University of Southampton
Faculty of Mathematical Studies

The Karlhede Classification and Derivative Bounds

by

Julian M Collins

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

MATHEMATICS

Doctor of Philosophy

THE KARLHEDE CLASSIFICATION AND DERIVATIVE BOUNDS

by Julian Marc Collins

This thesis describes the Karlhede classification of a spacetime, this classification providing a means of tackling the well known equivalence problem in general relativity.

The Karlhede classification is performed on a number of cylindrically symmetric and stationary axisymmetric spacetimes, and the results given. An invariant classification scheme is presented for vacuum type D and vacuum type N spacetimes, and a *canonical form* is derived for each of the resulting classes, these canonical forms forming an essential part of the Karlhede classification. In addition, the theoretical upper bound on the order of covariant derivative of the Riemann tensor required to perform a Karlhede classification is examined and reduced for a number of cases.

Acknowledgements

This research was carried out under a research studentship from the Science and Engineering Research Council.

I would like to give warm thanks to both my supervisor, Ray d'Inverno, and my advisor James Vickers for their generous contributions and encouragement — they were both a pleasure to work with.

I would also like to thank the whole relativity group for providing such a friendly environment in which to work.

Table of Contents

1
Introduction

§1. Exact Solutions and the Karlhede Classification

With the publication in 1980 of the book entitled 'Exact solutions of Einstein's field equations' [1], the result of many years work, the study of exact solutions in general relativity took a major step towards becoming a much more systematic enterprise. Until then the many solutions which had been discovered were spread throughout the literature, but with the publication of this book all solutions known at that time were presented systematically in a single text, with some of their elementary properties discussed.

At around the same time major advances were being made in the development of a practical procedure to provide a complete classification of a spacetime which would tackle the equivalence problem. This is a problem right at the heart of general relativity, because it arises from the fact that the theory allows arbitrary coordinate transformations as one of its fundamental properties. Because of these arbitrary coordinate transformations the algebraic form of a given metric tensor can vary enormously depending on the coordinate system in which it is expressed. Indeed, there are many incidents in the literature of the reporting of apparently new solutions, which turn out to be known solutions in different coordinates. This leads to the *equivalence problem*, the problem of determining whether two given metrics are genuinely different, or whether they just represent a single metric expressed in two different coordinate systems. The theoretical resolution of this problem was originally provided by Cartan [2], who showed that it is necessary to calculate the Riemann tensor and its successive covariant derivatives expressed in a frame with constant frame metric. He proved that in the worst case it would be necessary to continue up to the 10th covariant derivative. Cartan's procedure was refined by Karlhede [3] into a practical algorithm, largely by the introduction of so called *canonical forms* for the spacetimes. A canonical form is obtained by choosing a frame in which the Riemann tensor and its derivatives adopt a simple standardly defined form. Karlhede was also able to prove that the upper bound could, in fact, be reduced to seven in the worst case (in fact, Karlhede's proof did not cover the conformally flat case, but it was subsequently shown that the result could indeed be extended to this case as well).

The development of the Karlhede algorithm occurred at a time when computer algebra systems, and especially the system SHEEP [4], were being used for work on exact solutions. As a consequence, a project was set up to use Karlhede's algorithm to classify all known exact solutions, which were now conveniently at hand in the exact solutions book. Karlhede's classification was implemented in a computer program called CLASSI [4], which could perform large parts of the classification automatically. CLASSI was written in SHEEP, which was specifically designed for calculations in general relativity. A computer database of the solutions classified by CLASSI is gradually being built up [5], and at the time of writing it contains well over a hundred exact solutions (depending on the method of counting). When the database is complete it is hoped that it can be widely available to workers in the field of exact solutions, where it will be of great help in determining whether newly discovered solutions are indeed new or are just old solutions in new coordinates.

§2. The Organisation of the Thesis

The Karlhede classification is a kind of generalisation of the well known Petrov classification [6], in that the Petrov classification classifies only the Weyl tensor whereas the Karlhede classification classifies the Riemann tensor and its covariant derivatives. Indeed, CLASSI first performs a Petrov classification when running a Karlhede classification, because this is useful for calculating the *canonical forms* (see chapter 4). In chapter 2 of this thesis the Petrov classification is discussed and an algorithm for performing this classification is presented. This algorithm is due to Letniowski and McLenaghan [7], and it is very similar to an earlier algorithm of Åman et al [8]. The details of both of these algorithms have been thoroughly checked through and no new errors have been detected. Paper [7] identifies a few errors in [8] which have been confirmed. However, on checking the implementation in CLASSI of the Åman et al algorithm it was found that most of these errors had, in fact, been corrected in the code.

In chapter 3 the equivalence problem and its resolution in terms of the Karlhede classification are presented in great depth, together with some supporting appendices (appendices A and B). This discussion is largely based on a review by Karlhede [9], but it is included for completeness and many of the details are filled in and the subtleties are thoroughly discused. It is felt that this work is essential to a proper understanding of all subsequent work and, therefore, justifies the attention devoted to it.

In chapter 4 the practical operation of CLASSI is discussed and some approaches found to be useful when performing a Karlhede classification using CLASSI are examined. In particular, CLASSI has been used to classify the cylindrically symmetric and a number of the stationary axisymmetric vacuum solutions contained in [1], and the results of this classification work are presented. In addition, a few solutions and their generalisations have been investigated in more detail and the results discussed. A number of the stationary axisymmetric solutions in [1] were not successfully classified because the complexity of the metric caused the number of terms to multiply to such an extent that the computer algebra system was unable to cope. Although some further headway might be possible by clever choices of frame and carefully chosen algebraic substitutions, it is felt that a substantial number of these solutions may prove impossible to handle with presently available algebra systems.

As mentioned above, Karlhede has shown that in the worst case it is necessary to continue until the 7th covariant derivative in order to complete the Karlhede classification, which would entail calculating 3156 components. For complicated solutions each component would be expected to contain many terms, so the amount of computation required would become too much even for modern computer algebra systems. Therefore, the question of the upper bound on the order of covariant derivative required is an important question, not just from the theoretical perspective, but also in terms of assessing the viability of constructing the database. Each extra differentiation requires an extra $(n+1)(n+4)(n+5)$ components to be calculated, where n is the order of differentiation, so at higher orders even a reduction by one derivative represents a large computational saving. For example, a reduction from 7th order to 6th order represents a saving of 1056 components. In chapters 5 and 6 the question of the upper bound is addressed. All vacuum type D metrics are known explicitly as a result of some work by Kinnersley [10], and they have all been classified directly by CLASSI [11]. The result of this investigation is that it was found to be necessary to continue only up to the second covariant derivative. In chapter 5 the upper bound for this case is proved to be at most 3 by an indirect method which uses the field equations and Bianchi identities expressed in GHP formalism (Geroch-Held-Penrose [12]), together with some symmetry considerations. The importance of this approach is that it does *not* require any integration of the field equations, and can, therefore, be extended to cases other than vacuum type D where a complete set of solutions is not available. Karlhede's analysis shows that an upper bound of 7 only occurs for non-vacuum types D and N and vacuum type N, with all other cases having an upper bound of only 5 (Karlhede's analysis shows that vacuum type D has an upper bound of 5 because all vacuum type D solutions are known to have a 2-dimensional isometry group). In chapter 6 the upper bound is analysed for vacuum type N solutions using a similar approach to that for vacuum type D, except that NP formalism (Newman-Penrose formalism [13]) is used instead of GHP formalism. It is found that in the worst case the upper bound can be reduced from 7 to 6, although in other cases it can be reduced as far as the second covariant derivative.

In addition to lowering the upper bound, chapters 5 and 6 also analyse the canonical forms required for the Karlhede classification of vacuum type D and vacuum type N spacetimes. These canonical forms are the essential aspect of Karlhede's work which make it an effective practical procedure for tackling the equivalence problem, and are, therefore, of fundamental importance. All type D and type N vacuum spacetimes are split into a number of invariant classes, and a canonical form is derived for each class in turn. In addition to finding the canonical forms, the frame transformations required to fix them are also calculated.

In the final chapter, chapter 7, the upper bound for non-vacuum type D spacetimes is analysed. Because it is already known from chapter 6 that in the worst case it may still be necessary to continue up to the 6th covariant derivative, attention is focused on reducing the upper bound from 7 to 6. The approach here is similar to that for the vacuum cases in that it uses the field equations, the Bianchi identities and some symmetry considerations, but use is also made of crucial constraints on the form of the Ricci spinor if the worst case is to be realised.

§3. Future Work

Although the Karlhede algorithm has now been checked independently, its implementation in CLASSI has not. Moreover, the *canonical forms* reported in this thesis have not yet been implemented and this would seem to be a desirable addition.

On the theoretical side, the first project to be attempted should probably be the application of the techniques of chapter 7 to the case of non-vacuum type

N spacetimes. Some progress with this has been achieved already, although it has not yet proved possible to lower the upper bound. The approach could also be extended to solutions of Petrov types I, II and III. Karlhede lowers the upper bound to 5 for these cases. It seems that one of the main problems with using the techniques of chapters 5, 6 and 7 for these cases is that the symmetry considerations would not be effective because the invariance group of the Riemann tensor, which is the group of transformations which leaves its canonical form invariant, is zero dimensional (see chapter 3). One would then be reliant on reductions of the upper bound being achieved purely by analysis of the field equations and Bianchi identities.

Finally, it may well be the case that this indirect approach could be pushed even further in particular cases (especially by exploiting special symmetry conditions), which might lead to yet greater reductions in the upper bounds.

2
The Petrov Classification

§1. Introduction

The Petrov classification is an algebraic classification of the Weyl tensor $C_{\alpha\beta\gamma\delta}$, and it is most easily and elegantly described in the language of spinors. We shall not deal with spinors in this thesis but shall use much spinor theory, so the reader unfamiliar with this topic is advised to consult one of the many references dealing with it (for example [14]).

§2. The Classification

The starting point for the spinor description is to calculate the spinor equivalent of the Weyl tensor. The result is

$$C_{\alpha\beta\gamma\delta} \leftrightarrow \Psi_{ABCD}\epsilon_{W'X'}\epsilon_{Y'Z'} + \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{W'X'Y'Z'} \tag{2.1}$$

(the symbol $\leftrightarrow$ means 'is equivalent to' in the sense of spinor equivalents of tensors). The spinor Ψ_{ABCD} is a totally symmetric spinor called the Weyl spinor which determines and is determined by the Weyl tensor.

It can easily be proved that any totally symmetric spinor can be expressed as a symmetrised product of 1-spinors.

Proof :

Let $\phi_{AB\ldots K}$ be a totally symmetric spinor with p indices. Let ζ^A be an arbitrary spinor and consider the expression

$$\phi(\zeta) = \phi_{AB\ldots K}\zeta^A\zeta^B\ldots\zeta^K$$

This is a homogeneous polynomial of degree p in ζ^0, ζ^1 (for example if we take $\zeta^0 = 1$ and $\zeta^1 = z$, we simply have a polynomial in z). Such a polynomial can always be factorised into p linear factors. Thus

$$\phi(\zeta) = (\alpha_A\zeta^A)(\beta_B\zeta^B)\ldots(\pi_K\zeta^K)$$

or

$$(\phi_{AB\ldots K} - \alpha_A\beta_B\ldots\pi_K)\zeta^A\zeta^B\ldots\zeta^K = 0 \tag{2.2}$$

The left hand side of (2.2) will contain $p!/m!n!$ terms which contain m ζ^0s and n ζ^1s, where $m+n=p$. If we add all these terms together the resulting coefficient will, with a little thought, be seen to be

$$(p!/m!n!)(\phi_{(AB\ldots K)} - \alpha_{(A}\beta_B\ldots\pi_{K)})$$

with m of the indices 0 and n of them 1. This term, because it contains a different mix of ζ^0s and ζ^1s from all others and ζ^A is an arbitrary spinor, is independent of all others, so that (2.2) requires it to vanish. Therefore, we obtain

$$\phi_{AB\ldots K} = \alpha_{(A}\beta_B\ldots\pi_{K)} \tag{2.3}$$

where we have used the fact that $\phi_{AB\ldots K}$ is totally symmetric so that $\phi_{AB\ldots K} = \phi_{(AB\ldots K)}$. This argument can be repeated for all combinations of 0 and 1 in the indices, so equation (2.3) is our required result.

Clearly each of our spinors $\alpha_A\ldots\pi_k$, which are called *principal spinors*, is determined up to a complex scalar factor. It can be shown that each of them determines a real null vector, and hence a real null direction. Because the 1-spinors in (2.3) need not all be distinct, the decomposition (2.3), called the *canonical decomposition* of $\phi_{AB\ldots K}$, determines at least 1 and at most p real null directions, called the *principal null directions* of $\phi_{AB\ldots K}$. Thus the multiplicities of the principal null directions provide a means of classifying any symmetric spinor.

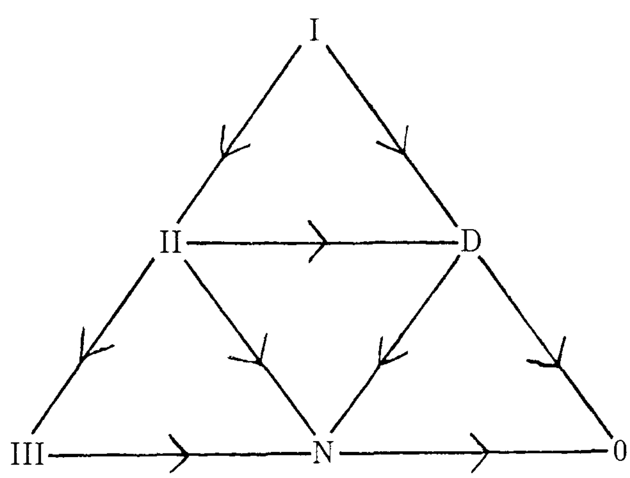
As has been stated following (2.1), the Weyl tensor determines and is determined by a totally symmetric 4-spinor Ψ_{ABCD} called the Weyl spinor, and thus we can classify the Weyl spinor, and hence the Weyl tensor, in terms of the multiplicities of the principal null directions. This gives Penrose's form of the Petrov classification [6]. The classification can be summarised by the following table :

Table of Petrov Types

Partition	Petrov type	Form of Ψ_{ABCD}
[1111]	I	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
[211]	II	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$
[22]	D	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}$
[31]	III	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$
[4]	N	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$
—	0	$\Psi_{ABCD} = 0$

The increasing algebraic specialisation as one goes down this table of Petrov types may be set out in the following diagram :

Penrose Diagram



The arrow points in the direction of increasing specialisation. Type I is sometimes referred to as algebraically general, the other types as algebraically special.

This all seems very nice but rather abstract. How do we determine the Petrov type in practice? A well known result from the theory of spinors is that the contraction of two spinors is zero if and only if they are proportional. Thus if we denote an arbitrary spinor by ζ^A, then it is proportional to a principal spinor if and only if

$$\Psi_{ABCD}\zeta^A\zeta^B\zeta^C\zeta^D = 0 \tag{2.4}$$

or

$$\Psi_{ABCD}\delta^A_E\zeta^E\delta^B_F\zeta^F\delta^C_G\zeta^G\delta^D_H\zeta^H = 0$$

Using the result that $\xi_{aA}\xi^{aB} = \delta^B_A$ for a dyad ξ^A_a, we can write this in terms of dyad components as

$$\Psi_{ABCD}\xi_{aE}\xi^{aA}\zeta^E\xi_{bF}\xi^{bB}\zeta^F\xi_{cG}\xi^{cC}\zeta^G\xi_{dH}\xi^{dD}\zeta^H = 0$$

$\Rightarrow$

$$\Psi^{abcd}\zeta_a\zeta_b\zeta_c\zeta_d = 0$$

$\Rightarrow$

$$\Psi_{abcd}\zeta^a\zeta^b\zeta^c\zeta^d = 0 \tag{2.5a}$$

Writing (2.5a) out in full it becomes

$$\Psi_0(\zeta^0)^4 + 4\Psi_1(\zeta^0)^3\zeta^1 + 6\Psi_2(\zeta^0)^2(\zeta^1)^2 + 4\Psi_3\zeta^0(\zeta^1)^3 + \Psi_4(\zeta^1)^4 = 0 \tag{2.5b}$$

where

$$\begin{aligned} \Psi_0 &= \Psi_{0000} \\ \Psi_1 &= \Psi_{0001} \\ \Psi_2 &= \Psi_{0011} \\ \Psi_3 &= \Psi_{0111} \\ \Psi_4 &= \Psi_{1111} \end{aligned} \tag{2.6}$$

are the Newman-Penrose scalars. Note that we have used the total symmetry of Ψ_{ABCD} to deduce the total symmetry of Ψ_{abcd} and hence obtain relations like $\Psi_{0001} = \Psi_{0010}$ e.t.c.. Assuming $\zeta^1 \neq 0$ then dividing (2.5b) by $(\zeta^1)^4$ and letting $\zeta^0/\zeta^1 = z$ one obtains the equation

$$\Psi_0 z^4 + 4\Psi_1 z^3 + 6\Psi_2 z^2 + 4\Psi_3 z + \Psi_4 = 0 \tag{2.7}$$

Distinct roots of this equation represent distinct ratios ζ^0/ζ^1 and hence distinct (i.e. non-proportional) spinors satisfying (2.4). Thus the multiplicities of the roots of (2.7) gives the multiplicities of the principal null directions of the Weyl spinor. If it happens that $\Psi_0 = 0$, then (2.7) will not be quartic, but the above analysis can still be performed by defining the roots above the order of the equation to be ∞.

In order to determine the multiplicities of the roots of (2.7) we need to study the theory of quartic equations [15]. We now summarise the results obtained. Defining :

$$
\begin{aligned}
I &\equiv \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3{\Psi_2}^2 \\
J &\equiv \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix} \\
G &\equiv {\Psi_0}^2\Psi_3 - 3\Psi_0\Psi_1\Psi_2 + 2{\Psi_1}^3 \\
H &\equiv \begin{vmatrix} \Psi_0 & \Psi_1 \\ \Psi_1 & \Psi_2 \end{vmatrix} \\
Z &\equiv {\Psi_0}^2 I - 12H^2 \\
D &\equiv I^3 - 27J^2
\end{aligned}
\tag{2.8}
$$

we have the conditions (iff $\equiv$ 'if and only if')

C1 : $D \neq 0$ iff 4 distinct roots.
C2 : $I = J = 0$ iff at least 3 equal roots.
C3 : $G = H = I = 0$ iff 4 equal roots.
C4 : $G = Z = 0$ iff 2 pairs of equal roots (not necessarily distinct).

Using this information the Petrov type can be determined as follows :

Type 0 : Iff $\Psi_0 \ldots \Psi_4$ are all zero.
Type I : Iff C1.

Type II : Iff not C1 and not C2 and not C4.

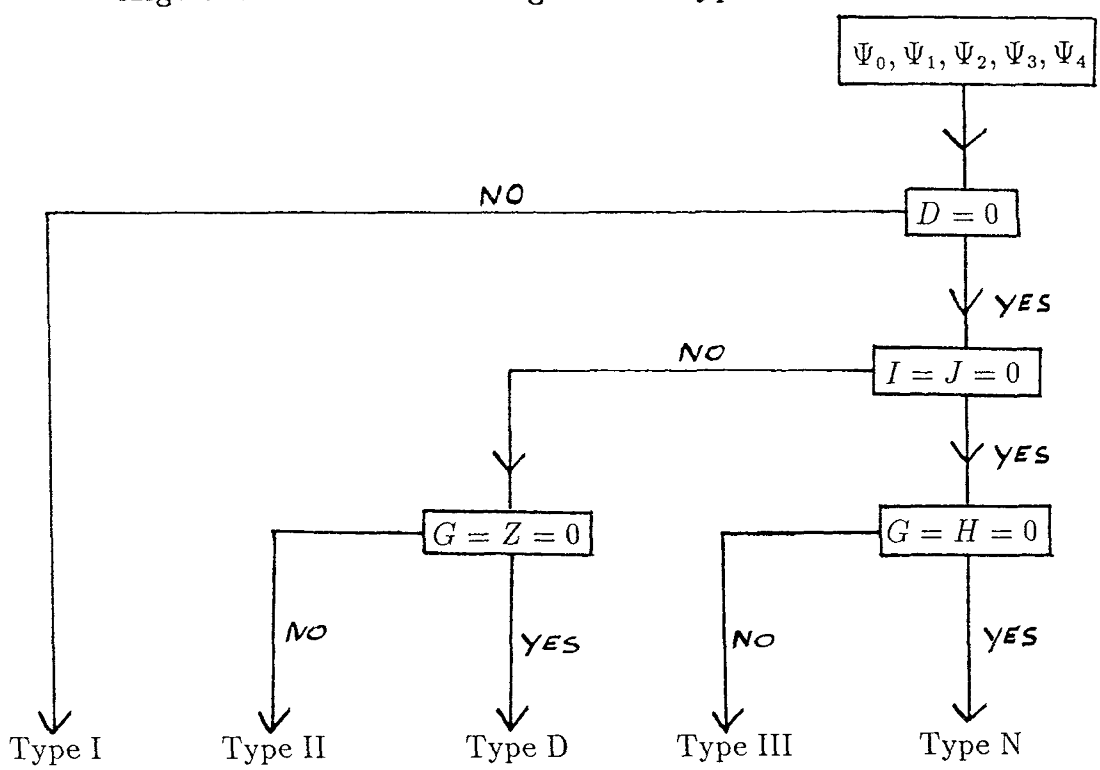
Type III : Iff C2 but not C3.

Type N : Iff C3.

Type D : Iff C4 but not C3.

d'Inverno and Russell-Clark [16] applied this logic directly in producing their algorithm for determining Petrov type from the Ψs. Their algorithm is represented in the following flow-diagram :

Algorithm for Determining Petrov Type from the Ψs



In the case where $\zeta^1 = 0$ we instead divide (2.5b) by $(\zeta^0)^4$, where we know that $\zeta^0 \neq 0$ (or else we would have $\zeta^A = 0$). In this case instead of (2.7) we obtain

$$\Psi_4 z^4 + 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0 \tag{2.9}$$

where we now have $z = \zeta^1/\zeta^0$. Thus if we interchange Ψ_0 with Ψ_4 and Ψ_1 with Ψ_3 in the definitions (2.8) then the algorithm proceeds as before. However, given only the set of Ψs how does one know which definitions (2.8) to use in the algorithm. The answer is that it does not matter. If one makes the substitution $z = 1/y$ in (2.7) and then multiplies through by y^4, then a quartic in y is obtained with the coefficients reversed. Since there is a $1-1$ correspondence between z and y we conclude that using the reversed coefficients in the algorithm will give exactly the same result for the multiplicities of the roots.

Note that in the case where $\Psi_0 = \Psi_4 = 0$, although the above algorithm can be used, d'Inverno and Russell-Clark present a much simplified algorithm.

§3. Advanced Algorithms

Although the logic of the d'Inverno and Russell-Clark algorithm is very clear, what is really required is an algorithm that minimises the amount of computation required to calculate the Petrov type. The general strategy behind these more complex algorithms is to delay the calculation of quantities of high order in the Ψs (for example D which is sixth order) until as late as possible, with the expectation that lower order quantities may determine the Petrov type earlier in the algorithm. It will be noted that the d'Inverno and Russell-Clark algorithm starts by calculating D. Here we shall present an algorithm due to F.W. Letniowski and R.G. McLenaghan [7]. The mathematics of this algorithm is largely the same as an earlier one due to Åman et al [8], but the presentation is much clearer. The Åman et al algorithm (with some minor corrections) is the one used by CLASSI and is compared to the Letniowski and McLenaghan algorithm in detail in [7].

The Letniowski and McLenaghan algorithm considers the thirty two different possible permutations of zero/non-zero Ψ_i, $i = 0, 1, 2, 3, 4$, assigns to each permutation a number n ranging from 0 to 31 and then analyses the Petrov type for each n. Let us introduce the symbol r_i where

$$r_i = 0 \quad \text{if } \Psi_i = 0$$

$$r_i = 1 \quad \text{if } \Psi_i \neq 0$$

Then n is defined by the equation

$$n = [(r_0 r_1 r_2 r_3 r_4)_{base\ 2}]_{base\ 10} \tag{2.10}$$

For example, if we have

$$\Psi_0 \neq 0,\ \ \Psi_1 = 0,\ \ \Psi_2 \neq 0,\ \ \Psi_3 = 0,\ \ \Psi_4 = 0 \tag{2.11}$$

then

$$n = [(10100)_{base\ 2}]_{base\ 10} = 20 \tag{2.12}$$

We introduce a shorthand whereby a non-zero Ψ_i is denoted by the letter N. Thus the permutation of Ψ_i in (2.11) would be denoted by N0N00. The following table shows how Petrov type is analysed in terms of the number n :

Table of Petrov Type Analysis by the Number n

$\Psi_0\Psi_1\Psi_2\Psi_3\Psi_4$ N=non-zero	n	**Analysis**
00000	0	type 0 by definition
0000N	1	type N, all four roots are ∞
000N0	2	type III, 3 roots ∞, 1 root 0
000NN	3	type III, 3 roots ∞, 1 root $= -\Psi_4/4\Psi_3$
00N00	4	type D, 2 roots ∞, 2 roots 0
00N0N	5	type II, 2 roots ∞, others $\pm(-\Psi_4/6\Psi_2)^{1/2}$
00NN0	6	type II, 2 roots ∞, 1 root 0, 1 root $-2\Psi_3/3\Psi_2$
00NNN	7	2 roots ∞, other 2 roots are equal iff the discriminant of the quadratic is 0 Thus if $2{\Psi_3}^2 - 3\Psi_2\Psi_4 = 0$ then type D, else type II
0N000	8	type III, 1 root ∞, 3 roots 0
0N00N	9	1 root ∞, $4\Psi_1 z^3 - \Psi_4 = 0$ has 3 distinct roots $\Rightarrow$ type I
0N0N0	10	type I, 1 root ∞, 1 root 0, others $\pm(-\Psi_3/\Psi_1)^{1/2}$

$\Psi_0\Psi_1\Psi_2\Psi_3\Psi_4$ N=non-zero	n	Analysis
0N0NN	11	$G = 2\Psi_1{}^3 \neq 0 \Rightarrow$ not type D or N $I = -4\Psi_1\Psi_3 \neq 0 \Rightarrow$ not type III $D = 0 \Leftrightarrow 27\Psi_4{}^2\Psi_1 + 64\Psi_3{}^3 = 0 \Leftrightarrow$ type II else type I
0NN00	12	same analysis as 6 with coefficients in reverse order
0NN0N	13	$G = 2\Psi_1{}^3 \neq 0 \Rightarrow$ not type D or N $I = 3\Psi_2{}^2 \neq 0 \Rightarrow$ not type III $D = 0 \Leftrightarrow \Psi_1{}^2\Psi_4 + 2\Psi_2{}^3 = 0 \Leftrightarrow$ type II else type I
0NNN0	14	1 root ∞, 1 root 0 the other 2 roots are equal iff the discriminant of the quadratic is 0 Thus if $9\Psi_2{}^2 - 16\Psi_1\Psi_3 = 0$ then type II, else type I
0NNNN	15	special case 1 (see later)
N0000	16	all 4 roots are 0, type N
N000N	17	$\Psi_0 z^4 + \Psi_4 = 0$ has 4 distinct roots $\Rightarrow$ type I
N00N0	18	same analysis as 9 with coefficients in reverse order

$\Psi_0\Psi_1\Psi_2\Psi_3\Psi_4$ N=non-zero	n	**Analysis**
N00NN	19	$G = \Psi_0{}^2\Psi_3 \neq 0 \Rightarrow$ not type D or N $I = \Psi_0\Psi_4 \neq 0 \Rightarrow$ not type III $D = 0 \Leftrightarrow \Psi_0\Psi_4{}^3 - 27\Psi_3{}^4 = 0 \Leftrightarrow$ type II else type I
N0N00	20	same analysis as 5 with coefficients in reverse order
N0N0N	21	results in a quadratic in z^2, each root of the quadratic gives 2 distinct roots of the quartic $\Rightarrow$ if the discriminant of the quadratic is 0 then type D, else type I i.e. $9\Psi_2{}^2 - \Psi_0\Psi_4 = 0 \Leftrightarrow$ type D, else type I
N0NN0	22	same analysis as 13 with coefficients in reverse order
N0NNN	23	special case 2 (see later)
NN000	24	same analysis as 3 with coefficients in reverse order
NN00N	25	same analysis as 19 with coefficients in reverse order
NN0N0	26	same analysis as 11 with coefficients in reverse order
NN0NN	27	special case 3 (see later)
NNN00	28	same analysis as 7 with coefficients in reverse order

$\Psi_0\Psi_1\Psi_2\Psi_3\Psi_4$ N=non-zero	n	Analysis
NNN0N	29	same analysis as 23 with coefficients in reverse order
NNNN0	30	same analysis as 15 with coefficients in reverse order
NNNNN	31	special case 4 (see later)

Special Case 1 $n = 15$ or 30 (with coefficients reversed)

In this case the quartic reduces to the cubic equation

$$4\Psi_1 z^3 + 6\Psi_2 z^2 + 4\Psi_3 z + \Psi_4 = 0$$

The quantity I reduces to

$$I = 3{\Psi_2}^2 - 4\Psi_1\Psi_3$$

The quantity J reduces to

$$J = -{\Psi_1}^2\Psi_4 + 2\Psi_1\Psi_2\Psi_3 - {\Psi_2}^3$$

If one defines

$$F_1 \equiv 2\Psi_2\Psi_3 - 3\Psi_1\Psi_4$$
$$F_2 \equiv 9\Psi_2\Psi_4 - 8{\Psi_3}^2$$

then the discriminant D is given by

$$\hat{D} \equiv -D/{\Psi_1}^2 = 3{F_1}^2 + 2IF_2$$

It is important to note that I, F_1 and F_2 are not independent since

$$\Psi_1 F_2 + 3\Psi_2 F_1 - 2\Psi_3 I = 0$$

$\Rightarrow$ it is not possible to have exactly 2 of I, F_1 and F_2 zero.

The algorithm for this case is given in figure 1.

Figure 1. Algorithm for Special Case 1

One root is ∞ but the other three are finite $\Rightarrow$ not type D or N

If $I = 0$ then $3J = \Psi_1 F_1$, hence if $F_1 = 0$ then $I = J = 0 \Rightarrow$ type III

else $\hat{D} = 3{F_1}^2 \neq 0 \Rightarrow$ type I

else $I \neq 0 \Rightarrow$ not type III

if $F_1 = 0$ then $F_2 \neq 0 \Rightarrow \hat{D} = 2IF_2 \neq 0 \Rightarrow$type I

if $F_2 = 0$ then $F_1 \neq 0 \Rightarrow \hat{D} = 3{F_1}^2 \neq 0 \Rightarrow$ type I

if $\hat{D}$ $(= 3{F_1}^2 + 2IF_2) = 0 \Rightarrow$ type II

else $\hat{D} \neq 0 \Rightarrow$ type I

Special Case 2 $n = 23$ or 29 (with coefficients reversed)

In this case the quartic reduces to the equation

$$\Psi_1 z^4 + 6\Psi_2 z^2 + 4\Psi_3 z + \Psi_4 = 0$$

The quantity I reduces to

$$I = \Psi_0\Psi_4 + 3{\Psi_2}^2$$

If $I = 0$ then

$$3J/\Psi_0 = 4\Psi_2\Psi_4 - 3{\Psi_3}^2 \equiv \hat{J}$$

If one defines $F_3 \equiv \Psi_0\hat{J} - 2\Psi_2 I$ then the discriminant D satisfies the relation

$$\tilde{D} \equiv D/\Psi_0 = \Psi_4 I^2 - 3\hat{J}F_3$$

The algorithm for this case is given in figure 2.

Figure 2. Algorithm for Special Case 2

G reduces to ${\Psi_0}^2\Psi_3 \neq 0 \Rightarrow$ not type N or D

If $I = 0$ then if $\hat{J} = 0 \Rightarrow I = J = 0 \Rightarrow$ type III

else $\tilde{D} = -3\Psi_0\hat{J}^2 \neq 0 \Rightarrow$ type I

else $I \neq 0 \Rightarrow$ not type III

if $\hat{J} = 0$ then $\tilde{D} = \Psi_4 I^2 \neq 0 \Rightarrow$ type I

if $F_3 = 0$ then $\tilde{D} = \Psi_4 I^2 \neq 0 \Rightarrow$ type I

if $\tilde{D}$ $(= \Psi_4 I^2 - 3\hat{J}F_3) = 0 \Rightarrow$ type II

else $\tilde{D} \neq 0 \Rightarrow$ type I

Special Case 3 $\quad n = 27$

In this case the quartic reduces to the equation

$$\Psi_0 z^4 + 4\Psi_1 z^3 + 4\Psi_3 z + \Psi_4 = 0$$

We have

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3$$

$$J = -\Psi_0\Psi_3^{\,2} - \Psi_1^{\,2}\Psi_4$$

$$G = \Psi_0^{\,2}\Psi_3 + 2\Psi_1^{\,3}$$

$$Z = \Psi_0^{\,3}\Psi_4 - 4\Psi_0^{\,2}\Psi_1\Psi_3 - 12\Psi_1^{\,4}$$

Let us define

$$U \equiv \Psi_0\Psi_4 + 2\Psi_1\Psi_3$$

$$V \equiv \Psi_0\Psi_3^{\,2} - \Psi_1^{\,2}\Psi_4$$

$$W \equiv \Psi_0\Psi_4 - 16\Psi_1\Psi_3$$

If $G = 0$ then

$$\Psi_3 Z = -2\Psi_1^{\,3} U \tag{2.13}$$

If $U = 0$ then

$$\Psi_4 G = -2\Psi_1 V \tag{2.14}$$

For type D we have the condition

$$G = Z = 0$$

But from (2.13) $G = 0$ and $Z = 0 \Rightarrow U = 0$
From (2.14) $U = 0$ and $G = 0 \Rightarrow V = 0$
So $G = Z = 0 \Rightarrow U = V = 0$
Also from (2.14) if $U = 0$ and $V = 0 \Rightarrow G = 0$
From (2.13) if $U = 0$ and $G = 0 \Rightarrow Z = 0$
Therefore, overall

$$Type\ D \Leftrightarrow G = Z = 0 \Leftrightarrow U = V = 0$$

The discriminant D reduces to

$$D \equiv I^3 - 27J^2 = WU^2 - 27V^2$$

If one examines the algebraic form of I, U and W one finds that at most 1 of them can be zero. Similarly, the algebraic forms of J and V shows that at most 1 of them can be zero.

The algorithm for this case is given in figure 3.

Figure 3. Algorithm for Special Case 3

$H = -\Psi_1{}^2 \neq 0 \Rightarrow$ not type N

If $V = 0$ then if $U = 0 \Rightarrow G = Z = 0 \Rightarrow$ type D

else if $W = 0 \Rightarrow D = 0 \Rightarrow$ type II

else $D = WU^2 \neq 0 \Rightarrow$ type I

else if $I = 0$ then if $J = 0 \Rightarrow$ type III

else $D = -27J^2 \neq 0 \Rightarrow$ type I

else if $J = 0$ then $D = I^3 \neq 0 \Rightarrow$ type I

else if D $(= I^3 - 27J^2) = 0 \Rightarrow$ type II

else $D \neq 0 \Rightarrow$ type I

Special Case 4 $n = 31$

In this case, all the coefficients Ψ_i are non-zero. Therefore, we have the full quartic

$$\Psi_0 z^4 + 4\Psi_1 z^3 + 6\Psi_2 z^2 + 4\Psi_3 z + \Psi_4 = 0 \tag{2.15}$$

or equivalently for $x \equiv \Psi_0 z + \Psi_1$

$$x^4 + 6Hx^2 + 4Gx + {\Psi_0}^2 I - 3H^2 = 0 \tag{2.16}$$

This new equivalent form is useful when considering the case $H \neq 0$ and $G = 0$, as it then has coefficients $1, 0, H, 0, {\Psi_0}^2 I - 3H^2$. Then, if ${\Psi_0}^2 I - 3H^2 = 0$, we have the case N0N00 which, from the table, is found to be type II. Otherwise, we have the case N0N0N, which is type D if ${\Psi_0}^2 I - 12H^2 = 0$ or type I otherwise.

The quantities defined below are used in the following analysis

$$H \equiv \Psi_0\Psi_2 - {\Psi_1}^2$$
$$F \equiv \Psi_0\Psi_3 - \Psi_1\Psi_2$$
$$A \equiv \Psi_1\Psi_3 - {\Psi_2}^2$$
$$E \equiv \Psi_0\Psi_4 - {\Psi_2}^2$$

Then we obtain

$$I = E - 4A$$
$$G = \Psi_0 F - 2\Psi_1 H$$
$$J = \Psi_4 H - \Psi_3 F + \Psi_2 A$$

Note that A, F and H are dependent since

$$\Psi_0 A - \Psi_1 F + \Psi_2 H = 0 \tag{2.17}$$

Type N Determination

The quartic (2.15) is of type N iff there exists 'a' such that it can be written

$$\Psi_0(z-a)^4 = \Psi_0 z^4 - 4\Psi_0 a z^3 + 6\Psi_0 a^2 z^2 - 4\Psi_0 a^3 z + \Psi_0 a^4 = 0 \qquad (2.18)$$

Then by equating the coefficients of (2.15) and (2.18) and performing a little algebra it can be shown that we have type N iff

$$H = F = E = 0 \qquad (2.19)$$

The algorithm for this case is given in figure 4.

Figure 4. Algorithm for Special Case 4

If $H = 0$ by (2.17) $\Rightarrow \Psi_0 A = \Psi_1 F$

then if $F = 0 \Rightarrow A = 0 \Rightarrow J = 0$ then if $E = 0 \Rightarrow$ type N

else $I = E \neq 0 \Rightarrow$ type I

else $F \neq 0 \neq A$, $G = \Psi_0 F \neq 0 \Rightarrow$ not type D or N and $\Psi_1 J = -AF \neq 0$

$\Rightarrow$ not type III

if $E = 0$ then $\Psi_1{}^3 D = -\Psi_0 F A^2(37\Psi_2{}^2 + 27\Psi_1\Psi_3)$

if $37\Psi_2{}^2 + 27\Psi_1\Psi_3 = 0$ then $D = 0 \Rightarrow$ type II

else $D \neq 0 \Rightarrow$ type I

else if $I = 0$ then since $J \neq 0 \Rightarrow D \neq 0 \Rightarrow$ type I

else if $D = 0$ then type II

else type I

else $H \neq 0$

if $I = 0$ then if $J = 0 \Rightarrow$ type III

else $D = -27J^2 \neq 0 \Rightarrow$ type I

else if $G = 0$ then if $Z = 0 \Rightarrow$ type D

else if $\Psi_0{}^2 I - 3H^2 = 0 \Rightarrow$ type II

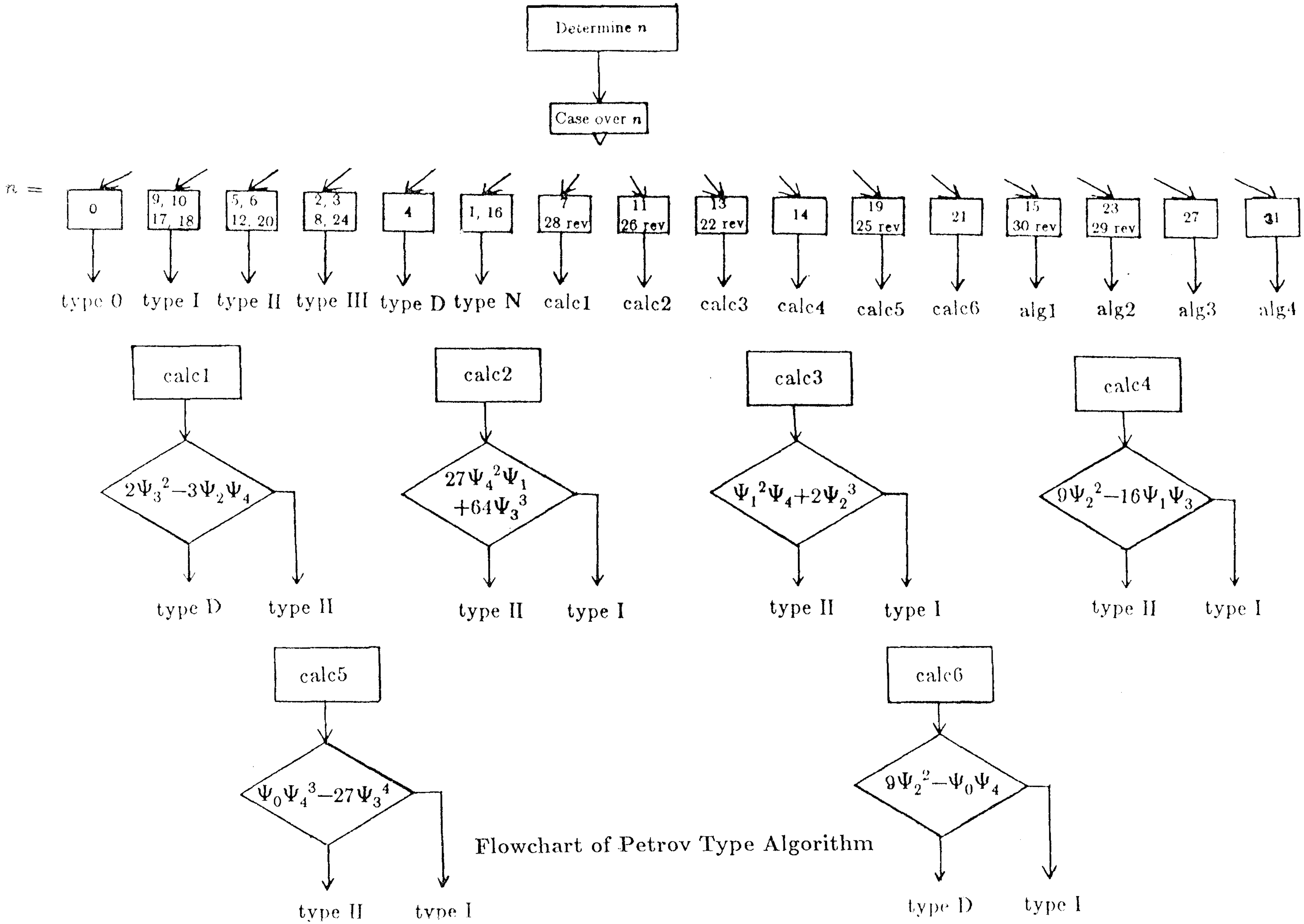
else type I

else if $J = 0$ then $D = I^3 \neq 0 \Rightarrow$ type I

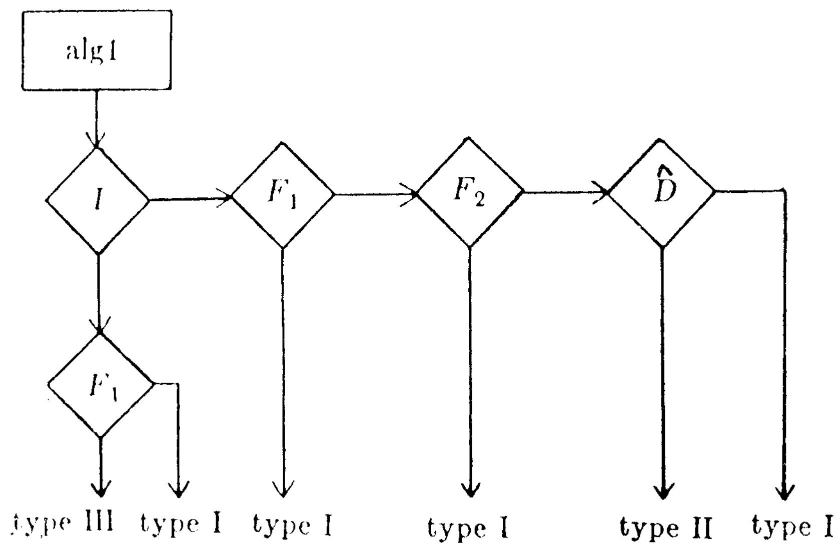
else if $D = 0$ then type II

else type I

The complete algorithm is presented in the following series of flow charts. The quantity appearing in a diamond shaped box is compared to 0 — the right direction from the box indicates that the test was false, the downward direction that it was true.



