The Impact of Network Topology and Market Structure on Pricing*

Ying-Ju Chen[†] Yves Zenou[‡] Junjie Zhou[§]

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Abstract

We develop a two-stage oligopolistic network competition model where, first, firms simultaneously determine their prices, and, second, consumers connected through a network determine their consumption. We show that firms price-discriminate consumers based on the consumer's network position. Denser networks (i.e., network topology) reduce prices, whereas increased competition (i.e., market structure) reduces prices only when competition is weak. However, the prices charged for the most influential consumers can increase with the number of firms when competition is very fierce and when there are strong network externalities. We also show that increasing competition always leads to lower firm profits, whereas equilibrium profits respond to the number of firms more sensitively when the network is denser or the externalities parameter is larger. Finally, we study the effects of network topology and market structure on price dispersion and determine the optimal network structure based on the perspective of both firms and consumers. We also extend the baseline model to allow for asymmetric firms and product compatibility.

Keywords: price discrimination, network position, market structure, spectral decomposition.

JEL classifications: D43, D85, L13, L14.

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[†]School of Business and Management (ISOM), The Hong Kong University of Science and Technology; e-mail: imchen@ust.hk.

[‡]Department of Economics, Monash University; e-mail: yves.zenou@monash.edu.

[§]Corresponding author: School of Economics and Management, Tsinghua University; e-mail: zhoujj03001@gmail.com.

1 Introduction

Competing firms account for network externalities among consumers when they make pricing decisions, while network externalities between them depend on the number and variety of products supplied by competing firms. Consider, for example, the market for gaming consoles. Consumers choose among different consoles, such as Switch (Nintendo), PlayStation (Sony), and Xbox (Microsoft). Their choices depend on the price of the console as well as network effects effects—those in their gaming community who choose to use each platform. These network effects depend on the number of firms that make consoles and the degree of differentiation among them. When competing firms make pricing decisions, they anticipate the network effects among consumers, which depend on product differentiation and competition.

This study examines the impact of market structure (i.e., product differentiation and competition) and the topology of the consumer network on firms' pricing decisions. Firms face the following trade-off: a lower price may benefit the firm by increasing demand, but, at the same time, a lower price is costly because it reduces the price-cost margins for existing demand. These effects are modulated by the degree of network externalities and the number of firms in the market. We show that firms price-discriminate based on consumers' positions in the network. The topology of the consumer network affects prices overall: when network effects are stronger or networks are denser, prices are lower. Moreover, the way that competition affects prices depends on the consumer network: when the network is regular, more competition always decreases prices, but when the network is not regular, increasing competition can increase prices charged to highly connected consumers.

In our model, each firm sells a differentiated product to consumers connected in a network of social interactions. Consumers derive utility from their own consumption as well as from the consumption of others who are directly linked to them in the network, who could be their friends, neighbors, or people who are connected to them online and with whom they have common interests. Thus, by interacting with other consumers in the network, each consumer receives positive spillovers, which we refer to as "local network effects." In the first stage, firms simultaneously determine potentially discriminatory prices, and, in the second stage, consumers determine their consumption of each product, based on the prices set by the firms.

We show that the price set by each firm is a decreasing function of the position of each consumer in the network in terms of Katz-Bonacich centrality, which means that firms price-discriminate consumers by charging central agents lower prices. This implies that highly connected individuals pay a lower price and can even be subsidized for consuming a product.[1] We also show that prices are lower when either the network becomes denser or the intensity of the network effects becomes stronger. Indeed, increasing network density or network externalities intensifies the price competition between firms for the central nodes in the network, who, in equilibrium, are compensated by being charged lower prices. We then study how a change in the market structure

[1] The idea of using celebrities, who have a high social value in markets, to influence consumers is a well-known marketing strategy (Knoll and Matthes, 2017).

(i.e., the number of firms L) affects prices. In regular networks (i.e., those in which all the nodes have the same degree), prices always decrease according to the number of firms. However, in non-regular networks, this is not necessarily true. In fact, in the latter, there is a non-monotonic relationship between prices and L. In particular, we show that for highly connected individuals, whose prices are negative (i.e., they are subsidized by firms to consume their goods), when L is large and network externalities are important, increasing L further increases rather than decreases prices. In other words, for these agents only, in a very competitive market, *increasing competition leads to an increase in prices.* Indeed, when competition is very fierce, it becomes less profitable for firms to subsidize these influential agents if competition further increases; in the limit, when L goes to infinity, firms will set a price (in our setting, a mark-up) of zero for highly connected individuals.

We then measure the amount of degree of price discrimination by the *price dispersion* in the market; that is, the maximal price differential among consumers. Price dispersion was found to be small in markets with monopolies and very stiff competition, and attains a maximum value in markets that include an intermediate number of firms. Some illustrative examples suggest that the extent of price dispersion can be significant; the maximal value is attained when there are two to four firms in the market.

Next, we examine the impact of market structure and network topology on firms' profitability. Intuitively, a firm's profit decreases as competition increases. However, it is less obvious that increasing the intensity of network effects does not necessarily increase firms' profits because it depends on the degree of competition. We show that it leads instead to a *rightward rotation* of the profit curve, which means that under low competition, when L is small, the firm's profit is higher when there are more network externalities, while, under fierce competition, when L is large, the opposite is true. In other words, the equilibrium profits respond to the number of firms more sensitively when the network is denser or the externalities parameter is larger. This unintended outcome is the result of two effects. The first effect is that a strong network intensity amplifies the benefits of network externalities, thereby leading to higher consumer consumption. The second effect is that it pushes the firms to lower their prices due to stronger competition, and this price effect can drive down the firm's profitability. When there are only a few firms, the first effect dominates the second one, and improvement in the network technology (i.e., when there are more network externalities) generates higher firm profits. When there are many firms, however, the price effect increases and the improvement in the network technology reduces the firms' profitability.

To show these results for non-regular networks, we reduce the complex problem of a general network structure to a series of sub-problems in regular networks, in which each sub-problem is easy to solve. This new technique is based on the spectral decomposition theorem, which allows us to decompose the equilibrium firms' profits into a weighted aggregation of several terms, where each term corresponds to a regular network which degree is replaced by the corresponding eigenvalue of the network adjacency matrix. These techniques are similar to those used in Galeotti, Golub, and Goyal (2020), which also applied the spectral decomposition theorem to simplify the optimal targeting interventions in networks. Here, we use them in the very different context of comparative statics in oligopoly networks.

To prove the robustness of our results, we consider four different extensions of our model. First, firms are heterogeneous, so products are asymmetric because of either their quality differences or their cost differences. We show that the prices are heterogeneous not only across consumers but also across firms, and that firms price-discriminate consumers based on their network position. Second, we incorporate in the utility function an additional benefit due to product compatibility. We show that product compatibility results in higher equilibrium prices and higher firm profits due to softer competition and enhanced demand multipliers. However, product compatibility may lead to lower consumption and lower consumer surplus due to endogenous price effects. Third, we consider an alternative setting in which firms compete in quantities (Cournot) instead of prices (Bertrand). We show that our main results hold; in particular, the fact that for very connected nodes, there is a non-monotonic relationship between prices and L. Finally, when the free entry of firms in the market is allowed, we show that an improvement in the network technology increases the equilibrium number of participating firms only when the fixed entry cost is high. Furthermore, we demonstrate how our analysis can be used to characterize optimal network structures from the perspectives of both firms and consumers.

1.1 Related literature

Our study is related to the game-on-network literature (for overviews, see Jackson (2008); Jackson and Zenou (2015); Bramoullé and Kranton (2016); Jackson et al. (2017)), particularly that which deals with pricing under imperfect competition in networks (see Bloch (2016) for a survey on targeting and pricing in networks).[2]

In the game-on-network literature, different aspects of imperfect competition with network effects have been addressed. In terms of pricing issues, Candogan et al. (2012), Bloch and Quérou (2013), Hu, Milner, and Wu (2015), Du, Cooper, and Wang (2016), Fainmesser and Galeotti (2016), Wang and Wang (2016), Leduc, Jackson, and Johari (2017), and Belhaj and Deroïan (2021) dealt with the monopoly case, whereas Chen, Zenou, and Zhou (2018a), Aoyagi (2018), and Fainmesser and Galeotti (2020) examined the duopoly framework.[3] Amir and Lazzati (2011) studied a general market structure (i.e., an oligopoly with L firms) with network effects. They considered oligopolistic competition among firms in a market characterized by positive (direct) network effects when the products of the firms are homogeneous and perfectly compatible so that the relevant network is industry-wide. One of their key results was that because of network effects, market prices need not

[2] There are early literature on network externalities with imperfect competition that either focused on the aggregate level of network externalities (e.g., Farrell and Saloner (1985); Katz and Shapiro (1985); for an overview, see Economides (1996)) or on the competitive pricing problem in the context of two-sided networks in which players on one side care about the aggregate contributions of those on the other side (see, for example, Caillaud and Jullien (2003); Armstrong (2006); Rochet and Tirole (2006); Birge et al. (2021)), which corresponds to a very specific network structure: the complete bipartite network. Tan and Zhou (2021) study effects of competition on pricing and entry in a general model of multi-sided platform competition. In our model, the network structure can be arbitrary.

[3] See also Ushchev and Zenou (2018), who modeled the substitutability between differentiated goods as a network and determined the equilibrium prices.

decrease with increased competition, a result that we also obtained in Proposition 2. However, our model differs greatly; in particular, Amir and Lazzati (2011) did not explicitly model the network topology.

To the best of our knowledge, our study is the first to investigate the interactions between the market structure and the network structure/topology in a general network oligopolistic competition model with an arbitrary number of firms, an arbitrary network structure, and a flexible degree of product differentiation.

As stated above, we also develop new techniques based on the spectral decomposition theorem to determine the comparative statics of our model. Similar techniques were used in Galeotti, Golub, and Goyal (2020), but their purpose was to simplify the optimal targeting interventions in networks. However, their study and ours have focused on different economic issues, highlighting the analytical advantages of looking at the problems through the angle of eigen-decomposition and the eigenvector space of the network matrix.

1.2 Comparison of duopolistic and oligopolistic competition

This study builds on our companion paper (Chen et al., 2018a), where we examined a network model of duopolistic pricing. In this section, we highlight new insights that a study of L competing firms (market structure) can contribute to the existing research.

First, in Section 4, we show the importance of considering a general market structure, especially to determine the impact of competition on equilibrium price and price dispersion. Indeed, when the number of firms increases from $L = 1$ to $L = 2$, the equilibrium prices always decrease (i.e., the standard competition effect). In Proposition 2, we show that the above price movement is no longer true when the number of firms further increases from 2 to L. In fact, we show that prices can increase with competition, especially when the network effect is strong, and the number of firms is large.

Second, contrary to the duopoly case (Chen et al., 2018a), in Proposition 4, we show that the impact of network effects δ on firms' equilibrium profits depends on L, and it is non-monotonic. Furthermore, in Proposition 5, we study the impact of L and δ on equilibrium profits in any *non-regular network*. We show that L has a negative impact on profit, whereas the effect is non-monotonic for δ. To show this result, we introduce a new technique based on the spectral decomposition theorem, which allows us to reduce the complex problem of a general network structure into a series of sub-problems with regular networks, in which each sub-problem is easy to solve. Finally, when $L > 2$, we show how the market structure and the network structure are determined when we allow for the free entry of firms into the market.

2 Model

Consider L firms that sell differentiated products to N consumers in a social network, where $L \geq 1$ and $N \geq 2$. Let $\mathcal{L} := \{1, 2, \cdots, L\}$ and $\mathcal{N} := \{1, 2, \cdots, N\}$ denote the set of firms and consumers, respectively. Each firm $l \in \mathcal{L}$ sells product variety l to all consumers (consumers love variety, so each of them consumes all goods),[4] for which the firms sets prices $\mathbf{p}^l = (p_1^l, \cdots, p_N^l)'$, since prices may differ across consumers, depending possibly on their network positions and other characteristics such as a_i^l (defined below). Each consumer $i \in \mathcal{N}$ has a quasi-linear utility, $x_i^0 + u_i(\mathbf{x}_i, \mathbf{x}_{-i})$, where x_i^0 is i's consumption of the numeraire good, $\mathbf{x}_i = (x_i^1, \cdots, x_i^L)' \in \mathbf{R}_+^L$ is i's consumption bundle of products offered by L firms, and $\mathbf{x}_{-i}$ is the consumption profile of consumers other than i.

Let us denote $\mathbf{G} = (g_{ij})_{n\times n}$ as the adjacency matrix that represents the network structure among consumers. In other words, $g_{ij} = 1$ if and only if i and j are directly connected and $g_{ij} = 0$. We also assume that $g_{ii} = 0$ (no self-loops) and that $g_{ij} = g_{ji} \in \{0, 1\}$ (i.e., an undirected and unweighted network).[5]

We adopt the following explicit functional form for the utility function:[6]

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) := \underbrace{\left(\sum_{l=1}^{L} a_i^l x_i^l - \frac{1}{2}\sum_{l=1}^{L}(x_i^l)^2 - \frac{1}{2}\sum_{l=1}^{L}\sum_{s\neq l}\beta x_i^s x_i^l\right)}_{:=v_i(\mathbf{x}_i)} + \delta \underbrace{\left(\sum_{l=1}^{L}\left(\sum_{j=1}^{N} g_{ij} x_i^l x_j^l\right)\right)}_{:=\eta_i(\mathbf{x}_i, \mathbf{x}_{-i})}. \tag{1}$$

The utility of consumer i consists of two terms. The first term, $v_i(\mathbf{x}_i)$, represents i's own consumption utility, and it only depends on $\mathbf{x}_i$, with β measuring the curvature of $v_i(\cdot)$ and the degree of substitution between the L products consumed. For each i, and for $s \neq l$, we have $\frac{\partial^2 u_i}{\partial x_i^l \partial x_i^s} = -\beta$, which is negative (positive) when β is positive (negative). Therefore, products are substitutable (complementary) when $\beta > (<)0$.

The second term in (1), $\eta_i(\mathbf{x}_i, \mathbf{x}_{-i})$, captures the *network effects* (the peer effects) enjoyed by consumer i from interacting with other consumers in network $\mathbf{G}$. These effects are scaled by the parameter δ, so that $\frac{\partial^2 \eta_i}{\partial x_i^l \partial x_j^l} = \delta g_{ij}$ for any $l \in \mathcal{L}$.[7] Therefore, i's utility depends on j's consumption

[4] Indeed, the utility function (1) is strictly concave in $\mathbf{x}_i$ under Assumption 2 below, which implies consumer i's *love for variety*.

[5] This is without loss of generality. Our model is flexible enough to allow for an arbitrary network structure $\mathbf{G}$, and achieve the same main results.

[6] The quadratic form in the private part of $v_i(\cdot)$ has been widely adopted in Industrial Organization, trade, and macroeconomics (see, for example, Singh and Vives (1984); Vives (2001); Ottaviano, Tabuchi, and Thisse (2002); Asplund and Nocke (2006); Foster, Haltiwanger, and Syverson (2008); Melitz and Ottaviano (2008); Syverson (2019)). Also, quasi-linear utility functions such as (1) is commonly adopted in many network models (e.g., Ballester, Calvó-Armengol, and Zenou (2006); Bramoullé and Kranton (2007); Bramoullé, Kranton, and d'Amours (2014)), especially in IO network models (e.g., Candogan, Bimpikis, and Ozdaglar (2012); Bloch and Quérou (2013); Fainmesser and Galeotti (2016, 2020); Chen, Zenou, and Zhou (2018a); Ushchev and Zenou (2018)).

[7] Note that, for $l \neq s$, $i \neq j$, $\frac{\partial^2 \eta_i}{\partial x_i^l \partial x_j^s} = 0$. In Section 6.2, we relax this assumption and analyze the case of *cross-product network effects* and thus, the degree of compatibility among products.

when i and j are directly connected, which reflects the *local* network effects among consumers in the network. In other words, the value to a given consumer of consuming a product increases when others directly connected to this consumer consume this product. The budget equation for consumer i is $x_i^0 + \sum_{l=1}^{L} p_i^l x_i^l = Y_i$, where p_i^l is the price that consumer i pays firm l for the consumption of x_i^l. As usual, we assume that the income Y_i is sufficiently large so that the nonnegativity constraints on consumptions $\{x_i^l\}$s and x_i^0 are not binding.[8]

Based on the price profile $\mathbf{p} = (\mathbf{p}^1, \cdots, \mathbf{p}^L)$ and the consumption profile $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_n)$,[9] firm l's profit is given by:

$$\Pi^l(\mathbf{p}^l, \mathbf{p}^{-l}; \mathbf{x}) := \sum_{i \in \mathcal{N}} (p_i^l - c_i) x_i^l,$$

where c_i is the marginal cost of serving consumer i.

Our first assumption is to focus on *ex ante* symmetric firms.[10]

Assumption 1. *For each consumer i, $a_i^l = a_i^s = a_i > c_i \geq 0$ for any l and s.*

Since each firm l is defined by its product l, this assumption is equivalent to having symmetric products. This implies that each consumer values all products in the same way but that different consumers value them differently. Let $\mathbf{a} = (a_1, \cdots, a_N)'$ denote the marginal utility vector. In the equilibrium analysis described in Section 3, we consider an arbitrary $\mathbf{a}$. In some analyses descrbed in Sections 4 and 5, we further restrict the model to the case where $a_i = a, c_i = 0, \forall i \in \mathcal{N}$ (see Assumption 3). While this is not without loss of generality, it best serves our purpose in this study because our main focus is network topology (see footnote 27 for a further discussion).

For ease of exposition, we focus on *substitutable* products ($\beta \geq 0$) with *positive network* effects ($\delta \geq 0$).[11] We assume that $\beta \in [0, 1)$, under which i's utility $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ in (1) is strictly concave in $\mathbf{x}_i$ (see Lemma A1 in the Online Appendix A for the proof). Let $\lambda_1(\mathbf{G})$ denote the largest eigenvalue of $\mathbf{G}$.

Assumption 2. *$\delta \geq 0$, $0 \leq \beta < 1$ and*

$$1 - \beta - \delta \lambda_1(\mathbf{G}) > 0. \tag{2}$$

Condition (2) guarantees the uniqueness of the consumption equilibrium for any price profile and the concavity of each firm's profit function in prices. This assumption imposes conditions on the product differentiation parameter β, the network effect parameter δ, and the spectral radius of $\mathbf{G}$ (which, based on the Perron-Frobenius theorem, is $\lambda_1(\mathbf{G})$, the largest eigenvalue of $\mathbf{G}$, since $\mathbf{G}$ is a matrix with non-negative entries) but does not depend on L, the number of firms. This condition is satisfied when δ is not too large, i.e., when $\delta < (1 - \beta)/\lambda_1(\mathbf{G})$. In many network papers, conditions

[8]Effectively, consumer i chooses $\mathbf{x}_i$ to maximize $u_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \sum_{l=1}^{L} p_i^l x_i^l$.

[9]Note that the dimension of $\mathbf{p}^l$ is N, whereas the dimension of $\mathbf{x}_i$ is L, so that both $\mathbf{p}$ and $\mathbf{x}$ have dimension NL.

[10]In Section 6.1, we relax this assumption and analyze the case of *ex ante asymmetric* firms.

[11]Similar techniques can be used for the case of negative network effects ($\delta < 0$) and/or the case of complementary products ($\beta < 0$), but the model interpretation and the resulting insights will be different.

that are analogous to (2) are imposed.[12]

We study the subgame perfect equilibrium of the following two-stage game: first, firms simultaneously choose their prices and, second, consumers simultaneously choose their consumption bundles. Network structure $\mathbf{G}$ and model parameters β, δ, and $\mathbf{a}$, are known and, in fact, are common knowledge among firms and consumers. Following the literature on industrial organization (Sutton, 2007), *market structure* is captured by both the number of firms L in the market and the degree of product differentiation β. Following the network literature (Jackson, 2008; Jackson and Zenou, 2015), *network topology* is captured by adjacency matrix $\mathbf{G}$.[13] We take as given the market structure L, β and the network structure $\mathbf{G}$ in much of the analysis, with the exception of Section F, where we study the free-entry of firms (so that the number of firms L is endogenized) and the optimal network structure. In addition to Section 6, throughout the paper, we maintain Assumptions 1 and 2 in all propositions, corollaries, and lemmas. Thus, we will not explicitly describe them.[14]

3 Equilibrium analysis

3.1 Consumption equilibrium

We first characterize consumers' simultaneous consumption decisions. Given the price profile $\mathbf{p} = (\mathbf{p}^1, \cdots, \mathbf{p}^L)$, with $\mathbf{p}^l = (p_1^l, \cdots, p_N^l)'$ for product l, each consumer $i = 1, \cdots, N$ chooses $\mathbf{x}_i$ to maximize $u_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \sum_{l=1}^L p_i^l x_i^l$, while taking as given the consumption decisions of all other consumers in the network. The consumption equilibrium (CE) $\mathbf{x}(\mathbf{p})$ is a Nash equilibrium of the second-stage consumption game among consumers in the network. When each consumer i makes her consumption decisions, she takes into account the following factors: her own preference a_i^l for each product l, the degree of substitution β between the different products, the number of firms L in the market, which determines the number of products consumed, the price p_i^l of each good l, and the local network effects δ (i.e., what her friends consume and how important their decisions are for i). Lemma 1 below shows that, under Assumptions 1 and 2, there exists a unique consumption equilibrium (CE) $\mathbf{x}(\mathbf{p})$.

