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REGULAR AND PERFECT SOLIDS

by

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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An n-solid is a compact convex subset B of E^m whose affine hull is n-dimensional, for some $m \geq n \geq -1$. The boundary of B is composed of faces which are solids of a lower dimension. A flag of B is a sequence $(A_0, \ldots, A_r)$ of distinct proper faces of B such that A_{j-1} is contained in A_j, $j = 1, \ldots, r$. A flag is said to be maximal if it is not contained in any other flag of B. If the symmetry group of B is transitive on the set of maximal flags of B, then we say that B is regular.

Two solids B and C are symmetry equivalent if the actions of their symmetry groups GB and GC on their face-lattices FB and FC, respectively, are equivalent. A solid B is said to be perfect if B is similar to C whenever B is symmetry equivalent to C.

The aim of this thesis is two-fold. First, the regular solids are classified. This classification is based on the projection of the adjoint action of a compact semisimple Lie group G on its Lie algebra $\mathfrak{g}$ to the Weyl group action of $\mathfrak{g}$. Secondly, a contribution to the solution of the more general problem of classifying perfect n-solids is given. The cases $n \leq 3$ are already completely understood. The case $n = 4$ is solved, thus proving Rostami's conjecture that all prime perfect 4-polytopes are Wythoffian up to polarity.

CONTENTS

PREFACE

The foundations for this work can be detected as far back as Ancient Greece, where some mathematicians chose to study examples of compact convex sets in terms of their symmetry properties. In particular, the so-called Platonic solids and Archimedean solids spring to mind.

The Platonic solids (also known as regular polyhedra) were extended to analogous figures in dimensions $n \geq 4$ in the nineteenth century by various mathematicians. Such figures are known as regular polytopes and were first classified in 1853 by Schläfli. The less symmetrical polyhedra also have analogous n-dimensional figures called polytopes.

The notion of regularity was generalized to that of perfection by Robertson [3] in 1981. A polytope P is said to be perfect if it has maximal symmetry properties in the sense that P cannot be deformed without changing its 'shape' or its symmetry group.

In 1993, Farran and Robertson [1] extended the classical concept of regularity from convex polytopes to convex solids (in other words, convex compact sets in general). A convex solid that is regular in this new sense is called a regular solid. Likewise, we have the notion of perfect solid.

This thesis is concerned with regular and perfect solids, and in particular with classifying them. These two concepts are clearly closely related, indeed the regular solids form a subset of the perfect solids. It is felt, however, that a clearer understanding is obtained by considering each in turn. Thus this thesis is divided into two parts. In part I attention is focused on regular solids, while in part II perfect solids are considered.

This work is type-faced using TEX. All diagrams are drawn using PicTEX.

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PART I

REGULAR SOLIDS

The notion of regularity may be defined as follows. We say that an **i-face** A_j of an n-dimensional solid B is a maximal i-dimensional solid contained in the boundary of B, $i \leq n$. A **flag** of B is a sequence $(A_0, \ldots, A_r)$ of distinct proper faces such that A_{j-1} is contained in A_j for each $j = 1, \ldots, r$. Then B is **regular** if its symmetry group acts transitively on the set of maximal flags of B, where a flag is **maximal** if it not contained in any other flag of B, that is, it is not a subsequence of any flag of B. As a consequence of the extension of regularity by Farran and Robertson [1], certain conjectures arise on the determination of the set of regular solids. Part I of this thesis proves these conjectures obtaining a complete classification of regular solids. In order to achieve this, a lot of introductory work is required, this is given in chapters 1, 2 and 3.

In chapter 1 we introduce the standard terminology and review some of the recent results in the symmetry theory of convex bodies. This includes the definition of a perfect solid. If $\mathcal{P}_R$ and $\mathcal{S}_R$ denote the sets of regular polytopes and regular solids respectively, then $\mathcal{P}_R$ is well-known and forms a subset of $\mathcal{S}_R$.

The classification of regular solids determines the set $\mathcal{S}_R$.

The classification theorem uses the projection $\rho : \mathcal{S}_R \to \mathcal{P}_R$ constructed by Farran and Robertson [1]. This projection and the work of Kostant [1] are closely related. In chapter 2, we summarize this work, refered to here as Kostant's construction, from which regular solids are obtained from regular polytopes.

In chapter 3, we consider the work of Dadok [1] on polar representations, where a representation π is said to be **polar** if the normal to a principal orbit of π cuts every orbit orthogonally. Dadok [1] classifies all irreducible polar representations by associating each such representation to a symmetric space representation. We also summarize the relevant symmetric space theory.

It is shown in chapter 4 that the inclusion of the symmetry group of any regular solid in $O(n)$ is an irreducible polar representation. Thus we can use Dadok's classification to associate each regular solid B to a certain symmetric space whose Weyl group enables us to assign a regular polytope P to B. This yields the projection $\rho : \mathcal{S}_R \to \mathcal{P}_R$ in a form that explains the analogy with Kostant's construction. We conclude part I with a complete description of $\mathcal{S}_R$ and give various examples of regular solids.

CHAPTER 1

PROPERTIES OF CONVEX BODIES

The following is a summary of some of the recent developments in the symmetry theory of convex bodies, based on work of Rostami [1], Pinto [1] and Pinto and Robertson [1]. The basic theory of convex sets is well established and a good exposition can be found in Lay [1]. Our main sources of information for the symmetry theory of convex bodies are Robertson [1] and Farran and Robertson [1], while Coxeter [1] provides detailed information on regular polytopes.

1. Convex sets

For any positive integer n, let Euclidean n-space be denoted by E^n and let d denote the usual metric on E^n.

A subset X of E^n is **convex** if for any points x, y of X, $tx + (1-t)y \in X$ for all $t \in \mathbb{R}, 0 \leq t \leq 1$. A subset X of E^n is an **affine space** if for any points x, y of X, $tx + (1-t)y \in X$ for all $t \in \mathbb{R}$. These two concepts, though closely related, are very different in character. Notice that any affine space is convex.

In particular E^n itself is convex.

It is well-known that the intersection of any family of convex sets is a convex set. However the union of two convex sets is not, in general, convex. For every subset X of E^n, we define the **convex hull** $conv(X)$ of X to be the intersection of all convex sets that contain X. Thus X is convex if and only if $X = conv(X)$. Similarly, we define the **affine hull** $aff(X)$ of a subset X of E^n to be the intersection of all affine subspaces that contain X. A trivial example of a convex set is the empty set, denoted by $\emptyset$.

The family of all convex subsets of E^n is a bounded lattice with zero $\emptyset$ and unit E^n, partially ordered by inclusion. The meet $X \wedge Y$ of two convex sets X and Y is $X \cap Y$ and the join $X \vee Y$ is $conv(X \cup Y)$. Likewise, the family of all affine subspaces of E^n is a bounded lattice with zero $\emptyset$ and unit E^n, partially ordered by inclusion. The meet of two affine subspaces X, Y is again their intersection while their join is $aff(X \cup Y)$.

2. Solids and polytopes

A compact convex subset of E^n is called a **convex body**, or **solid**. A solid B for which $aff(B)$ is a k-dimensional space is said to have **dimension** k, written $dim(B) = k$. We also refer to such a solid as a k-**solid**. The empty set is said to be of dimension -1. The affine subspaces of E^n of dimension k, $0 \leq k \leq n$, are also convex. Of these only the singletons, that is to say the affine subspaces of dimension 0, are compact convex sets. For any solid B, let ∂B denote the boundary of B in $aff(B)$. Then $B = conv(\partial B)$ (Lay [1]). The structure of ∂B may be analysed as follows.

Let $n \geq 1$ and let B be an n-solid in E^n. We define a **supporting hyperplane** of B in E^n as an affine $(n-1)$-plane Π in E^n such that (i) $\Pi \cap B \neq \emptyset$, and (ii) B lies entirely in one of the two closed half-spaces bounded by Π. Then for each such Π, the set $\Pi \cap B$ is convex and hence a j-solid for some j, $0 \leq j \leq n-1$. The set $\Pi \cap B$ is called a **j-face** of B. It is convenient to call $\emptyset$ and B the unique (-1)-face and n-face of B respectively. As usual a 0-face of B is called a **vertex** of B and we note that a nonempty solid always has at least one vertex. In fact, Minkowski [1] proved in 1911 the following important theorem.

Theorem 1:2.1 MINKOWSKI'S THEROEM

Every compact convex set in E^n is the closure of the convex hull of the set of its vertices.

Proof

See Jacobs [1].

The 1-faces and $(n-1)$-faces of B (if any) are called the **edges** and **facets** of B. It is customary in the case $n = 3$ to refer to the facets simply as the **faces** of B.

Among the most familiar examples of solids are convex plane polygons, convex polyhedra, and more generally polytopes. If we let $F_j B$ be the set of all j-faces of an n-solid B, then B is said to be a **polytope** or **n-polytope** if $F_0 B$ is finite. It then follows that $F_i B$ is finite and nonempty for all i, $0 \leq i \leq n$. Thus a polytope is the extension of the concept of polygon in two dimensions and of polyhedron in three dimensions to the case of n dimensions. Although $F_0 B$ is nonempty for $dim(B) \geq 0$, $F_0 B$ need not be finite. Take, for example, the n-disk $D^n = \{x \in E^n : |x| < r\}$ of radius r, for some $r \in \mathbb{R}^+$. Thus the family of polytopes forms a proper subset of the family of solids. The geometry of polytopes has been studied in great detail, most notably by Coxeter in recent decades (see for example, Coxeter [1] and Coxeter [2]).

We let $\mathcal{S}$ and $\mathcal{P}$ denote the set of all solids and the set of all polytopes, respectively. Let $\mathcal{S}^n$ and $\mathcal{P}^n$ denote the set of all n-solids and n-polytopes, respectively.

Let $\sigma_B = \{j : F_j B \neq \emptyset,\ 0 \leq j \leq n-1\}$. If B is an n-polytope then $\sigma_B = \{0, 1, \ldots, n-1\}$. However the converse is not true as we can quite easily see by considering a circular cone or the solid in figure 1.1. The vertex set $F_0 B$ is a subset of the set $ExtB$ of **extreme points** of B, that is to say the set of points x of B such that x is not the midpoint of any pair of distinct points of B. Note that $ExtB = Ext\partial B$. In all polytopes and many solids $F_0 B = ExtB$, although there are examples of solids where this is not true. For instance, consider a solid B given by the union of a rectangle $xyzw$ with a 2-disk D of diameter $d(x, y)$ such that the edge xy coincides with some diameter of D (see figure 1.1). Then the points x and y are extreme points of B but they are not vertices since a hyperplane supporting x or y also supports the edge wx or yz respectively.

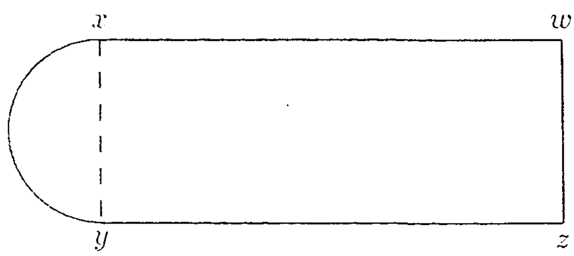


Figure 1.1

Note that $F_{-1}B = \{\emptyset\}$ and $F_n B = \{B\}$, for all solids B. We use the notation $A \triangleleft B$ to mean that A is a j-face of B for some j, $-1 \leq j \leq n$. The faces $\emptyset$ and B are called **improper** since $\emptyset \triangleleft A$ and $A \triangleleft B$ for any j-face A of B. All other faces of B are called **proper**.

Let FB denote the set of all faces of B, that is, if A is a solid then $A \in FB$ if and only if $A \triangleleft B$. Then FB is a bounded lattice with respect to $\triangleleft$, with unit B and zero $\emptyset$, graded by dimension. We call FB the **face-lattice** of B. Then for any $R, S \in FB$, the meet $R \wedge S$ is $R \cap S$ and the join $R \vee S$ is $conv(R \cup S)$. There is a basic equivalence relation between solids defined in terms of their face-lattices as follows. We say that the solids A and B are **combinatorially equivalent** or **face equivalent** if and only if there is a lattice isomorphism $\lambda : FA \to FB$. We denote this by $A \approx B$.

As an example, it is easy to see that any two triangles T_1, T_2 are face equivalent since the faces of T_i, $i = 1$ or 2, may be denoted as follows: let the vertices be labelled A, B, and C; and the edges be labelled a, b, and c such that $A \triangleleft b, c$. $B \triangleleft c, a$ and $C \triangleleft a, b$. Then T_1 and T_2 have the same face lattice (see figure 1.2).

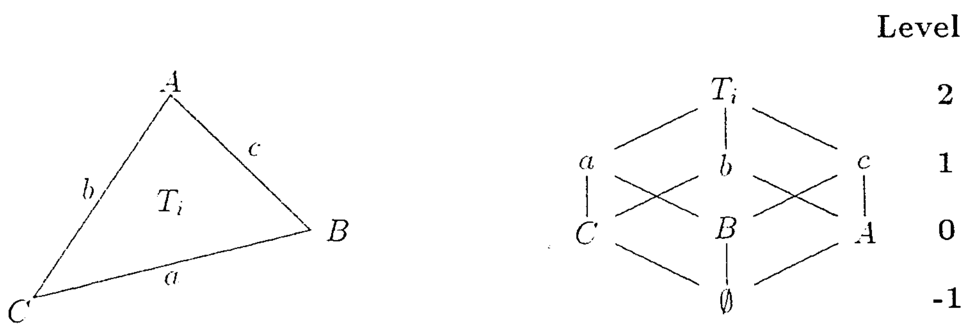


Figure 1.2 The face lattice of a triangle

A **flag** of B is a sequence $(A_0, \ldots, A_r)$ of distinct proper faces of B such that $A_0 \triangleleft \cdots \triangleleft A_r$. If $dim(A_s) = j_s$ then $0 \leq j_0 < j_1 < \cdots < j_r \leq n-1$. Such a flag is said to be **maximal** if it is not contained in any other flag of B, in other words, it is not a subsequence of any other flag of B. This concept of maximal flag is used to define regularity and is a reformulation suggested by A. J. Breda d'Azevedo. For the original definiton of a maximal flag see, for example, Farran and Robertson [1]. A flag of the form $(A_0, A_1, \ldots, A_{n-1})$, $dim(A_s) = s$, is called a **complete flag**. Any complete flag is, of course, maximal. If B is a polytope, then every maximal flag is complete. Let Φ_B denote the set of maximal flags of B.

3. Symmetry and similarity

The **symmetry group** $G = GB$ of an n-solid B in E^n is the set of all rigid transformations of E^n keeping B setwise fixed. Any isometry $g \in GB$ is called a **symmetry** of B. If O is the centroid of B, then GB is a subgroup of $\mathrm{O}(n)$. We assume that this is the case since otherwise $GB = T^{-1}(GB')T$ for some n-solid B' with centroid O, where T is a translation in E^n such that $T(B) = B'$. Since B is compact, so also is GB. We say that that $G\emptyset$ and $G\{0\}$ are trivial groups. A **fundamental region** for the action of G on B is an s-body D such that every point of B is in a G-orbit of some point of D and every G-orbit meets the relative interior of D in at most one point. It is noted that this is not the standard notion of a fundamental region. It is not obvious, in general, that such a fundamental region exists, however as we see in section 6 existence is shown in the area of our work.

The action of G on an n-solid B and hence on $E^n = \mathit{aff}(B)$ determines a stratification of E^n called the **G-stratification**. For any point $x \in B$, we let G_x denote the isotropy subgroup of G at x, that is, $G_x = \{g \in G : g(x) = x\}$. If we put $\mathit{fix}_x = \{y \in E^n : g(y) = y \text{ for all } g \in G_x\}$ and call this the **fixed point set** of x under G, then $\mathit{aff}(\mathit{fix}_x)$ is a linear subspace of E^n. We say that fix_x is of dimension m if $\mathit{aff}(\mathit{fix}_x)$ is m-dimensional. A point $x \in E^n$ is said to be in the j**-set** if and only if fix_x is j-dimensional. Each path component of the j-set is called a j**-stratum**. The fixed point set of B under G consists of all points of E^n held fixed by G and is denoted fix_B. It is trivial to note that for any nonempty solid B, the set fix_B is nonempty.

The effective action of GB on B induces an effective action α_B of GB on

FB, $\alpha_B : GB \times FB \to FB$. Thus we define an equivalence relation on $\mathcal{S}$ as follows. We say that two solids A and B are **symmetry equivalent**, denoted $A \simeq B$, if and only if there is an isometry f of E^n and a lattice isomorphism $\lambda : FA \to FB$ such that the diagram

$$\begin{tikzcd}
GA \times FA \arrow[r, "\alpha_A"] \arrow[d, "f_* \times \lambda"'] & FA \arrow[d, "\lambda"] \\
GB \times FB \arrow[r, "\alpha_B"] & FB
\end{tikzcd}$$

commutes, where $f_* : GA \to GB$ is the isomorphism given by $f_*(g) = f^{-1}gf$. Each symmetry equivalence class is called a **symmetry type** and the symmetry type to which B belongs is denoted $[B]$. If P is a polyhedron then $[P]$ can be realised as a topological manifold (see Robertson [1]), and the dimension of the symmetry type is called the **deficiency** of P, denoted $defP$.

A more general class of mapping of E^n is defined as follows. A map $s : E^n \to E^n$ such that for some $\lambda \in \mathbb{R}$, $d(x, y) = \lambda d(s(x), s(y))$ for all $x, y \in E^n$, is called a **similarity**. The group $S(n)$ of similarities of E^n is generated by reflections, rotations, translations and dilations and acts on $\mathcal{S}^n$ by mapping each n-solid to another that differs only in size and position in space. Two solids A and B are said to be **similar**, denoted $A \sim B$ if there exists a similarity $f \in S(n)$ such that $B = f(A)$. The equivalence relation $\approx$ is coarser than the relation $\simeq$ which in turn is coarser than the similarity relation $\sim$, as can be illustrated by considering the quadrilaterals shown in figure 1.3. Since we are interested in solids with regard to their metrical symmetry, we consider $\mathcal{S}/\sim$ rather than $\mathcal{S}$.

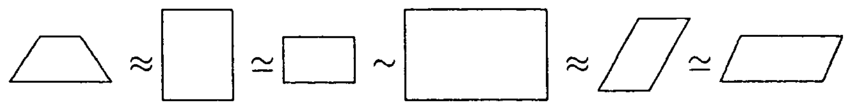


Figure 1.3 Similarity, symmetry equivalence and face equivalence

4. Duality and polarity

Following Robertson [1], we say that two solids A and B are **dual** or A is **dual** to B if there exists an anti-isomorphism $\alpha : FA \rightarrow FB$. In other words, there is a bijection $\alpha : FA \rightarrow FB$ such that for all faces $S, T \in FA$, we have $S \triangleleft T$ if and only if $\alpha(T) \triangleleft \alpha(S)$. If A and B are dual solids, we denote this by $A \wr B$. Duality $\wr$ is not an equivalence relation since, although it is symmetric, $\wr$ is neither reflexive nor transitive. Although it is false in general that $P \wr P$, there are some solids which are self-dual, for example, a triangle or indeed any simplex. It can be shown (for example, see Pinto [1]) that for any solid B, $B \wr A$ for some solid A. In fact, one such solid A dual to B is the polar of B. The **polar** B^* of a solid B is defined by $B^* = \{x \in \mathit{aff}(B) : \forall\, v \in F_0 B, \langle x - c, v - c\rangle \leq 1\}$, where c is the centroid of B. It is convenient to put $\emptyset^* = \emptyset$. Then it can be shown that $G(B) = G(B^*)$ for any solid.

Let P be a polytope with vertex set $\{v_1, \ldots, v_r\}$ and facet set $\{f_1, \ldots, f_s\}$. Suppose that c_i is the centroid of f_i. Then up to similarity the vertex set of P^* coincides with the set $\{c_1, \ldots, c_s\}$, and any facet of P^* has centroid v_i for some $i = 1, \ldots, r$. Moreover, if $\alpha : FP \rightarrow FP^*$ denotes the anti-isomorphism of face lattices, then for all $T \in FP$ and $g \in G$, we have $g \cdot \alpha(T) = \alpha(g \cdot T)$.

5. Products and coproducts

The product and coproduct are binary operations on $\mathcal{S}$ which are used to construct new solids from given solids.

We define the product of two solids by first considering the Cartesian product of E^m and E^n. We identify $E^m \times E^n$ with E^{m+n} by the isomorphism θ from $E^m \times E^n$ to E^{m+n} given by $\theta(x, y) = z$, where $z_i = x_i$, $i = 1, \ldots, m$ and $z = y_{m+j}$, $j = 1, \ldots, n$. Thus E^m and E^n are embedded in E^{m+n} as the orthogonal complements $E^m \times 0$ and $0 \times E^n$.

Let $A \subset E^m$ be an r-solid and $B \subset E^n$ be an s-solid, $r, s \geq 0$. Then $A \times B$, under the above identification, is an $(r+s)$-solid in E^{m+n} such that $A \times 0$ and $0 \times B$ are embedded as orthogonal subsets. The solid $A \times B$ is called the **product** of A with B and is denoted $A \square B$. We put $A \square \emptyset = \emptyset \square A = \emptyset$. The faces of $A \square B$ are of the form $U \square V$ where $U \in FA$ and $V \in FB$.

An n-solid B is said to be **$\square$-prime** if it is not isometric to $M \square N$ for some solids M and N with $dim(M) \geq 1$ and $dim(N) \geq 1$. Otherwise B is **$\square$-decomposable**. Then a **$\square$-decomposition** of B is a sequence $(B_1, \ldots, B_r)$ of solids B_i such that $dim(B_i) \geq 1$ and B is isometric to $B_1 \square \cdots \square B_r$. In such a case we put $B = B_1 \square \cdots \square B_r$. Such a decomposition is **complete** if and only if each B_i is $\square$-prime. It may be shown that a complete $\square$-decomposition of B is unique up to isometry and order. Two solids A and B are said to be **$\square$-coprime** if they have no common isometric $\square$-prime factors of positive dimension.

We now state some relations between the symmetry group of a solid and of its $\square$-prime powers. First recall (Rose [1], for example) that for any group G and for any integer $k \geq 1$, there is a **wreath product** $G \wr S_k$, defined as the group with underlying set $G^k \times S_k$ and product $*$, where

$$\big((g_1, \ldots, g_k), \sigma\big) * \big((h_1, \ldots, h_k), \tau\big) = (g_{\tau(1)} \cdot h_1, \ldots, g_{\tau(k)} \cdot h_k, \sigma \cdot \tau).$$

Then:

1. $G(\Box^k A)$ is isomorphic to $GA \wr S_k$, if A is a $\Box$-prime n solid, $n \geq 1$; and

2. $G(A \Box B)$ is isomorphic to $GA \times GB$, if A, B are $\Box$-coprime solids of positive dimension.

Thus we can write GB in terms of the symmetry group of the factors of its complete $\Box$-decomposition by collecting together isometric factors.

Associated to the product operator is an operation denoted $\diamond$ called the coproduct. The relationship between the two is explained in terms of polarity. The coproduct is defined as follows.

As above, let A and B be nonempty solids in E^m and E^n respectively. Suppose that the centroids of A and B are a and b respectively. Using the above identification of $E^m \times E^n$ with E^{n+m}, the **coproduct** $A \diamond B$ of A with B is defined to be the convex hull of $A \times \{b\} \cup \{a\} \times B$. In the case where A or B is empty, we put $A \diamond B = \emptyset$. If $dim(A) = r \geq 0$ and $dim(B) = s \geq 0$, then $dim(A \diamond B) = r + s$.

If A and B are any two solids then $(A \Box B)^* = (A^* \diamond B^*)$.

The concepts of $\diamond$-primeness and $\diamond$-decomposability follow in a similar manner to that for $\Box$.

We construct some low-dimensional solids to illustrate these operations. Let I be a line segment, T be a triangle, P be a pentagon and D a 2-disk. Then $T \Box I$ and $P \Box I$ are prisms with triangular and pentagonal base, respectively. The coproduct $T \diamond I$ and $P \diamond I$ are bipyramids with triangular and pentagonal bases, respectively. As an example, see figure 1.4 for the case using T. Similarly $D \Box I$ and $D \diamond I$ are the cylinder and the double cone, shown in figure 1.5. A more interesting solid is the 4-polytope given by the product $T \Box P$ (see figure 1.6) which has three pentagonal and five triangular prisms as its facets. The 2-faces of $T \Box P$ are triangles, rectangles and pentagons.

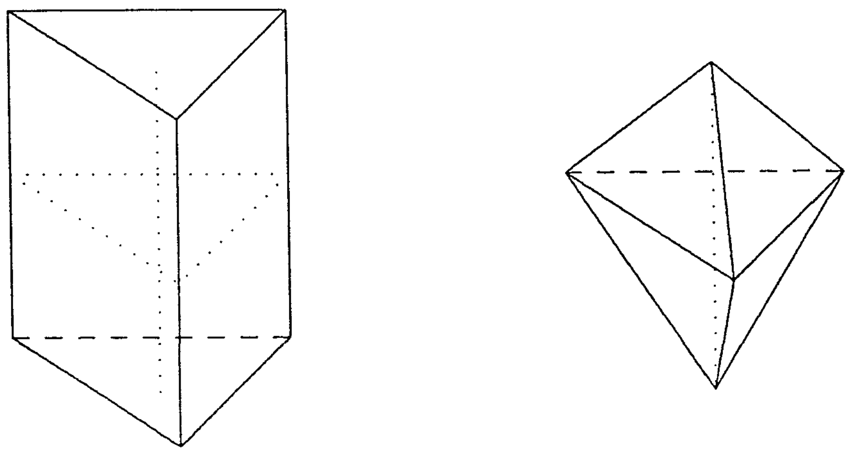


Figure 1.4 $T\Box I$ and $T\diamond I$

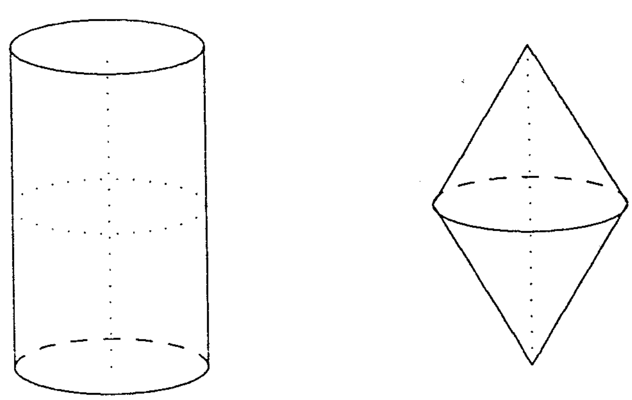


Figure 1.5 $D\Box I$ and $D\diamond I$

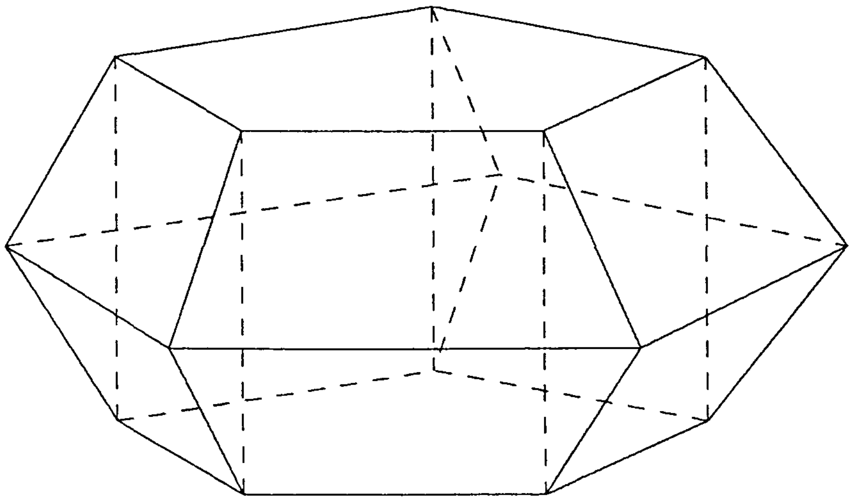


Figure 1.6 $T \square P$

6. Regularity

The theory of regular polytopes is well-known and a comprehensive study can be found in Coxeter [1]. In dimensions $-1, 0$, and 1, all solids are polytopes and are regular. A polygon is regular if it is both equiangular and equilateral. We recall that: (i) the **vertex figure** of an n-polytope P at a vertex v is the $(n-1)$-polytope given by the convex hull of the midpoints of all edges that emanate from v, if all such points lie in some $(n-1)$-space; and (ii) an n-polytope is said to be **regular** if its facets are regular and there is a regular vertex figure at every vertex.

In Farran and Robertson [1], the notion of regularity for convex polytopes was extended to convex bodies in general. This may be given as follows in a formulation using the new definition of maximal flag.

The action of GB on B induces an action of GB on the set FB of all j-faces of B for each j, and since $U \triangleleft V$ implies $g.U \triangleleft g.V$ for all $U \in F_iB, V \in F_jB, g \in GB$, there is an action of GB on ΦB, given by

$$g.(A_0, \ldots, A_r) = (g.A_0, \ldots, g.A_r)$$

for each $g \in G$ and each maximal flag $(A_0, \ldots, A_r)$ of B.

The n-solid B is said to be **regular** if GB is transitive on the maximal flags of B.

The definition of a regular polytope is consistent with this definition since the maximal flags of a polytope are complete flags. The regular 2-polytopes are the regular convex plane polygons. The regular 3-polytopes are the five classical regular polyhedra or Platonic solids (see figures 1.7 to 1.11), as described in Euclid Book XIII (see Heath [1]). The regular n-polytopes for $n \geq 4$ were classified by Schläfli in 1853. We describe this classification following Coxeter [1] using 'Schläfli symbols'. The **Schläfli symbol** $\{p, q, \ldots, u, v\}$ for a regular n-polytope is defined inductively using vertex figures. It is well-known that the vertex figure of a regular polytope P is itself regular and any two vertex figures of P are congruent to one another. Let a regular p-sided polygon be denoted by $\{p\}$. Let a regular polyhedron be denoted by $\{p, q\}$ if its faces are $\{p\}$ and its vertex figures are $\{q\}$, in other words there are q of the $\{p\}$'s around each vertex. A regular n-polytope whose $(n-1)$-faces are $\{p, q, \ldots, v\}$ and vertex figures are $\{q, \ldots, v, u\}$ is denoted by $\{p, q, \ldots, u, v\}$.

The classification of regular n-polytopes for $n \geq 2$ is given in terms of these symbols, as in table 1.1, where we rename the 'measure-polytope' and the 'cross-polytope' as defined in Coxeter [1], the **cube** and **cocube** respectively. These are denoted $\square_n$ and $\diamond_n$ respectively from their obvious decompositions. That is,

$$\square_n = I\square \ldots \square I = \square^n I,$$

$$\diamond_n = I\diamond \ldots \diamond I = \diamond^n I.$$

The more general problem of classifying regular n-solids, for each $n \in \mathbb{N}$, is discussed in chapters 2 to 4. We note that regular non-polytope solids do exist, the n-ball D^n being an obvious example.