Flowchart of Petrov Type Algorithm



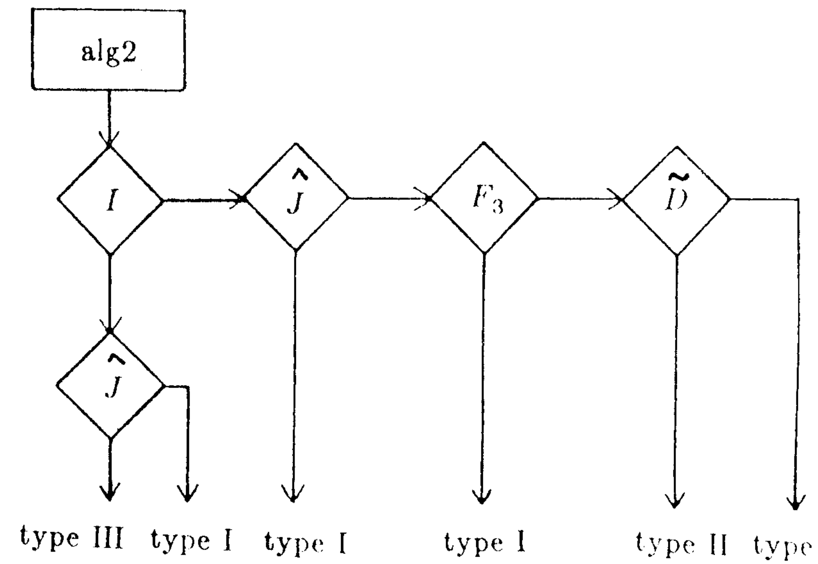
$$I = 3\Psi_2{}^2 - 4\Psi_1\Psi_3$$

$$F_1 = 2\Psi_2\Psi_3 - 3\Psi_1\Psi_4$$

$$F_2 = 9\Psi_2\Psi_4 - 8\Psi_3{}^2$$

$$\hat{D} = 3F_1{}^2 + 2IF_2$$

Case 0NNNN or NNNN0 reversed



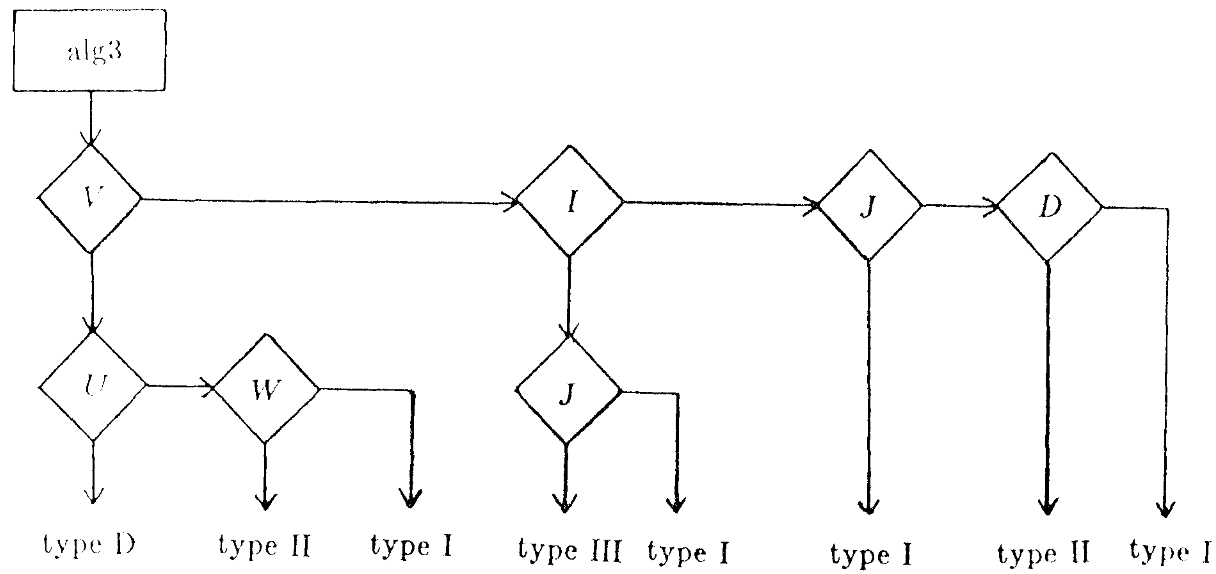
$$I = \Psi_0\Psi_4 + 3\Psi_2{}^2$$

$$\hat{J} = 4\Psi_2\Psi_4 - 3\Psi_3{}^2$$

$$F_3 = \Psi_0\hat{J} - 2\Psi_2 I$$

$$\tilde{D} = \Psi_4 I^2 - 3\hat{J}F_3$$

Case N0NNN or NNN0N reversed



$$U = \Psi_0\Psi_4 + 2\Psi_1\Psi_3$$

$$V = \Psi_0{\Psi_3}^2 - {\Psi_1}^2\Psi_4$$

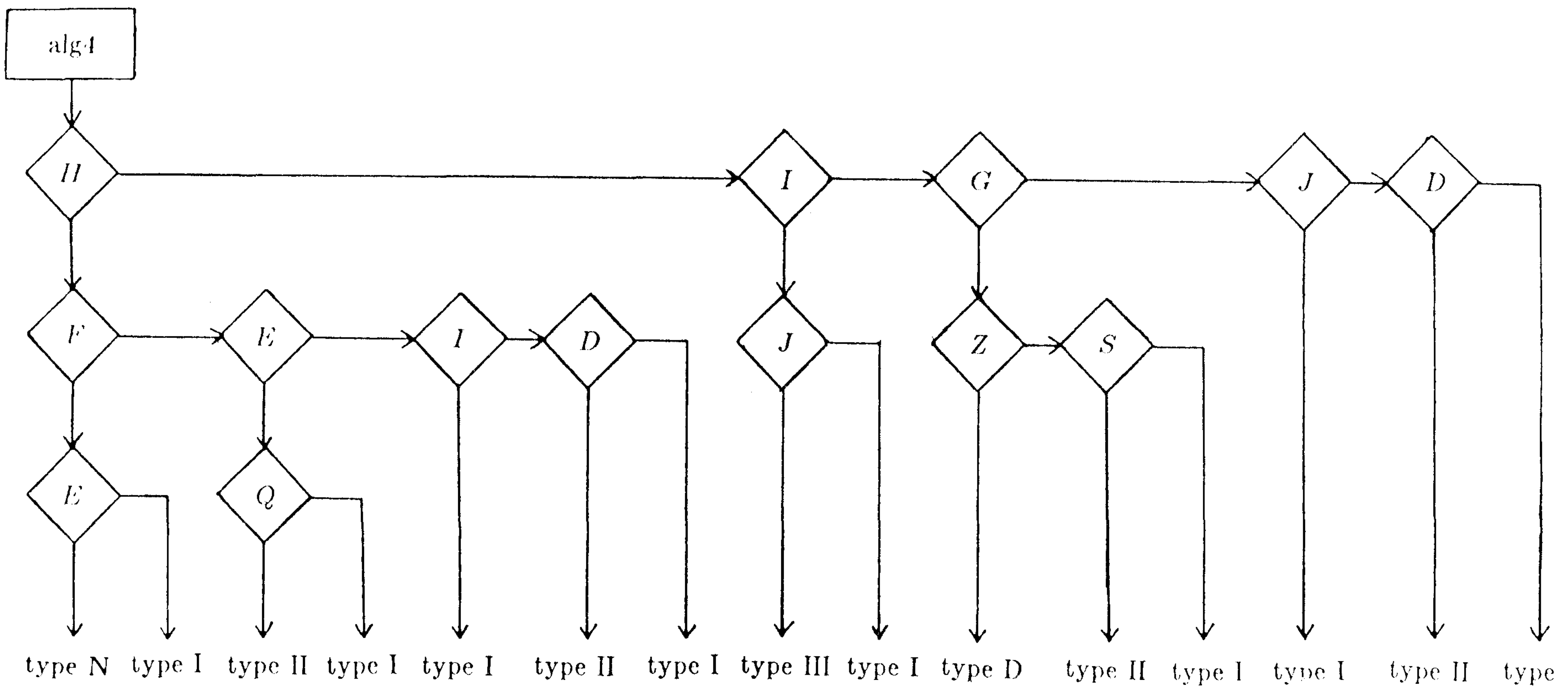
$$W = \Psi_0\Psi_4 - 16\Psi_1\Psi_3$$

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3$$

$$J = -\Psi_0{\Psi_3}^2 - {\Psi_1}^2\Psi_4$$

$$D = I^3 - 27J^2$$

Case NN0NN



$$H = \Psi_0\Psi_2 - \Psi_1{}^2$$

$$F = \Psi_0\Psi_3 - \Psi_1\Psi_2$$

$$E = \Psi_0\Psi_4 - \Psi_2{}^2$$

$$A = \Psi_1\Psi_3 - \Psi_2{}^2$$

$$I = E - 4A$$

$$Q = 37\Psi_2{}^2 + 27\Psi_1\Psi_3$$

$$J = \Psi_4 H - \Psi_3 F + \Psi_2 A$$

$$G = \Psi_0 F - 2\Psi_1 H$$

$$Z = \Psi_0{}^2 I - 12H^2$$

$$S = \Psi_0{}^2 I - 3H^2$$

$$D = I^3 - 27J^2$$

Case NNNNN

3
The Equivalence Problem

§1. Introduction

In this chapter the equivalence problem and its solution are discussed in detail as it is felt that an in depth understanding of this work is vital for a proper understanding of all subsequent work. The account given here closely follows a review by Karlhede [9], but concentrates on explaining as clearly as possible the various subtleties that arise. In this way it is hoped that the reader may be offered a short cut to a clear understanding of the problem, which for the author took a considerable amount of study.

We are given two metric tensor fields g and $\tilde{g}$ on manifolds M and $\tilde{M}$ where

$$g = \eta_{ij}\ \omega^i \otimes \omega^j \tag{3.1}$$

$$\tilde{g} = \eta_{ij}\ \tilde{\omega}^i \otimes \tilde{\omega}^j \tag{3.2}$$

(note that each manifold has the same constant frame metric η_{ij})

Then regions U and $\tilde{U}$ on M and $\tilde{M}$ respectively are said to be equivalent if and only if there exists a point-wise identification between points P in U and $\tilde{P}$ in $\tilde{U}$ such that

$$g_p = \tilde{g}_{\tilde{p}} \tag{3.3}$$

(g_p denotes the tensor at the point P)

If we express g_p and $\tilde{g}_{\tilde{p}}$ in terms of coordinates as

$$g_p = g_{\mu\nu} dx^\mu dx^\nu \tag{3.4}$$

$$\tilde{g}_{\tilde{p}} = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu \tag{3.5}$$

where x^μ and $\tilde{x}^\mu$ are local coordinates in U and $\tilde{U}$ respectively, and then express $d\tilde{x}^\mu$ in terms of dx^μ as

$$d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \tag{3.6}$$

we see that the condition $g_p = \tilde{g}_{\tilde{p}}$ gives

$$g_{\mu\nu} = \frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\, \tilde{g}_{\rho\sigma} \tag{3.7}$$

It is important to note that the investigation is for regions U and $\tilde{U}$ with local coordinates x^μ and $\tilde{x}^\mu$ and, therefore, determines whether the spaces are *locally* equivalent. As an example our investigation would find a plane and a cone to be equivalent because locally their geometries are the same.

§2. Preliminary Discussion of the Problem

Referring to (3.1) and (3.2) we see that if

$$\omega^i_p = \tilde{\omega}^i_{\tilde{p}} \tag{3.8}$$

then $g_p = \tilde{g}_{\tilde{p}}$. However, there exists a group of linear transformations of the 1-forms ω^i

$$\hat{\omega}^i = b^i{}_j\, \omega^j \tag{3.9}$$

which leave the frame metric invariant :

$$b^i{}_m \eta_{ij} b^j{}_n = \eta_{mn} \tag{3.10}$$

Then it is easily seen that these transformations $b^i{}_j$ also leave the metric, g, invariant

$$\begin{aligned} \hat{g} &= \eta_{ij}\ \hat{\omega}^i \otimes \hat{\omega}^j \\ &= \eta_{ij}\ b^i{}_m \omega^m \otimes b^j{}_n \omega^n \\ &= \eta_{mn}\ \omega^m \otimes \omega^n \\ &= g \end{aligned}$$

So we see that it is not necessary for the 1-forms to be equal to make the metrics equal, but only for them to be equal up to the transformations $b^i{}_j$. The transformations $b^i{}_j$ form a group G which has a continuous subgroup of rotations of dimension $n(n-1)/2$ where n is the dimension of M, together with a finite number of discrete transformations. In the case of $n = 4$ and η_{ij} the Lorentz metric, i.e. general relativity, then G is the six dimensional homogeneous Lorentz group, the continuous subgroup of rotations is the proper Lorentz group ($\mathcal{L}_+^\uparrow$) and the discrete transformations are space and time inversions. The analysis above enables us to state the following lemma :

Lemma 3.1a

Two geometries are equivalent if, and only if, there exists a point-wise identification $P = \tilde{P}$ of $P \in U$ and $\tilde{P} \in \tilde{U}$ and a transformation $b^i{}_j$ leaving η_{ij} invariant, such that

$$\tilde{\omega}^i_{\tilde{p}} = b^i{}_j \ \omega^j_p$$

Let $\omega^i(x^\mu, \epsilon^A, m)$ denote the basis of 1-forms at point P on the manifold with coordinate x^μ, with direction ϵ^A and m. x^μ are local coordinates in U, ϵ^A are the $n(n-1)/2$ parameters of the rotational subgroup of G and m are the discrete parameters in G. So one obtains all possible 1-forms at a point with frame metric η_{ij} by varying ϵ^A and m. In this language we can state the content of lemma (3.1a) as follows :

Lemma 3.1b

Two geometries are equivalent if, and only if, there is a correspondence

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu) \tag{3.11}$$

$$\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^B, x^\mu) \tag{3.12}$$

$$\tilde{m} = \tilde{m}(m) \tag{3.13}$$

which gives

$$\tilde{\omega}^i(\tilde{x}^\mu, \tilde{\epsilon}^A, \tilde{m}) = \omega^i(x^\mu, \epsilon^A, m) \tag{3.14}$$

A word about the m dependence — as stated before the number of discrete transformations in G is finite, so we can handle the m dependence by fixing m and investigating the existence of a solution for each value of $\tilde{m}$ in turn.

More generally we could have the relation

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu, \epsilon^A) \tag{3.15}$$

giving (3.14). However, for equivalence as we have defined it we need an identification of P in U and $\tilde{P}$ in $\tilde{U}$ realising $g_p = \tilde{g}_{\tilde{p}}$, where g_p and $\tilde{g}_{\tilde{p}}$ are independent of the choice of tetrad (because the tetrads are generated by the group G which keeps the metric constant). Therefore, the identification of U and $\tilde{U}$ has to be independent of the choice of tetrad, ruling out a relation like (3.15).

So we see that our investigation of the equivalence problem has reduced to the problem of determining the necessary and sufficient conditions for a solution of (3.14) to exist. In the next section we analyse a similar but simpler situation and then use our results to solve the real problem in §4.

§3. Analysis of the Simpler Problem

Statement of Simpler Problem :

Given two systems of n linearly independent 1-forms ω^i and $\tilde{\omega}^i$, defined on regions U and $\tilde{U}$ with local coordinates x^μ and $\tilde{x}^\mu$ respectively ($i, \mu = 1, 2, \ldots n$); when does there exist an identification of U and $\tilde{U}$, given by the relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, realising $\tilde{\omega}^i = \omega^i$?

This is a similar problem to the one which we have but it is simpler because we are only considering a point-wise identification of the regions U and $\tilde{U}$ to match up the 1-forms, not a rotation/discrete transformation as well.

Solution :

Take the exterior derivative of ω^i and $\tilde{\omega}^i$.

$$d\omega^i = \tfrac{1}{2} c^i_{kh}\, \omega^k \wedge \omega^h \qquad c^i_{kh} = -c^i_{hk} \tag{3.16a}$$

$$d\tilde{\omega}^i = \tfrac{1}{2} \tilde{c}^i_{kh}\, \tilde{\omega}^k \wedge \tilde{\omega}^h \qquad \tilde{c}^i_{kh} = -\tilde{c}^i_{hk} \tag{3.16b}$$

where $c^i_{kh} = c^i_{kh}(x^\mu)$, $\tilde{c}^i_{kh} = \tilde{c}^i_{kh}(\tilde{x}^\mu)$.

A) Let us first consider the special case where there are n functionally independent functions among c^i_{kh} (and $\tilde{c}^i_{kh}$).

Note : n functions $f_1, f_2, \ldots f_n$ are said to be functionally independent if and only if the vectors $df_1, df_2, \ldots df_n$ are linearly independent. The number of functionally independent components among the f_i is equal to the number of linearly independent vectors among the df_i.

Ai) To establish the necessary conditions for $\omega^i = \tilde{\omega}^i$ in this special case let us assume $\omega^i = \tilde{\omega}^i$ as our starting point and see what deductions arise from this.

If $\omega^i = \tilde{\omega}^i$ then $d\omega^i = d\tilde{\omega}^i$, so from equations (3.16)

$$\tilde{c}^i_{kh} = c^i_{kh} \tag{3.17}$$

Further differentiation gives

$$dc^i_{kh} = c^i_{kh|l}\ \omega^l \tag{3.18a}$$

$$d\tilde{c}^i_{kh} = \tilde{c}^i_{kh|l}\ \tilde{\omega}^l \tag{3.18b}$$

These derivatives must be equal from (3.17) so we obtain

$$\tilde{c}^i_{kh|l} = c^i_{kh|l} \tag{3.19}$$

So we see that a necessary condition for $\omega^i = \tilde{\omega}^i$ is for (3.17) and (3.19) to be compatible as equations relating $\tilde{x}^\mu$ and x^μ. If we continued the differentiation

$$dc^i_{kh|l} = c^i_{kh|lm}\omega^m \tag{3.20a}$$

$$d\tilde{c}^i_{kh|l} = \tilde{c}^i_{kh|lm}\tilde{\omega}^m \tag{3.20b}$$

we would obtain, using the fact that the derivatives must be equal from (3.19), that

$$\tilde{c}^i_{kh|lm} = c^i_{kh|lm} \tag{3.21}$$

However, in this special case we have n functionally independent functions among the c^i_{jk} ($\tilde{c}^i_{jk}$) which is of course the maximum number on an n dimensional manifold, so that the $c^i_{jk|l}$ ($\tilde{c}^i_{jk|l}$) must be functionally dependent on the c^i_{jk} ($\tilde{c}^i_{jk}$). For (3.19) to be compatible with (3.17) requires this functional dependence to be the same — i.e. the $c^i_{jk|l}$ must be the same function of the c^i_{jk} as the $\tilde{c}^i_{jk|l}$ are of the $\tilde{c}^i_{jk}$. The functional dependence enables us to express all the higher derivatives $c^i_{jk|lm}$ ($\tilde{c}^i_{jk|lm}$), $c^i_{jk|lmn}$ ($\tilde{c}^i_{jk|lmn}$) etc. in terms of c^i_{jk} ($\tilde{c}^i_{jk}$) (for example $c^i_{jk|lm}$ comes from differentiating $c^i_{jk|l}$ which, because of the functional dependence, is a function of c^i_{jk}. Therefore, $c^i_{jk|lm}$ is a function of c^i_{jk} and $c^i_{jk|l}$, but we can substitute for $c^i_{jk|l}$ using the functional dependence, giving $c^i_{jk|lm}$ as a function of only c^i_{jk}). Thus, assuming the functional dependence is the same for $c^i_{jk|l}$ and $\tilde{c}^i_{jk|l}$, we see that all untwiddled and twiddled higher derivatives will be the same function of c^i_{jk} and $\tilde{c}^i_{jk}$ respectively. Therefore, we see that compatibility of (3.17) with (3.19) guarantees compatibility of all higher derivatives, so that we need not concern ourselves with them.

Aii) In (Ai) we proved that the necessary condition for $\omega^i = \tilde{\omega}^i$ is compatibility of (3.17) and (3.19). We shall now show that compatibility of these equations is also a sufficient condition by assuming the compatibility as our starting point and showing this results in $\omega^i = \tilde{\omega}^i$.

Assuming (3.17) and (3.19) are compatible as relations between $\tilde{x}^\mu$ and x^μ, consider the set of equations

$$d\tilde{c}^i_{kh} - dc^i_{kh} = c^i_{kh|l}(\tilde{\omega}^l - \omega^l) = 0 \tag{3.22}$$

This set will contain n linearly independent equations in $\tilde{\omega}^l - \omega^l$ because n of the c^i_{kh} are functionally independent, which means, as stated before, that n of the vectors dc^i_{kh} are linearly independent. Let us denote the n linearly independent equations as

$$c^A_{|l}(\tilde{\omega}^l - \omega^l) = 0 \tag{3.23}$$

where A represents some combination of i, j, k in c^i_{jk} and runs from 1 to n. If we consider this as a matrix equation with $c^A_{|l}$ an $n \times n$ matrix then because of the linear independence of the vectors $c^A_{|l}$ they will form an $n \times n$ matrix of rank n, which will, therefore, have an inverse. This means that set (3.23) has *only* the trivial solution $\tilde{\omega}^l - \omega^l = 0$, which proves our desired result.

To summarise what we have found so far, in the case where there are n functionally independent functions amongst the c^i_{jk} $(\tilde{c}^i_{jk})$, compatibility of (3.17) and (3.19) is a necessary and sufficient condition for $\tilde{\omega}^i = \omega^i$. Note that because the set (3.17) contains n functionally independent relations it will yield a *unique* coordinate relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ giving $\tilde{\omega}^i = \omega^i$.

B) In (A) above we investigated the special case where n of the c^i_{jk} are functionally independent. We now want to consider the general case where there are $n_0 < n$ functionally independent components among the c^i_{jk}.

Bi) As in case (Ai) we first investigate the necessary conditions for $\tilde{\omega}^i = \omega^i$ by assuming at the outset this holds and investigating what consequences ensue. We proceed exactly as in (Ai) and, therefore, generate the set of equations

$$\begin{aligned} \tilde{c}^i_{kh} &= c^i_{kh} \\ \tilde{c}^i_{kh|l_1} &= c^i_{kh|l_1} \\ . &= . \\ . &= . \\ . &= . \\ . &= . \\ \tilde{c}^i_{kh|l_1 \ldots l_{p+1}} &= c^i_{kh|l_1 \ldots l_{p+1}} \end{aligned} \tag{3.24}$$

In case (Ai) it will be remembered that compatibility of the 0th and 1st derivatives guaranteed compatibility of all others so that we could stop at this stage, this being a result of the reasoning following (3.21). Using exactly the same reasoning again we see that set (3.24) need continue only to the $(p+1)$th derivative, which is the first derivative functionally dependent on lower derivatives. Therefore, we have established the result that compatibility of set (3.24) is a necessary condition for $\omega^i = \tilde{\omega}^i$.

Bii) We now want to determine the sufficient conditions, so once again we start by assuming that the set (3.24) are compatible.

Biia) First consider the case where the total number of independent components produced by going to the pth derivative is n. We again form the set of equations

$$d\tilde{c}^i_{kh} - dc^i_{kh} = c^i_{kh|l}(\tilde{\omega}^l - \omega^l) = 0 \tag{3.25a}$$

but because our n functionally independent components are now scattered among the first p derivatives, we obtain relations linearly independent of set (3.25a) from the set

$$\begin{aligned} d\tilde{c}^i_{kh|l} - dc^i_{kh|l} &= c^i_{kh|lm}(\tilde{\omega}^m - \omega^m) = 0 \\ . \quad &= \quad . \\ . \quad &= \quad . \\ . \quad &= \quad . \\ d\tilde{c}^i_{kh|l_1\ldots l_p} - dc^i_{kh|l_1\ldots l_p} &= c^i_{kh|l_1\ldots l_p m}(\tilde{\omega}^m - \omega^m) = 0 \end{aligned} \tag{3.25b}$$

Together (3.25a) and (3.25b) will contain n linearly independent equations for $\tilde{\omega}^i - \omega^i$ produced by differentiating the n functionally independent components among the c^i_{kh} and its first p derivatives. Therefore, just as in case (Aii), they give as the *only* solution the trivial solution $\tilde{\omega}^i - \omega^i = 0$. So for the case where continued differentiation does finally produce n functionally independent components we have the result that compatibility of the set (3.24) is a necessary and sufficient condition for $\tilde{\omega}^i = \omega^i$.

Biib) Now consider the case where continued differentiation never produces n functionally independent components. We can once again form the set (3.25), but the problem is that among the set there will only be $k < n$ linearly independent equations for the n unknown $\tilde{\omega}^i - \omega^i$, where k is the number of functionally independent components among the c^i_{kh} and its first p derivatives. Thus, the best that one will be able to do with the set (3.25) is to use it to express k of the $\tilde{\omega}^i - \omega^i$ as linear combinations of the other $n - k$. So with a suitable numbering we obtain

$$\tilde{\omega}^A - \omega^A = b^A_\alpha(\tilde{\omega}^\alpha - \omega^\alpha) \tag{3.26}$$

where A, B etc. run from $n - k + 1$ to n (i.e. k of them), and α, β etc. run from 1 to $n - k$ (i.e. $n - k$ of them).

So we have got to the stage where if we can just prove that compatibility of set (3.24) makes the $(n - k)$ $\tilde{\omega}^\alpha - \omega^\alpha$ zero, then we automatically have from (3.26) that $\tilde{\omega}^i - \omega^i = 0$ for i running from 1 to n. The argument will run in two stages :

1) For $\tilde{\omega}^\alpha - \omega^\alpha = 0$ it will turn out that a system of first order partial differential equations needs to be integrable. A proof by Cartan shows this to be the case.

2) It will then be necessary to show that the coordinate relation obtained from the system of first order partial differential equations is compatible with the one obtained from our starting point, the set (3.24). This will also turn out to be the case.

The final conclusion will be, therefore, that compatibility of set (3.24) is a *necessary and sufficient* condition for there to exist an identification of U and $\tilde{U}$ giving $\tilde{\omega}^i = \omega^i$ with $i = 1, \ldots, n$. The arguments involved in stages (1) and (2) above are fairly involved and are dealt with in appendix A so as not to interrupt the main flow of ideas.

Let us at least see explicitly how the requirement that $\tilde{\omega}^\alpha - \omega^\alpha = 0$ leads to a set of first order partial differential equations. In local coordinates we have

$$\tilde{\omega}^\alpha = \tilde{a}^\alpha_\mu \quad d\tilde{x}^\mu \tag{3.27a}$$

$$\omega^\alpha = a^\alpha_\mu \quad dx^\mu \tag{3.27b}$$

which gives the $n-k$ equations

$$\tilde{\omega}^\alpha - \omega^\alpha = \tilde{a}^\alpha_\mu \ d\tilde{x}^\mu - a^\alpha_\mu \ dx^\mu = 0 \tag{3.28}$$

These equations are linearly independent in the $d\tilde{x}^\mu$ because $\tilde{\omega}^\alpha$ are linearly independent, so we have $n-k$ linearly independent equations for n unknown $d\tilde{x}^\mu$. Therefore, we can solve for $n-k$ of them as linear combinations of the other k. With a suitable numbering one obtains

$$d\tilde{x}^\alpha = b^\alpha_\mu dx^\mu + c^\alpha_A d\tilde{x}^A \tag{3.29}$$

where once again A, B etc. run from $n-k+1$ to n (i.e. k of them), and α, β etc. run from 1 to $n-k$ (i.e. $n-k$ of them).

The integrability of equations (3.29) is discussed in appendix A, where it is shown that they can be integrated to the coordinate relation

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^\mu, \tilde{x}^A) \tag{3.30}$$

and that this coordinate relation is compatible with that emerging from set (3.24).

We see, therefore, that in general the n relations $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ providing the identification of U and $\tilde{U}$ giving $\tilde{\omega}^i = \omega^i$ are obtained from the set (3.24) (which gives k of them) together with the integral relations (3.30) (which gives $n-k$ of them). The relations (3.30) are not unique but depend on $n-k$ constants of integration (see appendix A), so there are $n-k$ continuous deformations of (3.30) which preserve $\tilde{\omega}^i = \omega^i$. There may also be discrete transformations which are not found in our analysis.

The entire analysis contained in this section has enabled us to reach the point where we can state the following theorem :

Theorem 3.1

Given two sets of n linearly independent 1-forms $\tilde{\omega}^i$ and ω^i defined on $\tilde{U}$ and U with local coordinates $\tilde{x}^\mu$ and x^μ respectively, then there exists a coordinate identification of $\tilde{U}$ and U, given by $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, giving $\tilde{\omega}^i = \omega^i$ *if and only if* the equations (3.24) are compatible. The $(p+1)$th derivative is the first one which is functionally dependent on lower derivatives (including the zeroth), so $p+1 \leq n$ (see below). The coordinate relations $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ depend on $n-k$ constants of integration, where k is the the number of functionally independent components in (3.24).

The reason that $p+1 \leq n$ can be understood as follows. If the c^i_{jk} are constants then their derivatives will be zero, so that the differentiation terminates at first order. Thus, in order for the process to continue beyond first order the c^i_{jk} must contain at least one functionally independent component. Subsequent differentiation must produce at least one new functionally independent component at each stage for the process to continue. However, in n dimensions there are at most n functionally independent components so by the $(n-1)$th derivative all n must have been produced, making the nth derivative the first to be dependent on lower derivatives.

§4. The Equivalence Theorem

Theorem 3.1 is essentially telling us that the c^i_{jk}, $c^i_{jk|l}$ etc. are the invariants that must be equal in order to have an identification of the two regions U and $\tilde{U}$ giving $\tilde{\omega}^i = \omega^i$. The actual identification map would be obtained by finding the coordinate relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ which makes the invariants equal (we call the c^i_{kh}, $c^i_{kh|l}$ etc. invariants in that they are frame dependent but *coordinate* invariant). However, the real problem that we have is not as stringent as the one we have just solved in that we do not require that $\tilde{\omega}^i = \omega^i$ but only that they are equal up to transformations $b^i{}_j$ of the group G.

To tackle this problem let us proceed according to Cartan [2], where the idea is to keep the rotational freedom of the tetrad. Therefore, the ω^i depend not only on the n position coordinates x^μ but also on the $n(n-1)/2$ rotation parameters ϵ^A. In a coordinate basis we have

$$\omega^i = a^i_\mu(x^\nu, \epsilon^A)\ dx^\mu \tag{3.31}$$

where i,j, etc. and μ, ν, etc. run from 1 to n, and A, B, etc. run from 1 to $n(n-1)/2$.

The connection 1-forms $\omega^i{}_j$ normally give the change in the tetrad between neighbouring points, but because of our rotational freedom they will now also contain a part which gives the change at a fixed point due to changes in the ϵ^A. So we write $\omega^i{}_j$ as

$$\omega^i{}_j = \omega^i_{(x)j} + \omega^i_{(\epsilon)j} \tag{3.32}$$

where $\omega^i_{(x)j}$ are the 'traditional' connection 1-forms defined by

$$d_x\omega^i = \omega^j \wedge \omega^i_{(x)j} \quad ; \qquad \omega^{ij}_{(x)} = -\omega^{ji}_{(x)} \tag{3.33}$$

where d_x stands for the exterior derivative in x^μ space. If the connection coefficients $\Gamma^i{}_{jk}$ are defined as

$$\Gamma^i{}_{jk} = < \omega^i, \nabla_k e_j > \tag{3.34}$$

where ∇_k is the covariant derivative and e_i are basis vectors whose dual basis 1-forms are ω^i, so that $< e_i, \omega^j >= \delta^j_i$, then for a symmetric connection

$$\omega^i_{(x)j} = \Gamma^i{}_{jk}\omega^k \tag{3.35}$$

This result (3.35) is derived in appendix B.

The $\omega^i_{(\epsilon)j}$ are defined by

$$d_\epsilon \omega^i = \omega^j \wedge \omega^i_{(\epsilon)j} \quad ; \qquad \omega^{ij}_{(\epsilon)} = -\omega^{ji}_{(\epsilon)} \tag{3.36}$$

where d_ϵ stands for the exterior derivative in ϵ^A space, and gives the change in the tetrad at a fixed point due to changes in ϵ^A. From equation (3.31) we see that $d_x\omega^i$ contains only terms like $dx^\mu \wedge dx^\nu$ whereas $d_\epsilon\omega^i$ contains only terms like $d\epsilon^A \wedge dx^\mu$. This implies, using equations (3.33) and (3.36), that in local coordinates $\{x^\mu,\, \epsilon^A\}$ we have

$$\omega^{ij}_{(x)} = a^{ij}_\mu(x^\nu, \epsilon^B)dx^\mu \tag{3.37a}$$

$$\omega^{ij}_{(\epsilon)} = a^{ij}_A(x^\nu, \epsilon^B)d\epsilon^A \tag{3.37b}$$

By antisymmetry there are $n(n-1)/2$ different 1-forms $\omega^{ij}_{(\epsilon)}$ and these must be linearly independent because they give the $n(n-1)/2$ parameter rotational subgroup of the group G which leaves the frame metric η_{ij} invariant. We therefore have $n(n-1)/2 + n = n(n+1)/2$ linearly independent 1-forms $\{\omega^i,\, \omega^{ij}\}$ spanning the dual tangent space to the $\{x^\mu,\, \epsilon^A\}$ space.

Recall that lemma 3.1b told us that two geometries are equivalent if, and only if, there is a correspondence

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu) \tag{3.38a}$$

$$\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^B, x^\mu) \tag{3.38b}$$

giving

$$\tilde{\omega}^i(\tilde{x}^\mu, \tilde{\epsilon}^A) = \omega^i(x^\mu, \epsilon^A) \tag{3.39}$$

(Remember that we ignore discrete transformations because as there is only a finite number of them they can be dealt with by keeping m fixed and investigating the existence of a solution for each $\tilde{m}$ in turn). From (3.33) and (3.36) we see that $\tilde{\omega}^i = \omega^i$ implies that $\tilde{\omega}^{ij} = \omega^{ij}$, so that we can say that equivalence requires a coordinate relation (3.38) giving

$$\tilde{\omega}^i(\tilde{x}^\mu, \tilde{\epsilon}^A) = \omega^i(x^\mu, \epsilon^A) \tag{3.40a}$$

$$\tilde{\omega}^{ij}(\tilde{x}^\mu, \tilde{\epsilon}^A) = \omega^{ij}(x^\mu, \epsilon^A) \tag{3.40b}$$

If we now consider $\{x^\mu, \epsilon^A\}$ as local coordinates on a region U^* of a $n(n+1)/2$ dimensional manifold M^* then, because the $n(n+1)/2$ 1-forms ω^i, ω^{ij} defined on U^* are linearly independent, the problem of determining whether there exists an identification of $\tilde{U}^*$ and U^* giving (3.40) is exactly the same problem as the one tackled in §3, the only thing having changed being the dimension of the problem, from n to $n(n+1)/2$. The identification obtained will be of the desired form required for the equivalence of regions U and $\tilde{U}$ (i.e. of the form (3.38)) because $\tilde{\omega}^i = \omega^i$ implies $\tilde{a}^i_{\ \mu}\, d\tilde{x}^\mu = a^i_{\ \mu}\, dx^\mu$, which can only be satisfied if $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ and not if $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu, \epsilon^A)$. Therefore, the analysis of §3 provides the solution of the equivalence problem.

The analysis of §3 led to theorem 3.1 which was essentially obtained by differentiating the original 1-forms and equating the results. Let us see what differentiation of our 1-forms ω^i, ω^{ij} gives.

$$\begin{aligned} d\omega^i &= d_x\omega^i + d_\epsilon\omega^i \\ &= \omega^k \wedge \omega^i_{(x)k} + \omega^k \wedge \omega^i_{(\epsilon)k} \\ &= \omega^k \wedge \omega^i_{\ k} \end{aligned} \tag{3.41}$$

The derivative of ω^{ij} is calculated in appendix B. The result is

$$d\omega^{ij} = \omega^{ik} \wedge \omega^j_{\ k} + \tfrac{1}{2} R^{ij}_{\ \ kl}\omega^k \wedge \omega^l \tag{3.42}$$

where R_{ijkl} are the frame components of the Riemann tensor for the symmetric connection $\Gamma^i_{\ jk}$, given by

$$R_{ijkl} = \Gamma_{ijl|k} - \Gamma_{ijk|l} + \Gamma_{ijm}(\Gamma^m_{\ \ kl} - \Gamma^m_{\ \ lk}) + \Gamma_{imk}\Gamma^m_{\ \ jl} - \Gamma_{iml}\Gamma^m_{\ \ jk} \tag{3.43}$$

So following the procedure used in §3, this time using the facts that $\tilde{\omega}^i = \omega^i$ and $\tilde{\omega}^{ij} = \omega^{ij}$, we see that the equation $\tilde{c}^i_{jk} = c^i_{jk}$ obtained from equating $d\tilde{\omega}^i$ and $d\omega^i$ has the analogue, obtained by equating $d\tilde{\omega}^i = d\omega^i$ and $d\tilde{\omega}^{ij} = d\omega^{ij}$

$$\tilde{R}_{ijkl} = R_{ijkl} \tag{3.44}$$

Next we differentiate R_{ijkl} (see appendix B)

$$dR_{ijkl} = R_{mjkl}\omega^m{}_i + R_{imkl}\omega^m{}_j + R_{ijml}\omega^m{}_k + R_{ijkm}\omega^m{}_l + R_{ijkl;m}\omega^m \qquad (3.45)$$

Then equating $d\tilde{R}_{ijkl}$ and dR_{ijkl} using $\tilde{\omega}^i = \omega^i$, $\tilde{\omega}^{ij} = \omega^{ij}$ and $\tilde{R}_{ijkl} = R_{ijkl}$ gives

$$\tilde{R}_{ijkl;m} = R_{ijkl;m} \qquad (3.46)$$

Differentiating $R_{ijkl;m}$ gives

$$\begin{aligned} dR_{ijkl;m} &= R_{njkl;m}\omega^n{}_i + R_{inkl;m}\omega^n{}_j + R_{ijnl;m}\omega^n{}_k \\ &\quad + R_{ijkn;m}\omega^n{}_l + R_{ijkl;n}\omega^n{}_m + R_{ijkl;mn}\omega^n \end{aligned} \qquad (3.47)$$

so we obtain

$$\tilde{R}_{ijkl;mn} = R_{ijkl;mn} \qquad (3.48)$$

and so on.

So it is seen that we end up with the result that equivalence is governed by the following theorem, which is essentially the same theorem as theorem 3.1 except applied to the $n(n+1)/2$ dimensional space with local coordinates $\{x^\mu, \epsilon^A\}$, where the c^i_{jk}, $c^i_{jk|l}$, etc. are replaced by the Riemann tensor and its covariant derivatives.

Theorem 3.2 — The Theorem of Equivalence

Two regions $\tilde{U}$ and U of two n-dimensional Riemannian manifolds are equivalent if, and only if, the set

$$\begin{aligned} \tilde{R}_{ijkl} &= R_{ijkl} \\ \tilde{R}_{ijkl;m} &= R_{ijkl;m} \\ . &= . \\ . &= . \\ . &= . \\ \tilde{R}_{ijkl;m_1m_2\ldots m_{p+1}} &= R_{ijkl;m_1m_2\ldots m_{p+1}} \end{aligned} \qquad (3.49)$$

is compatible as equations in $\tilde{x}^\mu, x^\mu, \tilde{\epsilon}^A, \epsilon^A$. The $(p+1)$th derivative is the first one which is functionally dependent on lower derivatives (including the zeroth), so $p+1 \leq n(n+1)/2$. The coordinate relations expressing $\tilde{x}^\mu, \tilde{\epsilon}^A$ as functions of x^μ, ϵ^A depend on $n(n+1)/2-k$ constants of integration, where k is the number of functionally independent components among $R_{ijkl},\ R_{ijkl;m}, \ldots,\ R_{ijkl;m_1\ldots m_p}$. This means there are $n(n+1)/2-k$ continuous deformations of the coordinate relations which preserve equivalence. There may also be discrete transformations which preserve equivalence but these are not found in the above analysis.

So we see that the set $\{R_{ijkl},\ R_{ijkl;m}, \ldots,\ R_{ijkl;m_1\ldots m_{p+1}}\}$ provides a complete invariant description of the geometry (invariant because the set are frame components of the Riemann tensor and are, therefore, frame dependent but coordinate invariant).

§5. How to Investigate Equivalence in Practice

In this section our aim is to use the theorem of equivalence 3.2 in order to develop a practical procedure for investigating the equivalence of metrics. We shall first clearly spell out the procedure used, leaving comments on it to afterwards.

The Procedure :

1) Choose a constant frame metric η_{ij} for the tetrad.
2) Calculate the tetrad components R_{ijkl} of the Riemann tensor in an arbitrary fixed tetrad with metric η_{ij}.
3) Determine H_0, the subgroup of G which leaves the R_{ijkl} invariant. In fact, this is a straightforward procedure as will be shown in §7. Note that H_0 may contain discrete transformations since G does.
4) Determine, up to a transformation in H_0, a standard tetrad by requiring that R_{ijkl} takes on a special form, called the *canonical form*. That this can always be performed for R_{ijkl} and all its derivatives is shown in [17] and it will

be done explicitly for R_{ijkl} in §7 (in spinor language).

5) Determine n_0, the number of functionally independent components among R_{ijkl} *in its canonical form.* Note that the R_{ijkl} are now only functions of x^μ because we are using a particular standard tetrad (up to transformations H_0 which leave the R_{ijkl} invariant). The ϵ^A dependence has been dealt with in (3) above.

6) Calculate $R_{ijkl;m_1}$ in the standard tetrad.

7) Determine H_1 the subgroup of H_0 which leaves R_{ijkl} and $R_{ijkl;m_1}$ invariant.

8) Determine among the earlier standard tetrads, up to a transformation in H_1, a new standard tetrad by stipulating a canonical form for $R_{ijkl;m_1}$.

9) Determine n_1, the number of components among R_{ijkl} and $R_{ijkl;m_1}$ *in their canonical forms* which are functionally independent.

10) If $dim(H_1) = dim(H_0)$ and $n_1 = n_0$ then the procedure terminates. Otherwise, we repeat the steps $6-9$ for $R_{ijkl;m_1m_2}$, $R_{ijkl;m_1m_2m_3}$, etc. until we reach the stage when $dim(H_{p+1}) = dim(H_p)$ and $n_{p+1} = n_p$, in which case the procedure terminates. The set $\{H_q,\ n_q,\ R_{ijkl;m_1m_2\dots m_q}\}$, $q = 0, 1, \dots, p+1$, classifies the solution.

Then given two metrics g and $\tilde{g}$ which we wish to compare for equivalence, we start by completing the above classification for each metric. The rest of the procedure is contained in the following steps :

11) If the two sequences $H_0, n_0;\ H_1, n_1; \dots;\ H_q, n_q$ for g and $\tilde{g}$ differ, then so do the metrics.

12) If the set of simultaneous algebraic equations $\tilde{R}_{ijkl} = R_{ijkl}$, $\tilde{R}_{ijkl;m_1} = R_{ijkl;m_1}, \dots,\ \tilde{R}_{ijkl;m_1m_2\dots m_q} = R_{ijkl;m_1m_2\dots m_q}$, *with the invariants in their canonical form*, admits a coordinate transformation $\tilde{x}^i = \tilde{x}^i(x^i)$, $i = 1, \dots, n$, as a solution then the metrics are equivalent, otherwise they are inequivalent.

Because the corresponding invariants ($\tilde{R}_{1111}$ and R_{1111} for instance) may differ wildly in their functional form, determining whether the coordinate identification exists can be a difficult problem. Indeed, since there is no constructive procedure for solving simultaneous algebraic equations (here we do not mean algebraic in the strict mathematical sense, our equations might be transcendental), step (12) is not algorithmic.

The reason why the procedure terminates when $dim(H_{p+1}) = dim(H_p)$, $n_{p+1} = n_p$, is firstly because we have no new x^μ functionally independent quantities as $n_{p+1} = n_p$ and secondly because we have no new ϵ^A functionally independent components because $dim(H_{p+1}) = dim(H_p)$. The second argument follows from the result that $dim(H_q) = n(n-1)/2 - m$, where m is the number of ϵ^A functionally independent components amongst the Riemann tensor and its first q derivatives. This result can be validated by recognising that the components of the Riemann tensor and its derivatives which are ϵ^A functionally independent can be used as new parameters for the group G, with the other components expressible *purely* in terms of these new parameters.

The key to understanding why this procedure tackles the equivalence problem lies in the use of the canonical form. The essence of what the equivalence theorem 3.2 is telling us is that in order for us to have $\tilde{\omega}^i = \omega^i$ up to the rotational freedom in the problem, we require that there exist frames in which the two sets of invariants R_{ijkl}, $R_{ijkl;m_1}$, etc. are equal, the actual identification map being provided by the coordinate relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ which gives this equality. This is the meaning of looking for an identification $\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^A, x^\nu)$ which satisfies equations (3.49). Once one realises that this is the geometrical meaning of the algebra the whole problem becomes much more transparent. By using the canonical form we pinpoint much more precisely (up to transformations in H_p) the frame which will enable these invariants to be made equal, by the coordinate identification $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$.

Even if the last step, step (12), cannot be tackled we still obtain a lot of information about the geometry before this difficulty arises. In §7 we shall see that steps (1) to (4) are equivalent, for empty space, $n = 4$ and η_{ij} the Lorentz metric, to the Petrov classification, in that the result depends uniquely on the multiplicities of the principal spinors in the spinor equivalent of the Weyl tensor. The procedure as a whole provides a kind of maximally generalised Petrov classification — maximal in the sense that we classify all the covariant derivatives of the Riemann tensor that are necessary to provide a complete description of the geometry, it works for non-empty spaces and spaces of arbitrary dimension n and frame metric η_{ij}. Note also that the classification works for any geometry, regardless of whether the metric satisfies any field equations, and is in this sense a purely geometrical classification. In the next section we shall see how the procedure, excluding step (12), enables us to investigate the dimension of the isometry group and its isotropy subgroup.

Finally let us make a brief comparison of the Karlhede procedure with the one suggested by Brans [17]. The two procedures are very similar, the main difference being that Brans first calculates the Riemann tensor and its covariant derivatives and then determines a canonical form for them, starting with the highest derivative. In this procedure we do this successively starting with the curvature tensor, which greatly simplifies the process, making the amount of computation required much more feasible.

In conclusion, we have reduced the equivalence problem to the problem of determining whether or not a finite set of algebraic equations is compatible or not as relations between $\tilde{x}^\mu$ and x^μ.

