Small Area Estimation for Spatially Correlated Populations – A Comparison of Direct and Indirect Model-Based Methods

Hukum Chandra, Nicola Salvati, Ray Chambers

Abstract

Linear mixed models underpin many small area estimation (SAE) methods. In this paper we investigate SAE based on linear models with spatially correlated small area effects where the neighbourhood structure is described by a contiguity matrix. Such models allow efficient use of spatial auxiliary information in SAE. In particular, we use simulation studies to compare the performances of model-based direct estimation (MBDE) and empirical best linear unbiased prediction (EBLUP) under such models. These simulations are based on theoretically generated populations as well as data obtained from two real populations (the ISTAT farm structure survey in Tuscany and the US Environmental Monitoring and Assessment Program survey). Our empirical results show only marginal gains when spatial dependence between areas is incorporated into the SAE model.

Southampton Statistical Sciences Research Institute
Methodology Working Paper M07/09

Small Area Estimation for Spatially Correlated Populations - A Comparison of Direct and Indirect Model-Based Methods

Hukum Chandra[1], Nicola Salvati[2] and Ray Chambers[3]

1. Southampton Statistical Sciences Research Institute
University of Southampton
Highfield, Southampton, SO17 1BJ, UK
Email: hchandra@soton.ac.uk

2. Dipartimento di Statistica e Matematica Applicata all'Economia
University of Pisa, Pisa, Italy
Email: salvati@ec.unipi.it

3. Centre for Statistical and Survey Methodology
University of Wollongong
Wollongong, NSW, 2522, Australia
Email: ray@uow.edu.au

Summary

Linear mixed models underpin many small area estimation (SAE) methods. In this paper we investigate SAE based on linear models with spatially correlated small area effects where the neighbourhood structure is described by a contiguity matrix. Such models allow efficient use of spatial auxiliary information in SAE. In particular, we use simulation studies to compare the performances of model-based direct estimation (MBDE) and empirical best linear unbiased prediction (EBLUP) under such models. These simulations are based on theoretically generated populations as well as data obtained from two real populations (the ISTAT farm structure survey in Tuscany and the US Environmental Monitoring and Assessment Program survey). Our empirical results show only marginal gains when spatial dependence between areas is incorporated into the SAE model.

Key words: Empirical Best Linear Unbiased Prediction, Spatial Models, Spatial EBLUP, Model-Based Direct Estimation

1. Introduction

Estimation of population characteristics for sub-national domains (or smaller regions) is an important objective for statistical surveys. In particular, geographically defined domains, e.g. regions, states, counties, wards and metropolitan areas can be of interest. Estimates for these domains based on the usual design-based approach to survey sampling inference are typically referred to as direct estimates in the literature. However, sample sizes are typically small (or even zero) within the domains/areas of interest, leading to large sampling variability for these direct estimators. An alternative approach that is now widely used in small area estimation is the so-called indirect or model-based approach. This uses auxiliary information for the small areas of interest and has been characterised in the statistical literature as 'borrowing strength' from the relationship between the target variables and the auxiliary information. A flexible and popular way of borrowing strength is based on the use of linear mixed models with area specific random effects, with estimation and inferences typically carried out using empirical best linear unbiased prediction (EBLUP - see Rao, 2003). An alternative approach, discussed in Chandra and Chambers (2005), is based on the use of model-based direct estimation (MBDE) within the small areas. In this case an estimate for a small area of interest corresponds to a weighted linear combination of the sample data for that area, with weights based on a population level version of the linear mixed model. These weights 'borrow strength' via this model, which includes random area effects. Provided the assumed small area model is true, the EBLUP is asymptotically the most efficient estimator for a particular small area. In practice however the 'true' model for the data is unknown and the EBLUP can be inefficient under misspecification. In such circumstances, Chandra and Chambers (2005) note that MBDE offers an alternative to potentially unstable EBLUP. In particular, MBDE is easy to implement, produces sensible estimates when the sample data exhibit patterns of variability that are inconsistent with the assumed model (e.g. contain too many zeros) and generates robust MSE estimates.

Model-based methods of small area estimation (SAE) are often based on assuming a linear mixed model, with area-specific random effects to account for between area variation beyond that explained by auxiliary variables included in the fixed part of the model. Although it is customary to assume that these random area effects are independent, in practice most small area boundaries are arbitrary and there appears to be no good reason why population units just one side of such a boundary should not generally be correlated with population units just on the other side. In particular, it is often reasonable to assume that the effects of neighbouring areas (defined, for example, by a contiguity criterion) are correlated, with the correlation decaying to zero as the distance between these areas increases (Pratesi and Salvati, 2005). That is, small area models should allow for spatial correlation of area random effects. See Cressie (1991).

In this paper we consider linear unit level small area models (Battese *et al.*, 1988) and we extend MBDE and EBLUP for SAE to account for spatial correlation between the small areas. We then contrast the performance of these two approaches via empirical studies. Our aim in doing so is to explore how much efficiency is gained by incorporating spatial correlation into SAE. The paper is organized as follows: In section 2 we review MBDE and EBLUP for SAE under random effect models with spatially independent area effects and discuss the extensions of these techniques to account for spatial dependence between the small areas. We define the resulting estimators for the small area means and their mean squared error estimators. In Section 3 we describe the design of our simulation studies and present empirical results. Besides using simulated population and sample data, we use two real data sets from the ISTAT farm structure survey (farm data) in Northern Tuscany and the Environmental Monitoring and Assessment Program (EMAP) survey of lakes in the north-east of the USA. Finally, in section 4 we provide some concluding remarks and identify further avenues of research.

