UNIVERSITY OF SOUTHAMPTON

Symmetries and Automorphisms of Compact Riemann Surfaces

Paul Daniel Watson

Submitted for the degree of
Doctor of Philosophy

Faculty of Mathematics

April 1995

ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. David Singerman for all his ideas and patience.

I am very grateful for the help of my fellow students at one time or another, especially Derek, Ismail, Timo, Manfred and Lawrence.

I would also like to thank the Engineering and Physical Science Research Council for their financial support, and my wife Rachel for her sustaining morale support.

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

Faculty of Mathematics

Doctor of Philosophy

SYMMETRIES AND AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

by Paul Daniel Watson

In this thesis we deal with compact Riemann surfaces, in fact mainly those uniformized by normal subgroups of Fuchsian triangle groups. A symmetry of such a surface is an anti-conformal involution mapping the surface to itself. Each symmetry is given a species which completely classifies its topological action on the surface. We examine an oversight in an important Theorem of Singerman's and try to mend it. In so doing we find a kind of symmetry not anticipated by Singerman. Chapter four contains the symmetry types of all Riemann surfaces with large cyclic automorphism groups. Harnack gave an upper bound on the number of curves fixed by a symmetry of a surface of a particular genus, the 'unexpected' symmetries mentioned above are the only symmetries of the surfaces in Chapter four to attain that bound. In Chapter five we give a similar treatment to those surfaces with large non-cyclic abelian automorphism groups. Harnack's bound is not attained by any symmetry of any of these surfaces. The Appendix chiefly accompanies Chapters four and five and looks in some detail at the inclusions between triangle groups and the NEC groups that contain them with index two.

In Chapter six we turn our attention to maps and hypermaps, lying on orientable, connected surfaces without boundary. Such objects can naturally be thought of as lying on those Riemann surfaces in the scope of this thesis. These surfaces, together with the maps and hypermaps themselves, are receiving much attention at the moment in connection with Belyi's Theorem, which implies that they are precisely the surfaces corresponding to the algebraic curves defined over algebraic number fields. The maps and hypermaps that we deal with are all regular and their face centres, vertices and edge centres are important in the geometric and combinatorial properties of the maps and hypermaps. We call these points 'geometric points'. Weierstrass points are right at the heart of Riemann surfaces and we determine whether the 'geometric' points of regular maps and hypermaps with abelian automorphism groups are Weierstrass points or not. Finally we calculate the weight at each of the 'geometric' points of all the regular maps of genus two, three, four and five.

CONTENTS

Introduction

Riemann surfaces were introduced as the "right" domain for certain functions, in order to make them one-to-one. The uniformization theorem of Poincaré, Klein and Koebe made it possible to prove that every connected Riemann surface, of genus greater than one, is conformally equivalent to the upper half plane $\mathcal{U}$, quotient some subgroup of PSL(2,$\mathbf{R}$) that acts discontinuously on $\mathcal{U}$. Furthermore, the quotient of the normalizer of the subgroup by the subgroup itself is isomorphic to the full group of conformal automorphisms of the surface. Fuchsian groups are subgroups of PSL(2,$\mathbf{R}$) that act properly discontinuous on $\mathcal{U}$ and they are precisely the groups of conformal isometries of $\mathcal{U}$ with the hyperbolic metric. These are very important in the study of Riemann surfaces. In Chapter one we detail some of these ideas and others.

A symmetry of a Riemann surface is defined to be an anti-conformal involution that maps the surface to itself. To each symmetry we give a species which consists of the number of curves it fixes on the surface and a plus or minus sign; plus if the quotient of the surface by the symmetry is orientable and minus if it is non-orientable. The species of a symmetry completely classifies its topological action on the surface. Non-Euclidean crystallographic (NEC) groups are properly discontinuous groups of isometries of the hyperbolic plane that may contain anti-conformal elements. Just as Fuchsian groups are useful when studying conformal automorphisms of Riemann surfaces, so NEC groups are useful in studying symmetries. If S is a compact Riemann surface and T a symmetry of S, then there is a proper NEC group Λ, such that $S/\langle T\rangle \simeq \mathcal{U}/\Lambda$. Hence, if we know the signature of Λ then we know the species of T. In Chapter two we present a Theorem of Hoare's [20] that gives us an algorithm to calculate the signature of a subgroup of a NEC group given the signature of the parent group and its action on the cosets of the subgroup. This is very useful in calculating the species of symmetries.

In Chapter three we look at an important Theorem of Singerman [35]. The Theorem considers surfaces uniformized by a surface group that is normal in some triangle group. It gives necessary and sufficient conditions, on the group of automorphisms of the surface that is isomorphic to the the triangle group quotient the surface group, for the surface to be symmetric. Such an automorphism group is called a large automorphism group. There is a slight oversight in the theorem which questions the necessity of the conditions. We highlight this and (almost) mend it. We also differentiate, with respect to a particular large automorphism group, between the symmetries of the surface.

We look at Riemann surfaces admitting large cyclic automorphism groups in Chapter four and those admitting large non-cyclic abelian groups in Chapter five. In both cases we employ Singerman's Theorem, plus amendment, to see what "kind" of symmetries the surface admits, and then use Hoare's Theorem to calculate the species of the symmetries. A surface may be uniformized by a surface group that is normal in two non-isomorphic triangle groups, and so the surface would admit two large automorphism groups, one contained within the other. These were the surfaces over-looked in [35]. In the last section of chapters four and five we pay particular attention to the symmetries of surfaces of this kind when one of the large automorphism groups is cyclic or non-cyclic abelian. We give the symmetry types of all surfaces that admit only abelian large automorphism groups. For those surfaces that admit a cyclic large group and another, which will necessarily be non-abelian, we also give all possible symmetry types. While for those admitting one non-cyclic abelian large group and another large group which is non-abelian we have only calculated the possible species of the symmetries.

The Appendix is chiefly for use with Chapters four and five although we also use it in Chapter six. In it we look at all the inclusions between triangle groups. For each inclusion we show that there is just one conjugacy class of subgroups with the given signature. For certain inclusions we give canonical generators of one of the subgroups with the specified signature in terms of canonical generators of the parent group, and give fundamental regions for these subgroups made up of fundamental regions for the parent group. We also make some comments about symmetries of surfaces uniformized by surface groups normal in these triangle groups that prove useful in Chapters four and five.

Maps are certain embeddings of certain graphs into certain surfaces. We only consider those on orientable, connected surfaces without boundary. These are classical objects [10]. Jones and Singerman [23], showed that it is natural to think of maps as lying on compact Riemann surfaces; indeed those uniformized by subgroups of triangle groups with one period equal to two. More recently hypermaps have been studied, [9]; these too can be thought of as lying on compact Riemann surfaces, those uniformized by subgroups of triangle groups (with no period necessarily equal to two). We will only be considering regular maps and hypermaps. These objects detail a finite set of points on their underlying surfaces which we call the geometric points of the map or hypermap. There is another finite set of points on the surface; the Weierstrass points. In Chapter six, for certain regular maps, we ask what the connection is between these two sets. To answer this question we mainly use results of Lewittes [24] and Harvey [18] on fixed points of automorphisms of compact Riemann surfaces, and other results derived from these. These are discussed in section two.

In the third section we look at those regular maps and hypermaps whose automorphism groups are abelian. We prove that the geometric points of these maps and hypermaps are all Weierstrass points, or that the Weierstrass points are a proper subset of the geometric points, or that the underlying surface is hyperelliptic and also carries a regular map with respect to which all the Weierstrass points are geometric. We also show that any regular map, whose automorphism group is abelian, lies on a hyperelliptic surface and the hyperelliptic involution is also an automorphism of the map, so we say the map itself is hyperelliptic. Thus the Weierstrass points are all geometric points of the map. It is a consequence of [37] that the Weierstrass points of any hyperelliptic surface carrying a regular map are geometric points of the map. Apart from one notable exception (ie. the hypermap of type $\{4,4,4\}$ on the Fermat curve of degree four whose automorphism group is isomorphic to $\mathbf{Z}_4 + \mathbf{Z}_4$), the Weierstrass points of a surface carrying a regular hypermap, which is not a map and whose automorphism group is abelian, are not all geometric points of the hypermap.

In the final section we examine all the regular maps of genus two, three, four and five, and determine the weight at each of the geometric points of the maps. We show that in each case, bar one map of genus five, the Weierstrass points of the underlying surfaces are all geometric points of the map or they are geometric points of another regular map which the surface carries. The numbers of geometric points of regular maps and hypermaps are small compared to the weight of all the Weierstrass points of the underlying surfaces for large genus. It seems, as might be expected, that the geometric points of regular maps and hypermaps of high genus fail to account for much of the total weight, (expect for hyperelliptic maps of course). However, because of the special nature of the Riemann surfaces that they lie on, particularly the maps, it may happen that the geometric points have especially high weights.

Chapter 1

Preliminaries

Section 1.1 NEC Groups

Let $\mathcal{U}$ denote the upper-half complex plane, $\{z \in \mathbf{C} : \operatorname{Im} z > 0\}$. If we define the metric on $\mathcal{U}$ to be

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

then $\mathcal{U}$, together with this metric, provides a model of the hyperbolic-plane. Geodesics of this metric are then just lines and arcs of circles which are perpendicular to the real axis.

Let $\mathcal{L}$ denote the group of all conformal and anti-conformal homeomorphisms of $\mathcal{U}$, and let $\mathcal{L}^+$ denote the subgroup of index two of all conformal homeomorphisms. A mapping is anti-conformal if it preserves angles while reversing orientation. $\mathcal{L}$ is precisely the group of isometries of $\mathcal{U}$ with the above metric.

$\mathcal{L}$ consists of elements of two kinds:

i) $$z \mapsto \frac{az+b}{cz+d}, \qquad a,b,c,d \in \mathbf{R},\ ad-bc=1.$$

ii) $$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}, \qquad a,b,c,d \in \mathbf{R},\ ad-bc=-1.$$

Elements of the first kind are the conformal homeomorphisms of $\mathcal{U}$ while the elements of the second kind are the anti-conformal homeomorphisms of $\mathcal{U}$.

$\mathcal{L}$ can be topologised as the subset of $\mathbf{R}^4$

$$\{(a,b,c,d) : ad-bc = \pm 1\}$$

by identifying $\pm(a, b, c, d)$ and taking the identification topology.

A discrete subgroup of $\mathcal{L}$ is called a *non-Euclidean crystallographic group* or NEC group. A NEC group is said to be a *proper* NEC group if it contains some anti-conformal elements and a NEC group is said to be *Fuchsian* if it contains conformal elements only. If Γ is a NEC group then we denote by Γ^+ the subgroup of all conformal elements of Γ; Γ contains Γ^+ with index one or two. We call Γ^+ the *canonical Fuchsian group* of Γ.

Elements of $\mathcal{L}$ can be classified according to their fixed point set when acting on $\mathbf{C}$ and their orientation.

Elements of the first kind preserve orientation and their fixed point set is determined by solving the equation

$$z = \frac{az + b}{cz + d}, \qquad ad - bc = 1.$$

There are three kinds:

i) If $|a + d| > 2$ then there are two fixed points, both of which are in $\mathbf{R} \cup \{\infty\}$ and the element is said to be *hyperbolic*.

ii) If $|a + d| = 2$ then there is a single fixed point which is in $\mathbf{R} \cup \{\infty\}$ and the element is said to be *parabolic*.

iii) If $|a + d| < 2$ then there are two (complex conjugate) fixed points, one of which is in $\mathcal{U}$, and the element is said to be *elliptic*.

When acting on $\mathcal{U}$, hyperbolic elements act like translations, elliptic elements as rotations about a point of $\mathcal{U}$ and parabolic elements can be thought of as rotations about a point in $\mathbf{R} \cup \{\infty\}$.

Elements of the second kind reverse orientation and their fixed point set is determined by solving the equation

$$z = \frac{a\bar{z} + b}{c\bar{z} + d}, \qquad ad - bc = -1.$$

There are two kinds:

i) If $a + d \neq 0$ then the fixed point set is a circle and we have a *reflection*.

ii) If $a + d = 0$ then there are just two fixed points, both of which are in $\mathbf{R} \cup \{\infty\}$ and we have a *glide reflection*.

Section 1.2 Fundamental Regions and Signatures of NEC Groups

A NEC Group Γ acts properly discontinuously on $\mathcal{U}$, that is each point $z \in \mathcal{U}$ has a neighbourhood V such that if $\gamma \in \Gamma$ and $V \cap \gamma V \neq \emptyset$ then γ fixes z. Hence the Γ-orbit of any point in $\mathcal{U}$ is a discrete subset of $\mathcal{U}$. If we give the set of all Γ-orbits the identification topology then we form the orbit (or quotient) space $\mathcal{U}/\Gamma$.

Definition 1.1 A *Surface* is a connected Hausdorff space on which there is an open covering by sets homeomorphic to open sets in $\mathbf{R}^2$.

Definition 1.2 A *Surface with boundary* is a connected Hausdorff space on which there is an open covering by sets which can be mapped homeomorphically on to relatively open sets of a closed half plane, and is not a surface.

We see that $\mathcal{U}/\Gamma$ is a surface with or without boundary, orientable or non-orientable depending on Γ. In fact the quotient space will have boundary if and only if Γ contains relections and will be non-orientable only if Γ contains glide reflections.

In this thesis only NEC groups with compact quotient spaces are considered and it is known that such groups contain no parabolic elements.

Definition 1.3 If Γ is a NEC group then a Γ-*fundamental region* is a closed subset F of $\mathcal{U}$ such that:

i) F contains at least one element of every orbit.

ii) F°, the interior of F, contains at most one element of every orbit.

iii) The hyperbolic area of $F \backslash F^\circ$ is zero.

F is not necessarily connected but a connected fundamental region can always be found. Let $p \in \mathcal{U}$ be such that $\gamma p \neq p$ for all $\gamma \in \Gamma$. Define F_p to be the set

$$F_p := \{z \in \mathcal{U} : d(z,p) \leq d(\gamma z, p) \quad \text{for all} \quad \gamma \in \Gamma\}$$

where $d(z,p)$ is the hyperbolic distance between z and p. F_p is called the *Dirichlet region* for Γ based at p and is a connected Γ-fundamental region. F_p is a convex hyperbolic polygon with a finite number of sides; such a fundamental region is said to be *regular*.

Let F be a Dirichlet polygon for Γ. Then there is a tessellation of $\mathcal{U}$ by F under Γ. Faces of this tessellation are said to be adjacent if they share a common edge. In fact the faces are in a one to one correspondence with the elements of Γ and Γ is generated by the elements that map any particular face to all the faces adjacent to it.

Wilkie [38], has shown that for every NEC group Γ with compact quotient space there is a canonical fundamental polygon from which a canonical presentation for Γ may be derived. This presentation is given by a set of generators

$$\begin{array}{lll} x_i & i=1,\dots,r & \\ c_{jk} & j=1,\dots,s & k=0,\dots,t_j \\ e_j & j=1,\dots,s & \\ a_p & p=1,\dots,g & \\ b_q & q=1,\dots,h & \end{array}$$

where $r \geq 0$, $j \geq 0$, $t_j \geq 0$, $g \geq 0$, and $h = 0$ or g,

with defining relators

$$\begin{array}{lll} x_i^{m_i} & i=1,\dots,r & \\ c_{jk}^2 & j=1,\dots,s & k=0,\dots,t_j \\ (c_{jk-1}c_{jk})^{n_{jk}} & j=1,\dots,s & k=1,\dots,t_j \\ c_{j0}e_jc_{jt_j}e_j^{-1} & j=1,\dots,s & \end{array}$$

where $n_{jk} \geq 2$, $m_i \geq 2$

and $$x_1x_2\cdots x_re_1e_2\cdots e_sD$$

for i) $D = a_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$ if $h = g \geq 0$

ii) $D = a_1^2\cdots a_g^2$ if $h = 0 < g$

The elements of the form x_i are elliptic, the c_{jk}'s are reflections and the elements e_j are usually hyperbolic but maybe elliptic. The numbers m_i are called the *proper periods* of Γ, while the n_{jk}'s are known as the *link periods* of Γ. The elements b_q are hyperbolic, when they exist, while the elements a_p are hyperbolic in the first case and glide-reflections in the second.

The quotient space $\mathcal{U}/\Gamma$ has genus g and is orientable in case (i) but non-orientable in case (ii). It is not difficult to see, from the canonical fundamental region, that the number of boundary components of the quotient space is simply s and the n_{jk} give the orders of branching on the boundaries.

To each NEC group with the above presentation we can associate a *signature,*

$$(g;+;[m_1,\dots,m_2];\{(n_{11},\dots,n_{1t_1}),\dots,(n_{s1},\dots,n_{st_s})\}) \tag{1.4}$$

in case (i) and

$$(g;-;[m_1,\dots,m_2];\{(n_{11},\dots,n_{1t_1}),\dots,(n_{s1},\dots,n_{st_s})\}) \tag{1.5}$$

in case (ii). The cycles $(n_{i1},\dots,n_{it_i})$, which may be empty, are called the *period cycles* of Γ.

When writing signatures of NEC groups indices may be employed to indicate repeated periods or empty period cycles.

$(g;+:[2,2,2,2];\{(\ ),(\ ),(\ )\})$ will usually be written as $(g;+;[2^{(4)}];\{(\ )^{(3)}\})$.

A Fuchsian group will have an orientable quotient space with no boundary and so it will have a signature of the form

$$(g;+;[m_1,\dots,m_r];\{\ \}),$$

which we may write as

$$(g;[m_1,\dots,m_r]).$$

If the genus g is zero this may be reduced further to

$$[m_1,\dots,m_r],$$

especially in the case when $r=3$ and the group is known as a triangle group.

Fuchsian groups with no periods are known as *surface groups*. These will have signature

$$(g;+;[\,];\{\ \}) \quad \text{or} \quad (g;—).$$

Throughout this thesis when we write NEC group or Fuchsian group we will actually mean NEC group or Fuchsian group with compact quotient space. Macbeath [25] found every group isomorphism between NEC groups can be realized geometrically. That is if Γ_1 and Γ_2 are NEC groups and $\varphi:\Gamma_1\to\Gamma_2$ is an isomorphism, then there is a homeomorphism ω, of $\mathcal{U}$ such that $\varphi(\gamma)=\omega\gamma\omega^{-1}$ for all $\gamma\in\Gamma$. He was then able to show that a signature for a NEC group is unique up to (a) permutation of the proper periods, (b) permutation of the period cycles,

(c) cyclic permutation of link periods in any period cycle and (d) in the case of (1.4) simultaneous inversion of all period cycles and in case (1.5) inversion of any number of period cycles. With this understanding the signature of a NEC group is unique.

Singerman [34] determined the hyperbolic area of a fundamental region of a NEC group. This depends only on the signature of the group and not on the fundamental region chosen and so we denote the hyperbolic area of a fundamental region for Γ by $\mu(\Gamma)$ unambiguously.

Theorem 1.7

Let Γ be a NEC group with signature (1.4). Then

$$\mu(\Gamma) = 2\pi\left(2g - 2 + s + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{j=1}^{s}\sum_{k=1}^{t_j}\left(1 - \frac{1}{n_{jk}}\right)\right).$$

If Γ has signature (1.5), then

$$\mu(\Gamma) = 2\pi\left(g - 2 + s + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) + \frac{1}{2}\sum_{j=1}^{s}\sum_{k=1}^{t_j}\left(1 - \frac{1}{n_{jk}}\right)\right).$$

Let Γ be a NEC group and $\Lambda < \Gamma$ be a subgroup of finite index, say n. Then a system of right cosets can be found and we can write

$$\Gamma = \Lambda\gamma_1 + \Lambda\gamma_2 + \cdots + \Lambda\gamma_n.$$

If F is a fundamental region for Γ then it can be shown that

$$\gamma_1 F \cup \gamma_2 F \cup \cdots \cup \gamma_n F$$

is a (compact) fundamental region for Λ. Now each $\gamma_i F$ is also a fundamental region for Γ and so we see

$$\frac{\mu(\Lambda)}{\mu(\Gamma)} = |\Gamma : \Lambda|$$

which is known as the *Riemann-Hurwitz formula*.

Theorem 1.8

Let Γ be a NEC group with sig (1.4), then Γ^+, the canonical Fuchsian group of Γ, has signature

$$(2g + s - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{st_s}).$$

If Γ has signature (1.5), then Γ^+ has signature

$$(g+s-1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{st_s}).$$

Section 1.3 Riemann Surfaces and their Uniformization

In this section we shall give the formal definition of Riemann Surfaces, which allows us to define holomorphic functions between them, notably on to the complex plane which give analytic functions and on to the Riemann sphere to give meromorphic functions. We shall also say how Riemann Surfaces can be thought of as equivalent and how they can be uniformized as quotients of $\mathcal{U}$ by Fuchsian groups.

Any surface S, is covered by a collection of open sets U_i, such that for each U_i there is a homeomorphism $\Phi_i : U_i \to V_i$ where V_i is an open subset of $\mathbf{C}$. The set of such pairs $\mathcal{A} = \{(U_i, \Phi_i)\}$ is called an *atlas* for S. If $s \in U_i$ we say (U_i, Φ_i) is a chart at s and that $z_i = \Phi_i(s)$ is a local coordinate (or local parameter) at s. The functions

$$\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$$

are called *coordinate transformation functions* and are defined whenever $U_i \cap U_j$ is non-empty. An atlas $\mathcal{A}$, on S is said to be *analytic* if all its coordinate transformation functions are analytic. We define two analytic atlases on S to be *compatible* if their union is an analytic atlas. Compatibility is an equivalance relation and an equivalance class of atlases is called a *complex structure* on S.

A *Riemann Surface* is a surface together with a complex structure. Usually we will denote a Riemann surface by the underlying surface only.

Examples i) The atlas of just one chart $\{\mathbf{C}, \mathrm{id} : \mathbf{C} \to \mathbf{C}\}$ on $\mathbf{C}$ is obviously analytic and is a representative of an equivalence class of analytic atlases which give a complex structure on $\mathbf{C}$ making it a Riemann surface.

ii) Consider $\Sigma = \mathbf{C} \cup \{\infty\}$ topologized as the one-point compactification of $\mathbf{C}$. There is an atlas on Σ consisting of just two charts (U_i, Φ_i) $i = 1, 2$ where $U_1 = \mathbf{C}$ with $\Phi_1 = \mathrm{id} : \mathbf{C} \to \mathbf{C}$ and $U_2 = \Sigma \backslash \{0\}$ with $\Phi_2 : \Sigma \backslash \{0\} \to \mathbf{C}$ given by $\Phi_2(z) = 1/z$ for $z \in \mathbf{C}$ and $\Phi_2(z) = 0$ for $z = \infty$. We have $U_1 \cap U_2 = \mathbf{C} \backslash \{0\}$ and

$$\Phi_1(U_1 \cap U_2) = \mathbf{C} \backslash \{0\}, \qquad \Phi_2(U_1 \cap U_2) = \mathbf{C} \backslash \{0\}$$

with

$$1/z = (\Phi_1 \circ \Phi_2^{-1})(z) = (\Phi_2 \circ \Phi_1^{-1})(z),$$

which is analytic on $\mathbf{C}\backslash\{0\}$. The resulting Riemann surface is called the *Riemann Sphere* and in fact there is just one complex structure on Σ.

iii) The atlas $\{(\mathcal{U}, \mathrm{id} : \mathcal{U} \to \mathcal{U})\}$ is again clearly analytic and gives a complex structure on $\mathcal{U}$ making it a Riemann surface.

Definition 1.9 Let S_1 and S_2 be two Riemann surfaces. Then a continuous function $f : S_1 \to S_2$ is called *holomorphic* if whenever (U, Φ) and (W, Ψ) are charts on S_1 and S_2 respectively, with $U \cap f^{-1}(W) \neq \emptyset$, the functions

$$\Psi \circ f \circ \Phi^{-1} : \Phi(U \cap f^{-1}(W)) \to \mathbf{C}$$

are analytic.

This definition is independent of the choices of atlases of charts for the complex structures on S_1 and S_2.

If $S_2 \subseteq \mathbf{C}$ in the above definition then f is said to be *analytic* and if $S_2 \subseteq \Sigma$ then f is said to be *meromorphic*.

If $f : S_1 \to S_2$ is a holomorphic homeomorphism then it can be shown that $f^{-1} : S_2 \to S_1$ is also a holomorphic homeomorphism and that local coordinates are transformed conformally by f and f^{-1}. We say f is a *conformal equivalence* and that S_1 and S_2 are *conformally equivalent*, written as $S_1 \simeq S_2$. Two conformally equivalent surfaces share the same analytic properties and are therefore indistinguishable in terms of their complex structure.

Definition 1.10 If S is a Riemann surface, then an *automorphism* of S is a conformal or anti-conformal homeomorphism $f : S \to S$.

We shall denote the group of all automorphisms of S by *Aut(S)* and the subgroup of conformal (orientation preserving) automorphisms by $Aut^+(S)$.

Theorem 1.11

If S is a connected Riemann surface, not conformally equivalent to the Riemann sphere Σ, the plane, the punctured plane $\mathbf{C}\backslash\{0\}$, or a torus, then S is conformally equivalent to $\mathcal{U}/K$ for some subgroup K of $\mathcal{L}^+$ which acts discontinuously on $\mathcal{U}$.

In the above theorem K is a Fuchsian surface group and will have signature $(g;—)$ where g is the genus of S. This theorem is a consequence of the uniformisation theorem of H. Poincaré, F. Klein and P. Koebe which says every simply connected Riemann surface is conformally equivalent to the Riemann Sphere or the complex plane or the upper half plane.

Suppose K_1 and K_2 are Fuchsian surface groups and $\Pi_{K_1} : \mathcal{U} \to \mathcal{U}/K_1$ and $\Pi_{K_2} : \mathcal{U} \to \mathcal{U}/K_2$ are the natural projections. Then ω, a homeomorphism of $\mathcal{U}$, is said to induce the homeomorphism $f : \mathcal{U}/K_1 \to \mathcal{U}/K_2$, if

$$[\omega z]_{K_2} = f\left([z]_{K_1}\right), \qquad \text{for all} \quad z \in \mathcal{U},$$

where $[z]_{K_1}$ is the K_1-orbit of the point $z \in \mathcal{U}$.
Or, equivalently, if the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\omega} & \mathcal{U} \\ {\scriptstyle \Pi_{K_1}}\downarrow & & \downarrow{\scriptstyle \Pi_{K_2}} \\ \mathcal{U}/K_1 & \xrightarrow{f} & \mathcal{U}/K_2 \end{array}$$

Clearly, if $\omega : \mathcal{U} \to \mathcal{U}$ is a homeomorphism such that $\omega K_1 \omega^{-1} = K_2$ then the mapping

$$f\left([z]_{K_1}\right) = [\omega z]_{K_2}$$

is well defined and f is a homeomorphism.

If there is a homeomorphism $f : \mathcal{U}/K_1 \to \mathcal{U}/K_2$, then $f\,\Pi_{K_1}$ is a homeomorphism from $\mathcal{U}$ to $\mathcal{U}/K_2$ as is Π_{K_2} and so by the theory of covering spaces there is a homeomorphism $\omega : \mathcal{U} \to \mathcal{U}$ that induces f. This homeomorphism ω is not uniquely defined as f is also induced by ωk for all $k \in K_1$. Since $f\,\Pi_{K_1} = \Pi_{K_2}\,\omega$, ω maps K_1-orbits of $\mathcal{U}$ to K_2-orbits of $\mathcal{U}$. Hence for each $k_1 \in K_1$ there is a $k_2 \in K_2$ such that $\omega k_1 \omega^{-1} = k_2$ and so $\omega K_1 \omega^{-1} = K_2$. Thus we have shown that every homeomorphism $\omega : \mathcal{U} \to \mathcal{U}$, obeying $\omega K_1 \omega^{-1} = K_2$, induces a homeomorphism from $\mathcal{U}/K_1$ to $\mathcal{U}/K_2$ and that every homeomorphism $f : \mathcal{U}/K_1 \to \mathcal{U}/K_2$ is induced by a homeomorphism of $\mathcal{U}$ to itself that takes K_1 to K_2 by conjugation.

Note that f above will be conformal if and only if ω is conformal.

Theorem 1.12

If K_1 and K_2 are Fuchian surface groups then $\mathcal{U}/K_1$ and $\mathcal{U}/K_2$ are conformally equivalent if and only if K_1 and K_2 are conjugate in $\mathcal{L}$.

If we let $K_1 = K_2 = K$ in the above then we see f is an automorphism of $\mathcal{U}/K$ if and only if ω is an element of the normalizer of K in $\mathcal{L}$.

Theorem 1.13

Let Γ be a NEC group with compact quotient space. Then the normaliser of Γ in $\mathcal{L}$ is also a NEC group with compact quotient space.

This follows from the well known result that the normalizer of a Fuchsian group in $\mathcal{L}^+$ is itself Fuchsian.

Theorem 1.14

Let S be a Riemann surface uniformized by some Fuchsian surface group K, then the group of automorphisms of S is isomorphic to $\mathcal{N}_{\mathcal{L}}(K)/K$.

Indeed every group of automorphisms of S is isomorphic to Γ/K where Γ is some Fuchsian group such that K is normal in Γ. In particular the group of all orientation preserving automorphisms of S, denoted by $Aut^+(S)$, is isomorphic to $\mathcal{N}_{\mathcal{L}^+}(K)$. Therefore a group G, acts as a group of automorphisms of S if and only if there is an epimorphism from a NEC group to G whose kernel is K. Hence

$$|G| = |\Gamma : K| = \frac{\mu(K)}{\mu(\Gamma)} = \frac{2\pi(2g-2)}{\mu(\Gamma)}$$

If Γ is Fuchsian then G will be a group of orientation preserving automorphisms and it can be seen that $\mu(\Gamma) \geq \pi/21$ with equality only when Γ has signature $[2, 3, 7]$. Thus the order of $Aut^+(S)$ is less than or equal to $84(g-1)$. This bound was first obtained by Hurwitz [21], who showed that it is attained when $g = 3$. It is now known that it is realized for infinitely many g, see [26]. If G is a finite group which acts as a group of orientation preserving automorphisms of a compact Riemann surface of genus $g \geq 2$ and the order of G is $84(g-1)$, then we call G a *Hurwitz group*. Wiman [39] found that $2(2g+1)$ is the upper bound for the order of a cyclic group of automorphisms of a compact Riemann surface of genus $g \geq 2$, this bound is attained for every $g \geq 2$. The cyclic group will be the image of a triangle group with signature $[2, 2g+1, 2(2g+1)]$.

Definition 1.15 A homomorphism from a NEC group to a finite group whose kernel is a surface group is called a *surface kernel homomorphism.*

Lemma 1.16

A homomorphism φ, from a NEC group Γ, on to a finite group, is a surface kernel homorphism if and only if it preserves the orders of the elements of Γ with finite order.

Harvey [17] used this result to find necessary and sufficient conditions on the signature of a Fuchsian group for there to be surface kernel homorphism from the Fuchsian group onto a cyclic group.

Theorem 1.17

Let Γ be a Fuchsian group with signature $(g; [m_1, \ldots, m_r])$. Then there is a surface kernel homomorphism from Γ on to a cyclic group of order N if and only if the following conditions are satisfied.

(i) $l.c.m.\{m_1, \ldots, \hat{m}_i, \ldots, m_r\} = M$ for all i, where $\hat{m}_i$ denotes the omission of m_i.

(ii) M divides N and if $g = 0$ then $M = N$.

(iii) $r \neq 1$ and if $g = 0$ then $r \geq 3$.

(iv) If M is even, then the number of periods divisible by the maximum power of 2 dividing M is even.

Harvey completely answered the question: Given a cyclic group, what is the minimum genus of a surface for which the cyclic group acts as a group of (conformal) automorphisms. Maclachlan tackled this problem for non-cyclic abelian groups in [27].

We now define a type of automorphism which will be considered in more detail later on.

Definition 1.18 An anti-conformal involution of a Riemann surface is called a *symmetry* and a Riemann surface that admits such an automorphism is said to be *symmetric*.

The fixed point set of a symmetry T, of S, is a union of disjoint smooth, simple, closed curves of S. We shall call these the *mirrors* of T. The topological action of T, on S, is fully described by the number of its mirrors, say k, and whether the quotient space $S/\langle T \rangle$ is orientable or not.

We define the *species* of T, $sp(T)$, to be

$$sp(T) = \begin{cases} +k & \text{if } S/\langle T \rangle \text{ is orientable;} \\ -k & \text{if } S/\langle T \rangle \text{ is non-orientable.} \end{cases}$$

In fact, the quotient is orientable if and only if the complement of the mirrors of T is not connected. Harnack found that if S has genus g then $0 \leq k \leq g+1$, both these bounds are attained for all g. Clearly any symmetries of S which are conjugate in $Aut(S)$ will have the same species. If T is fixed point free then $S/\langle T\rangle$ is necessarily non-orientable by a previous remark and we omit the minus sign in the species of T, just writing $sp(T) = 0$. The *symmetry type* of a symmetric Riemann surface is the unordered list of species associated with the conjugacy classes of symmetries that the Riemann surface admits. For example the Riemann sphere Σ, has only two conjugacy classes of symmetries, one represented by the antipodal map which is fixed point free and the other represented by the reflection $T : z \mapsto \bar{z}$ which has one mirror. It is easy to see that $\Sigma/\langle T\rangle$ is orientable and so the symmetry type of the Riemann sphere $st(\Sigma)$, is

$$st(\Sigma) = \{0, +1\}.$$

Bujalance and Singerman [6] obtained results for *hyperelliptic* Riemann surfaces (those surfaces admitting a conformal automorphism J, of order two such that the surface quotient $\langle J\rangle$ has genus zero). In particular they determined all possible symmetry types of compact Riemann surfaces of genus two.

Singerman [35] showed that the infinite family of Hurwitz groups that Macbeath found [26] are all symmetric and Broughton, Bujalance, Costa, Gamboa and Gromadzki [4], found the symmetry type of an infinite subset of these surfaces.

The case of a compact Riemann surface admitting two non-conjugate symmetries has also been considered [7], [8] and [22].

Section 1.4 Maps and Hypermaps

In this section we shall define maps and more importantly regular maps and indicate how Jones and Singerman [23] showed that every map is isomorphic to some canonical map on a compact Riemann surface. This means that maps can be thought of as lying on surfaces with constant curvature with respect to which the edges are geodesics, also automorphism groups of maps are isomorphic to automorphism groups of Riemann surfaces. We shall also define hypermaps and indicate the analagous results.

Let $\mathcal{E}$ be a non-empty set of topological spaces (edges) each of which is homeomorphic to the closed interval $I = [0,1]$ or the circle S^1 and let $\mathcal{V}$ (the set of vertices) be a subset of $\mathcal{G} = \bigcup_{e \in \mathcal{E}} e$ such that:

(i) If e is homeomorphic to S^1 then $|e \cap \mathcal{V}| = 1$ and if e is homeomorphic to I then $e \cap \mathcal{V}$ consists of one or two of the end points of e.

(ii) For all distinct $e_1, e_2 \in \mathcal{E}$ the intersection of $e_1 \backslash (e_1 \cap \mathcal{V})$ and $e_2 \backslash (e_2 \cap \mathcal{V})$ is empty.

(iii) For any $v \in \mathcal{V}$ the set $\{e \in \mathcal{E} : v \cap e\}$ is finite.

When these conditions are satisfied then $(\mathcal{G}, \mathcal{V})$ is said to be an *allowed* graph. This definition differs from the normal definition in that it allows loops, multiple edges and free edges.

Definition 1.19 A *map* $\mathcal{M}$, is an embedding of an allowed graph $(\mathcal{G}, \mathcal{V})$ into an orientable, connected surface S without boundry, such that $S \backslash \mathcal{M}$ is a collection of two-cells, known as faces.

Examples

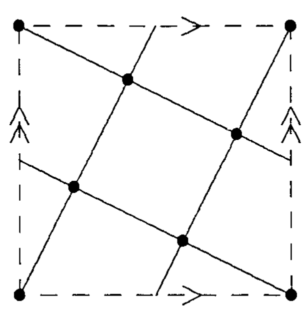


fig.1

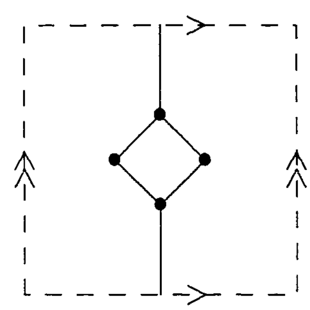


fig.2

The above figures are emmbeddings of allowed graphs in the torus, the first is a map while the second is not as the complement of this embedding consists of a disc and an annulus.

As in [23] we restrict our attention to the case when $\mathcal{G}$ is connected.

Maps can also be considered in purely algebraic terms. That is as quadruples (G, Ω, x, y), where Ω is a set and x, y are permutations of Ω, such that x is an involution and the group $G := \langle x,\ y \rangle$ is transitive on Ω. The set Ω can be thought of as the set of all pairs

$$\{(e, v) : e \in \mathcal{E}, v \in \mathcal{V}, e \cap v = v\}$$

known as *darts* or *brins*. The permutation x interchanges the two darts of each edge and loop and fixes the single dart of each free edge. The permutation y cyclically permutes the darts about each vertex according to the orientation of the underlying surface. These two approaches have formally been shown to be equivalent in [23].

The valency of a vertex is the number of edges it is incident with and the valency of a face is the number of edges that form its boundary. If the *least common multiple*, or *l.c.m*, of the valencies of the vertices and faces is m and n respectively, then the map is said to be of *type* $\{m, n\}$.

The *dual* of a map $\mathcal{M}$ is, as one would expect, obtained in the following way. Firstly, place a vertex in each face of the map and let these form the vertex set of the dual map. Secondly, for each edge e of the map adjoin an edge e' to the dual. If e is a common edge of two faces in $\mathcal{M}$ then e' connects the two associated vertices in the dual. If e is a free edge and hence only on the boundary of one face of $\mathcal{M}$ then e' is a loop at the associated vertex in the dual. We require that e' crosses no edge of $\mathcal{M}$ other than e and that it intersects other edges of the dual only in the vertices described above. Clearly if a map is of type $\{m, n\}$ then its dual is of type $\{n, m\}$.

Definition 1.20 If $\mathcal{M}_i = (\mathcal{G}_l, \mathcal{V}_i, S_i)$ $(i = 1, 2)$ are two maps then an *isomorphism* $\varphi' : \mathcal{M}_1 \to \mathcal{M}_2$ between them is a orientation preservimg homeomorphism $\varphi : S_1 \to S_2$, that obeys

$$\varphi^{-1}(\mathcal{G}_2) = \mathcal{G}_1, \qquad \varphi^{-1}(\mathcal{V}_2) = \mathcal{V}_1.$$

Definition 1.21 A map is said to be *regular* if its automorphism group is transitive on the darts of the map.

Thus regular maps can be thought of as a generalisation of the platonic solids.

A Fuchsian triangle group Γ, has signature of the form $[l, m, n]$, because it is Fuchsian we know $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. A triangle group can be constructed as follows. Consider the NEC group Γ_* generated by reflections in the side of a hyperbolic triangle with internal angles $\pi/l, \pi/m$ and π/n. This triangle is a fundamental region for Γ_* which has signature $(0; +; [\,]; \{(l, m, n)\})$ or just (l, m, n). Then Γ_*^+ is a triangle group with signature $[l, m, n]$. It is known that all triangle groups with a given signature are conjugate in $\mathcal{L}$ and so $\Gamma[l, m, n]$ is the canonical Fuchsian group of some *extended triangle group* $\Gamma_*[l, m, n]$.

Note that Γ_* preserves a tessellation of $\mathcal{U}$ by triangles with internal angles $\pi/l, \pi/m$ and π/n. Any two of these, that are non-congruent in Γ_*^+, are together a fundamental region for Γ and Γ also preserves the above tessellation. If $l = 2$ then Γ will preserve a tessellation of $\mathcal{U}$ by n-gons, m of which meet at a vertex (and a tessellation by m-gons, n of which meet at a vertex). This is the *universal map* of type $\{m, n\}$ of Jones and Singerman. If Λ is a subgroup of Γ above with finite index, then $S := \mathcal{U}/\Lambda$ is compact. Moreover S carries a map inherited from the universal map. This map will have type $\{r, s\}$ where r divides m and s divides n. The following thorem was proved in [23].

Theorem 1.22

For each map of type $\{m, n\}$ there is an associated subgroup M, of a triangle group $\Gamma[2, m, n]$ such that the map is isomorphic to the projection of the universal map of this type on to $\mathcal{U}/M$.

M is called the associated *map subgroup*. Another important result proved in the same paper is that a map is regular if and only if its associated map subgroup is normal in Γ, in which case M will necessarily be a surface group. Hence Γ/M is a group of automorphisms of S and the group of *all* orientation preserving automorphisms of the map.

The problem of enumerating regular maps has usually been tackled via their automorphism groups. It was seen that such a group can be generated by two elements, one of order m and one of order n, whose product has order two. Groups such as these are finite homomorphic images of Fuchsian triangle groups with signature $[2, m, n]$. In this way Coxeter and Moser [10] gave the automorphism groups of all the regular maps of genus two, Sherk gave those of genus three [30] and Garbe those of genus four, five, six [13] and seven [15]. Although all the automorphism groups of regular maps in these cases are known it is not clear how many maps there are, as it is possible that a group may act as an automorphism group for more than one regular map. Determining how many regular maps have a certain group G as their automorphism group is equivalent to calculating the number of surface kernel epimorphisms from triangle groups, with one period equal to two, to G with non-conjugate kernels. The regular maps of genus two have been throughly examined by Breda d'Azevedo and Jones [2], indirectly by Broughton [3] who determined all Riemann surfaces of genus two or three with non-trivial automorphism groups, and others. They have shown that there is only one regular for each group given in [10]. Breda d'Azevedo and Jones actually considered a wider class of objects called *hypermaps* of which maps are a subset.

Corn and Singerman [9] give the following definition of (topological) hypermaps, which strictly speaking precludes maps by our earlier definition but it is easy to see how they can be identified with maps in the appropriate cases.

A hypermap $\mathcal{H}$ on a compact orientable surface S is a triple (S, R, A), where R and A are closed subsets of S such that:

(i) $B = R \cap A$ is a non-empty finite set.

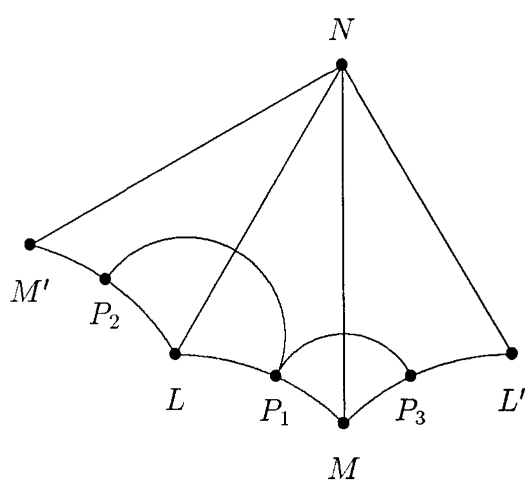
(ii) $R \cup A$ is connected.

(iii) Each component of R is homeomorphic to a closed disc and each component of A is homeomorphic to a closed disc.

(iv) Each component of $S \backslash (R \cup A)$ is homeomorphic to an open disc.

(Hypermaps can also be defined on non-compact orientable surfaces). The genus of $\mathcal{H}$ is the genus of S. The components of R are called *hyperedges*, the components of A the *hypervertices* and the components of $S \backslash (R \cup A)$ are called *hyperfaces*. The elements of B are called *bits* and a hypermap is said to be *regular* if its group of conformal automorphisms is transitive on the bits. If l and m are the least common multiples of the number of bits in the hyperedges and in the hypervertices respectively and n is the least common multiple of the number of hyperedges incident with each hyperface then the hypermap is said to be of *type* $\{l, m, n\}$.

We defined the universal map of type $\{m, n\}$, $(\frac{1}{m} + \frac{1}{n} < \frac{1}{2})$, via a Fuchsian triangle group with signature $[2, m, n]$. The *universal hypermap* of type $\{l, m, n\}$, $(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1)$, is defined via a Fuchsian triangle group with signature $[l, m, n]$ and can be constructed as follows.



Let F be a triangle in $\mathcal{U}$ with vertices L, M and N and internal angles $\pi/l, \pi/m$ and π/n, such that F is half a fundamental region for the above triangle group. F will tessellate $\mathcal{U}$ under the group generated by reflections in LM, MN and NL. If we reflect F in NL and MN we obtain the triangles $LM'N$ and $L'MN$. Let P_1 be a point on LM and P_2, P_3 be its images on LM' and ML'. The geodesics P_1P_2 and P_1P_3 intersect NL and MN at right angles. If we repeat this around L and M and for all the images of L and M under the above reflection group then we obtain a tessellation of $\mathcal{U}$ by regular l-gons, m-gons and $2n$-gons, see **[9]**. Thus we have a hypermap $(\mathcal{U}, \hat{R}, \hat{A})$ of type $\{l, m, n\}$ where $\hat{R}$ is the set of l-gons, $\hat{A}$ is the set of m-gons and the complement of $\hat{R} \cup \hat{A}$ is a union of $2n$-gons which are the hyperfaces. If $l = 2$ (or $m = 2$) then the l-gons (or m-gons) are just lines and so if we take P_1 to be M (or L) then we have a map in terms of the earlier definition. The hypermap constructed above is the universal hypermap of type $\{l, m, n\}$ in [9].

Most of the analogous thorems of maps also hold for hypermaps. For every hypermap $\mathcal{H}$, of type $\{l, m, n\}$, $(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1)$, there is an *associated hypermap subgroup* H, of a Fuchsian triangle group $\Gamma[l, m, n]$, such that $\mathcal{H}$ is isomorphic to the hypermap on the quotient $\mathcal{U}/H$ inherited from the universal hypermap of type $\{l, m, n\}$. Also, $\mathcal{H}$ is regular if and only if H is normal in Γ. Hence H is a surface group and Γ/H is isomorphic to the group of all conformal automorphisms of $\mathcal{H}$. Regular hypermaps of genus greater than one are then described by surface kernel homomorphisms from triangle groups onto finite groups.

Chapter 2

Subgroups of NEC Groups

The main aim of this chapter is to present a Theorem by A. H. M. Hoare concerning subgroups of cocompact NEC groups with finite index. Hoare's Theorem provides an algorithm to calculate the signature of the subgroup, given the signature of the big group and the action of its canonical generators on the cosets of the subgroup. This is an extension of a Theorem of Singerman, who given the same information about Fuchsian groups gives similar results. From now on, unless we say otherwise, when we write NEC group we mean cocompact NEC group.

Section 2.1 Finite Subgroups

Let Γ be a NEC group and let F be the fundamental region associated to the canonical presentation in Chapter One. Suppose $x \in \Gamma$ is elliptic and fixes $q \in \mathcal{U}$, then there is at least one point p, in F, such that p is an image of q under Γ. By the definition of F we know p must lie on the boundary of F and hence is the fixed point of a canonical elliptic generator of Γ. Suppose $c \in F$ is a reflection, then it fixes an arc l, in $\mathcal{U}$, and there is an arc segment k, in F, which is an image of a segment of l under Γ. Again, k must lie on the boundary of F and hence is fixed by a canonical reflection generator of Γ. These observations lead to the following lemma.

Lemma 2.1

Let Γ be a NEC group with signature (1.4) or (1.5). Then any element of finite order in Γ is conjugate to one of the following:

i) A power of x_i, for some $i \in \{1, \ldots, r\}$.

ii) A power of $c_{jk-1}c_{jk}$, for some $j \in \{1, \ldots, s\}$, $k \in \{1, \ldots, t_s\}$.

iii) c_{jk}, for some $j \in \{1, \ldots, s\}$, $k \in \{1, \ldots, t_s\}$.

Clearly any finite subgroup of Γ can only contain elements of this form.

Lemma 2.2

Let g be an elliptic element fixing $p \in \mathcal{U}$ and let h be another isometry of $\mathcal{U}$ that does not fix p. Then $[g, h]$, the commutator of g and h, is hyperbolic.

For a proof of this lemma see Theorem 7.39.2 of [1]. The previous two lemmas together prove the following theorem.

Theorem 2.3

Let Γ be a NEC group with signature (1.4) or (1.5). Then any finite subgroup is conjugate to a subgroup of $\langle x_i \rangle$ for some x_i, or a subgroup of $\langle c_{jk-1}, c_{jk} \rangle$ for some pair c_{jk-1}, c_{jk}, and hence is cyclic or dihedral.

Section 2.2 Singerman's Theorem

Singerman, in [32], considers Fuchsian groups which may contain parabolic elements or hyperbolic boundary elements so they may have non-compact quotient space and be of the first or second kind. We shall just state the theorem for cocompact Fuchsian groups.

Theorem 2.4

Let Γ be a Fuchsian group with signature $(g; [m_1, \ldots, m_r])$. Then Γ contains a subgroup Λ, of finite index N and signature

$$(h; [n_{11}, n_{12}, \ldots, n_{1p_1}, \ldots, n_{r1}, n_{r2}, \ldots, n_{rp_r}]),$$

if and only if:

(a) There exists a finite permutation group G, transitive on N points and an epimorphism $\Theta : \Gamma \to G$, satisfying the following.

The permutation $\Theta(x_i)$ has cycles of length m_i and precisely p_i other cycles whose lengths are $m_i/n_{i1}, \ldots, m_i/n_{ip_i}$.

(b) $\mu(\Lambda)/\mu(\Gamma) = N$.

Remark If Γ does contain a subgroup Λ, then the elliptic generators of Λ are powers of conjugates of the x_i's in Γ. Let $\alpha x_i^k \alpha^{-1}$ be a canonical generator of

Λ, for some $\alpha \in \Gamma$ and $1 \leq k \leq m_i$. Then $\alpha = \lambda g_j$, for some $\lambda \in \Lambda$ and $1 \leq j \leq N$, where $\Lambda g_1, \ldots, \Lambda g_N$ are a system of right Λ cosets. Hence $\Lambda g_j x_i^k = \Lambda g_j$ and so clearly, each elliptic generator of Λ is induced by the action of the x_i's on the Λ-cosets.

If there is a homomorphism $\Theta : \Gamma \to G$, as in the Theorem, then $\Theta^{-1}(Stab(1))$ is a subgroup with the required index and signature, where $Stab(1)$ is the stabilizer of 1 in G.

Example 1 Suppose Γ has signature $(0; [2, 3, 4, 12])$ and presentation

$$\langle x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^3 = x_3^4 = x_4^{12} = x_1 x_2 x_3 x_4 = 1 \rangle.$$

Consider the following epimorphism from Γ to a group of permutations which is transitive on eight points.

$$\begin{aligned} \Theta : x_1 &\longmapsto (1\,2)(3\,4)(5\,6)(7)(8) \\ x_2 &\longmapsto (1)(2)(3\,4\,5)(6\,7\,8) \\ x_3 &\longmapsto (1\,2\,3\,4)(5\,6)(7)(8) \\ x_4 &\longmapsto (1\,4\,6\,3)(2)(5\,8\,7) \end{aligned}$$

$\Theta(x_1)$ has three 2-cycles and two 1-cycles and so contributes three periods of order $1 = 2/2$ and two periods of order $2 = 2/1$ to the subgroup corresponding to the above action. $\Theta(x_2)$ has two 1-cycles and two 3-cycles and so contributes two periods of order $3 = 3/1$ and two of order $1 = 3/3$. $\Theta(x_3)$ has one 4-cycle, one 2-cycle and two 1-cycles and so contributes one period of order 1, one of order 2 and two of order 4. Finally, $\Theta(x_4)$ contributes one period of order 3, one of order 4 and one of order 12. Note that a proper period of order 1 just contributes a generator which is the identity to the group and so is omitted.

Hence Γ contains a subgroup Λ say, of index eight with signature

$$(g; [2, 2, 2, 3, 3, 3, 4, 4, 4, 12])$$

where g is determined by

$$\mu(\Lambda) = 8\mu(\Gamma)$$

and so

$$g = 2.$$

Example 2 Suppose Γ is a triangle group with signature $[2, 3, 2t]$, $(t > 3)$, and presentation

$$\langle x_1, x_2, x_3 \mid x_1^2 = x_2^3 = x_3^{2t} = x_1 x_2 x_3 = 1 \rangle.$$

Consider the epimorphism from Γ to a group of permutations on six points defined as follows.

$$\begin{aligned} \Theta : x_1 &\longmapsto (1\,2)(3\,4)(5\,6) \\ x_2 &\longmapsto (1\,3\,5)(2\,6\,4) \\ x_3 &\longmapsto (1\,6)(2\,3)(4\,5) \end{aligned}$$

The group generated by these permutations is clearly transitive on the six points on which it acts. Hence Γ contains a subgroup Λ, with index six and signature $(g; [t, t, t])$, where g is determined by the Riemann-Hurwitz formula and is in fact zero. This gives an example of an inclusion between triangle groups which will be important later.

Note that each $\Theta(x_i)$ is regular, i.e. consists of disjoint cycles of the same length, this is because Λ is a normal subgroup of Γ.