Let us define

$$\mathbf{M}^+ := [(1 + (L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1} \text{ and } \mathbf{M}^- := [(1-\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}. \tag{3}$$

Lemma 1 (Consumption Equilibrium)**.** *Suppose that Assumptions 1 and 2 hold. In the consumption game:*

[12] See, for example, Ballester, Calvó-Armengol, and Zenou (2006); Candogan, Bimpikis, and Ozdaglar (2012); Bloch and Quérou (2013); Bramoullé, Kranton, and d'Amours (2014); Fainmesser and Galeotti (2016); Chen, Zenou, and Zhou (2018a); Galeotti, Golub, and Goyal (2020).

[13] Note that δ captures the intensity of the network effects.

[14] In Appendix A, we provide the matrix notations used in this paper.

(i) Given any price profile $\mathbf{p}$*, there exists a unique consumption equilibrium* $\mathbf{x}(\mathbf{p}) = (\mathbf{x}^l(\mathbf{p}), l \in \mathcal{L})$.[15] *Furthermore, the demand vector of each firm* l, $\mathbf{x}^l(\mathbf{p})$*, is linear in* $\mathbf{p}$ *and given by:*

$$\mathbf{x}^l(\mathbf{p}) = \mathbf{M}^+ \left(\mathbf{a} - \frac{\sum_{s\in\mathcal{L}} \mathbf{p}^s}{L}\right) - \mathbf{M}^- \left(\mathbf{p}^l - \frac{\sum_{s\in\mathcal{L}} \mathbf{p}^s}{L}\right). \tag{4}$$

(ii) When each firm charges the same prices $\mathbf{p}$*, the unique CE is symmetric, and, for any* $l \in \mathcal{L}$*, is equal to:*

$$\mathbf{x}^l = \mathbf{x}^{sym}(\mathbf{p}) = \mathbf{M}^+(\mathbf{a} - \mathbf{p}) = [(1 + (L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}(\mathbf{a} - \mathbf{p}). \tag{5}$$

To understand Lemma 1, we first consider the average demands of firms by summing up (4) over L. We get:

$$\frac{1}{L}\sum_{s\in\mathcal{L}} \mathbf{x}^s = \mathbf{M}^+ \left(\mathbf{a} - \frac{1}{L}\sum_{s\in\mathcal{L}} \mathbf{p}^s\right). \tag{6}$$

We see that $\mathbf{M}^+$ measures the marginal reduction of the *average demands* of firms in a marginal increment of *average prices*.[16] Similarly, by taking the difference of (4) for l and s, the difference between the demands of the two firms l and s is equal to:

$$\mathbf{x}^l - \mathbf{x}^s = -\mathbf{M}^-(\mathbf{p}^l - \mathbf{p}^s). \tag{7}$$

In other words, $\mathbf{M}^-$ measures (the negative value of) the marginal reduction of the *demand difference* between the two firms for a marginal increment of *price differences* between the two firms.[17] Accordingly, two firms that charge the same prices must obtain equal demands; this is consistent with Lemma 1 (ii). Since $\mathbf{M}^+$ and $\mathbf{M}^-$ measure the marginal reductions, we term them *sensitivity matrices.*

Next, we link sensitivity matrices $\mathbf{M}^+$ and $\mathbf{M}^-$ to the underlying network structure and market structure. We introduce the following inverse Leontief matrix, which is well-known in the network literature:

$$\mathbf{M}(\mathbf{G}, \delta) = [\mathbf{I}_N - \delta\mathbf{G}]^{-1} = \mathbf{I}_N + \delta\mathbf{G} + \delta^2\mathbf{G}^2 + \cdots +,$$

where each entry m_{ij} of $\mathbf{M}$ represents the total number of walks from i to j in network $\mathbf{G}$, where each walk of length k is discounted by δ^k (the infinite sum converges when $\delta < 1/\lambda_1(G)$; see, for example, Ballester et al. (2006)). In fact, both sensitivity matrices are proportional to the inverse Leontief matrix $\mathbf{M}$, with some adjustments to the discount factor δ, according to a factor

[15] Here, $\mathbf{x}^l = (x_1^l, \cdots, x_N^l)'$ is the demand vector of firm l (which is different from $\mathbf{x}_i = (x_i^1, \cdots, x_i^L)'$, the consumption bundle of consumer i). So $(\mathbf{x}^l, l \in \mathcal{L})$ and $(\mathbf{x}_i, i \in \mathcal{N})$ are two equivalent representations of $\mathbf{x}$.

[16] It also implies that $-\frac{\partial\{\mathbf{x}^1+\cdots+\mathbf{x}^L\}}{\partial \mathbf{p}^s} = [(1 + (L-1)\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \mathbf{M}^+$.

[17] It also implies that $-\frac{\partial\{\mathbf{x}^l-\mathbf{x}^s\}}{\partial \mathbf{p}^l} = [(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \mathbf{M}^-$.

of $\frac{1}{1+(L-1)\beta}$ for $\mathbf{M}^+$ and a factor of $\frac{1}{1-\beta}$ for $\mathbf{M}^-$. Thus, we have:[18]

$$\begin{aligned}
\mathbf{M}^+ &= \frac{1}{1+(L-1)\beta}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{1+(L-1)\beta}\right) = \frac{1}{1+(L-1)\beta}\sum_{j\geq 0}\left(\frac{\delta}{1+(L-1)\beta}\right)^j \mathbf{G}^j, \\
\mathbf{M}^- &= \frac{1}{(1-\beta)}\mathbf{M}\left(\mathbf{G}, \frac{\delta}{1-\beta}\right) = \frac{1}{(1-\beta)}\sum_{j\geq 0}\left(\frac{\delta}{1-\beta}\right)^j \mathbf{G}^j.
\end{aligned}$$

Since $\beta \in [0,1)$, $\frac{1}{1+(L-1)\beta}$ is less than one, and it decreases with the number of firms L (and with the product differentiation parameter β). This decrease implies that the network structure has less impact on the sensitivity of the average demands with respect to prices when there are more firms or when products are more homogeneous. However, $\frac{1}{1-\beta}$ is greater than one, which implies that competition among substitutable products amplifies the role of network structure in shaping the sensitivity of the relative demand of firms with respect to price changes. Notably, the adjustment factor $1/(1-\beta)$ for $\mathbf{M}^-$ does not change with L.

In view of the interpretations of equations (6)–(7), we term $\mathbf{M}^+$ the *market expansion social multiplier*, and $\mathbf{M}^-$, the *business-stealing social multiplier*. This is because the former reflects the impact of the network structure on the aggregate demand of all firms (see (6) or (5)), and the latter refers to the impact of the network structure on the demand differences between two competing firms (see (7)). Interestingly, because $\frac{1}{1+(L-1)\beta} \leq 1 \leq \frac{1}{1-\beta}$, the business-stealing social multiplier is stronger than the market expansion social multiplier; that is,[19] $\mathbf{M}^- \succeq \mathbf{M}^+$.[20] Moreover, as the number of firms L increases, the market-expansion effect of the social network is diminished ($\mathbf{M}^+$ is weaker), whereas the business stealing effect of the social network $\mathbf{M}^-$ remains the same.[21]

The following corollary directly follows from Lemma 1 and our discussion above.

Corollary 1. *For $l \neq s \in \mathcal{L}$, we have:*

$$\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l} = -\frac{\mathbf{M}^+ + (L-1)\mathbf{M}^{-1}}{L} \preceq \mathbf{0}$$

and

$$\frac{\partial \mathbf{x}^s}{\partial \mathbf{p}^l} = -\frac{\mathbf{M}^+ - \mathbf{M}^{-1}}{L} \succeq \mathbf{0}.$$

Corollary 1 shows that linear combinations of $\mathbf{M}^+$ and $\mathbf{M}^-$ determine the sensitivity of a firm's demands with respect to its prices and those of its competitors. Since products are substitutable, if firm l raises its prices, l's demands decrease and the competitor s's demands increase; i.e., $\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l} \preceq \mathbf{0}$ and $\frac{\partial \mathbf{x}^s}{\partial \mathbf{p}^l} \succeq \mathbf{0}$. Interestingly, the "own-price" sensitivity $-\frac{\partial \mathbf{x}^l}{\partial \mathbf{p}^l}$ is a convex combination of $\mathbf{M}^+$ and

[18] The path-counting interpretations of $\mathbf{M}^+$ and $\mathbf{M}^-$ are similar to those of $\mathbf{M}$. Assumption 2 ensures that both infinite sums converge.

[19] Formally, $\mathbf{M}^- - \mathbf{M}^+ = \mathbf{M}^- L\beta\mathbf{M}^+ \succeq \mathbf{0}$.

[20] Given two matrices $\mathbf{H}$ and $\mathbf{D}$, we say that $\mathbf{H} \preceq (\succeq)\mathbf{D}$ if component-wise $h_{ij} \leq (\geq) d_{ij}$ for all i, j, where $\{h_{11}, ..., h_{mn}\}$'s are the components of the matrix $\mathbf{H}$ and $\{d_{11}, ..., d_{mn}\}$'s are $\mathbf{D}$'s components. We can define in a similar way (i.e., component-wise) the partial derivatives of matrices and vectors.

[21] Formally, $\partial\mathbf{M}^+/\partial L = -\mathbf{M}^+\beta\mathbf{M}^+ \preceq \mathbf{0}$ and $\partial\mathbf{M}^-/\partial L = \mathbf{0}$.

$\mathbf{M}^-$ but puts greater weight on $\mathbf{M}^{-1}$ and, therefore, less weight on $\mathbf{M}^+$ in a larger number of firms. These observations will shape the incentives of firms in the pricing stage.

3.2 Pricing equilibrium

We now characterize the pricing decisions of firms in the first stage. Suppose that there exists a symmetric equilibrium in the pricing stage in which all firms charge the same prices $\mathbf{p}^*$ (i.e., the price is the same for all firms but may be different for each consumer). We now derive the necessary conditions for sustaining $\mathbf{p}^*$ as a symmetric pricing equilibrium. Hence, the equilibrium demand of each firm would be $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$, as in part (ii) of Lemma 1. For example, if firm 1 deviates and lowers its price vector by $\Delta\mathbf{p}^1$, it must then satisfy the following *no-deviating* condition:[22]

$$\overbrace{\langle \Delta\mathbf{p}^1, \underbrace{\mathbf{M}^+(\mathbf{a}-\mathbf{p}^*)}_{=\mathbf{x}^* \text{ by Lemma 1}} \rangle}^{\text{marginal loss of lowering prices}} = \underbrace{\langle \overbrace{\frac{\mathbf{M}^+ + (L-1)\mathbf{M}^-}{L}\Delta\mathbf{p}^1}^{\Delta\mathbf{x}^1 \text{ by Corollary 1}}, (\mathbf{p}^* - \mathbf{c})\rangle}_{\text{marginal benefit of lowering prices}}. \tag{8}$$

Indeed, on one hand, if firm 1 lowers its prices by $\Delta\mathbf{p}^1$ for all its consumers, it increases its profits because all consumers will consume more of product 1, but this increase will depend on the network effects δ, the degree of substitution β between the different goods, and how this increase propagates through the network. All these aspects are captured by both $\mathbf{M}^+$, the market-expansion social multiplier, and $\mathbf{M}^-$, the business-stealing social multiplier. On the other hand, if the firm lowers its prices by $\Delta\mathbf{p}^1$, it reduces its profits because it loses money for each product sold to consumers. This is captured by $\mathbf{M}^+$ but not by $\mathbf{M}^-$ since the latter measures the business-stealing effect.

Equation (8) must hold for any $\Delta\mathbf{p}^1$ in $\mathbf{R}^N$, which implies the following identity:

$$\mathbf{M}^+(\mathbf{a}-\mathbf{p}^*) = \frac{\mathbf{M}^+ + (L-1)\mathbf{M}^-}{L}(\mathbf{p}^* - \mathbf{c}). \tag{9}$$

Solving (9) yields the $\mathbf{p}^*$ stated in Proposition 1.[23]

Proposition 1. *Suppose that Assumptions 1 and 2 hold. Then, there exists a unique equilibrium in the pricing stage in which all firms charge the same price $\mathbf{p}^*$, defined as follows:*

$$\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{(L-1)\beta}{2}[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{c}). \tag{10}$$

In equilibrium, each firm's profit is $\Pi^* = \langle \mathbf{x}^*, (\mathbf{p}^* - \mathbf{c})\rangle$, and the total consumer surplus is $\mathbf{CS}^* = \frac{L(1+\beta(L-1))}{2}\langle \mathbf{x}^*, \mathbf{x}^*\rangle$, where $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a}-\mathbf{p}^*) \in \mathbf{R}^N$ gives the equilibrium demand of each firm.

The pricing formula (10) in Proposition 1 is determined by the trade-off specified in (9): lowering prices enhances the firm's demands but is costly for the firm because it also reduces the

[22] $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^n x_i y_i$ is the inner product of $\mathbf{x}$ and $\mathbf{y}$.
[23] All proofs can be found in the Appendix.

price-cost margins of existing demands. The degree of the demand-enhancing effects and the size of the existing demands are captured by Corollary 1 and Lemma 1, respectively. The relative strength of these two forces determines the equilibrium prices in Proposition 1.[24]

It is easily verified that the pricing formula in (10) exhibits the following decomposition:

$$\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{(L-1)\beta}{2\left((2+(L-3)\beta)\right)} \underbrace{\left[\mathbf{I}_n - \frac{2\delta}{(2+(L-3)\beta)}\mathbf{G}\right]^{-1}(\mathbf{a}-\mathbf{c})}_{=\mathbf{b}(\mathbf{G}, \frac{2\delta}{(2+(L-3)\beta)}, \mathbf{a}-\mathbf{c})} \tag{11}$$

The first term does not depend on the network, whereas the second term is proportional to the Katz-Bonacich centralities[25] of nodes, where the discount factor δ is adjusted by a factor of $\frac{2}{(2+(L-3)\beta)}$. Unless $L = 1$ (the monopoly case) and/or $\beta = 0$ (independent products), equilibrium prices exhibit network-based discrimination; that ia, the price is a function of the consumer's position in the network. In particular, equation (11) shows that the more central a consumer is (in terms of Katz-Bonacich centrality), the lower the price the consumer will be charged for consuming a good. This is because a highly central consumer generates network externalities to her friends who are more likely to consume the good, which is considered by each firm. As shown in (11), this price discount depends on the intensity of the network effects δ, the degree of substitution between the goods β, the network structure $\mathbf{G}$, and the market structure L.

Our general model extends several existing models. The equilibrium prices in (10) are consistent with those in previous studies, where L, the number of firms, is either equal to 1 (indicating a monopoly) or 2 (indicating a duopoly). For example, the prices drop to $\frac{\mathbf{a}+\mathbf{c}}{2}$ in the case of a monopoly firm (Bloch and Quérou, 2013; Candogan, Bimpikis, and Ozdaglar, 2012)). Similarly, in the case of a duopoly, the prices are $\mathbf{p}^* = \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{\beta}{2}[(2-\beta)\mathbf{I}_n - 2\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{c})$ (Chen et al., 2018a). Since the monopoly prices are equal to[26] $\mathbf{p}^m = \frac{\mathbf{a}+\mathbf{c}}{2}$, the prices in (10) are always below the monopoly prices. However, this observation does not imply that competition always reduces prices, as we show in Proposition 2 below.

One interesting prediction is that, if $L = 1$, equilibrium prices do not depend on the network topology, whereas if $L > 1$, they depend on the network topology. These results can be explained by studying the trade-off between a firm's optimal value extraction from the more central nodes and the tendency of the firm to subsidize the more central nodes to influence the consumption of other less central consumers.

Monopoly versus oligopoly pricing. Bloch and Quérou (2013) and Candogan et al. (2012) show that the equilibrium price is given by $\mathbf{p}^{*m} = \frac{\mathbf{a}+\mathbf{c}}{2}$. The price $\mathbf{p}^{*m}$ is independent of any network effect and, in particular, of the position of consumers in the network. Indeed, when a firm sells network goods to a group of consumers, the firm faces two trade-offs in choosing optimal

[24] Proposition 1 shows that there is a unique symmetric equilibrium. In other words, non-symmetric pricing equilibria do not exist here. Relatedly, Proposition E6 in the Online Appendix shows that the pricing equilibrium is unique even with asymmetric firms.

[25] See Definition A1 in Appendix A for a formal definition of Katz-Bonacich centralities.

[26] The superscript m refers to the monopoly pricing.

prices, regardless of whether there is competition or not. On one hand, more central players obtain more network externalities from their neighbors. Hence, the firm has a strong incentive to charge a higher premium to such consumers in order to capture the surplus associated with it. On the other hand, the firm has an incentive to give such consumers larger discounts because they generate large network externalities from their neighbors. This is the logic of internalization of externalities in network games. For a monopoly firm, these two forces exactly cancel each other out, and thus, the firm charges a price of $(\mathbf{a}+\mathbf{c})/2$, which is independent of the network. For an oligopoly firm, the same two economic forces are in place, but they do not exactly cancel each other (see (9)), which results in network-dependent prices.

We now investigate the effects of the network structure and the market structure on equilibrium prices in Section 4 and on firms' profits in Section 5. We impose an additional assumption, as follows:

Assumption 3. $\mathbf{c} = \mathbf{0}$ *and* $\mathbf{a} = a\mathbf{1}_N$*.*

First, we normalize c_i to zero so that, in the analysis below, the mark-up $(p_i^t - c_i)$ is the same as the price. This normalization does not cause any loss of generality. Furthermore, to simplify the notations and without much loss of generality, we impose $a_i = a$ for every i, so that consumers have the same marginal utility for the products. Nonetheless, the network structure is arbitrary, and hence consumers are not necessarily located symmetrically.[27]

Example 1 (Regular networks)**.** *Under Assumptions 1, 2, and 3 (with $a = 1$), the common equilibrium price (mark-up) for regular networks with degree d (i.e., $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$) is given by:*[28]

$$p_i^* = p_j^* = p_{reg}^* := \frac{(1-\beta-\delta d)}{2+(L-3)\beta-2\delta d} = \frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}} \in (0, \frac{1}{2}]. \tag{12}$$

Moreover, each firm's equilibrium profit is equal to:

$$\Pi_{reg}^* := \frac{N(1-p_{reg}^*)p_{reg}^*}{1+(L-1)\beta-\delta d} = \frac{N\left[1+(L-2)\beta-\delta d\right](1-\beta-\delta d)}{\left[1+(L-1)\beta-\delta d\right]\left[2+(L-3)\beta-2\delta d\right]^2}. \tag{13}$$

4 The effects of market and network structure on equilibrium prices

We use the equilibrium characterization in Proposition 1 to study the effects of network topology and market structure on equilibrium prices. We first investigate the effects of the network structure and the strength of the network effects on equilibrium prices while fixing the number of firms L. In our Proposition E4 in Appendix E, we show that increasing the network density $\mathbf{G}$ or the strength of the network effects δ decreases the equilibrium price for all consumers. Since the equilibrium

[27] Under Assumption 3, the only heterogeneity between consumers is their network positions, which is our main focus.

[28] See Appendix B.1 for details.

price vector in Proposition 1 is decreasing in the Katz-Bonacich centrality measures of consumers in the network (see (11)), the result of Proposition E4 is straightforward. Indeed, increasing the network density or the network externalities intensifies the price competition among firms for the central nodes in the network, who, in equilibrium, are compensated with lower prices.