2. Small Area Estimation under Linear Models with Random Area Effects

2.1 Models with Spatially Independent Effects

To begin, let y_{ij} denote the value of the variable of interest for the j^{th} ($j = 1,\ldots,N_i$) unit in small area *i* ($i = 1,\ldots,m$) and let X_{ij} denote the vector of values of the *p* unit level auxiliary variables associated with this unit. We consider a nested error regression model of form

$$y_{ij} = X_{ij}^t\beta + u_i + e_{ij} \qquad (1)$$

where β is a vector of *p* unknown fixed effects, u_i is the random area effect associated with small area *i*, assumed to have mean zero and variance σ_u^2, and e_{ij} is an individual level random error with mean zero and variance σ_e^2. The two error terms are assumed to be mutually independent, both across individuals as well as across areas. In addition, it is often assumed that they are normally distributed. In matrix notation, (1) is expressed as

$$Y_i = X_i\beta + u_i 1_{N_i} + e_i \qquad (2)$$

where $Y_i = (y_{i1},\ldots.,y_{iN_i})^t$, $X_i = (X_{i1},\ldots,X_{iN_i})^t$ is a $N_i \times p$ matrix and $e_i = (e_{i1},\ldots,e_{iN_i})^t$. Here N_i is the number of population units in small area *i*. The covariance matrix of Y_i is $Var(Y_i) = V_i = \sigma_e^2 I_{N_i} + \sigma_u^2 1_{N_i} 1_{N_i}^t$, which depends on the vector $\theta = (\sigma_u^2, \sigma_e^2)^t$ of variance components of the model. Here 1_{N_i} is the unit vector of length N_i and I_{N_i} is the identity matrix of order N_i. Assuming (2) holds, the population mean of Y in area i is $\bar{Y}_i = \bar{X}_i\beta + u_i + \bar{e}_i$, where $\bar{X}_i = N_i^{-1}\sum_{j=1}^{N_i} x_j$ is assumed known.

Grouping the area-specific models (2) over the population leads to the population level model

$$Y = X\beta + Zu + e \tag{3}$$

where $Y = (Y_1^t, \dots, Y_m^t)^t$, $X = (X_1^t, ..., X_m^t)^t$, $Z = diag(Z_i = 1_{N_i}; 1 \le i \le m)$, $u = (u_1, \dots, u_m)^t$ and $e = (e_1^t, \dots, e_m^t)^t$. Since different areas are independent, the covariance matrix of Y has block diagonal structure given by $V = diag(V_i; 1 \le i \le m)$. We assume that X has full column rank p. In practice the variance components that define V are unknown and can be estimated from the sample data using methods described, for example, in Harville (1977). We denote these estimates by $\hat{\theta} = (\hat{\sigma}_u^2, \hat{\sigma}_e^2)^t$ and put a 'hat' on any quantity where these estimates are substituted for actual values. Thus $\hat{V} = diag(\hat{V}_i; 1 \le i \le m)$, with $\hat{V}_i = \hat{\sigma}_e^2 I_{N_i} + \hat{\sigma}_u^2 Z_i Z_i^t$.

Now consider the decomposition of Y, X, Z and V into sample and non-sample components so that X_s is the $n \times p$ matrix of sample values of the auxiliary variables, Z_s is the corresponding $n \times m$ matrix of sample components of Z and V_{ss} is the $n \times n$ covariance matrix associated with the n sample units that make up the $n \times 1$ sample vector Y_s. A subscript of r is used to denote corresponding quantities defined by the $N - n$ non-sample units, with V_{rs} denoting the $(N - n) \times n$ matrix defined by $Cov(Y_r, Y_s)$. In what follows we use 1_N, 1_s and 1_r to denote vectors of 1s of dimension N, n and $N - n$ respectively, with I_N, I_s and I_r denoting identity matrices of the same order. We use similar notation to denote restriction to small area level by introducing an extra subscript of i to denote the small area. For example, s_i corresponds to the set of n_i sample units in area i, r_i the corresponding set of $N_i - n_i$ non-sampled units, with associated variances and covariances $V_{iss} = \sigma_e^2 I_{is} + \sigma_u^2 Z_{is} Z_{is}^t$ and $V_{isr} = \sigma_u^2 Z_{is} Z_{ir}^t$.

Assuming (3) holds, the empirical best linear unbiased predictor (EBLUP) for the i^{th} small area mean $\bar{Y}_i$ is

$$\hat{\bar{Y}}_{i,EBLUP} = f_i \bar{Y}_{is} + (1 - f_i)\left\{ \bar{X}_{ir}^t \hat{\beta} + \hat{\gamma}_i (\bar{Y}_{is} - \bar{X}_{is}^t \hat{\beta}) \right\} \tag{4}$$

where $\hat{\beta} = \left(\sum_i X_{is}^t \hat{V}_{iss}^{-1} X_{is} \right)^{-1} \left(\sum_i X_{is}^t \hat{V}_{iss}^{-1} Y_{is} \right)$ is the empirical best linear unbiased estimator (EBLUE) of β, $f_i = N_i^{-1} n_i$, $\hat{\gamma}_i = \hat{\sigma}_u^2 \left(\hat{\sigma}_u^2 + n_i^{-1} \hat{\sigma}_e^2 \right)^{-1}$ is the shrinkage factor, $\bar{Y}_{is} = n_i^{-1} \sum_{s_i} y_j$ and $\bar{X}_{is} = n_i^{-1} \sum_{s_i} x_j$ are the sample means of Y and X for area i, while $\bar{X}_{ir} = (N_i - n_i)^{-1} (N_i \bar{X}_i - n_i \bar{X}_{is})$ is the corresponding mean of X for the $N_i - n_i$ non-sampled units in the area. An approximately unbiased estimator of the MSE of (4) under (3) is

$$M(\hat{\bar{Y}}_{i,EBLUP}) = (1 - f_i)^2 \left\{ g_{1i}(\hat{\theta}) + g_{2i}(\hat{\theta}) + 2 g_{3i}(\hat{\theta}) \right\} + N_i^{-1} (1 - f_i) \hat{\sigma}_e^2 \tag{5}$$

where

$$g_{1i}(\hat{\theta}) = \hat{\sigma}_u^2 \left(1 - \hat{\sigma}_u^2 Z_{is}^t \hat{V}_{iss}^{-1} Z_{is} \right)$$

$$g_{2i}(\hat{\theta}) = \left(\bar{X}_{ir} - \hat{c}_i^t X_{is} \right)^t \left(\sum_i X_{is}^t \hat{V}_{iss}^{-1} X_{is} \right)^{-1} \left(\bar{X}_{ir} - \hat{c}_i^t X_{is} \right)$$

$$g_{3i}(\hat{\theta}) = tr\left\{ (\nabla \hat{c}_i) V_{iss} (\nabla \hat{c}_i)^t \hat{C}(\hat{\theta}) \right\}$$

with $\hat{c}_i = \hat{\sigma}_u^2 \hat{V}_{iss}^{-1} Z_{is}$, $\nabla \hat{c}_i = \partial \hat{c}_i / \partial \hat{\theta} = \left(\partial \hat{c}_i / \partial \hat{\sigma}_u^2, \partial \hat{c}_i / \partial \hat{\sigma}_e^2 \right)^t$ and $\hat{C}(\hat{\theta})$ is the estimated asymptotic covariance matrix of $\hat{\theta}$ (i.e. the inverse of the observed information matrix for θ). For more details see Rao (2003, pp. 107-110).

