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FACULTY OF MATHEMATICAL STUDIES

Characters of Affine Kac-Moody Algebras

by

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ABSTRACT

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Kac-Moody algebras $\mathcal{G}(A)$ of rank r are Lie algebras associated with $n \times n$ generalised Cartan matrices A. If $n = r$ then $\mathcal{G}(A)$ is a complex simple finite-dimensional Lie algebra with finite Weyl group $\overline{W}$, but if $n = r + 1$ then $\mathcal{G}(A)$ is a complex infinite-dimensional affine Lie algebra with affine Weyl group W. This thesis is concerned with explicit calculations based on the use of W.

Manipulating the Weyl-Kac character formula for highest weight modules provides a means of expanding Weyl orbit sums in terms of irreducible characters. These expansions are inverted to obtain analytic weight multiplicity generating functions for level 1 and 2 modules for all affine algebras of rank 1 and 2. The orbit-character expansions and weight multiplicity generating functions are then used to obtain branching rule multiplicities for some affine embeddings.

On the other hand, the Weyl-Kostant-Liu character formula provides a means of expressing irreducible characters of an affine algebra in terms of irreducible characters of a simple finite-dimensional algebra. The key step is the identification of coset representatives $\{W : \overline{W}\}$ for each of the seven infinite series of affine Kac-Moody algebras indexed by their rank r. The proof is given in detail for $A_r^{(1)}$, while for the other affine algebras the results are expressed as conjectures which have been extensively verified by a computer program. Young diagrams are used to specify the action of the coset representatives on arbitrary weights as required in the character formula. This allows the computation of the irreducible characters to be done independently of the rank of the affine algebra. Since the weight multiplicities of finite-dimensional modules of the classical simple Lie algebras are polynomial in the rank this establishes that the weight multiplicities of irreducible highest weight modules of the seven infinite series of affine Kac-Moody algebras are also polynomial in the rank. Illustrative examples are given.

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CHAPTER 1

General Theory of Kac-Moody Algebras

1.1 Introduction

The classification of complex simple finite-dimensional Lie-algebras into four infinite sequence of classical Lie algebras, A_r, B_r, C_r and D_r, and five exceptional Lie algebras E_6, E_7, E_8, F_4 and G_2 was given by Cartan in his thesis of 1894 **[Ca]**. Since then finite-dimensional irreducible representations and modules of Lie algebras have been studied extensively by mathematicians and physicist alike. Their investigations have led to numerous methods and formulae for computing dimensions of irreducible modules, weight multiplicities, tensor product multiplicities and branching rule multiplicities. In this thesis, we extend some of these methods to representations of affine Kac-Moody algebras, working throughout over the field $\mathbb{C}$ of complex numbers.

The structure and representation theory of semisimple finite-dimensional Lie algebras have been discussed in many excellent text books, see e.g. **[H]** and **[J]**. A Lie algebra is called simple if it is non-abelian and has no proper ideals. A Lie algebra is said to be semisimple if it possesses no proper abelian ideals. Since every semisimple Lie algebra is a direct sum of simple Lie algebras, it is then sufficient to consider the structure of the latter. Each simple finite-dimensional Lie algebra $\mathcal{G}$ possesses a Cartan subalgebra $\mathcal{H}$ of dimension r which is the rank of the algebra $\mathcal{G}$. The structure of a simple Lie algebra of rank r is determined up to isomorphism by its root basis consisting of simple roots $\alpha_1, \ldots, \alpha_r$. A root is a vector lying in the dual space $\mathcal{H}^*$ of $\mathcal{H}$. The geometry of the root system is encoded in the Cartan matrix A or equivalently in the corresponding Dynkin diagram $S(A)$. However, the symmetry of the root system is best understood in terms of the Weyl group W, the group that is generated by fundamental reflections s_i in the hyperplanes perpendicular to the simple roots α_i.

Before the introduction of Kac-Moody Lie algebras, the standard approach to the construction of the simple finite-dimensional Lie algebras was to begin by defining simple algebras and then to proceed through various intermediate stages to the construction of the Cartan matrix A or Dynkin diagrams $S(A)$. It was then noted by Serre **[Se]** that every simple finite-dimensional Lie algebra $\mathcal{G}(A)$ can actually be constructed from a set of generators and relations which depend only on the entries in the Cartan matrix A. By weakening the conditions on the Cartan matrix A, Kac **[Kac1]** and independently Moody **[Mo1]** enquired whether similar constructions are still possible. Surprisingly the resulting Lie algebras which are now not neccessarily finite dimensional turn out to be more interesting than the original simple finite-dimensional Lie algebras. The defining matrix $A = (A_{ij})$ is called a generalised Cartan matrix (GCM) if $A_{ii} = 2$, A_{ij} is a nonpositive integer for $i \neq j$ and $A_{ij} = 0$ implies $A_{ji} = 0$. The Kac-Moody algebra $\mathcal{G}(A)$ associated with an $n \times n$ GCM A is the Lie algebra generated by the elements e_i, f_i, h_i $(i = 1, 2, \ldots, n)$ subject to the following defining relations:

$$[h_i, h_j] = 0\,;$$

$$[e_i, f_j] = \delta_{ij} h_i\,;$$

$$[h_i, e_j] = A_{ij} e_j\,;$$

$$[h_i, f_j] = -A_{ij} f_j\,;$$

$$(ad\, e_i)^{-A_{ij}+1}\, e_j = 0 \quad \text{for } i \neq j\,;$$

$$(ad\, f_i)^{-A_{ij}+1}\, f_j = 0 \quad \text{for } i \neq j\,;$$

for all $i, j = 1, 2, \ldots, n$. The vectors h_i lie in the Cartan subalgebra $\mathcal{H}$. Furthermore, the Kac-Moody algebra $\mathcal{G}(A)$ has the root space decomposition

$$\mathcal{G}(A) = \oplus_{\alpha \in \mathcal{H}^*} \mathcal{G}_\alpha\,,$$

where $\mathcal{G}_\alpha = \{x \in \mathcal{G}(A) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$. An element $\alpha \in \mathcal{H}^*$ is called a root if $\mathcal{G}_\alpha \neq 0$ and $dim\,\mathcal{G}_\alpha$ is called the multiplicity of α and is often written as $mult\,\alpha$.

The Kac-Moody algebra $\mathcal{G}(A)$ possesses a non singular invariant form only if the GCM A is symmetrisable i.e. there exists a diagonal matrix D such that DA is symmetric. Morever for each indecomposable GCM A, the Kac-Moody algebra $\mathcal{G}(A)$ belongs to one or other of the following three non intersecting classes **[Kac4]**:

a) if there exists a vector θ of positive integers such that all the components of the vector $A\theta$ are positive, then $\mathcal{G}(A)$ is a simple finite-dimensional Lie algebra;

b) if there exists a vector δ of positive integers such that $A\delta = 0$, then $\mathcal{G}(A)$ is an infinite-dimensional Lie algebra known as an affine Kac-Moody algebra;

c) if there exist a vector ϕ of positive integers such that all the components of the vector $A\phi$ are negative, then $\mathcal{G}(A)$ is an infinite-dimensional Lie algebra known as an indefinite Kac-Moody algebra.

The affine Kac-Moody algebras, sometimes known as Euclidean Lie algebras or just affine algebras were classified by Kac **[Kac1]** and Moody **[Mo1]** and they fall into one of the following classes: the untwisted algebras $A_r^{(1)}, B_r^{(1)}, C_r^{(1)}, D_r^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)}$ and the twisted algebras $A_{2r}^{(2)}, A_{2r-1}^{(2)}$, $D_{r+1}^{(2)}, E_6^{(2)}, D_4^{(3)}$. The centre of the affine algebra $\mathcal{G}(A)$ is one-dimensional **[Kac4]** and is spanned by the element K known as the canonical central element. The algebra $\mathcal{G}(A)/K$ is isomorphic to one of the following algebras:

(i) the loop algebra

$$\bar{\mathcal{G}} \otimes \mathbb{C}[t, t^{-1}]\,,$$

where $\bar{\mathcal{G}}$ is a simple finite-dimensional Lie algebra and $\mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials in t. This is the so called untwisted case.

(ii) the algebra

$$\oplus_{j\in\mathbb{Z}}\bar{\mathcal{G}}_j \otimes t^j ,$$

where $\bar{\mathcal{G}}_j$ is the eigenspace of a certain automorphism of $\bar{\mathcal{G}}$ of finite order m corresponding to the eigenvalue $e^{2\pi i j/m}$. In fact m can only equal to 2 or 3. This is the so called twisted case.

The structure of affine algebras are similar to those of simple finite-dimensional Lie algebras which permits one to generalise many results of the classical theory. However the theory of general Kac-Moody algebras is interesting not only because of the possibility of reproducing the results of the classical theory but mainly because the corresponding results for Kac-Moody algebras turn out to be directly connected with other topics in mathematics quite unrelated before.

Initially the Kac-Moody algebras attracted much attention because of the link between the affine algebras and Macdonald identities **[Ma]**. Macdonald discovered a remarkable product formula relating the Weyl group W and the positive roots Δ^+ of a certain kind of Lie algebra. Although cast in a slightly different form, Macdonald obtained in the framework of affine root systems the formula

$$\sum_{w\in W} \varepsilon(w)e^{-(\rho-w\rho)} = \prod_{\alpha\in\Delta+} (1-e^{-\alpha})^{dim\,\mathcal{G}_\alpha} ,$$

where $\rho - w\rho$ is the sum of the positive roots α such that $w^{-1}\alpha$ is negative. He used this formula to obtain identities for powers of Dedekind's eta-function, $\eta(\tau)^{dim\,\bar{\mathcal{G}}}$ where $\bar{\mathcal{G}}$ is a simple finite-dimensional Lie algebra. Kac **[Kac2]** later recognised Macdonald's unspecialised identity to be nothing other than the Weyl-Kac denominator identity for affine algebras and also established that the Macdonald identity was valid for the entire class of Kac-Moody algebras.

Representations of infinite-dimensional Kac-Moody algebras are difficult to construct explicitly even in the affine case. Inspired originally by the theory of relativistic strings, there is an extensive literature in which operator realisations of the affine algebras are discussed, see e.g **[GO]** also for other physical applications. However, our discussion of the representations of Kac-Moody algebras will largely be in terms of their characters and the related weight vectors. Much like a root, a weight is defined to be a linear functional $\lambda : \mathcal{H} \to \mathbb{C}$. A weight $\lambda \in \mathcal{H}^*$ is called integral if $\lambda(h_i) \in \mathbb{Z}$ and dominant if $\lambda(h_i) \geq 0$ for all i. Given a dominant integral weight λ of a Kac-Moody algebra $\mathcal{G}(A)$, there exist an irreducible module $V^\lambda = \oplus_{\mu \in \mathcal{H}^*} V^\lambda_\mu$ where

$$V^\lambda_\mu = \{v \in V^\lambda \mid h(v) = \mu(h)v \quad \text{for all} \quad h \in \mathcal{H}\}.$$

Such a module is called a highest weight module with highest weight λ. The dimension of the weight space V^λ_μ is referred to as the multiplicity of the weight μ. The character of this irreducible $\mathcal{G}(A)$- module is given by the Weyl-Kac character formula **[Kac2]**

$$ch\ V^\lambda = \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} / \prod_{\alpha \in \Delta+} (1 - e^{-\alpha})^{mult\ \alpha}.$$

This formula is a generalisation of the Weyl character formula of a simple finite-dimensional Lie algebra. Although the general formula is valid for an arbitrary Kac-Moody algebra, in the indefinite case the multiplicity of the roots and the exact structure of the Weyl group are unknown, leaving us with a purely formal expression.

The characters of the irreducible highest weight modules of affine algebra give rise to many interesting combinatorial identities **[FL]**, **[Kac3]**. The specialisation of the denominator identity for the simplest affine algebra $A_1^{(1)}$ leads to the famous Jacobi triple product identity, while the weight multiplicities of the fundamental weight module of $A_1^{(1)}$ module are the values of the classical partition function $p(n)$. Some weight multiplicity generating functions, known as string functions, can be found in an important paper **[KaP]** that relates affine algebras to the theory of theta functions. Using the classical transformation properties of theta functions Kac and Peterson have shown that the string functions are modular forms. Although their transformation laws have been established, obtaining explicit expression for the string functions is not an easy task. An explicit expression for all string functions is known only for the simplest affine algebra $A_1^{(1)}$ for which they are expressed in term of Hecke modular forms.

There are a number of methods available for computing weight multiplicities of the highest weight modules of simple finite-dimensional Lie algebras. Most of these methods can be extended to affine algebras but unlike the expression of string functions in terms of modular forms, the weight multiplicities can only be given numerically, with their values limited by 'depth'. Recently Begin and Sharp **[BS2]**, extending the work of Kass **[Kass]** on the affine algebra $A_1^{(1)}$ and of Patera and Sharp **[PS]** on simple finite-dimensional Lie algebras, developed a technique that allowed them to expand affine Weyl orbits in term of characters of irreducible representations. The weight multiplicities concerned can be read off from the inversion of this expansion. For the affine algebras of rank 1 and 2, they gave explicit Weyl orbit expansions in terms of characters of irreducible representations. Unfortunately, not much progress has been made in inverting even these expansions analytically.

Weights of irreducible highest weight modules are conjugate to dominant weights and their multiplicities are invariant under the action of the Weyl group. Therefore in order to specify all weight multiplicities it is sufficient to tabulate the multiplicities of dominant weights. Bremner, Moody and Patera **[BMP]** have published tables of dominant weights and their multiplicities in highest weight modules of simple finite-dimensional Lie algebras. These tables are extensive and extend up to rank 12 for some algebras. It was first reported by King **[King1]** that multiplicities of the dominant weights are in fact polynomials in the rank of the algebra for each of the sequences of the classical Lie algebras A_r, B_r, C_r and D_r. This polynomial dependence was later established explicitly by King and Plunkett **[KiP]** and Benkart, Britten and Lemire **[BBL]**.

As in **[BMP]** similar tables of dominant weights multiplicities but appropriate to the untwisted affine algebras have been published by Kass, Moody, Patera and Slansky **[KMPS]**. In order to extend these tables, it was first conjectured by Benkart and Kass **[BK]** that these weight multiplicities are again polynomial in the rank of the algebras. In the case of $A_r^{(1)}$ and for sufficiently large r it has been proven to be so by Benkart, Kang and Misra **[BKM1]** and they also later established this rank dependence of weight multiplicities up to depth 2 **[BKM2]**. The rank dependent expressions for weight multiplicities can be used to obtain root multiplicities of the hyperbolic Kac-Moody algebras $HA_r^{(1)}$ **[KM]**.

A problem which in applications appears quite often is to decompose irreducible modules of an algebra into those of a subalgebra. However, a knowledge of the subalgebras of affine algebras is nowhere near as extensive as that of simple finite-dimensional Lie algebras. Discussion for the conformal embeddings and their role in the context of two-dimensional conformal field theory can be found in the text by Fuchs **[F]**. Other explicit branching rules for embeddings of one affine algebra in another have been reported in **[BS1]**, **[Lu]**. It is also interesting to note that an affine algebra can be embedded in itself **[HKLP]**, **[LPS]**.

In the remaining part of this Chapter we give first some terminology appropriate to general Kac-Moody algebras before restricting our discussion to either the simple finite-dimensional Lie algebras or the affine algebras **[Kac4]**, **[KMPS]**. We begin with the definition and the classification of GCMs. With these we associate Dynkin diagrams and define the Kac-Moody algebras in term of generators and relations. The properties of highest weight modules and Weyl groups are then discussed. The main objects of interest are the Weyl-Kac and Weyl-Kostant-Liu character formulae **[Liu]** and the derivation of the method for expanding the orbit sums in term of irreducible characters **[PS]**.

In Chapter 2 we discuss representations of simple finite-dimensional Lie algebras. Since most of the results are classical we have omitted their proof. Our aim is to demonstrate some methods used in the context of simple finite-dimensional Lie algebras before extending the methods to affine algebras. Besides this we also discuss the relationship between the Young diagram notation for partitions and irreducible characters. We then consider the infinite series of characters obtained previously using the theory of Schur functions **[King2]**. We also give the modification rules that have to be taken into consideration when non standard labels are encountered **[King2]**.

In Chapter 3 we discuss the two common approaches to the construction of affine algebras. In the GCM approach we obtained all the conventions that will be employed. The central extension of a loop algebra approach is then considered in order to make the connection with simple finite-dimensional Lie algebras and also to obtain the roots and their multiplicities **[Co]**. Next we discuss the properties of affine Weyl groups, the partitioning of weight space into Weyl orbits and orbit-weight generating functions. Finally we give analytic expansions of affine orbit sums in term of affine irreducible characters for all level 1 and 2 modules of affine algebras of rank 1 and 2. Numerical inversion is then employed to illustrate the method of determining weight multiplicities. The algorithm developed here to compute weight multiplicities has been implemented for most affine algebras in the form of computer programs.

In Chapter 4 we spell out explicitly the Weyl-Kac denominator identities for all affine algebras of rank 1 and 2. With the help of these identities, we are able to rewrite and simplify the sum form of the orbit-character expansions given in Chapter 3 as product forms. Following the work of Kass **[Kass]** analytic expressions for some string functions are obtained when the matrix of string functions is of order less than 3. When the order of the matrix is greater than 2, the string functions are obtained by fitting product formulae to the weight multiplicities generated by our programs. The method exploits the modular characteristic of string functions.

Chapter 5 is concerned with an entirely new view of the relationship between the infinite series of characters based on Schur functions considered in Chapter 2 and the denominator of the Weyl-Kostant-Liu character formula. The idea behind the use of the Weyl-Kostant-Liu character formula is to transform the summation over affine Weyl group elements directly into irreducible characters of a simple finite-dimensional Lie algebra. The crucial step is the identification of an appropriate set $\{W : \overline{W}\}$ of right coset representatives of the affine Weyl group W with respect to the finite Weyl group $\overline{W}$. In this chapter we obtain the set $\{W : \overline{W}\}$ for all seven infinite series of rank dependent affine algebras but give a proof only for $A_r^{(1)}$. Although the others are left as conjectures, they have been extensively verified with a computer program and are in complete accord with the Schur function formulae. A Young diagrammatic method for computing the action of each right coset representative on weights is also given.

Chapter 6 is a consequence of Chapter 4 and 5. With the identification of the set $\{W : \overline{W}\}$ and the Young diagrammatic technique developed in Chapter 5 we give a decomposition procedure for expressing irreducible characters of affine algebras in terms of character of simple finite-dimensional Lie algebras up to any prescribed depth. The computations are done independently of the rank of the affine algebras. Illustrations are given for all seven infinite series of affine algebras with characters of particular irreducible representations being obtained up to depth 4. Since the weight multiplicities of the four infinite series of classical Lie algberas are polynomial in the rank, we have thereby established that the weight multiplicities of all seven infinite series of affine algebras are also polynomial in the rank. Examples illustrating the explicit calculation of this rank dependence are provided. In addition the analytic orbit-character and character-orbit expansions obtained in Chapter 4 are used following the method discussed in **[PS]**, to obtain analytic branching rule multiplicities for affine self embeddings and other maximal embeddings **[BS1]**.

Finally, in Chapter 7 we present some conclusions and recommendations on future developments associated with this work.

1.2 Kac-Moody algebra associated with generalised Cartan matrices

In the following sections we discuss some aspects of the general theory of Kac-Moody algebras. Unless specified, the proofs of the results can be found in the text by Kac **[Kac4]**. We begin with a definition of a complex Lie algebra.

Definition 1.1 A vector space $\mathcal{G}$ over the field $\mathbb{C}$ with a binary operation $[\cdot,\cdot]$ is called a Lie algebra if the following axioms are satisfied:

(L1) $[x, y]$ is a bilinear function of x and y;

(L2) $[x, x] = 0$ for all $x \in \mathcal{G}$;

(L3) Jacobi Identity: $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in \mathcal{G}$.

As has been noted by Serre **[Se]** and Gabber and Kac **[GK]**, we can construct a Lie algebra by the method of generators and relations given any generalised Cartan matrix(GCM).

Definition 1.2 An integral $n \times n$ matrix A of rank r is called a GCM if it satisfies the following conditions for all $i, j \in I = \{1, \ldots, n\}$:

(G1) $A_{ii} = 2$;

(G2) $A_{ij} \leq 0$ for $i \neq j$;

(G3) if $A_{ij} = 0$ then $A_{ji} = 0$.

The relation G3 implies that zeros appear symmetrically in A but in general the matrix A is not symmetric. A GCM is said to be symmetrisable if there exists a nonsingular diagonal matrix D such that DA is a symmetric matrix. The symmetrisability condition eliminates some infinite dimensional algebras that are difficult to study. Furthermore, in order to avoid direct products of algebra, the GCM will be assumed to be indecomposible i.e. that it cannot be brought into a block diagonal form by permuting rows and columns.

A matrix of the form A_{ij} where $i, j \in S \subset I$ is called a principal submatrix of A and is called proper if S is a proper subset of I. The determinant of a principal submatrix is called a principal minor. We then can make a distinction among the GCM as follows.

Definition 1.3 A GCM A is said to be of

(M1) finite type if all its principal minors are positive;

(M2) affine type if all its proper principal minors are positive and $\det A = 0$;

(M3) indefinite type if A is of neither finite nor affine type;

Although they are still the subject of active mathematical research, the theory of Lie algebras associated with cases M1 and M2 is well developed by now. However not many general results are known in the case of Lie algebras associated with M3 although some progress has been made in those special cases when A is of hyperbolic type **[KM]** i.e. when A is of indefinite type and all its proper principal submatrices are of finite or affine type.

To each GCM A we can associate a graph $S(A)$, called the Dynkin diagram of A as follows. The graph consists of n vertices labelled by i with $i = 1, 2, \ldots, n$ joined by edges or lines. If $A_{ij}A_{ji} \leq 4$ and $| A_{ij} | \geq | A_{ji} |$, the vertices i and j are connected by

$| A_{ij} |$ lines and these lines are equipped with an arrow pointing toward j if $| A_{ij} | > 1$.

In Tables 1.1 and 1.2 we give the Dynkin diagrams of all simple finite dimensional Lie algebras and affine algebras respectively. Here we adopt the Dynkin numbering system for simple roots and always assume that the enumeration of the roots begin from the leftmost vertex of $S(A)$. The numbers attached to the vertices in Table 1.2 are the level vector components (co-marks) whose definition will become apparent in Chapter 3. For the Dynkin diagram of simple finite-dimensional Lie algebras, the name consists of a letter (A - G) denoting the type and a numerical subscript denoting the rank of the algebra. For the affine algebras, the name consists of the name of the corresponding simple finite-dimensional Lie algebra from which it is derived together with a parenthetical superscript indicating the degree of the diagram automorphism used in its construction. The starting point of each sequence of Lie algebras is chosen both to avoid Lie algebras that are not simple and to eliminate the appearance of isomorphic algebras with different names. In particular, we have for simple finite-dimensional Lie algebras

$$A_1 \approx B_1 \approx C_1, \quad B_2 \approx C_2, \quad A_3 \approx D_3, \quad D_2 \approx A_1 \oplus A_1 .$$

Definition 1.4 A Kac-Moody Lie algebra associated with a GCM A is a vector space $\mathcal{G}(A)$ generated by $3n$ elements e_i, f_i, h_i with $i \in I$ satisfying the axioms L1 - L3 of a Lie algebra and for all $i, j \in I$ the additional relations:

(R1) $[h_i, h_j] = 0$;

(R2) $[e_i, f_j] = \delta_{ij} h_i$;

(R3) $[h_i, e_j] = A_{ji} e_j$;

(R4) $[h_i, f_j] = -A_{ji} f_j$;

(R5) $(ad\ e_i)^{-A_{ij}+1} e_j = (ad\ f_i)^{-A_{ij}+1} f_j = 0 \qquad \text{whenever} \quad i \neq j.$

The elements e_i, f_i and h_i are called the Chevalley generators. The relation R5 is

Table 1.1 : Dynkin diagrams of simple finite-dimensional Lie algebras.

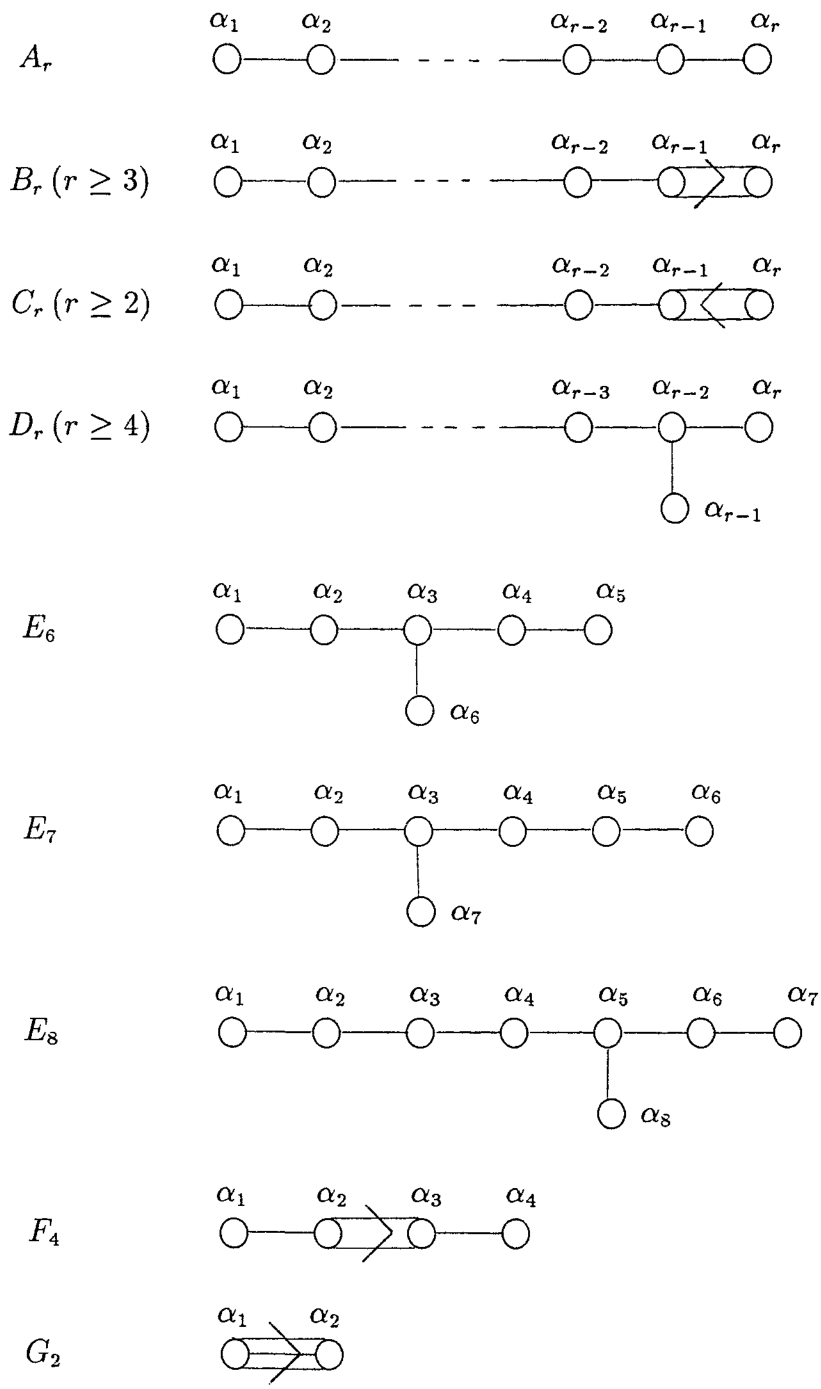


Table 1.2a : Dynkin diagrams of untwisted affine algebras.

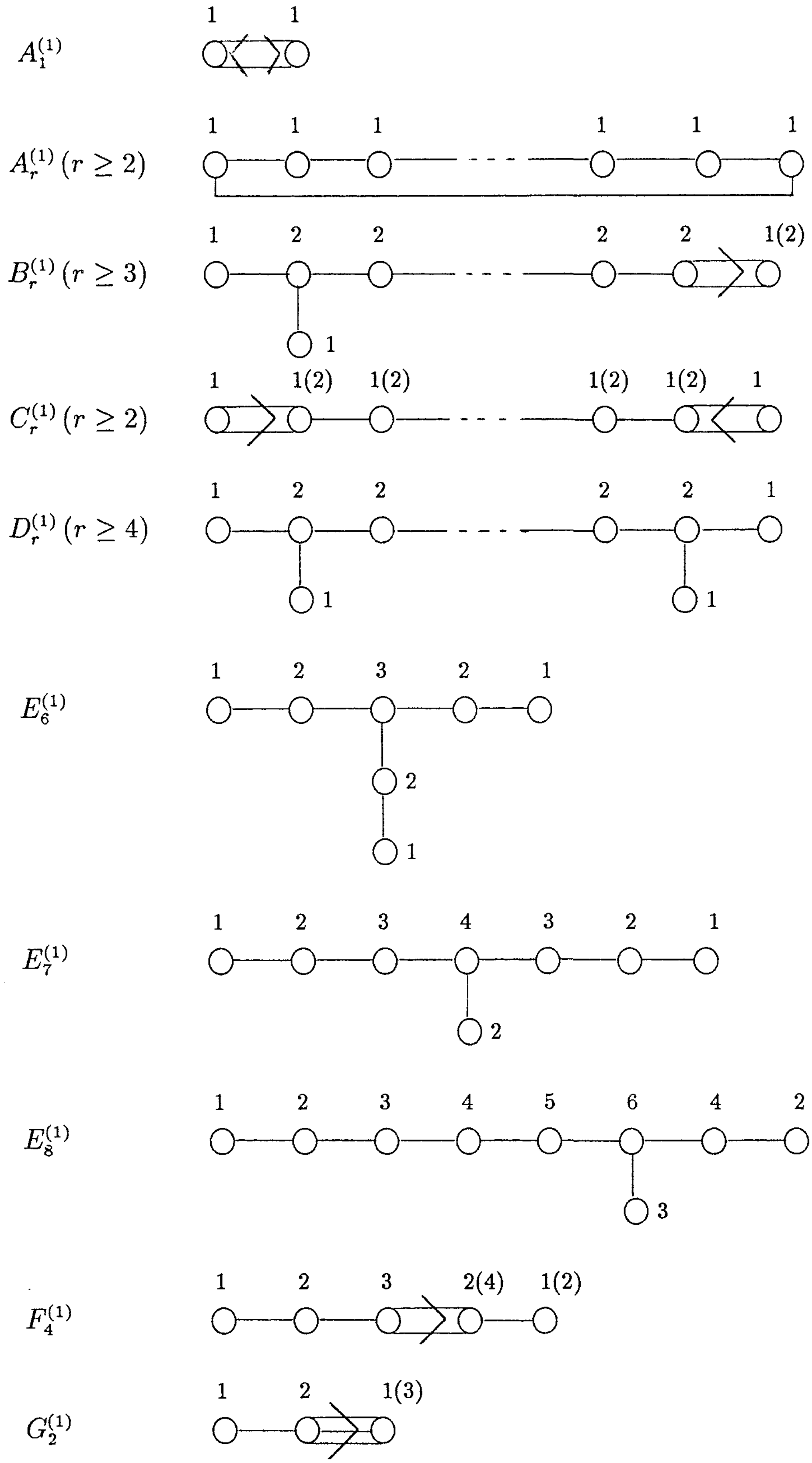
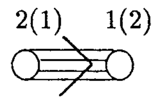
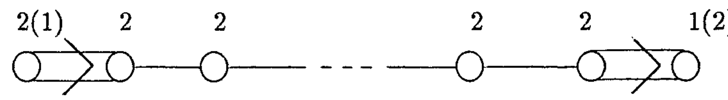


Table 1.2b : Dynkin diagrams of twisted affine algebras.

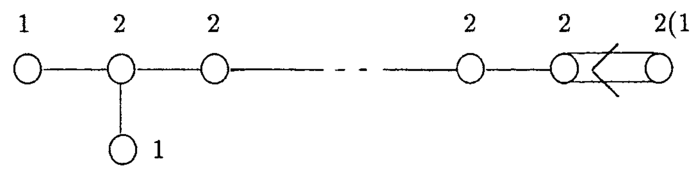
$A_2^{(2)}$



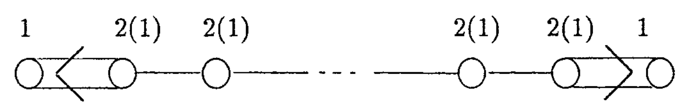
$A_{2r}^{(2)}\ (r \geq 2)$



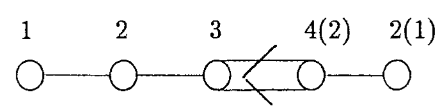
$A_{2r-1}^{(2)}\ (r \geq 3)$



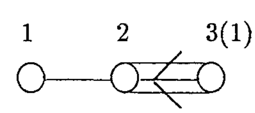
$D_{r+1}^{(2)}\ (r \geq 2)$



$E_6^{(2)}$



$D_4^{(3)}$



known as Serre's relation and the operator ad is defined as

$$(ad\, e_i)^m e_j = \overbrace{[e_i, \ldots, [e_i, [e_i}^{m \text{ times}}, e_j]] \ldots] \,.$$

The elements e_i and f_i for $i \in I$ generate subalgebras $\mathcal{N}_+$ and $\mathcal{N}_-$, respectively. Any commutator product $[x_1, [x_2, \ldots, [x_{t-1}, x_t] \ldots]]$ with $x_i = e_i, f_i$ or h_i where $i \in I$ can be expressed using the defining relations as a sum of commutators each involving only e's or only f's or only a sum of h's. We then have a direct sum of vector spaces or triangular decomposition

$$\mathcal{G}(A) = \mathcal{N}_- \oplus \mathcal{H} \oplus \mathcal{N}_+ \tag{1.1}$$

where the vectors h_i for $i \in I$ lie in the Cartan subalgebra $\mathcal{H}$. The dimension of $\mathcal{H}$ is given by

$$dim\, \mathcal{H} \;=\; 2n - r \,. \tag{1.2}$$

The centre K of the Kac-Moody algebra $\mathcal{G}(A)$, consists of elements of $\mathcal{H}$ commuting with all e_i and f_i and has dimension $n - r$. $K = 0$ if and only if A is nonsingular.

