UNIVERSITY OF SOUTHAMPTON
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THE HOMOLOGICAL GRADE OF A MODULE OVER A COMMUTATIVE RING

by

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ABSTRACT

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Throughout the thesis, 'a ring' means a 'commutative ring with identity', and 'R-module' means a 'unitary R-module'. The category of all R-modules is denoted by Mod-R and the annihilator of an R-module N is denoted by $ann_R N$. For an ideal A of a ring R and an R-module M, David Kirby and Hefzi A. Mehran, in a recent paper, define the homological grade of M in A, $hgr_R(A;M)$, to be $\inf\{n \mid Ext^n_R(R/A,M) \neq 0\}$. In the thesis, we consider the homological grade of M in A as the homological grade of M in the cyclic R-module R/A. We denote it by $hgr_R(R/A;M)$ and extend this definition to any arbitrary R-module N replacing R/A. The notion of an M-sequence in the work of D. Kirby and H.A. Mehran is adapted for use in new situations. For an R-module M and an exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of R-modules with N_2 finitely generated, it is proved that, $hgr_R(R/ann_R N_2; M) = hgr_R(N_2; M) = \min\{hgr_R(N_1; M), hgr_R(N_3; M)\}$ = the supremum of lengths of M-sequences in $ann_R N_2$. For R-modules N, M where N is finitely generated and for $r \leq hgr_R(N; M)$, it is shown that $Ext^r_R(N, M) \cong Hom_R(N, \ker d^r)$, where $\ker d^r$ is the r^{th} cosyzygy in the minimal injective resolution of M. If, in addition, $ann_R N$ is finitely generated, a relation between $hgr_R(N;M)$, $\text{proj dim}_R M$ and $hgr_R(N; R)$ is given. Dual notions of the homological grade and an M-sequence are given and most of the results concerning the homological grade are dualized. For an artinian R-module M and a finitely generated R-module N, it is proved that for every $n \leq cohgr_R(N; M)$ there exists a regular M-cosequence in $ann_R N$ of length n. Finally, if $(R,\mathcal{M})$ is a local (noetherian) ring, it is shown that for every $r \leq hgr_R(R/\mathcal{M};R)$ the contravariant left exact functor $Ext^r_R(-, Hom_R(\ker d^r, E(R/\mathcal{M})))$ is an exact functor in the category of all R-modules of finite length. And if, in addition, R is complete and $n = hgr_R(R/\mathcal{M};R)$ then, for a finitely generated R-module $N \neq 0$ of finite injective dimension, the R-module $Hom_R(\Omega_n,N)$ is finitely generated with finite projective dimension equal to $n - hgr_R(R/\mathcal{M};N)$, where $\Omega_n = Hom_R(\ker d^n, E(R/\mathcal{M}))$.

To the memory of my father,
my mother and my brother Abdullah.

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CHAPTER 1

INTRODUCTION AND SOME BASIC IDEAS

1.1 Introduction

In [16], D. Rees introduced a numerical character of a finitely generated module $N \neq 0$ over a noetherian ring R, called the grade of N. This is defined as the least integer k such that $\mathrm{Ext}_R^k(N,R) \neq 0$, or, equivalently; the maximal length of a regular R-sequence in the annihilator of N. When R is not necessarily noetherian and if J is an ideal of R and E an R-module such that $JE \neq E$, M.Hochster in [4] defined $g(J,E)$, the classical grade of J on E, as the supremum of lengths of E-sequences contained in J. Then, he defined a new notion of grade as follows. For an admissible pair (I,M); see [4] p.56; of an ideal I of R and an R-module M, $G(I,M) = \sup \{g(I \otimes_R B, M \otimes_R B) \mid B$ is a faithfully flat R-algebra $\}$. And in [14], D.G. Northcott considered the classical grade of an ideal A of R on an R-module E, $g_R\{A;E\}$, as the upper bound of the lengths of all finite E-sequences in A. Then he puts $G_R\{A;E\} = \lim_{m\to\infty} g_{R[x;m]}\{AR[x;m]; E[x;m]\}$ and calls it the true grade or polynomial grade of A on E, where $x_1,\ldots,x_m (= x;m)$ are indeterminates. Finally, D. Kirby and H.A.. Mehran, in [7], extended the notion of an E-sequence of length d in A as follows. For d = 0 the E-sequence is empty and for d = 1 it is a subset

$\{a_i \mid i \in I\} = \alpha \subseteq A$ such that $0:_E \sum_{i\in I} Ra_i = 0$. For $d > 1$, they define, inductively, an E-sequence of length d in A as a sequence $\alpha_1 = \{a_{i1} \mid i \in I_1\}, \ldots, \alpha_d = \{a_{id} \mid i \in I_d\}$ of subsets of A such that α_1 is an E-sequence of length 1 in A and $\alpha_2, \ldots, \alpha_d$ is a $[(\prod_{i\in I_1} E)/(a_{i1})E]$-sequence of length $d-1$ in A. Thus in the standard notion of E-sequence the sets α_j are singletons, and, in effect, in the extended notion of Hochster [4] and Northcott [14] the sets α_j are finite. It follows from the results of [7] that every maximal E-sequence has the same length (which may be infinite). This maximal length is $\inf\{n \mid \mathrm{Ext}_R^n(R/A,E) \neq 0\}$ which they call the homological grade of E in A and denote it by $\mathrm{hgr}_R(A;E)$.

The purpose of this thesis is to consider the homological grade of E in A as the homological grade of E in the cyclic R-module R/A and denote it by $\mathrm{hgr}_R(R/A;E)$, and extend this definition to any arbitrary R-module N replacing R/A in an attempt to bring together all the ideas and results mentioned above.

In section two of Chapter one, we quote some general results and definitions for reference. And in section (1.3), we prove two essential lemmas and their corollaries which we use frequently in this thesis.

In section one of Chapter two, we give a definition of the homological grade of M in N and investigate its properties. And in section (2.2), we adapt the notion, in [7], of an M-sequence and study the relation between the homological grade of M in N and the length of an M-sequence in the annihilator of N in R. Section (2.3) deals with the relation between the homological grade of M in a finitely generated R-module N and the first non-vanishing term in the complex $Hom_R(N, E_M)$, where E_M is the deleted minimal injective resolution of M.

In section one of Chapter Three, we give a definition of an M-cosequence in an ideal A of R and study the relation between the cohomological grade of M in N and the length of an M-cosequence in the annihilator of N. Section (3.3) is devoted to the relation between the homological grade and the cohomological grade of M in N and the injective and projective dimensions of M.

In section one of Chapter Four, we study the homological and cohomological grades of artinian R-modules. Finally, in section (4.2), we consider a local noetherian ring $(R, \mathcal{M})$ with a unique maximal ideal $\mathcal{M}$ and investigate the functors $Ext^r_R(-, Hom_R(\ker d^r, E(R/\mathcal{M})))$ and $Hom_R(Hom_R(\ker d^n, E(R/\mathcal{M})), -)$, where $r \leq n = hgr_R(R/\mathcal{M}; R)$, $\ker d^r$ is the r^{th} cosyzygy in the minimal injective resolution of R and $E(R/\mathcal{M})$ is the injective envelope of $R/\mathcal{M}$.

1.2 Some General Results and Definitions

In this section we will give some definitions and state some theorems which we will use throughout this thesis.

Definition 1.2.1 An R-module C is a cogenerator for Mod-R if for every R-module M and every $0 \neq m \in M$, there exists $f: M \to C$ with $f(m) \neq 0$. Or; for injective C equivalently; for every R-module M, $\mathrm{Hom}_R(M,C) = 0$ if and only if $M = 0$.

Theorem 1.2.2 There exists an injective cogenerator for Mod-R.

Proof Lemma 3.37 P. 79 of [17].

Definition 1.2.3 An "essential extension" of a module M is a module E containing M such that every non-zero submodule of E has a non-zero intersection with M. Or; equivalently; for every $0 \neq e \in E$ there exists $r \in R$ with $0 \neq re \in M$.

Definition 1.2.4 An "injective envelope" of an R-module M is an injective R-module which is an essential extension of M.

For the proof of the following theorems we refer to Chapter 2 Sections 5,6 of [13].

Theorem 1.2.5 Let M be an R-module. Then in Mod-R, we can construct an exact sequence $0 \to M \to E$, where E is an injective R-module.

Theorem 1.2.6 Let M be an R-module and E an injective extension module of M (i.e. $0 \to M \to E$ is an exact sequence). Then E contains a submodule which is an injective envelope of M.

Theorem 1.2.7 Let $f : M \simeq M'$ be an isomorphism of R-modules, E an injective envelope of M, and E' an injective envelope of M'. Then f can be extended to an isomorphism of E onto E'.

Remark Theorem 1.2.6 ensures that every R-module has an injective envelope, and Theorem 1.2.7 states that injective envelopes of an R-module are virtually unique.

From now on we shall therefore speak of "the injective envelope" of a module.

For an R-module M, one can construct an injective resolution of M; that is to say an infinite exact sequence, $0 \to M \xrightarrow{\epsilon} E^o \xrightarrow{d^o} E^1 \to \ldots \to E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \ldots$ where each E^i is an injective R-module. Further, one can construct an injective resolution in such a way that E^o is the injective envelope of M, E^1 is the injective envelope of E^o/M and for each $i \geq 1$, E^{i+1} is the injective envelope of $E^i/d^{i-1}(E^{i-1})$. Such an injective resolution is known as a "minimal injective resolution" of M.

For the construction of such injective resolutions and for the proof of the following theorem, we refer to Chapter 3, Section 7, p.79 of [13].

Theorem 1.2.8 Let the R-modules M and M′ have minimal injective resolutions

$$0 \to M \to E^0 \to E^1 \to E^2 \to \ldots.$$

and

$$0 \to M' \to E'^0 \to E'^1 \to E'^2 \to \ldots.$$

respectively. Further let $f : M \simeq M'$ be an isomorphism in Mod-R. Then it is possible to construct, in succession, R-isomorphisms $\phi_0 : E^0 \simeq E'^0$, $\phi_1 : E^1 \simeq E'^1$ and so on which are such that the diagram,

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & M & \longrightarrow & E^0 & \longrightarrow & E^1 & \longrightarrow & E^2 & \longrightarrow & \ldots. \\ & & \downarrow f & & \downarrow \phi_0 & & \downarrow \phi_1 & & \downarrow \phi_2 & & \\ 0 & \longrightarrow & M' & \longrightarrow & E'^0 & \longrightarrow & E'^1 & \longrightarrow & E'^2 & \longrightarrow & \ldots. \end{array}$$

is commutative.

Remark Theorem 1.2.8 shows that minimal injective resolutions of an R-module are essentially unique.

We end this section by stating and proving the following well-known theorem.

Theorem 1.2.9. Let N,M be R-modules, and let

$$0 \longrightarrow M \overset{\subseteq}{\longrightarrow} E^o \overset{d^o}{\longrightarrow} E^1 \overset{d^1}{\longrightarrow} E^2 \longrightarrow \dots \longrightarrow E^{n-1} \overset{d^{n-1}}{\longrightarrow} E^n \overset{d^n}{\longrightarrow} E^{n+1} \longrightarrow \dots$$

be an injective resolution of M. Then,

(i) $\mathrm{Ext}^n_R(N,M) \cong \mathrm{Ext}^r_R(N, \ker d^{n-r})$ for all $1 \leq r \leq n$

(ii) $\mathrm{Ext}^k_R(N, \ker d^m) \cong \mathrm{Ext}^{k+1}_R(N, \ker d^{m-1})$ for all $k \geq 1$, $m \geq 1$.

Proof (i) Since,

$$0 \longrightarrow M \overset{\subseteq}{\longrightarrow} E^o \overset{d^o}{\longrightarrow} E^1 \overset{d^1}{\longrightarrow} E^2 \longrightarrow \dots \longrightarrow E^{n-1} \longrightarrow E^n \longrightarrow E^{n+1} \longrightarrow \dots$$

is an injective resolution of M then for $1 \leq r \leq n$,

$$0 \longrightarrow \ker d^{n-r} \longrightarrow Q^o \overset{h^o}{\longrightarrow} Q^1 \overset{h^1}{\longrightarrow} Q^2 \longrightarrow \dots \longrightarrow Q^{r-1} \overset{h^{r-1}}{\longrightarrow} Q^r \overset{h^r}{\longrightarrow} Q^{r+1} \longrightarrow \dots$$

is an injective resolution of $\ker d^{n-r}$ where $Q^i = E^{n-r+i}$ and $h^i = d^{n-r+i}$ for $i \geq 0$. Now, deleting $\ker d^{n-r}$ from the above injective resolution of $\ker d^{n-r}$ and applying $\mathrm{Hom}_R(N,-)$, we obtain the following complex,

$$0 \longrightarrow \mathrm{Hom}_R\,(N,Q^o) \xrightarrow{\mathrm{Hom}_R(i_N,h^o)} \mathrm{Hom}_R(N,Q^1) \xrightarrow{\mathrm{Hom}_R(i_N,h^1)} \mathrm{Hom}_R(N,Q^2) \longrightarrow \ldots.$$

$$\longrightarrow \mathrm{Hom}_R(N,Q^{r-1}) \xrightarrow{\mathrm{Hom}_R(i_N,h^{r-1})} \mathrm{Hom}_R(N,Q^r) \xrightarrow{\mathrm{Hom}_R(i_N,h^r)} \mathrm{Hom}_R(N,Q^{r+1}) \longrightarrow \ldots.$$

Then, by definition of Ext;

$$\begin{aligned} \mathrm{Ext}_R^r(N,\ker d^{n-r}) &= \ker \mathrm{Hom}_R(i_N,h^r) \,/\, \mathrm{Im}\, \mathrm{Hom}_R\,(i_N,h^{r-1}) \\ &= \ker \mathrm{Hom}_R(i_N,d^n) \,/\, \mathrm{Im}\, \mathrm{Hom}_R\,(i_N,d^{n-1}) \\ &= \mathrm{Ext}_R^n\,(N,\ M)\ . \end{aligned}$$

(ii) By (i), we have, $\mathrm{Ext}_R^k(N,\ \ker d^m) = \mathrm{Ext}_R^k(N,\ \ker d^{m+k-k})$
$\cong \mathrm{Ext}_R^{m+k}(N,M) \cong \mathrm{Ext}_R^{k+1}(N,\ \ker d^{m+k-(k+1)}) = \mathrm{Ext}_R^{k+1}(N,\ \ker d^{m-1})$

1.3 <u>Some Basic New Results</u>

In this section, we will prove two Lemmas and their Corollaries which we will use frequently throughout this thesis.

<u>Lemma 1.3.1</u> Let N, M, M' be R-modules where N is finitely generated and M' is an essential extension of M. Then,

$$\mathrm{Hom}_R(N,\ M) = 0 \ \text{ if and only if } \ \mathrm{Hom}_R(N,\ M') = 0$$

Proof. If $\mathrm{Hom}_R(N, M') = 0$ then $\mathrm{Hom}_R(N,M) = 0$ is trivial for $M \subseteq M'$.

For the "only if" part, let $\mathrm{Hom}_R(N,M) = 0$. Now $N = \sum_{i=1}^{n} Rx_i$. Suppose $0 \neq f \in \mathrm{Hom}_R(N,M')$. Then there is $1 \leq i \leq n$ such that $0 \neq f(x_i) \in M'$ for $f \neq 0$. Without loss of generality we can take $i = 1$. Now $0 \neq f(x_1) \in M'$ and M' is an essential extension of M, so there exists $r_1 \in R$ with $0 \neq r_1 f(x_1) \in M$ (for $R\, f(x_1) \neq 0 \Rightarrow R\, f(x_1) \cap M \neq 0$). If $r_1 f(x_2) = 0$ take $r_2 = 1$, otherwise there exists $r_2 \in R$ with $0 \neq r_2 r_1\, f(x_2) \in M$. If $r_2 r_1\, f(x_3) = 0$ take $r_3 = 1$, otherwise there exists $r_3 \in R$ with $0 \neq r_3 r_2 r_1\, f(x_3) \in M$. Continue doing this until you choose r_n. Now let $r = r_n r_{n-1} \ldots r_2 r_1$. It is clear that $r\, f(x_i) \in M$ for $1 \leq i \leq n$.

Claim: $r\, f(x_i) \neq 0$ for some $1 \leq i \leq n$.

Proof of the Claim: If $r\, f(x_n) \neq 0$ we are done, otherwise $0 = r\, f(x_n) = r_n r_{n-1} \ldots r_2 r_1\, f(x_n)$. Hence, by the way we choose r_n, $r_n = 1$. Therefore $r = r_{n-1} \ldots r_2 r_1$. If $rf(x_{n-1}) \neq 0$, we are done, otherwise $0 = rf(x_{n-1}) = r_{n-1} r_{n-2} \ldots r_2 r_1$. Hence, by the way we choose r_{n-1}, $r_{n-1} = 1$. Therefore $r = r_{n-2} \ldots r_2 r_1$. If $rf(x_{n-2}) \neq 0$ we are done, otherwise $r_{n-2} = 1$ and $r = r_{n-3} \ldots r_2 r_1$. Continue this argument until either you find $i > 1$ with $rf(x_{n-(n-i)}) \neq 0$, hence we are done, or no such i, hence $r = r_1$. But in this case $rf(x_1) \neq 0$. This completes the proof of the claim.

Since $rf(x_i) \in M$ for each $1 \leq i \leq n$, $rf(N) \subseteq M$. Now define $g: N \rightarrow M$ such that $g(x) = rf(x)$. Then g is a well-defined R-homomorphism and

$g \neq 0$ for $g(x_i) = rf(x_i)$ and, by the claim, there exists $1 \leq i \leq n$ with $rf(x_i) \neq 0$. Therefore $0 \neq g \in Hom_R(N,M)$ which contradicts the assumption that $Hom_R(N,M) = 0$. Therefore, $Hom_R(N,M') = 0$. This completes the proof of the Lemma.

<u>Remark</u>: Lemma 1.3.1 may not be true if N is not finitely generated. For example, take $R = Z$ the ring of integers, $N = Q$ the rationals and $M = Z$. Then $M' = Q$ is an essential extension of $M = Z$ and $Hom_Z(Q, Z) = 0$. But $Hom_Z(Q, Q) \neq 0$.

The next Corollary is a Lemma due to D. Kirby with different proof.

<u>Corollary 1.3.2</u>: Let N,M be R-modules where N is finitely generated. Then $Hom_R(N,M) = 0$ if and only if $Hom_R(R/ann_RN,M) = 0$.