Under the population level linear mixed model (3), the sample weights that define the EBLUP for the population total of Y are

$$w_{EBLUP} = (w_{j,EBLUP}) = 1_s + \hat{H}^t \left(X^t 1_N - X_s^t 1_s \right) + \left(I_s - \hat{H}^t X_s^t \right) \hat{V}_{ss}^{-1} \hat{V}_{sr} 1_r \qquad (6)$$

where $\hat{H} = \left(\sum_i X_{is}^t \hat{V}_{iss}^{-1} X_{is} \right)^{-1} \left(\sum_i X_{is}^t \hat{V}_{iss}^{-1} \right)$. See Royall (1976). The model-based direct estimator (MBDE, see Chambers and Chandra, 2006) of the i^{th} small area mean is then defined as

$$\hat{\bar{Y}}_{i,MBD} = \sum_{j \in s_i} w_{j,EBLUP} y_j \Big/ \sum_{j \in s_i} w_{j,EBLUP} . \qquad (7)$$

A robust estimator (Chandra and Chambers, 2005; Royall and Cumberland, 1978) of the mean squared error of the MBDE (7) is

$$M(\hat{\bar{Y}}_{i,MBD}) = v(\hat{\bar{Y}}_{i,MBD}) + \left\{ b(\hat{\bar{Y}}_{i,MBD}) \right\}^2 \qquad (8)$$

where $v(\hat{\bar{Y}}_{i,MBD}) = \sum_{s_i} \lambda_j (y_j - x_j^t \hat{\beta})^2$, with $\lambda_j = N_i^{-2} \left\{ a_j^2 + (N_i - n_i)(n_i - 1)^{-1} \right\}$ and $a_j = \left(\sum_{s_i} w_k \right)^{-1} \left(N_i w_j - \sum_{s_i} w_k \right)$, is the estimate of the prediction variance of the MBDE, and $b(\hat{\bar{Y}}_{i,MBD}) = (\hat{\bar{X}}_{i,MBD} - \bar{X}_i)^t \hat{\beta}$ is the estimate of its prediction bias. Here $\hat{\bar{X}}_{i,MBD}$ denotes the weighted average of the sample values of the auxiliary variables in area i based on the EBLUP weights (6).

2.2 Models with Spatial Dependence

In order to take into account the correlation between neighbouring areas we consider the use of spatial models for random area effects (Cressie, 1991). In particular, we consider a linear regression model with spatial dependence in the error structure. In particular, we assume a Simultaneous Autoregressive (SAR) error process (Anselin, 1992), where the vector of random area effects $v = (v_i)$ satisfies

$$v = \rho W v + u . \qquad (9)$$

Here ρ is a spatial autoregressive coefficient, W is a proximity matrix of order m and $u \sim N(0, \sigma_u^2 I_m)$. Since $v = (I - \rho W)^{-1} u$ with $E(u) = 0$ and $Var(u) = \sigma_u^2 I_m$, we have $E(v) = 0$ and $Var(v) = \sigma_u^2 [(I_m - \rho W)(I_m - \rho W^t)]^{-1} = G$. The W matrix describes how random effects from neighbouring areas are related, whereas ρ defines the strength of this spatial relationship. The simplest way to define W is as a contiguity matrix. That is, the elements of W take non-zero values only for those pairs of areas that are adjacent. Generally, for ease of interpretation, this matrix is defined in row-standardized form; in which case ρ is called the spatial autocorrelation parameter (Banerjee *et al.*, 2004). Formally, the element w_{jk} of a contiguity matrix takes the value 1 if area j shares an edge with area k and 0 otherwise. In row-standardised form this becomes

$$w_{jk} = \begin{cases} d_j^{-1} & \text{if } j \text{ and } k \text{ are contiguous} \\ 0 & \text{otherwise} \end{cases}$$

where d_j is the total number of areas that share an edge with area j (including area j itself). Contiguity is the simplest but not necessarily the best specification of a spatial interaction matrix. It may be more informative to express this interaction in a more detailed way, e.g. as some function of the length of shared border between neighbouring areas or as a function of the distance between certain locations in each area. Furthermore, the concept of neighbours of a particular area can be defined not just in terms of contiguous areas, but also in terms of all areas within a certain radius of the area of interest. In the empirical evaluations reported later in this paper, however, we used simple contiguity (row-standardized) to define the spatial interaction between different areas.

In order to define the EBLUP in this situation, we replace (3) by a linear mixed model of form

$$Y = X\beta + Zv + e . \qquad (10)$$

Here the vector v is an m-vector of spatially correlated area effects that satisfy the SAR model (9), with $Var(e) = \sigma_e^2 I_N$ and $Var(v) = G$. This model can then be rewritten as

$$Y = X\beta + Z(I - \rho W)^{-1}u + e . \qquad (11)$$

It follows that the covariance matrix of Y is $Var(Y) = V = \sigma_e^2 I_N + ZGZ^t$. In practice, the vector of parameters $\theta = (\sigma_u^2, \sigma_e^2, \rho)^t$ is unknown. Replacing it with an asymptotically consistent estimator $\hat{\theta} = (\hat{\sigma}_u^2, \hat{\sigma}_e^2, \hat{\rho})^t$, and assuming that (11) holds, the spatial-EBLUP (SEBLUP) for the i^{th} small area mean $\bar{Y}_i$ is

$$\hat{\bar{Y}}_{i,SEBLUP} = f_i \bar{Y}_{is} + (1 - f_i)\left(\bar{X}_{ir}^t \hat{\beta}^s + m_i^t \hat{v}\right) \qquad (12)$$

where $\hat{\beta}^s = \left(X_s^t \hat{V}_{ss}^{-1} X_s\right)^{-1}\left(X_s^t \hat{V}_{ss}^{-1} Y_s\right)$ is the empirical BLU estimator of β under (11), m_i is the m-vector $(0,0,0,\ldots,1,\ldots,0,0)^t$ with the 1 in the i^{th} position and

$$\hat{v} = \hat{G} Z_s^t \hat{V}_{ss}^{-1}\left(Y_s - X_s \hat{\beta}^s\right).$$