Let $\alpha_i \in \mathcal{H}^*$ be n linear functionals defined on $\mathcal{H}$ as follows:

$$[h_i, e_j] = \alpha_j(h_i) e_j \equiv A_{ji} e_j \quad i, j \in I \,. \tag{1.3}$$

The dimension of the dual space $\mathcal{H}^*$ is the same as $\mathcal{H}$. When $n = r$, the elements h_i and α_i for $i \in I$ span $\mathcal{H}$ and $\mathcal{H}^*$ respectively, otherwise further elements are needed to complete both bases. The set of linear functionals α_i, $i \in I$ are called the simple roots of the Kac-Moody algebra $\mathcal{G}(A)$. The roots α_i and $-\alpha_i$ generate the root subspaces $\mathcal{G}_{\alpha_i} = \mathbb{C} e_i$ and $\mathcal{G}_{-\alpha_i} = \mathbb{C} f_i$ respectively. Other non-zero commutators of the form

$$[e_i, e_{i'}], \quad [e_i, [e_{i'}, e_{i''}]] \quad \text{etc.}$$

$$[f_i, f_{i'}], \quad [f_i, [f_{i'}, f_{i''}]] \quad \text{etc.}$$

belong to root subspaces $\mathcal{G}_\alpha$ for which the corresponding root $\alpha \in \mathcal{H}^*$ has the form

$$\alpha = \sum_{i=1}^{n} k_i \alpha_i$$

with integral coefficients all nonnegative or all nonpositive. Here $| k_i |$ is the number of times the generator e_i or f_i appears in the corresponding commutator. We call α positive (resp. negative) if $k_i \geq 0$ (resp. $k_i \leq 0$). By relation R1 of the Definition 1.4, it is sometimes convenient to regard $\mathcal{H}$ as being the subspace of $\mathcal{G}(A)$ corresponding to a zero root and to write $\mathcal{H} = \mathcal{G}_0$. We then have the following root space decomposition with respect to $\mathcal{H}$

$$\mathcal{G}(A) = \bigoplus_{\alpha \in \mathcal{H}^*} \mathcal{G}_\alpha \, , \tag{1.4}$$

where $\mathcal{G}_\alpha = \{x \in \mathcal{G}(A) \mid [h, x] = \alpha(h)x \quad \text{for all } h \in \mathcal{H}\}$ is the root subspace attached to α. The dimension of the root subspace $\mathcal{G}_\alpha$ is known as the multiplicity, *mult* α, of the root α. For a simple finite-dimensional Lie algebra, the multiplicity of a non-zero root is always unity.

Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ be sets of an n independent elements of $\mathcal{H}^*$ and $\mathcal{H}$ respectively. These basis vectors are related through a bilinear form on $\mathcal{H}^* \times \mathcal{H}$ defined by

$$\alpha_i(\alpha_j^\vee) \equiv < \alpha_i, \alpha_j^\vee >= A_{ij} \, . \tag{1.5}$$

We call the elements of Π and $\Pi^\vee$ simple roots and simple co-roots respectively. Let the root lattice and co-root lattice respectively then be

$$Q = \mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_n] \qquad \text{and} \qquad Q^\vee = \mathbb{Z}[\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee].$$

If A is a GCM then its transpose A^t is again a GCM. The algebras $\mathcal{G}(A)$ and $\mathcal{G}(A^t)$ are called dual to each other. If $Q^\vee$ is a co-root lattice of $\mathcal{G}(A)$ then $Q^\vee$ is the root lattice of $\mathcal{G}(A^t)$. We can also introduce a partial ordering $\geq$ on Q by setting

$$\lambda \geq \mu \text{ if } \lambda - \mu \in Q^+ = \mathbb{Z}^+[\alpha_1, \alpha_2, \dots, \alpha_n] \, . \tag{1.6}$$

The geometry of the root system of a simple Lie algebra is encoded in the Dynkin diagram which carries the relative lengths of the simple roots and the angles between them. We can speak of long and short roots. If all roots are equal in length then it is conventional to call them long. The arrows in the Dynkin diagrams of Tables 1.1 and 1.2 are pointing toward the short simple roots. We denote the set of all non-zero roots of $\mathcal{G}(A)$ by Δ, the set of positive roots by Δ^+ and the set of negative roots by Δ^-. Then by (1.1) and (1.4), we have

$$\Delta = \Delta^- \cup \Delta^+. \tag{1.7}$$

1.3 The Weyl group

Given a Kac-Moody algebra $\mathcal{G}(A)$, the Weyl group $W(A)$ or simply W is a group generated by fundamental reflections in the hyperplanes perpendicular to the simple roots. For each $i \in I$, the fundamental reflection s_i of the space $\mathcal{H}^*$ is defined by

$$s_i(\lambda) = \lambda - < \lambda, \alpha_i^\vee > \alpha_i\,. \tag{1.8}$$

This really defines a reflection in that it fixes the subspace known as the reflection hyperplane

$$H_{\alpha_i} = \{\lambda \in \mathcal{H}^* \mid < \lambda, \alpha_i^\vee >= 0\} \quad \text{for } i \in I\,, \tag{1.9}$$

and sends α_i to $-\alpha_i$.

If α is a root then $s_i(\alpha)$ is also a root. If a root $\beta = w(\alpha)$ for some $w \in W$ then we say β is W-conjugate to the root α. However, not every root is W-conjugate to a simple root. We define the set Δ_R of real roots to be the W-conjugate of the simple roots and the set Δ_I of imaginary roots to be $\Delta \backslash \Delta_R$. For simple finite-dimensional Lie algebras all roots are real but for affine algebras there exists imaginary roots which are not W-conjugate to any real root.

Next we fix an important element $\rho \in \mathcal{H}^*$ satisfying $< \rho, \alpha_i^\vee >= 1$, for all $1 \leq i \leq n$. In general this does not define ρ uniquely. However if $\mathcal{G}$ is simple finite-dimensional, ρ is actually equal to half the sum of the positive roots. With these definitions, we have in particular

$$s_i(\rho) = \rho - \alpha_i \,.$$

We also define the shifted (or dot) action of W on $\mathcal{H}^*$ by

$$w_\bullet\lambda = w(\lambda + \rho) - \rho \quad \text{for any } w \in W \text{ and } \lambda \in \mathcal{H}^* \,. \tag{1.10}$$

Observe that the action $\bullet$ is independent of any freedom that may exist in the choice of ρ.

Lemma 1.5. *The fundamental reflection s_i permutes the positive roots other than α_i.*

Proof Let $\alpha \in \Delta^+$ and $\alpha \neq \alpha_i$. If $\alpha = \sum_j k_j \alpha_j$ with $k_j > 0$ for some $j \neq i$, then

$$\begin{aligned} s_i(\alpha) &= \sum_j k_j(\alpha_j - A_{ji}\alpha_i) \\ &= \sum_{j \neq i} k_j \alpha_j - (\sum_{j \neq i} k_j A_{ji} + k_i)\alpha_i \end{aligned}$$

Since the coefficient of α_j is positive, this implies that $s_i(\alpha) \in \Delta_+$ □

A group such as the Weyl group with generators $s_1, \ldots, s_n$ and defining relations

$$s_i^2 = id \quad i \in I; \quad (s_i s_j)^{m_{ij}} = id \quad i, j \in I$$

is called a Coxeter group. For the Weyl group, the values of the m_{ij} are given by the following table **[Kac4]**:

Table 1.3 : The order of the element $s_i s_j$ of Coxeter groups $W(A)$

$A_{ij}A_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

We see that every element of the Weyl group can be written as a product of fundamental reflections $w = s_{i_1}s_{i_2}\dots s_{i_t}$. By Lemma 1.5 we have

$$\begin{aligned} s_i\Delta &= s_i(\Delta^+\backslash\{\alpha_i\} \cup \{\alpha_i\} \cup \{-\alpha_i\} \cup \Delta^-\backslash\{-\alpha_i\}) \\ &= \Delta^+\backslash\{\alpha_i\} \cup \{-\alpha_i\} \cup \{\alpha_i\} \cup \Delta^-\backslash\{-\alpha_i\}) \\ &= \Delta \end{aligned}$$

and hence $w \in W$ permutes the root system Δ.

For $i = 1, \dots, n$ the fundamental reflection s_i acts on $h \in \mathcal{H}$ as follows

$$s_i(h) = h - < \alpha_i, h > \alpha_i^\vee . \tag{1.11}$$

For $\lambda \in \mathcal{H}^*$ and $h \in \mathcal{H}$ we have

$$\begin{aligned} < s_i\lambda, h > &= < \lambda - < \lambda, \alpha_i^\vee > \alpha_i, h > \\ &= < \lambda, h > - < \lambda, \alpha_i^\vee >< \alpha_i, h > \\ &= < \lambda, h - < \alpha_i, h > \alpha_i^\vee > \\ &= < \lambda, s_i h > . \end{aligned}$$

More generally $< w\lambda, h > = < \lambda, w^{-1}h >$ which implies that the bilinear form $< \cdot, \cdot >$ is W-invariant.

Definition 1.6. The expression $w = s_{i_1}s_{i_2}\dots s_{i_t}$ is called reduced if t is minimal possible among all representations of $w \in W$. t is called the length of w and is denoted by $\ell(w)$. The parity of w is defined to be $\epsilon(w) = (-1)^{\ell(w)}$.

Since $w^{-1} = s_{i_t}s_{i_{t-1}}\dots s_{i_1}$, this implies that $\ell(w) = \ell(w^{-1})$.

Lemma 1.7. **[Kac4]** *Let $w = s_{i_1}\dots s_{i_t} \in W$ be of minimal length t. Then we have*

(a) $\ell(ws_i) < \ell(w)$ if and only if $w(\alpha_i) < 0$,

(b) $w(\alpha_{i_t}) < 0$.

Definition 1.8. **[Ko]** Define the following important set

$$\Phi_w = w\Delta^- \cap \Delta^+ = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) < 0\}.$$

Since Δ^+ and Δ^- are disjoint sets, then the set Φ_{id} is empty. However for $i \in I$,

$$\Phi_{s_i} = s_i\Delta^- \cap \Delta^+ = \{\alpha \in \Delta^+ \mid s_i(\alpha) < 0\}$$

and by Lemma 1.5, we have $\Phi_{s_i} = \{\alpha_i\}$.

Lemma 1.9. *If $\alpha_i \notin \Phi_w$ then $\Phi_{s_i w} = s_i\Phi_w \cup \{\alpha_i\}$.*

Proof

$$\begin{aligned} \alpha_i \in \Phi_{s_i w} \quad &\Leftrightarrow \quad (s_i w)^{-1}(\alpha_i) < 0 \\ &\Leftrightarrow \quad w^{-1}(\alpha_i) > 0 \\ &\Leftrightarrow \quad \alpha_i \notin \Phi_w \, . \end{aligned}$$

Hence α_i is in precisely one of Φ_w or $\Phi_{s_i w}$. Then by hypothesis $\alpha_i \in \Phi_{s_i w}$ so that $\Phi_w = w\Delta^- \cap (\Delta^+ \backslash \{\alpha_i\})$ and by Lemma 1.5 $s_i\Phi_w = s_i w\Delta^- \cap (\Delta^+ \backslash \{\alpha_i\})$. In addition $\alpha_i \in \Phi_{s_i w}$ and hence

$$\begin{aligned} \Phi_{s_i w} &= s_i w\Delta^- \cap \Delta^+ \\ &= (s_i w\Delta^- \cap \Delta^+ \backslash \{\alpha_i\}) \cup (\{\alpha_i\} \cap s_i w\Delta^-) \\ &= s_i\Phi_w \cup s_i w((s_i w)^{-1}\{\alpha_i\} \cap \Delta^-) \\ &= s_i\Phi_w \cup s_i w((s_i w)^{-1}\{\alpha_i\}) \\ &= s_i\Phi_w \cup \{\alpha_i\}. \end{aligned}$$

□

Proposition 1.10. $\ell(w) =$ *card* $\{\alpha \in \Delta^+ \mid w^{-1}(\alpha) < 0\} = \mid \Phi_w \mid$.

Proof We prove this Proposition by induction on the length of w. By definition, $\ell(id) = 0$ and $\ell(s_i) = 1$. The Proposition is trivial for $w = id$ and since $\Phi_{s_i} = \{\alpha_i\}$ the Proposition is also true for $w = s_i$. Assume that it is true for all $u \in W$ with $\ell(u) < \ell(w)$. Let $w = s_{i_1} \dots s_{i_t}$ have minimal length t. Then $w^{-1} = s_{i_t} \dots s_{i_1}$ also has minimal length t and by Lemma 1.7(b), $w^{-1}(\alpha_{i_1}) < 0$. Hence $\alpha_{i_1} \in \Phi_w = \{\alpha \in \Delta^+ \mid w^{-1}(\alpha) < 0\}$. From Lemma 1.9 we can then deduce that $\Phi_w = s_{i_1}\Phi_u \cup \{\alpha_{i_1}\}$ where

$w = s_{i_1}u$ and this implies that $|\Phi_w| = |\Phi_u| + 1$. On the other hand $\ell(u) = \ell(w) - 1$ and by induction $\ell(u) = |\Phi_u|$. Hence $\ell(w) = |\Phi_w|$. □

Lemma 1.9 and Proposition 1.10 implies that if $s_i w$ is a reduced form then $\ell(s_i w) = \ell(w)+1$. This tell us how to compute the set Φ_w with w of length t. Let $w = s_{i_1}s_{i_2}\dots s_{i_t}$. Then

$$\begin{aligned}\Phi_{s_{i_1}s_{i_2}\dots s_{i_t}} &= \{\alpha_{i_1}\} \cup s_{i_1}\Phi_{s_{i_2}\dots s_{i_t}} \\ &= \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2})\} \cup s_{i_1}s_{i_2}\Phi_{s_{i_3}\dots s_{i_t}} \\ &\vdots \\ &= \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1}s_{i_2}\dots s_{i_{t-1}}(\alpha_{i_t})\}.\end{aligned} \tag{1.12}$$

In general **[Liu]**, if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ then it follows directly from (1.12) that $\Phi_{w_1w_2} = \Phi_{w_1} \cup w_1\Phi_{w_2}$.

Proposition 1.11. $\rho - w(\rho) = \sum_{\alpha\in\Phi_w} \alpha$.

Proof Again we prove this Proposition by induction on the length of w. First $\rho - s_i(\rho) = \alpha_i$ and $\Phi_{s_i} = \{\alpha_i\}$. Hence it is true for $\ell(w) = 1$. Assume that it is true for all $\ell(u) < \ell(w)$. Let $w = s_{i_1}s_{i_2}\dots s_{i_t}$ be a reduced form for w and set $u = s_{i_2}\dots s_{i_t}$. This a minimal expression for u so that $\ell(u) = \ell(w) - 1$. Then

$$\begin{aligned}\rho - w(\rho) = \rho - s_{i_1}u(\rho) &= \rho - s_{i_1}\rho + s_{i_1}(\rho - u(\rho)) \\ &= \alpha_{i_1} + s_{i_1}\sum_{\alpha\in\Phi_u}\alpha\,.\end{aligned}$$

Hence by Lemma 1.9 $\rho - w(\rho) = \sum_{\alpha\in\Phi_{w=s_{i_1}u}} \alpha$. □

1.4 Highest weight modules

Definition 1.12 Let $\mathcal{G}$ be a Lie algebra over $\mathbb{C}$. A vector space V endowed with an operation $\mathcal{G}\times V \to V$ is called a $\mathcal{G}$-module if for all $x, y \in \mathcal{G}$, $v, w \in V$ and $a, b \in \mathbb{C}$ the following conditions are satisfied:

(M1) $(ax + by)\cdot v = a(x\cdot v) + b(y\cdot v)$;

(M2) $x \cdot (av + bw) = a(x \cdot v) + b(x \cdot w)$;

(M3) $[x, y] \cdot v = x \cdot y \cdot v - y \cdot x \cdot v$.

The dimension of a $\mathcal{G}$-module is the dimension of the underlying vector space. A $\mathcal{G}$-module is called irreducible if it has no proper $\mathcal{G}$-submodules.

An equivalent concept to the idea of a $\mathcal{G}$-module is a representation ψ of $\mathcal{G}$. By a representation ψ we meant a homomorphism of $\mathcal{G}$ into the general linear algebra of a vector space V. Given a representation $\psi : \mathcal{G} \to g\ell(V)$ the vector space V becomes a module of $\mathcal{G}$ via the action $x \cdot v = \psi(x)v$. Conversely, given a $\mathcal{G}$-module V, the same action defines a representation $\psi : \mathcal{G} \to g\ell(V)$.

A $\mathcal{G}$-module V is called $\mathcal{H}$-diagonalisable if

$$V = \bigoplus_{\lambda \in \mathcal{H}^*} V_\lambda$$

where $V_\lambda = \{v \in V \mid h(v) = \lambda(h)v \quad \text{for} \quad h \in \mathcal{H}\}$. V_λ is called a weight subspace, $\lambda \in \mathcal{H}^*$ is called a weight if $V_\lambda \neq \emptyset$ and the dimension of the weight subspace V_λ is called the multiplicity of λ and is denoted by *mult* λ(or *dim* V_λ). Viewing $\mathcal{G}(A)$ itself as a $\mathcal{G}(A)$-module, we see that the weights are the roots $\alpha \in \Delta$ (with weight subspace $\mathcal{G}_\alpha$) along with 0 (with weight subspace the Cartan subalgebra $\mathcal{H}$).

Let $\mathcal{G}(A) = \oplus_{\alpha \in \Delta \cup \{0\}} \mathcal{G}_\alpha$ be a root space decomposition with respect to $\mathcal{H}$ of a Kac-Moody algebra with GCM A and simple roots $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Let

$$P = \{\lambda \in \mathcal{H}^* \mid \lambda(\alpha_i^\vee) \in \mathbb{Z},\ i \in \{1, 2, \ldots, n\}\}$$

$$P^+ = \{\lambda \in P \mid \lambda(\alpha_i^\vee) \geq 0,\ i \in \{1, 2, \ldots, n\}\} .$$

The set P is called the weight lattice and the elements of P^+ are called dominant weights. Given an element $\Lambda \in P^+$, it is always possible to form an irreducible $\mathcal{G}(A)$-module V^Λ known as a highest weight module with highest weight Λ that satisfies the following properties **[Kac4]**, **[KMPS]**:

(a) V^Λ is $\mathcal{H}$-diagonalisable ;

(b) V_Λ^Λ is 1-dimensional and $\mathcal{G}_\alpha V_\Lambda^\Lambda = 0$ for all $\alpha \in \Delta^+$;

(c) $\mathcal{G}_\alpha V_\lambda^\Lambda \subset V_{\alpha+\lambda}^\Lambda$.

This irreducible highest weight module is determined up to isomorphism by its highest weight and up to isomorphism these modules are in one-to-one correspondence with the dominant weights of $\mathcal{G}(A)$.

It is convenient to introduce a set of fundamental weights $\Lambda_i \in \mathcal{H}^*$ for $i \in I$ such that $<\Lambda_i, \alpha_k^\vee>= \delta_{ik}$ for all $i, k \in I$ and a set of vectors $\delta_j \in \mathcal{H}^*$ for $j \in J = \{n+1, \ldots, n-r\}$ such that $<\delta_j, \alpha_k^\vee>= 0$ for all $j \in J$ and $k \in I$, where Λ_i for $i \in I$ and δ_j for $j \in J$ span $\mathcal{H}^*$. Then any vector $\lambda \in \mathcal{H}^*$ can be written in form

$$\lambda = \sum_{i=1}^{n} \lambda_i \Lambda_i + \sum_{i=n+1}^{n-r} n_i \delta_i \tag{1.13}$$

where the Dynkin labels λ_i are given by $\lambda_i =< \lambda, \alpha_i^\vee >$ for $i \in I$. In particular, in the case of a simple root α_k,

$$(\alpha_k)_i = A_{k,i} . \tag{1.14}$$

Denote the set of all weights of V^Λ by $P(\Lambda)$. Every element $\lambda \in P(\Lambda)$ is of the form $\lambda = \Lambda - \alpha$ for $\alpha \in Q^+$. The distinct weights of $P(\Lambda)$ written in Dynkin notation can be obtained from the highest weight $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ by applying the following algorithm **[KMPS]** :

(S1) Assign Λ to $P(\Lambda)$ and let $\lambda = \Lambda$;

(S2) For any positive Dynkin coordinate λ_i of λ assign to $P(\Lambda)$ the λ_i weights

$\lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \lambda - \lambda_i\alpha_i$ for $i = 1, \ldots, n$;

(S3) Repeat step S2, replacing λ by each new weight just found in S2.

The weights $\lambda \in P(\Lambda)$ can be partitioned into Weyl group orbits (W-orbit). The W-orbit of a weight λ is defined to be the set $\{w\lambda \mid w \in W\}$ and for each weight λ of W-orbit there exist a unique dominant weight $\lambda^+ \in P^+$ such that $\lambda = w'\lambda^+$ for some

$w' \in W$. Orbit labels are then taken to be the components of the highest weight of the orbit. If $\mu \in P^+$, define the orbit sum as

$$\Omega^\mu = \sum_{w \in \{W : W_\mu\}} e^{w\mu} \tag{1.15}$$

where $\{W : W_\mu\}$ denotes the set of left coset representatives of W with respect to the stabilizer $W_\mu = \{w \mid w\mu = \mu,\, w \in W\}$ of μ.

The set of weights of V^Λ is invariant under the action of the Weyl group W of $\mathcal{G}(A)$ and also $dim\, V^\Lambda_{w(\lambda)} = dim\, V^\Lambda_\lambda$ for all $w \in W$ and $\lambda \in P(\Lambda)$. Since each weight is conjugate under the Weyl group to a dominant weight, it suffices to determine only the multiplicities of $\mu \in P^+ \cap P(\Lambda)$.

1.5 The Weyl-Kac character formula

Let V^Λ be an irreducible highest weight module. The character of V^Λ is the formal exponential

$$ch\; V^\Lambda = \sum_{\lambda \in \mathcal{H}^*} (dim\; V^\Lambda_\lambda)\, e^\lambda\,, \tag{1.16}$$

where for $\lambda \in \mathcal{H}^*$ e^λ is the function $h \to e^{<\lambda,h>}$ on $\mathcal{H}$ converging absolutely on a nonempty open subset of $\mathcal{H}$ **[KaP]**. This definition means that a knowledge of the character of the irreducible highest weight module is equivalent to knowing its weight system and the multiplicity of each weight. In the case of simple finite-dimensional Lie algebras, Weyl has given a precise formula for this character and in the case of a general Kac-Moody algebra essentially the same formula was proven by Kac **[Kac2]**. The Weyl-Kac character formula is given by

$$ch\; V^\Lambda = \sum_{w \in W} \varepsilon(w) e^{w(\Lambda+\rho)-\rho} / \prod_{\alpha \in \Delta+} (1 - e^{-\alpha})^{mult\, \alpha}\,, \tag{1.17}$$

where $\rho \in \mathcal{H}^*$ is defined by $\rho = \sum_{i=1}^n \Lambda_i$. Setting $\Lambda = 0$ in the above character formula, we can deduce the following Weyl-Kac denominator identity

$$\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{mult\,\alpha}=\sum_{w\in W}\varepsilon(w)e^{w(\rho)-\rho}. \tag{1.18}$$

This then gives another form of the character formula:

$$ch\ V^{\Lambda}=\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)}/\sum_{w\in W}\varepsilon(w)e^{w(\rho)}. \tag{1.19}$$

Unfortunately, getting from the character formula to the weight multiplicities is not entirely straightforward because the character formula is a quotient of two alternating sums. However it can be reorganized to provide an effective way to compute the individual weight multiplicities.

In the case of a simple finite-dimensional Lie algebra there are a number of methods available for computing weight multiplicities. The Kostant formula provides a closed form expression for the multiplicity $mult\,\lambda$ for any weight λ of the irreducible module with highest weight Λ **[J]**:

$$mult\,\lambda=\sum_{w\in W}\varepsilon(w)P(\lambda+\rho-w(\Lambda+\rho))\,,$$

where $P(\mu)$ is the number of ways of writting μ as a linear combination of positive roots with nonnegative integers as coefficient. Alternatively the Racah formula **[R]** provides a recursion relation for the multiplicities of the weights:

$$mult\,\lambda=-\sum_{w\neq id}\varepsilon(w)\,mult\,(\lambda+\rho-w(\rho))\,.$$

Both of these formulae are a consequence of the Weyl-Kac character formula and depend on the generation of the Weyl group for the computation of the weight multiplicities. Another method of computing weight multiplicities due to Freudenthal is also a recursion formula but this time it avoids the Weyl group and can therefore handle Lie algebras of larger rank. This recursion relation is **[J]**

$$[(\Lambda+\rho\mid\Lambda+\rho)-(\lambda+\rho\mid\lambda+\rho)]\,mult\,\lambda=2\sum_{\alpha>0}\sum_{k>0}(\lambda+k\alpha\mid\alpha)mult\,(\lambda+k\alpha)\,.$$

It gives the multiplicity of a weight in terms of the multiplicities of the weights that are higher than it under a certain ordering. The use of the Freudenthal's formula can be made more efficient by exploiting the fact that weights conjugate under the Weyl group have the same multiplicities. Extensive tables of weight multiplicities have been tabulated **[BMP]** using this method.

Recently Patera and Sharp **[PS]** revived a method that can be traced back to Speiser **[Sp]** for computing weight multiplicities of a highest weight module and the branching rules of simple finite-dimensional Lie algebras. The idea is to write the orbit sum expansion of (1.15) in terms of irreducible characters. The orbit-character matrix of suitably ordered weights is triangular with ones on the diagonal and therefore can be easily inverted to obtain the character-orbit matrix whose components are the weight multiplicities.

Let $\lambda \in P^+$ and $dim\ V_\kappa^\lambda$ be the multiplicity of a weight κ of V^λ module. Then

$$\begin{aligned} ch\ V^\lambda &= \sum_\kappa (dim\ V_\kappa^\lambda) e^\kappa \\ &= \sum_{\mu \in P+} (dim\ V_\mu^\lambda) \sum_{w \in \{W:W_\mu\}} e^{w\mu} \\ &= \sum_{\mu \in P+} (dim\ V_\mu^\lambda)\, \Omega^\mu. \end{aligned} \tag{1.20}$$

The orbit sum Ω^μ can be expressed in terms of irreducible characters by inverting the weight multiplicity matrix $dim\ V_\mu^\lambda$ to give

$$\Omega^\mu = \sum_\lambda B_\lambda^\mu (ch\ V^\lambda)\,. \tag{1.21}$$

On substituting the Weyl-Kac character formula (1.19), this gives:

$$\Omega^\mu = \sum_\lambda B_\lambda^\mu \Big(\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)} \Big/ \sum_{w \in W} \varepsilon(w) e^{w(\rho)} \Big).$$

So that

$$\Omega^\mu \sum_{w \in W} \varepsilon(w) e^{w(\rho)} = \sum_\lambda B_\lambda^\mu \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}$$

and

$$\sum_{w'\in\{W:W_\mu\}} e^{w'\mu} \sum_{w\in W} \varepsilon(w) e^{w(\rho)} = \sum_\lambda B^\mu_\lambda \sum_{w\in W} \varepsilon(w) e^{w(\lambda+\rho)} .$$

However, the only dominant weight of $w(\lambda+\rho)$ is $\lambda+\rho$ so that B^μ_λ is the coefficient of $e^{\lambda+\rho}$ on both sides of this equation. Hence

$$B^\mu_\lambda = \sum_{w'\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w) \delta_{w'\mu+w\rho,\lambda+\rho} .$$

Furthermore, for a fixed $w' \in \{W : W_\mu\}$ and $w \in W$ there must exist $\hat{w} \in \{W : W_\mu\}$ such that $w^{-1}w'(\mu) = \hat{w}(\mu)$. Moreover for fixed $w \in W$ there is a one-to-one correspondence between w' and $\hat{w}$. Then

$$B^\mu_\lambda = \sum_{\hat{w}\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w) \delta_{w(\hat{w}\mu+\rho)-\rho,\lambda} . \qquad (1.22a)$$

Hence the elements of B^μ_λ for the expansion of the orbit sum in term of irreducible characters may then be obtained by adding ρ to each weight of the orbit of μ, reflecting each weight into the dominant sector, subtracting ρ and interpreting the result as a signed, positive or negative, coefficient of λ according to whether an even or odd number of elementary reflections is required. A reflected weight lying on a reflection hyperplane is ignored.

Alternatively,

$$\begin{aligned} B^\mu_\lambda &= \sum_{w'\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w) \delta_{w'\mu,\lambda+\rho-w\rho} \\ &= \sum_{w'\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w) \delta_{w^{-1}w'\mu,w^{-1}(\lambda+\rho)-\rho} \\ &= \sum_{\hat{w}\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w) \delta_{\hat{w}\mu,w(\lambda+\rho)-\rho} \\ &= \sum_{\hat{w}\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w) \delta_{\hat{w}\mu,w\bullet\lambda} \end{aligned} \qquad (1.22b)$$

where the dot action is as defined in (1.10). The interpretation of (1.22b) is that we plot the Weyl orbit of μ and the Weyl dot orbit of λ and look for their intersection weights. The sign of the parity of the Weyl dot orbit of λ is taken to be the sign of B^μ_λ.

Under the partial ordering (1.6) of the weight lattice, the matrix B^{μ}_{λ} is triangular and may be inverted to obtain the required weight multiplicities.

1.6 The Weyl-Kostant-Liu character formula

Let $U = \{1, 2, \ldots, u\} \subset I$. Consider the subalgebra $\mathcal{G}_U$ of $\mathcal{G}(A)$ generated by the elements e_i, f_i $(i = 1, \ldots, u)$ and $\mathcal{H}$. Denote by Δ_U^+ the set of positive roots generated by $\alpha_1, \alpha_2, \ldots, \alpha_u$ and let $\Delta_U^- = -\Delta_U^+$. Then much like (1.1) and (1.7) $\mathcal{G}_U$ has a triangular decomposition $\mathcal{G}_U = \mathcal{N}_U^- \oplus \mathcal{H} \oplus \mathcal{N}_U^+$ with $\Delta_U = \Delta_U^+ \cup \Delta_U^-$ as its root system **[Liu]**. For dominant integral weights let

$$P_U^+ = \{\lambda \in \mathcal{H}^* \mid < \lambda, \alpha_i^\vee > \geq 0, \quad i \in U\}. \tag{1.23}$$

Further let W_U be the Weyl group of $\mathcal{G}_U$ generated by $s_1, \ldots, s_u$ and let

$$W(U) = \{w \in W \mid \Phi_w \subset \Delta^+ \backslash \Delta_U^+\}. \tag{1.24}$$

The significance of this choice of $W(U)$ lies in the following lemma.

Lemma 1.13. *If $\lambda \in P^+$ and $w \in W(U)$ then $w(\lambda + \rho) - \rho \in P_U^+$.*

Proof For any $w \in W(U)$ and $i \in U$ we have $\alpha_i \in \Delta_U^+$ so that by (1.24) $\alpha_i \notin \Phi_w$. It then follows from Definition 1.8 that $w^{-1}(\alpha_i) > 0$ and this implies that in $\mathcal{H}$ space we should be able to write

$$w^{-1}(\alpha_i^\vee) = \sum_j k_j \alpha_j^\vee$$

with all coefficients k_j nonnegative integers. Then for any $\lambda \in P^+$ we have

$$< w(\lambda + \rho), \alpha_i^\vee > = < \lambda + \rho, w^{-1}(\alpha_i^\vee) > = < \lambda + \rho, \sum_j k_j \alpha_j^\vee > > 0$$

since $< \lambda, \alpha_j^\vee > \in \mathbb{Z}^+$ and $< \rho, \alpha_j^\vee > = 1$ for all j. Now since $< \rho, \alpha_i^\vee > = 1$ it follows that $< w(\lambda + \rho) - \rho, \alpha_i^\vee > \geq 0$ so that $w(\lambda + \rho) - \rho \in P_U^+$. □

Before we arrive at our next important result, we just state the following lemma [**Liu**] which shows that $W(U)$ is in fact $\{W : W_U\}$, the set of right coset representatives of W with respect to W_U.