<u>Proof</u>: For every $f \in Hom_R(N,M)$, $f(N) \subseteq (0 :_M ann_RN) \cong Hom_R(R/ann_RN,M)$. So, if $Hom_R(R/ann_RN,M) = 0$ then, $Hom_R(N,M) = 0$. For the "only if" part, let $Hom_R(N,M) = 0$. Let E(M) denote the injective envelope of M. By Theorem 1.2.6, E(M) exists and by the definition of an injective envelope, E(M) is an essential extension of M. Therefore, by Lemma 1.3.1, $Hom_R(N,E(M)) = 0$ for $Hom_R(N,M) = 0$. Now, let $A_i = ann_R Rx_i$ for $1 \leq i \leq n$ where $N = \sum_{i=1}^{n} R x_i$. We want to show that $Hom_R(R/A_i,M) = 0$ for each $1 \leq i \leq n$. So, suppose $Hom_R(R/A_i,M) \neq 0$ for some $1 \leq i \leq n$. Then there exists $0 \neq f \in Hom_R(R/A_i,M)$. Hence $0 \neq f \in Hom_R(R/A_i,E(M))$ for $M \subseteq E(M)$. Now define $g:R/A_i \to N$ such that $g(\bar{1}) = x_i$

($\bar{1} = 1 + A_i$, $A_i = \mathrm{ann}_R Rx_i$). Then g is a well-defined R-monomorphism. Now consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & R/A_i & \xrightarrow{g} & N \\ & 0 \neq f & \downarrow & & \\ & & E(M) & & \end{array}$$

Since $E(M)$ is an injective R-module, there exists an R-homomorphism, $\bar{f} : N \to E(M)$ such that $\bar{f}og = f$. But $f \neq 0$. So $0 \neq \bar{f} \in \mathrm{Hom}_R(N, E(M)) = 0$ which is a contradiction. Therefore $\mathrm{Hom}_R(R/A_i, M) = 0$ for $1 \leq i \leq n$. So, $(0 \underset{M}{:} A_i) = 0$ for $1 \leq i \leq n$. Hence $(0 \underset{M}{:} \prod_{i=1}^{n} A_i) = 0$. But, $(0 \underset{M}{:} \mathrm{Ann}_R N) = (0 \underset{M}{:} \cap A_i) \subseteq (0 \underset{M}{:} \prod_{i=1}^{n} A_i) = 0$. Hence $(0 \underset{M}{:} \mathrm{ann}_R N) = 0$. Therefore $\mathrm{Hom}_R(R/\mathrm{ann}_R N, M) \cong (0 \underset{M}{:} \mathrm{ann}_R N) = 0$. This completes the proof of the Corollary.

<u>Lemma 1.3.3</u>: Let N, M be R-modules with N finitely generated. Then $M \otimes_R N = 0$ if and only if $M \otimes_R (R/\mathrm{ann}_R N) = 0$.

<u>Proof</u>: Let E be an injective cogenerator for Mod-R. Such E exists by Theorem 1.2.2.

For the "only if part", let $M \otimes_R N = 0$. Then, $0 = \mathrm{Hom}_R(M \otimes_R N, E) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(M,E))$, and since N is finitely generated, then, by Corollary 1.3.2, $\mathrm{Hom}_R(R/\mathrm{ann}_R N, \mathrm{Hom}_R(M,E)) = 0$. Hence, $\mathrm{Hom}_R(M \otimes_R (R/\mathrm{ann}_R N), E) \cong \mathrm{Hom}_R(R/\mathrm{ann}_R N, \mathrm{Hom}_R(M,E)) = 0$. But E is an injective cogenerator for

Mod-R, so we must have $M \otimes_R (R/\mathrm{ann}_R N) = 0$. For the "if" part, let $M \otimes_R (R/\mathrm{ann}\, N) = 0$. Consider the exact sequence $0 \to \ker g \to \bigoplus_I R/\mathrm{ann}_R N \xrightarrow{g} N \to 0$ where $N = \sum_{i\in I} Rn_i$ and $g(\sum_{i\in I} \bar{r}_i) = \sum_{i\in I} r_i n_i$, $(\bar{r}_i = r_i + \mathrm{ann}_R N)$. Tensoring the above exact sequence with M, we obtain the exact sequence

$$M \otimes_R \ker g \to M \otimes_R (\bigoplus_I R/\mathrm{ann}_R N) \to M \otimes_R N \to 0.$$

But $M \otimes_R (\bigoplus_I R/\mathrm{ann}_R N) \cong \bigoplus_I (M \otimes_R R/\mathrm{ann}_R N) = 0$. Therefore, $M \otimes_R N = 0$.

We conclude this section with the following Corollary and a Remark on it.

<u>Corollary 1.3.4</u>: Let N,M be finitely generated R-modules. If $N \otimes_R M = 0$ then for any submodules L, K of N, M respectively and for every $n \geq 0$, $\mathrm{Tor}_R^n(L;K) = 0 = \mathrm{Tor}_R^n(K;L)$ and $\mathrm{Ext}_R^n(L,K) = 0 = \mathrm{Ext}_R^n(K,L)$.

<u>Proof</u>: Let $N \otimes_R M = 0$. Since N,M are finitely generated, then applying Lemma 1.3.3 twice, we have $R/\mathrm{ann}_R N \otimes_R R/\mathrm{ann}_R M = 0$ Hence,

$$0 = R/\mathrm{ann}_R N \otimes_R R/\mathrm{ann}_R M \cong R/(\mathrm{ann}_R N + \mathrm{ann}_R M).$$

Therefore, $R = \mathrm{ann}_R N + \mathrm{ann}_R M$. But for any submodules L, K of N, M respectively and for every $n \geq 0$, we have, $R = \mathrm{ann}_R N + \mathrm{ann}_R M \subseteq \mathrm{ann}_R \mathrm{Tor}_R^n(L, K) \cap \mathrm{ann}_R \mathrm{Tor}_R^n(K, L)$. Therefore, $\mathrm{Tor}_R^n(L, K) = 0 = \mathrm{Tor}_R^n(K, L)$. And, by the same argument, we can conclude that $\mathrm{Ext}_R^n(L,K) = 0 = \mathrm{Ext}_R^n(K,L)$.

Remark: The condition that both N, M are finitely generated in Corollary 1.3.4 is necessary. For example, take the ring of integers Z; then Q/Z, Z/nZ are Z-modules such that $Z/nZ \otimes_Z Q/Z = 0$. But $\mathrm{Tor}_Z^1(Z/nZ, Q/Z) \neq 0$. The latter is true, for if we tensor the exact sequence $0 \to Z \to Q \to Q/Z \to 0$ by Z/nZ we obtain a long exact sequence,

$$\ldots \longrightarrow \mathrm{Tor}_Z^1(Z/nZ, Q/Z) \longrightarrow Z/nZ \longrightarrow 0.$$

Hence $\mathrm{Tor}_Z^1(Z/nZ, Q/Z) \neq 0$ for $Z/nZ \neq 0$. Also, $\mathrm{Hom}_Z(Z/nZ, Q/Z) \cong (0 \underset{Q/Z}{:} nZ) \neq 0 = \mathrm{Hom}_Z(Q/Z, Z/nZ)$ and $\mathrm{Ext}_Z^1(Q/Z, Z/nZ) \neq 0$. The last two statements are true, for if we apply $\mathrm{Hom}_Z(-, Z/nZ)$ to the exact sequence $0 \to Z \to Q \to Q/Z \to 0$ we obtain the following long exact sequence,

$$0 \to \mathrm{Hom}_Z(Q/Z, Z/nZ) \to \mathrm{Hom}_Z(Q, Z/nZ) \to \mathrm{Hom}_Z(Z, Z/nZ) \to \mathrm{Ext}_Z^1(Q/Z, Z/nZ) \to \ldots$$

But

$$\mathrm{Hom}\ (Q, Z/nZ) \cong \mathrm{Hom}_Z(Q, \mathrm{Hom}_Z(Z/nZ, Z/nZ))$$

$$\cong \mathrm{Hom}_Z(Q \otimes_Z Z/nZ, Z/nZ) = 0 \quad (\text{for } Q \otimes_Z Z/nZ = 0)$$

Hence, we can conclude that:

$$\mathrm{Hom}_Z(Q/Z, Z/nZ) = 0 \text{ and } 0 \neq Z/nZ \cong \mathrm{Hom}_Z(Z, Z/nZ) \hookrightarrow \mathrm{Ext}_Z^1(Q/Z, Z/nZ).$$

CHAPTER 2

THE HOMOLOGICAL GRADE OF A MODULE

2.1 Definitions and Some Properties

For an ideal A of a commutative ring R with identity and a unitary R-module M, D. Kirby and H.A. Mehran in [7], define the homological grade of M in A to be

$$\inf\{n \mid \operatorname{Ext}_R^n(R/A, M) \neq 0\}$$

and denote it by $\operatorname{hgr}_R(A; M)$. In this chapter, we will consider the homological grade of M in A as the homological grade of M in the cyclic R-module R/A, and extend this definition to any arbitrary unitary R-module N. So, for any unitary R-modules, N, M, we define the homological grade of M in N to be

$$\inf\{n \mid \operatorname{Ext}_R^n(N; M) \neq 0\}$$

and denote it by $\operatorname{hgr}_R(N; M)$. Hence $0 \leq \operatorname{hgr}_R(N; M) \leq \infty$.

Most of this section is a generalization of Section 2 of [7].

Proposition 2.1.1: Let $L \subseteq N$, M be R-modules with N finitely generated and let $\mathrm{ann}_R N = \sum_{i\in I} Ra_i$. If $\mathrm{Hom}_R(N, M) = 0$ (So $(0 :_M \mathrm{ann}_R N) = 0$ by Corollary 1.3.2), then the exact sequence

$$0 \longrightarrow M \xrightarrow{f} \prod_{i\in I} M_i \xrightarrow{g} M' \longrightarrow 0,$$

where $M_i = M$ for all $i \in I$, $f(m) = (a_i m)$ for all $m \in M$ and M' is the cokernel of f, gives rise to an exact sequence,

$$0 \longrightarrow \prod_I \mathrm{Ext}_R^r(L, M_i) \xrightarrow{\mathrm{Ext}_R^r(id,g)} \mathrm{Ext}_R^r(L, M') \xrightarrow{\alpha_r} \mathrm{Ext}_R^{r+1}(L,M) \longrightarrow 0$$

for all $r \geq 0$, where α_r is the connecting homomorphism. Moreover $r < \mathrm{hgr}_R(L;M)$ if and only if $r \leq \mathrm{hgr}_R(L, M')$ and if $r < \mathrm{hgr}_R(L, M)$, α_r is an isomorphism.

Proof: Applying $\mathrm{Ext}_R(L, —)$ to the exact sequence $0 \to M \xrightarrow{f} \prod M_i \xrightarrow{g} M' \to 0$, we obtain a long exact sequence:

$$0 \to \cdots \to \mathrm{Ext}_R^r(L,M) \xrightarrow{\mathrm{Ext}_R^r(id,f)} \prod_{i\in I} \mathrm{Ext}_R^r(L,M_i) \xrightarrow{\mathrm{Ext}_R^r(id,g)} \mathrm{Ext}_R^r(L,M') \xrightarrow{\alpha_r} \mathrm{Ext}_R^{r+1}(L,M)$$

$$\xrightarrow{\mathrm{Ext}_R^{r+1}(id,f)} \prod_{i\in I} \mathrm{Ext}_R^{r+1}(L,M_i) \xrightarrow{\mathrm{Ext}_R^{r+1}(id,g)} \mathrm{Ext}_R^{r+1}(L,M') \longrightarrow \cdots .$$

Since for all $i \in I$, $a_i \in \mathrm{ann}_R N \subseteq \mathrm{ann}_R L$, the map

$$\mathrm{Ext}_R^r(\mathrm{id},f): \mathrm{Ext}_R^r(L,M) \longrightarrow \prod_{i\in I} \mathrm{Ext}_R^r(L, M_i)$$

is the zero map for all $r \geq 0$. Therefore, we obtain the short exact sequence of the statement of the proposition. And if $r < \mathrm{hgr}_R(L; M)$ then for all $s \leq r$, $\mathrm{Ext}_R^s(L, M) = 0$. Hence

$$\alpha_s : \mathrm{Ext}_R^s(L, M') \longrightarrow \mathrm{Ext}_R^{s+1}(L, M)$$

is an isomorphism for all $s \leq r$. Now, if $r < \mathrm{hgr}_R(L; M)$ then for all $s < r$,

$$\mathrm{Ext}_R^s(L, M') \cong \mathrm{Ext}_R^{s+1}(L, M) = 0,$$

since $s + 1 \leq r < \mathrm{hgr}_R(L; M)$. Therefore $r \leq \mathrm{hgr}_R(L; M')$. Conversely, if $r \leq \mathrm{hgr}_R(L, M')$ then for all $s < r$, $\mathrm{Ext}_R^s(L, M') = 0$. So, from the short exact sequence:

$$0 \longrightarrow \prod_{i\in I} \mathrm{Ext}_R^s(L, M_i) \xrightarrow{\mathrm{Ext}_R^s(\mathrm{id},g)} \mathrm{Ext}_R^s(L, M') \xrightarrow{\alpha_s} \mathrm{Ext}_R^{s+1}(L, M) \longrightarrow 0$$

we conclude that

$$0 = \mathrm{Ext}_R^s(L, M_i) = \mathrm{Ext}_R^{s+1}(L, M).$$

Therefore, $r < \mathrm{hgr}_R(L, M)$. This completes the proof of the proposition.

For $L \subseteq N$, M, R-modules with N finitely generated, the next proposition relates $hgr_R(N; M)$ to $hgr_R(L; M)$.

<u>Proposition 2.1.2</u>: Let $L \subseteq N$, M be R-modules with N finitely generated. Then $hgr_R(L; M) \geq hgr_R(N; M)$.

<u>Proof</u>: Proceed by induction on $r \geq 0$ to show $r \leq hgr_R(N; M)$ implies $r \leq hgr_R(L; M)$. For $r = 0$, $0 \leq hgr_R(L; M)$ is trivial. Now suppose it is true for $r \geq 0$. For $r + 1$, we have $0 < r+1 \leq hgr_R(N; M)$, hence $Hom_R(N, M) = 0$. So we can apply Proposition 2.1.1 and conclude that $r \leq hgr_R(N; M')$ since $r < r+1 \leq hgr_R(N; M)$. Therefore, by the induction step, $r \leq hgr(L; M')$. So again, by Proposition 2.1.1, $r < hgr_R(L; M)$. Therefore $r + 1 \leq hgr_R(L; M)$. This completes the induction and hence the proof of the proposition.

The next proposition relates $Ext^n_R(N/L, M)$ to $Ext^n_R(N, M)$ in terms of the projection $\pi : N \longrightarrow N/L$.

<u>Proposition 2.1.3</u>: Let $L \subseteq N$, M be R-modules where N is finitely generated. Then for all $n \leq hgr_R(L; M)$, the R-homomorphism,

$$Ext^n_R(\pi, id): Ext^n_R(N/L, M) \longrightarrow Ext^n_R(N,M)$$

is an injection which is an isomorphism when $n < \mathrm{hgr}_R(L; M)$.

Proof: Applying $\mathrm{Ext}_R(—, M)$ to the exact sequence

$$0 \longrightarrow L \xrightarrow{i} N \xrightarrow{\pi} N/L \longrightarrow 0 ,$$

we obtain a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^{n-1}(L, M) \xrightarrow{\alpha_{n-1}} \mathrm{Ext}_R^{n}(N/L, M) \xrightarrow{\mathrm{Ext}_R^n(\pi, \mathrm{id})} \mathrm{Ext}_R^{n}(N, M)$$

$$\xrightarrow{\mathrm{Ext}_R^n(i, \mathrm{id})} \mathrm{Ext}_R^{n}(L, M) \longrightarrow \cdots$$

So for $n \leq \mathrm{hgr}_R(L; M)$, $n-1 < \mathrm{hgr}_R(L; M)$. Hence, $\mathrm{Ext}_R^{n-1}(L, M) = 0$. Therefore,

$$\mathrm{Ext}_R^n(\pi, \mathrm{id}) : \mathrm{Ext}_R^n(N/L, M) \longrightarrow \mathrm{Ext}_R^n(N, M)$$

is injective. And if $n < \mathrm{hgr}_R(L; M)$, then $\mathrm{Ext}_R^n(L, M) = 0$. Therefore, $\mathrm{Ext}_R^n(\pi, \mathrm{id})$ is an isomorphism.

For an R-module M and a submodule L of a finitely generated R-module N, the following theorem relates the homological grade of M in N to the homological grade of M in L and N/L.

Theorem 2.1.4: Let $L \subseteq N$, M be R-modules with N finitely generated. Then

$$hgr_R(N; M) = \min\{hgr_R(L; M), hgr_R(N/L; M)\}$$

Proof: By Proposition 2.1.2, $hgr_R(N, M) \leq hgr_R(L; M)$ and, by Proposition 2.1.3,

$$Ext_R^n(\pi, id) : Ext_R^n(N/L, M) \longrightarrow Ext_R^n(N, M)$$

is injective for $n \leq hgr_R(N; M)$. So, if $r < hgr_R(N, M)$, then $r < hgr_R(N/L; M)$ hence $hgr_R(N; M) \leq hgr_R(N/L; M)$. Therefore

$$hgr_R(N; M) \leq \min\{hgr_R(L; M), hgr_R(N/L; M)\} .$$

Now if $hgr_R(N; M) < hgr_R(L; M)$, then, by Proposition 2.1.3,

$$Ext_R^n(\pi, id) : Ext_R^n(N/L, M) \longrightarrow Ext_R^n(N, M)$$

is an isomorphism for $n \leq hgr_R(N; M)$. Hence $hgr_R(N; M) = hgr_R(N/L; M)$. Therefore,

$$hgr_R(N; M) = \min\{hgr_R(L; M), hgr_R(N/L; M)\} .$$

Remark

For an exact sequence

$$0 \longrightarrow N' \xrightarrow{f} N \longrightarrow N'' \longrightarrow 0$$

of R-modules with N finitely generated and for any R-module M, we have

$$\mathrm{hgr}_R(N; M) = \min\{\mathrm{hgr}_R(N'; M),\ \mathrm{hgr}_R(N''; M)\} .$$

For $N' \cong f(N') \subseteq N$, $N'' \cong N/f(N')$.

We end this section with the following two Corollaries and a Remark on them.

Corollary 2.1.5: Let $L \subseteq N$, M be R-modules with N finitely generated; then, $\mathrm{Hom}_R(N, M) = 0$ if and only if $\mathrm{Hom}_R(L, M) = 0$ and $\mathrm{Hom}_R(N/L, M) = 0$.

Proof: By Theorem 2.1.4, we have

$$\mathrm{hgr}_R(N; M) = \min\{\mathrm{hgr}_R(L; M),\ \mathrm{hgr}_R(N/L; M)\} .$$

Therefore, $\mathrm{Hom}_R(N, M) = 0$ if and only if

$$0 < \mathrm{hgr}_R(N; M) = \min\{\mathrm{hgr}_R(L; M), \mathrm{hgr}_R(N/L; M)\} .$$

Hence, $\mathrm{Hom}_R(N, M) = 0$ if an only if $\mathrm{Hom}_R(L; M) = 0$ and $\mathrm{Hom}_R(N/L, M) = 0$.

<u>Corollary 2.1.6</u>: Let N, M be R-modules with N finitely generated, and let $\mathrm{ann}_R N \subseteq \mathrm{ann}_R M$. If $\mathrm{Hom}_R(N, M) = 0$ then $M = 0$.