Example 2. *Consider the kite network in Figure 1. Assume that Assumption 3 holds, with* $\mathbf{c} = \mathbf{0}$*,* $\mathbf{a} = a\mathbf{1}_N = \mathbf{1}_N$*, and* $\beta = 0.4$.[29] *The equilibrium prices for different network effects* δ *and for different numbers of firms* L *are easily calculated as follows*

$$\mathbf{p}^*(L=2, \delta=0.17) = \begin{pmatrix} 0.239 \\ 0.271 \\ 0.271 \\ 0.320 \end{pmatrix} \quad \textit{and} \quad \mathbf{p}^*(L=2, \delta=0.27) = \begin{pmatrix} -0.043 \\ 0.035 \\ 0.035 \\ 0.192 \end{pmatrix}$$

The above equations show that when there are two firms and the network effects are not too strong ($\delta = 0.17$)*, the firms charge all consumers a positive price. However, prices decrease with a consumer's connection; for example, consumer 1 (the hub) pays a price of* 0.239*, and consumer 4 (the spoke) pays* 0.320*. Interestingly, when the network effects increase to* $\delta = 0.27$*, the two firms find it profitable to subsidize consumer 1 (i.e., charge consumer 1 a negative price) and tax the other consumers (i.e., charge them a positive price). When we increase the competition from* $L = 2$ *to* $L = 10$*, we obtain:*

$$\mathbf{p}^*(L=10, \delta=0.17) = \begin{pmatrix} 0.034 \\ 0.061 \\ 0.061 \\ 0.091 \end{pmatrix} \quad \textit{and} \quad \mathbf{p}^*(L=10, \delta=0.27) = \begin{pmatrix} -0.034 \\ 0.010 \\ 0.010 \\ 0.066 \end{pmatrix}$$

The pattern is the same but because of fiercer competition, all prices decrease, which is good for consumers but not for consumer 1, whose absolute subsidy decreases when $\delta = 0.27$*. To further understand these results, let us plot the equilibrium prices for node 1 (the blue curve), node 2 (the green curve), and node 4 (the red curve) while varying the number of firms* L *(on the x-axis) and the network effects* δ*. The left panel (Figure 2) shows* $\delta = 0.17$*, the right panel (Figure 3) shows* $\delta = 0.27$*, and the solid black line shows* $\delta = 0$.[30]

In Figures 2 and 3, we see that by fixing δ *and* L*, consistent with Proposition 1, the most central node (node 1 in the kite) is charged the lowest price, whereas the least central node (node 4) is charged the highest price. After comparing the two figures, we find that all the curves indeed shift downward; that is, the price for each node is lower when* δ *increases from* 0.17 *to* 0.27*. In addition, the black curve* ($\delta = 0$) *is higher than all the other colored curves, which correspond to prices with strictly positive values of* δ*.*

Next, we explore the impact of competition L on prices while fixing $\mathbf{G}$. In fact, Example 2 has

[29] The largest eigenvalue λ_1 is about 2.17. Assumption 2 holds when $\delta < (1-\beta)/\lambda_1 \approx 0.276$.

[30] When $\delta = 0$, there is no price difference across different nodes: each node is charged the same price, which is equal to $a\frac{1-\beta}{2+(l-3)\beta}$.

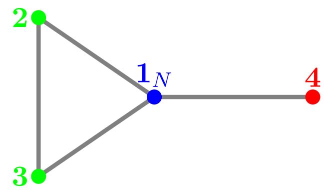

Figure 1: A kite with four nodes

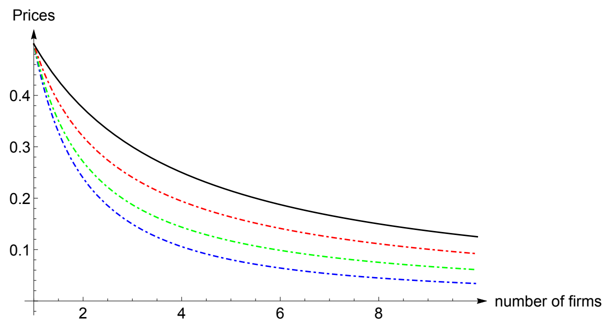

Figure 2: $\delta = 0.17$

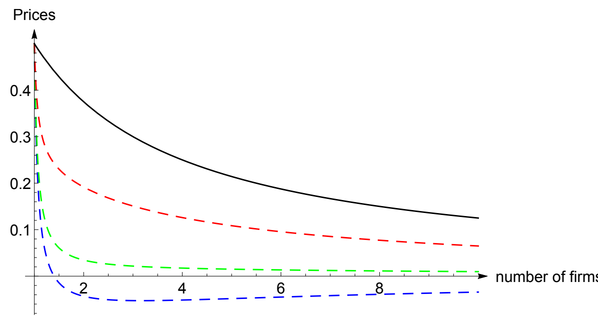

Figure 3: $\delta = 0.27$

already revealed several interesting observations on the effects of competition. As shown in Figure 2, when δ is equal to 0.17, the price for every node decreases with competition. In Figure 3, when δ is equal to 0.27, interestingly, the price for node 1 is not monotone in L.[31] In both figures, the prices eventually converge to zero as L becomes infinite. The following proposition demonstrates that these patterns hold in general settings.

Proposition 2. *Suppose that Assumptions 1, 2, and 3 hold. Let d_i denote the degree of node i.*

(i) If, $\forall i \in \mathcal{N}$, $1 - \beta - \delta d_i > 0$, then $\mathbf{p}^*$ decreases with L for any L.

(ii) Suppose that for some i, $1 - \beta - \delta d_i < 0$. Then p_i^* has a non-monotonic relationship with the number of firms L. In particular, $\partial p_i^*/\partial L < 0$ when L is small, and $\partial p_i^*/\partial L > 0$ for a sufficiently large L.

Consider Example 1 of *regular* networks. We have shown that the common equilibrium price $p^*_{reg} \in (0, \frac{1}{2}]$ is given by (12). Since, in regular networks (where $d_i = d = \lambda_1(\mathbf{G})$ for any i), condition (2) in Assumption 2 is reduced to $1 - \beta - \delta d_i > 0$, for all i, it can be seen that prices always decrease according to the number of firms L (see Appendix B.2 or equation (12)). Proposition 2 demonstrates, however, that the above price movement is not necessarily true in *non-regular* networks. First, consider part (i) of the proposition. The condition, $1 - \beta - \delta d_i > 0, \forall i$, holds when the network effects are very small (i.e., when δ is small). We obtain the standard competition effect:

[31] This result is reminiscent of that of Amir and Lazzati (2011), which showed that the price and the per-firm profits can both increase with the number of firms.

when L, the number of firms, increases, equilibrium prices decrease. In part (ii) of Proposition 2, this is not necessarily true, since for some consumers i, $1-\beta-\delta d_i<0$; that is, the network effects are large enough. Even when competition L increases, firms find it profitable to subsidize the consumers with the highest degrees (i.e., charge them negative prices) because they generate many network effects; that is, they influence many other consumers to consume more of the same product. Indeed, to prove the validity of Proposition 2(ii), we proceed as follows. If L is sufficiently large, then we have the case of perfect competition described in Appendix C, where the equilibrium price of agent i is given by[32]

$$p_i^* = \frac{1-\beta-\delta d_i}{\beta L} + \mathcal{O}(L^{-2}). \tag{14}$$

In (14), we see that, if $1-\beta-\delta d_i<0$ for consumer i, then p_i^* necessarily increases with a L for large L. The intuition regarding this result is as follows. When $1-\beta-\delta d_i<0$ (i.e., both δ and d_i are large), the markup on individual i is negative (i.e., the price is below marginal cost). Therefore, when the firm faces more competition (higher L), it subsidizes individual i less and less because its profit margins from non-central consumers gradually decrease.[33]

Now, let us consider the case when L is small. Note that the equilibrium price p_i^* under a monopoly ($L=1$) is strictly higher than that under an oligopoly ($L\geq 2$) (see (11)) and thus, p_i^* decreases with L for a small L. By combining both cases, we show the results of Proposition 2(ii).

In particular, when $\frac{1-\beta}{\max_j d_i} < \delta < \frac{1-\beta}{\lambda_1(\mathbf{G})}$,[34] the price for at least one consumer, such as i, who has the highest degree, has a non-monotonic relationship with L, which generates an interesting and counter-intuitive relationship between competition and prices: *greater competition (more firms) may increase prices for highly connected consumers if there are already many firms in the market.* Moreover, when L is large, for any node i with $1-\beta-\delta d_i<0$, the price is given by (14), which means that this price (or mark-up in our setting) is negative. In other words, when competition is fierce (i.e., when L is large), firms find it profitable to subsidize central consumers in consuming their products, whereas when there is very little competition, firms charge highly central consumers a positive price. Of course, as shown in Figures 2 and 3, this also depends on δ, the intensity of network externalities. If, as shown in Figure 2, δ is low, then firms always charge all consumers a positive price because they do not find it profitable to subsidize highly central consumers since they do not generate enough positive externalities to their friends. On the contrary, when δ is high enough, as shown in Figure 3, firms subsidize some influential consumers. Moreover, in both cases, when there is insufficient competition (i.e., when L is low), prices decrease as competition increases. When L is very large and δ is high enough, increasing competition raises prices for highly-central consumers because firms subsidize them less and less. Eventually prices converge to zero in the limit. See the price curve of node 1 (with the blue color) shown in Figure 3 in which the price

[32] See equation (C9) in Proposition C2 in Appendix C. Given a real-valued function f, we write $f(L)=\mathcal{O}(L^{-2})$ if $\limsup_{L\to\infty}\left|\frac{f(L)}{L^{-2}}\right|<\infty$.

[33] The *positive* relationship between the equilibrium price and L is only observed when the price (or markup) is *negative*. See the proof of Proposition 2 and that of Proposition C2 in Appendix C.

[34] Here, the upper bound on δ is required by Assumption 2, whereas the lower bound ensures that the condition stated in Proposition 2 (ii) is satisfied for at least one node i. For any connected non-regular network, $\lambda_1(\mathbf{G})<\max_i d_i$, and thus, this range for δ is always not empty.

increases with L only when the firm's markup is negative.

In equilibrium, firms charge prices depending on the positions of the nodes in the network. To evaluate the extent of such network-based price discrimination, we define $Disp(\cdot)$ as the maximal difference in equilibrium prices among consumers in the network.

Definition 1.

$$Disp(L) := \max_{i \neq j} |p_i^*(L) - p_j^*(L)|.$$

We obtain the following result:

Proposition 3. *$Disp(1) = 0$ (monopoly) and $\lim_{L\to\infty} Disp(L) = 0$ (perfect competition). Suppose that $\beta \neq 0$. For any non-regular network,*[35] *there exists an intermediate value $L^* > 1$ such that $Disp(L)$ is maximized.*

Proposition 3 demonstrates that the degree of price dispersion is *hump-shaped* according to the number of firms. When there is only one firm (i.e., a monopoly), there is no dispersion because the firm charges all consumers the same price. When the number of firms is infinite (i.e., perfect competition), all prices converge to zero and, again, there is no dispersion. The maximal dispersion occurs when L has an intermediate value L^*, which generates a non-monotonic relationship between price dispersion and competition. Thus, there is a tradeoff between weak competition (a low L), where firms mainly focus on attracting the "right" consumers by price discrimination, depending on their position in the network, and intense competition (a large L), where firms focus on prices that can steal (in terms of consumption) consumers from other firms. Therefore, when L is greater than 1 but is not too large, competition is not overly intense and thus, firms can price-discriminate their consumers depending on the consumer's position in the network. This is true until L reaches L^*, a value for which $Disp(\cdot)$, the maximal difference in equilibrium prices among consumers in the network, is the maximum. In this case, $Disp(\cdot)$ reaches its peak. Clearly, the higher δ, the network spillover effects, the higher $Disp(\cdot)$ is, since firms would like to subsidize central consumers as they significantly influence their neighbors, and firms would like to tax more the less central agents because they do not have many connections. As L exceeds L^*, $Disp(\cdot)$ decreases because the competition between firms is fiercer and the firms become increasingly preoccupied with inducing consumers to consume more of their products.

Example 2 (cont.) *Consider the kite network with four nodes shown in Figure 1. In Figure 4, we plot $Disp(L)$ for this network using different values of δ. We see that the maximal degree of dispersion may occur when the number of firms is reasonably small. In Figure 4, $L^* \approx 2$ for the dashed line ($\delta = 0.27$), $L^* \approx 3$ for the dot-dashed line ($\delta = 0.17$), and $L^* \approx 4$ for the solid line ($\delta = 0.07$). We also see that the price differences in the network can be significant. In the dashed curve, the maximal dispersion is about 0.25. (Note that the monopoly price is 0.5 as we set $a = 1$ and $c = 0$ in this example. See also Figure 3.)*

When the strength of the network effects is not large, we have the following simple characterization of the price dispersion.

[35] For regular networks, $Disp(L) = 0, \forall L$ as all firms charge the same price to all consumers.

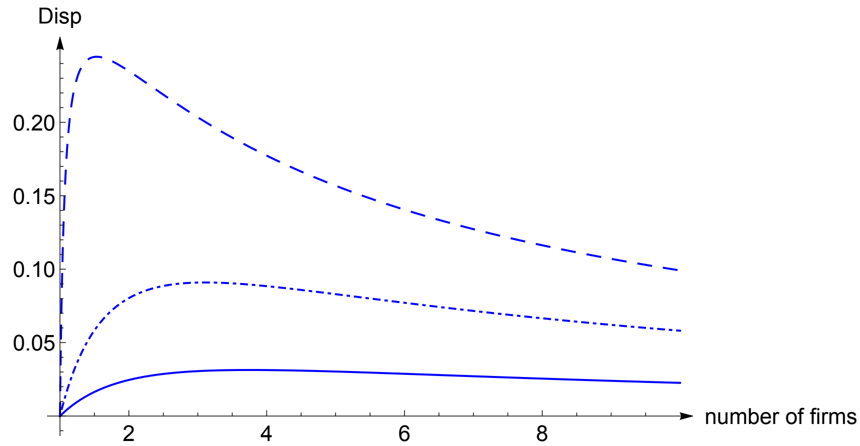

Figure 4: Price dispersion curve (y-axis) as a function of l (x-axis) for the kite network in Figure 1: dashed line ($\delta = 0.27$), dot-dashed line ($\delta = 0.17$) and solid line ($\delta = 0.07$).

Remark 1. *For a small δ,*

$$Disp(L) \approx \delta \frac{(L-1)\beta(d_{max} - d_{min})}{[2 + (L-3)\beta]^2}.$$

The maximal dispersion is about $\frac{(d_{max}-d_{min})}{8(1-\beta)}\delta$, obtained when $L^* = -1 + 2/\beta > 1$.

When δ is small, the Katz-Bonacich centrality is reduced to the simple counting of degrees of nodes in the network. In Remark 1, we show that the maximal dispersion increases with the difference between the maximal degree and the minimal degree, that is, the dispersion of the degrees in the network. In addition, the optimal L^* takes the simple form of $-1+2/\beta$, which decreases with the degree of product differentiation β. In Example 2, $\beta = 0.4$; thus, $L^* = 4$, which is consistent with the solid blue line indicating $\delta = 0.07$ in Figure 4.

Remark 2. *Proposition 3 holds under alternative measures of the variation of equilibrium prices across players. For example, a commonly used index is the* variance *of equilibrium prices:*

$$Var(L) := \sum_{i=1}^{n} |p_i^*(L) - \bar{p}^*(L)|^2,$$

where $\bar{p}^*(L) := \frac{\sum_{i=1}^{n} p_i^*(L)}{n}$ is the average price. It is easily verified that a counterpart of Proposition 3 holds if we replace $Disp(\cdot)$ with $Var(\cdot)$. Furthermore, for a small δ, $Var(L)$ takes the following simpler form:

$$Var(L) \approx \left(\frac{\delta(L-1)\beta}{[2+(L-3)\beta]^2}\right)^2 Var_{dg},$$

where $Var_{dg} = \sum_{i=1}^{n} |p_i - \bar{d}|^2$ with $\bar{d} = \frac{\sum_i d_i}{n}$ is the variance of the degrees in the network. The maximal variance of prices is obtained when $L^* = -1 + 2/\beta > 1$ (similar to Remark 1).

In summary, the results discussed in this section highlight the importance of considering the general market structure. The value of L qualitatively matters in the analysis of both price trend and price dispersion. When the number of firms increases from $L = 1$ to $L = 2$, the equilibrium prices always decrease. A naive conjecture would be that prices always decrease with competition. However, this observation regarding the relationship between prices and competition does not globally extend to the general setting if $L \geq 2$. As shown in Proposition 2, prices can easily increase with competition, especially when the network effect is strong and the number of firms is large. This finding suggests that we need to be cautious when drawing empirical implications for price trends in the presence of network effects.

Moreover, the degree of price dispersion/variance in non-regular networks can be significant, and the market structure that generates the largest degree of dispersion, L^*, although not equal to a monopoly, is not necessarily a duopoly. This finding further validates the role of local network effects in pricing strategies that interact with market competitiveness. Finally, for markets with a large number of competing firms, the structure of the network has little impact on equilibrium prices.

5 Effects of market structure and network topology on profits

In this section, we characterize the effects of network topology $\mathbf{G}$ and the number of firms L (market structure) on firms' profits. As expected, we show that the firm's profit curve, as a function of L, is downward sloping because of the intensified competition. However, increasing the network density does not lead to an overall upward shift of the firm's profit curve. This result is surprising because, other things being equal, a higher network density leads to a higher utility for each consumer since a consumer now benefits from other consumers' consumption. Instead, increasing the network density leads to a *rightward rotation* of the profit curve. More precisely, the equilibrium profits respond to the number of firms more sensitively when the externalities parameter δ is larger.

5.1 Regular networks

We first discuss regular networks (Example 1) and then proceed to describe general network structures. We assume that Assumption 3 holds and $\beta > 0$.

The firms' equilibrium profit is given by (13), which is the product of $(1 - p^*_{reg})p^*_{reg}$ and $\frac{N}{1+(L-1)\beta-\delta d}$. We study the impact of the number of firms L on the firm's profit. First, note that the price effect of L is negative:

$$\frac{\partial\{(1 - p^*_{reg})p^*_{reg}\}}{\partial L} = (1 - 2p^*_{reg})\frac{\partial p^*_{reg}}{\partial L} < 0$$

as p^*_{reg} decreases with L and p^*_{reg} lies between 0 and 1/2 (see (12)). Moreover, the social multiplier $\frac{1}{1+(L-1)\beta-\delta d}$ is weaker when L is larger. Since these two effects move in the same direction, we have:[36]

$$\frac{\partial \Pi^*_{reg}}{\partial L} < 0. \tag{15}$$

Next, we characterize the impact of δ on the firm's profit. The price effect of δ is also negative:

$$\frac{\partial \{(1 - p^*_{reg}) \times p^*_{reg}\}}{\partial \delta} = (1 - 2p^*_{reg}) \frac{\partial p^*_{reg}}{\partial \delta} < 0, \tag{16}$$

but the social multiplier effect is positive: $\frac{1}{1+(L-1)\beta-\delta d}$, and is stronger with a larger δ. Therefore, the net effect of δ on the firm's profit is determined by these two opposite effects. The following proposition characterizes the exact necessary and sufficient conditions under which one force dominates the other and the net effect can be determined.

Proposition 4. *Assume that Assumptions 1, 2, and 3 hold and consider a regular network with degree d. The sign of* $\frac{\partial \Pi^*_{reg}}{\partial \delta}$ *is positive when L is sufficiently small and negative when L is sufficiently large. More precisely, there exists a threshold* $\bar{L}$ *(depending on* β *and* δ*) such that*

$$\frac{\partial \Pi^*_{reg}}{\partial \delta} > (<) 0 \text{ if and only if } L < (>) \bar{L} := \chi\left(\frac{\beta}{1 - \delta d}\right),$$

where $\chi(.)$ *is the function defined in Lemma A3 in Appendix A.*[37]

Proposition 4 shows that the impact of δ on Π^*_{reg} is non-monotonic. More precisely, when the number of firms L is sufficiently small, increasing δ increases Π^*_{reg}; and when the number of firms L is sufficiently large, increasing δ decreases Π^*_{reg}. Remember that Π^*_{reg}, the equilibrium profit for regular networks, is the product of $(1 - p^*_{reg})p^*_{reg}$ (the price effect) and $\frac{N}{1+(L-1)\beta-\delta d}$ (the social multiplier effect). When L is relatively small, the equilibrium prices p^*_{reg} are close to the monopoly price 1/2 and, accordingly, $1 - 2p^*_{reg}$ is close to zero. Consequently, the price effect is negligible compared to the positive social multiplier effect. On the contrary, when L is large enough, the competition is so intense that the equilibrium price (or mark-up) is close to zero, and the negative price effect of δ dominates the positive social multiplier effect. Consequently, the firm's profit decreases with δ. Let us discuss some immediate consequences of Proposition 4. First, an improvement in the network technology, caused, for example, by an increase in δ, which makes peer effects stronger, does not always generate a higher firm profit. In fact, whether firms can profit from such a technology improvement or not critically depends on the degree of competition that they face. At the extreme end, a monopoly firm can extract some of the additional network benefits enjoyed by the consumers and, hence, earns a higher profit. The extent of such value extraction by firms is jeopardized by the intensified price competition. In fact, when competition is too strong, the firm's profit is reduced. This finding suggests potential incentives for firms not to improve their network technology. Second, when δ increases marginally, the profit curve, as a function of L, *rotates* instead of *shifts* globally (Figure 5). Furthermore, Proposition 4 explicitly identifies the critical $\bar{L} := \chi\left(\frac{\beta}{1-\delta d}\right)$ at which the profit curve rotates, where $\chi(.)$ is the function

[36] It is easy to show that Π^*_{reg} converges to zero as L approaches infinity.

[37] $\chi(z)$ is continuously differentiable and strictly decreases with z. See the proof of Lemma A3.

defined in Lemma A3 in Appendix A. $\chi(\cdot)$ is continuously differentiable and strictly decreasing. The graph of χ is plotted in Figure A1. Interestingly, the threshold $\bar{L}$ decreases with δ and β, as $\chi(\cdot)$ is decreasing.

