layers are flat but can be corrugated. It can be used to calculate stiffness of sandwich panel which its core is made from thin curved sheet, for e.g. sine-wave like core, semicircular, trapezoidal, triangular or elliptical arc like. CLPT allows to calculate **ABD** matrix by integration of the cardboard section's stiffness according to formulas [2, 3, 19]:

$$
\begin{aligned}
\mathbf{A}(x) &= \int \mathbf{Q} dz = t_{ls}\mathbf{Q}_{ls} + t_f \mathbf{Q}_f(\theta) + t_{li}\mathbf{Q}_{li}, \\
\mathbf{B}(x) &= \int \mathbf{Q} z dz = t_{ls}\mathbf{Q}_{ls} z_{ls} + t_f \mathbf{Q}_f(\theta) z_f + t_{li}\mathbf{Q}_{li} z_{li}, \\
\mathbf{D}(x) &= \int \mathbf{Q} z^2 dz = \mathbf{Q}_{ls}\left(z_{ls}^2 t_l s + \frac{t_{ls}^3}{12}\right) + \mathbf{Q}_f(\theta)\left(z_f^2 t_{vf} + \frac{t_f^3}{12}\right) + \mathbf{Q}_{li}\left(z_{li}^2 t_{li} + \frac{t_{li}^3}{12}\right),
\end{aligned}
\tag{11}
$$

where subscripts ls, li and f denote superior liner, interior liner and fluting respectively.

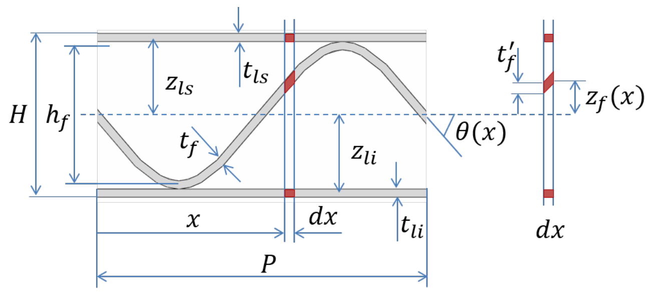

Figure 2: Cardboard section with characteristic CLPT properties annoted.

It is vital to understand that due to the fluting's shape, its material parameters for each section differs in global coordinate system (as local system is constantly rotating). Thus for each section dx fluting parameters must be rotated. If fluting's position is described with sine function:

$$
h(x) = \frac{h_f}{2} \sin\left(\frac{2\pi x}{P}\right),
$$

where: h_f is a distance between liners and P is a fluting's period, then rotation angle is given by:

$$
\theta(x) = \arctan\left(\frac{dh(x)}{dx}\right). \tag{12}
$$

Rotation of material is included by rotating local compliance matrix to global coordinate system and then inverting it to obtain matrix $\mathbf{Q}$. Compliance matrix must be built as full 3D matrix (with 9 parameters), rotated, reduced to standard 2D matrix and then inverted. It is suggested that while using this method all out-of-plane parameters should be set as very small in comparison to in-plane parameters (even for isotropic materials). Rotation of compliance matrix can be accomplished by matrix transformation:

$$
\mathbf{C}_{xyz} = \mathbf{T}\ \mathbf{C}_{123}\ \mathbf{T}^T, \tag{13}
$$

where xyz is global coordinate system, 123 is material's local coordinate stiffness and rotation matrix $\mathbf{T}$ is defined as:

$$\mathbf{T} = \begin{bmatrix} c^2 & 0 & s^2 & 0 & sc & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ s^2 & 0 & c^2 & 0 & -sc & 0 \\ 0 & 0 & 0 & c & 0 & -s \\ -2cs & 0 & 2cs & 0 & c^2-s^2 & 0 \\ 0 & 0 & 0 & s & 0 & c \end{bmatrix}, \tag{14}$$

and $c = cos(\theta), s = sin(\theta)$. Transformation matrix here differs from transformation matrix in (11) because parameters are rotated with respect to different axis.