<u>Proof</u>: Since $\mathrm{ann}_R N \subseteq \mathrm{ann}_R M$, we have an exact sequence,

$$0 \longrightarrow \mathrm{ann}_R M/\mathrm{ann}_R N \longrightarrow R/\mathrm{ann}_R N \longrightarrow R/\mathrm{ann}_R M \longrightarrow 0 .$$

Then, by the Remark on Theorem 2.1.4,

$$\mathrm{hgr}_R(R/\mathrm{ann}_R N; M) = \min\{\mathrm{hgr}_R(\mathrm{ann}_R M/\mathrm{ann}_R N; M), \mathrm{hgr}_R(R/\mathrm{ann}_R M; M)\} .$$

Now if $\mathrm{Hom}_R(N, M) = 0$ then, by Corollary 1.3.2, $\mathrm{Hom}_R(R/\mathrm{ann}_R N; M) = 0$. So,

$$0 < \mathrm{hgr}_R(R/\mathrm{ann}_R N; M) \leq \mathrm{hgr}_R (R/\mathrm{ann}_R M; M) .$$

Hence,

$$(0 \underset{M}{:} \mathrm{ann}_R M) \cong \mathrm{Hom}_R(R/\mathrm{ann}_R M, M) = 0 .$$

Therefore $M = 0$.

Remark:

Corollaries 2.1.4, 2.1.5 and 2.1.6 are not true when N is not finitely generated. For example let $(R, \mathcal{M})$ be a complete local (noetherian) ring and let $p \neq \mathcal{M}$ be a prime ideal of R and $E(R/p)$ its injective envelope. Consider the exact sequence:

$$0 \longrightarrow R/p \longrightarrow E(R/p) \longrightarrow E' \longrightarrow 0 \text{ ,}$$

where $E' = E(R/p)/(R/p)$. Then,

1) $\text{hgr}_R(R/p; R) < \text{hgr}_R(E(R/p); R) = \infty$. (See Corollary 4.2.2.)

2) $0 = \text{ann}_R\, E(R/\mathcal{M}) \subseteq \text{ann}_R\, R/p$ and $\text{Hom}_R(E(R/\mathcal{M}), R/p) = 0$. But $R/p \neq 0$.

2.2 The Extended M-Sequences

For N, M R-modules with N finitely generated, the aim of this section is to relate the homological grade of M in N to the length of an M-sequence in the annihilator of N ($\text{ann}_R\, N$).

Let A be an ideal of R and M an R-module. We will follow D. Kirby and H.A. Mehran in [7] and define an M-sequence in A as follows:

For $n = 0$ the M-sequence is empty, and for $n = 1$ it is a subset

$\{a_i \mid i \in I\} = \alpha \subseteq A$ such that $0 :_M \sum_{i \in I} Ra_i = 0$. For $n > 1$, we define, inductively, an M-sequence of length n in A as a sequence

$$\alpha_1 = \{a_{i1} \mid i \in I_1\} , \alpha_2 = \{a_{i2} \mid i \in I_2\}, \ldots, \alpha_n = \{a_{in} \mid i \in I_n\}$$

of subsets of A such that α_1 is an M-sequence of length 1 in A and $\alpha_2, \ldots, \alpha_n$ is a $[(\prod_{i \in I_1} M)/(a_{i1})M]$-sequence of length $n-1$ in A. Thus in the standard notion of M-sequences the sets α_j are singletons, and, in effect, the extended notion due to Hochster [4] and Northcott [14] the sets α_j are finite.

Let A be an ideal of R, M an R-module and let $\alpha_1, \ldots, \alpha_n$ be an M-sequence in A where

$$\alpha_i = \{a_{ij} \mid j \in J_\alpha^i\} \subseteq A .$$

We put

$$M_\alpha^o = M,\ M_\alpha^1 = (\prod_{j \in J_\alpha^1} M_\alpha^o)/(a_{1j})M_\alpha^o \quad ,$$

and

$$M_\alpha^i = (\prod_{j \in J_\alpha^i} M_\alpha^{i-1})/(a_{ij})M_\alpha^{i-1}$$

for $1 \leq i \leq n$, so that $\alpha_i, \alpha_{i+1}, \ldots, \alpha_n$ is an M_α^{i-1}-sequence in A of length $\left[n-(i-1)\right]$.

Theorem 2.2.1: Let N, M be R-modules with N finitely generated and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an M-sequence in $\mathrm{ann}_R N$. Then $n \leq \mathrm{hgr}_R(N; M)$ and if $n < \mathrm{hgr}_R(N; M)$, there exists $\alpha_{n+1} \subseteq \mathrm{ann}_R N$ such that $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}$ is an M-sequence in $\mathrm{ann}_R N$.

Proof: Proceed by induction on n. For $n = 0$, $n \leq \mathrm{hgr}_R(N; M)$ is trivial and if $0 < \mathrm{hgr}_R(N; M)$ then $\mathrm{Hom}_R(N, M) = 0$. Now since N is finitely generated then, by Corollary 1.3.2, $\mathrm{Hom}_R(R/\mathrm{ann}_R N, M) = 0$. Hence, $(0 \underset{M}{:} \mathrm{ann}_R N) = 0$ and $\alpha_1 = \mathrm{ann}_R N$ is an M-sequence in $\mathrm{ann}_R N$. Now assume it is true for $0 \leq n \leq m$. For $n = m + 1$, let $\alpha_1, \ldots, \alpha_{m+1}$ be an M-sequence in $\mathrm{ann}_R N$. Then $\alpha_2, \ldots, \alpha_{m+1}$ is an M^1_α-sequence in $\mathrm{ann}_R N$ of length m, where

$$M^1_\alpha = (\prod_{j \in J^1_\alpha} M)/(a_{1j})M .$$

So, by the induction hypothesis, $m \leq \mathrm{hgr}_R(N; M^1_\alpha)$ and if $m < \mathrm{hgr}_R(N; M^1_\alpha)$ there exists $\alpha_{m+2} \subseteq \mathrm{ann}_R N$ such that $\alpha_2, \ldots, \alpha_{m+1}, \alpha_{m+2}$ is an M^1-sequence in $\mathrm{ann}_R N$. Now for every

$$a_{1j} \in \alpha_1 = \{a_{1j} \mid j \in J^1_\alpha\} \subseteq \mathrm{ann}_R N$$

and for $r \geq 0$, we have a_{1j} annihilates $\mathrm{Ext}^r_R(N, M)$. So in the long exact sequence which results from applying $Ext_R(N, -)$ to the exact sequence

$$0 \longrightarrow M \xrightarrow{f} \prod_{j \in J_\alpha^1} M \xrightarrow{g} M_\alpha^1 \longrightarrow 0 ,$$

where $f(m) = (a_{1j}\, m)$ and $g((m_j)) = (m_j) + (a_{1j})M$, the homomorphism

$$Ext_R^r(id_N, f) : Ext_R^r(N, M) \longrightarrow Ext_R^r(N, \prod_{j \in J_\alpha^1} M)$$

is zero. Therefore, for $r \geq 0$ we obtain the following exact sequence,

$$0 \longrightarrow \prod_{j \in J_\alpha^1} Ext_R^r(N, M) \xrightarrow{Ext_R^r(id_N, g)} Ext_R^r(N, M_\alpha^1) \xrightarrow{\alpha_r} Ext_R^{r+1}(N, M) \longrightarrow 0$$

where α_r is the connecting homomorphism. Now for $r < m$, the central term of the above exact sequence is zero, therefore the first and the last terms are zero for each $r < m$ i.e. $Ext_R^s(N, M) = 0$ for each $s < m + 1$. Therefore $n = m + 1 \leq hgr_R(N, M)$ and if $n = m+1 < hgr_R(N, M)$ then for $r = m$, we have the first and the last terms in the above exact sequence are zeros, hence the central term is zero. Therefore $Ext_R^m(N, M_\alpha^1) = 0$. Hence $m < hgr_R(N, M_\alpha^1)$. So, by the induction step, there exists $\alpha_{m+2} \subseteq ann_R N$ such that $\alpha_2, \ldots, \alpha_{m+1}, \alpha_{m+2}$ is an M_α^1-sequence in $ann_R N$. Hence $\alpha_1, \alpha_2, \ldots, \alpha_{m+1}, \alpha_{m+2}$ is an M-sequence in $ann_R N$. This completes the induction and hence the proof of the theorem.

Corollary 2.2.2: Let N, M be R-modules where N is finitely generated. Then,

(i) If $n = \operatorname{hgr}_R(N; M)$ is finite, then every M-sequence in $\operatorname{ann}_R N$ has length $\leq n$, and can be extended to an M-sequence of length n.

(ii) If $\operatorname{hgr}_R(N; M) = \infty$, then no finite M-sequence in $\operatorname{ann}_R N$ is maximal.

Proof: Follows immediately from the Theorem.

The following Corollary shows that $\operatorname{hgr}_R(N; M)$ and $\operatorname{hgr}_R(R/\operatorname{ann}_R N; M)$ are equal when N is a finitely generated R-module.

Corollary 2.2.3: Let N, M be R-modules where N is finitely generated. Then

$$\operatorname{hgr}_R(N; M) = \operatorname{hgr}_R(R/\operatorname{ann}_R N; M) .$$

Proof: Straightforward from Corollary 2.2.2 and the fact that

$$\operatorname{ann}_R(R/\operatorname{ann}_R N) = \operatorname{ann}_R N.$$

We end this section with the following Proposition.

Proposition 2.2.4: Let N, M be R-modules where N is finitely generated, and let $n = \mathrm{hgr}_R(N; M)$ (so, by Theorem 2.2.1, $\mathrm{ann}_R N$ contains an M-sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of length n). Then $\mathrm{Ext}_R^n(N, M) \cong \mathrm{Ext}_R^{n-r}(N, M_R^r)$ for $0 \leq r \leq n$ where $\alpha_i = \{a_{ij} \mid j \in J_\alpha^i\} \subseteq \mathrm{ann}_R N$ and

$$M_\alpha^0 = M,\; M_\alpha^i = \Big(\prod_{j \in J_\alpha^i} M_\alpha^{i-1}\Big) \Big/ (a_{ij}) M_\alpha^{i-1} \quad (0 < i \leq n)$$

Proof: For $r = 0$, $\mathrm{Ext}_R^n(N, M) = \mathrm{Ext}_R^n(N, M_\alpha^0)$ for $M = M_\alpha^0$. If $r = 1$ then we have an exact sequence:

$$0 \longrightarrow M_\alpha^0 \xrightarrow{f_1} \prod_{J_\alpha^1} M_\alpha^0 \longrightarrow M_\alpha^1 \longrightarrow 0\,,$$

where $f_1(m) = (a_{1j} m)$. Now, since

$$\alpha_1 = \{a_{1j} \mid j \in J_\alpha^1\} \subseteq \mathrm{ann}_R N\,,$$

then applying $\mathrm{Hom}_R(N, -)$ to the above exact sequence, we obtain a long exact sequence:

$$\cdots \longrightarrow \prod_{J_\alpha^1} \mathrm{Ext}_R^{n-1}(N, M_\alpha^0) \longrightarrow \mathrm{Ext}_R^{n-1}(N, M_\alpha^1) \xrightarrow{\alpha_{n-1}} \mathrm{Ext}_R^n(N, M_\alpha^0)$$

$$\xrightarrow{\mathrm{Ext}_R^n(\mathrm{id}_N, f_1)} \prod_{J_\alpha^1} \mathrm{Ext}_R^n(N, M_\alpha^0) \longrightarrow \cdots\,.$$

But $\mathrm{Ext}_R^{n-1}(N, M_\alpha^o) = 0$ and the map $\mathrm{Ext}_R^n(id_N, f_1)$ is zero. Therefore

$$\mathrm{Ext}_R^{n-1}(N, M_\alpha^1) \cong \mathrm{Ext}_R^n(N, M_\alpha^o) = \mathrm{Ext}_R^n(N, M).$$

For $r = 2$, We have an exact sequence:

$$0 \longrightarrow M_\alpha^1 \xrightarrow{f_2} \prod_{J_\alpha^2} M_\alpha^1 \longrightarrow M_\alpha^2 \longrightarrow 0,$$

where $f_2(m) = (a_{2j}\, m)$, and since

$$\alpha_2 = \{a_{2j} \mid j \in J_\alpha^2\} \subseteq \mathrm{ann}_R\, N,$$

then applying $\mathrm{Hom}_R(N, -)$ to the above exact sequence we obtain:

$$\mathrm{Ext}_R^{n-2}(N, M_\alpha^2) \cong \mathrm{Ext}_R^{n-1}(N, M_\alpha^1) = \mathrm{Ext}_R^n(N, M).$$

And so on for every $0 \leq r \leq n$.

2.3 The Minimal Injective Resolution of a Module and the Homological Grade

In this section we will show that the homological grade of an R-module M in a finitely generated R-module N, is determined by the first non-vanishing term in the complex $\operatorname{Hom}_R(N, E_M)$, where

$$E_M : E^o \xrightarrow{d^o} E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \cdots$$

is a deleted minimal injective resolution of M (So that $0 \to M \to E_M$ is a minimal injective resolution of M).

Let M be an R-module and consider a minimal injective resolution of M,

$$0 \longrightarrow M \xrightarrow{\in} E^o \xrightarrow{d^o} E' \xrightarrow{d^1} \cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \cdots$$

<u>Theorem 2.3.1</u>: Let M and R be as above, and let N be a finitely generated R-module. Then the following are equivalent:

(i) $m \leq \operatorname{hgr}_R(N; M)$

(ii) For all $0 \leq r < m$, $\operatorname{Hom}_R(N, E^r) = 0$.

<u>Proof</u>: For (i) $\Rightarrow$ (ii), proceed by induction on m. For $m = 0$ it is trivial. For $m = 1 \leq \operatorname{hgr}_R(N; M)$, we have $\operatorname{Hom}_R(N, M) = 0$. But $M \cong \in (M)$, hence $\operatorname{Hom}_R(N, \in (M)) = 0$. And since N is finitely generated then, by Lemma 1.3.1, $\operatorname{Hom}_R(N, E^o) = 0$ for E^o is an essential extension of $\in(M)$.

Now suppose (i) $\Rightarrow$ (ii) for $m = k \geq 1$. For $m = k+1$, let $r < m$ then $r \leq k$. If $r < k$ then, by the induction hypothesis, $\mathrm{Hom}_R(N, E^r) = 0$. If $r = k$, we have

$$k < k + 1 = m \leq \mathrm{hgr}_R(N, M) ,$$

so $\mathrm{Ext}_R^k(N, M) = 0$. Applying $\mathrm{Hom}_R(N, -)$ to the exact sequence,

$$0 \longrightarrow \ker d^{k-1} \longrightarrow E^{k-1} \longrightarrow E^{k-1}/\ker d^{k-1} \longrightarrow 0 ,$$

we obtain the following long exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(N, \ker d^{k-1}) \longrightarrow \mathrm{Hom}_R(N, E^{k-1}) \longrightarrow \mathrm{Hom}_R(N, E^{k-1}/ \ker d^{k-1})$$
$$\longrightarrow \mathrm{Ext}_R^1(N, \ker d^{k-1}) \longrightarrow 0 .$$

But $\mathrm{Hom}_R(N, E^{k-1}) = 0$ by the induction hypothesis. So

$$\mathrm{Hom}_R(N, E^{k-1}/\ker d^{k-1}) \cong \mathrm{Ext}_R^1(N, \ker d^{k-1}) \cong \mathrm{Ext}_R^k(N, M) = 0 .$$

The last isomorphism is Theorem 1.2.9 (i). Therefore,

$$0 = \mathrm{Hom}_R(N, E^{k-1}/ \ker d^{k-1}) \cong \mathrm{Hom}_R(N, \mathrm{Im}\, d^{k-1}) .$$

But E^k is an essential extension of $\ker d^k = \operatorname{Im} d^{k-1}$, hence, by Lemma 1.3.1, $\operatorname{Hom}_R(N, E^k) = 0$ for N is finitely generated. Therefore, for each $r < m \leq k + 1$, $\operatorname{Hom}_R(N, E^r) = 0$ and the induction is complete.

(ii) ⇒ (i) is trivial for if $\operatorname{Hom}_R(N, E^r) = 0$ then, $\operatorname{Ext}^r_R(N, M) = 0$ simply because $\operatorname{Ext}^r_R(N, M)$ is a quotient of two submodules of $\operatorname{Hom}_R(N, E^r)$.

<u>Remark</u>: Theorem 2.3.1 may not be true if N is not finitely generated. For example let $(R, \mathcal{M})$ be a local (noetherian) ring and let $p \neq \mathcal{M}$ be any prime ideal of R and let $E(R/p)$ be the injective envelope of R/p. Consider the minimal injective resolution of $R/\mathcal{M}$,

$$0 \longrightarrow R/\mathcal{M} \longrightarrow E^o \longrightarrow E^1 \cdots$$

Then $0 < \operatorname{hgr}_R(E(R/p), R/\mathcal{M})$. But $\operatorname{Hom}_R(E(R/p), E^o) \neq 0$.

The next Corollary shows that $\operatorname{Ext}^n_R(N, M)$ and $\operatorname{Hom}_R(N, \ker d^n)$ are isomorphic R-modules for $n \leq \operatorname{hgr}_R(N; M)$ and N finitely generated.

<u>Corollary 2.3.2</u>: Let N, M be R-modules with N finitely generated, and let

$$0 \longrightarrow M \longrightarrow E^{o} \longrightarrow E^{1} \longrightarrow \cdots\cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^{n} \xrightarrow{d^{n}} E^{n+1} \longrightarrow \cdots$$

be a minimal injective resolution of M. Then for $n \leq hgr_R(N; M)$,

$$Ext_R^n(N, M) \cong Hom_R(N, \ker d^n) .$$

<u>Proof</u>: For $n < hgr_R(N; M)$, we have $Ext_R^n(N, M) = 0$. Also, by Theorem 2.3.1, $Hom_R(N, E^n) = 0$, hence $Hom_R(N, \ker d^n) = 0$. Therefore

$$0 = Ext_R^n(N, M) \cong Hom_R(N, \ker d^n).$$

For $n = hgr_R(N; M)$, we have

$$Ext_R^n(N, M) = \ker Hom_R(i_N, d^n) \,/\, Im\, Hom_R(i_N, d^{n-1})$$

But, by Theorem 2.3.1, $Hom_R(N, E^{n-1}) = 0$. So $Im\, Hom_R(i_N, d^{n-1}) = 0$. Therefore

$$Ext_R^n(N, M) \cong \ker Hom_R(i_N, d^n).$$

There is now left to show that $\ker Hom_R(i_N, d^n) = Hom_R(N, \ker d^n)$.

So, let $f \in \ker Hom_R(i_N, d^n)$, then $f \in Hom_R(N, E^n)$ and

$$d^n \circ f = d^n \circ f \circ i_N = Hom_R(i_N, d^n)\ (f) = 0 .$$

Hence $d^n(f(N)) = 0$ and $f(N) \subseteq \ker d^n$. Therefore $f \in Hom_R(N, \ker d^n)$.

Now let $g \in Hom_R(N, \ker d^n)$, then $g \in Hom_R(N, E^n)$ and $(d^n \circ g)(N) = d^n(g(N)) = 0$ for $g(N) \subseteq \ker d^n$. So,

$$0 = d^n \circ g = Hom_R(i_N, d^n)\ (g) .$$

Hence $g \in \ker Hom_R(i_N, d^n)$. Therefore,

$$\ker Hom_R(i_N, d^n) = Hom_R(N, \ker d^n) .$$

This completes the Proof of the Corollary.

The following Corollary together with 2.1.4 implies,

$$hgr_R(N \otimes_R L; M) \geq \max\{hgr_R(N; M), hgr_R(L; M)\}$$

for N, L finitely generated R-modules.