Lemma 1.14. *Every element $w \in W$ can be uniquely written as $w = \bar{u}v$ where $\bar{u} \in W_U$ and $v \in W(U)$.*

Theorem 1.15. *For $\Lambda \in P^+$*

$$ch\ V^\Lambda = \frac{\sum_{w\in\{W:W_U\}} \varepsilon(w)\, ch\ \overline{V}^{w(\Lambda+\rho)-\rho}}{\sum_{w\in\{W:W_U\}} \varepsilon(w)\, ch\ \overline{V}^{w(\rho)-\rho}} \tag{1.25}$$

where $ch\ \overline{V}^\mu$ is a formal character defined for all $\mu \in P_U^+$ by

$$ch\ \overline{V}^\mu = \frac{\sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}(\mu+\rho)-\rho}}{\sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}(\rho)-\rho}}.$$

Proof The Weyl-Kac character formula (1.19) and Lemma 1.14 imply

$$\begin{aligned}
ch\ V^\Lambda &= \frac{\sum_{w\in W} \varepsilon(w)\, e^{w(\Lambda+\rho)-\rho}}{\sum_{w\in W} \varepsilon(w)\, e^{w(\rho)-\rho}} \\
&= \frac{\sum_{v\in\{W:W_U\}} \varepsilon(v) \sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}v(\Lambda+\rho)-\rho}}{\sum_{v\in\{W:W_U\}} \varepsilon(v) \sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}v(\rho)-\rho}} \\
&= \frac{\sum_{v\in\{W:W_U\}} \varepsilon(v) (\sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}[v(\Lambda+\rho)-\rho+\rho]-\rho} / \sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}[\rho]-\rho})}{\sum_{v\in\{W:W_U\}} \varepsilon(v) (\sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}[v(\rho)-\rho+\rho]-\rho} / \sum_{\bar{u}\in W_U} \varepsilon(\bar{u}) e^{\bar{u}[\rho]-\rho})} \\
&= \frac{\sum_{v\in\{W:W_U\}} \varepsilon(v)\, ch\ \overline{V}^{v(\Lambda+\rho)-\rho}}{\sum_{v\in\{W:W_U\}} \varepsilon(v)\, ch\ \overline{V}^{v(\rho)-\rho}}
\end{aligned}$$

□

When $\mathcal{G}$ and $\mathcal{G}_U$ are both simple finite-dimensional Lie algebras this formula was first given by Kostant [**Ko**] and in the general case of Kac-Moody algebras it was proved by Liu [**Liu**]. Accordingly we shall refer to this important character formula as the Weyl-Kostant-Liu character formula. This character formula provides a means of expressing weight multiplicities of affine algebras in terms of known weight multiplicities of simple finite-dimensional Lie algebras. The idea behind its use is to transform summations over affine weights directly into irreducible characters of simple finite-dimensional Lie algebras. In general to be able to use the Weyl-Kostant-Liu character formula we must first be able to identify the elements of $\{W : W_U\}$. The following proposition **[Kang]** is very helpful in the explicit computation of $\{W : W_U\}$.

Proposition 1.16. *Let $w' = ws_k$ and $\ell(w') = \ell(w) + 1$. Then $w' \in \{W : W_U\}$ if and only if $w \in \{W : W_U\}$ and $w(\alpha_k) \in \Delta^+ \backslash \Delta_U^+$.*

Proof Let $\ell(w) = j$ with $w = s_{i_1} s_{i_2} \dots s_{i_j}$. Then by (1.12)

$$\begin{aligned}\Phi_{ws_k} &= \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1}s_{i_2}\dots s_{i_{j-1}}(\alpha_{i_j}), w(\alpha_k)\} \\ &= \Phi_w \cup \{w(\alpha_k)\}.\end{aligned}$$

Hence $\Phi_{w'} \subseteq \Delta^+ \backslash \Delta_U^+$ if and only if $\Phi_w \subseteq \Delta^+ \backslash \Delta_U^+$ and $w(\alpha_k) \in \Delta^+ \backslash \Delta_U^+$. Then, from (1.24) with $W(U) = \{W : W_U\}$ it follows that $w' \in \{W : W_U\}$ if and only if $w \in \{W : W_U\}$ and $w(\alpha_k) \in \Delta^+ \backslash \Delta_U^+$. □

More generally it can be shown that if $w' = w_1 w_2$ and $\ell(w') = \ell(w_1) + \ell(w_2)$ then $w' \in \{W : W_U\}$ if and only if $w_1 \in \{W : W_U\}$ and $w_1 \Phi_{w_2} \subseteq \Delta^+ \backslash \Delta_U^+$. The result follows from the fact that $\Phi_{w_1 w_2} = \Phi_{w_1} \cup w_1 \Phi_{w_2}$.

CHAPTER 2

Representations of Simple Finite-dimensional Lie Algebras

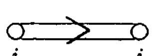
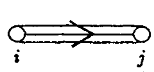
2.1 Root system and Weyl group

The complex simple finite-dimensional Lie algebras have been completely classified. The finite type GCM A that corresponds to any one of these algebras is the original Cartan matrix. Since $\det A \neq 0$ and $n = r$ then by (1.2) the dimension of $\mathcal{H}$ is r and the elements α_i and $\alpha_i^\vee$ for $i = 1, 2, \ldots, r$ span $\mathcal{H}^*$ and $\mathcal{H}$ respectively. The Killing form **[H]**, which involves taking a trace, provides the standard way to define a non-degenerate symmetric bilinear form for a simple finite-dimensional Lie algebra. We normalise a symmetric bilinear form $(\cdot \mid \cdot)$ on $\mathcal{H}^*$ so that $(\alpha \mid \alpha) = 2$ for all long roots and then

$$A_{ij} = \frac{2(\alpha_i \mid \alpha_j)}{(\alpha_j \mid \alpha_j)}. \tag{2.1}$$

For neighbouring nodes i and j of any Dynkin diagram, the data on lengths and angles is as set out below. The angle θ_{ij} between roots α_i and α_j is such that $\cos\theta_{ij} = (\alpha_i \mid \alpha_j)/\sqrt{(\alpha_i \mid \alpha_i)(\alpha_j \mid \alpha_j)}$. Arrows go from long to short roots.

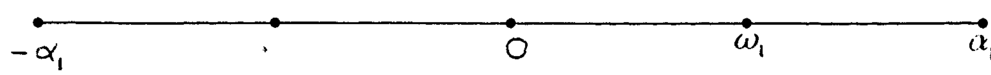
Table 2.1 : Data on neighbouring nodes and inner products.

Dynkin diagram	A_{ij}	Short root $(\alpha_i \mid \alpha_i)$	Long root $(\alpha_j \mid \alpha_j)$	θ_{ij}
	-1	2	2	$2\pi/3$
	-2	1	2	$3\pi/4$
	-3	2/3	2	$5\pi/6$

When $r \leq 2$, we can describe the root system Δ of a simple finite-dimensional Lie algebra by means of a picture as in Figure 2.1. The shaded region, in general a

Figure 2.1 : Roots and fundamental weights of A_1 , A_2 , C_2 and G_2

A_1



A_2

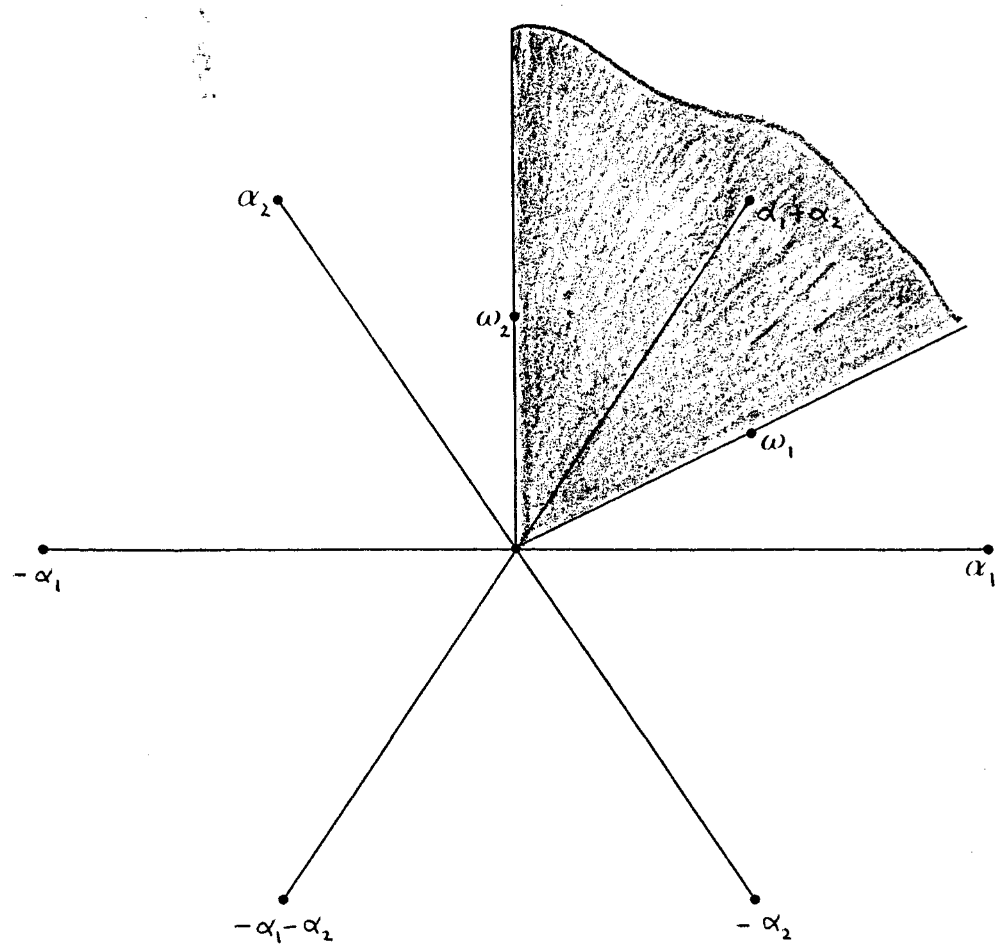
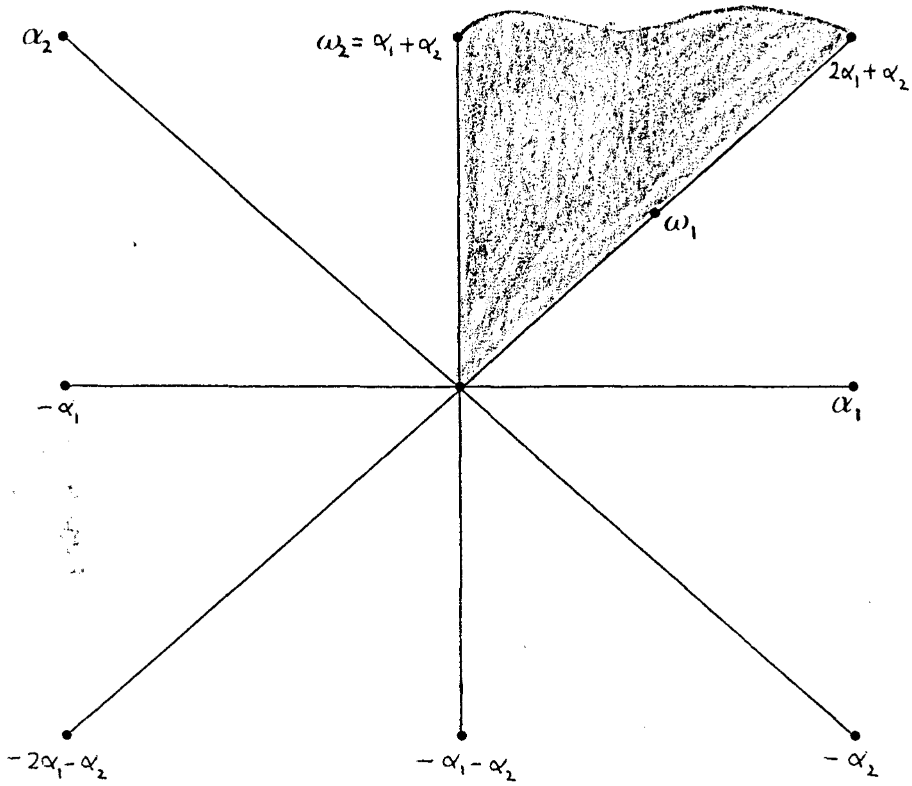
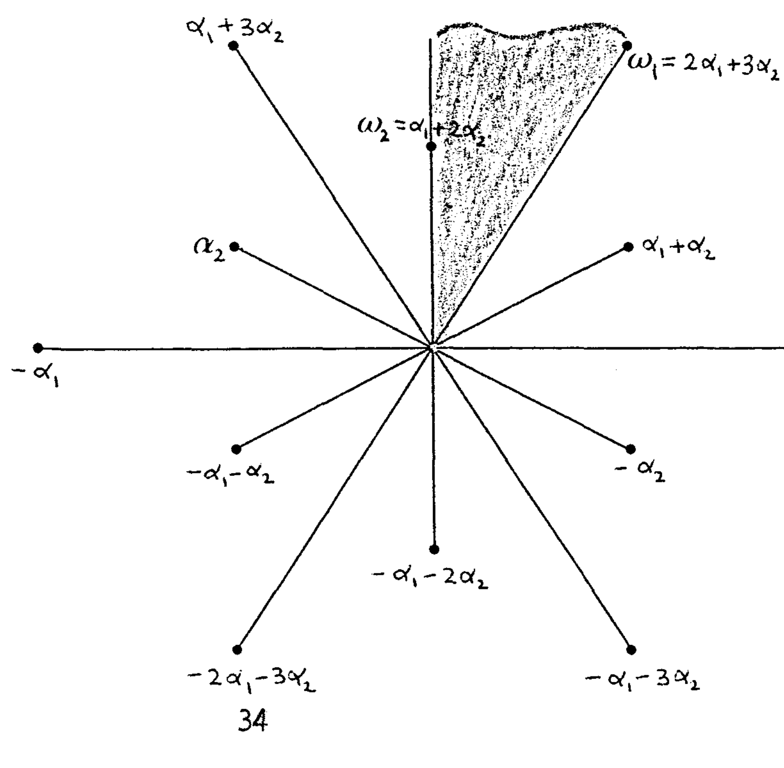


Figure 2.1 (cont.)

C2



G2



simplicial cone, is known as the dominant sector. Since Δ is finite there must exist a maximal root θ that satisfies $\theta - \alpha \in Q^+$ for all $\alpha \in \Delta^+$. In Table 2.2 we give the explicit values of the root θ [**BMP**]. It can be verified that $(\theta \mid \theta) = 2$ and hence θ is a long root.

Table 2.2 : Maximal long roots of simple finite-dimensional Lie algebras.

$\mathcal{G}$	θ
A_r	$\alpha_1 + \ldots + \alpha_r$
B_r	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_r$
C_r	$2\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-1} + \alpha_r$
D_r	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$
E_6	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$
E_7	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + 3\alpha_2$

The number of elements of the Weyl groups associated with a finite GCM is itself finite. For low rank algebras the Weyl groups can be obtained easily by treating them as Coxeter groups generated by fundamental reflections as given in Table 1.3. For example, the Weyl group $W(A_2)$ is given by

$$\{ id, s_1, s_2, s_2 s_1, s_1 s_2, s_1 s_2 s_1, \} \qquad (2.2)$$

However, for a higher rank algebras it is more efficient to generate the elements of the Weyl group by their action on the standard (or Euclidean) basis vectors $\epsilon_1, \ldots, \epsilon_n$ of $\mathbb{R}^n$. For example in the case of A_r, the reflection s_i permutes the subscripts i,$i+1$ and leaves other subscripts fixed. Thus, s_i corresponds to the transposition $(i, i+1)$ of the symmetric group S_{r+1} and the Weyl group $W(A_r)$ is isomorphic to S_{r+1}. The roots in the standard basis have the form $\epsilon_i - \epsilon_j$. If $\pi = \begin{pmatrix} 1\,2 \ldots r+1 \\ \pi_1\,\pi_2 \ldots \pi_{r+1} \end{pmatrix} \in S_{r+1}$ then π acts on the roots $\epsilon_i - \epsilon_j$ as follows

$$\pi(\epsilon_i - \epsilon_j) = \epsilon_{\pi_i} - \epsilon_{\pi_j}. \tag{2.3}$$

For easy reference, we give below for each classical simple finite-dimensional Lie algebra the relation between the root basis and the standard basis, all the roots in the standard basis, the order of Weyl group and the action of $w \in W$ in the standard basis. The complete set of data that includes the exceptional Lie algebras can be found, for example, in **[KQ]**.

Type A_r $(r \geq 1)$

Basis: $\alpha_i = \epsilon_i - \epsilon_{i+1} \quad 1 \leq i \leq r+1$

Roots: $\pm(\epsilon_i - \epsilon_j) \quad 1 \leq i < j \leq r+1$

Order of Weyl group: $(r+1)!$

Action of w: $(\epsilon_{\pi_1}, \epsilon_{\pi_2}, \ldots, \epsilon_{\pi_{r+1}})$

Parity of w: $(-1)^{\pi}$

Type B_r $(r \geq 3)$

Basis: $\alpha_i = \epsilon_i - \epsilon_{i+1}\ (1 \leq i \leq r-1), \quad \alpha_r = \epsilon_r$

Roots: $\pm\epsilon_i\ (1 \leq i \leq r), \quad \pm\epsilon_i \pm \epsilon_j\ (1 \leq i < j \leq r)$

Order of Weyl group: $2^r.r!$

Action of w: $(\sigma_1\epsilon_{\pi_1}, \sigma_2\epsilon_{\pi_2}, \ldots, \sigma_r\epsilon_{\pi_r}) \quad \sigma_i = \pm 1$

Parity of w: $\sigma(-1)^\pi$ where $\sigma = \sigma_1\sigma_2\ldots\sigma_r = \pm 1$

Type C_r $(r \geq 2)$

Basis: $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq r-1)$, $\quad \alpha_r = 2\epsilon_r$

Roots: $\pm 2\epsilon_i$ $(1 \leq i \leq r)$, $\quad \pm\epsilon_i \pm \epsilon_j$ $(1 \leq i < j \leq r)$

Order of Weyl group: $2^r.r!$

Action of w: $(\sigma_1\epsilon_{\pi_1}, \sigma_2\epsilon_{\pi_2}, \ldots, \sigma_r\epsilon_{\pi_r})$ $\quad \sigma_i = \pm 1$

Parity of w: $\sigma(-1)^\pi$ where $\sigma = \sigma_1\sigma_2\ldots\sigma_r = \pm 1$

Type D_r $(r \geq 4)$

Basis: $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \leq i \leq r-1)$, $\quad \alpha_r = \epsilon_{r-1} + \epsilon_r$

Roots: $\pm\epsilon_i \pm \epsilon_j$ $(1 \leq i < j \leq r)$

Order of Weyl group: $2^{r-1}.r!$

Action of w: $(\sigma_1\epsilon_{\pi_1}, \sigma_2\epsilon_{\pi_2}, \ldots, \sigma_r\epsilon_{\pi_r})$ $\quad \sigma_i = \pm 1 \quad$ where $\sigma_1\sigma_2\ldots\sigma_r = 1$

Parity of w: $(-1)^\pi$

2.2 Orbit-character expansion

Let the fundamental weights of the simple finite-dimensional Lie algebras be denoted by ω_i for $i = 1, \ldots, r$. Then (1.14) implies that $\alpha_i = \sum_{j=1}^r A_{ij}\omega_j$. As $\det A \neq 0$ we can express the fundamental weights in terms of simple roots. The inverses of the finite GCM are given in Table 2.3.

The weight system $P(\Lambda)$ for a given highest weight module V^Λ of a simple finite-dimensional Lie algebra $\mathcal{G}(A)$ lies entirely in one coset $\{P : Q\}$ of the weight lattice P with respect to the root lattice Q, called the congruence class. The number of congruence classes is $\mid P : Q \mid = \det A$, except for the case of D_r for which the number is $2\det A$. The class of a weight $\lambda \in P(\Lambda)$ is specified by an integer (or pair of integers in the case of D_r) defined in terms of the Dynkin components of λ and the components

Table 2.3 : The determinants $\det A$ and inverses A^{-1} of the GCM A of finite type.

1. A_r : $\det A = r + 1$

$$A^{-1} = \frac{1}{(r+1)} \begin{pmatrix} 1.r & 1.(r-1) & 1.(r-2) & \cdots & 1.3 & 1.2 & 1.1 \\ 1.(r-1) & 2.(r-1) & 2.(r-2) & \cdots & 2.3 & 2.2 & 2.1 \\ 1.(r-2) & 2.(r-2) & 3.(r-2) & \cdots & 3.3 & 3.2 & 3.1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1.3 & 2.3 & 3.3 & \cdots & (r-2).3 & (r-2).2 & (r-2).1 \\ 1.2 & 2.2 & 3.2 & \cdots & (r-2).2 & (r-1).2 & (r-1).1 \\ 1.1 & 2.1 & 3.1 & \cdots & (r-2).1 & (r-1).1 & r.1 \end{pmatrix}$$

2. B_r : $\det A = 2$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 2 & 4 & 4 & \cdots & 4 & 4 & 4 \\ 2 & 4 & 6 & \cdots & 6 & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-2) & 2(r-2) \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & 2(r-1) \\ 1 & 2 & 3 & \cdots & r-2 & r-1 & r \end{pmatrix}$$

3. C_r : $\det A = 2$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 6 & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-2) & r-2 \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & r-1 \\ 2 & 4 & 6 & \cdots & 2(r-2) & 2(r-1) & r \end{pmatrix}$$

4. D_r : $\det A = 4$

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 4 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 4 & 8 & 8 & \cdots & 8 & 4 & 4 \\ 4 & 8 & 12 & \cdots & 12 & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \cdots & 4(r-2) & 2(r-2) & 2(r-2) \\ 2 & 4 & 6 & \cdots & 2(r-2) & r & r-2 \\ 2 & 4 & 6 & \cdots & 2(r-2) & r-2 & r \end{pmatrix}$$

Table 2.3 (cont.)

5. E_6 : $\det A = 3$

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 1 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}$$

6. E_7 : $\det A = 2$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 2 & 4 \\ 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 3 \\ 4 & 8 & 12 & 9 & 6 & 3 & 7 \end{pmatrix}$$

7. E_8 : $\det A = 1$

$$A^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix}$$

8. F_4 : $\det A = 1$

$$A^{-1} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 2 \end{pmatrix}$$

9. G_2 : $\det A = 1$

$$A^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

of certain congruence vectors identified in Table 1 of [**BMP**]. To be more explicit for $\lambda = (\lambda_1, \ldots, \lambda_r)$, we tabulate the congruence classes for the algebras A_r, B_r, C_r, D_r, E_6 and E_7 in Table 2.4. For E_8, F_4 and G_2 there is only one congruence class since $\det A = 1$.

Table 2.4 : Congruence classes for the simple finite-dimensional Lie algebras.

Algebra	Class of λ
A_r	$(\lambda_1 + 2\lambda_2 + \ldots + r\lambda_r) \bmod r+1$
B_r	$\lambda_r \bmod 2$
C_r	$(\lambda_1 + 2\lambda_2 + \ldots + r\lambda_r) \bmod 2$
D_r	$(\lambda_{r-1} + \lambda_r, 2\lambda_1 + \ldots + 2(r-2)\lambda_{r-2} + (r-2)\lambda_{r-1} + r\lambda_r) \bmod(2,4)$
E_6	$(\lambda_1 + 2\lambda_2 + \lambda_4 + 2\lambda_5) \bmod 3$
E_7	$(\lambda_4 + \lambda_6 + \lambda_7) \bmod 2$

The weight space of any highest weight module of a simple finite-dimensional Lie algebra can be obtained by applying the algorithm discussed in Section 1.4. This weight space can be partitioned into W-orbits. For example, Figure 2.2a gives the weight space for the representation $\Lambda = (1,3)$ of the algebra A_2. The congruence class for the weights of this representation is 1. The dominant weights are $(1,3)$, $(2,1)$, $(0,2)$ and $(1,0)$ and their Weyl orbits are denoted respectively by $\triangle$, $\odot$, $\otimes$ and ∇.

In the interpretation of (1.22a) we have to add ρ to each weight and reflect it into

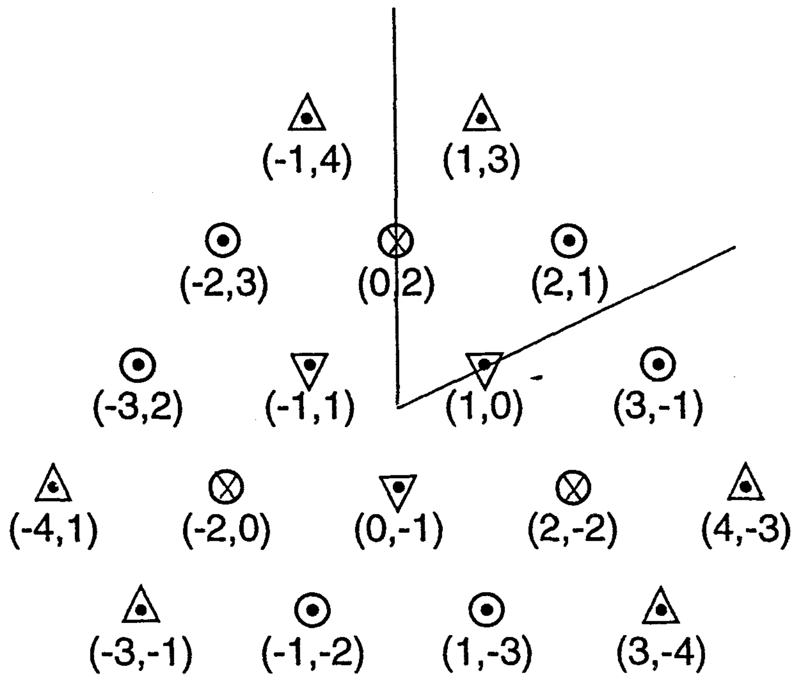


Figure 2.2a : Weyl orbits of P((1,3)) of A_2

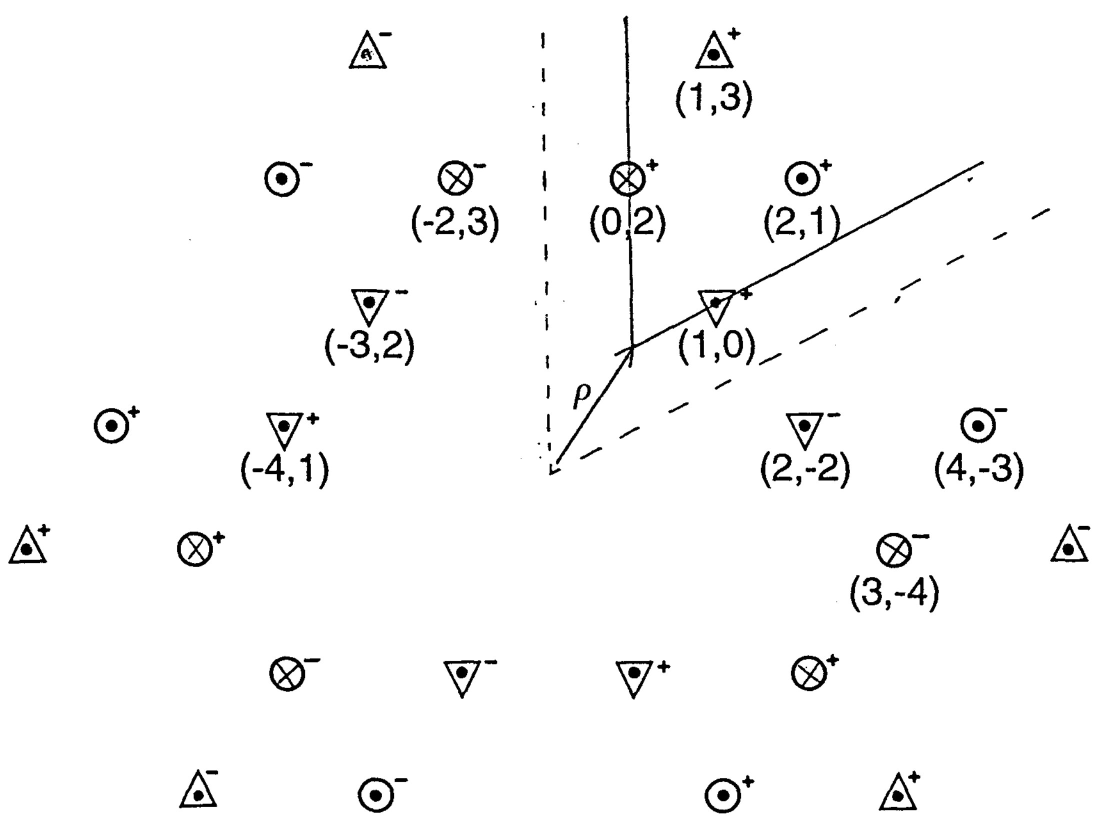


Figure 2.2b : Weyl dot orbits of P((1,3)) of A_2

the dominant sector. The elements of the Weyl orbit $\bigtriangledown$ of (1,0) give

$$\begin{aligned}(1,0)+(1,1) &= (2,1)\\(-1,1)+(1,1) &= (0,2)\\(0,-1)+(1,1)) &= (1,0).\end{aligned}$$

The elements of the Weyl orbit $\otimes$ of (0,2) give

$$\begin{aligned}(0,2)+(1,1) &= (1,3)\\s_2((2,-2)+(1,1)) &= (2,1)\\s_1((-2,0)+(1,1)) &= (1,0).\end{aligned}$$

The elements of the Weyl orbit $\odot$ of (2,1) give

$$\begin{aligned}(2,1)+(1,1) &= (3,2)\\s_1((-2,3)+(1,1)) &= (1,3)\\(3,-1)+(1,1)) &= (4,0)\\s_1((-3,2)+(1,1)) &= (2,1)\\s_2((1,-3)+(1,1)) &= (0,2)\\s_1s_2((-1,-2)+(1,1)) &= (1,0).\end{aligned}$$

The elements of the Weyl orbit $\triangle$ of (1,3) give

$$\begin{aligned}(1,3)+(1,1) &= (2,4)\\(-1,4)+(1,1) &= (0,5)\\s_2((4,-3)+(1,1)) &= (3,2)\\s_2s_1((-4,1)+(1,1)) &= (2,1)\\s_2((3,-4)+(1,1)) &= (1,3)\\s_2s_1((-3,-1)+(1,1)) &= (0,2).\end{aligned}$$

The reflected weights that lie on the reflection hyperplanes are to be ignored and ρ is subtracted from those that do not. The parity of the Weyl reflections is computed from the number of fundamental reflections s_i. This then gives the orbit sums of (1.21) as :

$$\begin{aligned}
\Omega^{(1,3)} &= ch\, V^{(1,3)} - ch\, V^{(2,1)} - ch\, V^{(0,2)} + ch\, V^{(1,0)}\,; \\
\Omega^{(2,1)} &= ch\, V^{(2,1)} - ch\, V^{(0,2)} - ch\, V^{(1,0)}\,; \\
\Omega^{(0,2)} &= ch\, V^{(0,2)} - ch\, V^{(1,0)}\,; \\
\Omega^{(1,0)} &= ch\, V^{(1,0)}\,.
\end{aligned} \tag{2.4}$$

Alternatively, as in the second interpretation of (1.22b), we may plot the corresponding Weyl dot orbit with their parities and look for intersection points with the original Weyl orbits. The Weyl dot orbits of $(1,3)$, $(2,1)$, $(0,2)$ and $(1,0)$ are given in Figure 2.2b. The parity factors $\varepsilon(w) = \pm$ are given as superscripts. On superimposing Figure 2.2a on Figure 2.2b, the parts of intersection which are labelled by their weights in Figure 2.2b define the same orbit-character expansion as in (2.4).

Under the partial ordering of (1.6) the orbit sum to irreducible character expansions can be written in matrix form as

$$\begin{pmatrix} \Omega^{(1,3)} \\ \Omega^{(2,1)} \\ \Omega^{(0,2)} \\ \Omega^{(1,0)} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ch\, V^{(1,3)} \\ ch\, V^{(2,1)} \\ ch\, V^{(0,2)} \\ ch\, V^{(1,0)} \end{pmatrix} .$$

Inverting the triangular transformation matrix we obtain :

$$\begin{aligned}
ch\, V^{(1,3)} &= \Omega^{(1,3)} + \Omega^{(2,1)} + 2\Omega^{(0,2)} + 2\Omega^{(1,0)}\,; \\
ch\, V^{(2,1)} &= \Omega^{(2,1)} + \Omega^{(0,2)} + 2\Omega^{(1,0)}\,; \\
ch\, V^{(0,2)} &= \Omega^{(0,2)} + \Omega^{(1,0)}\,; \\
ch\, V^{(1,0)} &= \Omega^{(1,0)}\,.
\end{aligned}$$

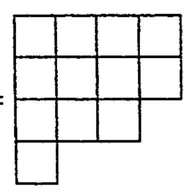
From the above equations, we can conclude that for the highest weight representation $(1,3)$ the elements of the Weyl orbits of (1,3) and (2,1) have multiplicity 1 and elements of the Weyl orbits of (0,2) and (1,0) have multiplicity 2. For the highest weight representation (2,1) the elements of the Weyl orbits of (2,1) and (0,2) have multiplicity 1 and elements of the Weyl orbit of (1,0) have multiplicity 2. While for the highest weight representations (0,2) and (1,0) all weights have multiplicity 1.

This technique may be extended to any simple finite-dimensional Lie algebra and requires for its implementation only a knowledge of the Weyl group action.

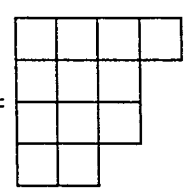
2.3 Partitions and characters

A partition ζ of a positive integer n is any finite sequence of positive integers $(\zeta_1 \zeta_2 \ldots \zeta_\ell)$ arranged in non-increasing order $\zeta_1 \geq \zeta_2 \geq \ldots \geq \zeta_\ell > 0$ such that $\zeta_1 + \zeta_2 + \ldots + \zeta_\ell = n$. The non-zero ζ_i form the parts of ζ and the number of parts $\ell = \ell(\zeta)$ is known as the length of ζ. It is convenient to denote a partition with repeated parts using exponents. For example, $(4^2 31)$ denotes the partition (4431).