§6. The Isometry Group

Definitions :

1) The isometry group I is the group of mappings $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ of the manifold onto *itself* which preserve the metric, i.e. the metric at the original point is the same as at the point reached under the mapping.

2) The isotropy group H at the point P is the subgroup of I which leaves P fixed.
3) The orbit T of a point P is the set of points into which it is mapped by the elements of I.

If we know that two regions U and $\tilde{U}$ of manifolds M and $\tilde{M}$ are equivalent then from the geometrical point of view the two regions are effectively the same region U of the same manifold M, with the coordinate relations giving equivalence, $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, identifying effectively the same points. The coordinate systems x^μ and $\tilde{x}^\mu$ are then different coordinate systems covering the same region of the same manifold. Variations in the coordinate relations giving equivalence map points in U into new points in $\tilde{U}$, and, therefore, from what has just been said can be viewed as mapping points in U into new points in U, i.e. they provide a mapping of the manifold onto itself. If these variations in the coordinate relations maintain equivalence, they will by definition map points to new points where the metric is the same, hence giving isometries. Variations in the coordinate relations giving equivalence, which maintain equivalence, are provided by the constants of integration obtained on integrating equations (3.29). As remarked in §3, these parameters give continuous deformations of the coordinate relations maintaining equivalence, so our analysis enables us to investigate only the continuous part of the isometry group — to find the discrete isometries we would need to identify discrete changes in the coordinate relations maintaining equivalence, which are not found in our analysis of the equivalence problem. In addition, because our investigation is purely local, we only find local isometries.

Bearing in mind these limitations, let us state the following theorem, considering its proof afterwards.

Theorem 3.3

Given an n-dimensional Riemannian manifold M, suppose that there are k

functionally independent components among R_{ijkl}, $R_{ijkl;m_1}, \ldots, R_{ijkl;m_1\ldots m_{p+1}}$, m of them functionally independent as functions of ϵ^A and t of them functionally independent as functions of x^μ. M then has an isometry group I with dimension

$$r = n(n+1)/2 - k \tag{3.50}$$

The isotropy subgroup H has dimension

$$s = n(n-1)/2 - m \tag{3.51}$$

and the dimension of the orbits T is

$$q = r - s = n - t \tag{3.52}$$

Proof :

Referring to appendix A, we see that if we use the functionally independent components of the Riemann tensor and its covariant derivatives as coordinates, then the coordinate relations giving equivalence take the form

$$\tilde{X}^I = X^I \tag{3.53}$$

$$\tilde{x}^b = \tilde{x}^b(x^\mu, c^\alpha) \tag{3.54}$$

$$\tilde{\epsilon}^M = \tilde{\epsilon}^M(x^\mu, \epsilon^A, c^\alpha) \tag{3.55}$$

where X^I represent the functionally independent components of the Riemann tensor and its covariant derivatives used as coordinates (so that with a suitable numbering of indices I runs from $n(n+1)/2 - k + 1$ to $n(n+1)/2$, b runs from 1 to $n - t$, M runs from 1 to $n(n-1)/2 - m$, μ runs from 1 to n, A runs from 1 to $n(n-1)/2$ and α runs from 1 to $\frac{1}{2}n(n+1) - k$).

As explained above varying the parameters in (3.54) provides isometries. However, we see that under these isometries it is only the $n - t$ coordinates $\tilde{x}^b$ which change, so that a point P is mapped into an $(n - t)$ dimensional region, the orbit, by these isometries.

Consider a point P in the manifold and assume that the geometry is analytic at P, by which we mean that the metric at points P' in a neighbourhood of P can be expressed in a Taylor series expansion about P. We take the case where the manifold M has dimension 4 and possesses a Lorentz metric, i.e. general relativity, although all our arguments in fact generalise to an arbitrary n dimensional manifold. If we choose geodesic normal coordinates for this neighbourhood then the Taylor expansion takes the form

$$g_{\mu\nu}(P') = \eta_{\mu\nu}(P) - \tfrac{1}{3}R_{\mu\nu\sigma\rho}(P)(x^\sigma - x_0^\sigma)(x^\rho - x_0^\rho) + \dots \tag{3.56}$$

where subsequent expansion coefficients are all functions only of the Riemann tensor and its covariant derivatives, and $\eta_{\mu\nu}$ is the Lorentz metric (diagonal (1,1,1,-1)). From (3.56) we see that the coordinate basis vectors at P, which are the tangent vectors to the coordinate lines, form a Lorentz frame — by applying a Lorentz transformation to this frame at P we produce another Lorentz frame at P which can in fact be associated with the coordinate basis vectors at P for another set of normal coordinates, in some way rotated with respect to the previous set. From equations (3.53) and (3.55) we see that a general Lorentz transformation will change the Riemann tensor and its covariant derivatives at P, but one using only the $n(n-1)/2 - m$ parameters ϵ^A will not, and will, therefore, keep the expansion coefficients in (3.56) for the new rotated normal coordinates the same. Therefore, the metric at a given coordinate location with respect to the rotated normal coordinates (i.e. a given $x'^\rho - x_0'^\rho$ for $\rho = 1$ to n, where x'^ρ is a rotated normal coordinate) will be the same as the metric at the same coordinate location with respect to the original normal coordinates (i.e. $x^\rho - x_0^\rho = x'^\rho - x_0'^\rho$). If we consider that instead of the normal coordinates having rotated about P, the manifold has rotated about P, that is we consider the transformation as active instead of passive, then we have identified isometries that leave the point P fixed, i.e. we have identified isotropies. Each ϵ^M in (3.55) will clearly give an independent isotropy, so that the dimension of the isotropy group is $n(n-1)/2 - m$. This completes the proof of theorem 3.3.

§7. Canonical Forms for the Weyl Spinor and its Invariance Group

In this section we restrict our attention to the case where the manifold M has dimension 4 and possesses a Lorentz metric, i.e. we have general relativity. Instead of working with the tetrad components of the Riemann tensor and considering proper homogeneous Lorentz transformations of the frame one can work with dyad components of the equivalent spinor and consider SL(2,$\mathbb{C}$) transformations. This is because of the following two important results [9] :

1) The tetrad components of a tensor in a Newman-Penrose null tetrad are the same as the dyad components of the equivalent spinor.
2) SL(2,$\mathbb{C}$) transformations of the dyad correspond to proper homogeneous Lorentz transformations ($\mathcal{L}_+^\uparrow$) of the Newman-Penrose null tetrad.

The relationship between the dyad $\{\zeta_0^A, \zeta_1^A\}$ and the Newman-Penrose null tetrad $\{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ is provided by the equations

$$\begin{aligned} l^\mu &= \sigma^\mu_{AB'}\zeta_0^A\bar{\zeta}_{0'}^{B'} \\ n^\mu &= \sigma^\mu_{AB'}\zeta_1^A\bar{\zeta}_{1'}^{B'} \\ m^\mu &= \sigma^\mu_{AB'}\zeta_0^A\bar{\zeta}_{1'}^{B'} \\ \bar{m}^\mu &= \sigma^\mu_{AB'}\zeta_1^A\bar{\zeta}_{0'}^{B'} \end{aligned} \tag{3.57}$$

The spinor equivalent of the Riemann tensor can be decomposed in the following way :

$$\begin{aligned} R_{AE'BF'CG'DH'} = {} & \Psi_{ABCD}\epsilon_{E'F'}\epsilon_{G'H'} + \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{E'F'G'H'} \\ & + \epsilon_{AB}\epsilon_{G'H'}\Phi_{CDE'F'} + \epsilon_{CD}\epsilon_{E'F'}\Phi_{ABG'H'} \\ & + \Lambda\{(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD})\epsilon_{E'F'}\epsilon_{G'H'} \\ & + \epsilon_{AB}\epsilon_{CD}(\epsilon_{E'H'}\epsilon_{F'G'} + \epsilon_{E'G'}\epsilon_{F'H'})\} \end{aligned} \tag{3.58}$$

where

$$\Psi_{ABCD} = \Psi_{(ABCD)} \quad , \quad \Phi_{ABC'D'} = \Phi_{(AB)(C'D')} = \bar{\Phi}_{ABC'D'} \tag{3.59}$$

Λ is essentially the Ricci scalar, the Ricci spinor, $\Phi_{ABC'D'}$, represents the trace-free Ricci tensor and the Weyl spinor, Ψ_{ABCD}, represents the Weyl tensor.

For a vacuum spacetime, in this decomposition of the Riemann spinor (3.58) it is only the Weyl spinor Ψ_{ABCD} which does not vanish. From chapter 2 it will be recalled that because the Weyl spinor is totally symmetric it can be written as a symmetrised product of 1-spinors, with the multiplicity of these principal spinors determining the Petrov type. Thus one obtains :

Petrov type	**Weyl spinor**
I	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
II	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$
D	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}$
III	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$
N	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$
0	$\Psi_{ABCD} = 0$

where α_A, β_A, γ_A and δ_A denote non-proportional spinors.

All SL(2,C) transformations can be represented as the product of three matrices as follows :

$$T = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{a} & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad , \qquad \lambda, a, b \in \mathbb{C} \tag{3.60}$$

Let us call the first matrix T_1, the second T_2 and the third T_3. Under T_1 the dyad $\{\zeta_0^A, \zeta_1^A\}$ transforms as

$$\begin{aligned} \zeta_0^A &\longrightarrow \lambda\zeta_0^A \\ \zeta_1^A &\longrightarrow \lambda^{-1}\zeta_1^A \end{aligned} \tag{3.61}$$

Under T_2 the dyad transforms as

$$\begin{aligned}\zeta_0^A &\longrightarrow \zeta_0^A \\ \zeta_1^A &\longrightarrow \zeta_1^A + \bar{a}\zeta_0^A\end{aligned} \tag{3.62}$$

Under T_3 the dyad transforms as

$$\begin{aligned}\zeta_0^A &\longrightarrow \zeta_0^A + b\zeta_1^A \\ \zeta_1^A &\longrightarrow \zeta_1^A\end{aligned} \tag{3.63}$$

By calculating the affect of these dyad transformations on the tetrad vectors via equation (3.57), one can gain a geometrical interpretation of the transformations they represent. Under T_1, writing $\lambda = re^{i\theta}$, the tetrad transforms as

$$\begin{aligned}l^\mu &\longrightarrow r^2 l^\mu \\ n^\mu &\longrightarrow r^{-2} n^\mu \\ m^\mu &\longrightarrow e^{2i\theta} m^\mu \\ \bar{m}^\mu &\longrightarrow e^{-2i\theta} \bar{m}^\mu\end{aligned} \tag{3.64}$$

Thus we see that T_1 represents a rotation in the $\{m, \bar{m}\}$ plane and a boost in the $\{l, n\}$ plane. Therefore, transformations T_1 are termed spin and boost transformations.

Under T_2 the tetrad transforms as

$$\begin{aligned}l^\mu &\longrightarrow l^\mu \\ n^\mu &\longrightarrow n^\mu + a\bar{m}^\mu + \bar{a}m^\mu + a\bar{a}l^\mu \\ m^\mu &\longrightarrow m^\mu + al^\mu \\ \bar{m}^\mu &\longrightarrow \bar{m}^\mu + \bar{a}l^\mu\end{aligned} \tag{3.65}$$

Thus we see that T_2 represents a rotation about the vector l^μ. Therefore, these transformations are termed null rotations.

Under T_3 the tetrad transforms as

$$\begin{aligned}
l^\mu &\longrightarrow l^\mu + b\bar{m}^\mu + \bar{b}m^\mu + b\bar{b}n^\mu \\
n^\mu &\longrightarrow n^\mu \\
m^\mu &\longrightarrow m^\mu + bn^\mu \\
\bar{m}^\mu &\longrightarrow \bar{m}^\mu + \bar{b}n^\mu
\end{aligned} \tag{3.66}$$

Thus we see that T_3 again represents a rotation, but this time about the n^μ vector. Therefore, these transformations are also called null rotations.

Let us examine how the components of the Weyl spinor transform under these transformations. It is easily calculated that under T_1 they transform as

$$\begin{aligned}
\Psi_0 &\longrightarrow \lambda^4\Psi_0 \\
\Psi_1 &\longrightarrow \lambda^2\Psi_1 \\
\Psi_2 &\longrightarrow \Psi_2 \\
\Psi_3 &\longrightarrow \lambda^{-2}\Psi_3 \\
\Psi_4 &\longrightarrow \lambda^{-4}\Psi_4
\end{aligned} \tag{3.67}$$

Under T_2 they transform as

$$\begin{aligned}
\Psi_0 &\longrightarrow \Psi_0 \\
\Psi_1 &\longrightarrow \Psi_1 + \bar{a}\Psi_0 \\
\Psi_2 &\longrightarrow \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0 \\
\Psi_3 &\longrightarrow \Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0 \\
\Psi_4 &\longrightarrow \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0
\end{aligned} \tag{3.68}$$

Under T_3 they transform as

$$\begin{aligned}
\Psi_0 &\longrightarrow \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 \\
\Psi_1 &\longrightarrow \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 \\
\Psi_2 &\longrightarrow \Psi_2 + 2b\Psi_3 + b^2\Psi_4 \\
\Psi_3 &\longrightarrow \Psi_3 + b\Psi_4 \\
\Psi_4 &\longrightarrow \Psi_4
\end{aligned} \tag{3.69}$$

Let us now consider the question of determining a *canonical form* for the Weyl spinor and determining its invariance group for each Petrov type in turn, beginning with Petrov type III.

Petrov Type III

For Petrov type III the Weyl spinor has the form

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)} \tag{3.70}$$

Let us choose as our dyad

$$\zeta_0^A = \alpha^A \tag{3.71a}$$

and

$$\zeta_1^A = \beta^A/\alpha_B\beta^B \tag{3.71b}$$

which clearly satisfies the dyad condition $\zeta_{0A}\zeta_1^A = 1$. With this dyad we can contract the Weyl spinor with at most one ζ_0^A for a non-zero result as otherwise we will contract two α^As giving zero. In addition, contracting with four ζ_1^As will also give zero as we will contract two β^As. So we have

$$\Psi_3 \neq 0 \quad , \quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0 \tag{3.72}$$

We must now consider the SL(2,C) transformations of the dyad which leave these components invariant. A little thought will convince one that any mixing of the dyad or swapping over of its basis vectors will change this pattern of zero/non-zero components so that we can only have the dyad transformation T_1 given by

$$T_1 = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad , \quad \lambda \in \mathbb{C} \tag{3.73}$$

From (3.67) we see that under this transformation the components in (3.72) transform as follows :

$$\tilde{\Psi}_3 = \lambda^{-2}\Psi_3 \ , \ \tilde{\Psi}_0 = \tilde{\Psi}_1 = \tilde{\Psi}_2 = \tilde{\Psi}_4 = 0 \tag{3.74}$$

where $\tilde{\Psi}$ refers to the transformed value. Thus we see that invariance requires that $\lambda = \pm 1$. These two SL(2,C) transformations generate proper homogeneous Lorentz transformations of the Newman-Penrose null tetrad via (3.57) as mentioned above. Clearly because (3.57) connects a tetrad vector with a product of two dyad vectors both the transformation of the dyad with $\lambda = +1$ and that with $\lambda = -1$ correspond to the identity transformation of the tetrad. Thus we have a zero dimensional invariance group and consequently, from theorem 3.3, the isotropy group is also zero dimensional.

Let us now consider a canonical form for the dyad components. Remember that the idea of a canonical form is that by insisting the components have a special form a special dyad is picked out, up to the dyad transformations that leave the components invariant. The canonical form is chosen so that the choice of dyads is as restrictive as possible. In this case it is obvious from (3.74) that this can be achieved by insisting, for convenience, that Ψ_3 is unity, with all the other components vanishing. Because the invariance group is zero dimensional, this canonical form determines a finite number of dyads and hence tetrads (in fact, 2 dyads and 1 tetrad).

Petrov Type II

For Petrov type II the Weyl spinor has the form

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)} \tag{3.75}$$

Let us choose as our dyad

$$\zeta_0^A = \alpha^A \tag{3.76a}$$

and

$$\zeta_1^A = \beta^A / \alpha_B \beta^B \tag{3.76b}$$

in which case the dyad components of the Weyl spinor clearly satisfy

$$\Psi_2, \Psi_3 \neq 0 \quad , \quad \Psi_0 = \Psi_1 = \Psi_4 = 0 \tag{3.77}$$

Once again a little thought will convince one that any mixing of the dyad or swapping over of its basis vectors will change this pattern of zero/non-zero components so that we can only have a dyad transformation of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad , \quad \lambda \in \mathbb{C} \tag{3.78}$$

Again from (3.67) we see that under this transformation the dyad components in (3.77) transform as follows :

$$\tilde{\Psi}_2 = \Psi_2 \ , \ \ \tilde{\Psi}_3 = \lambda^{-2}\Psi_3 \ , \ \ \tilde{\Psi}_0 = \tilde{\Psi}_1 = \tilde{\Psi}_4 = 0 \tag{3.79}$$

Thus once again invariance requires that $\lambda = \pm 1$, so we again have a zero dimensional invariance group (and hence a zero dimensional isotropy group).

Let us now consider a canonical form for the dyad components. In this case it is obvious from (3.79) that a convenient canonical form is obtained by insisting that Ψ_3 is unity, with all other components except Ψ_2 vanishing.

Petrov Type I

For Petrov type I the Weyl spinor has the form

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} \tag{3.80}$$

Remember that each of the principal spinors is determined only up to a complex scalar factor. Therefore, we can arrange that $\alpha_A\beta^A = 1$ and choose $\{\alpha^A, \beta^A\}$ as our dyad. In this dyad clearly Ψ_0 and Ψ_4 will both be zero as they will involve the contraction of two α^As together and two β^As together respectively. If we introduce the notation (α/β) to denote the contraction $\alpha_A\beta^A$, then the full set of components of the Weyl spinor are (up to a constant factor)

$$\begin{aligned}
\Psi_0 &= \Psi_4 = 0 \\
\Psi_1 &= 6(\alpha/\beta)(\beta/\alpha)(\gamma/\alpha)(\delta/\alpha) = -6(\gamma/\alpha)(\delta/\alpha) \neq 0 \\
\Psi_3 &= 6(\alpha/\beta)(\beta/\alpha)(\gamma/\beta)(\delta/\beta) = -6(\gamma/\beta)(\delta/\beta) \neq 0 \\
\Psi_2 &= 4[(\alpha/\beta)(\beta/\alpha)(\gamma/\alpha)(\delta/\beta) + (\alpha/\beta)(\beta/\alpha)(\gamma/\beta)(\delta/\alpha)] \\
&= -4[(\gamma/\alpha)(\delta/\beta) + (\gamma/\beta)(\delta/\alpha)]
\end{aligned} \tag{3.81}$$

In obtaining these expression we have used the result that for any two spinors $\alpha_A\beta^A = -\beta_A\alpha^A$, and the result that the four principal spinors in (3.80) are non-proportional and hence have a non-zero contraction with each other. Note that because Ψ_2 is a sum of two terms, it could possibly be zero.

First let us use a spin and boost transformation T_1 to make

$$\Psi_1 = \Psi_3 = (\Psi_1^u \Psi_3^u)^{1/2} \tag{3.82}$$

where Ψ_1^u and Ψ_3^u refer to the untransformed values given in (3.81), while keeping $\Psi_0 = \Psi_4 = 0$ and leaving Ψ_2 unchanged. If we now use a transformation of the form

$$T = \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} \tag{3.83}$$

it is easily calculated that the Ψs will transform as

$$\begin{aligned}
\Psi_0 &\longrightarrow 8\Psi_1 + 6\Psi_2 \\
\Psi_1 &\longrightarrow 0 \\
\Psi_2 &\longrightarrow -1/2\Psi_2 \\
\Psi_3 &\longrightarrow 0 \\
\Psi_4 &\longrightarrow -\tfrac{1}{2}\Psi_1 + \tfrac{3}{8}\Psi_2
\end{aligned} \tag{3.84}$$

Clearly Ψ_0 and Ψ_4 can only be zero if $\Psi_2 = -\frac{4}{3}\Psi_1$ or $\Psi_2 = \frac{4}{3}\Psi_1$ respectively. If we let $(\gamma/\alpha)(\delta/\beta) = X$ and $(\gamma/\beta)(\delta/\alpha) = Y$ then it is easily shown from (3.81) and (3.82) that this can only occur if $X = Y$. However, from the definitions of X and Y we see that $X = Y$ implies that $\gamma_1\delta_2 = \gamma_2\delta_1$ or

$$\gamma_1/\gamma_2 = \delta_1/\delta_2 \tag{3.85}$$

But we see that (3.85) contradicts our assumption that γ^A and δ^A are non-proportional spinors, so we can deduce that neither Ψ_0 or Ψ_4 transforms to zero under (3.83). Therefore, we can now use a spin and boost transformation to make $\Psi_0 = \Psi_4$, while keeping $\Psi_1 = \Psi_3 = 0$ and leaving Ψ_2 unchanged. Thus our canonical form for Petrov type I becomes

$$\Psi_0 = \Psi_4 \neq 0 \quad , \quad \Psi_1 = \Psi_3 = 0 \tag{3.86}$$

where Ψ_2 may or may not be zero.

From equations (3.67), (3.68) and (3.69) we see that the canonical form (3.86) fixes the parameters λ, a and b to certain discrete values, so that we again have a zero dimensional invariance group.

To summarise our results so far we see that because our invariance group is zero dimensional, defining a canonical form for Petrov types I, II and III defines a finite number of dyads (and hence tetrads). The analysis that follows shows that this is not the case for the remaining Petrov types — types D and N.

Petrov Type D

For Petrov type D the Weyl spinor has the form

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)} \tag{3.87}$$

Let us choose as our dyad

$$\zeta_0^A = \alpha^A \tag{3.88a}$$

and

$$\zeta_1^A = \beta^A / \alpha_B \beta^B \tag{3.88b}$$

in which case the dyad components of the Weyl spinor clearly satisfy

$$\Psi_2 \neq 0 \quad , \quad \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{3.89}$$

It is clear that the transformation

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \quad , \quad \lambda \in \mathbb{C} \tag{3.90}$$

will preserve this pattern of zeros/non-zeros, and that because of the symmetry between α^A and β^A in the Weyl spinor, a transformation swapping them over, i.e.

$$\begin{pmatrix} 0 & a \\ -1/a & 0 \end{pmatrix} \quad , \quad a \in \mathbb{C} \tag{3.91}$$

will also maintain this pattern. Any transformation other than (3.90) and (3.91) will involve mixing of the dyad vectors and will clearly change the pattern of zeros/non-zeros in the components (3.89). It is easily shown that transformations (3.90) and (3.91) leave the Weyl tensor components unchanged, so these two sets of transformations together represent the invariance group. However, transformation (3.91) can be written

$$\begin{pmatrix} 0 & a \\ -1/a & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.92}$$

so we may write the invariance group for type D as

$$I_D = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \ \lambda \in \mathbb{C} \quad ; \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.93}$$

Thus, because λ is complex, the invariance group is a 2-dimensional subgroup of SL(2,$\mathbb{C}$). From theorem 3.3 we see that this means that the maximum dimension of the isotropy group is two.

Because the pattern of components (3.89) limits us exactly to the invariance group I_D, it provides a convenient canonical form. However, because the invariance group is now 2-dimensional the canonical form limits us not to a unique dyad but to an infinite number of dyads.

Petrov Type N

For Petrov type N the Weyl spinor has the form

$$\Psi_{ABCD} = \alpha_A \alpha_B \alpha_C \alpha_D \tag{3.94}$$

Let us choose as our dyad

$$\zeta_0^A = \alpha^A \tag{3.95}$$

and ζ_1^A an arbitrary spinor satisfying $\zeta_{0A}\zeta_1^A = 1$. It is clear that any dyad component involving a contraction with α^A will vanish so that we obtain

$$\Psi_4 = 1 \quad , \quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \tag{3.96}$$

In order for these components to remain invariant under a dyad transformation it is clearly necessary that the transformation of ζ_0^A does not mix in any ζ_1^A. In addition we can transform ζ_1^A in such a way that we multiply it by ± 1 or $\pm i$ and add in any amount of ζ_0^A, as this will keep $\Psi_4 = 1$. The only SL(2,$\mathbb{C}$) transformations which satisfy these criteria are

$$\pm \begin{pmatrix} 1 & 0 \\ \bar{a} & 1 \end{pmatrix} \quad , \quad a \in \mathbb{C} \tag{3.97}$$

and

$$\pm i \begin{pmatrix} 1 & 0 \\ \bar{b} & -1 \end{pmatrix} \quad , \quad b \in \mathbb{C} \tag{3.98}$$

However, transformation (3.98) can be written

$$\pm i \begin{pmatrix} 1 & 0 \\ \bar{b} & -1 \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.99}$$

so the invariance group for type N can be written as

$$I_N = \pm \begin{pmatrix} 1 & 0 \\ \bar{a} & 1 \end{pmatrix}, \; a \in \mathbb{C} \quad ; \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.100}$$

which is a 2-dimensional subgroup of SL(2,$\mathbb{C}$). From theorem 3.3 we see that this means that the maximum dimension of the isotropy group is two.

Because the pattern of components in (3.96) limits us exactly to the invariance group I_N, it provides a convenient canonical form . However, because the invariance group is now 2-dimensional the canonical form limits us not to a unique dyad but to an infinite number of dyads.

We can use these results above to refine the upper bound on the number of covariant derivatives which need to be calculated in order to determine equivalence (remember that the equivalence theorem 3.2 set this upper bound at 10 for a 4-dimensional space). For Petrov types I, II and III the invariance group of Ψ_{abcd} is zero dimensional and so cannot change. If there are n_0 components among the Ψ_{abcd} which are functionally independent with respect to the coordinates x^μ, there are only $4 - n_0$ functionally independent components that remain to be generated. We must generate at least one new functionally independent component per differentiation, as otherwise the Karlhede algorithm terminates, so that after at most $4 - n_0$ differentiations all functionally independent components must have been generated, so we have

$$\textit{Petrov types I, II, III} \; : \; p + 1 \leq 5 - n_0 \tag{3.101}$$

So we see that in the worst case we only need to go to the fifth covariant derivative for these Petrov types.

For Petrov types D and N we can use similar arguments, the only difference being that the invariance group starts with dimension 2 and hence in the worst case could drop 1 dimension at each differentiation down to zero dimensional. Thus we have

$$\textit{Petrov types D and N} \; : \; p + 1 \leq 7 - n_0 \tag{3.102}$$

So we see that in the worst case we only need to go to the seventh covariant derivative for these Petrov types. It should be emphasised that these worst case values of 5 and 7 assume that :

1) The Ψ_{abcd} are constants (i.e. there are no functionally independent components).
2) The dimension of the invariance group and the number of functionally independent components do not both change on differentiating.
3) We produce at most one new functionally independent component on differentiating.

4) The dimension of the invariance group goes down by at most one dimension on differentiating.

So in actual calculations it seems very likely that less derivatives will be needed. In fact, for all calculations performed to date it has been found necessary to go up to at most the third covariant derivative.

If the solution possesses isometries then these upper bounds are reduced. If the dimension of the isometry group is r then this means, from theorem **3.3**, that the final dimension of the invariance group, $dim(H_P)$, plus the number of functionally independent components not produced (out of the 4 possible) equals r. Thus we have

$$\textit{Petrov types I, II and III} \;:\; p+1 \leq 5 - n_0 - r \tag{3.103}$$

$$\textit{Petrov types D and N} \;:\; p+1 \leq 7 - n_0 - r \tag{3.104}$$

These upper bounds will apply to non-vacuum as well as vacuum space-times. For the non-vacuum case it can be seen from the decomposition of the spinor equivalent of the Riemann tensor (3.58) that as well as the Weyl spinor we have the Ricci spinor $\Phi_{ABC'D'}$, which represents the trace-free Ricci tensor. Therefore, for the non-vacuum case any invariance group will have to keep the dyad components of $\Phi_{ABC'D'}$ invariant as well as the dyad components of the Weyl spinor, so the invariance group will either be of the same dimension as in the vacuum case or of smaller dimension. This means that the upper bound in the non-vacuum case will be the same as in the vacuum case.

The one case that has still not been considered is the conformally flat case, Petrov type 0, where the Weyl spinor vanishes. We shall not present the proof here, but it turns out that by proceeding in a similar manner to that above and considering the dimension of the invariance group of the Ricci spinor, it can be shown that the upper bound for this case is also seven. Thus we have

$$\textit{Petrov type } 0 \;:\; p+1 \leq 7 - n_0 - r \tag{3.105}$$

4
The Karlhede Classification of Cylindrically Symmetric and Stationary Axisymmetric Spacetimes

§1. CLASSI

The classification procedure developed in the previous chapter (i.e. steps 1 to 10) is generally referred to as the Karlhede classification. A suite of computer programs has been written to perform this classification on spacetime metrics, these programs being collectively called CLASSI [4]. Note that as discussed in the previous chapter, the Karlhede classification is algorithmic, although determining equivalence in fact involves a non-algorithmic step, so CLASSI does not determine equivalence. CLASSI is written in the computer algebra system SHEEP [4], which is written in the language LISP. In this section we give an account of the use of CLASSI to perform a Karlhede classification. For more details about CLASSI and SHEEP in general see [4].

Because tensorial symmetries become far simpler when expressed in terms of spinors, CLASSI performs the Karlhede classification in terms of the spinor equivalents of the Riemann tensor and its covariant derivatives, which as noted in the previous chapter is equivalent to using a Newman-Penrose null tetrad. An important question is what is the minimal set of components of the Riemann spinor and its covariant derivatives? More precisely, we wish to specify a smallest set of components of the Riemann spinor and its first n covariant derivatives, such that all components up to order n can be algebraically expressed (i.e. using sum, products and contractions) in terms of this set. The relations that enable one to find this algebraic dependence are the Ricci and Bianchi identities, for Siklos has shown that these are the only equations relevant in the most general case [18]. For the vacuum and electrovac cases, Penrose showed that only the totally symmetrised spinor derivatives of the

Weyl and Maxwell spinors are required [6]. However, for the general case more components are required, and a convenient minimal set has been determined by Åman and MacCallum [19]. Although this set is not unique it is convenient in that it is recursively defined (so that additional nth derivative quantities need not be calculated to determine the set for higher derivatives) and it contains only totally symmetrised spinors (which simplifies storage and retrieval algorithms). This minimal set is defined as follows :

Let the set $\nabla^n R$ contain the following :

1) The totally symmetrised spinor nth derivative of the Weyl spinor

$$\nabla_{(A}^{(X'}\nabla_B^{Y'}\ldots\nabla_C^{Z')}\Psi_{HKLM)} \tag{4.1}$$

2) The totally symmetrised spinor nth derivative of the Ricci spinor

$$\nabla_{(A}^{(X'}\nabla_B^{Y'}\ldots\nabla_C^{Z'}\Phi_{GH)}{}^{U'W')} \tag{4.2}$$

3) The totally symmetrised spinor nth derivative of the scalar curvature

$$\nabla_{(A}^{(X'}\nabla_B^{Y'}\ldots\nabla_{C)}^{Z')}\Lambda \tag{4.3}$$

4) Let us define $\Xi_{DEFW'}$ by

$$\Xi_{DEFW'} = \nabla^C{}_{W'}\Psi_{CDEF} = \nabla_{(D}{}^{Y'}\Phi_{EF)Y'W'} \tag{4.4}$$

Then for $n \geq 1$, the set $\nabla^n R$ contains the totally symmetrised $(n-1)$th derivative of $\Xi_{DEFW'}$

$$\nabla_{(A}^{(X'}\nabla_B^{Y'}\ldots\nabla_C^{Z'}\Xi_{DEF)}{}^{W')} \tag{4.5}$$

5) For $n \geq 2$, the d'Alembertian of all quantities in $\nabla^{n-2}R$, i.e.

$$\nabla_{X'}^A\nabla_A^{X'}Q \tag{4.6}$$

where Q is a member of $\nabla^{n-2}R$.

Then the minimal set for the Riemann tensor and all its derivatives up to the nth is provided by the $\nabla^r R$, for r running from 1 to n. How important is it to find a minimal set? Well, the nth derivative of the Riemann tensor contains in total 20×4^n components. It can be shown (see for example [18]) that the set $\nabla^n R$ contains $(n+1)(n+4)(n+5)$ components. Therefore, if one calculated the Riemann tensor and its first 7 derivatives (7 being the upper bound of Karlhede), the cumulative totals of components work out to be 436,900 and 3156 respectively.

CLASSI gives special names to the spinors contained in $\nabla^n R$. We have

$\nabla^{(X'}_{(A}\nabla^{Y'}_{B}\ldots\nabla^{Z')}_{C}\Psi_{HKLM)} \equiv$ DPSI, D2PSI, etc.
$\nabla^{(X'}_{(A}\nabla^{Y'}_{B}\ldots\nabla^{Z'}_{C}\Phi_{GH)}{}^{U'W')} \equiv$ DPHI, D2PHI, etc.
$\nabla^{(X'}_{(A}\nabla^{Y'}_{B}\ldots\nabla^{Z')}_{C)}\Lambda \equiv$ DLAMBDA, D2LAMBDA, etc.
$\nabla^{(X'}_{(A}\nabla^{Y'}_{B}\ldots\nabla^{Z'}_{C}\Xi_{DEF)}{}^{W')} \equiv$ XI, DXI, etc.
$\nabla^{A}{}_{X'}\nabla^{X'}{}_{A}Q \equiv$ APSI, APHI, etc.

Let us outline the procedure by which one performs a Karlhede classification using CLASSI, discussing various aspects in more detail subsequently.

1) Calculate a set of basis 1-forms for the metric (in any tetrad you like).
2) Input this set as IZUD to CLASSI.
3) Specify your input frame with (LORENTZ IFRAME) etc. (see [4]).
4) If the input frame, IFRAME, is not null, force calculations to be performed in a null tetrad by specifying (NULLT FRAME).
5) Calculate PSI in this null tetrad, and determine Petrov type.
6) Bring PSI into the *canonical form* for its Petrov type by calculating a suitable dyad transformation, and placing it in DYTRSP1.
7) Recalculate PSI in the new dyad, and check that it is in canonical form. Calculate PHI.
8) Determine the invariance group H_0 of PSI by checking its canonical form (and hence Petrov type — see §7 of chapter 3). If any invariance group remains,

and if it is possible to restrict PHI to a canonical form within this group, calculate the dyad transformation that will restrict PHI to its canonical form and place it in DYTRSP2.

9) Recalculate PSI and PHI in this new dyad. PSI should not be changed from its canonical form, since we are only allowed to perform dyad transformations in its invariance group in transforming PHI to canonical form. Check that PHI is in canonical form. Calculate LAMBD (note that LAMBD is used to denote the quantity LAMBDA, because LAMBDA has a very special meaning in all LISP systems).

10) Set $q = 0$.

11) Determine the number of functionally independent components n_q among the set $\nabla^n R$ *in its canonical form* for n running from 0 to q, and the dimension of the invariance group H_q of this complete set.

12) If $n_q = n_{q-1}$ and $dim(H_q) = dim(H_{q-1})$ then the classification terminates.

13) Otherwise set $q = q + 1$.

14) Further restrict the frame if possible, by putting $\nabla^q R$ into canonical form, placing the dyad transformation in DYTRSP3. Again note that that $\nabla^q R$ must be put into canonical form using only transformations in H_{q-1}, so that the set $\nabla^n R$ with n running from 0 to $q-1$ is not changed from its canonical form.

15) Go to (11).

Let us discuss further how one goes about finding the canonical forms and invariance groups for the spinors. It is convenient to consider three separate cases :

Case I : Vacuum Metric

In this case we only have the PSI at zeroth order, and, therefore, only derivatives of PSI appear at higher order. Let us list the canonical forms for PSI, which as discussed in §7 of the previous chapter, can be determined from the

Petrov type. The canonical forms are summarised in the following table :

Table of Canonical Forms for PSI

Petrov type	Canonical form	Abbreviation
I	$\Psi_4 = \Psi_0 \neq 0,\ \Psi_3 = \Psi_1 = 0$	X0–0X
II	$\Psi_4 = 1,\ \Psi_2 \neq 0,\ \Psi_3 = \Psi_1 = \Psi_0 = 0$	10X00
III	$\Psi_3 = 1,\ \Psi_4 = \Psi_2 = \Psi_1 = \Psi_0 = 0$	01000
D	$\Psi_2 \neq 0,\ \Psi_4 = \Psi_3 = \Psi_1 = \Psi_0 = 0$	00X00
N	$\Psi_4 = 1,\ \Psi_3 = \Psi_2 = \Psi_1 = \Psi_0 = 0$	10000

The abbreviations used for the canonical forms in the above table are a common shorthand in which the sequence of five characters represents the components $\Psi_4 \ldots \Psi_0$, with coincidences between components being reflected in the use of the same character to represent them (the blank entry for Petrov type I indicates that there are no conditions on Ψ_2). By checking the canonical form, CLASSI determines the invariance group H_0 for PSI, as discussed in §7 of the previous chapter. At first order the only new quantity will be DPSI, at second order D2PSI and so on. For Petrov types I, II and III the invariance group H_0 is zero dimensional so no new dyad transformations are allowed at higher order and the problem of finding a canonical form for DPSI, D2PSI etc. does not arise. The problem of finding a canonical form for DPSI, D2PSI etc. for Petrov types D and N, where H_0 is 2-dimensional, is discussed in [20] and [21] respectively.

Case II : Conformally Flat Metric

In the case of a conformally flat metric, PSI will vanish so that it is PHI which must be put into a canonical form. In fact, it is more convenient to use a spinor called CHI, which is built from PHI using the equation

$$\chi_{ABCD} = \tfrac{1}{4}\Phi_{(AB}{}^{E'F'}\Phi_{CD)E'F'} \tag{4.7}$$

as defined in [22]. This spinor CHI is called the Plebanski-Petrov spinor and is classified in a Plebanski-Petrov classification. The reason it is convenient to use this spinor is that it is totally symmetric and can, therefore, be classified in exactly the same way as the Weyl spinor using the Petrov classification programs of CLASSI. The canonical forms for CHI are exactly the same as for PSI, with Plebanski-Petrov type replacing Petrov type, and these canonical forms completely determine the canonical forms for PHI. By checking the canonical form, CLASSI determines the invariance group H_0 for PHI. At first order the new quantities will be DPHI and DLAMBDA, at second order D2PHI, D2LAMBDA and APHI and so on. At each stage of covariant differentiation the frame must be further restricted if possible, by putting the relevant spinors into canonical form.

This approach to the canonical form for PHI is fine as long as the spinor CHI does not vanish, when it will be necessary to use PHI directly. CHI only vanishes for three Segre types, but unfortunately two of these are physically interesting. The three Segre types involved are the perfect fluid, the tachyon fluid and the electromagnetic null fluid. The canonical forms for PHI for these Segre types are given in the following table, where unmentioned components of PHI are zero:

Table of Canonical Forms for PHI (CHI = 0)

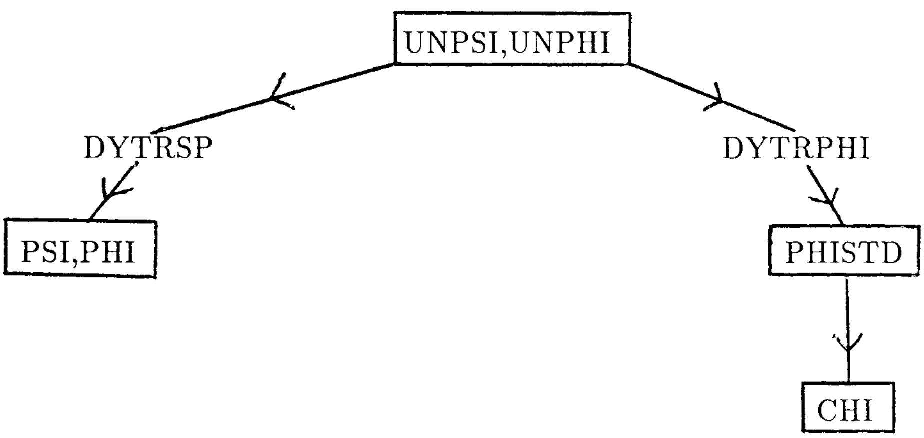
Segre Type	Canonical Form
Perfect fluid : A1[(111),1]	$\Phi_{22'} = 2\Phi_{11'} = \Phi_{00'} \neq 0$
Tachyon fluid : A1[1(11,1)]	$\Phi_{22'} = -2\Phi_{11'} = \Phi_{00'} \neq 0$
E.m. null fluid : A3[(11,2)]	$\Phi_{22'} \neq 0$

By checking the canonical form CLASSI determines the invariance group H_0 for PHI.

Case III : Non-vacuum and Non-conformally Flat Metric

For Petrov types D and N, determining a canonical form for PSI (in the same way as above, using the Petrov type, and putting the relevant dyad transformation in DYTRSP1) will still leave a 2-dimensional invariance group, so that one must then investigate if the dimension of this invariance group can be reduced by determining a canonical form for PHI. To determine the invariance group for PHI, and in fact to get a unique answer for the Segre type, CLASSI needs to have it in the canonical forms given previously for the conformally flat case. However, putting PHI into these canonical forms may disrupt the canonical form of PSI, so CLASSI provides a dyad transformation DYTRPHI which transforms CHI and a copy of PHI called PHISTD, but which does not affect PSI. The following diagram summarises the dyad transformations of CLASSI and the spinors they transform.

Dyad Transformation Mechanisms of CLASSI



UNPSI and UNPHI stand for the untransformed spinors, i.e. the spinors in the original null tetrad.
DYTRSP $\equiv$ DYTRSP1$\times$DYTRSP2$\times$DYTRSP3
DYTRPHI $\equiv$ DYTRPHI1$\times$DYTRPHI2$\times$DYTRPHI3
Note that DYTRSP in fact acts on all the spinors in $\nabla^n R$, for any n.

Thus by checking the canonical forms of CHI, or in the cases where it vanishes PHISTD, the invariance group of PHI is determined. If we denote the invariance group of PSI by I_Ψ and that of PHI by I_Φ, then the transformations $I_\Psi - I_\Phi$ can be used to determine a canonical form for PHI without disturbing PSI, this canonical form possibly being less restrictive than the conformally flat counterpart. The relevant dyad transformations are placed in DYTRSP2. The invariance group H_0 can be calculated as $I_\Psi \cap I_\Phi$. At first order the frame is further restricted if possible by finding a canonical form for DPSI, DPHI, DLAMBDA and XI, with the relevant dyad transformation being put in DYTRSP3. This process is continued at each subsequent stage of differentiation until the frame can be restricted no more. There is a package called DYTAUT which will find the dyad transformations required to put PSI, CHI and PHISTD into canonical form automatically, although at the time of writing it is not always effective, especially with Petrov type I, so the transformation must in many cases be calculated by hand.

In using CLASSI to classify a solution, apart from the substitution lists required for algebraic simplification (see [4]), the main interaction of the user is in transforming the dyad so as to put the various spinors into canonical form. As mentioned previously, the system possesses an automatic mechanism for finding the correct dyad transformation, and this works well for all Petrov types other than Petrov type I. For Petrov type I it is often no use, so the user has to experiment himself in order to find the required transformation. The 'Applications of SHEEP' manual of Jim Skea [4] states that a transformation of the form

$$\begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.8}$$

can be useful in order to transform Ψ_1 and Ψ_3 to zero (remember that the canonical form for Petrov type I is $\Psi_0 = \Psi_4 \neq 0$, $\Psi_2 \neq 0$, $\Psi_1 = \Psi_3 = 0$). Let us examine this in more detail. It is straightforward to calculate the effect of such a dyad transformation on PSI, and the result is

$$\begin{aligned}
\Psi_0 &\longrightarrow \Psi_0 + 4\Psi_1 + 6\Psi_2 + 4\Psi_3 + \Psi_4 \\
\Psi_1 &\longrightarrow -\tfrac{1}{2}\Psi_0 - \Psi_1 + \Psi_3 + \tfrac{1}{2}\Psi_4 \\
\Psi_2 &\longrightarrow \tfrac{1}{4}\Psi_0 - \tfrac{1}{2}\Psi_2 + \tfrac{1}{4}\Psi_4 \\
\Psi_3 &\longrightarrow -\tfrac{1}{8}\Psi_0 + \tfrac{1}{4}\Psi_1 - \tfrac{1}{4}\Psi_3 + \tfrac{1}{8}\Psi_4 \\
\Psi_4 &\longrightarrow \tfrac{1}{16}\Psi_0 - \tfrac{1}{4}\Psi_1 + \tfrac{3}{8}\Psi_2 - \tfrac{1}{4}\Psi_3 + \tfrac{1}{16}\Psi_4
\end{aligned}$$

Therefore, we see that Ψ_1 and Ψ_3 will vanish if initially $\Psi_0 = \Psi_4$ and $\Psi_1 = \Psi_3$. In the classification work which was carried out, the cases for which it was most difficult to find the correct dyad transformation indeed had $\Psi_0 = \Psi_4$ and $\Psi_1 = \Psi_3$, so the dyad transformation (4.8) enabled one to transform Ψ_1 and Ψ_3 to zero. A boost transformation,

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \tag{4.9}$$

clearly causes the spin coefficients to transform as

$$\begin{aligned}
\Psi_0 &\longrightarrow z^4\Psi_0 \\
\Psi_1 &\longrightarrow z^2\Psi_1 \\
\Psi_2 &\longrightarrow \Psi_2 \\
\Psi_3 &\longrightarrow z^{-2}\Psi_3 \\
\Psi_4 &\longrightarrow z^{-4}\Psi_4
\end{aligned}$$

Therefore, in these circumstances a boost can always be used to make $\Psi_0 = \Psi_4$ while keeping $\Psi_1 = \Psi_3 = 0$. One requires

$$z^4\Psi_0 = 1/z^4\Psi_4$$

$\Rightarrow$

$$z = (\Psi_4/\Psi_0)^{1/8} \tag{4.10}$$

In many cases the components of PSI will be complicated algebraic expressions and, because of simplification problems, the system will often still not recognise that $\Psi_0 = \Psi_4$ even after the correct dyad transformation has been applied. In such a case the only solution to the problem is to substitute for all of Ψ_0 by L say and all of Ψ_4 by M say, and to set $z = (M/L)^{1/8}$. This results in

$$\Psi_0 = \Psi_4 = (LM)^{1/2},$$

fixing the canonical form correctly.