Corollary 2.3.3: Let N, L, M be R-modules with N, L finitely generated. Then

$$hgr_R(N \otimes_R L; M) = hgr_R(R/(ann_R N + ann_R L); M) .$$

Proof: Since $ann_R N + ann_R L \subseteq ann_R(N \otimes_R L)$, we have an exact sequence,

$$0 \to ann_R(N \otimes_R L)/(ann_R N + ann_R L) \to R/(ann_R N + ann_R L) \to R/ann_R(N \otimes_R L) \to 0.$$

So, by the Remark on Theorem 2.1.4, we have:

$$\operatorname{hgr}_R(R/(\operatorname{ann}_R N + \operatorname{ann}_R L); M) \leq \operatorname{hgr}_R(R/\operatorname{ann}_R(N \otimes_R L); M) = \operatorname{hgr}_R(N \otimes_R L; M)$$

(Note: since L, N are finitely generated then $N \otimes_R L$ is finitely generated. Hence, by Corollary 2.2.3,

$$\operatorname{hgr}_R(N \otimes_R L; M) = \operatorname{hgr}_R(R/\operatorname{ann}_R(N \otimes_R L); M)) .$$

Now, consider a minimal injective resolution of M,

$$0 \longrightarrow M \xrightarrow{\in} E^o \xrightarrow{d^o} E^1 \longrightarrow \cdots\cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \cdots .$$

Let $r < \operatorname{hgr}_R(N \otimes_R L; M)$. Then, by Theorem 2.3.1, we have for all $s \leq r$, $\operatorname{Hom}_R(N \otimes_R L, E^s) = 0$. So,

$$\operatorname{Hom}_R(N, \operatorname{Hom}_R(L, E^s)) \cong \operatorname{Hom}_R(N \otimes_R L, E^s) = 0 .$$

Since N is finitely generated then, by Corollary 1.3.2, we have

$$\operatorname{Hom}_R(R/\operatorname{ann}_R N, \operatorname{Hom}_R(L, E^s)) = 0.$$

So,

$$\operatorname{Hom}_R(L, \operatorname{Hom}_R(R/\operatorname{ann}_R N, E^s)) \cong \operatorname{Hom}_R(L \otimes_R (R/\operatorname{ann}_R N), E^s))$$

$$\cong \operatorname{Hom}_R(R/\operatorname{ann}_R N, \operatorname{Hom}_R(L, E^s)) = 0.$$

Since L is finitely generated then, by the same Corollary, we have

$$\operatorname{Hom}_R(R/\operatorname{ann}_R L, \operatorname{Hom}_R(R/\operatorname{ann}_R N, E^s)) = 0.$$

Therefore,

$$\operatorname{Hom}_R(R/(\operatorname{ann}_R N + \operatorname{ann}_R L), E^s) \cong \operatorname{Hom}_R(R/\operatorname{ann}_R L \otimes_R R/\operatorname{ann}_R N, E^s)$$

$$\cong \operatorname{Hom}_R(R/\operatorname{ann}_R L, \operatorname{Hom}_R(R/\operatorname{ann}_R N, E^s)) = 0$$

for all $s \leq r < \operatorname{hgr}_R(N \otimes_R L; M)$. Then, by Theorem 2.3.1,

$$\operatorname{hgr}_R(N \otimes_R L; M) \leq \operatorname{hgr}_R(R/(\operatorname{ann}_R N + \operatorname{ann}_R L); M) .$$

Therefore

$$\operatorname{hgr}_R(N \otimes_R L; M) = \operatorname{hgr}_R(R/(\operatorname{ann}_R N + \operatorname{ann}_R L); M) .$$

The following Corollary includes

$$\operatorname{Ext}_R^n(N, M) \cong \operatorname{Hom}_R(N, \operatorname{Ext}_R^n(R/\operatorname{ann}_R N, M))$$

for $n \leq \operatorname{hgr}_R(N; M)$.

Corollary 2.3.4: Let N, L, M be R-modules where N, L are finitely generated. Then for

$$n \le \mathrm{hgr}_R(L;\, M),\ \mathrm{Ext}_R^n(N \otimes_R L,\, M) \cong \mathrm{Hom}_R(N,\, \mathrm{Ext}_R^n(L,\, M))\ .$$

Proof: By Corollary 2.3.3:

$$\mathrm{hgr}_R(L;\, M) \le \mathrm{hgr}_R(N \otimes_R L;\, M).$$

Now let

$$0 \longrightarrow M \xrightarrow{\in} E^o \xrightarrow{d^o} E^1 \longrightarrow \cdots\cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \cdots\cdots$$

be a minimal injective resolution of M. Let

$$n \le \mathrm{hgr}_R(L;\, M) \le \mathrm{hgr}_R(N \otimes_R L;\, M).$$

Then by Corollary 2.3.2,

$$\mathrm{Ext}_R^n(N \otimes_R L,\, M) \cong \mathrm{Hom}_R(N \otimes_R L,\, \ker d^n) \cong \mathrm{Hom}_R(N,\, \mathrm{Hom}_R(L,\, \ker d^n))$$

$$\cong \mathrm{Hom}_R(N,\, \mathrm{Ext}_R^n(L,\, M))\ .$$

For any R-modules N, M, the following Corollary relates $\mathrm{hgr}_R(N;\, M)$ to $\mathrm{hgr}_R(R/\mathrm{ann}_R\, N;\, M)$.

Corollary 2.3.5: Let N, M be R-modules. Then

$$\mathrm{hgr}_R(N;\ M) \geq \mathrm{hgr}_R(R/\mathrm{ann}_R\ N;\ M)\ .$$

Proof: Let $N = \sum_{i\in I} Rv_i$. Consider a minimal injective resolution of M

$$0 \longrightarrow M \longrightarrow E^o \xrightarrow{d^o} E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \xrightarrow{d^{n-1}} E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \cdots$$

Let $s < \mathrm{hgr}_R(R/\mathrm{ann}_R\ N;\ M)$. Now applying $\mathrm{Hom}_R(-,\ E^s)$ to the exact sequence,

$$0 \longrightarrow \ker f \longrightarrow \bigoplus_I R/\mathrm{ann}_R\ N \xrightarrow{f} N \longrightarrow 0\ ,$$

where

$$f(\sum_{i\in I} \bar{r}_i) = \sum_{i\in I} r_i v_i$$

and

$$\bar{r}_i = r_i + \mathrm{ann}_R\ N;$$

we obtain the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(N,\ E^s) \longrightarrow \prod_I \mathrm{Hom}_R(R/\mathrm{ann}_R\ N,\ E^s) \longrightarrow \mathrm{Hom}_R(\ker f,\ E^s) \longrightarrow 0\ ;$$

But since $s < \mathrm{hgr}_R(R/\mathrm{ann}_R\ N;\ M)$ then, by Theorem 2.3.1, we have

$Hom_R(R/ann_R N, E^S) = 0$. Hence, $Hom_R(N, E^S) = 0$. So, $Ext_R^S(N, M) = 0$ for all $s < hgr_R(R/ann_R N; M)$. Therefore, $hgr_R(R/ann_R N; M) \leq hgr_R(N; M)$.

CHAPTER 3

THE COHOMOLOGICAL GRADE OF A MODULE

3.1 Definitions and Some Properties

In this section we will dualize the definition of the homological grade of a unitary R-module M in a unitary R-module N, and define the cohomological grade of M in N, which we denote by $\mathrm{cohgr}_R(N; M)$, as follows:

$$\mathrm{cohgr}_R(N; M) = \inf\{n : \mathrm{Tor}_n^R(N, M) \neq 0\}$$

So, we have $0 \leq \mathrm{cohgr}_R(N; M) \leq \infty$.

To study some of the properties of the cohomological grade, we will start with the following proposition.

Proposition 3.1.2: Let $L \subseteq N$, M be R-modules with N finitely generated and let $\mathrm{ann}_R N = \sum_{i \in I} Ra_i$. If $N \otimes_R M = 0$ (so, $R/\mathrm{ann}_R N \otimes_R M = 0$ by Lemma 1.3.3), then the exact sequence,

$$0 \longrightarrow M' \xrightarrow{g} \bigoplus_{i \in I} M_i \xrightarrow{f} M \longrightarrow 0$$

where $M = M_i$ for all $i \in I$, $f(\sum_{i \in I} m_i) = \sum_{i \in I} a_i m_i$ and $M' = \ker f$,

gives rise to an exact sequence

$$0 \longrightarrow \mathrm{Tor}^R_{n+1}(L, M) \xrightarrow{\alpha_n} \mathrm{Tor}^R_n(L, M') \xrightarrow{\mathrm{Tor}^R_n(id,\ g)} \bigoplus_{i \in I} \mathrm{Tor}^R_n(L, M_i) \longrightarrow 0$$

for all $n \geq 0$ where α_n is the connecting homomorphism. Moreover, $n < \mathrm{cohgr}_R(L; M)$ if and only if $n \leq \mathrm{cohgr}_R(L; M')$ and if $n < \mathrm{cohgr}_R(L; M)$, α_n is an isomorphism.

<u>Proof</u>: Applying $\mathrm{Tor}_R(L, -)$ to the exact sequence:

$$0 \longrightarrow M' \xrightarrow{g} \bigoplus_I M_i \xrightarrow{f} M \longrightarrow 0 ,$$

we obtain the following long exact sequence:

$$\cdots\longrightarrow \mathrm{Tor}^R_{n+1}(L, \bigoplus_I M_i) \xrightarrow{\mathrm{Tor}^R_{n+1}(id,\ f)} \mathrm{Tor}^R_{n+1}(L, M) \xrightarrow{\alpha_n} \mathrm{Tor}^R_n(L, M')$$

$$\xrightarrow{\mathrm{Tor}^R_n(id,\ g)} \mathrm{Tor}^R_n(L, \bigoplus_I M_i) \xrightarrow{\mathrm{Tor}^R_n(id,\ f)} \mathrm{Tor}^R_n(L,M) \longrightarrow \cdots \longrightarrow \mathrm{Tor}^R_o(L,M) \longrightarrow 0 ,$$

where α_n is the connecting homomorphism. Since for each

$$i \in I,\ a_i \in \mathrm{ann}_R N \subseteq \mathrm{ann}_R L ,$$

we have a_i annihilates $\mathrm{Tor}_n^R(L, \bigoplus_I M_i)$ for $n \geq 0$. So for $n \geq 0$, the homomorphism:

$$\mathrm{Tor}_n^R(id, f) : \mathrm{Tor}_n^R(L, \bigoplus_I M_i) \longrightarrow \mathrm{Tor}_n^R(L, M)$$

is zero. Therefore, for each $n \geq 0$ we have the following short exact sequence

$$0 \longrightarrow \mathrm{Tor}_{n+1}^R(L, M) \xrightarrow{\alpha_n} \mathrm{Tor}_n^R(L, M') \xrightarrow{\mathrm{Tor}_n^R(id,\, g)} \mathrm{Tor}_n^R(L, \bigoplus_I M_i) \longrightarrow 0 .$$

But

$$\mathrm{Tor}_n^R(L, \bigoplus_I M_i) \cong \bigoplus_I \mathrm{Tor}_n^R(L, M_i)$$

Therefore, there results the short exact sequences of the statement of the Proposition. Now, when

$$n < \mathrm{cohgr}_R(L; M),\ \mathrm{Tor}_n^R(L, M_i) = 0.$$

Hence,

$$\alpha_r : \mathrm{Tor}_{r+1}^R(L, M) \longrightarrow \mathrm{Tor}_r^R(L, M')$$

is an isomorphism for $r \leq n$ and for

$$r < n,\ 0 = \mathrm{Tor}_{r+1}^R(L, M) \cong \mathrm{Tor}_r^R(L, M')$$

i.e. $n \le \mathrm{cohgr}_R\,(L, M')$. Conversely, if $n \le \mathrm{cohgr}_R(L, M')$, then for $r < n$, we have

$$\mathrm{Tor}^R_{r+1}(L, M) = 0 = \mathrm{Tor}^R_r(L, M).$$

i.e. $n < \mathrm{cohgr}_R(L; M)$.

For the R-modules $M, L \subseteq N$ where N is finitely generated, the next Proposition relates $\mathrm{cohgr}_R(N; M)$ to $\mathrm{cohgr}_R(L; M)$.

<u>Proposition 3.1.3</u>: Let $L \subseteq N, M$ be R-modules with N finitely generated. Then $\mathrm{cohgr}_R(N; M) \le \mathrm{cohgr}_R(L; M)$.

<u>Proof</u>: Proceed by induction on $n \ge 0$ to show $n \le \mathrm{cohgr}_R(N; M)$ implies $n \le \mathrm{cohgr}_R(L; M)$. For $n = 0$, $0 \le \mathrm{cohgr}_R(L; M)$ is trivial. Now suppose it is true for $n \ge 0$. For $n+1$ we have $0 < n + 1 \le \mathrm{cohgr}_R(N; M)$. Hence $N \otimes_R M = 0$. And since $n < n+1 \le \mathrm{cohgr}_R(N; M)$, then, by Proposition 3.1.2, we have $n \le \mathrm{cohgr}_R(N; M')$. Therefore, by the induction hypothesis, $n \le \mathrm{cohgr}_R(L, M')$. So again, by Proposition 3.1.2, $n < \mathrm{cohgr}_R(L; M)$. Therefore $n + 1 \le \mathrm{cohgr}_R(L; M)$. This completes the induction, and hence the Proof of the Proposition.

The following Proposition relates $\mathrm{Tor}^R_n(N/L, M)$ to $\mathrm{Tor}^R_n(N, M)$ in terms of the projection $\pi : N \longrightarrow N/L$.

Proposition 3.1.4: Let $L \subseteq N$, M be R-modules with N finitely generated. Then for all $n \leq \mathrm{cohgr}_R(L; M)$, the R-homomorphism

$$\mathrm{Tor}_n^R(\pi, \mathrm{id}) : \mathrm{Tor}_n^R(N, M) \longrightarrow \mathrm{Tor}_n^R(N/L, M)$$

is an epimorphism which is an isomorphism when $n < \mathrm{cohgr}_R(L; M)$.

Proof: Applying $\mathrm{Tor}^R(-, M)$ to the exact sequence,

$$0 \longrightarrow L \xrightarrow{i} N \xrightarrow{\pi} N/L \longrightarrow 0 ,$$

we obtain the following long exact sequence:

$$\cdots\cdot \longrightarrow \mathrm{Tor}_{n+1}^R(N/L, M) \xrightarrow{\alpha_{n+1}} \mathrm{Tor}_n^R(L, M) \xrightarrow{\mathrm{Tor}_n^R(i, \mathrm{id})} \mathrm{Tor}_n^R(N, M)$$
$$\xrightarrow{\mathrm{Tor}_n^R(\pi, \mathrm{id})} \mathrm{Tor}_n^R(N/L, M) \xrightarrow{\alpha_n} \mathrm{Tor}_{n-1}^R(L, M) \longrightarrow \cdots\cdot$$

So, for $n \leq \mathrm{cohgr}_R(L; M)$, we have $n - 1 < \mathrm{cohgr}_R(L; M)$. Hence, $\mathrm{Tor}_{n-1}^R(L, M) = 0$ and then,

$$\mathrm{Tor}_n^R(\pi, \mathrm{id}) : \mathrm{Tor}_n^R(N, M) \longrightarrow \mathrm{Tor}_n^R(N/L, M)$$

is an epimorphism. And if $n < \mathrm{cohgr}_R(L; M)$ then $\mathrm{Tor}_n^R(L, M) = 0$. Hence $\mathrm{Tor}_n^R(\pi, \mathrm{id})$ is an isomorphism.

Theorem 3.1.5: Let $L \subseteq N$, M be R-modules where N is finitely generated. Then, $\text{cohgr}_R(N; M) = \min\{\text{cohgr}_R(L; M), \text{cohgr}_R(N/L, M)\}$.

Proof: By Proposition 3.1.3, $\text{cohgr}_R(N; M) \leq \text{cohgr}_R(L; M)$. Now, if $\text{cohgr}_R(N; M) < \text{cohgr}_R(L; M)$, then, by Proposition 3.1.4,

$$\text{Tor}_n^R(\pi, \text{id}) : \text{Tor}_n^R(N, M) \longrightarrow \text{Tor}_n^R(N/L, M)$$

is an isomorphism for all $n \leq \text{cohgr}_R(N; M)$. Hence

$$\text{cohgr}_R(N; M) = \text{cohgr}_R(N/L; M) .$$

And if

$$\text{cohgr}_R(N; M) = \text{cohgr}_R(L; M) ,$$

then by Proposition 3.1.4, we have $\text{Tor}_n^R(\pi, \text{id})$ is an epimorphism for all $n \leq \text{cohgr}_R(N; M)$. Then we can conclude that $n \leq \text{cohgr}_R(N/L; M)$. Hence,

$$\text{cohgr}_R(N; M) \leq \text{cohgr}_R(N/L; M) .$$

Therefore,

$$\text{cohgr}_R(N; M) = \min\{\text{cohgr}_R(L; M), \text{cohgr}_R(N/L; M)\} .$$

Remark: For an exact sequence,

$$0 \longrightarrow N' \xrightarrow{f} N \longrightarrow N'' \longrightarrow 0 ,$$

of R-modules with N finitely generated and for any R-module M, we have

$$\mathrm{cohgr}_R(N; M) = \min \{\mathrm{cohgr}_R(N'; M), \mathrm{cohgr}_R(N''; M)\} .$$

This follows from Theorem 3.1.5 and the fact that

$$N' \cong f(N'), \; N'' \cong N/f(N') .$$

3.2 The Extended M-Cosequence

For an ideal A of R and an R-module M, the notion of an M-cosequence of length n in A can be extended as follows. For n = 0 the M-cosequence is empty and for n = 1 it is a subset $\{b_j \mid j \in J\} = \beta \subseteq A$ such that $(\sum_{j \in J} R b_j) M = M$.

For n > 1, we may define, inductively, an M-cosequence of length n in A as a sequence

$$\beta_1 = \{b_{1j} \mid j \in J_1^{\beta}\}, \beta_2 = \{b_{2j} \mid j \in J_2^{\beta}\}, \ldots, \beta_n = \{b_{nj} \mid j \in J_n^{\beta}\}$$

of subsets of A such that β_1 is an M-cosequence of length 1 in A and $\beta_2, \beta_3, \ldots, \beta_n$ is an M_1^β-cosequence of length n-1 in A where

$$M_1^\beta = \ker \left(\bigoplus_{j \in J_1^\beta} M \xrightarrow{f_1} M \longrightarrow 0 \right)$$

and

$$f_1\left(\sum_{j \in J_1^\beta} m_j \right) = \sum_{j \in J_1^\beta} b_{1j}\, m_j \; .$$

Thus in the notion of cosequence due to David J. Moore [11] the sets β_i are singletons.

The main theorem in this section is the next one. But before we state it, we need some notation.

Let A be an ideal of R, M and R-module and let $\beta_1, \beta_2, \ldots, \beta_n$ be an M-cosequence in A where $\beta_i = \{b_{ij} \mid j \in J_i^\beta\} \subseteq A$. Now let $M_o^\beta = M$ and for $1 \leq i \leq n$ Let

$$M_i^\beta = \ker \left(\bigoplus_{J_i^\beta} M_{i-1}^\beta \xrightarrow{f_i^\beta} M_{i-1}^\beta \longrightarrow 0 \right) ,$$

where

$$f_i^\beta\Big(\sum_{j\in J_i^\beta} m_{(i-1)j}\Big) = \sum_{j\in J_i^\beta} b_{ij}\, m_{(i-1)j}\ ,$$

so that $\beta_i, \beta_{i+1}, \ldots, \beta_n$ is an M_{i-1}^β-cosequence in A of length $(n-(i-1))$.