Each partition ζ of n may be associated with a Young diagram $F(\zeta)$ involving boxes in $\ell(\zeta)$ left-adjusted rows with the i-th row containing ζ_i boxes. The conjugate of a partition ζ is a partition ζ' whose Young diagram $F(\zeta')$ is obtained from $F(\zeta)$ by interchanging rows and columns. This definition gives for $\zeta = (4^2 31)$, the diagram

$F(4^2 31) =$ 

and its conjugate $\zeta' = (43^2 2)$ the diagram

$F(43^2 2) =$ 

Alternatively, we can represent a partition using Frobenius notation **[King2]**. Let the number of boxes in the leading diagonal of a Young diagram $F(\zeta)$ be the rank p of ζ. Let a_i be the number of boxes to the right of the leading diagonal in the i-th row and let b_i be the number of boxes below the leading diagonal in the i-th column. The partition ζ is then denoted in Frobenius notation by the array

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_p \\ b_1 & b_2 & \ldots & b_p \end{pmatrix},$$

where

$$a_1 > a_2 > \ldots > a_p \geq 0$$

$$b_1 > b_2 > \ldots > b_p \geq 0$$

$$\text{and} \quad \sum_{i=1}^{p} (a_i + b_i + 1) = n\,.$$

For example, the partition $(4^2 31)$ and its conjugate $(43^2 2)$ are denoted respectively, by

$$\begin{pmatrix} 3 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

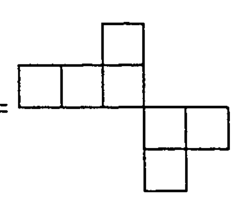
In general, if

$$(\zeta) = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix} = (\zeta_1\, \zeta_2 \ldots \zeta_{b_1+1})$$

then

$$\zeta_k = \begin{cases} a_k + k & k = 1, 2, \ldots, p \\ card\,\{i \mid b_i + i - k \geq 0\} & k = p+1, \ldots, b_1 + 1 \end{cases} \tag{2.5}$$

It is also useful to introduce other forms of Young diagram. In our case, we need what is called a composite Young diagram **[King2]**. For a partition ζ let $F(\bar{\zeta})$ be the diagram obtained by reflecting the Young diagram $F(\zeta)$ successively in its topmost and leftmost edges. Thus $F(\bar{\zeta})$ is right-adjusted with the lengths of the rows decreasing on passing up the diagram. The composite Young diagram $F(\bar{\zeta}; \eta)$ is constructed by adjoining $F(\bar{\zeta})$ and $F(\eta)$ corner to corner as in the following example:

$F(\overline{31}; 21) =$  .

An irreducible highest weight $\mathcal{G}(A)$-module can be indexed by its highest weight vector Λ which can be written either in the fundamental weight basis ω_i or in the standard basis ϵ_i. More generally, an arbitrary weight vector $\lambda \in \mathcal{H}^*$ can be written as

$$\lambda = \sum_{i=1}^{r} a_i \omega_i = \sum_{i=1}^{r} \lambda_i \epsilon_i\,. \tag{2.6}$$

The relationship between the Dynkin labels a_i and the partition labels λ_i described above are given in Table 2.5.

Let an indeterminate x_i denote the formal exponential e^{ϵ_i}. Then (2.6) gives $e^{\lambda} = x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_r^{\lambda_r}$. Further, let $x = (x_1, x_2, \ldots, x_N)$ signify the indeterminates and let $\lambda_N = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots, 0)$ with $\ell \leq N$ be a partition augmented by $N - \ell$ zeros. For the algebra A_r, ρ in the standard basis can be written as

$$\rho = r\epsilon_1 + (r-1)\epsilon_2 + \ldots + \epsilon_r + 0$$

where $\sum_{i=1}^{r+1} \epsilon_i = 0$. Then the Weyl character formula (1.19) and the isomorphism between the Weyl group $W(A_r)$ and the symmetric group S_{r+1} gives

$$\begin{aligned}
ch\, V^{\lambda} &= \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)} / \sum_{w \in W} \varepsilon(w) e^{w(\rho)} \\
&= \sum_{\pi \in S_{r+1}} \varepsilon(\pi) e^{\pi(\lambda+\rho)} / \sum_{\pi \in S_{r+1}} \varepsilon(\pi) e^{\pi(\rho)} \\
&= \sum_{\pi \in S_{r+1}} \varepsilon(\pi) e^{\pi(\lambda_1+r, \lambda_2+r-1, \ldots, \lambda_{r+1})} / \sum_{\pi \in S_{r+1}} \varepsilon(\pi) e^{\pi(r, r-1, \ldots, 0)} \\
&= \sum_{\pi \in S_{r+1}} \varepsilon(\pi) x_{\pi_1}^{\lambda_1+r} x_{\pi_2}^{\lambda_2+r-1} \ldots x_{\pi_{r+1}}^{\lambda_{r+1}} / \sum_{\pi \in S_{r+1}} \varepsilon(\pi) x_{\pi_1}^{r} x_{\pi_2}^{r-1} \ldots x_{\pi_{r+1}}^{0} \\
&= \det \mid x_j^{\lambda_i+r+1-i} \mid_{(r+1)\times(r+1)} / \det \mid x_j^{r+1-i} \mid_{(r+1)\times(r+1)} \\
&= \{\lambda\}(x_1, x_2, \ldots, x_{r+1}) \\
&= \{\lambda\}(x)_{N=r+1}.
\end{aligned}$$

The ratio of the two determinants as above is known famously as the Schur function, variously denoted by $s_\lambda(x_1, x_2, \ldots, x_N)$ or $\{\lambda\}(x_1, x_2, \ldots, x_N)$ **[King2]** and defined by:

$$s_\lambda(x_1, x_2, \ldots, x_N) = \frac{\det \mid x_j^{\lambda_i+N-i} \mid_{N\times N}}{\det \mid x_j^{N-i} \mid_{N\times N}}. \tag{2.7}$$

More generally, characters of the irreducible modules V^λ of the classical Lie algebras with highest weight vector $\lambda = \lambda_N$ are given by the following expressions **[Pr]**. Here i and j are row and column indices of the relevant determinants.

Table 2.5 : Relationship between Dynkin labels and partition labels.

Algebra	Dynkin label $(a_1, \ldots, a_r)$	Partition label $(\lambda_1, \ldots, \lambda_r)$
A_r	$a_1 = \lambda_1 - \lambda_2$ $a_2 = \lambda_2 - \lambda_3$ $\vdots$ $a_{r-1} = \lambda_{r-1} - \lambda_r$ $a_r = \lambda_r$	$\lambda_1 = a_1 + a_2 + \ldots + a_{r-1} + a_r$ $\lambda_2 = a_2 + \ldots + a_{r-1} + a_r$ $\vdots$ $\lambda_{r-1} = a_{r-1} + a_r$ $\lambda_r = a_r$
B_r	$a_1 = \lambda_1 - \lambda_2$ $a_2 = \lambda_2 - \lambda_3$ $\vdots$ $a_{r-1} = \lambda_{r-1} - \lambda_r$ $a_r = 2\lambda_r$	$\lambda_1 = a_1 + a_2 + \ldots + a_{r-1} + \frac{1}{2}a_r$ $\lambda_2 = a_2 + \ldots + a_{r-1} + \frac{1}{2}a_r$ $\vdots$ $\lambda_{r-1} = a_{r-1} + \frac{1}{2}a_r$ $\lambda_r = \frac{1}{2}a_r$
C_r	$a_1 = \lambda_1 - \lambda_2$ $a_2 = \lambda_2 - \lambda_3$ $\vdots$ $a_{r-1} = \lambda_{r-1} - \lambda_r$ $a_r = \lambda_r$	$\lambda_1 = a_1 + a_2 + \ldots + a_{r-1} + a_r$ $\lambda_2 = a_2 + \ldots + a_{r-1} + a_r$ $\vdots$ $\lambda_{r-1} = a_{r-1} + a_r$ $\lambda_r = a_r$
D_r	$a_1 = \lambda_1 - \lambda_2$ $a_2 = \lambda_2 - \lambda_3$ $\vdots$ $a_{r-2} = \lambda_{r-2} - \lambda_{r-1}$ $a_{r-1} = \lambda_{r-1} - \lambda_r$ $a_r = \lambda_{r-1} + \lambda_r$	$\lambda_1 = a_1 + a_2 + \ldots + a_{r-2} + \frac{1}{2}a_{r-1} + \frac{1}{2}a_r$ $\lambda_2 = a_2 + \ldots + a_{r-2} + \frac{1}{2}a_{r-1} + \frac{1}{2}a_r$ $\vdots$ $\lambda_{r-2} = a_{r-2} + \frac{1}{2}a_{r-1} + \frac{1}{2}a_r$ $\lambda_{r-1} = \frac{1}{2}a_{r-1} + \frac{1}{2}a_r$ $\lambda_r = -\frac{1}{2}a_{r-1} + \frac{1}{2}a_r$

A_r :

$$ch\,V^\lambda = \frac{\det \mid x_j^{\lambda_i+r+1-i} \mid_{(r+1)\times(r+1)}}{\det \mid x_j^{r+1-i} \mid_{(r+1)\times(r+1)}}$$
$$= \{\lambda\}(x_1, x_2, \ldots, x_{r+1}) \tag{2.8a}$$
$$= \{\lambda\}(x)_{N=r+1} \quad \text{with } x_1 x_2 \ldots x_{r+1} = 1\,.$$

B_r :

$$ch\,V^\lambda = \frac{\det \mid x_j^{\lambda_i+r+1/2-i} \pm x_j^{-(\lambda_i+r+1/2-i)} \mid_{(2r+1)\times(2r+1)}}{\det \mid x_j^{r+1/2-i} \pm x_j^{-(r+1/2-i)} \mid_{(2r+1)\times(2r+1)}}$$
$$= [\lambda](x_1, x_2, \ldots, x_r, x_1^{-1}, x_2^{-1}, \ldots, x_r^{-1}, 1) \tag{2.8b}$$
$$= [\lambda](x)_{N=2r+1}\,.$$

C_r :

$$ch\,V^\lambda = \frac{\det \mid x_j^{\lambda_i+r+1-i} - x_j^{-(\lambda_i+r+1-i)} \mid_{2r\times 2r}}{\det \mid x_j^{r+1-i} \pm x_j^{-(r+1-i)} \mid_{2r\times 2r}}$$
$$=< \lambda > (x_1, x_2, \ldots, x_r, x_1^{-1}, x_2^{-1}, \ldots, x_r^{-1}) \tag{2.8c}$$
$$=< \lambda > (x)_{N=2r}\,.$$

D_r :

$$ch\,V^\lambda = \frac{\det \mid x_j^{\lambda_i+r-i} - x_j^{-(\lambda_i+r-i)} \mid_{2r\times 2r}}{\det \mid x_j^{r+1-i} \pm x_j^{-(r+1-i)} \mid_{2r\times 2r}}$$
$$= [\lambda](x_1, x_2, \ldots, x_r, x_1^{-1}, x_2^{-1}, \ldots, x_r^{-1}) \tag{2.8d}$$
$$= [\lambda](x)_{N=2r}\,.$$

In the case of D_r there is a subtlety associated with the fact that for $\lambda_r \neq 0$ there are two inequivalent irreducible modules $[\lambda]_+$ and $[\lambda]_-$ with highest weights $(\lambda_1, \ldots, \lambda_{r-1}, \lambda_r)$ and $(\lambda_1, \ldots, \lambda_{r-1}, -\lambda_r)$ respectively.

In accordance with the composite Young diagram notation introduced before, the highest weight λ of an irreducible representation of A_r can also take the form **[King2]**

$$\lambda = (\bar{\zeta}; \eta) = (\eta_1, \eta_2, \ldots, \eta_p, 0, \ldots, 0, -\zeta_q, \ldots, -\zeta_2, -\zeta_1)\,,$$

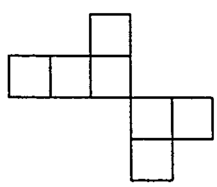
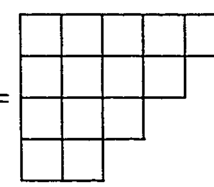
where η and ζ are partitions with $p = \ell(\eta)$, $q = \ell(\zeta)$ and $p + q \leq N = r + 1$. Its irreducible character is given by

$$ch\,V_N^{\bar{\zeta};\eta} = \frac{\sum_{\pi\in S_N} \varepsilon(\pi) x_{\pi_1}^{\eta_1+N-1} x_{\pi_2}^{\eta_2+N-2} \ldots x_{\pi_N}^{-\zeta_1}}{\sum_{\pi\in S_N} \varepsilon(\pi) x_{\pi_1}^{N-1} x_{\pi_2}^{N-2} \ldots x_{\pi_N}^{0}} = \{\bar{\zeta}; \eta\}(x)_{N=r+1}\,. \tag{2.8e}$$

When comparing this expression with (2.8a), which can also be written as

$$ch\, V_N^{\lambda} = \frac{\sum_{\pi \in S_N} \varepsilon(\pi) x_{\pi_1}^{\lambda_1+N-1} x_{\pi_2}^{\lambda_2+N-2} \dots x_{\pi_N}^{\lambda_N}}{\sum_{\pi \in S_N} \varepsilon(\pi) x_{\pi_1}^{N-1} x_{\pi_2}^{N-2} \dots x_{\pi_N}^{0}} = \{\lambda\}(x)_{N=r+1}$$

it can be deduced that $ch\, V_N^{\bar{\zeta};\eta} = (x_1 x_2 \dots x_N)^{-\zeta_1} ch\, V_N^{\lambda}$ where $\lambda = (\eta_1 + \zeta_1, \eta_2 + \zeta_1, \dots, -\zeta_2 + \zeta_1, 0)$. This then implies that $F(\lambda)$ can be obtained from $F(\bar{\zeta};\eta)$ by taking the complement in a column of length N of each of the ζ_1 columns which constitute $F(\zeta)$ and adjoining them to the remaining η_1 columns which constitute $F(\eta)$ **[King2]**. For example in the case of $N = 5$,

$F(\overline{31};21) =$  is equivalent to $F(5432) =$ 

The irreducible characters of the classical Lie algebras associated with Young diagrams labelled by partitions are said to be in standard form if the partitions satisfy the constraints given in Table 2.6.

Table 2.6 : Constraints for standard characters.

Algebra	Label	Constraints
A_r	$\{\lambda\}$	$\ell(\lambda) \leq r$
	$\{\bar{\zeta};\eta\}$	$\ell(\zeta) + \ell(\eta) \leq r + 1$
B_r	$[\lambda]$	$\ell(\lambda) \leq r$
C_r	$< \lambda >$	$\ell(\lambda) \leq r$
D_r	$[\lambda]$	$\ell(\lambda) < r$
	$[\lambda]_{\pm}$	$\ell(\lambda) = r$

However non-standard labels for characters may arise in certain computations. If this does happen then we have to apply modification rules **[King2]** to reduce a non-standard labelling to a standard one. The modification rules involve drawing the Young diagram $F(\lambda)$ associated with the non-standard labelling of the character and removing a continuous boundary strip of boxes of length h, starting at the foot of the first column and working up along the right boundary. The resulting diagram is denoted by $F(\lambda - h)$. If this diagram corresponds to a partition then $\lambda - h$ is identified with this partition, otherwise the corresponding character vanishes identically. A phase factor also occurs which is dependent upon the column c in which the strip removal ends. In the case of a composite Young diagram $F(\bar{\zeta};\eta)$ the procedure involves the removal of a pair of boundary strips. The modification rules appropriate to each classical Lie algebra is given below **[King2]**

Table 2.7 : Modification rules and striplengths

Algebra	Modification rule	Striplength h
A_r	$\{\bar{\zeta};\eta\} = (-1)^{c+\bar{c}+1}\{\overline{\zeta - h};\eta - h\}$	$\ell(\zeta) + \ell(\eta) - r - 2$
B_r	$[\lambda] = (-1)^{c-1}[\lambda - h]$	$2\ell(\lambda) - 2r - 1$
C_r	$<\lambda> = (-1)^{c} <\lambda - h>$	$2\ell(\lambda) - 2r - 2$
D_r	$[\lambda] = (-1)^{c-1}[\lambda - h]$	$2\ell(\lambda) - 2r$

It should be noted that if the strip removal is of length 0 then c is taken to be 1. In order to standardise any given character it may be necessary to repeat the strip removal procedure more than once.

2.4 Infinite series of characters

Using the theory of the Schur functions (2.7), King **[King2]** had obtained among others the following identities

$$\prod_{k=1}^{\infty}\prod_{i,j=1}^{N}(1-q^k x_i x_j^{-1})(1-q^k)^{-1}=\sum_{\zeta\in F}(-1)^{|\zeta|}q^{|\zeta|}\{\bar{\zeta};\zeta'\}(x)_N \tag{2.9a}$$

$$\prod_{k=1}^{\infty}\prod_{1\le i<j\le N}(1-q^k x_i x_j)=\sum_{\alpha\in A}(-1)^{|\alpha|/2}q^{|\alpha|/2}[\alpha](x)_N \tag{2.9b}$$

$$\prod_{k=1}^{\infty}\prod_{1\le i\le j\le N}(1-q^k x_i x_j)=\sum_{\gamma\in C}(-1)^{|\gamma|/2}q^{|\gamma|/2}<\gamma>(x)_N \tag{2.9c}$$

$$\prod_{k=1}^{\infty}\prod_{1\le i<j\le N}(1-q^k x_i x_j)\prod_{i=1}^{N}(1+q^k x_i)=\sum_{\alpha\in A}(-1)^{|\alpha|/2}q^{|\alpha|/2}<\alpha>(x)_{N-1} \tag{2.9d}$$

$$\prod_{k=1}^{\infty}\prod_{i=1}^{N}(1-q^k x_i)\prod_{1\le i<j\le N}(1-q^{2k}x_i x_j)\prod_{i=1}^{N}(1-q^{2k}x_i)^{-1}$$
$$=\sum_{\epsilon\in E}(-1)^{(|\epsilon|+p)/2}q^{|\epsilon|}[\epsilon](x)_N \tag{2.9e}$$

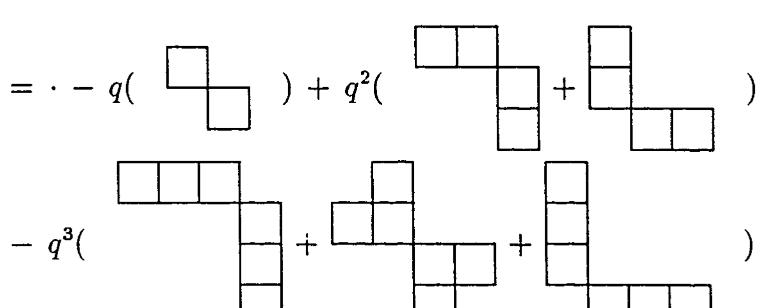
$$\prod_{k=1}^{\infty}\prod_{1\le i\le j\le N}(1-q^k x_i x_j)\prod_{i=1}^{N}(1+q^k x_i)^{-1}=\sum_{\gamma\in C}(-1)^{|\gamma|/2}q^{|\gamma|/2}[\gamma](x)_{N+1} \tag{2.9f}$$

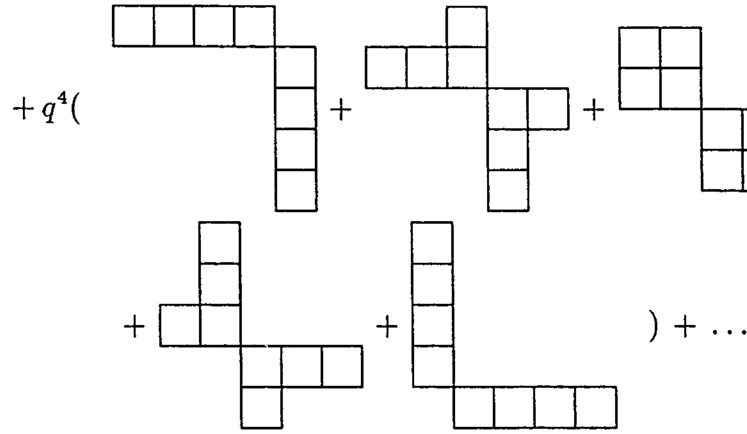
where A, C, E and F are the sets of partitions given in Frobenius notation by

$$\begin{aligned}
A=&\left\{\alpha\mid\alpha=\begin{pmatrix}a_1-1 & a_2-1 & \cdots\\ a_1 & a_2 & \cdots\end{pmatrix}\right\},\\
C=&\left\{\gamma\mid\gamma=\begin{pmatrix}a_1+1 & a_2+1 & \cdots\\ a_1 & a_2 & \cdots\end{pmatrix}\right\},\\
E=&\left\{\epsilon\mid\epsilon=\begin{pmatrix}a_1 & a_2 & \cdots\\ a_1 & a_2 & \cdots\end{pmatrix}\right\},\\
F=&\left\{\zeta\mid\zeta=\begin{pmatrix}a_1 & a_2 & a_3 & \cdots\\ b_1 & b_2 & b_3 & \cdots\end{pmatrix}\right\}.
\end{aligned} \tag{2.10}$$

The expansion of the right hand side of the above identities reveals that for specific values of N many of the terms involve characters with non standard labelling. To illustrate the role of modification rules in reducing non standard labelling to a standard labelling we expand a few terms of the right hand side of (2.9a) when $r=2$ so that $N=3$:

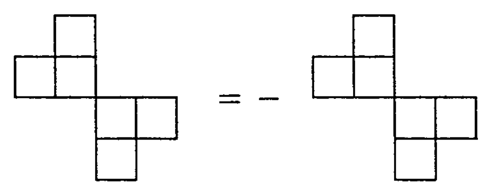
$$\sum_{\zeta\in F}(-1)^{|\zeta|}q^{|\zeta|}\{\bar{\zeta};\zeta'\}(x)_3=\cdot$$



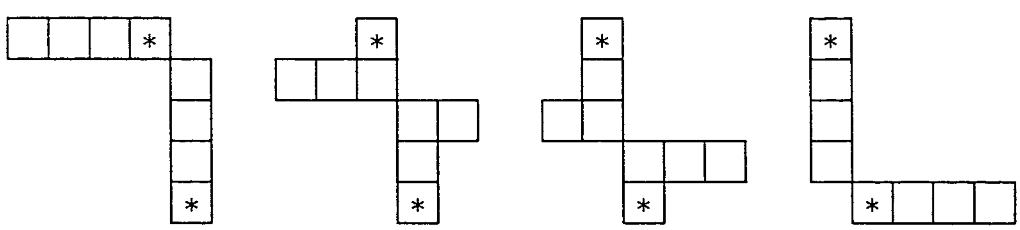


(2.11)

Only the first three terms correspond to standard labels. Consider first those terms for which $\ell(\zeta) + \ell(\zeta') = 4$. Since $r = 2$ the length of the strip removal is $h = 0$. The modification rule when applied to $F(\overline{21}; 21)$, for example, gives



since $c = \bar{c} = 1$. Hence the character that correspond to $F(\overline{21}; 21)$ is zero. Terms in the expansion (2.11) with $\ell(\zeta) + \ell(\zeta') = 5$ are



where the boxes fill with $*$'s denoted the boxes that will be removed under the modification rule. Hence the first few terms of the expansion for the RHS of (2.9a) in the case of A_2 with $N = 3$ takes the form

$$\sum_{\zeta \in F} (-1)^{|\zeta|} q^{|\zeta|} \{\bar{\zeta}; \zeta'\}(x)_3 = \{0\} - q\{\bar{1}; 1\} + q^2(\{\bar{2}; 1^2\} + \{\overline{1^2}; 2\}) - q^4(\{\bar{3}; 21\} + \{\overline{21}; 3\}) + \dots$$

In general the terms that survive are those that consist of Young diagrams which could be built from a core specified by $\{\bar{\zeta}; \zeta'\}$ with $\zeta \in F$ and $\ell(\zeta) + \ell(\zeta') \leq N$ by adding strips of length $(r+1)$ to this core in all possible ways such that each strips starts in the first row and their successive addition yields a Young diagram that correspond to a standard labelling **[King2]**. In (5.21) and (5.11) of **[King2]**, King has already obtained the expressions for the RHS of (2.9a) and (2.9b) in terms of standard characters (2.8e) of A_r and (2.8b) of B_r and D_r respectively, i.e.

$$\sum_{\xi\in F}(-1)^{|\xi|}q^{|\xi|}\{\bar{\xi};\xi'\}(x)_N = \sum_{\substack{\zeta\in F\\ \ell(\zeta)+\ell(\zeta')\leq N}}\sum_{s=0}^{\infty}\sum_{\substack{\mu\equiv\zeta \bmod N\\ \nu\equiv\zeta \bmod N}}(-1)^{|\zeta|+c+\bar{c}}\,q^{|\zeta|+r+\bar{r}-s}\,\{\bar{\nu};\mu\}(x)_N \tag{2.12a}$$

$$\sum_{\alpha\in A}(-1)^{|\alpha|/2}q^{|\alpha|/2}[\alpha](x)_N = \sum_{\substack{\alpha\in A\\ \ell(\alpha)\leq[N/2]}}\sum_{s=0}^{\infty}\sum_{\lambda\equiv\alpha \bmod (N-2)}(-1)^{|\alpha|/2+c}\,q^{|\alpha|/2+r}\,[\lambda](x)_N \tag{2.12b}$$

where in (2.12a) $F(\bar{\nu};\mu)$ is formed from the core diagram $F(\bar{\zeta};\zeta')$ by adding s pairs of boundary strips each of length N. The i-th strip added to $F(\zeta')$ starts at position $(1,r_i)$ and covers c_i columns, whilst the i-th strips added $F(\bar{\zeta})$ starts at the position $(1,\bar{r}_i)$ and cover $\bar{c}_i$ columns, $r=\sum_{i=1}^{s} r_i$, $c=\sum_{i=1}^{s} c_i$, $\bar{r}=\sum_{i=1}^{s}\bar{r}_i$ and $\bar{c}=\sum_{i=1}^{s}\bar{c}_i$, respectively. In (2.12b) $F(\lambda)$ is formed from the core diagram $F(\alpha)$ by adding s boundary strips each of length $N-2$. The i-th strip starts at position $(1,r_i)$ and covers c_i columns, $r=\sum_{i=1}^{s} r_i$ and $c=\sum_{i=1}^{s} c_i$. $N=2r+1$ in the case of B_r and $N=2r$ in the case of D_r.

To present these results and generalise them to the other cases (2.9c - 2.9f) we develop here a similar notation. Let $k=(m_1,m_2,\ldots,m_s)$ be an s-tuple with $m_1\leq m_2\leq\ldots\leq m_s$. Let $F(\lambda^s)$ (resp. $F(\overline{\nu^s};\mu^s)$ in the case of A_r) be the Young diagram formed from a core diagram $F(\beta)$ subject to certain restrictions by adding s strips (resp. pair of strips) each of length M starting at the first row of $F(\beta)$ and covering $m_1,m_2,\ldots,m_s$ columns successively. For each of the identities (2.9a - 2.9f) we tabulate their respective core $F(\beta)$ and strip length M in Table 2.8 below.

Table 2.8 : Core Young diagrams and strip length M.

Identity	Algebra	Core $F(\beta)$	Restriction	Strip length M
2.9a	A_r	$F(\bar{\zeta};\zeta')$	$\zeta \in F, a_1+b_1 \leq r-1$	$r+1$
2.9b	B_r	$F(\alpha)$	$\alpha \in A, a_1 \leq r-1$	$2r-1$
2.9b	D_r	$F(\alpha)$	$\alpha \in A, a_1 \leq r-2$	$2r-2$
2.9c	C_r	$F(\gamma)$	$\gamma \in C, a_1 \leq r-1$	$2r+2$
2.9d	C_r	$F(\alpha)$	$\alpha \in A, a_1 \leq r-1$	$2r$
2.9e	B_r	$F(\epsilon)$	$\epsilon \in E, a_1 \leq r-1$	$2r$
2.9f	B_r	$F(\gamma)$	$\gamma \in C, a_1 \leq r-1$	$2r+1$

Let M_{m_i} denote the i^{th} boundary strip added to β which begins at position $(1, n_i)$ and covers m_i columns. Further let the partition obtained at this stage be λ^i. Then $n_i = \lambda_1^i$, the first part of λ^i, and λ^i can be defined recursively as follows:

$$\begin{aligned} \lambda^0 &= \beta \\ \lambda^i &= \lambda^{i-1} + M_{m_i} , \end{aligned} \tag{2.13}$$

or equivalently

$$\lambda_j^i = \begin{cases} m_i + \lambda_{M+1-m_i}^{i-1} & j = 1 , \\ \lambda_{j-1}^{i-1} + 1 & j = 2, 3, \ldots, M+1-m_i , \\ \lambda_j^{i-1} & j = M+2-m_i, \ldots, \ell(\lambda^{i-1}) . \end{cases} \tag{2.14a}$$

In the case of A_r:

$$\mu_j^i = \begin{cases} m_i + \mu_{M+1-m_i}^{i-1} & j = 1 , \\ \mu_{j-1}^{i-1} + 1 & j = 2, 3, \ldots, M+1-m_i , \\ \mu_j^{i-1} & j = M+2-m_i, \ldots, \ell(\mu^{i-1}) ; \end{cases} \tag{2.14b}$$

$$\nu_j^i = \begin{cases} \overline{m_i} + \nu^{i-1}_{M+1-\overline{m_i}} & j = 1\ , \\ \nu^{i-1}_{j-1} + 1 & j = 2, 3, \ldots, M + 1 - \overline{m_i}\ , \\ \nu_j^{i-1} & j = M + 2 - \overline{m_i}, \ldots, \ell(\nu^{i-1})\ , \end{cases} \tag{2.14c}$$

where $(\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_s) = \bar{k}$ is also an s-tuple.

Proposition 2.1. *With the notation as in Table 2.8 and (2.14a - 2.14c), the standard character forms of the right hand sides of the identities (2.9a - 2.9f) take the form:*

$$\sum_{\theta \in F} (-1)^{|\theta|} q^{|\theta|} \{\bar{\theta}; \theta'\}(x)_{r+1}$$
$$= \sum_{\substack{\zeta \in F \\ \ell(\zeta)+\ell(\zeta') \le r+1}} \sum_{s=0}^{\infty} \sum_{\substack{k, \bar{k}, m_1 + \overline{m}_1 \ge r+3 \\ \zeta'_1 < m_1 \le r+1, \zeta_1 < \overline{m}_1 \le r+1}} (-1)^{|\zeta| + m + \bar{m}}\, q^{|\zeta| + n + \bar{n} - s} \{\overline{\nu^s}; \mu^s\}; \tag{2.15a}$$

$$\sum_{\theta \in A} (-1)^{|\theta|/2} q^{|\theta|/2} [\theta](x)_{2r+1} = \sum_{\substack{\alpha \in A \\ \ell(\alpha) \le r}} \sum_{s=0}^{\infty} \sum_{\substack{k \\ \alpha_1 < m_1 \le 2r-1}} (-1)^{|\alpha|/2 + m}\, q^{|\alpha|/2 + n} [\lambda^s]; \tag{2.15b}$$

$$\sum_{\theta \in A} (-1)^{|\theta|/2} q^{|\theta|/2} [\theta](x)_{2r} = \sum_{\substack{\alpha \in A \\ \ell(\alpha) \le r}} \sum_{s=0}^{\infty} \sum_{\substack{k \\ \alpha_1 < m_1 \le 2r-2}} (-1)^{|\alpha|/2 + m}\, q^{|\alpha|/2 + n} [\lambda^s]; \tag{2.15c}$$

$$\sum_{\theta \in C} (-1)^{|\theta|/2} q^{|\theta|/2} < \theta > (x)_{2r} = \sum_{\substack{\gamma \in C \\ \ell(\gamma) \le r}} \sum_{s=0}^{\infty} \sum_{\substack{k \\ \gamma_1 < m_1 \le 2r+2}} (-1)^{|\gamma|/2 + m}\, q^{|\gamma|/2 + n - s} < \lambda^s >; \tag{2.15d}$$

$$\sum_{\theta \in A} (-1)^{|\theta|/2} q^{|\theta|/2} < \theta > (x)_{2r} = \sum_{\substack{\alpha \in A \\ \ell(\alpha) \le r}} \sum_{s=0}^{\infty} \sum_{\substack{k \\ \alpha_1 < m_1 \le 2r}} (-1)^{|\alpha|/2 + m - s}\, q^{|\alpha|/2 + n} < \lambda^s >; \tag{2.15e}$$

$$\sum_{\theta \in E} (-1)^{(|\theta|+p)/2} q^{|\theta|} [\theta](x)_{2r+1} = \sum_{\substack{\epsilon \in E \\ \ell(\epsilon) \le r}} \sum_{s=0}^{\infty} \sum_{\substack{k \\ \epsilon_1 < m_1 \le 2r}} (-1)^{(|\epsilon|+p)/2 - m}\, q^{|\epsilon| + 2n - s} [\lambda^s]; \tag{2.15f}$$

$$\sum_{\theta\in A}(-1)^{|\theta|/2}q^{|\theta|/2}[\theta](x)_{2r+1} = \sum_{\substack{\gamma\in C\\ \ell(\gamma)\le r}}\sum_{s=0}^{\infty}\sum_{\substack{k\\ \gamma_1<m_1\le 2r+1}}(-1)^{|\gamma|/2+m-s}q^{|\gamma|/2+n-s}[\lambda^s], \quad (2.15g)$$

where $m = \sum_{i=1}^{s} m_i$ and $n = \sum_{i=1}^{s}\lambda_1^i$. In the case of A_r, $m = \sum_{i=1}^{s} m_i$, $\bar{m} = \sum_{i=1}^{s}\bar{m}_i$, $n = \sum_{i=1}^{s}\mu_1^i$ and $\bar{n} = \sum_{i=1}^{s}\nu_1^i$

Proof (2.15a) and (2.15b-c) are equivalent forms of (2.12a) and (2.12b) respectively. We shall give a proof for (2.15d) only as the remaining identities can be proved similarly.