Here $\hat{G} = \hat{\sigma}_u^2\left[\left(I_m - \hat{\rho}W\right)\left(I_m - \hat{\rho}W^t\right)\right]^{-1}$ and $\hat{V}_{ss} = \hat{\sigma}_e^2 I_n + Z_s\hat{\sigma}_u^2[(I_m - \hat{\rho}W)(I_m - \hat{\rho}W^t)]^{-1}Z_s^t$. When all random effects are normally distributed, the parameter vector θ can be estimated via maximum likelihood (ML) as well as restricted maximum likelihood (REML) (Pratesi and Salvati, 2005; Singh *et al.,* 2005; Petrucci and Salvati 2006). Numerical approximations to either the ML or REML estimators $\hat{\sigma}_u^2, \hat{\sigma}_e^2$ and $\hat{\rho}$ can be obtained via a two-step procedure. At the first step, the Nelder-Mead algorithm (Nelder and Mead, 1965) is used to approximate these estimates. The second step then uses these approximations as starting values for a Fisher scoring algorithm. This is necessary because the log-likelihood function has multiple local maxima (Pratesi and Salvati, 2005). In empirical studies reported in Section 3 we carried out parameter estimation via REML using the *lme* function in the *R* environment (Bates and Pinheiro, 1998).

Following the same approach as in Prasad and Rao (1990), an approximately unbiased estimator of the MSE of the SEBLUP (12) is given by

$$M(\hat{\bar{Y}}_{i,SEBLUP}) = (1-f_i)^2\left\{g_{1i}^{(s)}(\hat{\theta}) + g_{2i}^{(s)}(\hat{\theta}) + 2g_{3i}^{(s)}(\hat{\theta})\right\} + N_i^{-1}(1-f_i)\hat{\sigma}_e^2 \quad (13)$$

where

$$g_{1i}^{(s)}(\hat{\theta}) = m_i^t\left(\hat{G} - \hat{G}Z_s^t\hat{V}_{ss}^{-1}\hat{G}_s\right)m_i$$

$$g_{2i}^{(s)}(\hat{\theta}) = \left(\bar{X}_{ir} - \hat{c}_i^t X_s\right)^t\left(X_s^t\hat{V}_{ss}^{-1}X_s\right)^{-1}\left(\bar{X}_{ir} - \hat{c}_i^t X_s\right)$$

$$g_{3i}^{s}(\hat{\theta}) = tr\left\{(\nabla\hat{c}_i)V_{ss}(\nabla\hat{c}_i)^t\hat{C}(\hat{\theta})\right\}$$

with $\hat{c}_i = \hat{V}_{ss}^{-1}Z_s\hat{G}m_i$ and $\nabla\hat{c}_i = \partial\hat{c}_i/\partial\hat{\theta} = \left(\partial\hat{c}_i/\partial\hat{\sigma}_u^2, \partial\hat{c}_i/\partial\hat{\sigma}_e^2, \partial\hat{c}_i/\partial\hat{\rho}\right)^t$. Here, after dropping ‘hats’ for the sake of clarity,

$$\frac{\partial c_i}{\partial \sigma_u^2} = \frac{\partial V_{ss}^{-1} Z_s G}{\partial \sigma_u^2} m_i = \left\{ V_{ss}^{-1} Z_s \left(\frac{\partial G}{\partial \sigma_u^2} \right) + \left(\frac{\partial V_{ss}^{-1}}{\partial \sigma_u^2} \right) Z_s G \right\} m_i$$

$$= \left\{ V_{ss}^{-1} Z_s \left(\frac{\partial G}{\partial \sigma_u^2} \right) - \left(V_{ss}^{-1} \frac{\partial V_{ss}}{\partial \sigma_u^2} V_{ss}^{-1} \right) Z_s G \right\} m_i$$

$$= V_{ss}^{-1} Z_s D \left\{ I_s - Z_s^t V_{ss}^{-1} Z_s G \right\} m_i$$

where $D = \frac{\partial G}{\partial \sigma_u^2} = \frac{\partial \sigma_u^2 [(I_m - \rho W)(I_m - \rho W^t)]^{-1}}{\partial \sigma_u^2} = [(I_m - \rho W)(I_m - \rho W^t)]^{-1}$ and

$$\frac{\partial V_{ss}}{\partial \sigma_u^2} = \frac{\partial \left\{ \sigma_e^2 I_s + Z_s \sigma_u^2 [(I_m - \rho W)(I_m - \rho W^t)]^{-1} Z_s^t \right\}}{\partial \sigma_u^2} = Z_s D Z_s^t .$$

Similarly

$$\frac{\partial c_i}{\partial \sigma_e^2} = \frac{\partial V_{ss}^{-1} Z_s G}{\partial \sigma_e^2} m_i = V_{ss}^{-1} Z_s \left(\frac{\partial G}{\partial \sigma_e^2} \right) m_i + \left(\frac{\partial V_{ss}^{-1}}{\partial \sigma_e^2} \right) Z_s G m_i$$

$$= \left(-V_{ss}^{-1} \frac{\partial V_{ss}}{\partial \sigma_e^2} V_{ss}^{-1} \right) Z_s G m_i$$

$$= -\sigma_u^2 \left\{ V_{ss}^{-1} I_s V_{ss}^{-1} \right\} Z_s D m_i$$

since $G = \sigma_u^2 D$ and $\frac{\partial V_{ss}}{\partial \sigma_e^2} = I_s$. Finally