Example 1 (cont.) *In Figure 5, we plot Π^*_{reg} as a function of L for a regular network with $d = 2$ (for example, a ring network). We set $\delta = 0.2$ for the blue curve and $\delta = 0.3$ for the red curve. We see that the red curve is a rightward rotation of the blue curve, which means that under low competition (i.e., $L \leq 3.5$), the firm's profit is higher when δ is higher (i.e., when there are more network externalities), whereas, under fierce competition (i.e., when $L > 3.5$), the opposite is true. In other words, the equilibrium profits respond to the number of firms more sensitively when the externalities parameter δ is larger.*

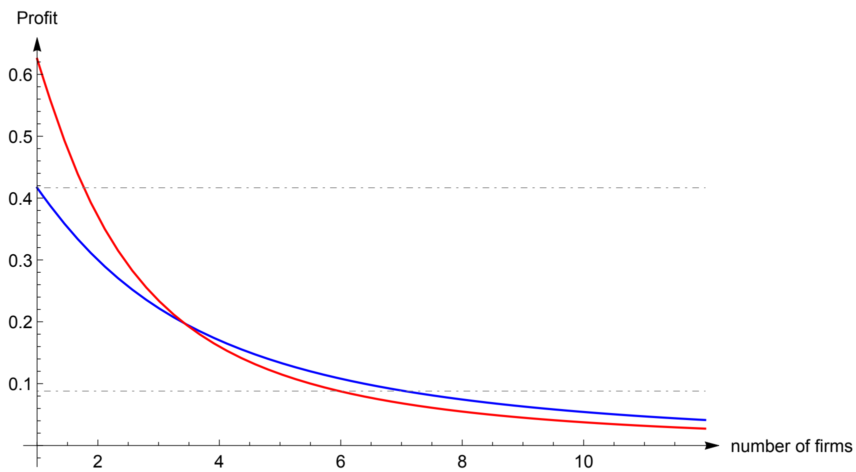

Figure 5: The Π^*_{reg} curve, as a function of L, for small δ values (blue curve; $\delta = 0.2$) and large δ values (red curve; $\delta = 0.3$)

Proposition 4 has several additional implications.

Corollary 2. *Suppose that Assumption 3 holds. Consider a regular network with degree d as a function of $\delta \in [0, \frac{1-\beta}{d})$.*[38] *Then,*

$$\Pi^*_{reg} \textit{ is } \begin{cases} \textit{monotonically increasing in } \delta, & \textit{if} \quad L = 1, \\ \textit{an inverted U-shaped curve of } \delta, & \textit{if} \quad 1 < L < \chi(\beta), \\ \textit{monotonically decreasing in } \delta, & \textit{if} \quad L \geq \chi(\beta). \end{cases}$$

This corollary shows that Π^*_{reg} has a non-monotonic relationship with δ depending on the market competition L. This is explained by the two main economic forces. On one hand, increasing δ decreases the equilibrium price p^*_{reg} because the firms compete more intensively in the pricing game (i.e., the demands are more elastic with a larger δ), which leads to uniformly lower prices

[38]The upper bound on δ makes sure that Assumption 2 holds.

(*the price effect*). This negatively affects profits. On the other hand, a rise in δ increases the social multiplier $\frac{1}{1+(L-1)\beta-\delta d}$ for a larger L, which positively affects profits (*the social multiplier effect*). In particular, when there is a monopolist ($L = 1$), increasing δ always increases profits, whereas when L is large enough, increasing δ always decreases profits. For intermediary values of L, there is an inverted U-shaped curve between δ and Π^*_{reg}. The following example illustrates this result:

Example 1 (cont.) *In Figure 6, we plot the curve of Π^*_{reg} as a function of δ under different market structures. For $L = 1$ (a monopoly; depicted by the blue dashed curve), the firm's profit always increases with δ. For $L = 2$ (a duopoly; depicted by the black dashed curve), the firm's profit first increases and then decreases with δ.*[39] *For $L = 5$ (an oligopoly; depicted by the red dashed curve), the profit always decreases with δ.*

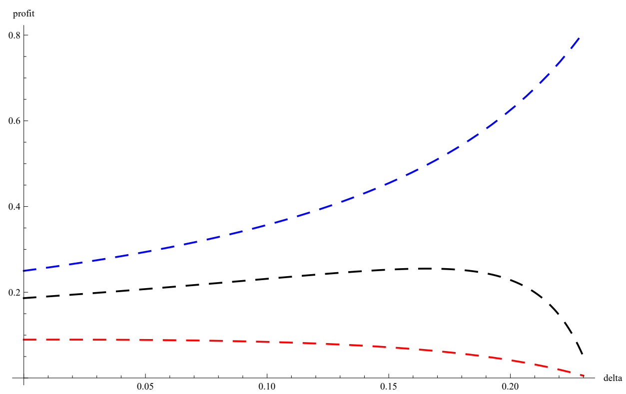

Figure 6: The profit Π^*_{reg} curve as a function of δ under different market structures: blue $L = 1$; black $L = 2$; red $L = 5$

We now study the effects of varying the degree d. Since $d\delta$ is a sufficient statistic of Π^*_{reg}, the next result immediately follows Proposition 4.

Corollary 3. *Consider a regular network with degree d. Then, the impact of d on firms' profit has the same sign as the impact of δ on firms' profit; that is, $sign\{\partial\Pi^*_{reg}/\partial d\} = sign\{\partial\Pi^*_{reg}/\partial\delta\}$.*

This corollary shows that increasing the degree d of a regular network (which then becomes a denser network) does not always increase firms' profits. In fact, increasing d leads to a rightward rotation similar to that observed when increasing δ. As a result, we obtain the same result as shown in Figure 5 if we fix $\delta = 0.2$ but increase d from 2 to 3.[40]

5.2 From regular networks to general networks: A technical contribution

The analysis of regular networks is particularly simple, as the equilibrium prices do not depend on the consumers' network position, and firms' profits and consumer surpluses are explicit functions

[39] $\partial\Pi^*_{reg}/\partial\delta$ vanishes only when $L = \chi(\beta/(1-\delta d))$, or equivalently when $\delta = \frac{1-\chi^{-1}(L)/\beta}{d}$.

[40] Note that, in Figure 6, d is fixed at 2, but δ increases from 0.2 to 0.3.

of the consumer's (common) degree instead of the entire vector of the Katz-Bonacich centralities. Here, we provide a new technique to reduce the analysis of general irregular networks to a series of regular graphs. A key aspect of this new technique is the *spectral decomposition* of the underlying network adjacency matrix, which is referred to as *principal component analysis* in Galeotti, Golub, and Goyal (2020). To illustrate this new technique, let us analyze the firm's profit. Note that the analysis of the firm's profit is similar to the analysis of the consumer surplus or the total welfare.

Let us demonstrate that the firms' equilibrium profit from any non-regular network $\mathbf{G}$ can be decomposed into a weighted sum of several terms, where each term corresponds to a regular network in which the degree d is replaced by an eigenvalue of $\mathbf{G}$, and where the positive weight is equal to the square of the inner product of the vector $a\mathbf{1}/\sqrt{N}$ and the associated eigenvector of $\mathbf{G}$. Mathematically, under Assumption 3, we establish the following identity for the equilibrium firm's profit:

$$\Pi^*(\mathbf{G};\beta,\delta,L) := \sum_{\lambda_i \in Spec(\mathbf{G})} \Pi^*_{reg}(\lambda_i,\beta,\delta,L) \times (\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}}\rangle)^2, \tag{17}$$

where $Spec(\mathbf{G}) = \{\lambda_1, \cdots, \lambda_N\}$ is the set of eigenvalues of $\mathbf{G}$, and $\mathbf{u}_i$ is the corresponding normalized eigenvector[41] associated with λ_i. These eigenvalues are monotonically ordered from λ_1, the largest eigenvalue, to λ_N, the lowest eigenvalue. The coefficient $(\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}}\rangle)^2$ is positive, unless $\mathbf{u}_i$ is orthogonal to $\mathbf{1}_N$.[42]

The identity in (17) follows a key lemma (Lemma 2), which is an application of the spectral decomposition theorem of the network matrix $\mathbf{G}$.

Lemma 2 (Localization lemma)**.** *Suppose that $f(z)$ is an analytical function on an interval that contains $Spec(\mathbf{G})$, so that $f(\mathbf{G})$ is well-defined.*[43] *Then, for any $\mathbf{v} \in \mathbf{R}^n$, we have the following equation*

$$\mathbf{v}' f(\mathbf{G})\mathbf{v} = \sum_{\lambda_i \in Spec(\mathbf{G})} f(\lambda_i)(\mathbf{v}'\mathbf{u}_i)^2.$$

In particular, if $f(z)$ is positive (nonnegative) at any $\lambda_i \in Spec(\mathbf{G})$, then $\mathbf{v}' f(\mathbf{G})\mathbf{v} > (\geq) 0$ for any $\mathbf{v} \in \mathbf{R}^n \backslash \{\mathbf{0}\}$.

Let us show how we use this Lemma in our framework. In Proposition D3 in Appendix D, under Assumption 3, we define the firm's equilibrium profit as $\mathbf{\Pi}^*(\mathbf{G};\beta,\delta,L) := \langle (a\mathbf{1}_N, \phi^{PT}(\mathbf{G})a\mathbf{1}_N\rangle$, where $\phi^{PT}(z) := \phi^{PT}(z;\beta,\delta,L) = \frac{(1+(L-2)\beta-\delta z)(1-\beta-\delta z)}{(1+(L-1)\beta-\delta z)(2+(L-3)\beta-2\delta z)^2}$ (see (D11)). In Lemma 2, the $f(.)$ function is given by $\phi^{PT}(.)$ and $\mathbf{v} = a\mathbf{1}_N$, which implies that $\mathbf{\Pi}^*(\mathbf{G};\beta,\delta,L) := \mathbf{v}' f(\mathbf{G})\mathbf{v}$. Lemma 2

[41] In other words, each $\mathbf{u}_i$ is of unit length, and $\mathbf{G}\mathbf{u}_i = \lambda_i \mathbf{u}_i$.

[42] An eigenvalue λ_i that satisfies $\langle \mathbf{u}_i, \mathbf{1}\rangle \neq 0$ is called a main eigenvalue of the network. The set of main eigenvalues is called the "main part of the spectrum". See Cvetković (1970).

[43] Since f is analytical, it has an infinite-series representation: $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for some c_k. We define $f(\mathbf{G})$ as $\sum_{k=0}^{\infty} c_k \mathbf{G}^k$, which, under the stated condition on f, is convergent. Lemma 2 is very general as it can applied to broad specifications of f, such as polynomials, or rational functions, as listed in (D11) for welfare analysis in this paper, and possibly many others.

implies that:

$$\mathbf{\Pi}^*(\mathbf{G};\beta,\delta,L) := \mathbf{v}'f(\mathbf{G})\mathbf{v} = \sum_{\lambda_i\in Spec(\mathbf{G})} f(\lambda_i)(\mathbf{v}'\mathbf{u}_i)^2 = \sum_{\lambda_i\in Spec(\mathbf{G})} \phi^{PT}(\lambda_i)(a\mathbf{1}'_N\mathbf{u}_i)^2.$$

Consider a regular network $\mathbf{G}$ with degree d. When $z = d$, it is easily verified (see (13)) that $\phi^{PT}(d) = \frac{1}{N}\Pi^*_{reg}(d;\beta,\delta,L)$. Recall that for a regular network, we have $\lambda_1 = d$ and $\mathbf{u}_1 = \mathbf{1}_N/\sqrt{N}$, and only one term in the summation in (17) is non-zero under Assumption 3. The reason is that all the eigenvectors $\mathbf{u}_i$ for $i \neq 1$ are orthogonal to $\mathbf{u}_1$; thus, the coefficient $(\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}}\rangle)^2 = 0$.[44] Therefore, $\sum_{\lambda_i\in Spec(\mathbf{G})} \phi^{PT}(\lambda_i)(a\mathbf{1}'_N\mathbf{u}_i)^2 = \sum_{\lambda_i\in Spec(\mathbf{G})} \Pi^*_{reg}(\lambda_i,\beta,\delta,L) \times (\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{N}}\rangle)^2$. This demonstrates (17).[45]

This identity greatly simplifies our analysis of the comparative statics results of the profit in a general network structure, as we can exploit the results in the case of regular networks, which are simpler and are fully studied in Appendix B.

The results for regular networks in the previous subsection qualitatively extend to settings with general network structures.[46,47]

Proposition 5. *For any network structure* $\mathbf{G}$*, let* $Spec(\mathbf{G}) = \{\lambda_1, \cdots, \lambda_N\}$ *denote the set of eigenvalues of* $\mathbf{G}$*.*

(i) The firm's profit Π^* *decreases with* L*, and it converges to zero as* $L \to \infty$*.*

(ii) Increasing δ *leads to higher (lower) profits when the market is sufficiently concentrated (competitive). More precisely,*

(a) $\partial\Pi^*/\partial\delta > 0$ *if* $L < \chi(\beta/(1-\delta\lambda_i))$ *for any* $\lambda_i \in Spec(\mathbf{G})$*,*[48]

(b) $\partial\Pi^*/\partial\delta < 0$ *if* $L > \chi(\beta/(1-\delta\lambda_i))$ *for any* $\lambda_i \in Spec(\mathbf{G})$*.*[49]

Let us illustrate how our new technique can be applied to prove Proposition 5. Remember that the firms' equilibrium profit can be written as shown in (17), where Π^*_{reg} is given by (13).

1. First, note that the coefficients $(\langle \mathbf{u}_i, \frac{a\mathbf{1}}{\sqrt{n}}\rangle)^2$ are nonnegative. Using (15), we can first show that $\partial\Pi^*_{reg}(\lambda_i,\beta,\delta,L)/\partial L < 0$ for any $\lambda_i \in Spec(\mathbf{G})$. Therefore, $\partial\Pi^*(\mathbf{G};\beta,\delta,L)/\partial L < 0$, which proves Proposition 5 (i).

[44] For a non-regular network, there are usually more non-zero summation terms in (17), as multiple coefficients can be positive.

[45] This identity should be viewed as a useful mathematical result rather than an economic identity, since, when the eigenvalue λ_i is negative (or not integer-valued), it does not make much sense to say that $\Pi^*_{reg}(\lambda_i,\beta,\delta,L)$ corresponds to a regular graph with degree λ_i.

[46] In Appendix D, we perform a similar analysis of the consumer surplus and total welfare.

[47] In Proposition E5 in Appendix E, we also investigate the impact of the network density on equilibrium profits.

[48] Equivalently, $L < \min_{\lambda_i\in Spec(\mathbf{G})} \chi(\beta/(1-\delta\lambda_i)) = \chi(\beta/(1-\delta\lambda_1))$ as $\chi(\cdot)$ is decreasing.

[49] Equivalently, $L > \max_{\lambda_i\in Spec(\mathbf{G})} \chi(\beta/(1-\delta\lambda_i)) = \chi(\beta/(1-\delta\lambda_N))$ as $\chi(\cdot)$ is decreasing, where λ_N is the smallest eigenvalue.

2. Second, similar to Proposition 4, we can prove that $\partial\Pi^*_{reg}(\lambda_i, \beta, \delta, L)/\partial\delta > (<)0$ if and only if $L < (>)\chi(\beta/(1 - d\lambda_i))$. Therefore, suppose that $L < \chi(\beta/(1 - \delta\lambda_i))$ for any $\lambda_i \in Spec(\mathbf{G})$. Then, we have $\partial\Pi^*(\mathbf{G}; \beta, \delta, L)/\partial\delta > 0$; this corresponds to case (a) in Proposition 5 (ii). Case (b) can be shown in a similar way.

The results of Proposition 5 are similar to those derived under regular networks, since they are based on the trade-off between the *interaction of the price effects of competition* and the *market-expansion social multiplier effects*. Indeed, Proposition 5 (i) is the counterpart of equation (15) and shows that competition drives down firms' profits. Proposition 5 (ii) generalizes Proposition 4 by showing that increasing δ leads to the rotation of the profit curve. In regular networks, these two effects take a simpler form due to the symmetry of the degree. The proofs are more involved for non-regular network structures, as both effects operate in the space of matrices due to the heterogeneity of the network positions of the nodes. Essentially, by using this new technique, we reduce the complex problem of a general network structure into a series of easy-to-solve sub-problems in regular networks.

As stated in Section 1.1, the techniques that we used (in particular, Lemma 2) complement those used recently by Galeotti, Golub, and Goyal (2020), who also used the spectral decomposition theorem to simplify the optimal targeted interventions in networks. In this study, we use them in a very different context to solve for comparative statics exercises of various welfare measures in oligopoly markets with network effects. We now discuss how Lemma 2 can be applied in the setting used by Galeotti, Golub, and Goyal (2020), who, in a special case, analyzed the following optimal targeting intervention problem:

$$\max_{\mathbf{a}} W(\mathbf{a}) := \mathbf{a}^T[\mathbf{I} - \delta\mathbf{G}]^{-2}\mathbf{a}, \quad s.t. \quad ||\mathbf{a} - \hat{\mathbf{a}}||^2 \leq C. \tag{18}$$

In the above equation, $W(\mathbf{a})$, the sum of the squares of the equilibrium action with the marginal utility vector $\mathbf{a}$,[50] measures the aggregate welfare; and $\hat{\mathbf{a}}$ is the initial marginal utility vector, and C, the budget of the planner. Applying Lemma 2 to $f(z) = 1/(1 - \delta z)^2$, we can reformulate the welfare $W(\mathbf{a})$ in (18) as:

$$W(\mathbf{a}) = \sum_{\lambda_i \in Spec(\mathbf{G})} \frac{1}{(1 - \delta\lambda_i)^2}(\mathbf{a}'\mathbf{u}_i)^2. \tag{19}$$

After changing the variables, $\tilde{a}_i = \mathbf{a}'\mathbf{u}_i, \forall i$, the objective function is further simplified to:[51]

$$W(\tilde{\mathbf{a}}) = W(\tilde{a}_1, \cdots, \tilde{a}_n) = \sum_{\lambda_i \in Spec(\mathbf{G})} \frac{1}{(1 - \delta\lambda_i)^2}\tilde{a}_i^2. \tag{20}$$

The neat form in (20) makes the optimal intervention much easier to solve and simpler to interpret. Galeotti, Golub, and Goyal (2020) also obtained (20), but they used a different technique. Note that the new choice variable $\tilde{\mathbf{a}}$ is not the original marginal vector $\mathbf{a}$ but its projection on

[50] In Ballester et al. (2006), the equilibrium action $\mathbf{x}^* = [\mathbf{I} - \delta\mathbf{G}]^{-1}\mathbf{a}$, and the aggregate welfare is $\frac{1}{2}||\mathbf{x}^T||^2 = \frac{1}{2}W(\mathbf{a})$.
[51] The budget constraint needs to be adjusted accordingly.

eigenvectors. Equation (20) shows that the coordinates in the eigenspace, $\tilde{\mathbf{a}}$, provide a more natural perspective in examining the optimal network targeting intervention problem. We expect our new technique based on Lemma 2 to be applicable to broader settings that involve networks.

Remark 3. *In this section, we mainly focused on the comparative statics of firms' profits with respect to L and δ. Our method is very general and can be applied to other welfare outcomes. As shown in Appendix D, we use similar techniques to study the comparative statics of the consumer surplus and total welfare. In particular, the decomposition of the consumer's surplus can be found in* (D14) *and that of total welfare in* (D15).

6 Extensions

We extend our baseline in four directions. First, we allow for heterogeneous firms. Second, we introduce partial product compatibility. Third, we consider an alternative setting in which firms compete in quantities instead of prices. Finally, we allow for the free entry of firms in the market and endogenize the network structure. In each extension, we derive the equilibrium prices and draw comparisons with the baseline model.

6.1 Heterogeneous firms and asymmetric products

We consider an extension in which products are asymmetric because of either their quality differences or their cost differences. To do so, we first relax the symmetric condition in Assumption 1, which imposes that for each consumer i, $a_i^l = a_i^s = a_i > c_i \geq 0$ for any l and s. Let a_i^l denote the marginal utility of consumer $i \in \mathcal{N}$ consuming product $l \in \mathcal{L}$, and let c_i^l denote the marginal cost of firm $l \in \mathcal{L}$ for serving consumer i. Let us define $\mathbf{a}^l = (a_1^l, \cdots, a_N^l)'$ and $\mathbf{c}^l = (c_1^l, \cdots, c_N^l)'$, $\bar{\mathbf{a}} = \frac{1}{L}\sum_{l \in \mathcal{L}} \mathbf{a}^l$, $\bar{\mathbf{c}} = \frac{1}{L}\sum_{l \in \mathcal{L}} \mathbf{c}^l$.

Proposition E6 in Appendix E.3 generalizes Proposition 1 by determining the equilibrium price in this extended model. The key matrices are $\mathbf{H}$ and $\mathbf{D}$, which are given by:

$$\begin{aligned} \mathbf{H} &= \frac{1}{2}\mathbf{I}_N - \frac{(L-1)\beta}{2}[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1} \text{ and} \\ \mathbf{D} &= \frac{1}{2}\mathbf{I}_N + \frac{\beta}{2}[(2+(2L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}. \end{aligned} \tag{21}$$

The result shows that the equilibrium price in (E16) can be decomposed into three additive components: the marginal costs $\mathbf{c}^l$ of firm l; the firm-independent average mark-ups $\mathbf{H}(\bar{\mathbf{a}} - \bar{\mathbf{c}})$, which depend on the average qualities $\bar{\mathbf{a}}$ and the average costs $\bar{\mathbf{c}}$; and the firm-specific adjustment term $\mathbf{D}((\mathbf{a}^l - \bar{\mathbf{a}}) - (\mathbf{c}^l - \bar{\mathbf{c}}))$, which reflects the asymmetry of each firm.[52] Clearly, when firms are homogeneous and products are symmetric (Assumption 1), we return to Proposition 1, since the firm-specific adjustment term $\mathbf{D}((\mathbf{a}^l - \bar{\mathbf{a}}) - (\mathbf{c}^l - \bar{\mathbf{c}}))$ vanishes.