We state the following theorem from Farran and Robertson [1].

THEOREM 1:6.1

Let B be a regular n-solid in E^n with centroid O. Let $(A_0, \ldots, A_r)$ be a maximal flag of B. Suppose O_i is the centroid of A_i for $i = 0, \ldots, r$. Then $conv(O, O_1, \ldots, O_r)$ is a fundamental region for the action of GB on B.

Since $\{O, O_1, \ldots, O_r\}$ is an affinely independent subset of E^n, it follows that $conv(O, O_1, \ldots, O_r)$ is an r-simplex with vertices $O, O_1, \ldots, O_r$. It can also be noted that $dim(fix_{O_i}) = 1$ for each $i = 1, \ldots, r$.

DIMENSION	SCHLÄFLI SYMBOL	DESCRIPTION
2	$\{p\}$	regular p-gon
3	$\{3,3\}$	regular tetrahedron $\triangle_3$
	$\{4,3\}$	cube $\square_3$
	$\{3,4\}$	regular octahedron $\diamond_3$
	$\{5,3\}$	regular dodecahedron
	$\{3,5\}$	regular icosahedron
4	$\{3,3,3\}$	4-simplex $\triangle_4$
	$\{4,3,3\}$	4-cube $\square_4$
	$\{3,3,4\}$	4-cocube $\diamond_4$
	$\{3,4,3\}$	24-cell
	$\{5,3,3\}$	120-cell
	$\{3,3,5\}$	600-cell
$n \geq 5$	$\{3,3,\ldots,3\}$	n-simplex $\triangle_n$
	$\{4,3,\ldots,3\}$	n-cube $\square_n$
	$\{3,\ldots,3,4\}$	n-cocube $\diamond_n$

Table 1.1

Classification of regular polytopes

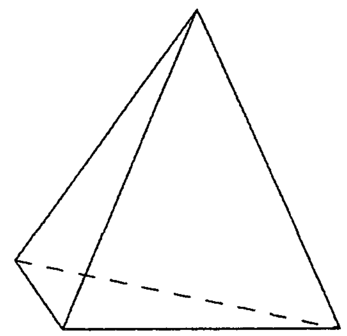


Figure 1.7 The tetrahedron

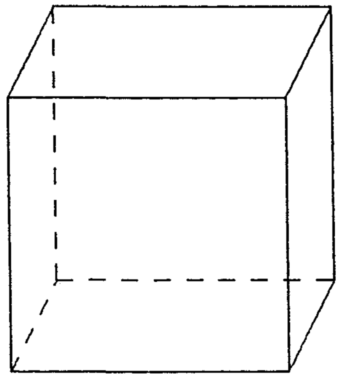


Figure 1.8 The cube

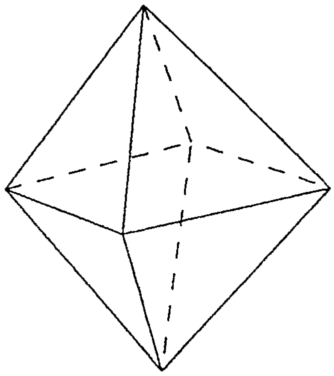


Figure 1.9 The octahedron or cocube

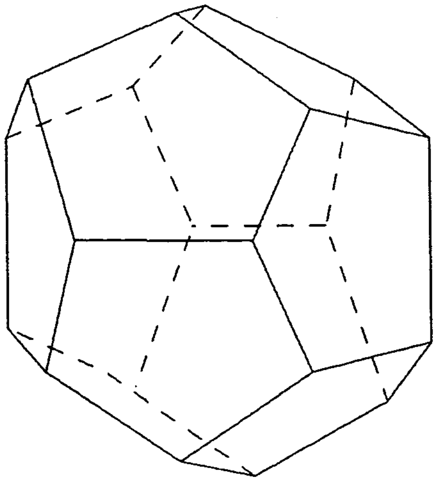


Figure 1.10 The dodecahedron

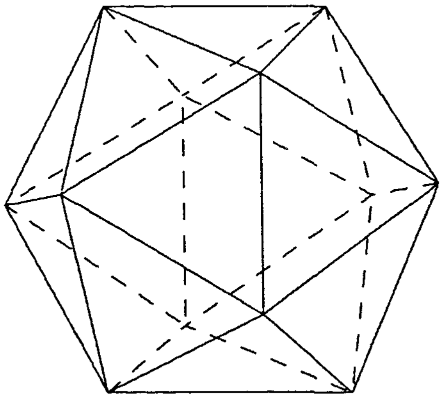


Figure 1.11 The icosahedron

7. Perfection

We may also study solids of $\mathcal{S}$ which are locally maximal in their symmetry behaviour. Such solids are called perfect and are defined in terms of geometry and topology, rather than in terms of groups. For this reason, it is considerably harder to handle this concept than it is for regularity.

A solid B is said to be **perfect** if and only if all solids symmetry equivalent to B are similar to B. Thus B is perfect if and only if all symmetry equivalent polytopes differ from B only in their size or position in space relative to B. Among the best known examples of perfect solids are the regular polytopes, most notably the five Platonic solids. More generally, all regular solids are perfect (Farran and Robertson [1]). However there are non-regular perfect solids as we shall see. The perfect n-polytopes have been classified up to $n = 3$ in Robertson [1]. Trivially in dimension $-1, 0$ and 1, all solids are perfect polytopes. In dimension 2, the perfect polygons coincide with the regular polygons. Finally in dimension 3, there are four non-regular perfect polyhedra in addition to the regular Platonic solids. These are the cuboctahedron, the icosadodecahedron, and their respective polars, the rhombic dodecahedron of the first kind † and the rhombic triacontrahedron (see figures 1.12 – 1.15). If we add the circular disk and the 3-ball to this list, we have a classification of perfect solids up to dimension 3. This classification shows that a polyhedron is perfect if and only if its symmetry group acts transitively on the set of edges proving one case of **Deicke's conjecture** (see Robertson [1]), since a polyhedron P is perfect if and only if $def P = 0$ (Robertson [1]). In 1987, this conjecture was proved by Rostami [1].

† For a discussion of the rhombic dodecahedron of the second kind see Appendix A.

THEOREM 1:7.1 ROSTAMI'S THEOREM

Let P be any polyhedron. Suppose that the action of GP on the set of edges of P has e orbits. Then $defP = e - 1$.

We now state some properties about perfect solids, whose proofs can be found in Pinto [1] and Robertson [1].

THEOREM 1:7.2

A solid B is perfect if and only if B^* is perfect.

THEOREM 1:7.3

A polytope P is perfect if and only if $P = \square^r Q$ for some $\square$-prime perfect polytope Q and for some integer $r \geq 1$.

THEOREM 1:7.4

Any n-solid is perfect if and only if it is a $\square$-prime power of a perfect solid.

THEOREM 1:7.5

Let B be an n-solid in E^n. Suppose B has symmetry group G and centroid c. If B is perfect then the fixed point set of B is the singleton $\{c\}$, that is $fix_B = \{c\}$.

Theorems 1:7.3 and 1:7.4 have obvious duals in terms of the coproduct.

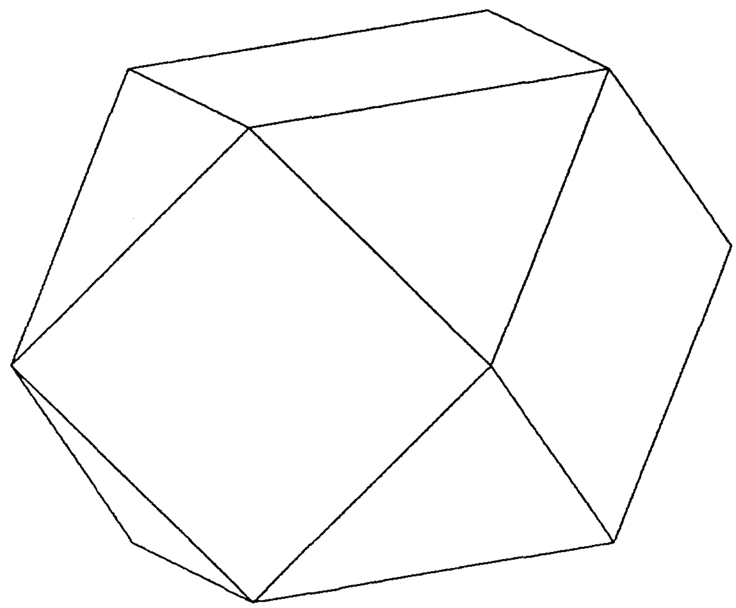


Figure 1.12 The cuboctahedron

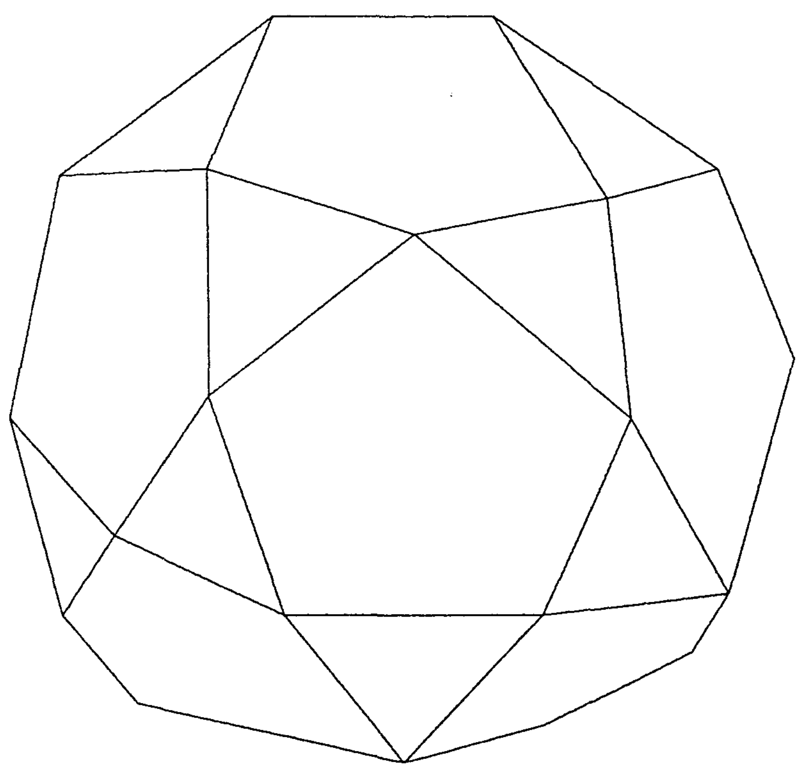


Figure 1.13 The icosidodecahedron

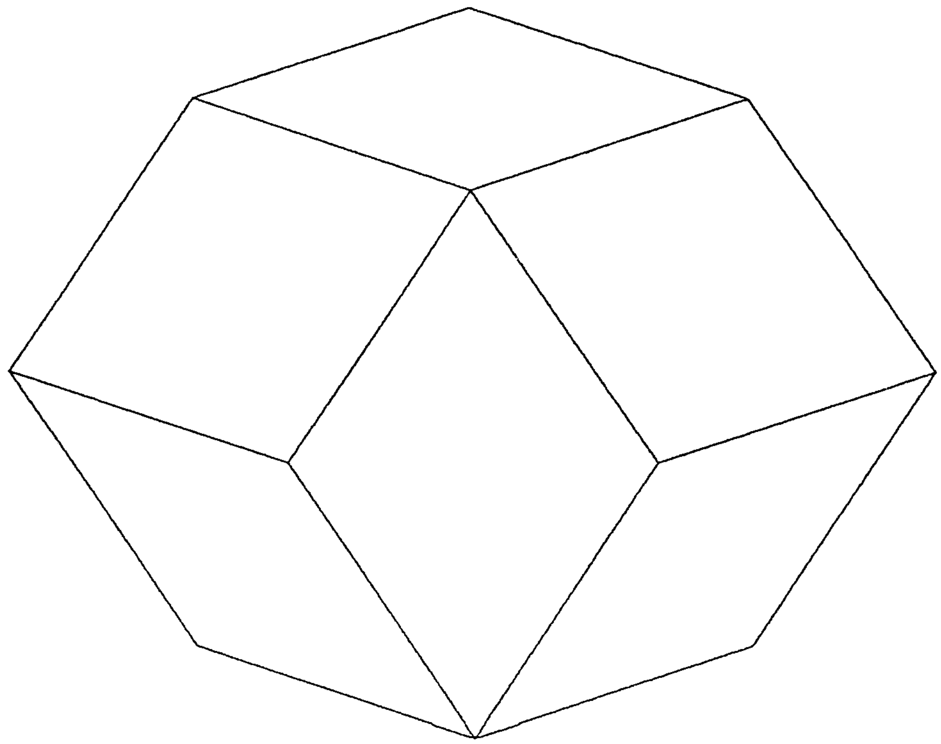


Figure 1.14 The rhombic dodecahedron of the first kind

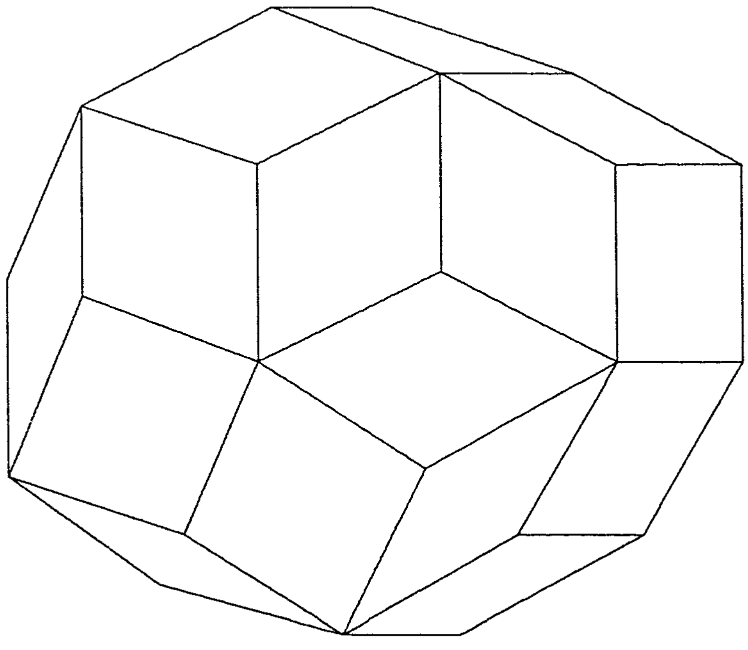


Figure 1.15 The rhombic triacontrahedron

8. Wythoffian polytopes

Let R be a regular n-polytope with symmetry group G. Then $R = conv(G.v)$ and $R^* = conv(G.c)$, where v is any vertex of R and c is the centroid of any facet of R. Let A be a complete flag of R. Then a fundamental region for the action of G on R is given by $conv(O, c_0 = v, c_1, \ldots, c_{n-1} = c)$, where c_i is the centroid of the i-face f_i contained in A. For each $0 \leq i \leq n-1$, the polytope $conv(G.c_i)$ is a perfect n-polytope R_i. We call R_i **Wythoffian**, since R_i can be derived from Wythoff's construction which is described below (see Coxeter [1] for full details). If P is a non-regular Wythoffian n-polytope we call P a **wythotope** or n**-wythotope**.

There is a well-established representation of various fundamental regions of Wythoffian polytopes and other polytopes by certain Coxeter graphs. The nodes of a Coxeter graph represent the walls of the fundamental region D (that is, the facets of D which contain O) or their respective reflections. Two nodes are joined by a branch whenever the corresponding walls are not perpendicular. Moreover, a branch between nodes i and j is marked with the integer a_{ij} to indicate the angle $\frac{\pi}{a_{ij}}$ $(a_{ij} \geq 3)$ between the two corresponding walls. It is usual to omit the a_{ij} on a branch if $a_{ij} = 3$. Coxeter graphs can also be used to represent degenerate polytopes (see Coxeter [1]). The various Wythoffian polytopes can now be represented by modifying these graphs, since for each wall of D there exists one vertex c_i of D not contained in that wall. Thus its corresponding node in the Coxeter graph is shaded to indicate the polytope P given by $conv(G.c_i)$. The procedure of determining P from vertices of a given fundamental region is called **Wythoff's construction**.

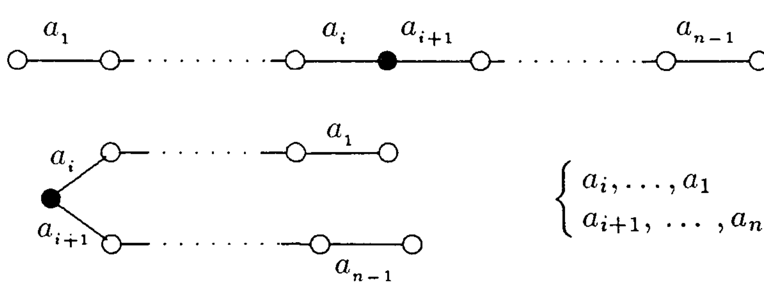
Suppose $P = R_i$ is a wythotope derived from a regular n-polytope $R = \{a_1, \ldots, a_{n-1}\}$, by taking the centroid of some i-face of R. Then we can also denote P by the Schläfli symbol

$$\left\{ \begin{array}{l} a_i, \ldots, a_1 \\ a_{i+1}, \ \cdots \ , a_{n-1} \end{array} \right\} \text{ or equivalently } \left\{ \begin{array}{l} a_{i+1}, \ \cdots \ , a_{n-1} \\ a_i, \ldots, a_1 \end{array} \right\},$$

as this polytope is a truncation of R (see Coxeter [1]). The facets of P are then $(n-1)$-polytopes of the form

$$\left\{ \begin{array}{l} a_i, \ldots, a_2 \\ a_{i+1}, \ \cdots \ , a_{n-1} \end{array} \right\} \text{ and } \left\{ \begin{array}{l} a_i, \ldots, a_1 \\ a_{i+1}, \ \cdots \ , a_{n-2} \end{array} \right\}.$$

Then for any Wythoffian polytope we can write out the Schläfli symbol for each i-face, $1 \leq i \leq n$. Schläfli symbols can also be used to represent degenerate polytopes, for example honeycombs, non-perfect truncations and non-convex polytopes (see Coxeter [1]). It can also be noted that for any polytope P given by a Schläfli symbol, P may be represented by a Coxeter graph, where the markings on the branches are given by the entries in the Schläfli symbol. For instance, if $P = R_i$ is given as above then the Coxeter graph of P is the graph in figure 1.16.



$$\left\{ \begin{array}{l} a_i, \ldots, a_1 \\ a_{i+1}, \ \cdots \ , a_{n-1} \end{array} \right\}$$

Figure 1.16 The Coxeter graph of $P = R_i$ and its Schläfli symbol

For example, the cuboctahedron $\left\{ \begin{array}{c} 3 \\ 4 \end{array} \right\} = \left\{ \begin{array}{c} 4 \\ 3 \end{array} \right\}$ is the truncation both of $\{4, 3\}$ and of $\{3, 4\}$. The faces of $\left\{ \begin{array}{c} 4 \\ 3 \end{array} \right\}$ are of the form $\{4\}$ and $\{3\}$, that is squares and equilateral triangles respectively. The icosidodecahedron $\left\{ \begin{array}{c} 3 \\ 5 \end{array} \right\} = \left\{ \begin{array}{c} 5 \\ 3 \end{array} \right\}$ is the truncation both of $\{5,3\}$ and of $\{3,5\}$. Thus each face of $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$ is either a $\{3\}$ or a $\{5\}$. The Coxeter graphs of these polyhedra are given in figure 1.17

Figure 1.17 The Coxeter graphs of $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$

It can be noted that the polars of $\left\{ \begin{matrix} 3 \\ 4 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$ (see figures 1.14 and 1.15, respectively), are perfect polyhedra which cannot be denoted by a Schläfli symbol. Likewise, in n dimensions there are perfect polytopes which have no Schläfli symbol. Rostami [1], however, stated the following conjecture for $n = 4$, which as we shall see in part II can be generalised for all n (also see Pinto [1]).

CONJECTURE 1:8.1

Let P be a prime 4-polytope. Then P is perfect if and only if P or P^* is Wythoffian.

CHAPTER 2

KOSTANT'S CONSTRUCTION

The introduction of the concept of regular solids immediately gives rise to the problem of classifying these objects. Robertson and Farran [1] constructed a process by which regular solids are obtained from regular polytopes. This construction is based on the adjoint action of a compact semisimple Lie group on its Lie algebra and is due to Kostant's work on convexity (Kostant [1]). The origins of this work are Schur [1] and Horn [1], see also Atiyah [1]. We summarize this process which is the fundamental idea in what follows.

1. Lie theory

We begin with some notation and definitions in Lie theory (for more information see Helgason [1] and Kostant [1]). Let B be a solid with centroid O. Then GB is a compact subgroup of $\mathrm{O}(n)$ and hence a semisimple Lie group. If GB is discrete and hence finite, we consider GB as a Lie group of zero dimension.

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$. Let $Ad : G \to GL(\mathfrak{g})$ denote the adjoint representation of G and $ad : \mathfrak{g} \to end\mathfrak{g}$ denote the adjoint representation of $\mathfrak{g}$, where $end\mathfrak{g}$ denotes the space of endomorphisms of $\mathfrak{g}$.

We recall some standard decompositions of semisimple Lie algebras.

A **Cartan subalgebra** of $\mathfrak{g}$ is a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and for each $H \in \mathfrak{h}$, the endomorphism $ad(H)$ of $\mathfrak{g}$ is semisimple.

THEOREM 2:1.1

Every semisimple Lie algebra over $\mathbb{C}$ contains a Cartan subalgebra.

Proof

See, for example, Helgason [1].

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Let α be a linear function on the complex vector space $\mathfrak{h}$. Then $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ is a linear subspace of $\mathfrak{g}$, where $[\ ,\ ]$ denotes the Lie bracket of $\mathfrak{g}$. If $\mathfrak{g}^\alpha \neq \{0\}$ then α is called a **root** and $\mathfrak{g}^\alpha$ is called a **root subspace**. Let Δ denote the set of all nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

THEOREM 2:1.2

(i) $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$ *(direct sum).*

(ii) $dim(\mathfrak{g}^\alpha) = 1$ *for each* $\alpha \in \Delta$.

(iii) The restriction of B to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate. For each linear form α on $\mathfrak{h}$, there exists a unique element $H_\alpha \in \mathfrak{h}$ such that $B(H, H_\alpha) = \alpha(H)$ for all $H \in h$, where B denotes the Killing form on $\mathfrak{g}$.

Proof

See Helgason [1].

Now let $\mathfrak{g}_0$ be a semisimple Lie algebra over $\mathbb{R}$, let $\mathfrak{g}$ be its complexification and σ the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. A direct decomposition $\mathfrak{g}_0 =$

$\mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$ into a subalgebra $\mathfrak{k}_0$ and a vector subspace $\mathfrak{p}_0$ is called a **Cartan decomposition** of $\mathfrak{g}_0$ if there exists a compact real form $\mathfrak{g}_k$ of $\mathfrak{g}$ such that

$$\sigma \cdot \mathfrak{g}_k \subset \mathfrak{g}_k;$$

$$\mathfrak{k} = \mathfrak{g}_0 \cap \mathfrak{g}_k;$$

$$\text{and} \quad \mathfrak{p}_0 = \mathfrak{g}_0 \cap (i\mathfrak{g}_k).$$

It is well-known (see, for example, Helgason [1]) that every $\mathfrak{g}_0$ has a Cartan decomposition and that any two Cartan decompositions of $\mathfrak{g}_0$ are conjugate under an inner automorphism of $\mathfrak{g}$.

A third decomposition, the Iwasawa decomposition, arises from the combination of the Cartan decomposition of a semisimple Lie algebra and the root space decomposition of its complexification. This decomposition is summarized as follows.

Let $\mathfrak{g}_0$ be a semisimple Lie algebra over $\mathbb{R}$ with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and let $\mathfrak{u} = \mathfrak{k}_0 + i\mathfrak{p}_0$. Let σ and τ denote the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}$, respectively. Let θ denote the automorphism $\theta = \sigma \cdot \tau$. We let $\mathfrak{h}_{\mathfrak{p}_0}$ be any maximal abelian subspace of $\mathfrak{g}_0$ and let $\mathfrak{h}_0$ be any maximal abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{h}_{\mathfrak{p}_0}$. Then $\theta(\mathfrak{h}_0) \subset \mathfrak{h}_0$ and we have the direct decomposition $\mathfrak{h}_0 = (\mathfrak{h} \cap \mathfrak{k}_0) + (\mathfrak{h}_0 \cap \mathfrak{p}_0)$ where $\mathfrak{h} \cap \mathfrak{p}_0 = \mathfrak{h}_{\mathfrak{p}_0}$. If $\mathfrak{h}$ denotes the subspace of $\mathfrak{g}$ generated by $\mathfrak{h}_0$ then it follows that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

The roots $\alpha_1, \ldots, \alpha_r$ are called **simple** if $(\alpha_1, \ldots, \alpha_r)$ is a basis of Δ and each root α of Δ can be written as $\alpha = \sum_{i=1}^{r} k_{\alpha_r}$ with integral coefficients k_{α_i} all nonnegative or all nonpositive. If all $k_{\alpha_i} \geq 0$ then α is called a **positive root**.

Let Δ^+ denote the set of positive roots of $\mathfrak{g}$. For each $\alpha \in \Delta$, let α^θ be defined by $\alpha^\theta(H) = \alpha(\theta H)$ where $H \in \mathfrak{h}$. Then the root α vanishes identically on $\mathfrak{a} = \mathfrak{h}_{\mathfrak{p}_0}$ if and only if $\alpha = \alpha^\theta$. We divide Δ^+ into the two classes:

$$P_+ = \{\alpha : \alpha \in \Delta^+, \alpha \neq \alpha^\theta\}$$

$$P_- = \{\alpha : \alpha \in \Delta^+, \alpha = \alpha^\theta\}.$$

THEOREM 2:1.3

Let $\mathfrak{n} = \sum_{\alpha \in P_+} \mathfrak{g}^\alpha$, *and suppose that* $\mathfrak{n}_0 = \mathfrak{g}_0 \cap \mathfrak{n}$ *and* $\mathfrak{s}_0 = \mathfrak{a} + \mathfrak{n}_0$. *Then* $\mathfrak{n}$ *and* $\mathfrak{n}_0$ *are nilpotent Lie algebras,* $\mathfrak{s}_0$ *is a solvable Lie algebra and* $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a} + \mathfrak{n}_0$ *is a direct vector space sum called an* **Iwasawa decomposition** *of* $\mathfrak{g}_0$.

Proof

See, for example, Helgason [1].

THEOREM 2:1.4

Let $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a} + \mathfrak{n}_0$ *be an Iwasawa decomposition of a semisimple Lie algebra* $\mathfrak{g}_0$ *over* $\mathbb{R}$. *Let* G *be any connected Lie group of* $\mathfrak{g}_0$ *and let* K, $A_{\mathfrak{p}}$, N *be the analytic subgroups of* G *with Lie algebras* $\mathfrak{k}_0$, $\mathfrak{a}$, *and* $\mathfrak{n}_0$, *respectively. Then the mapping* $\Phi : K \times A_{\mathfrak{p}} \times N \to G$ *defined by* $\Phi(k, a, n) = k \cdot a \cdot n$ *is an analytic diffeomorphism of the product manifold* $K \times A_{\mathfrak{p}} \times N$ *onto* G. *Accordingly,* $G = KA_{\mathfrak{p}}N$ *is called the* **Iwasawa decomposition** *of* G.

Proof

See Helgason [1].

Let G be a Lie group with semisimple Lie algebra $\mathfrak{g}$ such that $G = KAN$ and $\mathfrak{g} = \mathfrak{k}\mathfrak{a}\mathfrak{n}$ are Iwasawa decompositions of G and $\mathfrak{g}$ respectively. Let $exp : \mathfrak{g} \to G$ be the exponential mapping. If $a \in A$ then $x = log(a)$ is the unique element in $\mathfrak{a}$ such that $a = exp(x)$ (see Kostant [1]).

Finally, the **Weyl group** W associated to $(\mathfrak{a}, \mathfrak{g})$ is the finite group defined as the quotient $W = M'/M$, where M' and M are, respectively, the normalizer and centralizer of $A_{\mathfrak{p}}$ in K, that is,

$$M' = \{k \in K : k \cdot A_{\mathfrak{p}} \cdot k^{-1} \subset A_{\mathfrak{p}}\},$$

$$M = \{k \in K : k \cdot a \cdot k^{-1} = a \text{ for all } a \in A_{\mathfrak{p}}\}.$$

W acts on $\mathfrak{a}$ and A such that *exp*: $\mathfrak{a} \rightarrow A_{\mathfrak{p}}$ is a W-isomorphism. This definition from Kostant [1] is derived from the definition of the Weyl group of a symmetric space (see chapter 3).