Let us conclude this section by listing the commands which perform the various classifications.

A Petrov classification is performed by the command (PETROV).
A Plebanski-Petrov classification is performed by the command (PPETROV).
A Segre classification is performed by the command (SEGRE).
The Karlhede classification is performed by the command (CLASSIFY). Here it will be necessary that the requisite dyad transformations are included in the metric file. The commands (CLASSIFY0), (CLASSIFY1) etc. enable the user to step through each stage of the classification procedure, allowing the fixing of the tetrads along the way.

A classification summary can be obtained after (CLASSIFY) has been run, by using the command (CLASSISUM). The output of this command takes the form

$$ABC \qquad DE \qquad FGHI \qquad JKLM \tag{4.11}$$

A is the Segre type (e.g. e $\equiv$ e.m. non-null fluid, p $\equiv$ perfect fluid, r $\equiv$ e.m. null fluid), with 0 indicating vanishing Ricci spinor. B is 0 if LAMBD is zero and 1 if it is non-zero. C is the Petrov type. D is the dimension of the isometry group. E is the dimension of the isotropy group. F,G,H and I are the invariance group at each stage of differentiation, H_q (e.g. e $\equiv$ spin and boosts, n $\equiv$ null rotations, b $\equiv$ boosts, s $\equiv$ spin transformations). J,K,L and M are the number of functionally independent components at each stage of differentiation, n_q. A blank entry for F,G,H or I with a corresponding blank entry for J,K,L or M indicates that neither the dimension of the invariance group H_q nor the number n_q changes.

The output of (CLASSIFY) can be sent to a file by means of the command (OUTF 'file name' (CLASSIFY)) and similarly for the output of CLASSISUM.

§2. Stationary Axisymmetric Spacetimes

In 1980 a book entitled 'Exact solutions of Einstein's field equations' [1] was published, this book representing the result of many years work to compile a complete list of all exact solutions then known. A major effort is underway to use CLASSI to compile a computer database of all the exact solutions. The idea is that every exact solution will appear on the database together with a Karlhede classification of that solution. In §4 of this chapter we present the classification of the cylindrically symmetric metrics explicitly given in this book as well as a number of stationary axisymmetric vacuum solutions.

First let us consider stationary axisymmetric solutions. Such a solution possesses a Killing vector ζ^a whose orbits are timelike curves and a Killing vector ψ^a whose orbits are closed spacelike curves. In addition we require that these two Killing vectors commute, i.e.

$$[\zeta^a,\ \psi^a] = 0 \tag{4.12}$$

These spacetimes are of great interest in general relativity since they describe equilibrium configurations of axisymmetric rotating bodies. It has been shown [23] that for the cases of greatest physical importance, *asymptotically flat* stationary axisymmetric spacetimes, (4.12) is automatically satisfied.

The commutativity of ζ^a and ψ^a means that we can choose coordinates $\{x^0 = t,\ x^1 = \phi,\ x^2,\ x^3\}$ so that both $\zeta^a = (\frac{\partial}{\partial t})^a$ and $\psi^a = (\frac{\partial}{\partial \phi})^a$ are coordinate vector fields. In such a coordinate system the metric components will be independent of t and ϕ, so the metric will take the form

$$ds^2 = g_{\mu\nu}(x^2, x^3)dx^\mu dx^\nu \tag{4.13}$$

Thus we must solve for 10 unknown functions, $g_{\mu\nu}$, of two variables. We shall now show that by a further careful choice of coordinate system, a weak further assumption and some use of Einstein's equations, we can reduce the metric to one containing only four unknown functions for the non-vacuum case and one containing only three unknown functions for the vacuum case. In addition, the vacuum Einstein equations reduce to two equations for two of the unknown functions together with a quadrature for the third.

The first important simplification arises if the 2-dimensional subspaces of the tangent space at each point spanned by the vectors orthogonal to ζ^a and ψ^a are integrable, i.e. are tangent to 2-dimensional surfaces. This is the case if the conditions specified in the following theorem are satisfied, which is the case for all explicitly known stationary axisymmetric solutions.

Theorem 4.1

Let ζ^a and ψ^a be two commuting Killing fields such that (i) $\zeta_{[a}\psi_b\nabla_c\zeta_{d]}$ and $\zeta_{[a}\psi_b\nabla_c\psi_{d]}$ each vanishes at at least one point of the spacetime and (ii) $\zeta^a R_a{}^{[b}\zeta^c\psi^{d]} = \psi^a R_a{}^{[b}\zeta^c\psi^{d]} = 0$. Then the 2-planes orthogonal to ζ^a and ψ^a are integrable

The conditions of this theorem will be satisfied by a wide range of stationary, axisymmetric spacetimes of physical interest. Condition (i) will be satisfied for asymptotically flat spacetimes, since there must be a 'rotation axis' on which ψ^a vanishes. A wide variety of energy-momentum tensors will satisfy condition (ii) including vacuum, a perfect fluid with the 4-velocity in the plane spanned by ζ^a and ψ^a (i.e. circular flow) and a stationary axisymmetric electromagnetic field [24].

If the conditions of this theorem are satisfied, then we may introduce coordinates x^2, x^3 in one of the orthogonal 2-surfaces and 'carry' these coordinates to the rest of the spacetime along the integral curves of ζ^a and ψ^a. Then in the coordinate system $\{t, \phi, x^2, x^3\}$ the metric has the form

$$g_{\mu\nu} = \begin{pmatrix} -e^{2U} & W & 0 & 0 \\ W & X & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{23} & g_{33} \end{pmatrix} \tag{4.14}$$

The block of zero expresses the orthogonality of $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial x^3}$ with $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$. Thus theorem 4.1 allows us to reduce the number of unknown metric functions to 6. Without theorem 4.1 only two metric components in the 2×2 block could be set to zero by use of the coordinate freedom available in the choice of a coordinate system adapted to ζ^a and ψ^a.

We still have not specified how the coordinates x^2 and x^3 are to be chosen and we shall see that a clever choice of these coordinates leads to a further simplification of the metric. We define the scalar function ρ by

$$\rho^2 = e^{2U}X + W^2 \tag{4.15}$$

i.e. ρ^2 is equal to the negative of the determinant of the $t - \phi$ part of the metric. Assuming that $\nabla_a \rho \neq 0$, we choose ρ as one of the coordinates, x^2, of the

2-surface. We choose the other coordinate, $z = x^3$, so that $\nabla_a z$ is orthogonal to $\nabla^a \rho$. In the coordinates $\{t, \phi, \rho, z\}$ the metric now takes the form

$$g_{\mu\nu} = \begin{pmatrix} -e^{2U} & e^{2U}w & 0 & 0 \\ e^{2U}w & (\rho^2 e^{-4U} - w^2)e^{2U} & 0 & 0 \\ 0 & 0 & e^{2\gamma} & 0 \\ 0 & 0 & 0 & e^{2\gamma}\Lambda^2 \end{pmatrix} \tag{4.16}$$

where $w = We^{-2U}$, or

$$ds^2 = -e^{2U}(dt - wd\phi)^2 + e^{-2U}\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + \Lambda^2 dz^2) \tag{4.17}$$

Thus we have reduced the number of unknown metric functions to 4 — U, w, γ and Λ. Equation (4.17) is the general form of a stationary axisymmetric spacetime satisfying the conditions in theorem (4.1).

The form of the metric can be simplified further for the vacuum case, $R_{ab} = 0$. The Einstein equation

$$R^t{}_t + R^\phi{}_\phi = 0 \tag{4.18}$$

gives

$$D^a D_a \rho = 0 \tag{4.19}$$

where D_a is the covariant derivative in the 2-dimensional surface spanned by ρ and z with the induced metric $ds^2 = e^{2\gamma}(d\rho^2 + \Lambda^2 dz^2)$. Equation (4.19) leads to two important deductions : (a) If $\rho \neq$ constant, it can be shown that $\nabla_a \rho$ can vanish only at isolated points. Since our coordinates ρ, z will be well behaved except where $\nabla_a \rho = 0$, this shows that our coordinate system can break down only at isolated points. In fact, in many situations it is possible to show that $\nabla_a \rho \neq 0$ everywhere, so the coordinates ρ and z are globally well behaved [25]. (b) Evaluating (4.19) directly leads to the result that Λ is a function of z only. Thus if we transform z by the transformation $z \to \int \Lambda dz$, we set $\Lambda = 1$. Thus we have reduced the number of unknown metric functions to 3, and the metric now takes the remarkably simple form

$$ds^2 = -e^{2U}(dt - wd\phi)^2 + e^{-2U}[\rho^2 d\phi^2 + e^{2k}(d\rho^2 + dz^2)] \tag{4.20}$$

where $e^{2k} = e^{2\gamma}e^{2U}$.

The remaining vacuum Einstein equations, other than (4.19), can now be computed and the result is that we obtain four independent equations for the three unknown metric functions. The first two equations involve only U and w and can be formulated most succinctly by defining a 3-dimensional Euclidean space

$$d\tilde{s}^2 = \rho^2 d\phi^2 + d\rho^2 + dz^2 \tag{4.21}$$

and expressing the equations in terms of the flat covariant derivative operator, $\tilde{D}_a$, associated with this metric. In this way the first two equations may be viewed as equations for axisymmetric scalar fields U and w in 3-dimensional Euclidean space. The four equations are

$$0 = \tilde{D}^a(e^{-2U}\tilde{D}_a e^{2U} + \rho^{-2}e^{4U}w\tilde{D}_a w) \tag{4.22}$$

$$0 = \tilde{D}^a(\rho^{-2}e^{4U}\tilde{D}_a w) \tag{4.23}$$

$$\frac{\partial k}{\partial \rho} = \frac{1}{4}\rho e^{-4U}[(\frac{\partial e^{2U}}{\partial \rho})^2 - (\frac{\partial e^{2U}}{\partial z})^2] - \frac{1}{4}\rho^{-1}e^{4U}[(\frac{\partial w}{\partial \rho})^2 - (\frac{\partial w}{\partial z})^2] \tag{4.24}$$

$$\frac{\partial k}{\partial z} = \frac{1}{2}\rho e^{-4U}\frac{\partial e^{2U}}{\partial \rho}\frac{\partial e^{2U}}{\partial z} - \frac{1}{2}\rho^{-1}e^{4U}\frac{\partial w}{\partial \rho}\frac{\partial w}{\partial z} \tag{4.25}$$

The integrability condition for the last two equations, $\frac{\partial^2 k}{\partial z \partial \rho} = \frac{\partial^2 k}{\partial \rho \partial z}$ is satisfied by virtue of equations (4.22) and (4.23). Therefore, given a solution of (4.22) and (4.23), a solution of (4.24) and (4.25) always exists and is unique up to the addition of a constant. Thus, apart from the computation required to find k explicitly, the problem of solving for all stationary axisymmetric vacuum solutions has been reduced to solving (4.22) and (4.23) for the two axisymmetric functions U and w in 3-dimensional Euclidean space. This is a great simplification of the original problem which was to solve the full Einstein equations for the 10 unknown functions $g_{\mu\nu}$. However, the equations are still sufficiently difficult to solve that, with the exception of the static solutions discussed below, almost no solutions have been obtained by direct attack on equations (4.22) and (4.23) (in cases where solutions have been found it has often proved useful to work with coordinates other than ρ and z [26]).The equations can be reformulated by the introduction of potentials, as was first done by Ernst [27], and a few solutions of interest, the Tomimatsu-Sato solutions [28], have been found by direct study of these modified equations. Recent progress in methods for generating solutions [29] has produced algorithms for obtaining *all* asymptotically flat stationary axisymmetric vacuum solutions, but because the algebraic computations required in this procedure are so formidable, very few explicit solutions have so far been obtained.

For static axisymmetric vacuum solutions direct attack on equations (4.22) and (4.23) is completely successful in that *all* the static axisymmetric vacuum spacetimes are obtained by this approach. For the static case, assuming that the Killing vector ψ^a lies in the hypersurface orthogonal to ζ^a, we have $w = 0$, so (4.23) is trivially satisfied, and if we let $\chi = ln(e^{2U})$, equation (4.22) reduces to simply

$$\nabla^2 \chi = 0 \tag{4.26}$$

that is χ is an axisymmetric solution of the ordinary Laplace equation in 3-dimensional Euclidean space. Since all such solutions are explicitly known, all static, axisymmetric vacuum solutions of Einstein's equations can be explicitly obtained. This analysis of static, axisymmetric vacuum spacetimes was first carried out by Weyl [30], and the solutions are often referred to as the Weyl solutions. It should be noted that the properties of the solution of (4.26) do not translate in a simple way to the properties of the spacetime metric it generates. Specifically, the monopole solution of Laplace's equation does *not* generate a spherically symmetric spacetime, the Schwarzschild solution.

§3. Cylindrically Symmetric Spacetimes

A cylindrically symmetric solution possesses a Killing vector ζ^a whose orbits are spacelike curves and a Killing vector ψ^a whose orbits are closed spacelike curves. Again we require that the two Killing vectors commute. Therefore, the analysis of the metric form proceeds in the same way as the stationary axisymmetric case, except that we now call the coordinate adapted to the ζ^a Killing vector z. Thus the general cylindrically symmetric metric can be written as

$$ds^2 = e^{2U}(dz - w d\phi)^2 + e^{-2U}\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 - \Lambda^2 dt^2) \qquad (4.27)$$

which can be obtained from (4.17) by the substitutions

$$t \to z, \;\; z \to t \qquad (4.28a)$$

and the substitutions

$$\rho \to i\rho, \;\; e^{2U} \to -e^{2U}, \;\; e^{2\gamma} \to -e^{2\gamma} \qquad (4.28b)$$

which are required to obtain the correct signature for the metric. Therefore, locally at least, the solution of Einstein's field equations is formally the same problem for the cylindrically symmetric case as for the stationary axisymmetric case, although it should be noted that the substitutions (4.28) will in general take real stationary axisymmetric solutions into complex cylindrically symmetric solutions.

§4. Results

The following table summarises the results of the Karlhede classification which has been performed on the cylindrically symmetric and stationary axisymmetric vacuum solutions of the exact solutions book [1]. The metric files for these solutions are given at the end of the thesis, in appendix C. The left hand column of the table identifies the solution by its equation number in the exact solutions book, with a prefix $c-$ representing a cylindrically symmetric solution and a prefix $sa-$ representing a stationary axisymmetric vacuum solution. The right hand column is the output from (CLASSISUM), which has been described earlier in this chapter.

Table of Karlhede Classification of

Cylindrically Symmetric and Stationary Axisymmetric Solutions

Solution	Classification Summary
c-20.7	001 30 00−− 11−−
c-20.8	001 30 00−− 11−−
c-20.9a	e01 30 00−− 11−−
c-20.9b	e01 30 00−− 11−−
c-20.9c	e01 30 00−− 11−−
c-20.10	e0D 41 ebb− 111−
c-20.11	e01 30 00−− 11−−
c-20.12	e01 30 00−− 11−−
c-20.13	p11 30 00−− 11−−
c-20.14	p11 30 00−− 11−−
c-20.14(sp)	p11 30 00−− 11−−
c-20.18	p11 30 00−− 11−−
c-20.25	001 20 00−− 22−−
c-20.39	r01 20 00−− 22−−
c-20.41	r0N 20 n00− 022−
GH2	001 20 00−− 22−−
sa-18.2	001 20 00−− 22−−
sa-18.4	001 20 00−− 22−−
sa-18.8	00D 41 ess− 111−
sa-18.9	001 20 00−− 22−−
sa-18.23	002 20 000− 122−

Since the publication of the exact solutions book it has been discovered that c-20.12 is, in fact, not a solution of Einstein's equations. In addition, solutions c-20.9 require constraints on the values of the constants appearing in the solution in order to satisfy Einstein's equations.

Solution c-20.14(sp) is a specialisation of solution c-20.14 with $\zeta = \rho$ (see [1]). Solution GH2 is the general metric for a Lewis-type class of solutions discussed in G.Holmes's Ph.D thesis (1986), p.79, case (ii). Solutions c-20.9a and c-20.9b are, in fact, *locally* equivalent under the coordinate transformation $\phi \rightarrow z$.

Solutions c-20.9a and c-20.9c were investigated to discover which specialisations would give a more algebraically special Petrov type. In order for the Petrov type to be I, three of CLASSI's discriminants must be non-zero – they are called PSI4 (which is just Ψ_4), U10CRIT and U11CRIT. All possible specialisations of the three parameters c, d and m of the solutions which broke one or more of these conditions were found , and for both solutions c-20.9a and c-20.9c this resulted in four distinct solutions of a more algebraically special Petrov type – flat space, two of vacuum type D and one of non-vacuum type D. The results are summarised in the following tables.

i) Solution c-20.9a :

$$ds^2 = \rho^{2m^2} G^2 (d\rho^2 - dt^2) + \rho^2 G^2 d\phi^2 + G^{-2} dz^2$$

where

$$G = c\rho^m + d\rho^{-m}$$

Table of Specialisations of c-20.9a

Specialisation	Classification Summary
$m = 1$	e0D 41 ebb– 111–
$d = 0,\ m = -\frac{1}{2}$	00D 41 ess– 111–
$d = 0,\ m = -2$	00D 41 ebb– 111–
$m = 0$ or $c = 0,\ m = 1$	Flat space

ii) Solution c-20.9c :

$$ds^2 = \rho^{2m^2} G^2 (d\rho^2 + dz^2) + \rho^2 G^2 d\phi^2 - G^{-2} dt^2$$

where again

$$G = c\rho^m + d\rho^{-m}$$

Table of Specialisations of c-20.9c

Specialisation	Classification Summary
$m = 1$	e0D 41 ess– 111–
$c = 0,\ m = \frac{1}{2}$	00D 41 ebb– 111–
$c = 0,\ m = 2$	00D 41 ebb– 111–
$m = 0$ or $c = 0,\ m = 1$	Flat space

The solution c-20.9a with the specialisation $d = 0$, $m = -2$ is equivalent to the solution c-20.9c with the specialisation $c = 0$, $m = 2$ if $c = \pm 1$ and $d = \pm 1$ for the respective solutions.

The special case of the Weyl solutions with $e^{2U} = (\frac{x-1}{x+1})^{\delta}$, where x is a prolate spheroidal coordinate, has been investigated by Voorhees [31] and Zipoy [26]. For $\delta = 1$ we have the Schwarzschild solution, sa-18.8, and for $\delta = 2$ we have the Darmois solution [32], sa-18.9. The Schwarzschild solution is Petrov type D and the Darmois solution Petrov type I. An investigation was made into which values of δ give Petrov type D in general. This time, in order for the Petrov type to be I, CLASSI'S discriminants called B4CRIT, U8CRIT and U9CRIT must all be non-zero. It is found that the only values of δ which make the discriminants zero are $\delta = 0$, $\delta = 1$ and $\delta = -1$. The solution with $\delta = -1$ is equivalent to the Schwarzschild solution under the coordinate transformation $x \rightarrow -x$, the solution with $\delta = 0$ is flat space. Thus, one may conclude that the only solution in this class with Petrov type D is the Schwarzschild solution. Otherwise the Petrov type is always I, except when $\delta = 0$ in which case we have flat space.

5
The Karlhede Classification of Type D Vacuum Spacetimes

§1. Introduction

Following some work by Kinnersley [10], all type D vacuum spacetimes are split into 3 invariant classes. A *canonical form* is derived for each class in turn, these canonical forms forming an essential part of the Karlhede classification. In addition, the frame transformations required to obtain the canonical forms are calculated.

At the end of §7 of chapter 3 consideration is made of the upper bound on the order of covariant differentiation of the Riemann tensor required to perform a Karlhede classification of a spacetime metric, and hence tackle the equivalence problem. Equations (3.103), (3.104) and (3.105) tell us that, in the absence of isometries, it might be necessary to go as high as the 5th derivative for Petrov types I, II and III, and as high as the 7th derivative for Petrov types D, N and 0. Because all type D *vacuum* spacetimes admit at least a 2-dimensional isometry group, equation (3.104) tells us that the upper bound for these spacetimes is reduced to 5. The Karlhede classification is a purely geometrical classification in the sense that it works for any Riemannian manifold, regardless of whether the metric obeys any field equations, and these upper bounds are calculated without using the information contained in Einstein's field equations. In this chapter we use the information contained in the field equations and Bianchi identities, together with symmetry considerations, to reduce the upper bound for the vacuum type D case to 2 for two of our invariant classes, and to 3 for the third. In fact, Kinnersley [10] derived all type D vacuum solutions explicitly, so one can use CLASSI on these solutions directly. This has been done by Åman and Karlhede [11], and the result is that it is only necessary to go up to the second covariant derivative. However, the advantage of the approach in this chapter is that it does not require a complete set of solutions, and can, therefore, be extended to other cases where a complete set of solutions is not known. In the next chapter vacuum type N solutions are considered and in the final chapter non-vacuum type D solutions. In none of these cases is a complete set of solutions known.

In this chapter we work in the extended NP formalism (Newman-Penrose formalism [13]) called GHP formalism (Geroch-Held-Penrose formalism [12]). The advantage of using this formalism results from the fact that type D spacetimes have a Weyl tensor which admits spin and boost transformations as its invariance group (see [9]), and it is precisely these transformations that the GHP formalism respects. In this thesis we do not give an account of GHP formalism, so the reader unfamiliar with it will need to consult [12].

In §2 a canonical form for the Weyl spinor of a type D spacetime and the GHP equations for the vacuum case are given. In the following section the totally symmetrised spinor covariant derivative of the Weyl spinor, DPSI, is calculated, and by reference to the work by Kinnersley [10], the spacetimes are split into three different classes and a canonical form derived for each class in turn. In §4 D2PSI is calculated and the field equations, together with the result that all type D vacuum spacetimes admit at least a 2-dimensional isometry group, are used to show that the upper bound can in the worst case be reduced to three. In the following section, by considering the third covariant derivative, it is proved that the upper bound is three without having to use Kinnersley's result or the isometry group result. The final section summarises the main results of the chapter.

§2. Type D Vacuum Spacetimes

In a vacuum spacetime the Riemann tensor reduces to the Weyl tensor. We introduce a spin frame $\{o^A, \iota^A\}$ which satisfies

$$o_A \iota^A = 1$$

From now on we shall use dyad components of the Weyl spinor and its covariant derivatives, which is equivalent to using a Newman-Penrose null tetrad [9]. Because it is totally symmetric, the Weyl spinor has only five independent dyad components which, in standard notation, are labelled

$$\begin{aligned}
\Psi_0 &= \Psi_{0000} = \Psi_{ABCD}o^A o^B o^C o^D \\
\Psi_1 &= \Psi_{0001} = \Psi_{ABCD}o^A o^B o^C \iota^D \\
\Psi_2 &= \Psi_{0011} = \Psi_{ABCD}o^A o^B \iota^C \iota^D \\
\Psi_3 &= \Psi_{0111} = \Psi_{ABCD}o^A \iota^B \iota^C \iota^D \\
\Psi_4 &= \Psi_{1111} = \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D
\end{aligned}$$

Then the Weyl spinor of a type D spacetime has the *canonical form*

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0\ ; \qquad \Psi_2 \neq 0 \tag{5.1}$$

The proper Lorentz transformations form a six parameter invariance group on the spin frame at each point. The condition (5.1) is only preserved under the two parameter invariance subgroup H_0 defined by

$$H_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\ ; \qquad \lambda = re^{i\theta} \in \mathbb{C} \tag{5.2}$$

under which

$$\begin{aligned}
o^A &\longrightarrow \lambda o^A \\
\iota^A &\longrightarrow \lambda^{-1}\iota^A
\end{aligned}$$

i.e. H_0 is the subgroup of spin and boost transformations.

In GHP notation, the Bianchi identities in vacuum under assumption (5.1) become

$$3\kappa\Psi_2 = 0 \tag{5.3a}$$

$$Þ\Psi_2 = 3\rho\Psi_2 \tag{5.3b}$$

$$ð'\Psi_2 = 3\tau'\Psi_2 \tag{5.3c}$$

$$3\sigma'\Psi_2 = 0 \tag{5.3d}$$

$$3\kappa'\Psi_2 = 0 \tag{5.3e}$$

$$Þ'\Psi_2 = 3\rho'\Psi_2 \tag{5.3f}$$

$$ð\Psi_2 = 3\tau\Psi_2 \tag{5.3g}$$

$$3\sigma\Psi_2 = 0 \tag{5.3h}$$

These equations require

$$\sigma = \sigma' = \kappa = \kappa' = 0 \tag{5.4}$$

which also follows directly from the Goldberg-Sachs theorem [33]. The Bianchi identities, therefore, reduce to

$$Þ\Psi_2 = 3\rho\Psi_2 \tag{5.5a}$$

$$ð'\Psi_2 = 3\tau'\Psi_2 \tag{5.5b}$$

together with their primed versions (recall $\Psi_2' = \Psi_2$).

Under assumptions (5.1) and (5.4) the GHP vacuum field equations become

$$ð\rho = (\rho - \bar{\rho})\tau \tag{5.6a}$$

$$Þ\rho = \rho^2 \tag{5.6b}$$

$$Þ\tau = \rho(\tau - \bar{\tau}') \tag{5.6c}$$

$$ð\tau = \tau^2 \tag{5.6d}$$

$$Þ'\rho = \rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2 + ð'\tau \tag{5.6e}$$

together with their primed versions, and the commutators acting on a spin and boost weighted scalar of type $\{p, q\}$

$$\begin{aligned}(ÞÞ' - Þ'Þ)\eta_{pq} = {} & [(\bar{\tau} - \tau')ð + (\tau - \bar{\tau}')ð' \\ & - p(\Psi_2 - \tau\tau') - q(\bar{\Psi}_2 - \bar{\tau}\bar{\tau}')]\eta_{pq}\end{aligned} \tag{5.7a}$$

$$(Þð - ðÞ)\eta_{pq} = (\bar{\rho}ð - \bar{\tau}'Þ + q\bar{\rho}\bar{\tau}')\eta_{pq} \tag{5.7b}$$

$$\begin{aligned}(ðð' - ð'ð)\eta_{pq} = {} & [(\bar{\rho}' - \rho')Þ + (\rho - \bar{\rho})Þ' \\ & + p(\rho\rho' + \Psi_2) - q(\bar{\rho}\bar{\rho}' + \Psi_2)]\eta_{pq}\end{aligned} \tag{5.7c}$$

together with the equations obtained by applying prime, complex conjugation and both to (5.7b). We call equations (5.6) and (5.7) the GHP vacuum field equations because both sets together contain the same information as Einstein's vacuum field equations.

§3. First Covariant Derivative

The Bianchi identities in spinor form are

$$\epsilon^{AE}\Psi_{ABCD;EF'} = 0 \tag{5.8}$$

from which it follows that the first covariant derivative of the Weyl spinor is symmetric on all the unprimed indices, so that we do not need to symmetrise in order to obtain DPSI. Then, using a similar notation to that used for dyad components of the Weyl spinor, DPSI has 12 independent components labelled

$$(D\Psi)_{00'} = \Psi_{0000;00'} = \Psi_{ABCD;EF'}o^A o^B o^C o^D o^E \bar{o}^{F'} \tag{5.9a}$$

$$(D\Psi)_{10'} = \Psi_{0000;10'} = \Psi_{ABCD;EF'}o^A o^B o^C o^D \iota^E \bar{o}^{F'} \tag{5.9b}$$

$$(D\Psi)_{20'} = \Psi_{0001;10'} = \Psi_{ABCD;EF'}o^A o^B o^C \iota^D \iota^E \bar{o}^{F'} \tag{5.9c}$$

$$(D\Psi)_{30'} = \Psi_{0011;10'} = \Psi_{ABCD;EF'}o^A o^B \iota^C \iota^D \iota^E \bar{o}^{F'} \tag{5.9d}$$

$$(D\Psi)_{40'} = \Psi_{0111;10'} = \Psi_{ABCD;EF'}o^A \iota^B \iota^C \iota^D \iota^E \bar{o}^{F'} \tag{5.9e}$$

$$(D\Psi)_{50'} = \Psi_{1111;10'} = \Psi_{ABCD;EF'}\iota^A \iota^B \iota^C \iota^D \iota^E \bar{o}^{F'} \tag{5.9f}$$

$$(D\Psi)_{01'} = \Psi_{0000;01'} = \Psi_{ABCD;EF'}o^A o^B o^C o^D o^E \bar{\iota}^{F'} \tag{5.9g}$$

$$(D\Psi)_{11'} = \Psi_{0000;11'} = \Psi_{ABCD;EF'}o^A o^B o^C o^D \iota^E \bar{\iota}^{F'} \tag{5.9h}$$

$$(D\Psi)_{21'} = \Psi_{0001;11'} = \Psi_{ABCD;EF'}o^A o^B o^C \iota^D \iota^E \bar{\iota}^{F'} \tag{5.9i}$$

$$(D\Psi)_{31'} = \Psi_{0011;11'} = \Psi_{ABCD;EF'}o^A o^B \iota^C \iota^D \iota^E \bar{\iota}^{F'} \tag{5.9j}$$

$$(D\Psi)_{41'} = \Psi_{0111;11'} = \Psi_{ABCD;EF'}o^A \iota^B \iota^C \iota^D \iota^E \bar{\iota}^{F'} \tag{5.9k}$$

$$(D\Psi)_{51'} = \Psi_{1111;11'} = \Psi_{ABCD;EF'}\iota^A \iota^B \iota^C \iota^D \iota^E \bar{\iota}^{F'} \tag{5.9l}$$

Let us consider in detail the calculation of these components. We define a generic symbol ζ_a^A for the dyad $\{o^A,\ \iota^A\}$ by

$$\zeta_0^A = o^A \tag{5.10a}$$

$$\zeta_1^A = \iota^A \tag{5.10b}$$

together with

$$\bar{\zeta}_{0'}^{A'} = \bar{o}^{A'} \tag{5.10c}$$

$$\bar{\zeta}_{1'}^{A'} = \bar{\iota}^{A'} \tag{5.10d}$$

The first covariant derivative is then defined by

$$(D\Psi)_{\mu f'} = \Psi_{ABCD;EF'}\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'} \tag{5.11}$$

where μ of the unprimed dyad vectors are ζ_1^A 's. We use the Leibnitz property of covariant derivatives to write this as

$$(D\Psi)_{\mu f'} = (\Psi_{ABCD}\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D)_{;EF'}\zeta_e^E\bar{\zeta}_{f'}^{F'} - (\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D)_{;EF'}\zeta_e^E\bar{\zeta}_{f'}^{F'}\Psi_{ABCD}$$

and consider the three possible cases : $\mu = 5$, $\mu = 4$ and $\mu < 4$.

For the case $\mu = 5$ we can arrange that all 4 of the dyad vectors taken into the covariant derivative are ζ_1^A 's, so we have

$$(D\Psi)_{\mu f'} = (\Psi_4)_{;1f'} - 4\zeta_{1;1f'}^A\zeta_1^B\zeta_1^C\zeta_1^D\Psi_{ABCD} \tag{5.12}$$

where we have contracted the spinor covariant derivative indices and used the symmetry of the Weyl spinor. This contains a term $\zeta_{1;1f'}^A$ which can be expressed in terms of the spin coefficients $\Gamma_{abcd'}$ as follows :

$$\begin{aligned}\zeta_{1;1f'}^A &= \zeta_{1J;1f'}\epsilon^{AJ}\\ &= \zeta_{1J;1f'}\epsilon^{kl}\zeta_k^A\zeta_l^J\\ &= \Gamma_{1l1f'}\epsilon^{kl}\zeta_k^A\end{aligned} \tag{5.13}$$

Using this result in (5.12) gives

$$\begin{aligned}(D\Psi)_{\mu f'} &= (\Psi_4)_{;1f'} - 4\Gamma_{1l1f'}\epsilon^{kl}\zeta_k^A\zeta_1^B\zeta_1^C\zeta_1^D\Psi_{ABCD}\\ &= (\Psi_4)_{;1f'} - 4\Gamma_{111f'}\Psi_3 + 4\Gamma_{101f'}\Psi_4\end{aligned} \tag{5.14}$$

For the case $\mu = 4$ we can again arrange that all 4 of the dyad vectors taken into the covariant derivative are ζ_1^A 's, the only difference with the first case being that the remaining unprimed dyad vector is now a ζ_0^A instead of a ζ_1^A. In this case (5.12) will become

$$(D\Psi)_{\mu f'} = (\Psi_4)_{;0f'} - 4\zeta^A_{1;0f'}\zeta_1^B\zeta_1^C\zeta_1^D\Psi_{ABCD} \tag{5.15}$$

The steps now proceed in an exactly analogous way to the previous case, again expressing covariant derivatives of the dyad vectors in terms of spin coefficients using a calculation of the type (5.13). The final result is

$$(D\Psi)_{\mu f'} = (\Psi_4)_{;0f'} - 4\Gamma_{110f'}\Psi_3 + 4\Gamma_{100f'}\Psi_4 \tag{5.16}$$

For the final case $\mu < 4$ we can arrange that all μ of the ζ_1^A 's are taken into the covariant derivative so that the remaining dyad vectors are all ζ_0^A 's. Therefore, equation (5.12) now becomes

$$(D\Psi)_{\mu f'} = (\Psi_\mu)_{;0f'} - \mu\zeta^A_{1;0f'}[\zeta_b^B\zeta_c^C\zeta_d^D]\Psi_{ABCD} - (4-\mu)\zeta^A_{0;0f'}[\zeta_b^B\zeta_c^C\zeta_d^D]\Psi_{ABCD} \tag{5.17}$$

where a little counting will show that the first set of square parentheses contains $(\mu - 1)$ ζ_1^A 's and the second set contains μ ζ_1^A 's (in obtaining (5.17) the symmetry of the Weyl spinor has again been used). The calculation now proceeds as in the previous cases, again expressing covariant derivatives of the dyad vectors in terms of spin coefficients using a calculation of the type (5.13). The final result is

$$\begin{aligned}(D\Psi)_{\mu f'} &= (\Psi_\mu)_{;0f'} - \mu\Gamma_{110f'}\Psi_{\mu-1} + \mu\Gamma_{100f'}\Psi_\mu \\ &\quad - (4-\mu)\Gamma_{010f'}\Psi_\mu + (4-\mu)\Gamma_{000f'}\Psi_{\mu+1} \\ &= (\Psi_\mu)_{;0f'} - \mu\Gamma_{110f'}\Psi_{\mu-1} \\ &\quad + (2\mu-4)\Gamma_{100f'}\Psi_\mu + (4-\mu)\Gamma_{000f'}\Psi_{\mu+1}\end{aligned} \tag{5.18}$$

where, in obtaining the final equality in (5.18), we have used the symmetry of $\Gamma_{abcd'}$ on the first two indices.

For a type D vacuum spacetime (5.1) immediately tell us, referring to (5.14), (5.16) and (5.18), that $(D\Psi)_{\mu f'}$ vanishes for $\mu = 5$, $\mu = 4$ and $\mu = 0$. Using in addition (5.4), we see from (5.18) that $(D\Psi)_{\mu f'}$ also vanishes for $\mu = 1$. Therefore, from (5.18), using (5.1) and (5.4), the non-vanishing components for a type D vacuum solution in canonical form (5.1) are

$$(D\Psi)_{20'} = \text{Þ}\Psi_2 = 3\rho\Psi_2 \tag{5.19a}$$

$$(D\Psi)_{30'} = \eth'\Psi_2 = 3\tau'\Psi_2 \tag{5.19b}$$

$$(D\Psi)_{21'} = \eth\Psi_2 = 3\tau\Psi_2 \tag{5.19c}$$

$$(D\Psi)_{31'} = \text{Þ}'\Psi_2 = 3\rho'\Psi_2 \tag{5.19d}$$

where the second equality in each case follows from the Bianchi identities (5.5). The succinct nature of these equations illustrates how natural the GHP formalism is for this problem.

To summarise, the zeroth order covariant derivative of the Weyl tensor depends only on Ψ_2, i.e.

$$D^0\Psi = D^0\Psi(\Psi_2) \tag{5.20}$$

and the first covariant derivative depends additionally on ρ, ρ', τ and τ', i.e.

$$D\Psi = D\Psi(\Psi_2, \rho, \rho', \tau, \tau') \tag{5.21}$$

Under H_0, these spin coefficients transform as follows :

$$\rho \longrightarrow r^2\rho \tag{5.22a}$$

$$\rho' \longrightarrow r^{-2}\rho' \tag{5.22b}$$

$$\tau \longrightarrow e^{2i\theta}\tau \tag{5.22c}$$

$$\tau' \longrightarrow e^{-2i\theta}\tau' \tag{5.22d}$$

Following the work by Kinnersley [10], all type D vacuum spacetimes can be split into 3 invariant classes, depending on the zero/non-zero nature of the spin coefficients. Let us use equations (5.22) to determine the 1st order canonical forms and invariance groups for each of these classes.

Class I : $\rho \neq 0, \ \rho' \neq 0, \ \tau = 0, \ \tau' = 0$

From (5.22) it is seen that we can fix

$$|\rho| = |\rho'| \tag{5.23}$$

by setting

$$r = (|\rho'|/|\rho|)^{\frac{1}{4}} \tag{5.24}$$

However, since τ and τ' are zero, θ remains arbitrary. In this case, therefore, the dimension of H_1 is 1. Thus there still remains a 1-dimensional invariance subgroup generated by rotations.

Class II : $\rho = 0, \ \rho' = 0, \ \tau \neq 0, \ \tau' \neq 0$

From (5.22) it is seen that we can fix

$$Im(\tau) = 0 \tag{5.25}$$

by setting

$$\theta = -(Arg(\tau))/2 \tag{5.26}$$

(up to a multiple of $\pi/2$). However, since ρ and ρ' are zero, r remains arbitrary. In this case, therefore, the dimension of H_1 is again 1. Thus there still remains a 1-dimensional invariance subgroup generated by boosts.

Class III : $\rho \neq 0, \ \rho' \neq 0, \ \tau \neq 0, \ \tau' \neq 0$

In this case we can demand both conditions (5.23) and (5.25), which fixes the frame completely. In this case, therefore, the dimension of H_1 is zero, so that no frame transformations are permitted and (5.1), together with (5.23) and (5.25), completely specify the canonical form.

Since in each case the dimension of the invariance group has changed, we must continue to the next order of differentiation to continue the Karlhede classification.

§4. Second Covariant Derivative

We proceed as we did in §3 and compute the dyad components of the second covariant derivative of the Weyl spinor. In fact, it can be proved using the Bianchi and Ricci identities that, for the vacuum case, at all orders of covariant differentiation of the Weyl spinor only the symmetrised parts are algebraically independent [6]. Therefore, we shall first calculate the second covariant derivative and then symmetrise over its indices to obtain its symmetrised part, D2PSI.