[52]Note that the average of the third term in the prices over all firms must be zero, as the averages of $(\mathbf{a}^l - \bar{\mathbf{a}})$ and $(\mathbf{c}^l - \bar{\mathbf{c}})$ are zero.

We next discuss the implications of the network structure for the shaping of the equilibrium prices across firms and consumers in this extended model. First, we consider two firms s and t with possible asymmetric qualities and/or costs. Let us define $\Delta^{st} = (\mathbf{a}^s - \mathbf{a}^t) - (\mathbf{c}^s - \mathbf{c}^t)$, which measures the net differences between two products. Using (E16), we obtain:

$$(\mathbf{p}^{s*} - \mathbf{c}^s) - (\mathbf{p}^{t*} - \mathbf{c}^t) = \mathbf{D}\Delta^{st}.$$

Therefore, the difference in markups between firms s and t is linear in Δ^{st} and is multiplied by the matrix $\mathbf{D}$, which measures how the quality and cost differences of products between the two firms are mapped into differences in equilibrium prices in the pass-through matrix. Using (21), we write $\mathbf{D}\Delta^{st}$ as:

$$\mathbf{D}\Delta^{st} = \frac{1}{2}\Delta^{st} + \frac{\beta}{2((2+(2L-3)\beta))}\mathbf{b}(\mathbf{G}, \frac{2\delta}{(2+(2L-3)\beta)}, \Delta^{st}),$$

where the second term is proportional to the Katz-Bonacich centralities. In other words, the price differences between two asymmetric firms vary across consumers, which reflects their distinct network positions.

Taking averages over $l \in \mathcal{L}$ in (E16) yields the following expression for the average markups:

$$\bar{\mathbf{p}}^* - \bar{\mathbf{c}} = \frac{\sum_{l\in\mathcal{L}} \mathbf{p}^{l*}}{L} - \bar{\mathbf{c}} = \mathbf{H}(\bar{\mathbf{a}} - \bar{\mathbf{c}}),$$

which is linear in $(\bar{\mathbf{a}} - \bar{\mathbf{c}})$ with the sensitivity matrix $\mathbf{H}$. Furthermore, we can rewrite the average markups using the Katz-Bonacich centralities:

$$\bar{\mathbf{p}}^* - \bar{\mathbf{c}} = \mathbf{H}(\bar{\mathbf{a}} - \bar{\mathbf{c}}) = \frac{1}{2}(\bar{\mathbf{a}} - \bar{\mathbf{c}}) - \frac{(L-1)\beta}{2((2+(L-3)\beta))}\mathbf{b}(\mathbf{G}, \frac{2\delta}{(2+(L-3)\beta)}, (\bar{\mathbf{a}} - \bar{\mathbf{c}})),$$

which can be reduced to (11) under symmetric products. Thus, the analysis of average markups is exactly parallel to that in the baseline model with symmetric firms.

In summary, in asymmetric firms, we observe price heterogeneity across both consumers and firms. As illustrated above, matrices $\mathbf{H}$ and $\mathbf{D}$ play critical roles in shaping price dispersions across consumers and firms.[53]

[53] With the explicit equilibrium prices in Proposition E6, we could conduct various comparative statics, which boils down to the comparative statics of $\mathbf{H}$ and $\mathbf{D}$ with respect to the appropriate parameters. For brevity, we skip these straightforward exercises.

6.2 Product compatibility

We also extend our model by enriching the utility specification in (1) to incorporate the additional network benefit due to partial product compatibility.[54] We obtain:

$$\tilde{u}_i(\mathbf{x}_i, \mathbf{x}_{-i}) := \underbrace{\left(\sum_{l=1}^{L} a_i^l x_i^l - \frac{1}{2}\sum_{l=1}^{L}(x_i^l)^2 - \frac{1}{2}\sum_{l=1}^{L}\sum_{s\neq l}\beta x_i^s x_i^l\right)}_{:=v_i(\mathbf{x}_i)} + \underbrace{\delta\left(\sum_{l=1}^{L}\left(\sum_{j=1}^{N} g_{ij} x_i^l x_j^l\right)\right)}_{:=\eta_i(\mathbf{x}_i,\mathbf{x}_{-i})} + \underbrace{\mu\left(\sum_{1\leq s\neq t\leq L}\left(\sum_{j=1}^{N} g_{ij} x_i^s x_j^t\right)\right)}_{:=\tau_i(\mathbf{x}_i,\mathbf{x}_{-i})}. \tag{22}$$

Compared with (1), we have a third term where parameter μ measures the degree of compatibility among products. Indeed, when $\mu = 0$, we return to the baseline model without compatibility, where $\tau_i \equiv 0$. When $\mu = \delta$, the local network benefits enjoyed by consumers operate at the market level and not at the firm level, since the total network benefit enjoyed by consumer i is equal to $\eta_i + \tau_i = \delta \sum_j g_{ij} \left(\sum_{s\in\mathcal{L}} x_i^s\right)\left(\sum_{t\in\mathcal{L}} x_j^t\right)$, which depends on the aggregate consumptions, $\sum_{s\in\mathcal{L}} x_i^s$ and $\sum_{t\in\mathcal{L}} x_j^t$. We assume $0 \leq \mu \leq \delta$ to accommodate any degree of partial compatibility. Observe that the issue of compatibility is relevant only when there is some competition, i.e., when $L \geq 2$.

The detailed analysis of this extension to product compatibility is shown in Appendix E.4. In this section, we emphasize that the counterpart of Lemma 1 for demand characterizations and of Corollary 1 for price sensitivities hold virtually when we replace the market-expansion social multiplier $\mathbf{M}^+$ and the business-stealing social multiplier $\mathbf{M}^-$ defined in (3) by their counterparts, $\tilde{\mathbf{M}}^+$ and $\tilde{\mathbf{M}}^-$, respectively:

$$\tilde{\mathbf{M}}^+ = [(1+(L-1)\beta)\mathbf{I}_n - (\delta + (L-1)\mu)\mathbf{G}]^{-1} \text{ and } \tilde{\mathbf{M}}^- = [(1-\beta)\mathbf{I}_n - (\delta - \mu)\mathbf{G}]^{-1}. \tag{23}$$

Proposition E7 in Appendix E.4 determines the equilibrium price for each firm as well as the equilibrium profit and consumer surplus. When $\mu = 0$, we obtain Proposition 1. The effects of μ can be understood by inspecting its effects on the two main matrices, $\tilde{\mathbf{M}}^+$ and $\tilde{\mathbf{M}}^-$, since they fully determine the firms' incentives to undercut competitors and consumers' consumptions. Critically, improvement in product compatibility amplifies the market expansion social multiplier $\tilde{\mathbf{M}}^+$ but dampens the business-stealing social multiplier $\tilde{\mathbf{M}}^-$. Formally, $\partial\tilde{\mathbf{M}}^+/\partial\mu = (L-1)\tilde{\mathbf{M}}^+\mathbf{G}\tilde{\mathbf{M}}^+ \succeq \mathbf{0}$ and $\partial\tilde{\mathbf{M}}^-/\partial\mu = -\tilde{\mathbf{M}}^-\mathbf{G}\tilde{\mathbf{M}}^- \preceq \mathbf{0}$.

To illustrate our results, consider the regular network described in Example 1.[55] We analyze the effects of μ on prices and briefly discuss its effect on firms' profits and consumer surplus.

Example 3. *In a regular network with degree d, assume that $a_i = 1, c_i = 0, \forall i \in \mathcal{N}$. Based on*

[54] A similar analysis can be found in Amir and Lazzati (2011) but without an explicit network analysis.

[55] The general analysis can be done using the techniques developed in this study, but is beyond the scope of this study.

Proposition E7, the equilibrium price with partial compatibility is equal to:[56]

$$\tilde{p}_i^* = \tilde{p}_j^* = \tilde{p}_{reg}^* = \frac{1-(\beta-\mu d)-\delta d}{2+(L-3)(\beta-\mu d)-2\delta d}. \tag{24}$$

Direct calculation showed that $\tilde{p}_{reg}^*$ increases with μ.[57] *Indeed, by increasing μ, which is the degree of product compatibility, the price competition among firms is weaker and thus, leads to higher equilibrium prices.*

Furthermore, each consumer's equilibrium consumption of any product is $\tilde{x}_{reg}^* = \frac{(1-\tilde{p}_{reg}^*)}{1+(L-1)(\beta-\mu d)-\delta d}$.[58] *Thus, the total consumer surplus is equal to:*

$$CS_{reg}^* := N\frac{L(1+\beta(L-1))}{2}(\tilde{x}_{reg}^*)^2 = N\frac{L(1+\beta(L-1))}{2}\left[\frac{(1-\tilde{p}_{reg}^*)}{1+(L-1)(\beta-\mu d)-\delta d}\right]^2, \tag{25}$$

which is quadratic in $\tilde{x}_{reg}^*$.

Under a duopoly ($L=2$), we can show that

$$sign\left\{\frac{\partial CS_{reg}^*}{\partial \mu}\right\} = sign\left\{\frac{\partial \tilde{x}_{reg}^*}{\partial \mu}\right\} = sign\{1-\delta d-2(\beta-\mu d)\}.$$

In other words, CS_{reg}^* can either increase with μ or decrease with μ. For example, we set $d=2, \delta = 0.2$ and $\mu = 0.1$ and vary the product differentiation parameter β. For any $\beta \in (0, 0.5)$, $\frac{\partial CS_{reg}^*}{\partial \mu}$ is positive; and for any $\beta \in (0.5, 0.8)$, $\frac{\partial CS_{reg}^*}{\partial \mu}$ is negative.

To see the implication of this, we observe that increasing the product compatibility μ affects the consumer surplus, or equivalently, the consumption $\tilde{x}_{reg}^*$ through two channels. The first channel is through the obvious additional network benefit due to compatibility, which enhances the demand. The term $1/(1+(L-1)(\beta-\mu d)-\delta d)$ can be interpreted as the market expansion social multiplier, and it clearly increases with μ. The second channel is through a higher price due to softened competition (recall that $\tilde{p}_{reg}^*$ increases with μ). When β is small, the intensity of product competition is weak, and the first effect dominates the second effect so that the consumers are better off with improved product compatibility. When β is sufficiently large, the second price effect effect dominates, and the consumers are actually worse off with improved product compatibility.[59] *This observation shows the importance of considering the endogenous price effects when evaluating the effects of product compatibility.*

Next, we consider firms' profit. Since each firm's equilibrium profit is equal to:

$$\Pi_{reg}^* := N\tilde{x}_{reg}^*\tilde{p}_{reg}^* = \frac{N(1-\tilde{p}_{reg}^*)\tilde{p}_{reg}^*}{1+(L-1)(\beta-\mu d)-\delta d}. \tag{26}$$

[56] Here, the largest eigenvalue is d, so Assumption 2′ implies that $1+(L-1)(\beta-\mu d) > \delta d$ and $1-(\beta-\mu d) > \delta d$. Together, they imply that $1-\delta d > 0$ and $2+(L-3)(\beta-\mu d)-2\delta d > 0$.

[57] $\frac{\partial \tilde{p}_{reg}^*(\mu)}{\partial \mu} = \frac{d(1-\delta d)(L-1)}{[2+(L-3)(\beta-\mu d)-2\delta d]^2} > 0$ as $1-\delta d > 0$ by Assumption 2′, and $L \geq 2$.

[58] Here, each firm's equilibrium demand is $\tilde{\mathbf{x}}^* = \tilde{\mathbf{M}}^+(\mathbf{a}-\tilde{\mathbf{p}}^*) = \tilde{\mathbf{M}}^+(\mathbf{1}_N - \tilde{p}_{reg}^*\mathbf{1}_N) = \tilde{x}_{reg}^*\mathbf{1}_N$ because $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$.

[59] When $L \geq 3$, we can verify that $\frac{\partial CS_{reg}^*}{\partial \mu} > 0$, which implies that compatibility increases the consumer surplus.

we can show that Π^*_{reg} increases with μ, under a mild assumption.[60] *This is because $(1-\tilde{p}^*_{reg})\tilde{p}^*_{reg}$ increases with μ as $\tilde{p}^*_{reg}$ increases with μ and $\tilde{p}^*_{reg} \in (0, 1/2]$. Moreover, the market expansion social multiplier $1/(1+(L-1)(\beta-\mu d)-\delta d)$ also increases with μ. Both effects move in the same direction, which implies higher firm profits.*

In this example of a regular network, we observe that product compatibility results in higher equilibrium prices and higher firm profits due to weaker competition and enhanced demand multipliers. However, product compatibility may lead to lower consumption and lower consumer surplus due to endogenous price effects. Given the general insight developed here, we expect that analogous results would hold for general networks.

6.3 Cournot versus Bertrand competition

In this extension, we consider an alternative setting in which firms compete in quantities instead of prices. In the analysis of the consumption equilibrium in Lemma 1, we note the following one-to-one mapping between the price profile $\mathbf{p} = (\mathbf{p}^1, \cdots, \mathbf{p}^L)$ and the quantity profile $\mathbf{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^L)$:[61]

$$\begin{bmatrix} \mathbf{I}_N - \delta\mathbf{G} & \beta\mathbf{I}_N & \cdots & \beta\mathbf{I}_N \\ \beta\mathbf{I}_N & \mathbf{I}_N - \delta\mathbf{G} & \cdots & \beta\mathbf{I}_N \\ \vdots & \vdots & \ddots & \vdots \\ \beta\mathbf{I}_N & \beta\mathbf{I}_N & \cdots & \mathbf{I}_N - \delta\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^L \end{bmatrix} = \begin{bmatrix} \mathbf{a} - \mathbf{p}^1 \\ \vdots \\ \mathbf{a} - \mathbf{p}^L \end{bmatrix}. \tag{27}$$

Under Cournot competition, given firms' quantity choices $\mathbf{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^L)$, each firm l's profit is equal to $\pi^l_C(\mathbf{x}^l, \mathbf{x}^{-l}) = \langle \mathbf{x}^l, \mathbf{p}^l(\mathbf{x}^l, \mathbf{x}^{-l}))\rangle$, where $\mathbf{p}^l$ is given by (27).[62] The first-order conditions for firm l are given by:

$$\mathbf{0} = \frac{\partial \pi^l_C(\mathbf{x}^l, \mathbf{x}^{-l})}{\partial \mathbf{x}^l} = \mathbf{p}^l(\mathbf{x}^l, \mathbf{x}^{-l})) + \left(\frac{\partial \mathbf{p}^l(\mathbf{x}^l, \mathbf{x}^{-l})}{\partial \mathbf{x}^l}\right)' \mathbf{x}^l.$$

In a symmetric equilibrium with $\mathbf{x}^l = \mathbf{x}^*_C, \mathbf{p}^l = \mathbf{p}^*_C, \forall l \in \mathcal{L}$, the above equations reduce to

$$\mathbf{p}^*_C = (\mathbf{I}_N - \delta\mathbf{G})\mathbf{x}^*_C$$

Combining it with the consumption equilibrium condition $[(1+(L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]\mathbf{x}^*_C = (\mathbf{a} - \mathbf{p}^*_C)$ yields the following proposition:

Proposition 6. *Suppose that Assumptions 1, 2, and 3 hold. Under Cournot competition, there exists a unique equilibrium in which the (symmetric) equilibrium consumption vector and price vector are given by:*

$$\mathbf{x}^*_C = [(2+(L-1)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\mathbf{a}, \tag{28}$$

[60] This additional assumption, $\beta > \mu d$ makes sure that $\tilde{p}^*_{reg} \in (0, 1/2]$. Furthermore, if Assumption $2'$ holds for L, it must be the case that $\beta > \mu d$.

[61] The inverse mapping is given in the proof of Proposition E6 in the Appendix.

[62] Without loss of generality, we normalize $\mathbf{c} = \mathbf{0}$.

and

$$\mathbf{p}_C^* = [\mathbf{I}_N - \delta\mathbf{G}][(2 + (L-1)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\mathbf{a}. \tag{29}$$

Even though the equilibrium price under Cournot competition ($\mathbf{p}_C^*$ in (28)) differs from the price under Bertrand competition ($\mathbf{p}^*$ in (10)), it shares similar features. In particular, the price can be decomposed as follows

$$\mathbf{p}_C^* = \frac{\mathbf{a}}{2} - \frac{(L-1)\beta}{2(2+(L-1)\beta)}[\mathbf{I} - \frac{2\delta}{(2+(L-1)\beta)}\mathbf{G}]^{-1}\mathbf{a},$$

where the second term is proportional to the Katz-Bonacich centralities of nodes. Furthermore, when L is large, we have:

$$p_{i,C}^* = \frac{1-\delta d_i}{\beta L} + \mathcal{O}(L^{-2}). \tag{30}$$

In particular, it is possible that the price increases with L under the Cournot competition. We illustrate this observation using the same network shown in Figure 1 (i.e., a kite network with four nodes) with $\delta = 0.4$ and $\beta = 0.1$.[63] We plot the equilibrium prices for different nodes, as shown in Figure 7. The equilibrium price for node 1 (the lowest curve in blue) is not monotonic in L. This is similar to the results shown in Figure 3 regarding the Bertrand competition.

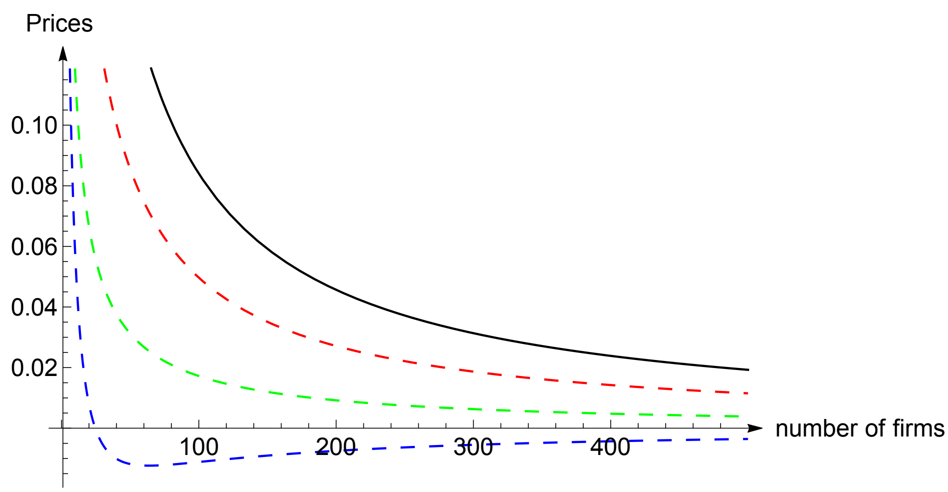

Figure 7: How equilibrium prices change with L under Cournot competition

6.4 Determinations of market structure and network topology

Here, we discuss the method by which the market structure and the network structure can be determined. First, we endogenize the market structure by allowing the free entry of firms into the market. Second, we endogenize the network structure by determining the firm's and the consumer's optimal network structure.

[63]Recall that the largest eigenvalue λ_1 is about 2.17, so Assumption 1 is satisfied.

6.4.1 Market structure with free entry

Suppose that the number of firms in the market is determined by a free-entry condition. We assume that each firm pays a fixed cost for entering the market. We show that when the fixed cost is sufficiently large, the equilibrium market structure is relatively concentrated. In this region, increasing the network density or the network technology level leads to more firms in the free-entry equilibrium. On the contrary, when the fixed cost is sufficiently low, the equilibrium market structure is relatively competitive. We reach a different region: increasing the network density or the network technology level leads to fewer firms in the free-entry equilibrium. The results are mainly driven by the rotation of the profit curve studied in the previous section.

Given a fixed entry fee $f > 0$, we define L^{FE*} as the number of firms in the free-entry equilibrium such that $\Pi^*(L^{FE*}) = f$. Since $\Pi^*(\cdot)$ is strictly decreasing in L, L^{FE*} is uniquely determined.[64] For ease of notation, we omit the other dependent variables (such as $\mathbf{G}, \beta$ and δ) in the definition of L^{FE*}. In Proposition B1 in Appendix B.3, for regular networks, we show that the equilibrium number of firms L^{FE*} decreases with δ when the entry cost f is sufficiently small and increases with δ when the entry cost f is sufficiently large.

Let us illustrate our result with the following example.