<u>Theorem 3.2.1</u>: Let N, M be R-modules with N finitely generated and let $\beta_1, \beta_2, \ldots, \beta_n$ be an M-cosequence in $\mathrm{ann}_R N$. Then $n \leq \mathrm{cohgr}_R(N; M)$ and if $n < \mathrm{cohgr}_R(N; M)$, there exists $\beta_{n+1} \subseteq \mathrm{ann}_R N$ such that $\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1}$ is an M-cosequence in $\mathrm{ann}_R N$.

<u>Proof</u>: Proceed by induction on n. For $n = 0$, $n \leq \mathrm{cohgr}_R(N; M)$ is trivial and if $0 < \mathrm{cohgr}_R(N; M)$, then $N \otimes_R M = 0$. So, by Lemma 1.3.3, $R/\mathrm{ann}_R N \otimes_R M = 0$. Hence $(\mathrm{ann}_R N)M = M$ and $\beta_1 = \mathrm{ann}_R N$ is an M-cosequence in $\mathrm{ann}_R N$. Now assume it is true for $0 \leq n \leq m$. For $n = m + 1$, let $\beta_1, \beta_2, \ldots, \beta_n$ be an M-cosequence in $\mathrm{ann}_R N$. Then $\beta_2, \ldots, \beta_n$ is an M_1^β-cosequence in $\mathrm{ann}_R N$ of length m where

$$M_1^\beta = \ker\Big(\oplus_{J_1^\beta} M_o^\beta \xrightarrow{f_1^\beta} M_o^\beta \longrightarrow 0\Big)\ ,$$

and $M_o^\beta = M$. So, by the induction hypothesis, $m \leq \mathrm{cohgr}_R(N; M_1^\beta)$ and if $m < \mathrm{cohgr}_R(N; M_1^\beta)$ there exists $\beta_{m+2} \subseteq \mathrm{ann}_R N$ such that $\beta_2, \ldots, \beta_{m+1}, \beta_{m+2}$

is an M_1^{β}-cosequence in $\mathrm{ann}_R N$. Now for every $b_{1j} \in \beta_1 \subseteq \mathrm{ann}_R N$ and for $r \geq 0$, we have b_{1j} annihilates $\mathrm{Tor}_r^R(N, M)$. So in the long exact sequence which results from applying $\mathrm{Tor}^R(N, —)$ to the exact sequence,

$$0 \longrightarrow M_1^{\beta} \xrightarrow{g} \bigoplus_{j \in J_1^{\beta}} M \xrightarrow{f_1^{\beta}} M \longrightarrow 0$$

where $f_1^{\beta}(\sum_{j \in J_1^{\beta}} m_j) = \sum_{j \in J_1^{\beta}} b_{1j} m_j$ and g is the inclusion, the homomorphism

$$\mathrm{Tor}_r^R(\mathrm{id}_N, f_1^{\beta}) : \mathrm{Tor}_r^R(N, \bigoplus_{j \in J_1^{\beta}} M) \longrightarrow \mathrm{Tor}_r^R(N, M)$$

is zero. Therefore, for $r \geq 0$, we obtain the following short exact sequence:

$$0 \longrightarrow \mathrm{Tor}_{r+1}^R(N, M) \xrightarrow{\alpha_r} \mathrm{Tor}_r^R(N, M_1^{\beta}) \xrightarrow{\mathrm{Tor}_r^R(\mathrm{id}_N, g)} \bigoplus_{j \in J_1^B} \mathrm{Tor}_r^R(N,M) \longrightarrow 0$$

where α_r is the connecting homomorphism. Now, for $r < m$, the central term of the above exact sequence is zero, therefore the first and the last terms are zero for each $r < m$. i.e. $\mathrm{Tor}_R^r(N, M) = 0$ for each $s < m+1$. Therefore $n = m + 1 \leq \mathrm{cohgr}_R(N, M)$. And if $n = m + 1 < \mathrm{cohgr}_R(N; M)$, then for $r = m$, we have the first and the last terms of the above exact sequence are zero. Hence the central term is zero. Consequently $m < \mathrm{cohgr}_R(N; M_1^{\beta})$. So, by induction, there exists $\beta_{m+2} \subseteq \mathrm{ann}_R N$ such that

$\beta_2, \ldots, \beta_{m+1}, \beta_{m+2}$ is an M_1^{β}-cosequence in $\mathrm{ann}_R N$. So, by definition, $\beta_1, \beta_2, \ldots, \beta_{m+1}, \beta_{m+2}$ is an M-cosequence in $\mathrm{ann}_R N$. This completes the induction and hence the Proof of the Theorem.

The following Corollary shows that $\mathrm{cohgr}_R(N; M)$ is the upper bound of the lengths of all M-cosequences in the annihilator of N.

Corollary 3.2.2: Let N, M be R-modules with N finitely generated. Then

(i) If $n = \mathrm{cohgr}_R(N; M)$ is finite, then every M-cosequence in $\mathrm{ann}_R N$ has length $\leq n$, and can be extended to an M-cosequence of length n.

(ii) If $\mathrm{cohgr}_R(N; M) = \infty$, then no finite M-cosequence in $\mathrm{ann}_R N$ is maximal.

Proof: Follows immediately from the Theorem.

Remark: Let N, M be as in Corollary 3.2.2. Then it follows immediately, from Corollary 3.2.2 and the fact that $\mathrm{ann}_R N = \mathrm{ann}_R(R/\mathrm{ann}_R N)$, that $\mathrm{cohgr}_R(N; M) = \mathrm{cohgr}_R(R/\mathrm{ann}_R N; M)$.

For a finitely generated R-module N and an R-module M, the next Proposition relates $\mathrm{cohgr}_R(N; M)$ to $\mathrm{hgr}_R(M; N)$.

<u>Proposition 3.2.3</u>: Let N, M be R-modules with N finitely generated; then $\mathrm{cohgr}_R(N; M) \leq \mathrm{hgr}_R(M; N)$.

<u>Proof</u>: Proceed by induction on $n \geq 0$ to show $n \leq \mathrm{cohgr}_R(N; M)$ implies $n \leq \mathrm{hgr}_R(M; N)$. For $n = 0$, $0 \leq \mathrm{hgr}_R(M; N)$ is trivial. For $n = 1$, we have $0 < \mathrm{cohgr}_R(N; M)$. So, by Theorem 3.2.1, there exists a subset $\beta_1 = \{b_{1j} \mid j \in J_1^{\beta}\}$ of $\mathrm{ann}_R N$ such that β_1 is an M-cosequence in $\mathrm{ann}_R N$. So, we have an exact sequence:

$$0 \longrightarrow M_1^{\beta} \xrightarrow{\ g\ } \bigoplus_{J_1^{\beta}} M \xrightarrow{\ f_1^{\beta}\ } M \longrightarrow 0$$

where $f_1^{\beta}(\sum_{j \in J_1^{\beta}} m_j) = \sum_{j \in J_1^{\beta}} b_{1j} m_j$, $M_1^{\beta} = \ker f_1^{\beta}$ and g is the inclusion map. Now for every $b_{1j} \in \beta_1 \subseteq \mathrm{ann}_R N$ and for $r \geq 0$, b_{1j} annihilates $\mathrm{Ext}_R^r(M, N)$. So, in the long exact sequence which results from applying $\mathrm{Hom}_R(-, N)$ to the above exact sequence, the homomorphism:

$$\mathrm{Ext}^r_R(f^\beta_1, \mathrm{id}_N) : \mathrm{Ext}^r_R(M, N) \longrightarrow \prod_{J^\beta_1} \mathrm{Ext}^r_R(M, N)$$

is zero. Therefore, we obtain the following short exact sequence:

$$0 \longrightarrow \prod_{J^\beta_1} \mathrm{Ext}^r_R(M, N) \xrightarrow{\mathrm{Ext}^r_R(g,\ \mathrm{id}_N)} \mathrm{Ext}^r_R(M^\beta_1, N) \xrightarrow{\alpha_r} \mathrm{Ext}^{r+1}_R(M, N) \longrightarrow 0 ,$$

where α_r is the connecting homomorphism. Also for $r = 0$ the homomorphism $\mathrm{Ext}^0_R(f^\beta_1, \mathrm{id}_N)$ is zero and at the same time is a monomorphism. Therefore $\mathrm{Hom}_R(M, N) = 0$ and hence $1 \le \mathrm{hgr}_R(M, N)$. Now assume it is true for $1 \le k \le n$. For $k = n + 1 \le \mathrm{cohgr}_R(N; M)$ we have $n \le \mathrm{cohgr}_R(N; M^\beta_1)$. So, by induction, $n \le \mathrm{hgr}_R(M^\beta_1; N)$. Hence for $r < n$, the central term of the above exact sequence is zero. Therefore the first and the last terms are zero. i.e., $n + 1 \le \mathrm{hgr}_R(M; N)$. This completes the induction and hence the Proof of the Proposition.

<u>Remark</u>

(i)a. The condition that N is finitely generated in Proposition 3.2.3 is necessary. For example consider the ring $R = k[x]$, where k is a field and consider the R-module $k[x^{-1}]$, where

$$x^n\, x^{-m} = \begin{cases} x^{-(m-n)} & \text{if } m \ge n \\ 0 & \text{if } m < n \end{cases} .$$

Then,

$$\operatorname{cohgr}_R(k[x^{-1}];\ k[x^{-1}]) > 0$$

but

$$\operatorname{hgr}_R(k[x^{-1}];\ k[x^{-1}]) = 0$$

b. Strict inequality may hold in Proposition 3.2.3, even if both N, M are finitely generated. For example, let R be a ring and let $x \in R$ be a non-unit non-zero devisor. Then

$$0 = \operatorname{cohgr}_R(R;\ R/Rx) < \operatorname{hgr}_R(R/Rx;\ R)$$

ii) If both N, M are finitely generated and if $N \otimes M = 0$ then by Corollary 1.3.4, we have

$$\operatorname{cohgr}_R(N;\ M) = \operatorname{cohgr}_R(M;\ N) = \operatorname{hgr}_R(N;\ M) = \operatorname{hgr}_R(M;\ N) = \infty$$

3.3 Injective and Projective Dimensions and the Homological and Cohomological Grades

From the definition of the projective and injective dimensions of an R-module and the definition of the homological and cohomological grades, it is clear that if $\operatorname{hgr}_R(N;\ M) \neq \infty$, then

$$\text{hgr}(N; M) \leq \min\{\text{proj dim}_R N, \text{Inj dim}_R M\}.$$

And if $\text{cohgr}_R(N; M) \neq \infty$, then

$$\text{cohgr}_R(N; M) \leq \min\{\text{proj dim }_R N, \text{proj dim }_R M\}.$$

In this section, we will try to relate $\text{hgr}_R(N; M)$ to the projective dimension of M, and $\text{cohgr}_R(N; M)$ to the injective dimension of M.

We start this section with the following proposition.

<u>Proposition 3.3.1</u> Let $A = \sum_{i=1}^{n} Ra_i$ be an ideal of R and let E be an injective R-module. If $\text{Hom}_R(R/A, E) = 0$, then $\text{cohgr}_R(R/A; E) = \infty$.

<u>Proof</u>: Since $\text{Hom}_R(R/A, E) = 0$, we have an exact sequence,

$$0 \longrightarrow E \xrightarrow{f} \prod_{i=1}^{n} E \xrightarrow{\pi} E' \longrightarrow 0,$$

where $f(e) = (a_i e)$ for every $e \in E$. And since E is an injective R-module, the above exact sequence splits. Now, since for every $r \geq 0$, $\text{Tor}_r^R(R/A, -)$ is an additive covariant functor, then for every $r \geq 0$, we have a split exact sequence.

$$0 \longrightarrow \text{Tor}_r^R(R/A, E) \xrightarrow{\text{Tor}_r^R(\text{id}, f)} \text{Tor}_r^R(R/A, \prod_{i=1}^{n} E)$$

$$\xrightarrow{\text{Tor}_r^R(\text{id}, \pi)} \text{Tor}_r^R(R/A, E') \longrightarrow 0$$

Now, for any projective resolution,

$$\cdots\longrightarrow P_m \xrightarrow{d_m} P_{m-1} \longrightarrow\cdots\longrightarrow P_1 \xrightarrow{d_1} P_o \xrightarrow{d_o} E \longrightarrow 0$$

of E, we obtain a projective resolution,

$$\cdots\longrightarrow \prod_{i=1}^{n} P_m \xrightarrow{\prod_{i=1}^{n} d_m} \prod_{i=1}^{n} P_{m-1} \longrightarrow\cdots\longrightarrow \prod_{i=1}^{n} P_1$$

$$\xrightarrow{\prod_{i=1}^{n} d_1} \prod_{i=1}^{n} P_o \xrightarrow{\prod_{i=1}^{n} d_o} \prod_{i=1}^{n} E \longrightarrow 0$$

of $\prod_{i=1}^{n} E$. And for every $r \geq 0$, the homomorphisms, $f_r : P_r \longrightarrow \prod_{i=1}^{n} P_r$ defined by $f(x) = (a_i\, x)$ form a chain of maps over f. So, the homomorphism, $Tor_r^R(id, f) : Tor_r^R(R/A, E) \longrightarrow Tor_r^R(R/A, \prod_{i=1}^{n} E)$ is a multiplication by (a_i) and hence a zero map. Therefore, $Tor_r^R(R/A, E) = 0$ for all $r \geq 0$. Hence, $cohgr_R(R/A; E) = \infty$.

The following Corollary includes $hgr_R(R/A; R) \leq Inj\ dim_R R$ where $A \neq R$ is a finitely generated ideal of R.

<u>Corollary 3.3.2</u>: Let M be an R-module and A a finitely generated ideal of R. If $Inj.dim_R\, M < hgr_R(R/A; M)$, then $cohgr_R(R/A; M) = \infty$.

Proof: Let Inj $\dim_R M < \text{hgr}_R(R/A; M)$. We will prove by induction on $n = \text{Inj} \dim_R M$ that $\text{cohgr}_R(R/A; M) = \infty$. If $n = 0$ then M is injective and since $\text{Hom}_R(R/A, M) = 0$, then, by Proposition 3.3.1, $\text{cohgr}_R(R/A; M) = \infty$. Now suppose it is true for $n \geq 0$. For $n+1$, let

$$0 \longrightarrow M \longrightarrow E^o \xrightarrow{d^o} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^n \xrightarrow{d^n} E^{n+1} \longrightarrow 0$$

be the minimal injective resolution of M. Then,

$$0 \longrightarrow \ker d^1 \longrightarrow Q^o \xrightarrow{h^o} Q^1 \xrightarrow{h^1} \cdots \longrightarrow Q^{n-1} \xrightarrow{h^{n-1}} Q^n \longrightarrow 0,$$

where $Q^i = E^{i+1}$, $h^i = d^{i+1}$ for all $0 \leq i \leq n$, is the minimal injective resolution of $\ker d^1$ and $\text{Inj} \dim_R \ker d^1 = n$. Now since $n+1 < \text{hgr}_R(R/A; M)$ then, by Theorem 2.3.1, we have $\text{Hom}_R(R/A, E^r) = 0$ for all $r \leq n + 1$. Hence for all

$$r \leq n, \; 0 = \text{Hom}_R(R/A, E^{r+1}) = \text{Hom}_R(R/A, Q^r).$$

Then again by Theorem 2.3.1, we have $\text{hgr}_R(R/A; \ker d^1) > n$. So, by induction, $\text{cohgr}_R(R/A; \ker d^1) = \infty$. Since $0 < n+1 < \text{hgr}_R(R/A; M)$, then $\text{Hom}_R(R/A, E^o) = 0$ and since E^o is an injective R-module then, by Proposition 3.3.1, $\text{cohgr}_R(R/A; E^o) = \infty$. Now, we have,

$$\text{cohgr}_R(R/A; E^o) = \text{cohgr}_R(R/A; \ker d^1) = \infty.$$

Therefore from the exact sequence

$$0 \longrightarrow M \longrightarrow E^{o} \longrightarrow \ker d^{1} \longrightarrow 0,$$

we can conclude that $\text{cohgr}_R(R/A;\ M) = \infty$. This completes the induction and hence the Proof of the Corollary.

Remark: If the ring R is noetherian, then the dual of Corollary 3.3.2 is true; i.e. if $\text{proj dim}_R\ M < \text{cohgr}_R(R/A;\ M)$ then $\text{hgr}_R(R/A,\ M) = \infty$.

For the next result, we need the following proposition.