The Weyl group W is identified with a subgroup of the symmetric group generated by reflections acting on some root system Γ. The Dynkin diagram of Γ completely determines W and is associated with a fundamental region D of W where the nodes indicate the walls of D and two walls are inclined at angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$ or $\frac{\pi}{6}$ depending whether there are 0, 1, 2 or 3 branches, respectively, joining the corresponding nodes. Thus a Dynkin diagram and a Coxeter graph provide the same information about D, as shown in the components of these graphs given in figure 2.1

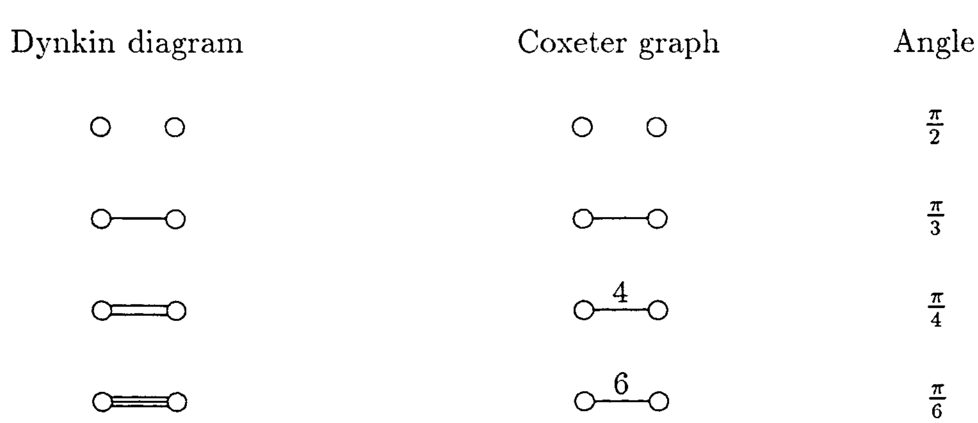


Figure 2.1 Components of a Dynkin diagram and Coxeter graph

The nodes in a Dynkin diagram represent the simple roots in the root space decomposition (see Helgason [1] and Humphreys [1]). Each simple root α_i is associated with a weight proportional to $\langle \alpha_i, \alpha_i \rangle$ where the scalar product $\langle \, , \, \rangle$ is a positive definite scalar product such as the Killing form. In each root system there are at most two weights, and accordingly each root is called long or short. It is usual to denote this in a Dynkin diagram by putting an arrow on the branch between a long and short root pointing to the short root. The root system is said to be **irreducible** if its graph is connected. The irreducible root systems are well-known and have been classified: the Dynkin diagram of any irreducible root system is one of the graphs in figure 2.2 (see Humphreys [1] for full details). The classification of the simple non-Abelian Lie algebras follows from the classification of irreducible root systems. The classical irreducible root systems are denoted A_l, B_l, C_l, D_l and give rise to the so-called classical simple Lie algebras. Likewise the exceptional irreducible root systems are denoted E_6, E_7, E_8, F_4 and G_2, giving rise to the exceptional simple Lie algebras. In order to avoid repetitions, the following restrictions are made on l:

$$A_l \ (l \geq 1); \quad B_l \ (l \geq 2); \quad C_l \ (l \geq 3) \quad \text{and } D_l \ (l \geq 4).$$

2. Kostant's convexity theorem

Let $G = KAN$ be an Iwasawa decomposition of a semisimple Lie group G, and let $\Phi : K \times A \times N \to G$ be the corresponding analytic diffeomorphism. Then Φ is trivially a bijection, for all $g \in G$, there exists a unique $\big(k(g), a(g), n(g)\big) \in K \times A \times N$ such that $g = k(g) \cdot a(g) \cdot n(g)$, and $a(g)$ is called the **a-component** of g.

Let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ where $\mathfrak{a}$, $\mathfrak{g}$ $(\mathfrak{a} \subset \mathfrak{g})$ are the Lie algebras of

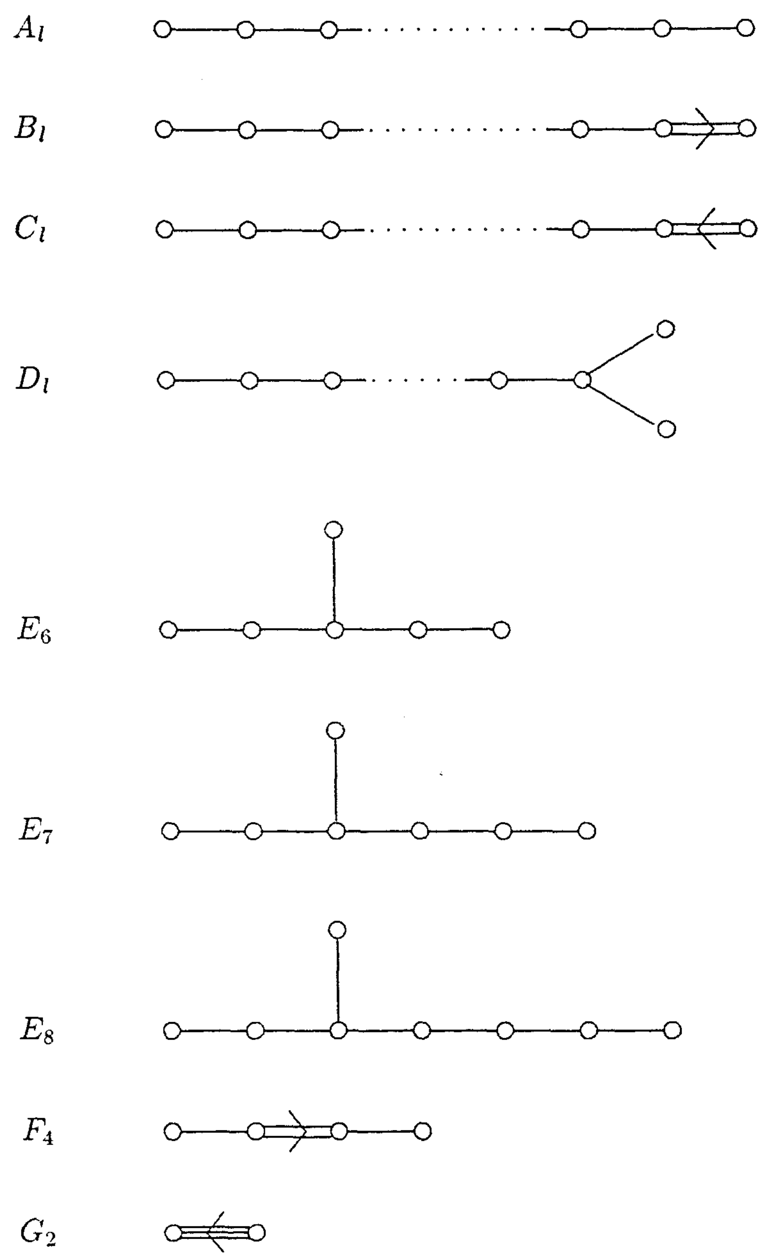


Figure 2.2 Dynkin diagrams of the irreducible rootsystems

A and G. Then for each $x \in \mathfrak{a}$, let $\mathfrak{a}(x) = conv(W(x))$ be the convex hull of the Weyl group orbit $W(x)$ and correspondingly, for $b \in A$ let $A(b) = exp\Big(\mathfrak{a}\big(log(b)\big)\Big)$.

THEOREM 2:2.1 KOSTANT'S CONVEXITY THEOREM

For any $b \in A, \quad A(b) = \{\mathfrak{a}(bv) : v \in K\}$.

Proof

See Kostant [1].

This theorem was reformulated by Pinto [1] by considering the following. Let K be a compact connected semisimple Lie group with Lie algebra $\mathfrak{k}$, let $\mathfrak{k}^{\mathbb{C}}$ denote the complexification of $\mathfrak{k}$ and let G be a semisimple Lie group with Lie algebra $\mathfrak{g} = (\mathfrak{k}^{\mathbb{C}})^{\mathbb{R}}$. Then let $G = KTN$ be the Iwasawa decomposition of G derived from the Iwasawa decomposition, $\mathfrak{g} = \mathfrak{k} + i\mathfrak{t} + \mathfrak{n}$, of $\mathfrak{g}$, where $\mathfrak{t}$ is the Lie algebra of a maximal torus T of K.

COROLLARY 2:2.2

Let K be a compact connected semisimple Lie group acting on its Lie algebra $\mathfrak{k}$ by the adjoint action. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus T of K and let W be the Weyl group of K acting on $\mathfrak{t}$. For every $x \in \mathfrak{k}$, the orthogonal projection of the K-orbit $K(x)$ onto $\mathfrak{t}$ coincides with the convex hull of the corresponding W-orbit $W(x')$, where $x' \in \mathfrak{t} \cap K(x)$.

Proof

See Pinto [1].

Let $K(x)$ denote the orbit of $x \in \mathfrak{k}$ given by the adjoint action of K on $\mathfrak{k}$. Suppose $\Pi : \mathfrak{k} \to \mathfrak{t}$ is the above orthogonal projection. Then

$$\Pi\Big(conv(K(x))\Big) = conv\Big(\Pi(K(x))\Big) = \Pi(K(x))$$

since $K(x)$ projects onto the convex set $conv(W(x'))$ for some $x' \in \mathfrak{t} \cap K(x)$. Alternatively, if $v^* \in M^* = \{g \in K : Ad(g)(i\mathfrak{t}) \subset i\mathfrak{t}\}$ is a representation of $v \in W$ then $Ad(v^*)(x') = v(x')$ by the definition of the action of W on $\mathfrak{t}$, so that $W(x') \subset K(x') = K(x)$. Thus $conv(W(x')) \subset conv(K(x) \cap \mathfrak{t})$ and therefore $\Pi(K(x)) = conv(K(x) \cap \mathfrak{t})$.

Hence each orbit $K(x)$ of the adjoint action of K on $\mathfrak{k}$ is associated with a polytope $P = conv(W(x'))$ such that $\Pi(P) = P$ (trivally $P = conv(W(x'))$ is a polytope since W is finite). This is the basis for Kostant's construction described in Farran and Robertson [1] which may be given as follows.

Let $G = K$ be any compact connected semisimple Lie group with Lie algebra $\mathfrak{g}$, and let T be a maximal torus in G with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. The adjoint action of G on $\mathfrak{g}$ is orthogonal with respect to a natural inner product in $\mathfrak{g}$, where the orthogonal projection is given by Π above. The quotient $N(T)/T$ may be identified with the Weyl group W of G acting on $\mathfrak{t}$ as a group generated by reflections, where $N(T) = \{g \in G : Ad(g)(\mathfrak{t}) = \mathfrak{t}\}$ is the isotropy subgroup of G at $\mathfrak{t}$. A fundamental region for the induced action of W on the unit sphere in $\mathfrak{t}$ is a spherical $(m-1)$-simplex Δ, where $m = dim(\mathfrak{t})$ is the rank of G. Then $P_x = conv(W(x))$ is an m-polytope (with vertex set $F_0 P_x = W(x)$) for each $x \in \Delta$. We put $B_x = conv(Ad(G)(x)) = conv(K(x))$, then B_x is an n-solid, where $n = dim(G)$, with extremal set $F_0 B_x = Ad(G)x$. Then G is a subgroup of the symmetry group of B_x and $\Pi(B_x) = P_x$.

It is possible to determine the face lattice FB_x of B_x in terms of FP_x. In particular, Farran and Robertson [1] proved the following.

THEOREM 2:2.3

P_x is a regular polytope if and only if B_x is a regular solid.

Proof

We give the proof that B_x is regular if P_x is regular in order to determine FB_x from FP_x and refer to Farran and Robertson [1] for the converse.

Suppose P_x is a regular polytope, then x is one of the vertices of $\triangle$ (we may suppose that the set F_0B_x lies on the unit sphere in $\mathfrak{t}$). Then there is a complete flag of P_x of the form $(A_0, A_1, \ldots, A_k)$ where $dim(A_i) = i$ and the centroid α_i of A_i is a vertex of $\triangle$, with $\alpha_0 = x$ (see theorem 1:6.1). Let G_i be the isotropy subgroup of G at α_i and let $B_i = G_i(A_i)$. Then B_i is a j-face of B_x with centroid α_i, where $j = j_i = dim(G) - dim(G_i) + i$. Since every j_i-face of B_x is of the form $g \cdot B_i$ and since $(B_0, B_1, \ldots, B_k)$ is a maximal flag in B_x, where $dim(B_i) = j_i$, it follows that B_x is regular.

The restriction ρ of the projection Π to the set of regular solids is a projection $\rho : \mathcal{S}_R \to \mathcal{P}_R$ where $dim(\rho(B)) = \#\sigma_B$ and $B = \rho(B)$ if and only if $B \in \mathcal{P}_R$. This process of deriving a regular solid B_x from a regular polytope P_x is called **Kostant's construction**.

This, however, does not yet give a classification of the regular solids as we have not determined $\rho^{-1}(P)$ for each regular polytope P.

CHAPTER 3

SYMMETRIC SPACE REPRESENTATIONS

The regular solids are classified by associating each regular solid with a certain symmetric space. The key to this association is the work of Dadok [1] on polar representations. In this chapter we summarize some symmetric space and polar representation theory in order to introduce the relevant notation and concepts. Symmetric spaces are studied in detail in various texts including Helgason [1], Loos [1] and Wolf [1].

1. Symmetric spaces

The study of symmetric spaces and Lie groups are closely related. Symmetric spaces, defined as Riemannian manifolds for which the curvature tensor is invariant under all parallel translations, were first studied by E. Cartan in the 1920s.

A Riemannian manifold M is a **Riemannian globally symmetric space** if each point $p \in M$ is an isolated fixed point of some involutive isometry of M.

This definition is equivalent to the original definition by E. Cartan. Let $I(M)$ be the set of all isometries of M. Then $I(M)$ is a group under composition and it can be shown that $I(M)$ is a Lie group (see, for example, Helgason [1]). Let $I_0(M)$ denote the identity component of $I(M)$.

THEOREM 3:1.1

Let M be a Riemannian globally symmetric space and let p_0 be any point in M. Let $H = I_0(M)$ and $K \subseteq H$ be the subgroup which leaves p_0 fixed. Then K is a compact subgroup of the connected group H and H/K is analytically diffeomorphic to M under the mapping $hK \to h \cdot p_0$ for $h \in H$.

Proof

See Helgason [1].

We may therefore denote a symmetric space by H/K for some Lie groups H, K where $K \subset H$. Then H is a Lie transformation group of H/K in the sense that the mapping $(h, g \cdot p_0) \to hg \cdot p_0$ is a differentiable mapping of $H \times H/K$ onto H/K. A symmetric space H/K is said to be of **compact type** or **noncompact type** according to whether it has positive or negative sectional curvature and there exists a duality between these two types (for more details, see Helgason [1]). The **rank** of H/K is the maximal dimension of a flat, totally geodesic submanifold of M (that is, the maximal dimension of a totally geodesic submanifold for which the curvature tensor vanishes identically).

The Weyl group of a symmetric space is defined as follows (see Helgason [1] or Loos [1]). Let H, K be Lie groups with Lie algebras $\mathfrak{h}$, $\mathfrak{k}$ respectively, such that H/K is a symmetric space. Let $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{h}$ and let $\mathfrak{a}$ denote an arbitrary maximal abelian subspace of $\mathfrak{p}$. Then the **Weyl group** of H/K is the Weyl group of $\mathfrak{h}^{\mathbb{C}}$ defined in Chapter 2, that is, the quotient of the normalizer of $i\mathfrak{a}$ in K by the centralizer of $i\mathfrak{a}$ in K, where $i^2 = -1$. In fact Kostant's definition in Chapter 2 (also see Kostant [1]) is derived from this definition. The same identifications with root systems and Dynkin diagrams hold as in Chapter 2. It then follows that the rank of H/K is given by $dim(\mathfrak{a})$.

The irreducible symmetric spaces are well-known and have been classified. Before we give the classification of irreducible symmetric spaces, we need some more definitions. We follow Helgason [1].

A pair (H, K) is said to be a **Riemannian symmetric pair** if

1) there exists an involutive analytic automorphism σ of H such that $(K_\sigma)_0 \subset H \subset K_\sigma$ where K_σ is the set of fixed points of σ and $(K_\sigma)_0$ is the identity component of (K_σ),

2) $Ad_H(K)$ is compact, where $Ad_H(K)$ is the group given by the adjoint action of K on H.

An **orthogonal symmetric Lie algebra** is a pair $(\mathfrak{h}, s)$ such that

1) $\mathfrak{h}$ is a Lie algebra over $\mathbb{R}$,

2) s is an involutive automorphism of $\mathfrak{h}$,

3) the set, $\mathfrak{k}$, of fixed points of s is a compactly embedded subalgebra of $\mathfrak{h}$.

For each Riemannian globally symmetric space H/K, the pair (H, K) is a Riemannian symmetric pair, and is associated with an orthogonal symmetric Lie algebra $(\mathfrak{h}, s)$. A symmetric space H/K is said to be **irreducible** if its associated orthogonal symmetric Lie algebra $(\mathfrak{h}, s)$ is irreducible, that is,

1) $\mathfrak{h}$ is semisimple and $\mathfrak{u}$ contains no ideals $\neq \{0\}$,

2) the algebra $ad_{\mathfrak{h}}(\mathfrak{u})$ acts irreducibly on $\mathfrak{e}$,

where $\mathfrak{u}$ and $\mathfrak{e}$ are eigenspaces of s for the eigenvalues $+1$ and -1, respectively.

The irreducible symmetric spaces are then classified using the following theorem of Helgason [1].

THEOREM 3:1.2

The irreducible orthogonal symmetric Lie algebras are of type

I $(\mathfrak{h}, s)$ *where* $\mathfrak{h}$ *is a compact simple Lie algebra and* s *is any involutive automorphism of* $\mathfrak{h}$.

II $(\mathfrak{h}, s)$ *where the compact algebra* $\mathfrak{h}$ *is the direct sum* $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ *of simple ideals which are interchanged by an involutive automorphism* s *of* $\mathfrak{h}$.

III $(\mathfrak{h}, s)$ *where* $\mathfrak{h}$ *is a simple, noncompact Lie algebra over* $\mathbb{R}$, *the complexification* $\mathfrak{h}^{\mathbb{C}}$ *is a simple Lie algebra over* $\mathbb{C}$ *and* s *is an involutive automorphism of* $\mathfrak{h}$ *such that the fixed points form a compactly embedded subalgebra.*

IV $(\mathfrak{h}, s)$ *where* $\mathfrak{h} = \mathfrak{g}^{\mathbb{R}}$ *for some simple Lie algebra* $\mathfrak{g}$ *over* $\mathbb{C}$ *and* s *is the conjugation of* $\mathfrak{h}$ *with respect to a maximal compactly embedded subalgebra.*

Furthermore,

$$(\mathfrak{h}, s) \text{ is of type III} \iff (\mathfrak{h}^*, s^*) \text{ is of type I}$$

$$\text{and} \quad (\mathfrak{h}, s) \text{ is of type IV} \iff (\mathfrak{h}^*, s^*) \text{ is of type II},$$

where $(\mathfrak{h}^*, s^*)$ *denotes the dual of* $(\mathfrak{h}, s)$.

The symmetric space H/K is said to be of type i, where i = I, II, III or IV, if its associated orthogonal symmetric Lie algebra is of type i. The symmetric spaces of type I and II are compact whereas those of type III and IV are noncompact. Then the Riemannian globally symmetric spaces of type IV are the spaces H/U where H is a connected Lie group whose Lie algebra is $\mathfrak{h}^{\mathbb{R}}$ where $\mathfrak{h}$ is a simple Lie algebra over $\mathbb{C}$, and U is a maximal compact subgroup of H. The metric on H/U is H-invariant and is uniquely determined (up to a factor) by this condition. The symmetric spaces of type III are given in table 3.1, which by duality classifies all irreducible symmetric spaces.

	Noncompact	Root system	Dimension
AI	$SL(n,\mathbb{R})/SO(n)$	A_{n-1}	$\frac{1}{2}(n-1)(n+2)$
AII	$SU^*(2n)/Sp(n)$	A_{n-1}	$(n-1)(2n+1)$
AIII	$SU(p,q)/S(U_p \times U_q)$	BC_q, C_q	$2pq$
BDI	$SO_0(p,q)/SO(p)\times SO(q)$	B_q, D_q	pq
DIII	$SO^*(2n)/U(n)$	$BC_q, D_q, q=[\frac{1}{2}n]$	$n(n-1)$
CI	$Sp(n,\mathbb{R})/U(n)$	C_n	$n(n+1)$
CII	$Sp(p,q)/Sp(p)\times Sp(q)$	BC_q, C_q	$4pq$
EI	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	E_6	42
EII	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(6)+\mathfrak{su}(2))$	F_4	40
EIII	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10)+\mathbb{R})$	B_2	32
EIV	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$	A_2	26
EV	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	E_7	70
EVI	$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12)+\mathfrak{su}(2))$	F_4	64
EVII	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6+\mathbb{R})$	C_3	54
EVIII	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	E_8	128
EIX	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7+\mathfrak{su}(2))$	F_4	112
FI	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3)+\mathfrak{su}(2))$	F_4	28
FII	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$	A_1	16
G	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2)+\mathfrak{su}(2))$	G_2	8

Table 3:1 The symmetric spaces of type III

The symmetric space denoted BDI by Helgason [1] is $SO_0(p,q)/SO(p) \times SO(q)$ where $q \leq p$. Such spaces include the case BI where $p+q$ is odd and DI where $p+q$ is even. The symmetric spaces AIII, DI, DIII and CII each can have different root systems depending on the values of p and q. For instance, the root system of AIII is BC_q if $q < p$ and C_q if $p = q$. For more details of the root system BC_q see Loos [1] and appendix B.

2. Polar representations

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\pi : G \rightarrow \mathrm{G}(V)$ be a representation of G on a real vector space V preserving an inner product $\langle\, , \rangle$. For each $v \in V$, the space $\mathfrak{g}{\cdot}v$ is the tangent space to the G-orbit through v. Thus we can define a linear **cross-section** $\mathfrak{a}_v$ of the G-orbits by

$$\mathfrak{a}_v = \{u \in V : \langle u, g \cdot v\rangle = 0\} = \{\ \mathfrak{g} \cdot v\}^{\perp}.$$

In other words, $\mathfrak{a}_v$ is normal to the G-orbit through v. It can be shown (see Dadok [1]) that $\mathfrak{a}_v$ meets every G-orbit.

If $\mathfrak{g}{\cdot}v$ is a principal orbit then $v \in V$ is called **regular**. The representation π is called **polar** if for some regular v and for any $u \in \mathfrak{a}_v$, $\langle\ \mathfrak{g} \cdot u,\ \mathfrak{a}_v\rangle = 0$. Thus the normal to a principal orbit of the action of a polar representation cuts every orbit orthogonally. Such a cross-section or normal is called a **Cartan subspace**. Obvious examples of polar representations are given by groups which act transitively on spheres, for instance, the orbits of the action of $\mathrm{O}(n)$ on E^n are $(n-1)$-spheres or the origin 0. Any normal to any and all of these orbits is a line through 0: see figure 3.1 for the case $n = 2$. The adjoint actions and the representations associated with symmetric spaces studied by Kostant and Rallis [1] are also polar. An example of a representation which is not polar is given by the action of $SU(1) = SO(2)$ acting on the 3-sphere S^3 lying E^4 which is identified with $\mathbb{C}^2$, the principal orbits of this action are great circles, therefore any normal is not orthogonal to every principal orbit. A full account of polar representations may be found in Dadok [1] and Dadok and Kac [1].

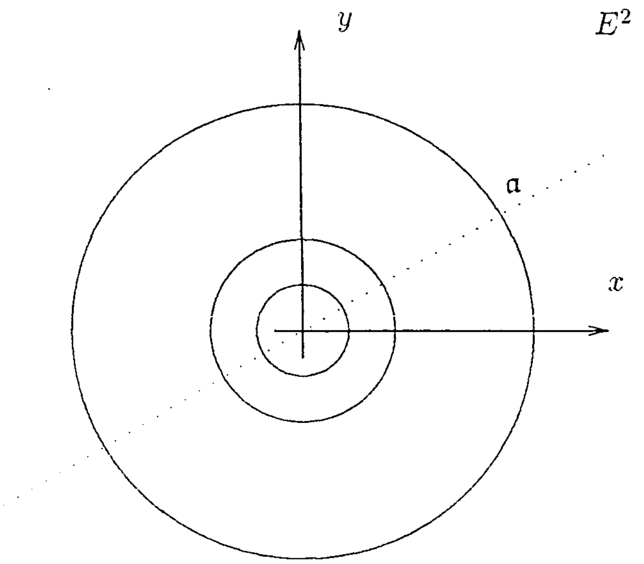


Figure 3.1 Principal orbits of the action of O(2) on E^2.

The irreducible polar representations were classified by Dadok [1] by associating each polar represenation to a symmetric space as follows.

Let G be a connected compact Lie group. Then a polar representation $\pi : G \to SO(V)$ is called a **symmetric space representation** if there exists:

(i) a real semisimple Lie algebra $\mathfrak{h}$ with Cartan decomposition $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$;

(ii) a Lie algebra isomorphism $A : \mathfrak{g} \to \mathfrak{k}$; and

(iii) a real vector space isomorphism $L : V \to \mathfrak{p}$

such that

$$L \cdot \pi(X)(y) = [A(X), y] \text{ for all } X \in \mathfrak{g}, y \in \mathfrak{p}.$$

By definition these representations are polar, and a Cartan subspace of such a representation is a maximal abelian subalgebra of $\mathfrak{p}$. Dadok [1] showed that almost all polar represenations give rise to symmetric space representations in the following proposition.

PROPOSITION 3:2.1

Let $\pi : G \rightarrow SO(V)$ be a polar representation of a connected Lie group G. Then there exists a connected Lie group $\tilde{G}$ with symmetric space representation $\tilde{\pi} : \tilde{G} \rightarrow SO(V)$ such that the G- and $\tilde{G}$-orbits in V coincide.

The symmetric space H/K associated to $\tilde{G}$ is of noncompact form and the representation $\tilde{\pi}$ may be given in terms of highest weight (see Dadok [1] and Humphreys [1]). The classification of polar representations is then given in terms of the noncompact symmetric spaces and a few exceptional cases, being the adjoint representations and the action of $Spin(7) \times SU(2)$ on $\mathbb{R}^{24}$. For full details see Dadok [1]. It is noted that all irreducible noncompact symmetric spaces with the exception of EII are associated with polar representations.

CHAPTER 4

CLASSIFICATION OF REGULAR SOLIDS

In this chapter, we describe the classification of regular solids, to appear in Madden and Robertson [1]. The regular solids arise from the study of both polar representations and Kostant's construction, the classification being given in terms of the symmetric space and polytope associated to each regular solid. We also give various examples to illustrate this connection in section 2.

1. Classification of regular solids

Let B be a regular n-solid in E^n, $n \geq 2$, with centroid O. Let $G = GB$ be the symmetry group of B and $\mathfrak{g}$ the Lie algebra of G. Let $\pi : G \to \mathrm{O}(n)$ be the representation given by inclusion. Recall that a representation is said to be **irreducible** if it has no proper invariant subspace. We show that π is irreducible by the following result.

PROPOSITION 1.1

Let B be a perfect n-solid in E^n. Then the action of GB on E^n is irreducible.

Proof

Let B be a perfect n-solid in E^n. Let B have symmetry group $G = GB$ and centroid O. Suppose G is reducible. Then there exist at least two orthogonal G-invariant subspaces of E^n of positive dimension. Let E^n be completely decomposed into non-trivial orthogonal G-invariant subspaces $A_1, \ldots, A_r$ in the sense that $E^n = A_1 \times \cdots \times A_r$ and A_j does not contain any non-trivial G-invariant subspaces, $1 \leq j \leq r$. Let $(e_1, \ldots, e_n)$ be an orthogonal basis for E^n compatible with the above decomposition, so that A_1 and A_2 are generated by $e_1, \ldots, e_{t_1}$ and $e_{t_1+1}, \ldots, e_{t_2}$ respectively, and A_j is generated by $e_{t_{j-1}+1}, \ldots, e_{t_j}$, $2 \leq j \leq r$, where $1 \leq t_1 \leq \cdots \leq t_r = n$. We write $x \in E^n$ as $x = (\tilde{x}_1, \ldots, \tilde{x}_r) \in A_1 \times \cdots \times A_r$.

For $j = 1, \ldots, r$, let L_j be a line through O in A_j such that $D_j = L_j \cap B$ has maximal length d_j. We may suppose that $d_1 \geq d_2 \geq \cdots \geq d_r$. Let $\delta : E^n \to E^n$ be the linear map defined as follows. For $j = 1, \ldots, r$, let $s_j = 1 + \epsilon_j$, such that $\epsilon_j > 0$ and $s_1 > s_2 > \cdots > s_r$. Then the action of δ on each A_j is a dilation by s_j and hence fixes each A_j setwise. Thus, $\delta(x_1, \ldots, x_n) = (s_1\tilde{x}_1, \ldots, s_r\tilde{x}_r)$.

Let $B_\epsilon = \delta(B) = \{\delta(x) : x \in B\}$, where $\epsilon = \{\epsilon_1, \ldots, \epsilon_r\} \in E^r$. Then δ induces an embedding of GB in GB_ϵ. But for $||\epsilon||$ sufficiently small, GB_ϵ is a subgroup of GB. Hence $GB = GB_\epsilon$. Also for $||\epsilon||$ sufficiently small, δ induces a face equivalence from B to B_ϵ (also see Pinto [1]). Hence, since B is perfect, there exists a similarity $f : E^n \to E^n$ such that $f(B) = B_\epsilon$. Let m be the scale of f so for all $x, y \in E^n$, $d\big(f(x), f(y)\big) = md(x, y)$.

Since B_ϵ has centroid O, f is the product of an orthogonal transformation and a dilation. If $x \in A_i$ for some $i = 1, \ldots, r$ then $f(x) \in A_j$ for some

$j = 1, \ldots, r$. Thus either $f(A_i) = A_i$ for all $i = 1, \ldots, r$, or there exists a linear subspace C_i of some A_i, $1 \leq dim(C_i) \leq dim(A_j)$, such that $f(C_i) \subset A_j$ and $1 \leq i \neq j \leq r$.

First suppose that $f(A_i) = A_i$ for all $i = 1, \ldots, r$. Let $x_j, y_j \in A_j$ and $x_k, y_k \in A_k$, for some $j < k$. Then $md(x_j, y_j) = d(f(x_j), f(y_j)) = d(\delta(x_j), \delta(y_j)) = s_j d(x_j, y_j)$. Hence $m = s_j$. However we also have $md(x_k, y_k) = d(f(x_k), f(y_k)) = d(\delta(x_k), \delta(y_k)) = s_k d(x_k, y_k)$. Hence $m = s_k$. Therefore $m = s_j > s_k = m$ which is a contradiction.

Now suppose that $f(C_i) \subset A_j$ for some maximal nontrivial subspace C_i of A_i. We may suppose $i < j$. Let $C_i^{\perp}$ be the orthogonal complement of C_i in A_i. Suppose that $0 < dim(C_i) < dim(A_i)$, and let $u \in C_i$ and $v \in C_i^{\perp}$. If $w = \lambda u + \mu v$. $\lambda \neq 0$ and $\mu \neq 0$. then $f(w) \notin A_j$, $j = 1 \ldots, r$. This is a contradiction, hence $dim(C_i) = dim(A_i)$. It then follows that $f(A_i) = A_j$ and $f(A_j) = A_k$ for some $k \neq j$. We may suppose that $k < j$. If $d_j = d_i$ or $d_j = d_k$, then $|f(D_j)| = d_i = d_j$ or $|f(D_j)| = d_k = d_j$, respectively, where $|T|$ denotes the length of the line segment T. Hence $m = 1$. This is a contradiction since $|\delta(D_i)| > d_i$. However if $d_i < d_j$ or $d_k < d_j$ then $|f(D_i)| = md_i > |f(D_j)| = md_j$ or $|f(D_k)| = md_k > |f(D_j)| = md_j$ which again is a contradiction.

COROLLARY 1.2

π is irreducible.

Proof

Let B be a regular solid. then B is a perfect solid (see section 1:1.7). The corollary then follows from proposition 1.1

We also note that theorem 1:7.5 follows from proposition 1.1.

PROPOSITION 1.3

π is polar.