Example 1 (cont.) *Consider regular networks of degree d and consider Figure 5. When f is higher than the profit at the rotation point ($L \approx 3.5$), the free-entry number of firms is higher for the red curve with a higher δ than for the blue curve with a lower δ. The reverse happens when f is lower than the profit at the intersection point. Indeed, for a marginal increase in δ, the rotation point occurs exactly when $L = L^* = \chi\left(\frac{\beta}{1-\delta d}\right)$ based on Proposition 4. Therefore, the threshold $\bar{f}$ in Proposition B1 is, in fact, equal to $\bar{f} = \Pi^*_{reg}(d; \beta, \delta, \chi\left(\frac{\beta}{1-\delta d}\right))$. In reality, the size of f depends on the institutional context, which varies from one industry to another. This provides several possible implications of the impact of technology or network improvement on market concentration with endogenous firm entry.*

Since the observation that increasing the network density or δ leads to similar rotations of the profit curve (see Proposition 5 (*ii*) and Proposition E5), a result (available upon request) similar to that in Proposition B1 holds in non-regular networks. Our unique result is that the equilibrium number of firms depends not only on the entry cost f and the degree of product substitution β, as is usually the case, but also on the network structure and the intensity of the network effects.

6.4.2 Optimal network structures

So far, we have treated the network structure as a given. We now fix the market structure L and discuss the optimal network structure from the perspectives of firms and consumers.[65]

[64] We treat L as a continuous variable. To make our problem interesting, we assume that f is below the monopoly profit, i.e., $f < \Pi^*(L)|_{L=1}$.

[65] Very few papers have examined optimal network design in network games. Exceptions include Belhaj, Bervoets, and Deroïan (2016), Hiller (2017), and König, Liu, and Zenou (2019).

As shown in Proposition F8 in Appendix F, we determine the optimal network structure. First, in Proposition F8 (i), we show that the *consumer*-optimal network is the complete network. Indeed, increasing the network density benefits consumers in two ways. First, the equilibrium prices decrease due to intensified price competition among firms (see Proposition E4). Second, the network multiplier becomes stronger, which generates greater network benefits for consumers. Formally, in the proof of Proposition F8 (i), we show that the equilibrium consumer's surplus is monotone in the equilibrium consumption of consumers, i.e, $u_i^* \propto (x_i^*)^2$. Recall that, in equilibrium, consumption is given by the multiplication of the market expansion by the difference between the marginal utility vector and the price vector, i.e., $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$. In a denser network, the multiplier $\mathbf{M}^+$ is stronger and prices are lower. The result that, regardless of the market structure, consumers' equilibrium utilities always increase with the network density, immediately follows.

Let us now consider the firms' perspective. Proposition F8 (ii) shows that firms' preferences over the network structure critically depend on the market structure. Indeed, increasing the network density again generates two forces that work in opposite directions. On one hand, higher density leads to lower equilibrium prices due to greater competition, which reduces profit margins and thus hurts firms' profitability. On the other hand, the demand-enhancing effects due to stronger network effects can benefit the firms. The dominant force depends on L, the number of existing competitors in the market. When L is low enough, the firm-optimal network is the complete network, whereas when L is high enough, the firm-optimal network is the empty network. Moreover, part (iib) of Proposition F8 highlights that adding a link between any pair of nodes can result in intensified price competition among firms, which may drive down their profits. This phenomenon occurs when the market is relatively concentrated in which consumers and firms hold opposite views of the optimal network structure. Note that even if the network is empty, firms' profits can still be strictly positive because each consumer's willingness to pay is positive based on the stand-alone consumption utility.

It is worth mentioning that in reality, neither firms nor consumers have full flexibility in adjusting network structures. Thus, the results of Proposition F8 should be regarded as benchmarks of optimal networks. Nonetheless, the proof of Proposition F8 reveals a deeper principle: we provide clear answers to the directions of changes in the firm's profit and the consumer's surplus when we add a link to an existing network.

7 Conclusion

In this study, we examined the interplay between the market structure (as measured by the number of firms in the market that sell differentiated products) and the network structure (as measured by the network topology) among consumers who make consumption decisions. We showed that prices are set lower when either the network becomes denser (by adding links) or the intensity of the network effects is stronger. This suggests that competition among firms is intensified due to the network effects among consumers. Moreover, we showed that price dispersion in the market is small for monopolies and in competitive cases (i.e., when there are many firms), and a maximum value is attained in the intermediate range. We also showed that when we increased the number of firms, prices initially decreased due to competition, but in some cases, they reverted to a rising trend when the number of firms increased.

We also found that increasing the intensity of the network effects did not shift the firms' profit curves but instead led to a rightward rotation. Improvement of network technology generated higher profits for a firm when there were only a few firms, but dampened the firm's profitability when there were many firms. This implies that if free entry were allowed, the improvement in the network technology would increase the equilibrium number of participating firms only when the entry cost was high. Finally, we characterized the optimal network structures from the perspectives of firms and consumers. Intriguingly, their rankings of network structures were consistent when the number of firms was small and the products were sufficiently differentiated. However, when there was a large number of firms or when the products were sufficiently homogeneous, the firms and consumers held completely opposite views of the optimal network structures.

Our model could be extended in several possible directions. First, we could extend our analysis to the impact of the merger of two networks resulting from a new link formed between the members of each network. An interesting question is the following: What would happen to the price dispersion in the new network compared with that in the original network? Second, our analysis relied on the common and perfect knowledge of the network structure held by consumers and firms. While this assumption is standard and holds for relatively small networks, relaxing it would lead to asymmetric network information and would allow us to study the interactions between network knowledge and the market structure. We leave these promising topics for future research.

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Appendix: Proofs of the results in the text

Proof of Lemma 1: Given the price profile, each consumer i maximizes

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \sum_{l=1}^{L} p_i^l x_i^l = \sum_{l=1}^{L}(a_i - p_i^l)x_i^l - \frac{1}{2}\sum_{l=1}^{L}(x_i^l)^2 - \frac{1}{2}\sum_{l=1}^{L}\sum_{s\neq l}\beta x_i^s x_i^l + \delta\sum_{l=1}^{L}\left(\sum_{j=1}^{N} g_{ij}x_i^l x_j^l\right).$$

With the presence of prices, the marginal utility a_i is reduced exactly by p_i^t for each product t. Note that this consumption game belongs to the family of network games with multi-dimensional strategy space studied in Chen et al. (2018b). Part (i) of this Lemma directly follows from Theorem 3 of Chen et al. (2018b), which characterizes the equilibrium in a general multi-activity network game. Given the functional forms of consumer utilities, the system determined by the first-order conditions for the underlying consumption equilibrium is linear both in $\mathbf{x}$ and $\mathbf{p}$. Under Assumption 2, the system has a unique solution $\mathbf{x}(\mathbf{p})$ (i.e., the CE), which linearly changes with prices as given in (4). Part (ii) directly follows from part (i). $\square$

Proof of Proposition 1: The FOC at the symmetric prices $\mathbf{p}^*$ is derived in the main text:

$$[(1+(L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{p}^*) = \frac{[(1+(L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1} + (L-1)[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}}{L}(\mathbf{p}^*-\mathbf{c}).$$

Solving this linear equation of $\mathbf{p}^*$ yields the symmetric prices in Proposition 1. We provide two equivalent expressions for the equilibrium prices $\mathbf{p}^*$ in (31). We will adopt the most convenient form in the analysis later.

$$\begin{aligned}\mathbf{p}^* &= [(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{a} + \left[(1+(L-2)\beta)\,\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{c} \\ &= \frac{\mathbf{a}+\mathbf{c}}{2} - \frac{(L-1)\beta}{2}[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}(\mathbf{a}-\mathbf{c}). \end{aligned} \tag{31}$$

Next we check the second-order conditions of firms' optimization problem. Recall that $\Pi^1 = \langle \mathbf{x}^1, \mathbf{p}^1 - \mathbf{c} \rangle$. Since $\mathbf{x}^1$ is linear in $\mathbf{p}^1$, the Hessian matrix of Π^1 with respect to $\mathbf{p}^1$ is just $-2\frac{\mathbf{M}^+ + (L-1)\mathbf{M}^-}{L}$ by Corollary 1, which is negative definite by Lemma A2. As a result, the profit function Π^1 is strictly concave in $\mathbf{p}^1$, and FOCs are sufficient for optimality.

Under this symmetric pricing equilibrium, for each firm the consumption vector is

$$\begin{aligned}
\mathbf{x}^* &= \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*) = [(1 + (L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}(\mathbf{a} - \mathbf{p}^*) \\
&= [(1 + (L-1)\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}[(2 + (L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}[(1 + (L-2)\beta)\mathbf{I}_N - \delta\mathbf{G}](\mathbf{a} - \mathbf{c}).
\end{aligned}$$

and the equilibrium profit is $\Pi = \langle \mathbf{x}^*, (\ \mathbf{p}^* - \mathbf{c}) \rangle$, which can be simplified to $\langle (\mathbf{a} - \mathbf{c}), \Phi(\mathbf{G})(\mathbf{a} - \mathbf{c}) \rangle$, where

$$\Phi(z) := \frac{(1 + (L-2)\beta - \delta z)(1 - \beta - \delta z)}{(1 + (L-1)\beta - \delta z)(2 + (L-3)\beta - 2\delta z)^2}.$$

□

Remark 4. *In the symmetric equilibrium, the consumption vector* $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$ *is positive as* $\mathbf{M}^+ \succeq \mathbf{0}$ *and* $\mathbf{a} - \mathbf{p}^* \succeq (\mathbf{a} - \mathbf{c})/2$. *Moreover we can show that there is no asymmetric pricing equilibrium. Hence, the symmetric pricing equilibrium stated in Proposition 1 is unique.*

Proof of Proposition 2: With Assumption 3, which assumes that $c = 0$ and $a_i = a$ for all i, the equilibrium prices are given by:

$$\mathbf{p}^* = a[(2 + (L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1}_N.$$

by using the formula in the first row of (31). We have

$$\frac{\partial \mathbf{p}^*}{\partial L} = -a\beta[(2 + (L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}[(2 + (L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1}_N.$$

For case (i), we have $1 - \beta - \delta d_i > 0$ for every i, so $\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1}_N \succeq \mathbf{0}$. Furthermore, $\left[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}\right]\mathbf{1} \succeq \mathbf{0}$ and $[(2 + (L-3)\beta)\mathbf{I}_n - 2\delta\mathbf{G}]^{-1} \succeq \mathbf{0}$. Thus, $\frac{\partial \mathbf{p}^*}{\partial L} \preceq \mathbf{0}$.

For case (ii), for any node i with $1 - \beta - \delta d_i < 0$, for large L, by equation (C9) in Proposition C2 in Appendix C, we have:

$$p_i^* = \frac{1 - \beta - \delta d_i}{\beta L} + \mathcal{O}(L^{-2}).$$

Clearly, $\partial p_i^*/\partial L > 0$ when L is sufficiently large (see (C10)). Moreover, the fact that $\partial p_i^*/\partial L < 0$ for small L follows from the observation that

$$\frac{\partial \mathbf{p}^*}{\partial L}|_{L=1} = -\frac{a\beta}{4}\underbrace{[(1-\beta)\mathbf{I}_N - \delta\mathbf{G}]^{-1}\mathbf{1}}_{\succeq \mathbf{0}} \preceq \mathbf{0}.$$

This proves the results. □

Proof of Proposition 3 : When $L = 1$, $p_i^* = p_j^* = \frac{a}{2}$, so the dispersion is zero. As $L \to \infty$, the equilibrium price p_i^* converges to zero for any $i \in \mathcal{N}$ by Proposition C2; again the dispersion is zero. Consequently, the maximal dispersion must occur when L takes some intermediate value $L^* > 1$. $\square$

Proof of Remark 1: The equilibrium price, by taking the Taylor series expansion of (10) with respect to δ, takes the following forms when δ is small:

$$\mathbf{p}^* = \frac{(1-\beta)\mathbf{a} + (1+(L-2)\beta)\mathbf{c}}{(2+(L-3)\beta)} - \delta\frac{(L-1)\beta}{(2+(L-3)\beta)^2}\mathbf{G}(\mathbf{a}-\mathbf{c}) + \mathcal{O}(\delta^2).$$

Under Assumptions 3, $p_i^* - p_j^* \approx \delta\frac{(L-1)\beta}{(2+(L-3)\beta)^2}(d_i - d_j)$ (we omit the higher-order terms of δ), implying $Disp(l) = \delta\frac{(L-1)\beta}{(2+(L-3)\beta)^2}(d_{max} - d_{min})$. The result about L^* just follows from the observation that:

$$\frac{(L-1)\beta}{(2+(L-3)\beta)^2} = \frac{(L-1)\beta}{((2-2\beta)+(L-1)\beta)^2} \leq \frac{(L-1)\beta}{4(2-2\beta)(L-1)\beta} = \frac{1}{8(1-\beta)}.$$

Here we use a simple fact: for $a, b > 0$, $(a+b)^2 \geq 4ab$ with equality when $a = b$. $\square$

Proof of Proposition 4: To prove Proposition 4, we use Lemma A3. Direct differentiation shows that

$$\begin{aligned}\frac{\partial \Pi_{reg}^*}{\partial \delta} &= \frac{dn\left[(-(5-2L-2L^2+L^3)\beta^3 - 6(L-2)\beta^2(1-d\delta) + 3(L-3)\beta(1-d\delta)^2 + 2(1-d\delta)^3)\right]}{(1+(L-1)\beta-\delta d)^2(2+(L-3)\beta-2\delta d)^3} \\ &= \underbrace{\left\{\frac{dn(1-d\delta)^3}{(1+(L-1)\beta-\delta d)^2(2+(L-3)\beta-2\delta d)^3}\right\}}_{>0} h\left(\left(\frac{\beta}{1-\delta d}\right), L\right)\end{aligned}$$

where $h(\cdot,\cdot)$ is defined in (A4) in Lemma A3. Therefore, $\frac{\partial \Pi_{reg}^*}{\partial \delta}$ has the same sign as that of $h(\left(\frac{\beta}{1-\delta d}\right), L)$. The rest just follows from Lemma A3. $\square$

Proof of Corollary 2: By Proposition 4,

$$\frac{\partial \Pi_{reg}^*}{\partial \delta} > (<)0 \text{ if and only if } L < (>)\chi\left(\frac{\beta}{1-\delta d}\right) \text{ if and only if } \frac{\beta}{1-\delta d} < (>)\chi^{-1}(L).$$

The rest follows immediately. $\square$

Proof of Corollary 3: Note that Π_{reg}^* depends on d and δ only through the product $d\delta$. Therefore, $\partial \Pi_{reg}^*/\partial d = \frac{\delta}{d}\partial \Pi_{reg}^*/\partial \delta$. The rest follows immediately. $\square$

Proof of Lemma 2: The first part of this Lemma is a direct application of the spectral theorem adapted to the symmetric matrix $\mathbf{G}$, so we omit the proof, which is standard. For the second part, when $f(\lambda_i) \geq 0$ for every λ_i, clearly $\mathbf{v}' f(G)\mathbf{v} \geq 0$. When $f(\lambda_i) > 0$, and $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v}'\mathbf{u}_i$ is non-zero for at least one i as these eigenvectors $\{\mathbf{u}_i\}$ form a basis of $\mathbf{R}^n$. As a result, $\mathbf{v}' f(G)\mathbf{v} > 0$. $\square$

Proof of Proposition 5: We first show identity (17). In the proof of Proposition 1, we show that the equilibrium profit, which equals $\langle \mathbf{M}^{+}(\mathbf{a} - \mathbf{p}^*), \mathbf{p}^* \rangle$, can be rewritten as

$$\Pi^*(\mathbf{G}; \beta, \delta, L) = \langle \mathbf{a}, \Phi^{PT}(\mathbf{G})\mathbf{a} \rangle,$$

where

$$\Phi^{PT}(z; \beta, \delta, L) := \frac{(1 + (L-2)\beta - \delta z)(1 - \beta - \delta z)}{(1 + (L-1)\beta - \delta z)(2 + (L-3)\beta - 2\delta z)^2}.$$

Note that when $z = d$, $\Phi^{PT}(d; \beta, \delta, L)$ exactly equals $\frac{1}{n} \times \Pi_{reg}(d; \delta, \beta, L)$ in (13). The identity then just follows from Lemma 2. Using Lemma A2 and the same argument in Section B, we can determine the signs of $\partial \Pi_{reg}(\lambda_i; \delta, \beta, L)/\partial L$ and $\partial \Pi_{reg}(\lambda_i; \delta, \beta, L)/\partial \delta$. Specifically, for any $\lambda_i \in Spec(\mathbf{G})$, we have

$$\partial \Pi_{reg}(\lambda_i; \delta, \beta, L)/\partial L < 0,$$

and

$$\frac{\partial \Pi_{reg}(\lambda_i; \delta, \beta, L)}{\partial \delta} > (<) 0 \text{ if and only if } L < (>) \chi \left(\frac{\beta}{1 - \delta \lambda_i} \right),$$

The rest of the proof follows from the discussion in the main text. $\square$

Online Appendix

A Matrix notation, Katz-Bonacich centrality and some preliminary results

Matrix notation. Let $\mathbf{A}'$ denote the transpose of matrix $\mathbf{A}$. $\mathbf{I}_n$ is the $n \times n$ identity matrix, $\mathbf{J}_{mn}$ is the $m \times n$ matrix with 1's, and $\mathbf{1}_n = \mathbf{J}_{n1}$ is a column vector with 1s:

$$\mathbf{I}_n = \begin{bmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{bmatrix}_{n\times n}, \quad \mathbf{J}_{mn} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{m\times n}, \quad \mathbf{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}.$$

The inner product of two column vectors $\mathbf{x} = (x_1, \cdots, x_n)'$ and $\mathbf{y} = (y_1, \cdots, y_n)'$ in $\mathbf{R}^n$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}'\mathbf{y} = \sum_i x_i y_i$. We use $\mathbf{0}$ to denote the zero matrix with suitable dimensions. For any two matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \preceq (\succeq)\mathbf{B}$ if component-wise $a_{ij} \leq (\geq) b_{ij}$ for all i, j. Consequently, $\mathbf{A}$ is a positive matrix if $\mathbf{A} \succeq \mathbf{0}$. A square symmetric matrix $\mathbf{A}$ is called positive definite if all of its eigenvalues are strictly positive.

Katz-Bonacich centrality. Let us define the Katz-Bonacich centrality. Denote by $\lambda_1(\mathbf{G})$ the spectral radius of matrix $\mathbf{G}$. Since $\mathbf{G}$ is a nonnegative matrix, by the Perron-Frobenius Theorem it is also equal to its largest eigenvalue.

Definition A1. *Assume $0 \leq \delta < 1/\lambda_1(\mathbf{G})$. Then, for any vector $\mathbf{a} = (a_1, \cdots, a_n)' \in \mathbf{R}^n$, the Katz-Bonacich centrality vector with weight $\mathbf{a}$ is defined as:*

$$\mathbf{b}(\mathbf{G}, \delta, \mathbf{a}) := \mathbf{M}(\mathbf{G}, \delta)\mathbf{a}, \tag{A1}$$

where

$$\mathbf{M}(\mathbf{G}, \delta) = [\mathbf{I} - \delta\mathbf{G}]^{-1} = \mathbf{I} + \sum_{k\geq 1} \delta^k \mathbf{G}^k. \tag{A2}$$

Let $b_i(\mathbf{G}, \delta, \mathbf{a})$ be the ith entry of $\mathbf{b}(\mathbf{G}, \delta, \mathbf{a})$. Let $m_{ij}(\mathbf{G}, \delta)$ be the ij entry of $\mathbf{M}(\mathbf{G}, \delta)$. Then,

$$b_i(\mathbf{G}, \delta, \mathbf{a}) = \sum_j m_{ij}(\mathbf{G}, \delta) a_j.$$

Some preliminary results. We would like now to present some results: Lemma A1 and Lemma A2, which will be used in the proofs in the Appendix.

Lemma A1. *Suppose $\beta \in [0, 1)$ and $L \geq 1$. Define the $L \times L$ matrix $\mathbf{\Psi}$ as:*

$$\mathbf{\Psi} = \begin{bmatrix} 1 & \beta & \cdots & \beta \\ \beta & 1 & \cdots & \beta \\ \vdots & \ddots & \ddots & \vdots \\ \beta & \cdots & \beta & 1 \end{bmatrix}_{L\times L} . \tag{A3}$$

Then (i) the matrix $\mathbf{\Psi}$ is positive definite. (ii) For $a > 0$, the function

$$v(\mathbf{x}) = a\mathbf{x}'\mathbf{1}_L - \frac{1}{2}\mathbf{x}'\mathbf{\Psi}\mathbf{x} = a(\sum_{t=1}^{L} x^t) - \frac{1}{2}\sum_{t=1}^{L}(x^t)^2 - \frac{\beta}{2}\sum_{t=1}^{L}\sum_{s\neq t} x^s x^t, \quad \mathbf{x} = (x^1, \cdots, x^L) \in \mathbf{R}^L$$

has a unique maximizer at $\mathbf{x}^* = \hat{x}\mathbf{1}_L$ with the maximum value $v(\mathbf{x}^*) = \frac{L(1+(L-1)\beta)}{2}(\hat{x})^2$, where $\hat{x} = \frac{a}{1+(L-1)\beta}$.