$$\frac{\partial c_i}{\partial \rho} = \frac{\partial V_{ss}^{-1} Z_s G}{\partial \rho} m_i = V_{ss}^{-1} Z_s \left(\frac{\partial G}{\partial \rho} \right) m_i + \left(\frac{\partial V_{ss}^{-1}}{\partial \rho} \right) Z_s G m_i$$

$$= \left\{ V_{ss}^{-1} Z_s A - V_{ss}^{-1} \left(\frac{\partial V_{ss}^{-1}}{\partial \rho} \right) V_{ss}^{-1} Z_s G \right\} m_i$$

$$= V_{ss}^{-1} Z_s A \left\{ I_s - Z_s^t V_{ss}^{-1} Z_s G \right\} m_i .$$

Here $\frac{\partial V_{ss}}{\partial \rho} = \frac{\partial \left\{ \sigma_e^2 I_s + Z_s G Z_s^t \right\}}{\partial \rho} = Z_s \left(\frac{\partial G}{\partial \rho} \right) Z_s^t = Z_s A Z_s^t$, with

$$A = \frac{\partial G}{\partial \rho} = \sigma_u^2 \left(\frac{\partial D}{\partial \rho} \right) = -\sigma_u^2 \left(D \frac{\partial D^{-1}}{\partial \rho} D \right) = -2\sigma_u^2 D \left(\rho W W^t - W \right) D$$

since $\frac{\partial D^{-1}}{\partial \rho} = \frac{\partial [(I - \rho W)(I - \rho W^t)]}{\partial \rho} = 2\left(\rho W W^t - W \right)$. We note that $\hat{C}(\hat{\theta}) = I^{-1}(\hat{\theta})$ is still the estimated asymptotic covariance matrix of $\hat{\theta}$ defined by the inverse of the information matrix $I(\hat{\theta})$ (Rao, 2003, pp. 107-110), with the $(i,j)^{th}$ element of $I(\hat{\theta})$ given by $\frac{1}{2}tr\left\{P\left(\frac{\partial V_{ss}}{\partial\theta_i}\right)P\left(\frac{\partial V_{ss}}{\partial\theta_j}\right)\right\}$ with $P = V_{ss}^{-1}\left\{I_s - X_s\left(X_s^tV_{ss}^{-1}X_s\right)^{-1}X_s^tV_{ss}^{-1}\right\}$.

Turning now to implementation of model-based direct estimation under (11) we note that the EBLUP sample weights (6) depend on the structure of the random area effects in the mixed model (3) only via the their sample and population covariance structure. Consequently, extension to more complex covariance structures requires only that $\hat{V}_{ss}^{-1}$ and $\hat{V}_{sr}$ be recomputed under these more complex models. When (11) holds, the corresponding spatial EBLUP weights $w_{SEBLUP} = (w_{j,SEBLUP})$ are therefore still given by (6), but where now $\hat{V}_{ss}^{-1} = \left\{\hat{\sigma}_e^2 I_s + Z_s\hat{\sigma}_u^2[(I_m - \hat{\rho}W)(I_m - \hat{\rho}W^t)]^{-1}Z_s^t\right\}^{-1}$ and $\hat{V}_{sr} = \hat{\sigma}_u^2 Z_s[(I_m - \hat{\rho}W)(I_m - \hat{\rho}W^t)]^{-1}Z_r^t$. The spatial-MBDE (denoted by SMBDE) of the i^{th} small area mean $\bar{Y}_i$ and the corresponding estimator of its mean squared error are then given by (7) and (8) respectively, with the weights (6) used there replaced by the spatial EBLUP weights w_{SEBLUP} defined above.

3. Empirical Evaluations

In this section we use simulation to illustrate the performance of the four different methods of SAE discussed in the previous section. These are the EBLUP and MBDE under the linear mixed model (3) with spatially independent area effects (see section 2.1) and the SEBLUP and SMBDE under the linear mixed model (10) with spatially dependent area effects (see section 2.2). We computed three measures of estimation performance using the estimates generated in our simulations. These are the relative bias (RB) and the relative root mean squared error (RRMSE), both expressed as percentages, of estimates of the small area means and the coverage rate of nominal 95 per cent confidence intervals for these means (for more details see Chandra and Chambers, 2005).

We carried out two types of simulation studies. The first used real data and design-based simulation to evaluate the performance of these methods in the context of a real population and realistic sampling methods. The second used model-based simulation to generate artificial populations, from which samples were then taken. The sample data obtained in each case were then used to contrast the performance of different methods of small area estimation. The populations underpinning the design-based simulations were based on two different data sets:

(i) *The ISTAT farm structure survey*. This is a sample of 529 farms from the farm structure survey in Tuscany carried out by ISTAT. Here we used these sample farms to generate a population of N = 22977 farms by sampling with replacement from the original sample of 529 farms with probabilities proportional to their sample weights. We drew 1000 independent stratified random samples from this (fixed) population, with total sample size in each draw equal to the original sample size (529) and with the small areas of interest defined by the 23 Local Economy Systems (LESs) of the North Tuscany region. Sample sizes within these areas were fixed to be the same as in the original sample. Note that these varied from 4 to 48. Our aim was to estimate average olive production (quintals) in each LES using utilized olive surface (hectares) as the auxiliary variable. The results from this simulation are set out in Tables 1 and 2.

(ii) *The Environmental Monitoring and Assessment Program (EMAP) survey*. The data, on which this population was based, was provided by the Space-Time Aquatic Resources Modelling and Analysis Program (STARMAP) at

Colorado State University. It consists of 551 measurements, taken between 1991 and 1996, from a sample of 349 of the 21,026 lakes located in the north-eastern United States. Here we define lakes grouped by 6-digit Hydrologic Unit Code (HUC) as our small areas of interest. Since three HUCS had sample sizes of one, these were combined with adjacent HUCS, leading to a total of 23 small areas. Sample sizes in these 23 areas varied from 2 to 45. A (fixed) population of size N = 21028 was then defined by sampling N times with replacement from the sample of 349 units, with probability proportional to a unit's sample weight. A total of 1000 independent stratified random samples of the same size as the original sample were selected from this simulated population, with HUC sample sizes fixed to be the same as in the original sample. The survey variable Y in this case was the Acid Neutralizing Capacity (ANC) of a lake - an indicator of the acidification risk of water bodies in water resource surveys - with elevation of the lake as the auxiliary variable X. Results from this simulation are set out in Tables 3 and 4.