Section 2.3 Hoare's Theorem

By canonical generators in the next theorem we mean generators which are associated to the signature of the group.

Theorem 2.5 [20]

Let Γ be a NEC group with signature of type (1.4) or (1.5) and let Λ be a subgroup with finite index.

Given the action of the canonical generators of Γ on the Λ-cosets, the signature of (and hence a presentation for), Λ can be determined as follows.

(i) The cosets fixed by canonical reflection generators of Γ correspond to the reflection generators of Λ.

(ii) If c and d are linked canonical reflection generators of Γ with link period n (*i.e.* cd is elliptic of order n), then c and d generate a dihedral group D_n. Let σ be an orbit of the Λ-cosets under D_n and let K be a coset in σ. If m is the least non-zero positive integer such that $K(cd)^m = K$, then either;

a) σ contains no cosets fixed by c or d, in which case σ has length $2m$ and gives an elliptic generator for Λ with order n/m, or

b) σ contains exactly two cosets fixed by c and d, one fixed by each if m is odd, two fixed by one and none by the other if m is even. σ has length m and the reflection generators of Λ corresponding to the two fixed cosets are linked with link period n/m. Each reflection generator of Λ appears in precisely two of these links, unless it is linked only to itself. This gives the period cycles of Λ, each coming from one of the period cycles of Γ.

(iii) If x is a canonical elliptic generator of Γ with period m say, then each orbit of Λ-cosets under x, of length n say, gives an elliptic generator for Λ of order m/n. Every elliptic generator of Λ is given in this way or by part (a) of (ii) above.

(iv) Λ has orientable quotient space if and only if the Λ-cosets in Γ divide into two classes (one of which may be empty), such that every canonical reflection generator of Γ either fixes a coset or takes it into the other class, every other orientation reversing canonical generator of Γ interchanges the classes and every orientation preserving canonical generator fixes the classes.

(v) The genus of the quotient space of Λ is given by the Riemann-Hurwitz formula

$$N\mu(\Gamma) = \mu(\Lambda)$$

where N is the index of Λ in Γ.

Remarks

(i) If d is a refection in Λ, then by (2.1) it must be conjugate to some canonical reflection generator c say, of Γ. Hence $d = \alpha c \alpha^{-1}$ for some $\alpha \in \Gamma$. Now $\alpha = \lambda g$ where g is a representitive for some Λ coset and $\lambda \in \Lambda$. Therefore $gcg^{-1} \in \Lambda$ and $\Lambda gc = \Lambda g$, so every reflection in Λ is conjugate to some reflection induced by one of the canonical reflection generator of Γ fixing a coset.

(ii) Part (ii) of the Theorem comes from applying the following lemma.

Lemma Let $D_n = \langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$ and $C_n = \langle\, r\, \rangle$ where $r = ab$. If H is a subgroup of D_n then either

1) $H \leq C_n$ and a and b fix no H-cosets, or

2) $H \nleq C_n$. Let m be the exponent of r modulo H. If m is odd, then a and b fix exactly one coset each. If m is even, then one of a and b fixes two cosets and the other none.

The application is to consider the group $\Gamma' := \langle c, d \rangle \simeq D_n$ generated by a pair of linked reflection generators of Γ. If $H = \Lambda \cap \Gamma'$, then the action of Γ' on the H-cosets is the same as that of H on the orbit of Λ under Γ'.

(iii) Part (iii) is just one way round of Singerman's Theorem combined with part (ii)(a) and (2.1).

(iv) This is part of Theorem 2 in [19].

(v) The links in (ii)(b) form chains which correspond to the period cycles of the subgroup, the link periods are also given by these links and are ordered up to choosing the direction of traversing the chains. Hoare indicates how to do this but the subgroups we will be concerned with have empty period cycles and so we omit that part of the Theorem.

(vi) The action of Γ on the Λ-cosets provides a homomorphism from Γ onto a group of permutations on N points which is transitive on these points. The stabilizers of these points lift to Λ and its conjugates in Γ. Hence Γ contains a subgroup of index N and of a certain signature if and only if there is a homomorphism from Γ onto a permutation group on N points which is transitive on these points and provides the right signature by the construction in the previous Theorem. Note that if the subgroup is normal then each canonical generator maps to a permutation with regular cycle structure. The converse is not true.

Example Let Γ have signature $(0; +; [6,6]; \{(5,8,12)\})$ and presentation with generators

$$x_1, x_2, e, c_0, c_1, c_2, c_3$$

and relators

$$x_1^6, x_2^6$$
$$c_0^2, c_1^2, c_2^2, c_3^2$$
$$(c_0c_1)^5, (c_1c_2)^8, (c_2c_3)^{12}$$
$$e^{-1}c_0ec_3$$
$$x_1x_2e.$$

Consider the following representation of Γ acting transitively on six points.

$$\begin{aligned}
x_1 &\longmapsto (1\,4)(2\,3\,6)(5)\\
x_2 &\longmapsto (1\,6\,2\,5\,4\,3)\\
e &\longmapsto (1\,2\,3)(4\,5\,6)\\
c_0 &\longmapsto (1\,2)(3\,4)(5)(6)\\
c_1 &\longmapsto (1\,3)(2\,6)(4)(5)\\
c_2 &\longmapsto (1\,4)(2\,6)(3\,5)\\
c_3 &\longmapsto (1\,5)(2\,3)(4)(6)
\end{aligned}$$

Let Λ be the stabilizer of a point, then we see x_1 induces three proper periods 2, 3 and 6 of Λ and x_2 induces no proper periods of Λ.

Λ will have six conjugacy classes of reflections, these are represented by the reflections induced by the fixed points of c_0, c_1 and c_3. Using the obvious notation we call these induced reflections $c_{05}, c_{06}, c_{14}, c_{15}, c_{34}, c_{36}$. The orbits of the dihedral groups are as follows.

Group	Orbits
$\langle c_0, c_1\rangle \simeq D_5$	$\{1,2,3,4,6\}, \{5\}$
$\langle c_1, c_2\rangle \simeq D_8$	$\{1,3,4,5\}, \{2,6\}$
$\langle c_2, c_3\rangle \simeq D_{12}$	$\{1,2,3,4,5,6\}$
$\langle e^{-1}c_0e, c_3\rangle \simeq D_1$	$\{1,5\}\{2,3\}, \{4\}, \{6\}$

The orbit $\{2,6\}$, of $\langle c_1, c_2\rangle$, and $\{2,3\}$, of $\langle e^{-1}c_0e, c_3\rangle$, have no points fixed by the generators and so give the periods 8 and 1 for Λ. The remaining orbits each contain two points fixed by the generators and so do give links of Λ. The links given between the reflection generators of Λ by the orbits above are as follows.

Group	Links
$\langle c_0, c_1\rangle$	$c_{05} \sim c_{15},\ c_{06} \sim c_{14}$
$\langle c_1, c_2\rangle$	$c_{14} \sim c_{15}$
$\langle c_2, c_3\rangle$	$c_{34} \sim c_{36}$
$\langle e^{-1}c_0e, c_3\rangle$	$c_{34} \sim c_{06},\ c_{36} \sim c_{05}$

Thus we obtain only one chain

$$c_{05} \sim c_{15} \sim c_{14} \sim c_{06} \sim c_{34} \sim c_{36} \sim c_{05}$$

and hence one period cycle $(5,2,1,1,2,1)$ or, after omitting the ones and performing a cyclic permutation, just $(2,2,5)$. Of course when there is only one period cycle the direction in which the link periods around the chain are read is not important.

The six points cannot be partioned as in part (iv) of the Theorem and hence the quotient space of Λ is non-orientable.

The genus of the quotient space of Λ is determined by the Riemann-Hurwitz formula and is nine. Hence the signature of Λ is

$$(9; -; [2,3,6,8]; \{(2,2,5)\}).$$

It is clear how most of the above links were determined but perhaps not so for the links $c_{34} \sim c_{06}, c_{36} \sim c_{05}$. Clearly the orbits of $D := \langle e^{-1}c_0e, c_3\rangle$ are just the orbits of c_3. Now $\{4\}$ is an orbit under D. Therefore c_3 and $e^{-1}c_0e$ each fix some coset Λg_4, for $g_4 \in \Gamma$. Hence $g_4c_3g_4^{-1}, g_4e^{-1}c_0eg_4^{-1}$ are conjugate to a pair of linked canonical reflection generators of Λ. We know $g_4c_3g_4^{-1}$ is conjugate to c_{34}, so we need to determine which of c_{05}, c_{06} is conjugate to $\hat{c}_0 := g_4e^{-1}c_0eg_4^{-1}$.

$$\Lambda g_4e^{-1}c_0eg_4^{-1} = \Lambda g_4 \quad \Leftrightarrow \quad \Lambda g_4e^{-1}c_0 = \Lambda g_4e^{-1}$$

Now $\Lambda g_4e^{-1} = \Lambda g_6$ for some $g_6 \in \Gamma$, where Λg_6 is the coset associated to the point 6, this is because $(4)e^{-1} = (6)$. Therefore $g_4e^{-1} = \lambda g_6$ for some $\lambda \in \Lambda$ and $\hat{c}_0 = \lambda g_6c_0g_6^{-1}\lambda$ which is conjugate to c_{06} in Λ.

In general, if $e^{-1}c_0e = c_i$ fixes the point k then we have the link $c_{ik} \sim c_{0j}$ where $j = (k)e^{-1}$.

Chapter 3

Symmetries

From chapter two we know that any Riemann surface S say, can be uniformized by some Fuchsian surface group K say, that is $S = \mathcal{U}/K$. We also know that any group of automorphisms of S say G, is isomorphic to some quotient Γ/K where Γ is a NEC group of which K is a normal subgroup. In [35] Singerman sought to determine whether a Riemann surface is symmetric by looking at groups of (conformal) automorphisms of the surface alone. He found that to do this, it is necessary to look only at the situation when the group of automorphisms lifts to a triangle group. One of the important theorems in this paper was proved using the assumption that whenever a symmetry of a surface is adjoined to such a group of automorphisms, the new group contains the original one with index two. This is not always the case as we shall show and we will also give an alternative version of the theorem in question.

Section 3.1 Large Automorphism Groups

Let Γ be a NEC group with compact quotient space, we denote by $R(\Gamma)$ the set of all isomorphisms $r : \Gamma \to \mathcal{L}$ such that $r(\Gamma)$ is discrete and $\mathcal{U}/r(\Gamma)$ is compact. We define an equivalence relation on $R(\Gamma)$ by saying $r_1, r_2 \in R(\Gamma)$ are equivalent if there exists some $g \in \mathcal{L}$ such that $r_1(\gamma) = gr_2(\gamma)g^{-1}$, for all $\gamma \in \Gamma$. $R(\Gamma)$ quotient this relation is called the *Teichmüller space* of Γ denoted by $T(\Gamma)$. If $d(\Gamma)$ is the dimension of $T(\Gamma)$ then it is known that for a Fuchsian group Γ, $d(\Gamma) = 6g-6+2r$, where g is the genus of $\mathcal{U}/\Gamma$ and r is the number of periods in the signature of Γ. It is also known that if Γ is a proper NEC group then $d(\Gamma) = \frac{1}{2}d(\Gamma^+)$.

Let $\alpha : \Gamma_1 \to \Gamma_2$ be a monomorphism between two NEC groups and let $r_1 \in R(\Gamma_1)$ and $r_2 \in R(\Gamma_2)$. Then $r_2 \circ \alpha \in R(\Gamma_1)$ and α induces an embedding of $R(\Gamma_2)$ in $R(\Gamma_1)$. Hence α also induces an embedding of $T(\Gamma_2)$ in $T(\Gamma_1)$ and so $d(\Gamma_2) \leq d(\Gamma_1)$. Using these ideas Singerman [35] proved the following lemma.

Lemma 3.1

Let Γ be a Fuchsian group which is not a triangle group. Then there exists a Fuchsian group Λ, isomorphic to Γ, such that Λ is not contained in any proper NEC group.

Let S be a Riemann surface and G a group of conformal automorphisms of S. Then there is a Fuchsian surface group K, such that $S := \mathcal{U}/K$ and a Fuchsian group Γ, such that $K \lhd \Gamma$ and $G \simeq \Gamma/K$; Γ is the lift of G. Lemma 3.1 tells us that, unless G lifts to a triangle group, there is a non-symmetric Riemann surface S', homeomorphic to S, and a group G', of automorphisms of S' isomorphic to G.

Hence to use automorphism groups of Riemann surfaces alone to look at symmetries we must confine our attention to groups which lift to triangle groups, such groups are said to be *large groups of automorphisms*. In this case the surface is normalised by a normal subgroup of a triangle group.

Section 3.2 Symmetries and Large Automorphism Groups

Theorem 3.2

Let S be a Riemann surface uniformized by the Fuchsian surface group K, and let G be a large group of automorphisms of S. Hence there is an epimorphism from a Fuchsian triangle group $\Gamma[l, m, n]$ with presentation

$$\langle x_1, x_2 \mid x_1^l = x_2^m = (x_1 x_2)^n = 1 \rangle$$

to G with kernel K. G is generated by X and Y, the images of x_1 and x_2. Then S is symmetric if G admits an automorphism α, obeying either

(i) $\alpha(X) = X^{-1}, \ \alpha(Y) = Y^{-1}$ or

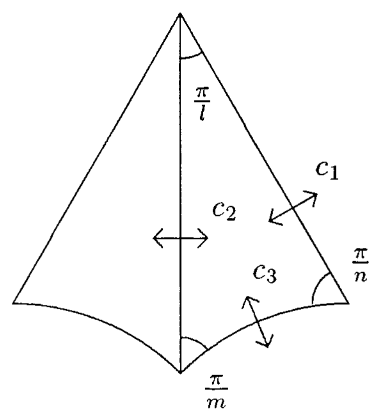
(ii) $\alpha(X) = Y^{-1}, \ \alpha(Y) = X^{-1}$.

This is one way round of Theorem 2 in [35]. We shall make a couple of remarks and then give the proof.

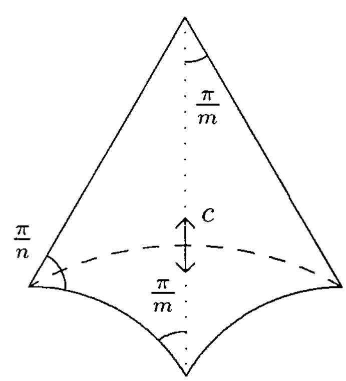
Let Γ be a Fuchsian triangle group with signature and presentation as described in the previous Theorem. If Γ is the canonical Fuchsian group for a proper NEC group Γ_* then by (1.8), Γ_* has one of two signatures. Indeed there is always a NEC group $\Gamma_*(l, m, n)$ with presentation

$$\langle c_1, c_2, c_3 \,|\, c_1^2 = c_2^2 = c_3^2 = (c_1c_2)^l = (c_2c_3)^m = (c_1c_3)^n = 1\rangle$$

such that $\Gamma_*^+ = \Gamma$. In fact this was how we constructed triangle groups in § 1.4. Γ_*^+ is generated by $c_1c_2 = x_1$ and $c_2c_3 = x_2$. Note that the automorphism of Γ defined by $\gamma \mapsto c_2\gamma c_2^{-1}$ for $\gamma \in \Gamma$ maps x_1 to x_1^{-1} and x_2 to x_2^{-1}. A fundamental region for Γ can be constructed from two fundamental regions of $\Gamma_*(l, m, n)$ as indicated below.



Hence any reflection in the side of a triangle with internal angles $\pi/l, \pi/m$ and π/n, which is half a fundamental region for Γ, extends Γ to $\Gamma_*(l, m, n)$.



If $l = m$, then there is a proper NEC group Γ_* with signature $(0; +; [m]; \{(n)\})$, or $([m], (n))$ for brevity, such that $\Gamma_*^+ = \Gamma$. Γ_* will have presentation

$$\langle x, c \,|\, x^m = c^2 = (xcx^{-1}c)^n = 1\rangle.$$

Γ_*^+ is generated by $x = x_1$ and $cx^{-1}c = x_2$ and the automorphism of Γ defined by $\gamma \mapsto c\gamma c$ for $\gamma \in \Gamma$ maps x_1 to x_2^{-1} and x_2 to x_1^{-1}. The above diagram shows how a fundamental region for Γ can be constructed from two fundamental regions for $\Gamma_*([m],(n))$. Hence, given a fundamental region for Γ of the above form, we can extend Γ to $\Gamma_*([m],(n))$ by adding a reflection similar to c.

These are the only possible proper NEC groups for which Γ is the canonical Fuchsian group.

Proof of Theorem 3.2

The automorphism α is of order two in both cases and so $H := \{I, \alpha\}$ is a group of automorphisms of G. Therefore, we can construct the semi-direct product of G by H, which we denote by F, by taking the ordered pairs $[h, g]$, $h \in H, g \in G$ together with the operation $[h_1, g_1].[h_2, g_2] = [h_1h_2, g_1^{h_2}g_2]$.

If α is as in the first case then there is an epimorphism

$$\varphi : \Gamma_*(l, m, n) \longmapsto F,$$

defined by

$$\varphi(c_1) = [\alpha, X^{-1}], \quad \varphi(c_2) = [\alpha, I], \quad \varphi(c_3) = [\alpha, Y].$$

If α is as in the second case then $l = m$ and there is an epimorphism

$$\varphi : \Gamma_*([m],(n)) \longmapsto F,$$

defined by

$$\varphi(x) = [I, X^{-1}], \quad \varphi(c) = [\alpha, I].$$

In each case $\varphi(\Gamma_*^+) \simeq G$ and the kernel of φ is K. Therefore Γ_*/K acts as a group of automorphisms of S and hence S is symmetric.

Now suppose S is symmetric and T is some symmetry of S. Let $G_* = \langle G, T\rangle$, so G_* lifts to a proper NEC group Γ_*, of which K is a normal subgroup. If G is normal in G_*, then $G_* = G \cup TG$ and Γ_*^+ is Γ (the lift of G), in which case Γ_* is one of the two NEC groups described above. Suppose G is not normal in G_*. Then Γ is not normal in Γ_* and so not of index two, therefore Γ is properly contained in Γ_*^+. Γ_*^+ must be a triangle group as the only groups to contain triangle groups are also triangle groups. In [33] Singerman gave all inclusions between Fuchsian triangle groups, we will consider the two cases of Γ being normal and not being normal in Γ_*^+ separately.

When Γ is normal in Γ_*^+ we have the following possibilities.

Γ	Γ_*^+	Γ_*	$\mid \Gamma_* : \Gamma \mid$
$[t,t,t]$	$[3,3,t]$	$(3,3,t)$	6
$[t,t,t]$	$[3,3,t]$	$([3],(t))$	6
$[t,t,t]$	$[2,3,2t]$	$(2,3,2t)$	12
$[t,t,u]$	$[2,t,2u]$	$(2,t,2u)$	4
$[2t,2t,t]$	$[2,2t,2t]$	$([2t],(2))$	4

We see from [5] that Γ is in fact normal in Γ_* for the first four cases and so, in these instances, Γ_* can not be the lift of G_*. We need to determine if Γ_* can be considered as the lift of G_* (G extended by a single symmetry of S) in the last case. Suppose K is normal in Γ_* then Γ_*/K acts as a group of automorphisms of S containing $G \simeq \Gamma/K$ with index four. Any reflection say c, in Γ_*, induces a symmetry say T, of S, and $\langle G, T\rangle$ lifts to $\langle \Gamma, c\rangle$. Γ_* contains no proper NEC group whose canonical Fuchsian group has the same signature as Γ and so $\langle \Gamma, c\rangle = \Gamma_*$. Thus, for a suitable K, Γ_* in the last case may occur as the lift of G extended by a single symmetry, and this symmetry extends G to a group containing G with index four. This case will be considered in more detail later on.

When Γ is not normal in Γ_*^+ we have the following possibilities.

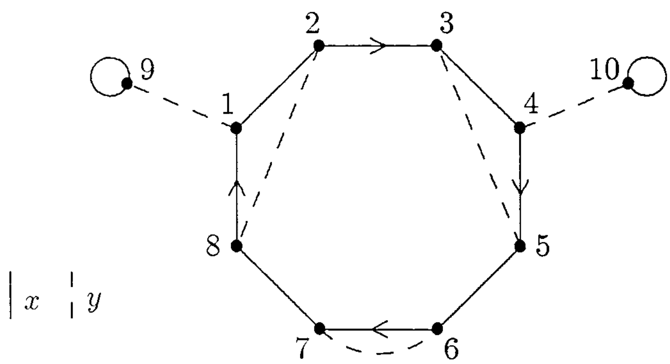
Γ	Γ_*^+	$\mid \Gamma_*^+ : \Gamma \mid$
$[7,7,7]$	$[2,3,7]$	24
$[2,7,7]$	$[2,3,7]$	9
$[3,3,7]$	$[2,3,7]$	8
$[4,8,8]$	$[2,3,8]$	12
$[3,8,8]$	$[2,3,8]$	10
$[9,9,9]$	$[2,3,9]$	12
$[4,4,5]$	$[2,4,5]$	6
$[n,4n,4n]$	$[2,3,4n]$	6
$[n,2n,2n]$	$[2,4,2n]$	4
$[3,n,3n]$	$[2,3,3n]$	4
$[2,n,2n]$	$[2,3,2n]$	3

In each of these cases there is only one possible signature for Γ_*. From [5] and [11] we know, that for each of the inclusions above, Γ_* contains a proper NEC group whose canonical Fuchsian group has the same signature as Γ. It can be seen that in all the above cases there is only one conjugacy class of groups isomorphic to Γ in Γ_*^+, and so Γ is the canonical Fuchsian group of some proper NEC subgroup Λ of Γ_*.

Example Consider the Fuchsian group

$$\Gamma'[2,3,8] = \langle x, y \,|\, x^8 = y^2 = (xy)^3 = 1 \rangle$$

and its subgroups of index ten with signature $[3,8,8]$. Let Γ be such a subgroup and consider the permutation representation of Γ' acting on the right Γ-cosets. Theorem 2.4 tells us x will act as a permutation with one eight cycle and two one cycles and that y will act as a permutation with five two cycles. Without loss of generality we may assume the eight cycle of x to be $(1\,2\,3\ldots 8)$. The permutation representation is transitive and so we may assume $(1\,9)$ is a cycle of y. Therefore, $xyxyx$ takes 8 to 2 and, as $(xy)^3 = 1$, we know $(2\,8)$ must be a cycle of y. Again, by transitivity, we know that y takes 10 to one of $\{1, 2, \ldots, 8\}$, it can not be 1, 2 or 8. Neither can it be 3 or 7 as $(xy)^3 = 1$ would imply that $(2\,4)$ or $(6\,8)$ are cycles of y.



Hence y takes 10 to 4, 5 or 6. If $(5\,10)$ is a cycle of y, then so is $(4\,6)$ and $(xy)^3$ then takes 2 to 7, which is not the case. Thus y takes 10 to 4 or 6, the coset graphs for these two possibilities are essentially the same and so there is only one conjugacy class of groups isomorphic to Γ in Γ'.

In the same way it can be shown that there is just one conjugacy class in Γ_* of subgroups isomorphic to Γ in all the above cases, see the Appendix. Hence Γ_* contains a subgroup Λ such that $|\Lambda : \Gamma| = 2$ and $\Lambda^+ = \Gamma$ in all the above. In which case Λ/K acts as a group of automorphisms of S that contains G, with index two, and also a symmetry.

Lemma 3.3

Let S be a symmetric, compact Riemann surface uniformized by K, and let G be a large group of automorphisms of S. Then either

(†) G lifts to $\Gamma[2n, 2n, n]$ and K is normal in $\Gamma_*([2n], (2))$ (which contains Γ), but is not normal in either of the two proper NEC groups for which Γ is the canonical Fuchsian group,

or S admits a symmetry which extends G to a group containing it with index two.

Theorem 3.4

Let S be a Riemann surface uniformized by some Fuchsian surface group K, and let G be a large automorphism group of S, generated by X and Y obeying

$$X^l = Y^m = (XY)^n = 1.$$

That is X and Y are induced by the canonical generators of the triangle group to which G lifts.

If S is symmetric, then either condition (†) holds or G admits an automorphism α, obeying

(i) $\alpha(X) = X^{-1}, \ \alpha(Y) = Y^{-1}$ or

(ii) $\alpha(X) = Y^{-1}, \ \alpha(Y) = X^{-1}$.

Proof G lifts to a triangle group $\Gamma[l, m, n]$ and Lemma 3.3 tells us S admits a group of automorphisms that lifts to a proper NEC group Γ_* that contains Γ with index two. Γ_* is one of the two groups mentioned earlier. If Γ_* has signature (l, m, n) and canonical generators c_1, c_2, c_3 then α is just the automorphism of S induced by c_2, acting, by conjugation, on G. If Γ_* has signature $([m], (n))$ and canonical generators c and x then α is induced in a similar fashion by c.

Theorems 3.2 and 3.4 together provide an amended version of Theorem 2 in [35].

In the previous list of signatures it can be seen that at least some (probably all) of the Γ_* contain reflections that extend Γ to Γ_*.

Example Let Γ be a NEC group with signature $(2, 3, 3n)$ and presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ac)^{3n} = 1\rangle.$$

Then Γ^+ has signature $[2,3,3n]$ and is generated by $x := ab$ and $y := bc$. There is an epimorphism from Γ^+ to a group of permutations on four points defined by

$$x \mapsto (1\,4)(2\,3), \quad y \mapsto (1\,2\,3)(\,4\,), \quad xy \mapsto (1\,4\,2)(\,3\,).$$

By (2.4) we know that Γ^+ contains a subgroup Λ, of index four, with signature $[3, n, 3n]$. Λ is generated by $p := xyx = acba$ and $q := yxy = bcac$. If $\Lambda_a := \langle \Lambda, a \rangle$, then

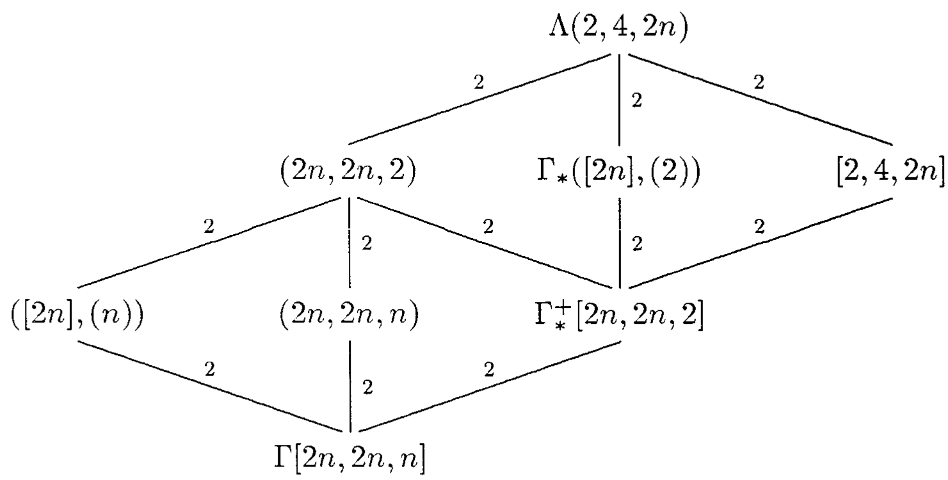
$$apa = cb = y^{-1} \in \Lambda_a \quad \Rightarrow \quad y \in \Lambda_a \quad \Rightarrow \quad x \in \Lambda_a \quad \Rightarrow \quad \Lambda_a = \Gamma.$$

So if K is a Fuchsian surface group normal in both Λ and Γ, then a induces a symmetry of $S := \mathcal{U}/K$ which extends the group of automorphisms $G \simeq \Lambda/K$, to a group containg G with index eight.

Section 3.3 Exceptional Cases

Although one way round of the proof of Singerman's Theorem does not account for all possibilities the result is true for the vast majority of cases and may even be correct in all cases. In this section we shall give the conditions that are required to find a counter example to this Theorem.

From Lemma 3.3 we are looking for a surface group K which is normal in a Fuchsian triangle group Γ with signature $[2n, 2n, n]$, normal in the proper NEC group Γ_* which contains Γ with index four and has signature $([2n],(2))$ but not normal in either of the two proper NEC groups for which Γ is the canonical Fuchsian group.



By considering the above subgroup lattice these conditions are equivalent to a surface subgroup of Γ being normal in Γ_* but not in Λ. It seems very difficult to decide whether such subgroups exist or to find one if they do.

An alternative method for finding a counter example is provided by the theory of maps and hypermaps. If such a surface group K exists, then $S := \mathcal{U}/K$ carries a regular hypermap of type $\{2n, 2n, n\}$ and a regular map of type $\{2n, 2n\}$. By considering the construction of the map and hypermap it can be seen that the vertices of the map correspond precisely to the hypervertices and hyperedges of the hypermap, and that the edges correspond to the incidences between the hypervertices and hyperedges, see [9]. Hence the map is bipartite. A map is said to be *reflexible* if there is a symmetry of the surface on which it lies that fixes a face or a vertex. (This is equivalent to the map subgroup also being normal in the extended triangle group). We want K to be normal in $\Gamma_*(2n, 2)$, so S will be symmetric, but not to be normal in any group with signature $(2n, 2n, 2)$. Therefore the map will be irreflexible or *chiral*, see [36].

In summary, we are looking for a regular chiral bipartite map of type $\{2n, 2n\}$ on a symmetric Riemann surface. Sherk [31] found an infinite family of regular chiral maps of type $\{6, 6\}$ and in [14] Garbe used the same idea to determine infinite families of regular chiral maps of type $\{6(2k-1), 6(2k-1)\}$, $(k \in \mathbb{N})$. Each of these maps is described by its conformal automorphism group. Such a group is generated by a pair r, s and has defining relations

$$r^{6(2k-1)} = s^{6(2k-1)} = (rs)^2 = r^6 s^6 = r^3 s r^{-3} s^{-1} = (r^{-2} s^{-2})^b (r^2 s^2)^c = 1.$$

Where b and c range over the positive integers and parametrise the maps of a particular type. We denote the above group by $G_{k,b,c}$, when $k = 1$ the orders of r and s are both six and we have the maps Sherk found. The above maps are reflexible if and only if $bc(b-c) = 0$. Our search is for regular chiral maps on symmetric Riemann surfaces and so by (3.4) we want $G_{k,b,c}$ to admit an automorphism taking r to s^{-1} and s to r^{-1}. We abuse our notation slightly and define the group G_k as having generators r, s and having as its relations the first five of $G_{k,b,c}$ plus their images under the above automorphism. (The images of the first four are equivalent to themselves).

$$G_k := \langle r, s \mid r^{6(2k-1)} = s^{6(2k-1)} = (rs)^2 = r^6 s^6 = r^3 s r^{-3} s^{-1} = s^3 r s^{-3} r^{-1} = 1 \rangle.$$

Garbe tells us that $|G_{k,b,c}| = 12(2k-1)(b^2 + bc + c^2)$ and it can be shown that $|G_k| = 48(2k-1)$. Thus for $bc(b-c) \neq 0$ we have $|G_{k,b,c}| \geq 84(2k-1) > |G_k|$. Hence $r \mapsto s^{-1}$, $s \mapsto r^{-1}$ is not an automorphism of $G_{k,b,c}$ and so these maps lie on non-symmetric Riemann surfaces. Therefore the maps of Sherk and Garbe do not provide us with any examples of the kind we require.

Chapter 4

Symmetries and Large Cyclic Groups

We are now going to use the results of the previous two chapters to determine the species of symmetries of certain types of Riemann surfaces. In this Chapter we will consider compact Riemann surfaces admitting large cyclic groups of automorphisms and in the next, compact Riemann surfaces admitting large non-cyclic abelian groups of automorphisms.

Section 4.1 Application of Hoare's Theorem

Let T be a symmetry of a compact Riemann surface S, uniformized by the Fuchsian surface group K. Then the group generated by T will lift to a proper NEC group Λ say, where Λ contains K with index two. Hence K is the canonical Fuchsian group of Λ, this means the signature of Λ is without proper periods and any period cycles are empty. That is, the signature of Λ is of the form $(h; \pm; [\,]; \{(\,)^k\})$. Now $\langle T\rangle \simeq \Lambda/K$ and $S = \mathcal{U}/K$, therefore $S' := S/\langle T\rangle = \mathcal{U}/\Lambda$. Thus S' is a surface of genus h with k boundary components and orientable or non-orientable according to the signature of Λ. The boundaries of S' correspond precisely to the curves fixed by T, and so the number of mirrors of T is k; the number of period cycles in the signature of Λ. As already noted we can also determine whether S' is orientable or not by looking at the signature of Λ and so the species of T is completely determined in this way.

Let S and T be as above and let G be a group of automorphisms of S that contains T. G will lift to some proper NEC group Γ. The action of G on the $\langle T\rangle$ cosets is the same as the action of Γ on the Λ cosets. Hence, given the action of

G on the $\langle T \rangle$ cosets, the signature of Γ and the homomorphism from Γ onto G we can, by Hoare's Theorem, determine the signature of Λ and thus the species of T.

Example 1 Consider the following epimorphism.

$$\varphi\colon \Gamma(0;+;[2,3];\{(2,3,6)\}) \longmapsto D_6 = \langle a,b \mid a^6 = b^2 = (ab)^2 = 1\rangle$$

x_1	a^3
x_2	a^2
e	a
c_0	b
c_1	a^3b
c_2	a^5b
c_3	a^4b

Note that φ preserves the orders of the canonical generators of Γ and so the kernel of φ is a surface group. Γ^+ will be mapped onto the cyclic subgroup of D_6 generated by a. There are two conjugacy classes of symmetries of $S := \mathcal{U}/K$, in D_6, one represented by b and the other by ab.

The $\Lambda := \varphi^{-1}\langle b\rangle$ cosets are represented by $1, e, e^2, \ldots, e^5$. If we label these 1 to 6, then the action of Γ on these cosets is given by

$$\begin{array}{ll}
x_1 \longmapsto (1\,4)(2\,5)(3\,6) & c_0 \longmapsto (1)(2\,6)(3\,5)(4) \\
x_2 \longmapsto (1\ 3\ 5)(2\ 4\ 6) & c_1 \longmapsto (1\ 4)(2\ 3)(5\ 6) \\
e \longmapsto (1\ 2\ 3\ 4\ 5\ 6) & c_2 \longmapsto (1\ 2)(3\ 6)(4\ 5) \\
 & c_3 \longmapsto (1\,3)(2)(4\,6)(5)
\end{array}$$

We denote the reflection generators of Λ, associated to the fixed points of the canonical reflection generators of Γ in this action, by c_{01}, c_{04} and c_{32}, c_{35}.

Actions	Links
$c_0c_1 \longmapsto (1\ 4)(2\,5)(3\,6)$	$c_{01} \sim c_{04}$
$c_2c_3 \longmapsto (1\,6\,5\,4\,3\,2)$	$c_{32} \sim c_{35}$
$ec_0e^{-1} \longmapsto (1\,5)(2\,4)(3)(6)$	$c_{01} \sim c_{32}, \quad c_{04} \sim c_{35}$

There is only one chain and so Λ has one empty period cycle. There is no partition of the Λ cosets as described in part (iv) of Theorem 2.5 and so $\mathcal{U}/\Lambda$ is non-orientable. The genus h, of this quotient is given by

$$h - 2 + 1 = 6\left(-2+1+\left(2-\frac{1}{2}-\frac{1}{3}\right)+\frac{1}{2}\left(3-\frac{1}{2}-\frac{1}{3}-\frac{1}{6}\right)\right)$$
$$h = 8.$$

In general we are not interested in the genus. Thus Λ has signature $(8; -; [\,]; \{(\,)\})$ and the species of b is -1. The same method shows that the species of ab is also -1.

We are only considering those compact Riemann surfaces that admit large cyclic or abelian groups of automorphisms. Using Theorems 3.3 and 3.4 we decide in what ways symmetries of these surfaces extend the large groups and obtain a homomorphism from a proper NEC group (with signature (l, m, n) or $([m], (n))$), onto the extended large group. Then the above procedure should yield the species of the conjugacy classes of symmetries. To find the symmetry type of these surfaces we have to make sure we eventually arrive at the full automorphism group of the surface. This corresponds to checking inclusions between triangle groups and determining whether homomorphisms extend to larger groups.

Example 2 Let Γ be a Fuchsian triangle group with signature $[12, 6, 4]$ and canonical generators x and y of orders 12 and 6. Then there is a homomorphism φ, from Γ onto $C_{12} := \langle r \,|\, r^{12} = 1\rangle$, defined by $\varphi(x) = r$ and $\varphi(y) = r^2$. Clearly C_{12} admits an automorphism that takes r to r^{-1}. We adjoin an element t, of order two, to C_{12} to get a group G, with presentation $\langle r, t \,|\, r^{12} = t^2 = (tr)^2 = 1\rangle$. G is dihedral and there is a homomorphism φ_*, from the extended triangle group $\Gamma_*(12, 6, 4)$, that contains Γ, onto G such that $\varphi_*|_\Gamma = \varphi$ and the kernels of both homomorphisms are the same. If the canonical generators of Γ_* are a, b and c, then we let $\varphi_*(a) = tr^{-1}$, $\varphi_*(b) = t$ and $\varphi_*(c) = tr^2$. G contains two conjugacy classes of symmetries of the associated Riemann surface S, represented by t and tr. The action of Γ_*, on the right $\Lambda := \varphi_*^{-1}(\langle t\rangle)$ cosets, is given by

$$\begin{aligned} a &\longmapsto (1\,12)(2\,11)(3\,10)(4\,9)(5\,8)(6\,7) \\ b &\longmapsto (1)(2\,12)(3\,11)(4\,10)(5\,9)(6\,8)(7) \\ c &\longmapsto (1\,3)(2)(4\,12)(5\,11)(6\,10)(7\,9)(8). \end{aligned}$$

We denote the reflections generators of Λ, induced by the fixed points of b and c, by b_1, b_7 and c_2, c_8. The orbits of ab, bc and ac provide the links

$$b_1 \sim b_7,\ b_7 \sim b_1,\ c_2 \sim c_8 \quad \text{and} \quad c_8 \sim c_2.$$

Hence there are two chains and Λ has two period cycles. By considering the above action we see that $\mathcal{U}/\Lambda$ is non-orientable and so $sp(t) = -2$. In the same way we find that $sp(tr) = -1$. Since Γ_* is maximal in $\mathcal{L}$, we can say that these are the only conjugacy classes of symmetries of S and so $st(S) = \{-1, -2\}$.

Example 3 If $\Gamma[8,8,2]$ has canonical generators x and y, then there is a homomorphism φ, from Γ onto $C_8 := \langle r \,|\, r^8 = 1\rangle$, defined by $\varphi(x) = r$ and $\varphi(y) = r^3$. We denote the kernel of φ by K. C_8 admits an automorphism taking r to r^{-3} and r^3 to r^{-1}. Let $G := \langle r, t \,|\, r^8 = t^2 = trtr^3 = 1\rangle$, there is a homomorphism φ_*, from a NEC group $\Gamma_*([8],(2))$, which contains Γ with index two, onto G. Γ_* has canonical presentation

$$\begin{aligned}\Gamma_* &= \langle x, e, c_0, c_1 \,|\, x^8 = c_0^2 = c_1^2 = (c_0c_1)^2 = e^{-1}c_0ec_1 = xe = 1\rangle \\ &= \langle x, c \,|\, x^8 = c^2 = (xcx^{-1}c)^2 = 1\rangle,\end{aligned}$$

where $c := c_0$. When we use Hoare's Theorem we really need to think of the first presentation. We can think of φ_* as φ extended and $\varphi_*(x) = r$ and $\varphi_*(c) = t$. G contains only two involutions not in C_8 and these are conjugate in G. Hence there is only one class of symmetries of $S := \mathcal{U}/K$, in G. If $\Lambda = \varphi_*^{-1}\langle t\rangle$, then the action of Γ_* on the Λ cosets is given by

$$\begin{aligned} x &\longmapsto (1\,2\,3\,\ldots\ldots\ldots\ldots\,8)\\ c\,(=c_0) &\longmapsto (1)(2\,6)(3)(4\,8)(5)(7)\\ xcx^{-1}\,(=c_1) &\longmapsto (1\,5)(2)(4)(3\,7)(6)(8).\end{aligned}$$

We represent the reflection generators of Λ, associated to the fixed points of c, by c_1, c_3, c_5 and c_7, and those associated to the fixed points of c^x are represented by c_2', c_4', c_6' and c_8'. The orbits of $c^xc = c_0c_1$, give us the links $c_1 \sim c_5$, $c_3 \sim c_7$ and $c_2' \sim c_6'$, $c_4' \sim c_8'$. The relation $xcx^{-1}c^x(= e^{-1}c_0ec_1) = 1$ gives the links $c_1 \sim c_2'$, $c_3 \sim c_4'$, $c_5 \sim c_6'$, $c_7 \sim c_8'$, see end of §2.3. Thus we have the chains $c_1 \sim c_5 \sim c_6' \sim c_2' \sim c_1$ and $c_3 \sim c_7 \sim c_8' \sim c_4' \sim c_3$. Therefore Λ has two empty period cycles. $\mathcal{U}/\Lambda$ is non-orientable by Theorem 2.5 and so $sp(t) = -2$.

Of course C_8 also admits an automorphism that takes r to r^{-1}, so C_8 can be extended to a (dihedral) group G', of automorphisms of S which lifts to an extended triangle group Γ_*, with signature $(8,8,2)$. G' contains two conjugacy classes of symmetries of S and it can be seen that one has species -1 and the other -2.

So K is normal in $\Gamma[8,8,2]$, $\Gamma_*([8],(2))$ and $\Gamma_*'(8,8,2)$. Hence, if $N_{\mathcal{L}}(K)$ is the normalizer of K in $\mathcal{L}$, then Γ is properly contained in $N_{\mathcal{L}}(K)^+$. By [33], $N_{\mathcal{L}}(K)^+$ has signature $[2,4,8]$ or $[2,3,8]$. Infact, see [10] table 9, K is normal in some $\Lambda[2,3,8]$ and, as S is symmetric and Λ is maximal, we know that the signature of $N_{\mathcal{L}}(K)$ must be $(2,3,8)$. The full group of automorphisms of S is

$$N_{\mathcal{L}}(K)/K \simeq \langle u, v, t \,|\, u^2 = v^3 = s^2 = (tu)^2 = (tv)^2 = (u(uv)^4)^2 = 1\rangle$$

and there are only two conjugacy classes of symmetries, of S, in this group. They are represented by s and sv. Thus, $st(S) = \{-1, -2\}$.

Section 4.2 Large Cyclic Groups

Riemann surfaces admitting large cyclic groups of automorphisms correspond to surface kernel homomorphisms from Fuchsian triangle groups onto cyclic groups. We shall now say how Harvey's Theorem (1.17) applies specifically to triangle groups.

If Γ is a Fuchsian triangle group with signature $[m_1, m_2, m_3]$, then there is a surface kernel homomorphism, from Γ onto a cyclic group of order N, if and only if

> $[m_1, m_2] = [m_1, m_3] = [m_2, m_3] = N$ and if N is even, then the number of periods divisible by the maximum power of two dividing N is two.

Here $[m_1, m_2]$ denotes the lowest common multiple of m_1 and m_2. If (m_1, m_2) denotes the greatest common factor of m_1 and m_2, then let $d = (m_1, m_2, m_3)$ and b_1, b_2, b_3 be such that

$$(m_1, m_2) = db_3, \quad (m_1, m_3) = db_2, \quad (m_2, m_3) = db_1.$$

Thus b_1, b_2 and b_3 are mutually coprime and $m_i = db_jb_k$ where $i, j, k \in \{1, 2, 3\}$ are mutually distinct. For the above conditions to be fully satisfied we require that if d is even, then one of the b_i's must also be even. Therefore, there is a surface kernel homomorphism, from $\Gamma[m_1, m_2, m_3]$ onto a cyclic group of order N, if and only if

> m_i is of the form db_jb_k where $d = (m_1, m_2, m_3)$, $i, j, k \in \{1, 2, 3\}$ are mutually distinct, $(m_i, m_j) = db_k$ and if $2|d$ then $2|b_i$ for some i.

Hence $N = db_1b_2b_3$. If K is the kernel of the above homomorphism and S is the Riemann surface uniformized by K, then S has a large cyclic group of automorphisms of order N.

Lemma 4.1

Let Γ be a Fuchsian group and let φ_1, φ_2 be two surface kernel homomorphisms from Γ onto a finite group G, with kernels K_1 and K_2 respectively. If φ_1 and φ_2 differ by an automorphism of G, then $K_1 = K_2$.

Proof Let α be an automorphism of G such that, for all γ in Γ, $\varphi_1(\gamma) = \alpha(\varphi_2(\gamma))$. Then

$$K_1 = \varphi_1^{-1}(\mathrm{id}_G) = (\varphi_2^{-1} \circ \alpha^{-1})(\mathrm{id}_G) = \varphi_2^{-1}(\mathrm{id}_G) = K_2.$$

If Γ is maximal then we have the following, stronger result.

Theorem 4.2

Let Γ be a *maximal* Fuchsian group and let φ_1, φ_2 be two surface kernel homomorphisms from Γ onto a finite group G, with kernels K_1, K_2 respectively. Then K_1 and K_2 are conjugate in $\mathcal{L}^+$ if and only if φ_1 and φ_2 differ by an automorphism of G.

The proof requires the following lemma.

Lemma 4.3

Let Γ be a maximal Fuchsian group and K_1, K_2 be two normal subgroups of Γ. If there is a $g \in \mathcal{L}^+$ such that $K_2 = gK_1g^{-1}$ then $K_1 = K_2$ and $g \in \Gamma$.

Proof Let $k_2 \in K_2$ and $\gamma \in \Gamma$.

$$\begin{aligned} \text{Then} \quad g\gamma g^{-1}k_2 g\gamma^{-1}g^{-1} &= g\gamma k_1\gamma^{-1}g^{-1} && \text{for some } k_1 \in K_1 \\ &= gkg^{-1} && \text{for some } k \in K_1 \\ &\in K_2. \end{aligned}$$

Thus $g\gamma g^{-1} \in N_{\mathcal{L}^+}(K_2)$.
As Γ is maximal we know that $N_{\mathcal{L}^+}(K_2) = \Gamma$ and so $g\gamma g^{-1} \in \Gamma$.
Therefore $g \in N_{\mathcal{L}^+}(\Gamma) = \Gamma$ and $K_1 = K_2$.

Proof of Theorem Suppose K_1 and K_2 are conjugate in $\mathcal{L}^+$ then by the previous lemma $K_1 = K_2$ and so φ_1 and φ_2 differ by an automorphism of G. Lemma 4.1 completes the proof.

If G is a cyclic group of order N generated by r, then the automorphisms of G are just of the form

$$r \longmapsto r^{\lambda} \quad \text{where } (\lambda, N) = 1, \text{ i.e. } \lambda \text{ is a unit in } \mathbf{Z}_N.$$

We denote the set of all units in $\mathbf{Z}_N$ by $U(N)$.

Let φ be a surface kernel homomorphism from $\Gamma[m_1, m_2, m_3]$ onto $C_N := \langle r \mid r^N = 1\rangle$. If x and y are canonical generators of Γ, of orders m_1 and m_2

respectively, then

$$\begin{aligned} \varphi(x) &= r^{k_1} & \quad \text{where} \quad (k_1, N) &= N/m_1 = b_1 \\ \varphi(y) &= r^{k_2} & (k_2, N) &= N/m_2 = b_2 \\ \varphi(xy) &= r^{k_1+k_2} & (k_1 + k_2, N) &= N/m_3 = b_3. \end{aligned}$$

Clearly, $r \mapsto r^{-1}$ defines an automorphism of C_N which maps r^{k_1} to r^{-k_1} and r^{k_2} to r^{-k_2}. Thus if K is the kernel of φ, then the Riemann surface $S := \mathcal{U}/K$, admits a symmetry that extends C_N to a dihedral group D_N and this D_N will lift to a proper NEC group $\Gamma_*(m_1, m_2, m_3)$, see §3.2.

Definition 4.4 Suppose G is a large group of automorphisms of a Riemann surface S, that lifts to some triangle group Γ. Then a symmetry of S, that extends G to a group that lifts to an extended triangle group containing Γ with index two, is said to be of the *first kind with respect to G.*

We now turn our attention to another kind of symmetry and ask when C_N admits an automorphism of the kind in (ii) of Theorem 3.2. For $r^{k_1} \mapsto r^{-k_2}$ and $r^{k_2} \mapsto r^{-k_1}$ to describe an automorphism of C_N we require that $k_1 \equiv k_2 \pmod N$ and so $m_1 = m_2$. Thus $b_1 = b_2 = 1$, Γ has signature $[db, db, d]$ and $N = db$. Hence we may assume that $\varphi(x) = r$ and that $\varphi(y) = r^k$ for some $k \in U(N)$. In this case $r \mapsto r^{-k}$, $r^k \mapsto r^{-1}$ is an automorphism of C_N if and only if $k^2 \equiv 1 \pmod N$. Note that the order of r^{k+1} must be d and so $(k+1, N) = b$.

Suppose $N = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}$, where $p_1, \dots, p_s$ are distinct odd primes and $\alpha \geq 0$, $\alpha_i > 0$. Then $k^2 \equiv 1 \pmod N$ if and only if

$$\begin{array}{lll} k \equiv \pm 1 & \pmod{p_i^{\alpha_i}} & i = 1, \dots, s \\ k \equiv 1 & \pmod{2^\alpha} & \text{if } \alpha = 1 \\ k \equiv \pm 1 & \pmod{2^\alpha} & \text{if } \alpha = 2 \\ k \equiv \pm 1,\ 2^{\alpha-1} \pm 1 & \pmod{2^\alpha} & \text{if } \alpha > 2. \end{array}$$

Thus there are $2^{s+\tau}$ solutions, modulo N, of $z^2 \equiv 1 \pmod N$. Here τ is 0 if $\alpha < 2$, 1 if $\alpha = 2$ and 2 if $\alpha > 2$. When k is as above S admits a symmetry u, such that

$$G := \langle C_N, u \rangle = \langle r, u \,|\, r^N = u^2 = urur^k = 1 \rangle$$

lifts to a proper NEC group $\Gamma_*([db], (d))$, that contains Γ with index two.

Definition 4.5 Let G be a large group of automorphisms of some surface, suppose a symmetry of the surface extends G to G'. If G' contains G with index two but does not lift to an extended triangle group, then the symmetry is said to be of the *second kind with respect to G*.

Assume that the surface S, uniformized by the kernel of φ above, admits a symmetry of the second kind w.r.t. C_N, so $N = db$ and $k^2 \equiv 1 \pmod{N}$. Let n_1 denote the product of the maximum odd prime power factors of N, for which k is congruent to $+1$ and n_2 the product of the maximum odd prime power factors of N, for which k is congruent to -1, so $N = 2^\alpha n_1 n_2$. Recall that $(k+1, N) = b$ and so

$$\begin{array}{llll}
\text{for } \alpha = 0, & d = n_1 & b = n_2 & \\
\text{for } \alpha = 1, & d = n_1 & b = 2n_2 & \\
\text{for } \alpha = 2, & d = 2n_1 & b = 2n_2 & \text{when } k \equiv +1 \pmod{4} \\
 & d = n_1 & b = 4n_2 & \text{when } k \equiv -1 \pmod{4} \\
\text{for } \alpha > 2, & d = 2^{\alpha-1} n_1 & b = 2n_2 & \text{when } k \equiv +1 \pmod{2^\alpha} \\
 & d = n_1 & b = 2^\alpha n_2 & \text{when } k \equiv -1 \pmod{2^\alpha} \\
 & d = 2^{\alpha-1} n_1 & b = 2n_2 & \text{when } k \equiv 2^{\alpha-1}+1 \pmod{2^\alpha} \\
 & d = 2n_1 & b = 2^{\alpha-1} n_2 & \text{when } k \equiv 2^{\alpha-1}-1 \pmod{2^\alpha}.
\end{array}$$

Hence

$$\left(d, \frac{N}{d}\right) = (d, b) = \begin{cases} 1 & \text{if } \alpha < 2; \\ 1 \text{ or } 2 & \text{if } \alpha \geq 2. \end{cases}$$

Conversely, if d and b are such that (d, b) is as above, then there is a k such that $k^2 \equiv 1 \pmod{N}$ and $(k+1, N) = b$. When it is clear with respect to which group a symmetry is of the first or second kind we will neglect to mention it.