Proof of Lemma A1: The eigenvalues of $\mathbf{\Psi}$: $1-\beta$ (with multiplicity $L-1$), and $1+(L-1)\beta$ (with multiplicity 1), are strictly positive, and hence $\mathbf{\Psi}$ is positive definite. The FOC of maximizing v is just $a\mathbf{1}_L = \mathbf{\Psi}\mathbf{x}^*$ which, by symmetry, leads to: $\mathbf{x}^* = a\mathbf{\Psi}^{-1}\mathbf{1}_L = \frac{a}{1+(L-1)\beta}\mathbf{1}_L$. Since $v(\cdot)$ is strictly concave by (i), $\mathbf{x}^*$ is the unique global maximizer. Substituting $\mathbf{x}^*$ into $v(\cdot)$ yields the maximum value. □

Lemma A2. *Under Assumption 2.*

(i) For any $\lambda_i \in Spec(\mathbf{G})$,

$$1-\beta-\lambda_i\delta > 0, \quad 1+(L-1)\beta-\lambda_i\delta > 0, \quad 1+(L-2)\beta-\lambda_i\delta > 0, \quad 2+(L-3)\beta-2\lambda_i\delta > 0.$$

(ii) The following matrices are symmetric and positive definite:

$$[(1-\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}, \; [(1+(L-1)\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}, \; [(1+(L-2)\beta)\mathbf{I}_n-\delta\mathbf{G}]^{-1}, \; [(2+(L-3)\beta)\mathbf{I}_n-2\delta\mathbf{G}]^{-1}.$$

Moreover, for each matrix above, every entry is nonnegative.

Proof of Lemma A2: (i) Note that λ_1 is the largest eigenvalue in $Spec(\mathbf{G})$. So $\lambda_i \leq \lambda_1$, implying $1-\beta-\lambda_i\delta \geq 1-\beta-\lambda_1\delta > 0$ by Assumption 2. Moreover,

$$1+(L-1)\beta-\lambda_i\delta \geq 1+(L-2)\beta-\lambda_i\delta = \underbrace{(1-\beta-\lambda_i\delta)}_{>0} + \underbrace{(L-1)\beta}_{\geq 0} > 0$$

and

$$2+(L-3)\beta-2\lambda_i\delta = (1-\beta-\lambda_i\delta)+(1+(L-2)\beta-\lambda_i\delta) > 0.$$

(ii) The eigenvalues of $[(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}$ are precisely $\frac{1}{1-\beta-\lambda_i\delta}$, where $\lambda_i \in Spec(\mathbf{G})$. Since $\frac{1}{1-\beta-\lambda_i\delta} > 0$ by part (i), $[(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1}$ is positive definite. The nonnegativity of the matrix follows from

$$[(1-\beta)\mathbf{I}_n - \delta\mathbf{G}]^{-1} = \frac{1}{1-\beta}\mathbf{M}(\mathbf{G}, \frac{\delta}{1-\beta}) = \frac{1}{1-\beta}\sum_{j\geq 0}(\frac{\delta}{1-\beta})^j\mathbf{G}^j \succeq \mathbf{0}.$$

The proofs for the other three matrices are similar. □

Lemma A3. *Define*

$$h(\beta, l) := 2 + 3(l-3)\beta - 6(l-2)\beta^2 - (l^3 - 2l^2 - 2l + 5)\beta^3. \tag{A4}$$

on the domain $\mathcal{O} = \{(\beta, l) \in R^2 | \beta \in [0,1], l \geq 1\}$*. There exists a continuously differentiable and strictly decreasing function* $\chi() : [0,1] \to [1,\infty)$ *of* β *with* $\chi(1) = 1$, $\lim_{\beta\to 0^+}\chi(\beta) = \infty$ *such that*

$$h(\beta, l) > (<) 0 \quad \textit{if and only if} \quad l < (>) \chi(\beta). \tag{A5}$$

Proof of Lemma A3: We complete the proof in several steps.

First, we show that for each $l \geq 1$, there exists a unique number $\beta^*(l) \in [0,1]$ with $g(\beta^*(l), l) = 0$. The existence of such a root β^* follows from Mean Value Theorem, as $g(0,l) = 2 > 0$, and $g(1,l) = -l(l-1)^2 \leq 0$. To show the uniqueness, we need to check the first and second derivatives of h:

$$h_\beta(\beta, l) = 3(l-3) - 12(l-2)\beta - 3(l^3 - 2l^2 - 2l + 5)\beta^2,$$

with $h_\beta(0,l) = 3(l-3)$, and

$$h_{\beta\beta}(\beta, l) = -12(l-2) - 6(l^3 - 2l^2 - 2l + 5)\beta.$$

Also note that for $l \geq 1$, the coefficient of β^3, $-(l^3 - 2l^2 - 2l + 5)$, of g is negative as the minimum value of $l^3 - 2l^2 - 2l + 5$ on $l \in [1,\infty)$ is about $0.73 > 0$ at $l^* \approx 1.72076$.

(i) When $l > 3$, $h_{\beta\beta}(\beta, l) < 0$ for any $\beta \in [0,1]$, so h is concave in β. Moreover, $h_\beta(0,l) = 3(l-3) > 0$, so h first increases , then decrease with β. Since $h(0,l) = 2 > 0$, h has a unique root on $[0,1]$.

(ii) When $2 \leq l \leq 3$, h_β is negative for any $\beta \in [0,1]$. Thus, h is strictly decreasing in β, which implies uniqueness.

(iii) When $1 < l < 2$, recall that $-h_\beta/3 = \big((l^3 - 2l^2 - 2l + 5)\beta^2 - 4(2-l)\beta + (3-l)\big)$, since the leading coefficient $(l^3 - 2l^2 - 2l + 5) > 0$, and the discriminant

$$\mathcal{D} := (-4(2-l))^2 - 4(l^3 - 2l^2 - 2l + 5)(3-l) = 4(l-1)^2 \underbrace{(1 + l^2 - 3l)}_{<0,\text{ given } l\in(1,2)} < 0,$$

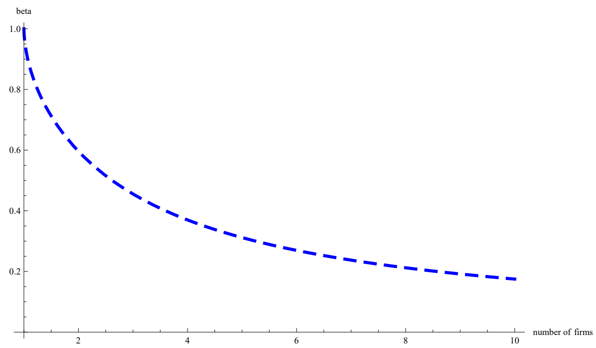

Figure A1: The cutoff curve $\beta^*(l)$: below (above) the curve, the function h in Lemma A3 is positive (negative).

So $\big((l^3-2l^2-2l+5)\beta^2-4(2-l)\beta+(3-l)\big)>0$ for any $\beta\in[0,1]$, and therefore h is strictly decreasing in β. This implies uniqueness.

(iv) When $l=1$, $h(\beta,1)=2(1-\beta)^3$, so the unique root is $\beta^*(1)=1$.

The analysis above also shows that

$$g(\beta,l)>(<0)\text{ if and only if }\beta<(>)\beta^*(l).$$

Clearly $\beta^*(l)$ is continuously differentiable in l. Moreover $\beta^*(1)=1$ and $\beta^*\to 0^+$ as $l\to\infty$.

Second, we show that for each fixed $\beta\in(0,1]$, there exists a unique $\chi^*(\beta)\geq 1$ such that $h(\beta,l^*(\beta))=0$. To this end, we write h as

$$h(\beta,l)=2(1-\beta)^3+3\beta(1-\beta)^2(l-1)-\beta^3(l-1)^2-\beta^3(l-1)^3.$$

Again the existence of such as $l^*(\beta)$ follows from mean value theorem, as $g(\beta,1)=2(1-\beta)^3\geq 0$ and $g(\beta,\infty)=-\infty$. To show uniqueness, we check that

$$h_{ll}(\beta,l)=-2\beta^3-6\beta^3(l-1)<0\ \ (\text{as } l\geq 1)$$

and

$$h_l(\beta,1)=3\beta(1-\beta)^2>0.$$

Therefore, $h(\beta,l)$ is concave in l, and it is positive for small l, first increases with l, and then decreases with l. So the solution to $\{l:h(\beta,l)=0\}$ is unique. Clearly, $\chi^*(\beta)$ is continuously differentiable in β. The analysis above also shows that

$$g(\beta,l)>(<0)\text{ if and only if }l<(>)\chi^*(\beta).$$

Third, combining with both observations about $\beta^*(l)$ and $\chi^*(\beta)$, it is immediate that $\beta^*(l)$ is the inverse function of $\chi^*(\beta)$, and vice versa. In particular, both curves are injective, and hence monotone. In Figure A1, we plot the cutoff curve $\beta^*(\cdot)$, which is indeed decreasing with l. The curve $\chi(\cdot)$ is just the inverse function of such $\beta^*(\cdot)$, and hence it decreases with β.[1] $\square$

B Regular networks

In this Appendix, we focus on regular graphs.

Definition B1. *A network* $\mathbf{G}$ *is called regular with degree* d *if every node has the same degree* d*, i.e.,* $\sum_{j\in\mathcal{N}} g_{ij} = d, \forall i \in \mathcal{N}$*. Equivalently,* $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$*.*

B.1 Equilibrium prices

For a regular network, $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$. Using (31), we obtain the following equilibrium prices:

$$\begin{aligned}\mathbf{p}^* &= a[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}[((1-\beta)\mathbf{I}_N - \delta\mathbf{G})\mathbf{1}_N \\ &= a\frac{(1-\beta-\delta d)}{2+(L-3)\beta-2\delta d}\mathbf{1}_N = a\frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}}\mathbf{1}_N.\end{aligned}$$

Set $a=1$ in (B6). Then, we obtain the following common equilibrium price (mark-up) for regular networks with degree d:

$$p_i^* = p_j^* = p_{reg}^* := \frac{(1-\beta-\delta d)}{2+(L-3)\beta-2\delta d} = \frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}} \in (0, \frac{1}{2}]. \tag{B6}$$

Indeed, since the Katz-Bonacich centrality measures are the same for all nodes in a regular network, every consumer gets the same equilibrium price. Note that $\frac{1}{1+(L-1)\beta-\delta d}\times(1-p_{reg}^*)$ is each consumer's equilibrium demand while $\frac{1}{1+(L-1)\beta-\delta d}$ is the corresponding market expansion social multiplier. Indeed, for a regular network, $\mathbf{G}\mathbf{1}_N = d\mathbf{1}_N$, and therefore

$$\mathbf{M}^+\mathbf{1}_N = \frac{1}{1+(L-1)\beta-\delta d}\mathbf{1}_N.$$

This term decreases with L and β, and increases with δ and d. For a regular network with degree d, each firm's equilibrium profit is then equal to:

$$\Pi_{reg}^* := \frac{n(1-p_{reg}^*)p_{reg}^*}{1+(L-1)\beta-\delta d}. \tag{B7}$$

[1] For $l = 1, 2, \cdots, 10$, $\beta^*(l)$ is about 1.0, 0.596072, 0.45541, 0.369902, 0.311759, 0.269522, 0.237411, 0.212159, 0.191773, 0.174969, respectively. Conversely, for $\beta = 0.1, 0.2, \cdots, 1.0$, the threshold $\chi^*(\beta)$ is about 18.5628, 8.57185, 5.24991, 3.59808, 2.61803, 1.97933, 1.54419, 1.25, 1.07078, 1., respectively.

B.2 The effects of market and network structure on equilibrium prices

By differentiating the equilibrium price in (12), we easily obtain:[2]

$$\frac{\partial p^*_{reg}}{\partial \delta} < 0, \quad \frac{\partial p^*_{reg}}{\partial d} < 0, \quad \frac{\partial p^*_{reg}}{\partial L} < 0.$$

Thus, for a regular network with degree d, only Proposition 2 (i) occurs and prices decrease with L for all nodes, i.e., $\frac{\partial p^*_{reg}}{\partial L} < 0$. Moreover, by Proposition C2, equilibrium prices (mark-ups in our setting) for a regular network always lie above zero.[3]

B.3 Equilibrium market structure with free entry

Proposition B1. *Consider a regular network with degree d. The number of firms L^{FE*} in the free-entry equilibrium decreases (increases) with δ when the entry fee f is sufficiently small (large). More precisely, there exists a threshold $\bar{f}$ such that*

$$\frac{\partial L^{FE*}}{\partial \delta} > (<) 0 \textit{ if and only if } f > (<) \bar{f}.$$

This result is a direct consequence of Corollary 2. Note that $\bar{f} = \Pi^*_{reg}(d; \beta, \delta, \chi\left(\frac{\beta}{1-\delta d}\right))$.

C Perfect competition

Regardless of the network structure, the equilibrium price must go to zero in the perfect competition limit.

Proposition C2. *For large L, we have:*

$$\mathbf{p}^* = \frac{(1-\beta)\mathbf{1}_n - \delta \mathbf{G}\mathbf{1}_n}{\beta L} + \mathcal{O}(L^{-2}) \tag{C8}$$

or equivalently, for each $i \in \mathcal{N}$,

$$p^*_i = \frac{1-\beta-\delta d_i}{\beta L} + \mathcal{O}(L^{-2}) \tag{C9}$$

As $L \to \infty$, p^*_i converges to 0, for any $\mathbf{G}, \delta, \beta$.

Proof of Proposition C2: Using the Taylor expansion of the equilibrium prices (10) in Proposition 1 at $L = \infty$, we obtain the following result $\mathbf{p}^* = \frac{(1-\beta)\mathbf{1}_n - \delta \mathbf{G}\mathbf{1}_n}{\beta L} + \mathcal{O}(L^{-2})$, which proves (C9). In particular, as $L \to \infty$, clearly we have $\mathbf{p}^* \to \mathbf{0}$. $\square$

[2] Indeed, since $\frac{1}{2+\frac{(L-1)\beta}{1-\beta-\delta d}}$ strictly decreases with δ, d and L, the results immediately follow.

[3] For a regular network with (common) degree d, $\lambda_1 = d$ and Assumption 2 reduces to $1-\beta-\delta d > 0$. So Assumption 2 rules out case (ii) in Proposition 2.

Recall that we normalize the marginal cost to be zero, and prices here are mark-ups. Also, for large L,

$$\frac{\partial p_i^*}{\partial L} \approx -\frac{(1-\beta-\delta d_i)}{\beta L^2}. \tag{C10}$$

D General welfare analysis

Define the following rational functions:

$$\begin{aligned}
\phi^{EX}(z;\beta,\delta,L) &:= \frac{(1+(L-2)\beta-\delta z)}{(1+(L-1)\beta-\delta z)(2+(L-3)\beta-2\delta z)}, \\
\phi^{PT}(z;\beta,\delta,L) &:= \frac{(1+(L-2)\beta-\delta z)(1-\beta-\delta z)}{(1+(L-1)\beta-\delta z)(2+(L-3)\beta-2\delta z)^2}, \\
\phi^{CS}(z;\beta,\delta,L) &:= \frac{L(1+(L-1)\beta)}{2}\left[\phi^{EX}(z;\beta,\delta,L)\right]^2, \\
\phi^{TW}(z;\beta,\delta,L) &:= \phi^{CS}(z;\beta,\delta,L)+L\phi^{PT}(z;\beta,\delta,L).
\end{aligned} \tag{D11}$$

Proposition D3. *Suppose Assumption 1 and 2 hold. Then, for any $L \geq 2$, the equilibrium consumption for each product is given by:*

$$\mathbf{x}^*(\mathbf{G};\beta,\delta,L) := \phi^{EX}(\mathbf{G};\beta,\delta,L)(\mathbf{a}-\mathbf{c})$$

while each firm's equilibrium profit is equal to:

$$\mathbf{\Pi}^*(\mathbf{G};\beta,\delta,L) := \langle(\mathbf{a}-\mathbf{c}),\phi^{PT}(\mathbf{G};\beta,\delta,L)(\mathbf{a}-\mathbf{c})\rangle.$$

Furthermore, the total consumer surplus is equal to:

$$\mathbf{CS}^*(\mathbf{G};\beta,\delta,L) := \langle(\mathbf{a}-\mathbf{c}),\phi^{CS}(\mathbf{G};\beta,\delta,L)(\mathbf{a}-\mathbf{c})\rangle.$$

while the total welfare, defined as the sum of the consumer surplus and the equilibrium profit, is given by:

$$\mathbf{TW}^*(\mathbf{G};\beta,\delta,L) := \langle(\mathbf{a}-\mathbf{c}),\phi^{TW}(\mathbf{G};\beta,\delta,L)(\mathbf{a}-\mathbf{c})\rangle.$$

The functions $\phi^{EX}(\cdot),\phi^{PT}(\cdot),\phi^{CS}(\cdot)$, and $\phi^{TW}(\cdot)$ are defined in (D11).

This Proposition directly follows from the equilibrium prices characterized in the main text, hence we omit the formal proof. Applying Lemma 2, we have the following decompositions of the consumer surplus and total welfare:

$$CS^*(\mathbf{G};\beta,\delta,L) := \sum_{\lambda_i\in Spec(\mathbf{G})} \phi^{CS}(\lambda_i;\beta,\delta,L)\times(\langle\mathbf{u}_i,\frac{(\mathbf{a}-\mathbf{c})}{\sqrt{N}}\rangle)^2 \tag{D12}$$

and

$$TW^*(\mathbf{G};\beta,\delta,L) := \sum_{\lambda_i \in Spec(\mathbf{G})} \phi^{TW}(\lambda_i;\beta,\delta,L) \times (\langle \mathbf{u}_i, \frac{(\mathbf{a}-\mathbf{c})}{\sqrt{N}} \rangle)^2 \tag{D13}$$

where $Spec(\mathbf{G}) = \{\lambda_1, \cdots, \lambda_N\}$ is the set of eigenvalues of $\mathbf{G}$, and $\mathbf{u}_i$ is the corresponding normalized eigenvector.

The comparative statics exercises of the consumer surplus and total welfare follow a similar analysis to the one we did for the firm's equilibrium profit in Section 5. Indeed, by direct differentiations of (D12) and (D13), we obtain:

$$\frac{\partial CS^*(\mathbf{G};\beta,\delta,L)}{\partial L} = \sum_{\lambda_i \in Spec(\mathbf{G})} \frac{\phi^{CS}(\lambda_i;\beta,\delta,L)}{\partial L} \times (\langle \mathbf{u}_i, \frac{(\mathbf{a}-\mathbf{c})}{\sqrt{N}} \rangle)^2, \tag{D14}$$

and

$$\frac{\partial TW^*(\mathbf{G};\beta,\delta,L)}{\partial L} = \sum_{\lambda_i \in Spec(\mathbf{G})} \frac{\partial \phi^{TW}(\lambda_i;\beta,\delta,L)}{\partial L} \times (\langle \mathbf{u}_i, \frac{(\mathbf{a}-\mathbf{c})}{\sqrt{N}} \rangle)^2. \tag{D15}$$

Therefore, we can analyze the effects of competition L on the consumer surplus and total welfare by computing the sign of $\frac{\phi^{CS}}{\partial L}$, $\frac{\phi^{CS}}{\partial L}$, respectively, at the eigenvalues. We first analyze welfare when the networks are regular.

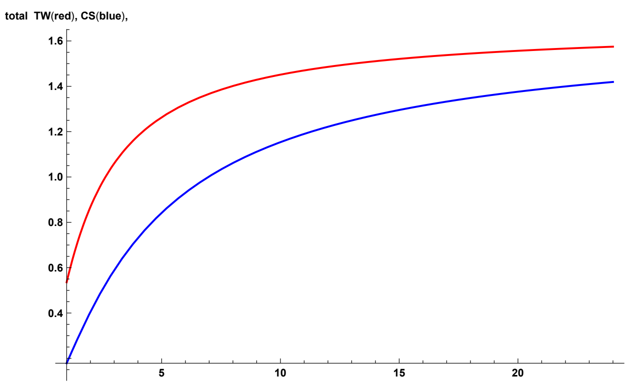

Figure D2: TW (in red)/CS (in blue) curves as a function of L.

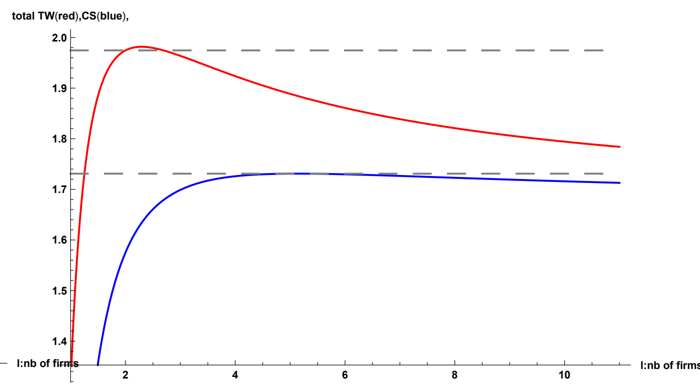

Figure D3: TW (in red)/CS (in blue) curves as a function of L.