Proposition 3.3.3: Let N be a finitely generated R-module such that $\text{ann}_R N$ is a finitely generated ideal of R. Let $\{M_i\}_{i \in I}$ be an arbitrary family of R-modules and put $K = \bigoplus_{i \in I} M_i$. Then for any integer $n \le \text{hgr}_R(N;\ K)$,

$$\text{Ext}_R^n(N,\ K) \cong \bigoplus_{i \in I} \text{Ext}_R^n(N,\ M_i)$$

and hence,

$$\text{hgr}_R(N;\ K) = \inf\{\text{hgr}_R(N,\ M_i)\ |\ i \in I\}\ .$$

Proof: Use induction on $n \ge 0$ to show $n \le \text{hgr}_R(N;\ K)$ implies

$$\text{Ext}_R^n(N,\ K) \cong \bigoplus_{i \in I} \text{Ext}_R^n(N,\ M_i).$$

When $n = 0$, then, since N is finitely generated,

$$\mathrm{Hom}_R(N,\ K) \cong \bigoplus_{i\in I} \mathrm{Hom}_R(N,\ M_i)\ .$$

Suppose it is true for $n \geq 0$. For $0 < n + 1 \leq \mathrm{hgr}_R(N;\ K)$, we have $\mathrm{Hom}_R(N,\ K) = 0$ and hence, $\mathrm{Hom}_R(N,\ M_i) = 0$ for all $i \in I$. Now, let $A = \mathrm{ann}_R\ N = \sum_{j=1}^{k} Ra_j$. Then, by Corollary 1.3.2,

$$\mathrm{Hom}_R(R/A,\ K) = 0 = \mathrm{Hom}_R(R/A,\ M_i)$$

for all $i \in I$. Hence, we have the following exact sequences:

$$0 \longrightarrow K \xrightarrow{f} \prod_{j=1}^{k} K \xrightarrow{\pi} K' \longrightarrow 0\ ,$$

$$0 \longrightarrow M_i \xrightarrow{f_i} \prod_{j=1}^{k} M_i \xrightarrow{\pi_i} M_i' \longrightarrow 0\ ,$$

for all $i \in I$, where $f(x) = (a_j x)$ for all $x \in K$ and $K' = \mathrm{coker}\ f$, and, for all $i \in I$, $f_i(x_i) = (a_j\ x_i)$ for all $x_i \in M_i$ and $M_i' = \mathrm{coker}\ f_i$. Then, by Proposition 2.1.1, $n \leq \mathrm{hgr}_R(N;\ K')$ and $\mathrm{Ext}_R^n(N,\ K') \cong \mathrm{Ext}_R^{n+1}(N,\ K)$, and for all $i \in I$, $n \leq \mathrm{hgr}_R(N;\ M_i)$ and $\mathrm{Ext}_R^n(N,\ M_i') \cong \mathrm{Ext}_R^{n+1}(N,\ M_i)$. Now, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{f} & \prod_{j=1}^{k} K & \xrightarrow{\pi} & K' & \longrightarrow & 0\ , \\
 & & \Big\| \alpha & & \Big\downarrow \beta & & \Big\downarrow \gamma & & \\
0 & \longrightarrow & \bigoplus_{i\in I} M_i & \xrightarrow{\oplus f_i} & \bigoplus_{i\in I} \prod_{j=1}^{k} M_i & \xrightarrow{\oplus \pi_i} & \bigoplus_{i\in I} M'_i & \longrightarrow & 0
\end{array}$$

where α is the identity map, β is the isomorphism

$$\beta((\sum_{i\in I} m_{ij})) = \sum_{i\in I} (m_{ij})$$

and γ is the R-homomorphism induced by α and β which makes the above diagram commute. Hence, we can conclude that γ is an isomorphism. Therefore

$$n \leq \text{hgr}_R(N;\ K') = \text{hgr}_R(N;\ \bigoplus_{i\in I} M'_i).$$

So, by induction,

$$\text{Ext}_R^n(N,\ K') \cong \text{Ext}_R^n(N,\ \bigoplus_{i\in I} M'_i) \cong \bigoplus_{i\in I} \text{Ext}_R^n(N,\ M'_i).$$

Therefore,

$$\text{Ext}_R^{n+1}(N,\ K) \cong \text{Ext}_R^n(N,\ K') \cong \bigoplus_{i\in I} \text{Ext}_R^n(N,\ M'_i) \cong \bigoplus_{i\in I} \text{Ext}_R^{n+1}(N,\ M_i).$$

This completes the induction. Now we have shown that for all

$$0 \leq n \leq hgr_R(N;\ K),\ Ext_R^n(N,\ K) \cong \bigoplus_{i\in I} Ext_R^n(N,\ M_i).$$

Hence, it is easy to conclude that

$$hgr_R(N;\ K) = \inf\{hgr_R(N;\ M_i) \mid i \in I\}\ .$$

For a noetherian ring R and finitely generated R-modules N, M, H. Matsumura, in [10], Theorem 16.9, p.132, and D. Rees, in [16] Theorem 1.1, p.29, have shown that if

$$\text{grade } N = k\ (\text{grade } N = \inf\{n \mid Ext_R^n(N,\ R) \neq 0\} = hgr_R(N;\ R))$$

and if $\text{proj dim}_R\ M = \ell < k$ then, $Ext_R^i(N,\ M) = 0$ for $i < k - \ell$. From this we can conclude that $k - \ell \leq hgr_R(N;\ M)$. Or,

$$hgr_R(N;\ M) + \text{proj dim}_R\ M \geq k = \text{grade } N = hgr_R(N;\ R).$$

The following Theorem generalizes the above result in the sense that we allow M to be any R-module and drop the noetherian condition on R, but impose some condition on N which is satisfied already in the noetherian case.

Theorem 3.3.4: Let N, M be R-modules such that N, $ann_R N$ are finitely generated. Then,

$$hgr(N;\ M) + \text{proj dim}_R M \geq hgr_R(N;\ R).$$

Proof: The Proof is trivial if proj $\dim_R M = \infty$. So, let proj $\dim_R M = n$. Now we use induction on n to prove the theorem. If $n = 0$ then, M is projective R-module and hence, we can find a free R-module G such that $G \cong M \oplus M'$ for some R-module M'. Hence,

$$\operatorname{Ext}_R^k(N, G) \cong \operatorname{Ext}_R^k(N, M) \oplus \operatorname{Ext}_R^k(N, M')$$

for all $k \geq 0$. Consequently, $\operatorname{hgr}_R(N; G) \leq \operatorname{hgr}_R(N; M)$. Therefore, by Proposition 3.3.3, $\operatorname{hgr}_R(N; R) = \operatorname{hgr}_R(N; G) \leq \operatorname{hgr}_R(N; M)$. So, the theorem is true for $n = 0$.

For $n > 0$, we can find an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is a free R-module. Hence, proj $\dim_R K = n - 1$. So, by induction,

$$\operatorname{hgr}_R(N; K) + \text{proj } \dim_R K \geq \operatorname{hgr}_R(N; R).$$

Now if $r < \operatorname{hgr}_R(N; R) - n$ then,

$$r + 1 < \operatorname{hgr}_R(N; R) - (n-1) = \operatorname{hgr}_R(N; R) - \text{proj } \dim_R K \leq \operatorname{hgr}_R(N; K) .$$

i.e. $r + 1 < \operatorname{hgr}_R(N; R) = \operatorname{hgr}_R(N; F)$ and $r + 1 < \operatorname{hgr}_R(N; K)$. Now, the short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0 ,$$

gives rise to a long exact sequence,

$$\cdots\cdots\longrightarrow \operatorname{Ext}_R^k(N, K) \longrightarrow \operatorname{Ext}_R^k(N, F) \longrightarrow \operatorname{Ext}_R^k(N, M) \longrightarrow \operatorname{Ext}_R^{k+1}(N, K) \longrightarrow \cdots$$

So, for $k \leq r$, $\operatorname{Ext}_R^k(N, F) = 0 = \operatorname{Ext}_R^{k+1}(N, K)$. Consequently $\operatorname{Ext}_R^k(N, M) = 0$ for all $k \leq r$. i.e. $r < \operatorname{hgr}_R(N; M)$. Therefore, $\operatorname{hgr}_R(N; R) - n \leq \operatorname{hgr}_R(N; M)$. Or $\operatorname{hgr}_R(N, M) + n \geq \operatorname{hgr}_R(N; R)$. This completes the induction and hence the Proof of the Theorem.

<u>Remark</u>: In the above Theorem, the condition, that $\operatorname{ann}_R N$ is finitely generated, is only imposed to ensure that $\operatorname{hgr}_R(N; R) = \operatorname{hgr}_R(N; F)$ for any free R-module $F \neq 0$.

Before we state the dual of Theorem 3.3.4, we need the following Proposition.

<u>Proposition 3.3.5</u>: Let A be a finitely generated ideal of R. Let $\{M_i\}_{i\in I}$ be an arbitrary family of R-modules and put $K = \prod_{i\in I} M_i$. Then for any integer $n \leq \operatorname{cohgr}_R(R/A; K)$, $\operatorname{Tor}_n^R(R/A, K) \cong \prod_{i\in I} \operatorname{Tor}_n^R(R/A, M_i)$ and hence,

$$\operatorname{coghr}_R(R/A; K) = \inf\{\operatorname{cohgr}_R(R/A, M_i) \mid i \in I\} .$$

<u>Proof</u>: The Proof is dual to the Proof of Proposition 3.3.3 if one realizes that

$$R/A \otimes_R \prod_{i\in I} M_i \cong \prod_{i\in I} (R/A \otimes_R M_i).$$

But this is always true, since R/A is finitely presented.

We end this section by stating the following Theorem which is dual to Theorem 3.3.4.

Theorem 3.3.6: Let N, M be R-modules such that N and $ann_R N$ are finitely generated. Then, for any injective cogenerator Q of Mod-R,

$$\text{cohgr}_R(N; M) + \text{inj dim}_R M \geq \text{cohgr}_R(N; Q).$$

CHAPTER 4

SOME APPLICATIONS

4.1 Homological and Cohomological Grade of Artinian Modules

In this section, we study the homological and cohomological grades of artinian R-modules. We show that, for a finitely generated R-module N and an artinian R-module M, if $\mathrm{Hom}_R(N, M) = 0$ then, $\mathrm{hgr}_R(N; M) = \mathrm{cohgr}_R(N; M) = \infty$. And for an injective cogenerator Q of Mod-R, we show that $\mathrm{Hom}_R(M, Q)$ behaves as a noetherian R-module in the sense that for all $n \leq \mathrm{hgr}_R(N; \mathrm{Hom}_R(M, Q))$, there exists a regular $\mathrm{Hom}_R(M, Q)$-sequence $a_1, \ldots a_n$ in $\mathrm{ann}_R N$ of length n.

Proposition 4.1.1: Let M be an artinian R-module and $B = \sum_{i=1}^{n} Rb_i$ be a finitely generated ideal of R. If $\mathrm{Hom}_R(R/B, M) = 0$ then $\mathrm{hgr}_R(R/B; M) = \infty$.

Proof: Let $\mathrm{Hom}_R(R/B, M) = 0$. Then, by Proposition 2.1.1, the exact sequence:

$$0 \longrightarrow M \xrightarrow{f} \prod_{i=1}^{n} M \xrightarrow{g} M' \longrightarrow 0 \qquad (1),$$

where $f(m) = (b_i m) \in \prod_{i=1}^{n} M$ for all $m \in M$ and M' is the cokernel of f, gives rise to an exact sequence:

$$0 \longrightarrow \prod_{i=1}^{n} \mathrm{Ext}_R^r(R/B, M) \xrightarrow{\mathrm{Ext}_R^r(id,\, g)} \mathrm{Ext}_R^r(R/B, M') \xrightarrow{\alpha_r} \mathrm{Ext}_R^{r+1}(R/B, M) \longrightarrow 0 \quad (2)$$

for all $r \geq 0$, where α_r is the connecting homomorphism, and if $r < \mathrm{hgr}_R(R/B, M)$, α_r is an isomorphism. Now, since Imf is a submodule of the artinian module $\prod_{i=1}^{n} M$ then, by [5] Proposition 3, p.55, we can conclude that there is an integer $k \geq 1$ such that

$$(\mathrm{Imf} :_{\prod_{i=1}^{n} M} B) \subseteq \mathrm{Imf} + (0 :_{\prod_{i=1}^{n} M} B^k).$$

But $(0 :_{\prod_{i=1}^{n} M} B^k) = 0$ for $(0 :_{\prod_{i=1}^{n} M} B) \cong \prod_{i=1} (0 :_M B) = 0$. Hence

$(\mathrm{Imf} :_{\prod_{i=1}^{n} M} B) = \mathrm{Imf}$. Therefore,

$$0 = (\mathrm{Imf} :_{\prod_{i=1}^{n} M} B)/\mathrm{Imf} \cong (0 :_{(\prod_{i=1}^{n} M)/\mathrm{Imf}} B) \cong \mathrm{Hom}_R(R/B, (\prod_{i=1}^{n} M)/\mathrm{Imf}) = \mathrm{Hom}_R(R/B, M').$$

Now, since $0 < \mathrm{hgr}_R(R/B, M)$,

$$0 = \mathrm{Hom}_R(R/B, M') \overset{\alpha_o}{\cong} \mathrm{Ext}_R^1(R/B, M).$$

Therefore, $\operatorname{Ext}_R^1(R/B, M') \overset{\alpha_1}{\cong} \operatorname{Ext}_R^2(R/B, M)$. But also M' is artinian and $\operatorname{Hom}_R(R/B, M') = 0$. Then, by the same argument, we can show that $\operatorname{Ext}_R^1(R/B, M') = 0$. Therefore,

$$\operatorname{Ext}_R^2(R/B, M) \cong \operatorname{Ext}_R^1(R/B, M') = 0.$$

So, for every $n \geq 0$, we can repeat the argument n times to conclude that $n < \operatorname{hgr}_R(N; M)$. Therefore, $\operatorname{hgr}_R(N; M) = \infty$.

For an artinian R-module M and a finitely generated ideal B of R, the following proposition includes,

$$\operatorname{cohgr}_R(R/B; M) \geq \operatorname{hgr}_R(R/B; M) .$$

<u>Proposition 4.1.2</u>: Let M be an artinian R-module and $B = \sum_{i=1}^{n} Rb_i$ an ideal of R. If $\operatorname{Hom}_R(R/B, M) = 0$ then, $\operatorname{cohgr}_R(R/B; M) = \infty$.

<u>Proof</u>: Let $\operatorname{Hom}_R(R/B, M) = 0$. Since BM is a submodule of the artinian R-module M then, by [5] Proposition 3, p.55, we can conclude that there is an integer $k \geq 1$ such that

$$M = (BM :_M B) \subseteq BM + (0 :_M B^k) .$$

But $(0 :_M B^k) = 0$ for $(0 :_M B) \cong \mathrm{Hom}_R(R/B, M) = 0$. Therefore $M = BM$. Then, by Proposition 3.1.2, the exact sequence:

$$0 \longrightarrow M' \xrightarrow{g} \bigoplus_{i=1}^{n} M \xrightarrow{f} M \longrightarrow 0 \qquad (1),$$

where $f(\sum_{i=1}^{n} m_i) = \sum_{i=1}^{n} b_i m_i$ and $M' = \ker f$, gives rise to an exact sequence:

$$0 \longrightarrow \mathrm{Tor}^R_{r+1}(R/B, M) \xrightarrow{\alpha_r} \mathrm{Tor}^R_r(R/B, M') \xrightarrow{\mathrm{Tor}^R_r(id, g)} \bigoplus_{i=1}^{n} \mathrm{Tor}^R_r(R/B, M) \longrightarrow 0$$

for all $r \geq 0$ where α_r is the connecting homomorphism and if $r <$ $\mathrm{cohgr}_R(R/B; M)$, α_r is an isomorphism. Now, since $(R/B) \otimes M \cong M/BM = 0$, $0 < \mathrm{cohgr}_R(R/B; M)$. Therefore,

$$\mathrm{Tor}^R_1(R/B, M) \overset{\alpha_o}{\cong} \mathrm{Tor}^R_o(R/B, M') \cong (R/B) \otimes_R M' .$$

But from the exact sequence (1), we can conclude that M' is an artinian R-module and $\mathrm{Hom}_R(R/B, M') = 0$. So, by the same argument, we can show that $BM' = M'$, or $(R/B) \otimes_R M' = 0$. Therefore,

$$\mathrm{Tor}^R_1(R/B, M) \cong (R/B) \otimes_R M' = 0.$$

Hence $1 < \mathrm{cohgr}_R(R/B; M)$ and

$$\mathrm{Tor}_R^2(R/B,\ M) \overset{\alpha_1}{\cong} \mathrm{Tor}_1^R(R/B,\ M').$$

But then, by the same argument as for M, we can show that $\mathrm{Tor}_1^R(R/B, M') = 0$. Therefore $\mathrm{Tor}_R^2(R/B,\ M) = 0$. So, for every $n \geq 0$, we can repeat the argument n times to conclude that $n < \mathrm{cohgr}_R(R/B;\ M)$. Therefore, $\mathrm{cohgr}_R(R/B;\ M) = \infty$.

In the following Theorem, we show that $\mathrm{hgr}_R(N;\ M) \in \{0, \infty\}$ for any finitely generated R-module N where M is an artinian R-module.

<u>Theorem 4.1.3</u>: Let N, M be R-modules where N is finitely generated and M is artinian. If $\mathrm{Hom}_R(N,\ M) = 0$, then,

$$\mathrm{hgr}_R(N;\ M) = \mathrm{cohgr}_R(N;\ M) = \infty .$$

<u>Proof</u>: Let $\mathrm{Hom}_R(N,\ M) = 0$. Then, by Corollary 1.3.2, $\mathrm{Hom}_R(R/\mathrm{ann}_R N,\ M) = 0$. Since M is artinian then, by [5] Lemma 3, p.54, there exists a finitely generated ideal $B \subseteq \mathrm{ann}_R N$ such that $\mathrm{Hom}_R(R/B,\ M) = 0$. Therefore, by Propositions 4.1.1, 4.1.2, we have $\mathrm{hgr}_R(R/B;\ M) = \mathrm{cohgr}_R(R/B;\ M) = \infty$. now, consider the exact sequence:

$$0 \longrightarrow \mathrm{ann}_R\, N/B \longrightarrow R/B \longrightarrow R/\mathrm{ann}_R\, N \longrightarrow 0 .$$

Then, by the Remarks on Theorem 2.1.4 and Theorem 3.1.5, we can conclude that

$$\mathrm{hgr}_R(R/\mathrm{ann}_R\, N; M) = \mathrm{cohgr}_R(R/\mathrm{ann}_R\, N; M) = \infty.$$

But since N is finitely generated then, by Corollary 2.2.3, and the Remark on Corollary 3.2.2, we have

$$\mathrm{hgr}_R(N; M) = \mathrm{hgr}_R(R/\mathrm{ann}_R\, N; M) = \infty$$

and

$$\mathrm{cohgr}_R(N; M) = \mathrm{cohgr}_R(R/\mathrm{ann}_R\, N; M) = \infty .$$

Remark: It is necessary to have N finitely generated in Theorem 4.1.3. For example, consider the artinian Z-module $Z_n = Z/nZ$ and the Z-module Q/Z, where Z is the ring of integers and Q its rational field. Then, $\mathrm{hgr}_Z(Q/Z; Z_n) = 1$.

Before we state the next Proposition, we need the following Definitions.

Definition: [C.f [6] p.571].

An R-module $M \neq 0$ is called coprimary if $r \in R$ and $r \notin \mathrm{rad}(\mathrm{ann}_R\, M)$ imply $M = rM$.

N.B.: If M is a coprimary R-module, then, by [6] Proposition 1, p.571, $\text{rad}(\text{ann}_R M)$ is a prime ideal of R.

Definition: [c.f. [6], p.571].

When the R-module M is coprimary and $P = \text{rad}(\text{ann}_R M)$, we say that M is P-coprimary and M belongs to P.

Definition: [c.f. [6], p.573].

A representation $M = N_1+....+N_k$ of an R-module as a sum of coprimary R-modules is called a normal coprimary decomposition of M when the prime ideals to which the N_i belong are distinct and $M \neq N_1+....+\hat{N_i}+....+N_k$ for $i = 1,....,k$.

Proposition 4.1.4: Let M be an artinian P-coprimary R-module. Then, for any ideal A of R, $AM = M$ if and only if $A \nsubseteq P$.

Proof: Let $AM = M$ and suppose $A \subseteq P$. Since M is artinian then, by [5] Lemma 3, p.54, there exists a finitely generated ideal $B \subseteq A$ such that $(0 \underset{M}{:} B^k) = (0 \underset{M}{:} A^k)$ for all $k \geq 0$. But since B is finitely generated and $B \subseteq A \subseteq P = \mathrm{rad}(\mathrm{ann}_R M)$ then, there exists an integer $n \geq 1$ such that $B^n \subseteq \mathrm{ann}_R M$. Hence $(0 \underset{M}{:} A^n) = (0 \underset{M}{:} B^n) = M$. And then, $A^n \subseteq \mathrm{ann}_R M$. So, $0 \neq M = AM = A^nM = 0$ which is a contradiction. Therefore, $A \nsubseteq P$.

For the only if part, let $A \nsubseteq P$; then there exists $a \in A$ such that $a \notin P$. Hence, by the Definition of coprimary R-module, $aM = M$. Therefore $AM = M$.

For any ideal A of a noetherian ring R and for any artinian R-module M, E. Matlis, in [9] Theorem 2, p.499, has shown that $AM = M$ if and only if there exists $a \in A$ such that $aM = M$. And D.G. Northcott, in [12], Proposition 2, p.290, has shown that if R is not necessarily noetherian but A is finitely generated then, the above result still holds.

In the following Corollary, we generalize the above result.

Corollary 4.1.5: Let M be an artinian R-module and A an ideal of R. If $AM = M$ then, there exists $a \in A$ such that $aM = M$.