Consider the Young diagram $F(\theta)$ associated with the partition

$$\theta \equiv \theta^1 = \begin{pmatrix} b_1+1 & b_2+1 & \cdots \\ b_1 & b_2 & \cdots \end{pmatrix} \in C$$

Then any boundary strip removal starting from the end of the first row, i.e. at position $(1, b_1+2)$ and ending at the bottom of the first column, i.e. at position $(b_1+1, 1)$, has length $2b_1+2$. The resulting Young diagram after removing this boundary strip corresponds to a partition $\theta^2 = \begin{pmatrix} b_2+1 & b_3+1 & \cdots \\ b_2 & b_3 & \cdots \end{pmatrix} \in C$. If $<\theta^1>$ corresponds to a standard label then we have the Proposition with $\gamma = \theta^1$ and $s = 0$. However if $<\theta^1>$ corresponds to a non standard labelling then by the modification rule of the Table 2.7, the boundary strip removal has length

$$h_1 = 2\ell - 2r - 2 = 2(b_1+1) - 2r - 2 = 2b_1 - 2r.$$

Hence the remaining part of the boundary strip has a length $M = 2b_1 + 2 - h_1 = 2r+2$. Assume that this remaining boundary strip starts at position $(1, n_s)$, i.e. $n_s = b_1+2$ and covers m_s columns. If $\theta - h_1$, does not correspond to a partition then the contribution to the character is zero. If on the otherhand $\theta - h_1$, corresponds to a partition then the modification rule boundary strip removal covers $c = n_s - m_s + 1$ columns so that the standardisation procedure gives

$$(-1)^{|\theta|/2}q^{|\theta|/2} < \theta > = (-1)^{|\theta^2|/2+b_1+1}(-1)^{n_s-m_s+1}q^{|\theta^2|/2+b_1+1} < \theta - h_1 >$$

$$= (-1)^{|\theta^2|/2+2n_s-m_s}q^{|\theta^2|/2+n_s-1} < \theta - h_1 > \qquad (A)$$

$$= (-1)^{|\theta^2|/2+m_s}q^{|\theta^2|/2+n_s-1} < \theta - h_1 > .$$

If $< \theta - h_1 >$ corresponds to a non standard labelling then we repeat the above procedure with $\theta^1 \to \theta^2 \to \theta^3$, $b_1 \to b_2$, $m_s \to m_{s-1}$, $n_s \to n_{s-1}$ and $h_1 \to h_2$, so that (A) further reduces to

$$(-1)^{|\theta|/2}q^{|\theta|/2} < \theta >$$

$$=(-1)^{|\theta^3|/2+b_2+1-m_s}(-1)^{n_{s-1}-m_{s-1}+1}q^{|\theta^3|/2+b_2+1+n_s-1} < \theta - h_1 - h_2 >$$

$$=(-1)^{|\theta^3|/2-m_{s-1}-m_s}q^{|\theta^3|/2+n_{s-1}+n_s-2} < \theta - h_1 - h_2 > .$$

For s number of applications of the modification rule, this procedure will define an s-tuple $k = (m_1, \ldots, m_s)$ where $M = 2r + 2 \geq m_s \geq m_{s-1} \geq \ldots \geq m_1 > \gamma_1$ that corresponds to columms covered successively by the remaining boundary strips. The standardisation procedure then gives

$$(-1)^{|\theta|/2}q^{|\theta|/2} < \theta >= (-1)^{|\theta^{s+1}|/2-\sum m_i}q^{|\theta^{s+1}|/2+\sum n_i-s} < \theta - \sum_{i=1}^{s} h_i > ,$$

where we assume $< \theta - \sum_{i=1}^{s} h_i >$ does not require further modification and $\theta^{s+1} = \gamma = \begin{pmatrix} a_1+1 & a_2+1 & \cdots & a_p+1 \\ a_1 & a_2 & \cdots & a_p \end{pmatrix} \in C$. Then $F(\theta - \sum_{j=1}^{s} h_j)$ corresponds to adding s boundary strips of length M and covering $m_1, \ldots, m_s$ columns successively to $F(\gamma)$, i.e. $F(\theta - \sum_{j=1}^{s} h_j) = F(\gamma^s)$.

Now let $\lambda^i = \gamma^s - \sum_{j=i+1}^{s} M_{m_j}$. Then

$$\lambda^{i-1} = \gamma^s - \sum_{j=i}^{s} M_{m_j} = \gamma^s - (M_{m_i} + \sum_{j=i+1}^{s} M_{m_j})$$

so that

$$\lambda^i = \lambda^{i-1} + M_{m_i} ,$$

as required.

Conversely, let $\gamma = \binom{a_1+1\, a_2+1 \cdots a_p+1}{a_1 \quad a_2 \quad \cdots a_p} \in C$ satisfies $\ell(\gamma) \leq r$. Let $k = (m_1, \ldots, m_s)$ $\gamma_1 < m_1 \leq \ldots \leq m_s \leq M$. First add a boundary strip of length $M = 2r+2$ to $F(\gamma)$ starting at position $(1, n_1')$, $n_1' > a_1 + 2$ and covering m_1 columns such that the resulting Young diagram $F(\lambda^1)$ corresponds to a partition. This boundary strip will end at position $(M - m_1 + 1, n_1' - m_1 + 1)$. Let $\lambda^0 = \gamma$ and $\lambda^i = \lambda^{i-1} + M_{m_i}$. This implies that $n_1' = \lambda_1^1$.

The boundary strip that can then be added to $F(\lambda^1)$ such that it extends from position $(M - m_1 + 2, \lambda_1^1 - m_1 + 1)$ to position $(\ell(\lambda^0) + 1, 1)$ has length

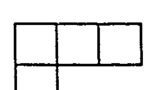
$$\lambda_1^1 + 1 + a_1 - M \leq 2\lambda_1^1 - M - 2 \quad \text{since } \lambda_1^1 \geq a_1 + 3.$$

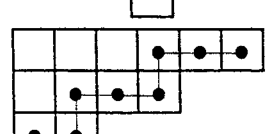
Choose a boundary strip of length $h_1' = 2\lambda_1^1 - M - 2$ as dictated by the modification rule for C_r and the fact that $M = 2r+2$ and add it to $F(\lambda^1)$ starting at $(M - m_1 + 2, \lambda_1^1 - m_1 + 1)$ and moving toward the left. Then the boundary strip will end at position $(\lambda_1^1 - 1, 1)$. The resulting Young diagram $F(\lambda^1 + h_1')$ now corresponds to a partition

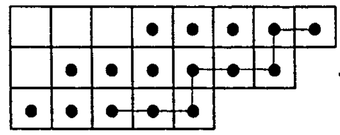
$$\gamma' = \begin{pmatrix} \lambda_1^1 - 1 & a_1 + 1 & a_2 + 1 & \ldots \\ \lambda_1^1 - 2 & a_1 & a_2 & \ldots \end{pmatrix} \in C.$$

with $< \gamma' > = < \lambda^1 + h_1' > = (-1)^{\lambda_1^1 - m_1 + 1} < \lambda^1 >$ under modification. This procedure can be repeated with boundary strips which cover $m_2, \ldots, m_s$ columns consecutively to give all possible $\theta \in C$ and characters $< \theta > (x)_{2r}$ as required in (2.15d). □

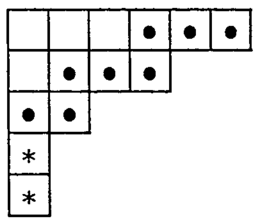
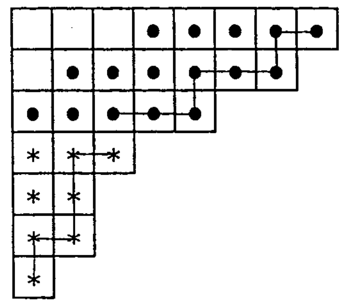
To illustrate (2.15d) consider a term of the expansion of the right hand side which comes from say $r = 3$, $\gamma = \binom{2}{1}$, $s = 2$, $k = (6, 6)$. Then applying (2.14a) successively diagramatically by adding 2 strips of length $M = 2r + 2 = 8$ each to $F(31)$ we obtain

$F(\lambda^0) = F(\gamma) =$  ,

$F(\lambda^1) =$  ,

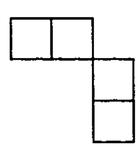
$F(\lambda^2) =$  .

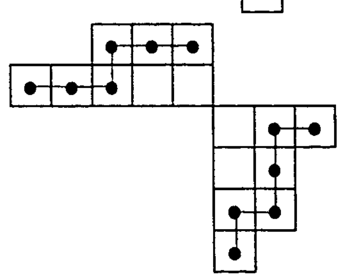
The first of these is standard and of the form $[\theta]$. The second and the third diagrams arises from the following non-standard terms of the form $[\theta]$, respectively,

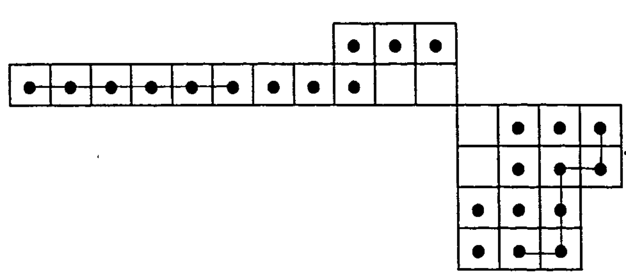
 and 

where the boxes fill with $*$'s are to be removed by the modification rules.

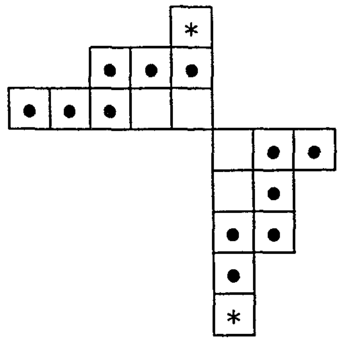
Next we illustrate (2.15a) by consider a term of the expansion of the right hand side which come from say $r = 5$, $\zeta = \binom{1}{0}$, $s = 2$, $k = (3,3)$ and $\bar{k} = (5,6)$. Then applying (2.14b) and (2.14c) successively diagrammatically by adding 2 pairs of strips of length $M = r + 1 = 6$ each to $F(\bar{2}; 1^2)$ we obtain

$F(\overline{\nu^0}; \mu^0) = F(\bar{2}; 1^2) =$ 

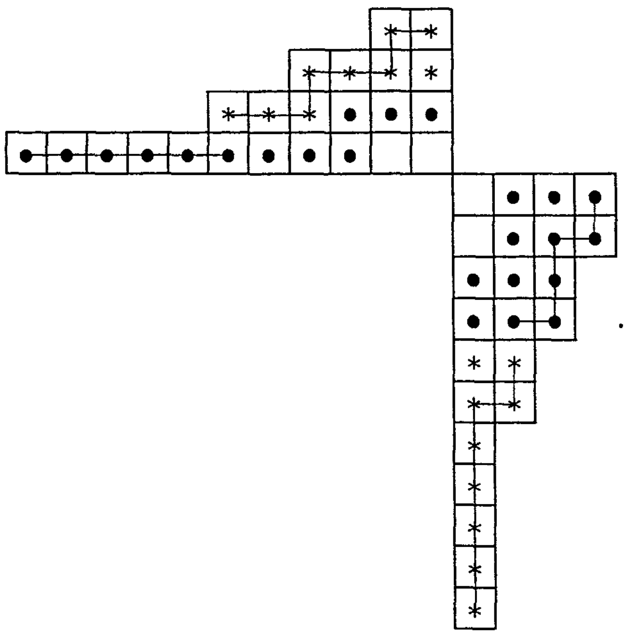
$F(\overline{\nu^1}; \mu^1) =$ 

$F(\overline{\nu^2}; \mu^2) =$  .

The first of these is standard and of the form $[\bar{\theta}; \theta']$. The second and the third diagrams arises from the following non-standard terms of the form $[\bar{\theta}; \theta']$, respectively,



and



CHAPTER 3

The Structure of Affine Algebras and their Modules

3.1 Generalised Cartan matrices and bilinear forms

The GCM of affine type is an $(r+1)\times(r+1)$ matrix of rank r. It is conventional to index the affine matrix $A = (A_{ij})$ with i,j running from $0,1,\ldots$ to r. The affine GCM are given in Appendix 1. Let $\mathcal{G}(A)$ be the Kac-Moody algebra associated with the matrix A. Let $\Pi = \{\alpha_0,\alpha_1,\ldots,\alpha_r\} \subset \mathcal{H}^*$ be the set of simple roots and let $\Pi^\vee = \{\alpha_0^\vee,\alpha_1^\vee,\ldots,\alpha_r^\vee\} \subset \mathcal{H}$ be the set of simple co-roots with

$$<\alpha_i,\alpha_j^\vee>= A_{ij} \quad \text{for}\, i,j=0,1,\ldots,r\,. \tag{3.1}$$

However from (1.2) $dim\,\mathcal{H} = r+2$, and hence the elements of Π and $\Pi^\vee$ do not span $\mathcal{H}^*$ and $\mathcal{H}$ respectively. In order to complete the bases we fix an element $d\in\mathcal{H}$ satisfying **[Kac4]**

$$<\alpha_i,d> = \delta_{0i} \quad \text{for } i=0,1,\ldots,r\,, \tag{3.2a}$$

and an element $\Lambda_0\in\mathcal{H}^*$ which satisfies the following conditions

$$\begin{aligned} <\Lambda_0,\alpha_i^\vee> &= \delta_{0i} \quad \text{for } i=0,1,\ldots,r \\ <\Lambda_0,d> &= 0\,. \end{aligned} \tag{3.2b}$$

The center of $\mathcal{G}(A)$ is one dimensional and is spanned by the canonical central element

$$K = \sum_{i=0}^{r} c_i^\vee \alpha_i^\vee\,, \tag{3.3a}$$

where the co-marks $c_i^\vee$'s are column linear dependence coefficients of A, i.e.

$$\sum_{j=0}^{r} A_{ij}c_j^\vee = 0\,. \tag{3.3b}$$

In the dual space, introduce a vector

$$\delta = \sum_{i=0}^{r} c_i\alpha_i\,, \tag{3.4a}$$

which is the smallest positive imaginary root. The integer marks c_i are chosen such that they form the row linear dependence coefficients for the affine matrix, i.e. the marks c_i satisfy

$$\sum_{i=0}^{r} c_i A_{ij} = 0. \tag{3.4b}$$

To fix the normalisation we choose marks and co-marks such that $min\{c_0, c_1, \ldots, c_r\} = min\{c_0^\vee, c_1^\vee, \ldots, c_r^\vee\} = 1$. In this normalisation $c_0 = 1$ in all cases. The integer co-marks are labelled on the Dynkin diagram of Table 1.2. If c_i differs from $c_i^\vee$, the corresponding c_i is given in a bracket beside $c_i^\vee$. The sums

$$h = \sum_{i=0}^{r} c_i \quad \text{and} \quad g = \sum_{i=0}^{r} c_i^\vee \tag{3.5}$$

are called the Coxeter number and the dual Coxeter number, respectively.

Since A is symmetrisable there must exist a non singular matrix D such that $S = DA$ is symmetric. The definition (3.4) of the imaginary root δ implies that $A^t\delta = 0$. Then we obtain successively:

$$(D^{-1}S)^t\,\delta = 0 \quad ;$$

$$S^t\,(D^{-1})^t\,\delta = 0 \quad ;$$

$$S\,D^{-1}\,\delta = 0 \quad ;$$

$$D\,A\,D^{-1}\,\delta = 0 \quad ;$$

$$A\,D^{-1}\,\delta = 0 \quad .$$

When compared with (3.3) we can deduce that $D^{-1}\,\delta = mK$ for some constant m. If we choose $m = 1$ then $D_{ii} = c_i/c_i^\vee$ and $D_{ii}^{-1} = c_i^\vee/c_i$. Since S is symmetric then $D_{ii}A_{ij} = D_{jj}A_{ji}$ and $A_{ij}D_{jj}^{-1} = A_{ji}D_{ii}^{-1}$.

We can now define non-degenerate symmetric bilinear forms $(\cdot \mid \cdot)$ on $\mathcal{H}$ and $\mathcal{H}^*$ as follows:

$$
\begin{aligned}
S_{ij} = (\alpha_i^\vee \mid \alpha_j^\vee) = D_{ii} A_{ij} = \frac{c_i}{c_i^\vee} A_{ij} \,; \\
(\alpha_i \mid \alpha_j) = A_{ij} D_{jj}^{-1} = \frac{c_j^\vee}{c_j} A_{ij} \,.
\end{aligned}
\tag{3.6}
$$

A consistent choice for an isomorphism $\nu : \mathcal{H} \to \mathcal{H}^*$ is

$$
\begin{aligned}
\nu(\alpha_i^\vee) &= \frac{c_i}{c_i^\vee} \alpha_i \quad \text{for } i = 0, 1, \ldots, r \\
\nu(K) &= \delta \\
\nu(d) &= \frac{1}{c_0^\vee} \Lambda_0 \,.
\end{aligned}
\tag{3.7}
$$

In general, for any coroot $\alpha^\vee \in \mathcal{H}$,

$$
\nu(\alpha^\vee) = \frac{2\alpha}{(\alpha \mid \alpha)} \,. \tag{3.8}
$$

Next we introduce the important element

$$
\theta = \delta - \alpha_0 = \sum_{i=1}^{r} c_i \alpha_i \,. \tag{3.9}
$$

We can then obtain the following relations involving θ:

$$
\theta^\vee = \frac{K}{c_0^\vee} - \alpha_0^\vee \quad \text{and} \quad \nu(\theta^\vee) = \frac{\theta}{c_0^\vee} \,. \tag{3.10}
$$

For easy reference we tabulate below the bilinear forms involving elements of $\mathcal{H}^*$ and $\mathcal{H}$.

Table 3.1a : Bilinear form on $\mathcal{H}^* \times \mathcal{H}$

$<\cdot,\cdot>$	$\alpha_j^\vee$	d	K	$\theta^\vee$
α_i	A_{ij}	δ_{i0}	0	$-A_{i0}$
Λ_0	δ_{0j}	0	$c_0^\vee$	0
δ	0	1	0	0
θ	$-A_{0j}$	0	0	2

Table 3.1b : Bilinear form on $\mathcal{H}^* \times \mathcal{H}^*$

$(\cdot \mid \cdot)$	α_j	Λ_0	δ	θ
α_i	$\frac{c_j^\vee}{c_j} A_{ij}$	$c_0^\vee \delta_{i0}$	0	$-c_0^\vee A_{i0}$
Λ_0	$c_0^\vee \delta_{0j}$	0	$c_0^\vee$	0
δ	0	$c_0^\vee$	0	0
θ	$-\frac{c_j^\vee}{c_j} A_{0j}$	0	0	$2c_0^\vee$

Table 3.1c : Bilinear form on $\mathcal{H} \times \mathcal{H}$

$(\cdot \mid \cdot)$	$\alpha_j^\vee$	d	K	$\theta^\vee$
$\alpha_i^\vee$	$\frac{c_i}{c_i^\vee} A_{ij}$	$\frac{1}{c_0^\vee} \delta_{i0}$	0	$-\frac{c_i}{c_i^\vee} A_{i0}$
d	$\frac{1}{c_0^\vee} \delta_{0j}$	0	1	0
K	0	1	0	0
$\theta^\vee$	$-\frac{1}{c_0^\vee} A_{0j}$	0	0	$\frac{2}{c_0^\vee}$

3.2 Construction of affine algebras

Starting with a GCM A provides one way to construct affine algebras. Another way to construct them is through an extension of the well known simple finite-dimensional Lie algebras. This is particular useful if we want to identify the structure of the affine algebras in terms of their simple finite-dimensional Lie subalgebras. Our aim in this section is to obtain the roots of all affine algebras and their multiplicities. Let us first construct the untwisted affine algebras, i.e. the affine algebras with parenthetical superscript (1).

Let $\bar{\mathcal{G}}$ be a simple complex finite-dimensional Lie algebra and $\mathbb{C}[t, t^{-1}]$ the ring of Laurent polynomials in t. A complex untwisted affine algebra $\mathcal{G}$ may be constructed as an extention of a loop algebra $\bar{\mathcal{G}} \otimes \mathbb{C}[t, t^{-1}]$ as

$$\mathcal{G} = (\bar{\mathcal{G}} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}K \oplus \mathbb{C}d, \tag{3.11}$$

with the bracket operation defined on $\mathcal{G}$ as follows:

$$\begin{aligned} &[(x \otimes t^i) + pK + \mu d,\ (y \otimes t^j) + qK + \mu' d] \\ &= [x, y] \otimes t^{i+j} + j\mu(y \otimes t^j) - i\mu'(x \otimes t^i) + i\delta_{i+j,0}(x \mid y)K\,, \end{aligned} \tag{3.12}$$

where $(\cdot \mid \cdot)$ is the Killing bilinear form on $\bar{\mathcal{G}}$. It can be verified that the above commutator is antisymmetric and satisfies the Jacobi identity. The element K lies in the centre of $\mathcal{G}$ and d acts on the elements of the loop algebra in the same way as the differential operator $t\frac{\partial}{\partial t}$.

We identify $\bar{\mathcal{G}}$ with the subalgebra $\bar{\mathcal{G}} \otimes id$ of $\mathcal{G}$ and let $h_i = \alpha_i^\vee, e_i, f_i$ for $i = 1, 2, \ldots, r$ be the Chevalley generators of $\bar{\mathcal{G}}$. If θ is the highest root of $\bar{\mathcal{G}}$ then its expression is given by (3.9) and we can choose $f_\theta \in \bar{\mathcal{G}}_\theta$ and $e_\theta \in \bar{\mathcal{G}}_{-\theta}$ such that **[Kac4]**

$$[e_\theta, f_\theta] = -\theta^\vee\,. \tag{3.13}$$

Let $e_0 = e_\theta \otimes t$ and $f_0 = f_\theta \otimes t^{-1}$ then it can be deduced from (3.13) that $[e_0, f_0] = K - \theta^\vee$. Let $\alpha_0^\vee = K - \theta^\vee$, then for $i = 0, 1, \ldots, r$ $\quad \alpha_i^\vee, e_i, f_i$ are the generators of $\mathcal{G}$ and they generate the matrix $A = (<\alpha_i, \alpha_j^\vee>)_{i,j=0}^r$ which coincides with the untwisted affine GCM.

Let $\bar{\mathcal{H}}$ be the Cartan subalgebra of $\bar{\mathcal{G}}$. For $\bar{h} \in \bar{\mathcal{H}}$, corresponding elements $h \in \mathcal{H}$ of the Cartan subalgebra of $\mathcal{G}$ are given by

$$h = \bar{h} \otimes t^0 + pK + \mu d\,. \tag{3.14}$$

Let δ be the linear functional on $\mathcal{H}$ defined by [Co]

$$\begin{aligned} \delta(\alpha_i^\vee \otimes t^0) &= 0 \quad \text{for } i = 1, \ldots, r \\ \delta(K) &= 0 \\ \delta(d) &= 1 \end{aligned} \tag{3.15}$$

which are consistent with the conditions on the imaginary root δ in Table 3.1a. The bracket operation of (3.12) then gives

$$\begin{aligned} [h, \bar{e}_\alpha \otimes t^j] =& [\bar{h} \otimes t^0 + pK + \mu d, \bar{e}_\alpha \otimes t^j] \\ =& [\bar{h}, \bar{e}_\alpha] \otimes t^j + j\mu(\bar{e}_\alpha \otimes t^j) \\ =& (\alpha(h) + j\delta(h))\bar{e}_\alpha \otimes t^j \end{aligned}$$

and similarly

$$[h, \bar{h}_\alpha \otimes t^j] = j\delta(h)(\bar{h}_\alpha \otimes t^j). \tag{3.16}$$

Hence $\bar{e}_\alpha \otimes t^j$ corresponds to a root $\alpha + j\delta$ and $\bar{h}_\alpha \otimes t^j$ corresponds to a root $j\delta$. However, there are r linearly independent elements $\bar{h}_\alpha \otimes t^j$ that can correspond to the root $j\delta$ and hence the multiplicity for the root $j\delta$ is r.

Next we construct the twisted affine algebras. Again let $\bar{\mathcal{G}}$ be a simple finite-dimensional Lie algebra and let τ be a symmetry of the corresponding Dynkin diagram. Non-trivial symmetries are admitted only by the Dynkin diagrams of the algebras A_r, D_r, E_6 and D_4. For all of these algebras, except D_4, there is only one non-trivial symmetry τ [Co] and this satisfies $\tau^2(\alpha_i) = \alpha_i$ for $i = 1, \ldots, r$. But for D_4 there is also a symmetry τ of order 3.

Let σ be the automorphisms of $\bar{\mathcal{G}}$ which correspond to the symmetries of Dynkin diagrams. If $\sigma^m = 1$ for $m = 2$ or 3 then we have the decomposition of $\bar{\mathcal{G}}$ into a direct sum of eigenspaces of σ **[Kac4]**

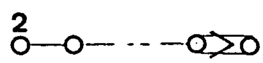
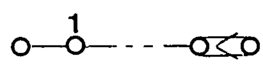
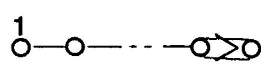
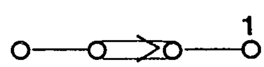
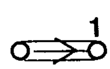
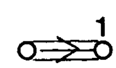
$$\bar{\mathcal{G}} = \bar{\mathcal{G}}_0 + \bar{\mathcal{G}}_1 \quad (\text{ or } \quad \bar{\mathcal{G}} = \bar{\mathcal{G}}_0 + \bar{\mathcal{G}}_1 + \bar{\mathcal{G}}_2) \tag{3.17}$$

and they satisfy

$$[\bar{\mathcal{G}}_0, \bar{\mathcal{G}}_0] \subset \bar{\mathcal{G}}_0, \quad [\bar{\mathcal{G}}_0, \bar{\mathcal{G}}_1] \subset \bar{\mathcal{G}}_1, \quad [\bar{\mathcal{G}}_0, \bar{\mathcal{G}}_2] \subset \bar{\mathcal{G}}_2.$$

We observe that the space $\bar{\mathcal{G}}_0$ is a subalgebra of $\bar{\mathcal{G}}$ and that $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ are $\bar{\mathcal{G}}_0$-modules. In fact these $\bar{\mathcal{G}}_0$-modules are irreducible, and $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ are equivalent $\bar{\mathcal{G}}_0$-modules. For each $\bar{\mathcal{G}}$ the corresponding $\bar{\mathcal{G}}_0$ is given in Table 3.2. Its construction in term of the generators h_i, e_i, f_i for each algebra $\bar{\mathcal{G}}$ can be found, for example, in **[Kac4]**.

Table 3.2 : Underlying information for the construction of twisted algebras

m	$\mathcal{G}$	$\bar{\mathcal{G}}_0$	$\bar{\mathcal{G}}_0$ − module $\bar{\mathcal{G}}_1$	$\bar{\mathcal{G}}_0$ − module $\bar{\mathcal{G}}_2$	$dim\,\bar{\mathcal{G}}_1$
2	A_{2r}	B_r			$2r^2 + 3r$
2	A_{2r-1}	C_r			$2r^2 - r - 1$
2	D_{r+1}	B_r			$2r + 1$
2	A_2	A_1			5
2	E_6	F_4			26
3	D_4	G_2			7

Let $\bar{\mathcal{H}}_0$ be the Cartan subalgebra of $\bar{\mathcal{G}}_0$ and let α_i denote the associated simple roots of $\bar{\mathcal{G}}_0$. The $\bar{\mathcal{G}}_0$-modules, $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ have highest weight $\theta = \sum_i c_i\alpha_i$ given in Table 3.2 in term of fundamental weights. The weight space decomposition of $\bar{\mathcal{G}}_p$ takes the form:

$$\bar{\mathcal{G}}_p = \sum_{\beta\in\Delta_p} \bar{\mathcal{G}}_{p,\beta} + \bar{\mathcal{G}}_{p,0} \quad \text{for}\, p = 1, 2 \tag{3.18}$$

where $\bar{\mathcal{G}}_{p,0}$ is the subspace corresponding to zero weight and Δ_p is the set of non-zero weights of $\bar{\mathcal{H}}_0$ on $\bar{\mathcal{G}}_p$.

With all this notation, the corresponding twisted affine algebra is defined as

$$\mathcal{G}^{(m)} = \sum_{p=0}^{m-1} \sum_{j\in\mathbf{Z}} (\bar{\mathcal{G}}_{j=p\,mod\,m} \otimes t^j) \oplus \mathbf{C}K \oplus \mathbf{C}d \tag{3.19}$$

where K and d are as defined in (3.11). Let δ be a functional on $\mathcal{H}$ as in (3.15). Then the root system Δ of $\mathcal{G}^{(m)}$ is given by **[KV]**

$$\Delta = \{\alpha + j\delta \mid \alpha \in \Delta_p,\, j \in \mathbf{Z},\, j \equiv p \bmod m\} \cup \{j\delta \mid j \in \mathbf{Z},\, j \neq 0\}.$$

Here Δ_p is the root system Δ_0 of the algebra $\bar{\mathcal{G}}_0$ if $p = 0$ and the weight system Δ_p of (3.17) if $p \neq 0$.

Let us consider by way of an example, the determination of the roots of the twisted algebra $\mathcal{G} = A_4^{(2)}$. From (3.17) and Table 3.2 we have

$$A_4 \supset \bar{\mathcal{G}}_0 + \bar{\mathcal{G}}_1 = B_2 + \bar{\mathcal{G}}_1\,.$$

We then choose the Cartan subalgebra $\bar{\mathcal{H}}_0$ in B_2 and the roots

$$0,\ \pm\alpha_1,\ \pm\alpha_2,\ \pm(\alpha_1+\alpha_2),\ \pm(\alpha_1+2\alpha_2)$$

with respect to $\bar{\mathcal{H}}_0$. All these roots have multiplicity one except the zero root which has multiplicity 2.

If ω_i is the fundamental weight of simple finite-dimensional Lie algebras then from Table 3.2 the B_2-module $\bar{\mathcal{G}}_1$ has highest weight $2\omega_1 = 2\alpha_1 + 2\alpha_2$. Then from Figure 3.1, the rest of the weights can be computed to be

$$\pm(2\alpha_1+2\alpha_2),\ \pm(\alpha_1+2\alpha_2),\ \pm(\alpha_1+\alpha_2),\ \pm 2\alpha_2,\ \pm\alpha_1,\ \pm\alpha_2,\ 0\,.$$

All the weights have multiplicity 1 except the weight 0 which has multiplicity 2. The twisted affine algebra is then given

$$A_4^{(2)} = \sum_{j\in\mathbf{Z}} B_2 \otimes t^{2j} \oplus \sum_{j\in\mathbf{Z}} \bar{\mathcal{G}}_1 \otimes t^{2j-1} \oplus (\mathbf{C}K + \mathbf{C}d).$$

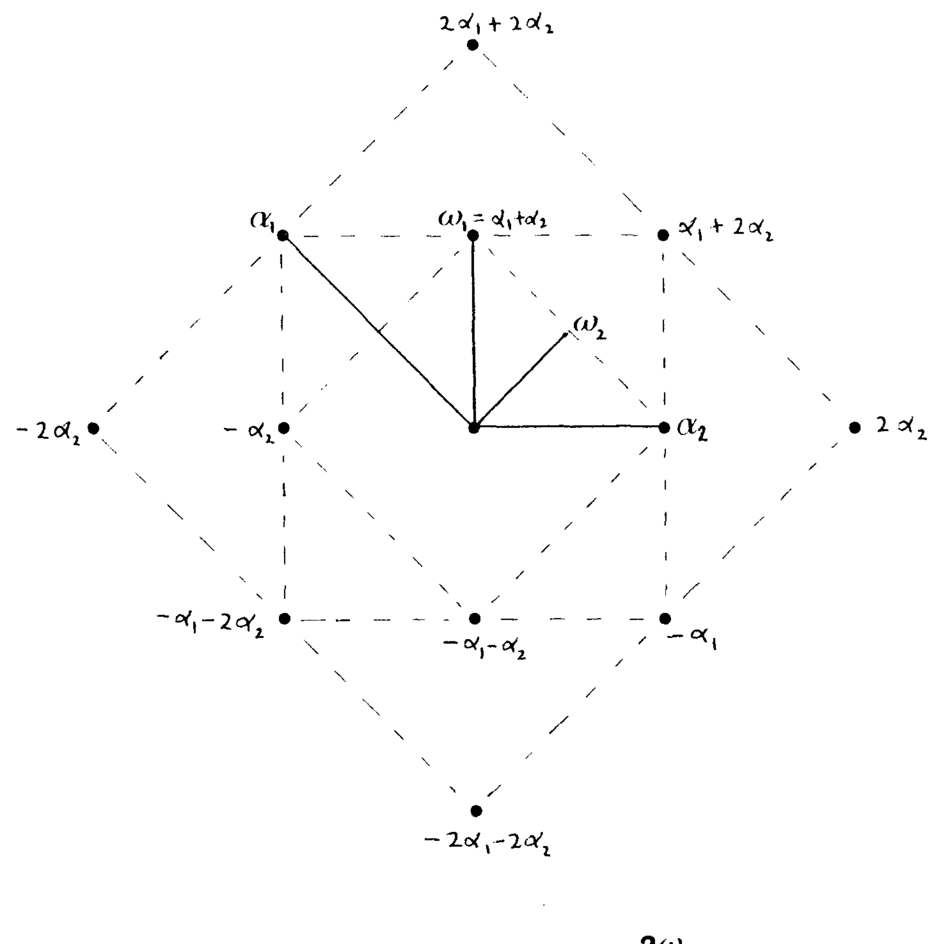


Figure 3.1 : Weight diagram of the $V^{2\omega_1}$ module of B_2

The roots of $A_4^{(2)}$ can then be read off as

$$\begin{array}{rl} j\delta & \text{with multiplicity } 2 \\ \pm\alpha_1 \pm j\delta & \alpha_1 \text{ long} \\ \pm\alpha_2 \pm j\delta & \alpha_2 \text{ short} \\ \pm(\alpha_1+\alpha_2) \pm j\delta & \alpha_1+\alpha_2 \text{ short} \\ \pm(\alpha_1+2\alpha_2) \pm j\delta & \alpha_1+2\alpha_2 \text{ long} \\ \pm(2\alpha_1+2\alpha_2) \pm (2j+1)\delta & 2\alpha_1+2\alpha_2 \text{ very long} \\ \pm 2\alpha_2 \pm (2j+1)\delta & 2\alpha_2 \text{ very long} \end{array}$$

It has been shown how the construction of the twisted affine algebra $\mathcal{G}^{(m)}$, $m = 2$ or 3, involves a non-trivial automorphism of the Dynkin diagram of $\bar{\mathcal{G}}$. Analogously, we can think of the untwisted affine algebras $\mathcal{G}^{(1)}$ as involving a trivial automorphism of the Dynkin diagram of $\bar{\mathcal{G}}$.