It was clear that a symmetry of the first kind would extend C_N to D_N. Now we consider briefly if C_N, extended by a symmetry of the second kind, can be expressed in simple terms. Let

$$G_{N,k} := \langle r, u \mid r^N = u^2 = urur^k = 1 \rangle.$$

Firstly, we look for normal subgroups. Let $H_c = \langle u, r^c \rangle$ where c divides N. Then H_c will be normal in $G_{N,k}$ if and only if $r^{-1}ur = ur^{k+1} \in H$. That is, $r^{k+1} \in \langle r^c \rangle$. Thus H_c is normal in $G_{N,k}$ if and only if $c \mid (k+1, N)$ and

$$(k+1, N) = \begin{cases} n_2 & \text{if } \alpha = 0; \\ 2n_2 & \text{if } \alpha = 1; \\ 2n_2, 4n_2 & \text{if } \alpha = 2 \text{ and } k = \pm 1 \pmod{4}; \\ 2n_2, 2^\alpha n_2, 2n_2, 2^{\alpha-1} n_2 & \text{if } \alpha > 2 \text{ and } k = \pm 1,\ 2^\alpha \pm 1 \pmod{2^\alpha}. \end{cases}$$

Now H_c will be dihedral if and only if $ur^cu = r^{-c}$ and this is the case if and only if $c(k-1) \equiv 0 \pmod N$. This is equivalent to

$$\begin{array}{rl} n_2 \mid c & \text{if } \alpha = 0,\ 1, \\ n_2 \mid c,\ \ 2n_2 \mid c & \text{if } \alpha = 2 \text{ and } k = \pm 1 \pmod 4, \\ n_2 \mid c,\ \ 2^{\alpha-1}n_2 \mid c,\ \ 2n_2 \mid c,\ \ 2^{\alpha-1}n_2 \mid c & \text{if } \alpha > 2 \text{ and } k = \pm 1,\ 2^\alpha \pm 1 \pmod{2^\alpha}. \end{array}$$

Clearly any subgroup of C_N is a normal subgroup of $G_{N,k}$. Using the above conditions on c we can determine when H_c is a dihedral, normal subgroup of $G_{N,k}$ and we see that the following isomorphisms hold.

$$\begin{array}{ll} \text{for } \alpha = 0 & G_{N,k} \simeq H_{n_2} \times \langle r^{n_1} \rangle \simeq D_d \times C_b, \\ \text{for } \alpha = 1 & G_{N,k} \simeq H_{2n_2} \times \langle r^{n_1} \rangle \simeq D_d \times C_b, \\ \text{for } \alpha = 2 & G_{N,k} \simeq \begin{cases} H_{n_2} \times \langle r^{4n_1} \rangle \simeq D_{2d} \times C_{b/2}, & k \equiv +1 \pmod 4; \\ H_{4n_2} \times \langle r^{n_1} \rangle \simeq D_d \times C_b, & k \equiv -1 \pmod 4. \end{cases} \\ \text{for } \alpha > 2 & G_{N,k} \simeq \begin{cases} H_{n_2} \times \langle r^{2^\alpha n_1} \rangle \simeq D_{2d} \times C_{b/2}, & k \equiv +1 \pmod{2^\alpha}; \\ H_{2^\alpha n_2} \times \langle r^{n_1} \rangle \simeq D_d \times C_b, & k \equiv -1 \pmod{2^\alpha}. \end{cases} \end{array}$$

Thus, if k is congruent to ± 1 for all maximum prime powers dividing N, then a symmetry of the second kind extends C_N to the direct product of a dihedral group and a cyclic group. If 2^α is the maximum power of 2 dividing N, $\alpha > 2$, and $k \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}$, then $G_{N,k}$ does not have such a simple structure.

Section 4.3 Symmetries of the First Kind

We have seen that a symmetry of the first kind extends a cyclic group to a dihedral group. Hence we have a surface kernel homomorphism φ, from some extended triangle group $\Gamma_*(m_1, m_2, m_3)$, onto a dihedral group D_N such that φ maps $\Gamma_*^+[m_1, m_2, m_3]$ onto C_N; the cyclic subgroup of D_N of index two. Suppose Γ_* has canonical generators a, b and c. Then $x := ab$ and $y := bc$ are generators of $\Gamma := \Gamma_*^+$, and we may assume that

$$\begin{array}{ll} \varphi : \Gamma_* \longmapsto & D_N = \langle r, t \mid r^N = t^2 = (tr)^2 = 1 \rangle \\ a & r^{k_1} t = t r^{-k_1} \\ b & t \\ c & t r^{k_2} \\ x & r^{k_1} \\ y & r^{k_2} \end{array}$$

for $(k_1, N) = m_1$, $(k_2, N) = m_2$ and $(k_1 + k_2, N) = m_3$. If K is the kernel of φ, then the symmetries of $S := \mathcal{U}/K$ are the involutions in $D_N \backslash C_N$. If N is odd there is only one conjugacy class of symmetries in D_N and if N is even there are two conjugacy classes, represented by t and tr.

Suppose N is odd and let $L := \langle t \rangle$. The right L cosets maybe represented by $1, r, r^2, \ldots, r^{-1}$, and $r^{k_1}t$, t, tr^{k_2} each fix exactly one L coset. Thus a, b and c each fix exactly one $\Lambda := \varphi^{-1}(L)$ coset. As N is odd so are m_1, m_2 and m_3. Hoare's Theorem tells us that the cosets fixed by a and b are in the same orbit under $\langle a, b \rangle$, the cosets fixed by b and c are in the same orbit under $\langle b, c \rangle$ and the cosets fixed by a and c are in the same orbit under $\langle a, c \rangle$. Therefore Λ has exactly one empty period cycle and $sp(t) = \pm 1$.

Suppose N is even. Let 2^α be the maximum power of two dividing N. We may assume 2^α divides m_1 and m_2 but not m_3. Thus we have two cases to consider, when m_3 is odd and when it is even.

Suppose m_3 is odd. Now m_3 is the order of $r^{k_1+k_2}$, this is $N/(k_1 + k_2, N)$ and so $k_1 + k_2$ must be even (in fact divisible by 2^α). Of course k_1 and k_2 are odd and so tr^{-k_1} and tr^{k_2} are conjugate in D_N, the other conjugacy class of symmetries is represented by t. Let $L' = \langle tr \rangle$, so tr^{-k_1} and tr^{k_2} fix two L' cosets each and t fixes none. Thus a and c each fix two $\Lambda' := \varphi^{-1}(L')$ cosets and b fixes none. Hoare's Theorem tells us that, because m_3 is odd, there is a coset fixed by a and one fixed by c in the same orbit under $\langle a, c \rangle$, while the other coset fixed by a is in the same orbit as the other coset fixed by c. m_1 and m_2 are both even and so the two cosets fixed by a lie in a single orbit of $\langle a, b \rangle$ and the two cosets fixed by c lie in a single orbit of $\langle b, c \rangle$. Therefore, Λ' has but one period cycle and $sp(tr) = \pm 1$. Let $L'' = \langle t \rangle$. Then a and c fix no $\Lambda'' := \varphi^{-1}(L'')$ cosets while b fixes two. These two will be in the same orbit under $\langle a, b \rangle$ and $\langle b, c \rangle$, thus Λ'' has only one period cycle and $sp(t) = \pm 1$. So when N is even, and one of the periods of Γ is not, then D_N contains two non-conjugate symmetries of S, each with one mirror.

Suppose m_3 is even. Then $k_1 + k_2$ is even as 2^α does not divide m_2, and k_1 and k_2 are still odd. tr^{-k_1} and tr^{k_2} will each fix two $\langle tr \rangle$ cosets and t will fix none. Thus a and c fix two $\varphi^{-1}(\langle tr \rangle)$ cosets each while b fixes none as before. However, because m_1, m_2 and m_3 are all even, the cosets fixed by a lie in the same orbits under $\langle a, b \rangle$ and $\langle a, c \rangle$ while the cosets fixed by c lie in the same orbits under $\langle b, c \rangle$ and $\langle a, c \rangle$. Hence there are two distinct chains of links and tr has two mirrors. t fixes only two $\langle t \rangle$ cosets and, because m_1 and m_2 are even, t has only one mirror. Therefore, when all the periods of Γ are even, D_N contains two non-conjugate classes of symmetries, one with one mirror and one with two.

The symmetries of surfaces that arise from surface kernel homomorphisms from proper NEC groups onto dihedral groups were studied in [8] and [22], where a useful graphical method has been developed to find the number of mirrors of these symmetries.

For each of the above symmetries, we want to determine whether the quotient by the symmetry is orientable or not. We do this by looking at Schreier coset graphs. If H is a subgroup of D_N generated by a symmetry, say T, then we denote by $\mathcal{G} := (D_N, H, \{tr^{-k_1}, t, tr^{k_2}\})$ the coset graph of H under the generating set $\{tr^{-k_1}, t, tr^{k_2}\}$. Thus $\mathcal{G}$ is precisely the coset graph of the subgroup $\Lambda := \varphi^{-1}(H)$, of Γ_*, under the generating set $\{a, b, c\}$. If $\widehat{\mathcal{G}}$ denotes the graph $\mathcal{G}$ minus loops, then Hoare's Theorem tells us that $\mathcal{U}/\Lambda\,(= S/\langle T\rangle)$ is orientable if and only if $\widehat{\mathcal{G}}$ is bipartite.

Let N be odd and let $H = \langle t\rangle$. We represent the H cosets by $1, r, \ldots, r^{-1}$. Of course, the vertices of the Schreier coset graph $\mathcal{G} = (D_N, H, \{tr^{-k_1}, t, tr^{k_2}\})$, correspond to these cosets. We shall endeavour to find a circuit of length three in $\widehat{\mathcal{G}}$, showing it not to be bipartite and showing that t has negative species.

Consider the element $t.tr^{-k_1}.tr^{k_2} = tr^{k_1+k_2}$ of D_N, this fixes a single H coset r^i, where $2i \equiv k_1+k_2 \pmod N$. Without lose of generality we may assume k_1+k_2 to be even and so $i = (k_1+k_2)/2$. If we start at the vertex in $\mathcal{G}$ corresponding to the H coset r^i, then $t.tr^{-k_1}.tr^{k_2}$ represents a circuit of length three in $\mathcal{G}$. If this circuit contains no loops then it is a circuit in $\widehat{\mathcal{G}}$.

$Hr^i.t = Hr^{-i}$ and so t fixes Hr^i if and only if $2i = k_1 + k_2 \equiv 0 \pmod N$ and this is the case if and only if $m_3 = N/(k_1+k_2, N) = 1$.
$Hr^{-i}.tr^{-k_1} = Hr^{(-k_1+k_2)/2}$ so tr^{-k_1} fixes Hr^{-i} if and only if $k_2 \equiv 0 \pmod N$, if and only if $m_2 = 1$.
Finally, $Hr^{(-k_1+k_2)/2}.tr^{k_2} = Hr^i$ and so tr^{k_2} fixes $Hr^{(-k_1+k_2)/2}$ if and only if $k_1 \equiv 0 \pmod N \Leftrightarrow m_2 = 1$. Thus we have found a circuit of odd length in $\widehat{\mathcal{G}}$ and so $sp(t) = -1$.

Let N be even. As before, we assume that if 2^α is the greatest power of 2 dividing N, then 2^α divides m_1 and m_2 but not m_3. We know that D_N has two conjugacy classes of symmetries of the associated Riemann surface, one represented by t and the other by tr. We have shown that tr has one mirror when m_3 is even and two when m_3 is odd, while t has one mirror in both cases. Let $L_1 = \langle t\rangle$, $L_2 = \langle tr\rangle$ and let $\mathcal{G}_i = (D_N, L_i, \Phi)$, where $\Phi = \{tr^{-k_1}, t, tr^{k_2}\}$, $i = \{1, 2\}$. Now k_1 and k_2 are both odd and so, with slight changes, the above argument shows that

$\widehat{\mathcal{G}}_1$ also has a circuit of length three when N is even and so again $sp(t) = -1$.

To find a circuit of odd length in $\mathcal{G}_2$, without loops, it is necessary to consider circuits of length five. (This is because we need an element of the form tr^{2i+1} to fix a L_2 coset and this element must be a word of odd length in Φ such that each element of Φ occurs once in any substring of length three of this element.) The element $t.tr^{-k_1}.tr^{k_2}.t.tr^{-k_1} = tr^{k_2}$ in D_N fixes two L_2 cosets, one of which is represented by $r^{(k_2-1)/2}$. We now look at the circuit in $\mathcal{G}_2$ represented by the above word in Φ, starting at the vertex associated to the coset $L_2r^{(k_2-1)/2}$, to see when it is loop free. We enumerate the cosets $0, 1, \dots, N-1$ where i corresponds to L_2r^i.

$t\colon \frac{1}{2}(k_2-1) \longmapsto \frac{1}{2}(-k_2-1)$, this represents a loop if and only if $k_2 \equiv 0 \pmod{N}$, if and only if $m_2 = 1$.

$tr^{-k_1}\colon \frac{1}{2}(-k_2-1) \longmapsto \frac{1}{2}(k_2-1)-k_1$, is a loop if and only if $k_1 \equiv k_2 \pmod{N}$, only if $m_1 = m_2$.

$tr^{k_2}\colon \frac{1}{2}(k_2-1)-k_1 \longmapsto \frac{1}{2}(k_2-1)+k_1$, is a loop if and only if $2k_1 \equiv 0 \pmod{N}$, if and only if $m_1 = 2$.

$t\colon \frac{1}{2}(k_2-1)+k_1 \longmapsto \frac{1}{2}(-k_2-1)-k_1$, is a loop if and only if $2k_1 + k_2 \equiv 0 \pmod{N}$, if and only if $k_1 + k_2 \equiv -k_1 \pmod{N}$, only if $m_1 = m_3$.

$tr^{-k_1}\colon \frac{1}{2}(-k_2-1)-k_1 \longmapsto \frac{1}{2}(k_2-1)$, is a loop if and only if $k_1 + k_2 \equiv 0 \pmod{N}$, only if $m_3 = 1$.

We require all the m_i's to be greater than one and by our hypothesis m_1 and m_3 are not equal as 2^α $(\alpha > 0)$ divides m_1 but not m_3. Hence the above circuit in $\mathcal{G}$ is also a circuit in $\widehat{\mathcal{G}}$ unless (i) $k_1 \equiv k_2 \pmod{N}$ or (ii) $m_1 = 2$. Now if m_1 is indeed two, then the circuit in $\mathcal{G}$ starting at $\frac{1}{2}(-k_1-1)$ represented by $t.tr^{k_2}.tr^{-k_1}.t.tr^{k_2} = tr^{-k_1}$ is without loops unless $m_2 = 2$, (this can be seen by the same procedure as above). However $[2,2,m]$ is not the signature of a Fuchsian group and so $\widehat{\mathcal{G}}$ is not bipartite unless, (possibly) $k_1 \equiv k_2 \pmod{N}$.

We now show that if k_1 and k_2 are equal, then $\widehat{\mathcal{G}}$ is bipartite and so tr does have positive species. If they are equal we may assume, without lose of generality, that $k_1 = k_2 = 1$, and so $\varphi(a) = tr^{-1}$, $\varphi(b) = t$ and $\varphi(c) = tr$. Recall that we enumerated the $\langle tr \rangle$ right cosets $1, r, \dots, r^{-1}$ by $0, 1, , \dots, N-1$. We partition the cosets in to two sets;

$$A := \{0, 1, \dots, (N/2)-1\}, \quad B := \{N/2, (N/2)+1, \dots, N-1\}.$$

If $i \in A$ then, acting by right multiplication, tr^{-1} maps i to $N-i-2$ which is in

B unless $i = (N/2) - 1$, in which case i is fixed by tr^{-1}. t maps i to $N - i - 1$ which is always in B. tr maps i to $N - i$ if $i > 0$, which is also in B, and fixes i when $i = 0$. Thus we see that tr^{-1}, t, tr map cosets in A to B or fix them, they act on cosets in B in a similar way. Hence the species of tr in this case is positive. Note that if $k_1 \equiv k_2 \pmod{N}$, then $m_1 = m_2$ and $m_3 = m/(k_1 + k_2, m) = m/2$, so C_N lifts to a group with signature of the form $[2n, 2n, n]$.

We now summarize these results.

Theorem 4.6

Let S be a compact Riemann surface which admits a large cyclic automorphism group of order N. S is uniformized by some Fuchsian surface group K and there is a surface kernel homomorphism φ, from a Fuchsian triangle group $\Gamma[m_1, m_2, m_3]$ onto a $\mathbf{Z}_N$, such that the kernel is K. Let x and y be canonical generators of Γ, and let their respective images under φ be k_1 and k_2. Then S is symmetric and always admits a symmetry (of the first kind) that extends the large cyclic group to a dihedral group D_N, and this lifts to an extended triangle group $\Gamma_*(m_1, m_2, m_3)$. The table below gives the number of conjugacy classes of these symmetries in D_N and their species; these depend on the signature of Γ and the homomorphism φ.

$\Gamma[m_1, m_2, m_3]$, φ	No. of conjugacy classes of symmetries in D_N	Species of conjugacy classes
Each m_i odd	1	$\{-1\}$
m_1, m_2 even, m_3 odd and $k_1 \not\equiv k_2 \pmod{N}$	2	$\{-1, -1\}$
$m_1 = m_2 = 2m_3$, m_3 odd and $k_1 \equiv k_2 \pmod{N}$	2	$\{-1, +1\}$
Each m_i even and $k_1 \not\equiv k_2 \pmod{N}$	2	$\{-1, -2\}$
$m_1 = m_2 = 2m_3$, m_3 even and $k_1 \equiv k_2 \pmod{N}$	2	$\{-1, +2\}$

Riemann surfaces that attain Wiman's bound on the order of an automorphism correspond to surface kernel homomorphisms from triangle groups with signature $[2, 2n + 1, 2(2n + 1)]$, $(n > 1)$, onto cyclic groups of order $2(2n + 1)$.

Hence surfaces of this nature only admit symmetries of the first kind with species -1. Although these triangle groups are not maximal they are the normalizers of the surface groups that uniformize the surfaces with this property, see case 14 of §4.5. Thus the surfaces for which Wiman's bound is attained have symmetry type $\{-1, -1\}$.

Section 4.4 Symmetries of the Second Kind

We now consider symmetries that extend large cyclic groups to groups that lift to proper NEC groups with signature $([m], (n))$. Let φ be a surface kernel homomorphism from a triangle group $\Gamma[m, m, n]$ onto a cyclic group of order m. If the cyclic group is generated by r and x, y are canonical generators of Γ, both of order m, then we may assume that $\varphi(x) = r$ and $\varphi(y) = r^k$, for some k. Hence, $(k, m) = 1$ and $(k+1, m) = m/n$. We are assuming that φ extends to some NEC group $\Gamma_*([m], (n))$, containing Γ with index two, and so, by §4.2 p41, $k^2 \equiv 1 \pmod{m}$. If x and c are canonical generators for Γ_*, then the group G, generated by $r := \varphi(x)$ and $u := \varphi(c)$ has presentation

$$G := \langle r, u \mid r^m = u^2 = urur^k = 1 \rangle.$$

Recall that if $k^2 \equiv 1 \pmod{m}$, then $m = 2^\alpha n_1 n_2$ where n_1 and n_2 are both odd and coprime such that $k \equiv 1 \pmod{n_1}$, $k \equiv -1 \pmod{n_2}$ and, if $\alpha > 0$, $k \equiv \pm 1 \pmod{2^\alpha}$, or, if $\alpha > 2$, $k \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}$.

Now $\varphi(xcx^{-1}) = rur^{-1} = ur^{-(k+1)}$. If $L = \langle u \rangle$, then $1, r, r^2, \ldots, r^{-1}$ represents a system of right L cosets in G. Hence, $1, x, x^2, \ldots, x^{-1}$ represents a system of right $\Lambda := \varphi^{-1}(L)$ cosets in Γ_*. The action of the canonical generators of Γ_* on these cosets, acting by right multiplication, is

$$\begin{aligned} x : x^i &\longmapsto x^{i+1}, \\ c : x^i &\longmapsto x^{-ki}, \\ xcx^{-1} : x^i &\longmapsto x^{-ki-(k+1)}. \end{aligned}$$

Note that x permutes the cosets as a m cycle. Now c fixes coset x^i, or i for ease, if and only if $i(k+1) \equiv 0 \pmod{m}$ and, as $(k+1, m) = m/n$, this is the case if and only if $n \mid i$. Hence c fixes $b := m/n$ cosets; $0, n, 2n, \ldots, n(b-1) = m - n$. We denote the induced reflection generators of Λ by $c_0, c_n, c_{2n}, \ldots, c_{m-n}$. We see that c^x fixes i if and only if $(i+1)(k+1) \equiv 0 \pmod{m}$, if and only if $n \mid i+1$. Thus c^x also fixes b cosets; $n-1, 2n-1, \ldots, bn-1 = m-1$ and we denote the induced reflection generators of Λ by $c^x_{n-1}, c^x_{2n-1}, \ldots, c^x_{m-1}$.

We now need to determine the links between these reflection generators. The relation $xcx^{-1} = c^x$, gives the links

$$c_0 \sim c^x_{m-1},\ c_n \sim c^x_{n-1},\ c_{2n} \sim c^x_{2n-1},\ \ldots\ldots,\ c_{m-n} \sim c^x_{m-n-1}.$$

See end of §2.3 and example 3 of §4.1.

By part (ii) of Hoare's Theorem, when an orbit of $\langle c^x, c\rangle$ contains a point fixed by either c or c^x, the orbit merely corresponds to a cycle of $c^x c$. Hence we look at the cycles of $c^x c$ to find the other links. We see that $\varphi(c^x c) = r^{(k+1)}$ which is of order n. Suppose n is even then we know each cycle of $c^x c$ contains two points fixed by c or two fixed by c^x or no points fixed by either. Therefore, if c fixes x^i, then for some $\lambda \in \mathbf{Z}$, $0 < \lambda < n$, we know that c also fixes $x^{i+\lambda(k+1)}$. This is true if and only if

$$\begin{aligned} & i + \lambda(k+1) \equiv -ik + -\lambda k(k+1) \pmod{m} \\ \Leftrightarrow \quad & 2\lambda(k+1) \equiv 0 \pmod{m}. \end{aligned}$$

Thus $\lambda(k+1) \equiv m/2 \pmod{m}$. Similarly, if c^x fixes x^j, then j and $j + m/2$ are in the same cycle of $c^x c$. Hence, when n is even we have the following links.

$$\begin{array}{lll} c_0 & \sim c_{\frac{m}{2}} & \qquad c^x_{n-1} \sim c^x_{\frac{m}{2}+n-1} \\ c_n & \sim c_{\frac{m}{2}+n} & \qquad c^x_{2n-1} \sim c^x_{\frac{m}{2}+2n-1} \\ \vdots & \quad \vdots & \qquad \quad \vdots \qquad\quad \vdots \\ c_{\frac{m}{2}-n} & \sim c_{m-n} & \qquad c^x_{\frac{m}{2}-1} \sim c^x_{m-1} \end{array}$$

Putting these links together with the ones previously obtained we form the following chains.

$$\begin{gathered} c_0 \sim c_{\frac{m}{2}} \sim c^x_{\frac{m}{2}-1} \sim c^x_{m-1} \sim c_0 \\ c_n \sim c_{\frac{m}{2}+n} \sim c^x_{\frac{m}{2}+n-1} \sim c^x_{n-1} \sim c_n \\ \vdots \\ c_{in} \sim c_{\frac{m}{2}+in} \sim c^x_{\frac{m}{2}+in-1} \sim c^x_{in-1} \sim c_{in} \\ \vdots \end{gathered}$$

There are $2b/4 = b/2 = m/2n$ chains and so u is a symmetry of $S := \mathcal{U}/K$, $K := ker(\varphi)$, with $\frac{m}{2n}$ mirrors.

Suppose n is odd. If a cycle of $c^x c$ contains a point fixed by c, then it must also contain a point fixed by c^x and vice versa. Therefore, if c fixes i then c^x must fix $i + \lambda(k+1)$, for some λ, $0 < \lambda < n$. This holds if and only if

$$\begin{aligned} & i + \lambda(k+1) \equiv -ki - \lambda k(k+1) - (k+1) \pmod{m} \\ \Leftrightarrow \quad & i(1+k) + 2\lambda(k+1) + (k+1) \equiv 0 \pmod{m} \\ \Leftrightarrow \quad & (2\lambda+1)(k+1) \equiv 0 \pmod{m} \\ \Leftrightarrow \quad & n \,|\, 2\lambda + 1. \end{aligned}$$

For ease we denote c_i by $[i]$ and c_j^x by $[j]'$, so we have links of the form $[i] \sim [i + \lambda(k+1)]'$. Thus with these links and those from the relation $xcx^{-1} = c^x$, we have the following chains.

$$\begin{aligned} [i] \sim \; & [i+\lambda(k+1)]' \sim [i+\lambda(k+1)+1] \sim [i+2\lambda(k+1)+1]' \sim \\ & [i+2\lambda(k+1)+2] \sim [i+3\lambda(k+1)+2]' \sim [i+3\lambda(k+1)+3] \sim \\ & [i+4\lambda(k+1)+3]' \sim [i+4\lambda(k+1)+4] \sim \cdots\cdots \end{aligned}$$

As, in total, there are only $2b = 2m/n$ points fixed by c or c^x, after $2b$ links we will surely arrive back at c_i or $[i]$. However, suppose the minimum number of links required to return to $[i]$ is p. Then clearly p is even, p divides $2b$ and

$$\frac{p}{2}(\lambda(k+1)+1) \equiv 0 \pmod{m}.$$

We want n to be odd and so $(k+1, m) = 2^\alpha n_2$, therefore $(\lambda(k+1)+1, 2^\alpha n_2) = 1$. Thus if m divides $\frac{p}{2}(\lambda(k+1)+1)$, then $2^\alpha n_2$ divides $p/2$ and so $2^{\alpha+1} n_2 = 2b$ divides p. Hence when n is odd there is only one chain and u has but one mirror.

The question of the sign of the species of u is a simple one in this case. We saw that x cyclically permutes the Λ cosets in a m cycle. Therefore, because x is conformal, we see, by (2.5)(iv), that for $S/\langle u \rangle$ to be orientable c would have to fix every coset and this is never the case. This could only happen if m divides $k+1$ in which case n would be one. Hence the species of u is negative.

Lemma 4.7

Let H be a group of automorphisms of a symmetric compact Riemann surface that contains at least one symmetry. Hence, H lifts to a proper NEC group, say Δ. If $v \in H$ is a symmetry of X that is not conjugate to any of the symmetries induced by the canonical reflection generators of Δ, then $sp(v) = 0$.

Proof Let K be the surface group that uniformizes S and let $\delta \in \Delta$ induce v. By the hypothesis, δ is a glide reflection, as any reflection in Δ is conjugate to one of the canonical reflection generators. Hence, v fixes a point of S, if and only if there is a point $p \in \mathcal{U}$ and a $k \in K$ such that $\delta(p) = k(p)$. This is true if and only if $k^{-1}\delta$, which also induces v, is a reflection. Hence v is fixed point free.

We now determine the number of conjugacy classes of (symmetries) involutions in $G\backslash\langle r\rangle$. The elements of $G\backslash\langle r\rangle$ can all be written in the form ur^i. Such an element is an involution if and only if $i(k-1) \equiv 0 \pmod{m}$, as $(ur^i)^2 = r^{i-ki}$. Hence, there are $(k-1,m)$ involutions in $G\backslash\langle r\rangle$. We see that u conjugates ur^i to ur^{-ik} and that r conjugates ur^i to $ur^{i-(k+1)}$. If ur^i is an involution, then, as noted above, $i(k-1) \equiv 0 \pmod{m}$ and so $ur^{-ik} = ur^i$. Therefore the conjugacy class of ur^i is

$$\{ur^i, ur^{i+(k+1)}, ur^{i+2(k+1)}, \ldots\ldots, ur^{i-(k+1)}\}$$

and so contains $m/(k+1,m)$ elements. The number of conjugacy classes of symmetries of S in G, is the number of symmetries in G, divided by the length of a conjugacy class, (they all have the same length).

$$\frac{(k-1,m)(k+1,m)}{m} = \begin{cases} 1 & \text{if } \alpha = 0; \\ 2 & \text{if } \alpha > 0 \text{ and } k \equiv \pm 1 \pmod{2^\alpha}; \\ 1 & \text{if } \alpha > 2 \text{ and } k \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}. \end{cases}$$

We know that a NEC group with signature $([m],(n))$ has only one conjugacy class of reflections. Hence, by Lemma 4.7 when there are two conjugacy classes of symmetries of S in G above, one of them must have zero species. Note that when $n = m/(k+1,m)$ is odd $k \equiv -1 \pmod{2^\alpha}$ and so the number of conjugacy classes of symmetries in G, when n is odd, is one if m is odd and two if m is even.

We now summarize these results.

Theorem 4.8

Let φ be a surface kernel homomorphism from a Fuchsian group $\Gamma[m,m,n]$, onto a cyclic group C of order m. Let k be such that $\varphi(x)^k = \varphi(y)$, where x and y are canonical generators of Γ both of order m. Hence $(k,m) = 1$ and $(k+1,m) = m/n$. Let S be the surface uniformized by the kernel of φ. Then C acts as a large group of automorphisms of S and S admits a symmetry of the second kind with respect to C if and only if $k^2 \equiv 1 \pmod{m}$. (Such a k exists if and only if $(n, m/n) = 1$ when $4 \not| \, m$ and 1 or 2 otherwise.) If G denotes the extension of C by such a symmetry, then the table gives the number of conjugacy classes of symmetries in G and their species according to the signature of Γ and

φ. The genus g of S, is given by the Riemann Hurwitz formula; $g = \frac{m}{2}(1 - \frac{1}{n})$.

$\Gamma[m, m, n]$, φ	No. of conj. classes of symmetries in G	Species of conj. classes
m odd	1	$\{-1\}$
m even, n odd [so $k \equiv -1 \pmod{2^\alpha}$]	2	$\{0, -1\}$
m and n even and $k \equiv +1 \pmod{2^\alpha}$	2	$\{0, -\frac{m}{2n}\}$
m and n even and $k \not\equiv \pm 1 \pmod{2^\alpha}$	1	$\{-\frac{m}{2n}\}$

2^α is the greatest power of 2 dividing m.

Now $n \geq 2$ and so $g \geq \frac{m}{4}$. Therefore, if we fix g, to maximize $\frac{m}{2n}$ we must let n equal two, in which case $g = \frac{m}{4}$ and $\frac{m}{2n} = g$. Hence Harnack's bound is not attained on compact Riemann surfaces that admit large cyclic automorphism groups by symmetries of the first or second kind with respect to these large groups.

Section 4.5 Symmetry Types

We know that if a Riemann surface admits a large cyclic automorphism group, then it also admits a symmetry of the first kind with respect to this large group. If such a surface also admits a symmetry of the second kind with respect to this large group, then clearly the cyclic group is not the full group of conformal automorphisms of the surface. Let S be a surface that admits a large cyclic automorphism group G, and symmetries of the first and second kind with respect to G. Let G_1 denote the extension of G by a symmetry of the first kind, G_2 the extension of G by a symmetry of the second kind and G' the full automorphism group of S. Then, possibly, two symmetries that are non-conjugate in G_1 are conjugate in G', or perhaps a symmetry from G_1 is conjugate to a symmetry from G_2 in G'. Furthermore, G' may contain symmetries that are non-conjugate to those in G_1 or G_2. Alternatively, a surface admitting a large cyclic group may not admit symmetries of the second kind w.r.t. the cyclic group and yet the cyclic group may still be properly contained in the full group of conformal automorphisms of the surface.

Therefore, to determine the symmetry type of surfaces that admit a large cyclic group, which is not the full conformal automorphism group, we need to consider the inclusions between Fuchsian triangle groups. Given such an inclusion $\Gamma < \Lambda$, let S be a surface uniformised by a surface group K, which is normal in Γ, such that Γ/K is cyclic and finite. We must determine if K is also normal in Λ. Thus given a surface kernel homomorphism φ, from Γ onto a cyclic group G, we need to determine if φ extends to Λ. That is; is there a surface kernel homomorphism from Λ onto a finite group G', containing G, such that the kernel is K and the restriction of this homomorphism to Γ is φ? Of course if the homomorphism does not extend, then G is the full group of conformal automorphisms and the symmetry type of the surface is one of those in the table at the end of §4.3.

In the Appendix all triangle group inclusions are considered. Information relating the conjugacy classes of reflections in the proper NEC groups containing Γ, with index two, and those in the proper NEC groups containing Λ, with index two, is also given. Once we find surface kernel homomorphisms onto cyclic groups that do extend we can use these results and those in the previous sections to determine the symmetry types of the associated surfaces. The requirements of Theorem 1.17 mean we only have to consider cases 1–4, 7, 9 and 10–14, with some restrictions on the first three and last two cases. Note that in these cases the Appendix tells us that the symmetries of the surfaces with non-zero species are all conjugate to symmetries of the first or second kind w.r.t. the cyclic group except in case twelve.

1. The Appendix tells us we can choose canonical generators x and y for $\Lambda[2, m, 2n]$ of orders 2 and m respectively, such that $X := y$ and $Y := xyx$ are canonical generators for $\Gamma[m, m, n]$, both of order m. Therefore given a surface kernel epimorphism φ, from Γ to a finite group, we see that φ extends to Λ if and only if $\varphi(X) \mapsto \varphi(Y)$ is an automorphism of order two of the finite group. Thus if the finite group is cyclic, generated by r and $\varphi(X) := r$, $\varphi(Y) := r^k$, then φ extends to Λ if and only if $k^2 \equiv 1 \pmod{m}$. This is precisely the same condition for the associated surface to admit symmetries of the second kind w.r.t. the cyclic group.

If k is as above, then the extension of φ maps Λ to the group

$$G := \langle r, s \,|\, r^m = s^2 = srsr^{-k} = 1 \rangle,$$

by taking x to s and y to r. We know the associated surface is symmetric and so admits a symmetry of the first kind t, w.r.t. G, that conjugates s and r to their inverses. We denote by G_*, the group generated by these three automorphisms.

The elements of G_* are

$$\{1, r, \dots, r^{-1}, s, sr, \dots, sr^{-1}, t, tr, \dots\dots\dots, tsr^{-1}\}.$$

The involutions (symmetries) in $G_*\backslash G$ are of the form tr^i, for all i, or tsr^i for $i(k-1) \equiv 0 \pmod{m}$. Now

$$r^{-1}tr = tr^2, \quad str^is = tr^{ik}, \quad ttr^it = tr^{-i}, \quad rtsr^ir^{-1} = tsr^{i-k-1},$$
$$stsr^is = tsr^{ik} = tsr^i \quad \text{and} \quad ttsr^it = tsr^{-i}.$$

Hence the symmetries of the form tr^i form one conjugacy class in G_* if m is odd and two if m is even. The symmetries of the form $tsr^{\lambda d}$, where $d = m/(k-1, m)$ and λ a constant, split in to classes of order $(k+1, m) = n$. Hence, as there are $(k-1, m)$ such symmetries there are $(k-1, m)/(k+1, m)$ conjugacy classes of such symmetries. We recall the notation of §4.2.

$$\frac{(k-1,m)}{(k+1,m)} = \begin{cases} 1 & \text{if } \alpha = 0 \text{ or, } \alpha > 2 \text{ and } k \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}; \\ 2 & \text{if } \alpha > 0 \text{ and } k \equiv \pm 1 \pmod{2^\alpha}. \end{cases}$$

The symmetries tr^i are of the first kind w.r.t. $\langle r \rangle$, while $tsr^{\lambda d}$ are of the second kind or conjugate to those of the second kind. Thus the classes and species of symmetries in G_* are

$$\begin{array}{cl}
-1,\ -1 & \text{if } m \text{ odd;} \\
0,\ -1,\ -1,\ -1 & \text{if } m \text{ even, } n \text{ odd (so } k \equiv -1 \pmod{2^\alpha}\text{);} \\
0,\ -1,\ -1,\ +1 & \text{if } m = 2n,\ n \text{ odd and } k = 1; \\
0,\ -1,\ -2,\ -\frac{m}{2n} & \text{if } m, n \text{ even and } 1 \neq k \equiv 1 \pmod{2^\alpha}; \\
0,\ -1,\ +2,\ -\frac{m}{2n} & \text{if } m = 2n,\ n \text{ even and } k = 1; \\
-1,\ -2,\ -\frac{m}{2n} & \text{if } m, n \text{ even and } k \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}.
\end{array}$$

We will show, in our treatment of inclusion twelve, that in the third and fifth cases G_* is not the full group of automorphisms of the surface, while in the other instances it clearly is. This is because $\Lambda[2, m, 2n]$ is maximal when $m \neq 2n$.

2. Let x and y be canonical generators of $\Lambda[3, 3, m]$, both of order three, such that $X := xy$ and $Y := yx$ are canonical generators of $\Gamma[m, m, m]$. Such generators exist by the method in the Appendix using Schreier transversals. Note that,

$$xXx^{-1} = (XY)^{-1}, \; xYx^{-1} = X \quad \text{and so} \quad x(XY)^{-1}x^{-1} = Y.$$

Therefore, a homomorphism φ, from Γ onto a finite group G, extends to Λ if and only if

$$\varphi(X) \mapsto \varphi(XY)^{-1},\ \varphi(Y) \mapsto \varphi(X) \quad \text{and} \quad \varphi(XY)^{-1} \mapsto \varphi(Y)$$

describes an automorphism of the finite group. In our case the finite group G is cyclic, of order m, generated by r say and we may assume φ maps X to r and Y to r^k. Recall that for this to be a surface kernel homomorphism m must be odd. Let ψ be a mapping of G defined by

$$\psi(r) = r^{-(k+1)}, \quad \psi(r^k) = r, \quad \psi(r^{(k+1)}) = r^{-k}.$$

Then this is an automorphism of G if and only if $k^2 + k + 1 \equiv 0 \pmod{m}$. Note that if $k^2 \equiv 1 \pmod{m}$ as well, then $m = 3$, and $\Gamma[3,3,3]$ is not Fuchsian. For low k the possibilities are as follows.

k	2	3	4	5	6	7	8	9	10	11
k^2+k+1	7	13	21	31	43	57	73	91	111	133
Possible m	7	13	7	31	43	19, 57	73	91	37, 111	7, 19
genus	3	6	3	15	21	9, 28	36	45	18, 55	3, 9

The genus here is that of the associated surface. Suppose that φ does indeed extend to $\Lambda[3,3,m]$. If G' is the image of Λ, under φ, then it has presentation

$$G' = \langle r, s \,|\, r^m = s^3 = srs^{-1}r^{k+1} = 1 \rangle,$$

where $s := \varphi(x)$ and $s^2r := \varphi(y)$. We know that the associated surface S, is symmetric and that it admits symmetries of the first kind w.r.t. G but none of the second kind. Thus, any symmetry that normalizes G' is of the second kind w.r.t. G', see Appendix. Therefore φ extends to some $\Delta([3],(m))$ but to no $(3,3,m)$. Now $\Delta([3],(m))$ is generated by x and a reflection C such that $CxC = y^{-1}$. Hence if G_* is the image of Δ, under φ, then G_* has presentation

$$G_* = \langle r, s, t \,|\, r^m = s^3 = t^2 = srs^{-1}r^{k+1} = tsts^2r = 1 \rangle,$$

where $t := \varphi(C)$. Note that $(tr)^2 = 1$. The only symmetries in G_* are of the form tr^i, for all i. We see that r conjugates t to tr^{-2} and so, because m is odd, there is just one conjugacy class of symmetries in G_*. This class is represented by a symmetry of the first kind w.r.t. G and so has species -1, see §4.3. In fact, as we shall see in the next case, G_* is the full automorphism group of S and so the symmetry type of S is just $\{-1\}$.

3. If a surface kernel homomorphism from a $[m,m,m]$, onto a cyclic group, extends to a $[2,3,2m]$, then it must extend to a $[2,m,2m]$ and to a $[3,3,m]$. This requires there to be a k such that $k^2 \equiv 1 \pmod{m}$ and $k^2+k+1 \equiv 0 \pmod{m}$, see the previous two cases. Thus m must be three, in which case $[m,m,m]$ is not Fuchsian. Hence there is no such extension, and so in the previous case G_* is indeed the full automorphism group of S. This is because the only Fuchsian group to contain a $[3,3,m]$, has signature $[2,3,2m]$.

4. If $\Gamma[7,7,7]$ is generated by cannonical generators x and y then it is well known that the surface associated to the surface kernel homomorphism, from Γ onto $\mathbf{Z}_7$, that maps x to 1 and y to 2 is the Klein surface. It is also well known that this surface has symmetry type $\{-1\}$.

Note that, up to an automorphism of $\mathbf{Z}_7$ there are only two surface kernel homomorphisms from Γ onto $\mathbf{Z}_7$, only the one above extends to $\Lambda[2,3,7]$.

7. If a surface kernel homomorphism from $\Gamma[4,8,8]$, onto a cyclic group of order eight, extends to a homomorphism from $\Lambda[2,3,8]$ onto a finite group G', then the order of G' must be 96 and the associated surface S must carry a regular map of type $\{3,8\}$. S would have genus three, so the map it carries should appear in Sherk's list [30] and such a map does indeed appear. The group of conformal automorphisms of the map is given as

$$\langle r,s \mid r^8 = s^2 = (rs)^3 = (srs^{-1}r^{-1})^3 = 1\rangle.$$

Canonical generators x and y, of orders 8 and 2 respectively, can be chosen for $\Lambda[2,3,8]$ such that $X := yxy$ and $Y := x^4yxyx^{-4}$ are canonical generators, both of order eight, for Γ, see Appendix. Clearly, $\varphi : x \mapsto r,\ y \mapsto s$ is a surface kernel homomorphism from Λ onto $\langle r,s\rangle$. If we denote by G, the image of Γ under this homomorphism, then it can be seen that G is cyclic of order eight and $r^4srsr^{-4} = (srs)^5$. There are two inequivalent surface kernel homomorphisms from Γ onto C_8, one characterized by $k=1$ and the other by $k=5$. The surface associated to the epimorphism φ, can therefore be thought of as an extension of the second epimorphism from Γ to C_8. (The first, leads to a hyperelliptic surface, and extends to a $[2,4,8]$ as we shall see later). By §4.4 a symmetry of the second kind w.r.t. G has species -1 and by §4.3 the two classes of symmetries in G extended by a symmetry of the first kind have species -1 and -2. The Appendix tells us $\mathrm{Aut}(S)$ contains at most two conjugacy classes of symmetries with non-zero species, both of which are represented by symmetries of the first or second kind w.r.t. G. From remarks in §3.2 $\mathrm{Aut}(S)$ is $\langle r, s\rangle$ extended by a symmetry t, such that tr and ts are both involutions. Thus it can be seen that there are three conjugacy classes of symmetries in $\mathrm{Aut}(S)$ and so S has symmetry type $\{0, -1, -2\}$.

9. If a surface kernel homomorphism from $\Gamma[9,9,9]$, onto a cyclic group of order nine were to extend to $\Lambda[2,3,9]$, we would expect to find a regular map of type $\{3,9\}$ on a surface of genus four. The automorphism group of the map would be of order 108. There is no such map in Garbe's list [13]. Thus no such homomorphism extends in this way.

11. For all $n \geq 2$ there are surface kernel homomorphisms from $\Gamma[4n, 4n, n]$ onto C_{4n}, the cyclic group of order $4n$ generated by r say. Let φ be one such homomorphism and let $\varphi(g_1) := r$, $\varphi(g_2) := r^k$, where g_1 and g_2 are canonical generators for Γ, both of order $4n$.

Clearly, if φ extends to $\Lambda[2,3,4n]$ it must also extend to some $[2, 2n, 4n]$ and so $k^2 \equiv 1 \pmod{4n}$, see case one. Thus we require $4n$ to have the form $2^\alpha n_1 n_2$ where n_1, n_2 are odd and coprime such that $k \equiv 1 \pmod{n_1}$, $k \equiv -1 \pmod{n_2}$ and $k \equiv \pm 1 \pmod{2^\alpha}$ or $k \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}$ if $\alpha > 2$, see §4.2. The order of $\varphi(g_1 g_2) = r^{k+1}$ is n and so $(k+1, 4n) = 4$. This is the case if and only if $n_2 = 1$ and (i) n is odd, $k \equiv -1 \pmod 4 \Rightarrow k = 2n+1$, *or* (ii) $n \equiv 2 \pmod 4$, so $\alpha = 3$ and $k \equiv 3 \pmod 8 \Rightarrow k = n+1$ or $3n+1$. Of course these are only necessary conditions for φ to extend to Λ.

The Appendix tells us, that when φ does extend, there are one or two classes of symmetries with non-zero species and these are represented by symmetries of the first and second kind w.r.t. C_{4n}. §4.3 says that C_{4n} extended by a symmetry of the first kind has two conjugacy classes of symmetries with species -1 and -1 if n is odd, and -1 and -2 if n is even. C_{4n} extended by a symmetry of the second kind has two classes of symmetries with species 0 and -1, if n is odd and one class with species -2, if n is even. Thus if φ does indeed extend to Λ then the associated surface has

(i) one or two classes of symmetries with species -1, at least one class with species 0, and possiblly others all with zero species if n is odd, *or*

(ii) one class with species -1, another with species -2 and possiblly other classes, all with species 0 if n is even.

Here are a couple of examples to show that some of these extensions do occur.

(a) There is a surface kernel homomorphism φ, from $\Lambda[2,3,8]$ onto

$$G' := \langle r, s \mid r^8 = s^2 = (rs)^3 = (r^4 s)^3 = 1\rangle$$

defined by $\varphi(x) := r$ and $\varphi(y) := s$, where x and y are canonical generators for Λ of orders eight and two respectively. The subgroup Γ, of Λ, generated by $g_1 := yxy$ and $g_2 := x^2yxyx^{-2}$ has signature $[8,8,2]$, see Appendix. Indeed, g_1 and g_2 are canonical generators for Γ, both of order eight, and it can be seen that $\varphi(g_2) = \varphi(g_1)^3$. Thus $\varphi(\Gamma)$ is cyclic of order eight. The full group of automorphisms G_*, of the associated surface S, is G extended by a symmetry t such that tr and ts are involutions. It can be seen that G_* has two classes of symmetries, thus the symmetry type of S is $\{-1,-2\}$.

(b) There is a surface kernel homomorphism ψ, from $\Lambda[2,3,12]$ onto

$$G' := \langle r, s \mid r^{12} = s^2 = (rs)^3 = sr^4s^{-1}r^{-4} = 1\rangle$$

defined by $\psi(x) := r$ and $\psi(y) := s$, where x and y are canonical generators for Λ of orders twelve and two respectively. The subgroup Γ, of Λ, generated by $g_1 := yxy$ and $g_2 := x^2yxyx^{-2}$ has signature $[12,12,3]$, see Appendix, and g_1, g_2 are canonical generators for Γ, both of order twelve. We see that $\psi(g_2) = \psi(g_1)^7$ and so $\psi(\Gamma)$ is cyclic of order twelve. If S is the associated surface then $\mathrm{Aut}(S)$ is G' extended by a symmetry t such that tr and ts are involutions. $\mathrm{Aut}(S)$ can be shown to have three classes of symmetries, represented by rt, t and rsr^2str. $\mathrm{Aut}(S)$ lifts to $\Lambda_*(2,3,4n)$ which is generated by reflections a, b and c such that $ab = x$, $bc = y$, and rt and t are the images of a and b. Thus precisely two of the classes of symmetries have non-zero species and so the symmetry type of S is $\{0,-1,-1\}$.

It may well be that the symmetry type is precisely $\{0,-1,-1\}$ when n is odd, and $\{-1,-2\}$ when n is even, for all extensions. However all we can say for sure is the symmetry type is (i) $\{0^p,-1\}$ or $\{0^r,-1,-1\}$ for some $p \geq 1$ when n is odd, or (ii) $\{0^q,-1,-2\}$ for some $q \geq 0$ when n is even. Here 0^p denotes the occurrence of 0, p times in the symmetry type.

12. Suppose x and y are canonical generators for $\Gamma[2n,2n,n]$, both of order $2n$. Let φ be a surface kernel homomorphism from Γ onto C_{2n}, which is generated by r, defined by $\varphi(x) := r$ and $\varphi(y) := r^k$. We see that for φ to extend to $\Lambda[2,4,2n]$ it must also extend to some $\Delta[2n,2n,2]$. Case one tells us that this happens if and only if $k^2 \equiv 1 \pmod{2n}$. Thus we can write $2n$ as $2^\alpha n_1 n_2$ where n_1 and n_2 are both odd and coprime, see §4.2. We require $(k+1, 2n) = 2$, therefore $n_2 = 1$ and $k \equiv 1 \pmod{2^\alpha}$ or, if $\alpha > 2$, $k \equiv 2^{\alpha-1} + 1 \pmod{2^\alpha}$. Hence $k = 1$ or $k = n+1$, note that the second case can occur only when four divides n. The two possibilities for k describe two epimorphisms that do not differ by just an automorphism of C_{2n} and so we have to consider two epimorphisms and two extensions.

$$\begin{array}{ccc} \varphi_1 : \Gamma & \longmapsto & C_{2n} \\ x & & r \\ y & & r \end{array} \qquad\qquad \begin{array}{ccc} \varphi_2 : \Gamma & \longmapsto & C_{2n} \\ x & & r \\ y & & r^{n+1} \end{array}$$

We can choose canonical generators x_1 and y_1 for $\Delta[2,2n,2n]$ of orders 2 and $2n$ respectively, such that $x = y_1$ and $y = x_1 y_1 x_1$, see Appendix, and we let $z_1 = (x_1 y_1)^{-1}$. Let G_1 denote the image of Δ under φ_1 and G_2 the image under φ_2. Then

$$G_1 := \langle r, s \,|\, r^{2n} = s^2 = srsr^{-1} = 1 \rangle \quad \text{and} \quad G_2 := \langle r, s \,|\, r^{2n} = s^2 = srsr^{n-1} = 1 \rangle.$$

$$\begin{array}{ccc} \varphi_1 : \Delta & \longmapsto & G_1 \\ x_1 & & s \\ y_1 & & r \\ z_1 & & sr^{-1} \end{array} \qquad\qquad \begin{array}{ccc} \varphi_2 : \Delta & \longmapsto & G_2 \\ x_1 & & s \\ y_1 & & r \\ z_1 & & sr^{n-1} \end{array}$$

In the same way we can choose canonical generators x_2 and y_2 for Λ, of orders 2 and $2n$ respectively, such that $y_1 = y_2$ and $z_1 = x_2 y_2 x_2$.

Hence φ_1 extends to Λ if and only if, G_1 admits an automorphism of order two that maps $\varphi_1(y_1) = r$ to $\varphi_1(z_1) = sr^{-1}$. Clearly sr^{-1} is of order $2n$ and this mapping takes $srsr^{-1}$ to the identity, so G_1 does admit such an automorphism and φ_1 does extend. Note that such an automorphism fixes s.

Similarly, φ_2 extends to Λ if and only if $r \mapsto sr^{n-1}$, $sr^{n-1} \mapsto r$ describes an automorphism of G_2. Note that

$$sr^{n-1}.r^{n+1} = s \longmapsto r(sr^{n-1})^{n+1} = rsr^{n-1}(sr^{n-1}sr^{n-1})^{n/2} = s(r^{n-2})^{n/2} = sr^n$$

because 4 divides n. This mapping is indeed an automorphism of G_2 and so φ_2 also extends to Λ.

The Appendix tells us that the surfaces associated to these extensions will admit a class of symmetries that are not conjugate to any symmetry that normalizes C_{2n} and so we will now seek to calculate its species. From the Appendix we know that the class will be represented by a symmetry of the second kind w.r.t. G_1 in the first case and G_2 in the second, however we will work in the full automorphism group so as to determine the number of conjugacy classes of symmetries with species zero.

We deal with φ_1 first. The image of Λ, under φ_1, is G_1 extended by an involution $u := \varphi_1(x_2)$ such that $uru = sr^{-1}$ and so $usu = s$. Note that $\varphi_1(y_2) = r$.