Proof: Let $AM = M$. If $M = 0$ then the Proof is trivial. So, let $M \neq 0$. Then, by [6], Proposition 4(b), p.572 and Theorem 1, p.573, M has a normal coprimary decomposition say, $M = N_1 + \dots + N_k$, where N_i is P_i-coprimary. If $A \subseteq P_j$ for some $1 \leq j \leq k$ then, as in the Proof of Proposition 4.1.4, we can find an integer $n \geq 1$ such that $A^n \subseteq \mathrm{ann}_R N_j$. Now, since $M = AM$ then,

$$M = A^n M = \sum_{i=1}^{k} A^n N_i = \sum_{\substack{i=1 \\ i \neq j}}^{k} A^n N_i \subseteq \sum_{\substack{i=1 \\ i \neq j}} N_i$$

which contradicts the irredundancy of the N_i's. Therefore $A \nsubseteq P_i$ for all $1 \leq i \leq k$. So, by [13], Theorem 13, p.179, there exists $a \in A$ such that $a \notin P_i$ for all $1 \leq i \leq k$. Hence, by Proposition 4.1.4,

$$aM = \sum_{i=1}^{k} a\, N_i = \sum_{i=1}^{k} N_i = M .$$

For an R-module M, D.G. Northcott in [12], defined an R-cosequence on M of length $n \geq 1$ to be a sequence $a_1, \dots, a_n$ of elements of R such that for all $1 \leq i \leq n$, $a_i M^{i-1} = M^{i-1}$, where $M^i = (0 :_M Ra_1 + \dots + Ra_i)$ and $M^o = M$. And for an ideal A of R, he defined the cograde of A on M ($\mathrm{cogr}\{A; M\}$) to be the upper bound of the lengths of all finite R-cosequences on M whose members are contained in A. We will refer to such sequence as a regular M-cosequence in A to distinguish it from an M-cosequence.

The following theorem relates $\text{cohgr}_R(R/A;\ M)$ to $\text{cogr}\{A;\ M\}$, where M is an artinian R-module and A any ideal of R.

<u>Theorem 4.1.6</u>: Let M be an artinian R-module. Then for any ideal A of R, $\text{cohgr}_R(R/A;\ M) = \text{cogr}\{A;\ M\}$.

<u>Proof</u>: Since every regular M-cosequence in A is an M-cosequence in A then, we have $\text{cogr}_R\{A;\ M\} \leq \text{cohgr}_R(R/A;\ M)$. Hence, if $\text{cogr}_R\{A;\ M\} = \infty$ then the Proof is trivial. So, let $\text{cogr}_R\{A;\ M\} = n$. Now, use induction on $n = \text{cogr}_R\{A;\ M\}$. For $n = 0$, we have $aM \neq M$ for all $a \in A$. Hence, by Corollary 4.1.5, $AM \neq M$. Therefore, $\text{cohgr}_R(R/A;\ M) = 0$. Assume it is true for $n \geq 0$. For $n + 1 = \text{cogr}_R\{A;\ M\}$, let $a_1, a_2, \ldots, a_{n+1}$ be a maximal regular M-cosequence in A. Then $a_2, \ldots, a_{n+1}$ is a maximal regular $(0 \underset{M}{:} a_1)$-cosequence in A. So, by induction,

$$\text{cohgr}_R(R/A;\ (0 \underset{M}{:} a_1)) = \text{cogr}\{A;\ (0 \underset{M}{:} a_1)\} = n$$

And from the exact sequence:

$$0 \longrightarrow (0 \underset{M}{:} a_1) \longrightarrow M \xrightarrow{a_1} M \longrightarrow 0\ ,$$

we can conclude that

$$\text{cohgr}_R(R/A;\ M) = \text{cohgr}_R(R/A;\ (0 \underset{M}{:} a_1)) + 1 = n + 1 .$$

This completes the induction and hence the Proof of the Theorem.

For an artinian R-module M and an injective cogenerator Q of Mod-R, the following Corollary shows that $\text{Hom}_R(M, Q)$ behaves as a noetherian R-module in some sense.

<u>Corollary 4.1.7</u>: Let A be an ideal of R and M an artinian R-module and let Q be an injective cogenerator of Mod-R. Then for every integer $0 \leq n \leq \text{hgr}_R(R/A;\ \text{Hom}_R(M, Q))$, there exists a regular $\text{Hom}_R(M, Q)$-sequence $a_1, \ldots, a_n$ in A of length n.

<u>Proof</u>: For $n = 0$, the empty sequence is a regular M-cosequence of length 0. So, we may assume that $n \geq 1$. Since Q is an injective R-module then, by [2], Proposition 5.1, p.120, we have for every $r \geq 0$,

$$\text{Ext}^r_R(R/A,\ \text{Hom}_R(M, Q)) \cong \text{Hom}_R(\text{Tor}^R_r(R/A,\ M),\ Q).$$

And since Q is an injective cogenerator of Mod-R, we conclude that

$$\mathrm{hgr}_R(R/A;\ \mathrm{Hom}_R(M,\ Q)) = \mathrm{cohgr}_R(R/A;\ M).$$

Hence, by Theorem 4.1.6,

$$\mathrm{hgr}_R(R/A;\ \mathrm{Hom}_R(M,\ Q)) = \mathrm{cogr}\{A;\ M\}.$$

So, if

$$1 \leq n \leq \mathrm{hgr}_R(R/A;\ \mathrm{Hom}_R(M,\ Q)) = \mathrm{cogr}\{A;\ M\}$$

then, there exists a regular M-cosequence $a_1,\ldots,a_n$ in A of length n. Now for every $1 \leq i \leq n$,

$$(R/\sum_{k=1}^{i-1} Ra_k) \otimes_R \mathrm{Hom}_R(M,\ Q) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R/\sum_{k=1}^{i-1} Ra_k,\ M),\ Q) = \mathrm{Hom}_R(M^{i-1},\ Q),$$

where $\sum_{k=1}^{i-1} R\,a_k = 0$ for $i = 1$. Hence

$$\mathrm{Hom}_R(R/R\,a_i,\ (R/\sum_{k=1}^{i-1} Ra_k) \otimes_R \mathrm{Hom}_R(M,\ Q)) \cong \mathrm{Hom}_R(R/Ra_i,\ \mathrm{Hom}_R(M^{i-1},\ Q))$$

$$\cong \mathrm{Hom}_R((R/Ra_i) \otimes_R M^{i-1},\ Q) = 0.$$

Therefore, $a_1,\ldots,a_n$ is a regular $\mathrm{Hom}_R(M,\ Q)$-sequence in A.

4.2 Further Applications

In this section, we say $(R, \mathcal{M})$ is a quasi local ring when R is a commutative ring with a unique maximal ideal $\mathcal{M}$. If, in addition, R is noetherian, we say that $(R, \mathcal{M})$ is a local ring. For a local ring $(R, \mathcal{M})$ and for every integer $r \le \mathrm{hgr}_R(R/\mathcal{M}; R)$, the aim in this section is to show that the contravariant functor $\mathrm{Ext}_R^r(-, \mathrm{Hom}_R(\ker d^r, E(R/\mathcal{M})))$ is an exact functor on the category of all R-modules of finite length, where $\ker d^r$ is the r^{th} cosyzygy in the minimal injective resolution of R and $E(R/\mathcal{M})$ is the injective envelope of $R/\mathcal{M}$. Hence, if $\mathrm{Hom}_R(\ker d^r, E(R/\mathcal{M}))$ is finitely generated, it is a Gorenstein R-module. Also, we show that if $(R, \mathcal{M})$ is a complete local ring and if $n = \mathrm{hgr}_R(R/\mathcal{M}; R)$ then, for every finitely generated R-module $N \neq 0$ of finite injective dimension, the R-module $M = \mathrm{Hom}_R(\mathrm{Hom}_R(\ker d^n, E(R/\mathcal{M})), N)$ is finitely generated of finite projective dimension equal to $\mathrm{hgr}_R(R/\mathcal{M}; R) - \mathrm{hgr}_R(R/\mathcal{M}; N)$, and M, N have the same support.

We start with the following Lemma which we will use throughout this section.

<u>Lemma 4.2.1</u>: Let $(R, \mathcal{M})$ be a quasi-local ring (i.e., R is a commutative ring with a unique maximal ideal $\mathcal{M}$), and let $L \neq 0$ be an R-module such that $\mathcal{M}^k \subseteq \mathrm{ann}_R L \subseteq \mathcal{M}$ for some $k \geq 1$. Then, for any R-module M, $\mathrm{hgr}_R(L; M) = \mathrm{hgr}_R(R/\mathcal{M}; M)$ and $\mathrm{cohgr}_R(L; M) = \mathrm{cohgr}_R(R/\mathcal{M}; M)$.

Proof: Since $\mathcal{M}^k \subseteq \mathrm{ann}_R L \subseteq \mathcal{M}$ then, by [7], Theorem 2.2, p.115 and Proposition 2.3, p.116, for any R-module M, we can conclude that $\mathrm{hgr}_R(R/\mathrm{ann}_R L;\ M) = \mathrm{hgr}_R(R/\mathcal{M};\ M)$. Therefore, by Corollary 2.2.3, we have:

$$\mathrm{hgr}_R(L;\ M) = \mathrm{hgr}_R(R/\mathrm{ann}_R L;\ M) = \mathrm{hgr}_R(R/\mathcal{M};\ M)\ .$$

Now, if $E = E(R/\mathcal{M})$ is the injective envelope of $R/\mathcal{M}$ then, for any R-modules N, M and for all $i \geq 0$, we have, by [2], Proposition 5.1, p.120:

$$\mathrm{Ext}_R^i(N,\ \mathrm{Hom}_R(M,\ E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_i^R(N,\ M),\ E)\ .$$

Since $E = E(R/\mathcal{M})$ is an injective cogenerator of R then, it is easy to conclude that for all $i \geq 0$, $\mathrm{Ext}_R^i(N,\ \mathrm{Hom}_R(M,\ E)) = 0$ if and only if $\mathrm{Tor}_R^i(N,\ M) = 0$. i.e., $\mathrm{hgr}_R(N;\ \mathrm{Hom}_R(M,\ E)) = \mathrm{cohgr}_R(N;\ M)$. Therefore, from the above argument, we have

$$\mathrm{cohgr}_R(L;M) = \mathrm{hgr}_R(L;\ \mathrm{Hom}_R(M,\ E)) = \mathrm{hgr}_R(R/\mathcal{M};\ \mathrm{Hom}_R(M,E)) = \mathrm{cohgr}_R(R/\mathcal{M};\ M).$$

This completes the proof of the Lemma.

Corollary 4.2.2: Let $(R,\ \mathcal{M})$ be a complete local (noetherian) ring and let $P \neq \mathcal{M}$ be any prime ideal of R and $E(R/P)$ its injective envelope. Then for any artinian R-module M and any finitely generated R-module N,

$$\text{cohgr}_R(M;\ E(R/P)) = \text{hgr}_R(E(R/P);\ N) = \infty.$$

<u>Proof</u>: Let $P \neq \mathcal{M}$ be a prime ideal of R. Then, $\text{Hom}_R(R/\mathcal{M},\ R/P) = 0$. So, by Lemma 1.3.1, $\text{Hom}_R(R/\mathcal{M},\ E(R/P)) = 0$ and, by Proposition 3.3.1, $\text{coghr}_R(R/\mathcal{M};\ E(R/P)) = \infty$. Hence, by Lemma 4.2.1, we have for any R-module $L \neq 0$ of finite length,

$$\text{cohgr}_R(L;\ E(R/P)) = \text{cohgr}_R(R/\mathcal{M};\ E(R/P)) = \infty\ .$$

Let M be an artinian R-module.

<u>Claim</u>: $M = \bigcup_{k\geq 1} (0 \underset{M}{:} \mathcal{M}^k)$ and for all $k \geq 1$, $(0 \underset{M}{:} \mathcal{M}^k)$ has finite length.

<u>Proof of the Claim</u>: We have $\bigcup_{k\geq 1} (0 \underset{M}{:} \mathcal{M}^k) \subseteq M$. And if $x \in M$ then, Rx has finite length. Hence, there exists an integer $k \geq 1$ such that $\mathcal{M}^k x = 0$. Therefore $M = \bigcup_{k\geq 1} (0 \underset{M}{:} \mathcal{M}^k)$.

Now for every $k \geq 1$, $(0 \underset{M}{:} \mathcal{M}^k) \,/\, (0 \underset{M}{:} \mathcal{M}^{k-1})$ is an artinian R-module annihilated by $\mathcal{M}$. So, it is a finitely generated $R/\mathcal{M}$-vector space. Hence it has finite length as an R-module. So, by induction, $(0 :_M \mathcal{M}^k)$ has finite length and the claim is proved.

By the above argument, we have for all $i \geq 0$,

$$\mathrm{Tor}_i^R(M, E(R/P)) \cong \mathrm{Tor}_i^R(\varinjlim_k (0 :_M \mathcal{M}^k), E(R/P)) \cong \varinjlim_k \mathrm{Tor}_i^R((0 :_M \mathcal{M}^k), E(R/P)) = 0$$

Therefore, $\mathrm{cohgr}_R(M; E(R/P)) = \infty$.

Now let N be any finitely generated R-module and $E = E(R/\mathcal{M})$ be the injective envelope of $R/\mathcal{M}$. Then, by [8], Corollary 4.3 (3), p.528, $\mathrm{Hom}_R(N, E)$ is an artinian R-module and $N \cong \mathrm{Hom}_R(\mathrm{Hom}_R(N, E), E)$. Hence, by the above argument; $\mathrm{cohgr}_R(\mathrm{Hom}_R(N, E); E(R/P)) = \infty$, i.e. for every $i \geq 1$, $\mathrm{Tor}_i^R(E(R/P), \mathrm{Hom}_R(N, E)) = 0$. Now for every $i \geq 1$, we have, by [1], Proposition 5.1, p.120,

$$\mathrm{Ext}_R^i(E(R/P), N) \cong \mathrm{Ext}_R^i(E(R/P), \mathrm{Hom}_R(\mathrm{Hom}_R(N, E), E))$$

$$\cong \mathrm{Hom}_R(\mathrm{Tor}_i^R(E(R/P), \mathrm{Hom}_R(N, E)), E) = 0$$

i.e $\mathrm{hgr}_R(E(R/P), N) = \infty$. Therefore,

$$\mathrm{cohgr}_R(M, E(R/P)) = \mathrm{hgr}_R(E(R/P); N) = \infty.$$

This completes the Proof of the Corollary.

Before we state and prove the next result, we have to digress for a moment.

Let R be a commutative noetherian ring with identity and M an R-module and consider the minimal injective resolution of M.

$$0 \longrightarrow M \longrightarrow E^{o}(M) \longrightarrow E^{1}(M) \longrightarrow \cdots\cdots \longrightarrow E^{i}(M) \longrightarrow \cdots$$

For every $P \in \text{spec}(R)$, let $E(R/P)$ denote the injective envelope of R/P. H.Bass in [1] defines cardinals $\mu^{i}(P, M)$ by the equation

$$E^{i}(M) \cong \bigoplus_{P \in \text{spec}(R)} \mu^{i}(P, M)\, E(R/P),$$

where $\oplus \mu E$ denotes a direct sum of μ copies of E. And he shows, [1], Lemma 2.7, p.11, that for all $P \in \text{spec}(R)$ and all $i \geq 0$,

$$\mu^{i}(P, M) = \dim_{R_P/PR_P} \text{Ext}^{i}_{R_P}(R_P/P R_P, M_P),$$

and that if M is finitely generated then $\mu^{i}(P, M) < \infty$.

<u>Theorem 4.2.3</u>: Let $(R, \mathcal{M})$ be a local (noetherian) ring and M an R-module. And let,

$$0 \longrightarrow M \longrightarrow E^{o} \xrightarrow{d^{o}} E^{1} \xrightarrow{d^{1}} \cdots\cdots \longrightarrow E^{k} \xrightarrow{d^{k}} E^{k+1} \longrightarrow \cdots$$

be the minimal injective resolution of M. Then for any finitely generated R-module N and for all $r \geq 0$ and all $0 \leq k \leq hgr_R(N; M)$, we have the following:

(i) $$Tor_r^R(N, \ker d^k) \cong \begin{cases} 0 & \text{if } r < k \\ Tor_{r-k}^R(N, M) & \text{for } r \geq k \end{cases}$$

(ii) $$Ext_R^r(N, Hom_R(\ker d^k, E(R/\mathcal{M}))) \cong \begin{cases} 0 & \text{for } r < k \\ Ext_R^{r-k}(N, Hom_R(M, E(R/\mathcal{M}))) \\ \text{for } r \geq k \end{cases}$$

where $E(R/\mathcal{M})$ is the injective envelope of $R/\mathcal{M}$.

<u>Proof</u>: If $N = 0$, then the Proof is trivial. So, assume $N \neq 0$.

To prove (i), use induction on k with $0 \leq k \leq hgr_R(N; M)$. For $k = 0$, we have $\ker d^o \cong M$. Hence for all

$$r \geq k = 0, \ Tor_r^R(N, \ker d^o) \cong Tor_{r-o}^R(N, M) \ .$$

Now suppose it is true for $0 \leq k \leq hgr_R(N; M)$. For $0 < k+1 \leq hgr_R(N;M)$, we have $0 \leq k < k+1 \leq hgr_R(N; M)$. Hence, by Theorem 2.3.1,

$\mathrm{Hom}_R(N, E^k) = 0$. Then, by Corollary 1.3.2, $\mathrm{Hom}_R(R/\mathrm{ann}_R N, E^k) = 0$, and, by Proposition 3.3.1, $\mathrm{cohgr}_R(R/\mathrm{ann}_R N; E^k) = \infty$. Therefore, by the Remark on Corollary 3.2.2, $\mathrm{cohgr}_R(N; E^k) = \infty$. Now applying $\mathrm{Tor}_R(N, -)$ to the exact sequence

$$0 \longrightarrow \ker d^k \longrightarrow E^k \longrightarrow \ker d^{k+1} \longrightarrow 0 ,$$

we obtain the following long exact sequence:

$$\cdots \longrightarrow \mathrm{Tor}^R_{r+1}(N, E^k) \longrightarrow \mathrm{Tor}^R_{r+1}(N, \ker d^{k+1}) \longrightarrow \mathrm{Tor}^R_r(N, \ker d^k)$$
$$\longrightarrow \mathrm{Tor}^R_r(N, E^k) \longrightarrow \mathrm{Tor}^R_r(N, \ker d^{k+1}) \longrightarrow \cdots$$

Since $\mathrm{cohgr}_R(N, E^k) = \infty$, then $\mathrm{Tor}^R_r(N, E^k) = 0$ for all $r \geq 0$. So, from the above exact sequence, we can conclude that

$$\mathrm{Tor}^R_r(N, \ker d^k) \cong \mathrm{Tor}^R_{r+1}(N, \ker d^{k+1}) .$$

Now, if $r + 1 < k + 1$ then $r < k$ and, by induction

$$0 = \mathrm{Tor}^R_r(N, \ker d^k) \cong \mathrm{Tor}^R_{r+1}(N, \ker d^{k+1}) .$$

And if $r + 1 \geq k + 1$ then $r \geq k$ and, by induction,

$$\mathrm{Tor}^R_{r-k}(N, M) \cong \mathrm{Tor}^R_r(N, \ker d^k) \cong \mathrm{Tor}^R_{r+1}(N, \ker d^{k+1}) .$$

Hence,

$$\mathrm{Tor}^R_{r+1}(N, \ker d^{k+1}) \cong \mathrm{Tor}^R_{r-k}(N, M) = \mathrm{Tor}^R_{(r+1)-(k+1)}(N, M) .$$

This completes the induction and hence the Proof of (i).