The calculation of the second covariant derivative follows much the same pattern as that of the first covariant derivative. The second covariant derivative is defined by

$$(D^2\Psi)_{\mu f';gh'} = \Psi_{ABCD;EF';GH'}[\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E]\bar{\zeta}_{f'}^{F'}\zeta_g^G\bar{\zeta}_{h'}^{H'} \tag{5.27}$$

where μ of the unprimed dyad vectors in the square parentheses are ζ_1^A 's. The Leibnitz property then gives

$$\begin{aligned}(D^2\Psi)_{\mu f';gh'} = &(\Psi_{ABCD;EF'}\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'})_{;GH'}\zeta_g^G\bar{\zeta}_{h'}^{H'}\\ &-(\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'})_{;GH'}\zeta_g^G\bar{\zeta}_{h'}^{H'}\Psi_{ABCD;EF'}\end{aligned}$$

The right hand side of this equation has μ ζ_1^A 's inside both sets of parentheses and, therefore, $(5-\mu)$ ζ_0^A 's. Hence, contracting the spinor covariant derivative indices and using the symmetry of the Weyl spinor gives

$$\begin{aligned}(D^2\Psi)_{\mu f';gh'} = &[(D\Psi)_{\mu f'}]_{;gh'}\\ &-\mu\zeta^A_{1;gh'}[\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'}]\Psi_{ABCD;EF'}\\ &-(5-\mu)\zeta^A_{0;gh'}[\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'}]\Psi_{ABCD;EF'}\\ &-\bar{\zeta}^{F'}_{f';gh'}[\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E]\Psi_{ABCD;EF'}\end{aligned} \tag{5.28}$$

A little counting shows that the second set of square parentheses contains $(\mu - 1)$ ζ_1^A *'s* and the third set μ ζ_1^A *'s*, as do the final set. All the types of term in (5.28) have occurred previously except the term $\bar{\zeta}^{F'}_{f';gh'}$. This term can be expressed in terms of spin coefficients as follows :

$$\begin{aligned}
\bar{\zeta}^{F'}_{f';gh'} &= \overline{(\zeta^F_f)}_{;gh'} \\
&= \overline{(\zeta^F_{f;g'h})} \\
&= \overline{(\zeta_{fJ;g'h}\epsilon^{FJ})} \\
&= \overline{(\zeta_{fJ;g'h}\epsilon^{kl}\zeta^F_k\zeta^J_l)} \\
&= \overline{(\Gamma_{flhg'}\epsilon^{kl}\zeta^F_k)} \\
&= \bar{\Gamma}_{f'l'h'g}\bar{\epsilon}^{k'l'}\bar{\zeta}^{F'}_{k'}
\end{aligned} \tag{5.29}$$

If we now substitute in (5.28) using results (5.13) and (5.29) then we obtain

$$\begin{aligned}
(D^2\Psi)_{\mu f';gh'} &= [(D\Psi)_{\mu f'}]_{;gh'} \\
&\quad - \mu\Gamma_{1lgh'}\epsilon^{kl}\zeta^A_k[\zeta^B_b\zeta^C_c\zeta^D_d\zeta^E_e\bar{\zeta}^{F'}_{f'}]\Psi_{ABCD;EF'} \\
&\quad - (5-\mu)\Gamma_{0lgh'}\epsilon^{kl}\zeta^A_k[\zeta^B_b\zeta^C_c\zeta^D_d\zeta^E_e\bar{\zeta}^{F'}_{f'}]\Psi_{ABCD;EF'} \\
&\quad - \bar{\Gamma}_{f'l'h'g}\bar{\epsilon}^{k'l'}\bar{\zeta}^{F'}_{k'}[\zeta^A_a\zeta^B_b\zeta^C_c\zeta^D_d\zeta^E_e]\Psi_{ABCD;EF'} \\
&= [(D\Psi)_{\mu f'}]_{;gh'} \\
&\quad - \mu\Gamma_{11gh'}(D\Psi)_{(\mu-1)f'} \\
&\quad + (2\mu-5)\Gamma_{10gh'}(D\Psi)_{\mu f'} \\
&\quad + (5-\mu)\Gamma_{00gh'}(D\Psi)_{(\mu+1)f'} \\
&\quad - \bar{\Gamma}_{f'1'h'g}(D\Psi)_{\mu 0'} \\
&\quad + \bar{\Gamma}_{f'0'h'g}(D\Psi)_{\mu 1'}
\end{aligned} \tag{5.30}$$

For a type D vacuum spacetime (5.19) immediately tell us, referring to (5.30), that $(D\Psi)_{\mu f'}$ vanishes for $\mu = 5$, $\mu = 4$ and $\mu = 0$. Therefore, from (5.30), using (5.4) and (5.19), the non-vanishing components for a type D vacuum solution in canonical form (5.1) are

$$(D^2\Psi)_{10';10'} = 12\rho^2\Psi_2 \tag{5.31a}$$

$$(D^2\Psi)_{10';11'} = 12\rho\tau\Psi_2 \tag{5.31b}$$

$$(D^2\Psi)_{11';10'} = 12\rho\tau\Psi_2 \tag{5.31c}$$

$$(D^2\Psi)_{11';11'} = 12\tau^2\Psi_2 \tag{5.31d}$$

$$(D^2\Psi)_{20';00'} = 3(\text{Þ}\rho)\Psi_2 + 9\rho^2\Psi_2 \tag{5.31e}$$

$$(D^2\Psi)_{20';10'} = 3(\eth'\rho)\Psi_2 + 18\rho\tau'\Psi_2 \tag{5.31f}$$

$$(D^2\Psi)_{20';01'} = 3(\eth\rho)\Psi_2 + 9\rho\tau\Psi_2 + 3\tau\bar{\rho}\Psi_2 \tag{5.31g}$$

$$(D^2\Psi)_{20';11'} = 3(\text{Þ}'\rho)\Psi_2 + 9\rho\rho'\Psi_2 + 9\tau\tau'\Psi_2 + 3\tau\bar{\tau}\Psi_2 \tag{5.31h}$$

$$(D^2\Psi)_{21';00'} = 3(\text{Þ}\tau)\Psi_2 + 9\rho\tau\Psi_2 + 3\rho\bar{\tau}'\Psi_2 \tag{5.31i}$$

$$(D^2\Psi)_{21';10'} = 3(\eth'\tau)\Psi_2 + 9\tau\tau'\Psi_2 + 9\rho\rho'\Psi_2 + 3\rho\bar{\rho}'\Psi_2 \tag{5.31j}$$

$$(D^2\Psi)_{21';01'} = 3(\eth\tau)\Psi_2 + 9\tau^2\Psi_2 \tag{5.31k}$$

$$(D^2\Psi)_{21';11'} = 3(\text{Þ}'\tau)\Psi_2 + 18\tau\rho'\Psi_2 \tag{5.31l}$$

$$(D^2\Psi)_{30';00'} = 3(\text{Þ}\tau')\Psi_2 + 18\rho\tau'\Psi_2 \tag{5.31m}$$

$$(D^2\Psi)_{30';10'} = 3(\eth'\tau')\Psi_2 + 9\tau'^2\Psi_2 \tag{5.31n}$$

$$(D^2\Psi)_{30';01'} = 3(\eth\tau')\Psi_2 + 9\tau\tau'\Psi_2 + 9\rho\rho'\Psi_2 + 3\rho'\bar{\rho}\Psi_2 \tag{5.31o}$$

$$(D^2\Psi)_{30';11'} = 3(\text{Þ}'\tau')\Psi_2 + 9\rho'\tau'\Psi_2 + 3\rho'\bar{\tau}\Psi_2 \tag{5.31p}$$

$$(D^2\Psi)_{31';00'} = 3(\text{Þ}\rho')\Psi_2 + 9\rho\rho'\Psi_2 + 9\tau\tau'\Psi_2 + 3\tau'\bar{\tau}'\Psi_2 \tag{5.31q}$$

$$(D^2\Psi)_{31';10'} = 3(\eth'\rho')\Psi_2 + 9\tau'\rho'\Psi_2 + 3\tau'\bar{\rho}'\Psi_2 \tag{5.31r}$$

$$(D^2\Psi)_{31';01'} = 3(\eth\rho')\Psi_2 + 18\tau\rho'\Psi_2 \tag{5.31s}$$

$$(D^2\Psi)_{31';11'} = 3(\text{Þ}'\rho')\Psi_2 + 9\rho'^2\Psi_2 \tag{5.31t}$$

$$(D^2\Psi)_{40';00'} = 12\tau'^2\Psi_2 \tag{5.31u}$$

$$(D^2\Psi)_{40';01'} = 12\rho'\tau'\Psi_2 \tag{5.31v}$$

$$(D^2\Psi)_{41';00'} = 12\rho'\tau'\Psi_2 \tag{5.31w}$$

$$(D^2\Psi)_{41';01'} = 12\rho'^2\Psi_2 \tag{5.31x}$$

We now have to consider the problem of symmetrising the second covariant derivative of the Weyl tensor to obtain its symmetrised part, D2PSI. Using a similar notation to that used before the components of D2PSI are labelled

$$(D^2\Psi)_{\mu\nu'} \quad ; \quad \mu = 0,1,2,3,4,5,6 \ , \ \nu = 0,1,2) \tag{5.32}$$

this representing the symmetrisation of the component with μ of the unprimed and ν of primed indices equal to 1. Thus, for example

$$(D^2\Psi)_{31'} = \Psi_{(0001;10';11')} \tag{5.33}$$

where the unprimed and primed indices are symmetrised independently. It is readily confirmed that there are $7 \times 3 = 21$ independent components. The formula for calculating $(D^2\Psi)_{\mu\nu'}$ from the $(D^2\Psi)_{\mu f';gh'}$ is

$$\begin{aligned}(D^2\Psi)_{\mu\nu'} = \tfrac{1}{12}[&\mu(D^2\Psi)_{(\mu-1)f';1h'} + (6-\mu)(D^2\Psi)_{\mu f';0h'} \\ &+ \mu(D^2\Psi)_{(\mu-1)h';1f'} + (6-\mu)(D^2\Psi)_{\mu h';0f'}]\end{aligned} \tag{5.34}$$

where μ of the unprimed and ν of the primed indices are 1. The reason this formula is correct is that the process of symmetrising over the unprimed indices can be broken down into (i) placing each of the 6 unprimed indices in turn into the g slot and (ii) each time symmetrising over all the other unprimed indices. There will be μ different 1s that can be inserted in the g slot but only $6-\mu$ different 0s that can be inserted there, thus giving the weightings in (5.34). The symmetrising over the 2 primed indices is trivial. From (5.34), using equations (5.31), the non-vanishing components of D2PSI for a type D vacuum solution in canonical form (5.1) are

$$(D^2\Psi)_{20'} = 2(\text{Þ}\rho)\Psi_2 + 10\rho^2\Psi_2 \tag{5.35a}$$

$$\begin{aligned}(D^2\Psi)_{21'} = &(\text{Þ}\tau)\Psi_2 + 10\rho\tau\Psi_2 + (\eth\rho)\Psi_2 + \bar{\rho}\tau\Psi_2 \\ &+ \rho\bar{\tau}'\Psi_2\end{aligned} \tag{5.35b}$$

$$(D^2\Psi)_{22'} = 2(\eth\tau)\Psi_2 + 10\tau^2\Psi_2 \tag{5.35c}$$

$$(D^2\Psi)_{30'} = \frac{3}{2}(\text{Þ}\tau')\Psi_2 + \frac{3}{2}(\eth'\rho)\Psi_2 + 18\rho\tau'\Psi_2 \tag{5.35d}$$

$$\begin{aligned}(D^2\Psi)_{31'} = &\frac{3}{4}(\text{Þ}\rho')\Psi_2 + \frac{3}{4}(\text{Þ}'\rho)\Psi_2 + 9\rho\rho'\Psi_2 + \frac{3}{4}(\eth\tau')\Psi_2 \\ &+ \frac{3}{4}(\eth'\tau)\Psi_2 + 9\tau\tau'\Psi_2 \\ &+ \frac{3}{4}\tau\bar{\tau}\Psi_2 + \frac{3}{4}\tau'\bar{\tau}'\Psi_2 + \frac{3}{4}\bar{\rho}\rho'\Psi_2 + \frac{3}{4}\rho\bar{\rho}'\Psi_2\end{aligned} \tag{5.35e}$$

$$(D^2\Psi)_{32'} = \frac{3}{2}(\text{Þ}'\tau)\Psi_2 + \frac{3}{2}(\eth\rho')\Psi_2 + 18\tau\rho'\Psi_2 \tag{5.35f}$$

$$(D^2\Psi)_{40'} = 2(\eth'\tau')\Psi_2 + 10\tau'^2\Psi_2 \tag{5.35g}$$

$$\begin{aligned}(D^2\Psi)_{41'} = &(\text{Þ}'\tau')\Psi_2 + (\eth'\rho')\Psi_2 + \rho'\bar{\tau}\Psi_2 \\ &+ \bar{\rho}'\tau'\Psi_2 + 10\rho'\tau'\Psi_2\end{aligned} \tag{5.35h}$$

$$(D^2\Psi)_{42'} = 2(\text{Þ}'\rho')\Psi_2 + 10\rho'^2\Psi_2 \tag{5.35i}$$

In summary

$$D^2\Psi = D^2\Psi(\Psi_2, \rho, \tau, , \text{Þ}\rho, \text{Þ}'\rho, \eth\rho, \eth'\rho, \text{Þ}\tau, \text{Þ}'\tau, \eth\tau, \eth'\tau, \quad + \, primes) \tag{5.36}$$

that is the second covariant derivative is a functional of the zeroth order quantity Ψ_2, the first order quantities ρ, ρ', τ and τ', together with the derivatives with respect to Þ, Þ$'$, $\eth$ and $\eth'$ of all first order quantities. Thus the only possibly new information arises from the GHP operators acting on the first order quantities. To investigate these terms we use the vacuum field equations (5.6) and (5.7). Using equations (5.6) to substitute into equations (5.35) we obtain

$$(D^2\Psi)_{20'} = 12\rho^2\Psi_2 \tag{5.35a bis}$$

$$(D^2\Psi)_{21'} = 12\rho\tau\Psi_2 \tag{5.35b bis}$$

$$(D^2\Psi)_{22'} = 12\tau^2\Psi_2 \tag{5.35c bis}$$

$$(D^2\Psi)_{30'} = \frac{3}{2}(\text{Þ}\tau')\Psi_2 + \frac{3}{2}(\eth'\rho)\Psi_2 + 18\rho\tau'\Psi_2 \tag{5.35d bis}$$

$$\begin{aligned}(D^2\Psi)_{31'} = &\frac{3}{4}(\text{Þ}'\rho)\Psi_2 + \frac{3}{4}(\eth\tau')\Psi_2 + \frac{3}{4}(\eth'\tau)\Psi_2 \\ &+ \frac{3}{4}(\text{Þ}\rho')\Psi_2 + 9\rho\rho'\Psi_2 + 9\tau\tau'\Psi_2\end{aligned}$$

$$+\frac{3}{4}\tau\bar{\tau}\Psi_2+\frac{3}{4}\bar{\rho}\rho'\Psi_2+\frac{3}{4}\rho\bar{\rho}'\Psi_2+\frac{3}{4}\tau'\bar{\tau}'\Psi_2 \qquad (5.35e\ bis)$$

$$(D^2\Psi)_{32'}=\frac{3}{2}(Þ'\tau)\Psi_2+\frac{3}{2}(ð\rho')\Psi_2+18\tau\rho'\Psi_2 \qquad (5.35f\ bis)$$

$$(D^2\Psi)_{40'}=12\tau'^2\Psi_2 \qquad (5.35g\ bis)$$

$$(D^2\Psi)_{41'}=12\rho'\tau'\Psi_2 \qquad (5.35h\ bis)$$

$$(D^2\Psi)_{42'}=12\rho'^2\Psi_2 \qquad (5.35i\ bis)$$

where all GHP operator terms other than

$$ð'\rho,\ \ Þ\tau',\ \ Þ'\rho,\ \ ð'\tau$$

and their primes have been expressed in terms of first order spin coefficients. In addition, equation (5.6e) enables us to relate $Þ'\rho$ to $ð'\tau$ and its prime relates $Þ\rho'$ to $ð\tau'$. However, equations (5.6) provide no information about $ð'\rho$ and $Þ\tau'$ and their primes — for these it is necessary to use the commutators (5.7). Taking the complex conjugate of (5.7b) and acting with it on the $\{0,0\}$ weighted quantity Ψ_2 we obtain

$$(Þð'-ð'Þ)\Psi_2=(\rho ð'-\tau'Þ)\Psi_2$$

Using the Bianchi identities (5.5) this gives

$$3Þ(\tau'\Psi_2)-3ð'(\rho\Psi_2)=0$$

which, on using (5.5) again, leads to

$$3\Psi_2(Þ\tau'-ð'\rho)=0 \qquad (5.37)$$

The vanishing of the quantity in parentheses, together with its prime, provides us with relations between the required quantities.

In summary

$$D^2\Psi=D^2\Psi(\Psi_2,\rho,\rho',\tau,\tau',Þ'\rho,Þ\rho',ð'\rho,ð\rho') \qquad (5.38a)$$

where, from (5.6e) and (5.37), we have the relationships

$$Þ'\rho = \rho\bar{\rho}' - \tau\bar{\tau} - \Psi_2 + ð'\tau \tag{5.38b}$$

$$ð'\rho = Þ\tau' \tag{5.38c}$$

and their primes. We return to the three classes of §3.

Class I : $\rho \neq 0,\ \rho' \neq 0,\ \tau = 0,\ \tau' = 0$

From (5.38), all components of $D^2\Psi$ can be expressed algebraically in terms of first order quantities. Therefore, we always retain a 1-dimensional invariance group and (5.1), together with (5.23), completely specify the canonical form. As no new quantities are introduced at this order, Karlhede's procedure terminates at 2nd order.

Class II : $\rho = 0,\ \rho' = 0,\ \tau \neq 0,\ \tau' \neq 0$

Again, from (5.38), all components of $D^2\Psi$ can be expressed algebraically in terms of first order quantities. Therefore, we always retain a 1-dimensional invariance group and (5.1), together with (5.25), completely specify the canonical form. As no new quantities are introduced at this order, Karlhede's procedure again terminates at 2nd order.

Class III : $\rho \neq 0,\ \rho' \neq 0,\ \tau \neq 0,\ \tau' \neq 0$

We use the result that all type D vacuum spacetimes admit at least a two dimensional isometry group I [10]. Combining this with the result (3.50) of chapter 3, we see that, since the dimension of H_1 is zero, we have $n_q \leq 2$, i.e. there are at most *two* functionally independent components in total. One of these is provided by Ψ_2, since it cannot be constant (as the Bianchi identities (5.5) would then imply that $\rho = \rho' = \tau = \tau' = 0$, which contradicts our assumptions). In the worst case no new functionally independent components will be produced at first order, as the reduction of the dimension of the invariance group in going from $q = 0$ to $q = 1$ would still allow the Karlhede procedure to continue. However, in going from $q = 1$ to $q = 2$ the dimension of the invariance group remains unchanged (namely zero) and, therefore, if the procedure is to continue a new functionally independent component *must* be produced at this stage, so that we now have them both. Therefore, the Karlhede procedure terminates at $q = 3$.

§5. Third Covariant Derivative

In the previous section we used the result that all type D vacuum space-times admit at least a 2-dimensional isometry group to arrive at our conclusion that the Karlhede algorithm terminates at at most the third covariant derivative. In this section we consider the third covariant derivative explicitly to obtain this result without having to use the isometry group result. We shall not actually calculate the third covariant but deduce its form by some simple arguments.

The third covariant derivative of the Weyl spinor is defined by the equation

$$(D^3\Psi)_{\mu f';gh';lm'} = \Psi_{ABCD;EF';GH';LM'}[\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E]\bar{\zeta}_{f'}^{F'}\zeta_g^G\bar{\zeta}_{h'}^{H'}\zeta_l^L\bar{\zeta}_{m'}^{M'} \tag{5.39}$$

where μ of the dyad vectors in square parentheses are ζ_1^A's. All the dyad vectors other than ζ_l^L and $\bar{\zeta}_{m'}^{M'}$ can now be taken into the covariant derivative to obtain

$$\begin{aligned}(D^3\Psi)_{\mu f';gh';lm'} = {} & [(D^2\Psi)_{\mu f';gh}]_{;lm'} - \mu\zeta^A_{1;lm'}\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'}\zeta_g^G\bar{\zeta}_{h'}^{H'}\Psi_{ABCD;EF';GH'} \\ & - (5-\mu)\zeta^A_{0;lm'}\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'}\zeta_g^G\bar{\zeta}_{h'}^{H'}\Psi_{ABCD;EF';GH'} \\ & - \bar{\zeta}^{F'}_{f';lm'}\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\zeta_g^G\bar{\zeta}_{h'}^{H'}\Psi_{ABCD;EF';GH'} \\ & - \zeta^G_{g;lm'}\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'}\bar{\zeta}_{h'}^{H'}\Psi_{ABCD;EF';GH'} \\ & - \bar{\zeta}^{H'}_{h';lm'}\zeta_a^A\zeta_b^B\zeta_c^C\zeta_d^D\zeta_e^E\bar{\zeta}_{f'}^{F'}\zeta_g^G\Psi_{ABCD;EF';GH'}\end{aligned} \tag{5.40}$$

The vital part of the argument now comes from inspecting equations (5.13) and (5.29). These equations show that in order for a term of (5.40) to contain a $(D^2\Psi)_{\mu f';gh'}$ then it must contain a $\Gamma_{abcd'}$ of the form Γ_{01--} or Γ_{10--}, or a

$\bar{\Gamma}_{a'b'c'd}$ of the form $\bar{\Gamma}_{0'1'--}$ or $\bar{\Gamma}_{1'0'--}$ (this is because such Γs *replace* the dyad vector which is being 'used up' in the covariant derivative). If we now apply this argument to (5.40), using (5.13) and (5.29), and assuming that x of ζ_g^G are ζ_1^As and that y of $\bar{\zeta}_{f'}^{F'}$ and $\bar{\zeta}_{h'}^{H'}$ are $\bar{\zeta}_{1'}^{A'}$s, then it will take the form

$$
\begin{aligned}
(D^3\Psi)_{\mu f';gh';lm'} = {} & [(D^2\Psi)_{\mu f';gh'}]_{;lm'} + \mu\Gamma_{10lm'}(D^2\Psi)_{\mu f';gh'} \\
& - (5-\mu)\Gamma_{01lm'}(D^2\Psi)_{\mu f';gh'} + x\Gamma_{10lm'}(D^2\Psi)_{\mu f';gh'} \\
& - (1-x)\Gamma_{01lm'}(D^2\Psi)_{\mu f';gh'} + y\bar{\Gamma}_{1'0'm'l}(D^2\Psi)_{\mu f';gh'} \\
& - (2-y)\bar{\Gamma}_{0'1'm'l}(D^2\Psi)_{\mu f';gh'} + \ldots
\end{aligned}
\tag{5.41}
$$

where the remaining terms do not contain $(D^2\Psi)_{\mu f';gh'}$. $(D^2\Psi)_{\mu f';gh'}$ is a spin and boost weighted scalar of type $\{p,q\}$ where $p = (5-\mu)+(1-x)-\mu-x = 6-2\mu-2x$ and $q = (2-y)-y = 2-2y$. So (5.41) can be written

$$
\begin{aligned}
(D^3\Psi)_{\mu f';gh';lm'} = {} & [(D^2\Psi)_{\mu f';gh'}]_{;lm'} \\
& - p\Gamma_{10lm'}(D^2\Psi)_{\mu f';gh'} - q\bar{\Gamma}_{1'0'm'l}(D^2\Psi)_{\mu f';gh'} + \ldots
\end{aligned}
\tag{5.42}
$$

where we have used the symmetry $\Gamma_{10lm'} = \Gamma_{01lm'}$ and $\bar{\Gamma}_{1'0'm'l} = \bar{\Gamma}_{0'1'm'l}$. Equation (5.42) now becomes

$$(D^3\Psi)_{\mu f';gh';lm'} = \text{Þ}(D^2\Psi)_{\mu f';gh'} + \ldots \quad ; \quad lm' = 00' \tag{5.43a}$$

$$(D^3\Psi)_{\mu f';gh';lm'} = \eth(D^2\Psi)_{\mu f';gh'} + \ldots \quad ; \quad lm' = 01' \tag{5.43b}$$

$$(D^3\Psi)_{\mu f';gh';lm'} = \eth'(D^2\Psi)_{\mu f';gh'} + \ldots \quad ; \quad lm' = 10' \tag{5.43c}$$

$$(D^3\Psi)_{\mu f';gh';lm'} = \text{Þ}'(D^2\Psi)_{\mu f';gh'} + \ldots \quad ; \quad lm' = 11' \tag{5.43d}$$

Thus, we see that the spin coefficients β, β', ϵ and ϵ', occur in exactly the right combinations and proportions for them to be included in either a Þ, $\eth$, $\eth'$ or Þ$'$ GHP derivative operator. It is clear that this result can, in fact, be generalised to the dyad components of the covariant derivative of any spinor. Now, using (5.38a), we see that this means that the only new terms which will occur at third order are Þ, $\eth$, Þ$'$ and $\eth'$ acting on Þρ', Þ$'\rho$, $\eth\rho'$ and $\eth'\rho$. Of these 16 new terms the 8 $\eth'$Þρ', Þ$'$Þρ', ÞÞ$'\rho$, $\eth$Þ$'\rho$, $\eth'\eth\rho'$, Þ$'\eth\rho'$, Þ$\eth'\rho$ and $\eth\eth'\rho$ can be expressed in terms of lower order quantities by first using the commutators (5.7) and then the field equations (5.6). For example,

$$\text{Þ}\text{Þ}'\rho = \text{Þ}'\text{Þ}\rho + (\bar{\tau} - \tau')\eth\rho + (\tau - \bar{\tau}')\eth'\rho - (\Psi_2 - \tau\tau')\rho - (\bar{\Psi}_2 - \bar{\tau}\bar{\tau}')\rho$$
$$= 2\rho\text{Þ}'\rho + (\bar{\tau} - \tau')\eth\rho + (\tau - \bar{\tau}')\eth'\rho - (\Psi_2 - \tau\tau')\rho - (\bar{\Psi}_2 - \bar{\tau}\bar{\tau}')\rho$$

The remaining 8 terms $\text{Þ}\text{Þ}\rho'$, $\eth\text{Þ}\rho'$, $\eth'\text{Þ}'\rho$, $\text{Þ}'\text{Þ}'\rho$, $\text{Þ}\eth\rho'$, $\eth\eth\rho'$, $\eth'\eth'\rho$ and $\text{Þ}'\eth'\rho$ can be expressed in terms of lower order quantities by alternately using the commutators (5.7) and equation (5.38b) or (5.38c), and finally the field equations (5.6). For example

$$\text{Þ}\text{Þ}\rho' \leftrightarrow \text{Þ}\eth\tau' \leftrightarrow \eth\text{Þ}\tau' \leftrightarrow \eth\eth'\rho \leftrightarrow \eth'\eth\rho \leftrightarrow \eth'[(\rho - \bar{\rho})\tau]$$

and

$$\eth\text{Þ}\rho' \leftrightarrow \text{Þ}\eth\rho' \leftrightarrow \text{Þ}\text{Þ}'\tau \leftrightarrow \text{Þ}'\text{Þ}\tau \leftrightarrow \text{Þ}'[(\tau - \bar{\tau}')\rho]$$

where $\leftrightarrow$ implies equality modulo lower order quantities.

So we see that the only apparently new quantities at third order can, in fact, be expressed in terms of lower order quantities. Therefore, we can indeed show that for type D vacuum spacetimes it is necessary for the Karlhede algorithm to continue to at most the third covariant derivative, without requiring the isometry group result. The other result which was used on the way to our conclusion that the upper bound is 3 is the splitting into only 3 classes at the end of §3. In the proof above that we can express all quantities occurring at third order in terms of lower order quantities, we have made no assumptions about the zero/non-zero nature of the spin coefficients, and therefore it applies for any conceivable case. Therefore, we could obtain our result that the upper bound is 3 without knowing that the 3 classes of §3 are the only ones allowed. This is important because it is only by integrating the field equations that one finds that there are only these three classes, and it is nice that we can obtain our result without requiring any integration of the field equations.

§6. Summary of Results

The main results of this chapter are summarised in the following table :

Table of Results

Class	I	II	III
Invariant Characterisation	$\rho \neq 0$ $\rho' \neq 0$ $\tau = 0$ $\tau' = 0$	$\rho = 0$ $\rho' = 0$ $\tau \neq 0$ $\tau' \neq 0$	$\rho \neq 0$ $\rho' \neq 0$ $\tau \neq 0$ $\tau' \neq 0$
Canonical Form :			
Zeroth Order	$\Psi_2 \neq 0$ $\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_3 = 0$ $\Psi_4 = 0$	$\Psi_2 \neq 0$ $\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_3 = 0$ $\Psi_4 = 0$	$\Psi_2 \neq 0$ $\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_3 = 0$ $\Psi_4 = 0$
1st Order	$\|\rho\| = \|\rho'\|$	$Im(\tau) = 0$	$\|\rho\| = \|\rho'\|$ $Im(\tau) = 0$
Upper Bound	2	2	3

As mentioned previously, all type D vacuum metrics have been explicitly obtained by Kinnersley [10]. For completeness let us list which of these metrics fall into each of the above classes. The 'related metrics' are metrics which differ only in the choice of sign for constants of integration.

Class I : Schwarzschild (plus two related metrics) and NUT (plus two related metrics).

Class II : 'B' (plus two related metrics) and 'Rotating B' (plus two related metrics).

Class III : Kerr-NUT (plus five related metrics), 'C' and 'Twisting C'.

6
The Karlhede Classification of Type N Vacuum Spacetimes

§1. Introduction

In this chapter all type N vacuum spacetimes are split into various invariant classes and a *canonical form* is derived for each class in turn, as well as the frame transformation required to obtain this canonical form.

The work begun in the previous chapter in reducing the upper bound on the order of covariant differentiation of the Riemann tensor required for a complete Karlhede classification is continued in this chapter. It is shown that for a type N vacuum spacetime the Karlhede upper bound of 7 can be reduced to 2, 4, 5 or 6, depending on the invariant class. The analysis is carried out in spinor language using NP formalism (Newman-Penrose formalism [13]), and uses the simplifications of the NP equations that result from having the spacetime in its canonical form.

In §2 a canonical form for the Weyl spinor of a type N spacetime and the NP equations for the vacuum case are given. §3 introduces the invariant classes into which type N vacuum spacetimes naturally split, and derives the canonical forms at first order. §4 makes a general consideration of functional independence and derives the second order canonical forms. The following section analyses the problem of the upper bound on derivatives of the Weyl tensor required in the Karlhede classification for each class in turn. The final section summarises the main results of the chapter.

§2. Zeroth Order

The Weyl spinor of a type N spacetime has the canonical form

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \; ; \qquad \Psi_4 = 1 \tag{6.1}$$

The proper Lorentz transformations form a six parameter invariance group on the spin frame at each point. The condition (6.1) is preserved only under the *two* parameter invariance subgroup H_0 defined by

$$H_0 = \begin{pmatrix} 1 & 0 \\ \bar{a} & 1 \end{pmatrix} ; \qquad a \in \mathbb{C} \tag{6.2}$$

under which

$$\begin{aligned} o^A &\longrightarrow o^A \\ \iota^A &\longrightarrow \iota^A + \bar{a} o^A \end{aligned}$$

i.e. H_0 is the subgroup of null rotations.

In Newman-Penrose notation, the Bianchi identities for a type N vacuum spacetime become, using (6.1)

$$\kappa \Psi_4 = 0 \tag{6.3a}$$

$$(4\epsilon - \rho)\Psi_4 = 0 \tag{6.3b}$$

$$\sigma \Psi_4 = 0 \tag{6.3c}$$

$$(4\beta - \tau)\Psi_4 = 0 \tag{6.3d}$$

These equations require

$$\kappa = \sigma = 0 \tag{6.4a}$$

$$4\epsilon = \rho \tag{6.4b}$$

$$4\beta = \tau \tag{6.4c}$$

with (6.4a) also following directly from the Goldberg-Sachs theorem. From now on, in order to simplify equations, ρ and τ will be used instead of ϵ and β respectively.

For later reference the NP vacuum field equations and commutator equations are written out for a type N vacuum spacetime. The NP vacuum field equations become, using (6.1) and (6.4)

$$D\rho = \tfrac{5}{4}\rho^2 + \tfrac{1}{4}\rho\bar{\rho} \tag{6.5a}$$

$$D\tau = \tfrac{5}{4}\rho\tau + \bar{\pi}\rho - \tfrac{1}{4}\bar{\rho}\tau \tag{6.5b}$$

$$D\alpha - \tfrac{1}{4}\bar{\delta}\rho = \tfrac{1}{2}\rho\alpha + \tfrac{1}{4}\bar{\rho}\alpha - \tfrac{1}{16}\bar{\tau}\rho + \tfrac{5}{4}\rho\pi \tag{6.5c}$$

$$D\tau - \delta\rho = \tfrac{3}{4}\bar{\rho}\tau - \bar{\alpha}\rho + \bar{\pi}\rho \tag{6.5d}$$

$$D\gamma - \tfrac{1}{4}\Delta\rho = \tfrac{5}{4}\tau\pi - \tfrac{1}{2}\gamma\rho + \tau\alpha + \alpha\bar{\pi} + \tfrac{1}{4}\tau\bar{\tau} - \tfrac{1}{4}\bar{\gamma}\rho - \tfrac{1}{4}\gamma\bar{\rho} \tag{6.5e}$$

$$D\lambda - \bar{\delta}\pi = \tfrac{1}{4}\rho\lambda + \tfrac{1}{4}\bar{\rho}\lambda + \pi^2 + \alpha\pi - \tfrac{1}{4}\bar{\tau}\pi \tag{6.5f}$$

$$D\mu - \delta\pi = \tfrac{3}{4}\bar{\rho}\mu + \pi\bar{\pi} - \tfrac{1}{4}\rho\mu - \pi\bar{\alpha} + \tfrac{1}{4}\pi\tau \tag{6.5g}$$

$$D\nu - \Delta\pi = \pi\mu + \bar{\tau}\mu + \bar{\pi}\lambda + \tau\lambda + \gamma\pi - \bar{\gamma}\pi - \tfrac{3}{4}\rho\nu - \tfrac{1}{4}\bar{\rho}\nu \tag{6.5h}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \bar{\mu}\lambda - 3\gamma\lambda + \bar{\gamma}\lambda + 3\alpha\nu + \pi\nu - \tfrac{3}{4}\bar{\tau}\nu - \Psi_4 \tag{6.5i}$$

$$\delta\rho = \tfrac{5}{4}\tau\rho + \bar{\alpha}\rho - \bar{\rho}\tau \tag{6.5j}$$

$$\delta\alpha - \tfrac{1}{4}\bar{\delta}\tau = \tfrac{5}{4}\mu\rho + \alpha\bar{\alpha} + \tfrac{1}{16}\tau\bar{\tau} - \tfrac{1}{2}\alpha\tau + \gamma\rho - \gamma\bar{\rho} - \tfrac{1}{4}\rho\bar{\mu} \tag{6.5k}$$

$$\delta\lambda - \bar{\delta}\mu = \rho\nu - \bar{\rho}\nu + \mu\pi - \bar{\mu}\pi + \mu\alpha + \tfrac{1}{4}\mu\bar{\tau} + \lambda\bar{\alpha} - \tfrac{3}{4}\lambda\tau \tag{6.5l}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu - \bar{\nu}\pi + \tfrac{1}{4}\tau\nu - \bar{\alpha}\nu \tag{6.5m}$$

$$\delta\gamma - \tfrac{1}{4}\Delta\tau = \tfrac{1}{2}\tau\gamma - \bar{\alpha}\gamma + \tfrac{5}{4}\mu\tau - \tfrac{1}{4}\rho\bar{\nu} + \tfrac{1}{4}\tau\bar{\gamma} + \alpha\bar{\lambda} \tag{6.5n}$$

$$\delta\tau = \bar{\lambda}\rho + \tfrac{5}{4}\tau^2 - \tau\bar{\alpha} \tag{6.5o}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \tfrac{3}{4}\bar{\tau}\tau - \alpha\tau + \gamma\rho + \bar{\gamma}\rho \tag{6.5p}$$

$$\Delta\alpha - \bar{\delta}\gamma = \tfrac{5}{4}\rho\nu - \tfrac{5}{4}\tau\lambda + \bar{\gamma}\bar{\alpha} - \bar{\mu}\bar{\alpha} - \tfrac{3}{4}\tau\gamma \tag{6.5q}$$

The commutator equations become, using (6.4)

$$(\Delta D - D\Delta)\phi = [(\gamma + \bar{\gamma})D + \tfrac{1}{4}(\rho + \bar{\rho})\Delta - (\tau + \bar{\pi})\bar{\delta} - (\bar{\tau} + \pi)\delta]\phi \tag{6.6a}$$

$$(\delta D - D\delta)\phi = [(\bar{\alpha} + \tfrac{1}{4}\tau - \bar{\pi})D - (\tfrac{3}{4}\bar{\rho} + \tfrac{1}{4}\rho)\delta]\phi \tag{6.6b}$$

$$(\delta\Delta - \Delta\delta)\phi = [-\bar{\nu}D + (\tfrac{3}{4}\tau - \bar{\alpha})\Delta + \bar{\lambda}\bar{\delta} + (\mu - \gamma + \bar{\gamma})\delta]\phi \tag{6.6c}$$

$$(\bar{\delta}\delta - \delta\bar{\delta})\phi = [(\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta - (\bar{\alpha} - \tfrac{1}{4}\tau)\bar{\delta} - (\tfrac{1}{4}\bar{\tau} - \alpha)\delta]\phi \tag{6.6d}$$

§3. First Order

The calculation of the first covariant derivative for a vacuum type N vacuum spacetime in canonical form (6.1) is exactly the same as the corresponding calculation given in chapter 5 for a type D vacuum spacetime. From (5.14), (5.16) and (5.18) one obtains for the non-vanishing components, using (6.1) and (6.4),

$$(D\Psi)_{40'} = \rho \tag{6.7a}$$

$$(D\Psi)_{50'} = 4\alpha \tag{6.7b}$$

$$(D\Psi)_{41'} = \tau \tag{6.7c}$$

$$(D\Psi)_{51'} = 4\gamma \tag{6.7d}$$

where the same notation as in chapter 5 has been used.

To summarise, at zeroth order there is just the constant $\Psi_4 = 1$, whereas the first covariant derivative depends on ρ, α, τ and γ, i.e.

$$D\Psi = D\Psi(\rho, \alpha, \tau, \gamma) \tag{6.8}$$

Under H_0 these spin coefficients transform as follows :

$$\rho \longrightarrow \rho \tag{6.9a}$$

$$\alpha \longrightarrow \alpha + \tfrac{5}{4}\bar{a}\rho \tag{6.9b}$$

$$\tau \longrightarrow \tau + a\rho \tag{6.9c}$$

$$\gamma \longrightarrow \gamma + a\alpha + \tfrac{5}{4}\bar{a}\tau + \tfrac{5}{4}a\bar{a}\rho \tag{6.9d}$$

Let us now consider the possible zero/non-zero nature of these spin coefficients for type N vacuum metrics, using equations (6.9) to determine the first order canonical forms and invariance groups for the various invariant classes so obtained.

Class I : $\rho \neq 0$

From (6.9c) it is seen that τ can always be transformed to zero by setting $a = -\tau/\rho$, which fixes the frame completely.

Class II : $\rho = 0,\, \tau = 0$

From equations (6.9) it is seen that the only transformation remaining is (6.9d) which reduces to

$$\gamma \quad \longrightarrow \quad \gamma + a\alpha$$

This class is divided into two subclasses :

IIa) $\alpha \neq 0$

The remaining transformation can be used to set $\gamma = 0$ by setting $a = -\gamma/\alpha$, which fixes the frame completely.

IIb) $\alpha = 0$

None of the spin coefficients can be transformed at all so the frame cannot be fixed any further i.e. the full 2-d null rotations remain.

Class III : $\rho = 0,\, \tau \neq 0$

Again from equations (6.9) it is seen that the only transformation remaining is (6.9d) which reduces to

$$\gamma \quad \longrightarrow \quad \gamma + (a\alpha + \tfrac{5}{4}\bar{a}\tau) \tag{6.10}$$

We shall denote

$$\alpha = x \exp(ix') \tag{6.11a}$$

$$\tau = y \exp(iy') \tag{6.11b}$$

$$a = z \exp(iz') \tag{6.11c}$$

Then the term in parentheses in equation (6.10) has real part

$$zx \cos(x' + z') + \tfrac{5}{4} zy \cos(y' - z') \tag{6.12a}$$

and imaginary part

$$zx \sin(x' + z') + \tfrac{5}{4} zy \sin(y' - z') \tag{6.12b}$$

This class is divided into two subclasses :

IIIa) $|\alpha| \neq \frac{5}{4}|\tau| \Rightarrow x \neq \frac{5}{4}y$

In this case equations (6.12a) and (6.12b) can be set equal to arbitrary values to yield two *linearly independent* equations for the two unknowns z and z'. Therefore, we may transform both the real and the imaginary parts of γ to zero, fixing the value of a and hence the frame completely.

IIIb) $|\alpha| = \frac{5}{4}|\tau| \Rightarrow x = \frac{5}{4}y$

In this case we can use the trigonometric formulas for the sum of two cosines and the sum of two sines to write (6.12a) as

$$\tfrac{5}{2}zy\cos(\tfrac{1}{2}(x'+y'))\cos(\tfrac{1}{2}(x'-y')+z') \tag{6.12a bis}$$

and (6.12b) as

$$\tfrac{5}{2}zy\sin(\tfrac{1}{2}(x'+y'))\cos(\tfrac{1}{2}(x'-y')+z') \tag{6.12b bis}$$

The two equations are now seen to be *linearly dependent* for the two unknowns z and z' and we further divide this class into three subclasses :

IIIbi) $\bar{\alpha} = -\frac{5}{4}\tau \Rightarrow x'+y' = (2n+1)\pi$

In this case the real part (6.12a bis) is zero for all z, z' whereas the imaginary part can have any value depending on z, z'. Therefore, we can transform the imaginary part of γ to zero but not the real part. Because (6.12b bis) is only one equation for the two unknowns z and z' the frame is fixed up to a 1-dimensional invariance subgroup H_1.

IIIbii) $\bar{\alpha} = \frac{5}{4}\tau \Rightarrow x'+y' = 2n\pi$

In this case the imaginary part (6.12b bis) is zero for all z, z' whereas the real part can have any value depending on z, z'. Therefore, we can transform the real part of γ to zero but not the imaginary part. Because (6.12a bis) is only one equation for the two unknowns z and z' the frame is fixed up to a 1 dimensional invariance subgroup H_1.

IIIbiii) $\bar{\alpha} \neq -\frac{5}{4}\tau$, $\bar{\alpha} \neq \frac{5}{4}\tau \Rightarrow x' + y' \neq (2n+1)\pi$, $x' + y' \neq 2n\pi$

In this case we may start with either the real or imaginary part and give it a particular value, fixing the frame up to a 1 dimensional invariance subgroup H_1 as above. However, we see from equations (6.12a bis) and (6.12b bis) that the ratio of the imaginary to the real part is $\tan(\frac{1}{2}(x'+y'))$, so that a particular value for either the real or imaginary part fixes the other uniquely. Therefore, in this case generically we may transform either the real or the imaginary part ot γ to zero, but not both.

Let us examine the 1-dimensional set of values that the group parameter a assumes on fixing the three canonical forms in Class IIIb), as this will prove important later. Consider first the canonical form $Re(\gamma) = 0$. From equation (6.12a bis) this gives

$$\frac{5}{2}zy\cos(\tfrac{1}{2}(x'+y'))\cos(\tfrac{1}{2}(x'-y')+z') = -Re(\gamma) \tag{6.13}$$

If we now write

$$(x'-y')/2 = \theta \tag{6.14a}$$

$$-2Re(\gamma)/5y\cos(\tfrac{1}{2}(x'+y')) = f \tag{6.14b}$$

then equation (6.13) becomes

$$z\cos(z'+\theta) = f \tag{6.15}$$

Expanding this equation one can obtain the connection between the real and imaginary parts of a that it implies. Writing $a = s + it$ one obtains

$$t = ms + c \tag{6.16}$$

where

$$m = \cot(\theta) \tag{6.17a}$$

$$c = -f/\sin(\theta) \tag{6.17b}$$

Thus, at each point on the manifold the canonical form chosen fixes a to lie on a straight line, but as m and c are functions of the coordinates this straight line will vary from point to point on the manifold. In the case where the canonical form is $Im(\gamma) = 0$ one obtains exactly the same results except that now

$$f = \frac{-2Im(\gamma)}{5y\sin(\frac{1}{2}(x' + y'))}$$

§4. Second Order

Let us proceed as we did in §3 and compute the dyad components of the second covariant derivative of the Weyl spinor. In fact, as discussed in chapter 5, it can be proved using the Bianchi and Ricci identities that at all orders of covariant differentiation of the Weyl spinor only the symmetrised parts are algebraically independent so that there will be only $7 \times 3 = 21$ independent components (see [6]). However, in our subsequent analysis there is no advantage to working with symmetrised components, so we shall not symmetrise to save additional work. Thus, using a similar notation to that used before, the components are labelled

$$(D^2\Psi)_{\mu a';bc'} \quad (\mu = 0,1,2,3,4,5\ ;\ a,b,c = 0,1) \tag{6.18}$$

For example

$$\begin{aligned}(D^2\Psi)_{00';00'} &= \Psi_{0000;00';00'} \\ &= \Psi_{ABCD;EF';GH'}o^A o^B o^C o^D o^E \bar{o}^{F'} o^G \bar{o}^{H'}\end{aligned} \tag{6.19}$$

The calculation of this second covariant derivative for a type N vacuum space-time in canonical form (6.1) is exactly the same as the corresponding calculation for a type D vacuum spacetime. From (5.30) the non-vanishing components are, using (6.4) and (6.7),

$$(D^2\Psi)_{30';10'} = 2\rho^2 \tag{6.20a}$$

$$(D^2\Psi)_{30';11'} = 2\rho\tau \tag{6.20b}$$

$$(D^2\Psi)_{31';10'} = 2\rho\tau \tag{6.20c}$$

$$(D^2\Psi)_{31';11'} = 2\tau^2 \tag{6.20d}$$

$$(D^2\Psi)_{40';00'} = D\rho + \tfrac{3}{4}\rho^2 - \tfrac{1}{4}\bar{\rho}\rho \tag{6.20e}$$

$$(D^2\Psi)_{40';10'} = \bar{\delta}\rho + 7\alpha\rho - \tfrac{1}{4}\bar{\tau}\rho \tag{6.20f}$$

$$(D^2\Psi)_{40';01'} = \delta\rho + \tfrac{3}{4}\tau\rho - \bar{\alpha}\rho + \tau\bar{\rho} \tag{6.20g}$$

$$(D^2\Psi)_{40';11'} = \Delta\rho + 3\gamma\rho + 4\alpha\tau - \bar{\gamma}\rho + \bar{\tau}\tau \tag{6.20h}$$

$$(D^2\Psi)_{41';00'} = D\tau + \tfrac{3}{4}\rho\tau - \bar{\pi}\rho + \tfrac{1}{4}\bar{\rho}\tau \tag{6.20i}$$

$$(D^2\Psi)_{41';10'} = \bar{\delta}\tau + 3\alpha\tau + 4\gamma\rho - \bar{\mu}\rho + \tfrac{1}{4}\bar{\tau}\tau \tag{6.20j}$$

$$(D^2\Psi)_{41';01'} = \delta\tau + \tfrac{3}{4}\tau^2 - \bar{\lambda}\rho + \bar{\alpha}\tau \tag{6.20k}$$

$$(D^2\Psi)_{41';11'} = \Delta\tau + 7\gamma\tau - \bar{\nu}\rho + \bar{\gamma}\tau \tag{6.20l}$$

$$(D^2\Psi)_{50';00'} = 4D\alpha - 5\pi\rho + 5\alpha\rho - \bar{\rho}\alpha \tag{6.20m}$$

$$(D^2\Psi)_{50';10'} = 4\bar{\delta}\alpha - 5\lambda\rho + 20\alpha^2 - \bar{\tau}\alpha \tag{6.20n}$$

$$(D^2\Psi)_{50';01'} = 4\delta\alpha - 5\mu\rho + 5\tau\alpha - 4\alpha\bar{\alpha} + 4\gamma\bar{\rho} \tag{6.20o}$$

$$(D^2\Psi)_{50;11'} = 4\Delta\alpha - 5\nu\rho + 20\gamma\alpha - 4\bar{\gamma}\alpha + 4\gamma\bar{\tau} \tag{6.20p}$$

$$(D^2\Psi)_{51';00'} = 4D\gamma - 5\pi\tau + 5\rho\gamma - 4\bar{\pi}\alpha + \bar{\rho}\gamma \tag{6.20q}$$

$$(D^2\Psi)_{51';10'} = 4\bar{\delta}\gamma - 5\lambda\tau + 20\alpha\gamma - 4\bar{\mu}\alpha + \bar{\tau}\gamma \tag{6.20r}$$

$$(D^2\Psi)_{51';01'} = 4\delta\gamma - 5\mu\tau + 5\gamma\tau - 4\bar{\lambda}\alpha + 4\gamma\bar{\alpha} \tag{6.20s}$$

$$(D^2\Psi)_{51';11'} = 4\Delta\gamma - 5\nu\tau + 20\gamma^2 - 4\bar{\nu}\alpha + 4\gamma\bar{\gamma} \tag{6.20t}$$

In summary

$$D^2\Psi = D^2\Psi(D^1\Psi, D(D^1\Psi), \delta(D^1\Psi), \bar{\delta}(D^1\Psi), \Delta(D^1\Psi), \pi, \lambda, \mu, \nu) \tag{6.21}$$

that is, from (6.8), the second covariant derivative is a functional of the first order quantities ρ, α, τ and γ, the derivatives with respect to D, δ, $\bar{\delta}$ and Δ of all these quantities, and the new spin coefficients π, λ, μ and ν.