Proof

Recall (Bolton [1]), that a **transnormal system** in a complete connected Riemannian manifold M is a partition of M into foils (nonempty connected submanifolds) such that any geodesic of M cuts the foils orthogonally at none or all of its points. Then by Bolton [1], the orbits of π form a transnormal system in E^n and hence π is polar.

Alternatively, let x lie on a principal orbit of this action of π and let the tangent plane of this action at x be denoted by $\mathfrak{g}\cdot x$. By similarity, we can assume x lies on the boundary of B, and hence in a proper face F of B. Since B is regular, each flag of F is in a maximal flag of B. By Dadok [1], $\mathfrak{a}_x = \{u \in E^n : \langle u, \mathfrak{g}\cdot x\rangle = 0\}$ is a Cartan subspace of E^n. Then by the transitivity of G on ΦB, for any v on a principal orbit, $\mathfrak{a}_v = g \cdot \mathfrak{a}_x$ for some $g \in G$. Hence π is polar by Dadok [1].

By Dadok [1], we have associated to the given n-solid B a noncompact symmetric space H/K, where H and K have Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$, respectively, such that:

$$\mathfrak{h} = \mathfrak{k} + \mathfrak{p};$$

$$\mathfrak{k} \simeq \mathfrak{g};$$

$$\text{and} \quad \mathfrak{p} \simeq E^n.$$

Hence given an n-solid B and its symmetry group, H/K and therefore H are determined from the classification of noncompact symmetric spaces (see table 3.1).

Let W be the Weyl group of H/K and $\mathfrak{a}$ be a Cartan subspace of π, that is, a maximal abelian subspace of $\mathfrak{p}$. Then the intersection of a G-orbit with $\mathfrak{a}$ is a single W-orbit (see Dadok [1]) and W is the symmetry group of a regular polytope P such that $\Pi(B) = P$ where Π is the projection in Kostant's construction (see chapter 2 and Farran and Robertson [1]).

The root system and hence Dynkin diagram (or Coxeter graph) determined by the Weyl group of a symmetric space is one of the following:

$$\begin{array}{rll} A_l & l \geq 1, & E_6, \\ B_l & l \geq 2, & E_7, \\ BC_l & l \geq 3, & E_8, \\ C_l & l \geq 3, & F_4 \\ D_l & l \geq 4, & G_2. \end{array}$$

Of these root systems only A_l, B_l, C_l, D_l, F_4 and G_2 are derived from a Weyl group corresponding to the symmetry group a regular polytope P. We call such root systems **regular**. The polytope P is determined by taking the convex hull of a W-orbit of a point x in the fundamental region D of W corresponding to an end node of the Coxeter graph, that is, by Wythoff's construction on an end node of the Coxeter graph. Any other polytopes arising from Wythoff's construction, that is, those from nodes which are not end nodes, are not regular (see section 1.8). Polytopes also arise by Wythoff's construction on the Coxeter graphs which are not associated to any regular polytope, namely, the root systems E_6, E_7 and E_8. We shall call these polytopes **Gosset polytopes** or **Gossetopes** since they were first studied by Gosset [1].

Therefore we are only interested in symmetric spaces whose root system is regular. As usual, using the Coxeter graph of the root system, the Schläfli symbol of P can be read directly. The root system D_l is a special case as it has three end nodes, from Coxeter [1], we make the identifications in figure 4.1 when using Wythoff's construction on any of the nodes in the long branch of

D_l. The polytope arising from Wythoff's construction using either node in the short branch is the alternation $h\square_l$ of $\square_l$, that is, the polytope derived from $\square_l$ by taking the convex hull of its alternate vertices. In general,

$$h\square_k = \left\{ \begin{matrix} 3 \\ 3, 3, \ldots, 3 \end{matrix} \right\},$$

for instance, $h\square_3 = \triangle_3$ (note that $D_3 = A_3$ by section 2.1) and $h\square_4 = \diamond_4$ are given in figure 4.2.

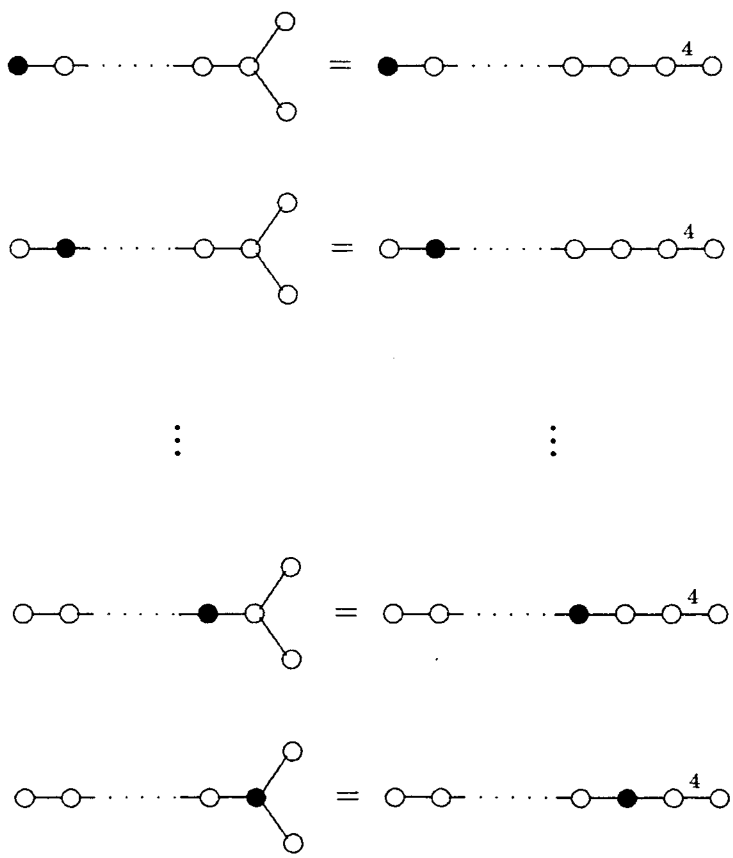


Figure 4.1 Equivalent polytopes from D_l

A geometric interpretation of this process of deriving P from H/K may be given as follows. The Cartan subspace $\mathfrak{a}$ of $E^n \simeq \mathfrak{p}$ is a linear subspace of E^n and hence intersects B in a nonempty subset. In fact $\mathfrak{a}$ is generated by the centroids of faces of a maximal flag of B (see proposition 1.2). Hence, by theorem 2:2.3 and Kostant's construction, the set $P = B \cap \mathfrak{a}$ is a regular polytope. Then $P = \rho(B)$, where ρ is the projection of chapter 2.

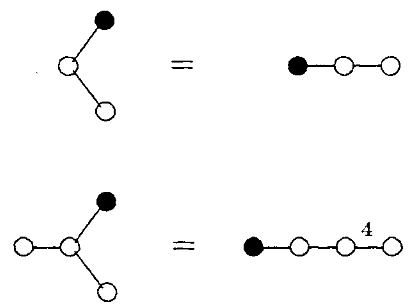


Figure 4.2 Alternation polytopes of $\square_3$ and $\square_4$.

The dimension of the regular polytope associated to a regular solid and the dimension of the regular solid are respectively given by the rank and dimension of the associated symmetric space. The symmetric spaces from which regular polytopes are derived by Wythoff's construction are given in table 4.1 and 4.2, along with the associated regular polytopes and the above mentioned dimensions. Thus we obtain a classification of regular solids such that

(1) If $P = \{p\}$ for $p \neq 3, 4, 6$, or $P = \{3,5\}$, $\{5,3\}$, $\{3,3,5\}$, or $\{5,3,3\}$, then $\rho^{-1}(P)$ consists of only P itself.

(2) All regular solids that are not polytopes are specified in tables 4.1 and 4.2 in terms of their dimension, and the associated symmetric spaces and polytopes. (Of course, the regular polytopes have symmetry groups which are 0-dimensional Lie subgroups of $O(n)$.)

(3) In addition to the above, there are regular solids associated with the adjoint representation of simple Lie groups (with the exception of E_6, E_7 and E_8) whose corresponding symmetric space is of type IV.

	H/K	$dim(B)$	$dim(\rho(B))$	$\rho(B)$
AI	$SL(n,\mathbb{R})/SO(n)$	$\frac{1}{2}(n-1)(n+2)$	$(n-1)$	$\triangle_{n-1}$
AII	$SU^*(2n)/Sp(n)$	$(n-1)(2n+1)$	$(n-1)$	$\triangle_{n-1}$
$AIII$	$SU(p,q)/S(U_p \times U_q)$ $q \leq p$	$2pq$	q	$\square_q, \diamond_q$
BI	$SO_0(p,q)/SO(p) \times SO(q)$ $p+q$ odd, $q < p$	pq	q	$\square_q, \diamond_q$
DI	$SO_0(p,q)/SO(p) \times SO(q)$ $p+q$ even	pq	q $q < p$	$\square_q, \diamond_q$
			$p = q$	$\diamond_q$
$DIII$	$SO^*(2n)/U(n)$	$n(n-1)$	$[\frac{1}{2}n] = q$	$\square_q, \diamond_q$
CI	$Sp(n,\mathbb{R})/U(n)$	$n(n+1)$	$n = q$	$\square_q, \diamond_q$
CII	$Sp(p,q)/Sp(p) \times Sp(q)$ $q \leq p$	$4pq$	q	$\square_q, \diamond_q$

Table 4.1 Regular solids derived from the classical noncompact symmetric spaces and their associated regular polytopes.

H/K		$dim(B)$	$dim(\rho(B))$	$\rho(B)$
$EIII$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + \mathbb{R})$	32	2	$\Box_2$
EIV	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$	26	2	$\triangle_2$
EVI	$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12)+\mathfrak{su}(2))$	64	4	24-cell
$EVII$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 + \mathbb{R})$	54	3	$\Box_3, \Diamond_3$
EIX	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7+\mathfrak{su}(2))$	112	4	24-cell
FI	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3)+\mathfrak{su}(2))$	28	4	24-cell
FII	$(\mathfrak{f}_{4(-20}, \mathfrak{so}(9))$	16	1	$\triangle_1$
G	$(\mathfrak{g}_{2(2)},\mathfrak{su}(2)+\mathfrak{su}(2))$	8	2	Hexagon

Table 4.2 Regular solids derived from the exceptional noncompact symmetric spaces and their associated regular polytopes.

2. Some examples

With the classification of regular solids complete, it is interesting to locate some familiar regular solids in tables 4.1 and 4.2.

A well-known family of regular solids is the family of balls D^n, $D^n = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 \leq 1\}$, where the case $n = 1$ is the line segment I which is the only solid (and hence polytope) up to similarity in E^1, and $n = 2$ is the disk. Clearly $\sigma_{D^n} = \{0\}$ for all n, therefore $\rho(D^n) = I$ for all n. Let $B = D^n$, then $GB = O(n)$, the connected component of the identity of GB is $G = SO(n)$, so we take $\mathfrak{g} = \mathfrak{so}(n)$. The only symmetric space H/K such that $\text{rank}(H/K) = 1$ and $\mathfrak{k} = \mathfrak{so}(n)$ is the space BDI, that is, $H/K = SO_0(p,q)/SO(p) \times SO(q)$ where $p = n$ and $q = 1$. We check that $dim(H/K) = dim(B) = pq = n$, and the action of $SO(n) \times SO(1)$ on I is indeed D^n.

Another family of regular solids are the Veronese solids, of which the 5-dimensional solid is best-known (see Robertson [2]). Let $B = \mathcal{V}$ be this 5-solid, given by the following. Let $\mathcal{M}$ denote the real projective plane and $p \in \mathcal{M}$ have homogeneous coordinates x, y, z with $x^2 + y^2 + z^2 = 1$. Then $\mathcal{V}$ is given by the image of a smooth embedding $v : \mathcal{M} \to E^6$, where

$$v(p) = (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy).$$

From Farran and Robertson [1], we see that the symmetry group G of $\mathcal{V}$ is isomorphic to $SO(3)$ and the polytope associated to $\mathcal{V}$ by Kostant's construction is an equilateral triangle T. The set of faces of $\mathcal{V}$ consist of vertices and 2-disks only. From tables 4.1 and 4.2, we see that there is only one symmetric space such that $\mathfrak{k} = \mathfrak{so}(3)$ with rank 2, namely $H/K = SL(3,\mathbb{R})/SO(3)$ (which is of type AI). The solid given by action of the isotropy subgroup of G at the midpoint of an edge of T is a 2-disk, which agrees with Farran and Robertson [1].

In both these cases the polytope derived from B by this classification agrees with that found geometrically in Farran and Robertson [1]

We can also check the classification of perfect non-polytope 2 and 3-solids by looking for all symmetric spaces of dimension 2 and 3, respectively, in tables 4.1 and 4.2. Using the identifications $AI(n = 1) = AIII(p = q = 1) = BDI(p = 2, q = 1) = CI(n = 1)$ from Helgason [1] (see also appendix B) the only such symmetric spaces are the spaces $SO_0(2,1)/SO(2)\times SO(1)$ and $SO_0(3,1)/S0(3)\times SO(1)$. Thus we have agreement with Farran and Robertson [1].

PART II

PERFECT SOLIDS

Let $\mathcal{S}_P$ and $\mathcal{P}_P$ denote the sets of perfect solids and perfect polytopes respectively. It has been noted that $\mathcal{S}_R$ is a proper subset of $\mathcal{S}_P$ and it is well-known that $\mathcal{P}_R$ is a proper subset of $\mathcal{P}_P$. The projection $\rho : \mathcal{S}_R \to \mathcal{P}_R$ can be extended to $\rho : \mathcal{S}_P \to \mathcal{P}_P$ by restricting the projection Π given by Kostant's construction to the set of perfect solids. The perfect solids have yet to be classified, one of the obstacles being that there is no theorem for perfect solids analogous to theorem 2:2.3. Moreover the perfect polytopes have not been classified. However the results given in part I provide many examples of non-regular perfect solids whose associated polytope is a truncation of a regular polytope in the following sense.

THEOREM II:1

Let P be a polytope given by Wythofff's construction on some node of the Dynkin diagram of an irreducible root system. Then P is perfect.

Proof

This result can be read into the analysis of Robinson [1], who of course was writing about quite different ideas. The key fact is that for any vertex transitive polytope P whose symmetry group is the Weyl group of such a root system, the fixed point set of any vertex has dimension 1 if and only if the vertex is a vertex of the fundamental region of P.

The polytopes of theorem II:1 are the wythoffian polytopes and the Gosset polytopes. It is thought that these along with their polars are the only prime perfect polytopes. In fact Pinto [1] generalised Rostami's conjecture (conjecture 1:8.1) for n-dimensions to the following.

CONJECTURE II:2

Let P be a perfect polytope. Then P or P^* is a $\square$-power of some prime perfect polytope Q, where Q is given by theorem II:1.

In part II, we are concerned mainly with perfect polytopes but in doing so we work towards a classification of perfect solids. We make some progress towards a general classification of perfect polytopes in chapters 5 and 6. In chapter 5 we consider transitivity on i-faces for each dimension $i \leq n$ for certain perfect n-polytopes. Deformations of polytopes to nearby polytopes are introduced in chapter 6. This allows us to determine the dimensions of fixed point sets of vertices of perfect polytopes. We also consider G-stratifications of E^n, where G is the symmetry group of a regular polytope.

In chapter 7 we consider conjecture II:2. In dimension $n \geq 6$, the Gosset polytopes need to be considered. The symmetry groups of these polytopes are not associated to any regular polytope. Therefore we concentrate on the case $n = 4$. We make several conjectures on the orbit vector of perfect poytopes that agree with conjecture II:2.

In chapter 8 we consider the angle formed by adjacent i-faces of a polytope, $i = 1, 2$. By considering perfect 0- and 3-transitive perfect 4-polytopes we prove some of the conjectures in chapter 7. Rostami's conjecture and a classification of perfect 4-polytopes then follow. This leads to a classification of perfect 4-solids.

CHAPTER 5

TRANSITIVITY

The regular polytopes form a more restricted family than that of the perfect polytopes. As we shall see, the condition that a polytope P is regular imposes restrictions on the transitivity properties of the proper faces of P. In this chapter we are concerned with the transitivity properties of polytopes and the number of face orbits of P in each dimension where a proper face exists. A more general study of transitivity in solids may be found in Farran and Robertson [1], from which some ideas used here have been derived.

1. Complete Transitivity

We start with two definitions from Farran and Robertson [1]. For any $0 \leq i \leq n-1$, an n-polytope P is said to be i**-transitive** if GP acts transitively on F_iP. If P is i-transitive for all $0 \leq i \leq n-1$, then we call P **completely transitive**. It is noted in Farran and Robertson [1] that a polytope P is completely transitive if and only if P is regular. Clearly every regular polytope is completely transitive as any i-face of P can be embedded in a maximal flag, for any $0 \leq i \leq n-1$, the transitivity of the maximal flags then ensures the transitivity of the i-faces. A proof that a completely transitive polytope is regular will be given here. This proof makes use of an alternative definition of regularity, which is provided by the following lemma.

LEMMA 5:1.1

Let P_m be an m-polytope in E^m such that P_m has congruent regular facets and all the vertices of P_m lie on a sphere. Then P_m is regular.

Proof

For $m = 2$. P_m is an equilateral polygon with common length λ. Suppose that the vertices of P_m lie on a circle of radius r. Let v_1, v_2, v_3 be any three successive vertices of P_2 so that v_1v_2, v_2v_3 are edges of P_2. Then since v_1, v_2, v_3 all lie on some circle. the length of the chord v_1v_3 is determined by r and λ. Hence the internal angle of P_2 at v_2 is independent of the choice of v_2. Thus P_2 is equiangular as well as equilateral and is therefore regular.

For $m = 3$. P_3 is a polyhedron with congruent regular polygons as faces. The regular-faced polyhedra have been classified by Johnson [1]. Of the polyhedra with congruent regular faces. only the Platonic solids satisfy the condition that the vertex set lies on a 2-sphere. Therefore P_3 is regular.

We now argue by induction. Suppose any k-polytope P_k satisfying the above conditions is regular. Let P_{k+1} be any $(k+1)$-polytope with congruent regular facets such that all the vertices of P_{k+1} lie on some sphere. Then all the edges of P_{k+1} are equal. Thus the vertex figure Q of P_{k+1} at some vertex v is given by the convex hull of the midpoints of all edges of P_{k+1} emanating from v.

Now Q is a k-polytope. Since the facets of Q are the vertex figures of the facets of P_{k+1}, the facets of Q are congruent and regular. We now show that the vertices of Q lie on a sphere. Consider the 1-dimensional space τ containing v such that τ and $\mathit{aff}(Q)$ are orthogonal. Suppose x denotes the point of intersection of τ and $\mathit{aff}(Q)$ and let $|ab| = d(a,b)$. Since $|vv_i| = |vv_j|$ for any vertices v_i, v_j of Q, the triangles given by the vertices v, v_i, x and v, v_j, x are congruent right-angled triangles. Hence $|xv_i| = |xv_j|$ for all v_i, v_j, that is, all vertices of Q lie on a sphere with centre x. Therefore Q is regular.

Then P_{k+1} has regular facets and regular vertex figures and hence P_{k+1} is regular.

Therefore by induction all such P_m are regular. Thus the above definition of regularity for polytopes agrees with that found in Coxeter [1] and hence with section 1.6.

We note that congruent regular facets is not a sufficient condition for regularity. For instance in dimension 2, a rhombus is not regular. In dimension 3, there are five polyhedra, called deltapolyhedra, each with regular triangular faces which are not regular (see Williams [1] or Cundy and Rollet [1]). Two such polyhedra are the double cones on a triangle and a pentagon.

PROPOSITION 5:1.2

Any completely transitive n-polytope is regular.

Proof

Let P be a completely transitive n-polytope in E^n. Let f be a 2-face of P. Then f is an equilateral k-polygon, for some $k \geq 3$. The convex hull $conv(f \cup O)$ of f and O is a cone with f as base and with k triangular faces. Each triangle is given by the points O, v_i, v_{i+1}, where v_i, v_{i+1} are the endpoints of an edge of f. Since P is 0-transitive, $|Ov_i| = |Ov_{i+1}|$. Hence all the triangles are congruent and isosceles. Thus f is a regular polygon. Since P is 2-transitive all 2-faces of P are congruent.

Suppose that the m-faces of P are regular for some m , $3 \leq m \leq n-2$. Let f_{m+1} be an $(m+1)$-face of P. Then f_{m+1} is an $(m+1)$-polytope with regular m-faces. Since P is m-transitive, the m-faces are congruent. Since P is 0-transitive the vertices of f_{m+1} lie on a sphere. Hence by lemma 1.2, f_{m+1} is regular. Since P is $(m+1)$-transitive, all $(m+1)$-faces are congruent.

Therefore the facets of P are regular and congruent. Since P is 0-transitive, all vertices of P lie on a sphere, therefore P is regular. This completes the proof, hence P is regular if and only if P is completely transitive.

It is noted in Farran and Robertson [1] that this property of a polytope being completely transitive if and only if it is regular does not carry over into the more general setting of the solids. As an example, consider the 3-solid $B = D^1 \square D^2$, then B is completely transitive and $\sigma_B = \{0, 1, 2\}$. However B is not regular as GB is not transitive on the maximal flags of B.

2. The orbit vector

Let P be an n-polytope. Then we say the **orbit vector** of P is the n-vector $\theta(P) = (\theta_0, \ldots, \theta_{n-1})$, where θ_i is the number of orbits of i-faces in F_iP, for each $i = 0, \ldots, n-1$. under the action of GP. Then by the above a polytope P is regular if and only if $\theta(P) = (1, \ldots, 1)$. It is possible to express the orbit vector of any $\diamond$-prime polytope in terms of the orbit vectors of the prime polytopes in its $\square$-decomposition.

Suppose that $P = A \square B$, where A and B are $\diamond$-prime $\square$-coprime polytopes. Suppose that $dim(A) = m \geq 1$ and $dim(B) = n \geq 1$ and the orbit vectors of A

and B are given by $\theta(A)$ and $\theta(B)$, respectively, where

$$\theta(A) = (a_0, \ldots, a_{m-1})$$

and

$$\theta(B) = (b_0, \ldots, b_{n-1}).$$

It is convenient to put $a_m = b_n = 1$, and to refer to $(a_0, \ldots, a_m)$ and $(b_0, \ldots, b_n)$ as the **extended orbit vectors** of A and B respectively.

The product $A \square B$ is an $(m+n)$-polytope with symmetry group $GA \times GB$ if $A \neq B$. A k-face of $A \square B$ is of the form $f_i^A \square f_j^B$ where $f_i^A \in F_i A$, $f_j^B \in F_j B$ and $i + j = k$. Therefore $\theta(A \square B) = (\theta_0, \ldots, \theta_{m+n-1})$ where $\theta_k = \sum_{i+j=k} a_i b_j$ for $0 \leq i \leq m, 0 \leq j \leq n$.

Now if $P = A \square A$ where A is $\square$-prime, then $G = G(A \square A) = Z_2 \wr GA$. Therefore the $(i+j)$-faces of $A \square A$ given by $(f_i \square f_j)$ and $(f_j \square f_i)$, for some $f_i \in F_i A$ and $f_j \in F_j A$, lie in the same orbit. First suppose that $i + j = k$ is odd, then there are $\frac{k+1}{2}$ distinct pairs (i, j), for $i, j \geq 0$, such that $i + j = j + i = k$. Therefore the number of k-face orbits is $\sum_{i+j=k} \frac{k+1}{2} a_i a_j$ for $0 \leq i, j \leq m$ if k is odd. If $i + j = k$ is even, then there are $k/2$ distinct pairs (i, j) for $i, j \geq 0$ such that $i + j = j + i = k$ and $i \neq j$. If $i = j$, then the number of k-face orbits is a_i. Therefore the number of k-face orbits is $a_{k/2} + \sum_{\substack{i+j=k \\ i \neq j}} \frac{k}{2} a_i a_j$ for $0 \leq i, j \leq m$ if k is even. Therefore $\theta(A \square A) = (\theta_0, \ldots, \theta_{2m-1})$ where

$$\theta_k = \begin{cases} \sum_{i+j=k} \frac{k+1}{2} a_i a_j, & \text{if } k \text{ odd;} \\ a_{k/2} + \sum_{\substack{i+j=k \\ i \neq j}} \frac{k}{2} a_i a_j, & \text{if } k \text{ even.} \end{cases}$$

We illustrate these facts with two simple examples. First recall that a **semiregular polygon** is an equiangular polygon whose edges are of alternatively equal length. Suppose that I is an interval. Let $\{6\}$ and S_6 be hexagons, where $\{6\}$ is regular and S_6 is strictly semiregular. Then the extended orbit vectors of I, $\{6\}$ and S_6 are $(1,1)$, $(1,1,1)$ and $(1,2,1)$, respectively. Then $\theta(\{6\}\square I) = (1,2,2)$ and $\theta(S_6\square I) = (1,3,3)$, see figure 5.1. There are 3 face orbits in the latter product, representatives of which are given by a hexagon S_6 and two rectangles.

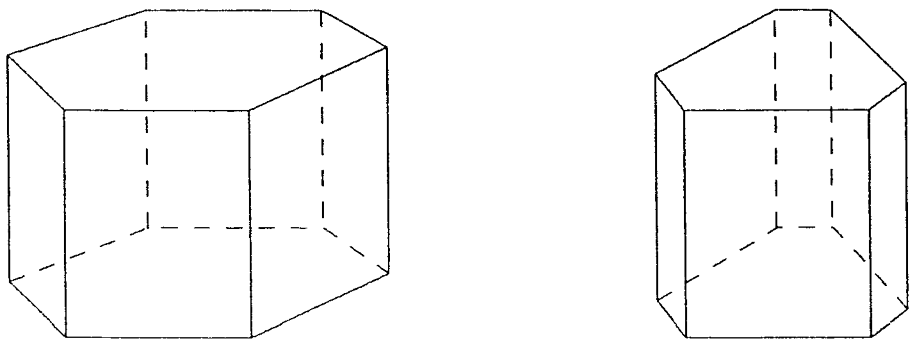


Figure 5.1

We study the orbit vectors of the known perfect polytopes, namely those perfect polytopes arising from Wythoff's construction and their polars. First we consider the wythotopes. Let R be a regular n-polytope and, for some $i = 1,\ldots,n-2$, let W_i^R be the wythotope given by taking the convex hull of the centroids of all i-faces of R. In other words, if R is the regular n-polytope $\{a_1,\ldots,a_{n-1}\}$ then W_i^R is the n-wythotope $\left\{\begin{matrix} a_i,\ldots,a_1 \\ a_{i+1},\ \ldots\ ,a_{n-1}\end{matrix}\right\}$ given by figure 1.16.

Since R is regular, (i) the polar R^* is regular and (ii) the fixed point sets of centroids of i-faces of R and the fixed point sets of centroids of $(n-1-i)$-faces of R^* coincide, for each $i = 0,\ldots,n-1$. Therefore we may assume that $i \leq n/2$, since otherwise $W_i^R \sim W_j^{R^*}$ where $j = n-1-i \leq n/2$.

PROPOSITION 5:2.1

Let R be a regular n-polytope and $i \leq \frac{n}{2}$. Then the orbit vector of W_i^R is

$$(1, 1, 2, 3, \ldots, i, i+1, \ldots, i+1, i, i-1, \ldots, 4, 3, 2).$$

Proof

Let $R = \{a_1, \ldots, a_{n-1}\}$, so that $W_i^R = \left\{\begin{array}{l} a_i, \ldots, a_1 \\ a_{i+1}, \ldots, a_{n-1} \end{array}\right\}$. Then the facets of W_i^R are of type $F_1 = \left\{\begin{array}{l} a_i, \ldots, a_1 \\ a_{i+1}, \ldots, a_{n-2} \end{array}\right\}$ and $F_2 = \left\{\begin{array}{l} a_i, \ldots, a_2 \\ a_{i+1}, \ldots, a_{n-1} \end{array}\right\}$. A facet of type F_1 is given by the convex hull of the centroids of all i-faces of some facet $\{a_1, \ldots, a_{n-2}\}$ of R. A facet of type F_2 is given by the convex hull of the centroids of all i-faces of R containing some vertex of R. Since G is transitive on the maximal flags there are exactly two orbits of facets of W_i^R, namely those of type F_1 and F_2 respectively. Thus an m-face of W_i^R is given by the intersection of a suitable number of facets of W_i^R of types F_1 and F_2. We shall consider all such intersections and show that all m-faces of W_i^R given by the intersection of j facets of type F_1 and k facets of type F_2 lie in one orbit.

case (i). The intersection of j facets of type F_1, $1 \leq j \leq n-i-1$, $m = n-j$.

Let $f = \{a_1, \ldots, a_{n-j-1}\}$ be an $(n-j)$-face of R. Then f contains r i-faces of R where $r \geq n-j-1$ (since R is n-dimensional). Thus the convex hull of the centroids of the r i-faces is an $(n-j)$-face $\tilde{f} = \left\{\begin{array}{l} a_i, \ldots, a_1 \\ a_{i+1}, \ldots, a_{n-j-1} \end{array}\right\}$ since $n-j \geq i+1$. We note that f is the intersection of at least $n-j$ facets of R, therefore $\tilde{f}$ is an $(n-j)$-face of W_i^R. Hence a suitable choice of j facets of type F_1 intersect in an $(n-j)$-face of W_i^R.

case (ii). The intersection of j facets of type F_1,$n-i \leq j \leq n-1$,$m = n-j$.

The intersection of j facets of R is an s-face of R, where $s \leq n-(n-i) = i$. Thus the convex hull of the centroids of all i-faces in such an intersection is at most a point . Therefore there are no m-faces, $m \geq 1$, of R given by j facets of type F_1 for $j \geq n-i$.

case(iii). The intersection of k facets of type F_2, $k \leq i$, $m = n - k$.

By polarity, the convex hull of the centroids of i-faces in the intersection of k facets of type F_2 is similiar to the convex hull of the centroids of $(n - i - 1)$-faces in the intersection of k facets of type F_1^* of R^*. By (i) this is $(n-k)$-dimensional if and only if the intersection of the relative facets of R^* are $(n-k)$-faces $\{a_{n-1}, \ldots, a_k\}$ and $n - i - 1 \leq n - k - 1$. Thus the convex hull of the centroids of all $(n - i - 1)$-faces in this intersection is an $(n-k)$-polytope $\left\{\begin{matrix} a_{n-(n-i-1)}, & \cdots, a_{n-1} \\ a_i, \ldots, a_k & \end{matrix}\right\} = \left\{\begin{matrix} a_{i+1}, & \cdots, a_{n-1} \\ a_i, \ldots, a_k & \end{matrix}\right\}$. So the intersection of k suitable facets of type F_2 is an $(n-k)$-face $\left\{\begin{matrix} a_i, \ldots, a_k & \\ a_{i+1}, & \cdots, a_{n-1} \end{matrix}\right\}$ of P_i, $1 \leq k \leq i$.

case (iv). The intersection of k facets of type F_2,$k \geq i+1$, $m = n - k$.