Thus
$$\begin{aligned}\varphi_1(\Lambda) &= \langle r, s, u \mid r^{2n} = s^2 = u^2 = srsr^{-1} = urur s = 1\rangle \\ &= \langle r, u \mid r^{2n} = u^2 = (ru)^4 = rur^2ur = 1\rangle \\ &= \langle u, v \mid u^2 = v^4 = (uv)^{2n} = uv^2uv^2 = 1\rangle\end{aligned}$$

where $v := ur$. The full group of automorphisms G_*, of the associated surface S, will lift to a $\Lambda_*(2,4,2n)$. Λ_* will be generated by reflections a, b and c such that $ab = x_2$ and $bc = x_2y_2$. Note that a, b and c here are not the same as those in the Appendix. However the Appendix does tell us that the symmetry induced by the canonical reflection generator of Λ_*, that is associated to the link periods two and four, represents the class we are interested in. In this case it is $\varphi_1(b)$, In fact, if $\varphi_1(b) = t$ then

$$G_* := \langle u, v, t \mid u^2 = v^4 = t^2 = (uv)^{2n} = uv^2uv^2 = (tu)^2 = (tv)^2 = 1\rangle.$$

Therefore, $trt = tuvt = uv^{-1} = ur^{-1}u = rs$ and $tst = tv^2t = s$. Let $L := \langle t\rangle$, so the right L cosets are represented by the elements of $\varphi_1(\Lambda)$, which are

$$\{1, r, \ldots, r^{-1}, s, sr, \ldots, sr^{-1}, u, ur, \ldots\ldots\ldots, usr^{-1}\}.$$

We know that $\varphi_1(a) = tu$ and $\varphi_1(c) = tv$ are not conjugate to t. Therefore, we only need to look at the action of t, on the L cosets, for the reflection generators of $\varphi_1^{-1}(L)$. Note that $tr^it = (rs)^i = s^ir^i$. Hence

$$\begin{aligned}Lr^it &= Ls^ir^i & &= Lr^i & &\iff & &2 \mid i \\ Lsr^it &= Ls^{i+1}r^i & &= Lsr^i & &\iff & &2 \mid i \\ Lur^it &= Lus^ir^i & &= Lur^i & &\iff & &2 \mid i \\ Lusr^it &= Lus^ir^i & &= Lusr^i & &\iff & &2 \mid i\end{aligned}$$

Therefore t fixes the following $4n$ cosets.

$$1, r^2, \ldots, r^{-2}, s, sr^2, \ldots, sr^{-2}, u, ur^2, \ldots, ur^{-2}, us, usr^2, \ldots, usr^{-2}.$$

We must now determine how the associated reflection generators of $\varphi_1^{-1}(L)$ are linked. $\varphi_1(ab) = u$, which is of order two, thus if g represents a coset fixed by t then the associated reflection generator is linked to that of gu. Therefore we have the links

$$1 \sim u,\ r^2 \sim ur^{-2},\ \ldots\ , r^2 \sim ur^{2i} \quad \text{and} \quad s \sim us,\ sr^2 \sim usr^{-2},\ \ldots\ , sr^{-2} \sim usr^2.$$

$\varphi_1(bc) = v = ur$, which is order four, thus the cycles of v acting on the L cosets are all four cycles and if g represents a coset fixed by t then gv^2 is the other coset fixed by t in the same v cycle. This is because $\varphi_1^{-1}(L)$ has no proper periods and only empty period cycles. Note that $v^2 = s$ and s is in the centre of G_*. Therefore we have the following links.

$$1 \sim s,\ r^2 \sim sr^2,\ \ldots\ , r^{-2} \sim sr^{-2} \quad \text{and} \quad u \sim us,\ ur^2 \sim usr^2,\ \ldots\ , ur^{-2} \sim usr^{-2}.$$

Putting these links together we form the following chains.

$$\begin{gathered} 1 \sim u \sim us \sim s \sim 1 \\ r^2 \sim ur^{-2} \sim usr^{-2} \sim sr^2 \sim r^2 \\ \vdots \\ r^{-2} \sim ur^2 \sim usr^2 \sim sr^{-2} \sim r^{-2} \end{gathered}$$

Hence t has $4n/4 = n$ mirrors. The Riemann Hurwitz formula tells us that the genus g, of the surface is $n-1$ and so $n = g+1$ which is Harnack's bound. Therefore $sp(t) = +(g+1)$, as a symmetry that attains the bound of Harnack necessarily separates the surface. At the end of the last section we remarked that symmetries of the first or second kind, w.r.t. a large cyclic automorphism group, never attain Harnack's bound, now we have shown that there is an infinite family of surfaces with such automorphism groups that do admit symmetries attaining this bound.

We now consider how many conjugacy classes of symmetries there are in G_*.

$$\begin{aligned} (tr^i)^2 &= (rs)^i r^i = s^i r^{2i} &&= 1 \quad \Leftrightarrow i = 0 \text{ or } i = n \text{ and } 2\,|\,n \\ (tsr^i)^2 &= s(rs)^i sr^i = s^i r^{2i} &&= 1 \quad \Leftrightarrow i = 0 \text{ or } i = n \text{ and } 2\,|\,n \\ (tur^i)^2 &= u(rs)^i ur^i = (sr^{-1}s)^i r^i &&= 1 \quad \text{for all } i \\ (tsur^i)^2 &= us(rs)^i usr^i = s(sr^{-1}s)^i sr^i &&= 1 \quad \text{for all } i \end{aligned}$$

The symmetries are the elements tur^i and $tusr^i$ (for all i), t, ts and, if n is even, tr^n, tsr^n. We see that

$$\begin{aligned} r.tur^i.r^{-1} &= tur^{i-2} & r.tusr^i.r^{-1} &= tusr^{i-2} \\ u.tur^i.u &= tus^ir^{-i} & u.tusr^i.u &= tus^{i+1}r^{-i} \\ t.tur^i.t &= tus^ir^i & t.tusr^i.t &= tus^{i+1}r^i \\ r.t.r^{-1} &= ts & r.tr^n.r^{-1} &= tsr^n. \end{aligned}$$

Thus the conjugacy classes are

$$\begin{gathered} \{tu, tur^2, \ldots, tur^{-2}\} \quad \{tus, tusr^2, \ldots, tusr^{-2}\} \\ \{tur, tur^3, \ldots, tur^{-1}, tusr, tusr^3, \ldots, tusr^{-1}\} \\ \{t, ts\} \quad \text{and} \quad \{tr^n, tsr^n\} \quad \text{if } n \text{ is even.} \end{gathered}$$

Hence G_* has four classes when n is odd and five when n is even. Recall that $\varphi_1(a) = tu$ and $\varphi_1(c) = tur$, and are therefore non-conjugate. Hence G_* contains three classes of symmetries with non-zero species, two of which are represented by symmetries of the first or second kind w.r.t. C_{2n}. Thus we are in a position to use results from §4.3 to write down the symmetry type.

$$st(S) = \begin{cases} \{0, -1, +1, +(g+1)\} & \text{if } n \text{ is odd;} \\ \{0, 0, -1, +2, +(g+1)\} & \text{if } n \text{ is even.} \end{cases}$$

Now we consider the extension of φ_2. The image of Λ, under φ_2, is G_2 extended by an involution $u := \varphi_2(x_2)$, such that $uru = sr^{n-1}$ (and so $usu = sr^n$). Note that $\varphi_2(y_2) = r$. Thus

$$\varphi_2(\Lambda) = \langle r, s, u \mid r^{2n} = s^2 = u^2 = srsr^{n-1} = urur^{n+1}s = 1\rangle.$$

The full group of automorphisms G_*, of the associated surface S, will lift to a $\Lambda_*(2, 4, 2n)$, generated by reflections a, b and c such that $ab = x_2$ and $bc = x_2y_2$. We are interested in $\varphi_2(b)$ and if $\varphi_2(b) = t$, then $tut = u$, $turt = (ur)^{-1}$. Therefore $trt = sr$ and

$$G_* := \langle r, s, u, t \mid r^{2n} = s^2 = u^2 = t^2 = srsr^{n-1} = urur^{n+1}s = (tu)^2 = (tr)^2s = 1\rangle.$$

Let $L := \langle t\rangle$, so the right L cosets are represented by the elements of $\varphi_2(\Lambda)$. We know that $\varphi_2(a) = tu$ and $\varphi_2(c) = tur$ are not conjugate to t and so we only need to look at the action of t on the L cosets for the reflection generators of $\varphi_2^{-1}(L)$. Note that $tr^it = (sr)^i$.

$$Lr^i.t = L(sr)^i = \begin{cases} Lsr(srsr)^{(i-1)/2} = Lsrr^{(n+2)(i-1)/2} & \text{if } i \text{ is odd;} \\ L(srsr)^{i/2} = Lr^{(n+2)i/2} = Lr^{i+ni/2} & \text{if } i \text{ is even.} \end{cases}$$

Hence t fixes r^i if and only if 4 divides i. As t commutes with u and s we see that altogether t fixes $2n$ cosets and they are

$$1, r^4, \ldots, r^{-4}, s, sr^4, \ldots, sr^{-4}, u, ur^4, \ldots, ur^{-4}, us, usr^4, \ldots, usr^{-4}.$$

$\varphi_2(ab) = u$ and $r^{4\lambda}u = ur^{-4\lambda}$, so we have the links

$$1 \sim u,\ r^4 \sim ur^{-4},\ \ldots\ , r^{-4} \sim ur^4,$$
$$s \sim usr^n,\ sr^4 \sim usr^{n-4},\ \ldots\ , sr^{-4} \sim usr^{n+4}.$$

$\varphi_2(bc) = ur$, which is of order four, and $(ur)^2 = sr^n$. As $r^{4\lambda}sr^n = sr^{n+4\lambda}$ we have the links

$$1 \sim sr^n,\ r^4 \sim sr^{n+4},\ \ldots\ , r^{-4} \sim sr^{n-4},$$
$$u \sim usr^n,\ ur^4 \sim usr^{n+4},\ \ldots\ , ur^{-4} \sim usr^{n-4}.$$

These links combine to form the following chains

$$1 \sim u \sim usr^n \sim s \sim r^n \sim ur^n \sim us \sim sr^n \sim 1,$$
$$r^4 \sim ur^{-4} \sim usr^{n-4} \sim sr^4 \sim r^{n+4} \sim ur^{n-4} \sim usr^{-4} \sim sr^{n+4} \sim r^4,$$
$$\vdots$$
$$r^{-4} \sim ur^4\ usr^{n+4} \sim sr^{-4} \sim r^{n-4} \sim ur^{n+4} \sim usr^4 \sim sr^{n-4} \sim r^{-4}.$$

There are $2n/8 = n/4$ chains. A rather tedious argument shows that

$$tu.tur.tu.t.tur.t.tur.tu.tur.tu.tur.t.tur = t$$

represents a circuit, without loops, of odd length in the coset graph of L in G_*, over $\{tu, t, tur\}$. Hence, $sp(t) = -n/4 = -(g+1)/4$, where g is the genus S.

It can be seen that G_* has three conjugacy classes of symmetries represented by t, tu and tur when $n \equiv 4 \pmod 8$ and four classes represented by t, tu, tur

and $tr^{n/2}$ when $n \equiv 0 \pmod 8$. Using the above results and those in the tables of the previous sections we see that

$$st(S) = \begin{cases} \{-1,-2,-n/4\} & \text{when } n \equiv 0 \pmod 8; \\ \{0,-1,-2,-n/4\} & \text{when } n \equiv 4 \pmod 8). \end{cases}$$

Note that if $n = 4$, then φ_2 extends further, to a $[2,3,8]$, see case seven. The symmetry type in this case is $\{0,-1,-2\}$. Hence, in this instance, the two classes of symmetries with species $-1 = -n/4$, in G_* above, are in fact conjugate in $\mathrm{Aut}(S)$.

13. Suppose φ is a surface kernel homomorphism from $\Gamma[3,n,3n]$ onto C_{3n}, of course there is such a homomorphism if and only if n and three are coprime. Let X and Y be canonical generators for Γ of orders $3n$ and 3 respectively. We may assume that $\varphi(X) = r$, where r generates C_{3n}. Then, as $\varphi(XY)$ must be of order n, we see that

$$\varphi(Y) = \begin{cases} r^{2n} & \text{if } n \equiv 1 \pmod 3; \\ r^{n} & \text{if } n \equiv 1 \pmod 3). \end{cases}$$

The genus g, of the associated surface S, is given by $g = n - 1$. If φ extends to some $\Lambda[2,3,3n]$, then S carries a regular map of type $\{3,3n\}$ whose (conformal) automorphism group is of order $12n$. Hence, if such an extension were possible for $n = 4$, 5, 7 or 8, we would expect to find the corresponding regular map in Sherk's list [30], or in the lists of Garbe [13], [15]. For $n = 5$, 7 or 8 there is no such map. However, Sherk lists a map of type $\{3,12\}$, on a surface of genus three, whose automorphism group G, has the following presentation

$$\begin{aligned} G := &\langle r, s \mid r^{12} = s^3 = (rs)^2 = sr^3s^{-1}r^{-3} = 1\rangle \\ = &\langle u, v \mid u^{12} = v^2 = (uv)^3 = u^3vu^{-3}v^{-1} = 1\rangle. \end{aligned}$$

Here $u := r^{-1}$ and $v := rs$. Let x and y be canonical generators for $\Lambda[2,3,3n]$, of orders $3n$ and 2 respectively, such that $X = yxy$ and $Y = xyx^{-2}$, the Appendix says such generators exist. Clealy, when $n = 4$, $x \mapsto u$ and $y \mapsto v$ defines a surface kernel homomorphism from Λ onto G, and it maps X, Y to vuv, uvv^{-2}. It can be shown that vuv and uvv^{-2} do indeed generate a cyclic group and $uvv^{-2} = (vuv)^8$, so for $n = 4$ there is a homomorphism that does extend.

Let us now consider the general case. Suppose a homomorphism φ, does extend to Λ and $G := \varphi(\Lambda)$. Then G is generated by $u := \varphi(x)$ and $v := \varphi(y)$, the orders of which are $3n$ and 2 and the order of their product is 3. Now $\varphi(\Gamma)$ will be generated by vuv and uvu^{-2}. By the above remarks we see that

$$\varphi(Y) = uvu^{-2} = \begin{cases} vu^{2n}v & \text{if } n \equiv 1 \pmod 3; \\ vu^{n}v & \text{if } n \equiv -1 \pmod 3. \end{cases}$$

Hence $$\varphi(XY)^{-1} = u^3 = \begin{cases} vu^{n-1}v & \text{if } n \equiv 1 \pmod 3; \\ vu^{2n-1}v & \text{if } n \equiv -1 \pmod 3. \end{cases}$$

Suppose $n \equiv 1 \pmod 3$, then $u^3 = vu^{n-1}v$. Thus $vu^3v = u^{n-1}$ and so

$$vu^{n-1}v = (vu^3v)^{(n-1)/3} = u^{(n-1)^2/3}.$$

Now $u^{(n-1)^2/3} = u^3$ if and only if $(n-1)^2/3 \equiv 3 \pmod{3n}$, if and only if $\left(\frac{n-1}{3}\right)^2 \equiv 1 \pmod n$. If we let $c := (n-1)/3$, then we require $c^2 \equiv 1 \pmod{3c+1}$. Thus if p is an odd prime factor of $3c+1$, then $c \equiv \pm 1 \pmod p$.

(i) If $c \equiv 1 \pmod p$, then $p \mid c-1$ and $p \mid 3c+1$ which implies $p \mid 4$.

(ii) If $c \equiv -1 \pmod p$, then $p \mid c+1$ and $p \mid 3c+1$ which implies $p \mid 2$.

Thus n has no odd prime divisors, and so $3c+1 = 2^\alpha$ for some α. We require $c \equiv \pm 1 \pmod{2^\alpha}$ or $c \equiv 2^{\alpha-1} \pm 1 \pmod{2^\alpha}$, if $\alpha > 2$. However, $c = (2^\alpha - 1)/3$ and so these conditions are satisfied only when $\alpha = 2$ and $n = 4$. By a similar argument it can be shown that when $n \equiv -1 \pmod 3$, no extension is possible.

Hence there are no extensions of the above nature except when $n = 4$. In this case the full group of automorphisms of S, is G extended by a symmetry t, such that tu and tv are both involutions. It can easily be seen that this group contains only two conjugacy classes of symmetries, one represented by t and the other by tu, and so the symmetry type of S is $\{-1, -1\}$.

14. Let φ be a surface kernel homomorphism from $\Gamma[2, n, 2n]$ onto C_{2n}, so n is odd and φ is unique up to an automorphism of C_{2n}. If X and Y are canonical generators of Γ, of orders $2n$ and n, then we may assume $\varphi(X) = r$ and $\varphi(Y) = r^{n-1}$, where r generates C_{2n}. Suppose φ extends to $\Lambda[2, 3, 2n]$ and $G := \varphi(\Lambda)$. We can find canonical generators x and y of Λ, of orders $2n$ and 2, such that $X = yxy$ and $Y = x^2$, see Appendix. Now G will be generated by $u := \varphi(x)$ and $v := \varphi(y)$, and $u^{2n} = v^2 = (uv)^3 = 1$. Thus $\varphi(X) = vuv$, $\varphi(Y) = u^2$ and so by the uniqueness of φ, we see that $\varphi(X)^{n-1} (= vu^{n-1}v) = \varphi(Y) (= u^2)$. Thus $vu^2v = u^{n-1}$ and so $vu^{n-1}v = (vu^2v)^{(n-1)/2} = u^{(n-1)^2/2} = u^2$. Now,

$(n-1)^2/2 \equiv 2 \pmod{2n}$ if and only if $c^2 \equiv 1 \pmod{n}$, where $c = (n-1)/2$. Let p be any prime divisor of $n = 2c+1$, then $c \equiv \pm 1 \pmod{p}$. Suppose $c \equiv 1 \pmod{p}$, then p divides $c-1$ and $2c+1$, thus p divides 3 and so $p = 3$. In the same way we see that if $c \equiv -1 \pmod{p}$, then p divides 1. This argument also applies to the maximum power of a prime divisor of n and so $n = 3$ is the only possibility, in which case Γ is not Fuchsian. Therefore there are no surface kernel homomorphisms of this kind that extend, for any n.

We now summarize the results of the chapter and list all possible symmetry types of compact Riemann surfaces with large cyclic automorphism groups. In the following S denotes the surface, g its genus, K the surface group that uniformizes S, C the large cyclic group, Γ the lift of C, G the full group of conformal automorphisms of S, Λ the lift of G, G_* the group of all automorphisms of S and Λ_* the lift of G_*.

Theorem 4.9

If $C = G$, that is $\Gamma = \mathcal{N}_{\mathcal{L}}(K)$, then S only admits symmetries of the first kind with respect to C, and the symmetry type of S appears in the following table.

$\Gamma[m_1, m_2, m_3]$	Symmetry Type
m_i odd	$\{-1\}$
m_1, m_2 even, m_3 odd	$\{-1, -1\}$
m_i even	$\{-1, -2\}$

Note that the table in Theorem 4.6 has two entries containing positive symmetries. In these cases Γ has signature of the form $[2n, 2n, n]$ and we have shown in case 12 that $C \neq G$.

Theorem 4.10

Let $C < G$. Then the signature is one of those in the following table. At least one of the periods of Γ is equal to the order of C. Let x and y be canonical generators of Γ such that the order of x is the order of C and the order of y is the next largest period of Γ. Let $\varphi : \Gamma \mapsto C$, be a homomorphism with kernel K, and let k be such that $\varphi(x)^k = \varphi(y)$. Then the symmetry type of S is given in the following table.

<table>
<tr><th colspan="2">Γ</th><th>Λ_*</th><th colspan="2">φ</th><th>Symmetry Type</th><th>g</th></tr>
<tr><td rowspan="4">$[m, m, n]$
$n \mid m$</td><td>m odd</td><td rowspan="4">$(2, m, 2n)$</td><td colspan="2" rowspan="2">$k^2 \equiv 1\ (m)$</td><td>$\{-1, -1\}$</td><td rowspan="4">$\frac{m}{2}(1 - \frac{1}{n})$</td></tr>
<tr><td>m even
n odd</td><td>$\{0, -1, -1, -1\}$</td></tr>
<tr><td rowspan="2">m, n even
$(\frac{n}{m}, n) \leq 2$</td><td rowspan="2">$k^2 \equiv 1\ (m)$</td><td>$k \equiv 1\ (2^\alpha)$</td><td>$\{0, -1, -2, -\frac{m}{2n}\}$</td></tr>
<tr><td>$k \equiv 2^{\alpha-1} \pm 1\ (2^\alpha)$</td><td>$\{-1, -2, -\frac{m}{2n}\}$</td></tr>
<tr><td colspan="2">$[m, m, m]$, m odd</td><td>$([3], (m))$</td><td colspan="2">$k^2 + k + 1 \equiv 0\ (m)$</td><td>$\{-1\}$</td><td>$\frac{(m-1)}{2}$</td></tr>
<tr><td colspan="2">$[7, 7, 7]$</td><td>$(2, 3, 7)$</td><td colspan="2">$k \neq 1$</td><td>$\{-1\}$</td><td>3</td></tr>
<tr><td colspan="2">$[4, 8, 8]$</td><td>$(2, 3, 8)$</td><td colspan="2">$k \neq 5$</td><td>$\{0, -1, -2\}$</td><td>3</td></tr>
<tr><td rowspan="3">$[4n, 4n, n]$</td><td>n odd</td><td rowspan="3">$(2, 3, 4n)$</td><td colspan="2">$k = 2n + 1$</td><td>$\{0^p, -1^q\}$</td><td rowspan="3">$2n - 2$</td></tr>
<tr><td rowspan="2">$n \equiv 2\ (4)$</td><td colspan="2">$n \equiv 2\ (8), \quad k = n + 1$</td><td rowspan="2">$\{0^r, -1, -2\}$</td></tr>
<tr><td colspan="2">$n \equiv 6\ (8), \quad k = 3n + 1$</td></tr>
<tr><td rowspan="4">$[2n, 2n, n]$</td><td>n odd</td><td rowspan="4">$(2, 4, 2n)$</td><td colspan="2" rowspan="2">$k = 1$</td><td>$\{0, -1, +1, +n\}$</td><td rowspan="4">$n - 1$</td></tr>
<tr><td>n even</td><td>$\{0, 0, -1, +2, +n\}$</td></tr>
<tr><td>$n \equiv 0\ (8)$</td><td colspan="2" rowspan="2">$k = n + 1$</td><td>$\{-1, +2, -n/4\}$</td></tr>
<tr><td>$n \equiv 4\ (8)$</td><td>$\{0, -1, +2, -n/4\}$</td></tr>
<tr><td colspan="2">$[3, 4, 12]$</td><td>$(2, 3, 12)$</td><td colspan="2">$k = 3, \quad \varphi$ is unique</td><td>$\{-1, -1\}$</td><td>3</td></tr>
</table>

In the table q is one or two, $p \geq 1$ and $r \geq 0$. Since p and r are the numbers of conjugacy classes of involutions in $G_* \backslash G$, less one or two, the only obvious upper bound on them is of the order $8n$.

Chapter 5

Symmetries and Large Abelian Groups

In the first section we consider surface kernel homomorphisms from Fuchsian triangle groups, onto finite non-cyclic (two generator) abelian groups. We determine necessary and sufficient conditions on the signature of the triangle group and on the abelian group for such an epimorphism to exist. To each of these homomorphisms there is associated a compact Riemann surface which admits a large non-cyclic abelian group of automorphisms. In the second and third sections, we find the species of symmetries of the first and second kind w.r.t. the abelian group. Finally, in the fourth section, we see that there is an infinite family of the above surfaces that admit symmetries that are not conjugate to those in sections two or three. We calculate the species of these symmetries, and so completely determine all possible species of symmetries of compact Riemann surfaces admitting large non-cyclic abelian automorphism groups. We refrain from calculating all possible symmetry types, content with finding the symmetry types of all but three or four infinite families and three or four particular examples of these surfaces.

Section 5.1 Large non-Cyclic Abelian Groups

Let x_1, x_2 and x_3 be canonical generators of the Fuchsian triangle group $\Gamma[m_1, m_2, m_3]$, of orders m_1, m_2 and m_3. Suppose φ is a surface kernel homomorphism from Γ onto a non-cyclic abelian group G_N, of order N. As in the case of cyclic groups, we must have $[m_1, m_2] = [m_1, m_3] = [m_2, m_3]$. Therefore, if $d = (m_1, m_2, m_3)$ and $db_1 = (m_2, m_3)$, $db_2 = (m_1, m_3)$, $db_3 = (m_1, m_2)$, then b_1, b_2 and b_3 are mutually coprime and $m_1 = db_2b_3$, $m_2 = db_1b_3$ and $m_3 = db_1b_2$.

Clearly G_N will be generated by any two of $\varphi(x_1)$, $\varphi(x_2)$ or $\varphi(x_3)$. If u_1 is the order of the intersection $\langle\varphi(x_2)\rangle \cap \langle\varphi(x_3)\rangle$, then u_1 divides $(m_2, m_3) = db_1$. If we define u_2 and u_3 in the same way, then u_2 divides db_2, u_3 divides db_3 and

$$N = \frac{m_2 m_3}{u_1} = \frac{m_1 m_3}{u_2} = \frac{m_1 m_2}{u_3}.$$

Hence b_1 divides u_1, b_2 divides u_2, b_3 divides u_3, and so $N = d^2 b_1 b_2 b_3 / h$, for some proper divisor h, of d.

If p is a prime divisor of N, then the Sylow-p subgroup of G_N is either cyclic or two generator abelian. We distinguish between three different types of prime divisors of N.

(i) Those that are coprime to $b_1 b_2 b_3$.

(ii) Those that divide d and $b_1 b_2 b_3$.

(iii) Those that are coprime to d.

Let ϵ be the exponent of p in the prime decomposition of N. Then the Sylow-p subgroup of G_N, is isomorphic to $\mathbf{Z}_{p^\sigma} + \mathbf{Z}_{p^\tau}$ for some σ and τ that sum to ϵ. We may assume that $\sigma \geq \tau$.

If p is of type (i), then $\epsilon = 2\alpha - \nu$, where α and ν are the exponents of p in d and h. For φ to preserve orders we must have $\sigma \geq \alpha$ and so, for φ to also be onto, we must have $\sigma = \alpha$ and $\tau = \alpha - \nu$.

If p is of type (ii), then $\epsilon = 2\alpha - \nu + \omega$ where ω is the exponent of p in the decomposition of $b_1 b_2 b_3$. For φ to preserve orders we must have $\sigma \geq \alpha + \omega$. Thus, for φ to also be onto, we must have $\sigma = \alpha + \omega$ and $\tau = \alpha - \nu$.

If p is a prime of type (iii), then ϵ is just ω and so, for φ to preserve orders, we must have $\sigma = \omega$.

We arrange the prime decomposition of N as follows

$$N = p_1^{\epsilon_1} \dots p_k^{\epsilon_k} \dots p_a^{\epsilon_a} \dots p_b^{\epsilon_b},$$

where $p_1, \dots, p_b$ are all distinct primes, $k \leq a \leq b$ and $\epsilon_i > 0$. The first k primes are of type (i), note that k may be zero. For $k < i \leq a$, p_i is of type (ii) and $\epsilon_i = 2\alpha_i - \nu_i + \omega_i$, where α_i, ν_i and ω_i are the exponents of p_i in d, h and $b_1 b_2 b_3$ respectively. For $a < i \leq b$, p_i is of type (iii) and so divides exactly one of the b_i's and is coprime to d. From the previous remarks about Sylow subgroups we see that

$$\begin{aligned} G_N \simeq & (\mathbf{Z}_{p_1^{\alpha_1}} + \mathbf{Z}_{p_1^{\alpha_1 - \nu_1}}) + \ldots\ldots + (\mathbf{Z}_{p_k^{\alpha_k}} + \mathbf{Z}_{p_k^{\alpha_k - \nu_k}}) + \\ & \left(\mathbf{Z}_{p_{k+1}^{\alpha_{k+1}+\omega_{k+1}}} + \mathbf{Z}_{p_{k+1}^{\alpha_{k+1}-\nu_{k+1}}}\right) + \ldots\ldots + \left(\mathbf{Z}_{p_a^{\alpha_a+\omega_a}} + \mathbf{Z}_{p_a^{\alpha_a-\nu_a}}\right) + \\ & \mathbf{Z}_{p_{a+1}^{\epsilon_{a+1}}} + \ldots\ldots + \mathbf{Z}_{p_b^{\epsilon_b}} \\ \simeq & \mathbf{Z}_M + \mathbf{Z}_{d/h}, \end{aligned}$$

where $M = db_1b_2b_3 = [m_1, m_2] = [m_1, m_3] = [m_2, m_3]$.

Theorem 5.1

Let $\Gamma[m_1, m_2, m_3]$ be a Fuchsian triangle group. Let $M = [m_1, m_2, m_3]$ and $d = (m_1, m_2, m_3)$. Then there is a surface kernel homomorphism from Γ onto a non-cyclic abelian group G, of order N, if and only if the following conditions hold.

(i) $[m_1, m_2] = [m_1, m_3] = [m_2, m_3] = M$.

(ii) $N = dM/h$ for some proper divisor $h \neq d$, of d.

(iii) $G \simeq \mathbf{Z}_M + \mathbf{Z}_{d/h}$.

(iv) If d is even then h is odd or M/d is even.

Proof We have already shown the necessity of conditions (i)-(iii). To show that (i)-(iv) are sufficent we construct a surface kernel homomorphism of the required nature. If p is a prime factor of N, then $\mathbf{Z}_{p^\sigma} + \mathbf{Z}_{p^\tau}$ is the Sylow p subgroup of G_N, where σ and τ are the exponents of p in the decomposition of M and d/h respectively. We build up a homomorphism by specifying its restriction to the Sylow subgroups. To this end we define φ_p, a homomorphism from Γ onto $\mathbf{Z}_{p^\sigma} + \mathbf{Z}_{p^\tau}$, the only requirement being that the orders of $\varphi_p(x)$, $\varphi_p(y)$ and $\varphi_p(xy)$ are the powers of p in the prime decompositions of m_1, m_2 and m_3.

If p is of type (i) , then $\sigma = \alpha$ and $\tau = \alpha - \nu$. We let $\varphi_p(x) = (1, 0)$ and $\varphi_p(y) = (1, 1)$, then $\varphi_p(x)$, $\varphi_p(y)$ and $\varphi_p(xy)$ are all of order p^α, unless $p = 2$. If $p = 2$ and h is odd, then $\nu = 0$ and $\varphi_p(x) = (1, 0)$, $\varphi_p(y) = (0, 1)$ describes a satisfactory homomorphism. However, if $p = 2$ and h is even, then $\alpha - \nu < \alpha$ and a satisfactory homomorphism can not be found. (This shows the necessity of the fourth condition).

If p is of the type (ii), then $\sigma = \alpha + \omega$ and $\tau = \alpha - \nu$. Without loss of generality we may assume that p divides b_3. In this case we let $\varphi_p(x) = (1, 0)$ and $\varphi_p(y) = (p^\omega - 1, 1)$. Thus $\varphi_p(x)$ and $\varphi_p(y)$ both have order $p^{\alpha+\omega}$, while $\varphi_p(xy)$

has order p^α.

If p is of type (iii), then $\sigma = \omega$ and $\tau = 0$. Again we assume that p divides b_3 and so we let $\varphi_p(x) = (1)$ and $\varphi_p(y) = (-1)$.

By the Chinese Remainder Theorem, there is a unique x_1 modulo M that is congruent to the first coordinate of $\varphi_p(x)$, modulo p^ϵ, for all p dividing N. Similarly there is a unique x_2, modulo d/h, that is congruent to the second coordinate (if there is one), of $\varphi_p(x)$, modulo p^ϵ, for all p dividing d/h. If we define y_1 and y_2 in a similar way from the $\varphi_p(y)$'s, then $\varphi(x) = (x_1, x_2)$, $\varphi(y) = (y_1, y_2)$ defines a surface kernel homomorphism from Γ onto G.

Example Let x and y be canonical generators of $\Gamma[600, 840, 1050]$ of orders 600 and 840. Now $(600, 840, 1050) = 30$ and $[600, 840, 1050] = 126,000$. The Theorem tells us that there are surface kernel homomorphisms from Γ onto $\mathbf{Z}_M + \mathbf{Z}_r$ where $r \in \{2, 3, 5, 6, 10, 15\}$. We will take the case when $r = 10$. Hence $N = 2^4.3.5^3.7$, the only prime of the first type is 3, those of the second type are 2 and 5, and 7 is the only one of the third type. Let x_1 and x_2 obey the following.

$$\begin{array}{ll} x_1 = 1 \pmod 3 & \\ x_1 = 1 \pmod 8 & x_2 = 0 \pmod 2 \\ x_1 = 4 \pmod{25} & x_2 = 1 \pmod 5 \\ x_1 = 0 \pmod 7 & \end{array}$$

Then the Chinese Remainder Theorem tells us that $x_1 \equiv 1729 \pmod M$ and $x_2 \equiv 6 \pmod{10}$. If y_1 and y_2 obey the following.

$$\begin{array}{ll} y_1 = 1 \pmod 3 & \\ y_1 = 3 \pmod 8 & y_2 = 1 \pmod 2 \\ y_1 = 5 \pmod{25} & y_2 = 0 \pmod 5 \\ y_1 = 1 \pmod 7 & \end{array}$$

Then $y_1 \equiv 1555 \pmod M$ and $y_2 \equiv 5 \pmod{10}$. Therefore, $x \mapsto (1729, 6)$ and $y \mapsto (1555, 5)$ describes a surface kernel homomorphism from Γ onto $\mathbf{Z}_M + \mathbf{Z}_{10}$.

Section 5.2 Symmetries of the First Kind

Let φ be a surface kernel homomorphism from $\Gamma[m_1, m_2, m_3]$, onto a non-cyclic abelian group G, of order N. If x_1 and x_2 are canonoical generators of Γ, of orders m_1 and m_2, then $r := \varphi(x_1)$ and $s := \varphi(x_2)$ generate G. We use the notation of the previous section and so $m_1 = db_2b_3$, $m_2 = db_1b_3$, $m_3 = db_1b_2$,

$M = [m_1, m_2, m_3] = db_1b_2b_3$ and $N = Md/h$, where h is some proper divisor of d. The b_i's are mutually coprime and $d = (m_1, m_2, m_3)$. Theorem 5.1 tells us that $G \simeq \mathbf{Z}_M + \mathbf{Z}_f$, for $f := d/h$. The orders of r and s are m_1 and m_2 and so the order of the intersection $\langle r \rangle \cap \langle s \rangle$, must be hb_3. Thus $\langle r^{fb_2} \rangle = \langle s^{fb_1} \rangle$. Let η be the least positive integer such that $r^{fb_2} = s^{\eta f b_1}$, then $(\eta, hb_3) = 1$. We know that $(rs)^{m_3} = 1$, and so

$$r^{m_3} = r^{db_1b_2} = (r^{fb_2})^{hb_1} = (s^{\eta f b_1})^{hb_1} = s^{\eta d b_1^2} = s^{-m_3} = s^{-db_1b_2}.$$

Therfore $\eta d b_1^2 \equiv -db_1b_2 \pmod{m_2}$, and so b_3 divides $\eta b_1 + b_2$. We actually require the order of rs to be m_3, hence $(\eta b_1 + b_2, hb_3) = b_3$. With this understanding G has the presentation

$$G = \langle r, s \,|\, r^{m_1} = s^{m_2} = rsr^{-1}s^{-1} = r^{fb_2}s^{-\eta f b_1} = 1 \rangle.$$

The elements of G are

$$\{1, r, r^2, \ldots, r^{-1}, s, rs, \ldots, r^{-1}s, \ldots\ldots, s^{fb_1-1}, rs^{fb_1-1}, \ldots, r^{-1}s^{fb_1-1}\}.$$

Symmetries of the first kind (w.r.t. G, of the associated compact Riemann surface) exist if and only if $r \mapsto r^{-1}$, $s \mapsto s^{-1}$ describes an automorphism of G, see §3.2. This is always the case. Let G_* denote the extention of G by such a symmetry. Then G_* lifts to $\Gamma_*(m_1, m_2, m_3)$. Γ_* is generated by three reflections a, b and c such that $ab = x_1$ and $bc = x_2$. Hence φ will extend to Γ_*. If $t := \varphi(b)$, then $(tr)^2 = (ts)^2 = 1$ and $\varphi(a) = tr^{-1}$, $\varphi(c) = ts$. The symmetries in G_* are the involutions in $G_* \backslash G$, clearly tr^is^j is an involution for all i and j. Note that

$$r.tr^is^j.r^{-1} = tr^{i-2}s^j, \qquad s.tr^is^j.s^{-1} = tr^is^{j-2}, \qquad t.tr^is^j.t = tr^{-i}s^{-j}.$$

Therefore, if the m_i's are all odd, then there is just one conjugacy class of symmetries in G_*. Suppose m_1, m_2 are even and m_3 is odd, then d is odd and b_3 is even. Thus ts^{fb_1-1} is conjugate to ts^{fb_1+1}, which is equal to $tr^{\psi f b_2}s$, where ψ is the inverse of η modulo hb_3. Hence there are two classes of symmetries in G_*, one contains those elements of the form $tr^{2i}s^{2j}$ and $tr^{2i+1}s^{2j+1}$, the other contains those elements of the form $tr^{2i}s^{2j+1}$ and $tr^{2i+1}s^{2j}$. If the m_i are all even, then d is even and we see that when f is odd the classes are the same as in the previous case, while when f is even there are four classes represented by t, tr, ts and trs.

The $\langle tr^{-1}\rangle$, $\langle t\rangle$ and $\langle ts\rangle$ right cosets are represented by the elements of G. We now consider the actions of $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$, by right multiplication, on these cosets.

On the $\langle tr^{-1}\rangle$ cosets;

$$\begin{aligned} tr^{-1} &: r^i s^j \longmapsto r^{-i}s^{-j} \\ t &: r^i s^j \longmapsto r^{-i+1}s^{-j} \\ ts &: r^i s^j \longmapsto r^{-i+1}s^{-j+1} \end{aligned}$$

On the $\langle t\rangle$ cosets;

$$\begin{aligned} tr^{-1} &: r^i s^j \longmapsto r^{-i-1}s^{-j} \\ t &: r^i s^j \longmapsto r^{-i}s^{-j} \\ ts &: r^i s^j \longmapsto r^{-i}s^{-j+1} \end{aligned}$$

On the $\langle ts\rangle$ cosets;

$$\begin{aligned} tr^{-1} &: r^i s^j \longmapsto r^{-i-1}s^{-j-1} \\ t &: r^i s^j \longmapsto r^{-i}s^{-j-1} \\ ts &: r^i s^j \longmapsto r^{-i}s^{-j} \end{aligned}$$

(1) Suppose the m_i's are all odd. Then there is only one class of symmetries and so we need only determine the species of t. t fixes the $\langle t\rangle$ coset represented by r^is^j, $0 \le j < fb_1$, if and only if $r^{2i}s^{2j} = 1$. Hence fb_1 must divide $2j$ and so, because fb_1 is odd, j must be zero. Therefore t only fixes one coset, namely 1. This means that tr^{-1} and ts each fix precisely one coset and, because the m_i are all odd, t will only have one mirror. The word $tr^{-1}.ts.t.tr^{-1}.ts = t$, represents a circuit in the Schreier coset graph of $\langle t\rangle$ in G_* over $\{tr^{-1}, t, ts\}$, that passes through the vertex associated to the coset 1. When the m_i are odd it can be seen to be loop free and so $sp(t) = -1$.

(2) Suppose m_1 and m_2 are even and m_3 is odd, so d is odd and b_3 is even. Then there are two classes of symmetries, one represented by t and the other by ts. Note that tr^{-1} is conjugate to ts.

Firstly we consider the action of G_* on the $\langle t\rangle$ cosets and determine the species of t. fb_1 is still odd and so t can only fix r^is^j when $j = 0$. Hence t fixes only two cosets, 1 and $r^{m_1/2}$. Therefore, because m_1 and m_2 are even, the two associated reflection generators of $\varphi^{-1}(\langle t\rangle)$ are linked and t only has one mirror. The arguement that shows t is non-separarting in case one can be adapted to show that t is also non-separarting here.

We now look at the action of G_* on the $\langle tr^{-1}\rangle$ cosets. tr^{-1} will fix r^is^j $(0 \le j < fb_1)$, if and only if $r^{2i}s^{2j} = 1$. This is the case only if fb_1 divides $2j$ and as fb_1 is odd, j must be zero. Hence tr^{-1} fixes only two cosets, 1 and $r^{m_1/2}$, similarly ts must fix only two cosets. m_1 is even and so the two reflection generators of $\varphi^{-1}(\langle tr^{-1}\rangle)$ associated to the cosets fixed by tr^{-1} are linked. m_2 is also even and so the cosets fixed by ts also give rise to linked reflection generators. However, m_3 is odd and so there are links between these two pairs of generators. Therefore tr^{-1} only has one mirror. In the associated Schreier coset graph $t.ts.tr^{-1}.t.ts = tr^{-1}$, represents a circuit of odd length that passes through 1. This circuit is without loops when m_3 is odd and so $sp(tr^{-1}) = -1$.

(3) Suppose m_1, m_2 and m_3 are all even, so d is even. If M/d is even then we assume that it is b_3 which is even.

(i) If f is odd, then there are two conjugacy classes of symmeteries in G_*, $\{t\}$ and $\{tr^{-1}, ts\}$. Clearly tr^{-1} and ts will fix no $\langle t\rangle$ cosets, and t will only fix two $\langle t\rangle$ cosets, 1 and $r^{m_1/2}$. This is because fb_1 and fb_2 are still odd. Hence, because m_1 and m_3 are even, t only has one mirror.

Similarly, tr^{-1} only fixes two $\langle tr^{-1}\rangle$ cosets, 1 and $r^{m_1/2}$. Therefore ts will fix exactly two of these cosets and, because the m_i are even, tr^{-1} has two mirrors. We consider whether t and tr^{-1} separate after we have dealt with the case when f is even.

(ii) If f is even, then there are four classes of symmetries in G_*, $\{t\}$, $\{tr^{-1}\}$, $\{ts\}$ and $\{t\}$. On the $\langle t\rangle$ cosets: t fixes r^is^j, $0 \le j < fb_1$, if and only if $r^{2i}s^{2j} = 1$. This is the case only if fb_1 divides $2j$, and so j is 0 or $fb_1/2$. When $j = 0$ we must have $i = 0$ or $m_1/2$. If $j = fb_1/2$, then $r^{2i}s^{2j} = 1$ only if i is congruent to $-\psi fb_2/2$ or $(m_1 - \psi fb_2)/2$ modulo m_1, where $\psi\eta \equiv 1 \pmod{hb_3}$. Thus t fixes exactly four $\langle t\rangle$ cosets. If t fixes r^is^j, then we denote the associated reflection generator of $\varphi^{-1}(\langle t\rangle)$ by (i,j). Hence the reflection generators are

$$(0,0),\quad (m_1/2,0),\quad (-\psi fb_2/2, fb_1/2)\ \text{ and }\ ((m_1-\psi fb_2)/2, fb_1/2).$$

Now $tr^{-1}.t = r$ and the r orbits of cosets give the following links.

$$(0,0)\sim(m_1/2,0),\quad (-\psi fb_2/2, fb_1/2)\sim((m_1-\psi fb_2)/2, fb_1/2).$$

The other links are obtained by looking at the $t.ts = s$ orbits, in particular the action of $s^{m_2/2}$ on the cosets fixed by t. We see that $s^{m_2/2} = r^{m_1/2}$ if and only if fb_1 divides $m_2/2 (= fb_1hb_3/2)$, if and only if hb_3 is even. Thus $s^{m_2/2}$ links $(0,0)$ to $(r^{m_1/2}, 0)$ if and only if hb_3 is even. Therefore, t has two mirrors if hb_3 is even and one otherwise.

Similarly, tr^{-1} has two mirrors if hb_2 is even and one otherwise, and ts has two mirrors if hb_1 is even and one otherwise.

The element $tr^{-1}.ts.t.tr^{-1}.ts = t$, represents a circuit in the Schreier coset graph of $\langle t \rangle$ in G_* over $\{tr^{-1}, t, ts\}$, that passes through 1. The circuit is loop free unless $r^2s^2 = 1$ and so, unless $m_3 = 2$, the species of t is negative. Therefore the above circuit is loop free unless $d = 2$ and $b_1 = b_2 = 1$. If this is case, then $m_1 = m_2 = 2b_3$, $m_3 = 2$ and the epimorphism φ, from Γ to G, is unique up to an automorphism of G. The $\langle t \rangle$ cosets are $1, r, r^2, \ldots, r^{-1}, s, rs, \ldots, r^{-1}s$ and the action of G_* on these cosets is described by the following.

$$tr^{-1} : \begin{array}{ccc} r^i & \longmapsto & r^{-i-1} \\ r^i s & & r^{-i+1}s \end{array} \qquad t : \begin{array}{ccc} r^i & \longmapsto & r^{-i} \\ r^i s & & r^{-i+2}s \end{array} \qquad ts : \begin{array}{ccc} r^i & \longmapsto & r^{-i}s \\ r^i s & & r^{-i} \end{array}$$

Thus, we can partition the cosets in to two sets,

$$\begin{array}{rl} & \{1, r, r^2, \ldots, r^{m_1/2-1}, rs, r^2s, \ldots, r^{m_1/2}s\} \\ \text{and} & \{r^{-1}, r^{-2}, \ldots, r^{m_1/2}, s, r^{-1}s, r^{-2}s \ldots, r^{m_1/2+1}s\}, \end{array}$$

such that tr^{-1}, t and ts either take cosets from one set to the other, or they fix them. Hence t separates if and only if m_3 is two. In this case Γ will have signature $[2b, 2b, 2]$ and t will have species $+1$ if b is odd and $+2$ if b is even.

The element $t.ts.tr^{-1}.t.ts = tr^{-1}$, represents a circuit in the Schreier coset graph of $\langle tr^{-1} \rangle$ in G_* over $\{tr^{-1}, t, ts\}$, through 1. This can be seen to be loop free unless $s^2 = 1$, that is $m_2 = 2$. When m_2 is two then the coset graph can also be seen to be bipartite. Hence the species of tr^{-1} is positive if and only if G lifts to $[2b, 2, 2b]$, in which case $sp(tr^{-1})$ will be $+1$ if b is odd and $+2$ if b is even.

Consideration of the element $t.tr^{-1}.ts.t.tr^{-1} = ts$, in a similar manner to the above, shows that ts separates if and only if G lifts to $[2, 2b, 2b]$. Again, the epimorphism φ, is unique up to an automorphism of G and $sp(ts)$ is $+1$ if b is odd and $+2$ if b is even.

Note that at most one class of symmetries in G_* has positive species.

We now summarize the results in this section.

Theorem 5.2

Let φ be a surface kernel homomorphism from $\Gamma[m_1, m_2, m_3]$ onto a non-cyclic abelian group G, of order N. (Thus $m_1 = db_1b_2$, $m_2 = db_1b_3$, $m_3 = db_1b_2$, where $d = (m_1, m_2, m_3)$ and b_1, b_2, b_3 are mutually coprime, and for a proper factor f of d, $N = fdb_1b_2b_3$.) The Riemann surface S that is uniformized by the kernel of φ

admits symmetries of the first kind w.r.t G. If G_* denotes the extension of G by such a symmetry, then the following table gives the number of conjugacy classes of symmetries in G_* and their species. The results are independent of φ.

<table>
<tr><td colspan="4">$\Gamma[m_1, m_2, m_3]$</td><td>Classes</td></tr>
<tr><td colspan="4">m_i odd</td><td>$\{-1\}$</td></tr>
<tr><td colspan="4">m_1, m_2 even, m_3 odd</td><td>$\{-1\}$</td></tr>
<tr><td rowspan="6">m_i even</td><td colspan="3">f odd</td><td>$\{-1, -2\}$</td></tr>
<tr><td rowspan="5">f even</td><td rowspan="2">$f = d = 2$,
$M/d = b_i$</td><td>b_i odd</td><td>$\{0, -1, -1, +1\}$</td></tr>
<tr><td>b_i even</td><td>$\{0, -1, -1, +2\}$</td></tr>
<tr><td rowspan="3">otherwise</td><td>h, M/d odd</td><td>$\{0, -1, -1, -1\}$</td></tr>
<tr><td>h odd, M/d even</td><td>$\{0, -1, -1, -2\}$</td></tr>
<tr><td>h even</td><td>$\{0, -2, -2, -2\}$</td></tr>
</table>

Note that $\{0, -2, -2, +2\}$ and $\{0, -1, +1, -2\}$ do not appear in the table. The first does not because for G_* to contain three classes of symmetries that have two mirrors h must be even, but h must be one for there to be a symmetry that separates in G_*. The reason the second does not occur is that for G_* to contain a separating symmetry G must lift to $[2b, 2b, 2]$. If G_* also contains a symmetry with two mirrors, then b must be even and the class of symmetries that has two mirrors will necessarily be the class that separates.

When G_* contains separating symmetries, that is G lifts to $\Gamma[2b, 2b, 2]$, then G is isomorphic to $\mathbf{Z}_{2b} + \mathbf{Z}_2$ and, as we shall see later, G_* is not the full automorphism group of the surface. However, as most Fuchsian triangle groups are maximal, most Riemann surfaces admitting large non-cyclic abelian automorphism groups have one of the symmetry types in the previous table, excluding those symmetry types containing any positive species.

Section 5.3 Symmetries of the Second Kind

To even consider symmetries of the second kind w.r.t. a large non-cyclic abelian group G, we must have G lifting to some $\Gamma[db, db, d]$. As before we let φ be a surface kernel homomorphism from Γ onto G, the associated Riemann surface S, is uniformized by $ker(\varphi)$. G will then have presentation

$$\langle r, s \mid r^{db} = s^{db} = rsr^{-1}s^{-1} = r^f s^{-\eta f} = 1 \rangle,$$

where f is a proper divisor of d, and r and s are the images of canonical generators x_1 and x_2 of Γ, both of order db. We have seen that $(\eta, hb) = 1$ and $(\eta+1, hb) = b$ where $h := d/f$. From §4.2, we know that S admits symmetries of the second kind w.r.t. G if and only if, $r \mapsto s^{-1}$, $s \mapsto r^{-1}$ describes an automorphism of G. Clearly this is the case if and only if $s^{-f} r^{\eta f} = s^{f(1-\eta^2)} = 1$, if and only if $\eta^2 \equiv 1 \pmod{hb}$.

Suppose p is an odd prime factor of hb, if p^ϵ is the maximum power of p dividing hb then we want $\eta \equiv \pm 1 \pmod{p^\epsilon}$. However, $(\eta + 1, hb) = b$ and so p^ϵ divides h or b. If p^ϵ divides b then $\eta \equiv -1 \pmod{p^\epsilon}$ and if p^ϵ divides h then $\eta \equiv 1 \pmod{p^\epsilon}$. Now suppose hb is even and that α, ν and ω are the exponents of two in the decompositions of hb, h and b respectively. Then $\eta \equiv \pm 1 \pmod{2^\alpha}$ or $2^{\alpha-1} \pm 1 \pmod{2^\alpha}$ if $\alpha > 2$. If $\eta \equiv 1 \pmod{2^\alpha}$, then $(\eta + 1, 2^\alpha) = 2$ which implies $\omega = 1$. If $\eta \equiv -1 \pmod{2^\alpha}$, then $(\eta + 1, 2^\alpha) = 2^\alpha$ which implies $\omega = \alpha$ and $\nu = 0$. If $\eta \equiv 2^{\alpha-1} + 1 \pmod{2^\alpha}$, then $(\eta + 1, 2^\alpha) = 2$ which implies $\omega = 1$ and $\nu = \alpha - 1$. If $\eta \equiv 2^{\alpha-1} - 1 \pmod{2^\alpha}$, then $(\eta + 1, 2^\alpha) = 2^{\alpha-1}$ which implies $\omega = \alpha - 1$ and $\nu = 1$. Hence given h and b, such an η exists only if $(h, b) = 1$ or 2. It is also quite clear from the above that given such a pair an η, satisfying $(\eta, hb) = 1$, $(\eta + 1, hb) = b$ and $\eta^2 \equiv 1 \pmod{hb}$, can be constructed.

We assume that S does indeed admit a symmetry of the second kind and we let G_* denote G extended by such a symmetry. G_* lifts to some $\Gamma_*([db], (d))$, which is generated by a rotation x and a reflection c such that $x_1 = x$ and $x_2 = cx^{-1}c$. G_* will have presentation

$$\langle r, s, t \,|\, r^{db} = s^{db} = t^2 = rsr^{-1}s^{-1} = r^f s^{-\eta f} = trts = 1\rangle,$$

where $t := \varphi(c)$. We want to determine the species of t and so we use Theorem 2.5, remembering that the canonical generators of Γ_* are strictly x, xcx^{-1} and c. Note that $\varphi(xcx^{-1}) = tr^{-1}s^{-1}$ and $\varphi(xcx^{-1}c) = rs$. We now determine the action of r, $tr^{-1}s^{-1}$ and t on the right $\langle t\rangle$ cosets. The cosets are represented by the elements of G,

$$1, r, r^2, \ldots, r^{-1}, s, rs, r^2s, \ldots, r^{-1}s, \ldots, s^{f-1}, rs^{f-1}, r^2s^{f-1}, \ldots, r^{-1}s^{f-1}.$$

We shall sometimes abuse this notation by refering to the coset r^is^j, for some j greater than $f - 1$, meaning the coset which is equivalent to it in the above representation. The actions are as follows.

$$r : r^is^j \mapsto r^{i+1}s^j \qquad t : r^is^j \mapsto r^{-j}s^{-i} \qquad tr^{-1}s^{-1} : r^is^j \mapsto r^{-j-1}s^{-i-1}$$

Hence t fixes $r^i s^j$ if and only if $(rs)^{i+j} = 1$, if and only if d divides $i + j$. Thus t fixes fb cosets,

$$1, r^d, r^{2d}, \ldots, r^{-d}, r^{-1}s, r^{d-1}s, \ldots, r^{-d-1}s, \ldots\ldots$$
$$\ldots, r^{-f+1}s^{f-1}, r^{d-f+1}s^{f-1}, \ldots, r^{-d-f+1}s^{f-1}.$$

Similarly, $tr^{-1}s^{-1}$ fixes $r^i s^j$ if and only if d divides $i + j + 1$ and so fixes

$$r^{-1}, r^{d-1}, r^{2d-1}, \ldots, r^{-d-1}, r^{-2}s, r^{d-2}s, \ldots, r^{-d-2}s, \ldots\ldots$$
$$\ldots, r^{-f}s^{f-1}, r^{d-f}s^{f-1}, \ldots, r^{-d-f}s^{f-1}.$$

We must now determine the links between the reflection generators of $\varphi^{-1}(\langle t\rangle)$, induced by $\varphi(c)$ and $\varphi(xcx^{-1})$ fixing these $\langle t\rangle$ cosets. We denote the reflection generator corresponding to the coset $r^i s^j$, fixed by t or $tr^{-1}s^{-1}$, by (i, j). The (i, j) that are associated to t are such that d divides $i + j$, while those associated to $tr^{-1}s^{-1}$ are such that d divides $i + j + 1$, this is how we distinguish between them. The relation $x^{-1}.xcx^{-1}.x = c$, provides us with the following links, see end of §2.3.

$$\begin{array}{ccc}
(0,0) \sim (-1,0) & (d-1,1) \sim (d-2,1) & \cdots\cdots\cdots \\
(d,0) \sim (d-1,0) & (2d-1,1) \sim (2d-2,1) & \cdots\cdots\cdots \\
\vdots & \vdots & \\
(-d,0) \sim (-d-1,0) & (-d-1,1) \sim (-d-2,1) & \cdots\cdots\cdots
\end{array}$$

That is, if (i, j) is associated to t, so $d\,|\,i + j$, then it is linked to $(i - 1, j)$ which is associated to $tr^{-1}s^{-1}$, and vice versa. We must now look at the cycles of $\varphi(xcx^{-1}c) = rs$ for the other links. In doing this we must distinguish between the case when d is odd and the case when d is even.