If we let $X_{N(r)}^{(m)}$ be the affine algebra generated by $\alpha_i^\vee, e_i, f_i$ $i = 0, 1, \ldots, r$ and Y_r be the subalgebra of $X_{N(r)}^{(m)}$ generated by $\alpha_i^\vee, e_i, f_i$ $i = 1, \ldots, r$. Then $Y_r \simeq \bar{\mathcal{G}}$ in the case of an untwisted affine algebra and $X_{N(r)} \supset Y_r \simeq \bar{\mathcal{G}}_0$ in the case of a twisted affine algebra. Equivalently we can identify Y_r with the simple finite-dimensional Lie algebra $\mathcal{G}(\bar{A})$ whose GCM $\bar{A}$ is obtained from A by deleting the zeroth row and column. Let $\mathcal{H}$ be the Cartan subalgebra of $X_{N(r)}^{(m)}$ and $\bar{\mathcal{H}}_0 = Y_r \cap \mathcal{H}$ and by (3.14) we have orthogonal direct sum of subspaces as follows:

$$\mathcal{H} = \bar{\mathcal{H}} \oplus (\mathbb{C}K + \mathbb{C}d) \quad \text{and} \quad \mathcal{H}^* = \bar{\mathcal{H}}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0)\,.$$

Let $\Delta \subset \mathcal{H}^*$ be the root system and $\bar{\Delta} = \Delta \cap \mathbb{C}[\alpha_1, \ldots, \alpha_r]$. Denote by $\bar{\Delta}_s$ and $\bar{\Delta}_\ell$ the sets of short and long roots, respectively, in $\bar{\Delta}$. A closer observation of the Dynkin diagram of $A_{2r}^{(2)}$ or our previous example reveals that in Δ there exist also roots of length twice that of the short roots but longer than the long roots. With our convention we have the following results on the real roots of the affine algebras [**Kac4**].

Proposition 3.1.

a) $\Delta_{re} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta},\ n \in \mathbb{Z}\}$ if $m = 1$.

b) $\Delta_{re} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta}_s,\ n \in \mathbb{Z}\} \cup \{\alpha + 2n\delta \mid \alpha \in \bar{\Delta}_\ell,\ n \in \mathbb{Z}\}$ if $m = 2$ but not $A_{2r}^{(2)}$

c) $\Delta_{re} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta}_s,\ n \in \mathbb{Z}\} \cup \{\alpha + 3n\delta \mid \alpha \in \bar{\Delta}_\ell,\ n \in \mathbb{Z}\}$ if $m = 3$.

d) $\Delta_{re} = \{\alpha + n\delta \mid \alpha \in \bar{\Delta}_s,\ n \in \mathbb{Z}\} \cup \{\alpha + n\delta \mid \alpha \in \bar{\Delta}_\ell,\ n \in \mathbb{Z}\}$
$\cup\{2\alpha + (2n-1)\delta \mid \alpha \in \bar{\Delta}_s,\ n \in \mathbb{Z}\}$ for $A_{2r}^{(2)}$.

All of these real roots have multiplicity 1. The multiplicity of the imaginary root $n\delta$ is given by the following Proposition [**Co**].

Proposition 3.2. *The multiplicity of the non-zero imaginary roots are as follows*

(a) For all untwisted algebras or $A_{2r}^{(2)}$

$$\operatorname{mult} n\delta = r\,.$$

(b) For $A_{2r-1}^{(2)}$

$$\operatorname{mult}\ n\delta = \begin{cases} r & \text{if } n \text{ is even,} \\ r-1 & \text{if } n \text{ is odd.} \end{cases}$$

(c) For $D_{r+1}^{(2)}$

$$\operatorname{mult}\ n\delta = \begin{cases} r & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

(d) For $E_6^{(2)}$

$$\operatorname{mult}\ n\delta = \begin{cases} 4 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

(e) For $D_4^{(3)}$

$$\operatorname{mult}\ n\delta = \begin{cases} 2 & \text{if } n \bmod 3 = 0, \\ 1 & \text{if } n \bmod 3 = 1 \text{ or } 2. \end{cases}$$

3.3 The affine Weyl group

Let W and $\overline{W}$ be the Weyl groups generated by $s_0, s_1, \ldots, s_r$ and $s_1, \ldots, s_r$ respectively. The element s_i acts on $\mathcal{H}^*$ as in (1.8) and on $\mathcal{H}$ as in (1.11). In particular we can see from Table 3.1a $s_i(\delta) = \delta$ and $s_i(K) = K$.

Let $\lambda, \lambda' \in \mathcal{H}^*$. Then the mapping (3.7) implies

$$\begin{aligned}(s_i(\lambda) \mid \lambda') &= (\lambda \mid \lambda') - < \lambda, \alpha_i^\vee > (\alpha_i \mid \lambda') \\ &= (\lambda \mid \lambda') - < \lambda, \alpha_i^\vee > \frac{c_i^\vee}{c_i} < \lambda', \alpha_i^\vee > \\ &= (\lambda \mid \lambda') - < \lambda', \alpha_i^\vee > (\lambda \mid \alpha_i) \\ &= (\lambda \mid s_i(\lambda')).\end{aligned}$$

More generally for any $w \in W$ we have $(w\lambda \mid \lambda') = (\lambda \mid w^{-1}\lambda')$. Hence the bilinear form $(\cdot \mid \cdot)$ is also W-invariant.

Let the lattice M for each affine algebra be defined as follows **[Kac4]**

$$M = \begin{cases} \bar{Q} & \text{if } A \text{ is symmetric or } m = 2 \text{ or } 3, \\ \nu(\bar{Q}^\vee) & \text{otherwise }, \end{cases} \tag{3.20a}$$

or more explicitly as

$$M = \begin{cases} \mathbb{Z}[\alpha_1, \ldots, \alpha_r] & \text{for } A_r^{(1)},\ D_r^{(1)},\ E_r^{(1)} \text{ and twisted algebras}, \\ \mathbb{Z}[\alpha_1, \ldots, \alpha_{r-1}, 2\alpha_r] & \text{for } B_r^{(1)}, \\ \mathbb{Z}[2\alpha_1, \ldots, 2\alpha_{r-1}, \alpha_r] & \text{for } C_r^{(1)}, \\ \mathbb{Z}[\alpha_1, \alpha_2, 2\alpha_3, 2\alpha_4] & \text{for } F_4^{(1)}, \\ \mathbb{Z}[\alpha_1, 3\alpha_2] & \text{for } G_2^{(1)}. \end{cases} \tag{3.20b}$$

For $\alpha \in M$ define an endomorphism t_α on $\mathcal{H}^*$ as follows **[Kac4]**

$$t_\alpha(\lambda) = \lambda + < \lambda, K > \alpha - ((\lambda \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha) < \lambda, K >)\delta. \tag{3.21}$$

Then we have the following Lemma **[Kac4]**.

Lemma 3.3. *The endomorphism t_α satisfies the following relations*

(a) $t_\alpha t_\beta = t_{\alpha+\beta}, \qquad \alpha, \beta \in M$

(b) $t_{w(\alpha)} = w t_\alpha w^{-1}, \qquad w \in \bar{W}$

Proof

(a)

$$\begin{aligned}
t_\alpha t_\beta(\lambda) &= \lambda + <\lambda, K> \alpha - ((\lambda \mid \alpha) + \frac{1}{2} \mid \alpha \mid^2 < \lambda, K>)\delta \\
&\quad + <\lambda, K> (\beta + <\beta, K> \alpha - ((\alpha \mid \beta) + \frac{1}{2}(\alpha \mid \alpha) < \beta, K>)\delta) \\
&\quad - ((\lambda \mid \beta) + \frac{1}{2}(\beta \mid \beta) < \lambda, K>)(\delta + <\delta, K> \alpha - (\delta \mid \alpha)\delta \\
&\qquad - \frac{1}{2}(\alpha \mid \alpha) < \delta, K > \delta) \\
&= \lambda + <\lambda, K> \alpha - ((\lambda \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha) < \lambda, K>)\delta \\
&\quad + <\lambda, K> \beta - <\lambda, K> (\alpha \mid \beta)\delta - ((\lambda \mid \beta) + \frac{1}{2}(\beta \mid \beta) < \lambda, K>)\delta \\
&= \lambda + <\lambda, K> (\alpha + \beta) - ((\lambda \mid \alpha + \beta) - \frac{1}{2}(\alpha \mid \alpha) < \lambda, K>)\delta \\
&\quad - <\lambda, K> (\alpha \mid \beta)\delta - \frac{1}{2}(\beta \mid \beta) < \lambda, K>)\delta \\
&= \lambda + <\lambda, K> (\alpha + \beta) - ((\lambda \mid \alpha + \beta) - \frac{1}{2}(\alpha + \beta \mid \alpha + \beta) < \lambda, K>)\delta \\
&= t_{\alpha+\beta}(\lambda).
\end{aligned}$$

(b) Considering the facts that $w(K) = K$, $w(\delta) = \delta$ and both bilinear forms $< \cdot, \cdot >$ and $(\cdot \mid \cdot)$ are W-invariant,

$$\begin{aligned}
t_\alpha w^{-1}(\lambda) &= w^{-1}(\lambda) + <\lambda, K> \alpha - ((\lambda \mid w(\alpha)) + \frac{1}{2} \mid \alpha \mid^2 < \lambda, K>)\delta \\
w t_\alpha w^{-1}(\lambda) &= \lambda + <\lambda, K> w(\alpha) - ((\lambda \mid w(\alpha)) + \frac{1}{2} \mid \alpha \mid^2 < \lambda, K>)\delta \\
&= t_{w(\alpha)}.
\end{aligned}$$

□

Lemma 3.3(a) and (3.21) imply that t_α acts like a translation on $\mathcal{H}^*$.

Recall that $\theta = \delta - \alpha_0 \in \bar{\Delta}^+$. θ is the highest long root of $\bar{\mathcal{G}}$ in the case of untwisted algebras or is the highest short root of $\bar{\mathcal{G}}_0$ in the case of twisted algebras.

Then $\nu(\theta^\vee) = \theta$ except in the case of $A_{2r}^{(2)}$ where $\nu(\theta^\vee) = \theta/c_0^\vee = \theta/2$. Analogous to the definition of the fundamental reflections, we can define a reflection s_α with respect to a real root α by

$$s_\alpha(\lambda) = \lambda - <\lambda, \alpha^\vee> \alpha, \quad \lambda \in \mathcal{H}^* . \tag{3.22}$$

Since $<\alpha, \alpha^\vee> = 2$ then s_α sends α to $-\alpha$. If $\alpha = \theta/c_0^\vee$ then (3.22) and (3.10) imply that

$$s_{\theta/c_0^\vee}(\lambda) = \lambda - <\lambda, \frac{K}{c_0^\vee} - \alpha_0^\vee> \theta . \tag{3.23}$$

Further action by s_0 gives

$$\begin{aligned} s_0 s_{\theta/c_0^\vee}(\lambda) =& \lambda - <\lambda, \alpha_0^\vee> \alpha_0 \\ & - <\lambda, \frac{K}{c_0^\vee} - \alpha_0^\vee> (\theta - <\theta, \alpha_0^\vee> \alpha_0) \\ =& \lambda - \frac{1}{c_0^\vee} <\lambda, K> (\alpha_0 + \delta) + <\lambda, \alpha_0^\vee> \delta . \end{aligned}$$

However from (3.21), (3.10) and the fact that $(\theta \mid \theta) = 2c_0^\vee$

$$\begin{aligned} t_{\nu(\theta^\vee)}(\lambda) &= \lambda + <\lambda, K> \nu(\theta^\vee) - ((\lambda \mid \nu(\theta^\vee)) + \frac{1}{2}(\nu(\theta^\vee) \mid \nu(\theta^\vee)) <\lambda, K>)\delta \\ &= \lambda - \frac{1}{c_0^\vee} <\lambda, K> (\alpha_0 + \delta) + <\lambda, \alpha_0^\vee> \delta . \end{aligned}$$

Hence for each affine algebra we can write

$$t_{\nu(\theta^\vee)} = s_0 s_{\theta/c_0^\vee} , \tag{3.24}$$

where $s_{\theta/c_0^\vee}$, which does not contain the fundamental reflection s_0, satisfies (3.23). In Table 3.3 we list explicitly the reflection $s_{\theta/c_0^\vee}$ in terms of fundamental reflections for each affine algebra. These expression are obtained from Table 1 of [Mo2] by adding certain conjugates. In fact $s_{\theta/c_0^\vee} = w s_i w^{-1}$ for any w and i such that $w(\alpha_i) = \theta/c_0^\vee$. The length of each tabulated expression for $s_{\theta/c_0^\vee}$ is minimal in the sense that it satisfies Proposition 1.10. The rank independent cases in Table 3.3 can be verified directly. For the rank dependent cases we will give the proof for just $A_r^{(1)}$. The proof for the other cases is similar.

Table 3.3 : The reflection $s_{\theta/c_0^\vee}$ in terms of fundamental reflections.

Algebra	$s_{\theta/c_0^\vee}$
$A_1^{(1)}, A_2^{(2)}$	s_1
$A_r^{(1)}, A_{2r}^{(2)}$	$s_1 s_2 \ldots s_{r-1} s_r s_{r-1} \ldots s_2 s_1$
$B_r^{(1)}, A_{2r-1}^{(2)}$	$s_2 s_3 \ldots s_{r-1} s_r s_{r-1} \ldots s_2 s_1 s_2 \ldots s_{r-1} s_r s_{r-1} \ldots s_3 s_2$
$C_r^{(1)}, D_{r+1}^{(2)}$	$s_1 s_2 \ldots s_{r-1} s_r s_{r-1} \ldots s_2 s_1$
$D_r^{(1)}$	$s_2 s_3 \ldots s_{r-2} s_r s_{r-1} \ldots s_2 s_1 s_2 \ldots s_{r-1} s_r s_{r-2} \ldots s_3 s_2$
$E_6^{(1)}$	$s_6 s_3 s_4 s_2 s_3 s_5 s_4 s_1 s_2 s_3 s_6 s_3 s_2 s_1 s_4 s_5 s_3 s_2 s_4 s_3 s_6$
$E_7^{(1)}$	$s_1 s_2 s_3 s_4 s_5 s_7 s_3 s_4 s_2 s_3 s_6 s_5 s_4 s_7 s_3 s_2 s_1$ $s_2 s_3 s_7 s_4 s_5 s_6 s_3 s_2 s_4 s_3 s_7 s_5 s_4 s_3 s_2 s_1$
$E_8^{(1)}$	$s_1 s_2 s_3 s_4 s_5 s_6 s_8 s_5 s_4 s_3 s_7 s_6 s_5 s_4 s_2 s_3 s_8 s_5 s_6 s_4 s_5 s_7 s_6 s_8 s_5 s_4 s_3 s_2 s_1$ $s_2 s_3 s_4 s_5 s_8 s_6 s_7 s_5 s_4 s_6 s_5 s_8 s_3 s_2 s_4 s_5 s_6 s_7 s_3 s_4 s_5 s_8 s_6 s_5 s_4 s_3 s_2 s_1$
$F_4^{(1)}, E_6^{(2)}$	$s_1 s_2 s_3 s_2 s_4 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_2 s_1$
$G_2^{(1)}, D_4^{(3)}$	$s_1 s_2 s_1 s_2 s_1$

Proposition 3.4. *For the algebra* $A_r^{(1)}$ $\ell(s_\theta) = 2r - 1$ *and* $s_\theta = w s_r w^{-1}$ *with* $w = s_1 s_2 \ldots s_{r-1}$.

Proof: If α is a root then $< \alpha, K >= 0$. Then (3.21) implies that

$$t_\theta(\alpha) = \alpha - (\alpha \mid \theta)\delta$$

so that (3.24) further gives

$$s_\theta(\alpha) = s_0(\alpha) - (\alpha \mid \theta)\delta \tag{3.25}$$

where $\theta = \sum_{i=1}^r \alpha_i$ and $\delta = \sum_{i=0}^r \alpha_i$. The set of positive roots for A_r is

$$\{\alpha_i + \alpha_{i+1} + \ldots + \alpha_j \mid 1 \le i \le j \le r\}. \tag{3.26}$$

From (3.25) we can show that

$$s_\theta(\alpha_i) = \begin{cases} -(\alpha_2 + \ldots + \alpha_r) \in \bar{\Delta}^- & i = 1, \\ \alpha_i & i = 2, \ldots, r-1, \\ -(\alpha_1 + \ldots + \alpha_{r-1}) \in \bar{\Delta}^- & i = r. \end{cases}$$

While for $i < j$

$$s_\theta(\alpha_i + \ldots + \alpha_j) = \begin{cases} -(\alpha_1 + \ldots + \alpha_r) \in \bar{\Delta}^- & i = 1, j = r, \\ -(\alpha_{j+1} + \ldots + \alpha_r) \in \bar{\Delta}^- & i = 1, j = 2, \ldots r-1, \\ \alpha_i + \ldots + \alpha_j & 2 \leq i < j \leq r-1, \\ -(\alpha_1 + \ldots + \alpha_{i-1}) \in \bar{\Delta}^- & i = 2, \ldots r-1, j = r. \end{cases}$$

Hence by Proposition 1.10, $\ell(s_\theta) = 1 + 1 + 1 + (r-2) + (r-2) = 2r - 1$. Finally, the element $w = s_1 s_2 \ldots s_{r-1}$ is simply the permutation $(1, r)$. The action of this element on $\alpha_r = \epsilon_r - \epsilon_{r+1}$ gives $\epsilon_1 - \epsilon_{r+1} = \theta$ as required. □

Lemma 3.3 implies that the operations t_α with $\alpha \in M$ forms an abelian group known as the group of translations T. This group is generated by $wt_{\nu(\theta^\vee)}w^{-1}$ for $w \in \overline{W}$. With this result we are then able to express the affine Weyl group W in term of the finite Weyl group $\overline{W}$.

Theorem 3.5. *The affine Weyl group W is the semidirect product of a finite Weyl group $\overline{W}$ and the group of translations T.*

Proof: First recall that if N and H are subgroups of a group G then G is said to be a semidirect product of N by H whenever

(a) N is a normal subgroup in G,

(b) $G = NH$ and

(c) $N \cap H = id$.

It is clear that $\overline{W}$ and T are subgroups of W and by Lemma 3.3(b), T is a normal subgroup in W.

In all cases shown above, the translation $t_{\nu(\theta^\vee)}$ can be written as $s_0 s_\theta$ where s_θ does not contain the fundamental reflection s_0.

$$s_0 s_\theta = t_{\nu(\theta^\vee)}$$

$$s_0 = t_{\nu(\theta^\vee)} s_\theta^{-1} = t_{\nu(\theta^\vee)} s_\theta \in T\overline{W}$$

and trivially $T\overline{W}$ contains $\overline{W}$ and hence all the fundamental reflections $s_1, \ldots, s_r$. Thus $W = T\overline{W}$.

For each $\alpha \in M$, the translation t_α contains the fundamental reflection s_0 but no element of $\overline{W}$ contains s_0. Hence $T \cap \overline{W} = id$. □

In the process of obtaining $t_{\nu(\theta^\vee)}$, we have also identified some of the other fundamental translations i.e. t_α's where the α's form the basis for the lattice M. Since $t_{w(\nu(\theta^\vee))} = w t_{\nu(\theta^\vee)} w^{-1}$, for certain $\alpha \in M$ we just need to identify $w \in \overline{W}$ such that $\alpha = w(\theta/c_0^\vee)$. In other instances we can only express α in the form $\alpha = \sum_{w \in \overline{W}} w(k\theta)$ with $k \in \frac{1}{c_0^\vee}\mathbb{Z}$. For example in the case of $G_2^{(1)}$,

$$t_{\alpha_1} = t_{s_2 s_1(\theta)} = s_2 s_1 t_\theta s_1 s_2$$

$$= s_2 s_1 s_0 s_1 s_2 s_1$$

but

$$t_{3\alpha_2} = t_{\theta - 2 s_2 s_1(\theta)} = t_{\theta - 2\alpha_1} = t_\theta (t_{\alpha_1})^{-2}$$

$$= s_0 s_1 s_2 s_0 s_1 s_2 s_1 s_2 s_1 s_0 s_1 s_2$$

$A_r^{(1)}$:

$$\alpha_i = s_{i-1} s_{i-2} \ldots s_1 s_{i+1} s_{i+2} \ldots s_r(\theta) \text{ for } i = 1, \ldots, r$$

$B_r^{(1)}$:

$$\alpha_i = s_{i-1} s_{i-2} \ldots s_1 s_{i+1} s_{i+2} \ldots s_{r-1} s_r s_{r-1} \ldots s_3 s_2(\theta) \text{ for } i = 1, \ldots, r-1$$

$$2\alpha_r = s_{r-1} s_{r-2} \ldots s_2(\theta) + s_{r-1} s_{r-2} \ldots s_2 s_1 s_2 \ldots s_{r-1} s_r s_{r-1} \ldots s_2(\theta)$$

$C_r^{(1)}$

$$2\alpha_i = s_{i-1} s_{i-2} \ldots s_1(\theta) + s_{i+1} s_{i+2} \ldots s_{r-1} s_r s_{r-1} \ldots s_1(\theta) \text{ for } i = 1, \ldots, r-1$$

$$\alpha_r = s_{r-1} s_{r-2} \ldots s_1(\theta)$$

$D_r^{(1)}$

$$\alpha_i = s_{i-1}s_{i-2}\ldots s_1 s_{i+1}s_{i+2}\ldots s_{r-1}s_r s_{r-2}s_{r-3}\ldots s_2(\theta) \text{ for } i = 1,\ldots,r-1$$

$$\alpha_r = s_{r-2}s_{r-3}\ldots s_1 s_{r-1}s_{r-2}\ldots s_3 s_2(\theta)$$

$E_6^{(1)}$

$$\alpha_1 = s_2s_3s_4s_6s_3s_5s_4s_2s_3s_6(\theta)$$

$$\alpha_2 = s_3s_6s_4s_3s_5s_4s_1s_2s_3s_6(\theta)$$

$$\alpha_3 = s_6s_4s_2s_3s_5s_4s_1s_2s_3s_6(\theta)$$

$$\alpha_4 = s_3s_6s_2s_3s_5s_4s_1s_2s_3s_6(\theta)$$

$$\alpha_5 = s_4s_3s_6s_2s_3s_1s_2s_4s_6s_3s_6(\theta)$$

$$\alpha_6 = s_1s_3s_4s_2s_3s_5s_4s_1s_2s_3s_6(\theta)$$

$E_7^{(1)}$

$$\alpha_1 = s_2s_3s_4s_5s_7s_3s_4s_2s_3s_6s_5s_4s_7s_3s_2s_1(\theta)$$

$$\alpha_2 = s_1s_3s_4s_5s_7s_3s_4s_2s_3s_6s_5s_4s_7s_3s_2s_1(\theta)$$

$$\alpha_3 = s_2s_1s_4s_5s_7s_3s_4s_2s_3s_6s_5s_4s_7s_3s_2s_1(\theta)$$

$$\alpha_4 = s_3s_2s_1s_5s_7s_3s_4s_2s_3s_6s_5s_4s_7s_3s_2s_1(\theta)$$

$$\alpha_5 = s_4s_3s_2s_1s_7s_3s_4s_2s_3s_6s_5s_4s_7s_3s_2s_1(\theta)$$

$$\alpha_6 = s_5s_4s_3s_2s_1s_7s_3s_2s_4s_3s_5s_4s_7s_3s_2s_1(\theta)$$

$$\alpha_7 = s_3s_7s_3(\theta)$$

$E_8^{(1)}$

$$\alpha_1 = s_2s_3s_4s_5s_6s_7s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_2 = s_3s_4s_5s_6s_7s_1s_6s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_3 = s_4s_5s_6s_7s_2s_1s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_4 = s_5s_6s_7s_3s_2s_1s_6s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_5 = s_6s_7s_4s_3s_2s_1s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_6 = s_7s_5s_4s_3s_2s_1s_6s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_7 = s_6s_5s_4s_3s_2s_1s_8s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$$\alpha_8 = s_5s_4s_3s_2s_1s_7s_6s_7s_5s_4s_3s_6s_5s_4s_2s_3s_8s_5s_6s_4s_5s_8s_7s_6s_5s_4s_3s_2s_1(\theta)$$

$F_4^{(1)}$

$$\alpha_1 = s_2s_3s_4s_2s_3s_2s_1(\theta)$$

$$\alpha_2 = s_1s_3s_4s_2s_3s_2s_1(\theta)$$

$$2\alpha_3 = s_1s_2s_4s_3s_2s_1(\theta) - s_1s_3s_4s_2s_3s_2s_1(\theta)$$

$$2\alpha_4 = s_3s_2s_1(\theta) - s_4s_3s_2s_1(\theta)$$

$G_2^{(1)}$

$$\alpha_1 = s_2s_1(\theta)$$

$$3\alpha_2 = \theta - 2s_2s_1(\theta)$$

$A_{2r}^{(2)}$

$$\alpha_i = s_{i-1}\ldots s_1s_{i+1}\ldots s_r(\theta/2) + s_{i-1}\ldots s_1s_{i+1}\ldots s_rs_{r-1}\ldots s_2s_1(\theta/2)$$

$$\alpha_r = s_{r-1}\ldots s_2s_1(\theta/2)$$

$A_{2r-1}^{(2)}$

$$\alpha_i = s_{i-1}s_{i-2}\ldots s_1s_{i+1}s_{i+2}\ldots s_{r-1}s_rs_{r-1}\ldots s_3s_2(\theta) \text{ for } i = 1,\ldots,r-1$$

$$\alpha_r = s_{r-1}s_{r-2}\ldots s_2(\theta) + s_{r-1}s_{r-2}\ldots s_2s_1s_2\ldots s_{r-1}s_rs_{r-1}\ldots s_2(\theta)$$

$D_{r+1}^{(2)}$

$$\alpha_i = s_{i-1}s_{i-2}\ldots s_1(\theta) + s_{i+1}s_{i+2}\ldots s_{r-1}s_rs_{r-1}\ldots s_1(\theta) \text{ for } i = 1,\ldots,r-1$$

$$\alpha_r = s_{r-1}s_{r-2}\ldots s_1(\theta)$$

$E_6^{(2)}$

$$\alpha_1 = s_2s_3s_4s_2s_3s_2s_1(\theta)$$

$$\alpha_2 = s_1s_3s_4s_2s_3s_2s_1(\theta)$$

$$\alpha_3 = s_1s_2s_4s_3s_2s_1(\theta) - s_1s_3s_4s_2s_3s_2s_1(\theta)$$

$$\alpha_4 = s_3s_2s_1(\theta) - s_4s_3s_2s_1(\theta)$$

$D_4^{(3)}$

$$\alpha_1 = s_2 s_1(\theta)$$

$$\alpha_2 = \theta - 2 s_2 s_1(\theta)$$

3.4 Highest weight modules

We shall study the highest weight modules V^Λ of affine algebras in the same way as we have done for those of the simple finite-dimensional Lie algebras. In affine algebras, it is convenient to express the weights of V^Λ in Dynkin notation i.e. with respect to the basis $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_r, \delta\}$. In term of this basis the simple roots can be expressed as follows

$$\begin{aligned} \alpha_0 &= \sum_{j=0}^{r} A_{0j}\Lambda_j + \delta, \\ \alpha_i &= \sum_{j=0}^{r} A_{ij}\Lambda_j \quad i = 1, \ldots, r. \end{aligned} \tag{3.27}$$

From (3.3b) and (3.4b) we can deduce respectively that

$$c_i^\vee = -c_0^\vee \sum_{j=1}^{r} \bar{A}_{ij}^{-1} A_{j0} \quad \text{for } i = 1, \ldots, r \tag{3.28a}$$

and

$$c_j^\vee = -\sum_{i=1}^{r} A_{0i}\bar{A}_{ij}^{-1} \quad \text{for } j = 1, \ldots, r. \tag{3.28b}$$

With the help of the relations (3.3b), (3.4b), (3.28a) and (3.28b) it can be shown that

$$\begin{pmatrix} 0 & 1 & c_1 & \ldots & c_r \\ 1 & 0 & 0 & \ldots & 0 \\ c_1^\vee/c_0^\vee & 0 & & & \\ \vdots & \vdots & \bar{A}^{-1} & & \\ c_r^\vee/c_0^\vee & 0 & & & \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & A_{00} & A_{01} & \ldots & A_{0r} \\ 0 & A_{10} & & & \\ \vdots & \vdots & \bar{A} & & \\ 0 & A_{r0} & & & \end{pmatrix} = I$$

and

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & A_{00} & A_{01} & \ldots & A_{0r} \\ 0 & A_{10} & & & \\ \vdots & \vdots & \bar{A} & & \\ 0 & A_{r0} & & & \end{pmatrix} \begin{pmatrix} 0 & 1 & c_1 & \ldots & c_r \\ 1 & 0 & 0 & \ldots & 0 \\ c_1^\vee/c_0^\vee & 0 & & & \\ \vdots & \vdots & \bar{A}^{-1} & & \\ c_r^\vee/c_0^\vee & 0 & & & \end{pmatrix} = I$$

where I is the identity matrix. Hence from the inversion of the (3.27) we obtain

$$\begin{aligned} \Lambda_i &= \frac{c_i^\vee}{c_0^\vee}\Lambda_0 + \sum_{j=1}^{r}(\bar{A}^{-1})_{ij}\alpha_j \quad ; i = 1,\ldots,r \\ &\equiv \frac{c_i^\vee}{c_0^\vee}\Lambda_0 + \bar{\Lambda}_i \quad ; i = 0,1,\ldots,r \end{aligned} \tag{3.29}$$

where $\bar{\Lambda}_0 = 0$. For $\rho = \sum_{i=0}^{r}\Lambda_i$ and $\bar{\rho} = \sum_{i=1}^{r}\bar{\Lambda}_i$ this gives

$$\rho = \sum_{i=0}^{r}(\frac{c_i^\vee}{c_0^\vee}\Lambda_0 + \bar{\Lambda}_i) = \frac{g}{c_0^\vee}\Lambda_0 + \bar{\rho} \tag{3.30}$$

Lemma 3.6. *Any weight $\lambda \in \mathcal{H}^*$ can be written as*

$$\lambda = \sum_{i=0}^{r}\lambda_i\Lambda_i + n\delta$$

where $\lambda_i =< \lambda, \alpha_i^\vee >$ and $n = \frac{1}{c_0^\vee}(\lambda \mid \Lambda_0)$.