The derivative terms can be expressed in terms of spin coefficients by means of the NP vacuum field equations (6.5). Using these equations to substitute for the derivative terms in (6.20) gives

$$(D^2\Psi)_{30';10'} = 2\rho^2 \tag{6.20a bis}$$

$$(D^2\Psi)_{30';11'} = 2\rho\tau \tag{6.20b bis}$$

$$(D^2\Psi)_{31';10'} = 2\rho\tau \tag{6.20c bis}$$

$$(D^2\Psi)_{31';11'} = 2\tau^2 \tag{6.20d bis}$$

$$(D^2\Psi)_{40';00'} = 2\rho^2 \tag{6.20e bis}$$

$$(D^2\Psi)_{40';10'} = \bar{\delta}\rho + 7\alpha\rho - \tfrac{1}{4}\bar{\tau}\rho \tag{6.20f bis}$$

$$(D^2\Psi)_{40';01'} = 2\rho\tau \tag{6.20g bis}$$

$$(D^2\Psi)_{40';11'} = \Delta\rho + 3\gamma\rho + 4\alpha\tau - \bar{\gamma}\rho + \bar{\tau}\tau \tag{6.20h bis}$$

$$(D^2\Psi)_{41';00'} = 2\tau\rho \tag{6.20i bis}$$

$$(D^2\Psi)_{41';10'} = \bar{\delta}\tau + 3\alpha\tau + 4\gamma\rho - \bar{\mu}\rho + \tfrac{1}{4}\bar{\tau}\tau \tag{6.20j bis}$$

$$(D^2\Psi)_{41';01'} = 2\tau^2 \tag{6.20k bis}$$

$$(D^2\Psi)_{41';11'} = \Delta\tau + 7\gamma\tau - \bar{\nu}\rho + \bar{\gamma}\tau \tag{6.20l bis}$$

$$(D^2\Psi)_{50';00'} = 4D\alpha - 5\pi\rho + 5\alpha\rho - \bar{\rho}\alpha \tag{6.20m bis}$$

$$(D^2\Psi)_{50';10'} = 4\bar{\delta}\alpha - 5\lambda\rho + 20\alpha^2 - \bar{\tau}\alpha \tag{6.20n bis}$$

$$(D^2\Psi)_{50';01'} = 4\delta\alpha - 5\mu\rho + 5\tau\alpha - 4\alpha\bar{\alpha} + 4\gamma\bar{\rho} \tag{6.20o bis}$$

$$(D^2\Psi)_{50;11'} = 4\Delta\alpha - 5\nu\rho + 20\gamma\alpha - 4\bar{\gamma}\alpha + 4\gamma\bar{\tau} \tag{6.20p bis}$$

$$(D^2\Psi)_{51';00'} = 4D\gamma - 5\pi\tau + 5\rho\gamma - 4\bar{\pi}\alpha + \bar{\rho}\gamma \tag{6.20q bis}$$

$$(D^2\Psi)_{51';10'} = 4\bar{\delta}\gamma - 5\lambda\tau + 20\alpha\gamma - 4\bar{\mu}\alpha + \bar{\tau}\gamma \tag{6.20r bis}$$

$$(D^2\Psi)_{51';01'} = 4\delta\gamma - 5\mu\tau + 5\gamma\tau - 4\bar{\lambda}\alpha + 4\gamma\bar{\alpha} \tag{6.20s bis}$$

$$(D^2\Psi)_{51';11'} = 4\Delta\gamma - 5\nu\tau + 20\gamma^2 - 4\bar{\nu}\alpha + 4\gamma\bar{\gamma} \tag{6.20t bis}$$

In addition, equations (6.5) provide connections between the derivative terms. The relevant equations are

$$D\alpha - \tfrac{1}{4}\bar{\delta}\rho = \tfrac{1}{2}\rho\alpha + \tfrac{1}{4}\bar{\rho}\alpha - \tfrac{1}{16}\bar{\tau}\rho + \tfrac{5}{4}\rho\pi \qquad (6.5c\ bis)$$

$$D\tau - \delta\rho = \tfrac{3}{4}\bar{\rho}\tau - \bar{\alpha}\rho + \bar{\pi}\rho \qquad (6.5d\ bis)$$

$$D\gamma - \tfrac{1}{4}\Delta\rho = \tfrac{5}{4}\tau\pi - \tfrac{1}{2}\gamma\rho + \tau\alpha + \alpha\bar{\pi} + \tfrac{1}{4}\tau\bar{\tau} - \tfrac{1}{4}\bar{\gamma}\rho - \tfrac{1}{4}\gamma\bar{\rho} \qquad (6.5e\ bis)$$

$$\delta\alpha - \tfrac{1}{4}\bar{\delta}\tau = \tfrac{5}{4}\mu\rho + \alpha\bar{\alpha} + \tfrac{1}{16}\tau\bar{\tau} - \tfrac{1}{2}\alpha\tau + \gamma\rho - \gamma\bar{\rho} - \tfrac{1}{4}\rho\bar{\mu} \qquad (6.5k\ bis)$$

$$\delta\gamma - \tfrac{1}{4}\Delta\tau = \tfrac{1}{2}\tau\gamma - \bar{\alpha}\gamma + \tfrac{5}{4}\mu\tau - \tfrac{1}{4}\rho\bar{\nu} + \tfrac{1}{4}\tau\bar{\gamma} + \alpha\bar{\lambda} \qquad (6.5n\ bis)$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \tfrac{3}{4}\bar{\tau}\tau - \alpha\tau + \gamma\rho + \bar{\gamma}\rho \qquad (6.5p\ bis)$$

$$\Delta\alpha - \bar{\delta}\gamma = \tfrac{5}{4}\rho\nu - \tfrac{5}{4}\tau\lambda + \bar{\gamma}\bar{\alpha} - \bar{\mu}\bar{\alpha} - \tfrac{3}{4}\tau\gamma \qquad (6.5q\ bis)$$

Equation (6.5d bis) gives nothing new but is consistent with equations (6.5b) and (6.5j). Inspection of the remaining equations shows that there is functional dependence amongst the following groups of derivatives

$$D\alpha, \bar{\delta}\rho \qquad (6.22a)$$

$$D\gamma, \Delta\rho, \bar{\delta}\tau, \delta\alpha \qquad (6.21b)$$

$$\delta\gamma, \Delta\tau \qquad (6.22c)$$

$$\Delta\alpha, \bar{\delta}\gamma \qquad (6.22d)$$

$$\bar{\delta}\alpha \qquad (6.22e)$$

$$\Delta\gamma \qquad (6.22f)$$

with (6.22e) and (6.22f) following from the fact that equations (6.5) provide no information about the derivatives $\bar{\delta}\alpha$ and $\Delta\gamma$. In summary, it has been shown that

$$D^2\Psi = D^2\Psi(D^1\Psi, D\alpha, D\gamma, \delta\gamma, \Delta\alpha, \bar{\delta}\alpha, \Delta\gamma, \pi, \lambda, \mu, \nu) \qquad (6.23)$$

Let us now consider the second order canonical forms for our various classes. For classes I, IIa and IIIa the frame is completely fixed at first order, so the specification of canonical forms is complete. For Class IIb a 2-dimensional invariance group remains at first order, and for Class IIIb a 1-dimensional invariance group remains. Let us consider each of these classes in turn.

Class IIb : $\rho = 0, \tau = 0, \alpha = 0$

From equations (6.20 bis) we see that the only potentially new functionally independent information at second order is $D\gamma$, $\delta\gamma$, $\bar{\delta}\gamma$ and $\Delta\gamma$. However, on substituting $\rho = \tau = \alpha = 0$ into equations (6.5e), (6.5n) and (6.5q) one obtains

$$D\gamma = 0 \tag{6.24a}$$

$$\delta\gamma = 0 \tag{6.24b}$$

$$\bar{\delta}\gamma = 0 \tag{6.24c}$$

Therefore, from equations (6.24a), (6.24b) and (6.24c) it is seen that all of the potentially new functionally independent information is zero except $\Delta\gamma$. Can this term be used to fix the frame any further. It is easily verified that under H_0 the NP derivative operator Δ transforms as

$$\Delta \quad \longrightarrow \quad (\Delta + a\bar{\delta} + \bar{a}\delta + a\bar{a}D) \tag{6.25}$$

It is seen from equation (6.9d) that γ remains unchanged under H_0 so we have

$$\begin{aligned} \Delta\gamma \quad &\longrightarrow \quad (\Delta + a\bar{\delta} + \bar{a}\delta + a\bar{a}D)\gamma \\ &= \quad \Delta\gamma \end{aligned} \tag{6.26}$$

using the fact that $D\gamma = \delta\gamma = \bar{\delta}\gamma = 0$. So we see that $\Delta\gamma$ remains unchanged under H^0 so that at second order we still have the full 2-d null rotations as the invariance group.

It can be readily calculated using extensions of the calculations shown in chapter 5 that at third and higher orders of covariant differentiation the only non-zero component will be the highest labelled one and it will contain as potentially new functionally independent information only a term of the form $\Delta\Delta\Delta\ldots\gamma$. For example, at fifth order we shall have, using an obvious generalisation of our previous labelling, the component $(D^5\Psi)_{51';11';11';11';11'}$ as our only non-vanishing one, and it will contain the term $\Delta\Delta\Delta\Delta\gamma$ as the only source of potentially new functionally independent information. It will now be proved by induction that, using an obvious shorthand, $\Delta^n\gamma$ is invariant under the group H_0 of 2-d null rotations, for any n.

Suppose that $\Delta^{(n-1)}\gamma$ is invariant under H_0, then under H_0 $\Delta^n\gamma$ will transform as

$$\begin{aligned} \Delta^n\gamma \quad &\longrightarrow \quad (\Delta + a\bar{\delta} + \bar{a}\delta + a\bar{a}D)\Delta^{(n-1)}\gamma \\ &= \quad \Delta^n\gamma + a\bar{\delta}\Delta^{(n-1)}\gamma + \bar{a}\delta\Delta^{(n-1)}\gamma + a\bar{a}D\Delta^{(n-1)}\gamma \end{aligned} \tag{6.27}$$

From the commutator equations (6.6a), (6.6c) and the complex conjugate of (6.6c) it is seen that the NP derivative operators D, δ and $\bar{\delta}$ can be moved through a line of Δ to the right. However, from equations (6.24a), (6.24b) and (6.24c) we see that $D\gamma = \delta\gamma = \bar{\delta}\gamma = 0$. Therefore, equation (6.27) becomes

$$\Delta^n\gamma \quad \longrightarrow \quad \Delta^n\gamma$$

Thus, we have shown that if $\Delta^{(n-1)}\gamma$ is unchanged under H_0 then so is $\Delta^n\gamma$. However, we know from equation (6.26) that $\Delta\gamma$ is unchanged under H_0 so we have by induction that $\Delta^n\gamma$ is unchanged for any n. Thus the canonical form for this class cannot be restricted any further, so we shall retain the 2-d null rotation group H_0 as our invariance group at all orders.

Class IIIb : $\rho = 0, \tau \neq 0, |\alpha| = \frac{5}{4}|\tau|$

From equations (6.20 bis) we see that the potentially new functionally independent information at second order is π, λ, μ and ν together with the NP derivatives of the first order spin coefficients α, τ and γ. Under null rotations the new spin coefficients π, λ, μ and ν transform, using (6.1), (6.4) and the class condition $\rho = 0$, as follows :

$$\begin{aligned}
\pi &\longrightarrow \pi + D\bar{a} && (6.28a)\\
\lambda &\longrightarrow \lambda + \bar{a}(\pi + 2\alpha) + \bar{\delta}\bar{a} + \bar{a}D\bar{a} && (6.28b)\\
\mu &\longrightarrow \mu + a\pi + \tfrac{1}{2}\bar{a}\tau + \delta\bar{a} + aD\bar{a} && (6.28c)\\
\nu &\longrightarrow \nu + a\lambda + \bar{a}(\mu + 2\gamma) + a\bar{a}(2\alpha + \pi) && \\
&\qquad + \tfrac{3}{2}\bar{a}^2\tau + \Delta\bar{a} + a\bar{\delta}\bar{a} + \bar{a}\delta\bar{a} + a\bar{a}Da && (6.28d)
\end{aligned}$$

The NP derivative operators transform as

$$\begin{aligned}
D &\longrightarrow D && (6.29a)\\
\delta &\longrightarrow \delta + aD && (6.29b)\\
\bar{\delta} &\longrightarrow \bar{a}D && (6.29c)\\
\Delta &\longrightarrow \Delta + a\bar{\delta} + \bar{a}\delta + a\bar{a}D && (6.29d)
\end{aligned}$$

In order to fix the canonical form further at second order one requires a second order quantity that transforms under null rotations but which does not contain derivatives of the transformation parameter a in the transformation equation. From equations (6.28) we see that the transformations of π, λ, μ and ν are not suitable, and from equations (6.9) we see that only a transformation of an NP derivative operator acting on α or τ could be suitable. However from equations (6.5b) and (6.5c), on substituting the class condition $\rho = 0$, we see that one obtains $D\tau = D\alpha = 0$. Thus, from equations (6.29), we see that we must consider only the transformations of $\Delta\tau$ or $\Delta\alpha$, or the quantity is fixed under null rotations. It proves convenient to use the transformation of $\Delta\tau$:

$$\Delta\tau \longrightarrow \Delta\tau + (a\bar{\delta} + \bar{a}\delta)\tau \qquad (6.30)$$

This equation can be split into the transformation of the real part of $\Delta\tau$ by a real amount and the transformation of the imaginary part of $\Delta\tau$ by an imaginary amount. It will turn out that the frame can only be further restricted in a way consistent with the fixing at first order if one has a second order canonical form which restricts only the numerical value of the real part of $\Delta\tau$. Thus, from equation (6.30), it is seen that to transform the real part of $\Delta\tau$ by a real amount requires that τ is not purely imaginary. Let us prove that this is the case.

Adding equation (6.5o) and equation (6.5p), remembering that $\rho = 0$, one obtains

$$(\delta + \bar{\delta})\tau = \tfrac{5}{4}\tau^2 + \tfrac{3}{4}\bar{\tau}\tau + \tau(\alpha - \bar{\alpha}) \tag{6.31}$$

If τ were purely imaginary then it is readily confirmed by considering the real/imaginary nature of the left and right hand sides of equation (6.31), that this is only possible if both sides are zero. Subtracting equation (6.5p) from equation (6.5o) gives

$$(\delta - \bar{\delta})\tau = \tfrac{5}{4}\tau^2 - \tfrac{3}{4}\bar{\tau}\tau - \tau(\alpha + \bar{\alpha}) \tag{6.32}$$

The same considerations as above will readily reveal that the only way that τ can be purely imaginary is if α is also purely imaginary. Writing τ as $\tau = iy$, the class condition $|\alpha| = \frac{5}{4}|\tau|$ means that the condition that α is purely imaginary gives $\alpha = \pm\frac{5}{4}iy$. Substituting these expressions for τ and α into the previous condition expressed in the form that the right hand side of equation (6.31) is zero gives

$$-\tfrac{5}{4}y^2 \pm \tfrac{5}{2}y^2 + \tfrac{3}{4}y^2 = -\tfrac{1}{2}y^2 \pm \tfrac{5}{2}y^2 = 0$$

which is clearly a contradiction. Thus we may conclude that τ is definitely not purely imaginary.

Going back to equation (6.30) and choosing $Re(\Delta\tau) = 0$ as our canonical form gives

$$(a\bar{\delta} + \bar{a}\delta)Re(\tau) = -Re(\Delta\tau) \tag{6.33}$$

From equation (6.16) $a = s + it = s + i(ms + c)$. Substituting this in equation (6.33) yields after some algebra

$$s = \frac{-Re(\Delta\tau) + (ic\delta - ic\bar{\delta})Re(\tau)}{(\delta + \bar{\delta})Re(\tau) + (im\bar{\delta} - im\delta)Re(\tau)} \tag{6.34}$$

(note that m and c are real functions)

We now see the reason why one can only restrict the numerical value of the $Re(\Delta\tau)$ to be consistent with our canonical conditions at first order. Inspecting equation (6.34) one sees that if it did not contain only $Re(\tau)$ then s, which is defined to be real, would in fact be complex. All that remains is to check that the denominator of equation (6.34) is not zero, for otherwise the attempt at fixing the canonical form clearly does not work. Substituting for $\delta\tau$, $\bar{\delta}\tau$, m and c from equations (6.5o), (6.5p), (6.17a) and (6.17b) respectively, and substituting for τ and α from equations (6.11) (using the class condition $x = \frac{5}{4}y$), shows after much algebra that the denominator of equation (6.34) evaluates to $2y^2$.

Therefore, to summarise, the standard form at second order is $Re(\Delta\tau) = 0$ which fixes the value of the group parameter a and hence the frame completely.

§5. Upper Bounds

In this final section we use the simplifications to the NP field equations and the commutator equations which result from using the canonical forms discussed above, to consider the upper bound on the order of covariant differentiation of the Weyl tensor required in the Karlhede classification of each of our classes.

Class I : $\rho \neq 0$

As shown in §3 $\rho \neq 0$ implies that τ can be set to zero, fixing the frame completely at first order. Let us examine the reduction of equations (6.5) which occurs on setting $\tau = 0$. It is seen from equation (6.5b) that as $\rho \neq 0$ then $\pi = 0$, and from equation (6.5o) that $\lambda = 0$. Substituting these zeros into equations (6.5) gives

$$D\rho = \tfrac{5}{4}\rho^2 + \tfrac{1}{4}\rho\bar{\rho} \qquad (6.35a)$$

$$D\alpha - \tfrac{1}{4}\bar{\delta}\rho = \tfrac{1}{2}\rho\alpha + \tfrac{1}{4}\bar{\rho}\alpha \tag{6.35b}$$

$$\delta\rho = \bar{\alpha}\rho \tag{6.35c}$$

$$D\gamma - \tfrac{1}{4}\Delta\rho = -\tfrac{1}{4}\bar{\gamma}\rho - \tfrac{1}{2}\gamma\rho - \tfrac{1}{4}\gamma\bar{\rho} \tag{6.35d}$$

$$D\mu = \tfrac{3}{4}\bar{\rho}\mu - \tfrac{1}{4}\rho\mu \tag{6.35e}$$

$$D\nu = -\tfrac{3}{4}\rho\nu - \tfrac{1}{4}\bar{\rho}\nu \tag{6.35f}$$

$$\bar{\delta}\nu = -3\alpha\nu + \Psi_4 \tag{6.35g}$$

$$\delta\rho = \bar{\alpha}\rho \tag{6.35h}$$

$$\delta\alpha = \tfrac{5}{4}\mu\rho + \alpha\bar{\alpha} + \gamma\rho - \gamma\bar{\rho} - \tfrac{1}{4}\rho\bar{\mu} \tag{6.35i}$$

$$\bar{\delta}\mu = \bar{\rho}\nu - \mu\alpha - \rho\nu \tag{6.35j}$$

$$\delta\nu - \Delta\mu = \mu^2 + \gamma\mu + \bar{\gamma}\mu - \bar{\alpha}\nu \tag{6.35k}$$

$$\delta\gamma = -\bar{\alpha}\gamma - \tfrac{1}{4}\rho\bar{\nu} \tag{6.35l}$$

$$\Delta\rho = -\rho\bar{\mu} + \gamma\rho + \bar{\gamma}\rho \tag{6.35m}$$

$$\Delta\alpha - \bar{\delta}\gamma = \tfrac{5}{4}\rho\nu + \bar{\gamma}\bar{\alpha} - \bar{\mu}\bar{\alpha} \tag{6.35n}$$

In principle, the worst case could occur if everything were constant at first order, when it might be necessary to continue right up to the sixth order derivative. However, from equation (6.35c) it is seen that putting ρ as a constant implies that $\alpha = 0$. In addition, from equation (6.35l) it is seen that putting γ as a constant and $\alpha = 0$ implies that $\nu = 0$. Now inspection of equation (6.35g) shows that $\nu = 0$ gives an inconsistency ($\Psi_4 = 1$). Therefore, the worst possible case is, in fact, not permitted. Thus, there must be a non-constant component at first order, which means that at worst it could be necessary to continue to the fifth covariant derivative.

Class II : $\rho = 0,\ \tau = 0$

IIa) $\alpha \neq 0$

As shown in §3 we can set $\gamma = 0$, fixing the frame completely at first order. Let us examine the reduction of equations (6.5) which occurs on setting $\rho =$

$\tau = \gamma = 0$. It is seen from equation (6.5e) that as $\alpha \neq 0$ then $\pi = 0$, and from equation (6.5n) that $\lambda = 0$. Substituting these zeros in equations (6.5) gives

$$D\alpha = 0 \tag{6.36a}$$

$$D\mu = 0 \tag{6.36b}$$

$$D\nu = 0 \tag{6.36c}$$

$$\bar{\delta}\nu = \Psi_4 - 3\alpha\nu \tag{6.36d}$$

$$\delta\alpha = \alpha\bar{\alpha} \tag{6.36e}$$

$$\bar{\delta}\mu = -\mu\alpha \tag{6.36f}$$

$$\delta\nu - \Delta\mu = \mu^2 - \bar{\alpha}\nu \tag{6.36g}$$

$$\Delta\alpha = -\bar{\mu}\bar{\alpha} \tag{6.36h}$$

In principle, the worst case could occur if everything were constant at first order, when it might be necessary to continue right up to the sixth order derivative. However, putting α as constant in equation (6.36e) gives $0 = \alpha\bar{\alpha}$ which is a contradiction as for this class $\alpha \neq 0$. Therefore, the worst possible case is, in fact, not permitted. Thus, there must be a non-constant component at first order, which means that at worst it could be necessary to continue to the fifth covariant derivative. Clearly, the worst case now will be where all spin coefficients are non-constant, as here all the derivatives of these spin coefficients could produce new functionally independent information. Let us examine this case in detail.

The only spin coefficients which are non-zero are α, μ and ν. At first order only α occurs, whereas at second order the NP derivatives of α occur as well as μ and ν. From equations (6.36) it is seen that we know how to express all of the derivatives of α in terms of spin coefficients except $\bar{\delta}\alpha$, about which there is no information. However, information about this term can be obtained by applying $\bar{\delta}$ to the complex conjugate of equation (6.36h) and using the complex conjugate of commutator equation (6.6c). Thus one has

$$\bar{\delta}\Delta\bar{\alpha} = -[(\bar{\delta}\mu)\alpha + (\bar{\delta}\alpha)\mu]$$

Using the complex conjugate of commutator (6.6c) the left hand side can be initially transformed to a double derivative term $\Delta\bar{\delta}\bar{\alpha}$ and the three single derivative terms $D\bar{\alpha}$, $\Delta\bar{\alpha}$ and $\bar{\delta}\bar{\alpha}$. On using equations (6.36) all of these terms reduce to combinations of spin coefficients, as does the only remaining derivative term $\bar{\delta}\mu$. Thus one obtains an expression for $\bar{\delta}\alpha$ in terms of spin coefficients only. The exact result is

$$\mu\bar{\delta}\alpha = \mu\alpha^2 + \bar{\mu}\bar{\alpha}^2 - \bar{\mu}\alpha\bar{\alpha} \tag{6.37}$$

Thus it is seen that at second order everything can be expressed in terms of μ and ν. At third order one will obtain the NP derivatives of μ and ν. From equations (6.36) it is seen that $D\mu$, $\bar{\delta}\mu$, $D\nu$ and $\bar{\delta}\nu$ can be expressed purely in terms of the lower order quantities α, μ and ν. In addition, there is a relationship between $\Delta\mu$ and $\delta\nu$. However, there is no information about $\delta\mu$ and $\Delta\nu$. Information about $\delta\mu$ can be obtained by a similar calculation to that which lead to equation (6.37), this time applying δ to the complex conjugate of equation (6.36h) and using commutator (6.6c). The result is that one obtains an equation connecting $\delta\mu$ and $\Delta\mu$. However, one is still left with the problem of finding information about $\Delta\nu$ and examination of the equations (6.36) and the commutators (6.6) shows that no information about this term can be obtained. Thus at third order the only possible new information comes from the term $\Delta\nu$ and the functionally dependent group $\Delta\mu$, $\delta\mu$ and $\delta\nu$.

From equations (6.36a), (6.36b) and (6.36c) one sees that α, μ and ν are all independent of one of our four coordinates (the l^μ tetrad vector is taken to lie along one of the coordinate lines). All derivatives of these quantities will also be independent of this coordinate as can be proved by considering the NP operator D acting on the derivative, and commuting the D through the expression from left to right using the commutators (6.6a) and (6.6b). Thus, only three of the four coordinates appear in the covariant derivatives and, therefore, at most 3 functionally independent components can be produced. Thus, the Karlhede algorithm must terminate at fourth order.

IIb) $\alpha = 0$

In §4 it was shown that the full 2-dimensional null rotations remain as the invariance group throughout the Karlhede classification. Thus, for the Karlhede algorithm to continue it will be necessary to produce at least one new functionally independent component on each differentiation. Thus it might be thought that the upper bound would be the fifth covariant derivative. However, in this case the simplifications of the NP field equations and commutators are sufficient to reduce this upper bound still further by means of the following considerations.

Substituting $\rho = \tau = \alpha = 0$ into equations (6.5) one obtains

$$D\gamma = 0 \tag{6.38a}$$

$$D\lambda - \bar{\delta}\pi = \pi^2 \tag{6.38b}$$

$$D\mu - \delta\pi = \pi\bar{\pi} \tag{6.38c}$$

$$D\nu - \Delta\pi = \pi\mu + \bar{\pi}\lambda + \gamma\pi - \bar{\gamma}\pi \tag{6.38d}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \bar{\mu}\lambda - 3\gamma\lambda + \bar{\gamma}\lambda + \pi\nu - \Psi_4 \tag{6.38e}$$

$$\delta\lambda - \bar{\delta}\mu = \mu\pi - \bar{\mu}\pi \tag{6.38f}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu - \bar{\nu}\pi \tag{6.38g}$$

$$\delta\gamma = 0 \tag{6.38h}$$

$$\bar{\delta}\gamma = 0 \tag{6.38i}$$

Now from equations (6.38a), (6.38h) and (6.38i) it follows that γ is independent of three of the coordinates. The derivatives which occur at higher orders, $\Delta\gamma$, $\Delta\Delta\gamma$ etc. will also be independent of these three coordinates as can be shown by acting on them with D, δ or $\bar{\delta}$ and using the commutators (6.6a) and (6.6c) to commute the derivative operator through the expression from left to right. Thus only one of the four coordinates of this specially chosen coordinate system can ever appear in the covariant derivatives and hence only one functionally independent component can ever be produced. This component must be the γ appearing at first order, so the Karlhede algorithm will terminate at second order.

Class III : $\rho = 0,\ \tau \neq 0$

IIIa) $|\alpha| \neq \frac{5}{4}|\tau|$

As shown in §3 we can set $\gamma = 0$, fixing the frame completely at first order. Let us examine the reduction of equations (6.5) which occurs on setting $\rho = \gamma = 0$.

$$D\tau = 0 \tag{6.39a}$$

$$D\alpha = 0 \tag{6.39b}$$

$$0 = \tfrac{5}{4}\tau\pi + \tau\alpha + \alpha\bar{\pi} + \tfrac{1}{4}\tau\bar{\tau} \tag{6.39c}$$

$$D\lambda - \bar{\delta}\pi = \pi^2 + \alpha\pi - \tfrac{1}{4}\bar{\tau}\pi \tag{6.39d}$$

$$D\mu - \delta\pi = \pi\bar{\pi} - \pi\bar{\alpha} + \tfrac{1}{4}\pi\tau \tag{6.39e}$$

$$D\nu - \Delta\pi = \pi\mu + \bar{\tau}\mu + \bar{\pi}\lambda + \tau\lambda \tag{6.39f}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \bar{\mu}\lambda + 3\alpha\nu + \pi\nu - \tfrac{3}{4}\bar{\tau}\nu - \Psi_4 \tag{6.39g}$$

$$\delta\alpha - \tfrac{1}{4}\bar{\delta}\tau = \alpha\bar{\alpha} + \tfrac{1}{16}\tau\bar{\tau} - \tfrac{1}{2}\alpha\tau \tag{6.39h}$$

$$\delta\lambda - \bar{\delta}\mu = \mu\pi - \bar{\mu}\pi + \mu\alpha + \tfrac{1}{4}\mu\bar{\tau} + \lambda\bar{\alpha} - \tfrac{3}{4}\lambda\tau \tag{6.39i}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} - \bar{\nu}\pi + \tfrac{1}{4}\tau\nu - \bar{\alpha}\nu \tag{6.39j}$$

$$-\tfrac{1}{4}\Delta\tau = \tfrac{5}{4}\mu\tau + \alpha\bar{\lambda} \tag{6.39k}$$

$$\delta\tau = \tfrac{5}{4}\tau^2 - \tau\bar{\alpha} \tag{6.39l}$$

$$-\bar{\delta}\tau = -\tfrac{3}{4}\bar{\tau}\tau - \alpha\tau \tag{6.39m}$$

$$\Delta\alpha = -\tfrac{5}{4}\tau\lambda - \bar{\mu}\bar{\alpha} \tag{6.39n}$$

In principle, the worst case could occur if everything were constant at first order, when it might be necessary to continue right up to the sixth order derivative. However, on putting τ as constant in equations (6.39l) and (6.39m) one obtains

$$0 = \tfrac{5}{4}\tau^2 - \tau\bar{\alpha} \tag{6.39l bis}$$

$$0 = -\tfrac{3}{4}\bar{\tau}\tau - \alpha\tau \tag{6.39m bis}$$

Equation (6.39l bis) gives $\bar{\alpha} = \frac{5}{4}\tau$. Substituting this value into (6.39m bis) then gives $2\tau\bar{\tau} = 0$ which is impossible because $\tau \neq 0$. Therefore, the worst possible case is, in fact, not permitted. Thus, there must be a non-constant component at first order, which means that at worst it could be necessary to continue to the fifth covariant derivative.

IIIb) $|\alpha| = \frac{5}{4}|\tau|$

In this case it has been shown that at first order one can make either the imaginary part or the real part of γ equal to zero but not both, fixing the frame up to a 1-dimensional invariance subgroup. At second order one can set the real part of $\Delta\tau$ equal to zero, fixing the frame completely. On setting $\rho = 0$ in equations (6.5) they reduce to

$$D\tau = 0 \tag{6.40a}$$

$$D\alpha = 0 \tag{6.40b}$$

$$D\gamma = \tfrac{5}{4}\tau\pi + \tau\alpha + \alpha\bar{\pi} + \tfrac{1}{4}\tau\bar{\tau} \tag{6.40c}$$

$$D\lambda - \bar{\delta}\pi = \pi^2 + \alpha\pi - \tfrac{1}{4}\bar{\tau}\pi \tag{6.40d}$$

$$D\mu - \delta\pi = \pi\bar{\pi} - \pi\bar{\alpha} + \tfrac{1}{4}\pi\tau \tag{6.40e}$$

$$D\nu - \Delta\pi = \pi\mu + \bar{\tau}\mu + \bar{\pi}\lambda + \tau\lambda + \gamma\pi - \bar{\gamma}\pi \tag{6.40f}$$

$$\Delta\lambda - \bar{\delta}\nu = -\mu\lambda - \bar{\mu}\lambda - 3\gamma\lambda + \bar{\gamma}\lambda + 3\alpha\nu + \pi\nu - \tfrac{3}{4}\bar{\tau}\nu - \Psi_4 \tag{6.40g}$$

$$\delta\alpha - \tfrac{1}{4}\bar{\delta}\tau = \alpha\bar{\alpha} + \tfrac{1}{16}\tau\bar{\tau} - \tfrac{1}{2}\alpha\tau \tag{6.40h}$$

$$\delta\lambda - \bar{\delta}\mu = \mu\pi - \bar{\mu}\pi + \mu\alpha + \tfrac{1}{4}\mu\bar{\tau} + \lambda\bar{\alpha} - \tfrac{3}{4}\lambda\tau \tag{6.40i}$$

$$\delta\nu - \Delta\mu = \mu^2 + \lambda\bar{\lambda} + \gamma\mu + \bar{\gamma}\mu - \bar{\nu}\pi + \tfrac{1}{4}\tau\nu - \bar{\alpha}\nu \tag{6.40j}$$

$$\delta\gamma - \tfrac{1}{4}\Delta\tau = \tfrac{1}{2}\tau\gamma - \bar{\alpha}\gamma + \tfrac{5}{4}\mu\tau + \tfrac{1}{4}\tau\bar{\gamma} + \alpha\bar{\lambda} \tag{6.40k}$$

$$\delta\tau = \tfrac{5}{4}\tau^2 - \tau\bar{\alpha} \tag{6.40l}$$

$$\bar{\delta}\tau = \tfrac{3}{4}\bar{\tau}\tau + \alpha\tau \tag{6.40m}$$

$$\Delta\alpha - \bar{\delta}\gamma = -\tfrac{5}{4}\tau\lambda + \bar{\gamma}\bar{\alpha} - \bar{\mu}\bar{\alpha} - \tfrac{3}{4}\tau\gamma \tag{6.40n}$$

In principle, the worst case could occur if everything were constant at first and second order, when it might be necessary to continue right up to the seventh order derivative. However, by exactly the same argument as used above for Class IIIa, it is not in fact permitted to have everything constant at first order. The worst case will now be when only one new functionally independent component is produced at first order, none at second order, and then only one more at each subsequent order. The Karlhede algorithm would then not terminate until sixth order.

§6. Summary of results

The main results of this chapter are summarised in the following table :

Table of Results

Class	I	IIa)	IIb)	IIIa)	IIIb)
Invariant Characterisation	$\rho \neq 0$	$\rho = 0$ $\tau = 0$ $\alpha \neq 0$	$\rho = 0$ $\tau = 0$ $\alpha = 0$	$\rho = 0$ $\tau \neq 0$ $\lvert\alpha\rvert \neq \frac{5}{4}\lvert\tau\rvert$	$\rho = 0$ $\tau \neq 0$ $\lvert\alpha\rvert = \frac{5}{4}\lvert\tau\rvert$
Canonical Form :					
Zeroth Order	$\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_2 = 0$ $\Psi_3 = 0$ $\Psi_4 = 1$	$\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_2 = 0$ $\Psi_3 = 0$ $\Psi_4 = 1$	$\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_2 = 0$ $\Psi_3 = 0$ $\Psi_4 = 1$	$\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_2 = 0$ $\Psi_3 = 0$ $\Psi_4 = 1$	$\Psi_0 = 0$ $\Psi_1 = 0$ $\Psi_2 = 0$ $\Psi_3 = 0$ $\Psi_4 = 1$
1st Order	$\tau = 0$	$\gamma = 0$		$\gamma = 0$	$Re(\gamma) = 0$ or$Im(\gamma) = 0$
2nd Order					$Re(\Delta\tau) = 0$
Upper Bound	5	4	2	5	6

All vacuum type N spacetimes have been split into four classes by C. McIntosh [34]. Let us consider these four classes and relate them to the classes discussed in this chapter :

Class MI : pp-wave metric

These are plane-fronted gravitational waves with parallel rays, as defined in [1]. The metric form is

$$g = 2dudv - 2d\zeta d\bar{\zeta} + 2[f(u,\zeta) + \bar{f}(u,\bar{\zeta})]du^2 \tag{6.41}$$

where $f(u,\zeta)$ is an arbitrary function of u and ζ. These solutions fall into Class II of the present chapter.

Class MII : Rotating, plane-fronted wave metric

These are again plane-fronted gravitational waves, as defined by Kundt [35]. The metric form is

$$g = 2dudv - 2d\zeta d\bar{\zeta} + 2(z+\bar{z})[f(Y,z) + \bar{f}(Y,\bar{z})]dY^2 \tag{6.42}$$

where $f(Y,z)$ is an arbitrary function of Y and z and

$$z = \zeta - Yv, \quad 0 = (u - Y\zeta) - Y(\bar{\zeta} - Yv) \tag{6.43}$$

These solutions fall into Class III of the present chapter.

Class MIII : The Robinson-Trautman metric

Metrics in this class are type N examples of a wider class of vacuum metrics given by Robinson and Trautman [36]. Robinson-Trautman vacuum solutions are defined to be vacuum solutions admitting a geodesic, shearfree, twistfree and diverging null congruence. The metric form is

$$g = -2v^2 d\zeta d\bar{\zeta} + 2[\epsilon(u)\, du + v(P_\zeta d\zeta + P_{\bar{\zeta}} d\bar{\zeta}) + Pdv]du \tag{6.44}$$

where $P(u,\zeta,\bar{\zeta})$ satisfies

$$P^2[ln(P)]_{\zeta\bar{\zeta}} = \epsilon(u) \tag{6.45}$$

These solutions fall into Class I of the present chapter, but additionally require that the imaginary part of ρ is zero.

Class MIV : The twisting case

This is a class of solutions with non-zero twist, the Hauser metric [37] being the best known solution. The metric form of the Hauser metric is rather complicated and will not be presented here. These solutions also fall into Class I of the present chapter, but additionally require that the imaginary part of ρ is non-zero.

7
Lowering the Upper Bound for Type D Non-Vacuum Spacetimes

§1. Introduction

From chapter 3 equations (3.103), (3.104) and (3.105) we see that the upper bound on the order of covariant derivative of the Riemann tensor required to perform a complete Karlhede classification of a spacetime is 5 for Petrov types I, II and III, and 7 for Petrov types D, N and 0. Thus in the worst case it may be necessary to continue up to the seventh covariant derivative. In chapter 5 this upper bound was reduced to the third covariant derivative for vacuum type D metrics, and in chapter 6 it was reduced to the sixth covariant derivative for vacuum type N metrics. Therefore, we see that at best the upper bound can only be reduced to 6 for a general spacetime by the methods used in this thesis. However, in order for an upper bound of six to be true in general it still remains to show that it is not necessary to continue as high as the seventh covariant derivative for *non-vacuum* type D and type N spacetimes. In this chapter this is proved for the case of non-vacuum type D spacetimes using some simple considerations involving the transformation properties of quantities appearing in the non-vacuum case, the Bianchi identities and the field equations. Work is under way to use a similar approach to prove the result for non-vacuum type N spacetimes.

§2. Lowering the Upper Bound

From our discussion at the end of chapter 3, it is seen that the following conditions *all* have to be satisfied for it to be necessary to continue up to the seventh covariant derivative :

C1) The Weyl spinor, Ricci spinor and Λ must all be constants.

C2) The invariance group at zeroth order H_0 must have dimension 2.

C3) The dimension of the invariance group and the number of functionally independent components must not both change on differentiating.
C4) We must produce at most 1 new functionally independent component on differentiating.
C5) The dimension of the invariance group must go down by at most 1 dimension on differentiating.

The Ricci spinor $\Phi_{ABC'D'}$ has the following symmetries

$$\Phi_{ABC'D'} = \Phi_{(AB)(C'D')} = \bar{\Phi}_{ABC'D'} \tag{7.1}$$

It is easily verified that these symmetries reduce the number of independent components to 6. Because of the symmetry on the primed and unprimed indices, we can label these components using a similar notation to that used previously for the Weyl spinor. Thus, we have

$$\begin{aligned} \Phi_{00'} &= o^A o^B \bar{o}^{C'} \bar{o}^{D'} \Phi_{ABC'D'} \\ \Phi_{01'} &= o^A o^B \bar{o}^{C'} \bar{\iota}^{D'} \Phi_{ABC'D'} \\ \Phi_{02'} &= o^A o^B \bar{\iota}^{C'} \bar{\iota}^{D'} \Phi_{ABC'D'} \\ \Phi_{11'} &= o^A \iota^B \bar{o}^{C'} \bar{\iota}^{D'} \Phi_{ABC'D'} \\ \Phi_{12'} &= o^A \iota^B \bar{\iota}^{C'} \bar{\iota}^{D'} \Phi_{ABC'D'} \\ \Phi_{22'} &= \iota^A \iota^B \bar{\iota}^{C'} \bar{\iota}^{D'} \Phi_{ABC'D'} \end{aligned} \tag{7.2}$$

For a type D spacetime the *canonical form* for the Weyl spinor is

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \quad ; \qquad \Psi_2 \neq 0 \tag{7.3}$$

and, for the vacuum case, the invariance group a zeroth order, H_0, is the 2-dimensional spin and boost transformations,

$$H_0 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad ; \qquad \lambda = re^{i\theta} \in \mathbb{C} \tag{7.4}$$

Under these transformations, the components $\Phi_{ab'}$ transform as

$$\Phi_{00'} \longrightarrow r^4\Phi_{00'} \tag{7.5a}$$

$$\Phi_{01'} \longrightarrow r^2 e^{2i\theta}\Phi_{01'} \tag{7.5b}$$

$$\Phi_{02'} \longrightarrow e^{4i\theta}\Phi_{02'} \tag{7.5c}$$

$$\Phi_{11'} \longrightarrow \Phi_{11'} \tag{7.5d}$$

$$\Phi_{12'} \longrightarrow r^{-2} e^{2i\theta}\Phi_{12'} \tag{7.5e}$$

$$\Phi_{22'} \longrightarrow r^{-4}\Phi_{22'} \tag{7.5f}$$

From these equations we see that in order for the invariance group H_0 to remain 2-dimensional for the *non-vacuum* case, it is necessary that all $\Phi_{ab'}$ other than $\Phi_{11'}$ vanish. Therefore, if the Karlhede classification is going to continue up to the seventh covariant derivative, condition (2) above requires that

$$\Phi_{00'} = \Phi_{01'} = \Phi_{02'} = \Phi_{12'} = \Phi_{22'} = 0 \tag{7.6}$$

Using the fact that everything must be constant at zeroth order (condition (1)), together with equations (7.3) and (7.6), the Bianchi identities become

$$3\kappa\Psi = 2\kappa\Phi \tag{7.7a}$$

$$3\rho\Psi = -2\rho\Phi \tag{7.7b}$$

$$3\tau\Psi = 2\tau\Phi \tag{7.7c}$$

$$3\sigma\Psi = -2\sigma\Phi \tag{7.7d}$$

together with their primes (here $\Psi = \Psi_2$ and $\Phi = \Phi_{11'}$). The contracted Bianchi identities become

$$(\rho + \bar{\rho})\Phi = 0 \tag{7.8a}$$

$$(\tau + \bar{\tau}')\Phi = 0 \tag{7.8b}$$

together with their primes. The field equations (excluding the commutators) become

$$\eth\rho - \eth'\sigma = (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa \tag{7.9a}$$

$$Þ\rho - ð'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa \tag{7.9b}$$

$$Þ\sigma - ð\kappa = 0 \tag{7.9c}$$

$$Þ\tau - Þ'\kappa = \rho(\tau - \bar{\tau}') + \sigma(\bar{\tau} - \tau') \tag{7.9d}$$

$$ð\tau - Þ'\sigma = \rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' \tag{7.9e}$$

$$Þ'\rho - ð'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \Psi - 2\Lambda \tag{7.9f}$$

together with their primes (where we have used (7.8a) and (7.8b), assuming that $\Phi \neq 0$). It is seen from equations (7.7) and their primes that if any of κ, κ', τ and τ' are non-zero then $\Phi = \frac{3}{2}\Psi$, whereas if any of ρ, ρ', σ and σ' are non-zero then $\Phi = -\frac{3}{2}\Psi$. As we have a type D spacetime, we know that $\Psi \neq 0$, so $\Phi = \frac{3}{2}\Psi$ and $\Phi = -\frac{3}{2}\Psi$ are in contradiction. Therefore, there are only 3 possible cases :

Case 1) $\kappa = \kappa' = \tau = \tau' = \rho = \rho' = \sigma = \sigma' = 0$
Case 2) $\kappa = \kappa' = \tau = \tau' = 0$ and at least one of ρ, ρ', σ and σ' is non-zero.
Case 3) $\rho = \rho' = \sigma = \sigma' = 0$ and at lease one of κ, κ', τ and τ' is non-zero.

It is important to note that $\Phi \neq 0$ for cases 2 and 3 — for case 2 $\Phi = -\frac{3}{2}\Psi$ and for case 3 $\Phi = \frac{3}{2}\Psi$.