Suppose d is odd. We see that if (i, j) is associated to t then it is linked to $(i + \frac{d-1}{2}, j + \frac{d-1}{2})$, which is associated to $tr^{-1}s^{-1}$. This is because the cosets $r^i s^j$ and $r^{i+(d-1)/2}s^{j+(d-1)/2}$ lie in the same orbit under rs. Putting these links together with those above, we form the following chain.

$$(0,0) \sim (\tfrac{d-1}{2}, \tfrac{d-1}{2}) \sim (\tfrac{d+1}{2}, \tfrac{d-1}{2}) \sim (d, d-1) \sim (d+1, d-1) \sim (\tfrac{3d+1}{2}, \tfrac{3d-3}{2}) \cdots$$

Let $2n$ be the length of the above chain, then clearly $2n$ divides $2fb$, the total number of cosets fixed by t and $tr^{-1}s^{-1}$. After $2n$ links we arrive at

$$(n(\tfrac{d-1}{2} + 1), n(\tfrac{d-1}{2})) = (n(\tfrac{d+1}{2}), n(\tfrac{d-1}{2})).$$

For this to be $(0,0)$, f must divide $n(\frac{d-1}{2})$, but $(f, \frac{d-1}{2}) = 1$ as f is a factor of d. Therefore f divides n and

$$(n(\tfrac{d+1}{2}), n(\tfrac{d-1}{2})) = (n(\tfrac{d+1}{2} + \eta\tfrac{d-1}{2}), 0) = (n(\tfrac{d-1}{2}(\eta+1)+1), 0).$$

This is $(0,0)$ if and only if db divides $n(\frac{d-1}{2}(\eta+1)+1)$. We know f divides n and so this is the case if and only if hb divides $\frac{n}{f}(\frac{d-1}{2}(\eta+1)+1)$. Since b divides $\eta+1$, b must divide n/f and so $2n = 2fb$. Hence there is only one chain and t has only one mirror.

Suppose d is even, then if (i,j) is associated to t, it is linked to $(i+\frac{d}{2}, j+\frac{d}{2})$ which is also associated to t. Similarly, if (k,l) is associated to $tr^{-1}s^{-1}$, it is linked to $(k+\frac{d}{2}, l+\frac{d}{2})$ which is also associated to $tr^{-1}s^{-1}$. Putting these together with the links above we obtain the following chains.

$$\begin{gathered}(0,0) \sim (\tfrac{d}{2}, \tfrac{d}{2}) \sim (\tfrac{d}{2}-1, \tfrac{d}{2}) \sim (d-1, d) \sim (d,d) = (0,0)\\ (d,0) \sim (\tfrac{3d}{2}, \tfrac{d}{2}) \sim (\tfrac{3d}{2}-1, \tfrac{d}{2}) \sim (2d-1, d) \sim (2d,d) = (d,0)\\ \vdots\end{gathered}$$

Hence there are $2fb/4 = fb/2$ chains and so t has $fb/2$ mirrors. Note that if g is the genus of S, then $fdb[1-1/db-1/db-1/d] = 2g-2$, therefore $(d-1)fb/2 - f + 1 = g$. Substituting in $fb/2 = g+1$, we see that t attains Harhack's bound if and only if $(g+1)(d-2) = f-2$, and as $g > 1$, $2 \leq f \leq d$, we must have $d = f = 2$. So we see that $sp(t) = +(g+1)$ if and only if Γ has signature $[2b, 2b, 2]$. In this case, up to isomorphism, there is only one finite abelian group that is the image of Γ under a surface kernel epimorphism and the epimorphism is unique up to an automorphism of the abelian group.

Clearly Γ_* contains only one conjugacy class of reflections and so G_* will contain only one class of symmetries with non-zero species. We now determine the number of classes of all symmetries in G_*. Consider, $(tr^is^j)^2 = r^{i-j}s^{j-i}$. Therefore, tr^is^j is an involution if and only if f divides $i-j$ and $s^{(i-j)(\eta-1)} = 1$, if and only if $f \mid i-j$ and hb divides $\frac{i-j}{f}(\eta-1)$. Now,

$$(\eta-1, hb) = \begin{cases} 2h & \text{if } \alpha > 0,\ \eta \equiv \pm 1 \pmod{2^\alpha};\\ h & \text{otherwise.}\end{cases}$$

Here 2^α is the maximum power of two dividing hb. Hence there are $2h$ possibilities for $i-j$ in the first instance and h in the second. Therefore, G_* contains $2h.f = 2d$

symmetries, f for each possible $i-j$, in the first instance. These are

$$t, trs, tr^2s^2, \dots, tr^{f-1}s^{f-1}, tr^{fb/2}, tr^{fb/2+1}s, \dots, tr^{fb/2+f-1}s^{f-1},$$
$$tr^{fb}, tr^{fb+1}s, \dots, tr^{fb+f-1}s^{f-1}, \dots\dots, tr^{-fb/2}, tr^{-fb/2+1}s, \dots, tr^{-fb/2+f-1}s^{f-1},$$

or

$$t, trs, tr^2s^2, \dots, tr^{d-1}s^{d-1} \quad \& \quad tr^{fb/2}, tr^{fb/2+1}s, tr^{fb/2+2}s^2, \dots, tr^{fb/2+d-1}s^{d-1}.$$

In the second instance G_* contains $h.f = d$ involutions,

$$t, trs, tr^2s^2, \dots, tr^{f-1}s^{f-1}, tr^{fb}, tr^{fb+1}s, \dots, tr^{fb+f-1}s^{f-1}, \dots$$
$$\dots, tr^{-fb}, tr^{-fb+1}s, \dots, tr^{-fb+f-1}s^{f-1}$$

or

$$t, trs, tr^2s^2, \dots, tr^{d-1}s^{d-1}.$$

G_* is generated by r, s and t, and

$$r^{-1}.trs.r = tr^{i+1}s^{j+1}, \quad s^{-1}.trs.s = tr^{i+1}s^{j+1}, \quad t.trs.t = tr^{-j}s^{-i}.$$

Note that if tr^is^j is a symmetry then $tr^{-j}s^{-i} = tr^is^j$. Hence the order of each conjugacy class of symmetries in G_* is $d = Ord(rs)$. Thus G_* contains two classes of symmetries, $\{t\}$ and $\{tr^{fb/2}\}$, in the first instance, and one in the second.

We must now decide whether $S/\langle t\rangle$ is orientable or not. If it is, then the $\langle t\rangle$ cosets can be partioned into two sets such that $\varphi(x) = r$ fixes the sets and $\varphi(c) = t$, $\varphi(xcx^{-1}) = tr^{-1}s^{-1}$ take cosets from one set to the other or fix them. If r preserves the partition, then, for all i and j, r^is^c and r^js^c are in the same set of the partition. Now $r^{-1}.t = ts$, $r^{-2}st = tr^{-1}s^2$ and $r^{-1} \neq s$, $r^{-2}s \neq r^{-1}s^2$. Therefore cosets r^i and r^js are in different sets of the partition, as are r^js and r^ks^2. This implies that r^i and r^ks^2 are in the same set of the partition, but $r^{-2}.t = ts^2$ and so, unless $r^{-2} = s^2$, t does not separate. When $r^2s^2 = 1$ then $d = 2$ and we know that t does separate, as this is the case when t has $g+1$ mirrors.

We now summarize these results.

Theorem 5.3

Let φ be a surface kernel homomorphism from $\Gamma[db, db, d]$ onto a non-cyclic finite abelian group G of order $N = fdb$, for f a proper factor of d. If x_1 and x_2 are canonical generators of Γ, both of order db, let η be the least positive integer such that $\varphi(x_1)^f = (\varphi(x_2)^f)^\eta$. The associated Riemann surface S, admits symmetries of the second kind w.r.t. G if and only if $\eta^2 \equiv 1 \pmod{hb}$, $(h := d/f)$, and such an η exists if and only if $(h, b) = 1$ or 2. Suppose this is the case and that G_* is G extended by such a symmetry, then the species of the conjugacy classes of symmetries in G_* are given in the following table. α is the exponent of two in the prime decomposition of hb.

<table>
<tr><th colspan="3">$\Gamma[db, db, d],\ \varphi$</th><th>Classes</th></tr>
<tr><td rowspan="2">d odd</td><td colspan="2">b odd</td><td>$\{-1\}$</td></tr>
<tr><td colspan="2">b even ($\Rightarrow \eta \equiv -1 \ (2^\alpha)$)</td><td>$\{0, -1\}$</td></tr>
<tr><td rowspan="4">d even</td><td colspan="2">h, b odd</td><td>$\{-fb/2\}$</td></tr>
<tr><td colspan="2">h odd, b even ($\Rightarrow \eta \equiv -1 \ (2^\alpha)$)</td><td>$\{0, -fb/2\}$</td></tr>
<tr><td rowspan="2">h even
($\Rightarrow b$ even)</td><td>$\eta \equiv 1 \ (2^\alpha)$</td><td>$\{0, -fb/2\}$</td></tr>
<tr><td>otherwise</td><td>$\{-fb/2\}$</td></tr>
<tr><td rowspan="2">$d = 2 \Rightarrow h = 1$</td><td colspan="2">b odd</td><td>$\{+(g+1)\}$</td></tr>
<tr><td colspan="2">b even ($\Rightarrow \eta \equiv -1 \ (2^\alpha)$)</td><td>$\{0, +(g+1)\}$</td></tr>
</table>

Looking at this table and the one in the previous section, we see that the only surfaces to admit symmetries of the first or second kind that separate are those that arise via epimorphims from some $[2b, 2b, 2]$, onto $\mathbf{Z}_{2b} + \mathbf{Z}_2$. These surfaces also admit a large cyclic automorphism group and are covered in case twelve of §4.5. The symmetry type of such a surface can be seen to be $\{0, -1, +1, +(g+1)\}$ when g is odd, and $\{0, 0, -1, +2, +(g+1)\}$ when g is even.

Section 5.4 Other Symmetries

If Γ and Λ are Fuchsian triangle groups such that Γ is contained in Λ, then this inclusion appears in [33]. There are only eight particular inclusions or infinite families of inclusions for which there is a surface kernel epimorphism from Γ to a finite non-cyclic abelian group. These inclusions are 1-4, 7, 9, 11 and 12 in the Appendix. If Λ_* is a proper NEC group containing Λ above, then the Appendix tells us that every reflection in Λ_* is conjugate to a reflection that normalizes Γ except for one of these inclusions, (No. 12). Therefore, if φ is a surface kernel epimorphism from Γ to an abelian group of the above nature, that extends to Λ, then all the symmetries of the associated surface, with non-zero species, are conjugate to symmetries of the first or second kind, with non-zero species, w.r.t.

the abelian group unless $\Gamma < \Lambda$ is inclusion twelve. That is unless Λ has signature $[2, 2d, 4]$ and contains $\Gamma[2d, 2d, d]$ with index four. In this case if φ extends to Λ, then it must extend to some $\Delta[2, 2d, 2d]$, $\Gamma \lhd \Delta \lhd \Lambda$. If G' denotes $\varphi(\Delta)$, then the Appendix tells us that there is a symmetry of the second kind w.r.t. G', with non-zero species, which may not be conjugate to any of the symmetries of the first or second kind w.r.t $\varphi(\Gamma)$.

In this section we find necessary and sufficient conditions on $\Gamma[2d, 2d, d]$, the abelian group and φ for φ to extend to $\Lambda[2, 2d, 4]$. Then we calculate the species of symmetries of the second kind w.r.t. $\varphi(\Delta)$ and show that in fact one class of these symmetries, with non-zero species, contains no symmetries conjugate to any symmetry of the first or second kind w.r.t. $\varphi(\Gamma)$.

Let x_1 and y_1 be canonical generators of Γ, both of order db. If φ takes Γ to G, then G is isomorphic to $\mathbf{Z}_{2d} + \mathbf{Z}_f$ for some proper factor f, of d, and G has the following presentation.

$$G := \langle r, s \,|\, r^{2d} = s^{2d} = rsr^{-1}s^{-1} = r^f s^{-\eta f} = 1 \rangle$$

Here $r := \varphi(x_1)$, $s := \varphi(y_1)$ and η is the least positive integer such that $(\eta, 2h) = 1$ and $(\eta + 1, 2h) = 2$, $h := d/f$. $\Delta[2, 2d, 2d]$ has canonical gnerators x_2 and y_2, of orders 2 and $2d$, such that $x_1 = y_2$ and $y_1 = x_2 y_2 x_2$. Thus $x_2 x_1 x_2 = y_1$ and so φ extends to Δ if and only if $r \mapsto s$, $s \mapsto r$ is an automorphism of G. This is the case if and only if $\eta^2 \equiv 1 \pmod{2h}$. We assume η satisfies this condition, if $u := \varphi(x_2)$ then $G' := \varphi(\Delta)$ has presentation

$$G' = \langle r, s, u \,|\, r^{2d} = s^{2d} = u^2 = rsr^{-1}s^{-1} = r^f s^{-\eta f} = urus^{-1} = 1 \rangle.$$

Hence $\varphi(y_2) = r$ and if $z_2 = (x_2 y_2)^{-1}$, then $\varphi(z_2) = r^{-1}u = us^{-1}$. Now $\Lambda[2, 2d, 4]$ contains Δ with index two and so has canonical generators x_3 and y_3, of orders 2 and $2d$, such that $y_2 = y_3$ and $z_2 = x_3 y_3 x_3$. Therefore $x_3 y_2 x_3 = z_2$ and so φ extends to Λ if and only if $r \mapsto r^{-1}u$ is an automorphism, of order two, of G'. Let ϑ denote this mapping. If ϑ is an automorphism, then $\vartheta(u) = \vartheta(r.r^{-1}u) = r^{-1}u.r = urs^{-1}$ and $\vartheta(s) = \vartheta(uru) = urs^{-2}$. However, since ϑ is an involution and $urs^{-1} = sr^{-1}u$, we see that $\vartheta(s)r = u$. We have already shown that $\vartheta(s) = urs^{-2}$ and so we require $urs^{-2} = ur^{-1}$. This is true only if $r^2 s^{-2} = 1$, so $f = 2$ and $\eta = 1$. Thus necessary conditions for φ to extend to Λ are $f = 2$, which implies d is even, and $\eta = 1$. In fact these conditions can also be seen to be sufficient and from now on we shall assume that $f = 2$ and $\eta = 1$.

Under the above circumstances the associated surface S, admits symmetries of the second kind w.r.t. G'. Thus φ extends to $\Delta_*([2d],(2))$, which is generated by a rotation x and a reflection C such that $y_2 = x$ and $z_2 = Cx^{-1}C$. If $t := \varphi(C)$, then $t\varphi(y_2)t = \varphi(z_2)^{-1}$, that is $trt = ur$. Therefore,

$$tut = turr^{-1}t = u \quad \text{and} \quad tst = turut = us.$$

Recall that, strictly speaking, the canonical generators of Δ_* are x, xCx^{-1} and C. Thus we look at the actions of r, tu and t on the right $\langle t\rangle$ cosets to determine the species of t. The cosets are represented by the elements of G',

$$1, r, r^2, \ldots, r^{-1}, s, rs, \ldots, r^{-1}s, u, ur, ur^2, \ldots, ur^{-1}, us, urs, \ldots, ur^{-1}s.$$

$$t\colon r^i \mapsto (ur)^i = \begin{cases} ur(urur)^{(i-1)/2} = ur^{(i+1)/2}s^{(i-1)/2} & i \text{ odd;} \\ (urur)^{i/2} = r^{i/2}s^{i/2} & i \text{ even.} \end{cases}$$

$$t\colon r^i s \mapsto (ur)^i us = \begin{cases} (urur)^{(i-1)/2}urus = r^{(i+3)/2}s^{(i-1)/2} & i \text{ odd;} \\ (urur)^{i/2}us = ur^{i/2}s^{i/2+1} & i \text{ even.} \end{cases}$$

Note that t commutes with u and so the above also indicates the action of t on the cosets ur^i and $ur^i s$. Thus t fixes r^i and ur^i if and only if 4 divides i, and t fixes $r^i s$ and $ur^i s$ if and only if 4 divides $i+1$. Hence t fixes $2d$ cosets.

$$tu\colon r^i \mapsto (ur)^i u = \begin{cases} (urur)^{(i-1)/2}uru = r^{(i-1)/2}s^{(i+1)/2} & i \text{ odd;} \\ (urur)^{i/2}u = ur^{i/2}s^{i/2} & i \text{ even.} \end{cases}$$

$$tu\colon r^i s \mapsto (ur)^i usu = \begin{cases} ur(urur)^{(i-1)/2}r = ur^{(i+3)/2}s^{(i-1)/2} & i \text{ odd;} \\ (urur)^{i/2}usu = r^{i/2+1}s^{i/2} & i \text{ even.} \end{cases}$$

Therefore tu fixes r^i and ur^i if and only if 4 divides $i+1$, $r^i s$ and $ur^i s$ if and only if 4 divides $i+2$ and so, tu also fixes $2d$ cosets.

It now remains for us to calculate the links between the reflection generators of $\varphi^{-1}(\langle t\rangle)$ associated to these fixed cosets. We also use the coset representitives to represent the associated reflection generators. Thus if t fixes $u^i r^j s^k$, $i,k \in \{0,1\}$, $0 \le j < 2d$, then the reflection generator $u^i r^j s^k$ is linked to $u^i r^{j+1} s^k$, which is associated to coset $u^i r^{j+1} s^k$ fixed by tu. This is because the action of r on the cosets is simply $r\colon u^i r^j s^k \mapsto u^i r^{j+1} s^k$, see end of §2.3. Hence

$$\begin{array}{cccc}
1 \sim r^{-1} & u \sim ur^{-1} & r^3 s \sim r^2 s & ur^3 s \sim ur^2 s \\
r^4 \sim r^3 & ur^4 \sim ur^3 & r^7 s \sim r^6 s & ur^7 s \sim ur^6 s \\
\vdots & \vdots & \vdots & \vdots \\
r^{-4} \sim r^{-5} & ur^{-4} \sim ur^{-5} & r^{-1} s \sim r^{-2} s & ur^{-1} s \sim ur^{-2} s
\end{array}$$

The remaining links come from the orbits, or cycles, of $tut = u$. We know u is an involution and so the links it induces will link generators associated to cosets fixed by t to generators also associated to cosets fixed by t, or generators associated to cosets fixed by tu to generators associated to cosets fixed by tu. Note that u commutes with r^{4i}, and so we have the links

$$r^{4i} \sim ur^{4i}, \quad r^{4i-1}s \sim ur^{4i-1}s, \quad r^{4i-1} \sim ur^{4i-2}s, \quad r^{4i-2}s \sim ur^{4i-1}.$$

Putting all these links together we form the following chains.

$$1 \sim r^{-1} \sim ur^{-2}s \sim ur^{-1}s \sim r^{-1}s \sim r^{-2}s \sim ur^{-1} \sim u \sim 1,$$
$$r^4 \sim r^3 \sim ur^2s \sim ur^3s \sim r^3s \sim r^2s \sim ur^3 \sim ur^4 \sim r^4,$$
$$\vdots$$
$$r^{4i} \sim r^{4i-1} \sim ur^{4i-2}s \sim ur^{4i-1}s \sim r^{4i-1}s \sim r^{4i-2}s \sim ur^{4i-1} \sim ur^{4i} \sim r^{4i},$$
$$\vdots$$
$$r^{-4} \sim r^{-5} \sim ur^{-6}s \sim ur^{-5}s \sim r^{-5}s \sim r^{-6}s \sim ur^{-5} \sim ur^{-4} \sim r^{-4}.$$

Hence there are $4d/8 = d/2$ chains and t has $d/2$ mirrors. If g is the genus of S, then $4d[1 - \frac{1}{2d} - \frac{1}{2d} - \frac{1}{d}] = 2g - 2$, so $g = 2d - 3$ and $d/2 = (g+3)/4$.

Under r there are four orbits of $\langle t \rangle$ cosets, these are $\{r^i\}$, $\{r^is\}$, $\{ur^i\}$ and $\{ur^is\}$. However t maps cosets from each of these orbits to cosets in the remaining three orbits. For instance t maps coset r to ur, r^2 to rs and r^3 to ur^2s. Hence, by (2.3)(iv), t does not separate and so $sp(t) = -(g+3)/4$.

Clearly t does not attain Harnack's bound, unlike in the cyclic case. By considering the tables in the previous sections, we can see that the associated surface S, admits at least one class of symmetries with zero species and two or three with non-zero species. When $d/2$ is odd the non-zero species are -1, -2 and $-d/2$. When $d/2$ is even there is one class with species $-d/2$ and the others have species -2. If one works in the full automorphism group, then the symmetry type can be calculated exactly. If d is congruent to two modulo four, then $st(S) = \{0, -1, -2, -\frac{1}{4}(g+3)\}$. If d is congruent to zero modulo four and is greater than four, then $st(S) = \{0, -2, -2, -\frac{1}{4}(g+3)\}$. When $d = 4$, $\Lambda[2, 2d, 4]$ is not maximal but is contained in some $[2, 3, 8]$. In this case $g = 5$ and we see that the surface kernel epimorphism from $\Gamma[4, 8, 8]$ to $\mathbf{Z}_8 + \mathbf{Z}_2$ does extend to this $[2, 3, 8]$. Hence the automorphism group of S has order 384. By working in this group we can see that S admits just three classes of symmetries and has symmetry type $\{0, -2, -2\}$.

Chapter 6

Weierstrass points and Regular Maps

The set of Weierstrass points on a compact Riemann surface is a finite set of "special" points. In the first chapter we saw that maps and hypermaps can naturally be thought of as lying on compact Riemann surfaces. By the nature of the construction of these objects, we see that the vertices (hypervertex centres), face centres (hyperface centres) and edge centres (hyperedge centres) also form a finite set of "special" points on the surface. We call this set the set of *geometric* points of the map or hypermap. In this chapter we shall be looking at certain *regular* maps and hypermaps and asking what the coincidence is between the geometric points of these objects and the Weierstrass points on the underlying surfaces. When a Weierstrass point is also a geometric point, with respect to some map, we will say that it is itself *geometric* with respect to that map.

In the first section we give the necessary definitions and some background results. In §6.2 we present the theorems that we will use directly. In §6.3 we consider those regular maps and hypermaps whose automorphism groups are abelian. We show that the Weierstrass points of surfaces carrying such *maps* are all geometric, with respect to the maps, and the surfaces themselves are hyperelliptic. Finally, in §6.4, we look at all the regular maps of genus two, three, four and five and determine (in most cases), whether the Weierstrass points are geometric.

Section 6.1 Weierstrass points

For a compact Riemann surface S, recall that a function $f: S \mapsto \Sigma$, is called meromorphic if, for every chart (U, Φ) on S, the function $f \circ \Phi^{-1} : \Phi(U) \mapsto \Sigma$ is meromorphic. Thus a meromorphic function on S can be thought of as a set of local meromorphic functions $f_i\colon \Phi_i(U_i) \mapsto \Sigma$, where $\mathcal{A} = \{U_i, \Phi_i\}$ is an atlas that gives rise to the complex structure on S. f is said to have a *pole* (*zero*) at $P \in S$ of order n, if $f \circ \Phi_i^{-1}$ has a pole (zero) at $\Phi_i(P)$ of order n, where (U_i, Φ_i) is any chart at P.

When we differentiate the set of functions $\{f \circ \Phi_i^{-1}\}$, we get another set of meromorphic functions $\{(f \circ \Phi_i^{-1})'\}$. However, if $P \in U_i \cap U_j$, then

$$\begin{aligned}(f \circ \Phi_i^{-1})'\Phi_i(P) &= (f \circ \Phi_j^{-1} \circ \Phi_j \circ \Phi_i^{-1})'\Phi_i(P) \\ &= (f \circ \Phi_j^{-1})'(\Phi_j \circ \Phi_i^{-1})\Phi_i(P).(\Phi_j \circ \Phi_i^{-1})'\Phi_i(P) \\ &= (f \circ \Phi_j^{-1})'\Phi_j(P).(\Phi_j \circ \Phi_i^{-1})'\Phi_i(P).\end{aligned}$$

Hence $(f \circ \Phi_i^{-1})'\Phi_i(P)$ and $(f \circ \Phi_j^{-1})'\Phi_j(P)$ differ by the derivative of the coordinate transformation function at $\Phi_i(P)$. Thus the derivative of a meromorphic function is not necessarily a meromorphic function, but a collection of meromorphic functions on the local coordinates.

Definition 6.1 A *meromorphic differential*, on a Riemann surface with atlas $\mathcal{A} = \{(U_i, \Phi_i)\}$, is a collection of meromorphic functions $\eta_i : \Phi_i(U_i) \mapsto \Sigma$, such that if $P \in U_i \cap U_j$, then $\eta_i(\Phi_i(P)) = \eta_j(\Phi_j(P)).(\Phi_j \circ \Phi_i^{-1})'(\Phi_i(P))$.

Clearly if η and ζ are meromorphic differentials on S with $\zeta \not\equiv 0$, then $\eta_i(\Phi_i(P))/\zeta_i(\Phi_i(P)) = \eta_j(\Phi_j(P))/\zeta_j(\Phi_j(P))$. Hence η/ζ is a meromorphic function on S.

Let $M(S)$ denote the vector space of meromorphic functions on S, and $D(S)$ the space of meromorphic differentials on S. If g is the genus of S, then it is known that the subspace of analytic differentials, denoted by $A(S)$, has dimension g.

Definition 6.2 A *divisor* on S is a symbol $P_1^{\alpha_1}P_2^{\alpha_2}\dots P_k^{\alpha_k}$, where $P_i \in S$, $\alpha_i \in \mathbf{Z}$ and k is finite.

If $A = \prod_{P \in S} P^{\alpha(P)}$ and $B = \prod_{P \in S} P^{\beta(P)}$ are divisors on S, then we define their product and inverses by, $AB = \prod_{P \in S} P^{\alpha(P)+\beta(P)}$ and $A^{-1} = \prod_{P \in S} P^{-\alpha(P)}$. Thus the set of divisors on S, denoted by $Div(S)$, with the above product and inverses is a group. The *degree* of A, or $deg(A)$, is defined as $deg(A) := \sum_{P \in S} \alpha(P)$.

To each $f \in M(S)$ we associate a divisor $(f) = \prod_{P \in S} P^{ord_P f}$, where $ord_P f$ is n if f has a zero of order n at P, $-n$ if f has a pole of order n at P and 0 otherwise.

If $f \in M(S)$, then the divisor (f) is said to be ***principal***. The principal divisors form a subgroup of $Div(S)$.

Similarly, to each $\eta \in D(S)$ we can also associate a divisor $(\eta) = \prod_{P \in S} P^{ord_P \eta}$. We define an equivalence relation on $Div(S)$ by saying that the divisors A and B are equivalent if AB^{-1} is principal. Hence, if $\eta, \zeta \in D(S)\backslash\{0\}$, then $\eta/\zeta \in M(S)$ and so $(\eta) \sim (\zeta)$.

We say that the divisor A is *integral*, or that $A \geq 0$, if $\alpha(P) \geq 0$ for all $P \in S$. Furthermore, we say $A \geq B$ if $AB^{-1} \geq 0$.

Definition 6.3 If $A \in Div(S)$, then we define $L(A)$ to be the space of $f \in M(S)$ such that $(f) \geq A$ and we let $r(A)$ denote the dimension of $L(A)$. We let $\Omega(A)$ be the space of $\eta \in D(S)$ obeying $(\eta) \geq A$, and $i(A)$ denote the dimension of $\Omega(A)$.

Note that $\Omega(1) = A(S)$ and so $i(1) = g$.

Theorem 6.4

For $A \in Div(S)$, $r(A)$ and $i(A)$ depend only on the class of A, and if $\eta \not\equiv 0$ is any meromorphic differential, then $i(A) = r(A(\eta^{-1}))$.

For a proof of this theorem and other results stated here, see [12].

Theorem 6.5 (Riemann-Roch)

If $A \in Div(S)$ is integral, then $r(A^{-1}) = deg(A) - g + 1 + i(A)$.

This is a very meaningful result and has lots of applications, not least in the study of Weierstrass points. We now define Weierstrass points via the following theorem, which is itself an application of the Riemann-Roch Theorem.

Theorem 6.6 (The Weierstrass gap Theorem)

If S is a compact Riemann surface of genus g, then for each point $P \in S$, there are precisely g integers

$$1 = \gamma_1 < \gamma_2 < \cdots < \gamma_g < 2g,$$

such that there does not exist a function $f \in M(S)$, analytic on $S\backslash\{P\}$ with a pole of order γ_i at P.

The above sequence is called the *gap sequence* at P, while its complement in the natural numbers is the set of *non-gaps* at P. The non-gaps form a semi-group.

We denote the first g non-gaps by

$$1 < \alpha_1 < \alpha_2 < \cdots < \alpha_g = 2g.$$

For each α_j, there is a meromorphic function with a pole of order α_j at P and which is analytic everywhere else.

Definition 6.7 A point $P \in S$, is a *Weierstrass* point if $\alpha_1 \leq g$ or equivalently, if $\gamma_g > g$. That is, there exists a function in $M(S)$ that is analytic on $S \backslash \{P\}$, and has a pole of order less than or equal to g at P.

Let W denote the set of Weierstrass points on S. To each point of S we ascribe a *weight* according to its gap sequence. The weight of P, denoted by w_P, is given by $w_P = \sum_{i=1}^{g}(\gamma_i - i)$, where $\gamma_1, \ldots, \gamma_g$ are the gaps at P. This is zero unless $P \in W$. The total weight of points on S is known to be $g(g-1)(g+1) = g^3 - g$. It is also known that, for each P on S, $w_P \leq \frac{1}{2}g(g-1)$, with equality if and only if $\alpha_1 = 2$, in which case S is hyperelliptic. Thus if $\omega = \sharp W$, then $2(g+1) \leq \omega \leq g^3 - g$. When S is hyperelliptic all its Weierstrass points have weight $\frac{1}{2}g(g-1)$.

Thus γ is a gap at $P \in S$ if and only if $L(P^{-\gamma}) - L(P^{-\gamma+1}) = \emptyset$, that is, $r(P^{-\gamma}) - r(P^{-\gamma+1}) = 0$. The Riemann-Roch Theorem tells us that

$$r(P^{-\gamma}) - r(P^{-\gamma+1}) = i(P^{\gamma}) - i(P^{\gamma-1}) + 1.$$

Hence, γ is a gap at P if and only if there is an $\eta \in A(S)$ with a zero of order $\gamma - 1$ at P. For each $P \in S$ a basis for $A(S)$, $\theta_1, \theta_2, \ldots, \theta_g$, can be constructed such that θ_i has a zero of order $\gamma_i - 1$ at P, where γ_i is the i^{th} gap at P.

Section 6.2 Automorphisms

We have previously remarked that the automorphisms of a surface S, fix W setwise, and so $Aut(S)$ acts as a group of permutations on W. In fact this representation can be seen to be faithful if S is not hyperelliptic. Lewittes [24], considered the representation of (conformal) automorphisms on $D(S)$, in particular $A(S)$, and on spaces of q-differentials which we shall not discuss. Let $\theta \in D(S)$ at $P \in S$ be given locally by $\theta = (a_k z^k + \ldots)dz$, [$z$ is a parametrization of $\Phi(U)$, where (U, Φ) is some chart at P]. Let $h \in Aut^+(S)$ and let $Q = h^{-1}(P)$. Suppose θ at Q is locally given by $\theta = (b_j u^j + \ldots)du$, for some local parameter u. Since h^{-1} is conformal, it is given locally at P by $u = F(z)$, where $F'(0) \neq 0$. We define $h(\theta)$ to be the differential which at P is locally given by $h(\theta) = (b_j(F(z))^j + \ldots)F'(z)dz$; that is, $h(\theta) = \theta h^{-1}$. The mapping $\theta \mapsto h(\theta)$, is a linear transformation of $D(S)$

that maps $A(S)$ to itself. Let $R(h)$ denote the linear transformation determined by h acting on $A(S)$. Lewittes has shown that this representation of $Aut^+(S)$ is faithful.

Automorphisms of S can act on $M(S)$ in the same way that they act on $D(S)$ and $A(S)$. Let $H = \langle h \rangle$, a meromorphic function or differential which is fixed by h is said to be H-*invariant*. We let $M(S)^H$, $D(S)^H$ and $A(S)^H$ denote the subspaces of $M(S)$, $D(S)$ and $A(S)$ that are H-invariant. If $\tilde{S}$ is the quotient space S/H, with the obvious complex structure, then Lewittes proved that $M(S)^H \simeq M(\tilde{S})$, $D(S)^H \simeq D(\tilde{S})$ and $A(S)^H \simeq A(\tilde{S})$. This was done by "lifting" differentials from $D(\tilde{S})$ to $D(S)^H$ and "lowering" those in $D(S)^H$ to $D(\tilde{S})$, [meromorphic functions were considered to be 0-differentials].

Lewittes then used these ideas to calculate a formula for the multiplicities of the eigenvalues of $R(h)$, which we shall outline. If h is of order N, then so is $R(h)$ which is therefore diagonalisable. Hence its eigenvalues are powers of $\epsilon := e^{2\pi i/N}$. Let n_k denote the multiplicity of ϵ^k as an eigenvalue of $R(h)$. Thus n_0 is the multiplicity of one and is equal to the dimension of $A(S)^H$, which is $\tilde{g}$; the genus of $\tilde{S}$ and the dimension of $A(\tilde{S})$. Let $\pi_H : S \mapsto \tilde{S}$ be the branched analytic covering whose branch points are precisely the points fixed by either h or its powers. We denote these points by $P_1, P_2, \ldots, P_t$. Lewittes only considered the case when N is prime and so each P_i has branch order $N-1$, this is because H has no non-trivial proper subgroups when N is prime. Now n_k is the dimension of the space of analytic differentials θ, such that $h(\theta) = \epsilon^k \theta$. This can be seen to be the dimension of a space of H-invariant meromorphic functions with certain properties. These functions can be "lowered" into $M(\tilde{S})$ and the Riemann-Roch Theorem then gives an expression for the dimension of this space in terms of the action of h at all its fixed points. At P_i, h^{-1} is locally $z \mapsto \eta_i z$, where η_i is some primitive N^{th} root of unity. If $\epsilon^k = \eta_i^{p_i}$, $1 \le p_i \le N$, then Lewittes showed that

$$n_k = \tilde{g} - 1 + \sum_{i=1}^{t} \left(1 - \frac{p_i}{N}\right), \qquad 0 < k < N.$$

If N is not prime, then there may be branch points of π_H with branching orders less than $N-1$. Let $\{P_{11}, P_{12}, \ldots, P_{1r_1}\}, \ldots, \{P_{t1}, P_{t2}, \ldots, P_{tr_t}\}$ be the t orbits of branch points, with branching orders $N/r_1 - 1, \ldots, N/r_t - 1$ and let $N_i = N/r_i$. Suppose h^{-r_i}, at P_{ij}, is locally $z \mapsto \eta_i z$, where η_i is a primitive N_i^{th} root of unity. Note that the action of h^{-r_i} is locally the same at each of the points

$P_{i1}, \ldots, P_{ir_i}$. The expression for the multiplicities of the eigenvalues, not equal to one, generalizes to

$$n_k = \tilde{g} - 1 + \sum_{i=1}^{t} \left(1 - \frac{p_i}{N_i}\right), \qquad \text{where } \epsilon^{kr_i} = \eta_i^{p_i}, \ 1 \leq p_i \leq N_i.$$

We now describe and make use of some results of Harvey, [32]. If S is uniformized by the surface group K, then H will lift to a Fuchsian group Γ, such that $K \triangleleft \Gamma$ and $\Gamma/K \simeq H$. For each $\gamma \in \Gamma$, we define the mapping $\gamma' : S \mapsto S$ by $\gamma'(Kz) = K\gamma z$. This is an automorphism of S and so $\gamma' = h^{-\xi}$ for some $0 \leq \xi < N$. By these means we define a homomorphism φ, from Γ onto $\mathbf{Z}_N \simeq H$. If $\gamma' = h^{-\xi}$, then we let $\varphi(\gamma) = \xi$, and so the kernel is K. Let x_i be a canonical elliptic generator of Γ, of order N_i, and let ξ_i be the integer between zero and N such that $x_i' = h^{-\xi_i}$. If z_i is the unique point of $\mathcal{U}$ fixed by x_i, then $Kz_i, h(Kz_i), \ldots, h^{r_i-1}(Kz_i)$ are the points of S fixed by x_i' and its powers alone. These are precisely the points fixed by every element of $\langle h^{r_i} \rangle$ and by no other power of h, which we have called $P_{i1}, P_{i2}, \ldots, P_{ir_i}$. It can be assumed that x_i, at z_i, is locally a rotation of $2\pi i/N_i$. Thus $h^{-\xi_i}(= x_i')$, at P_{ij}, is locally $z \mapsto \epsilon^{r_i} z$. Hence, h^{-r_i} at P_{ij} is locally $z \mapsto \epsilon^{r_i \sigma_i} z$, where $\xi_i \sigma_i \equiv r_i \pmod N$ and $0 < \sigma_i < N_i$. Therefore $\eta_i = \epsilon^{r_i \sigma_i}$ and so $\epsilon^{kr_i} = (\epsilon^{r^i \sigma_i})^{p_i}$. Thus $k \equiv \sigma_i p_i \pmod{N_i}$. If $\hat{\sigma}_i$ denotes the multiplicative inverse of σ_i modulo N_i, then, by definition, $\xi_i = \hat{\sigma}_i \frac{N}{N_i}$. Therefore we may rewrite the above expression for n_k $(0 < k < N)$, as

$$n_k = \tilde{g} - 1 + \sum_{i=1}^{t} \left(1 - \left\langle \frac{k\xi_i}{N} \right\rangle \right).$$

Here $\langle\,\rangle$ denotes the fractional part of a non-integer and one for an integer.

Example Consider the surface kernel epimorphism $\varphi : \Gamma[7,7,7] \mapsto \mathbf{Z}_7$, defined by $\varphi(x_1) = 1$, $\varphi(x_2) = 2$ and $\varphi(x_3) = 4$. The surface S uniformized by the kernel is known to be Klein's surface of genus three. In the above notation $\xi_1 = 1$, $\xi_2 = 2$, $\xi_3 = 4$, $\sigma_1 = 1$, $\sigma_2 = 4$, $\sigma_3 = 2$ and $N_1 = N_2 = N_3 = 7$. Therefore S admits an automorphism h, of order seven (associated to -1 in $\mathbf{Z}_7$), fixing exactly three points P_1, P_2 and P_3 (associated to x_1, x_2 and x_3 respectively). Thus, h^{-1} at P_i is given locally by $z \mapsto \epsilon^{\sigma_i} z$, where $\epsilon = e^{2\pi i/7}$. We now calculate the n_k's for $R(h)$.

$$\begin{aligned}
n_0 = 0, \qquad n_1 &= -1 + (1 - \tfrac{1}{7}) + (1 - \tfrac{2}{7}) + (1 - \tfrac{4}{7}) = 1 \\
n_2 &= -1 + (1 - \tfrac{2}{7}) + (1 - \tfrac{4}{7}) + (1 - \tfrac{1}{7}) = 1 \\
n_3 &= -1 + (1 - \tfrac{3}{7}) + (1 - \tfrac{6}{7}) + (1 - \tfrac{5}{7}) = 0 \\
n_4 &= -1 + (1 - \tfrac{4}{7}) + (1 - \tfrac{1}{7}) + (1 - \tfrac{2}{7}) = 1
\end{aligned}$$

Hence the eigenvalues of $R(h)$ are ϵ, ϵ^2 and ϵ^4.

Now we give Lewittes' expression for the eigenvalues of $R(h)$ in terms of the gaps at a particular fixed point of h and the local action of h at this fixed point. We mentioned, at the end of the last section, that for each $P \in S$ we may construct a basis for $A(S)$, $\theta_1, \theta_2, \ldots, \theta_g$, such that θ_i has a zero of order $\gamma_i - 1$ at P, where γ_i is the i^{th} gap at P. Lewittes considered such a basis and "normalized" it still further. In his basis, θ_i at P is locally $\theta_i = (z^{\gamma_i - 1} + \ldots)dz$ and the coefficient of $z^{\gamma_j - 1}$ is zero for $j > i$. If h^{-1} at P is locally $z \mapsto \eta z$, then

$$h(\theta_i) = ((\eta z)^{\gamma_i - 1} + \ldots)\eta dz = R(h)(\theta_i) = \sum_{j=1}^{g} c_{ij}\theta_j$$

where $R(h) = \{c_{ij}\}$ is non-singular. Since $h(\theta_i)$ has the same order zero at P as θ_i does, $c_{ij} = 0$ for $j < i$. By construction θ_i, and so $h(\theta_i)$, have no $z^{\gamma_j - 1}$ term for $j > i$. Hence $c_{ij} = 0$ for $j > i$, and so $c_{ii} = \eta^{\gamma_i}$.

Theorem 6.8 (Lewittes)

Let h be an automorphism of a compact Riemann surface S that fixes a point $P \in S$. If h^{-1} is locally $z \mapsto \eta z$ at P and the gaps at P are $\gamma_1, \ldots, \gamma_g$, then the eigenvalues of $R(h)$ are $\eta^{\gamma_1}, \ldots, \eta^{\gamma_g}$.

Example 1 Let us go back to the previous example. At P_1, h^{-1} is locally $z \mapsto \epsilon z$, where $\epsilon = e^{2\pi i/7}$. Thus, if the gaps at P_1 are γ_1, γ_2 and γ_3, then the eigenvalues of $R(h)$ are ϵ^{γ_1}, ϵ^{γ_2} and ϵ^{γ_3}. However, our previous calculations show the eigenvalues of $R(h)$ to be ϵ, ϵ^2 and ϵ^4. Hence one of the gaps at P_1 is congruent to 1 modulo 7, one is congruent to 2 and one is congruent to 4. Since $7 > 2g$ there is no ambiguity and the gaps at P_1 are $\{1, 2, 4\}$, so $w_{P_1} = 1$. At P_2, h^{-1} is locally $z \mapsto \epsilon^4 z$. Now $\epsilon = (\epsilon^4)^2$ $[= (\epsilon^{\sigma_2})^{\xi_2}]$, so $\epsilon^2 = (\epsilon^4)^4$ and $\epsilon^4 = (\epsilon^4)$. Hence the gaps at P_2 are also $\{1, 2, 4\}$. The gaps at P_3 are again $\{1, 2, 4\}$. These three points are in the same orbit under $Aut(S)$, indeed this orbit contains 24 points and they are all Weierstrass points with weight one. The genus of the surface is three and $3^3 - 3 = 24$, so these are all the Weierstrass points. In fact, they correspond to the face centers of the regular map of type $\{3, 7\}$ that lies on Klein's surface.

Note that the order of h above was greater than $2g$ and so we could calculate the gaps explicitly. In general this method gives the gaps at a fixed point of h, modulo the order of h. When the order of h is "small" with respect to g then it may still be uncertain as to what the gaps actually are.

Example 2 Consider the following surface kernel epimorphism.

$$\begin{array}{lcl} \varphi : \Gamma[2,n,2n] \longmapsto & \mathbf{Z}_{2n} & (n,2)=1 \\ x_1 & n & \sigma_1 = 1 \\ x_2 & n-1 & \sigma_2 = n-2 \\ x_3 & 1 & \sigma_3 = 1 \end{array}$$

Let S denote the surface uniformized by the kernel, and let h be the automorphism of S associated to -1 in $\mathbf{Z}_{2n}$. Hence, for $R(h)$,

$$n_0 = 0, \quad n_1 = -1 + (1 - \tfrac{n}{2n}) + (1 - \tfrac{n-1}{2n}) + (1 - \tfrac{1}{2n}) = 1,$$
$$n_2 = -1 + (1 - \tfrac{2n}{2n}) + (1 - \tfrac{2n-2}{2n}) + (1 - \tfrac{2}{2n}) = 0, \ldots$$

If k is even, then $\langle kn/2n \rangle = 1$ and $\langle k(n-1)/2n \rangle = 2n - k$. If k is odd, then $\langle kn/2n \rangle = n$ and $\langle k(n-1)/2n \rangle = n - k$ when $k < n$, 1 when $k = n$ and $2n - k$ when $k > n$. Hence, $n_k = 1$ if and only if k is odd and less than n. Thus the eigenvalues of $R(h)$ are ϵ, $\epsilon^3, \ldots, \epsilon^{n-2} = \epsilon^{2g-1}$, where g is the genus of S and $g = (n-1)/2$. Since $\xi_3 = 1$ the gaps at P_3 are $\{1, 3, \ldots, 2g-1\}$ and S is hyperelliptic.

Let $L = \langle \varphi(x_1) \rangle$, so $|\mathbf{Z}_{2n} : L| = n$ and $\varphi(x_2), \varphi(x_2)^2, \ldots, \varphi(x_2)^n$, are L coset representitives. The action of Γ on the $\Lambda := \varphi^{-1}(L)$ cosets is therefore given by

$$x_1 \mapsto (1)(2)\ldots(n), \quad x_2 \mapsto (1\,2\ldots n), \quad x_3 \mapsto (1\,n\ldots 2).$$

Thus Λ has signature $[2^{(n+1)}]$ or $[2^{(2g+2)}]$; n of the canonical elliptic generators will be conjugate to x_1 and one to x_3^n. In the restriction of φ to Λ, combined with the isomorphism from $\{0, n\}$ onto $\mathbf{Z}_2$, all the elliptic elements are mapped to one. So, for $R(h^n)$, $n_0 = 0$ and $n_1 = g$. Hence the $2g+2$ fixed points of h^n, one of which is the single point fixed by h, are the (hyperelliptic) Weierstrass points of S.

Example 3 Let φ be the surface kernel epimorphism from $\Gamma[2, 2n, 2n]$ to $\mathbf{Z}_{2n}$, (n even), defined by $\varphi(x_1) = n$, $\varphi(x_2) = n - 1$ and $\varphi(x_3) = 1$. Again we let S be the associated surface and h the automorphism of S associated to -1 in $\mathbf{Z}_{2n}$. We now calculate the eigenvalues of $R(h)$.

$$\begin{aligned} n_k &= -1 + \left(1 - \left\langle \frac{nk}{2n} \right\rangle\right) + \left(1 - \left\langle \frac{(n-1)k}{2n} \right\rangle\right) + \left(1 - \left\langle \frac{k}{2n} \right\rangle\right) \\ &= \begin{cases} 1 & \text{if } k \text{ is odd and } k < n; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Hence the eigenvalues of $R(h)$ are ϵ, $\epsilon^3, \ldots, \epsilon^{n-1}$. If g is the genus of S, then the Riemann-Hurwitz formula tells us that $g = n/2$, and so $n - 1 = 2g - 1$. Therefore, as $\xi_3 = 1$, the gaps at P_3 are $\{1, 3, \ldots, 2g-1\}$ and S is hyperelliptic.

If $M = \langle\varphi(x_1)\rangle$, then $|\mathbf{Z}_{2n} : M| = n$ and $\varphi(x_2), \varphi(x_2)^2 \ldots, \varphi(x_2)^n$ are M coset representitives. Hence the action of Γ on the $\Delta := \varphi^{-1}(M)$ cosets is

$$x_1 \mapsto (1)(2)\ldots(n), \quad x_2 \mapsto (1\,2\ldots n), \quad x_3 \mapsto (1\,n\ldots 2).$$

The signature of Δ is $[2^{(n+2)}]$; n of its conjugacy classes of elliptic elements will be conjugate to x_1, one to x_2 and the other to x_3. The restriction of φ to Δ followed by the unique isomorphism from M to $\mathbf{Z}_2$ maps all the elliptic generators of Δ to 1. Thus, $n_0 = 0$ and $n_1 = g$ for $R(h^n)$, and so the gaps at the $n+2 = 2g+2$ fixed points of h^n are $\{1, 3, \ldots, 2g-1\}$. These are all the (hyperelliptic) Weierstrass points of S.

Lewittes showed that if an automorphism fixes five or more points, then they are all Weierstrass points. Schoeneberg had shown that a fixed point of an automorphism is a Weierstrass point provided, the integer part of the genus of the surface divided by the order of the automorphism, is not equal to the genus of the quotient of the surface by the group generated by the automorphism. In [28] Maclachlan improved on both of these theorems using the results of Lewittes and Harvey outlined in this section.

Theorem 6.9 (Maclachlan)

Let S be a compact Riemann surface of genus $g > 1$, and let h be an automorphism of S of order N. Let the Fuchsian group Γ be the lift of $\langle h\rangle$; that is, if S is uniformized by K, then $K \triangleleft \Gamma$ and $\Gamma/K \simeq \langle h\rangle$. If h fixes a point which is not a Weierstrass point, then Γ has one of the following signatures

(i) $\left(\frac{g}{N}; [N, N]\right)$, or

(ii) $\left(\frac{g - N + 1}{N}; [N, N, N, N]\right)$, or

(iii) $(\tilde{g}; [N, N_1, N_2])$, where $2N\tilde{g} = 2g - 1 - N + \dfrac{N_1 + N_2}{(N_1, N_2)}$ and $[N_1, N_2] = N$.

We also state a related theorem of Guerrero's [16], In fact it is a further improvement on part (iii) of Maclachlan's Theorem.

Theorem 6.10 (Guerrero)

Let S, g, h and Γ be as in (6.9), but now suppose h fixes exactly one point, $P \in S$. Then P is a Weierstrass point unless h has order 6 and Γ has signature of the form $(\tilde{g}; [2,3,6])$. (Note that $\tilde{g} > 0$, else Γ is not Fuchsian.)

Section 6.3 Regular Maps and Hypermaps with Abelian Automorphism Groups

See §1.4.

Theorem 6.11

The underlying (compact Riemann) surface of a regular map whose automorphism group is abelian is hyperelliptic, and the Weierstrass points of the surface are all geometric points of the map.

Proof Firstly we consider those regular maps whose automorphism groups are cyclic. These correspond to surface kernel homomorphisms from Fuchsian triangle groups with signature $[2, m, n]$, onto cyclic groups. By (1.17) the only triangle groups for which such homomorphisms exist are those with signature $[2, n, 2n]$ (n odd), or $[2, 2n, 2n]$ (n even), and in both cases the homomorphism will map them to $\mathbf{Z}_{2n}$. The canonical generator of order two must map to n in each case. In the first case we may assume that the canonical generator of order $2n$ maps to 1 and so the generator of order n must be taken to $n-1$. In the second case we may assume that one of the canonical generators of order $2n$ maps to 1 and so the other must be taken to $n-1$. Hence, in both cases the epimorphism is unique up to an automorphism of $\mathbf{Z}_{2n}$. These two possibilities were taken as examples two and three in the last section. There we showed the surface uniformized by the kernel to be hyperelliptic; the Weierstrass points to be the edge centres and the face centre (or vertex) in the first case, and the edge centres, face centre and vertex in the second case.