Total stiffness of the section is now obtained by integrating the section's stiffness over interval of the fluting's period:

$$\begin{aligned} \mathbf{A}_h &= \frac{1}{P}\int_0^P \mathbf{A}(x)dx, \\ \mathbf{B}_h &= \frac{1}{P}\int_0^P \mathbf{B}(x)dx, \\ \mathbf{D}_h &= \frac{1}{P}\int_0^P \mathbf{D}(x)dx, \end{aligned} \tag{15}$$

This method though simple and straight-forward to implement provide good homogenization in elastic region. It requires some numerical integration scheme for e.g. mid-point rule with fine integration step to achieve appropriate accuracy. Unfortunately this method is only suitable for sandwich panels which are periodic only in one direction.

3.2 Homogenization using Hill-Mandel condition

Homogenization technique based on the energy-equivalence was proposed in [16, 4]. It was suggested that Finite Element (FE) model of RVE (Representative Volume Element) can be used to obtain equivalent elastic properties of a corrugated cardboard [4]. RVE is a minimal structure that periodically repeats itself in whole sandwich structure. For example for corrugated cardboard it would be element of length equal to the fluting's period. Such RVE can be modeled using shell elements to obtain structure's stiffness. Provided none of internal nodes are loaded following equation is fulfilled:

$$\bar{\mathbf{K}}\boldsymbol{u}_e = \mathbf{F}_e, \tag{16}$$

where $\bar{\mathbf{K}}$ is the stiffness matrix of external (boundary) degrees of freedom (DoFs) which is obtained using static condensation.

Overall stiffness matrix of structure can be arranged as:

$$\begin{bmatrix} \mathbf{K}_{ee} & \mathbf{K}_{ei} \\ \mathbf{K}_{ie} & \mathbf{K}_{ii} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_e \\ \boldsymbol{u}_i \end{bmatrix} = \begin{bmatrix} \mathbf{F}_e \\ \mathbf{0} \end{bmatrix}. \tag{17}$$

Transforming this equation to (16) leads to following formula for $\bar{\boldsymbol{K}}$:

$$\bar{\mathbf{K}} = \mathbf{K}_{ee} - \mathbf{K}_{ei}\mathbf{K}_{ii}^{-1}\mathbf{K}_{ie}. \tag{18}$$

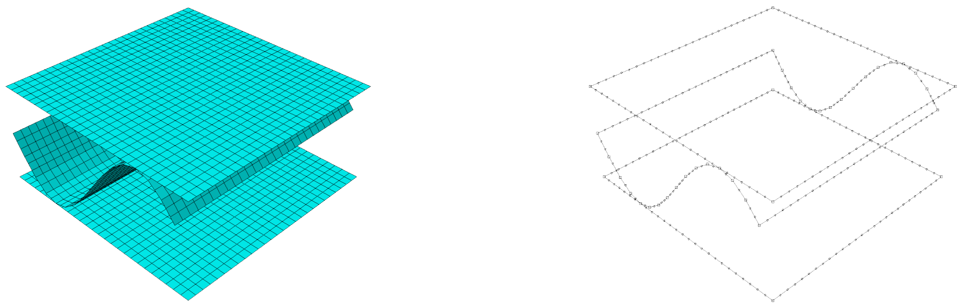

Figure 3: RVE and it's external nodes used in Energy-equivalence homogenisation technique

Assuming vector of external DoFs displacements is known the total energy stored in system is given by:

$$E = \frac{1}{2}\boldsymbol{u}_e^T\mathbf{F}_e. \tag{19}$$

Having assumed Kirchhoff-Love's Plate Theory and constant strain distribution over element one can show that the displacement of the node is related to strain of that node with formula:

$$\begin{bmatrix} \vdots \\ \begin{bmatrix} u_x \\ u_y \\ u_z \\ \phi_x \\ \phi_y \end{bmatrix}_j \\ \vdots \end{bmatrix}^T = \begin{bmatrix} \vdots \\ [\boldsymbol{A}_e]_j \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix}_j \\ \vdots \end{bmatrix}, \tag{20}$$

or simply:

$$\boldsymbol{u}_j^T = [\boldsymbol{A}_e]_j\,\boldsymbol{\kappa}_j, \tag{21}$$

where $[\mathbf{A}_e]_j$ represents displacement distribution over element given the strains which for shell elements is represented by:

$$[\mathbf{A}_e]_j = \begin{bmatrix} x^j & 0 & \frac{y^j}{2} & x^j z^j & 0 & \frac{y^j z^j}{2} \\ 0 & y^j & \frac{x^j}{2} & 0 & y^j z^j & \frac{x^j z^j}{2} \\ 0 & 0 & 0 & -\frac{x^j x^j}{2} & -\frac{y^j y^j}{2} & -\frac{x^j y^j}{2} \\ 0 & 0 & 0 & 0 & -y^j & -\frac{x^j}{2} \\ 0 & 0 & 0 & x^j & 0 & \frac{y^j}{2} \end{bmatrix}. \tag{22}$$