Proof

$$\begin{aligned} < \lambda, \alpha_j^\vee > &= \sum_{i=0}^{r}\lambda_i < \Lambda_i, \alpha_j^\vee > + n < \delta, \alpha_j^\vee > \\ &= \lambda_j \end{aligned}$$

From (3.29),

$$\begin{aligned} \lambda &= \sum_{i=0}^{r}(\frac{c_i^\vee}{c_0^\vee}\Lambda_0 + \bar{\Lambda}_i) + n\delta \\ &= \frac{1}{c_0^\vee}(\sum_{i=0}^{r}\lambda_i c_i^\vee)\Lambda_0 + \sum_{i=1}^{r}\lambda_i\bar{\Lambda}_i + n\delta. \end{aligned}$$

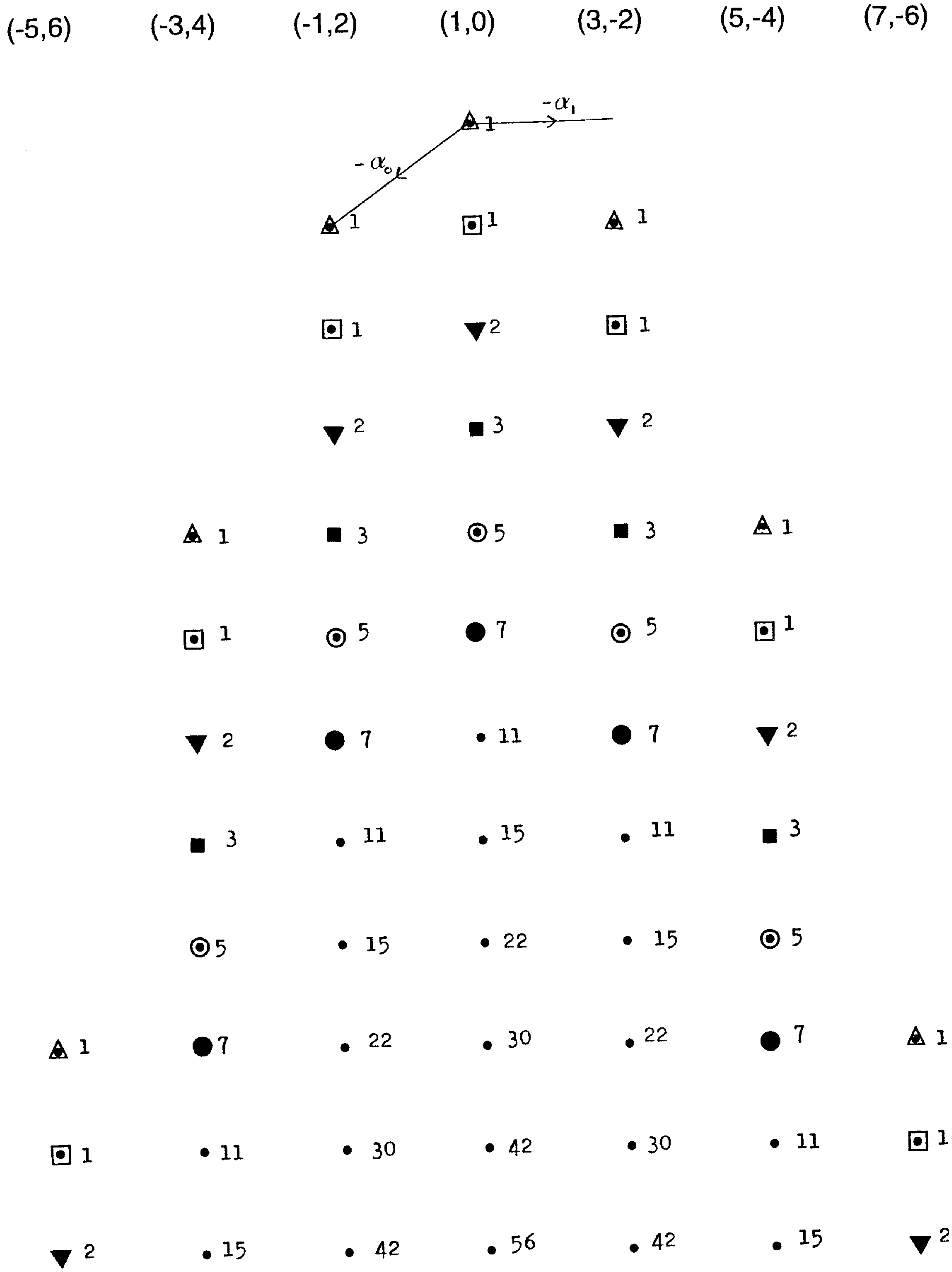
Hence

$$\begin{aligned} (\lambda \mid \Lambda_0) &= \sum_{i=1}^{r}\lambda_i(\bar{\Lambda}_i \mid \Lambda_0) + n(\delta \mid \Lambda_0) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\lambda_i(\bar{A}^{-1})_{ij}(\alpha_j \mid \Lambda_0) + nc_0^\vee \\ &= nc_0^\vee . \end{aligned}$$

□

Let us begin by studying the simplest representation of an affine algebra, i.e. the weight system of the highest weight module of $A_1^{(1)}$ with highest weight Λ_0. Applying the algorithm developed in Section 1.4 we obtain the weight diagram as in Figure 3.2. In contrast to the case of simple finite-dimensional Lie algebras this time the

Figure 3.2: Weight diagram of the V^{Λ_0} module of $A_1^{(1)}$



procedure does not terminate. Any weight of V^{Λ_0} can be written as $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + n\delta \equiv (\lambda_0, \lambda_1; -n)$. The highest weight is assigned the null depth 0. All other weights can be obtained from the highest weight $(1,0;0)$ by subtracting linear combination of fundamental roots $\alpha_0 = (2,-2;-1)$ and $\alpha_1 = (-2,2;0)$. For each α_0 subtracted, the depth is increased by one unit. In Figure 3.2 the numbers next to the weights are their multiplicities which are the values of the partition function $p(n)$. The $\triangle$ signify the weights in the first Weyl orbit, $\square$ those in the second orbit, etc. The weight system of V^{Λ_0} is the union of infinitely many Weyl group orbits and each orbit itself is infinite. Weights in the same Weyl orbit have the same multiplicities.

In general, the most striking feature of any affine weight system is the appearance of weights of the form $\lambda - n\delta$ where λ is an element of the weight lattice $P(\Lambda)$. That is we have strings of the form

$$\lambda_m,\ \lambda_m - \delta,\ \lambda_m - 2\delta,\ \ldots$$

where λ_m is the highest weight in the string and is called a maximal weight. We denote the set of maximal weights by P_{max} and we have $W \cdot P_{max} = P_{max}$. The weight system of the highest weight module V^{Λ} is then completely determined by the Weyl orbits of $\mu^+ \in P_{max} \cap P^+$, i.e the Weyl orbits of the maximal dominant weights.

The weights of $P(\Lambda)$ can be further organised into affine congruence classes. Each congruence class involves two invariants **[KMPS]**. The first one is the level $L(\lambda)$ of a weight λ defined by

$$L(\lambda) = <\lambda, K> = \sum_{i=0}^{r} \lambda_i c_i^\vee\,. \tag{3.31}$$

The level is constant for all $\lambda \in P(\Lambda)$ since all the roots have level zero. The second invariant is the finite congruence class of the underlying simple finite-dimensional Lie algebra as defined in Table 2.4.

For a weight λ, it is sometime convenient to use an $(r+1)$-tuple incomplete Dynkin label $(\lambda_0, \lambda_1, \ldots, \lambda_r)$. When it is necessary to make a distinction we shall attach a null depth d relative to Λ to give $(\lambda_0, \lambda_1, \ldots, \lambda_r)_d$ as the complete weight labelling. If two weights λ and λ' lie on the same Weyl orbit then the null depth of λ' relative to λ is the number of times the simple root α_0 is subtracted from λ in reaching λ'. Below we give an explicit formula **[KiW]** for computing the null depth λ relative to a dominant weight λ^+.

Theorem 3.7. *Let $\lambda \in P$ and $\lambda^+ \in P^+$ lie in the same Weyl orbit. The null depth of λ relative to λ^+ is given by*

$$d = \frac{1}{2L(\lambda)} \sum_{i=1}^{r} \sum_{j=1}^{r} \bar{G}_{ij}(\lambda_i\lambda_j - \lambda_i^+\lambda_j^+)$$

where $\bar{G} = \bar{S}^{-1}$ and $\bar{S}_{ij} = (\alpha_i^\vee \mid \alpha_j^\vee) = \frac{c_i}{c_i^\vee} A_{ij}$.

Proof: Let $\lambda = \lambda^+ - \sum_{i=0}^{r} k_i\alpha_i$. Then the relative null depth required is k_0. Consider

$$\begin{aligned}
\lambda_j^+ - \lambda_j &= <\lambda^+ - \lambda, \alpha_j^\vee> = <\sum_{i=0}^{r} k_i\alpha_i, \alpha_j^\vee> \\
&= k_0 <\alpha_0, \alpha_j> + \sum_{i=1}^{r} k_i A_{ij} \\
&= k_0 <\delta - \sum_{i=1}^{r} c_i\alpha_i, \alpha_j^\vee> + \sum_{i=1}^{r} k_i A_{ij} \\
&= -k_0 \sum_{i=1}^{r} c_i A_{ij} + \sum_{i=1}^{r} k_i A_{ij} \\
&= \sum_{i=1}^{r} \frac{c_i^\vee}{c_i}(k_i - k_0 c_i)\bar{S}_{ij} \\
\bar{G}_{j\ell}(\lambda_j - \lambda_j^+) &= \sum_{i=1}^{r} \frac{c_i^\vee}{c_i}(k_0 c_i - k_i)\bar{S}_{ij}\bar{G}_{j\ell} \\
\sum_{j=1}^{r} \bar{G}_{j\ell}(\lambda_j - \lambda_j^+) &= \frac{c_\ell^\vee}{c_\ell}(k_0 c_\ell - k_\ell)
\end{aligned}$$

However $\bar{S}$ is symmetric and so is $\bar{G}$. Then we have

$$\sum_{j=1}^{r} \bar{G}_{ij}(\lambda_i + \lambda_i^+)(\lambda_j - \lambda_j^+) = \frac{c_i^\vee}{c_i}(k_0 c_i - k_i)(\lambda_i + \lambda_i^+)$$

$$\sum_{i,j=1}^{r} \bar{G}_{ij}(\lambda_i\lambda_j - \lambda_i^+\lambda_j^+) = k_0 \sum_{i=1}^{r} c_i^\vee(\lambda_i + \lambda_i^+) - \sum_{i=1}^{r} \frac{k_i c_i^\vee}{c_i}(\lambda_i + \lambda_i^+)$$

The facts that the bilinear form is symmetric and W-invariant gives,

$$\begin{aligned} 0 &= (\lambda^+ \mid \lambda^+) - (\lambda \mid \lambda) = (\lambda^+ + \lambda \mid \lambda^+ - \lambda) \\ &= (\lambda^+ + \lambda \mid \sum_{i=0}^{r} k_i\alpha_i) \\ &= k_0 c_0^\vee(\lambda_0^+ + \lambda_0) + \sum_{i=1}^{r} \frac{k_i c_i^\vee}{c_i}(\lambda_i^+ + \lambda_i) \\ k_0 c_0^\vee(\lambda_0^+ + \lambda_0) &= -\sum_{i=1}^{r} \frac{k_i c_i^\vee}{c_i}(\lambda_i^+ + \lambda_i) \end{aligned}$$

These implies that

$$\begin{aligned} \sum_{i,j=1}^{r} \bar{G}_{ij}(\lambda_i\lambda_j - \lambda_i^+\lambda_j^+) &= k_0 \sum_{i=1}^{r} c_i^\vee(\lambda_i + \lambda_i^+) + k_0 c_0^\vee(\lambda_0^+ + \lambda_0) \\ &= k_0 \sum_{i=0}^{r} c_i^\vee(\lambda_i + \lambda_i^+) \\ &= k_0(L(\lambda^+) + L(\lambda)) = 2k_0 L(\lambda) \end{aligned}$$

Hence

$$k_0 = \frac{1}{2L(\lambda)} \sum_{i=1}^{r}\sum_{j=1}^{r} \bar{G}_{ij}(\lambda_i\lambda_j - \lambda_i^+\lambda_j^+) .$$

□

Explicit values for the symmetric matrix $\bar{G}$ for all affine algebras are given in Appendix 2.

3.5 Orbit sums

Recall from (1.21) and (1.22) that the relation between the orbit sum and the irreducible character is given by $\Omega^\mu = \sum_\lambda B_\lambda^\mu \, ch\, V^\lambda$ where

$$B_\lambda^\mu = \begin{cases} \sum_{\hat{w}\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w)\delta_{w(\hat{w}\mu+\rho)-\rho,\lambda} & \text{first interpretation,} \\ \sum_{\hat{w}\in\{W:W_\mu\}} \sum_{w\in W} \varepsilon(w)\delta_{\hat{w}\mu,w\bullet\lambda} & \text{second interpretation.} \end{cases}$$

In the affine case, the first interpretation of (1.21) is suitable for computional purposes. However although lengthy, the second interpretation of (1.21) may be used to obtain analytic expressions for B_λ^μ.

Consider a weight ν that lies on the intersection of the Weyl orbit of μ and the Weyl dot orbit of λ where $\mu = (\mu_0^+, \mu_1^+, \ldots, \mu_r^+)_{d_{\mu\nu}}$ and $\lambda = (\lambda_0^+, \lambda_1^+, \ldots, \lambda_r^+)_{d^\rho_{\lambda\nu}}$. Then the null depth of λ relative to μ is $d = d_{\mu\nu} - d^\rho_{\lambda\nu}$ where

$$\begin{aligned} d_{\mu\nu} &= \frac{1}{2L(\mu)} \sum_{i,j=1}^{r} \bar{G}_{ij}(\nu_i\nu_j - \mu_i^+\mu_j^+)\,, \\ d^\rho_{\lambda\nu} &= \frac{1}{2L(\lambda+\rho)} \sum_{i,j=1}^{r} \bar{G}_{ij}((\nu_i+1)(\nu_j+1) - (\lambda_i^+ + 1)(\lambda_j^+ + 1))\,. \end{aligned} \tag{3.32}$$

We can then interprete the elements of the matrix B^μ_λ as

$$B^\mu_\lambda = \sum_{\nu \in \Upsilon} \varepsilon(w_{\lambda\nu}) \tag{3.33}$$

where Υ is the intersection set of the Weyl orbit of $\mu = (\mu_0^+, \mu_1^+, \ldots, \mu_r^+)_0$ and the Weyl dot orbit of $\lambda = (\lambda_0^+, \lambda_1^+, \ldots, \lambda_r^+)_d$, while $\varepsilon(w_{\lambda\mu}) = 1$(resp. -1) if the number of fundamental reflections required to reach ν from λ is even (resp. odd). Since any $w \in W$ can be written as $w = t_\alpha \bar{w}$ with $t_\alpha \in T$ and $\bar{w} \in \overline{W}$, and the parity of t_α is even then the parity of w is the same as the parity of $\bar{w}$.

Before we proceed with explicit calculations it is of the utmost importance that we identify first a set of coset representatives $\{W : W_\mu\}$ such that we do not double count terms appearing in the Weyl orbit of μ. Each $w \in W$ can be written in the form $t_\alpha \bar{w}$ with $t_\alpha \in T$ for some $\alpha \in M$ and $\bar{w} \in \overline{W}$. Two terms $\bar{w}(\mu)$ and $\bar{w}'(\mu)$ of the $\overline{W}$-orbit of μ are said to be equivalent if there exists $\alpha \in M$ such that $\bar{w}(\mu) = t_\alpha \bar{w}'(\mu)$. In such a case it follows from (3.21) and the fact that $L(\bar{w}'(\mu)) = L(\mu)$,

$$\bar{w}(\mu) = \bar{w}'(\mu) + L(\mu)\alpha - ((\bar{w}'(\mu) \mid \alpha) + \frac{1}{2}(\alpha \mid \alpha)L(\mu))\delta$$

where the last term necessarily vanishes since $\bar{w}(\mu)$ and $\bar{w}'(\mu)$ both have null depth 0 relative to μ. Hence

$$\bar{w}(\mu) - \bar{w}'(\mu) \in L(\mu)M\,. \tag{3.34}$$

Thus reduces the generation of the complete Weyl orbit of μ to that of finding a complete set of inequivalent terms $\bar{w}(\mu)$ and applying translations to these.

For example, in the case of $A_2^{(1)}$ and the dominant weight $\lambda = (0, \lambda_1, \lambda_2)$, we have

$$s_2(\lambda) = \lambda - \lambda_2 \alpha_2$$

$$s_1 s_2(\lambda) = \lambda - (\lambda_1 + \lambda_2)\alpha_1 - \lambda_2 \alpha_2$$

The lattice M in this case is $m\alpha_1 + n\alpha_2$ with $m, n \in \mathbf{Z}$ and $L(\lambda) = \lambda_1 + \lambda_2$. Hence

$$s_2(\lambda) - s_1 s_2(\lambda) = (\lambda_1 + \lambda_2)\alpha_1 \in (\lambda_1 + \lambda_2)M = L(\lambda)M.$$

so that $s_2(\lambda)$ and $s_1 s_2(\lambda)$ are equivalent. However when the dominant weight is $(\lambda_0, 0, \lambda_2)$ we obtain

$$s_2(\lambda) - s_1 s_2(\lambda) = \lambda_2 \alpha_1 \notin (\lambda_0 + \lambda_2)M$$

so that $s_2(\lambda)$ and $s_1 s_2(\lambda)$ are not equivalent.

In Table 3.4a-3.4d we tabulate $w' \in \{\overline{W} : W_\lambda\}$ such that no two $w'(\lambda)$ are equivalent. Thus, for example from Table 3.4b, the set of coset representatives $\{W : W_{(0,\lambda_1,\lambda_2)}\}$ is given by

$$\{\, t_{m\alpha_1+n\alpha_2},\ t_{m\alpha_1+n\alpha_2} s_1,\ t_{m\alpha_1+n\alpha_2} s_2 \ \mid\ m, n \in \mathbf{Z} \} \,.$$

As discussed by Patera and Sharp [**PS**], for simple finite-dimensional Lie algebras, the complete weight content of a Weyl orbit of a dominant weight λ may be obtained from a corresponding orbit-weight generating function. The same principle applies to the affine algebras. In the affine algebra case the orbit-weight generating functions take the form

$$H(A, \Lambda) = \sum_{w \in \{W : W_\mu\}} w \prod_{i=0}^{r} (1 - A_i \Lambda_i)^{-1} (1 - \Delta\delta)^{-1} \tag{3.35}$$

where $A = (A_0, A_1, \ldots, A_r, \Delta)$ are dummy labels which carry the affine orbit labels $\mu = (\mu_0, \mu_1, \ldots, \mu_r)_{d_\mu}$ as exponents and $\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_r, \delta)$ are also dummy labels which carry affine weight labels $\nu = (\nu_0, \nu_1, \ldots, \nu_r)_{d_\nu}$ as exponents. Thus:

$$H(A, \Lambda) = \sum_{\mu,\nu} A_0^{\mu_0} A_1^{\mu_1} \ldots A_r^{\mu_r} \Delta^{d_\mu} \Lambda_0^{\nu_0} \Lambda_1^{\nu_1} \ldots \Lambda_r^{\nu_r} \delta^{d_\nu}$$

Table 3.4a : Left coset representatives of $\overline{W}$ with respect to W_λ for $A_1^{(1)}$ and $A_2^{(2)}$.

λ	$w' \in \{\overline{W} : W_\lambda\}$
(λ_0, λ_1)	$id,\ s_1$
$(\lambda_0, 0)$	id
$(0, \lambda_1)$	id

Table 3.4b : Left coset representatives of $\overline{W}$ with respect to W_λ for $A_2^{(1)}$.

λ	$w' \in \{\overline{W} : W_\lambda\}$
$(\lambda_0, \lambda_1, \lambda_2)$	$id,\ s_1,\ s_2,\ s_2s_1,\ s_1s_2,\ s_1s_2s_1$
$(0, \lambda_1, \lambda_2)$	$id,\ s_1,\ s_2$
$(\lambda_0, 0, \lambda_2)$	$id,\ s_2,\ s_1s_2$
$(\lambda_0, \lambda_1, 0)$	$id,\ s_1,\ s_2s_1$
$(0, 0, \lambda_2)$	id
$(0, \lambda_1, 0)$	id
$(\lambda_0, 0, 0)$	id

Table 3.4c : Left coset representatives of $\overline{W}$ with respect to W_λ for $C_2^{(1)}$, $A_4^{(2)}$ and $D_3^{(2)}$.

λ	$w' \in \{\overline{W} : W_\lambda\}$
$(\lambda_0, \lambda_1, \lambda_2)$	id, s_1, s_2, s_2s_1, s_1s_2, $s_1s_2s_1$, $s_2s_1s_2$, $s_2s_1s_2s_1$
$(0, \lambda_1, \lambda_2)$	id, s_1, s_2, s_1s_2
$(\lambda_0, 0, \lambda_2)$	id, s_2, s_1s_2, $s_2s_1s_2$
$(\lambda_0, \lambda_1, 0)$	id, s_1, s_2s_1, $s_1s_2s_1$
$(0, 0, \lambda_2)$	id
$(0, \lambda_1, 0)$	id, s_1

Table 3.4d : Left coset representatives of $\overline{W}$ with respect to W_λ for $G_2^{(1)}$ and $D_4^{(3)}$.

λ	$w' \in \{\overline{W} : W_\lambda\}$
$(\lambda_0, \lambda_1, \lambda_2)$	id, s_1, s_2, s_2s_1, s_1s_2, $s_1s_2s_1$, $s_2s_1s_2$, $s_2s_1s_2s_1$, $s_1s_2s_1s_2$, $s_1s_2s_1s_2s_1$, $s_2s_1s_2s_1s_2$, $s_2s_1s_2s_1s_2s_1$
$(0, \lambda_1, \lambda_2)$	id, s_1, s_2, s_1s_2, s_2s_1, $s_2s_1s_2$
$(\lambda_0, 0, \lambda_2)$	id, s_2, s_1s_2, $s_2s_1s_2$, $s_1s_2s_1s_2$, $s_2s_1s_2s_1s_2$
$(\lambda_0, \lambda_1, 0)$	id, s_1, s_2s_1, $s_1s_2s_1$, $s_2s_1s_2s_1$, $s_1s_2s_1s_2s_1$
$(0, 0, \lambda_2)$	id, s_2
$(0, \lambda_1, 0)$	id, s_1, s_2s_1

where the relative depth $d_{\mu\nu} = d_\mu - d_\nu$, so that the factor $(1-\Delta\delta)^{-1}$ is redundant.

The sum in (3.35) is over the left coset representatives which it should be emphasised operate only on the weights carried by Λ. It should also be noted that we have abused the notation by denoting dummy variables and fundamental weights by the same symbols. We shall give a derivation of the orbit-weight generating function of $A_1^{(1)}$ in order to illustrate the notation and the method.

First note that

$$\frac{1}{(1-A_0\Lambda_0)(1-A_1\Lambda_1)} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(A_0\Lambda_0)^i(A_1\Lambda_1)^j$$

and a general 2-tuple (λ_0, λ_1) can be classified as one or other of

$$(0,0),\ (i,0),\ (0,j) \text{ and } (i,j)$$

where $i, j \neq 0$. Let $\lambda = i\Lambda_0 + j\Lambda_1$ be a weight. Then the set of left coset representatives $\{W : W_\lambda\}$ that can be associated with various i and j can be obtained from Table 3.4a. By (3.21)

$$t_{n\alpha_1}(i\Lambda_0 + j\Lambda_1) = (i - 2n(i+j))\Lambda_0 + (j + 2n(i+j))\Lambda_1 - (nj + n^2(i+j))\delta$$

$$t_{n\alpha_1}s_1(i\Lambda_0 + j\Lambda_1) = (i + 2j - 2n(i+j))\Lambda_0 + (-j + 2n(i+j))\Lambda_1 - (-nj + n^2(i+j))\delta\,.$$

The Weyl group for $A_1^{(1)}$ is $\{t_{n\alpha_1}, t_{n\alpha_1}s_1 \mid n \in \mathbb{Z}\}$ so that from Table 3.4a the orbit-weight generating function (3.35) can be expanded to

$$\begin{aligned} H(A,\Lambda) = &\sum_{n\in\mathbb{Z}} t_{n\alpha_1}(1-\Delta\delta)^{-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(A_0\Lambda_0)^i(A_1\Lambda_1)^j \\ &+ \sum_{n\in\mathbb{Z}} t_{n\alpha_1}s_1(1-\Delta\delta)^{-1}\Big(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(A_0\Lambda_0)^i(A_1\Lambda_1)^j \\ &\qquad - \sum_{i=0}^{\infty}(A_0\Lambda_0)^i - \sum_{j=1}^{\infty}(A_1\Lambda_1)^j\Big) \end{aligned}$$

$$H(A,\Lambda) = \sum_{n\in\mathbb{Z}}(1-\Delta\delta)^{-1}\Big(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} A_0^i A_1^j \Lambda_0^{i-2n(i+j)}\Lambda_1^{j+2n(i+j)}\delta^{nj+n^2(i+j)}$$
$$+\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} A_0^i A_1^j \Lambda_0^{i+2j-2n(i+j)}\Lambda_1^{-j+2n(i+j)}\delta^{-nj+n^2(i+j)}$$
$$-\sum_{i=0}^{\infty} A_0^i \Lambda_0^{i-2ni}\Lambda_1^{2ni}\delta^{n^2 i} - \sum_{j=1}^{\infty} A_1^j \Lambda_0^{2j-2nj}\Lambda_1^{-j+2nj}\delta^{-nj+n^2 j}\Big)$$
$$= \sum_{n\in\mathbb{Z}}(1-\Delta\delta)^{-1}\Big(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})^i (A_1\Lambda_0^{-2n}\Lambda_1^{1+2n}\delta^{n^2+n})^j$$
$$+\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})^i (A_1\Lambda_0^{2-2n}\Lambda_1^{-1+2n}\delta^{n^2-n})^j$$
$$-\sum_{i=0}^{\infty}(A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})^i - \sum_{j=1}^{\infty}(A_1\Lambda_0^{2-2n}\Lambda_1^{-1+2n}\delta^{n^2-n})^j\Big)$$

This can then be simplified to the rational form

$$H(A,\Lambda) = \sum_{n\in\mathbb{Z}}\frac{1}{(1-\Delta\delta)}\Big(\frac{1}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})(1-A_1\Lambda_0^{-2n}\Lambda_1^{1+2n}\delta^{n^2+n})}$$
$$+\frac{1}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})(1-A_1\Lambda_0^{2-2n}\Lambda_1^{-1+2n}\delta^{n^2-n})}$$
$$-\frac{1}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})} - \frac{A_1\Lambda_0^{2-2n}\Lambda_1^{-1+2n}\delta^{n^2-n}}{(1-A_1\Lambda_0^{2-2n}\Lambda_1^{-1+2n}\delta^{n^2-n})}\Big) \tag{3.36a}$$
$$= \sum_{n\in\mathbb{Z}}\frac{1}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})(1-A_1\Lambda_0^{-2n}\Lambda_1^{1+2n}\delta^{n^2+n})(1-\Delta\delta)}$$
$$+\sum_{n\in\mathbb{Z}}\frac{A_0A_1\Lambda_0^{3-4n}\Lambda_1^{-1+4n}\delta^{2n^2-n}}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{2n}\delta^{n^2})(1-A_1\Lambda_0^{2-2n}\Lambda_1^{-1+2n}\delta^{n^2-n})(1-\Delta\delta)}.$$

In the following we give the remaining orbit-weight generating functions for all affine algebras of rank 1 and 2. The parity of the Weyl element used to obtain the terms are denoted by superscript + or − of $(1-\Delta\delta)$. Otherwise specify, m and n are always assume to be integers.

The orbit-weight generating function for $A_2^{(2)}$ is

$$\sum_n\Big[\frac{1}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{4n}\delta^{n^2})(1-A_1\Lambda_0^{-n}\Lambda_1^{1+2n}\delta^{(n^2+n)/2})(1-\Delta\delta)^+}$$
$$+\quad\frac{A_0A_1\Lambda_0^{2-3n}\Lambda_1^{-1+6n}\delta^{(3n^2-n)/2}}{(1-A_0\Lambda_0^{1-2n}\Lambda_1^{4n}\delta^{n^2})(1-A_1\Lambda_0^{1-n}\Lambda_1^{-1+2n}\delta^{(n^2-n)/2})(1-\Delta\delta)^-}\Big] \tag{3.36b}$$

The orbit-weight generating function for $A_2^{(1)}$ is

$$\sum_{m,n}\Big[\frac{1}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_1\delta^m)(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^2\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-m+n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^-}$$
$$+\frac{A_1A_2Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^2\delta^{-m+2n}}{(1-A_1Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^-} \qquad (3.36c)$$
$$+\frac{A_0A_2Q^2\Lambda_0^2\Lambda_1\Lambda_2^{-1}\delta^{m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_1\delta^m)(1-A_2Q\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-\Delta\delta)^+}$$
$$+\frac{A_1A_2Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-1}\delta^{2m-n}}{(1-A_1Q\Lambda_1\delta^m)(1-A_2Q\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-\Delta\delta)^+}$$
$$+\frac{A_0A_1Q^2\Lambda_0^3\Lambda_2^{-1}\delta^{-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-A_2Q\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-\Delta\delta)^-}$$
$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_1^{-1}\delta^{-m}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\delta^{-m})(1-\Delta\delta)^+}$$
$$+\frac{A_0A_1A_2Q^3\Lambda_0^5\Lambda_1^{-1}\Lambda_2^{-1}\delta^{-m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\delta^{-m})(1-\Delta\delta)^-}$$

where $Q=\Lambda_0^{-m-n}\Lambda_1^{2m-n}\Lambda_2^{-m+2n}\delta^{m^2-mn+n^2}$.

The orbit-weight generating function for $C_2^{(1)}$ is

$$\sum_{m,n}\Big[\frac{1}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_1\delta^m)(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^+}$$
$$+\frac{A_1Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^-}$$
$$+\frac{A_0A_2Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-1}\delta^{2m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_1\delta^m)(1-A_2Q\Lambda_1^2\Lambda_2^{-1}\delta^{2m-n})(1-\Delta\delta)^-}$$
$$+\frac{A_1A_2Q^2\Lambda_1^3\Lambda_2^{-1}\delta^{3m-n}}{(1-A_1Q\Lambda_1\delta^m)(1-A_2Q\Lambda_1^2\Lambda_2^{-1}\delta^{2m-n})(1-\Delta\delta)^-} \qquad (3.36d)$$
$$+\frac{A_0A_1Q^2\Lambda_0^2\Lambda_1\Lambda_2^{-1}\delta^{m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-A_2Q\Lambda_1^2\Lambda_2^{-1}\delta^{2m-n})(1-\Delta\delta)^+}$$
$$+\frac{A_1A_2Q^2\Lambda_0^3\Lambda_1^{-3}\Lambda_2^2\delta^{-3m+2n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-A_2Q\Lambda_0^2\Lambda_1^{-2}\Lambda_2\delta^{-2m+n})(1-\Delta\delta)^+}$$
$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_1^{-2}\Lambda_2\delta^{-2m+n}}{(1-A_0Q\Lambda_0)(1-A_2Q\Lambda_0^2\Lambda_1^{-2}\Lambda_2\delta^{-2m+n})(1-\Delta\delta)^+}$$
$$+\frac{A_0A_1Q^2\Lambda_0^3\Lambda_1^{-1}\delta^{-m}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0^2\Lambda_1^{-1}\delta^{-m})(1-A_2Q\Lambda_0^2\Lambda_1^{-2}\Lambda_2\delta^{-2m+n})(1-\Delta\delta)^-}$$
$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_2^{-1}\delta^{-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-A_2Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_1A_2Q^3\Lambda_0^5\Lambda_1^{-1}\Lambda_2^{-1}\delta^{-m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q\Lambda_0^2\Lambda_1^{-1}\delta^{-m})(1-A_2Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-\Delta\delta)^+}]$$

where $Q=\Lambda_0^{-2m}\Lambda_1^{4m-2n}\Lambda_2^{-2m+2n}\delta^{2m^2-2mn+n^2}$.

The orbit-weight generating function for $A_4^{(2)}$ is

$$\sum_{m,n}[\frac{1}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q\Lambda_2\delta^{n/2})(1-\Delta\delta)^+}$$

$$+\frac{A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^2\delta^{-m+n}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^2\delta^{-m+n})(1-A_2Q\Lambda_2\delta^{n/2})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^3\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{m-n/2}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{m-n/2})(1-\Delta\delta)^-}$$

$$+\frac{A_1A_2Q^3\Lambda_1^2\Lambda_2^{-1}\delta^{2m-n/2}}{(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{m-n/2})(1-\Delta\delta)^-} \qquad (3.36e)$$

$$+\frac{A_0A_1Q^4\Lambda_0^2\Lambda_1\Lambda_2^{-2}\delta^{m-n}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1\Lambda_2^{-2}\delta^{m-n})(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{m-n/2})(1-\Delta\delta)^+}$$

$$+\frac{A_1A_2Q^3\Lambda_0^2\Lambda_1^{-2}\Lambda_2^3\delta^{-2m+3n/2}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^2\delta^{-m+n})(1-A_2Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n/2})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_2Q^3\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-m+n/2}}{(1-A_0Q^2\Lambda_0)(1-A_2Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n/2})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^4\Lambda_0^3\Lambda_1^{-1}\delta^{-m}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_0^2\Lambda_1^{-1}\delta^{-m})(1-A_2Q\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+n/2})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^3\Lambda_0^2\Lambda_2^{-1}\delta^{-n/2}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1\Lambda_2^{-2}\delta^{m-n})(1-A_2Q\Lambda_0\Lambda_2^{-1}\delta^{-n/2})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_1A_2Q^5\Lambda_0^4\Lambda_1^{-1}\Lambda_2^{-1}\delta^{-m-n/2}}{(1-A_0Q^2\Lambda_0)(1-A_1Q^2\Lambda_0^2\Lambda_1^{-1}\delta^{-m})(1-A_2Q\Lambda_0\Lambda_2^{-1}\delta^{-n/2})(1-\Delta\delta)^+}]$$

where $Q=\Lambda_0^{-m}\Lambda_1^{2m-n}\Lambda_2^{-2m+2n}\delta^{(2m^2-2mn+n^2)/2}$.