Example 1 (cont.) *Under Assumption 3, we plot the consumer surplus and total welfare for the regular networks.*

(i) When the network effect is relatively small,[4] *both consumer surplus and total welfare are increasing in L; see Figure D2.*

(ii) When the network effect is relatively large,[5] *both consumer surplus and total welfare are inverted U-shaped in L; see Figure D3. Interestingly, as illustrated by Figure D3, perfect*

[4] In Figure D2, we set $a = 1, \delta d = 0.3$, $\beta = 0.3$

[5] In Figure D3, we set $a = 1, \delta d = 0.63$, $\beta = 0.3$

competition does not maximize the consumer surplus, as the consumer surplus first increases in L, then decreases in L. These non-monotonicity results are likely to be driven by strong network effects and market fragmentation effects so that an oligopoly market structure, instead of perfect competition, can generate the highest consumer surplus.

Using the results for regular networks, and the decompositions in (D14) and (D15), we can easily derive the welfare comparative-statics results for *non-regular* networks following the similar steps used in Section 5 for firms' profit.

E Additional results

E.1 Impact of network density and/or the strength of network effects on equilibrium prices

Proposition E4. *Suppose that Assumption 3 holds and the number of firms L is fixed. Then, increasing network density $\mathbf{G}$ or the strength of network effects δ decreases the equilibrium price for all consumers. Formally, suppose $\mathbf{G}' \succeq \mathbf{G}''$, and $\delta' \geq \delta''$. Then we have*

$$\mathbf{p}^*(\mathbf{G}', \delta') \preceq \mathbf{p}^*(\mathbf{G}'', \delta'').$$

Proof of Proposition E4: With Assumption 3, which assumes that $c = 0$ and $a_i = a$ for all i, the equilibrium prices are given by:

$$\mathbf{p}^* = \frac{1}{2}\mathbf{1}_N - \frac{(L-1)\beta}{2}[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1}\mathbf{1}_N,$$

by using the formula in the first row of (31). Note that

$$[(2+(L-3)\beta)\mathbf{I}_N - 2\delta\mathbf{G}]^{-1} = \frac{1}{(2+(L-3)\beta)}\sum_{j\geq 0}\left(\frac{2\delta}{2+(L-3)\beta}\right)^j \mathbf{G}^j.$$

When $\mathbf{G}' \succeq \mathbf{G}''$, and $\delta' \geq \delta''$, clearly,

$$\begin{aligned} [(2+(L-3)\beta)\mathbf{I}_N - 2\delta'\mathbf{G}']^{-1} &= \frac{1}{(2+(L-3)\beta)}\sum_{j\geq 0}\left(\frac{2\delta'}{2+(L-3)\beta}\right)^j \mathbf{G}'^j \\ \succeq \quad & \frac{1}{(2+(L-3)\beta)}\sum_{j\geq 0}\left(\frac{2\delta''}{2+(L-3)\beta}\right)^j \mathbf{G}''^j = [(2+(L-3)\beta)\mathbf{I}_N - 2\delta''\mathbf{G}'']^{-1}. \end{aligned}$$

Consequently, $\mathbf{p}^*(\mathbf{G}', \delta') \preceq \mathbf{p}^*(\mathbf{G}'', \delta'')$. □

E.2 Impact of network density on equilibrium profits

Proposition E5. *Given two network structures* $\mathbf{G}'$ *and* $\mathbf{G}''$ *with* $\mathbf{G}' \succeq \mathbf{G}''$*, there exist cutoffs* $\bar{L}$ *and* $\underline{L}$ *such that*[6]

(i) For any $L < \underline{L}$*,* $\quad \Pi^*(\mathbf{G}'; \beta, \delta, l) \geq \Pi^*(\mathbf{G}''; \beta, \delta, L)$;

(ii) For any $L > \bar{L}$*,* $\quad \Pi^*(\mathbf{G}'; \beta, \delta, L) \leq \Pi^*(\mathbf{G}''; \beta, \delta, L)$.

Proof of Proposition E5: We assume $\mathbf{G}' \succ \mathbf{G}''$. When $L = 1$, $\Pi^*(\mathbf{G}; \beta, \delta, L = 1) = \frac{1}{4}\mathbf{a}'[\mathbf{I}_N - \delta\mathbf{G}]^{-1}\mathbf{a}$ is monotone in $\mathbf{G}$; in other words, $\Pi^*(\mathbf{G}'; \beta, \delta, L = 1) > \Pi^*(\mathbf{G}''; \beta, \delta, L = 1)$ as $\mathbf{G}' \succ \mathbf{G}''$. By continuity, we obtain part (i). Using the following Taylor series expansion of Φ^{PT} at $L = \infty$:

$$\Phi^{PT}(z; \beta, \delta, L) = \frac{(1 - \beta - \delta z)}{\beta^2}\frac{1}{L^2} + \mathcal{O}(L^{-3}),$$

we obtain

$$\Pi^*(\mathbf{G}; \beta, \delta, L) = \langle \mathbf{a}, \frac{(1 - \beta)\mathbf{I}_N - \delta\mathbf{G})}{\beta^2}\frac{1}{L^2}\mathbf{a}\rangle + \mathcal{O}(L^{-3}).$$

Therefore,

$$\Pi^*(\mathbf{G}'; \beta, \delta, L) - \Pi^*(\mathbf{G}''; \beta, \delta, L) = \underbrace{\langle \mathbf{a}, \frac{\delta(\mathbf{G}'' - \mathbf{G}')}{\beta^2}\mathbf{a}\rangle}_{<0, \text{ as } \mathbf{G}' \succ \mathbf{G}''} \times \frac{1}{L^2} + \mathcal{O}(L^{-3}).$$

So, for sufficiently large L, $\Pi^*(\mathbf{G}'; \beta, \delta, L) < \Pi^*(\mathbf{G}''; \beta, \delta, L)$, which proves part (ii). $\square$

Increasing network density generates a similar rotation of the profit curve, as shown by Proposition 3 (for regular networks) and Propositions 5 and E5 (for general networks). As above, these results are due to the fact that there is a trade-off between the price effect and the social multiplier effect. When L, the number of firms, is small, the latter dominates the former and firms benefit from an increase in network externalities δ and in network density. When competition becomes very fierce because L is large, increasing δ or network density decreases profit because the negative impact on prices is stronger than the positive network effect.

E.3 Heterogeneous firms and asymmetric products

Here we present the equilibrium prices with asymmetric firms.

Proposition E6. *Suppose that Assumption 2 holds. Then, there exists a unique equilibrium in the pricing stage in which each firm* $l \in \mathcal{L}$ *chooses* $\mathbf{p}^l$ *such that:*

$$\mathbf{p}^{l*} = \mathbf{c}^l + \mathbf{H}(\bar{\mathbf{a}} - \bar{\mathbf{c}}) + \mathbf{D}((\mathbf{a}^l - \bar{\mathbf{a}}) - (\mathbf{c}^l - \bar{\mathbf{c}})) \tag{E16}$$

where $\mathbf{H}$ *and* $\mathbf{D}$ *are given by* (21).

[6] These thresholds depend on $\mathbf{G}'$, $\mathbf{G}''$, β and δ.

Proof of Proposition E6: With asymmetric firms, we can characterize the consumption equilibrium (CE), for a fixed price profile $\mathbf{p}$, as follows

$$\mathbf{x}(\mathbf{p}) = \begin{bmatrix} \mathbf{x}^1(\mathbf{p}) \\ \vdots \\ \mathbf{x}^L(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \cdots & \mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B} & \mathbf{B} & \cdots & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 - \mathbf{p}^1 \\ \vdots \\ \mathbf{a}^L - \mathbf{p}^L \end{bmatrix} \tag{E17}$$

where

$$\mathbf{A} = \frac{\mathbf{M}^+ + (L-1)\mathbf{M}^{-1}}{L}, \quad \mathbf{B} = \frac{\mathbf{M}^+ - \mathbf{M}^{-1}}{L}.$$

Here $\mathbf{M}^+$ and $\mathbf{M}^-$ are defined in (3). This result generalizes Lemma 1. The proof directly follows from Theorem 3 of Chen et al. (2018b), hence is omitted. Given this CE, we study each firm l's optimal price decisions. Each firm l's profit is $\pi^l(\mathbf{p}^l, \mathbf{p}^{-l}) = \langle \mathbf{x}^l(\mathbf{p}^l, \mathbf{p}^{-l}), (\mathbf{p}^l - \mathbf{c}^l)\rangle$, where $\mathbf{x}^l$ is given above. The FOCs for firm l yield are

$$\mathbf{x}^l(\mathbf{p}^l, \mathbf{p}^{-l}) = -\left(\frac{\partial \mathbf{x}^l(\mathbf{p}^l, \mathbf{p}^{-l})}{\partial \mathbf{p}^l}\right)' (\mathbf{p}^l - \mathbf{c}^l).$$

Collectively, the equilibrium price profile $\mathbf{P}^*$ solves the following:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \cdots & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \cdots & \mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B} & \mathbf{B} & \cdots & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 - \mathbf{p}^{1*} \\ \vdots \\ \mathbf{a}^L - \mathbf{p}^{L*} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{p}^{1*} - \mathbf{c}^1 \\ \vdots \\ \mathbf{p}^{L*} - \mathbf{c}^L \end{bmatrix}. \tag{E18}$$

The solution $\mathbf{p}^*$ to (E18) is unique as the coefficient matrix is positive definite, hence invertible. Though this linear system is of high dimensionality, it can be solved by exploiting the special structure of the underlying coefficient matrix. First, we note that (E18) implies that

$$\underbrace{(\mathbf{A} + (L-1)\mathbf{B})}_{=\mathbf{M}^+}(\bar{\mathbf{a}} - \bar{\mathbf{p}}^*) = \mathbf{A}(\bar{\mathbf{p}}^* - \bar{\mathbf{c}}) = \frac{\mathbf{M}^+ + (L-1)\mathbf{M}^{-1}}{L}(\bar{\mathbf{p}}^* - \bar{\mathbf{c}}),$$

where $\bar{\mathbf{p}}^* = \frac{\sum_{l\in\mathcal{L}} \mathbf{p}^{l*}}{L}$ is the average price. Solving it yields

$$\bar{\mathbf{p}}^* - \bar{\mathbf{c}} = \mathbf{H}(\bar{\mathbf{a}} - \bar{\mathbf{c}}), \tag{E19}$$

where $\mathbf{H} = \mathbf{M}^+[\mathbf{M}^+ + \mathbf{A}]^{-1}$. After some algebra, $\mathbf{H}$ can be simplified to the form given in (21). Also, (E18) implies that, for two firms $s \neq t$,

$$\underbrace{(\mathbf{A} - \mathbf{B})}_{=\mathbf{M}^-}((\mathbf{a}^s - \mathbf{p}^{s*}) - (\mathbf{a}^t - \mathbf{p}^{t*})) = \mathbf{A}((\mathbf{p}^{s*} - \mathbf{c}^s) - (\mathbf{p}^{t*} - \mathbf{c}^t)).$$

After some tedious calculations, we obtain

$$(\mathbf{p}^{s*} - \mathbf{c}^s) - (\mathbf{p}^{t*} - \mathbf{c}^t) = \mathbf{D}\underbrace{(\mathbf{a}^s - \mathbf{a}^t) - (\mathbf{c}^s - \mathbf{c}^t))}_{=\Delta^{st}}, \tag{E20}$$

where $\mathbf{D} = \mathbf{M}^-[\mathbf{M}^- + \mathbf{A}]^{-1}$, after some algebra, is reduced to the form given in (21). Combining (E19) and (E20) yields the equilibrium prices as given in (E16). □

E.4 Product compatibility

We first state a similar condition as Assumption 2 to make sure $\tilde{\mathbf{M}}^+$ and $\tilde{\mathbf{M}}^-$ are well-defined.

Assumption $2'$ *$\delta \geq 0$, $0 \leq \beta < 1$, $0 \leq \mu \leq \delta$, and*[7]

$$(1 + (L-1)\beta) - (\delta + (L-1)\mu)\lambda_1(\mathbf{G}) > 0, \quad 1 - \beta - (\delta - \mu)\lambda_1(\mathbf{G}) > 0. \tag{E21}$$

We have the following characterization of the symmetric equilibrium prices, which is a generalization of Proposition 1.

Proposition E7. *Suppose Assumptions 1 and 2' hold. With partial product compatibility, there exists a unique equilibrium in the pricing stage in which all firms charge the same price $\tilde{\mathbf{p}}^*$ defined as follows:*

$$\tilde{\mathbf{p}}^* = \frac{\mathbf{a} + \mathbf{c}}{2} - \frac{(L-1)(\beta\mathbf{I}_n - \mu\mathbf{G})}{2}[(2 + (L-3)\beta)\mathbf{I}_n - (2\delta + (L-3)\mu)\mathbf{G}]^{-1}(\mathbf{a} - \mathbf{c}). \tag{E22}$$

In equilibrium, each firm's profit is $\tilde{\Pi}^* = \langle \tilde{\mathbf{x}}^*, (\tilde{\mathbf{p}}^* - \mathbf{c})\rangle$, and the total consumer surplus is $\frac{L(1+\beta(L-1))}{2}\langle \tilde{\mathbf{x}}^*, \tilde{\mathbf{x}}^*\rangle$, where $\tilde{\mathbf{x}}^* = \tilde{\mathbf{M}}^+(\mathbf{a} - \tilde{\mathbf{p}}^*) \in \mathbf{R}^N$ is the each firm's demand in equilibrium.

Proof of Proposition E7: The first order conditions for the consumer i's consumption decision, $\max_{\mathbf{x}_i} \tilde{u}_i(\mathbf{x}_i, \mathbf{x}_{-i}) - \langle \mathbf{p}_i, \mathbf{x}_i\rangle$ yields

$$a_i^s - p_i^s - x_i^{s*} - \beta \sum_{t \neq s} x_i^{t*} + \delta \sum_{j \neq i} g_{ij} x_j^{s*} + \mu \sum_{t \neq s} \sum_{j \neq i} g_{ij} x_j^{t*} = 0 \tag{E23}$$

for all $i \in \mathcal{N}$, $s \in \mathcal{L}$. The consumption equilibrium is the solution to the above linear system. Note that by Assumption 1, $a_i^s = a_i$ for any s. Summing over s in (E23) and simplifying yield

$$\mathbf{a} - \bar{\mathbf{p}} - (1 + \beta(L-1))\bar{\mathbf{x}}^* + \delta\mathbf{G}\bar{\mathbf{x}}^* + \mu(L-1)\mathbf{G}\bar{\mathbf{x}}^* = \mathbf{0},$$

[7]Note that (E21) immediately implies that $1 - \delta\lambda_1(\mathbf{G}) > 0$.

or

$$\bar{\mathbf{x}}^* = \underbrace{[(1+(L-1)\beta)\mathbf{I}_n - (\delta+(L-1)\mu)\mathbf{G}]^{-1}}_{=\tilde{\mathbf{M}}^+}(\mathbf{a}-\bar{\mathbf{p}}).$$

Here $\bar{\mathbf{p}}$ is the average price, and $\bar{\mathbf{x}}^*$ is the average consumption. Comparing the conditions for firm s and firm t in (E23) yields,

$$\mathbf{x}^t - \mathbf{x}^s = -\underbrace{[(1-\beta)\mathbf{I}_n - (\delta-\mu)\mathbf{G}]^{-1}}_{=\mathbf{M}^-}(\mathbf{p}^t - \mathbf{p}^s).$$

Define

$$\tilde{\mathbf{A}} = \frac{\tilde{\mathbf{M}}^+ + (L-1)\tilde{\mathbf{M}}^{-1}}{L}, \quad \tilde{\mathbf{B}} = \frac{\tilde{\mathbf{M}}^+ - \tilde{\mathbf{M}}^{-1}}{L}.$$

where $\tilde{\mathbf{M}}^+$ and $\tilde{\mathbf{M}}^-$ are defined in (23). Then we have $\tilde{\mathbf{A}} + (L-1)\tilde{\mathbf{B}} = \tilde{\mathbf{M}}^+$ and $\tilde{\mathbf{A}} - \tilde{\mathbf{B}} = \tilde{\mathbf{M}}^-$. For a fixed price profile $\mathbf{P}$, we can fully characterize the consumption equilibrium (CE) as follows

$$\mathbf{x}(\mathbf{p}) = \begin{bmatrix} \mathbf{x}^1(\mathbf{p}) \\ \vdots \\ \mathbf{x}^L(\mathbf{p}) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} & \cdots & \tilde{\mathbf{B}} \\ \tilde{\mathbf{B}} & \tilde{\mathbf{A}} & \cdots & \tilde{\mathbf{B}} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{B}} & \tilde{\mathbf{B}} & \cdots & \tilde{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 - \mathbf{p}^1 \\ \vdots \\ \mathbf{a}^L - \mathbf{p}^L \end{bmatrix}. \tag{E24}$$

Using a similar argument as the proof of Proposition 1 and condition 9, a symmetric price $\tilde{\mathbf{p}}^*$ must satisfy the following conditions

$$\tilde{\mathbf{M}}^+(\mathbf{a}-\mathbf{p}^*) = \frac{\tilde{\mathbf{M}}^+ + (L-1)\tilde{\mathbf{M}}^-}{L}(\mathbf{p}^* - \mathbf{c}). \tag{E25}$$

Solving it yields the $\tilde{\mathbf{p}}^* = \mathbf{c} + \tilde{\mathbf{M}}^+[\tilde{\mathbf{M}}^+ + \frac{\tilde{\mathbf{M}}^+ + (L-1)\tilde{\mathbf{M}}^-}{L}]^{-1}(\mathbf{a}-\mathbf{c})$, which can be simplified to the form stated in (E22) in Proposition E7. The formulae for firm profit and aggregate consumer surplus follow immediately. □

F Optimal network structures

Proposition F8. *Let $\mathcal{G}$ denote the set of networks with n nodes: $\mathcal{G} = \{\mathbf{G} \in \{0,1\}^{n\times n} : g_{ii} = 0, g_{ij} = g_{ji}\}$.*

(i) Under any market structure, among networks in $\mathcal{G}$, the consumer-optimal network is the complete network.

(ii) Fix the number of firms L. Then, the firm-optimal network is given by:

(iia) When $L < \chi(\beta)$, there exists $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$, the firm-optimal network is the complete network.

(iib) When $L > \chi(\beta)$, there exists $\bar{\bar{\delta}} > 0$ such that, for any $0 < \delta < \bar{\bar{\delta}}$, the firm-optimal network is the empty network.

Proof of Proposition F8: (i) By Lemma A1, each consumer's equilibrium utility is equal to: $\frac{L(1+(L-1)\beta)}{2}(x_i^*)^2$, for each i, while the equilibrium consumption $\mathbf{x}^* = (x_1^*, \cdots, x_n^*)$ is given by $\mathbf{x}^* = \mathbf{M}^+(\mathbf{a} - \mathbf{p}^*)$. The rest just follows from the discussions in the main text.[8]

(ii) When δ is small, each firm's equilibrium profit is equal to:

$$\begin{aligned}\Pi^*(\mathbf{G}) &= \frac{(1+(L-2)\beta)(1-\beta)}{(1+(L-1)\beta)(2+(L-3)\beta)^2}\langle \mathbf{a}, \mathbf{a}\rangle \\ &\quad + \delta\frac{2+3(L-3)\beta - 6(L-2)\beta^2 - (L^3-2L^2-2L+5)\beta^3}{(1+(L-1)\beta)^2(2+(L-3)\beta)^3}\langle \mathbf{a}, \mathbf{G}\mathbf{a}\rangle + \mathcal{O}(\delta^2)\end{aligned}$$

which follows from the following Taylor series expansion of Φ^{PT} at $\delta = 0$:

Consider two networks $\mathbf{G}'$ and $\mathbf{G}''$,

$$\Pi^*(\mathbf{G}') - \Pi^*(\mathbf{G}'') = \delta \frac{\overbrace{2+3(L-3)\beta - 6(L-2)\beta^2 - (L^3-2L^2-2L+5)\beta^3}^{:=h(\beta,L)}}{\underbrace{(1+(L-1)\beta)^2(2+(L-3)\beta)^3}_{>0}}\langle \mathbf{a}, (\mathbf{G}' - \mathbf{G}'')(\mathbf{a})\rangle + \mathcal{O}(\delta^2)$$

Note that $\langle \mathbf{a}, (\mathbf{G}' - \mathbf{G}'')(\mathbf{a})\rangle = \sum_{i,j} a_i a_j (g'_{ij} - g''_{ij})$ is positive whenever $\mathbf{G}' \succ \mathbf{G}''$ (i.e., $g'_{ij} \geq g''_{ij}$ for any i, j, with strict inequality for some i, j). The sign of $h(\beta, L)$ is given by Lemma A3. When $L < \chi(\beta)$, $h(\beta, L)$ is positive, and therefore increasing network density generates a higher firm profit: $\Pi^*(\mathbf{G}') - \Pi^*(\mathbf{G}'') > 0$ for small δ, whenever $\mathbf{G}' \succ \mathbf{G}''$. This shows case (iia). The proof of case (iib) is similar. □

[8] By the same argument, consumer surpluses increase with δ for any $\mathbf{G}$ and L.