Note that $i + 1$ vertices of R either define a unique i-face of R, or don't all lie in any one i-face of R. Hence the convex hull of the centroids of all i-faces containing those vertices is at most a point. Therefore there are no m-faces, $m \geq 1$, of W_i^R given by k facets of type F_2 for $k \geq i + 1$.

case (v). The intersection of j facets of type F_1 with k facets of type F_2, $m = n - j - k$.

By (ii) and (iv) we may assume $j \leq n - i - 1$, and $k \leq i$. By the repetitive use of Coxeter [1] (also see section 1:8), we see that any m-face, $1 \leq m \leq n-1$, of W_i^R is of the form

$$\begin{cases} \{a_i, \ldots, a_{i-m}\}, & \text{where } m \leq i; \\ \left\{\begin{matrix} a_i, \ldots, a_{i-r} & \\ a_{i+1}, & \cdots, a_s \end{matrix}\right\}, & \text{where } m = r + s + 2, r \leq i, s \leq n - i\ ; \\ \{a_{i+1}, \ldots, a_{i+m-1}\}, & \text{where } m \leq n - i. \end{cases}$$

We also note that the intersection of a facet f_1 of type F_1 with a facet f_2 of type F_2 of W_i^R is $(n-2)$-dimensional if and only if the vertex of R from which f_2 is given is contained in the facet of R from which f_1 is given. Thus if f

is the $(n-j)$-face of W_i^R given by the intersection of j facets of type F_1 and $v_1, \ldots, v_k$ are the vertices from which the k facets of type F_2 are defined, then the intersection of these $j+k$ facets is $(n-j-k)$-dimensional if and only if the $(n-j)$-face of R corresponding to f contains the vertices $v_1, \ldots, v_k$.

Since R is j-transitive, the $(n-j)$-faces of W_i^R given by the intersections of facets of type F_1 only all lie in one orbit, if they exist. Hence by polarity, the $(n-k)$-faces of W_i^R given by the intersections of facets of type F_2 only all lie in one orbit, if they exist. Let D_v^q be the set of all i-faces of some q-face f_q of R containing some vertex v such that $v \triangleleft f_q$ and $q \geq i$. Since G is transitive on the maximal flags of R, G is transitive on the sets D_v^q for all q-faces f_q such that $v \triangleleft f$, $q \geq i$. Therefore G is transitive on the m-faces of W_i^R given by j facets of type F_1 and k facets of type F_2, $m = n-j-k$, for given j, k.

Thus to determine the face orbit vector of W_i^R, we need only count the number of combinations ϕ of $j+k = n-m$ for each $1 \leq m \leq n-1$ subject to the conditions $1 \leq j \leq n-i-1$, $1 \leq k \leq i$, and $i \leq n/2$. These results are tabulated below.

$1 \leq m \leq i$	$\phi = m$
$i+1 \leq m \leq n-i$	$\phi = i+1$
$n-i \leq m \leq n-1$	$\phi = n-m+1$

By the definition of W_i^R, W_i^R is 0-transitive. Hence

$$(1, 1, 2, 3, \ldots, i, i+1, \ldots, i+1, i, i-1, \ldots, 4, 3, 2)$$

is the face orbit vector of W_i^R.

COROLLARY 5:2.2

Let W be the polar of a n-Wythotope W_i^R, for some $1 \leq i \leq n-2$. Then

$$\theta(W) = (2, 3, \ldots, i, i+1, \ldots, i+1, i, i-1, \ldots, 2, 1, 1).$$

Proof

Follows immediately by polarity and the inversion of face lattices.

As an example we can consider the Wythotope $W = W_3^8 = \left\{ \begin{matrix} 3,3,4 \\ 3,3,3,3 \end{matrix} \right\}$. We form a lattice of face orbits of W, see figure 5.2. Each face orbit is denoted by a representative of that orbit and the lattice is graded by dimension. The lattice is partially ordered by $<$, where $A < B$ if $A \lhd g.B$ for some $g \in GW$. Then the number of k-face orbits is given by the number of elements in the k^{th} level of the lattice of face orbits. We find that $\theta(W) = (1,1,2,3,4,4,3,2)$, by counting the number of elements in each level of the lattice. In this example, there are different orbits of faces which are congruent. For instance, there are two orbits of tetrahedral faces of W. Thus it is not sufficient just to count the number of different faces when computing $\theta(W)$.

Thus the orbit vector $\theta(P)$ of any perfect polytope P derived from a Wythoffian polytope can be computed if the decomposition of P is known. It is easy to see that $(\theta_0, \theta_1, \theta_{n-2}, \theta_{n-1}) = (1,1,x,2)$ or $(2,x,1,1,)$ for some x, that is, either P or P^* is vertex and edge transitive.

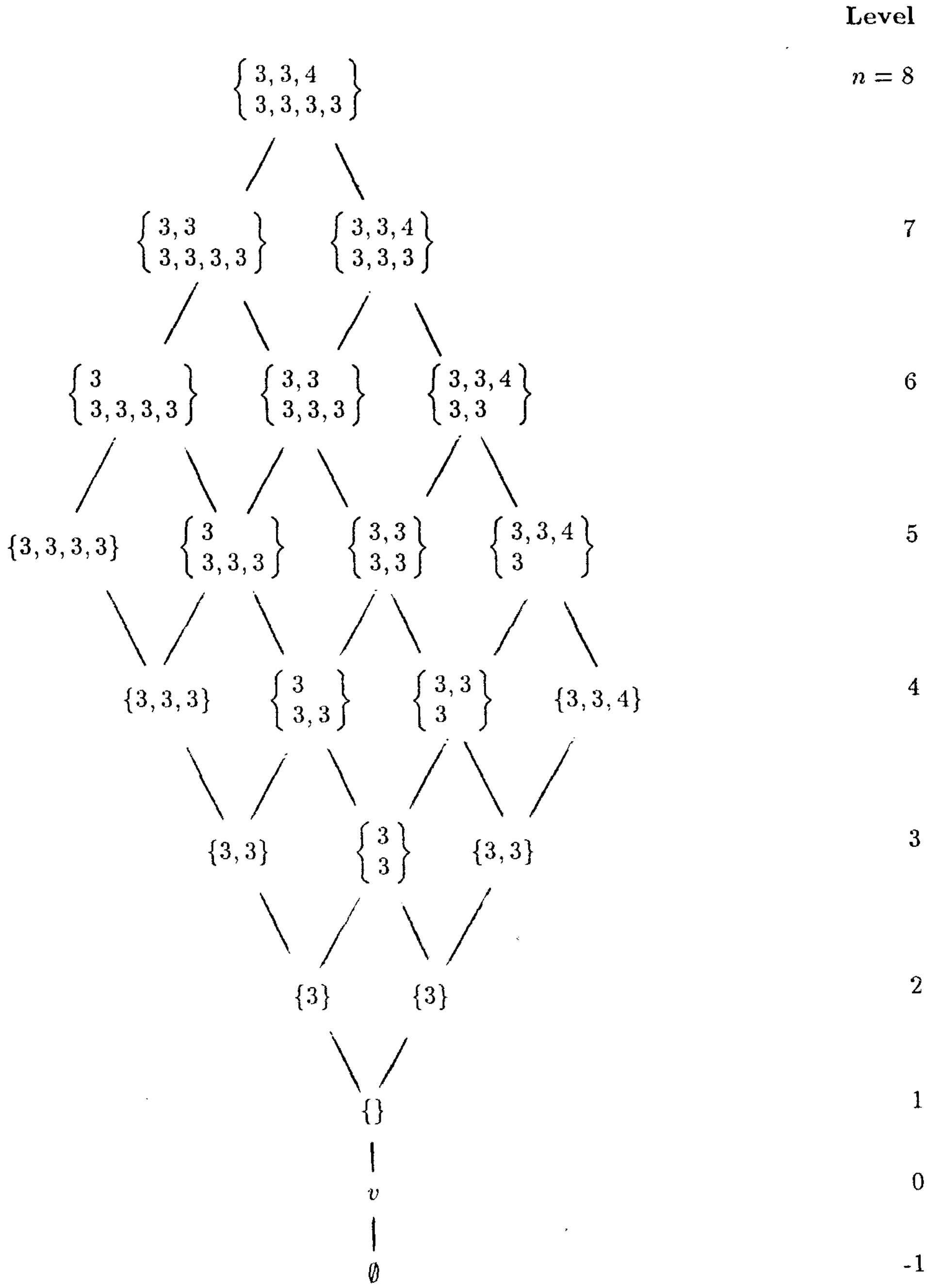


Figure 5:2. Lattice of face orbits of W_3^s.

Besides the family of perfect polytopes obtained above from regular polytopes, there are fourteen other perfect polytopes derived by Wythoff's construction, namely the Gosset polytopes. Such polytopes are of dimension 6, 7 or 8. If the shaded node is an end node then these polytopes are denoted K_{ij} according to the lengths of the branches, for example, see figure 5.3. The orbit vectors of Gosset polytopes are more difficult to compute since the corresponding Coxeter graph is not a simple chain because one of the nodes is attached to 3 branches. Although it is not necessary for present purposes to compute these orbit vectors, it is useful to consider the Gosset polytopes in order to understand some of the difficulties in the general classification problem .

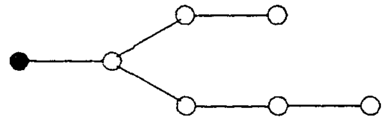


Figure 5.3 The polytope $1_{23} = \left\{3 \begin{matrix} 3,3 \\ 3,3,3 \end{matrix}\right\}$

CHAPTER 6

FIXED POINT SETS OF VERTICES OF A PERFECT POLYTOPE

In this chapter, we explore the restrictions that are imposed on the fixed point set of a vertex of some polytope by the condition of perfection. We start by considering how a general polytope P may be deformed into a symmetry equivalent polytope. Such deformations depend on the dimensions of the fixed point sets of vertices of P. The dimensions of such sets are determined for perfect polytopes. We then conclude by examining perfect n-polytopes whose G-stratification of E^n is isomorphic to that of some regular polytope.

1. Deformations of a polytope

A solid S is perfect if and only if every nearby solid that is symmetry equivalent to S is similar to S. For this reason we study maps between such solids and S, particularly when S is a polytope. To do this we consider deformations of solids and for this purpose we introduce some elementary category theory.

Let S be the space of all solids. Then S can be regarded as the set of objects in a category $\mathfrak{S}$ of face-maps between solids. The morphisms or face-maps are defined as follows. Let B be an n-solid, $n \geq 0$. We first need to give each point x in B a sequence of radial coordinates (similar to barycentric coordinates) in terms of the centroids of a flag of B. Let f_1 be the face of B of lowest dimension j_1 such that $x \in f_1$. If $x = c_1$ (the centroid of f_1), then we say x is given by the n coordinates $(0, \ldots, 0, 1, 0, \ldots, 0)$ where the $(n+1-j_1)^{th}$ entry is 1. If $x \neq c_1$, there is a unique straight line through points c_1 and x oriented from c_1 towards x. This line meets ∂f_1in a unique point x_1 at or beyond x. Thus $x = t_1 x_1 + (1-t_1)c_1$ for some unique $t_1 \in I = [0,1]$, $t_1 \neq 0$. Let f_2 be the face of B of lowest dimension $j_2 < j_1$ such that $x_1 \in f_2$. If $x_1 = c_2$ (the centroid of f_2), then x_1 is given by coordinates similar to c_1 above but having the only non-zero entry 1 at the $(n-j)^{th}$ position. If $x_1 \neq c_2$, then we repeat the above process on x_1 in terms of x_2 and c_3. After a finite number of steps x_k is the centroid c_k of a face of B. Therefore we can associate x with an affine sum of c_i (or x_i), $i = 1, \ldots, k$, that is.

$$
\begin{aligned}
x &= t_1 x_1 + (1-t_1)c_1 \\
&= t_1\big(t_2 x_2 + (1-t_2)c_2\big) + (1-t_1)c_1 \\
&= (1-t_1)c_1 + t_1(1-t_2)c_2 + t_1 t_2 x_2 \\
&\ \vdots \\
&= (1-t_1)c_1 + t_1(1-t_2)c_2 + \cdots + t_1(1-t_2)\cdots(1-t_{k-1})c_{k-1} + t_1 t_2 \cdots t_k x_k \\
&= (1-t_1)c_1 + t_1(1-t_2)c_2 + \cdots + t_1(1-t_2)\cdots(1-t_{k-1})c_{k-1} + t_1 t_2 \cdots t_k c_k.
\end{aligned}
$$

Hence $x = \{X_1, \ldots, X_n\}$ such that $X_i = t_i$ if $j_i = i$ and $X_i = 0$ otherwise. Then a **face-map** $f : B_1 \to B_2$ between two n-solids B_1, B_2 is a map which takes the centroids of faces of B_1 to centroids of faces of B_2 such that the image of any point x written as an affine sum above in term of c_i is the affine sum of the images of c_i, that is,

$$f(x) = (1-t_1)f(c_1) + t_1(1-t_2)f(c_2) + \cdots + t_1(1-t_2)\cdots(1-t_{k-1})f(c_{k-1})$$
$$+ t_1 t_2 \cdots t_k f(c_k).$$

It follows at once that any face-map is continous, that the composition of face-maps $f : B_1 \to B_2$ and $g : B_2 \to B_3$ is a face-map $g \circ f : B_1 \to B_3$, and that for any nonempty solid B, the identity 1_B is a face-map. It is convenient to regard $\mathcal{S}(\emptyset, B)$, $\mathcal{S}(B, \emptyset)$ and $\mathcal{S}(\emptyset, \emptyset)$ as singletons.

Suppose next that $\mathfrak{M}$ denotes the category of similarities between solids. Then $\mathfrak{M}$ is a subcategory of $\mathfrak{S}$. Thus: every object in $\mathfrak{M}$ is an object in $\mathfrak{S}$ (in fact, both have the same objects, namely S); every similarity between solids is a face-map; and the composition operation is the same in both categories. We can define a notion of symmetry equivalence between solids which is coarser than similarity and finer than face-isomorphism, as follows. Let B_1, B_2 be solids. Then a face-isomorphism $f : B_1 \to B_2$ is said to be a **symmetry equivalence** if and only if there is a group isomorphism $f_* : GB_1 \to GB_2$ such that for all $g \in GB_1$, $f_*(g) \circ f = f \circ g$. If such an f exists, we say that B_1 is symmetry equivalent to B_2 and write $B_1 \simeq B_2$.

We now define a deformation of an n-solid B in terms of a connected (continuous) path in the space of solids. A **deformation** is a continous map, $\delta : B \times I \to E^n$, such that for all $t \in I$, $\delta(B, t) = B_t$ is an n-solid and $f_t : B \to B_t$, $f_t(x) = \delta(x, t)$, is a symmetry equivalence. We can assume that $GB = GB_t$ for all $t \in I$. If B is an n-polytope P, then since $P = conv(F_0P)$ we may define $f_t : P \to P_t$ in terms of a set of paths $\{\delta(v, I) : v \in F_0P\}$. It follows that each such path $\delta(v, I)$ lies in the fixed point set fix_v of v. Since we are interested in the difference between perfect and imperfect polytopes, we consider only deformations involving non-similar polytopes, that is to say, B_t is not similar to B, for some $t \in I$. If δ is such a deformation for some solid B, we call $f_t : B \to \delta(B, t)$ a

D-path from B. Thus a polytope P is perfect if and only if there are no D-paths from P. The above formulation of the notion of face-map and the definition of symmetry equivalence in terms of categories has been devised by Professor S. A. Robertson.

2. Vertex orbits of a perfect polytope

Since no D-paths exist from a perfect polytope P, we explore the conditions this imposes on the fixed point sets of vertices and of centroids of facets of P. First we need to prove a simple proposition.

PROPOSITION 6:2.1

Let P be an $(n-1)$-transitive n-polytope. Then every facet of P contains a fundamental region of GP for ∂P.

Proof

Let f be any facet of P. Since P is $(n-1)$-transitive, every point of P is in a G-orbit of some point of f. If the isotropy subgroup of the centroid c of f is trivial then f is a fundamental region D by definition. If the isotropy subgroup of c is nontrivial. then any fundamental region D for the action of G_c on f is a fundamental region for the action of G on ∂P. In either case, $D \subseteq f$.

COROLLARY 6:2.2

If P is a vertex transitive n-polytope, then there exists a fundamental region for ∂P containing only one vertex of P.

Proof

Since P^* is an $(n-1)$-transitive n-polytope, there is a fundamental region for $GP = GP^*$ in ∂P^* containing only one centroid of a facet of P^*. The radial projection of this region to ∂P from the centroid of P is then a fundamental region in ∂P of the required type.

Now we examine the dimensions of fixed point sets of the vertices of a perfect n-polytope. We first consider the case when a polytope is both 0-transitive and $(n-1)$-transitive.

LEMMA 6:2.3

Let P be a 0-transitive $(n-1)$-transitive n-polytope. Then P is perfect if and only if $dim(fix_v) = dim(fix_c) = 1$, where v is any vertex of P and c is the centroid of any facet of P.

Proof

Let P be a 0- and $(n-1)$-transitive n-polytope. First suppose that P is perfect. Then P cannot be deformed into a symmetry equivalent non-similar polytope. Suppose that $dim(fix_v) \neq 1$. By corollary 2.2, there exists a fundamental region D containing only one vertex v of P. Thus we can define a D-path δ_t from P by mapping v to $w = v + \epsilon$. Since $dim(fix_v) \neq 1$, δ_t may be chosen such that the points O, v and w are not collinear. Then $P_t = \delta_t(P) = conv(G.w)$ is a symmetry equivalent non-similar polytope. This is a contradiction if P is perfect. Therefore $dim(fix_v) = 1$ and hence $dim(fix_c) = 1$ by polarity.

Conversely suppose $dim(fix_v) = dim(fix_c) = 1$. Let D be a fundamental region of P containing only one vertex v of P. We shall consider all deformations of P to nearby polytopes by paths of v. The paths that take v off its fixed point set are not D-paths as the resulting polytopes have different number of vertices to P. Such polytopes cannot be symmetry equivalent to P. A path keeping v on its fixed point set results in a polytope similar to P, since the effect is a dilation. Hence any deformation of P to a symmetry equivalent polytope is a similarity, and so P is perfect. By the same argument, since P^* is perfect, it is necessary for $dim(fix_c) = 1$.

It is not a sufficent condition for perfection that a polytope be both vertex transitive and facet transitive. As an example, we can consider the anti-prism of a line segment. Such a polyhedron P is tetrahedral in shape. If the four congruent triangular faces are not regular, then P is a non-perfect vertex and facet transitive polytope. See figure 6.1 for a picture of such a tetrahedron along with its net.

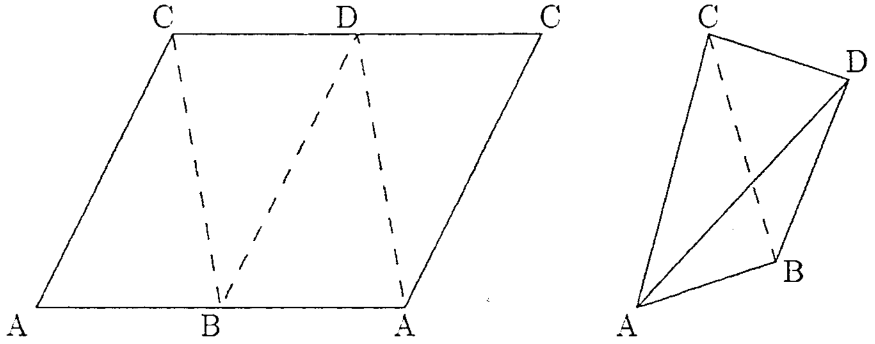


Figure 6.1 The net of a non-perfect tetrahedron T, $\theta(T) = (1,3,1)$ which is vertex and facet transitive

In fact, what we have shown in the proof of lemma 6:2.3 is the following, which we give as a corollary.

COROLLARY 6:2.4

If P is a perfect vertex- or facet-transitive n-polytope then $dim(fix_v) = 1$ or $dim(fix_c) = 1$ respectively.

Now let P be any polytope with vertex orbits given by $G.v_1, \ldots, G.v_m$ for some $m \geq 1$. Then let $P_i = conv(G.v_i)$ denote the polytope given by the convex hull of the i^{th} orbit. Clearly the polytope derived from a vertex transitive polytope P gives rise to $P_i = P$ for all $v_i \in F_0P$. We explore the fixed point set of a vertex of any perfect polytope, that is, one in which vertex transitivity is not assumed, by studying the polytopes P_i.

PROPOSITION 6:2.5

Let P be a perfect n-polytope. Then P_i is an n-polytope, for all i.

Proof

Suppose $dim(P_i) = k$ for some i, where $0 \leq k \leq n-2$. Then G holds a k-dimensional subspace $aff(P_i)$ setwise (or, if $k \geq 0$, pointwise) fixed. It then follows that G is reducible or that $fix_B \neq O$. This is a contradiction by theorem 1:7.5. Hence $dim(P_i) = n$.

It is easy to see that the converse of this proposition is false by considering any non-perfect vertex transitive polyhedron. These have been classified in Robertson [1] and are given with their deficiencies. For instance, consider the general antiprism given in figure 6.1 which has deficiency 2.

PROPOSITION 6:2.6

If P is perfect then P_i is perfect.

Proof

Suppose P_i is not perfect for some i. Then there exists a deformation δ of P_i and a D-path $\delta_t : P_i \rightarrow \delta(P_i, t)$ for some $t \in I$ such that the paths $\{\delta(v, I) : v \in F_0P_i\}$ are given by $G.\delta(v_i, I)$. Since $dim(fix_{v_i}) \neq 1$, we can assume that O, v_i and $\delta_t(v_i) \in fix_{v_i}$ are not collinear. Then there exists a deformation δ' of P_i defined by paths of vertices given by $G.\{s\delta_t(v_i) + (1-s)v_i : s \in I\}$ such that $s\delta_t(v_i) + (1-s)v_i \in fix_{v_i}$ for all $s \in I$. Let $\delta'_s : P_i \rightarrow \delta'(P_i, s)$ be a D-path of δ'. We may consider δ'_s acting on some fundamental region D of the action of GP containing only one vertex v_i of P_i.

Since any point x in P may be given using barycentric coordinates in terms of vertices of P, the deformation δ' induces a deformation of P such that the D-paths on the vertices $\{G.v_j : j \notin G.v_i\}$ are radial projections $v_j \to (1+\epsilon)v_j$ for some ϵ and the D-paths on $\{G.v_i\}$ are given by δ'_s. This is a contradiction to the hypothesis that P is perfect.

We note that the converse of this proposition is false. We can find numerous counter-examples by taking P to be the convex hull of the union of a regular polytope R with $(\tilde{R})^*$, where $\tilde{R}$ is a dilation of R, such that P is not a Wythotope. For instance see figure 6.2, where $conv(G.P_1)$ is a cube and $conv(G.P_2)$ is an octahedron. The polytope P is not perfect as each of the two vertex orbits can be displaced independently of the other one. At a suitable dilation of P_i, $i=1$ or 2, P is a rhombic dodecahedron of the first kind (this occurs when the two triangles common to any edge of P_1 become co-planar). In such a case $G.v_1$ and $G.v_2$ have the same symmetry group $G = GP$.

COROLLARY 6:2.7

Let P be a perfect n-polytope. Then $dim(fix_c) = dim(fix_v) = 1$ for any vertex v of P and for the centroid c of any facet of P.

Proof

This follows from proposition 2.6 and corollary 2.4.

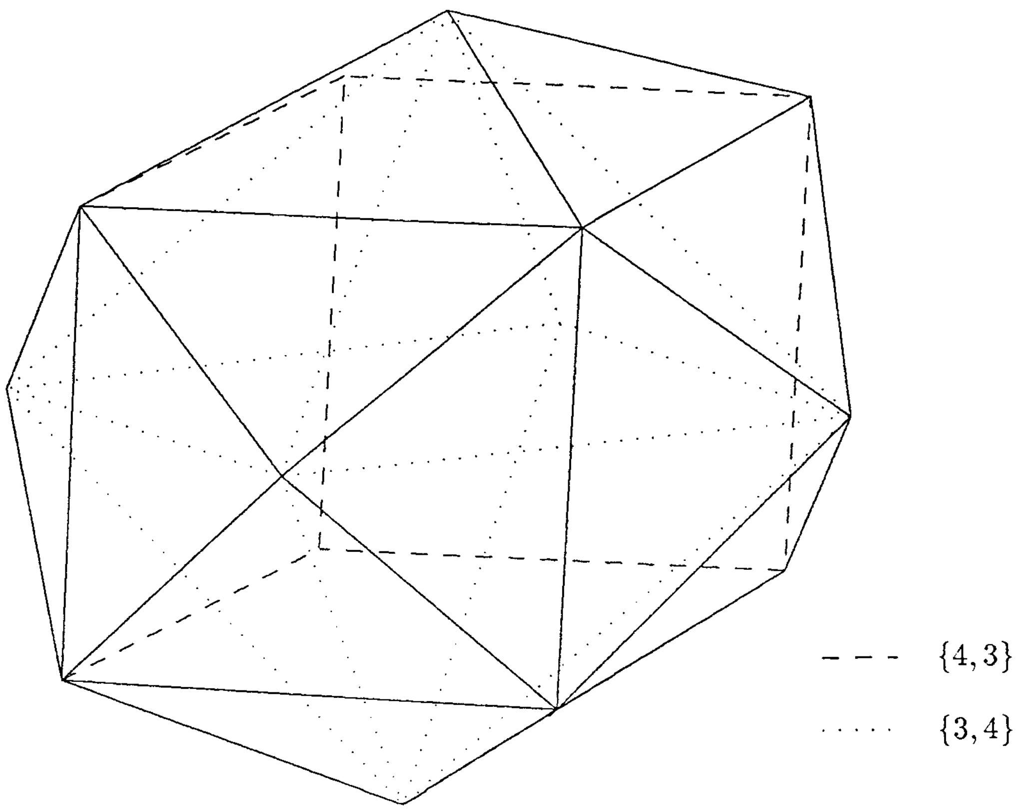


Figure 6.2 $conv(\square_3 \cup \diamond_3)$

The fixed point sets of the vertices of an n-polytope and of its polar form part of the G-stratification of E^n. We consider the case in which a given perfect polytope has the same G-stratification as a regular polytope, in other words both polytopes have the isomorphic symmetry groups.

First we give a simple definition. Let P be an n-polytope. Then we say that two i-faces, $1 \leq i \leq n-1$, f_1 and f_2 of P are **adjacent** if $(f_1 \cap f_2)$ is an $(i-1)$-face of P. Two vertices v_1 and v_2 are said to be **adjacent** if the line segment $v_1 v_2$ is an edge of P.

PROPOSITION 6:2.8

Let P be an n-polytope such that GP is the symmetry group of a regular n-polytope. If P is perfect then either P or P^* is Wythoffian.

Proof

Clearly if P or P^* is Wythoffian, then P is perfect. Let P and R be n-polytopes with the same symmetry group G such that P is perfect and R is regular. Suppose that there are r vertex orbits and s facet orbits in P. Then P^* is perfect with s vertex orbits and r facet orbits, and $GP^* = G$. We may suppose therefore that $s \geq r$. It may be helpful to consider the polyhedron in figure 6.2 as an example during this proof.

A fundamental region D of R (and hence of P) is the n-simplex $\triangle$ whose vertices are $O, c_0, \ldots, c_{n-1}$ where c_i is the centroid of an i-face A_i in a maximal flag of R. Suppose $P = conv(G.v_1 \cup \cdots \cup G.v_r)$ where $v_j \in D$, $1 \leq j \leq r$. Since P is perfect, $dim(fix_{v_j}) = 1$. Hence for each $j = 1, \ldots, r$, v_j lies on the ray α_i from O to c_i for some $i = 0, \ldots, n-1$. Without loss of generality, suppose $\{v_1, \ldots, v_r\}$ is labelled such that if $v_i \in \alpha_j$ and $v_{i+1} \in \alpha_k$ then $j < k$. Each $P_j = conv(G.v_j)$ is Wythoffian and is derived from a regular polytope similar to R (or R^*).

We note that the fixed point sets of centroids of facets of a wythotope W derived from R coincide with the rays $G.\alpha_0$ and $G.\alpha_{n-1}$. We also note that the facets f_1 and f_2 with centroids c_1 and c_2 on α_0 and α_{n-1}, respectively, are adjacent with common $(n-2)$-face f' such that the centroids of f', f_1 and f_2 are coplanar with O.

If $r = 1$, then P is, by definition, Wythoffian. Suppose $r \geq 2$. Therefore (1) there are t polytopes P_t, $1 \leq t \leq r$, that have a facet f_t with centroid γ_t on α_0 and (2) the intersection $(\alpha_0 \cap \partial P)$ is either a vertex of P or the centroid of a facet of P.

First suppose $(\alpha_0 \cap \partial P)$ is a vertex v of P. Then $conv(G.v)$ is regular and similar to R. Thus $v = v_1$. Let x be such that $1 < x \leq r$ and $|O\gamma_x| > |O\gamma_t|$ for all $1 \leq t \leq r$. Then x is unique. For suppose that f_y is a facet of P_y for some

y such that $\gamma_y = \gamma_x$. Let x and y be representatives of the vertice orbits of f_x and f_y respectively, such that x and y both lie in the same fundamental region. Therefore x and y are centroids of faces in some maximal flag of μR for some dilation μ. Thus P_x and P_y are derived from the same polytope μR. This is a contradiction since either $P_x \subset P_y$ or $P_y \subset P_x$, according as $x > y$ or $x < y$.

We also show that P_x is not regular. For if P_x is regular then P_x is similar to R^* and $x = r$. If $r > 2$, then there exists i, $1 \leq i \leq r$ such that $G.v_i \subset F_0P$. We may suppose that the convex hull P_i of $G.v_i$ is derived from νR^* for some positive real number $\nu > 0$. Then the centroid γ_i of some facet of P_i is on α_0. Since $\nu > 0$, $|O\gamma_i| > |O\gamma_x|$, which contradicts our hyphothesis. Now suppose $r = 2$, in other words, $P = conv(\mu R \cup \nu R^*)$ for some μ. Since the centroids of i-faces of R are centroids of $(n-i)$-faces of R^*, it follows that P is $(n-1)$-transitive. This contradicts $r \geq s$. Therefore P_x is a wythotope.