In our model-based simulations we again used the data from the EMAP survey, but this time based the population model underlying our simulations on variance components obtained by fitting a linear mixed effects model to these data. In particular, we generated a population of size $N = 21028$, with the same small area (HUC) population sizes as before. We used a sample size $n = 349$ and constrained the small area sample sizes to be the same as in the EMAP survey. These population and sample sizes were kept fixed in all our simulations. The model used to generate the population corresponded to a nested error regression model with random area effects for neighbouring areas distributed according to a SAR spatial correlation structure. This was of the form

$$y_{ij} = 1000 - 3x_{ij} + v_i + e_{ij}$$

where the x_{ij} values were generated from the uniform distribution on [10, 700], $v = (v_i) = (I_m - \rho W)^{-1}u$ was an *m*-vector of spatially correlated area effects with $u = (u_i)$ an *m*-vector of independent realisations from $N(0, \sigma_u^2)$ and the e_{ij} were individual error terms distributed as $N(0, \sigma_e^2)$. Using estimates derived from the linear mixed model fitted to the original EMAP survey data, we put $\sigma_u^2 = 265000$ and $\sigma_e^2 = 125000$, with intra area effect, $\gamma = \sigma_u^2 / (\sigma_u^2 + \sigma_e^2) = 0.68$. The row standardised SAR neighbourhood structure matrix *W* used in the simulations was kept fixed and corresponded to contiguous HUCs in the EMAP survey data set. Population data for four values of ρ (0.05, 0.25, 0.50 and 0.75) were generated and simple random samples selected from each small area, with a total of 1000 combinations independently simulated. The results from this simulation are set out in Table 5.

Table 1 shows the relative bias and relative root mean squared error for small area estimates calculated using the four different methods of small area estimation (EBLUP, MBDE, SEBLUP and SMBDE) based on repeated sampling from the simulated Northern Tuscany population. Corresponding coverage rates for nominal 95% intervals for area means generated by these four approaches are set out in Table 2. Tables 3 and 4 show analogous results for repeated sampling from the simulated EMAP population. Note that the estimated value of the spatial autocorrelation parameter ρ in the original ISTAT farm survey data was quite small ($\hat{\rho} = 0.025$), while for the EMAP survey data this estimate was considerably larger ($\hat{\rho} = 0.50$). Table 5 shows the average values of relative bias and relative root mean squared error, both expressed in percentage terms, and average coverage rates for the different methods generated under the model based simulations. All averages in Table 5 are over the 23 small areas of interest.

The results set out in Table 1 show that both EBLUP and SEBLUP are very unstable in a few small areas (e.g. regions 3, 6 and 14), due mainly to there being little or no variability in the variable of interest in these areas. In such situations, the SEBLUP seems to perform worse than the EBLUP. In contrast, the MBDE and SMBDE methods appear unaffected by such behaviour. Since the average values of performance measures are influenced by outlying estimates, we compare different methods using the median values of their area-specific performance measures. From Table 1 we see that the median relative bias of MBDE is smaller than that of EBLUP. In contrast, the median relative root mean squared error of EBLUP is smaller than that of MBDE. The median relative bias and median relative root mean squared error of SEBLUP is marginally smaller than that of EBLUP. However, these values are almost same for MBDE and SMBDE. Table 2 shows that average coverage rates increase when estimation methods are based on a spatial model (SEBLUP and SMBDE).

The results in Table 3 show that in region 1 (with sample size 2) all methods are very unstable, while in regions 2 and 3 (both with samples of size 3) EBLUP and SEBLUP are unstable. As noted earlier, EBLUP in these regions is affected by lack of variability in the data whereas MBDE is influenced by the presence of outlying values (see Chambers and Chandra, 2006). Although the estimated spatial autocorrelation is relatively higher for the EMAP data compared to the Northern Tuscany data, the simulation results for the EMAP data (Tables 3 and 4) are similar to those for the Northern Tuscany data (Tables 1 and 2). In both cases we see that the overall gain from introduced spatial dependence into small area estimation is rather small.

Finally, in Table 5 we show the performance of the different methods when population (and sample) data follow the assumed model. Here, we considered four different values ($\rho = 0.05, 0.25, 0.50, 0.75$) for the spatial autocorrelation parameter ρ and a W matrix that characterises the neighbourhood structure of the small areas in terms of the contiguity characteristics of the sampled lakes in the EMAP data. As in Chambers and Chandra (2006) we note that when the assumed model is correct, estimation via EBLUP dominates estimation via MBD. These results also show that in this case the gain in small area estimation from taking account of the spatial correlation of random effects remains marginal for the MBD estimator for all values of ρ and only improves the performance of the EBLUP for large values of this parameter.

4. Concluding Remarks

This paper presents results from an initial exploration of the use of unit level models with spatially correlated area effects in small area estimation. In particular, we show how the EBLUP and MBD methods of estimation can be adapted for this situation. However, our empirical results, based both on real data as well as on simulated data under the spatial model, indicate that the gains from inclusion of spatial structure in small area estimation do not appear to be large. This is especially true for model-based direct estimation based on this structure (SMBDE), where the extra spatial information seems to have very little impact on the distribution of the SEBLUP weights that characterise this method of estimation.

There are many issues that still need to be explored in the context of using unit level models with spatially distributed area effects in small area estimation. The most important of these is identification of situations where inclusion of spatial information does have an impact, and the most appropriate way of then including this spatial information in the small area modelling process. An important practical issue in this regard relates to the computational burden in fitting spatial models to survey data. With the large data sets common in survey applications it can be extremely difficult to fit spatial models without access to high-end computational facilities. Although spatial information is becoming increasingly available in environmental, epidemiological and economic applications, there has been comparatively little work carried out on how to efficiently use this information. A further issue relates to the link between the survey data and the spatial information. In this paper we have assumed that all areas have sample units. In many situations this is not true, with survey data available only from a sample of areas. However, we often have spatial information for all areas. Saei and Chambers (2005) have explored the use of this spatial information in order to efficiently estimate the characteristics of the so-called 'out of sample' areas. Finally, we note that the spatial models considered in this paper have been based on neighbourhoods defined by contiguous areas. It is easy to see that this is just one way of introducing spatial dependence between area effects, and several other options remain to be investigated.