For the Proof of (ii), we have, by [2], Proposition 5.1, p.120, for all $r \geq 0$ and all $0 \leq k \leq \mathrm{hgr}_R(N; M)$,

$$\mathrm{Ext}^r_R(N, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_r(N, \ker d^k), E(R/\mathcal{M}))$$

But, by (i),

$$\mathrm{Tor}^R_r(N, \ker d^k) \cong \begin{cases} 0 & \text{if } r < k \\ \mathrm{Tor}^R_{r-k}(N, M) & \text{if } r \geq k . \end{cases}$$

Therefore, $\mathrm{Ext}^r_R(N, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) = 0$ for $r < k$. And for $r \geq k$,

$$\begin{aligned} \mathrm{Ext}^r_R(N, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_r(N, \ker d^k), E(R/\mathcal{M})) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_{r-k}(N, M), E(R/\mathcal{M})) \\ &\cong \mathrm{Ext}^{r-k}_R(N, \mathrm{Hom}_R(M, E(R/\mathcal{M}))) . \end{aligned}$$

This completes the Proof of (ii).

For a local (noetherian) ring R with a unique maximal ideal $\mathcal{M}$, if

$$0 \longrightarrow R \longrightarrow E^o \xrightarrow{d^o} E' \xrightarrow{d^1} \cdots \longrightarrow E^k \xrightarrow{d^k} E^{k+1} \longrightarrow \cdots$$

is the minimal injective resolution of R then, for $k \leq \mathrm{hgr}_R(R/\mathcal{M}; R)$, the following Corollary includes, $\mu^i(\mathcal{M}, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) = \delta_{ik}$, where δ_{ik} is the Kronecker delta and $E(R/\mathcal{M})$ is the injective envelope of $R/\mathcal{M}$.

<u>Corollary 4.2.4</u>: Let $(R, \mathcal{M})$ be as above and consider the above minimal injective resolution of R. Then for every finitely generated R-module N and for all $r \geq 0$ and all $0 \leq k \leq \mathrm{hgr}_R(N; R)$, we have the following:

(i) $$\mathrm{Tor}_r^R(N, \ker d^k) \cong \begin{cases} 0 & \text{if } r \neq k \\ N & \text{for } r = k \end{cases}$$

(ii) $$\mathrm{Ext}_R^r(N, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) \cong \begin{cases} 0 & \text{if } r \neq k \\ \mathrm{Hom}_R(N, E(R/\mathcal{M})) & \text{if } r = k \end{cases}$$

Proof: Taking $M = R$ in Theorem 4.2.3, we have

$$\mathrm{Tor}_r^R(N, \ker d^k) \cong \begin{cases} 0 & \text{if } r < k \\ \mathrm{Tor}_{r-k}^R(N, R) & \text{for } r \geq k\ . \end{cases}$$

But R is a flat R-module, hence $\mathrm{Tor}_{r-k}^R(N, R) = 0$ for $r > k$ and for $r = k$,

$$\mathrm{Tor}_{r-k}^R(N, R) = \mathrm{Tor}_o^R(N, R) \cong N \otimes R \cong N.$$

This proves (i). Similarly, for (ii), we have, by Theorem 4.2.3 (ii),

$$\mathrm{Ext}_R^r(N, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) = 0$$

for $r < k$. And for $r \geq k$,

$$\mathrm{Ext}_R^r(N, \mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))) \cong \mathrm{Ext}_R^{r-k}(N, \mathrm{Hom}_R(R, E(R/\mathcal{M}))).$$

But $\mathrm{Hom}_R(R, E(R/\mathcal{M})) \cong E(R/\mathcal{M})$ is an injective R-module, hence

for $r > k$, $$\mathrm{Ext}_R^{r-k}(N, \mathrm{Hom}_R(R, E(R/\mathcal{M}))) = 0$$

and for $r = k$,

$$\mathrm{Ext}^{0}_{R}(N, \mathrm{Hom}_{R}(R, E(R/\mathcal{M}))) \cong \mathrm{Hom}_{R}(N, E(R/\mathcal{M})).$$

This completes the Proof of (ii).

For the next result, we need to recall the definition of a Gorenstein module. For a commutative noetherian ring R, R.Y.Sharp in [19] has introduced a special class of R-modules of finite injective dimension under the name of Gorenstein modules, and he defines them as follows:

Definition: [c.f. [19], p.123].

A non-zero finitely generated R-module M is Gorenstein if and only if the Cousin Complex for M provides a minimal injective resolution for M.

For the construction and the properties of the Cousin Complex for an R-module M, see [18] (2.6) p.344, and (2.7) p.345. And for the characterization of Gorenstein modules, see [19], Theorem 3.6, p.124. Also, see Theorem 3.11, in the case when R is a local ring.

Corollary 4.2.5: Let $(R, \mathcal{M})$ be a local (noetherian) ring and,

$$0 \longrightarrow R \longrightarrow E^{0} \xrightarrow{d^{0}} E^{1} \xrightarrow{d^{1}} \cdots \longrightarrow E^{k-1} \longrightarrow E^{k} \xrightarrow{d^{k}} E^{k+1} \longrightarrow \cdots$$

be the minimal injective resolution of R. Then for every $0 \le k < hgr_R(R/\mathcal{M}; R)$, the contravariant functor $F_k = Ext_R^k(—, Hom_R(\ker d^k, E(R/\mathcal{M})))$ is a non-zero exact functor on the category of all R-modules of finite length. Hence if $Hom_R(\ker d^k, E(R/\mathcal{M}))$ is finitely generated, then it is a Gorenstein R-module and R is a Cohen-Macaulay ring of Krull dimension k.

<u>Proof</u>: Let $0 \le k \le hgr_R(R/\mathcal{M}; R)$. Then, by Lemma 4.2.1, $k \le hgr_R(L; R)$ for every R-module $L \neq 0$ of finite length. Now, let $\Omega_k = Hom_R(\ker d^k, E(R/\mathcal{M}))$ so that $F_k = Ext_R^k(-, \Omega_k)$, and let $0 \to L_1 \to L_2 \to L_3 \to 0$ be an exact sequence of R-modules of finite length. Applying $Hom_R(-, \Omega_k)$ to the above exact sequence, we obtain the following long exact sequence,

$$\cdots \to Ext_R^{k-1}(L_1, \Omega_k) \to Ext_R^k(L_3, \Omega_k) \to Ext_R^k(L_2, \Omega_k) \to Ext_R^k(L_1, \Omega_k)$$

$$\to Ext_R^{k+1}(L_3, \Omega_k) \to \cdots$$

But, by Corollary 4.2.4, we have $Ext_R^{k-1}(L_1, \Omega_k) \cong Hom_R(Tor_{k-1}^R(L_1, \ker d^k), E(R/\mathcal{M})) = 0$. And $Ext_R^{k+1}(L_3, \Omega_k) \cong Hom_R(Tor_{k+1}^R(L_3, \ker d^k), E(R/\mathcal{M})) = 0$. So, we have an exact sequence

$$0 \to Ext_R^k(L_3, \Omega_k) \to Ext_R^k(L_2, \Omega_k) \to Ext_R^k(L_1, \Omega_k) \to 0$$

Therefore F_k is an exact functor on the category of all R-modules of finite length. Hence if $\mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))$ is finitely generated, then, by [3], Theorem 3.8, p.203, $\mathrm{Hom}_R(\ker d^k, E(R/\mathcal{M}))$ is a Gorenstein R-module and k = Krull dimension of R. And, by [19] Theorem 3.11 (vii), p.127, R is Cohen-Macaulay. This completes the Proof of the Corollary.

If $(R, \mathcal{M})$ is a complete local (noetherian) ring and M is any R-module, the following theorem shows, in particular, that for every finitely generated R-module $N \neq 0$ of finite injective dimension (so, $\mathrm{inj\ dim}_R N = \mathrm{hgr}_R(R/\mathcal{M}; R)$.) and for all $t > \mathrm{hgr}_R(R/\mathcal{M}; R) - \mathrm{hgr}_R(R/\mathcal{M}; M)$, $\mathrm{Ext}_R^t(M, N) = 0$. Moreover, if $\mathrm{hgr}_R(R/\mathcal{M}; M) > \mathrm{hgr}_R(R/\mathcal{M}; R)$ then, $\mathrm{Ext}_R^t(M, N) = 0$ for all $t \geq 0$.

<u>Theorem 4.2.6</u>: Let $(R, \mathcal{M})$ be a complete local (noetherian) ring. Let M be any R-module and

$$0 \longrightarrow M \longrightarrow E^o \xrightarrow{d^o} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^{k-1} \xrightarrow{d^{k-1}} E^k \xrightarrow{d^k} E^{k+1} \longrightarrow \cdots$$

be the minimal injective resolution of M. Then for any finitely generated R-module N and for every $0 \leq k \leq \mathrm{hgr}_R(R/\mathcal{M}; M)$,

$$\mathrm{hgr}_R(M; N) + k = \mathrm{hgr}_R(\ker d^k, N)$$

and for all $t \geq 0$, $\operatorname{Ext}_R^t(M, N) \cong \operatorname{Ext}_R^{t+k}(\ker d^k, N)$.

<u>Proof</u>: For $0 = k \leq \operatorname{hgr}_R(R/\mathcal{M}; M)$, we have $M \cong \ker d^o$ and the Proof is trivial. So, assume that $0 < \operatorname{hgr}_R(R/\mathcal{M}; M)$. By Theorem 2.3.1 (ii), we have for all $0 \leq i < \operatorname{hgr}_R(R/\mathcal{M}; M)$, $\operatorname{Hom}_R(R/\mathcal{M}, E^i) = 0$. But

$$E^i \cong \bigoplus_{P \in \operatorname{spec}(R)} \mu^i(P, M)\, E(R/P) .$$

Hence, we can conclude that

$$E^i \cong \bigoplus_{\substack{P \in \operatorname{spec}(R) \\ P \neq \mathcal{M}}} \mu^i(P, M)\, E(R/P) .$$

Now, by Corollary 4.2.2, we have for all

$$\mathcal{M} \neq P \in \operatorname{spec}(R),\ \operatorname{hgr}_R(E(R/P); N) = \infty.$$

Therefore, it is easy to conclude that $\operatorname{hgr}_R(E^i, N) = \infty$ for all $0 \leq i < \operatorname{hgr}_R(R/\mathcal{M}; M)$. Now applying $\operatorname{Hom}_R(\text{—}, N)$ to the exact sequence:

$$0 \longrightarrow \ker d^i \longrightarrow E^i \longrightarrow \ker d^{i+1} \longrightarrow 0 ,$$

we obtain the following long exact sequence:

$$\cdots \longrightarrow Ext^t_R(\ker d^{i+1}, N) \longrightarrow Ext^t_R(E^i, N) \longrightarrow Ext^t(\ker d^i, N)$$

$$\longrightarrow Ext^{t+1}_R(\ker d^{i+1}, N) \longrightarrow Ext^{t+1}_R(E^i, N) \longrightarrow \cdots$$

But, for $0 \le i < hgr_R(R/\mathcal{M}; M)$, $hgr_R(E^i, N) = \infty$. Hence, from the above long exact sequence, we can conclude that $Ext^t_r(\ker d^i, N) \cong Ext^{t+1}(\ker d^{i+1}, N)$ for all $t \ge 0$. i.e. for all

$$0 \le i < hgr_R(R/\mathcal{M}; M),\ hgr_R(\ker d^i, N) = hgr_R(\ker d^{i+1}, N) - 1$$

Therefore,

$$\begin{aligned} hgr_R(M; N) = hgr_R(\ker d^o, N) &= hgr_R(\ker d^1, N) - 1 \\ &= hgr_R(\ker d^2, N) - 2 \\ &= hgr_R(\ker d^{i+1}, N) - (i+1). \end{aligned}$$

Or, for all $0 \le k \le hgr_R(R/\mathcal{M}; M)$, $hgr_R(M, N) = hgr_E(\ker d^k; N) - k$. i.e. $hgr_R(M; N) + k = hgr_R(\ker d^k; N)$, and for all $t \ge 0$,

$$\begin{aligned} Ext^t_R(M; N) \cong Ext^t_R(\ker d^o, N) &\cong Ext^{t+1}_R(\ker d^1, N) \\ &\cong Ext^{t+2}_R(\ker d^2, N) \\ &\cong Ext^{t+(k-1)}_R(\ker d^{k-1}, N) \cong Ext^{t+k}_R(\ker d^k, N). \end{aligned}$$

This completes the Proof of the Theorem.

If $(R, \mathcal{M})$ is a Cohen-Macaulay local ring of Krull dimension m and if R is a homomorphic image of a Gorenstein local ring $(\Lambda, \mathcal{N})$ of Krull dimension n, R.Y. Sharp in [20] has shown that the R-module $\Omega = \mathrm{Ext}_{\Lambda}^{n-m}(R, \Lambda)$ is a Gorenstein R-module which satisfies the following property: for every finitely generated R-module $N \neq 0$ of finite injective dimension, the R-module $\mathrm{Hom}_R(\Omega, N)$ is finitely generated of finite projective dimension equal to $\mathrm{hgr}_R(R/\mathcal{M}; R) - \mathrm{hgr}_R(R/\mathcal{M}; N)$. For a complete local ring $(R, \mathcal{M})$ and for $n = \mathrm{hgr}_R(R/\mathcal{M}, R)$, the following Corollary shows that the R-module, $\Omega_n = \mathrm{Hom}_R(\ker d^n, E(R/\mathcal{M}))$ satisfies the above property, where $\ker d^n$ is the n^{th} cosyzygy in the minimal injective resolution of R and $E(R/\mathcal{M})$ is the injective envelope of $R/\mathcal{M}$.

<u>Corollary 4.2.7</u>: Let $(R, \mathcal{M})$ be a complete local ring and $n = \mathrm{hgr}_R(R/\mathcal{M}; R)$. If

$$0 \longrightarrow R \longrightarrow E^o \xrightarrow{d^o} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^n \xrightarrow{d^n} E^{n+1} \longrightarrow \cdots$$

is the minimal injective resolution of R and if $\Omega_n = \mathrm{Hom}_R(\ker d^n, E(R/\mathcal{M}))$ then, for every finitely generated R-module $N \neq 0$, of finite injective dimension the R-module $\mathrm{Hom}_R(\Omega_n, N)$ is finitely generated and

$$\text{proj dim}_R \, \text{Hom}_R(\Omega_n, N) = \text{hgr}_R(R/\mathcal{M}; R) - \text{hgr}_R(R/\mathcal{M}; N)$$

and $\text{supp Hom}_R(\Omega_n, N) = \text{supp } N$.

Proof: Let $\Omega_n = \text{Hom}_R(\ker d^n, E(R/\mathcal{M}))$ and consider the minimal injective resolution of Ω_n,

$$0 \longrightarrow \Omega_n \longrightarrow Q^o \xrightarrow{h^o} Q^1 \xrightarrow{h^1} \cdots \longrightarrow Q^n \xrightarrow{h^n} Q^{n+1} \longrightarrow \cdots$$

By Corollary 4.2.4 (ii), we have for all $i \geq 0$:

$$\text{Ext}^i_R(R/\mathcal{M}, \Omega_n) \cong \begin{cases} 0 & \text{if } i \neq n \\ \text{Hom}_R(R/\mathcal{M}, E(R/\mathcal{M})) \cong R/\mathcal{M} & \text{for } i = n . \end{cases}$$

Hence, by [1], Lemma 2.7, p.11, $\mu^i(\mathcal{M}, \Omega_n) = \begin{cases} 0 & \text{if } i \neq n \\ 1 & \text{if } i = n \end{cases}$

So, for all $i \geq n + 1$, $\text{Hom}_R(R/\mathcal{M}, Q^i) = 0$. Now, since

$$0 \longrightarrow \ker h^{n+1} \longrightarrow Q^{n+1} \longrightarrow Q^{n+2} \longrightarrow \cdots$$

is the minimal injective resolution of $\ker h^{n+1}$ and $\text{Hom}_R(R/\mathcal{M}, Q^i) = 0$ for all $i \geq n+1$ then; by Theorem 2.3.1, we have $\text{hgr}_R(R/\mathcal{M}; \ker h^{n+1}) = \infty$. So, $n + 1 < \text{hgr}_R(R/\mathcal{M}; \ker h^{n+1})$. Hence, by Theorem 4.2.6, we have for all $t \geq 0$, $\text{Ext}^t_R(\ker h^{n+1}, N) \cong \text{Ext}^{t+(n+1)}_R(\ker h^{2(n+1)}, N) = 0$ for inj $\dim_R N = n$. So, applying $\mathrm{Hom}_R(\text{—}, N)$ to the exact sequence,

$$0 \longrightarrow \ker h^n \longrightarrow Q^n \longrightarrow \ker h^{n+1} \longrightarrow 0 ,$$

we can conclude that $\mathrm{Ext}_R^t(\ker h^n, N) \cong \mathrm{Ext}_R^t(Q^n, N)$. But, since

$$\mu^n(\mathcal{M}, \Omega_n) = 1,\ Q^n \cong E(R/\mathcal{M}) \quad \bigoplus_{P \in \mathrm{spec}(R) \setminus \mathcal{M}} \mu^n(P, \Omega\ ) E(R/P) .$$

Hence, by Corollary 4.2.2, we can conclude that,

$$\mathrm{Ext}_R^n(\ker h^n, N) \cong \mathrm{Ext}_R^n(Q^n, N) \cong \mathrm{Ext}_R^n(E(R/\mathcal{M}), N) .$$

Now, since $n = \mathrm{hgr}_R(R/\mathcal{M}; \Omega_n)$ by Theorem 4.2.6,

$$\mathrm{Hom}_R(\Omega_n, N) \cong \mathrm{Ext}_R^n(\ker h^n, N) \cong \mathrm{Ext}_R^n(E(R/\mathcal{M}), N) .$$

Now, the Proof of the Corollary follows from [15] Theorem 4.10, p.66.

We end this thesis with the following Corollary.

<u>Corollary 4.2.8</u>: Let $(R, \mathcal{M})$ be a complete local ring and let:

$$0 \longrightarrow R \longrightarrow E^o \xrightarrow{d^o} E^1 \xrightarrow{d^1} \cdots \longrightarrow E^k \xrightarrow{d^k} E^{k+1} \longrightarrow \cdots$$

be the minimal injective resolution of R. If N is any finitely generated R-module then, for every $k \le \mathrm{hgr}_R(R/\mathcal{M}; R)$ and for all $t \ge 0$,

$$\mathrm{Ext}_R^t(\ker d^k, N) \cong \begin{cases} 0 & \text{if } t \neq k \\ N & \text{if } t = k \end{cases}$$

Proof: By Theorem 4.2.6, we have for all $t \geq 0$,

$$\mathrm{Ext}_R^{t+k}(\ker d^k, N) \cong \mathrm{Ext}_R^t(R, N) .$$

hence, if $t \neq 0$, $\mathrm{Ext}_R^t(R, N) = 0$ and if $t = 0$,

$$N \cong \mathrm{Hom}_R(R, N) \cong \mathrm{Ext}_R^k(\ker d^k, N) .$$

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