Here x^j, y^j, z^j are coordinates of j-th node of FE model. Substituting (16) and (21) into (18) leads to:

$$E = \frac{1}{2}\boldsymbol{u}_e^T\bar{\mathbf{K}}\boldsymbol{u}_e = \frac{1}{2}\boldsymbol{\kappa}^T\mathbf{A}_e^T\bar{\mathbf{K}}\mathbf{A}_e\boldsymbol{\kappa} \tag{23}$$

On the other hand internal energy of shell subjected to bending can be represented with:

$$E = \frac{1}{2}\boldsymbol{\kappa}^T\left[\mathbf{ABD}\right]\boldsymbol{\kappa}\{AREA\} \tag{24}$$

Imposing equality on both of those energy representations leads to matrix [**ABD**] which describes homogenized shell stiffness and can be used to obtain the effective elastic parameters:

$$[\mathbf{ABD}] = \frac{\mathbf{A}_e^T \bar{\mathbf{K}} \mathbf{A}_e}{\{AREA\}} \tag{25}$$

This method requires building FE model of RVE and extracting assembled global stiffness matrix. In fact one does not have to define forces or boundary conditions or calculate model, simply extraction of stiffness matrix and numbers of external nodes suffice.

4 Results

In order to verify proposed methods few tests were conducted. Firstly, certain sandwich panels of known constituents were homogenized, then response of homogeneous and structural samples were compared in simple bending and uniaxial tensile tests. Such procedure may show effectiveness of homogenization process. As different methods are compared, similar outcome is expected to be found in all of them. Morever, if homogeneous and structural response is similar, one can assume that simplified model can give desired results and can be used instead of typical structural model. Furthermore, accuracy of both CLPT and Hill-Mandel homogenization could be compared.
The panel under consideration is presented in figure 4, it is symmetrical three-layered steel panel with internal layer formed in circular shape. Its properties are given in Tables 1 and 2. The panel was homogenized in [5] by means of Asymptotic Homogenization (AH) which is widely accepted method for numerous homogenization problems, thus results can be compared with those given in literature.

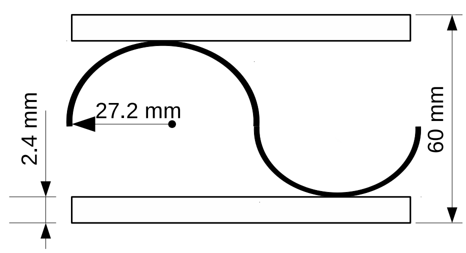

Figure 4: Geometry of analysed sandwich panel

Table 1: Properties of constituents of sandwich panel used in experiment

	t [mm]	E [GPa]	ν [-]	G [GPa]
Top Liner	2.4	210	0.3	85
Fluting	0.8	210	0.3	85
Bottom Liner	2.4	210	0.3	85

Table 2: Geometric characteristic of fluting

Wavelength of fluting [mm]	Height of core [mm]
109.6	57

In order to obtain the General Stiffness Matrix using CLPT, simple numerical integration scheme was used, namely midpoint rule. RVE's section was divided into 5000 short intervals and then simple numerical integration was carried according to algorithm presented in chapter 3.1.

In order to implement Hill-Mandel condition's based homogenization, RVE was built in code prepared in MATLAB environment and meshed with quadrilateral isoparametric shell elements (S4) which size was approximately 5 mm. The Global Stiffness Matrix and coordinates of external nodes were extracted and the General Stiffness Matrix was obtained according to algorithm presented in 3.2.

Table 3: Effective parameters of steel sandwich panel obtained from membrane stiffness matrices for homogenized thickness of 5 mm.

Method	E_1 [GPa]	E_2 [GPa]	ν	G [GPa]
CLPT	205.59	255.94	0.241	81.64
Hill-Mandel	205.57	254.80	0.2426	90.65
Reference (AH)	205.4	254.4	0.242	85.76

Table 4: Effective parameters of steel sandwich panel obtained from bending stiffness matrices for homogenized thickness of 5 mm.