Now we look at those regular maps whose automorphism groups are non-cyclic abelian. By (5.1) these can only arise via surface kernel epimorphisms from groups with signature of the form $[2, 2n, 2n]$, to groups isomorphic to $\mathbf{Z}_{2n} + \mathbf{Z}_2$. Let Γ have such a signature and φ be a surface kernel epimorphism from Γ to

$$G := \langle a, b \mid a^{2n} = b^{2n} = aba^{-1}b^{-1} = (ab)^2 = 1 \rangle \simeq \mathbf{Z}_{2n} + \mathbf{Z}_2,$$

defined by $\varphi(x_1) = (ab)^{-1}$, $\varphi(x_2) = a$ and $\varphi(x_3) = b$. It is easy to see that φ is unique upto an automorphism of G. Let $L = \langle (ab)^{-1} \rangle$, so $|G : L| = 2n$ and

$1, a, a^2, \ldots, a^{2n-1}$ are L coset representatives. The action of Γ on the $\Lambda := \varphi^{-1}(L)$ cosets is therefore given by

$$x_1 \mapsto (1)(2)\ldots(n) \quad x_2 \mapsto (1\,2\ldots n) \quad x_3 \mapsto (1\,n\ldots 2).$$

Thus Λ has signature $[2^{(2n)}]$. The Riemann-Hurwitz formula tells us that the genus g, of the associated surface, is given by $g = n - 1$. It can easily be verified that the gaps at the $2n = 2g + 2$ fixed points of $(ab)^{-1}$, which correspond to the edge centres of the map, are $\{1, 3, \ldots, 2g - 1\}$. Hence the surface is hyperelliptic and the Weierstrass points are geometric. □

The theorem in [37] tells us that if the underlying surface of a regular map is hyperelliptic, then the map automorphism group contains the hyperelliptic involution, (we say that such a map is itself hyperelliptic). It is a trivial consequence of this theorem that the set of geometric points of a regular map on a hyperelliptic Riemann surface contains *all* the Weierstrass points of the underlying surface.

We now consider those regular *hypermaps* whose automorphism group is abelian, and try to determine when the geometric points are also Weierstrass points. First we look at the cyclic case.

Suppose that φ is a surface kernel epimorphism from $\Gamma[m_1, m_2, m_3]$ to $\mathbf{Z}_M$, we return to our earlier notation, so $\langle\varphi(x_i)\rangle = \langle b_i\rangle$. If $L_i = \langle\varphi(x_i)\rangle$, then $|\mathbf{Z}_M : L_i| = b_i$ and $\varphi(x_j), \varphi(x_j)^2, \ldots, \varphi(x_j)^{b_i}$, for $j \neq i$, are L_i coset representitives. Hence the action of Γ on the $\Lambda_i := \langle\varphi^{-1}(L_i)\rangle$ cosets is given by

$$x_i \mapsto (1)(2)\ldots(b_i) \quad x_j \mapsto (1\,2\ldots b_i) \quad x_k \mapsto (1\,b_i\ldots 2),$$

where $i, j, k \in \{1, 2, 3\}$ are mutually distinct. Thus $[m_i^{(b_i)}, db_k, db_j]$ is the signature of Λ_i. Let S be the surface uniformized by the kernel of φ, let $g > 1$ be the genus of S and let h be the automorphism of S associated to -1 in $\mathbf{Z}_M$. This is the automorphism of S induced by any element in $\varphi^{-1}(-1)$. Provided the signature of Λ_i is not one of those in (6.9), or is of the form $[m_i, N_1, N_2]$ with $m_i > N_1$ and $m_i > N_2$, then the point(s) of S fixed solely by h^{b_i} and its powers are Weierstrass points. (The second possibility is included because of (6.10)). This corresponds to one "type" of the geometric points being Weierstrass points, hypervertex centres, hyperedge centres or hyperface centres. We now determine when the signature of Λ_i does indeed have the form of one of those in (6.9).

The signature of Λ_1 can never take the form $(\frac{g}{m_1}; [m_1, m_1])$. This is because the genus of $\mathcal{U}/\Lambda_1$ is zero and $g/m_1 \neq 0$.

If Λ_1 has signature of the form $(\frac{g-m_1+1}{m_1};[m_1^{(4)}])$, then

(i) $db_2 = db_3 = 1$ and $b_1 = 4$, but then Γ would have signature $[4,4]$ which is not Fuchsian. Or

(ii) $db_2 = 1$ and $db_3 = m_1$ or vice versa, and $b_1 = 3$. In which case Γ would have signature $[b_3, 3b_3, 3]$, $(b_3, 3) = 1$ and $b_3 > 2$. Hence Λ_1 has signature $[b_3^{(4)}]$, three of the conjugacy classes of elliptic elements of order b_3 will be conjugate to x_1 and the other to x_2. Thus φ restricted to Λ_1, follwed by the isomorphism from L_1 to $\mathbf{Z}_{m_1}$ that maps $\varphi(x_1)$ to one, takes three of the four canonical generators to 1 and the other to $b_3 - 3$. Hence, for $R(h^{\xi_1})$,

$$n_0 = 0 \quad \text{and} \quad n_1 = -1 + 3\left(1 - \left(\tfrac{1}{b_3}\right)\right) + \left(1 - \left(\tfrac{b_3-3}{b_3}\right)\right) = 2.$$

Since $g = b_3 - 1$, $g+2$ is a gap at the three points fixed by $\langle h^{\xi_1}\rangle$ alone, and so these are indeed Weierstrass points. Note that $\Lambda_2 = \Gamma$, and so by (6.10), the fixed point of h is a Weierstrass point. Furthermore the signature of Λ_3 is $[3^{(b_3+1)}]$, which does not appear in (6.9), so all the geometric points of the hypermap are Weierstrass points despite the signature of Λ_1 appearing in (6.9). Or

(iii) $db_2 = db_3 = m_1$ and $b_1 = 2$, in which case Γ has signature $[d, 2d, 2d]$, $(d > 2)$. Note that $\Lambda_2 = \Lambda_3 = \Gamma$, and the signature of Γ also appears in (6.9) and is not ruled out by (6.10).

If Λ_1 has signature of the form $(\tilde{g}; [m_1, N_1, N_2])$, then

(i) $db_2 = db_3 = 1$ and $b_1 = 3$, but then the Γ would not be Fuchsian. Or

(ii) $db_2 = 1$ and $db_3 > 1$ or vice versa, and $b_1 = 2$. In this case Γ would have signature $[b_3, 2b_3, 2]$. We know from (6.11) that the points fixed by $\langle h^2\rangle$ are not Weierstrass points while the other geometric points constitute all the Weierstrass points. Or

(iii) $b_1 = 1$, db_2, $db_3 \neq 1$, and so Γ will have signature of the form $[db_2b_3, db_3, db_2]$. Hence $\Lambda_1 = \Gamma$, Λ_2 has signature $[db_3^{(b_2+1)}, d]$ and Λ_3 has signature $[db_2^{(b_3+1)}, d]$. Without loss of generality we may assume $1 \leq b_2 \leq b_3$. If $b_2 > 1$, then $b_3 > 2$ and $db_2b_3 > db_3 > db_2$, and so, by (6.10), the single fixed point of h is a Weierstrass point. The signature of Λ_2 only appears in (6.9) if $b_2 = 1$ and $d > 1$, or $b_2 = 2$ and $d = 1$. The second possibility is covered in (6.11). Similarly the signature of Λ_3 only appears in (6.9) if $b_3 = 1$ and $d > 1$, or

$b_3 = 2$ and $d = 1$. Under our hypothesis that $b_2 < b_3$ the second possibility cannot occur.

Hence, in (iii), if $b_2 > 1$ ($\Rightarrow b_3 > 2$), then all of the geometric points are Weierstrass points, or, in the case when $b_2 = 2$ and $d = 1$, so that Γ has the signature $[2b_3, b_3, 2]$, all the Weierstrass points are geometric.

We need now only consider the case when $b_2 = 1$, so Γ has signature $[db, db, d]$, $\Lambda_1 = \Lambda_2 = \Gamma$ and Λ_3 has signature $[d^{(b+2)}]$. Hence the points fixed by $\langle h^b \rangle$ alone are certainly Weierstrass points when $b > 2$. For $R(h)$,

$$n_d = -1 + (1 - \langle \varphi(x_1)d/bd \rangle) + (1 - \langle \varphi(x_2)d/bd \rangle) + (1 - \langle \varphi(x_3)d/bd \rangle).$$

Now $\varphi(x_3)$ is of order d and so $(1 - \langle \varphi(x_3)d/bd \rangle) = 0$, which implies $n_d = 0$. This is the case whatever $\varphi(x_1)$ and $\varphi(x_2)$ are. In fact, if we consider $R(h^{\xi_1})$, then n_d is still zero, this is also the case for $R(h^{\xi_2})$. Therefore d is a non-gap at both fixed points of h. The genus of S is given by $g = b(d-1)/2$ and so $d < g$ whenever $b > 2$. Thus when $b > 2$ all the geometric points are Weierstrass points.

We are now going to consider the two remaining cases, when $b = 1$, and $b = 2$ in some detail.

a) When $b = 2$, Γ has signature $[2d, 2d, d]$ and $g = d - 1$. Let $\varphi : \Gamma \mapsto \mathbf{Z}_{2d}$, be defined by $\varphi(x_1) = 1$, $\varphi(x_2) = \alpha$ and $\varphi(x_3) = \beta$, so $(\alpha, 2d) = 1$ and $(\beta, 2d) = 2$. First we make a useful observation. If $r_1 + r_2 + r_3 \equiv 0 \pmod{N}$ and $0 < r_i < N$, then

$$r_1/N + r_2/N < 1 \;\Rightarrow\; \sum_{i=1}^{3} r_i/N = 1,$$

$$\text{and} \qquad r_1/N + r_2/N > 1 \;\Rightarrow\; \sum_{i=1}^{3} r_i/N = 2.$$

The genus of $\mathcal{U}/\Gamma$ is zero and so, for $R(h)$, $n_0 = 0$.

$$n_1 = -1 + 3 - \frac{1}{2d} - \frac{\alpha}{2d} - \frac{\beta}{2d} = 1,$$

$$n_2 = -1 + 3 - \frac{2}{2d} - \left\langle \frac{2\alpha}{2d} \right\rangle - \left\langle \frac{2\beta}{2d} \right\rangle = 1, \quad \text{as } 2/2d + \langle 2\alpha/2d \rangle < 1.$$

$$n_d = n_{g+1} = -1 + 2 - \left\langle \frac{d}{2d} \right\rangle - \left\langle \frac{d}{2d} \right\rangle = 0, \quad \text{as } \alpha \text{ is odd and } (\beta, 2d) = 2.$$

If the fixed point of h, associated to x_1, is not a Weierstrass point then $n_k = 1$ for $1 \le k \le g$. Now we determine the conditions for this to be the case.

$$\left\langle \tfrac{(d-1)\alpha}{2d} \right\rangle = \begin{cases} \frac{d-\alpha}{2d} & \text{if } \alpha < d; \\ \frac{3d-\alpha}{2d} & \text{if } \alpha > d, \end{cases} \qquad \text{and } \tfrac{d-1}{2d} + \tfrac{3d-\alpha}{2d} > 1, \text{ as } \alpha + 1 < 2d.$$

Hence $n_{d-1} = n_g = 1$ if $\alpha < d$, and 0 if $\alpha > d$. Note that $\alpha \neq 2d/\lambda$ for any divisor λ of $2d$. We assume $\alpha < d$, so

$$\left\langle \frac{(d-2)\alpha}{2d} \right\rangle = \begin{cases} \frac{d-2\alpha}{2d} & \text{if } \alpha < d/2; \\ \frac{3d-2\alpha}{2d} & \text{if } \alpha > d/2. \end{cases} \quad \text{and } \frac{d-2}{2d} + \frac{3d-2\alpha}{2d} > 1 \text{ else } \alpha > d-1.$$

Hence $n_{d-2} = 1$ if $\alpha < d/2$, and 0 if $\alpha > d/2$, so we assume $\alpha < d/2$.

$$\left\langle \frac{(d-3)\alpha}{2d} \right\rangle = \begin{cases} \frac{d-3\alpha}{2d} & \text{if } \alpha < d/3; \\ \frac{3d-3\alpha}{2d} & \text{if } \alpha > d/3, \end{cases} \quad \begin{array}{l} \text{and } \frac{d-3}{2d} + \frac{3d-3\alpha}{2d} < 1 \Leftrightarrow \alpha > \frac{2d}{3} - 1, \\ \text{but } \alpha < \frac{d}{2} \text{ and } \frac{2d}{3} - 1 < \frac{d}{2} \Leftrightarrow d < 6. \end{array}$$

We know $n_1 = n_2 = 1$ and so we are only interested in the case when $d - 3 > 2$, that is $d > 5$. In which case $n_{d-3} = 1$ if $\alpha < d/3$, and 0 if $\alpha > d/3$, so we require $\alpha < d/3$. Suppose that $\alpha < d/(j-1)$, for some $3 < j < d-2$, then

$$\left\langle \frac{(d-j)\alpha}{2d} \right\rangle = \begin{cases} \frac{d-j\alpha}{2d} & \text{if } \alpha < d/j; \\ \frac{3d-j\alpha}{2d} & \text{if } \alpha > d/j. \end{cases}$$

Since $\alpha < d/(j-1)$, we see that $3d - j\alpha > 0$. Now

$$\frac{d-j}{2d} + \frac{3d-j\alpha}{2d} < 1, \text{ for } \alpha > \frac{d}{j}, \quad \Leftrightarrow \quad \alpha > \frac{2d}{j} - 1.$$

However, $\alpha < \frac{d}{j-1}$ and so $\frac{2d}{j} - 1 < \frac{d}{j-1}$ if and only if $d\frac{j-2}{j(j-1)} < 1$, if and only if $\frac{d}{j-1} < \frac{j}{j-2} \leq 2$. Note that $\frac{2d}{j} - 1 > 1$ and so there is no integer k, such that $\frac{2d}{j} - 1 < k < \frac{d}{j-1}$. Thus under the above hypothesis, that is $\alpha < \frac{d}{j-1}$,

$$n_{d-j} = \begin{cases} 1 & \text{if } \alpha < d/j; \\ 0 & \text{if } \alpha > d/j. \end{cases}$$

If $\alpha > 1$, then $\exists\ j$, $2 \leq j \leq d-3$ such that $d/j < \alpha < d/(j-1)$, in which case $n_{d-j} = 0$, unless $d/(d-3) \geq 2$ or $d \leq 6$. We know $d \geq 3$. When $d = 3$, φ is essentially unique and $\alpha = 1$. In the two cases, $d = 4$ and $d = 5$, there are essentially two epimorphisms from Γ to $\mathbf{Z}_{2d}$. In each case one maps x_1 and x_2 to the same element, (1 if you like). The other does not and in these cases the fixed points of h can be seen to be Weierstrass points.

In general, since $\varphi(x_1) = \xi_1 = 1$, the gaps at the fixed point of h associated to x_1, are precisely the k for which $n_k = 1$ in $R(h)$. Thus, if $\alpha \neq 1$ the fixed point of h associated to x_1 is certainly a Weierstrass point. Furthermore, if we follow φ by the automorphism of $\mathbf{Z}_{2d}$ that maps α to 1 and 1 to $\bar{\alpha}$, where $\alpha\bar{\alpha} \equiv 1 \pmod{2d}$, we can apply the same arguement to show that the fixed point of h associated to x_2 is also a Weierstrass point provided $\bar{\alpha} \neq 1$. This process is equivalent to looking at the eigenvalues of $R(h^{\alpha})$.

Consider $\Lambda_3[d,d,d,d]$; two of its classes of elliptic elements are conjugate to x_3, one to x_2 and the other to x_1. In fact the Reidemeister-Schreier method shows that x_3, $x_3^{x_1}$, x_1^2 and x_2^2 are canonical generators for Λ_3. Thus $\varphi(x_3) = \varphi(x_3^{x_1}) = \beta$, $\varphi(x_1^2) = 2$ and $\varphi(x_2^2) = 2\alpha$. If we restrict φ to Λ_3 and follow it by the isomorphism from L_3 to $\mathbf{Z}_d$ that maps β to 1, then x_3, $x_3^{x_1}$ map to 1 and x_1^2, x_2^2 map to some τ and λ, and $1+1+\tau+\lambda = d$ or $2d$. In the first case, for $R(h^2)$, $n_1 = 2$ and so $\epsilon^{d+1} = \epsilon^{g+2}$ ($\epsilon := e^{2\pi i/d}$), is an eigenvalue of $R(h^2)$. Therefore, $g+2$ is a gap at the points fixed by $\langle h^2 \rangle$ alone. In the second case, $\tau = \lambda = d-1$ and $n_k = 3 - k/d - k/d - (d-k)/d - (d-k)/d = 1$ for $0 < k < d$, and so we can not say whether the fixed points of h^2 are Weierstrass points or not. Of course if $\tau = \lambda = d-1$, then $2\varphi(x_1) \equiv 2\varphi(x_2) \equiv (d-1)\varphi(x_3) \pmod{2d}$, and so $\varphi(x_2) \equiv \varphi(x_1)$ or $\varphi(x_1) + d \pmod{2d}$. If $\varphi(x_2) = d+1$, then $\beta = d-2$ and so d must be even, but $2\varphi(x_2) \equiv (d-1)\varphi(x_3) \pmod{2d}$ implies $2d+2 \equiv (d-1)(d-2) \equiv d+2 \pmod{2d}$.

In conclusion, when the images of the two canonical generators of order $2d$ of Γ, are not the same under the epimorphism φ, then all the geometric points are Weierstrass points. It can easily be seen that $\varphi(x_1) = \varphi(x_2)$ is a sufficient condition for φ to "exetnd" to $\varphi : \Delta[2d, 2d, 2] \mapsto \mathbf{Z}_{2d} + \mathbf{Z}_2$, where $|\Delta : \Gamma| = 2$. We have already considered this case and know the Weierstrass points to be the edge centres of the associated regular map. These points are not geometric points of the hypermap, see case two of the Appendix.

b) When $b = 1$, Γ has signature $[d,d,d]$ (d odd), and the genus of the surface is given by $g = (d-1)/2$. We may assume that $\varphi(x_1) = 1$, $\varphi(x_2) = \alpha$ and $\varphi(x_3) = \beta$, where $(\alpha, d) = (\beta, d) = 1$ and $1 + \alpha + \beta = d$. As before, for $R(h)$, $n_0 = 0$ and $n_1 = 1$. Now $n_2 = 0$ if and only if $\alpha = \beta = (d-1)/2$, in this case $\xi_1 = 1$ and so 2 is a non-gap at the fixed point of h associated to x_1. Thus the surface is hyperelliptic. In fact, φ mapping any two of the x_i's onto the same element of $\mathbf{Z}_d$ is precisely the condition for φ to "extend" to $\Delta[2,d,2d]$. In this case $\varphi(\Delta) \simeq \mathbf{Z}_{2d}$ and we have already seen that the Weierstrass points of the surface are the edge centres and face centre (or vertex) of the associated regular *map*. By the construction of fundamental regions for Γ, see Appendix, only one of the geometric points of the regular hypermap will be a Weierstrass point.

We now assume $1 < \alpha < (d-1)/2$ and so exclude the above case. Recall that,

$$n_k = 2 - \left\langle \frac{k}{d} \right\rangle - \left\langle \frac{k\alpha}{d} \right\rangle - \left\langle \frac{k\beta}{d} \right\rangle.$$

For each k there is some λ, $0 \leq \lambda \leq k$, such that $(\lambda-1)d/k < \alpha < \lambda d/k$, else $(\alpha, d) > 1$. Therefore $\langle k\alpha/d \rangle = (k\alpha - (\lambda-1)d)/d$, and so $\langle k/d \rangle + \langle k\alpha/d \rangle > 1$ if and only if $\alpha > \lambda d/k - 1$. Thus $n_k = 0$ if and only if there is a λ, $1 \leq \lambda \leq k$, such that $\lambda d/k - 1 < \alpha < \lambda d/k$. Note that if for a particular λ, k divides λd, then clearly α is not in the interval $(\lambda d/k - 1, \lambda d/k)$, so we need only consider those $\lambda \leq k-1$.

Lemma 6.12

Let d be an odd positive integer and let $a \in \mathbf{Z}$ be such that $1 < a < \frac{d-1}{2}$ and $(a, d) = 1$. Then there is a pair k, $\lambda \in \mathbf{Z}$, $1 < k \leq \frac{d-1}{2}$, $1 \leq \lambda \leq k-1$ such that a is in the open interval $(\frac{\lambda d}{k} - 1, \frac{\lambda d}{k})$.

Proof First we prove the result for even a. If $k = \frac{d-1}{2}$, then $\frac{d}{k} = 2\frac{2}{d-1}$ and $2\lambda < \frac{\lambda d}{k} < 2\lambda + 1$ for $1 \leq \lambda < k$. Hence, if a is even, then, when $k = \frac{d-1}{2}$ and $\lambda = \frac{a}{2}$, $a \in (\frac{\lambda d}{k} - 1, \frac{\lambda d}{k})$.

Now we need to prove the result for odd a. If $\lfloor\ \rfloor$ denotes the integer part, then $\lfloor \frac{\lambda d}{k} \rfloor = 2\lambda + \lfloor \frac{\lambda(d-2k)}{k} \rfloor$ and $\lfloor \frac{\lambda(d-2k)}{k} \rfloor = 1$ if and only if $1 \leq \frac{\lambda(d-2k)}{k} < 2$, if and only if $\frac{\lambda d}{2\lambda+2} < k \leq \frac{\lambda d}{2\lambda+1}$. There will certainly be an integer in the interval $(\frac{\lambda d}{2\lambda+2}, \frac{\lambda d}{2\lambda+1}]$ if $\frac{\lambda d}{2\lambda+1} - \frac{\lambda d}{2\lambda+2} \geq 1$, and this is so if and only if $\lambda \leq \frac{d-6}{4} - \frac{1}{2\lambda}$. Thus, for $\lambda > 1$, $\frac{1}{2\lambda} \leq \frac{1}{4}$ and so $\frac{d-6}{4} - \frac{1}{2\lambda} \leq \frac{d-7}{4}$, provided $d \geq 15$. Thus for $1 < \lambda \leq \frac{d-7}{4}$ there is a k such that $\lfloor \frac{\lambda d}{k} \rfloor = 2\lambda + 1$, note that $\frac{\lambda d}{k} = 2\lambda + 1$ only if $2\lambda + 1$ is not coprime to d.

We do require $k > \lambda$ and so we must check that at least one of the integers in $(\frac{\lambda d}{2\lambda+2}, \frac{\lambda d}{2\lambda+1}]$ is greater than λ. Therefore, $\lambda < \frac{\lambda d}{2\lambda+2}$ if and only if $2\lambda + 2 < d$ which is clearly always the case for the k we are constructing.

Also note that if $d > 5$, then $(\frac{d}{4}, \frac{d}{3}]$, $(\lambda = 1)$, always contains an integer greater than one. Hence, if $d > 5$, then there is an integer $k > 1$, such that $3 = \lfloor \frac{d}{k} \rfloor = 2.1 + 1$. Thus if a is any positive integer coprime to d such that $1 < a < \frac{d-3}{d}$, then for $\lambda = \frac{a-1}{2}$ and $k \in (\frac{\lambda d}{2\lambda+2}, \frac{\lambda d}{2\lambda+1}]$, we have shown that $a \in (\frac{\lambda d}{k} - 1, \frac{\lambda d}{k})$.

We have proved the lemma except when $\frac{d-3}{2}$ is odd and coprime to d, and when $d < 15$. Hence, when $k = \frac{d-3}{2}$ and $\lambda = \frac{d-5}{4}$, we see that $\frac{\lambda d}{k} - 1 < \frac{d-3}{2} < \frac{\lambda d}{k}$ unless $d < 9$. It only remains to verify the result for $d < 15$ which is left as an exercise for the reader. $\square$

The above lemma tells us that for each $1 < \alpha < \frac{d-1}{2}$, there is a $1 < k < \frac{d-1}{2} = g$ such that, for $R(h)$, $n_k = 0$. Hence the fixed point of h associated to x_1 is a Weierstrass point. By considering $R(h^{\xi_3})$ and $R(h^{\xi_3})$ we can see that when no two of $\varphi(x_1)$, $\varphi(x_2)$ and $\varphi(x_3)$ are equal, then the three fixed points of h are all Weierstrass points. [Incidentally the above lemma helps to prove that if an automorphism of a compact Riemann surface fixes exactly three points and its powers fix no additional points, then either one or three of the fixed points are Weierstrass points.]

We now determine when the geometric points of a regular hypermap, whose automorphism group is non-cylic abelian, are also Weiestrass points. Let

$$\varphi : \Gamma[m_1, m_2, m_3] \longmapsto G \simeq \mathbf{Z}_M + \mathbf{Z}_f$$

be a surface kernel epimorphism where M is the lowest common multiple of the m_i's and $f > 1$ is a factor of the highest common factor d, of the m_i's. Let S be the surface that is uniformized by the kernel of φ, so S carries a regular hypermap of type $\{m_1, m_2, m_3\}$ whose automorphism is G and has genus $g > 1$. Note that G is generated by any two of $\varphi(x_1)$, $\varphi(x_2)$ and $\varphi(x_3)$. Hence, if $L_i = \langle\varphi(x_i)\rangle$, then $|G : L_i| = fb_i$ and the order of the intersection $L_i \cap L_j$, is hb_k, where $h := d/f$ and i, j, $k \in \{1, 2, 3\}$ are mutally distinct. Therefore $\varphi(x_j), \varphi(x_j)^2, \ldots, \varphi(x_j)^{fb_i}$ $(j \neq i)$, are L_i coset representitives. The action of Γ on the $\Lambda_i := \varphi^{-1}(L_i)$ cosets is given by

$$x_1 \mapsto (1)(2)\ldots(fb_i) \qquad x_2 \mapsto (1\,2\ldots fb_i) \qquad x_3 \mapsto (1\,fb_i\ldots 2).$$

Thus Λ_i has signature $[m_i^{(fb_i)}, hb_k, hb_j]$.

We must now consider when the signature of Λ_i appears in (6.9). Clearly, Λ_i cannot have signature of the form $[\frac{g}{m_i}; m_i, m_i]$, as the genus of $\mathcal{U}/\Lambda_i$ is zero.

Note that $hb_k \neq m_i \neq hb_j$, as $h < d$, and so Λ_i only has signature of the form $[\frac{g-m_1+1}{m_1} : m_1^{(4)}]$ if $fb_i = 4$ and $hb_jb_k = 1$. In which case $f = d$ and $f = b_i = 2$ or $f = 4$ and $b_i = 1$. However $[2, 4, 4]$ is not the signature of a Fuchsian group and so the only possibility is $\Gamma[4, 4, 4]$; Λ_1, Λ_2 and Λ_3 will all have signature $[4, 4, 4, 4]$. It can easily be seen that the twelve fixed points in all, of $\varphi(x_1)$, $\varphi(x_2)$ and $\varphi(x_3)$, are all Weierstrass points of weight two. The genus is three and so the twelve geometric points of the hypermap are precisely the Weierstrass points. In fact the surface here corresponds to the fermat curve of degree four; we shall say more about this later.

If Λ_1 has signature of the form $[\tilde{g}; m_1, N_1, N_2]$, then

(i) $fb_1 = 3$ ($\Rightarrow f = 3$, $b_1 = 1$) and $hb_2 = hb_k = 1$, but then Γ would have signature $[3, 3, 3]$ which is not Fuchsian. Or

(ii) $fb_1 = 2$ ($\Rightarrow f = 2$, $b_1 = 1$) and one of hb_2, hb_3 is one while the other is greater than one. Hence $h = 1$, so $d = f = 2$ and we assume that $b_2 = 1$, $b_3 > 1$. In which case Γ has signature $[2b_3, 2b_3, 2]$, $G \simeq \mathbf{Z}_{2b_3} + \mathbf{Z}_2$ and Λ_1 has signature $[2b_3, 2b_3, b_3]$, as does Λ_2. We have shown, in (6.11), that the corresponding surface is hyperelliptic and that the Weierstrass points are precisely the edge centres of the regular map on the surface, and so the face centres and vertecies are not Weierstrass points.

Thus we have shown that all the geometric points of regular hypermaps, whose automorphism groups are non-cyclic abelian, are Weierstrass points unless the hypermap is of type $\{2b, 2b, 2\}$ (and so is a map, its automorphism group is isomorphic to $\mathbf{Z}_{2b} + \mathbf{Z}_2$), in which case the Weierstrass points form a subset of the geometric points. We summarize our findings in the next theorem.

Theorem 6.13

The geometric points of a regular hypermap with abelian automorphism group are *all* Weierstrass points unless:

(i) The hypermap is a *map* of type $\{b, 2b\}$ b odd, and its automorphism group is isomorphic to $\mathbf{Z}_{2b}$, or it is a *map* of type $\{2b, 2b\}$ and its automorphism group is isomorphic to $\mathbf{Z}_{2b} + \mathbf{Z}_2$. In both cases the underlying surfaces are hyperelliptic and the Weierstrass points form a proper subset of the geometric points of the maps.

(ii) The hypermap is of type $\{d, d, d\}$, with automorphism group isomorphic to $\mathbf{Z}_d$, and lies on a hyperelliptic surface. In this case only one of the geometric points is a Weierstrass point.

(iii) The hypermap is of type $\{d, 2d, 2d\}$, with automorphism group isomorphic to $\mathbf{Z}_{2d}$, and lies on a hyperelliptic surface. In this case none of the geometric points are Weierstrass points.

Remark In cases (ii) and (iii) there are hypermaps of the same type and with isomorphic automorphism groups that do *not* lie on hyperelliptic surfaces. The geometric points of these hypermaps are all Weierstrass points.

Now we ask when the set of geometric points of a regular hypermap, whose automorphism group is abelian, contains *all* the Weierstrass points of the underlying surface. Note that the number of Weierstrass points is at least $2g+2$, and so we need only consider hypermaps with at least $2g+2$ geometric points.

As always we look at the cyclic case first. These hypermaps correspond to a surface kernel epimorphisms from some $\Gamma[m_1, m_2, m_3]$ to $\mathbf{Z}_M$, where, in terms of our earlier notation, $M = db_1b_2b_3$. We let σ denote the number of geometric points of the hypermap, and so $\sigma = M/m_1 + M/m_2 + M/m_3 = b_1 + b_2 + b_3$. The Riemann-Hurwitz formula tells us that $M - \sigma = 2g - 2$. We are only interested in the case when $\sigma \geq 2g+2$ which implies $M - \sigma = 2g - 2 \leq \sigma - 4$, and so want $M + 4 \leq 2\sigma$ or equivalently

$$db_1b_2b_3 + 4 \leq 2(b_1 + b_2 + b_3).$$

We now assume this to be true and determine what restrictions this implies. Without lose of generality we also assume that $b_1 \leq b_2 \leq b_3$.

(i) Let $d = 1$, then $b_1b_2b_3 + 4 \leq 2(b_1 + b_2 + b_3)$.

1) $b_1 = 1 \Rightarrow b_2b_3 + 2 \leq 2(b_2 + b_3)$.

$\Rightarrow$ Possible solutions are: $(b_1, b_2, b_3) = (1, 1, n),\ n \geq 1;$
$(1, 2, n),\ n > 2;$
$(1, 3, 4).$

2) $b_1 > 1$, $\quad b_1b_2b_3 + 4 - 2(b_1 + b_2 + b_3) = b_1[b_2b_3 - \frac{2b_2}{b_1} - \frac{2b_3}{b_1} + \frac{4}{b_1} - 2]$

$$= b_1[(b_2 - \tfrac{2}{b_1})(b_3 - \tfrac{2}{b_1}) + \tfrac{4}{b_1} - 2 - \tfrac{4}{b_1^2}]$$

If $b_1 > 1$, then $2 < b_2 < b_3$ and $-2 < \frac{4}{b_1} - 2 - \frac{4}{b_1^2} < 0$, while $(b_2 - \frac{2}{b_1})$, $(b_3 - \frac{2}{b_1}) > 0$. Thus there are no solutions for $b_1 > 1$.

(ii) Let $d = 2$, then $b_1b_2b_3 + 2 \leq b_1 + b_2 + b_3$.

1) $b_1 = 1 \Rightarrow b_2b_3 + 1 \leq b_2 + b_3$.

$\Rightarrow$ Only possible solution is: $(b_1, b_2, b_3) = (1, 1, n),\ n \geq 1.$

[As $b_2b_3 + 1 - b_2 - b_3 = (b_2 - 1)(b_3 - 1) > 0$ unless b_2 or b_3 is one.]

2) $b_1 > 1$, $\quad b_1b_2b_3 + 2 - (b_1 + b_2 + b_3) = b_1[(b_2 - \frac{1}{b_1})(b_3 - \frac{1}{b_1}) + \frac{2}{b_1} - 1 - \frac{1}{b_1^2}]$.

As $1 < b_1 < b_2 < b_3$, we see that $-1 < \frac{2}{b_1} - 1 - \frac{1}{b_1^2} < 0$ and $(b_2 - \frac{1}{b_1})$, $(b_3 - \frac{1}{b_1}) > 0$. Hence, when $b_1 > 1$ there are no solutions.

(iii) Let $d > 2$.

$$db_1b_2b_3 + 4 - 2(b_1 + b_2 + b_3) = db_1[b_2b_3 - \tfrac{2b_2}{db_1} - \tfrac{2b_3}{db_1} + \tfrac{4}{db_1} - \tfrac{2}{d}]$$

$$= db_1[(b_2 - \tfrac{2}{db_1})(b_3 - \tfrac{2}{db_1}) + \tfrac{4}{db_1} - \tfrac{2}{d} - \tfrac{4}{(db_1)^2}]$$

Let $k = \frac{4}{db_1} - \frac{2}{d} - \frac{4}{(db_1)^2}$,

then $-\frac{2}{3} \leq -\frac{2}{d} < k < \frac{4}{db_1} \leq \frac{4}{3b_1}$.

If $b_1 = 1$, then $[\quad] \geq (1-\frac{2}{d})^2 + \frac{2}{d} - \frac{4}{d^2} = 1 - \frac{2}{d} > 0$.

If $b_1 > 1$, then $2 < b_2 < b_3$ and $-\frac{2}{3} < k < \frac{2}{3}$,

while $(b_2 - \frac{2}{db_1})$, $(b_3 - \frac{2}{db_1}) > \frac{8}{3}$.

Hence there are no solutions when $d > 2$.

Therefore the only instances when $\sigma \geq 2g+2$ are:

a) $d = 1$, $b_1 = b_2 = 1$ and $b_3 = n$, $n \geq 1$;

b) $d = 1$, $b_1 = 1$, $b_2 = 2$ and $b_3 = n$, $n > 2$ and $(n, 2) = 1$;

c) $d = 1$, $b_1 = 1$, $b_2 = 3$ and $b_3 = 4$;

d) $d = 2$, $b_1 = b_2 = 1$ and $b_3 = n$, $n \geq 1$.

The first case gives rise to a non-Fuchsian signature.

In the second case Γ has signature $[2n, n, 2]$, so we have a regular map whose automorphism group is cyclic. We have covered this in (6.11) and the Weierstrass points are a subset of the geometric points.

In (c) Γ has signature $[12, 4, 3]$ and the epimorphism to $\mathbf{Z}_{12}$ is essentially unique. It is not hard to see that the total weight of the geometric points is twelve, half the total weight of the Weierstrss points.

Finally, in (d), Γ has signature $[2n, 2n, 2]$ and again this is covered in (6.11). The Weierstrass points are precisely the geometric points.

We now treat the non-cyclic abelian case. Given a surface kernel epimorphism from $\Gamma[m_1, m_2, m_3]$ to $\mathbf{Z}_M + \mathbf{Z}_f$, the number of geometric points of the associated hypermap is now $\sigma = fM/m_1 + fM/m_2 + fM/m_3 = f(b_1 + b_2 + b_3)$. We may use the Riemann-Hurwitz formula to show we are only interested in hypermaps for which $fM + 4 \leq 2\sigma$, and so we assume that this is true and that $1 \leq b_1 \leq b_2 \leq b_3$. Note that $2 \leq f \leq d$.

(i) Let $d = 2$ ($\Rightarrow f = 2$) $\Rightarrow b_1 b_2 b_3 + 1 \leq b_1 + b_2 + b_3$

1) $b_1 = 1 \Rightarrow b_2 b_3 \leq b_2 + b_3$, only solutions are $b_1 = b_2 = 1$ and $b_3 \geq 1$.

2) $b_1 > 1$ ($\Rightarrow 2 < b_2 < b_3$)

$b_1 b_2 b_3 + 1 - (b_1 + b_2 + b_3) = b_1[(b_2 - \frac{1}{b_1})(b_3 - \frac{1}{b_1}) + \frac{1}{b_1} - 1 - \frac{1}{b_1^2}]$

If $k = \frac{1}{b_1} - 1 - \frac{1}{b_1^2}$, then $-1 < k < -\frac{1}{2}$

while $(b_2 - \frac{1}{b_1})$, $(b_3 - \frac{1}{b_1}) > \frac{5}{4}$.

Hence there are no solutions in this case.

(ii) Let $d \geq 2$.

$db_1b_2b_3 - 2(b_1 + b_2 + b_3) + \frac{4}{f} = db_1[(b_2 - \frac{2}{db_1})(b_3 - \frac{2}{db_1}) + \frac{4}{fdb_1} - \frac{2}{d} - \frac{4}{d^2b_1^2}]$

1) $b_1 = 1 \Rightarrow [\quad] < 0$, if and only if $db_2b_3 - 2(b_2 + b_3 + 1) + \frac{4}{f} < 0$,

if and only if $d = 3$ ($\Rightarrow f = 3$) and $b_1 = b_2 = b_3 = 1$, or

$d = 3$ ($\Rightarrow f = 3$) and $b_1 = b_1 = 1$ and $b_3 = 2$, or

$d = 4$ ($\Rightarrow f = 2$ or 4) and $b_1 = b_2 = b_3 = 1$, or

$d = 5$ ($\Rightarrow f = 5$) and $b_1 = b_2 = b_3 = 1$.

2) $b_1 > 1$. If $k = \frac{4}{fdb_1} - \frac{2}{d} - \frac{4}{d^2b_1^2}$, then $-\frac{1}{2} < k < \frac{1}{2}$,

but $2 < b_2 < b_3$ and so $(b_2 - \frac{2}{db_1})$, $(b_3 - \frac{2}{db_1}) > \frac{5}{2}$.

Hence there are no solutions in this case.

The instances for which $\sigma \geq 2g + 2$ are as follows.

a) $d = f = 2$, $b_1 = b_2 = 1$ and $b_3 = n$, $n \geq 1$.

b) $d = f = 3$, $b_1 = b_2 = b_3 = 1$.

c) $d = f = 3$, $b_1 = b_2 = 1$ and $b_3 = 2$.

d) $d = 4$, $f = 2$ and $b_1 = b_2 = b_3 = 1$.

e) $d = f = 4$ and $b_1 = b_2 = b_3 = 1$.

f) $d = f = 5$ and $b_1 = b_2 = b_3 = 1$.

The hypermaps that arise from (a) are in fact maps and so are covered in (6.11); the Weierstrass points are a subset of the geometric points.

The signature in (b) is not Fuchsian and so there are no hypermaps of this type with genus greater that one.

The signature in (c) is $[6, 6, 3]$ and the automorphism group of the hypermap is $\mathbf{Z}_6 + \mathbf{Z}_3$. The genus is four but it can be seen that the total weight of the geometric points is only 42.

In (d), Γ has signature $[4, 4, 4]$ but there is no surface kernel epimorphism from Γ to $\mathbf{Z}_4 + \mathbf{Z}_2$.

The underlying surfaces in (e) and (f) are the Fermat curves of degrees four and five respectively. We have met the degree four case earlier in this section and have seen that the Weierstrass points in that case are precisely the geometric points of the hypermap. The gaps at the geometric points on the hypermap on the degree five curve can be seen to be $\{1, 2, 3, 6, 7, 11\}$, and so the total weight of the geometric points is 120. The surface is of genus six and so the total weight of Weierstrass points is 210. However the automorphism group of a Fermat curve of degree $N \geq 4$, is of order $6N^2$ and the surface carries a regular map of type

$\{3, 2N\}$; the kernel of the essentially unique epimorphism from some $\Lambda[N, N, N]$ to $\mathbf{Z}_N + \mathbf{Z}_N$ that uniformizes the Fermat curve of degree N is also a normal subgroup of some $\Delta[2, 3, 2N]$ that contains Λ with index six. When $N \geq 5$ the edge centres of the map, (fixed points of automorphisms of order two), can also be seen to be Weierstrass points. For the degree five case the edge centres and face centres of the map constitute all the Weierstrass points, while for degree greater than five this is not the case. It is still not known whether the fixed points of automorphisms of order three (the vertices of the map), are Weierstrass points or not. Unfortunately, this is an example of the ambiguity that sometimes arises when using Lewittes' methods. For a survey of the results stated here about Fermat curves see [29]. Note that the geometric points of the hypermap are precisely the vertices of the map, see the third case in the Appendix.

These calculations prove the following.

Theorem 6.14

The set of geometric points of a regular hypermap whose automorphism group is abelian contains all the Weierstrass points of the underlying surface if and only if

(i) the hypermap is in fact a map, or

(ii) it is the hypermap of type $\{4, 4, 4\}$ whose automorphism group is isomorphic to $\mathbf{Z}_4 + \mathbf{Z}_4$. [The hypermap lies on the Fermat curve of degree four].

Section 6.4 Weierstrass points and Regular Maps of Low Genus

By the genus of a map we mean the genus of the underlying surface. In this section we look at the regular maps of genus two, three, four and five listed by Coxeter and Moser [10], Sherk [30] and Garbe [13]. In most cases we determine the weight of the geometric points of the maps. We do this by using the results of Lewittes and Harvey presented in §6.2, and by relying quite heavily on counting arguments. As we are considering maps of low genus the orders of their automorphism groups are relatively large compared to their genus and to the total weight of the Weierstrass points of the underlying surfaces. That is; $84(g-1)$ is large compared to $g^3 - g$ for low g.

We will see that sometimes a surface carries more than one regular map, this corresponds to a surface group being normal in two Fuchsian groups $\Gamma[2, m, n]$ and

$\Delta[2, p, q]$, where Δ contains Γ with finite index. When this is the case we may be able to determine the weight of the geometric points of the "smaller" map by calculating the weight of the geometric points of the "larger" map and using the fundamental regions constructed in the Appendix.

The results will be presented on the tables given in [10], [30] and [13], where the maps are denoted by their type and other "appendages". For example, we see $\{4, 10 \,|\, 2\}$ denotes a certain map of type $\{4, 10\}$ and genus four, and $\{4 \cdot 2, 8\}_4$ denotes a certain map of type $\{8, 8\}$ and genus five. The tables list the number of faces, vertices and edges, and we will usually indicate the weight of the the associated geometric points by adding indecies to these numbers. If the map is of type $\{m, n\}$, then G is generated by two elements r and s of orders n and m, whose product is of order two. Of course the underlying surface of the map is uniformized by the kernel of a epimorphism φ, from some $\Gamma[n, m, 2]$ to G that maps x, y and z, canonical generators of orders n, m and 2 for Γ, to r, s, $(rs)^{-1}$. Our tables will be incomplete in the sense that we shall only detail one map from each pair of dual maps, of course the face centres and vertices of the dual are the vertices and face centres of the original map, while the edge centres are the same in each case. So we are really only considering automorphism groups of regular maps.

Here is an example of a regular map of genus three.

Map Type	Number of Faces	Number of Vertices	Number of Edges	G	Order of G	
$\{2 \cdot 2, 2 \cdot 6\}$	2^3	6^3	12	$\langle 6, 2 \,|, 2 : 2 \rangle \simeq \mathcal{C}_4 \times \mathcal{D}_3$	24	H

The two face centres and six vertices all have weight three while the 12 edge centres are not Weierstrass points. Thus the underlying surface is hyperelliptic, indicated by the H on the right, and the Weierstrass points are precisely the face centres and vertices of the map.

Regular maps of genus two

It is known that all the compact Riemann surfaces of genus two are hyperelliptic and so they have a total of six Weierstrass points, each with weight one. The orders of the automorphism groups of the regular maps of genus two are all greater than six. Thus the Weierstrass points of the underlying surfaces of these maps are all geometric points with respect to these maps, as a non-geometric point is in an orbit of length greater than six under the map automorphism group. Hence we are looking for six of the geometric points in each case to be the Weierstrass points. Clearly each "kind" of geometric point, face centres, vertices or edge centres, all have the same weight. Therefore, we can easily see which of the geometric points are the Weierstrass points.

Map Type	Number of Faces	Number of Vertices	Number of Edges	G	Order of G
$\{8,8\}_{1,0}$	1^1	1^1	4^1	C_8	8
$\{5,10\}_2$	1^1	2	5^1	C_{10}	10
$\{6,6\}_2$	2	2	6^1	$C_6 \times C_2$	12
$\{4,8\}_{1,1}$	2^1	4^1	8	$\langle -2,4 \mid 2\rangle$	16
$\{4,6 \mid 2\}$	4	6^1	12	$(4,6 \mid 2,2)$	24
$\{3,4+4\}$	6^1	16	24	$\langle -3,4 \mid 2\rangle$	48

Regular maps of genus three

When the genus g, is three, then $g^3 - g = 24$. Half of the maps in the following table have automorphism groups of order greater than 24. This immediately tells us that, in these cases, the Weierstrass points are all geometric. In the last two cases, because of the large numbers of geometric points, we are able to see straight away the weight of these points. The first three maps listed have abelian automorphism groups, so lie on hyperelliptic surfaces and the proof of (6.11) tells us the weight of the geometric points in these cases. The remaining maps require some direct calculations.

For presentations of the groups in (i)-(v), (vii) and (x)-(xii) see [10], presentations for the remaining groups appear in [30]

iv) $G = \langle 2,2 \mid 2\rangle = \langle r,s \mid r^2 s^{-2} = (rs)^2 = 1\rangle$.

Now $(rs)^2 = 1 \Rightarrow srs^{-1} = r^{-1}s^{-2} = r^{-3} \Rightarrow r = s^2 r s^{-2} = r^9$

$\Rightarrow r^8 = s^8 = 1$.

So $G = \{1, r, \ldots, r^7, s, rs, \ldots, r^7 s\}$ and $r^i s = s r^{-3i}$.

If $L = \langle r \rangle$, then $|G : L| = 2$ and L coset representitives are 1 and s. Thus the action of G on the L cosets is given by

$$r \mapsto (1)(2), \qquad s \mapsto (1\,2), \qquad (rs)^{-1} \mapsto (1\,2).$$

Hence $\Lambda := \varphi^{-1}(L)$ has signature $[8,8,4]$. Canonical generators for Λ are r, $srs^{-1} = r^5$ and r^2, so for $R(r^{-1})$,

$$
\begin{aligned}
n_0 = 0 \qquad & n_1 = -1 + (1-\tfrac{1}{8}) + (1-\tfrac{5}{8}) + (1-\tfrac{2}{8}) = 1 \\
& n_2 = -1 + (1-\tfrac{2}{8}) + (1-\tfrac{2}{8}) + (1-\tfrac{4}{8}) = 1 \\
& n_3 = -1 + (1-\tfrac{3}{8}) + (1-\tfrac{7}{8}) + (1-\tfrac{6}{8}) = 0 \\
& n_4 = -1 + (1-\tfrac{4}{8}) + (1-\tfrac{4}{8}) + (1-\tfrac{8}{8}) = 0 \\
& n_5 = -1 + (1-\tfrac{5}{8}) + (1-\tfrac{1}{8}) + (1-\tfrac{2}{8}) = 1
\end{aligned}
$$

Thus the gaps at the two fixed points of r, the face centres, are $\{1,2,5\}$. By symmetry, these are also the gaps at the two vertices.

Map Type	Number of Faces	Number of Vertices	Number of Edges	G	Order of G	
$\{12,12\}_{1,0}$	1^3	1^3	6^3	$\mathcal{C}_{12}$	12	H
$\{7,14\}_2$	1^3	2	7^3	$\mathcal{C}_{14}$	14	H
$\{8,8\}_2$	2	2	8^3	$\mathcal{C}_8 \times \mathcal{C}_2$	16	H
$\{4\cdot 2, 4\cdot 2\}$	2^2	2^2	8^2	$\langle 2,2\|2\rangle$	16	
$\{2\cdot 2, 2\cdot 6\}$	2^3	6^3	12	$\langle 6,2\|2;2\rangle \simeq \mathcal{C}_4 \times \mathcal{D}_3$	24	H
$\{2\cdot 3, 2\cdot 3\}$	4^3	4^3	12	$\langle 3,3\|2;2\rangle \simeq \mathcal{A}_4 \times \mathcal{C}_2$	24	H
$\{2\cdot 2, 8\}$	4	8^3	16	$\langle\langle 2,8\|2;2\rangle\rangle \simeq (8,4\|2,2)$	32	H
$\{4, 4\cdot 2\}$	4^2	8^2	16	$\langle\langle 2,4\|2\rangle\rangle$	32	
$\{3, 4\cdot 3\}$	4^2	16^1	24	$\langle\langle 2,3\|3\rangle\rangle$	48	
$\{4, 2\cdot 3\}$	8^3	12	24	$\langle\langle 2,4\|3;2\rangle\rangle \simeq \mathcal{S}_4 \times \mathcal{C}_2$	48	H
$\{3,8\}_6$	12^2	32	48	$(2,3,8;3)$	96	
$\{3,7\}_8$	24^1	56	84	$(2,3,7;4) \simeq LF(2,7)$	168	

If $M = \langle rs \rangle$, then $|G:M| = 8$ and M coset representitives are $1, r, \ldots, r^7$. The action of G on the M cosets is

$$r \mapsto (1\,2\,\ldots\,8), \quad s \mapsto (1\,4\,3\,6\,5\,8\,7\,2), \quad rs \mapsto (1)(2\,6)(3)(4\,8)(5)(7).$$

Hence $\Lambda' := \varphi^{-1}(M)$ has signature $[1;2,2,2,2]$. Any surface kernel epimorphism from Λ' to $\mathbf{Z}_2$ maps all elliptic elements to 1. Thus, for $R(rs)$, $n_0 = 1$ and $n_1 = 2$, so the gaps at the edge centres are $\{1,2,3\}$ or $\{1,2,5\}$. This is rather inconclusive.

Fortunately $\Gamma[8,8,2]$ is not maximal, but is contained in a $[8,4,2]$. If the kernel of φ is also normal in a group with this signature, then the surface also carries a regular map of type $\{4,8\}$ whose automorphism group has order 32. So we need to see if cases (vii) or (viii) can be considered to be an "extension" of (iv). We shall now investigate all the possible extensions (or inclusions). We look at the inclusions in [32] and consider the signatures of the lifts of the map automorphism groups.

The map automorphism group G_5, in (v) lifts to some $\Gamma[12,4,2]$ and G in (i) lifts to a $[12,12,2]$. If x and y are canonical generators for G_5 of orders 12 and 4, then x, x^y and y^2 are canonical generators of some $\Lambda[12,12,2]$. Now

$$G_5 = \langle r, s \,|\, r^{12} = s^4 = (rs)^2 = [r, s^2] = [r^6, s] = 1 \rangle,$$

and if $r = \varphi(x_1)$, $s = \varphi(x_2)$, then $\varphi(x) = r$, $\varphi(x^y) = r^5$ and $s^2 = r^6$. Thus $\varphi(\Lambda) \simeq \mathbf{Z}_{12}$ and the underlying surfaces in (i) and (v) are conformally equivalent. This tells us that the map in (v) is hyperelliptic and that, by construction of fundamental regions for Γ and Λ, the two face centres and six vertices in (v) are the Weierstrass points.

From [32], each $[8,8,2]$ is contained with index two in a $[8,4,2]$, which in turn is contained in some $[8,3,2]$ with index three. Note that (iii) and (iv) are both of type $\{8,8\}$ and have automorphism groups of order 16. There are two regular maps of type $\{4,8\}$, (vii) and (viii), with automorphism groups of order 32, but only one of type $\{3,8\}$, (xi), with automorphism group of order 96. If $\Gamma[8,4,2]$ has canonical generators x and y of orders eight and four, then x, x^y and y^2 are canonical generators of a subgroup Λ, with signature $[8,8,2]$. In

$$G_7 = \langle r, s \,|\, r^8 = s^4 = (rs)^2 = (r^{-1}s)^2 = 1 \rangle,$$

we see that $[r, srs^{-1}] = 1$. Hence $\varphi(\Lambda)$ is abelian and so isomorphic to $\mathbf{Z}_8 + \mathbf{Z}_2$. Thus the underlying surfaces in (iii) and (vii) are conformally equivalent, so the map in (vii) lies on a hyperelliptic surface and the Weierstrass points are precisely the eight vertices of the map. We know the Weierstrass points in case (xi) have weight two and so the underlying surface is not hyperelliptic and not the one in (iii) and (vii). In

$$G_8 = \langle r, s \,|\, s^4 = (rs)^2 = [r^2, s] = 1 \rangle,$$

the subgroup generated by r and srs^{-1} is certainly not abelian but it is of order 16 and $r^2 = sr^2s^{-1}$. So we see that it is isomorphic to G_4. Therefore the maps in

(iv) and (viii) do lie on essentially the same surface. We now ask is it the surface in (xi)?

$$G_{11} = \langle r, s \,|\, r^8 = s^3 = (rs)^2 = (rsr^{-1}s^{-1})^3 = 1\rangle$$

We have already noted that G_{11} is the image of some $\Gamma[8,3,2]$, under a surface kernel epimorphism. Γ contains a subgroup of index three and signature $[8,4,2]$ whose canonical generators map to r, sr^2s^{-1} and $sr^{-1}s$ under this epimorphism. It can be seen that $H := \langle r, sr^2s^{-1}\rangle$, has order 32 and $[r^2, sr^2s^{-1}] = 1$. Hence H is isomorphic to G_8, so the maps in (iv), (viii) and (xi) all lie on essentially the same surface. The Weierstrass points in (xi) are the 24 face centres, by the construction of fundamental regions detailed in the Appendix, the Weierstrass points in (viii) are the four face centres and eight vertices, and in (iv) they are all the geometric points.