References

Anselin, L. (1992). *Spatial Econometrics. Method and Models*. Kluwer Academic Publishers, Boston.

Battese, G., Harter, R. and Fuller, W. (1988). An Error-Components Model for Prediction of County Crop Areas using Survey and Satellite Data, *Journal of the American Statistical Association*, **83**, 28-36.

Banerjee, S., Carlin, B. and Gelfand, A. (2004). *Hierarchical Modelling and Analysis for Spatial Data.* Chapman and Hall, New York.

Bates, D.M. and Pinheiro, J.C. (1998). Computational Methods for Multilevel Models. http://franz.stat.wisc.edu/pub/NLME/.

Chambers, R.L. and Chandra, H. (2006). Improved Direct Estimators for Small Areas. Submitted for publication.

Chandra, H. and Chambers, R.L. (2005). Comparing EBLUP and C-EBLUP for Small Area Estimation. *Statistics in Transition*, **7**, 637-648.

Cressie, N. (1991). Small-area prediction of undercount using the general linear model. *Proceedings of Statistics Canada Symposium 90: Measurement and Improvement of Data Quality, Ottawa: Statistics Canada* pp. 93–105.

Harville, D.A. (1977). Maximum Likelihood Approaches to Variance Component Estimation and to Related Problems. *Journal of the American Statistical Association*, **72**, 320–338.

Nelder, J. and Mead, R. (1965). A Simplex Method for Function Minimization, *Computer Journal*, **7**, 308–313.

Prasad, N.G.N. and Rao, J.N.K. (1990). The Estimation of the Mean Squared Error of Small Area Estimators. *Journal of the American Statistical Association*, **85**, 163-171.

Petrucci, A. and Salvati, N. (2004). Small Area Estimation Considering Spatially Correlated Errors: the Unit Level Random Effects Model, *Working Paper 2004/10*, Dipartimento di Statistica“G. Parenti”, Firenze.

Pratesi, M. and Salvati, N. (2005). Small Area Estimation: the EBLUP Estimator with Autoregressive Random Area Effects, *Report No. 261*, Department of Statistics and Mathematics, University of Pisa. To appear *in Statistical Methods & Applications*.

Petrucci, A. and Salvati, N. (2006). Small Area Estimation for Spatial Correlation in Watershed Erosion Assessment. *Journal of Agricultural, Biological and Environmental Statistics*, **11**, 169-182.

Rao, J.N.K. (2003). *Small Area Estimation*. New York: Wiley.

Royall, R.M. (1976). The Linear Least-Squares Prediction Approach to Two-Stage Sampling. *Journal of the American Statistical Association*, **71**, 657-664.

Royall, R.M. and Cumberland, W.G. (1978). Variance Estimation in Finite Population Sampling. *Journal of the American Statistical Association*,**73**, 351-358.

Saei, A. and Chambers, R. (2005). Out of Sample Estimation for Small Areas using Area Level Data. *Methodology Working Papers, M05/11*, Southampton Statistical Sciences Research Institute. (http://eprints.soton.ac.uk/14327)

Singh, B.B., Shukla, G.K. and Kundu, D. (2005). Spatio-Temporal Models in Small Area Estimation. *Survey Methodology*, **31**, 183-195.

Table 1 Relative Bias and Relative Root Mean Squared Errors generated by design based simulations using Northern Tuscany data. Regions are arranged in order of increasing population size.

	Relative Bias (%)				Relative Root Mean Squared Error (%)			
Regions	EBLUP	SEBLUP	MBDE	SMBDE	EBLUP	SEBLUP	MBDE	SMBDE
1	4.11	10.39	-3.03	-3.45	13.67	17.41	48.61	47.63
2	32.18	70.11	0.31	0.31	45.35	90.33	31.16	31.16
3	278.07	415.79	1.34	1.34	302.70	463.39	20.76	20.76
4	64.39	93.43	0.01	0.01	69.06	112.79	38.50	38.50
5	-4.90	-7.72	-1.23	-1.25	13.28	14.83	22.53	22.54
6	70.18	71.76	-9.47	-11.35	93.05	93.50	34.92	34.08
7	36.94	44.49	-0.13	-0.17	51.99	57.17	21.31	21.29
8	9.08	1.79	-1.43	-1.84	21.91	22.40	36.36	36.22
9	24.60	24.53	-0.73	-1.22	31.20	31.42	37.50	37.25
10	41.54	42.31	-2.13	-2.66	60.12	60.31	47.03	46.75
11	11.07	12.94	-0.42	-0.53	17.00	18.34	14.27	14.19
12	-3.90	-9.87	0.18	0.17	13.10	16.94	19.05	19.05
13	-15.11	-14.54	-0.47	-0.67	28.68	28.49	35.80	35.66
14	116.59	140.97	1.14	1.13	128.10	152.73	32.85	32.84
15	5.25	1.26	0.49	0.35	11.52	13.60	32.49	32.27
16	-8.17	-7.73	-1.73	-2.47	21.48	21.53	25.43	25.09
17	8.53	7.52	0.60	0.52	18.01	17.58	23.87	23.80
18	7.46	8.38	-1.59	-1.95	19.63	19.84	23.51	23.35
19	5.29	9.43	0.69	0.50	15.36	17.85	31.53	31.44
20	9.81	6.74	-0.02	-0.03	21.61	20.80	16.48	16.48
21	-5.89	-10.25	0.76	0.61	30.53	33.84	42.26	42.23
22	-9.14	-9.67	0.31	0.15	17.71	17.82	22.26	22.27
23	-11.42	-12.33	0.49	0.46	20.62	21.02	23.52	23.53
Mean	28.98	38.68	-0.70	-0.96	46.33	59.30	29.65	29.49
Median	8.53	8.38	-0.02	-0.03	21.61	21.53	31.16	31.16

Table 2 Coverage rates generated by design based simulations using Northern Tuscany data. Intervals are defined by the small area mean estimate plus or minus twice their corresponding estimated root mean squared error. Regions are arranged in order of increasing population size.