Method	E_1 [TPa]	E_2 [TPa]	ν	G [TPa]
CLPT	81.21	91.30	0.267	32.52
Hill-Mandel	81.01	88.84	0.273	34.29
Reference (AH)	81.07	89.68	0.271	30.89

As seen from Table 4 results acquired using both of presented methods have good agreement with reference data. The biggest discrepancy is found in in-plane shearing modulus G, ranging more than 10%. Additionally, it is noted that even though material is isotropic in terms of constitutive relationship, corrugated structure is not. This is due to fact that fluting does not carry much loading in direction of corrugation while it gives additional stiffness for the other in-plane direction. Results provide evidence that for any effective stiffness, effective parameters derived from membrane and bending stiffnesses will differ (here thickness was taken as 5 mm). Table 6 provides evidence that eq. (8) indeed gives good estimation of effective thickness as effective parameters acquired from $\mathbf{A}$ and $\mathbf{D}$ differ only by factor of approximately 10%. Given this error, it is suggested to use average value from $\mathbf{A}$ and $\mathbf{D}$ values.

Table 5: Effective parameters of steel sandwich panel obtained from membrane stiffness matrices for effective thickness.

Method	t[mm]	E_1 [GPa]	E_2 [GPa]	ν	G [GPa]
CLPT	97.46	10.55	13.13	0.241	4.19
Hill-Mandel	96.63	10.64	13.19	0.2426	4.69

Table 6: Effective parameters of steel sandwich panel obtained from bending stiffness matrices for effective thickness.

Method	t[mm]	E_1 [TPa]	E_2 [TPa]	ν	G [TPa]
CLPT	97.46	10.97	12.33	0.267	4.39
Hill-Mandel	96.63	11.23	12.31	0.273	4.75

Second part of verification of the two methods were standard bending and tension tests. Samples of sizes 300x100 mm prepared both in machine direction (MD) and cross direction (CD) directions (two main in-plane directions for corrugated cardboard) were tested in three-point bending tests and uniaxial tension tests. For given type of cardboard (KLSKL595C [4]) effective parameters were obtained using both methods.

The homogeneous models were also tested in standard tests in order to compare characteristic displacements, which are presented in table 9.

Table 7: Properties of constituents of KLSKL595C corrugated paperboard used in experiment

	t [mm]	E_1 [MPa]	E_2 [MPa]	ν_{12} [-]	G_{12} [MPa]
Top Liner	0.29	3326	1694	0.34	859
Fluting	0.3	2614	1532	0.32	724
Bottom Liner	0.29	3326	1694	0.34	859

Table 8: Geometric characteristic of fluting

Wavelength of fluting [mm]	Height of core [mm]
8	3.8

Table 9: Difference of response of structural sample against homogeneous plates

Sample	Bending MD sample [mm]	Bending CD sample [mm]	Elongation MD sample [mm]	Elongation CD sample [mm]
Structural	6.06	9.82	0.101	0.122
CLPT	6.31	9.12	0.90	0.128
Hill-Mandel	6.30	9.13	0.092	0.130

5 Conclusions

In this work basic concepts of homogenization in elastic regime were presented. Two techniques namely Classical Laminated Plate Theory and Hill-Mandel approach (energy-equivalence method) were discussed with reference to cardboard. It was shown that both methods provides reasonable approximation to structural model, although their application is limited to elastic regime. Implementation of both methods is straightforward for corrugated plates but Hill-Mandel approach is more tedious as it requires building Finite Elements representation of RVE, while CLPT relies only on simple numerical integration scheme. Moreover, both methods are not limited to sine-like core, in fact core can have any shape (including continuum core - e.g. made from foam).

Proposed methods were tested against structural response in simple bending and uniaxial tests in both MD and CD directions. The biggest error was noted for tension test in MD estimated as 10%. Moreover, both methods provided very similar results even though acquired effective parameters differed by factor of 5-10%. This is satisfactory approximation given that the FE model is reduced by factor up to 100 times (depending on sample's size).

In order to include more complicated mechanical properties such as perforations or plastic action of constituents other methods must be used. Among the most prominent methods one might mention Asymptotic Homogenization [20, 7, 15, 5], or Multi-Scale Homogenization. Both of these methods can offer full range of mechanical behaviour but they require much more sophisticated algorithms and much more computational power. This is the reason why for simpler applications CLPT is recommended.

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