If x is an extreme point of P, the boundary of f_x is a part of the boundary of P. Then any $(n-2)$-face of f_x is an $(n-2)$-face of P. $Conv(f_x \cup v)$ is an n-cone with base f_x. Then with the exception of f_x, the facets of $conv(f_x \cup v)$ are facets of P. Let f' be an $(n-2)$-face of f_x such that the centroid c' of f' is coplanar with the points of α_0 and α_{n-1}. Then $f^* = conv(f' \cup v)$ is a facet of P. Since P is perfect, the centroid c^* of f^* lies in a 1-dimensional fixed point set. Therefore $c^* = c_i$ for some i, $1 \leq i \leq n-2$. Then O, c_0, c_i and c_{n-1} are coplanar which is a contradiction if $(A_0, \ldots, A_{n-1})$ is a maximal flag of R.

Now if $(\alpha \cup \partial P)$ is the centroid c of a facet f of P, then the $(n-2)$-faces of f are $(n-2)$-faces of P and $f \in F_{n-1}P_i$ for some $1 \leq i \leq r$. Let f' be the $(n-2)$-face of f whose centroid is coplanar with O, c_0 and c_{n-1}. Then there exist a facet f^* of P containing f' and a facet f_j of some P_j, $1 \leq j \neq i \leq r$, whose centroid lies in α_0 such that either $v \in F_0 f_j$ and $v \in F_0 f^*$ or $\tilde{f} \in F_{n-2} f_j$ and $\tilde{f} \in F_{n-2} f^*$. In either case the centroid of f^* is coplanar with O, c_0 and

c_{n-1}. This is a contradiction by the above.

Hence $r = 1$ and P is Wythoffian. Note that if $s \leq r$ then the proof shows $s = 1$ and P^* is Wythoffian.

CHAPTER 7

PERFECT POLYTOPES

One problem in proving Pinto's conjecture II:2 is dealing with the perfect polytopes, such as the Gosset polytopes, that are not associated to any regular polytope. If Pinto's conjecture is correct then such polytopes occur only in dimensions $n \geq 6$. For this reason we concentrate on the case $n = 4$ (in other words, Rostami's conjecture 1:8.1), although some results are found for the general case. We first consider necessary conditions for a perfect polytope to be regular, in the form of a conjecture. This leads to some interesting results on the orbit vector of a perfect polytope and points towards a classification in dimensions 4 and 5.

1. Regularity for perfect polytopes

There are many examples of non-regular prime perfect polytopes which are transitive only on either the vertices or the facets. Take, for example, the wythotopes or their polars respectively. The following conjecture is based on consideration of all presently known perfect polytopes.

CONJECTURE 7:1.1

Every prime perfect 0-transitive $(n-1)$-transitive n-polytope is regular.

It is noted that the Gosset polytopes do not contradict this statement as no such polytope is facet transitive. We also note from chapter 5 that this conjecture is true if Pinto's conjecture is true. It has not yet been possible to prove conjecture 1.1 in the general case but a proof for n=4 is given in chapter 8. The cases $n < 4$ are already familiar. We explore some of the metric properties of such polytopes

PROPOSITION 7:1.2

Let P be a prime perfect 0-transitive $(n-1)$-transitive n-polytope. Let f be an i-face with centroid c and f' an $(i+1)$-face with centroid c', such that $f \triangleleft f'$. Then (1) $|cv_j| = |cv_k|$ for all v_j, $v_k \in F_0 f$, and (2) cc' is perpendicular to f.

Proof

We shall prove this inductively by considering an i-face of P to be given by the intersection of $(n-i)$ mutually adjacent facets of P. We assume P to be $\square$-prime, since otherwise the $\square$-decomposition results in a decomposition of E^n into orthogonal subspaces. Also if $P = \square^r A$ for some integer r and some polytope A, then P is perfect 0- and $(n-1)$-transitive if and only if A is perfect 0- and $(n-1)$-transitive by proposition 1:7.4 and chapter 5.

$(n-i=1)$. Let f_r be a facet of P with centroid c_r and let v be any vertex of f_r. Then by corollary 6:2.7, $dim(fix_{c_r}) = dim(\text{fix}_v) = 1$. Therefore $aff(f_r)$ and the line Oc_r are orthogonal. In particular, $\angle v_j c_r O = \pi/2$ for all $v_j \in F_0 f_r$. Since P is 0-transitive and $(n-1)$-transitive, we have $|Ov_j| = |Ov_k|$ and $|Oc_r| = |Oc_s|$

for all $v_r, v_s \in F_0P$ and $f_r, f_s \in F_{n-1}P$, respectively. Therefore $|c_r v_j| = |c_r v_k|$ for all $v_j, v_k \in F_0 f_r$. Hence the proposition is true for $n - i = 1$.

We also check the case $n - i = 2$. Let $f = f_1 \cap f_2$ be an $(n-2)$-face with centroid c, where f_1 and f_2 are facets with centroids c_1 and c_2, respectively. Then for any $x \in f$, $\angle Oc_1x = \angle Oc_2x = \pi/2$. Then $|c_1x| = |c_2x|$. Therefore cc_1 and cc_2 are perpendicular to f. Let v_1, v_2 be any two vertices of f, then $\angle c_1cv_1 = \angle c_1cv_2 = \pi/2$ and $|c_1v_2| = |c_1v_2|$. Thus we have $|cv_1| = |cv_2|$. Hence the proposition is true for $n - i = 2$.

$(n - i = h + 1)$. Suppose the proposition is true for all $n - i = 1, \ldots, h$. Let f be an $(n - h - 1)$-face of P common to the facets $f_1, \ldots, f_r$, $r \geq h + 1$, with centroids $c_1, \ldots, c_r$ respectively. Let f have centroid c. Then f is the intersection of s $(n - h)$-faces $\alpha_1, \ldots, \alpha_s$ of P, where α_i is given by the intersection of h suitably chosen facets f_j and $s \geq h + 1$. Let α_i have centroid γ_i. Then for any $x \in f$, $\angle O\gamma_i x = \angle O\gamma_j x = \pi/2$ for $i, j = 1, \ldots, s$. Therefore f lies in the intersection of the perpendicular bisectors of the points γ_i, γ_j for all $i, j = 1, \ldots, s$, $i \neq j$. Therefore f is perpendicular to $c\gamma_i$ for $i = 1, \ldots, s$. Now let v_1, v_2 be vertices of f. Then $v_1, v_2 \triangleleft \alpha_i$ for $i = 1, \ldots, s$. Hence $|v_1\gamma_i| = |v_2\gamma_i|$. Therefore $|v_1c| = |v_2c|$. Thus the proposition is true for $n - i = h + 1$.

Therefore by induction the proposition is true for $n - i = 1, \ldots, n - 1$, that is, it is true for all i-faces, $i = n - 1, \ldots, 1$.

We now give two corollaries of Proposition 2.1.

COROLLARY 7:1.3

Let P be a prime perfect 0-transitive $(n-1)$-transitive n-polytope. Let f be an i-face with centroid c. Then $|cc_r| = |cc_s|$ for all facets f_r, f_s such that $f \triangleleft f_r, f_s$.

Proof

This follows by polarity.

COROLLARY 7:1.4

Let P be a prime perfect 0-transitive $(n-1)$-transitive n-polytope. Let f be an i-face, $1 \leq i \leq n-2$, with centroid c. Suppose that $f \lhd f'$ for some facet f' with centroid c'. Then the line cc' is perpendicular to f.

Proof

The case $i = n-2$ follows from proposition 1.2. Suppose $i < n-2$. Let α_j be an j-face with centroid γ_j, where $i+1 \leq j \leq n-2$, such that $f \lhd \alpha_{i+1} \lhd \cdots \lhd \alpha_{n-2} \lhd f'$. Then $c'\gamma_{n-2}$ is perpendicular to $\gamma_{n-2}\gamma_{n-3}$ by proposition 1.2. Hence $\gamma_{k+1}\gamma_k$ is perpendicular to $\gamma_k\gamma_{k-1}$ by a repetitive argument, $i \leq k \leq n-3$. Thus cc' is perpendicular to $\mathit{aff}(f)$.

Thus each i-face f of prime perfect 0-transitive $(n-1)$-transitive n-polytope P, $1 \leq i \leq n$, is such that the set $F_0 f$ lies on some $(i-1)$-sphere with centre at the centroid of f. Also we have all of the centroids of the i-faces of some facet f' of P lying on some n-sphere with centre at the centroid of f'.

Suppose we let f_j, f_k be i-faces of P with centroids c^i_j, c^i_k and let f_s, f_t be r-faces with centroids c^r_s, c^r_t, such that $f_s \lhd f_j$ and $f_t \lhd f_k$. Then in view of lemma 5:1.1 and the above, the proof of conjecture 1.1 would follow if it were possible to prove that $|c^i_j c^r_s| = |c^i_k c^r_t|$ for all $0 \leq r < i \leq n$. However we have not been able to prove this property.

2. Conjectures on orbit vectors of perfect polytopes

We consider the orbit vector of any prime perfect polytope P with the assumption that conjecture 1.1 is true. In particular, we explore what restrictions are imposed on the transitivity of vertices and facets of P.

We begin with some notation. Let P be any n-polytope. For any vertex v of P, let $E_v = \{e \in F_1P : v \lhd e\}$ be the set of all edges of P emanating from v. Let $V_v = \{v_i \in F_0P : v_i \lhd e \text{ for some } e \in E_v\}$ be the set of all vertices of P adjacent to v. Now we suppose that P is perfect and let $P_i = conv(G.v_i)$ for some $v_i \in F_0P$. Then by proposition 6:2.6, each P_i is a perfect 0-transitive polytope.

PROPOSITION 7:2.1

Let P be a prime perfect facet transitive n-polytope. If P is not vertex transitive, then either (a) $\theta(P) = (2, 1, \theta_2, \ldots, \theta_{n-2}, 1)$ or (b) $dim(aff(G.v_i \cap f)) < n-1$ for some facet f of P and some vertex v_i of P.

Proof

Let P be as stated, and suppose that P is not vertex transitive. Since P is $(n-1)$-transitive we may suppose that $v_i \lhd f$. Suppose $aff(G.v_i \cap f)$ is an $(n-1)$-dimensional space. Then $f' = conv(G.v_i \cap f)$ is an $(n-1)$-polytope contained within f and $f \neq f'$. Thus the closure of the complement of f' in f is a collection of (n-1)-polytopes $\alpha_1, \ldots, \alpha_r$ such that $f = f' \cup \alpha_1 \cup \cdots \cup \alpha_r$, for some $r \geq 1$. Let $v \in V_{v_i}$, $v \notin G.v_i$. Thus v and v_i are endpoints of some edge e of f. It may be useful to consider the polyhedron given in figure 7.1 during this proof.

Then if $(f \cap V_v) \not\subseteq G.v_i$, there is a vertex v^* adjacent to v, that is $v^* \in V_v$, such that $v^* \notin G.v_i$. In this case, both v and v^* belong to one of the polytopes α_j. Now α_j and f' share an $(n-2)$-face β. However β is an $(n-2)$-face of P_i and hence belongs to a unique facet f_1 of P_i other than f'. Therefore each ray $0v$ and $0v^*$ intersect the interior of f_1 in a distinct point, but these points have 1-dimensional fixed point sets, which is a contradiction. Therefore $f \cap V_v \subset G.v_i$ and hence $V_v \subset G.v_i$ for all $v \notin G.v_i$. Similarly we have $V_{v_i} \subset G.v$ for $v_i \notin G.v$.

Thus f contains vertices from two orbits and each edge of P has one endpoint in $G.v_i$. Let $v_i, v \lhd e$ for some edge e. Then $G_v.e$ generates E^n since $dim(fix_v) = 1$, and likewise for $G_{v_i}.e$. Thus $conv(G.e)$ is an n-polytope Q such that $F_0P = F_0Q$. Hence $P = Q$ and $G.e = F_1P$. Since P is facet transitive we conclude that the face-orbit vector of P is given by $\theta(P) = (2, 1, \theta_2, \ldots, \theta_{n-2}, 1)$.

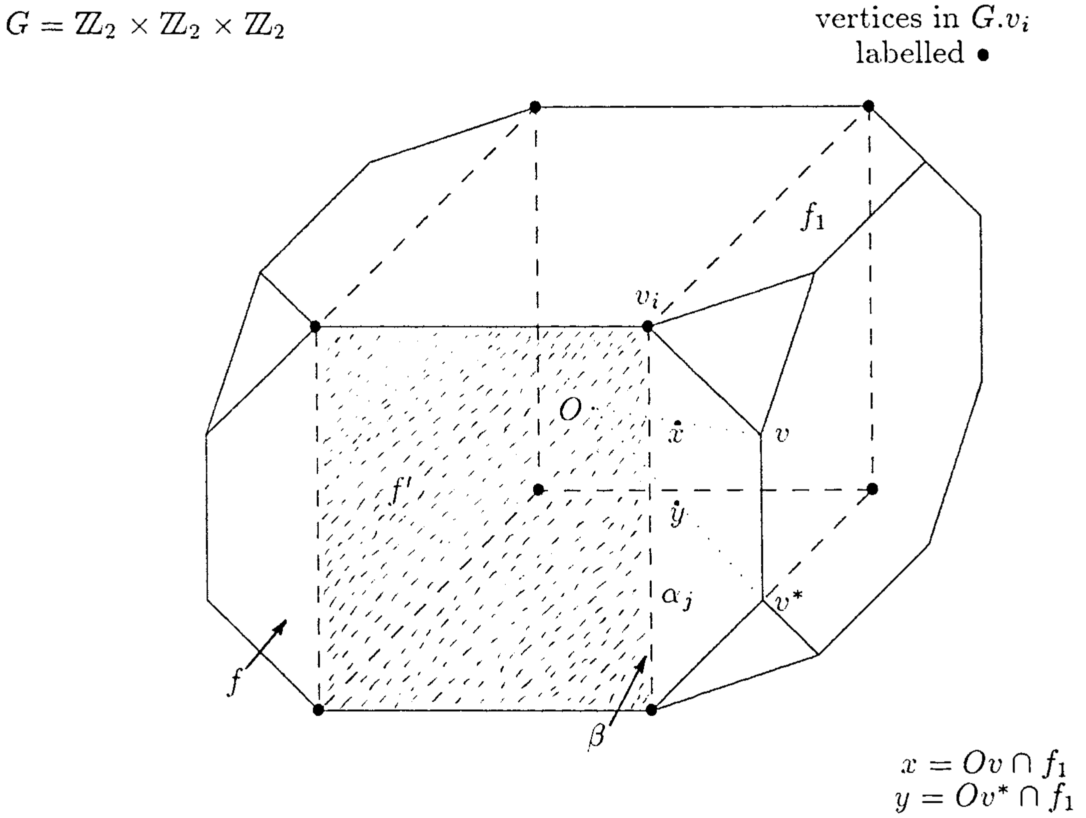


Figure 7.1 A distorted truncated cube with three vertex orbits

PROPOSITION 7:2.2

Let P be a prime perfect facet transitive n-polytope. As a consequence of conjecture 1.1 either $conv(G.v)$ is a prime perfect n-polytope P^i such that

$\theta(P^i) = (1, \theta_1, \ldots, \theta_{n-3}, 1, 2)$ *or* P^* *is Wythoffian, where* v *is a vertex of* P *or of* P^*.

Proof

Let $P = P^1$ have m_1 distinct vertex orbits given by $G.v_1, \ldots, G.v_{m_1}$. Let $P_i^1 = conv(G.v_i)$. If the supporting hyperplanes of a facet f of P^1 and of a facet of P_i^1 coincide, in other words, $f' \subset f$ for some facet f' of P_i^1, then $dim(aff(G.v_i \cap f)) = n - 1$. Therefore either $\theta(P^1) = (2, 1, \theta_2, \ldots, \theta_{n-2}, 1)$ or the facets of P_i^1 are only given by $conv(G.v_i \cap V_v)$ for some vertex $v \notin G.v_i$ of P^1 by proposition 2.1. Then in the latter case $P^2 = (P_i^1)^*$ is a prime perfect facet transitive n-polytope with m_2 distinct vertex orbits, where $m_2 < m_1$ since there are at most $m_1 - 1$ vertex orbits to which v can belong. After finitely many repetitions of this procedure, and writing $P^{k+1} = (P_i^k)^*$, we obtain P^j such that $\theta(P^j) = (2, 1, \theta_2, \ldots, \theta_{n-2}, 1)$ or $m_j = 1$. If $m_j = 1$ then P^j is a prime perfect vertex transitive facet transitive n-polytope and hence regular by conjecture 1. Otherwise P is Wythoffian by proposition 6:2.8.

The general polytope Q with $\theta(Q) = \theta(P^j)$ above has facets with strict $(n-2)$-transitivity conditions as we now see.

LEMMA 7:2.3

Let P *be an* n*-polytope with two facet orbits and one* $(n-2)$*-face orbit. Then the isotropy subgroup at the centroid* c *of any facet is transitive on the* $(n-2)$*-faces of that facet, and* $dim(fix_c) = 1$.

Proof

Let P be an $(n-2)$-transitive n-polytope with two facet orbits. Let f_1, f_2 be adjacent facets. Then the intersection $f_1 \cap f_2$ is an $(n-2)$-face and hence a representative for the orbit $F_{n-2}P$. Since any $(n-2)$-face of P belongs to just two facets of P, $f_1 \notin G.f_2$. For since otherwise there would be only one facet orbit. Hence all facets adjacent to any given facet f_j lie in the orbit $G.f_i$ such that $f_j \notin G.f_i$. Let $\alpha, \alpha^* \triangleleft f_j$ be any two $(n-2)$-faces. Then $\alpha \triangleleft f$ and $\alpha^* \triangleleft f^*$ for some facets $f, f^* \in G.f_i$. Since P is $(n-2)$-transitive there exists $g \in G$ such that $g(\alpha) = \alpha^*$. If $g(f) = f_j$ then $f_j \in G.f_i$ which is a contradiction, therefore $g(f) = f^*$. Hence g keeps f_j setwise fixed. Therefore G_{c_j} is transitive on the $(n-2)$-faces of f_j, where c_j is the centroid of f_j. The affine hull of the centroids of all $(n-2)$-faces of f_j is an $(n-1)$-dimensional space kept setwise fixed by G_{c_j}. Hence $dim(fixc_i) = 1$.

COROLLARY 7:2.4

Let P be an edge-transitive n-polytope with two vertex orbits. Then the isotropy subgroup of any vertex v is transitive on all vertices adjacent to v.

Proof

Follows immediately from the above by polarity.

3. Conjectures on a classification for n = 4 and 5

We conjecture a classification of the perfect polytopes up to dimension 5 based on conjecture 1.1. We consider perfect polytopes whose orbit vectors are given by $\theta(P) = (1, \theta_1, \ldots, \theta_{n-3}, 1, 2)$. It seems unlikely that such polytopes exist in dimensions $n \geq 4$ due to the $(n-2)$-transitivity. We first consider a simple lemma.

LEMMA 7:3.1

Let R be a regular n-polytope with symmetry group G. Let c be the centroid of any facet f and v be any vertex in f. Then any symmetry g is a product of finitely many elements of $G_c \cup G_v$.

Proof

This follows immediately since the stratification of G_c and G_v defines the boundary of a fundamental region for the action of G on R. Alternatively, consider any symmetry g of R. Then we can assume that g is a reflection since otherwise g is a product of reflections. Here, a reflection is a transformation g that holds an $(n-1)$-space H pointwise fixed and takes $x \notin H$ to $-x$ with respect to H, in other words g has eigenvalues equal to -1 and 1 of multiplicity 1 and $(n-1)$ respectively. Any reflective hyperplane H then contains $n-1$ of the centroids of faces from a maximal flag. Therefore H contains the centroid of a facet or a vertex, hence the reflection is in the isotropy subgroup of that point.

Recall that the k-faces, $0 \leq k \leq n-2$, of any polytope Q may be defined by the intersection of a suitable number of suitable facets of Q. We use this fact to define a new polytope from a perfect polytope, in order to prove the following proposition.

PROPOSITION 7:3.2

Let P be an n-polytope with orbit vector $\theta(P) = (1, \theta_1, \ldots, \theta_{n-3}, 1, 2)$. As a consequence of conjecture 1.1, P is perfect if and only if $n = 3$.

Proof

If $n = 3$ then $\theta(P) = (1, 1, 2)$. Therefore P is perfect by Robertson [1]. Let $n \geq 4$, and suppose that P is perfect. Let f_1, f_2 be any two adjacent facets of P. Let c_j be the centroid of f_j, $j = 1, 2$ and c_0 any vertex of P. Then $dim(fix_{c_i}) = 1$ and G_{c_i} is not reducible on $aff(f_i)$ by the proof of proposition 2.2, for $i = 0, 1, 2$, where f_0 is a facet of P^* with centroid on fix_{c_0}. By lemma 7:2.3, $f_1 \notin G.f_2$.

If $aff(f_1)$ and $aff(f_2)$ are perpendicular, then all facets adjacent to one another are perpendicular. In this case, any vertex v is given by the intersection of n mutually perpendicular hyperplanes. Now v is contained in some facet in $G.f_1$ and some facet in $G.f_2$. Since $dim(fix_v) = 1$, it follows that G_v is reducible on the hyperplane orthogonal to fix_v. This is a contradiction by the above. Therefore $aff(f_1)$ and $aff(Oc_2)$ intersect in some point x. By lemma 2.3, G_{c_j} is transitive on the facets adjacent to f_j for $j = 1, 2$. Since $x \in fix_{c_2}$, $dim(fix_x) = 1$ and G_x is transitive on the facets adjacent to f_2, and hence on their affine hulls. Let $F_0Q = \{y \in E^n : y = g.x \text{ for some } g \in G\}$ and let $Q = conv(F_0Q)$. Then $y \in F_0Q$ is such that $y \in fix_{g(c_2)}$ and G_y is transitive on the affine hulls of all facets adjacent to $g(f_2)$, for some $g \in G$. Let $h = aff(f_1)$, then $G_y(h) = \{H \subset E^n : H = g(h) \text{ for some } g \in G_y, y \in F_0Q\}$ is the set of all hyperplanes supporting facets in $G.f_1$. Now G_{c_1} is transitive on the set F_{f_1} of facets adjacent to f_1, and hence on the fixed point sets of centroids of facets in F_{f_i}. Therefore the affine hull of $(G_{c_1}.x)$ is the hyperplane h and the convex hull is a facet f' of Q. Clearly G_{c_1} is transitive on the points of $G_{c_1}.x$. Therefore the centroids of f_1 and f' coincide and $h = aff(f_1) = aff(f')$. Thus Q is a vertex transitive facet transitive n-polytope, such that the fixed point set of the centroid of any facet or any vertex is 1-dimensional. Therefore Q is perfect. By conjecture 1.1 Q is regular. Obviously, if we considered the supporting hyperplanes of facets in $G.f_2$, we would get the polar of Q as our polytope. Let G be the symmetry group of P and $G(Q)$ be the symmetry group of Q. Then $G(Q) \subseteq G$ by the above and lemma 3.1.

Suppose P^* has vertex orbits $G.v_1$ and $G.v_2$. Then let $(P^*)_i$ be the convex hull of the $G.v_i$, $i = 1, 2$. Let $Q = Q_0$, then we can consider the perfect n-polytopes $(P^*)_1$ and $(P^*)_2$. Both of these polytopes are 0 and $(n-2)$-transitive and have two facet orbits. $(P^*)_1$ and $(P^*)_2$ have vertex-set given by the orbits of c_1 and c_2, respectively, and have facet-set given by the union of orbits of c_0 with c_2 and c_0 with c_1, respectively. For $(P^*)_1$ and $(P^*)_2$, we can define regular polytopes Q_1 and Q_2, respectively, in a fashion similar to the above, such that Q_i has vertex orbit $G.c_j$, where $i \neq j$ and $i, j = 0, 1, 2$. Now $\{f_0 Q_i, f_{n-1} Q_i\}$ is independent of $i = 0, 1, 2$, where $f_r P$ denotes the number of r-faces in P. Hence for each $i, j = 0, 1, 2$, Q_i is similar to Q_j or to Q_j^*, since any two regular n-polytopes R and R' are similar if and only if $f_0 R = f_0 R'$. If Q_0 and Q_1, say, have the same vertex set $G.c_2$, then the fixed point sets of c_0 and C_1 coincide. This is a contradiction to the face-structure of P if P is perfect. Hence $n = 3$.

We now are able to state Pinto's conjecture for the case $n = 4$ or 5 which is true if conjecture 1.1 is true.

CLASSIFICATION CONJECTURE 7:3.3

Let P be a prime perfect n-polytope, $n = 4$ or 5, then P or P^* is Wythoffian.

Proof

Suppose P is a prime perfect n-polytope. Let P_i $= conv(G.v_i)$. Then P_i is a vertex transitive perfect n-polytope. Therefore P_i is Wythoffian or $\theta(P) = (1, 1, 2)$, in which case P is a 3-wythotope since the perfect polyhedra are classified. In any case P has the symmetry group of a regular polytope and hence P is Wythoffian.

CHAPTER 8

DIMENSION 4

We investigate sums of angles formed by adjacent faces of a polytope. In view of proposition 7:1.2, this gives some useful information on the 2-faces of certain perfect polytopes. In section 2 we prove conjecture 7:1.1 for $n = 4$, and hence classify the prime perfect 4-polytopes. This proves Rostami's conjecture 1:8.1 and leads to a classification of the perfect 4-solids.

1. Angle sums

We explore the restrictions on the angle between two adjacent i-faces of some $(i + 1)$-face of an n-polytope. We are specifically concerned with the case $i + 1 = n = 4$.

Suppose v is a vertex of some polygon R. Then the two edges of R containing v form an angle θ called the **interior angle** of R at v. It is trivial to note that because R is convex $\theta < \pi$. Now let v be any vertex of some polyhedron P. Suppose the number of faces of P containing v is q. Then at v there are q

interior angles. These q angles must total less than 2π since otherwise v is not an extreme point of P. Thus we have an angle sum inequality at v. We can also find an angle sum inequality at an edge of some 4-polytope as follows. Let e be any edge with midpoint ϵ contained in a 3-face f of some 4-polytope P'. Then the subspace of $\mathit{aff}(f)$ orthogonal to $\mathit{aff}(e)$ is a 2-space which cuts the boundary of f such that the intersection is locally the boundary of a polygon α with vertex ϵ. The interior angle of α at ϵ is called the **dihedral angle** ϕ of e in f. We have $\phi < \pi$ if e is an edge of f. Likewise the subspace of $\mathit{aff}(P')$ orthogonal to $\mathit{aff}(e)$ is a 3-space which cuts $\partial P'$ such that the intersection is locally the boundary of a polyhedron β. Thus the sum of the dihedral angles at any edge e of the 3-faces containing e is less than 2π. For example, suppose P' is the 4-cube (see figure 8.1). Then α is a square and β is a cube.

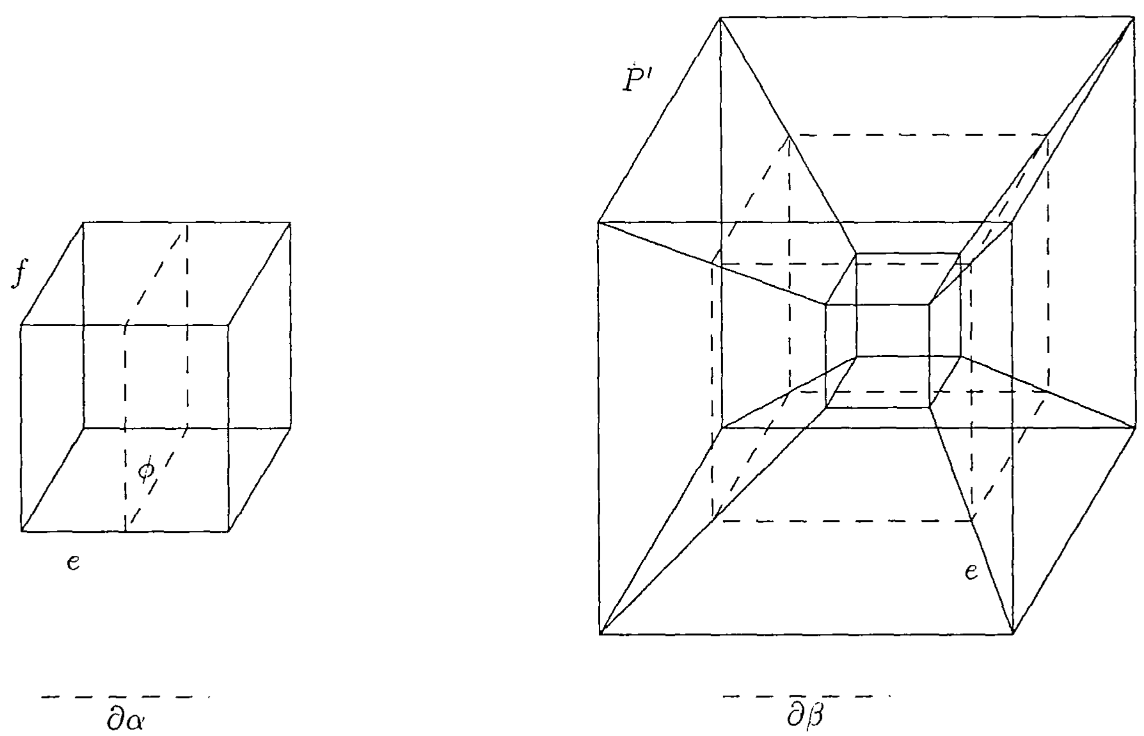


Figure 8.1

Now let P be a 0-transitive $(n-1)$-transitive prime perfect n-polytope. Then every 2-face of P is a circumscribed polygon Q by proposition 7:1.2 in the following sense: if c is the centroid of Q then the set F_0Q lies on a circle C with centre c. Suppose $v_1, \ldots, v_q$ are the vertices of Q such that v_iv_{i+1} and v_qv_1 are the edges of Q, where $i = 1, \ldots, q-1$. Let θ_i be the interior angle of Q at v_i. Then it is useful to prove the following geometric result.

LEMMA 8:1.1

Let Q be a q-gon, $q \geq 3$, circumscribed by a circle C with centre c. Suppose Q has centroid c and θ is the minimum interior angle of Q. If $\theta < \frac{\pi}{2}$ then Q is an equiangular triangle and if $\theta = \frac{\pi}{2}$ then Q is an equilateral quadrilateral (in other words, a square or rectangle).

Proof

Suppose $q \geq 4$ and $\theta_2 = \theta \leq \frac{\pi}{3}$. Then Q contains the triangle $\Delta = \Delta v_1v_2v_3$. The complement of Δ in Q is a $(q-1)$-gon Q'. Since $\theta \leq \frac{\pi}{3}$, there exists a diameter D of C such that the centroids of both Δ and Q' lie in one of the semicircles of C defined by D. Therefore c is not the centroid of Q. Now suppose that $\frac{\pi}{3} < \theta < \frac{\pi}{2}$. Let D_i be the diameter of C given by the point v_i, for $i = 1, 3$. Then Q is divided into four regions A_1, A_2, A_3 and A_4, where D_1, D_3 define the regions A_1, A_3 and A_2 contains the vertex v_2, see figure 8.2. Let a_i denote the area of the region A_i. Clearly $a_2 < a_4$ if θ is a minimum interior angle and $q \geq 4$. However, $a_1 + a_4 = a_2 + a_3$ and $a_1 + a_2 = a_3 + a_4$ if c is the centroid of Q. Thus $a_2 = a_4$. This is a contradiction. Hence $q = 3$. Then by lemma 5:1.1, Q is regular.