There is one more pair of maps, (vi) and (x), that are candidates for lying on the same surface.

$$G_{10} = \langle\langle 2,4 \,|\, 3;2\rangle\rangle = \langle r, s \,|\, r^6 = s^4 = (rs)^2 = [r^3, s] = 1\rangle, \qquad |G_{10}| = 48.$$

If $H = \langle r, srs^{-1}, s^2\rangle$, then H lifts to a $[6,6,2]$. Clearly $(srs^{-1})^3 = r^3$, and so $H \simeq G_6 = \langle 3,3 \,|\, 2,2\rangle$. Thus the maps in (x) and (vi) lie on essentially the same surface.

We have yet to determine the Weierstrass points in (vi) or (x), but once we have done so for one, then we should know them for both. It will then only remain to determine the Weierstrass points in (ix).

We will now calculate the weights of the geometric points in (vi). Recall that

$$G_6 = \langle r, s \,|\, r^6 = s^6 = (rs)^2 = r^3s^{-3} = [r^3, s] = [r, s^3] = 1\rangle,$$

and so, by symmetry, the weight of the face centres and vertices is the same. If $L = \langle r\rangle$, then $|G_6 : L| = 4$ and L must lift to a Fuchsian group Λ, with signature $[k : 6^{(\alpha)}, 3^{(\beta)}, 2^{(\gamma)}]$, where $\alpha > 0$. The Rieman-Hurwitz formula tells us that

$$2k - 2 + \alpha\tfrac{5}{6} + \beta\tfrac{2}{3} + \gamma\tfrac{1}{2} = 4(1 - \tfrac{1}{6} - \tfrac{1}{6} - \tfrac{1}{2}).$$

Hence the only possible signatures for Λ are $[6,3,3,2]$ and $[6,6,2,2]$. There is essentially only one surface kernel epimorphism from Λ to $\mathbf{Z}_6$ in each case. If Γ has the first signature then the eight face centres and vertices each have weight two, leaving a weight of eight to be found from the 12 edge centres or non-geometric points. Clearly this is not possible and so Γ has signature $[6,6,2,2]$, in which case it can be seen that the surface is hyperelliptic and the eight Weierstrass points are the face centres and vertices in (vi), and the eight face centres in (x). Alternatively we could have determined the signature of Λ by looking at the action of r and s on the L cosets.

Finally we look at case (ix). $G_9 = \langle r, s \mid s^3 = (rs)^2 = [r^3, s] = 1\rangle$. If $M = \langle r\rangle$ and Δ is the lift of M, then Δ has signature of the form

$$[k; 12^{(\alpha)}, 6^{(\beta)}, 4^{(\gamma)}, 3^{(\delta)}, 2^{(\epsilon)}],$$

where $\alpha > 0$ and

$$2k - 2 + \alpha\tfrac{11}{12} + \beta\tfrac{5}{6} + \gamma\tfrac{3}{4} + \delta\tfrac{2}{3} + \epsilon\tfrac{1}{2} = 4(1 - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{12}).$$

The only possible signatures for Δ are $[12,12,2]$ or $[12,4,3]$. From [32] we know it can not be the first. There is a unique surface kernel epimorphism from $\Delta[12,4,3]$ to $\mathbf{Z}_{12}$ and the fixed point of r can be seen to have weight two. Hence the four face centres all have weight two and this implies that the 16 vertices are the remaining Weierstrass points and have weight one.

This completes our analysis of the genus three case.

Regular maps of genus four

We know that if G is the automophism group of a regular map of type $\{m, n\}$, then it is generated by a pair r and s of orders n and m, whose product is of order two. If the genus of the map is greater than one then there are other defining relations for G. Garbe specifies the automorphism groups of the maps in his lists by giving the necessary extra defining relations. Again the first three maps in the following table have abelian automorphism groups and so we know the weight of the geometric points from (6.11). For $g = 4$, $g^3 - g = 60$, so the Weierstrass points of the underlying surfaces in cases (ix), (xi) and (xii) must *all* be geometric with respect to these maps. For regular maps of genus two and three we have seen that the sets of geometric points always contain the Weierstrass points of the underlying surface. We can see from the table below that this is not true for regular maps of genus four. The total weight of the geometric points in cases (iv), (vii) and (x) is less than 60, indeed in (x) none of the geometric points are Weierstrass points.

However, we shall show that the underlying surfaces in these cases carry another regular map with respect to which the Weierstrass points are *all* geometric.

Map Type	Number of Faces	Number of Vertices	Number of Edges	Extra Relation	Order of G	
$\{16,16\}_{1,0}$	1^6	1^6	8^6	r^7s^{-1}	16	H
$\{9,18\}_2$	1^6	2	9^6	r^8s^{-1}	18	H
$\{10,10\}_2$	2	2	10^6	r^2s^2	20	H
$\{3\cdot 2, 3\cdot 4\}$	2^4	4^4	12^1	r^4s^{-2}	24	
$\{4,16\}$	2^6	8^6	16	r^8s^{-2}	32	H
$\{4,10\,\|\,2\}$	4	10^6	20	$(rs^{-1})^2$	40	H
$\{6,6\,\|\,2\}$	6^2	6^2	18	$(rs^{-1})^2$	36	
$\{3\cdot 2,6\}$	6^4	6^3	18^1	$[r^2,s]$	36	
$\{3,3\cdot 4\}$	6^4	24	36^1	$[r^4,s]$	72	
$\{5,5\,\|\,3\}$	12	12	30	$(rs^{-1})^3$	60	
$\{4,6\}_4$	12^2	18	36^1	$(r^2s^2)^2$	72	
$\{4,5\}_6$	24	30	60^1	$(r^2s^2)^3$	120	

First we determine when the underlying surfaces of different maps are conformal equivalent.

If $\Gamma[n,4,2]$ has canonical generators x and y of orders n and 4, then x, yxy^{-1} and y^2 are canonical generators of a subgroup with signature $[n,n,2]$ and index two.

If $\Delta[2n,3,2]$ has canonical generators x and y of orders $2n$ and 3, then x, yx^2y^{-1} and $yx^{-1}y$ are canonical generators of a subgroup with signature $[2n,n,2]$ and index three.

The map in (v) is of type $\{4,16\}$ and

$$G_5 = \langle r,s \,|\, r^{16} = s^4 = (rs)^2 = r^8s^{-2} = 1\rangle.$$

Let $H_5 = \langle r, srs^{-1}, s^2\rangle$. Since $srs^{-1} = r^7$ and $s^2 = r^8$, $H_5 \simeq \mathbf{Z}_{16} \simeq G_1$ and so the maps in (i) and (v) lie on essentially the same surface. Therefore the map in (v) is hyperelliptic and the Weierstrass points are the two face centres and eight vertices.

$$G_6 = \langle r, s \,|\, r^{10} = s^4 = (rs)^2 = (rs^{-1})^2 = 1 \rangle$$

If $H_6 = \langle r, srs^{-1}, s^2 \rangle$, then $|H_6| = 20$ and, as $[r, srs^{-1}] = 1$, H_6 is abelian. Hence $H_6 \simeq \mathbf{Z}_{10} + \mathbf{Z}_2 \simeq G_3$. Therefore the map in (vi) is also hyperelliptic and the Weierstrass points are the 10 vertices.

$$G_9 = \langle r, s \,|\, r^{12} = s^3 = (rs)^2 = [r^4, s] = 1 \rangle$$

If $H_9 = \langle r, sr^2s^{-1}, sr^{-1}s \rangle$, then $|H_9| = 24$ and, as $(sr^2s^{-1})^2 = r^4$, $H_9 \simeq G_4$.

$$G_{11} = \langle r, s \,|\, r^6 = s^4 = (rs)^2 = (r^2s^2)^2 = 1 \rangle$$

If $H_{11} = \langle r, srs^{-1}, s^2 \rangle$, then $|H_{11}| = 36$. Since $(r(srs^{-1})^{-1})^2 = 1$, $H_{11} \simeq G_7$ (not G_8).

$$G_{12} = \langle r, s \,|\, r^5 = s^4 = (rs)^2 = (r^2s^2)^3 = 1 \rangle$$

If $H_{12} = \langle r, srs^{-1}, s^2 \rangle$, then $|H_{12}| = 60$. Since $(r(srs^{-1})^{-1})^3 = 1$, $H_{12} \simeq G_{10}$.

We have shown that although there are 12 regular maps of genus four, they essentially lie on only seven surfaces. This helps us alot in determining the Weierstrass points, which we do now. We return to looking upon the map automorphism groups us images under a surface kernel epimorphism φ, from some Fuchsian group $\Gamma[n, m, 2]$, where x and y are canonical generators of Γ and $\varphi(x) = r$ and $\varphi(y) = s$.

(viii) The extra relation that defines G_8 is $[r^2, s] = 1$ and $|G_8| = 36$. Let $L = \langle r \rangle$, so $|G_8 : L| = 6$ and $1, s, \ldots, s^5$ are coset representitives. The action of G_8 on the cosets is

$$r \mapsto (1)(2\,6)(3\,5)(4), \qquad s \mapsto (1\,2\,3\,4\,5\,6), \qquad rs \mapsto (1\,2)(3\,6)(4\,5).$$

Hence L lifts to a Fuchsian group Λ, with signature $[6, 6, 3, 3]$ and has canonical generators x, y^3xy^{-3}, yx^2y^{-1} and $y^2x^2y^{-2}$. If we denote by ψ the restriction of φ to Λ followed by the epimorphism from Λ to $\mathbf{Z}_6$ that maps r to 1, then $\psi(x) = 1$, $\psi(y^3xy^{-3}) = 1$, $\psi(yx^2y^{-1}) = 2$ and $\psi(y^2x^2y^{-2}) = 2$. This is because $[r^2, s] = 1$

and so $sr^2s^{-1} = r^2$, $s^2r^2s^{-2} = r^2$. Hence for $R(r^{-1})$,

$$\begin{aligned} n_0 = 0 \qquad n_1 &= -1 + (1 - \tfrac{1}{6}) + (1 - \tfrac{1}{6}) + (1 - \tfrac{2}{6}) + (1 - \tfrac{2}{6}) = 2 \\ n_2 &= -1 + (1 - \tfrac{2}{6}) + (1 - \tfrac{2}{6}) + (1 - \tfrac{4}{6}) + (1 - \tfrac{4}{6}) = 1 \\ n_3 &= -1 + (1 - \tfrac{3}{6}) + (3 - \tfrac{1}{6}) + (1 - \tfrac{6}{6}) + (1 - \tfrac{6}{6}) = 0 \\ n_4 &= -1 + (1 - \tfrac{4}{6}) + (1 - \tfrac{4}{6}) + (1 - \tfrac{2}{6}) + (1 - \tfrac{2}{6}) = 1 \end{aligned}$$

Thus the gaps at the fixed points of r are $\{1, 2, 4, 7\}$ and so the weight of the six face centres is four. Further analysis of this kind shows that the weight of each of the vertices is three. We are lacking 18 from the total weight of all the points of the surface and so the edge centres must each have weight one.

(iv), (ix) The action of G_4 on the $\langle r\rangle$ cosets is clearly given by $r \mapsto (1)(2)$ and $s \mapsto (1\ 2)$. Thus $\Lambda := \varphi^{-1}(\langle r\rangle)$ has signature $[12, 12, 3]$. There is essentially only one surface kernel epimorphism from Λ to $\mathbf{Z}_{12}$, so it is easily verified that the weight of each of the two face centres is four. As the maps in (iv) and (ix) lie on the same surface, by looking at fundamental regions, we see that the two face centres and four vertices in (iv) together are the six face centres in (ix) and so all have weight four. Hence we are lacking in weight by 36. The order of G_9 is greater than 60 and so the 36 edge centres must have weight one and be the remaining Weierstrass points. Only 12 of the edge centres in (ix) are geometric points in (iv); the 12 edge centres, and so these have weight one.

(vii), (xi) The extra relation needed to define G_7 is $(rs^{-1})^2 = 1$ and so, by symmetry, the weight of the six face centres is the same as the weight of the six vertices. If $\Lambda = \varphi^{-1}(\langle r\rangle)$, then it can be shown that Λ has signature $[6, 6, 3, 3]$ and canonical generators x, y^3xy^{-3}, $y^4x^2y^{-4}$ and $y^5x^2y^{-5}$. Since $s^3rs^{-3} = r^5$, $s^4r^2s^{-4} = r^2$ and $s^5r^2s^{-5} = r^4$, the weight of each of the fixed points of r is two. Thus the total weight of the face centres and vertices is 24 and we are lackig 36 from the total weight. Hence the 18 edge centres have weight two or there are 36 non-geometric points with weight one, this is one orbit of points under G_7, each with trivial stabilizer. We need to look at case (xi) to know for sure. The extra relation in G_{11} is $(r^2s^2)^2 = 1$ and if $\Delta = \varphi^{-1}(\langle s\rangle)$, then, by looking at the actions of r and s on the $\langle s\rangle$ cosets, we see that Δ has signature $[1; 4, 4]$. Hence, for $R(s^{-1})$, $n_0 = n_1 = n_2 = n_3 = 1$. Therefore the gaps at the 18 vertices are $\{1, 2, 3, 4\}$ or $\{1, 2, 4, 7\}$. The latter implies that the weight of each of the edge centres is four, which clearly is not possible. Hence the vertices in (xi) are not Weierstrass points while the edge centres are and each has weight one. Thus the edge centres in (vii) are not Weierstrass points and there are 36 non-geometric points with respect to the map in (vii), each with weight one.

(x), (xii) When we look at the order of G_{12} and the numbers of geometric points in (xii) then it is clear that the 24 face centres are certainly not Weierstrass points, while the vertices have weight two and the edge centres weight zero or, the vertices have weight zero and the edge centres have weight one. Let Λ be the lift of $\langle s \rangle$, then Λ has signature $[k; 4^{(\alpha)}, 2^{(\beta)}]$ where $\alpha > 0$ is even and

$$2k - 2 + \alpha\tfrac{3}{4} + \beta\tfrac{1}{2} = 30(1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{5}) = \tfrac{3}{2}.$$

Hence the only possible signatures for Λ are $[1; 4, 4]$, $[4^{(4)}, 2]$ and $[4, 4, 2^{(2)}]$. There is essentially only one surface kernel epimorphism from a group with the last signature to $\mathbf{Z}_4$ and it is not difficult to see that were Λ to have this signature then the vertices would each have weight 6. There are two surface kernel epimorphisms from a group with the second signature to $\mathbf{Z}_4$, one would give the weight of the vertices to be one and the other three. Hence the signature of Λ must be $[1; 4, 4]$. So the gaps at the vertices are $\{1, 2, 3, 4\}$ or $\{1, 2, 4, 7\}$, our previous remarks tells us that they must be $\{1, 2, 3, 4\}$. Therfore the Weierstrass points in (xii) are the 60 edge centres, while the Weierstrass points in (x) are all non-geometric with respect to this map. This can be seen by constructing fundamental regions for some $\Gamma[5, 5, 2]$ from fundametal regions of some $\Delta[5, 4, 2]$ that contains Γ with index two.

This concludes our analysis of the regular maps of genus four.

Regular maps of genus five

We analyse the regular maps of genus five in much the same way as we did those of genus four: First determining when different maps lie on the same Riemann surface and then using these "inclusions" in our calculations. Again there are maps whose set of geometric points does not include all the Weierstrass points of the underlying surface; cases (iv), (vi) and (vii). The surfaces in (vi) and (vii) each carry another regular map with respect to which all the Weierstrass points are geometric. However we shall show that this is not the case in (iv). The asterisk on the number of edges in (iv) indicates that we are uncertain as to the weight of the edge centres, our methods only give us a range of possible weights for these geometric points. The range is such that the total weight of all the points on the surface, which is 120 for surfaces of genus five, can not be accounted for by just the geometric points of the map.

Map Type	Number of Faces	Vertices	Edges	Extra Relations	Order of G	
$\{20,20\}_{1,0}$	1^{10}	1^{10}	10^{10}	$r^9 s^{-1}$	20	H
$\{11,22\}_2$	1^{10}	2	11^{10}	$r^{10} s^{-1}$	22	H
$\{12,12\}_2$	2	2	12^6	$r^2 s^2$	24	H
$\{3\cdot 2, 3\cdot 5\}$	2^5	5^4	15^*	$r^{10} s^{-2}$	30	
$\{4,20\}_{1,1}$	2^{10}	10^{10}	20	$r^{10} s^{-2}$	40	H
$\{4\cdot 2,8\}_4$	4^5	4^5	16^5	$[r^2,s]$, $(r^2 s^2)^2$	32	
$\{8,8\}_4^{(r^2)^s=r^{-2}}$	4^3	4^3	16	$(r^2)^s r^2$, $(r^2 s^2)^2$	32	
$\{4,12\,\vert\,2\}$	4	12^{10}	24	$(rs^{-1})^2$	48	H
$\{6,6\}_{2,0}$	8^3	8^3	24^3	$(r^2 s^{-1})^2$, $(r^2 s^2)^2$	48	
$\{4,8\}^{[r^2,s^2]=1}$	8^5	16^5	32	$[r^2,s^2]$	64	
$\{4,8\,\vert\,4\}^{r^{-2}s^2r^2=s^{-2}}$	8^3	16	32^3	$(r^2)^{s^2} r^2$, $(rs^{-1})^4$	64	
$\{3,2\cdot 5\}$	12^{10}	40	60	$[r^5,s]$	120	H
$\{5,5\}_4$	16	16	40^3	$(r^2 s^2)^2$	80	
$\{4,6\}^{(r^2s^3)^2=1}$	16^3	24^3	48	$(r^3 s^2)^2$	96	
$\{3,8\}^{[s^2,r^{-1}s^4r]=1}$	24^5	64	96	$[r^2, s^{-1}r^4 s]$	192	
$\{4,5\,\vert\,4\}$	32	40^3	80	$(rs^{-1})^4$	160	

$$G_5 = \langle r,s \mid r^{20} = s^4 = (rs)^2 = r^{10}s^{-2} = 1\rangle$$

If $H_5 = \langle r, srs^{-1}, s^2\rangle$, then $|H_5| = 20$ and its lift has signature $[20,20,2]$. Since $srs^{-1} = r^9$ and $s^2 = r^{10}$, $H_5 \simeq \mathbf{Z}_{20} \simeq G_1$. Hence the map in (v) is hyperelliptic and the Weierstrass points are the two face centres and ten vertices.

$$G_8 = \langle r,s \mid r^{12} = s^4 = (rs)^2 = (rs^{-1})^2 = 1\rangle$$

If $H_8 = \langle r, srs^{-1}, s^2\rangle$, then $|H_8| = 24$. Since $[r, srs^{-1}] = 1$, H_5 is abelian and so $H_5 \simeq \mathbf{Z}_{12} + \mathbf{Z}_2 \simeq G_3$. Hence the map in (viii) is hyperelliptic and the 12 vertices are the Weierstrass points.

$$G_{15} = \langle r,s \mid r^8 = s^3 = (rs)^2 = [r^2, s^{-1}r^4 s] = 1\rangle$$

If $H_{15} = \langle r, sr^2s^{-1}, sr^{-1}s\rangle$, then $|H_{15}| = 64$. Since $[r^2, sr^4s^{-1}] = 1$, H_{15} is isomorphic to G_{10} (not G_{11}).

$$G_{11} = \langle r, s, \,|\, r^8 = s^4 = (rs)^2 = s^2r^2s^{-2}r^2 = (rs^{-1})^4 = 1\rangle$$

If $H_{11} = \langle r, srs^{-1}, s^2\rangle$, then $|H_{11}| = 32$.
Now, $(r^2(srs^{-1})^2)^2 = (srs^{-1})r^2(srs^{-1})^{-1}r^2 = 1$ but $[r^2, srs^{-1}] \neq 1$, so H_{11} is ismorphic to G_7 (not G_6).

$$G_{10} = \langle r, s, \,|\, r^8 = s^4 = (rs)^2 = [r^2, s^2] = 1\rangle$$

If $H_{10} = \langle r, srs^{-1}, s^2\rangle$, then $|H_{10}| = 32$. Now, $(r^2(srs^{-1})^2)^2 = [r^2, srs^{-1}] = 1$, so H_{10} is ismorphic to G_6.

$$G_{14} = \langle r, s \,|\, r^6 = s^4 = (rs)^2 = (r^3s^2)^2 = 1\rangle$$

If $H_{14} = \langle r, srs^{-1}, s^2\rangle$, then $|H_{14}| = 48$ and it can be seen that H_{14} is isomorphic to G_9.

$$G_{16} = \langle r, s \,|\, r^5 = s^4 = (rs)^2 = (rs^{-1})^4 = 1\rangle$$

If $H_{16} = \langle r, srs^{-1}, s^2\rangle$, then $|H_{16}| = 96$ and it can also be seen that H_{16} is isomorphic to G_{13}.

Thus we have shown that the 16 regular maps of genus five lie on essentially only nine surfaces. We know three of these are hyperelliptic and so we now only need to determine the Weierstrass points on the remaining six surfaces.

(iv) Note that the extra relation for G_4 in [13] is incorrect and should read $r^{10}s^{-2}$. The subgroup $L := \langle r\rangle$ has index two in G_4. The action of G_4 on the cosets is given by $r \mapsto (1)(2)$ and $s \mapsto (1\ 2)$. Hence $\Lambda := \varphi^{-1}(L)$ has signature $[15, 15, 3]$. There is essentially only one surface kernel epimorphism from Λ to $\mathbf{Z}_{15}$, this is described by $x_1 \mapsto 1$, $x_2 \mapsto 4$ and $x_3 \mapsto 10$, where x_1, x_2 and x_3 are canonical generators of Λ. Hence, for $R(r^{-1})$, $n_1 = n_2 = n_4 = n_5 = n_8 = 1$ and so the gaps at the two face centres are $\{1, 2, 4, 5, 8\}$. If $M = \langle s\rangle$, then $1, r, r^2, r^3, r^4$ are M coset representitives and the action of G_4 on the M cosets is given by

$$r \mapsto (1\ 2\ 3\ 4\ 5), \qquad s \mapsto (1)(2\ 5)(3\ 4), \qquad rs \mapsto (1\ 5)(2\ 4)(3).$$

Hence M lifts to a group with signature $[6, 3, 3, 3, 2]$, there is essentially only one surface kernel epimorphism from a group with this signature to $\mathbf{Z}_6$. Furthermore it can be seen that for $R(s^{-1})$, $n_1 = n_2 = n_4 = n_5 = n_7 = 1$ and so the weight of each the five vertices is four. Finally, $1, r, \ldots, r^{14}$ are representitives of the $\langle rs\rangle$ cosets and by looking at the action of r and s on these cosets we see that $\langle rs\rangle$ lifts to a group with signature $[2; 2, 2, 2, 2]$. Thus, for $R(rs)$, $n_0 = 2$ and $n_1 = 3$. This information fails to give us the gaps at the edge centres, it only tells us that the gap sequence at the edge centres is one of the following.

$$\{1,2,3,4,5\} \qquad \{1,2,3,4,7\} \qquad \{1,2,3,4,9\} \qquad \{1,2,3,5,6\}$$
$$\{1,2,4,5,7\} \qquad \{1,2,3,6,7\}$$

Hence we only know that the weight of each of the 15 edge centres is 0, 2 or 4. Therefore the total weight of the geometric points is 30, 60 or 90, and so there is at least one orbit of non-geometric Weierstrass points under G_4, possibly three. Note that any Fuchsian group with signature $[15, 6, 2]$ is maximal and so the underlying surface here admits no other regular maps.

(vi), (x), (xv) The extra relations to define G_6 are $[r^2, s] = (r^2s^2)^2 = 1$. If $L = \langle r\rangle$, then $1, s, s^2, s^3$ are L coset representitives and the action of G_6 on these cosets is given by

$$r \mapsto (1)(2\,4)(3), \qquad s \mapsto (1\,2\,3\,4), \qquad rs \mapsto (1\,2)(3\,4).$$

Thus L lifts to some $\Lambda[8, 8, 4, 2]$. Furthermore Λ has canonical generators that map to r, r, r^2 and r^4. Hence the weight of each face centre is five. The four face centres and four vertices in (vi) are the eight face centres in (x), and the 16 edge centres in (vi) are the 16 vertices in (x). Furthermore, the eight face centres and 16 vertices in (x) are the 24 face centres in (xv). Hence all the geometric points in (vi) have weight 5, as do the face centres and vertices in (x), and the face centres in (xv). So the Weierstrass points of the underlying surface of these maps are all geometric with respect to each map.

(vii), (xi) Let L denote the subgroup of G_7 generated by r, so L has order eight and $1, s, s^2, s^3$ are L coset representitives. The action of G_7 on these cosets is

$$r \mapsto (1)(2\,4)(3), \qquad s \mapsto (1\,2\,3\,4), \qquad rs \mapsto (1\,2)(3\,4).$$

This can be seen by analysing the group or by looking at coset graphs, as combinatorial arguments show L must lift to a group Λ, with signature $[8, 8, 4, 2]$. Thus Λ has canonical generators that map to r, $s^2rs^{-2} = r^5$, $s^3r^2s^{-3} = r^2$ and r^4. Hence, for $R(r^{-1})$, $n_1 = n_2 = n_3 = n_5 = n_7 = 1$ and so the gaps at the fixed points of r are $\{1,2,3,5,7\}$. The four face centres and four vertices in (vii) correspond to the eight face centres in (xi), and so these eight points each have weight three.

We now turn our attention to G_{11}. If $M := \langle s \rangle$ lifts to $\Delta[k; 4^{(\alpha)}, 2^{(\beta)}]$, then $\alpha > 0$ is even and

$$2k - 2 + \alpha\tfrac{3}{4} + \beta\tfrac{1}{2} = 16(1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8}) = 2.$$

Thus Δ has one of the following signatures $[1; 4, 4, 2]$, $[4^{(4)}, 2, 2]$ or $[4, 4, 2^{(5)}]$. There is a unique surface kernel epimorphism from a group with the first signature to $\mathbf{Z}_4$ and if Δ is such a group then our methods tell us that the weight of each of the 16 vertices is zero or four. There are essentially two surface kernel epimorphisms from a group with the second signature to $\mathbf{Z}_4$. One would imply the weight of the vertices to be three and the other would imply it to be five. Finally, there is a unique surface kernel epimorphism from a group with the last signature to $\mathbf{Z}_4$ and the associated surface is hyperelliptic. Therefore the weight of the vertices is either zero, three, four or five. Hence the total weight from the vertices and face centres in (xi) is either 24, 72, 88 or 104, leaving 96, 48, 32 or 16 to come from the 32 edge centres and non-geometric points. Hence the vertices must have weight zero or four so Δ has signature $[1; 4, 4, 2]$, this implies that the 32 edge centres have weight one or three.

Similarly we can show that $\langle rs \rangle$ must lift to a group with signature $[2; 2^{(4)}]$, or $[1; 2^{(8)}]$, or $[2^{(12)}]$. If it is the first, then the edge centres have weight zero, two or four. If it is the second, then the edge centres have weight three or five and it can not be the last as the associated surface would be hyperelliptic. Notice that one is not an option for the weight of the edge centres, so it must be three and this implies that the vertices, (which are the edge centres in (vii)), are not Weierstrass points.

(ix), (xiv) Let L denote the subgroup of G_{14} generated by r, so $|G_{14} : L| = 16$. The action of r, s and rs on the L cosets tells us that L lifts to a group with signature $[6, 6, 3, 2, 2]$. There is essentially only one surface kernel epimorphism from such a group to $\mathbf{Z}_6$ and calculations show that the weight of each of the face centres is three. It can also be seen that $\langle s \rangle$ lifts to a group with signature $[4^{(4)}, 2, 2]$. There are two distinct surface kernel epimorphisms from such a group to $\mathbf{Z}_4$, one implies the vertices have weight three and the other five. Hence the 16 face centres and 24 vertices each have weight three in (xiv), and these are all the

Weierstrass points of the surface. Thus all the geometric points in (ix) each have weight three.

The remaining cases can also be dealt with using coset actions and counting arguments.

Garbe has also given lists of all the regular maps of genus six and seven. However our methods prove to be unsucessful for almost half the maps of genus six and so we have not included such partial results.

Appendix

In [33] Singerman gave all inclusions between Fuchsian triangle groups. In this appendix we look at each of these. For each inclusion $\Gamma < \Lambda$, we will show that every subgroup of Λ, that is isomorphic to Γ, is in fact conjugate to Γ in Λ. This is done by showing that the coset graph in each case is unique up to a relabling of the vertices and this merely corresponds to conjugation in Λ. For some cases we determine canonical generators for Γ in terms of canonical generators of Λ and construct a fundamental region for Γ out of fundamental regions of Λ.

In all but one of the cases Λ is the canonical Fuchsian group of only one proper NEC group, an extended triangle group Λ_*. Γ is often contained in two proper NEC groups with index two and using [5] and [11] we see when these are contained in Λ_*. We have already shown that not every reflection in Λ_* necessarily extends Γ to a group containing it with index two, and now we ask: Is every reflection in Λ_* conjugate to a reflection that normalizes Γ? This is the case for most inclusions but not all. (The reason we construct fundamental regions in some cases is to determine which conjugacy classes of reflections in Λ_* are represented by reflections that normalize Γ.) This question is closely related to the following question: Given a large group of automorphisms G, of a compact Riemann surface, are all the symmetries of this surface conjugate to a symmetry of the first or second kind with respect to G? Again the answer is "usually". However there are interesting instances when this is not the case, providing exceptional surfaces. The results here are used in chapters 4 and 5 to find all symmetries with non zero species of Riemann surfaces with large cyclic and non-cyclic abelian groups and hence to determine the symmetry type of these surfaces.

Suppose Λ_* is an extended triangle group with signature (l, m, n) and presentation

$$\Lambda_* = \langle a, b, c \,|\, a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ac)^n = 1 \rangle.$$

If l is odd, then $(ab)^l = (ab)^{\frac{(l-1)}{2}} a(ba)^{\frac{(l-1)}{2}} b = 1$ and so $(ab)^{\frac{(l-1)}{2}} a(ba)^{\frac{(l-1)}{2}} = b$. Therefore a and b are conjugate in Λ_*.

Now consider the epimorphism θ, from Λ_* to

$$H := \langle r, s, t \,|\, r^2 = s^2 = t^2 = (rs)^l = (st)^m = (rt)^n = [r, s] = [s, t] = [r, t] = 1 \rangle,$$

defined by $\theta(a) = r$, $\theta(b) = s$ and $\theta(c) = t$. Here $[r, s]$ denotes the commutator of r and s. We see that if l is odd, then $rs = 1$ and so $r = s$. H is abelian and so r is conjugate (equal) to s if and only if l is odd or m and n are both odd. Our previous calculations show that under these circumstances a is conjugate to b in Λ_*. Clearly a can only be conjugate to b, in Λ, if r is conjugate to s in H. Hence a is conjugate to b if and only if l is odd or m and n are both odd. Thus Λ_* has three conjugacy classes of reflections if l, m and n are all even, two if one is odd and the others even and one if two or more are odd.

We now give a brief outline of the Reidemeister-Schreier method. This is an algorithm for finding a presentation of a subgroup, with finite index, in terms of generators of the parent group, given the action of the parent group on the cosets of the subgroup. Let Γ be a subgroup of finite index in Λ and let Φ be a set of generators of Λ that do indeed generate Λ. A right *Schreier Transversal* Σ, for Γ in Λ, over Φ is a set of words over Φ such that there is a one to one correspondence between the members of Σ and the right Γ cosets, and every initial segment of a word in Σ (reading left to right), is also in Σ. If $\phi \in \Phi$ and $\sigma \in \Sigma$, then there is a unique $\alpha \in \Sigma$ such that $\Gamma\sigma\phi = \Gamma\alpha$, and of course $\sigma\phi\alpha^{-1}$ is then in Γ. In fact the set of all such elements

$$\{\sigma\phi\alpha^{-1} : \sigma, \alpha \in \Sigma,\ \phi \in \Phi,\ \Gamma\sigma\alpha = \Gamma\alpha\},$$

is a set of generators for Γ, some of which may be redundant. These are know as Shreier generators. The defining relations are derived by writing the defining relations of Λ, for Φ and their conjugates under Σ, in terms of this set.

Note that if Λ above is a NEC group and F is a fundamental region for Λ, then by [19] we see that

$$\bigcup_{\sigma \in \Sigma} \sigma F$$

is a connected fundamental region for Γ.

Example Let $D := \langle a, b \,|\, a^6 = b^2 = (ab)^2 = 1\rangle$ and let $a \mapsto (1\,2)(3\,4)$, $b \mapsto (1\,3)(2\,4)$ describe the action of D on the right cosets of some subgroup C, of index four. Then a right Schreier transversal for C is $\{1, a, b, ab\}$.

Σ	a		b		a^6	b^2	$(ab)^2$
1	$-$		$-$		g_1^3	g_4	g_3g_4
a	a^2	g_1	$-$		g_1^3	g_5	$g_1g_2g_5$
b	$bab^{-1}a^{-1}$	g_2	b^2	g_4	$(g_3g_2)^3$	g_4	$g_2g_5g_1$
ab	$abab^{-1}$	g_3	ab^2a^{-1}	g_5	$(g_2g_3)^3$	g_5	g_3g_5

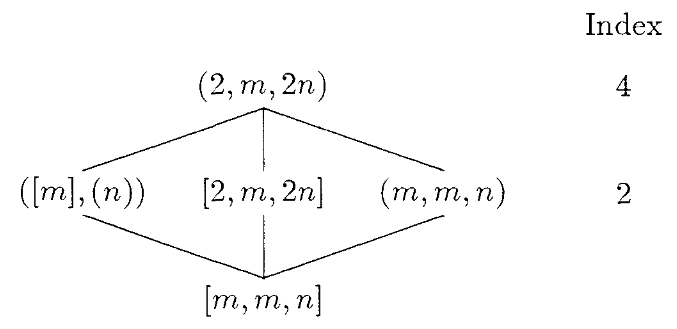
The column on the left just lists the members of Σ. The entries under a are the Schreier generators of C of the form $\sigma a\alpha^{-1}$, where $C\sigma a = C\alpha$ and σ, $\alpha \in \Sigma$. We see the action of a on the cosets of C is such that $3a = 4$, therefore $Cb.a = Cab$ and so $b.a.(ab)^{-1}$ appears under a and across from b. Note that if $\sigma a \in \Sigma$, then we will just get an identity generator which we omit from the table. We perform the same procedure for b and label the generators that we obtain g_1 to g_5. The entries under the defining relations of D are these relations and their conjugates under Σ written in terms of the g_i's. For instance

$$b.(ab)^2.b^{-1} = baba = bab^{-1}a^{-1}.ab^2a^{-1}.a^2 = g_2g_5g_1.$$

Hence $\quad C := \langle g_1, \ldots, g_5 \,|\, g_1^3 = (g_2g_3)^3 = g_4 = g_5 = g_3g_4 = g_1g_2g_5 = 1\rangle$
$$= \langle g_1\rangle = \langle a^2\rangle.$$

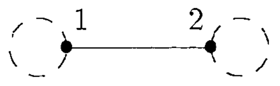
We now proceed with our analysis of the triangle group inclusions. In our subgroup lattices we merely write signatures of Fuchsian and NEC groups. We are able to do this without ambiguity because all triangle groups of a certain signature are conjugate in $\mathcal{L}$.

1. $[m, m, n] \triangleleft [2, m, 2n]$, index 2.



Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[m,m,n]$ and another large group of automorphisms that lifts to $\Lambda[2,m,2n]$. Then, depending on the parity of m, S may have up to two or three classes of symmetries with non-zero species in $\mathrm{Aut}(S)$. S admits symmetries of the first and second kind w.r.t. G and we will determine which of these represent the classes in $\mathrm{Aut}(S)$ mentioned above.

Coset Graph: Clearly if $\Lambda[2,3,2n] = \langle x, y \mid x^2 = y^m = (xy)^{2n} = 1\rangle$, then x must permute the two $\Gamma[m,m,n]$ cosets while y fixes both.

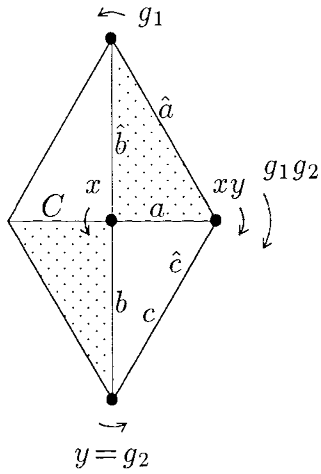


A right Schreier transversal for Γ, in Λ, over $\{x,y\}$ is $\Sigma := \{1, x\}$, and

$$\Gamma = \langle g_1, g_2 \mid g_1^m = g_2^m = (g_1 g_2)^n = 1\rangle,$$

where $g_1 = xyx$ and $g_2 = y$.

Let a, b and c be reflections such that $ab = x$ and $bc = y$, let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_1$ and $\hat{b}\hat{c} = \bar{g}_2$, and let C be the reflection such that $Cg_2C = g_1^{-1}$.

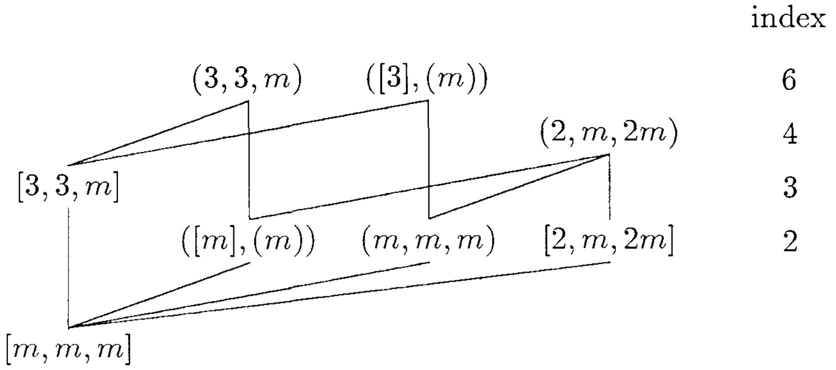


Then in $\Lambda_*(2,m,2n) := \langle a, b, c\rangle$, we see that

$$\hat{a} \sim \hat{c} \sim c, \;\; \hat{b} \sim b \;\; \text{and} \;\; C \sim a.$$

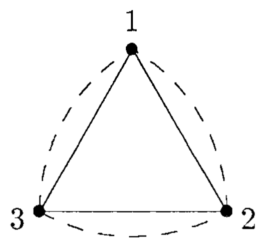
Let us suppose that S has the maximum number of classes of symmetries with non-zero species, two if m is odd and three if m is even. Then these are represented by a symmetry of the second kind induced by C and (i) any symmetry of the first kind w.r.t. G if m is odd, or (ii) two symmetries of the first kind w.r.t. G if m is even, one induced by $\hat{a}$ or $\hat{c}$ and the other induced by $\hat{b}$. In any case, the classes of symmetries of S, with non-zero species, are represented by those of the first or second kind w.r.t. G.

2. $[m,m,m] \triangleleft [3,3,m]$, index 3.

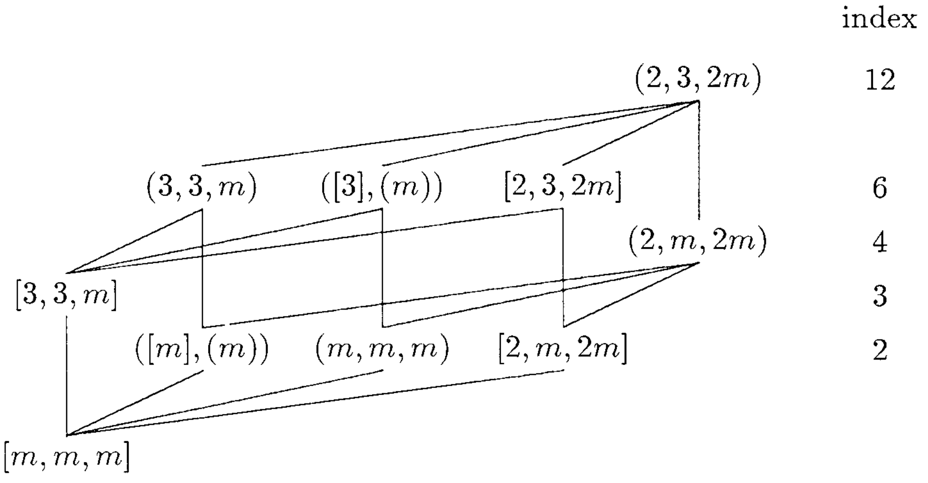


Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[m,m,m]$ and whose full group of conformal automorphisms lifts to $\Lambda[3,3,m]$. Then $\mathrm{Aut}(S)$ lifts to $(3,3,m)$ or $([3],(m))$ and in both cases S will have only one class of symmetries with non-zero species. This class will be represented by a symmetry of the second kind w.r.t. G in the first case and one of the first kind in the second case.

Coset Graph: If $\Lambda[3,3,m] = \langle x, y \mid x^3 = y^3 = (xy)^m = 1 \rangle$, we may assume that, for a suitable enumeration of the $\Gamma[m,m,m]$ cosets, the action of x on these cosets is $(1\,2\,3)$ and hence y must act as $(1\,3\,2)$.

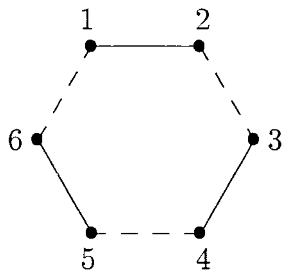


3. $[m,m,m] \triangleleft [2,3,2m]$, index 6.



Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[m,m,m]$ and another group of automorphisms that lifts to $\Lambda[2,3,2m]$. S will have one or two conjugacy classes of symmetries with non-zero species in $\mathrm{Aut}(S)$ and admits symmetries of the first and second kind w.r.t. G.

Coset Graph: If $\Lambda[2,3,2m] = \langle x, y \mid x^{2m} = y^2 = (xy)^3 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[2,n,2n]$ cosets, the action of x on these cosets is $(1\ 2)(3\ 4)(5\ 6)$. xy fixes no cosets therefore $1y \neq 2$ and so we assume $2y = 3$. Thus $4y = 1$, 5 or 6. If $4y = 1$ then $1(xy)^3 = 3$.



Hence $4y = 5$ or 6, we may assume that it is 5, as by symmetry they are equivalent, and so $6y = 1$.

$$\begin{array}{ll} x & (1\,2)(3\,4)(5\,6) \\ y \longmapsto & (1\,6)(2\,3)(4\,5) \\ xy & (1\,3\,5)(2\,6\,4) \end{array}$$

A right Schreier transversal for Γ in Λ over $\{x, y\}$ is

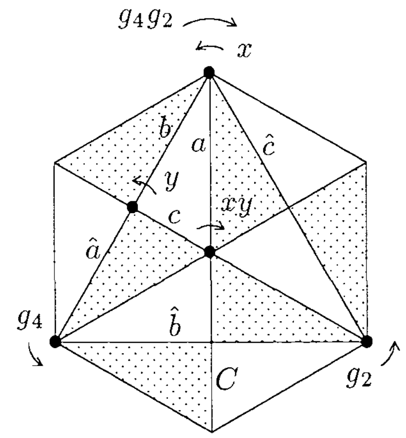
$$\Sigma := \{1, x, xy, xyx, xyxy, y\}.$$

Using the Reidemeister-Schreier method we determine a presentation and a fundamental region for Γ.

Σ	x		y		x^{2m}	y^2	$(xy)^3$
1	$-$		$-$		g_1^m	g_7	g_3
x	x^2	g_1	$-$		g_1^m	g_5	$g_1g_4g_2$
xy	$-$		xy^2x^{-1}	g_5	g_2^m	g_5	g_3
xyx	$xyx^2y^{-1}x^{-1}$	g_2	$-$		g_2^m	g_6	$g_2g_1g_4$
$xyxy$	$xyxyxy^{-1}$	g_3	$xyxy^2x^{-1}y^{-1}x^{-1}$	g_6	$(g_3g_4)^m$	g_6	g_3
y	$yxy^{-1}x^{-1}y^{-1}x^{-1}$	g_4	y^2	g_7	$(g_4g_3)^m$	g_7	$g_4g_2g_1$

Hence $$\Gamma = \langle g_4, g_2 \mid g_4^m = g_2^m = (g_4g_2)^m = 1\rangle.$$

Let a, b and c be reflections such that $ab = x$ and $bc = y$, let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_4$ and $\hat{b}\hat{c} = \bar{g}_2$, and let C be the reflection such that $Cg_2C = g_4^{-1}$.

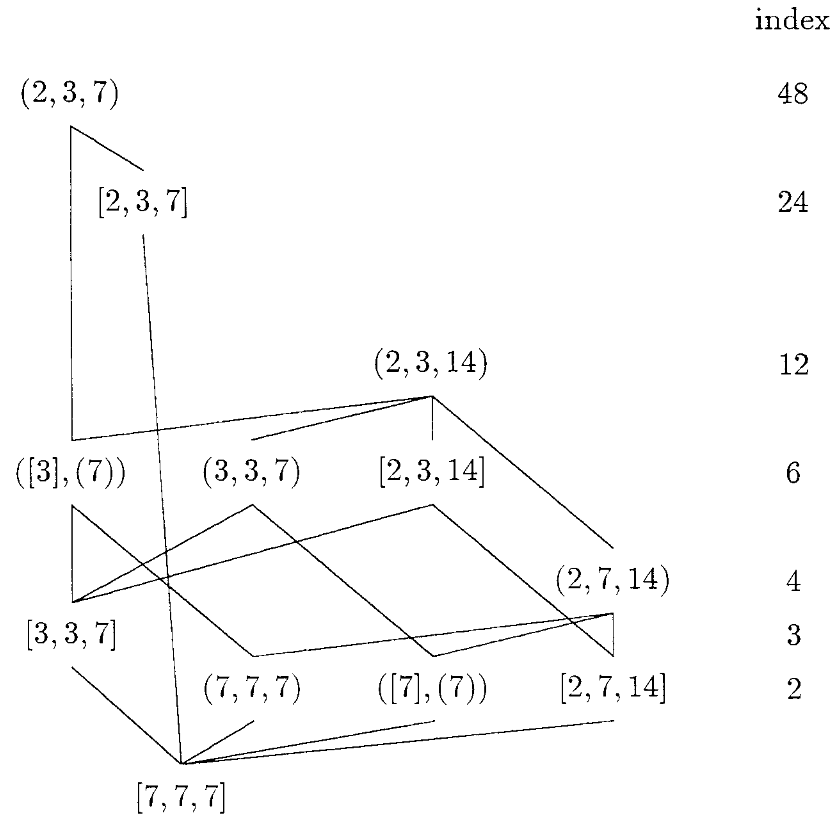


Then in $\Lambda_*(2,3,2m) := \langle a, b, c\rangle$, we see that

$$a \sim c,\ \hat{a} \sim \hat{b} \sim \hat{c} \sim b \ \text{ and } \ C \sim a.$$

Hence, if S does have two classes of symmetries with non-zero species, then one is represented by a symmetry of the first kind and the other by a symmetry of the second kind w.r.t. G. In any case, the classes of symmetries of S, with non-zero species, are represented by those of the first or second kind w.r.t. G.

4. $[7,7,7] < [2,3,7]$, index 24.



Let K be a surface group normal in some $\Gamma[7,7,7]$ and $\Lambda[2,3,7]$ and let G be a large group of automorphisms of $S := \mathcal{U}/K$ that lifts to $\Gamma[7,7,7]$. If S is symmetric then there is just one conjugacy class of symmetries with non-zero species in $\mathrm{Aut}(S)$, this is because $\Lambda(2,3,7)$ contains only one conjugacy class of reflections. Hence, the class of symmetries of S, with non-zero species, is represented by a symmetry of the first kind with respect to G. Note that S admits no symmetries of the second kind w.r.t. G and this is because no $([7],(7))$ is contained in any $(2,3,7)$.

Coset Graph: Let $\langle x, y \mid x^7 = y^2 = (xy)^3 = 1\rangle$ be a presentation for $\Lambda[2,3,7]$. By Theorem 2.4 we know what the cycle structures of x, y and xy are when they act on the cosets; x must have three seven cycles and three one cycles, y must have twelve three cycles and xy must have eight three cycles. Hence y and xy fix no cosets. We may assume that on the $\Gamma[7,7,7]$ cosets x acts as

$$(1,\ldots,7)(8,\ldots,14)(15,\ldots,21)(22)(23)(24)$$

and, because transitivety is required, $1y = 22$. Then $7(xy)^3 = 7$ implies $2y = 7$.

Suppose $23y \in \{3,4,5,6\}$. If $23y = 3$, then $2(xy)^3 = 2$ which implies $2\,y = 4$

and, as $2y = 7$, this is not the case. For a similar reason $23\,y \neq 6$ either. Now $23y = 4$ implies $3y = 5$ and hence $6xyxyx = 6$, which means $6y = 6$ and this is certainly not the case. In the same way we see that $23y \neq 5$. Hence $23y \notin \{3, 4, 5, 6\}$, therefore we may assume that $23y = 8$, $24y = 15$ and so $14y = 9$, $21y = 16$.

If $3y = 4$ then $3xy = 3$, if $3y = 6$ then $2(xy)^3 = 6$ and if $3y = 5$ then $6xyxyx = 6$, which implies $6y = 6$. Thus $3y \notin \{1, \dots, 7\}$ and similarly neither are $4y$, $5y$ or $6y$. This implies that the rest of the transpositions of y map cosets from each seven cycle of x to cosets in a different seven cycle of x.

Without lose of generality we may assume that $3y \in \{10, 11, 12, 13\}$. As $3y = 13$ implies that $2xyxyx = 10$, we see $3y \neq 13$. Now

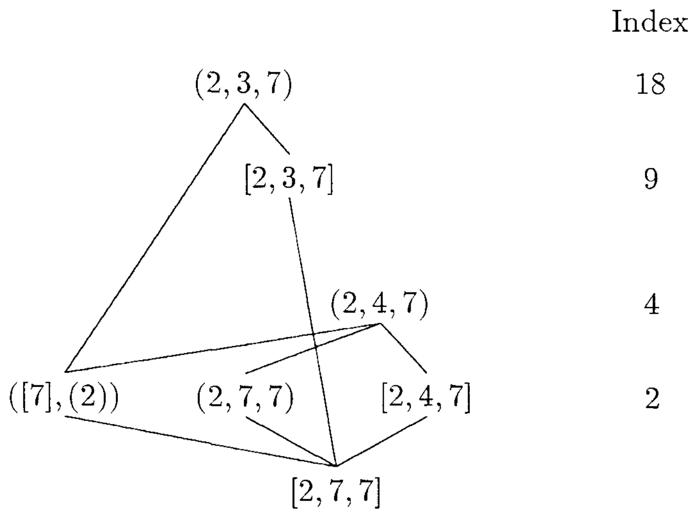
$$2(xy)^3 = 2 \;\Rightarrow\; 2xyxy = 2y^{-1}x^{-1} = 6 \;\Rightarrow\; 3yx = 6y \;\Rightarrow\; 6y \neq 10, \quad \text{else} \quad 9y = 3.$$

If $3y = 10$, then $6y = 11$. For the same reasons that $3y$ and $6y$ belong to the same cycle of x so must $10y$ and $13y$. Therefore, if $3y = 10$, then $13y \in \{4, 5\}$. Similarly $17y$, $20y$ both lie in $\{1, \dots, 7\}$ or $\{8, \dots, 14\}$ but if $3y = 10$, then there is no room for them both. Hence $3y \neq 10$ and similarly $3y \neq 12$.

Thus $3y = 11$ and $6y = 12$. Therefore, $\{10, 13\}y = \{4, 5\}$ or $\{19, 18\}$, if it is the former, then there will be nowhere for y to take $\{17, 20\}$ and so $10y = 18$, $13y = 19$ which implies $17y = 4$, $20y = 5$.

$$\begin{array}{ll}
x & (1, \dots\dots\dots, 7)(8, \dots\dots\dots, 14)(15, \dots\dots\dots, 21)(22)(23)(24) \\
y \longmapsto & (1\ 22)(2\ 7)(3\ 11)(4\ 17)(5\ 20)(6\ 12)(8\ 23)(9\ 14)(13\ 19)(15\ 24)(16\ 21) \\
xy & (1\ 7\ 22)(3\ 17\ 10)(4\ 20\ 16)(5\ 12\ 19)(8\ 14\ 24)(9\ 18\ 13)(15\ 16\ 21)
\end{array}$$

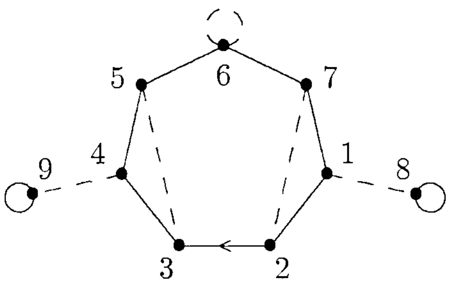
5. $[2, 7, 7] < [2, 3, 7]$ index 9.