	Coverage rates			
Regions	EBLUP	SEBLUP	MBDE	SMBDE
1	1.00	1.00	1.00	1.00
2	1.00	1.00	0.99	0.99
3	1.00	1.00	1.00	1.00
4	1.00	1.00	0.97	0.97
5	1.00	1.00	0.95	0.95
6	0.50	0.51	1.00	1.00
7	1.00	1.00	1.00	1.00
8	1.00	1.00	0.97	0.97
9	0.98	0.99	0.98	0.98
10	0.99	1.00	0.97	0.97
11	1.00	1.00	0.99	0.99
12	1.00	1.00	0.94	0.94
13	0.65	0.68	0.91	0.91
14	1.00	1.00	1.00	1.00
15	1.00	1.00	0.99	0.99
16	0.49	0.50	0.91	0.91
17	1.00	1.00	0.97	0.98
18	0.98	0.98	0.98	0.98
19	1.00	1.00	0.97	0.98
20	1.00	1.00	1.00	1.00
21	0.98	0.98	0.63	0.63
22	0.91	0.93	0.93	0.93
23	0.99	0.99	0.92	0.92
Mean	0.93	0.94	0.95	0.96

Table 3 Relative Bias and Relative Root Mean Squared Errors generated by design based simulations using EMAP data. Regions are arranged in order of increasing population size.

	Relative Bias (%)				Relative Root Mean Squared Error (%)			
Regions	EBLUP	SEBLUP	MBDE	SMBDE	EBLUP	SEBLUP	MBDE	SMBDE
1	173.18	190.85	-8.85	-8.72	224.28	236.92	263.32	263.46
2	-17.18	-96.23	1.52	1.60	160.02	189.91	35.26	35.34
3	1044.89	1166.54	0.14	0.14	1056.36	1188.52	37.61	37.61
4	19.86	20.07	0.00	0.00	24.48	24.79	9.61	9.61
5	-12.08	-12.28	-0.39	-0.39	14.52	14.70	11.87	11.87
6	3.24	24.29	-0.32	-0.31	64.38	69.67	31.95	31.94
7	-0.28	-2.50	1.56	1.62	42.42	42.11	51.34	51.35
8	5.49	4.73	0.28	0.27	29.70	28.93	33.24	33.24
9	-8.07	-8.49	-0.08	-0.07	17.09	17.16	16.56	16.56
10	0.81	-0.25	0.94	1.01	22.54	22.34	27.23	27.24
11	7.19	4.51	0.51	0.57	35.00	34.51	30.32	30.38
12	-4.40	-5.07	-0.70	-0.66	21.48	21.48	25.31	25.31
13	-3.44	-0.23	0.87	0.87	32.10	31.77	34.46	34.47
14	3.07	4.10	0.29	0.48	23.90	24.18	22.51	22.51
15	-1.20	-1.09	0.30	0.33	11.65	11.59	12.48	12.47
16	22.12	25.14	-0.17	-0.18	49.42	50.92	38.06	38.06
17	4.15	2.53	0.22	0.25	16.44	16.22	11.21	11.22
18	-0.86	-3.29	0.36	0.36	10.28	10.83	7.25	7.25
19	0.38	1.51	1.93	2.12	15.50	15.57	20.47	20.52
20	-1.30	-2.10	-0.57	-0.58	17.36	17.43	16.83	16.83
21	2.15	1.49	1.17	1.27	12.25	12.16	14.96	14.98
22	4.85	2.90	-0.02	-0.03	15.82	15.29	13.83	13.83
23	-0.41	0.49	0.73	0.88	12.64	12.64	15.23	15.27
Mean	54.01	57.29	12.00	0.04	83.90	91.72	33.95	33.97
Median	0.81	1.49	0.28	0.27	22.54	22.34	22.51	22.51

Table 4 Coverage rates generated by design based simulations using EMAP data. Intervals are defined by the small area mean estimate plus or minus twice their corresponding estimated root mean squared error. Regions are arranged in order of increasing population size.

	Coverage rates			
Regions	EBLUP	SEBLUP	MBDE	SMBDE
1	1.00	1.00	0.98	0.99
2	1.00	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00
4	1.00	1.00	1.00	1.00
5	1.00	1.00	1.00	1.00
6	1.00	1.00	1.00	1.00
7	0.83	0.82	0.86	0.86
8	0.98	0.98	0.62	0.64
9	0.60	0.60	1.00	1.00
10	0.92	0.92	0.95	0.96
11	1.00	1.00	1.00	1.00
12	0.77	0.77	0.96	0.96
13	0.74	0.76	0.72	0.70
14	1.00	1.00	1.00	1.00
15	0.75	0.75	0.98	0.98
16	1.00	1.00	1.00	1.00
17	1.00	1.00	1.00	1.00
18	1.00	1.00	1.00	1.00
19	1.00	1.00	0.96	0.97
20	0.78	0.77	0.97	0.97
21	1.00	1.00	1.00	1.00
22	1.00	1.00	1.00	1.00
23	0.85	0.85	0.96	0.95
Mean	0.92	0.92	0.95	0.96

Table 5 Average Relative Bias (ARB,%), average Relative Root Mean Squared Error (ARRMSE, %) and average Coverage Rate (ACR) generated by model-based simulations. All averages are over the 23 small areas of interest.

		ρ			
Criteria	Methods	0.05	0.25	0.50	0.75
ARB, %	EBLUP	-8.22	-4.96	-25.56	45.84
	SEBLUP	-4.51	-5.13	-20.35	41.09
	MBDE	-35.25	5.02	9.80	69.16
	SMBDE	-35.12	5.03	9.64	68.74
ARRMSE,%	EBLUP	448.91	305.26	258.18	932.95
	SEBLUP	451.18	305.31	256.38	911.47
	MBDE	921.80	622.98	531.09	1924.81
	SMBDE	921.71	622.97	531.13	1922.11
ACR	EBLUP	0.95	0.94	0.95	0.96
	SEBLUP	0.95	0.94	0.95	0.95
	MBDE	0.98	0.98	0.98	0.98
	SMBDE	0.98	0.98	0.98	0.98