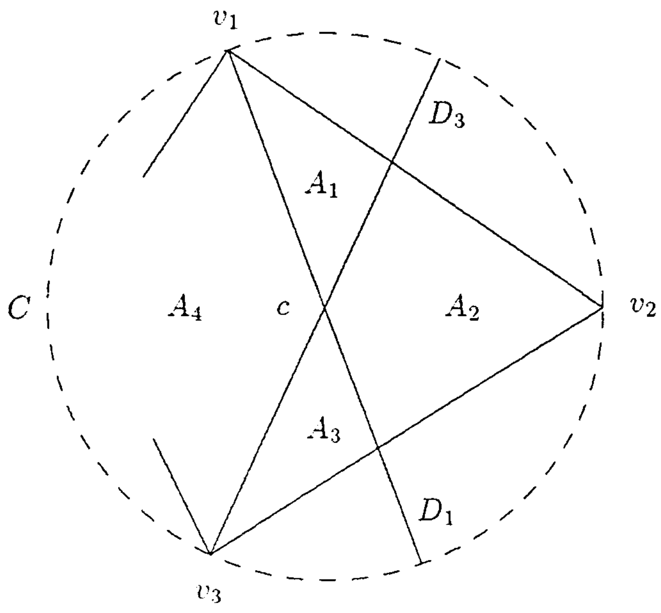


Figure 8.2

Let Q be a q-gon with smallest interior angle $\theta_2 = \frac{\pi}{2}$. Then v_1 and v_3 lie on a diameter. Clearly $q \neq 3$. Suppose $q \geq 5$. The centroid c of Q is given by $\sum_i v_i = c$. However $v_1 + v_3 = c$. Thus $v_2 + v_4 + \cdots + v_q = c$. Therefore $Q' = conv(v_2, v_4, \ldots, v_q)$ is a $(q-2)$-gon with centroid c and minimum interior angle less than $\frac{\pi}{2}$. Thus Q' is an equilateral triangle and hence $q = 5$.

Let $A_1 = conv(v_2, v_3, v_4)$ and $A_2 = conv(v_1, v_2, v_5)$ be regions such that $Q = A_1 + Q' + A_2$, see figure 8.3. Let D be a diameter through v_5. Let B_1, B_2 be the regions defined by D such that $A_1 = B_1 + B_2$. Suppose a_i and b_i denote the areas of A_i and B_i respectively. Then $a_1 = a_2$ and $b_1 = A_2 + b_2$. This is a contradiction since $a_1 = b_1 + b_2$. Hence $q = 4$. It then follows that Q is equiangular.

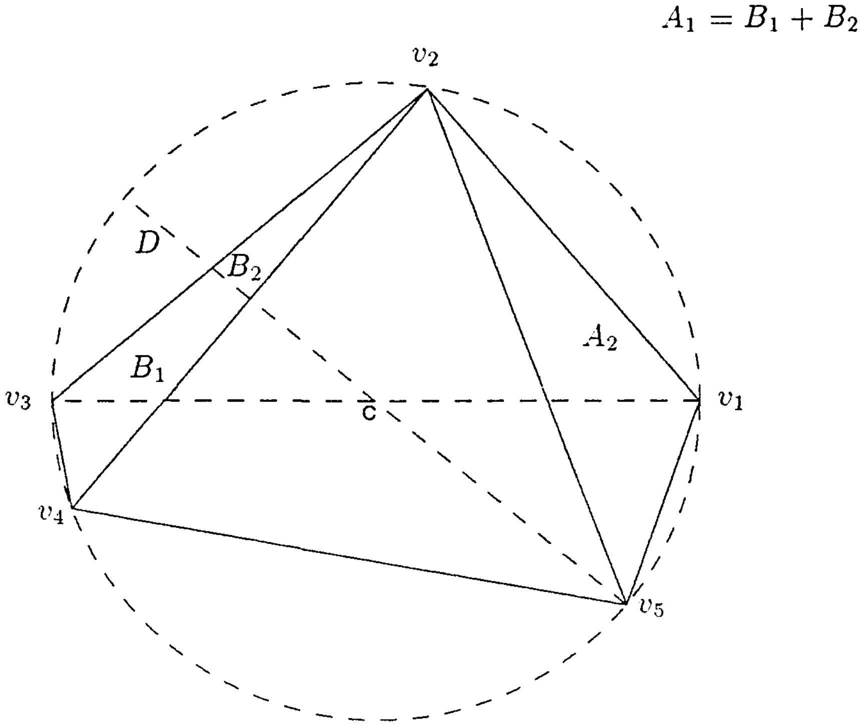


Figure 8.3

Now suppose P is a prime perfect 0-transitive $(n-1)$-transitive n-polytope. Let $val_f(v)$ be the valency of v in f where v is some vertex of f and $f \in FP$. If $f \in F_3P$ then $val_f(v) = 3$, 4 or 5 in order to preserve the interior angle sum inequality at v. In fact if $val_f(v) \neq 3$, then $v \lhd t \lhd f$, where t is some equilateral triangular face. Moreover we have the following simple lemma.

LEMMA 8:1.2

Suppose P is a prime perfect 0-transitive $(n-1)$-transitive n-polytope with vertex v such that $val_f(v) = 5$ where $f \in F_3P$. Then the edges of f containing v are all of the same length and at least four of the 2-faces of f containing v are equilateral triangles.

Proof

Suppose v has valency 5 in f. Let $\phi_1, \ldots, \phi_5$ be the interior angles of the 2-faces of f at v. Then $\phi_1 + \cdots + \phi_5 < 2\pi$ and $\phi_i = \frac{\pi}{3}$ or $\phi_i \geq \frac{\pi}{2}$. Therefore at least four of the ϕ_i are angles of $\frac{\pi}{3}$, in other words, there are four equilateral triangles at v in f. Hence all edges emanating from v in f are of equal length.

Proposition 7:1.2, suggests that the 2-faces of a 0-transitive $(n-1)$-transitive perfect n-polytope should be either regular or semiregular.

2. Perfect 4-polytopes

Let P be a 0-transitive 3-transitive prime perfect 4-polytope. Let f be a facet of P. Then $val_f(v) = 3$, 4 or 5 for any vertex v of f. The case $val_f(v) = 5$ is quite interesting as is shown by the following lemma.

LEMMA 8:2.1

Let P be a 0-transitive 3-transitive prime perfect 4-polytope. Let f be a facet with vertex v such that $val_f(v) = 5$. Then one of the 2-faces of f containing v is not a triangle

Proof

Let $val_f(v) = 5$ for some vertex v. Then there are at least four triangular 2-faces containing v in f by lemma 1.2. Suppose that all the 2-faces of f containing v are triangles. Then the 2-faces of f containing v are equilateral triangles by proposition 7:1.2. Let S be the sphere with centre s containing the vertex set $F_0 f$. Then there exists an icosahedron α such that $F_0\alpha \subset S$, $v \in F_0\alpha$ and the 2-faces of f containing v are 2-faces of α.

Let v' be a vertex adjacent to v. If $val_f(v') = 3$ then f is a pentagonal cone . This is a contradiction of proposition 7:1.2 since the centroid of such a cone does not coincide with s. If $val_f(v') = 4$ then v' is contained in either (1) two triangular faces and two quadrilateral faces or (2) three triangular faces and one m-gonal face. In (1), f is a 'lantern' with 12 vertices, 25 edges and 12 faces. There are possible three dihedral angles, e_1, e_2, or e_3, of an edge e of f depending whether e belongs to 0, 1 or 3 triangular faces respectively. See figure 8.4. These dihedral angles may be calculated. We find that $e_1 = \frac{3\pi}{5}$ and e_2, $e_3 > \frac{\pi}{3}$. This is a contradiction of the dihedral angle sum inequality at an edge contained in a triangular face. In (2), the facet contains at least two vertices of valency 5. This then leads to a contradiction to proposition 7:1.2.

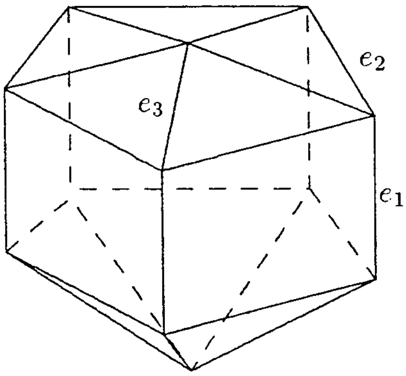


Figure 8.4

Therefore $val_f(v') = 5$ and f is an icosahedron by a similar argument. Hence P has congruent regular facets. Thus by lemma 5:1.1 P is regular. This is a contradiction as no 4-polytopes exist with only icosahedral facets.

The lantern described in the proof of this lemma is derived from a pentagonal prism $5 \square I$, for some interval I. We call a polyhedron an **M-lantern** if it is derived from an m-gonal prism in a similar fashion.

We now prove conjecture 7:1.1 for the case $n = 4$ by considering the dihedral angle sums of edges of a 4-polytope. The proof also uses Euler's relation:

$$v - e + f - h = 0,$$

where $v = f_oP$, $e = f_1P$, $f = f_2P$ and $h = f_3P$ are respectively the total number of 0-, 1-, 2- and 3-faces of P. In such a case, we say the **face vector** of P is the vector $f(P) = (v, e, f, h)$.

THEOREM 8:2.2

A prime perfect 0*-transitive* 3*-transitive* 4*-polytope is regular.*

Proof

For any facet f of P, $G(f) = F_3P$. Suppose c is the centroid of f. Since P is perfect fix_c is an 1-dimensional subspace of E^4. Thus c is the only fixed point in $\mathit{aff}(f)$ under G_c.

We consider the action of G_c on $\mathit{aff}(f)$ in terms of orbits $G_c.v_i$ of vertices v_i of f. We show that f is regular using the classification of vertex transitive polyhedra (see Robertson [1] and Robertson and Carter [1]).

First suppose that $f_i = conv(G_c.v_i)$ is of dimension 1 or 2 for some vertex v_i. In other words, v_i lies in some G_c-invariant subspace of $\mathit{aff}(f)$. Since the triangular and quadrilateral faces of f are equiangular, the facet f is of the form $M\diamond I$ or is an M-lantern, for some m-gon M and interval I. The valency $val_f(v)$ of any vertex is 3, 4 or 5. Then the vertex set F_0f lies on a sphere S containing F_0Q for some $Q = \{3,3\}$, $\{3,4\}$ or $\{3,5\}$. The cases $val_f(v) = 3$ and $val_f(v) = 5$ are contradictions by proposition 7:1.2 and lemma 2.1, respectively. Therefore $val_f(v) = 4$. Thus $f = \{3,4\}$ and hence by lemma 5:1.1, P is regular. This is a contradiction since the isotropy subgroup of the centroid of a facet f of such a polytope is irreducible on $\mathit{aff}(f)$.

Therefore $f_i = conv(G_c.v_i)$ is a polyhedron, where $v_i \in F_0f$. There are four possible cases:

1. G_c reducible on $\mathit{aff}(f)$ and $f = f_i$;

2. G_c reducible on $\mathit{aff}(f)$;

3. G_c irreducible on $\mathit{aff}(f)$ and $f = f_i$;

and 4. G_c irreducible on $\mathit{aff}(f)$.

In each case we consider dihedral angle sums and Euler's relation on the various polyhedra to show that the non-regular polyhedra give rise to contradictions. Euler's relation for a polyhedron Q, where $f(Q) = (v', e'f')$ is as follows.

$$v' - e' + f' = 2.$$

Case 1.

Suppose that G_c is reducible on $\mathit{aff}(f)$ and $f = f_i$. Then f is one of the five families of prisms, which are labelled in table 8.1 (see Robertson [1] for more details).

LABEL	DESCRIPTION
C_m	right prism on regular m-gon
D_m	anti-prism on regular m-gon
E_m	skew prism on regular m-gon
F_m	right prism on semi-regular $2m$-gon
G_m	antiprism on semi-regular $2m$-gon

Table 8.1 The five families of prisms

We first note that the dihedral angle ϕ_{ij} between faces f_i and f_j of C_m is given by

$$\phi_{ij} = \begin{cases} \frac{\pi}{2}, & \text{if } f_i \text{ or } f_j \text{ is not a quadrilateral;} \\ \frac{\pi}{m}(m-2), & \text{if } f_i \text{ and } f_j \text{ are both quadrilaterals.} \end{cases}$$

See, for example, Cundy and Rollett [1]. The dihedral angles of F_m coincide with the dihedral angles of C_{2m}.

Consider $f = C_m$. Suppose $m = 3$, so f has 6 vertices, 9 edges and 5 faces. Any edge is common to only three 2-faces (and hence three facets) as we now show. Suppose e is an edge belonging to some triangular face T of f. In any facet there are no adjacent triangular faces. Hence e is common to at least two quadrilateral faces, two of which belong to facets containing T. If e is contained in another triangular face, then the dihedral angle sum at e is 2π, which is a contradiction. Suppose therefore e is contained in another quadrilateral face f'. Then there are two facets common to f' such that the edge of f' adjacent to e is contained in two triangular faces. This is a contradiction by the above. Hence each edge of P is contained in two quadrilateral faces and one triangular face. Now if $m \geq 4$ then $\phi_{ij} \geq \frac{\pi}{2}$. Therefore each edge of P is common to (at most) 3 facets and 3 faces of P. Let e be an edge common to the 2-faces f_1, f_2 and f_3. Since f_i and f_j, $i \neq j$, are adjacent in some facet, only one of the f_i is an m-gon, $i = 1, 2$ or 3. Therefore each edge is common to two quadrilateral faces and one m-gonal face. For any $m \geq 3$, let $f(P) = (v, e, f, h)$ be the face vector of P. Suppose that the valency of a vertex v_i of P is q and x is the number of facets common with v_i. Then $(m+2)h = 2f$, $mh = e$, $2mh = xv$ and $2e = qv$. Therefore by Euler's relation we have

$$\frac{2mh}{x} - mh + \frac{(m+2)h}{2} - h = 0.$$

Therefore $x = 4$ and likewise $q = 4$ for all $m \geq 3$. Hence P is of the form $\{m\} \square s(\{m\})$, for some similarity s. If s is not the identity then P is not facet transitive. However if s is the identity P is not prime. Both cases are contradictions.

In the case $f = F_m$, any dihedral angle is at least $\frac{\pi}{2}$. Therefore there are three 2-faces and three facets common with each edge of P. A similar argument to that for C_m shows that each edge is contained in one $2m$-gonal face and two quadrilateral faces. Using the above notation we have $(2m+2)h = 2f$, $2mh = e$, $4mh = xv$ and $2e = qv$. By Euler's relation, $x = q = 4$. For each facet f, some vertex v of f is contained in three edges, at least two of which are not equal if f is a semiregular prism. Then whatever the length of the fourth edge of P containing v , P is not facet transitive. This is a contradiction.

The 3- and 4-sided faces of E_m and G_m are not equiangular. Hence $f \neq E_m$ or G_m by lemma 1.1.

Suppose $f = D_m$. Then f has $2m$ vertices, $4m$ edges and $(2m+2)$ faces. If $m = 3$ then f is a regular octahedron and $P = \{3,4,3\}$, which is a contradiction if G_c is reducible. Suppose $m \geq 4$. Since the triangular faces of f are regular, f is an Archimedean anti-prism. Thus the dihedral angles of f are

$$\theta = sec^{-1}\ \sqrt{3}\{cosec\frac{\pi}{m} + cot\frac{\pi}{m}\}$$
$$\phi = cos^{-1}\frac{1}{3}\{1 - 4cos\frac{\pi}{m}\},$$

where the dihedral angle between two triangular faces is θ. Since $\theta, \phi \geq \frac{\pi}{2}$, each edge of P is common to 3 facets. Let $f(P) = (v, e, f, h)$. Then $3e = 4mh$, $f = (m+1)h$ and $vx = 2mh$, where x is the number of facets common to v. By Euler's relation $x = 6$. Likewise the valency of v is 8. Therefore a facet f^* of P^* has 6 vertices and 8 faces. Thus $f^* = \{3,4\}$. It then follows that $P = \{3,4,3\}$. This is a contradiction.

Case 2

Suppose $f = conv(f_1 \cup \cdots \cup f_r)$ such that $v_i \notin G_c \cdot v_j$ for $i \neq j$, $i, j = 1, \ldots, r$, where f_i is a 'case 1' polyhedron . Then the axis of rotation of each f_i coincides with every other such axis. Hence each f_i is a prism on a regular m-gon or a semiregular m-gon for some fixed m. It follows that no such f exists since the triangular and quadrilateral faces of f are equiangular and the dihedral angles of f are at least those found in each f_i. Moreover in the case of $f_i = F_m$ or G_m, the valency of a vertex of f would exceed 4, which by lemma 2.1 is a contradiction.

Case 3

Therefore G_c is irreducible on *aff*(f) and f is one of the twenty-four vertex transitive non-prism polyhedra, see Robertson [1] and appendix C. These polyhedra have deficiency 0, 1 or 2. A 0-transitive polyhedron f with deficiency 0 or 1 has dihedral angles which are invariant under any deformation δ such that $f \simeq \delta(f)$. If f has deficiency 2 then $val_f(v) = 5$ for some vertex v or f has δ-invariant dihedral angles. The dihedral angles of such polyhedra can be calculated and are given in table 8.2, see also Cundy and Rollett [1].

If each dihedral angle θ of f is such that $\theta \geq \frac{2\pi}{3}$ then the polyhedra comprising of $G(f)$ cannot be 'folded' into a 4-polytope. Thus f is not a polyhedron of the form C, E, F, I, M, O, P, V or W. The polyhedra of the forms J, K and R have irregular triangular faces, since otherwise they would be icosahedra. Likewise $f = U$ has trapezium faces. Hence f is not of any of these forms by proposition 7:1.2. If f is of the form S or T then by lemma 1.2, the triangular 2-faces of f are equilateral. Therefore f is Archimedean and every dihedral angle is greater than $\frac{2\pi}{3}$. This is impossible by the above. We also note that with the exception of A and H, $\theta \geq \frac{\pi}{2}$. Hence each edge of $f \neq A$, H is common to three 2-faces and three facets of P.

If $f = A$, B, D, or G then P has congruent regular facets. Hence P is regular by lemma 5:1.1 and proposition 7:1.2. If $f = A$ then P is the 4-simplex, the 4-cocube or the 600-cell. If $f = B$, D or G then P is the 24-cell, the 4-cube or the 120-cell respectively.

We now check that f is not one of the remaining polyhedra. Let $f(P) = (v, e, f, h)$ denote the face vector of P and $q = val_P(v)$ be the valency of some vertex v.

	Name	Dihedral angles	(approx.)
A	Tetrahedron	$2sin^{-1}\sqrt{\frac{1}{3}}$	$70°32'$
B	Octahedron	$2sin^{-1}\sqrt{\frac{2}{3}}$	$109°28'$
C	Cuboctahedron	$\pi - sin^{-1}\sqrt{\frac{2}{3}}$	$125°26'$
D	Cube	$\frac{\pi}{2}$	$90°$
E	Icosahedron	$2sin^{-1}(\frac{2}{\sqrt{3}}sin\frac{3\pi}{10})$	$138°11'$
F	Icosidodecahedron		$142°37'$
G	Dodecahedron		$116°34'$
H	Truncated tetrahedron	$2sin^{-1}\sqrt{\frac{1}{3}}, 2sin^{-1}\sqrt{\frac{2}{3}}$	$70°32', 109°28'$
I		$\pi - sin^{-1}\sqrt{\frac{2}{3}}$	$125°26'$
J	†		
K	†		
L	Truncated octahedron	$2sin^{-1}\sqrt{\frac{2}{3}}, \pi - sin^{-1}\sqrt{\frac{2}{3}}$	$109°28', 125°26'$
M	Rhombicuboctahedron	$\frac{3\pi}{4}, \frac{\pi}{2} + sin^{-1}\sqrt{\frac{2}{3}}$	$135°, 144°44'$
N	Truncated cube	$\frac{\pi}{2}, \pi - sin^{-1}\sqrt{\frac{2}{3}}$	$90°, 125°26'$
O	Truncated icosahedron	$2sin^{-1}(\frac{2}{\sqrt{3}}sin\frac{3\pi}{10}), 142°37'$	$138°11', 142°37'$
P	Rhombiicosidodecahedron		$148°17', 159°6'$
Q	Truncated dodecahedron		$116°34', 142°37'$

† These polyhedra have irregular triangular faces.

Table 8.2 The vertex transitive polyhedra

	Name	Dihedral angles	(approx.)
R	†		
S	Snub cube	**	$142°59', 153°14'$
T	Snub dodecahedron	**	$152°56', 164°11'$
U	‡		
V	Rhombitruncated cuboctahedron	as I and M	each $> \frac{\pi}{3}$
W	Rhombitruncated icosidodecahedron		each $> \frac{\pi}{3}$
X		$\frac{\pi}{2} + sin^{-1}\sqrt{\frac{1}{3}}, \pi - 2sin^{-1}\sqrt{\frac{1}{3}}$	$125°16', 109°28'$

** Dihedral angles given when polyhedra are Archimedean.

† These polyhedra have irregular triangular faces.

‡ These polyhedra have trapezium faces.

Table 8.2 (continued)

(i) $f = H$ The truncated tetrahedron : $f(H) = (12, 18, 8)$

The dihedral angles are $\theta = sin^{-1}\sqrt{\frac{1}{\ }}$ and $\phi = 2sin^{-1}\sqrt{\frac{2}{3}}$. We note that $\theta + \phi = \pi$. Thus any edge is common to three faces and three facets of P. Then $18h = 3e$, $16h = 2f$ and $qv = 2e$. Then by Euler's relation we have

$$\begin{aligned} v - e + f - h &= \frac{36h}{3q} - 6h + 8h - h \\ &= \frac{12h}{q} + h = 0. \end{aligned}$$

Hence $q < 0$ which is a contradiction.

(ii) $f = L$ The truncated octahedron: $f(L) = (24, 36, 14)$

Each edge is common with three 2-faces and three facets. Thus $qv = 2e$,

$3e = 36h$ and $2f = 14h$. Then by Euler's relation we have

$$\frac{2e}{q} - 12h + 7h - h = \frac{24h}{q} - 5h = 0.$$

Therefore $q = \frac{24}{5}$ which is impossible.

(iii) $f = N$ The truncated cube: $f(N) = (24, 36, 14)$ Each edge is common to three 2-faces and three facets. Thus $qv = 2e$, $3e = 36h$ and $2f = 14h$. Therefore by Euler's relation we have

$$\frac{24h}{q} - 12h + 7h - h = 0$$

$$\text{and so} \qquad q = 4.$$

Hence v is contained in four facets. Consider the 0-transitive 3-transitive prime perfect 4-polytope P^*. A facet f^* of P^* only has four vertices so f^* is a tetrahedron. Therefore f^* is of the form A and hence P^* is regular. Thus $P = \{4, 3, 3\}$ or $\{3, 3, 3\}$, which is a contradiction.

(iv) $f = Q$ The truncated dodecahedron: $f(Q) = (60, 90, 32)$

The sum of any three dihedral angles of Q exceeds 2π which is impossible.

(v) $f = X$: $f(X) = (24, 36, 14)$

Each edge is common to three 2-faces and three facets. Thus $qv = 2e$, $3e = 36h$ and $2f = 14$. Thus we get a contradiction as in the case $f = N$.

Thus there are no 0-transitive 3-transitive prime perfect 4-polytopes whose facets are vertex transitive with the exception of the regular 4-polytopes.

Case 4

Suppose that f is not vertex transitive. If $conv(f_i \cup f_j) \subseteq f$ then the stratifications of the symmetry groups (or subgroups) of f_i and f_j coincide. Therefore we check that f is not the convex hull of two polyhedra with symmetry groups G_i, G_j such that $G_i \subseteq G_j$. Recall from case 3 that $f_i \neq C$, E, F, I, J, K,

M, O, P, R, S, T, U, V or W. Also note that $val_f(v) = 3$ or 4 for any vertex v of f. Thus if $val_{f_i}(v) = 4$ for some f_i then this would lead to a contradiction in $val_f(v)$ if new edges were introduced at v apon taking convex hulls.

$f = conv(A \cup A^*)$

Such a polyhedron has $val_f(v) = 6$ for some vertex or f is a cube. In the latter case $P = \{4, 3, 3\}$ which is a contradiction since the facets of P are vertex transitive.

$f = conv(A \cup B)$

This is possible only if for each vertex v of B, there exists a midpoint ϵ of an edge of A such that v, ϵ and the centroid of f are collinear. In such a case the resulting polyhedron has a vertex (coinciding with a vertex of A) of valency 6.

$f = conv(A \cup X)$

Such a polyhedron has irregular triangular faces or trapezium faces which contradicts lemma 1.1.

$f = conv(B \cup D)$

Such a polyhedron is either a cuboctahedron which has already been discussed or is given in figure 6.2. In the latter case, the triangular faces are regular, therefore the sphere containing $F_0 f$ also contains the vertices of an octahedron B. This is a contradiction.

$f = conv(B \cup L)$

Then B coincides with the octahedron which is truncated to form L. Thus $f = B$, which by case 3 is a contradiction.

$f = conv(B \cup N)$

Such a polyhedron has a vertex of valency 8, which is a contradiction.

$f = conv(D \cup L)$

Such a polyhedron has a vertex of valency 6, which is a contradiction.

$f = conv(D \cup N)$

Since the G_c-stratifications of $aff(f)$ coincide, this union results in a cube D or a truncated cube N.

$f = conv(L \cup N)$

Such a polyhedron has a vertex of valency 5, which is a contradiction.

The case where $f_i = G$ is impossible since the polyhedra with the same symmetry group as G have already been excluded.

Thus a 0-transitive 3-transitive prime perfect 4-polytope does not have a facet of the form $f = conv(f_i \cup f_j)$. Since such a facet has a vertex of valency four we conclude that f is vertex transitive since otherwise $val_f(v) \geq 5$ for some vertex v of f.

Thus we have shown that no non-regular 0-transitve 3-transitive prime perfect 4-polytope exists.

COROLLARY 8:2.3

The prime perfect 4-polytopes are given by the Wythoffian 4-polytopes and their polars.

Proof

Follows from theorem 2.2 and section 7:3.

3. Perfect 4-solids

With the classification of perfect 4-polytopes complete, we turn our attention to the perfect 4-solids. Let G be the symmetry group of a perfect non-polytope 4-solid. Then $G \subseteq O(4)$ and $dim(G) \geq 1$. There are three known perfect 4-solids which are not polytopes, the 4-ball D^4, the product of the disk with itself $D^2 \square D^2$, and its polar $D^2 \diamond D^2$. By considering the compact subgroups of O(4), we show that $G = \mathrm{O}(2) \wr Z_2$ or $G = \mathrm{O}(4)$. In other words, G is the symmetry group of $D^2 \square D^2$ or D^4.

First suppose that G is not the direct product of compact subgroups of $O(4)$ or a wreath product in O(4). If $dim(G) < 4$ then there exists a linear subspace C, $dim(C) = 4 - dim(G)$, held pointwise fixed by G. Thus $fix_B = C$ which contradicts theorem 1:7.5. If $dim(G) \geq 4$, then $G = O(4)$ or $G = \mathrm{SO}(4)$ (which has the same orbits as O(4)). The only 4-solid with such symmetry is D^4.

Now suppose G is the direct product $J \times K$ of the compact subgroups J, K of O(4). Then the inclusion $\pi : G \to \mathrm{O}(4)$ is a reducible representation. This is a contradiction to proposition 4:1.1. In any case, there is a decomposition of B, for instance, $B = B_1 \square B_2$, where $B_1 \neq B_2$, such that $G = GB_1 \times GB_2$. This is a contradiction to theorem 1:7.4.

Suppose now that $G = GK \wr S_r$, $r > 1$, where K is a $\square$-prime i-solid, $i < 4$. Then $B = \square^r K$ and $ri = 4$. Thus $(i, r) = (1, 4)$ or $(2, 2)$. If $i = 1$, then B is a polytope and $dim(G) = 0$, which is a contradiction. Hence $G = GK \wr S_2 = GK \wr Z_2$ and K is a perfect 2-solid. In other words $G = \mathrm{O}(2) \wr Z_2$ and $K = D^2$.

We now check that there are no perfect solids whose symmetry group is O(4) or $\mathrm{O}(2) \wr Z_2$ other than those mentioned above. The fundamental region of the action of O(4) on S^3 is a radial line. Hence D^4 is the only solid with symmetry group $O(4)$. Now consider the action of $G = \mathrm{O}(2) \wr Z_2$ on S^3. The fundamental region of this action may be given as follows. Let E^4 be identified with the orthogonal product $E^2 \times E^2$. Then $C_1 = E^2 \times 0$ and $C_2 = 0 \times E^2$ are two non-intersecting linked great circles of S^3. Let v_i be any point on C_i, $i = 1, 2$. Then there exists a great circle C of S^3 through v_i, $i = 1, 2$, which intersects C_1

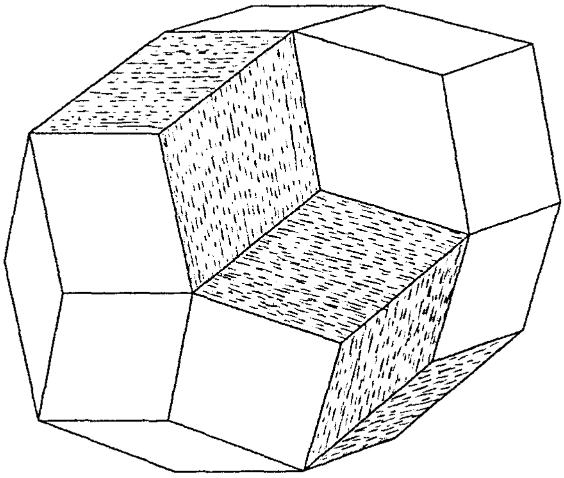
and C_2 each in one further point. Then C is divided into four equal arcs. If one of these arc is subdivided equally into two, then the result is an arc A subtended by an angle of $\frac{\pi}{4}$ at O. Suppose that the endpoints of A are x and y, such that $x \in C_i$, $i = 1$ or 2. Then the convex hull of A and O is a fundamental region D of the action of G such that $conv(G.x) = D^2 \diamond D^2$ and $conv(G.y) = D^2 \square D^2$. We may assume that a point v on A is given by $e^{i\theta}$, $0 \leq \theta \leq \frac{\pi}{4}$, where $v = x$ if $\theta = 0$ and $v = y$ if $\theta = \frac{\pi}{4}$. Then there is a one-parameter family of face equivalent 4-solids given by the convex hull of the action of G on a point v on A. These 4-solids are face equivalent since $dim(fix_v) = 4$ and v may be mapped along a D-path by a deformation in a similar fashion to that in chapter 5. The solids in this family are not perfect as v may be deformed by changing θ. Similarly a solid given by the convex hull of the action of G on a number of points of D is not perfect. Thus the only perfect 4-solids are those already mentioned.