Let K be a surface group normal in some $\Gamma[2,7,7]$ and $\Lambda[2,3,7]$, and let G be a large group of automorphisms of $S := \mathcal{U}/K$, that lifts to $\Gamma[2,7,7]$. Then as in case four, if S is symmetric there is only one conjugacy class of symmetries with non-zero species in $\mathrm{Aut}(S)$. Hence, this class is represented by a symmetry of the second kind w.r.t. G. Note that S admits no symmetries of the first kind w.r.t. G as $\Lambda(2,3,7)$ contains no $(2,7,7)$.

Coset Graph: Let $\Lambda[2,3,7] = \langle x, y \mid x^7 = y^2 = (xy)^3 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[2,7,7]$ cosets, x acts as $(1,\ldots,7)(8)(9)$ and $1y = 8$, y will fix one coset only and xy must fix none.

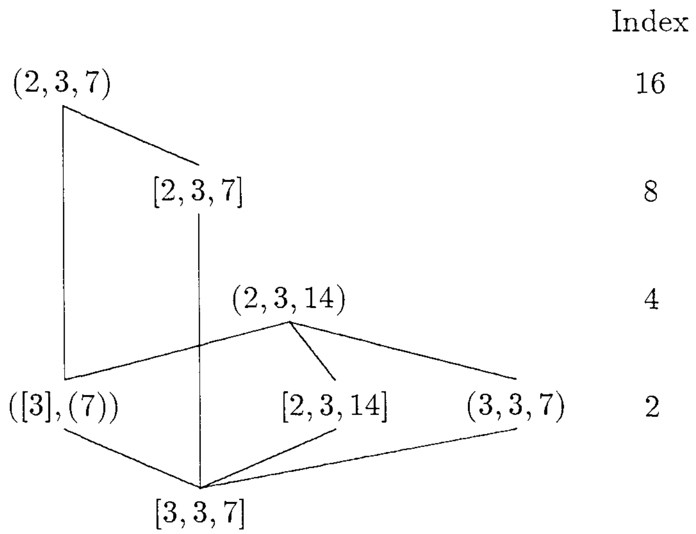
Now $7xyxyx = 2$ and so $2y = 7$. Hence $9y \in \{3,4,5,6\}$, by symmetry 3 and 6 are equivalent, as are 4 and 5. If $9y = 3$ then for the same reason that $2y = 7$ we require $2y = 4$.



Therefore we may assume that $9y = 4$, which implies $3y = 5$ and so $6y = 6$.

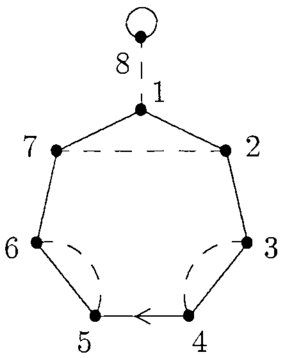
$$\begin{array}{rl} x & (1,\ldots\ldots,7)(8)(9) \\ y \longmapsto & (1\,8)(2\,7)(3\,5)(4\,9)(6) \\ xy & (1\,7\,8)(2\,5\,6)(3\,9\,4) \end{array}$$

6. $[3,3,7] < [2,3,7]$, index 8.



Suppose K is a surface group normal in some $\Gamma[3,3,7]$ and $\Lambda[2,3,7]$, and that $S := \mathcal{U}/K$ is symmetric. If G is a large group of automorphisms of S that lifts to $\Gamma[3,3,7]$, then the single conjugacy class of symmetries of S, with non-zero species in $\mathrm{Aut}(S)$, is represented by a symmetry of the second kind w.r.t. G. Furthermore, S admits no symmetries of the first kind w.r.t. G.

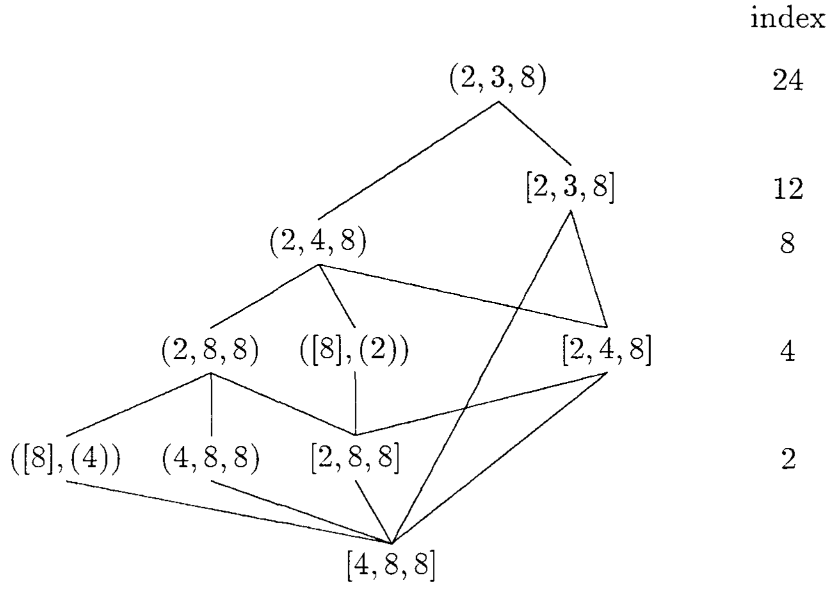
Coset Graph: Let $\Lambda[2,3,7] = \langle x, y \,|\, x^7 = y^2 = (xy)^3 = 1 \rangle$, we may assume that, for a suitable enumeration of the $\Gamma[3,3,7]$ cosets, x acts as $(1, \ldots, 7)(8)$ and $1y = 8$. We know that y must fix no cosets while xy must fix exactly two.



We know $7(xy)^3 = 7$ and as $7xyxyx = 2$ we see that $2y = 7$. Therefore $3y = 4$, 5 or 6. If $3y = 6$ then $2(xy)^3 = 6$, if $3y = 5$ then $4y$ must be 6 and so $3(xy)^3 = 5$. Hence $3y = 4$ and $5y = 6$.

$$\begin{array}{lc} x & (1 \ldots\ldots 7)(8) \\ y \longmapsto & (1\,8)(2\,7)(3\,4)(5\,6) \\ xy & (1\,7\,8)(2\,4\,6)(3)(5) \end{array}$$

7. $[4,8,8] < [2,3,8]$, index 12.



Let a surface group K, be normal in some $\Gamma[4,8,8]$ and $\Lambda[2,3,8]$. Let $S = \mathcal{U}/K$ be symmetric and G be a large group of automorphisms of S that lifts to $\Gamma[4,8,8]$. Then S has one or two conjugacy classes of symmetries with non-zero species in $\mathrm{Aut}(S)$ and S admits symmetries of the first and second kinds w.r.t. G. If t_1 is a symmetry of the first kind and t_2 is a symmetry of the second kind w.r.t. G, then $\langle G, t_1\rangle$ contains up to three conjugacy classes of symmetries with non-zero species and $\langle G, t_2\rangle$ contains one. Hence, if $\langle G, t_1\rangle$ does contain three classes of symmetries with non-zero species, then some of the classes of symmetries of the first and second kind w.r.t. G are certainly conjugate in $\mathrm{Aut}(S)$. This is because any $(2,3,8)$ contains only two conjugacy classes of reflections. We use fundamental regions to determine which are conjugate and if the two possible classes in $\mathrm{Aut}(S)$ mentioned above are represented by thoses symmetries of the first or second kind w.r.t. G.

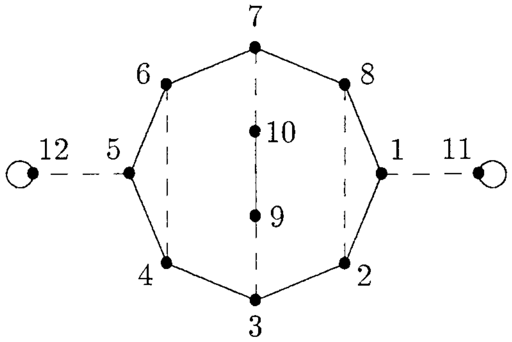
Coset Graph: Let $\Lambda[2,3,8] = \langle x, y \,|\, x^8 = y^2 = (xy)^3 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[4,8,8]$ cosets, the action of x on these cosets is

$$(1,\ldots,8)(9\ 10)(11)(12)$$

and $1y = 11$. Of course y and xy fix no cosets.

We see that $8xyxyx = 2$ and so $2y = 8$. Now $12y \in \{3,\ldots,7\}$ or $\{9,10\}$. If $12y \in \{9,10\}$, then $12xyxyx \in \{1,\ldots,8\}$ but $12y^{-1}$ is not. Therefore $12y \in$

$\{3,\ldots,7\}$, in fact $12y = 4$, 5 or 6, as $12y = 3$ implies $2y = 4$ and $12y = 7$ implies $8y = 6$, neither of which is the case. Suppose $12y = 4$, then $3y = 5$ and we may assume that $9y = 6$ and $10y = 7$, but then $6(xy)^3 = 10$. By symmetry, 4 and 6 are equivalent, hence $12y = 5$, $4y = 6$, $3y = 10$ and $7y = 10$.



$$\begin{array}{lc} x & (1,\ldots\ldots,8)(9\ 10)(11)(12) \\ y \longmapsto & (1\ 11)(2\ 8)(3\ 9)(4\ 6)(5\ 12)(7\ 10) \\ xy & (1\ 8\ 11)(2\ 9\ 7)(3\ 6\ 10)(4\ 12\ 5) \end{array}$$

A right Schreier transversal for Γ in Λ over $\{x,y\}$ is

$$\Sigma := \{1, x, x^2, \ldots, x^7, x^2y, x^6y, y, x^4y\}.$$

We now employ the Reidemeister-Schreier method to find a presentation for Γ.

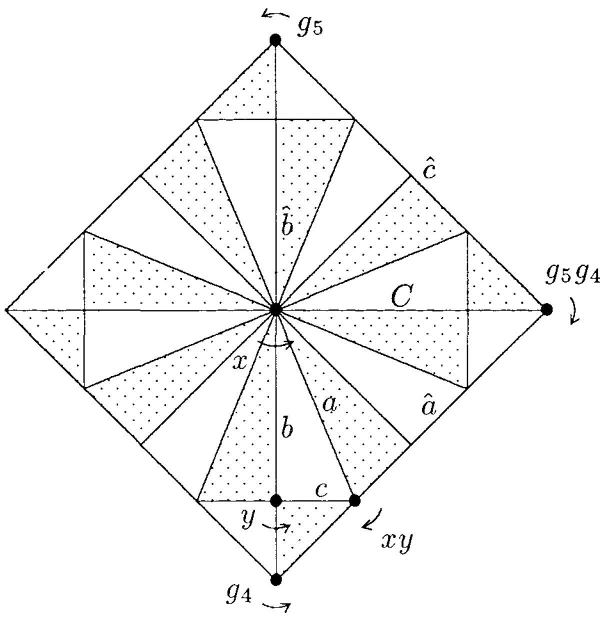
Σ	x		y		x^8	y^2	$(xy)^3$
1	$-$		$-$		g_1	g_{12}	g_6g_4
x	$-$		xyx^{-7}	g_6	g_1	g_6g_9	g_2g_9
x^2	$-$		$-$		g_1	g_{10}	g_7g_3
x^3	$-$		x^3yx^{-5}	g_7	g_1	g_7g_8	g_5g_8
x^4	$-$		$-$		g_1	g_{13}	g_8g_5
x^5	$-$		x^5yx^{-3}	g_8	g_1	g_8g_7	g_3g_7
x^6	$-$		$-$		g_1	g_{11}	g_9g_2
x^7	x^8	g_1	x^7yx^{-1}	g_9	g_1	g_9g_6	g_4g_6
x^2y	$x^2yxy^{-1}x^{-6}$	g_2	$x^2y^2x^{-2}$	g_{10}	$(g_2g_3)^4$	g_{10}	g_2g_9
x^6y	$x^6yxy^{-1}x^{-2}$	g_3	$x^6y^2x^{-6}$	g_{11}	$(g_3g_2)^4$	g_{11}	g_3g_7
y	yxy^{-1}	g_4	y^2	g_{12}	g_4^8	g_{12}	g_4g_6
x^4y	$x^4yxy^{-1}x^{-4}$	g_5	x^4yx^{-4}	g_{13}	g_5^8	g_{13}	g_5g_8

Hence $\Gamma = \langle g_1,\ldots,g_{13} \mid g_1 = (g_2g_3)^4 = g_4^8 = g_5^8 = g_{10} = g_{11} = g_{12} =$

$$g_{13} = g_6g_9 = g_7g_8 = g_4g_6 = g_2g_9 = g_3g_7 = g_5g_8 = 1\rangle$$

$$= \langle g_4, g_5 \mid g_4^8 = g_5^8 = (g_4g_5)^4 = 1\rangle.$$

Using Σ we construct a connected fundamental region for Γ out of fundamental regions for Λ.

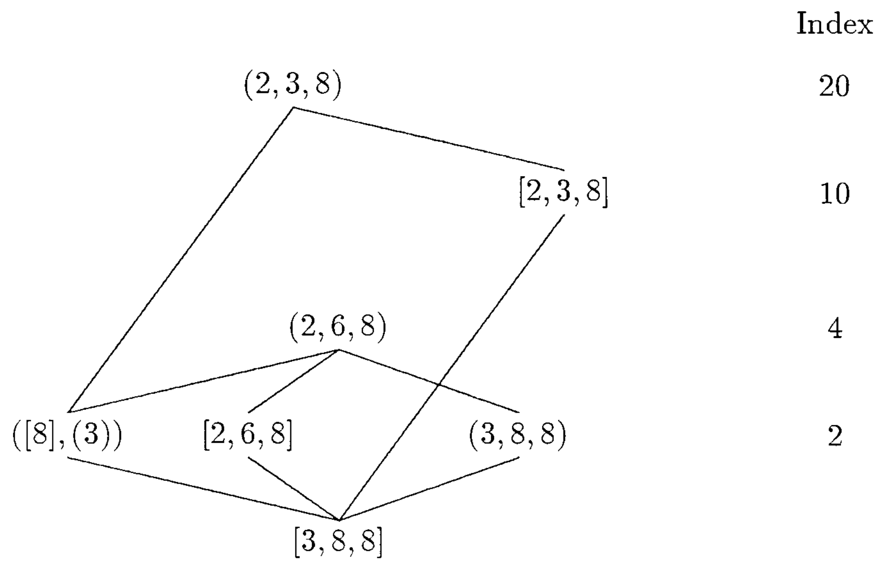


Let a, b and c be reflections such that $ab = x$ and $bc = y$, let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_4$ and $\hat{b}\hat{c} = g_5$, and let C be the reflection such that $Cg_4C = g_5^{-1}$. Then in $\Lambda_*(2,3,8) := \langle a, b, c \rangle$, we see that

$$a \sim c, \ \hat{a} \sim \hat{c} \sim c \ \text{ and } \ \hat{b} \sim C \sim b.$$

Thus the classes of reflections in $\Lambda_*(2,3,8)$ are represented by $\hat{a}$ or $\hat{c}$ and $\hat{b}$ or C. Hence the classes of symmetries of S above, with non-zero species, are represented by symmetries of the first and second kind w.r.t. G.

8. $[3,8,8] < [2,3,8]$, index 10.



Note that a $(2,3,8)$ contains two conjugacy classes of reflections while a $([8],(3))$ contains only one and so the classes in $(2,3,8)$ are not both represented by reflections in $([8],(3))$. Thus, if a surface group K is normal in some $\Gamma[3,8,8]$ and $\Lambda(2,3,8)$, then $S := \mathcal{U}/K$ may admit two conjugacy classes of symmetries in $\mathrm{Aut}(S)$ with non-zero species. Only one of which would be represented by a symmetry of the first or second kind w.r.t. to the large automorphism group G, that lifts to $\Gamma[3,8,8]$.

Coset Graph: Let $\Lambda[2,3,8] = \langle x, y \mid x^8 = y^2 = (xy)^3 = 1\rangle$, so, for a suitable enumeration of the $\Gamma[3,8,8]$ cosets, the actions of x and y on the cosets are

$$\begin{array}{rl} x & (1,\ldots\ldots,8)(9)(10) \\ y \longmapsto & (1\,9)(2\,8)(3\,5)(4\,10)(6\,7) \\ xy & (1\,8\,9)(2\,5\,7)(3\,10\,4)(6). \end{array}$$

See the example in §3.2.

A right Schreier transversal for Γ in Λ over $\{x,y\}$ is

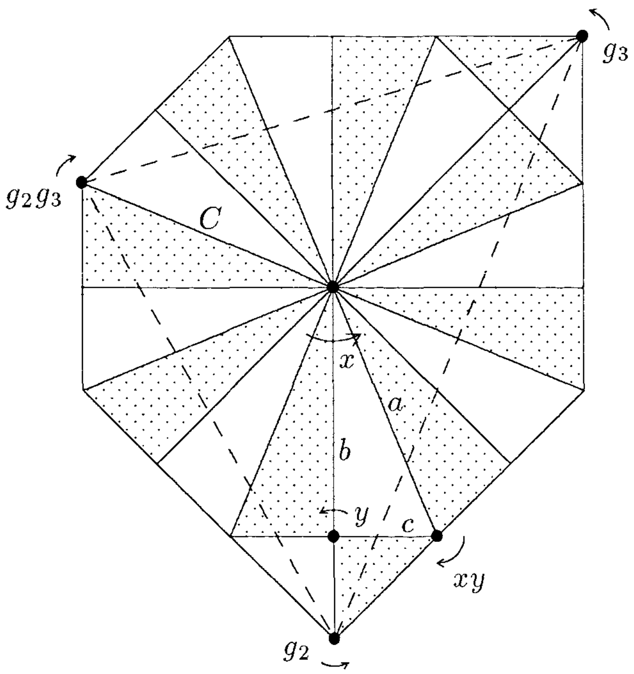
$$\Sigma := \{1, x, x^2, \ldots, x^7, y, x^3y\}.$$

Using the Reidemeister-Schreier method we determine a presentation and a fundamental region for Γ.

Σ	x		y		x^8	y^2	$(xy)^3$
1	$-$		$-$		g_1	g_{10}	g_4g_2
x	$-$		xyx^{-7}	g_4	g_1	g_4g_9	$g_5g_7g_9$
x^2	$-$		x^2yx^{-4}	g_5	g_1	g_5g_6	g_3g_6
x^3	$-$		$-$		g_1	g_{11}	g_6g_3
x^4	$-$		x^4yx^{-2}	g_6	g_1	g_6g_5	$g_7g_9g_5$
x^5	$-$		x^5yx^{-6}	g_7	g_1	g_7g_8	g_8^3
x^6	$-$		x^6yx^{-5}	g_8	g_1	g_8g_7	$g_9g_5g_7$
x^7	x^8	g_1	x^7y^{-1}	g_9	g_1	g_9g_4	g_2g_4
y	yxy^{-1}	g_2	y^2	g_{10}	g_2^8	g_{10}	g_2g_4
x^3y	$x^3yxy^{-1}x^{-3}$	g_3	$x^3y^2x^{-3}$	g_{11}	g_3^8	g_{11}	g_3g_6

Hence
$$\begin{aligned}\Gamma &= \langle g_1,\dots,g_{11} \mid g_1 = g_2^8 = g_3^8 = g_{10} = g_4g_9 = g_5g_6 = \\ &\qquad g_{11} = g_7g_8 = g_2g_4 = g_5g_7g_9 = g_3g_6 = 1\rangle \\ &= \langle g_2, g_3 \mid g_2^8 = g_3^8 = (g_2g_3)^3 = 1\rangle.\end{aligned}$$

Let a, b and c be reflections such that $ab = x$ and $bc = y$, and let C be the reflection such that $Cg_3C = g_2^{-1}$.

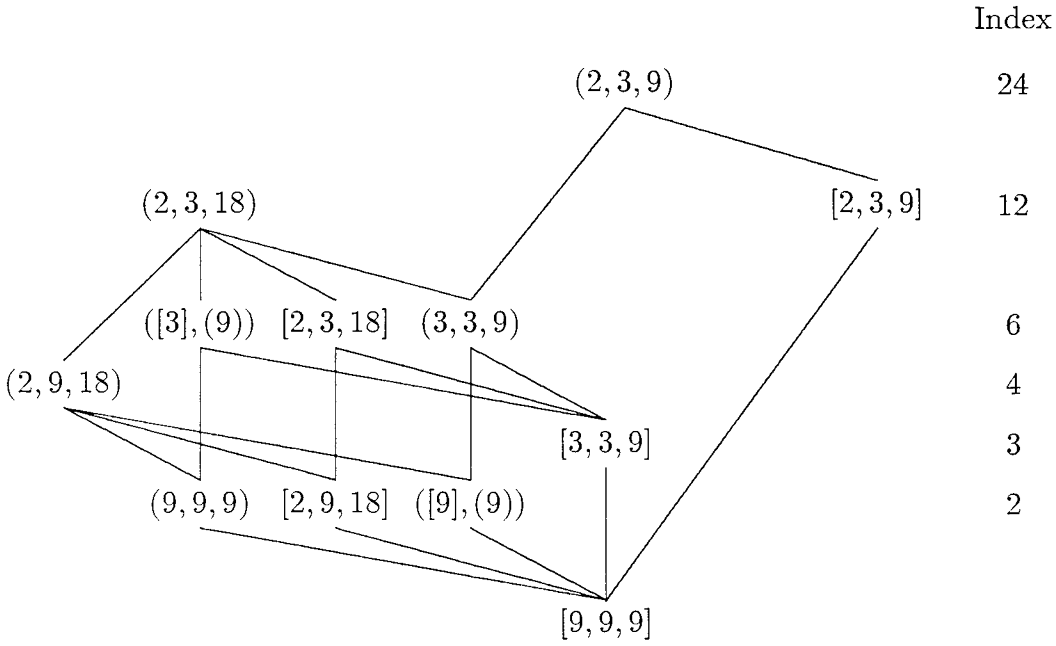


Then in $\Lambda_*(2,3,8) := \langle a,b,c\rangle$, we see that

$$a \sim c \text{ and } C \sim a.$$

Hence, if S above does admit two classes of symmetries with non-zero species, then one class is represented by the symmetry induced by C, which is of the second kind w.r.t. G, and the other is represented by the symmetry induced by b which extends G to $\mathrm{Aut}(S)$.

9. $[9,9,9] < [2,3,9]$, index 12.

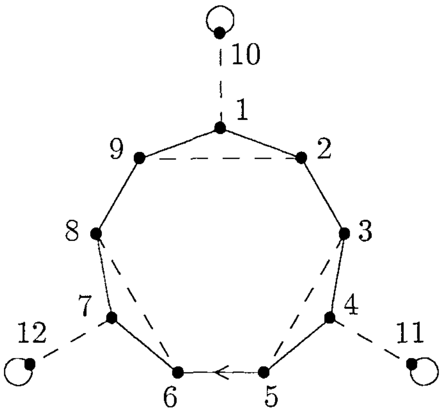


Let S be a Riemann surface admitting a large automorphism group G, that lifts to $\Gamma[9,9,9]$, and a large automorphism group that lifts to $\Lambda[2,3,9]$. If S is symmetric, then it only has one conjugacy class of symmetries with non-zero species in $\mathrm{Aut}(S)$. This class is therefore represented by a symmetry of the second kind w.r.t G and S admits no symmetries of the first kind w.r.t G.

Coset Graph: Let $\Lambda[2,3,9] = \langle x, y \mid x^9 = y^2 = (xy)^3 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[9,9,9]$ cosets, the action of x on these cosets is

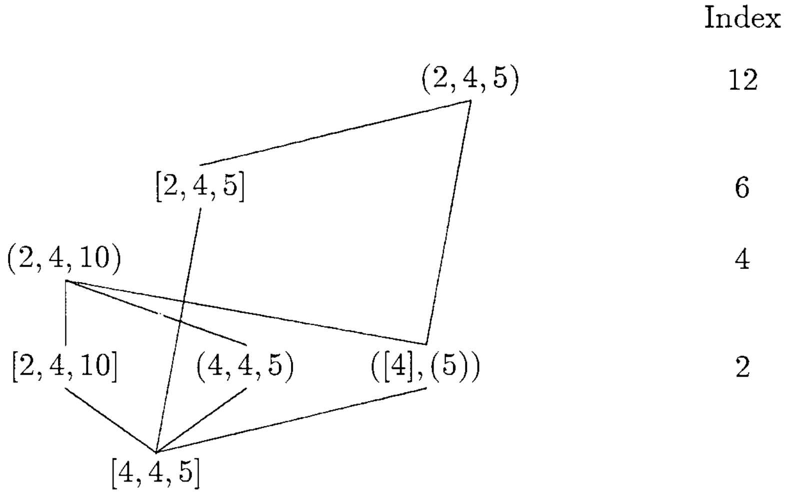
$$(1,\ldots,9)(10)(11)(12)$$

and $10y = 1$. Hence $2y = 9$ and $11y \in \{4,5,6,7\}$. If $11y = 5$ or 6, then there will be no "room" for y to connect 12 to $\{1,\ldots 9\}$. Therefore we may assume that $11y = 4$, as by symmetry 4 and 7 are equivalent, and so $3(xy)^3 = 3$ implies that $3y = 5$. Thus $12y = 6, 7$ or 8, if $12y = 6$ then $5y = 7$, if $12y = 8$ then $9y = 7$ and so $12y = 7,\ 6y = 8$.



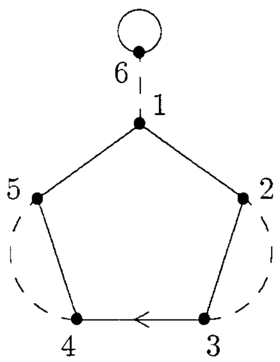
$$\begin{array}{rl} x & (1, \ldots\ldots, 9)(10)(11)(12) \\ y \longmapsto & (1\ 10)(2\ 9)(3\ 5)(4\ 11)(6\ 8)(7\ 12) \\ xy & (1\ 9\ 10)(2\ 5\ 8)(3\ 11\ 4)(6\ 12\ 7) \end{array}$$

10. $[4,4,5] < [2,4,5]$, index 6.



Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[4,4,5]$ and another group of automorphisms that lifts to $\Lambda[2,4,5]$. Then S admits no symmetries of the first kind w.r.t G and only one class of symmetries of the second kind. Since $\Lambda(2,4,5)$ has two classes of reflections S may have two conjugacy classes of symmetries in $\mathrm{Aut}(S)$ with non-zero species, in which case only one of the classes would be represented by a symmetry of the second kind w.r.t. G.

Coset Graph: Let $\Lambda[2,4,5] = \langle x, y \mid x^5 = y^2 = (xy)^4 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[4,4,5]$ cosets, the action of x on these cosets is $(1\ldots5)(6)$ and $6y = 1$. If $2y = 5$ then $5(xy)^4 = 6$, if $2y = 4$ then $3y = 5$ and $1(xy)^4 = 5$.



Thus $2y = 3$ and $4y = 5$.

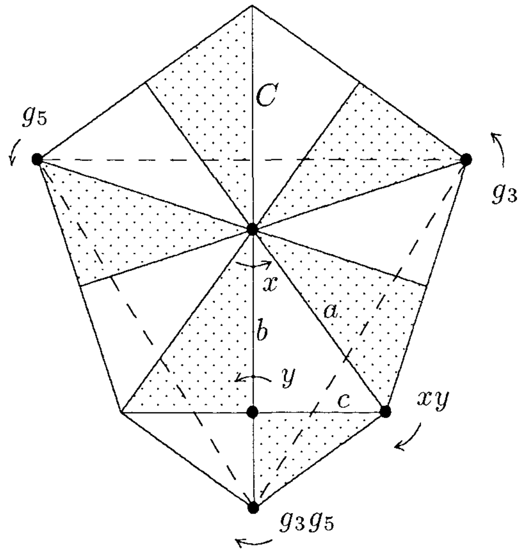
$$\begin{array}{rl} x & (1\ldots5)(6) \\ y \longmapsto & (1\,6)(2\,3)(4\,5) \\ xy & (1\,3\,5\,6)(2)(4) \end{array}$$

A right Schreier transversal for Γ in Λ over $\{x, y\}$ is $\Sigma := \{1, x, \dots, x^5, y\}$.

Σ	x		y		x^5	y^2	$(xy)^4$
1	$-$		$-$		g_1	g_7	$g_3g_5g_2$
x	$-$		xyx^{-2}	g_3	g_1	g_3g_4	g_4^4
x^2	$-$		x^2yx^{-1}	g_4	g_1	g_4g_3	$g_5g_2g_3$
x^3	$-$		x^3yx^{-4}	g_5	g_1	g_5g_6	g_6^4
x^4	x^5	g_1	x^4yx^{-3}	g_6	g_1	g_6g_5	$g_2g_3g_5$
y	yxy^{-1}	g_2	y^2	g_7	g_2^5	g_7	$g_2g_3g_5$

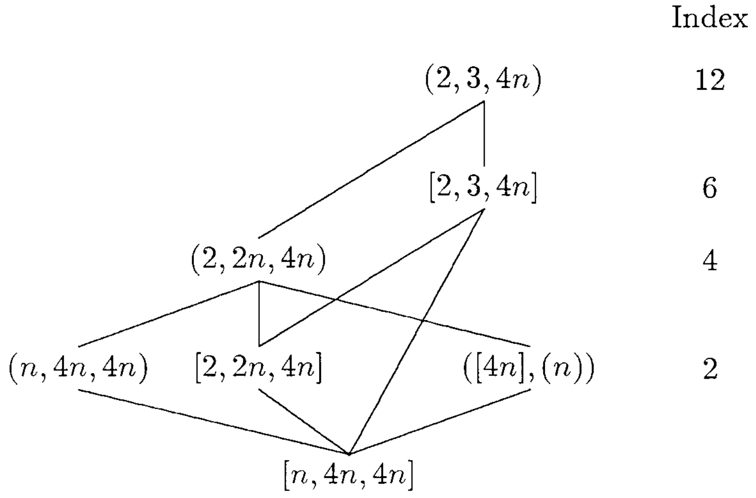
$$\begin{aligned}\text{Hence}\quad \Gamma &= \langle g_1, \dots, g_7 \mid g_1 = g_7 = g_2^5 = g_3g_4 = g_5g_6 = g_2g_3g_5 = g_4^4 = g_6^4 = 1\rangle \\ &= \langle g_3, g_5 \mid g_3^4 = g_5^4 = (g_3g_5)^5 = 1\rangle.\end{aligned}$$

Let a, b and c be reflections such that $ab = x$ and $bc = y$, and let C be the reflection such that $Cg_3C = g_5^{-1}$.



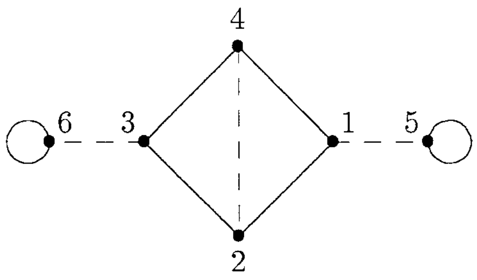
Then in $\Lambda_*(2,4,5) := \langle a, b, c\rangle$, we see that $a \sim b \sim C$ and the other class of reflections is represented by c.

11. $[n,4n,4n] < [2,3,4n]$, index 6.



Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[n,4n,4n]$ and another group of automorphisms that lifts to $\Lambda[2,3,4n]$. Then S admits symmetries of the first and second kind w.r.t G and has one or two conjugacy classes of symmetries in $\mathrm{Aut}(S)$ with non-zero species.

Coset Graph: Let $\Lambda[2,3,4n] = \langle x, y \mid x^{4n} = y^2 = (xy)^3 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[n,4n,4n]$ cosets, the action of x on these cosets is $(1\,2\,3\,4)(5)(6)$ and $5y = 1$. Hence $4xyxyx = 2$ and, as $(xy)^3 = 1$, we know that $2y = 4$ and so $6y = 3$.



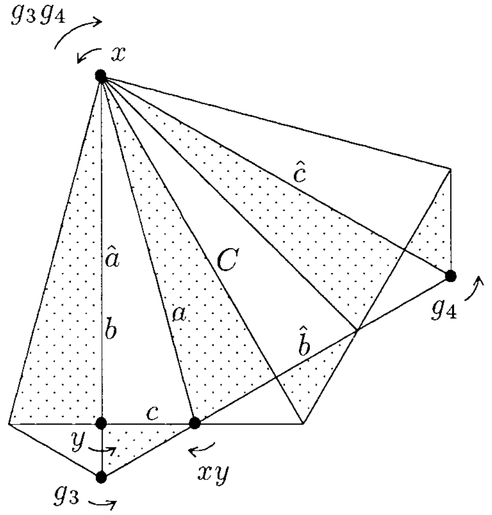
$$\begin{array}{rl} x & (1\,2\,3\,4)(5)(6) \\ y \longmapsto & (1\,5)(2\,4)(3\,6) \\ xy & (1\,4\,5)(2\,6\,3) \end{array}$$

A right Schreier transversal for Γ in Λ over $\{x, y\}$ is $\Sigma := \{1, x, x^2, xy, y, x^2y\}$. Using the Reidemeister-Schreier method we determine a presentation and a fundamental region for Γ.

Σ	x		y		x^{4n}	y^2	$(xy)^3$
1	$-$		$-$		$(g_1g_2)^n$	g_6	g_2g_3
x	$-$		$-$		$(g_1g_2)^n$	g_5	g_4g_1
x^2	$x^3y^{-1}x^{-1}$	g_1	$-$		$(g_1g_2)^n$	g_7	g_1g_4
xy	xyx	g_2	xy^2x^{-1}	g_5	$(g_2g_1)^n$	g_5	g_2g_3
y	yxy^{-1}	g_3	y^2	g_6	g_3^{4n}	g_6	g_3g_2
x^2y	$x^2yxy^{-1}x^{-2}$	g_4	$x^2y^2x^{-2}$	g_7	g_4^{4n}	g_7	g_4g_1

$$\begin{aligned}\text{Hence }\ \Gamma &= \langle g_1,\ldots,g_7\,|\,(g_1g_2)^n = g_3^{4n} = g_4^{4n} = g_5 = g_6 = g_7 = g_2g_3 = g_1g_4 = 1\rangle \\ &= \langle g_3, g_4\,|\,g_3^{4n} = g_4^{4n} = (g_3g_4)^n = 1\rangle.\end{aligned}$$

Let a, b and c be reflections such that $ab = x$ and $bc = y$, let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_3$ and $\hat{b}\hat{c} = g_4$, and let C be the reflection such that $Cg_4C = g_3^{-1}$.



Then in $\Lambda_*(2,3,4n) := \langle a,b,c\rangle$, we see that

$$a \sim c,\ \ \hat{a} \sim \hat{c} \sim C \sim b \ \text{ and }\ \hat{b} \sim c.$$

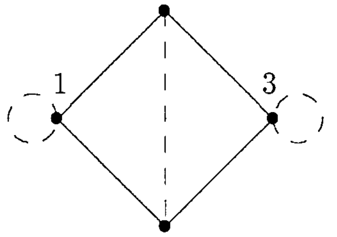
Thus the classes of reflections in $(2,3,4n)$ are represented by one of $\hat{a}$, $\hat{c}$ or C and $\hat{b}$. Hence the classes of symmetries of S with non-zero species are represented by symmetries of the first and second kind w.r.t. G. Indeed, if S does have two such classes, then one is represented by the symmetry induced by $\hat{a}$ and the other by the symmetry induced by $\hat{b}$.

12. $[n, 2n, 2n] < [2, 4, 2n]$, index 4.

See §3.3 for the subgroup lattice in this case.

Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[n, 2n, 2n]$ and another group of automorphisms that lifts to $\Lambda[2, 4, 2n]$. Then S admits symmetries of the first and second kind w.r.t G and has up to three conjugacy classes of symmetries in $\mathrm{Aut}(S)$ with non-zero species.

Coset Graph: Let $\Lambda[2,4,2n] := \langle x, y \mid x^4 = y^{2n} = (xy)^2 = 1\rangle$. We know that x fixes no $\Gamma[n, 2n, 2n]$ cosets while y fixes two and interchanges the other two. We may assume that, for a suitable enumeration of the cosets, the action of x is $(1\,2\,3\,4)$ and $1y = 1$.



Now $4xyx = 2$ and so, as $(xy)^2 = 1$, $2y = 4$. Hence y fixes 3.

$$\begin{array}{rl} x & (1\,2\,3\,4) \\ y \longmapsto & (1)(2\,4)(3) \\ xy & (1\,4)(2\,3) \end{array}$$

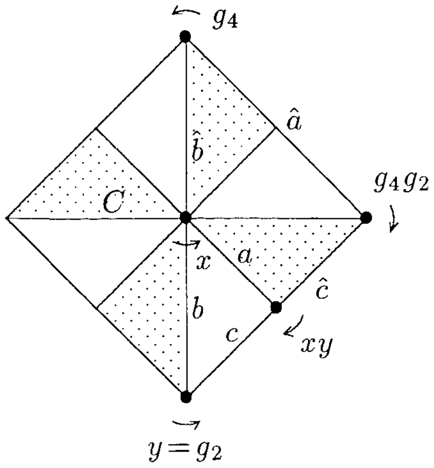
A right Schreier transversal for Γ in Λ over $\{x, y\}$ is $\Sigma := \{1, x, x^2, x^3\}$.

Σ	x		y		x^4	y^{2n}	$(xy)^2$
1	$-$		y	g_2	g_1	g_2^{2n}	$g_3g_1g_2$
x	$-$		xyx^{-3}	g_3	g_1	$(g_3g_5)^n$	g_4g_5
x^2	$-$		x^2yx^{-2}	g_4	g_1	g_4^{2n}	g_5g_4
x^3	x^4	g_1	x^3yx^{-1}	g_5	g_1	$(g_5g_3)^n$	$g_1g_2g_3$

Hence $\quad \Gamma = \langle g_1, \ldots, g_5 \mid g_1 = g_2^{2n} = g_4^{2n} = (g_3g_5)^n = g_4g_5 = g_1g_2g_3 = 1\rangle$
$= \langle g_2, g_4 \mid g_2^{2n} = g_4^{2n} = (g_4g_2)^n = 1\rangle.$

Let a, b and c be reflections such that $ab = x$ and $bc = y$, let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_4$ and $\hat{b}\hat{c} = g_2$, and let C be the reflection such that

$Cg_2C = g_4^{-1}$.

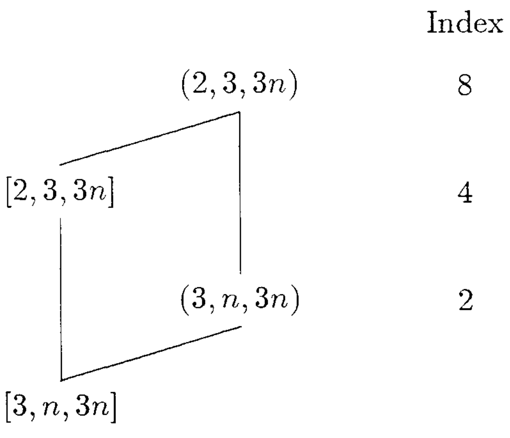


Then in $\Lambda_*(2,4,2n) := \langle a,b,c \rangle$, we see that

$$\hat{a} \sim \hat{c} \sim c, \quad \text{and} \quad \hat{b} \sim C \sim b.$$

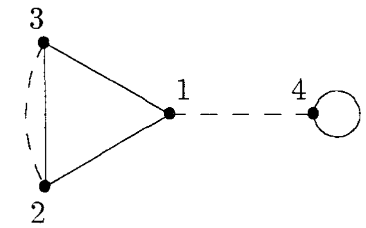
Thus only two of the three classes of reflections in $\Lambda_*(2,4,2n)$ are represented by reflections that normalize Γ. The "missing reflection" is in $([2n],(2))$, see §3.3. Hence if S above does have three classes of symmetries with non-zero species in $\mathrm{Aut}(S)$, then only two of them are represented by symmetries of the first or second kind w.r.t. G.

13. $[3,n,3n] < [2,3,3n]$, index 4.



Let S be a symmetric Riemann surface with a large group of automorphisms G, that lifts to $\Gamma[3,n,3n]$ and another group of automorphisms that lifts to $\Lambda[2,3,3n]$. Then S has one (if n is odd) or possibly two (if n is even), classes of symmetries in $\mathrm{Aut}(S)$ with non-zero species. Clearly any symmetry of S that normalizes G must be of the first kind and G extended by such a symmetry contains one (if n is odd) or possibly two (if n is even), classes of symmetries with non-zero species.

Coset Graph: Let $\Lambda[2,3,3n] = \langle x, y \,|\, x^{3n} = y^2 = (xy)^3 = 1\rangle$. We see that, on the $\Gamma[3,n,3n]$ cosets, x acts as a three-cycle and y as two two-cycles. Hence for a suitable enumeration of the cosets we may assume that x acts as $(1\,2\,3)(4)$ and $4y = 1$. Thus $3xyxyx = 2$ and so $2y = 3$.



$$\begin{array}{rl} x & (1\,2\,3)(4) \\ y \longmapsto & (1\,4)(2\,3) \\ xy & (1\,3\,4)(2) \end{array}$$

A right Schreier transversal for Γ in Λ over $\{x, y\}$ is $\Sigma := \{1, x, x^2, y\}$.

Σ	x		y		x^{3n}	y^2	$(xy)^3$
1	$-$		$-$		$(g_1g_2)^n$	g_5	g_2g_3
x	$x^2y^{-1}x^{-1}$	g_1	$-$		$(g_1g_2)^n$	g_4	g_1^3
xy	xyx	g_2	xy^2x^{-1}	g_4	$(g_2g_1)^n$	g_4	g_2g_3
y	yxy^{-1}	g_3	y^2	g_5	g_3^{3n}	g_5	g_3g_2

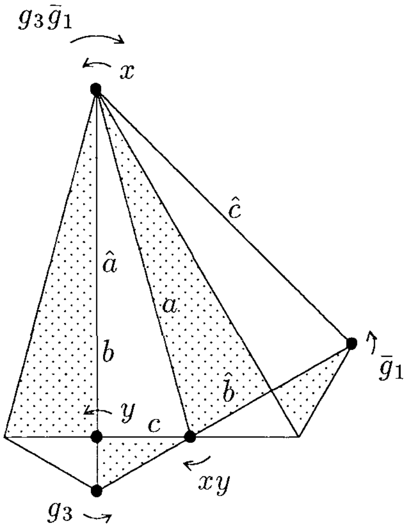
Hence
$$\begin{aligned} \Gamma &= \langle g_1, \ldots, g_5 \,|\, (g_1g_2)^n = g_3^{3n} = g_4 = g_5 = g_1^3 = g_2g_3 = 1\rangle \\ &= \langle g_3, \bar{g}_1 \,|\, (g_3)^{3n} = \bar{g}_1^3 = (g_3\bar{g}_1)^n = 1\rangle. \end{aligned}$$

Where $\bar{g}_1 = g_1^{-1} = xyx^{-2}$ and $g_3 = yxy$.

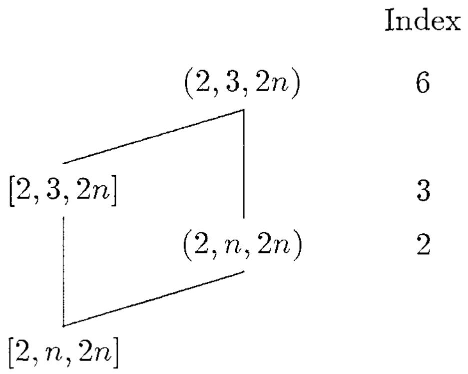
Let a, b and c be reflections such that $ab = x$ and $bc = y$, let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_3$ and $\hat{b}\hat{c} = \bar{g}_1$. Then in $\Lambda_*(2,3,3n) := \langle a, b, c\rangle$, we see that

$$a \sim c,\ \hat{a} \sim b,\ \hat{b} \sim c \text{ and } \hat{c} \sim a.$$

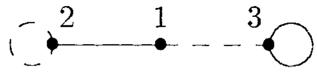
Hence the classes of symmetries of S, with non-zero species, are represented by symmetries of the first or second kind w.r.t. G.



14. $[2, n, 2n] < [2, 3, 2n]$, index 3.



Coset Graph: Let $\Lambda[2, 3, 2n] = \langle x, y \mid x^{2n} = y^2 = (xy)^3 = 1\rangle$, we may assume that, for a suitable enumeration of the $\Gamma[2, n, 2n]$ cosets, the action of x on these cosets is $(1\,2)(3)$ and $3y = 1$.



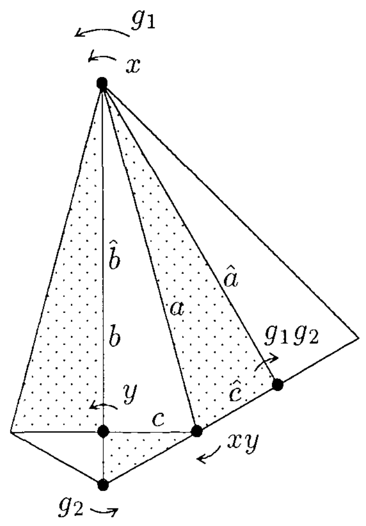
$$\begin{array}{rl} x & (1\,2)(3) \\ y \longmapsto & (1\,3)(2) \\ xy & (1\,2\,3) \end{array}$$

A right Schreier transversal for Γ in Λ over $\{x, y\}$ is $\Sigma := \{1, x, y\}$.

Σ	x		y		x^{2n}	y^2	$(xy)^3$
1	–		–		g_1^n	g_4	$g_3g_1g_2$
x	x^2	g_1	xyx^{-1}	g_3	g_1^n	g_3^2	$g_1g_2g_3$
y	yxy^{-1}	g_2	y^2	g_4	g_2^{2n}	g_4	$g_2g_3g_1$

Hence $$\Gamma = \langle g_1, g_2 \,|\, g_1^n = g_2^{2n} = (g_1 g_2)^2 = 1 \rangle.$$

Let a, b and c be reflections such that $ab = x$ and $bc = y$, and let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be reflections such that $\hat{a}\hat{b} = g_1$ and $\hat{b}\hat{c} = g_2$.



Then in $\Lambda_*(2,3,2n) := \langle a, b, c \rangle$, we see that

$$a \sim c, \;\; \hat{a} \sim c \;\; \text{and} \;\; \hat{b} \sim \hat{c} \sim b.$$

Thus, if a symmetric Riemann surface S, has a large automorphism group G, that lifts to a $[2, n, 2n]$ and another automorphism group that lifts to a $[2, 3, 2n]$, then S has one or two classes of symmetries in $\mathrm{Aut}(S)$ with non-zero species. In either case these classes are represented by symmetries of the first kind w.r.t. G.

REFERENCES

1. A. F. Beardon, *The geometry of discrete groups*, (Springer-Verlag, 1983, Graduate text in Mathematics; 91).

2. A. J. Breda d' Azevedo and G. A. Jones, Rotary hypermaps of genus two, [Preprint, University of Southampton].

3. S. A. Broughton, Classifying finite group actions on surfaces of low genus, *J. Pure and Appl. Algebra* **69** *(1990), 233–270.*

4. S. A. Broughton, E. Bujalance, A. F. Costa, J. M. Gamboa and G. Gromadzki, Symmetries of Riemann surfaces on which PSL(2,q) acts as a Hurwitz group, *J. Pure and Appl. Math.* [To Appear].

5. E. Bujalance, Normal NEC signatures, *Illinois J. Math. (3)* **26** *(1982), 519–530.*

6. E. Bujalance and D. Singerman, The symmetry type of a Riemann surface, *Proc. Lon. Math. Soc. (3)* **51** *(1985), 501–519.*

7. E. Bujalance and A. F. Costa, Acombinatorial approach to the symmetries of M and M-1 Riemann surfaces, *Discrete Groups, Lon. Math. Soc. Lecture Note Series, 173, 16–25.*

8. E. Bujalance, A. F. Costa and D. Singerman, Applications of Hoare's theorem to symmetries of Riemann surfaces, *Annales Academiæ Scientiarum Fennicæ, Series A. 1. Mathematica* **18** *(1993), 307–322.*

9. D. Corn and D. Singerman, Regular hypermaps, *Europ. J. Combinatorics* **9** *(1988), 337–351.*

10. H. S. M. Coxeter and W. O. J. Moser, *Generators and relations of discrete groups*, (Springer-Verlag, 1980, Fourth Edition).

11. J. L. Estv́ez, Pares de signaturas NEC no normales, *Actas 15 Jornadas Luso-Espantiolas matemática, U. Évora 1990.*

12. H. M. Farkas and I. Kra, *Riemann Surfaces*, (Springer-Verlag, 1992, Second Edition).

13. D. Garbe, Über die regularien zerlegungen geschlossener orientierbarer flächen, *J.reine angew. Math.* **237** *(1969), 39–55.*

14. D. Garbe, A Generalization of the regular maps of type $\{4,4\}_{b,c}$ and $\{3,6\}_{b,c}$, *Canad. Math. Bull. (12)* **3** *(1969), 293–297.*

15. D. Garbe, A remark on non-symmetric compact Riemann surfaces, *Archiv Der Math.* **30** *(1978), 435–437.*

16. I. Guerrero, Automorphisms of compact Riemann surfaces and Weierstrass points, *Riemann Surfaces and Related Topics*, (Prin. Uni. Press 1980).

17. W. J. Harvey, Cyclic groups of automorphisms of compact Riemann surafces, *Quart. J. Math. Oxford (2)* **17** *(1966), 86–97.*

18. W. J. Harvey, On branch loci in Teichmuller space, *Trans. Amer. Math. Soc.* **153** *(1971), 387–399.*

19. A. H. M. Hoare and D. Singerman, The orientability of subgroups of plane groups, *Groups St. Andrews 1981, L.M.S. Lecture Notes 71, 221–227.*

20. A. H. M. Hoare, Subgroups of NEC groups and finite permutation groups, *Quart. J. Oxford (2)* **41** *(1990), 45–59.*

21. A. Hurwitz, Uber algebraische gebilde mit endeutigen transformationen in sich, *Math. Ann.* **41** *(1892), 403–442.*

22. M. Izquierdo and D. Singerman, Pairs of symmetries of Riemann surfaces, [In Preparation].

23. G. A. Jones and D. Singerman, Theory of maps on orientable surfaces, *Proc. Lon. Math. Soc. (3)* **37** *(1978), 273–307.*

24. J. Lewittes, Automorphisms of compact Riemann surfaces, *Amer. J. Math.* **84** *(1963), 734–752.*

25. A. M. Macbeath, The classification of non-Euclidean plane crystallographic groups, *Canad. J. Math.* **19** *(1967), 1192–1205.*

26. A. M. Macbeath, Generators of linear fractional groups, *Proc. Symp. pure Math., Houston (1967).*

27. C. Maclachlan, Abelian groups of automorphisms of compact Riemann surfaces, *Proc. of the Lon. Math. Soc. (3)* **15** *(1964), 669–712.*

28. C. Maclachlan, On Schoeneberg's theorem, *Glas. Math. J.* **14** *(1973), 202–204.*

29. D. E. Rohrlich, Some remarks on Weierstrass points, *Number theory related to Fermat's last theorem, Prog. in Math.* **26** *(1982), 71–78.*

30. F. A. Sherk, The regular maps on a surface of genus three, *Canad. J. Math.* **11** *(1959), 452–480.*

31. F. A. Sherk, A family of regular maps of type $\{6,6\}$, *Canad. Math. Bull. (1)* **5** *(1962), 13–20.*

32. D. Singerman, Subgroups of Fuchsian groups and finite permutation groups, *Bull. Lon. Math. Soc.* **2** *(1970), 319–323.*

33. D. Singerman, Finitely maximal Fuchsian groups, *J. Lon. Math. Soc. (2)* **6** *(1972) 29–38.*

34. D. Singerman, On the structure of non-Euclidean crystallographic groups, *Proc. of the Camb. Phil. Soc.* **76** *(1974), 233–240.*

35. D. Singerman, Symmetries of Riemann surfaces with large automorphism group, *Math. Ann.* **210** *(1974), 17–32.*

36. D. Singerman, Reflexibility and symmetry, *Table ronde "Cartes es hypercartes", I.B.P, Universite Paris VII (1992).*

37. D. Singerman, Hyperelliptc maps and surfaces, [Submitted].

38. H. C. Wilkie, On non-Euclidean crystallographic groups, *Math. Zeit.* **91** *(1966), 87–102.*

39. A. Wiman, Uber die hyperelliptischen curven und diejenigen vom geschlechte p=3, welch eindeutigen transformationen in sich zulassen, *Bihang Kongl. Svenska Vetenskaps-Akademiens Handlinger (Stockholm 1895–6).*