Let us now calculate the first covariant derivative of the Weyl spinor and Ricci spinor. Note that because Λ is a scalar, assuming it is constant implies that its covariant derivative is zero. From equations (5.14), (5.16) and (5.18), we see that under condition (7.3), the non-vanishing components of $(D\Psi)_{\mu f'}$ are

$$(D\Psi)_{10'} = 3\kappa\Psi \tag{7.10a}$$

$$(D\Psi)_{11'} = 3\sigma\Psi \tag{7.10b}$$

$$(D\Psi)_{20'} = D\Psi \tag{7.10c}$$

$$(D\Psi)_{21'} = \delta\Psi \tag{7.10d}$$

$$(D\Psi)_{30'} = 3\tau'\Psi \tag{7.10e}$$

$$(D\Psi)_{31'} = 3\rho'\Psi \tag{7.10f}$$

The calculation of the covariant derivative of the Ricci spinor follows the same pattern as the calculation of the covariant derivative of the Weyl spinor. Using a similar notation to that used for the covariant derivative of the Weyl spinor, the covariant derivative of the Ricci spinor is defined by

$$(D\Phi)_{\mu\nu';ef'} = \Phi_{ABC'D';EF'}[\zeta_a^A\zeta_b^B\bar{\zeta}_{c'}^{C'}\bar{\zeta}_{d'}^{D'}]\zeta_e^E\bar{\zeta}_{f'}^{F'} \tag{7.11}$$

where μ of the unprimed dyad vectors and ν of the primed dyad vectors in square parentheses are ζ_1^As and $\bar{\zeta}_{1'}^{A'}$s respectively. We use the Leibnitz property of covariant derivatives to write this as

$$(D\Phi)_{\mu\nu';ef'} = (\Phi_{ABC'D'}\zeta_a^A\zeta_b^B\bar{\zeta}_{c'}^{C'}\bar{\zeta}_{d'}^{D'})_{;ef'} - (\zeta_a^A\zeta_b^B\bar{\zeta}_{c'}^{C'}\bar{\zeta}_{d'}^{D'})_{;ef'}\Phi_{ABC'D'} \tag{7.12}$$

Using the symmetries (7.1) of $\Phi_{ABC'D'}$, (7.12) becomes

$$\begin{aligned}(D\Phi)_{\mu\nu';ef'} &= (\Phi_{\mu\nu'})_{;ef'} \\ &\quad - \mu\zeta^A_{1;ef'}\zeta_b^B\bar{\zeta}_{c'}^{C'}\bar{\zeta}_{d'}^{D'}\Phi_{ABC'D'} - (2-\mu)\zeta^A_{0;ef'}\zeta_b^B\bar{\zeta}_{c'}^{C'}\bar{\zeta}_{d'}^{D'}\Phi_{ABC'D'} \\ &\quad - \nu\bar{\zeta}^{C'}_{1';ef'}\zeta_a^A\zeta_b^B\bar{\zeta}_{d'}^{D'}\Phi_{ABC'D'} - (2-\nu)\bar{\zeta}^{C'}_{0';ef'}\zeta_a^A\zeta_b^B\bar{\zeta}_{d'}^{D'}\Phi_{ABC'D'}\end{aligned} \tag{7.13}$$

If we now substitute in (7.13) using (5.13) and (5.29) then it becomes

$$\begin{aligned}(D\Phi)_{\mu\nu';ef'} &= (\Phi_{\mu\nu'})_{ef'} \\ &\quad - \mu\Gamma_{11ef'}\Phi_{(\mu-1)\nu'} + (2\mu-2)\Gamma_{10ef'}\Phi_{\mu\nu'} \\ &\quad + (2-\mu)\Gamma_{00ef'}\Phi_{(\mu+1)\nu'} - \nu\bar{\Gamma}_{1'1'ef'}\Phi_{\mu(\nu-1)'} \\ &\quad + (2\nu-2)\bar{\Gamma}_{1'0'ef'}\Phi_{\mu\nu'} + (2-\nu)\bar{\Gamma}_{0'0'ef'}\Phi_{\mu(\nu+1)'}\end{aligned} \tag{7.14}$$

Using (7.6), we see from (7.14) that $(D\Phi)_{\mu\nu';ef'}$ is definitely zero for all cases other than i) $a=1$, $b=1$, ii) $a=1$, $b=0$, iii) $a=1$, $b=2$, iv) $a=0$, $b=1$ and v) $a=2$, $b=1$. For these cases we obtain from (7.14), using (7.6),

$$(D\Phi)_{11';00'} = D\Phi_{11'} \tag{7.15a}$$

$$(D\Phi)_{11';01'} = \delta\Phi_{11'} \tag{7.15b}$$

$$(D\Phi)_{11';10'} = \bar{\delta}\Phi_{11'} \tag{7.15c}$$

$$(D\Phi)_{11';11'} = \Delta\Phi_{11'} \tag{7.15d}$$

$$(D\Phi)_{10';00'} = 2\bar{\kappa}\Phi_{11'} \tag{7.15e}$$

$$(D\Phi)_{10';01'} = 2\bar{\rho}\Phi_{11'} \tag{7.15f}$$

$$(D\Phi)_{10';10'} = 2\bar{\sigma}\Phi_{11'} \tag{7.15g}$$

$$(D\Phi)_{10';11'} = 2\bar{\tau}\Phi_{11'} \tag{7.15h}$$

$$(D\Phi)_{12';00'} = 2\bar{\tau}'\Phi_{11'} \tag{7.15i}$$

$$(D\Phi)_{12';01'} = 2\bar{\sigma}'\Phi_{11'} \tag{7.15j}$$

$$(D\Phi)_{12';10'} = 2\rho'\Phi_{11'} \tag{7.15k}$$

$$(D\Phi)_{12';11'} = 2\bar{\kappa}'\Phi_{11'} \tag{7.15l}$$

$$(D\Phi)_{01';00'} = 2\kappa\Phi_{11'} \tag{7.15m}$$

$$(D\Phi)_{01';01'} = 2\sigma\Phi_{11'} \tag{7.15n}$$

$$(D\Phi)_{01';10'} = 2\rho\Phi_{11'} \tag{7.15o}$$

$$(D\Phi)_{01';11'} = 2\tau\Phi_{11'} \tag{7.15p}$$

$$(D\Phi)_{21';00'} = 2\tau'\Phi_{11'} \tag{7.15q}$$

$$(D\Phi)_{21';01'} = 2\rho'\Phi_{11'} \tag{7.15r}$$

$$(D\Phi)_{21';10'} = 2\sigma'\Phi_{11'} \tag{7.15s}$$

$$(D\Phi)_{21';11'} = 2\kappa'\Phi_{11'} \tag{7.15t}$$

Let us now consider each of our three cases in turn.

Case 1) $\kappa = \kappa' = \tau = \tau' = \rho = \rho' = \sigma = \sigma' = 0$

From equations (7.10) and (7.15) we see that everything vanishes at first order so that the Karlhede algorithm terminates at first order.

Case 2) $\kappa = \kappa' = \tau = \tau' = 0$ and at least one of ρ, ρ', σ and σ' is non-zero.

From equations (7.10) and (7.15) we see that the only non-vanishing components at first order are :

$$(D\Psi)_{11'} = 3\sigma\Psi \tag{7.16a}$$

$$(D\Psi)_{31'} = 3\rho'\Psi \tag{7.16b}$$

and

$$(D\Phi)_{10';01'} = 2\bar{\rho}\Phi_{11'} \tag{7.16c}$$

$$(D\Phi)_{10';10'} = 2\bar{\sigma}\Phi_{11'} \tag{7.16d}$$

$$(D\Phi)_{12';01'} = 2\bar{\sigma}'\Phi_{11'} \tag{7.16e}$$

$$(D\Phi)_{12';10'} = 2\bar{\rho}'\Phi_{11'} \tag{7.16f}$$

$$(D\Phi)_{01';01'} = 2\sigma\Phi_{11'} \tag{7.16g}$$

$$(D\Phi)_{01';10'} = 2\rho\Phi_{11'} \tag{7.16h}$$

$$(D\Phi)_{21';01'} = 2\rho'\Phi_{11'} \tag{7.16i}$$

$$(D\Phi)_{21';10'} = 2\sigma'\Phi_{11'} \tag{7.16j}$$

The field equations become

$$\eth\rho - \eth'\sigma = 0 \tag{7.17a}$$

$$Þ\rho = \rho^2 + \sigma\bar{\sigma} \tag{7.17b}$$

$$Þ\sigma = 0 \tag{7.17c}$$

$$Þ'\sigma = -\rho'\sigma + \bar{\sigma}'\rho \tag{7.17d}$$

$$Þ'\rho = \rho\bar{\rho}' + \sigma\sigma' - \Psi - 2\Lambda \tag{7.17e}$$

together with their primes.

Under spin and boost transformations (7.4) the spin coefficients ρ, ρ', σ and σ' transform as

$$\rho \longrightarrow r^2\rho \tag{7.18a}$$

$$\rho' \longrightarrow r^{-2}\rho' \tag{7.18b}$$

$$\sigma \longrightarrow r^2 e^{4i\theta}\sigma \tag{7.18c}$$

$$\sigma' \longrightarrow r^{-2} e^{-4i\theta}\sigma' \tag{7.18d}$$

Equations (7.18c) and (7.18d) enable one to fix r by requiring a particular modulus for the transformed quantity. In addition, they fix θ by requiring that the transformed quantity is, say, real (imaginary will, of course, do equally as well). Therefore, unless both σ and σ' are zero condition (C5) will be broken, in that the dimension of the invariance group will reduce from 2 to 0. Thus, in order for it to remain possible for one to require the seventh covariant derivative, one must have $\sigma = \sigma' = 0$. The field equations (7.17) now become

$$\eth\rho = 0 \tag{7.17a bis}$$

$$Þ\rho = \rho^2 \tag{7.17b bis}$$

$$Þ'\rho = \rho\bar{\rho}' - \Psi - 2\Lambda \tag{7.17e bis}$$

together with their primes. Taking the complex conjugate of (7.17a bis), and using (7.8a) (remembering that $\Phi = -\frac{3}{2}\Psi \neq 0$) one obtains

$$\eth'\rho = 0 \tag{7.19}$$

where we have used the fact that $\bar{\eth} = \eth'$.

At second order the only new quantities which will appear are the GHP operators $Þ$, $Þ'$, $\eth$ and $\eth'$ acting on ρ and ρ'. From equations (7.17 bis) and (7.19), together with their primes, we see that all these apparently new quantities can, in fact, be expressed in term of lower order quantities. Therefore, the Karlhede algorithm will terminate at second order.

Case 3) $\rho = \rho' = \sigma = \sigma' = 0$ and at lease one of κ, κ', τ and τ' is non-zero.

From equations (7.10) and (7.15) we see that the only non-vanishing components at first order are :

$$(D\Psi)_{10'} = 3\kappa\Psi \tag{7.20a}$$

$$(D\Psi)_{30'} = 3\tau'\Psi \tag{7.20b}$$

and

$$(D\Phi)_{10';00'} = 2\bar{\kappa}\Phi_{11'} \tag{7.20c}$$

$$(D\Phi)_{10';11'} = 2\bar{\tau}\Phi_{11'} \tag{7.20d}$$

$$(D\Phi)_{12';00'} = 2\bar{\tau}'\Phi_{11'} \tag{7.20e}$$

$$(D\Phi)_{12';11'} = 2\bar{\kappa}'\Phi_{11'} \tag{7.20f}$$

$$(D\Phi)_{01';00'} = 2\kappa\Phi_{11'} \tag{7.20g}$$

$$(D\Phi)_{01';11'} = 2\tau\Phi_{11'} \tag{7.20h}$$

$$(D\Phi)_{21';00'} = 2\tau'\Phi_{11'} \tag{7.20i}$$

$$(D\Phi)_{21';11'} = 2\kappa'\Phi_{11'} \tag{7.20j}$$

The field equations become

$$\eth'\kappa = \bar{\kappa}\tau + \tau'\kappa \tag{7.21a}$$

$$\eth\kappa = 0 \tag{7.21b}$$

$$þ\tau - þ'\kappa = 0 \tag{7.21c}$$

$$\eth\tau = \tau^2 + \kappa\bar{\kappa}' \tag{7.21d}$$

$$\eth'\tau = \tau\bar{\tau} + \kappa\kappa' - \Psi - 2\Lambda \tag{7.21e}$$

together with their primes.

Under spin and boost transformations (7.4) the spin coefficients τ, τ', κ and κ' transform as

$$\tau \longrightarrow e^{2i\theta}\tau \tag{7.22a}$$

$$\tau' \longrightarrow e^{-2i\theta}\tau' \tag{7.22b}$$

$$\kappa \longrightarrow r^4 e^{2i\theta}\kappa \tag{7.22c}$$

$$\kappa' \longrightarrow r^{-4} e^{-2i\theta}\kappa' \tag{7.22d}$$

Equations (7.22c) and (7.22d) enable one to fix r by requiring a particular modulus for the transformed quantity. In addition, they fix θ by requiring that the transformed quantity is, say, real. Therefore, unless both κ and κ' are zero condition (C5) will be broken, in that the dimension of the invariance group will reduce from 2 to 0. Thus, in order for it to remain possible for one to require the seventh covariant derivative, one must have $\kappa = \kappa' = 0$. The field equations (7.21) now become

$$Þ\tau = 0 \tag{7.21c bis}$$

$$ð\tau = \tau^2 + \kappa\bar{\kappa}' \tag{7.21d bis}$$

$$ð'\tau = \tau\bar{\tau} + \kappa\kappa' - \Psi - 2\Lambda \tag{7.21e bis}$$

together with their primes. Taking the complex conjugate of the prime of (7.21c bis), and using (7.8b) (remembering that $\Phi = \frac{3}{2}\Psi \neq 0$) one obtains

$$Þ'\tau = 0 \tag{7.23}$$

where we have used the fact that $\bar{Þ}' = Þ'$.

At second order the only new quantities which will appear are the GHP operators $Þ$, $Þ'$, $ð$ and $ð'$ acting on τ and τ'. From equations (7.21 bis) and (7.23), together with their primes, we see that all these apparently new quantities can, in fact, be expressed in term of lower order quantities. Therefore, the Karlhede algorithm will terminate at second order.

In conclusion, therefore, we see that for non-vacuum type D spacetimes it is, in fact, not possible for the Karlhede algorithm to require the seventh covariant derivative.

Appendix A

We need to show that there is a solution of (3.29) of the form (3.30) which is compatible with the coordinate relations obtained from equations (3.24).

Let W be a $2n$-dimensional space with coordinates $\{\tilde{x}^\mu, x^\mu\}$. Let

$$\hat{\omega}^\alpha = \tilde{\omega}^\alpha - \omega^\alpha = \tilde{a}^\alpha_\mu d\tilde{x}^\mu - a^\alpha_\mu dx^\mu \tag{A1}$$

where α, β etc. run from 1 to $n-k$. Therefore, solving (3.29), which is derived from (3.28), is equivalent to finding the submanifolds $V \subset W$ such that

$$\hat{\omega}^\alpha|_V = 0 \tag{A2}$$

where $\hat{\omega}^\alpha|_V$ means the 'restriction of $\hat{\omega}^\alpha$ to V', in the sense that $\hat{\omega}^\alpha$ only acts on vectors tangent to V. V will only exist (i.e. a solution of (3.29) will only exist) if the vectors X such that

$$< \hat{\omega}^\alpha, X >= 0 \tag{A3}$$

'knit' together in such a way as to be tangent to some submanifold V.

According to Cartan [2], the condition for this 'knitting' together is that

$$d(\tilde{\omega}^\alpha - \omega^\alpha) = \theta^\alpha_\beta \wedge (\tilde{\omega}^\beta - \omega^\beta) \tag{A4}$$

where θ^α_β are arbitrary 1-forms. The exterior derivative must be taken in W but will be the same as in (3.16) because ω^i ($\tilde{\omega}^i$) are independent of $\tilde{x}^\mu$ (x^μ). Using (3.16), (3.17) and (3.26) we prove that (A4) is indeed satisfied

$$\begin{aligned}
d(\tilde{\omega}^\alpha - \omega^\alpha) &= \tfrac{1}{2} c^\alpha_{kh}(\tilde{\omega}^k \wedge \tilde{\omega}^h - \omega^k \wedge \omega^h) \\
&= \tfrac{1}{4} c^\alpha_{kh}[(\tilde{\omega}^k + \omega^k) \wedge (\tilde{\omega}^h - \omega^h) + (\tilde{\omega}^k - \omega^k) \wedge (\tilde{\omega}^h + \omega^h)] \\
&= \tfrac{1}{4} c^\alpha_{\beta\gamma}[(\tilde{\omega}^\beta + \omega^\beta) \wedge (\tilde{\omega}^\gamma - \omega^\gamma) - (\tilde{\omega}^\gamma + \omega^\gamma) \wedge (\tilde{\omega}^\beta - \omega^\beta)] \\
&\quad + \tfrac{1}{2} c^\alpha_{\beta A}[b^A_\gamma(\tilde{\omega}^\beta + \omega^\beta) \wedge (\tilde{\omega}^\gamma - \omega^\gamma) - (\tilde{\omega}^A + \omega^A) \wedge (\tilde{\omega}^\beta - \omega^\beta)] \\
&\quad + \tfrac{1}{4} c^\alpha_{AB}[b^B_\beta(\tilde{\omega}^A + \omega^A) \wedge (\tilde{\omega}^\beta - \omega^\beta) - b^A_\gamma(\tilde{\omega}^B + \omega^B) \wedge (\tilde{\omega}^\gamma - \omega^\gamma)] \\
&= \theta^\alpha_\beta \wedge (\tilde{\omega}^\beta - \omega^\beta)
\end{aligned}$$

V will have dimension $2n$ - (number of constraints in (A2)). So we have

$$dim(V) = 2n - (n-k) = n+k \tag{A5}$$

In addition, V will not be unique but there will be an $n-k$ parameter family of Vs. This arises because the number of orthogonal normal directions to a given V is $2n$ - (dimension of V) $= 2n-(n+k) = n-k$, and each orthogonal normal direction will parameterise a set of Vs — the initial vector X which is 'knit' together with the others to form the submanifold may lie at any initial point along the normal directions.

To show how the above analysis works in practice consider the following simple example.

Example :

$n = 1$, $k = 0$, coordinates $\{x, \tilde{x}\}$, $x > 0$, $\tilde{x} > 0$
1-forms $\omega = x\ dx$, $\tilde{\omega} = -\tilde{x}\ d\tilde{x}$
($k = 0$ because in 1-d we only have c^1_{11} which by antisymmetry must be zero)

So (A2) becomes $\tilde{x}\ d\tilde{x} + x\ dx = 0$, which on integration yields $\tilde{x}^2 + x^2 = c$ or $\tilde{x} = f(x, c)$.

Thus we obtain an $n+k = 1$ dimensional solution submanifold with $n-k = 1$ parameter (c) parameterising different solution submanifolds, exactly as expected. In this case the solution submanifolds are concentric circles with the parameter c giving their radius, the radial direction being the only normal direction.

We now need to show that the solution (3.30) is compatible with the coordinate relations that are obtained from the set of equations (3.24). This is achieved by reperforming the steps that led from (3.24) to (3.30), but in a special coordinate system that makes the compatibility of the coordinate relations obtained with those obtained from (3.24) obvious. The argument is as follows :

Introduce a new coordinate system $\{\tilde{x}', x'\}$ such that the k functionally independent relations among the set (3.24) become

$$\tilde{x}'^{A} = x'^{A} \tag{A6}$$

where A runs from $n-k+1$ to n as before. What we have done here is simply to let the functionally independent components act as a new coordinate system, which we are at liberty to do because they are functionally independent. Differentiating we obtain

$$d\tilde{x}'^{A} = dx'^{A} \tag{A7}$$

where

$$d\tilde{x}'^{A} = \tilde{c}'^{A}_{|i}\tilde{\omega}^{i} \tag{A8.i}$$

$$dx'^{A} = c'^{A}_{|i}\omega^{i} \tag{A8.ii}$$

Suppose $\tilde{x}'^{A} = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_x}$ for example. Then

$$d\tilde{x}'^{A} = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_x i}\tilde{\omega}^{i} \tag{A9.i}$$

As $\tilde{x}'^{A} = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_x}$ then $x'^{A} = c^{i_0}_{k_0h_0|l_1\ldots l_x}$ so

$$dx'^{A} = c^{i_0}_{k_0h_0|l_1\ldots l_x i}\omega^{i} \tag{A9.ii}$$

Comparing (A8.i) and (A8.ii) with (A9.i) and (A9.ii) shows

$$\tilde{c}'^{A}_{|i} = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_x i} \tag{A10.i}$$

$$c'^{A}_{|i} = c^{i_0}_{k_0h_0|l_1\ldots l_x i} \tag{A10.ii}$$

But from equations (3.24) we know that

$$\tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_x i} = c^{i_0}_{k_0h_0|l_1\ldots l_x i}$$

so that

$$\tilde{c}'^A_{|i} = c'^A_{|i} \tag{A11}$$

Because the $\tilde{x}'^A$ are functionally independent, the $d\tilde{x}'^A$ are linearly independent. So equations (A8.i) represent k linearly independent equations in the n $\tilde{\omega}^i$. Thus, as before, we can express some k of the $\tilde{\omega}^i$ as linear combinations of the other $n-k$ (c.f. (3.26)). With a suitable numbering one obtains

$$\tilde{\omega}^A = b^A_\alpha \tilde{\omega}^\alpha + d^A_B d\tilde{x}'^B \tag{A12.i}$$

where A, B etc. run from $n-k+1$ to n, and α, β etc. run from 1 to $n-k$. Because of the equality (A11) the corresponding equation for ω^A will contain exactly the same expansion coefficients b^A_α and d^A_B. So we obtain

$$\omega^A = b^A_\alpha \omega^\alpha + d^A_B dx'^B \tag{A12.ii}$$

Now subtracting (A12.ii) from (A12.i) we obtain, using (A7)

$$\tilde{\omega}^A - \omega^A = b^A_\alpha(\tilde{\omega}^\alpha - \omega^\alpha) \tag{A13}$$

What we have performed so far is merely what was performed in §3 to obtain (3.26), except it has been performed in our new coordinate system. Continuing as in that section, we investigate the solution of the equation

$$\tilde{\omega}^\alpha - \omega^\alpha = 0 \tag{A14}$$

which, from (A13), will if satisfied give $\tilde{\omega}^i - \omega^i = 0$ for i running from 1 to n. In our new coordinate system $\{x', \tilde{x}'\}$ this equation becomes

$$\tilde{a}'^\alpha_\beta d\tilde{x}'^\beta + \tilde{a}'^\alpha_B d\tilde{x}'^B - a'^\alpha_\beta dx'^\beta - a'^\alpha_B dx'^B = 0 \tag{A15}$$

However, and this point is crucial, from (A12.i) we see that $\{\tilde{\omega}^\alpha, d\tilde{x}'^A\}$ span the cotangent space and hence represent n linearly independent 1-forms. Expanding the $\tilde{\omega}^\alpha$ in terms of the coordinates we have

$$\tilde{\omega}^\alpha = \tilde{a}'^\alpha_\beta d\tilde{x}'^\beta + \tilde{a}'^\alpha_B d\tilde{x}'^B \tag{A16.i}$$

We rewrite this equation as

$$\tilde{\omega}^\alpha - \tilde{a}'^\alpha_B d\tilde{x}'^B = \tilde{a}'^\alpha_\beta d\tilde{x}'^\beta \qquad (A16.ii)$$

Because $\{\tilde{\omega}^\alpha, d\tilde{x}'^A\}$ are n linearly independent 1-forms the left hand sides of (A16.ii) are linearly independent for different values of α, and so, therefore, are the right hand sides. This means that the $\tilde{a}'^\alpha_\beta$ form a non-singular matrix. If we call its inverse $(\tilde{a}'^{-1})^\alpha_\beta$, i.e.

$$(\tilde{a}'^{-1})^\alpha_\beta \tilde{a}'^\beta_\gamma = \delta^\alpha_\gamma \qquad (A17)$$

we obtain, by multiplying (A15) through by the inverse $(\tilde{a}'^{-1})^\gamma_\alpha$,

$$d\tilde{x}'^\gamma = -(\tilde{a}'^{-1})^\gamma_\alpha \tilde{a}'^\alpha_B d\tilde{x}'^B + dx'^\gamma + (\tilde{a}'^{-1})^\gamma_\alpha a'^\alpha_B dx'^B \qquad (A18)$$

The integrability conditions (A4) will again be satisfied by virtue of (A13), which enables us to keep reexpressing quantities in terms of $\tilde{\omega}^\alpha - \omega^\alpha$ so as to produce result (A4).Thus (A18) can be integrated to give

$$\tilde{x}'^\alpha = \tilde{x}'^\alpha(x'^\alpha, x'^A, \tilde{x}'^A) \qquad (A19)$$

Now, because we have been working in this special coordinate system in which equations (3.24) have the exceptionally simple form given in (A6), we can immediately see that the coordinate relations needed to make $\tilde{\omega}^\alpha - \omega^\alpha = 0$, given in this special coordinate system by (A19), are completely compatible with the coordinate relations which come from the set of equations (3.24), given in this special coordinate system by (A6).

Appendix B

§1. Proof of Equation (3.35)

We have the connection coefficients defined as

$$\Gamma^i{}_{jk} = <\omega^i, \nabla_k e_j> \qquad (B1)$$

Expanding this equation we obtain

$$\Gamma^i{}_{jk} = \omega^i_\sigma e^\mu_k e^\sigma_{j;\mu} = -\omega^i_{\sigma;\mu} e^\mu_k e^\sigma_j \qquad (B2)$$

The exterior derivative of the 1-form ω^i is given by

$$d_x\omega^i = \omega^i_{\mu,\nu} dx^\nu \wedge dx^\mu \qquad (B3)$$

For a symmetric connection (i.e. $\Gamma^\mu{}_{\nu\sigma} = \Gamma^\mu{}_{\sigma\nu}$) we can replace the partial derivative in (B3) by a covariant derivative because the additional term containing the symmetric $\Gamma^\sigma{}_{\mu\nu}$ so introduced will disappear when contracted with the antisymmetric wedge product. Thus (B3) can be rewritten as

$$d_x\omega^i = \omega^i_{\mu;\nu} dx^\nu \wedge dx^\mu \qquad (B4)$$

If we now multiply (B2) by $\omega^k_\rho \omega^j_\nu$, using the result that $e^\sigma_i \omega^i_\mu = \delta^\sigma_\mu$, then we obtain

$$\omega^i_{\nu;\rho} = -\Gamma^i{}_{jk}\omega^k_\rho \omega^j_\nu \qquad (B5)$$

Substituting this result in (B4) gives

$$d_x\omega^i = \Gamma^i{}_{jk}\omega^j \wedge \omega^k \qquad (B6)$$

Comparing (B6) with (3.33) then gives

$$\omega^i_{(x)j} = \Gamma^i{}_{jk}\omega^k \qquad (B7)$$

which is our desired result.

§**2.**

In this section we derive a number of formulae which will be needed in the calculation of $d\omega^{ij}$.

Let $\omega^i_{(f)}(x^\mu)$ be a fixed tetrad at x^μ. An arbitrary tetrad at x^μ with the same frame metric η_{ij} is obtained by rotating $\omega^i_{(f)}$, as discussed in §2 of chapter 3. Thus we have

$$\omega^i = b^i{}_j \omega^j_{(f)} \tag{B8a}$$

$$\omega^i_{(f)} = (b^{-1})^i{}_j \omega^j \tag{B8b}$$

where $b^i{}_j (b^{-1})^j{}_k = (b^{-1})^i{}_j b^j{}_k = \delta^i_k$. The change in ω^i when the tetrad is rotated by $\delta\epsilon^A$ is

$$\delta\omega^i = b^i{}_{j,A} \delta\epsilon^A \omega^j_{(f)} = (b^{-1})^j{}_k b^i{}_{j,A} \delta\epsilon^A \omega^k \tag{B9}$$

Since the transformations $b^i{}_j$ preserve η_{ij} we have

$$\begin{aligned} 0 = \delta g &= \eta_{ij}\ \delta\omega^i \otimes \omega^j + \eta_{ij}\ \omega^i \otimes \delta\omega^j \\ &= \delta\epsilon^A [(b^{-1})^l{}_k b_{jl,A} + (b^{-1})^l{}_j b_{kl,A}]\ \omega^k \otimes \omega^j \end{aligned}$$

and hence, since $\delta\epsilon^A$ is arbitrary,

$$(b^{-1})^l{}_k b_{jl,A} = -(b^{-1})^l{}_j b_{kl,A} \tag{B10}$$

As in §4 of chapter 3, we define $\omega^i_{(\epsilon)j} = a^i{}_{jA} d\epsilon^A$ as

$$d_{(\epsilon)}\omega^i = \omega^j \wedge \omega^i_{(\epsilon)j} \tag{B11}$$

From (B8) we have

$$d_{(\epsilon)}\omega^i = (b^{-1})^j{}_k b^i{}_{j,A}\ d\epsilon^A \wedge \omega^k \tag{B12}$$

Comparison of this equation with (B11) then gives

$$\omega^i_{(\epsilon)j} = -(b^{-1})^k{}_j b^i{}_{k,A} d\epsilon^A \tag{B13}$$

$$a^i{}_{jA} = -(b^{-1})^k{}_j b^i{}_{k,A} \tag{B14}$$

§3. Calculation of $d\omega^{ij}$ and dR_{ijkl}

Using the results in §4 of chapter 3 we obtain

$$
\begin{aligned}
d\omega^{ij} &= (d_{(x)} + d_{(\epsilon)})(\omega^{ij}_{(x)} + \omega^{ij}_{(\epsilon)}) \\
&= (d_{(x)} + d_{(\epsilon)})(\Gamma^{ij}{}_k\omega^k + \omega^{ij}_{(\epsilon)}) \\
&= \Gamma^{ij}{}_{k|l}\, \omega^l \wedge \omega^k + \Gamma^{ij}{}_k(\omega^m \wedge \Gamma^k{}_{ml}\omega^l) \\
&\quad + a^{ij}_{A|k}\, \omega^k \wedge d\epsilon^A + \Gamma^{ij}{}_{k,A}\, d\epsilon^A \wedge \omega^k \\
&\quad + \Gamma^{ij}{}_k\, \omega^m \wedge a^k{}_{mA} d\epsilon^A + a^{ij}_{A,B}\, d\epsilon^B \wedge d\epsilon^A \\
&= (\Gamma^{ij}{}_{k|l} + \Gamma^{ij}{}_m\Gamma^m{}_{lk})\, \omega^l \wedge \omega^k \\
&\quad + (\Gamma^{ij}{}_{k,A} - \Gamma^{ij}{}_l a^l{}_{kA} - a^{ij}_{A|k})\, d\epsilon^A \wedge \omega^k + a^{ij}_{A,B}\, d\epsilon^B \wedge d\epsilon^A \qquad (B15)
\end{aligned}
$$

and

$$
\begin{aligned}
\omega^{im} \wedge \omega_m{}^j &= (\Gamma^{im}{}_l\omega^l + a^{im}_A d\epsilon^A) \wedge (\Gamma_m{}^j{}_k\omega^k + a_m{}^j{}_B d\epsilon^B) \\
&= \Gamma^{im}{}_l\Gamma_m{}^j{}_k\, \omega^l \wedge \omega^k \\
&\quad + (\Gamma^{im}{}_l a_m{}^j{}_A - \Gamma_m{}^j{}_l a^{im}_A)\, \omega^l \wedge d\epsilon^A + a^{im}_A a_m{}^j{}_B\, d\epsilon^A \wedge d\epsilon^B \, (B16)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d\omega^{ij} + \omega^{im} \wedge \omega_m{}^j &= (\Gamma^{ij}{}_{k|l} + \Gamma^{ij}{}_m\Gamma^m{}_{lk} + \Gamma^{im}{}_l\Gamma_m{}^j{}_k)\, \omega^l \wedge \omega^k \\
&\quad + (\Gamma^{ij}{}_{l,A} - a^{ij}_{A|l} + a^{im}_A\Gamma_m{}^j{}_l + a^{jm}_A\Gamma^i{}_{ml} - a^k{}_{lA}\Gamma^{ij}{}_k)\, d\epsilon^A \wedge \omega^l \\
&\quad + (a^{ij}_{A,B} - a^{im}_A a_m{}^j{}_B)\, d\epsilon^B \wedge d\epsilon^A \qquad (B17)
\end{aligned}
$$

By using (B14) a fairly long but straightforward calculation shows that

$$
a^{ij}_{[A,B]} - a^i{}_{m[A} a^{mj}_{B]} = -(b^{-1})^{kj} b^i{}_{k,[AB]} = 0 \qquad (B18)
$$

Therefore, we see that because the wedge product is antisymmetric, the last term in (B17) vanishes. To handle the second term we calculate $\Gamma^{ij}{}_{l,A}$. From (B2) and (3.31), using the result that $\omega^i = \eta^{ij} e_j$, we have

$$
\Gamma^i{}_{jk} = -a^i_{\mu;\nu} a^\mu_j a^\nu_k \qquad (B19)
$$

This formula can be used to calculate the change in $\Gamma^{i}{}_{jk}$ under an infinitesimal rotation.

$$\begin{aligned}\delta\Gamma^{ijk} &= -(\delta(a^{i}_{\mu;\nu})a^{j\mu}a^{k\nu} + a^{i}_{\mu;\nu}\delta a^{j\mu}a^{k\nu} + a^{i}_{\mu;\nu}a^{j\mu}\delta a^{k\nu}) \\ &= -[(\delta a^{i}_{\mu})_{;\nu}a^{j\mu}a^{k\nu} + a^{i}_{\mu;\nu}\delta a^{j\mu}a^{k\nu} + a^{i}_{\mu;\nu}a^{j\mu}\delta a^{k\nu}] \qquad (B20)\end{aligned}$$

Let us calculate δa^{i}_{μ} from (B9). Writing (B9) in a coordinate basis using (3.31) gives

$$\delta\omega^{i} = (b^{-1})^{j}{}_{k}b^{i}{}_{j,A}\delta\epsilon^{A}a^{k}_{\mu}dx^{\mu} = \delta a^{i}_{\mu}dx^{\mu}$$

$\Rightarrow$

$$\delta a^{i}_{\mu} = (b^{-1})^{j}{}_{k}b^{i}{}_{j,A}\delta\epsilon^{A}a^{k}_{\mu} = -a^{i}{}_{kA}\delta\epsilon^{A}a^{k}_{\mu} \qquad (B21)$$

Multiplying (B19) by $a^{j}_{\sigma}a^{k}_{\rho}$ and using the result $e^{\sigma}_{i}\omega^{i}_{\mu} = \delta^{\alpha}_{\mu}$ gives

$$a^{i}_{\mu;\nu} = -\Gamma^{i}{}_{jk}a^{j}_{\mu}a^{k}_{\nu} \qquad (B22)$$

Using (B21) and (B22) in (B20) gives

$$\begin{aligned}\delta\Gamma^{ijk} &= -a^{j}{}_{lA}\Gamma^{ilk}\delta\epsilon^{A} - a^{k}{}_{lA}\Gamma^{ijl}\delta\epsilon^{A} + (a^{il}_{A}a_{l\mu}\delta\epsilon^{A})_{;\nu}a^{j\mu}a^{k\nu} \\ &= [(a^{il}_{A}a_{l\mu})_{;\nu}a^{j\mu}a^{k\nu} - a^{j}{}_{lA}\Gamma^{ilk} - a^{k}{}_{lA}\Gamma^{ijl}]\delta\epsilon^{A} \\ &= (a^{il}{}_{A,\nu}a_{l\mu}a^{j\mu}a^{k\nu} + a_{l\mu;\nu}a^{il}_{A}a^{j\mu}a^{k\nu} - a^{j}{}_{lA}\Gamma^{ilk} - a^{k}{}_{lA}\Gamma^{ijl})\delta\epsilon^{A} \\ &= (a^{ij}_{A,\nu}a^{k\nu} + a_{l\mu;\nu}a^{il}_{A}a^{j\mu}a^{k\nu} - a^{j}{}_{lA}\Gamma^{ilk} - a^{k}{}_{lA}\Gamma^{ijl})\delta\epsilon^{A} \qquad (B23)\end{aligned}$$

If we now use result (B22) to substitute for $a_{l\mu;\nu}$, we obtain after some simplification the final result

$$\delta\Gamma^{ijk} = (a^{ij}_{A,\nu}a^{k\nu} + a^{i}{}_{lA}\Gamma^{ljk} - a^{j}{}_{lA}\Gamma^{ilk} - a^{k}{}_{lA}\Gamma^{ijl})\delta\epsilon^{A} \qquad (B24)$$

so that

$$\Gamma^{ij}{}_{k,A} = a^{ij}_{A|k} - a^{i}{}_{lA}\Gamma^{lj}{}_{k} - a^{j}{}_{lA}\Gamma^{il}{}_{k} - a_{klA}\Gamma^{ijl} \qquad (B25)$$

where we have used $a^{ij}_{A|l} = a^{ij}_{A,\nu}a^{\nu}_{l}$. Therefore, we see that the second term in (B17) also vanishes. If we use the antisymmetry property of the wedge product, (B17) can now be written as

$$\begin{aligned} d\omega^{ij} + \omega^{im} \wedge \omega_m{}^j = (\Gamma^{ij}{}_{[k|l]} + \Gamma^{ij}{}_m \Gamma^m{}_{[lk]} \\ + \tfrac{1}{2}\Gamma^{im}{}_l \Gamma_m{}^j{}_k - \tfrac{1}{2}\Gamma^{im}{}_k \Gamma_m{}^j{}_l)\ \omega^l \wedge \omega^k \\ = \tfrac{1}{2} R^{ij}{}_{kl}\ \omega^k \wedge \omega^l \end{aligned} \qquad (B26)$$

Finally let us calculate the exterior derivative of the Riemann tensor in frame.

$$\begin{aligned} dR_{ijkl} &= R_{ijkl,\mu} dx^\mu + R_{ijkl,A} d\epsilon^A \\ &= R_{ijkl|m}\omega^m + R_{ijkl,A} d\epsilon^A \end{aligned} \qquad (B27)$$

So we see that we must calculate $R_{ijkl|m}$ and $R_{ijkl,A}$. We have

$$R_{ijkl|m} = R_{ijkl,\mu} a^\mu_m = R_{ijkl;\mu} a^\mu_m \qquad (B28)$$

where the last equality follows from the fact that R_{ijkl} is a scalar, so that its partial derivative is equal to its covariant derivative.

$$\begin{aligned} R_{ijkl;\mu} a^\mu_m &= (R_{\alpha\beta\gamma\delta} a^\alpha_i a^\beta_j a^\gamma_k a^\delta_l)_{;\mu} a^\mu_m \\ &= R_{ijkl;m} + R_{\alpha\beta\gamma\delta} a^\mu_m (a^\alpha_{i;\mu} a^\beta_j a^\gamma_k a^\delta_l \\ &\quad + a^\alpha_i a^\beta_{j;\mu} a^\gamma_k a^\delta_l + a^\alpha_i a^\beta_j a^\gamma_{k;\mu} a^\delta_l + a^\alpha_i a^\beta_j a^\gamma_k a^\delta_{l;\mu}) \\ &= R_{ijkl;m} + (a^\alpha_{i,\mu} + \Gamma^\alpha{}_{\sigma\mu} a^\sigma_i) a^\mu_m R_{\alpha jkl} + (a^\beta_{j,\mu} + \Gamma^\beta{}_{\sigma\mu} a^\sigma_j) a^\mu_m R_{i\beta kl} \\ &\quad + (a^\gamma_{k,\mu} + \Gamma^\gamma{}_{\sigma\mu} a^\sigma_k) a^\mu_m R_{ij\gamma l} + (a^\delta_{l,\mu} + \Gamma^\delta{}_{\sigma\mu} a^\sigma_l) a^\mu_m R_{ijk\delta} \end{aligned} \qquad (B29)$$

In order to handle the second term in each of the brackets, let us take another look at covariant differentiation.

$$\begin{aligned} \nabla_i(e_j) &= \Gamma^m{}_{ji} e_m \\ &= \nabla_i (a^\mu_j \tfrac{\partial}{\partial x^\mu}) \\ &= a^\mu_{j|i} \tfrac{\partial}{\partial x^\mu} + \Gamma^m{}_{\mu i} e_m a^\mu_j \\ &= a^\mu_{j|i} a^m_\mu e_m + \Gamma^m{}_{\mu i} a^\mu_j e_m \end{aligned}$$

$\Rightarrow$

$$\Gamma^m{}_{ji} = a^\mu_{j|i} a^m_\mu + \Gamma^m{}_{\mu i} a^\mu_j \tag{B30}$$

Applying this result to the second term in each bracket and simplifying then gives

$$\begin{aligned} R_{ijkl|m} &= R_{ijkl;m} + a^\alpha_{i|m} R_{\alpha jkl} + a^\beta_{j|m} R_{i\beta kl} + a^\gamma_{k|m} R_{ij\gamma l} \\ &\quad + a^\delta_{l|m} R_{ijk\delta} + (\Gamma^n{}_{im} - a^\mu_{i|m} a^n_\mu) R_{njkl} + (\Gamma^n{}_{jm} - a^\mu_{j|m} a^n_\mu) R_{inkl} \\ &\quad + (\Gamma^n{}_{km} - a^\mu_{k|m} a^n_\mu) R_{ijnl} + (\Gamma^n{}_{lm} - a^\mu_{l|m} a^n_\mu) R_{ijkn} \\ &= R_{ijkl;m} + \Gamma^n{}_{im} R_{njkl} + \Gamma^n{}_{jm} R_{inkl} + \Gamma^n{}_{km} R_{ijnl} + \Gamma^n{}_{lm} R_{ijkn} \end{aligned} \tag{B31}$$

If we now turn our attention to $R_{ijkl,A}$ we have under an infinitesimal rotation

$$\begin{aligned} \delta R_{ijkl} &= R_{\alpha\beta\gamma\delta} \delta a^\alpha_i a^\beta_j a^\gamma_k a^\delta_l + R_{\alpha\beta\gamma\delta} a^\alpha_i \delta a^\beta_j a^\gamma_k a^\delta_l \\ &\quad + R_{\alpha\beta\gamma\delta} a^\alpha_i a^\beta_j \delta a^\gamma_k a^\delta_l + R_{\alpha\beta\gamma\delta} a^\alpha_i a^\beta_j a^\gamma_k \delta a^\delta_l \end{aligned}$$

If we now use (B21) to substitute for δa^α_i, δa^β_j etc. we obtain

$$\begin{aligned} \delta R_{ijkl} &= -R_{\mu jkl} a_{inA} a^{n\mu} \delta\epsilon^A - R_{i\mu kl} a_{jnA} a^{n\mu} \delta\epsilon^A \\ &\quad - R_{ij\mu l} a_{knA} a^{n\mu} \delta\epsilon^A - R_{ijk\mu} a_{lnA} a^{n\mu} \delta\epsilon^A \\ &= (R_{njkl} a^n{}_{iA} + R_{inkl} a^n{}_{jA} + R_{ijnl} a^n{}_{kA} + R_{ijkn} a^n{}_{lA}) \delta\epsilon^A \end{aligned}$$

$\Rightarrow$

$$R_{ijkl,A} = R_{njkl} a^n{}_{iA} + R_{inkl} a^n{}_{jA} + R_{ijnl} a^n{}_{kA} + R_{ijkn} a^n{}_{lA} \tag{B32}$$

If we now substitute in (B27) using (B31) and (B32) we obtain, using (3.35) and (3.37b),

$$dR_{ijkl} = R_{ijkl;m} \omega^m + R_{mjkl} \omega^m{}_i + R_{imkl} \omega^m{}_j + R_{ijml} \omega^m{}_k + R_{ijkm} \omega^m{}_l \tag{B33}$$