APPENDIX A: A second rhombic dodecahedron

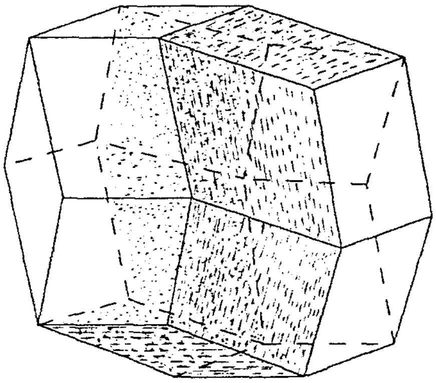
As noted in Coxeter [1], the rhombic dodecahedron of the second kind was discovered by Bilinski [1] in 1960. This polyhedron can be derived from a rhombic triacontahedron (see figure A:1) in the following way.

Recall (Coxeter [1]) that a **zone** of faces of a polyhedron P with parallelogram faces is a collection of all the faces which have two sides equal and parallel to some given edge e. Such a P is called a **zonahedron**. Thus the edges of a zone are m edges parallel to e (including e) and $(m-1)$ pair-wise parallel edges, for some m. The removal of any zone from ∂P results in two pieces of surface and the loss of the m parallel edges. These can be brought together by the identification of two parallel edges e_1 and e_2, one from each piece of surface ($e_1,\ e_2 \triangleleft f$ for some $f \in F_2P$). This gives an identification of the remaining $(m-2)$ pair-wise parallel edges from the zone. The result is a zonahedron with m less faces. If P is a rhombic triacontahedron, then $m = 10$. The resulting surface is the surface of a rhombic icosahedron, $Z(P)$ say. The surface of the rhombic dodecahedron of the second kind $Z^2(P)$ is given by repeating this process on $Z(P)$. The faces of this polyhedron have the same shape as those of P.

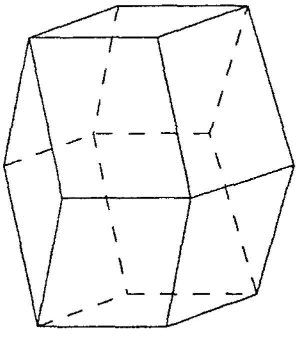
We can see that $Z^2(P)$ is not a perfect polyhedron by considering the interior angles of the faces at each vertex. See figure A:2 where a different perspective is given and the interior angles at a vertex are given by α and β. Moreover, the symmetry group G of $Z^2(P)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (the symmetry group of a 'matchbox-shaped' cuboid). The orbit vector of $Z^2(P)$ is $(4, 4, 3)$. Hence by Rostami [1], $Z^2(P)$ has deficiency 3.



a: A zone of a rhombic triacontrahedron



b: A zone of a rhombic icosahedron (derived from **a**)



c: A rhombic dodecahedron (derived from **b**)

Figure A:1 a $\rightarrow$ c

Construction of Bilinski's rhombic dodecahedron by removal of zones

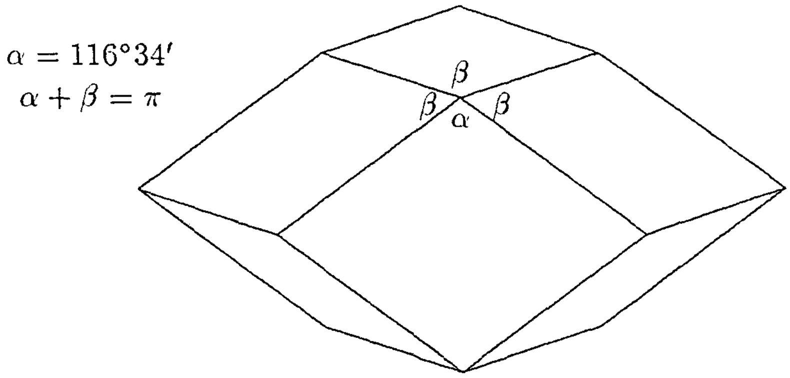


Figure A.2 The rhombic dodecahedron of the second kind

APPENDIX B

1. The root system $\mathbf{BC_q}$

The notation BC_q for a root system of rank q is used by Loos [1]. In terms of graphs, the Dynkin diagrams of BC_q, B_q and C_q are all the same. However BC_q is a non-reduced root system given by $B_q \cup C_q$. (Recall Loos [1] that a root system R is reduced if $\alpha \in R$ and $c\alpha \in R$ then $c = \pm 1$.) If $e_1, \ldots, e_q$ is the usual basis of E^q and $\epsilon_1, \ldots, \epsilon_q$ is the dual basis to $(E^q)^*$, then

$$BC_q \simeq \{\pm\epsilon_i, \pm 2\epsilon_i, \pm\epsilon_i \pm \epsilon_j : i \neq j\}.$$

2. Symmetric space isomorphisms

The restrictions on l on the classical root systems A_l, B_l, C_l and D_l give rise to some overlaps in table 3:1 for small n. These may be given in terms of the following isomorphisms, which appear in Helgason [1].

(i) **AI** $(n = 2)$ = **AIII** $(p = q = 1)$ = **BDI** $(p = 2, q = 1)$ = **CI** $(n = 1)$.

Corresponding isomorphisms:

$$\mathfrak{su}(2) \approx \mathfrak{so}(4) = \mathfrak{sp}(1),$$
$$\mathfrak{sl}(2, \mathbb{R}) \approx \mathfrak{su}(1,1) \approx \mathfrak{so}(2,1) \approx \mathfrak{sp}(1, \mathbb{R}).$$

(ii) **BDI** $(p = 3, q = 2)$ = **CI** $(n = 2)$.

Corresponding isomorphisms:

$$\mathfrak{so}(5) \approx \mathfrak{sp}(2),$$
$$\mathfrak{so}(3,2) \approx \mathfrak{sp}(2, \mathbb{R}).$$

(iii) **BDI** $(p = 4, q = 1)$ = **CII** $(p = q = 1)$.

Corresponding isomorphisms:

$$\mathfrak{so}(5) \approx \mathfrak{sp}(2), \quad \mathfrak{so}(4) \approx \mathfrak{sp}(1)\times \mathfrak{sp}(1),$$
$$\mathfrak{so}(4,1) \approx \mathfrak{sp}(1,1)$$

(iv) **AI** $(n = 4)$ = **BDI** $(p = q = 3)$.

Corresponding isomorphisms:

$$\mathfrak{su}(4) \approx \mathfrak{so}(6), \quad \mathfrak{so}(4) \approx \mathfrak{so}(3)\times \mathfrak{so}(3),$$
$$\mathfrak{sl}(4, \mathbb{R}) \approx \mathfrak{so}(3,3).$$

(v) **AII** ($n = 2$) = **BDI** ($p = 5, q = 1$).

Corresponding isomorphisms:

$$\mathfrak{su}(4) \approx \mathfrak{so}(6), \quad \mathfrak{sp}(2) \approx \mathfrak{so}(5),$$
$$\mathfrak{su}^*(4) \approx \mathfrak{so}(5,1).$$

(vi) **AIII** ($p = q = 3$) = **BDI** ($p = 4, q = 2$).

Corresponding isomorphisms:

$$\mathfrak{su}(4) \approx \mathfrak{so}(6),$$
$$\mathfrak{su}(2,2) \approx \mathfrak{so}(4,2).$$

(vii) **AIII** ($p = 3, q = 1$) = **DIII** ($n = 3$).

Corresponding isomorphisms:

$$\mathfrak{su}(4) \approx \mathfrak{so}(6),$$
$$\mathfrak{su}(3,1) \approx \mathfrak{so}^*(6).$$

(viii) **BDI** ($p = 6, q = 2$) = **DIII** ($n = 4$).

Corresponding isomorphisms:

$$\mathfrak{su}(4) \approx \mathfrak{so}(6),$$
$$\mathfrak{so}^*(8) \approx \mathfrak{so}(6,2).$$

(ix) **BDI** ($p = 3, q = 1$) = $\mathfrak{a}_n$ ($n = 1$).

Corresponding isomorphisms:

$$\mathfrak{so}(4) \approx \mathfrak{su}(2) \times \mathfrak{su}(2),$$
$$\mathfrak{so}(3,1) \approx \mathfrak{sl}(2, \mathbb{C}).$$

(x) **BDI** ($p = 2, q = 2$) = **AI** ($n = 2$)$\times$ **AI** ($n = 2$).

Corresponding isomorphisms:

$$\mathfrak{so}(4) \approx \mathfrak{su}(2) \times \mathfrak{su}(2),$$
$$\mathfrak{so}(2,2) \approx \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}).$$

(xi) **DIII** ($n = 2$) = **AI** ($n = 2$).

Corresponding isomorphisms:

$$\mathfrak{so}(4) \approx \mathfrak{su}(2) \times \mathfrak{su}(2),$$
$$\mathfrak{so}^*(4) \approx \mathfrak{su}(2) \times \mathfrak{sl}(2, \mathbb{R})$$

APPENDIX C: Vertex transitive polyhedra

In this appendix we give some information about the vertex transitive polyhedra classified in Robertson and Carter [1] (see also Robertson [1]). The non-prism 0-transitive polyhedra are given in table 8.2.

It is useful to note the symmetry groups of such polyhedra. We follow the notation of Robertson [1]. Suppose P is a 0-transitive polyhedron with symmetry group $G = \Gamma(P)$. Then let $\Gamma_+(P)$ be the subgroup of $\Gamma(P)$ consisting of all rotations of $\Gamma(P)$. Let $\Gamma_*(P)$ denote the subgroup $\langle j, \Gamma_+(P)\rangle$ where j is the reflection in O given by the matrix $-I_3 = diag\{-1,-1,-1\}$. The symmetry groups of the polyhedra $A \to X$ are given in table C:1

Symmetry group	Polyhedra
$\Gamma(A)$	A, H, I, X
$\Gamma_*(A)$	J, K
$\Gamma_+(A)$	R
$\Gamma(B)$	B, C, D, L, M, N,V
$\Gamma_*(B)$	U
$\Gamma_+(B)$	S
$\Gamma(E)$	E, F, G, O, P, Q, W
$\Gamma_+(E)$	T

Table C:1

The polyhedra $A \to G$ have deficiency 0. Examples of these polyhedra can be found in chapter 1. The polyhedra $H \to Q$ have deficiency 1, while the rest have deficiency 2. Examples of the polyhedra of deficiency 1 or 2 are given in figures C:1 to C:17. Two interesting polyhedra are those of type J and K, which may be derived as follows. Let D be a cube. Then it is possible to embed an icosahedron I in D such that the midpoints of six edges $e_1, \ldots, e_6$ of I coincide with the centroids of the faces of D. Then I $= conv(e_1, \ldots, e_6)$. Now let the length of e_i, $i = 1, \ldots, 6$, be increased in size by an amount to a new edge e_i' such that the midpoints of e_i and e_i' coincide such that the new edges remain in the relative interior of D. Then the convex hull of these edges is K. If the length of e_i is decreased in a similar fashion, the resulting polyhedron is J. At each vertex of J and K there are exactly two non-adjacent regular triangular faces.

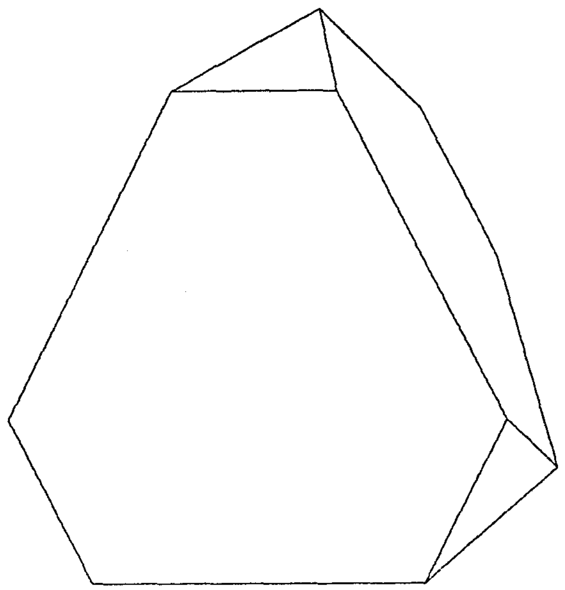


Figure C:1 **H**

The truncated tetrahedron

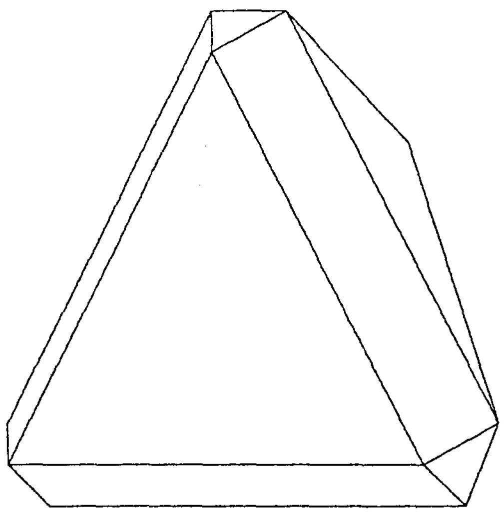


Figure C:2 **I**

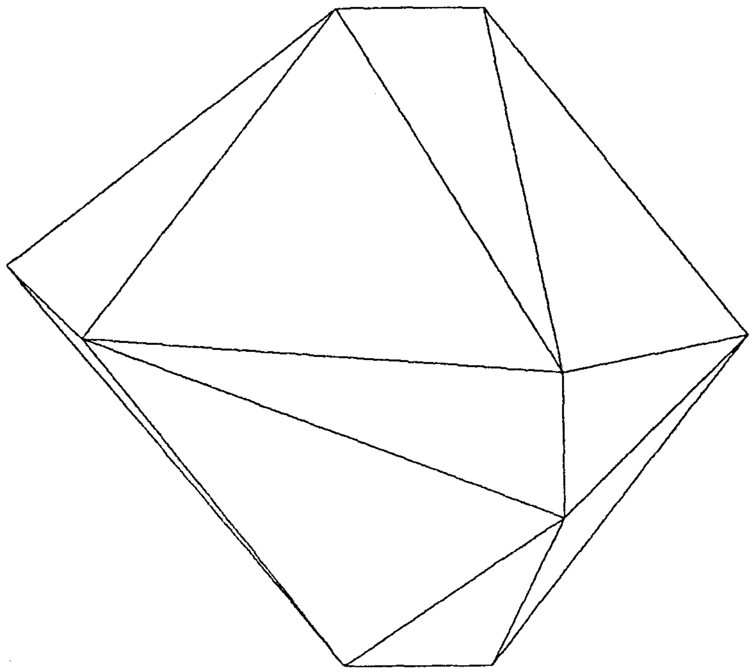


Figure C:3 J

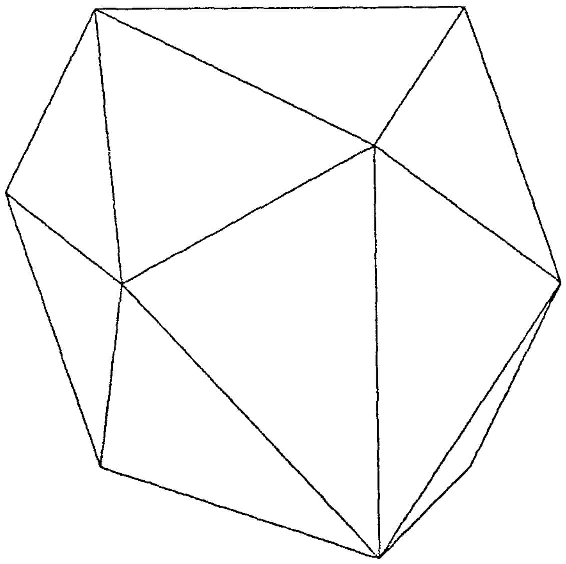


Figure C:4 K

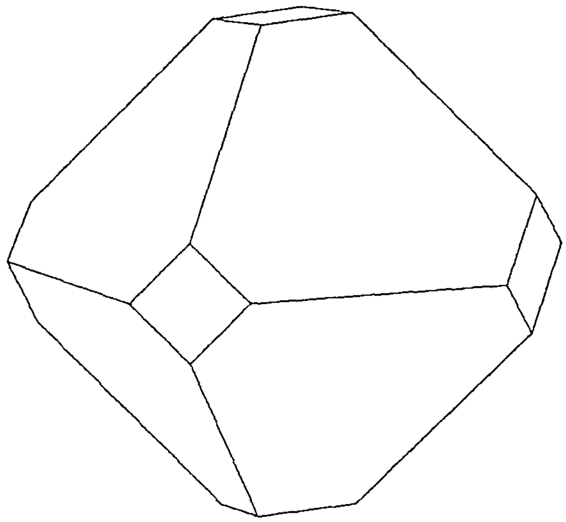


Figure C:5 L

The truncated octahedron

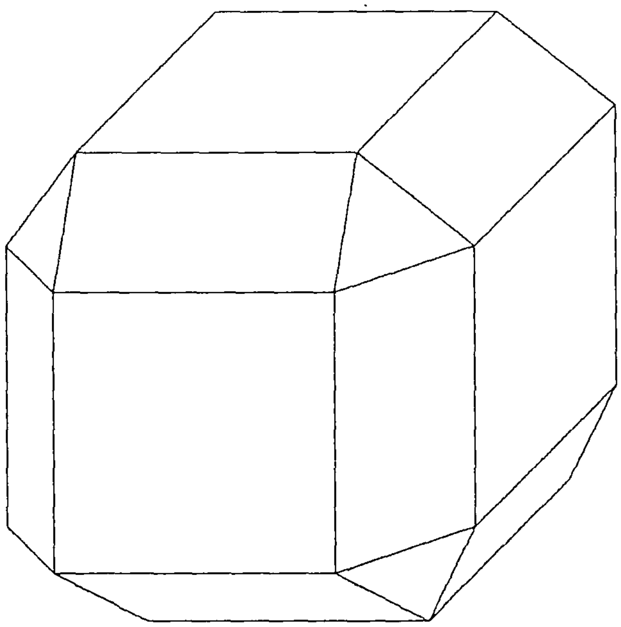


Figure C:6 M

The rhombicuboctahedron

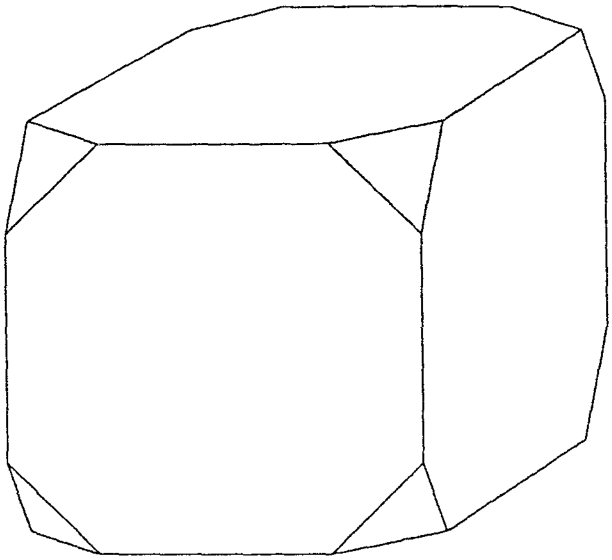


Figure C:7 **N**

The truncated cube

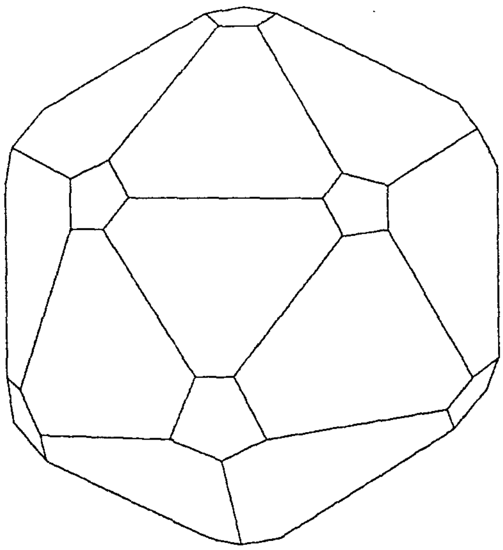


Figure C:8 **O**

The truncated icosahedron

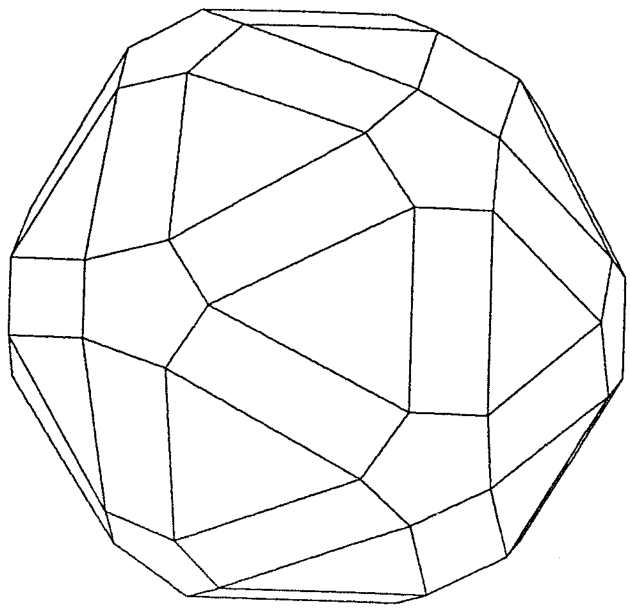


Figure C:9 **P**

The rhombicosidodecahedron

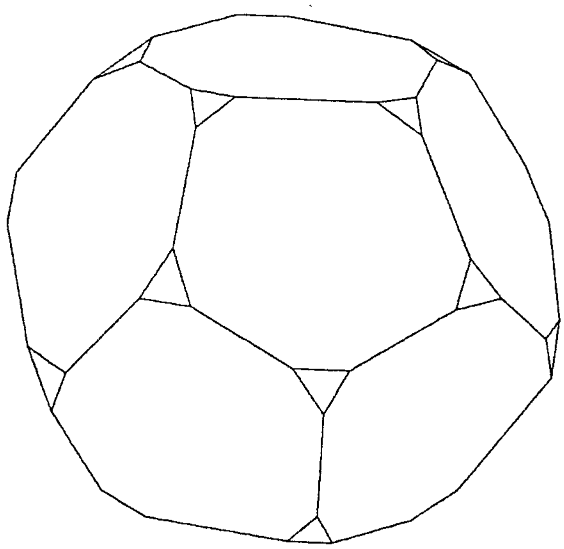


Figure C:10 **Q**

The truncated dodecahedron

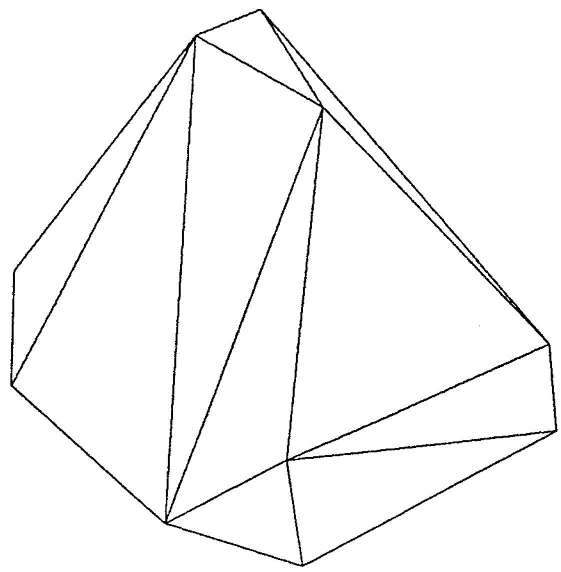


Figure C:11 **R**

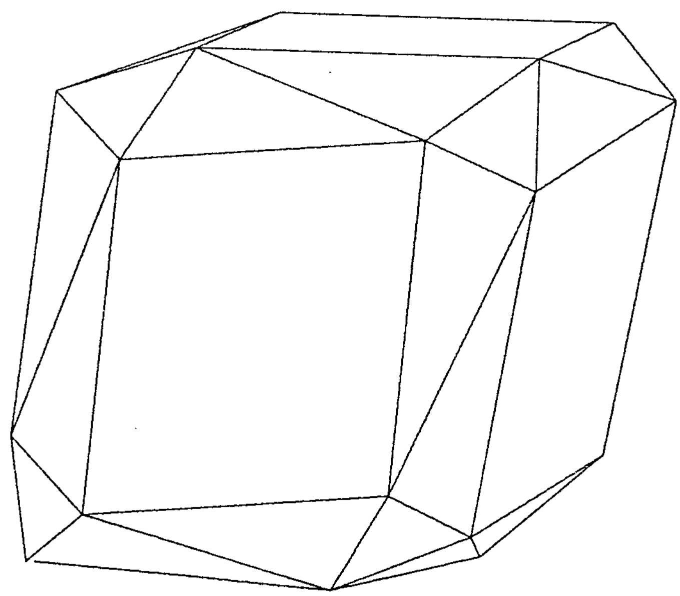


Figure C:12 **S**

The snub cube

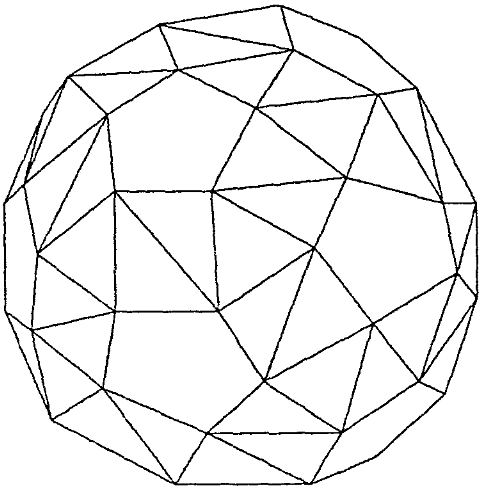


Figure C:13 **T**

The snub dodecahedron

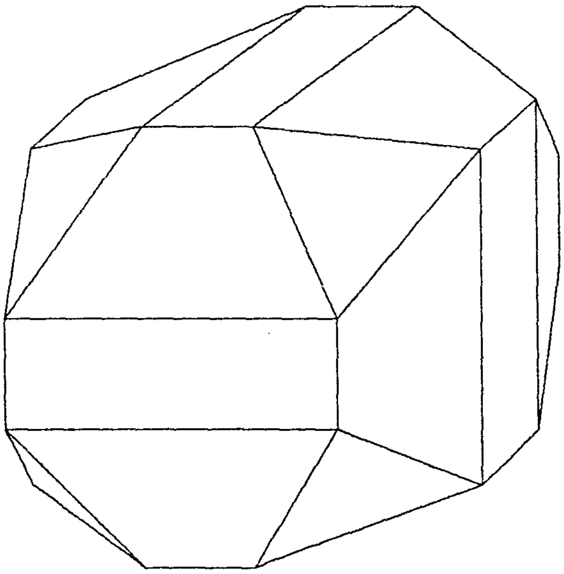


Figure C:14 **U**

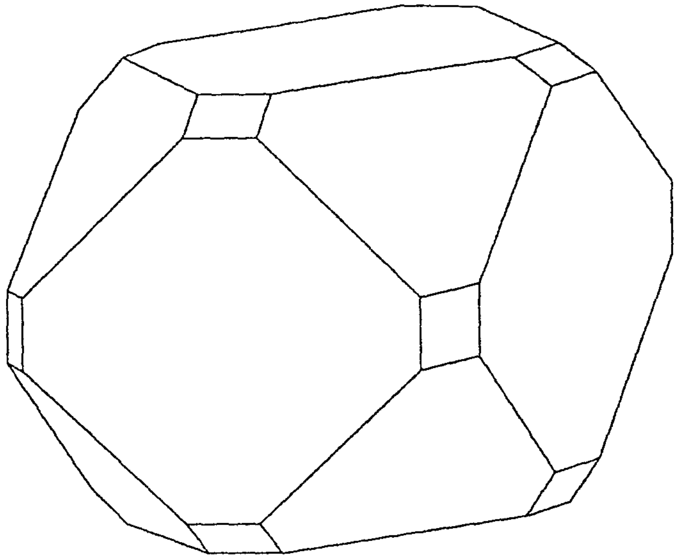


Figure C:15 V

The rhombitruncated cuboctahedron

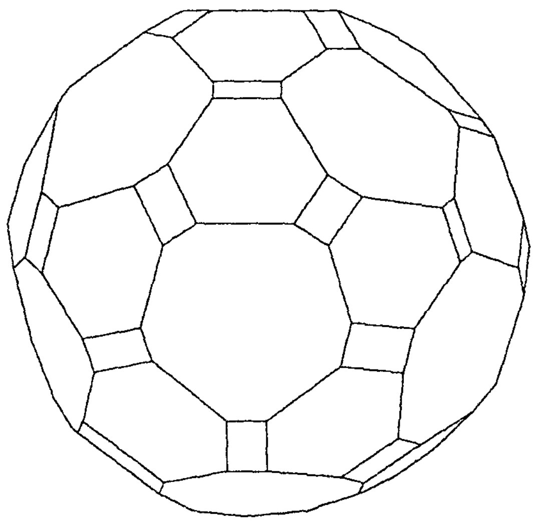


Figure C:16 W

The Rhombitruncated icosidodecahedron

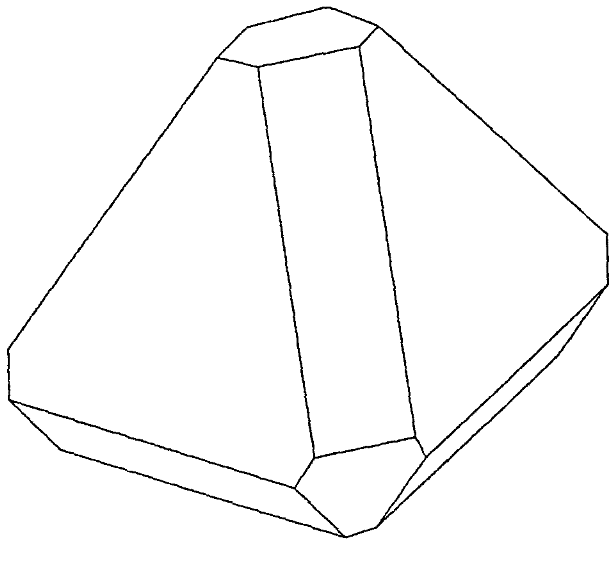


Figure C:17 X

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