Appendix C
Metric Files

```
(TITLE "C7.DIA
        GENERAL STATIONARY CYLINDRICALLY SYMMETRIC VACUUM FIELD.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.7)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       SO9 5NH

(OFF ALL) (ON ESUBS SUBPOT POTSIM NOZERO)

(PRELOAD DIAINP DYTRSP)

(VARS T F P Z) % t , phi , rho , z

(RPL IZUD)
 EXP(U) $  EXP(U)*A  $  0             $  0             $
 0      $  P*EXP(-U) $  0             $  0             $
 0      $  0         $  P^(N^2/4-1/4) $  0             $
 0      $  0         $  0             $  P^(N^2/4-1/4) $

(RPL U A C S Q R)
 (LOG(P*S))/2 $
 C*P^N/N/Y/S+B $
 I*N*(X*Y)^(1/2) $
 X*P^N+Y*P^(-N) $
 X^(1/2)*P^(N/2)-I*Y^(1/2)*P^(-N/2) $
 X^(1/2)*P^(N/2)+I*Y^(1/2)*P^(-N/2) $

(FUNS (U P) (A P) (X) (Y) (N) (C) (B) S Q R)

(NEWSUL 5 RIESUL (DIFF 2))
 U $  :U $
 A $  :A $
 C $  :C $
 Y $  P^N(S-X*P^N) $
 X $  S*P^(-N)-Y*P^(-2N) $

(USESUL RIESUL RIE)

(SETSUB ESUL)
 S*P^(-N)-Y*P^(-2N) $  X $
```

```
(NEWSUL 3 PSISUL)
 Q $  :Q $
 R $  :R $
 S $  :S $

(USESUL PSISUL PSI)

(REDSIMP PSI)

(RPL DYTRSP1)
 (Q/R)^(1/4) $  0           $
 0           $  (R/Q)^(1/4) $
```

```
(TITLE "C8.DIA
        STATIC CYLINDRICALLY SYMMETRIC VACUUM FIELD.
        LEVI-CIVITA (1917-19) SOLUTION.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.8)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P F Z) % t , rho , phi , z

(FUNS (M))

(RPL GD)
 P^M $  P^(M^2-M) $  P^(1-M) $  P^(M^2-M) $
```

```
(TITLE "C9a.DIA
        STATIC CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL FIELD.
        ANGULAR MAGNETIC FIELD (axial current).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.9a)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P F Z) % t , rho , phi , z

(FUNS (C) (D) (M))

(RPL G)
 C*P^M+D*P^(-M) $

(FUNS G)

(NEWSUL 2 RIESUL)
 1/G^2 $  :G^2/G^4 $
 1/G^3 $  :G/G^4 $

(USESUL RIESUL RIE)

(NEWSUL JC1SUL)
 1/G^4 $  :G/G^5 $

(USESUL JC1SUL JC1)

(NEWSUL PSIDSUL)
 1/G^5 $  :G/G^6 $

(USESUL PSIDSUL PSID PHID)

(NEWSUL DPSIDSUL)
 1/G^7 $  :G/G^8 $
```

```
(USESUL DPSIDSUL DPSID DPHID APSI XID APHI)

(RPL GD)
 P^(M^2)*G $  P^(M^2)*G $  P*G $  G^(-1) $
```

```
(TITLE "C9b.DIA
        STATIC CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL FIELD.
        LONGITUDINAL MAGNETIC FIELD (angular current).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.9b)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P Z F) % t , rho , z , phi

(FUNS (C) (D) (M))

(RPL G)
 C*P^M+D*P^(-M) $

(FUNS G)

(NEWSUL 2 RIESUL)
 1/G^2 $  :G^2/G^4 $
 1/G^3 $  :G/G^4 $

(USESUL RIESUL RIE)

(NEWSUL JC1SUL)
 1/G^4 $  :G/G^5 $

(USESUL JC1SUL JC1)

(NEWSUL PSIDSUL)
 1/G^5 $  :G/G^6 $

(USESUL PSIDSUL PSID PHID)

(NEWSUL DPSIDSUL)
 1/G^7 $  :G/G^8 $
```

```
(USESUL DPSIDSUL DPSID DPHID APSI XID APHI)

(RPL GD)
 P^(M^2)*G $  P^(M^2)*G $  P*G $  G^(-1) $
```

```
(TITLE "C9c.DIA
        STATIC CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL FIELD.
        RADIAL ELECTRIC FIELD (axial charge distribution).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.9c)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P F Z)  % t , rho , phi , z

(FUNS (C) (D) (M))

(RPL G)
 C*P^M+D*P^(-M) $

(FUNS G)

(NEWSUL 2 RIESUL)
 1/G^2 $  :G^2/G^4 $
 1/G^3 $  :G/G^4 $

(USESUL RIESUL RIE)

(NEWSUL JC1SUL)
 1/G^4 $  :G/G^5 $

(USESUL JC1SUL JC1)

(NEWSUL PSIDSUL)
 1/G^5 $  :G/G^6 $

(USESUL PSIDSUL PSID PHID)

(RPL GD)
 G^(-1) $  P^(M^2)*G $  P*G $  P^(M^2)*G $
```

```
(TITLE "C10.DIA
        STATIC CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL FIELD.
        LONGITUDINAL MAGNETIC FIELD (angular current).
        THE MELVIN SOLUTION (MELVIN (1964)).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
             Exact Solutions of Einstein's Field Equations.
             Equation (20.10)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS T P Z F) % t , rho , z , phi

(FUNS (B))

(RPL G)
 B^2/4*P+1/P $

(FUNS G)

(NEWSUL 2 RIESUL)
 1/G^2 $  :G^2/G^4 $
 1/G^3 $  :G/G^4 $

(USESUL RIESUL RIE)

(NEWSUL JC1SUL)
 1/G^4 $  :G/G^5 $

(USESUL JC1SUL JC1)

(NEWSUL PSIDSUL)
 1/G^5 $  :G/G^6 $

(USESUL PSIDSUL PSID PHID)

(NEWSUL DPSIDSUL)
 1/G^7 $  :G/G^8 $
```

```
(USESUL DPSIDSUL DPSID DPHID APSI XID APHI)

(RPL GD)
 P*G $  P*G $  P*G $  G^(-1) $

(RPL DYTR1)
 -1   $  1    $
 -1/2 $  -1/2 $
```

```
(TITLE "C11.DIA
        STATIC CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL FIELD.
        THE CHITRE SOLUTION (1975).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.11)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS T P F Z) % t , rhp , phi , z

(RPL IZUD)
 C*P^(-2/9)*EXP(1/2*A^2*P^(2/3)) $  0 $  0 $  0 $
 0 $  C*P^(-2/9)*EXP(1/2*A^2*P^(2/3)) $  0 $  0 $
 0 $  0 $  P^(2/3) $  0 $
 0 $  0 $  A*P $  P^(1/3) $

(RPL X)
 1/54C^(-2)*E^(-A^2*P^(2/3))*P^(-14/9)*(-4P^(1/3)*A*I+6*A^3*P*I-2-3A^2*P^(2/3)) $

(RPL Y)
 1/54C^(-2)*E^(-A^2*P^(2/3))*P^(-14/9)*(4P^(1/3)*A*I-6*A^3*P*I-2-3A^2*P^(2/3))$

(FUNS (A) (C) X Y)

(NEWSUL 2 UNPSISUL)
 E^(-A^2*P^(2/3))*C^(-2)*(-1/18*A^2*P^(-8/9)-1/27P^(-14/9)-1/9*A^3*P^(-5/9)*I
 +2/27*A*P^(-11/9)*I) $
 Y $
 E^(-A^2*P^(2/3))*C^(-2)*(-1/18*A^2*P^(-8/9)-1/27P^(-14/9)+1/9*A^3*P^(-5/9)*I
 -2/27*A*P^(-11/9)*I) $
 X $

(USESUL UNPSISUL UNPSI)
```

```
(NEWSUL 2 JC1SUL)
 SQRT X $  :X/SQRT X $
 SQRT Y $  :Y/SQRT Y $

(USESUL JC1SUL JC1)

(RPL DYTRSP1)
 (X/Y)^(1/8) $  0           $
 0           $  (Y/X)^(1/8) $
```

```
(TITLE "C12.DIA
        STATIONARY CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL FIELD.
        THE WILSON SOLUTION (1968).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.12)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS W X Y Z) % x1 , x2 , x3 , x4

(RPL IZUD)
 0 $  0         $  -SQRT(3)*LOG(W) $  1 $
 0 $  W^(-1/2) $  0               $  0 $
 0 $  0         $  1               $  0 $
 1 $  0         $  0               $  0 $

(RPL M L N)
 (((6-SQRT 3)/(6+SQRT 3))^(1/8))/SQRT 2 $
 6+SQRT 3 $
 6-SQRT 3 $

(FUNS (L)(M)(N))

(NEWSUL 3 PSISUL)
 M $  :M $
 6+SQRT 3 $  L $
 SQRT 3 $  6-N $

(USESUL PSISUL PSI)

(SETNSUB 3 ESUL)
 1/8W^(-2)*N^(3/2)*L^(-1/2)-3/2W^(-2)*N^(1/2)*L^(-1/2) $
 -1/8W^(-2)*N^(1/2)*L^(1/2) $
 L $  :L $
 N $  :N $
```

```
(RPL DYTRSP1)
 1    $  1   $
 -1/2 $  1/2 $

(RPL DYTRSP2)
 M $  0   $
 0 $  1/M $
```

```
(TITLE "C13.DIA
        STATIONARY CYLINDRICALLY SYMMETRIC DUST SOLUTION.
        van STOCKUM class of dust solutions specialised to cylindrically
        symmetry.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.13)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO  POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS T P F Z) % t , rho , phi , z

(RPL IZUD)
1 $  0               $  A*P^2 $  0               $
0 $  0               $  P     $  0               $
0 $  0               $  0     $  E^(-1/2A^2*P^2) $
0 $  E^(-1/2A^2*P^2) $  0     $  0               $

(RPL X Y)
 1 -2*A*P $
 1 +2*A*P $

(FUNS (A) X Y)

(NEWSUL 2 UNPSISUL)
 A^3*P*E^(A^2*P^2) $  A^2*E^(A^2*P^2)*(1-X)/2 $
 -A^3*P*E^(A^2*P^2) $  A^2*E^(A^2*P^2)*(1-Y)/2 $

(USESUL UNPSISUL UNPSI)

(NEWSUL 2 JC1SUL)
 SQRT X $  :X/SQRT X $
 SQRT Y $  :Y/SQRT Y $

(USESUL JC1SUL JC1)

(RPL DYTRSP1)
 (X/Y)^(1/8) $ 0           $
 0           $ (Y/X)^(1/8) $
```

```
(TITLE "C14.DIA
        GENERAL STATIC CYLINDRICALLY SYMMETRIC PERFECT FLUID SOLUTION.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.14)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P F Z) % t , rho , phi , z

(FUNS (B P) (G P) (D P))

(RPL GD)
 EXP(D) $  1 $   EXP(B) $  EXP(G) $

(NEWSUL 2 RIESUL)
 G&P&P $  B&P&P+(B&P)^2+B&P*D&P-(G&P)^2-G&P*D&P $
 D&P&P $  -(D&P)^2+B&P*G&P+G&P*D&P-(B&P)^2-B&P&P $

(USESUL RIESUL RIE PSID XI)
```

```
(TITLE "C14(SP).DIA
        STATIC CYINDRICALLY SYMMETRIC PERFECT FLUID SOLUTION.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.14).(special case where exp(eta)=rho)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P F Z) % t , rho , phi , z

(FUNS (B P) (G P) (D P))

(RPL GD)
 EXP(D) $  1 $  EXP(B) $  EXP(G) $

(NEWSUL 4 RIESUL (DIFF 1))
 D&P&P $  -3(D&P)^2+(D&P)/P $
 B&P $  1/P+G&P $
 G&P&P $  6(G&P)^2+(G&P)/P $
 D&P $  -2G&P $

(USESUL RIESUL RIE PSID XI PHID DLAMBDA)

(NEWSUL JC1SUL)
 G&P&P $  6(G&P)^2+(G&P)/P $

(USESUL JC1SUL JC1)
```

```
(TITLE "C18.DIA
        STATIONARY CYLINDRICALLY SYMMETRIC PERFECT FLUID WITH RIGID ROTATION.
        THE KRASINSKI SOLUTION (1978).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.18)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       SO9 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS T X Y Z) % t , x ,y , z

(RPL IZUD)
 1/H $  0              $  X/H                $  0                       $
 0   $  0              $  0                  $  SQRT(B)/SQRT(P)*H^(3/2) $
 0   $  0              $  SQRT(F)/SQRT(B)/H $  0                       $
 0   $  (F*P*H)^(-1/2) $  0                  $  0                       $

(FUNS (A) (B) (F X) (P X) (H X) (U X) (L X) (M X))

(RPL L M)
 -1/16*B*H*P+1/16*SQRT B*SQRT F*H*P&X-3/16SQRT B*SQRT F*P*H&X+1/32*P*F&X&X
 +1/32H*F&X*P&X-3/32P*F&X*H&X $
 -B*H*P-SQRT B*SQRT F*H*P&X+3SQRT B*SQRT F*P*H&X+1/2H*P*F&X&X+1/2H*F&X*P&X-
 3/2P*F&X*H&X $

(NEWSUL 2 UNPSISUL)
 -1/16*B*H*P+1/16*SQRT B*SQRT F*H*P&X-3/16SQRT B*SQRT F*P*H&X+1/32H*P*F&X&X
 +1/32H*F&X*P&X-3/32P*F&X*H&X $
 L $
 -B*H*P-SQRT B*SQRT F*H*P&X+3SQRT B*SQRT F*P*H&X+1/2H*P*F&X&X+1/2H*F&X*P&X-
 3/2P*F&X*H&X $
 M $

(USESUL UNPSISUL UNPSI)
```

```
(NEWSUL 3 UNPHISUL (DIFF 2))
 P&X $  P*(5H&X/H-F&X/F+B*X/F) $
 H $  U^(1/3) $
 U&X&X $   (U&X(F&X-B*X)+3/4U(F&X&X-F&X^2/F+B*X*F&X/F-B))/F $

(USESUL UNPHISUL UNPHI)

(RPL DYTR1)
 1    $  1   $
 -1/2 $  1/2 $

(RPL DYTRSP1)
 (L/M)^(1/8) $  0            $
 0           $  (L/M)^(-1/8) $

(RPL DYTRPHI1)
 2^(-1/2) $  0      $
 0        $  SQRT 2 $
```

```
(TITLE "C25.DIA
        CYLINDRICALLY SYMMETRIC VACUUM FIELD.
        EINSTEIN-ROSEN WAVES.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.25)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS T P F Z) % t , rho , phi , z

(RPL GD)
 E^(K-U) $  E^(K-U) $  E^(-U)*P $  E^(U) $

(FUNS (U P T)(K P T)(L P T)(M P T))

(RPL L M)
 E^(-2K+2U)*(P*U&T^3-3P*U&T^2*U&P+3P*U&T*U&P^2-P*U&P^3+3/2U&T^2-3U&T*U&P
 +3/2U&P^2-U&T&P+U&P&P+1/2/P*U&P $
 E^(-2K+2U)*(-P*U&T^3-3P*U&T^2*U&P-3P*U&T*U&P^2-P*U&P^3+3/2U&T^2+3U&T*U&P
 +3/2U&P^2+U&T&P+U&P&P+1/2/P*U&P $

(NEWSUL 2 UNPSISUL)
 E^(-2K+2U)*(P*U&T^3-3P*U&T^2*U&P+3P*U&T*U&P^2-P*U&P^3+3/2U&T^2-3U&T*U&P
 +3/2U&P^2-U&T&P+U&P&P+1/2/P*U&P $
 L $
 E^(-2K+2U)*(-P*U&T^3-3P*U&T^2*U&P-3P*U&T*U&P^2-P*U&P^3+3/2U&T^2+3U&T*U&P
 +3/2U&P^2+U&T&P+U&P&P+1/2/P*U&P $
 M $

(USESUL UNPSISUL UNPSI)

(NEWSUL 3 RIESUL (DIFF 1))
 K&P $  P*(U&P^2+U&T^2) $
 K&T $  2P*U&P*U&T $
 U&T&T $  (P*U&P)&P/P $
```

```
(USESUL RIESUL RIE)

(RPL DYTRSP1)
 (L/M)^(1/8) $  0            $
 0           $  (L/M)^(-1/8) $
```

```
(TITLE "C39.DIA
        CYLINDRICALLY SYMMETRIC PURE RADIATION FIELD.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.39)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       SO9 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP)

(VARS T P F Z) % t , rho , phi , z

(FUNS (U P T) (K P T))

(RPL GD)
 E^(K-U) $  E^(K-U) $  P*E^(-U) $  E^(U) $

(NEWSUL 2 RIESUL (DIFF 2))
 K&P $  P*(U&P+U&T)^2 - K&T $
 U&P&P $  U&T&T-U&P/P $
```

```
(USESUL RIESUL RIE PHI PHISTD LAMBD)

(NEWSUL 3 UNPSISUL)
 -P*(E**(-2*K+2*U))*(DF(U,T))**3-3*P
*(E**(-2*K+2*U))*(DF(U,T))**2*DF(U,P)-3*P*(E**
(-2*K+2*U))*DF(U,T)*(DF(U,P))**2-P*(E**(-2*K+
2*U))*(DF(U,P))**3+3/2*(E**(-2*K+2*U))*(DF(U,
T))**2+3*(E**(-2*K+2*U))*DF(U,T)*DF(U,P)+3/2*
(E**(-2*K+2*U))*(DF(U,P))**2+(E**(-2*K+2*U))*
DF(U,T,2)+(E**(-2*K+2*U))*DF(U,T,P)-1/2*P**(-
1)*(E**(-2*K+2*U))*DF(U,P) $
 L $

 1/3*P*(E**(-2*K+2*U))*DF(U,T)*DF(U,
T,2)+1/3*P*(E**(-2*K+2*U))*DF(U,T)*DF(U,T,P)+
1/3*P*(E**(-2*K+2*U))*DF(U,P)*DF(U,T,2)+1/3*P
*(E**(-2*K+2*U))*DF(U,P)*DF(U,T,P)-1/6*(E**(-
2*K+2*U))*DF(K,T,2)-1/6*(E**(-2*K+2*U))*DF(K,
T,P)+1/2*(E**(-2*K+2*U))*(DF(U,T))**2-1/2*(E**
(-2*K+2*U))*(DF(U,P))**2+1/2*P**(-1)*(E**(-2*
K+2*U))*DF(U,P) $
 M $

 P*(E**(-2*K+2*U))*(DF(U,T))**3+P*(E
**(-2*K+2*U))*(DF(U,T))**2*DF(U,P)-P*(E**(-2*
K+2*U))*DF(U,T)*(DF(U,P))**2-P*(E**(-2*K+2*U)
)*(DF(U,P))**3-2*(E**(-2*K+2*U))*DF(K,T)*DF(U
,T)+2*(E**(-2*K+2*U))*DF(K,T)*DF(U,P)+3/2*(E**
(-2*K+2*U))*(DF(U,T))**2-(E**(-2*K+2*U))*DF(U
,T)*DF(U,P)+3/2*(E**(-2*K+2*U))*(DF(U,P))**2+
(E**(-2*K+2*U))*DF(U,T,2)-(E**(-2*K+2*U))*DF(
U,T,P)-P**(-1)*(E**(-2*K+2*U))*DF(K,T)-1/2*P**
(-1)*(E**(-2*K+2*U))*DF(U,P) $
 N $

(USESUL UNPSISUL UNPSI)

(FUNS (L P T) (M P T) (N P T))

(LOAD DYTRSP)

(RPL DYTRSP1)
 (N/L)^(1/8) $  0           $
 0           $  (L/N)^(1/8) $
```

```
(TITLE "C41.DIA
        CYLINDRICALLY SYMMETRIC PURE RADIATION FIELD.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (20.41)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS T P F Z) % t , rho , phi , z

(FUNS (K P T))

(RPL GD)
 E^(K) $  E^(K) $  P $  1 $

(NEWSUL 4 RIESUL)
 K&P $  -K&T $
 K&P&T $  -K&T&T $
 K&P&P $  K&T&T $
 K&P&T&T $ -K&T&T&T $

(USESUL RIESUL RIE PSID JC1 JC2 PHID D2PSI APSI DXI D2PHI APHI)

(RPL DYTRSP1)
 (-1/P*E^(-2K)*K&T)^(1/4) $  0                         $
 0                         $  (-1/P*E^(-2K)*K&T)^(-1/4) $
```

```
(TITLE "GH2.DIA
        CYLINDRICALLY SYMMETRIC RADIATIVE VACUUM SOLUTION.
        Ref: J.Holmes PhD thesis (1986),p.79,case(ii)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(RPL IZUD)
 EXP(G-S) $  0        $  0        $  0      $
 0        $  EXP(G-S) $  0        $  0      $
 0        $  0        $  P*EXP(-S) $  0      $
 0        $  0        $  X*EXP(S)  $  EXP(S) $

(VARS T P F Z) % t , rho , phi , z

(FUNS (G P T)(S P T)(X P T)(W P T)(H P T)(A)(B))

(NEWSUL 7 RICSUL (DIFF 2))
 S $  (LOG(A*P*COSH(H)))/2 $
 X $  (SINH(H)/COSH(H))/A+B $
 G $  (LOG(COSH(H)))/2+(LOG(P))/4+W/4 $
 W&P $  (H&P^2+H&T^2)*P $
 W&T $ (2H&P*H&T)*P $
 H&P&P $  H&T&T-(H&P)/P $
 SINH(H) $  (COSH(H)^2-1)^(1/2) $

(USESUL RICSUL RIC)
```

```
(NEWSUL 3 UNPSISUL)
 -(E**(-2*G+2*S))*DF(G,T)*DF(S,T)-(E
**(-2*G+2*S))*DF(G,T)*DF(S,P)-(E**(-2*G+2*S))
*DF(G,P)*DF(S,T)-(E**(-2*G+2*S))*DF(G,P)*DF(S
,P)+(E**(-2*G+2*S))*(DF(S,T))**2+2*(E**(-2*G+
2*S))*DF(S,T)*DF(S,P)+(E**(-2*G+2*S))*(DF(S,P
))**2+1/2*(E**(-2*G+2*S))*DF(S,T,2)+(E**(-2*G
+2*S))*DF(S,T,P)+1/2*(E**(-2*G+2*S))*DF(S,P,2
)+1/2*P**(-1)*(E**(-2*G+2*S))*DF(G,T)+1/2*P**
(-1)*(E**(-2*G+2*S))*DF(G,P)-1/4*P**(-2)*(E**
(-2*G+6*S))*(DF(X,T))**2-1/2*P**(-2)*(E**(-2*
G+6*S))*DF(X,T)*DF(X,P)-1/4*P**(-2)*(E**(-2*G
+6*S))*(DF(X,P))**2-1/2*I*P**(-1)*(E**(-2*G+4
*S))*DF(G,T)*DF(X,T)-1/2*I*P**(-1)*(E**(-2*G+
4*S))*DF(G,T)*DF(X,P)-1/2*I*P**(-1)*(E**(-2*G
+4*S))*DF(G,P)*DF(X,T)-1/2*I*P**(-1)*(E**(-2*
G+4*S))*DF(G,P)*DF(X,P)+3/2*I*P**(-1)*(E**(-2
*G+4*S))*DF(S,T)*DF(X,T)+3/2*I*P**(-1)*(E**(-
2*G+4*S))*DF(S,T)*DF(X,P)+3/2*I*P**(-1)*(E**(
-2*G+4*S))*DF(S,P)*DF(X,T)+3/2*I*P**(-1)*(E**
(-2*G+4*S))*DF(S,P)*DF(X,P)+1/4*I*P**(-1)*(E**
(-2*G+4*S))*DF(X,T,2)+1/2*I*P**(-1)*(E**(-2*G
+4*S))*DF(X,T,P)+1/4*I*P**(-1)*(E**(-2*G+4*S)
)*DF(X,P,2)-1/4*I*P**(-2)*(E**(-2*G+4*S))*DF(
X,T)-1/4*I*P**(-2)*(E**(-2*G+4*S))*DF(X,P) $
 L $

1/2*(E**(-2*G+2*S))*(DF(S,T))**2-1/
2*(E**(-2*G+2*S))*(DF(S,P))**2+1/2*P**(-1)*(E
**(-2*G+2*S))*DF(S,P)+1/8*P**(-2)*(E**(-2*G+6
*S))*(DF(X,T))**2-1/8*P**(-2)*(E**(-2*G+6*S))
*(DF(X,P))**2+1/2*I*P**(-1)*(E**(-2*G+4*S))*DF
(S,T)*DF(X,P)-1/2*I*P**(-1)*(E**(-2*G+4*S))*DF
(S,P)*DF(X,T)+1/4*I*P**(-2)*(E**(-2*G+4*S))*DF
(X,T) $
 M $
```

```
 -(E**(-2*G+2*S))*DF(G,T)*DF(S,T)+(E
**(-2*G+2*S))*DF(G,T)*DF(S,P)+(E**(-2*G+2*S))
*DF(G,P)*DF(S,T)-(E**(-2*G+2*S))*DF(G,P)*DF(S
,P)+(E**(-2*G+2*S))*(DF(S,T))**2-2*(E**(-2*G+
2*S))*DF(S,T)*DF(S,P)+(E**(-2*G+2*S))*(DF(S,P
))**2+1/2*(E**(-2*G+2*S))*DF(S,T,2)-(E**(-2*G
+2*S))*DF(S,T,P)+1/2*(E**(-2*G+2*S))*DF(S,P,2
)-1/2*P**(-1)*(E**(-2*G+2*S))*DF(G,T)+1/2*P**
(-1)*(E**(-2*G+2*S))*DF(G,P)-1/4*P**(-2)*(E**
(-2*G+6*S))*(DF(X,T))**2+1/2*P**(-2)*(E**(-2*
G+6*S))*DF(X,T)*DF(X,P)-1/4*P**(-2)*(E**(-2*G
+6*S))*(DF(X,P))**2+1/2*I*P**(-1)*(E**(-2*G+4
*S))*DF(G,T)*DF(X,T)-1/2*I*P**(-1)*(E**(-2*G+
4*S))*DF(G,T)*DF(X,P)-1/2*I*P**(-1)*(E**(-2*G
+4*S))*DF(G,P)*DF(X,T)+1/2*I*P**(-1)*(E**(-2*
G+4*S))*DF(G,P)*DF(X,P)-3/2*I*P**(-1)*(E**(-2
*G+4*S))*DF(S,T)*DF(X,T)+3/2*I*P**(-1)*(E**(-
2*G+4*S))*DF(S,T)*DF(X,P)+3/2*I*P**(-1)*(E**(
-2*G+4*S))*DF(S,P)*DF(X,T)-3/2*I*P**(-1)*(E**
(-2*G+4*S))*DF(S,P)*DF(X,P)-1/4*I*P**(-1)*(E**
(-2*G+4*S))*DF(X,T,2)+1/2*I*P**(-1)*(E**(-2*G
+4*S))*DF(X,T,P)-1/4*I*P**(-1)*(E**(-2*G+4*S)
)*DF(X,P,2)-1/4*I*P**(-2)*(E**(-2*G+4*S))*DF(
X,T)+1/4*I*P**(-2)*(E**(-2*G+4*S))*DF(X,P) $
 N $

(USESUL UNPSISUL UNPSI)

(FUNS (L P T) (M P T) (N P T))

(LOAD DYTRSP)

(RPL DYTRSP1)

 (N/L)^(1/8) $  0           $
 0           $  (L/N)^(1/8) $
```

```
(TITLE "SA2.DIA
        GENERAL STATIONARY AXISYMMETRIC STATIC VACUUM SOLUTION.
        WEYL'S CLASS.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (18.2)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS P Z T F) % rho , z , t , phi

(RPL IZUD)
 0        $  0        $  EXP(U) $  0          $
 EXP(K-U) $  0        $  0      $  0          $
 0        $  EXP(K-U) $  0      $  0          $
 0        $  0        $  0      $  P*EXP(-U)  $

(FUNS (U P Z) (K P Z))

(NEWSUL 3 RIESUL (DIFF 2))
 K&P   $  P*(U&P^2 - U&Z^2) $
 K&Z   $  2P*U&P*U&Z        $
 U&Z&Z $  -U&P&P - 1/P*U&P  $

(USESUL RIESUL RIE)

(NEWSUL 2 UNPSISUL)
 4*P*(E**(-2*K+2*U))*(DF(U,P))**3-12
*P*(E**(-2*K+2*U))*DF(U,P)*(DF(U,Z))**2-6*(E**
(-2*K+2*U))*(DF(U,P))**2+6*(E**(-2*K+2*U))*(DF
(U,Z))**2-4*(E**(-2*K+2*U))*DF(U,P,2)-2*P**(-
1)*(E**(-2*K+2*U))*DF(U,P)-12*I*P*(E**(-2*K+2
*U))*(DF(U,P))**2*DF(U,Z)+4*I*P*(E**(-2*K+2*U
))*(DF(U,Z))**3+12*I*(E**(-2*K+2*U))*DF(U,P)*
DF(U,Z)+4*I*(E**(-2*K+2*U))*DF(U,P,Z) $
 X $
```

```
 1/4*P*(E**(-2*K+2*U))*(DF(U,P))**3-
3/4*P*(E**(-2*K+2*U))*DF(U,P)*(DF(U,Z))**2-3/
8*(E**(-2*K+2*U))*(DF(U,P))**2+3/8*(E**(-2*K+
2*U))*(DF(U,Z))**2-1/4*(E**(-2*K+2*U))*DF(U,P
,2)-1/8*P**(-1)*(E**(-2*K+2*U))*DF(U,P)+3/4*I
*P*(E**(-2*K+2*U))*(DF(U,P))**2*DF(U,Z)-1/4*I
*P*(E**(-2*K+2*U))*(DF(U,Z))**3-3/4*I*(E**(-2
*K+2*U))*DF(U,P)*DF(U,Z)-1/4*I*(E**(-2*K+2*U)
)*DF(U,P,Z) $
 Y $

(USESUL UNPSISUL UNPSI)

(FUNS (X P Z) (Y P Z))

(RPL DYTR1)
 I    $  1      $
 -1/2 $  -1/2*I $

(RPL DYTRSP1)
 (Y/X)^(1/8) $  0           $
 0           $  (X/Y)^(1/8) $
```

```
(TITLE "SA4.DIA
        STATIONARY AXISYMMETRIC STATIC VACUUM SOLUTION.
        WEYL'S CLASS.
        CHAZY(1924), CURZON(1924).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (18.4)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(ON NOZERO POTSIM ESUBS)

(PRELOAD DIAINP DYTRSP)

(VARS R H T F) % r , theta , t , phi

(RPL IZUD)
 0        $  0          $  EXP(U) $  0                  $
 EXP(K-U) $  0          $  0      $  0                  $
 0        $  R*EXP(K-U) $  0      $  0                  $
 0        $  0          $  0      $  R*(SIN H)*EXP(-U) $

(RPL U K)
 -M/R $
 -1/2M^2*(SIN H)^2/R^2 $

(FUNS (M) (U R) (K R H))

(SETNSUB 2 ESUL)
 U $  :U $
 K $  :K $

(NEWSUL RIESUL)
 (SIN H)^2 $  1 - (COS H)^2 $

(USESUL RIESUL RIE)

(RPL DYTR1)
 I    $  1      $
 -1/2 $  -1/2*I $
```

```
(RPL X Y)
 -4*M**3*R**(-5)*(E**(-M**2*R**(-2
)*(COS(H))**2+M**2*R**(-2)-2*M*R**(-1)))*(COS
(H))**2+4*M**3*R**(-5)*(E**(-M**2*R**(-2)*(COS
(H))**2+M**2*R**(-2)-2*M*R**(-1)))-6*M**2*R**
(-4)*(E**(-M**2*R**(-2)*(COS(H))**2+M**2*R**(
-2)-2*M*R**(-1)))+6*M*R**(-3)*(E**(-M**2*R**(
-2)*(COS(H))**2+M**2*R**(-2)-2*M*R**(-1)))+4*
I*M**3*R**(-5)*(E**(-M**2*R**(-2)*(COS(H))**2
+M**2*R**(-2)-2*M*R**(-1)))*COS(H)*SIN(H) $

 -1/4*M**3*R**(-5)*(E**(-M**2*R**(
-2)*(COS(H))**2+M**2*R**(-2)-2*M*R**(-1)))*(COS
(H))**2+1/4*M**3*R**(-5)*(E**(-M**2*R**(-2)*(
COS(H))**2+M**2*R**(-2)-2*M*R**(-1)))-3/8*M**
2*R**(-4)*(E**(-M**2*R**(-2)*(COS(H))**2+M**2
*R**(-2)-2*M*R**(-1)))+3/8*M*R**(-3)*(E**(-M**
2*R**(-2)*(COS(H))**2+M**2*R**(-2)-2*M*R**(-1
)))-1/4*I*M**3*R**(-5)*(E**(-M**2*R**(-2)*(COS
(H))**2+M**2*R**(-2)-2*M*R**(-1)))*COS(H)*SIN
(H) $

(FUNS (X R H) (Y R H))

(NEWSUL 2 UNPSISUL)
 :X $  X $
 :Y $  Y $

(USESUL UNPSISUL UNPSI)

(RPL DYTRSP1)
 (Y/X)^(1/8) $  0           $
 0           $  (X/Y)^(1/8) $
```

```
(TITLE "SA8.DIA.
        STATIONARY AXISYMMETRIC STATIC VACUUM SOLUTION.
        WEYL'S CLASS.
        THE SCHWARZSCHILD SOLUTION IN PROLATE SPHEROIDAL COORDINATES.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (18.8)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       S09 5NH

(OFF ALL) (ON NOZERO POTSIM ESUBS)

(PRELOAD DIAINP)

(VARS T X H F) % t, x, theta (using y = cos(H)), phi

(RPL GD) % D = delta
 (A/B)^(D/2) $
 K*(B/A)^(D/2)*(C*G)^(1/2)*(A*B/C/G)^(D^2/2)*(A*B)^(-1/2) $
 K*(B/A)^(D/2)*(C*G)^(1/2)*(A*B/C/G)^(D^2/2) $
 K*(B/A)^(D/2)*(A*B)^(1/2)*SIN(H) $

(RPL A B C G)
 X-1 $
 X+1 $
 X-COS(H) $
 X+COS(H) $

(FUNS (K) (M) (D) A B C G)

(SETNSUB 2 ESUL)
 D $  1 $
 K $  M $
```

```
(TITLE "SA9.DIA.
        STATIONARY AXISYMMETRIC STATIC VACUUM SOLUTION.
        WEYL'S CLASS.
        THE DARMOIS SOLUTION IN PROLATE SPHEROIDAL COORDINATES.
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (18.9)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       SO9 5NH

(OFF ALL) (ON NOZERO POTSIM ESUBS)

(PRELOAD DIAINP DYTRSP)

(VARS T X H F) % t, x, theta (using y = cos(H)), phi

(RPL GD) % D = delta
 (A/B)^(D/2) $
 K*(B/A)^(D/2)*(C*G)^(1/2)*(A*B/C/G)^(D^2/2)*(A*B)^(-1/2) $
 K*(B/A)^(D/2)*(C*G)^(1/2)*(A*B/C/G)^(D^2/2) $
 K*(B/A)^(D/2)*(A*B)^(1/2)*SIN(H) $

(RPL A B C G)
 X-1 $
 X+1 $
 X-COS(H) $
 X+COS(H) $

(FUNS (K) (M) (D) A B C G)

(SETNSUB 2 ESUL)
 D $  2 $
 K $  M/2 $

(NEWSUL 9 RIESUL)
 1/A $  (:A)^2/A^3 $
 1/A^2 $  :A/A^3 $
 1/A^(3/2) $  :A/A^(5/2) $
```

```
 1/B^5 $  (:B)^2/B^7 $
 1/B^6 $  :B/B^7 $
 1/B^(11/2) $  :B/B^(13/2) $
 C $  :C $
 G $  :G $
 COS(H) $  (1-(SIN(H))^2)^(1/2) $

(USESUL RIESUL RIE)

(NEWSUL 2 B4CRITSUL)
 1/A^5 $  :A/A^6 $
 1/B^13 $  :B/B^14 $

(USESUL B4CRITSUL B4CRIT)

(RPL DYTR1)
 1   $  I   $
 I/2 $  1/2 $

(RPL L N P)
 48*A**(-3)*B**(-7)*M**(-2)*X**7-96
*A**(-3)*B**(-7)*M**(-2)*X**6+240*A**(-3)*B**
(-7)*M**(-2)*X**5*(SIN(H))**2-144*A**(-3)*B**
(-7)*M**(-2)*X**5-288*A**(-3)*B**(-7)*M**(-2)
*X**4*(SIN(H))**2+288*A**(-3)*B**(-7)*M**(-2)
*X**4+336*A**(-3)*B**(-7)*M**(-2)*X**3*(SIN(H
))**4-480*A**(-3)*B**(-7)*M**(-2)*X**3*(SIN(H
))**2+144*A**(-3)*B**(-7)*M**(-2)*X**3-288*A**
(-3)*B**(-7)*M**(-2)*X**2*(SIN(H))**4+576*A**
(-3)*B**(-7)*M**(-2)*X**2*(SIN(H))**2-288*A**
(-3)*B**(-7)*M**(-2)*X**2+144*A**(-3)*B**(-7)
*M**(-2)*X*(SIN(H))**6-336*A**(-3)*B**(-7)*M**
(-2)*X*(SIN(H))**4+240*A**(-3)*B**(-7)*M**(-2
)*X*(SIN(H))**2-48*A**(-3)*B**(-7)*M**(-2)*X-
96*A**(-3)*B**(-7)*M**(-2)*(SIN(H))**6+288*A**
(-3)*B**(-7)*M**(-2)*(SIN(H))**4-288*A**(-3)*
B**(-7)*M**(-2)*(SIN(H))**2+96*A**(-3)*B**(-7
)*M**(-2)-96*I*A**(-5/2)*B**(-13/2)*M**(-2)*X
**4*SIN(H)*(-(SIN(H))**2+1)**(1/2)-192*I*A**(
-5/2)*B**(-13/2)*M**(-2)*X**2*(SIN(H))**3*(-(
SIN(H))**2+1)**(1/2)+192*I*A**(-5/2)*B**(-13/
2)*M**(-2)*X**2*SIN(H)*(-(SIN(H))**2+1)**(1/2
)-96*I*A**(-5/2)*B**(-13/2)*M**(-2)*(SIN(H))**
5*(-(SIN(H))**2+1)**(1/2)+192*I*A**(-5/2)*B**
(-13/2)*M**(-2)*(SIN(H))**3*(-(SIN(H))**2+1)**
(1/2)-96*I*A**(-5/2)*B**(-13/2)*M**(-2)*SIN(H
)*(-(SIN(H))**2+1)**(1/2) $
```

```
 4*A**(-3)*B**(-7)*M**(-2)*X**7-8*A**
(-3)*B**(-7)*M**(-2)*X**6+12*A**(-3)*B**(-7)*
M**(-2)*X**5*(SIN(H))**2-12*A**(-3)*B**(-7)*M
**(-2)*X**5-24*A**(-3)*B**(-7)*M**(-2)*X**4*(
SIN(H))**2+24*A**(-3)*B**(-7)*M**(-2)*X**4+12
*A**(-3)*B**(-7)*M**(-2)*X**3*(SIN(H))**4-24*
A**(-3)*B**(-7)*M**(-2)*X**3*(SIN(H))**2+12*A
**(-3)*B**(-7)*M**(-2)*X**3-24*A**(-3)*B**(-7
)*M**(-2)*X**2*(SIN(H))**4+48*A**(-3)*B**(-7)
*M**(-2)*X**2*(SIN(H))**2-24*A**(-3)*B**(-7)*
M**(-2)*X**2+4*A**(-3)*B**(-7)*M**(-2)*X*(SIN
(H))**6-12*A**(-3)*B**(-7)*M**(-2)*X*(SIN(H))
**4+12*A**(-3)*B**(-7)*M**(-2)*X*(SIN(H))**2-
4*A**(-3)*B**(-7)*M**(-2)*X-8*A**(-3)*B**(-7)
*M**(-2)*(SIN(H))**6+24*A**(-3)*B**(-7)*M**(-
2)*(SIN(H))**4-24*A**(-3)*B**(-7)*M**(-2)*(SIN
(H))**2+8*A**(-3)*B**(-7)*M**(-2) $

 3*A**(-3)*B**(-7)*M**(-2)*X**7-6*
A**(-3)*B**(-7)*M**(-2)*X**6+15*A**(-3)*B**(-
7)*M**(-2)*X**5*(SIN(H))**2-9*A**(-3)*B**(-7)
*M**(-2)*X**5-18*A**(-3)*B**(-7)*M**(-2)*X**4
*(SIN(H))**2+18*A**(-3)*B**(-7)*M**(-2)*X**4+
21*A**(-3)*B**(-7)*M**(-2)*X**3*(SIN(H))**4-30
*A**(-3)*B**(-7)*M**(-2)*X**3*(SIN(H))**2+9*A
**(-3)*B**(-7)*M**(-2)*X**3-18*A**(-3)*B**(-7
)*M**(-2)*X**2*(SIN(H))**4+36*A**(-3)*B**(-7)
*M**(-2)*X**2*(SIN(H))**2-18*A**(-3)*B**(-7)*
M**(-2)*X**2+9*A**(-3)*B**(-7)*M**(-2)*X*(SIN
(H))**6-21*A**(-3)*B**(-7)*M**(-2)*X*(SIN(H))
**4+15*A**(-3)*B**(-7)*M**(-2)*X*(SIN(H))**2-
3*A**(-3)*B**(-7)*M**(-2)*X-6*A**(-3)*B**(-7)
*M**(-2)*(SIN(H))**6+18*A**(-3)*B**(-7)*M**(-
2)*(SIN(H))**4-18*A**(-3)*B**(-7)*M**(-2)*(SIN
(H))**2+6*A**(-3)*B**(-7)*M**(-2)+6*I*A**(-5/
2)*B**(-13/2)*M**(-2)*X**4*SIN(H)*(-(SIN(H))**
2+1)**(1/2)+12*I*A**(-5/2)*B**(-13/2)*M**(-2)
*X**2*(SIN(H))**3*(-(SIN(H))**2+1)**(1/2)-12*
I*A**(-5/2)*B**(-13/2)*M**(-2)*X**2*SIN(H)*(-
(SIN(H))**2+1)**(1/2)+6*I*A**(-5/2)*B**(-13/2
)*M**(-2)*(SIN(H))**5*(-(SIN(H))**2+1)**(1/2)
-12*I*A**(-5/2)*B**(-13/2)*M**(-2)*(SIN(H))**
3*(-(SIN(H))**2+1)**(1/2)+6*I*A**(-5/2)*B**(-
13/2)*M**(-2)*SIN(H)*(-(SIN(H))**2+1)**(1/2) $
```

```
(NEWSUL 3 UNPSISUL)
 :L $  L $
 :N $  N $
 :P $  P $

(USESUL UNPSISUL UNPSI)

(RPL DYTRSP1)
 (P/L)^(1/8) $  0           $
 0           $  (L/P)^(1/8) $

(FUNS (L X H) (N X H) (P X H))
```

```
(TITLE "SA23.DIA
        STATIONARY AXISYMMETRIC VACUUM SOLUTION.
        VAN STOCKUM SUBCLASS (1937).
        Ref: D.Kramer , H.Stephani , M.MacCallum & E.Herlt.
        Exact Solutions of Einstein's Field Equations.
        Equation (18.23)")

% This file was written by :
%  Julian Collins
%   Department of Mathematics
%    Southampton University
%     Highfield
%      Southampton
%       SO9 5NH

(OFF ALL) (ON NOZERO POTSIM)

(PRELOAD DIAINP DYTRSP)

(VARS P Z T F)  % rho , z , t , phi

(RPL IZUD)
 0        $ 0        $ 0           $ (P/W)^(1/2)  $
 0        $ 0        $ (P*W)^(1/2) $ -(P/W)^(1/2) $
 P^(-1/4) $ 0        $ 0           $ 0            $
 0        $ P^(-1/4) $ 0           $ 0            $

(FUNS (W P Z))

(NEWSUL RIESUL (DIFF 2))
 W&Z&Z $  -W&P&P - W&P/P $
(USESUL RIESUL RIE)

(RPL DYTR1)
 0 $  -1 $
 1 $  0  $

(RPL X)
 -P**(1/2)*W**(-1)*DF(W,P,2)-5/4*P**
(-1/2)*W**(-1)*DF(W,P)+I*P**(1/2)*W**(-1)*DF(
W,P,Z)+3/4*I*P**(-1/2)*W**(-1)*DF(W,Z) $

(FUNS (X P Z))

(NEWSUL UNPSISUL)
 :X $  X $
```

```
(USESUL UNPSISUL UNPSI)

(RPL DYTRSP1)
 X^(1/4) $  0        $
 0       $  X^(-1/4) $
```

References

[1] D. Kramer, H. Stephani, E. Herlt and M. MacCallum (1980) *Exact Solutions of Einstein's Field Equations* ed. E. Schmutzer (Cambridge University Press).

[2] E. Cartan (1946) *Leçons sur la Geometrie des Espaces de Riemann* (Gauthier-Villars, Paris).

[3] A. Karlhede (1980) *Gen. Rel. Grav.* **12** 693.

[4] J. Skea (1988) *Applications of SHEEP, Tech. Rep.* School of Mathematical Sciences, Queen Mary College, London.

[5] J.E. Åman, R.A. d'Inverno, G.C. Joly and M.A.H. MacCallum (1985) *Progress on the Equivalence Problem, Proceedings of EUROCAL '85* ed. B.F. Caviness (Springer-Verlag, Lecture Notes in Computer Science) **204**.

[6] R. Penrose (1960) *Annals of Physics* **10** 171.

[7] F.W. Letniowski & R.G. McLenaghan (1988) *Gen. Rel. Grav.* **20** 463.

[8] J.E. Åman, R.A. d'Inverno, G.C. Joly & M.A.H. MacCallum (1984) *Proceedings of the 1984 European Conference on Symbolic and Algebraic Manipulation* (Springer-Verlag, Lecture Notes in Computer Science) **174**.

[9] A. Karlhede (1979) *A Review of the Equivalence Problem, Tech. Rep.* University of Stockholm.

[10] W.M. Kinnersley (1969) *Type D Gravitational Fields* Ph.D. thesis, California Institute of Technology.

[11] J.E. Åman & A. Karlhede (1981) *Preprint* Institute of Theoretical Physics, University of Stockholm.

[12] R. Geroch, A. Held & R. Penrose (1973) *J. Math. Phys.* **14** 874.

[13] E. Newman & R. Penrose (1962) *J. Math. Phys.* **3** 566.

[14] A. Trautman, F.A.E. Pirani & H. Bondi (1960) *Lectures on General Relativity* (Prentice-Hall, Inc. Englewood Cliffs, New Jersey) **1**.

[15] J.W. Archbold *Algebra* (Sir Isaac Pitman & Sons Ltd) 186.

[16] R.A. d'Inverno & R.A. Russell-Clark (1971) *J. Math. Phys.* **12** 1258.

[17] C.H. Brans (1965) *J. Math. Phys.* **6** 94.

[18] S.T.C. Siklos (1984) *Relativistic Astrophysics and Cosmology* ed. X. Fustero & E. Verdaguer (Singapore : World Scientific) pp 201-48.

[19] M.A.H. MacCallum & J.E. Åman (1986) *Class. Quantum Grav.* **3** 1133.

[20] J.M. Collins, R.A. d'Inverno & J.A. Vickers (1990) *Class. Quantum Grav.* **7** 2005.

[21] J. Collins (1991) *The Karlhede Classification of Type N Vacuum Spacetimes, Preprint* Faculty of Mathematical Studies, Southampton University.

[22] C.B.G. McIntosh, J.M. Foyster & A. Lun (1981) *J. Math. Phys.* **22** 2920.

[23] B. Carter (1970) *Commun. Math. Phys.* **17** 233.

[24] B. Carter (1969) *J. Math. Phys.* **10** 70.

[25] B. Carter (1973) *Black Hole Equilibrium States* ed. C. DeWitt & B.S. DeWitt (New York : Gordon & Breach, Black Holes).

[26] D. Zipoy (1966) *J. Math. Phys.* **7** 1137.

[27] F.J. Ernst (1968) *Phys. Rev.* **167** 1175.

[28] A. Tomimatsu & H. Sato (1972) *Phys. Rev. Lett.* **29** 1344.

[29] J. Hauser & F.J. Ernst (1981) *J. Math. Phys.* **22** 1051.

[30] H. Weyl (1917) *Ann. Physik* **54** 117.

[31] B.H. Voorhees (1970) *Phys. Rev.* **D2** 2119.

[32] G. Darmois (1927) *Les équations de la gravitation einsteinienne* Mémorial des sciences mathématique, Fasc. XXV (Gauthier-Villars, Paris).

[33] J.N. Goldberg & R.K. Sachs (1962) *Acta Phys. Polon., Suppl.* **22** 13.

[34] C.B.G. McIntosh (1985) *Class. Quantum Grav.* **2** 87.

[35] W. Kundt (1961) *Z. Phys.* **163** 77.

[36] I. Robinson and A. Trautman (1962) *Proc. R. Soc.* **A 265** 463.

[37] I. Hauser (1974) *Phys. Rev. Lett.* **18** 1112.