The orbit-weight generating function for $D_3^{(2)}$ is

$$\sum_{m,n}[\frac{1}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_1\delta^{2m})(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^+}$$

$$+\frac{A_1Q^2\Lambda_0^2\Lambda_1^{-1}\Lambda_2^2\delta^{-2m+2n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^2\Lambda_1^{-1}\Lambda_2^2\delta^{-2m+2n})(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^2\Lambda_0\Lambda_1\Lambda_2^{-1}\delta^{2m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_1\delta^{2m})(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{2m-n})(1-\Delta\delta)^-}$$

$$+\frac{A_1A_2Q^3\Lambda_1^2\Lambda_2^{-1}\delta^{4m-n}}{(1-A_1Q^2\Lambda_1\delta^{2m})(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{2m-n})(1-\Delta\delta)^-} \qquad (3.36f)$$

$$+\frac{A_0A_1Q^3\Lambda_0^3\Lambda_1\Lambda_2^{-2}\delta^{2m-2n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^2\Lambda_1\Lambda_2^{-2}\delta^{2m-2n})(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{2m-n})(1-\Delta\delta)^+}$$

$$+\frac{A_1A_2Q^3\Lambda_0^4\Lambda_1^{-2}\Lambda_2^3\delta^{-4m+3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^2\Lambda_1^{-1}\Lambda_2^2\delta^{-2m+2n})(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-2m+n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_1^{-1}\Lambda_2\delta^{-2m+n}}{(1-A_0Q\Lambda_0)(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-2m+n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^3\Lambda_0^5\Lambda_1^{-1}\delta^{-2m}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^4\Lambda_1^{-1}\delta^{-2m})(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-2m+n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_2^{-1}\delta^{-n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^2\Lambda_1\Lambda_2^{-2}\delta^{2m-2n})(1-A_2Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_1A_2Q^4\Lambda_0^7\Lambda_1^{-1}\Lambda_2^{-1}\delta^{-2m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^4\Lambda_1^{-1}\delta^{-2m})(1-A_2Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-\Delta\delta)^+}\Big]$$

where $Q=\Lambda_0^{-2m}\Lambda_1^{2m-n}\Lambda_2^{-2m+2n}\delta^{2m^2-2mn+n^2}$.

The orbit-weight generating function for $G_2^{(1)}$ is

$$\sum_{m,n}\Big[\ \frac{1}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^+}$$

$$+\frac{A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^3\delta^{-m+3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^3\delta^{-m+3n})(1-A_2Q\Lambda_2\delta^n)(1-\Delta\delta)^-}$$

$$+\frac{A_2Q\Lambda_1\Lambda_2^{-1}\delta^{m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-\Delta\delta)^-}$$

$$+\frac{A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-3}\delta^{2m-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-3}\delta^{2m-3n})(1-A_2Q\Lambda_1\Lambda_2^{-1}\delta^{m-n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_2Q^2\Lambda_0^2\Lambda_1^{-1}\Lambda_2^2\delta^{-m+2n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2^3\delta^{-m+3n})(1-A_2Q\Lambda_0\Lambda_1^{-1}\Lambda_2^2\delta^{-m+2n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^3\Lambda_0^4\Lambda_1^{-2}\Lambda_2^3\delta^{-2m+3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1^{-2}\Lambda_2^3\delta^{-2m+3n})(1-A_2Q\Lambda_0\Lambda_1^{-1}\Lambda_2^2\delta^{-m+2n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^2\Lambda_0^2\Lambda_1\Lambda_2^{-2}\delta^{m-2n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-3}\delta^{2m-3n})(1-A_2Q\Lambda_0\Lambda_1\Lambda_2^{-2}\delta^{m-2n})(1-\Delta\delta)^-}$$

$$+\frac{A_1A_2Q^3\Lambda_0^2\Lambda_1^3\Lambda_2^{-5}\delta^{3m-5n}}{(1-A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-3}\delta^{2m-3n})(1-A_2Q\Lambda_0\Lambda_1\Lambda_2^{-2}\delta^{m-2n})(1-\Delta\delta)^-} \qquad (3.36g)$$

$$+\frac{A_0A_1Q^3\Lambda_0^4\Lambda_1\Lambda_2^{-3}\delta^{m-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1\Lambda_2^{-3}\delta^{m-3n})(1-A_2Q\Lambda_0\Lambda_1\Lambda_2^{-2}\delta^{m-2n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_1^{-1}\Lambda_2\delta^{-m+n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1^{-2}\Lambda_2^3\delta^{-2m+3n})(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^3\Lambda_0^5\Lambda_1^{-1}\delta^{-m}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^4\Lambda_1^{-1}\delta^{-m})(1-A_2Q\Lambda_0^2\Lambda_1^{-1}\Lambda_2\delta^{-m+n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^2\Lambda_0^3\Lambda_2^{-1}\delta^{-n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1\Lambda_2^{-3}\delta^{m-3n})(1-A_2Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_1A_2Q^4\Lambda_0^7\Lambda_1^{-1}\Lambda_2^{-1}\delta^{-m-n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^4\Lambda_1^{-1}\delta^{-m})(1-A_2Q\Lambda_0^2\Lambda_2^{-1}\delta^{-n})(1-\Delta\delta)^+}]$$

where $Q=\Lambda_0^{-m}\Lambda_1^{2m-3n}\Lambda_2^{-3m+6n}\delta^{m^2-3mn+3n^2}$.

The orbit-weight generating function for $D_4^{(3)}$ is

$$\sum_{m,n}[\quad\frac{1}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q^3\Lambda_2\delta^{3n})(1-\Delta\delta)^+}$$

$$+\frac{A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+3n})(1-A_2Q^3\Lambda_2\delta^{3n})(1-\Delta\delta)^-}$$

$$+\frac{A_2Q^3\Lambda_1^3\Lambda_2^{-1}\delta^{3m-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_1\delta^m)(1-A_2Q^3\Lambda_1^3\Lambda_2^{-1}\delta^{3m-3n})(1-\Delta\delta)^-}$$

$$+\frac{A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-1}\delta^{2m-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-1}\delta^{2m-3n})(1-A_2Q^3\Lambda_1^3\Lambda_2^{-1}\delta^{3m-3n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_2Q^4\Lambda_0^4\Lambda_1^{-3}\Lambda_2^2\delta^{-3m+6n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^{-1}\Lambda_2\delta^{-m+3n})(1-A_2Q^3\Lambda_0^3\Lambda_1^{-3}\Lambda_2^2\delta^{-3m+6n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^3\Lambda_0^4\Lambda_1^{-2}\Lambda_2\delta^{-2m+3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1^{-2}\Lambda_2\delta^{-2m+3n})(1-A_2Q^3\Lambda_0^3\Lambda_1^{-3}\Lambda_2^2\delta^{-3m+6n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^4\Lambda_0^4\Lambda_1^3\Lambda_2^{-2}\delta^{3m-6n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-1}\delta^{2m-3n})(1-A_2Q^3\Lambda_0^3\Lambda_1^3\Lambda_2^{-2}\delta^{3m-6n})(1-\Delta\delta)^-}$$

$$+\frac{A_1A_2Q^5\Lambda_0^4\Lambda_1^5\Lambda_2^{-3}\delta^{5m-9n}}{(1-A_1Q^2\Lambda_0\Lambda_1^2\Lambda_2^{-1}\delta^{2m-3n})(1-A_2Q^3\Lambda_0^3\Lambda_1^3\Lambda_2^{-2}\delta^{3m-6n})(1-\Delta\delta)^-} \qquad (3.36h)$$

$$+\frac{A_0A_1Q^3\Lambda_0^4\Lambda_1\Lambda_2^{-1}\delta^{m-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1\Lambda_2^{-1}\delta^{m-3n})(1-A_2Q^3\Lambda_0^3\Lambda_1^3\Lambda_2^{-2}\delta^{3m-6n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_2Q^4\Lambda_0^7\Lambda_1^{-3}\Lambda_2\delta^{-3m+3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1^{-2}\Lambda_2\delta^{-2m+3n})(1-A_2Q^3\Lambda_0^6\Lambda_1^{-3}\Lambda_2\delta^{-3m+3n})(1-\Delta\delta)^+}$$

$$+\frac{A_0A_1Q^3\Lambda_0^5\Lambda_1^{-1}\delta^{-m}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^4\Lambda_1^{-1}\delta^{-m})(1-A_2Q^3\Lambda_0^6\Lambda_1^{-3}\Lambda_2\delta^{-3m+3n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_2Q^4\Lambda_0^7\Lambda_2^{-1}\delta^{-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^3\Lambda_1\Lambda_2^{-1}\delta^{m-3n})(1-A_2Q^3\Lambda_0^6\Lambda_2^{-1}\delta^{-3n})(1-\Delta\delta)^-}$$

$$+\frac{A_0A_1A_2Q^6\Lambda_0^{11}\Lambda_1^{-1}\Lambda_2^{-1}\delta^{-m-3n}}{(1-A_0Q\Lambda_0)(1-A_1Q^2\Lambda_0^4\Lambda_1^{-1}\delta^{-m})(1-A_2Q^3\Lambda_0^6\Lambda_2^{-1}\delta^{-3n})(1-\Delta\delta)^+}]$$

where $Q = \Lambda_0^{-m}\Lambda_1^{2m-3n}\Lambda_2^{-m+2n}\delta^{m^2-3mn+3n^2}$.

For the purpose of illustration let us obtain weight multiplicities for the affine algebra $A_2^{(1)}$ whose highest weight has level 2 and there are mixing of orbits. The dominant weights $2\Lambda_1 = (0,2,0)$ and $\Lambda_0 + \Lambda_2 = (1,0,1)$ have the same level and are in the same affine congruence class. The Weyl orbit of $\mu = (0,2,0)$ can be obtained by picking up the coefficient of A_1^2 in the expansion of the orbit-weight generating function (3.36c) namely $Q^2\Lambda_1^2\delta^{2m}$ where $Q = \Lambda_0^{-m-n}\Lambda_1^{2m-n}\Lambda_0^{-m+2n}\delta^{m^2-mn+n^2}$. This orbit consists of weights $\nu = (\nu_0,\nu_1,\nu_2)_{d_{\mu\nu}}$. However only two of the components of ν are independent because of the constancy of the level. In fact $\nu_0 = L(\mu) - \nu_1 - \nu_2$ and in the following we shall not need to write down ν_0 explicitly. Hence the Weyl orbit of $\mu = (0,2,0)$ is

$$\{(\nu_0,\nu_1,\nu_2)_{d_{\mu\nu}} \mid \nu_1 = 4m-2n+2,\ \nu_2 = -2m+4n, \quad d_{\mu\nu} = 2\Gamma + 2m \quad \}.$$

where $\Gamma = m^2 - mn + n^2$.

Similarly, the Weyl orbit of $\mu = (1,0,1)$ is obtained by picking out the coefficients of A_0A_2 and can be shown to be

$$\{\nu \mid \nu_1 = 4m-2n,\ \nu_2 = -2m+4n+1,\ d_{\mu\nu} = 2\Gamma + n \quad \}$$
$$\bigcup\{\nu \mid \nu_1 = 4m-2n+1,\ \nu_2 = -2m+4n-1,\ d_{\mu\nu} = 2\Gamma + m - n \quad \}$$
$$\bigcup\{\nu \mid \nu_1 = 4m-2n-1,\ \nu_2 = -2m+4n,\ d_{\mu\nu} = 2\Gamma - m \quad \}.$$

The Weyl dot orbit of λ can also be computed similarly. But this time the level is increased to $L(\lambda+\rho) = 5$ and at the same time we have to take into consideration the parity $\varepsilon(w_{\lambda\nu})$ of $t_{m\alpha_1+n\alpha_2}\bar{w}$. First we obtain the Weyl orbit of $(1,3,1)$ by picking up the coeffients of $A_0A_1^3A_2$ and then have to subtract ρ from ν. The Weyl dot orbit of $\lambda = (0,2,0)$ is then

$$\{\nu \mid \nu_1 = 10m - 5n + 2,\ \nu_2 = -5m + 10n,\ d^{\rho}_{\lambda\nu} = 5\Gamma + 3m + n,\ \varepsilon = +1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n - 4,\ \nu_2 = -5m + 10n + 3,\ d^{\rho}_{\lambda\nu} = 5\Gamma - 3m + 4n,\ \varepsilon = -1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n + 3,\ \nu_2 = -5m + 10n - 2,\ d^{\rho}_{\lambda\nu} = 5\Gamma + 4m - n,\ \varepsilon = +1\}$$

$$\bigcup\{(\nu \mid \nu_1 = 10m - 5n,\ \nu_2 = -5m + 10n - 5,\ d^{\rho}_{\lambda\nu} = 5\Gamma + m - 4n,\ \varepsilon = -1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n - 5,\ \nu_2 = -5m + 10n + 2,\ d^{\rho}_{\lambda\nu} = 5\Gamma - 4m + 3n,\ \varepsilon = +1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n - 2,\ \nu_2 = -5m + 10n - 4),\ d^{\rho}_{\lambda\nu} = 5\Gamma - m - 3n,\ \varepsilon = -1\}.$$

The Weyl dot orbits of $\lambda = (1,0,1)$ are obtained by picking up the coefficients of $A_0^2 A_1 A_2^2$ and subtracting ρ,

$$\{\nu \mid \nu_1 = 10m - 5n,\ \nu_2 = -5m + 10n + 1,\ d^{\rho}_{\lambda\nu} = 5\Gamma + m + 2n,\ \varepsilon = +1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n - 2,\ \nu_2 = -5m + 10n + 2,\ d^{\rho}_{\lambda\nu} = 5\Gamma - m + 3n,\ \varepsilon = -1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n + 2,\ \nu_2 = -5m + 10n - 3,\ d^{\rho}_{\lambda\nu} = 5\Gamma + 3m - 2n,\ \varepsilon = +1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n + 1,\ \nu_2 = -5m + 10n - 4,\ d^{\rho}_{\lambda\nu} = 5\Gamma + 2m - 3n,\ \varepsilon = -1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n - 4,\ \nu_2 = -5m + 10n,\ d^{\rho}_{\lambda\nu} = 5\Gamma - 3m + n,\ \varepsilon = +1\}$$

$$\bigcup\{\nu \mid \nu_1 = 10m - 5n - 3,\ \nu_2 = -5m + 10n - 2,\ d^{\rho}_{\lambda\nu} = 5\Gamma - 2m - n,\ \varepsilon = -1\}$$

Let $\Upsilon(\mu,\lambda)$ denote the intersection of the Weyl orbit of μ and the Weyl dot orbit of λ. The null depth of λ relative to μ is $d = d_{\mu\nu} - d^{\rho}_{\lambda\nu}$. For illustration, consider the intersection of the Weyl orbit of $\mu = (0,2,0)$ and the second subset of the Weyl dot orbit of (0,2,0) given above, i.e. we must have

$$4m_1 - 2n_1 + 2 = 10m_2 - 5n_2 - 4 \text{ and } -2m_1 + 4n_1 = -5m_2 + 10n_2 + 3.$$

In matrix form this can be written as

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2m_1 - 5m_2 \\ 2n_1 - 5n_2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2m_1 - 5m_2 \\ 2n_1 - 5n_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

This then implies that

$$m_1 = 5m + 1, \quad m_2 = 2m + 1, \quad m \in \mathbf{Z}$$

$$n_1 = 5n, \quad n_2 = 2n, \quad n \in \mathbf{Z}$$

Then

$$\begin{aligned} d &= 2((5m+1)^2 - (5m+1)(5n) + (5n)^2) + 2(5m+1) \\ &\quad - 5((2m+1)^2 - (2m+1)(2n) + (2n)^2) - 3(2m+1) + 4(2n) \\ &= 30\Gamma + 16m - 8n + 2 \end{aligned}$$

Continuing with the other subsets we obtained the intersection sets $\Upsilon((0,2,0),\ (0,2,0))$ as follows

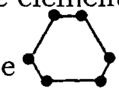
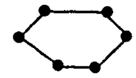
$$\begin{aligned} &\{(\nu_0, 20m-10n, -10m+20n)_d \mid d = 30\Gamma + 4m - 2n,\ \varepsilon = +1\} \\ &\bigcup \{(\nu_0, 20m-10n+6, -10m+20n-2)_d \mid d = 30\Gamma + 16m - 8n + 2,\ \varepsilon = -1\} \\ &\bigcup \{(\nu_0, 20m-10n-2, -10m+20n+8)_d \mid d = 30\Gamma - 8m + 22n + 4,\ \varepsilon = +1\} \\ &\bigcup \{(\nu_0, 20m-10n, -10m+20n+10)_d \mid d = 30\Gamma - 2m + 28n + 8,\ \varepsilon = -1\} \\ &\bigcup \{(\nu_0, 20m-10n+10, -10m+20n+2)_d \mid d = 30\Gamma + 28m + 4n + 10,\ \varepsilon = +1\} \\ &\bigcup \{(\nu_0, 20m-10n+8, -10m+20n+6)_d \mid d = 30\Gamma + 22m + 16n + 12,\ \varepsilon = -1\}. \end{aligned}$$

Similarly it can be shown that $\Upsilon((0,2,0),\ (1,0,1))$ is

$$\begin{aligned} &\{(\nu_0, 20m-10n+10, -10m+20n-4)_d \mid d = 30\Gamma + 28m - 14n + 6,\ \varepsilon = +1\} \\ &\bigcup \{(\nu_0, 20m-10n+18, -10m+20n-8)_d \mid d = 30\Gamma + 52m - 26n + 22,\ \varepsilon = -1\} \\ &\bigcup \{(\nu_0, 20m-10n+2, -10m+20n+12)_d \mid d = 30\Gamma + 4m + 34n + 14,\ \varepsilon = +1\} \\ &\bigcup \{(\nu_0, 20m-10n+16, -10m+20n-4)_d \mid d = 30\Gamma + 46m - 14n + 18,\ \varepsilon = -1\} \\ &\bigcup \{(\nu_0, 20m-10n+6, -10m+20n+10)_d \mid d = 30\Gamma + 16m + 28n + 16,\ \varepsilon = +1\} \\ &\bigcup \{(\nu_0, 20m-10n+12, -10m+20n-2)_d \mid d = 30\Gamma + 34m - 8n + 10,\ \varepsilon = -1\}. \end{aligned}$$

Then (3.33) and (1.21) imply that the orbit sum of $(0,2,0)$ is given by

$$\begin{aligned} \Omega^{(0,2,0)_0} = \sum_{m,n} [\, & ch\, V^{(0,2,0)_{30\Gamma+4m-2n}} - ch\, V^{(0,2,0)_{30\Gamma+16m-8n+2}} + ch\, V^{(0,2,0)_{30\Gamma-8m+22n+4}} \\ & - ch\, V^{(0,2,0)_{30\Gamma-2m+28n+8}} + ch\, V^{(0,2,0)_{30\Gamma+28m+4n+10}} - ch\, V^{(0,2,0)_{30\Gamma+22m+16n+12}} \\ & + ch\, V^{(1,0,1)_{30\Gamma+28m-14n+6}} - ch\, V^{(1,0,1)_{30\Gamma+52m-26n+22}} + ch\, V^{(1,0,1)_{30\Gamma+4m+34n+14}} \\ & - ch\, V^{(1,0,1)_{30\Gamma+46m-14n+18}} + ch\, V^{(1,0,1)_{30\Gamma+16m+28n+16}} - ch\, V^{(1,0,1)_{30\Gamma+34m-8n+10}} \,]. \end{aligned} \tag{3.37}$$

Geometrically we can visualise Weyl orbits of rank 1 and 2. For the Weyl orbits of $\mu = (0,2,0)$ and $(1,0,1)$ given previously we may plot them as in Figure 3.3. The symbols $\bullet$ specify the Weyl orbit of $(1,0,1)_0$ and the symbols $\spadesuit$ specify the Weyl orbit of $(0,2,0)_0$. The number next to the elements are the null depths $d_{\mu\nu}$. The elements of the Weyl dot orbit of $(1,0,1)_0$ are the vertices of the hexagons of the shape  and the Weyl dot orbit of $(0,2,0)_0$ are the vertices of the hexagons of shape  .

An alternative method of obtaining the orbit sum expansion for μ is to add ρ to each weight of the Weyl orbit of μ, reflecting into the dominant sector, subtracting ρ and interpreting the result as a signed, positive or negative, coefficient of λ according to the parity. A reflected weight lying on a reflection hyperplane is ignored. When computing the orbit sums numerically we must truncate at a certain depth. This truncation depth is determined by reflecting some neighbouring elements into the dominant sector. In Figure 3.3, the neighbouring elements that we should consider are those that lie in the upper part since these elements tend to have a lower depth and a negative zeroth Dynkin component. These neighbouring elements, among others, includes

$$(-7,-2,11)_{17}, \quad (-8,0,10)_{16}, \quad (-12,8,6)_{24}\,.$$

Reflecting these weights into the dominant sector, we obtain

$$s_2s_0s_2s_1s_0((-7,-2,11;-17)+\rho)-\rho = (0,2,0;-9)$$

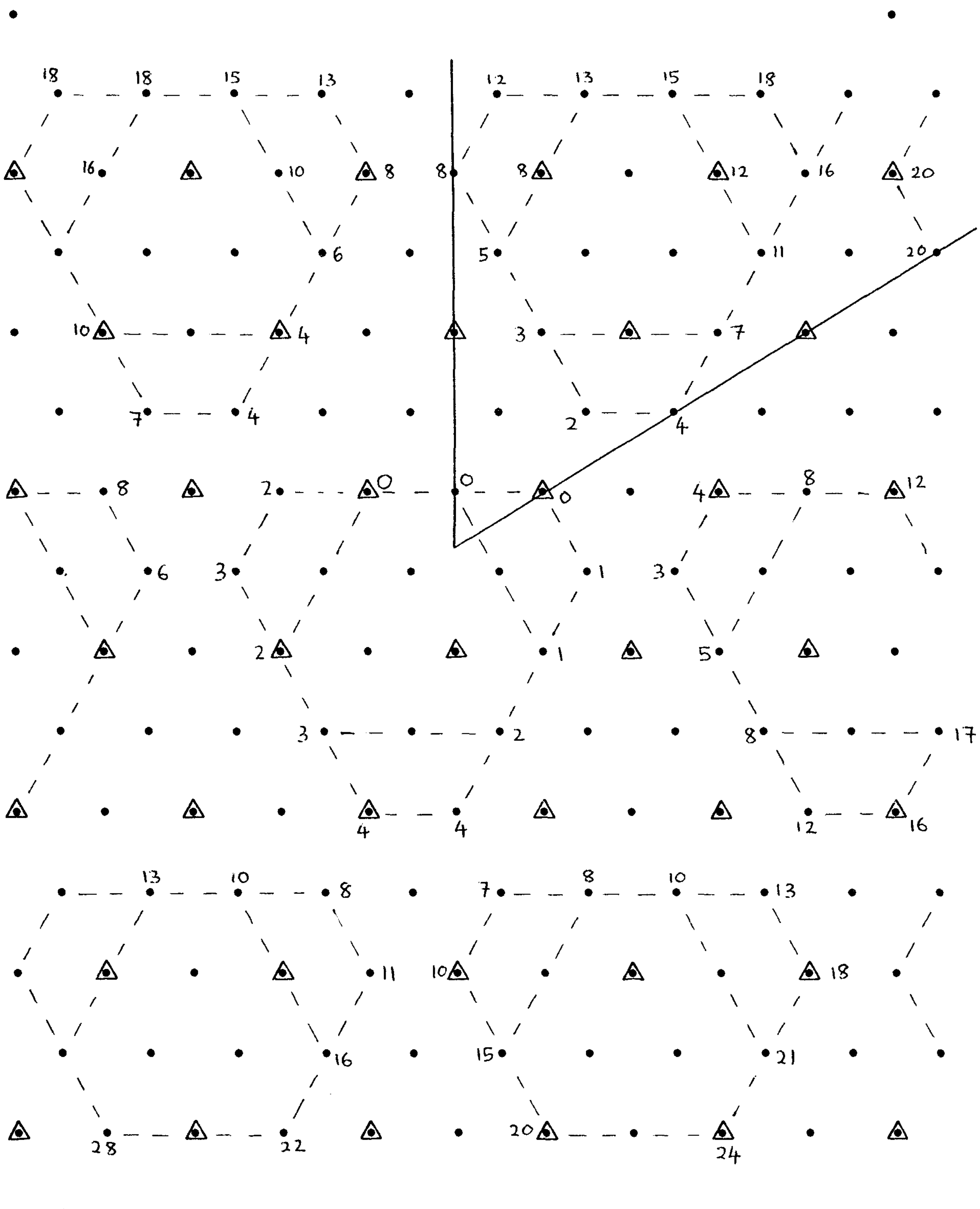
$$s_0s_2s_1s_0((-8,0,10;-16)+\rho)-\rho = (0,2,0;-8)$$

$$s_0s_1s_2s_1s_0((-12,8,6;-24)+\rho)-\rho = (0,2,0;-12)$$

Hence the weight lattice in Figure 3.3 will gives result accurate until depth 7. Applying similar reflections to other weights on the hexagons, we obtain

$$\Omega^{(020)_0} = ch\,V^{(020)_0} - ch\,V^{(101)_0} - ch\,V^{(020)_2} + 2ch\,V^{(101)_2} - 2ch\,V^{(020)_4} - ch\,V^{(101)_6} + \dots \tag{3.38a}$$

Figure 3.3 : Orbits of (0,2,0) and (1,0,1) modules of $A_2^{(1)}$.



$$\Omega^{(101)_0} = ch\,V^{(101)_0} - 2ch\,V^{(020)_1} - 2ch\,V^{(101)_1} + ch\,V^{(020)_2} + 2ch\,V^{(101)_2}$$
$$+ 2ch\,V^{(020)_3} + 2ch\,V^{(020)_4} - ch\,V^{(101)_4} - 2ch\,V^{(020)_5} - 2ch\,V^{(101)_5} \quad (3.38b)$$
$$- ch\,V^{(020)_6} - 2ch\,V^{(101)_6} + 2ch\,V^{(020)_7} + 2ch\,V^{(101)_7} + \dots$$

Other non-maximal orbit sums $\Omega^{(0,2,0)_k}$ and $\Omega^{(1,0,1)_k}$ can be obtained directly as

$$\Omega^{(020)_k} = ch\,V^{(020)_k} - ch\,V^{(101)_k} - ch\,V^{(020)_{k+2}} + 2ch\,V^{(101)_{k+2}} + \dots$$
$$\Omega^{(101)_k} = ch\,V^{(101)_k} - 2ch\,V^{(020)_{k+1}} - 2ch\,V^{(101)_{k+1}} + ch\,V^{(020)_{k+2}} + \dots$$

In matrix form this can be written as

$$\begin{pmatrix} \Omega^{(020)_0} \\ \Omega^{(101)_0} \\ \Omega^{(020)_1} \\ \Omega^{(101)_1} \\ \vdots \\ \Omega^{(020)_7} \\ \Omega^{(101)_7} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & \dots \\ 0 & 1 & -2 & -2 & \dots & 2 & 2 & \dots \\ 0 & 0 & 1 & -1 & \dots & 0 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots & -1 & -2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & \dots \end{pmatrix} \begin{pmatrix} ch\,V^{(020)_0} \\ ch\,V^{(101)_0} \\ ch\,V^{(020)_1} \\ ch\,V^{(101)_1} \\ \vdots \\ ch\,V^{(020)_7} \\ ch\,V^{(101)_7} \end{pmatrix}$$

The multiplicity matrix is upper triangular with 1's on the diagonal and can be easily inverted. The inversion will gives the expression of irreducible characters in term of the orbit sums whose coefficients are the weights multiplicities.

$$\begin{pmatrix} ch\,V^{(020)_0} \\ ch\,V^{(101)_0} \\ ch\,V^{(020)_1} \\ ch\,V^{(101)_1} \\ \vdots \\ ch\,V^{(020)_7} \\ ch\,V^{(101)_7} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 4 & \dots & 522 & 740 & \dots \\ 0 & 1 & 2 & 4 & \dots & 636 & 908 & \dots \\ 0 & 0 & 1 & 1 & \dots & 256 & 365 & \dots \\ 0 & 0 & 0 & 1 & \dots & 300 & 441 & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & \dots & 0 & 1 & \dots \end{pmatrix} \begin{pmatrix} \Omega^{(020)_0} \\ \Omega^{(101)_0} \\ \Omega^{(020)_1} \\ \Omega^{(101)_1} \\ \vdots \\ \Omega^{(020)_7} \\ \Omega^{(101)_7} \end{pmatrix}$$

As in the case of the orbit sums the expansion of the irreducible characters $ch\,V^{(0,2,0)_0}$ and $ch\,V^{(1,0,1)_0}$ determine the expansion of $ch\,V^{(0,2,0)_k}$ and $ch\,V^{(1,0,1)_k}$ respectively, i.e. the first two rows of the inverse matrix determine the rest. The modules of $V^{(0,2,0)_0}$ and $V^{(0,2,0)_k}$ are isomorphic. Hence if the highest weight representation is $(0,2,0)$, then the first row of the above inverse matrix gives the following weight multiplicities of the dominant weights up to depth 7.

depth	(020)	(101)
0	1	1
1	2	4
2	8	12
3	20	32
4	52	77
5	116	172
6	256	365
7	522	740

If the highest weight representation is $(1,0,1)$, then the second row gives the following weight multiplicities of the dominant weights.

depth	(020)	(101)
0	0	1
1	2	4
2	7	13
3	22	36
4	56	89
5	136	204
6	300	441
7	636	908

These results are in agreement with the tabulation given by **[KMPS]** for level 2 modules of $A_2^{(1)}$.

Using a similar algorithm we have written a computer program to calculate weight multiplicities of heighest weight representations of the affine algebras $A_r^{(1)}$, $B_r^{(1)}$, $C_r^{(1)}$, $D_r^{(1)}$, $G_2^{(1)}$, $A_{2r}^{(2)}$, $D_{r+1}^{(2)}$ and $D_4^{(3)}$. The program runs successfully for low rank algebras. In the case of higher rank algebras we have to consider a Weyl group of large order which grows factorially with rank and a large weight lattice which grows exponentially with rank. This places a practical bound on the calculations. In Appendix 3 we tabulate some weight multiplicities of level 2 modules of twisted affine algebras of rank 2.

To obtain analytic results for the weight multiplicities we have to introduce a dummy variable $q = e^{-\delta}$ which carries as its exponent the depth of the irreducible character **[Kass]**, i.e. we shall write in general $ch\, V^{(\lambda_0,\ldots,\lambda_r)_d}$ as $ch\, V^{(\lambda_0,\ldots,\lambda_r)_0} q^d$. For example, the previous orbit-character expansions (3.38) can be written as

$$\begin{aligned}\Omega^{(020)_0} &= ch\, V^{(020)_0}(1 - q^2 - 2q^4 + \ldots) \\ &\quad + ch\, V^{(101)_0}(-1 + 2q^2 - q^6 + \ldots) \\ \Omega^{(101)_0} &= ch\, V^{(020)_0}(-2q + q^2 + 2q^3 + 2q^4 - 2q^5 - q^6 + 2q^7 + \ldots) \\ &\quad + ch\, V^{(101)_0}(1 - 2q + 2q^2 - q^4 - 2q^5 - 2q^6 + 2q^7 + \ldots)\end{aligned}$$

In general for each particular affine congruence class, we need to consider

$$\Omega^{\mu} = \sum_{\lambda} (ch\, V^{\lambda})\, \kappa^{\mu}_{\lambda} \tag{3.39}$$

where μ and λ are maximal dominant weights. For example, from (3.37) in the case of level 2 modules of $A_2^{(1)}$, the analytic expressions for $\kappa^{(020)}_{(020)}$ and $\kappa^{(020)}_{(101)}$ are

$$\begin{aligned}\kappa^{(020)}_{(020)} = \sum_{m,n} [&q^{30\Gamma+4m-2n} - q^{30\Gamma+16m-8n+2} + q^{30\Gamma-8m+22n+4} \\ &- q^{30\Gamma-2m+28n+8} + q^{30\Gamma+28m+4n+10} - q^{30\Gamma+22m+16n+12}] \\ \kappa^{(020)}_{(101)} = \sum_{m,n} [&q^{30\Gamma+28m-14n+6} - q^{30\Gamma+52m-26n+22} + q^{30\Gamma+4m+34n+14} \\ &- q^{30\Gamma+46m-14n+18} + q^{30\Gamma+16m+28n+16} - q^{30\Gamma+34m-8n+10}].\end{aligned}$$

In Appendix 4 we tabulate some analytic expressions for κ^{μ}_{λ} in the case of level 1 and 2 modules of the affine algebras of rank 1 and 2. Although given with different parametrisations, some of these expressions can be inferred from or checked against the work of Begin and Sharp **[BS1]**. Inverting the matrices of the q-series analytically extends the work of Begin and Sharp to give the required expansion of irreducible characters

$$ch\, V^{\mu} = \sum_{\lambda} (\Omega^{\lambda})\, \sigma^{\mu}_{\lambda}\,. \tag{3.40}$$

This will be discussed in the next chapter.

CHAPTER 4

Weight Multiplicity Generating Functions

4.1 String functions and modular forms

Let V^Λ be a highest weight module of an affine Kac-Moody algebra $\mathcal{G}(A)$. Let λ be a maximal weight and $dim\, V^\Lambda_{\lambda-n\delta}$ denote the multiplicity of the weight $\lambda - n\delta$. A string function σ^Λ_λ is defined as the weight generating function

$$\sigma^\Lambda_\lambda = \sum_{n=0}^{\infty} dim\, V^\Lambda_{\lambda-n\delta} e^{-n\delta} . \tag{4.1}$$

Since any weight λ of V^Λ is conjugate to a dominant weight $\lambda^+ \in P^+$, we know all the string functions and hence all the weight multiplicities as soon as we know σ^Λ_λ for all maximal dominant weights λ^+.

Although σ^Λ_λ is not really a 'function', it can be turned into a genuine function that is defined and converges in the upper half complex plane $H = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$ by replacing $e^{-\delta}$ with $e^{2\pi i\tau}$ to give

$$\sigma^\Lambda_\lambda(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau} = \sum_{n=0}^{\infty} a_n q^n , \tag{4.2}$$

where $a_n = dim\, V^\Lambda_{\lambda-n\delta}$ and $q = e^{2\pi i\tau}$. This string function can further be turned into a modular function by multiplying with a certain power of q known as the modular characteristic

$$s(\Lambda, \lambda) = \frac{(\Lambda+\rho \mid \Lambda+\rho)}{2(L+g)} - \frac{(\rho \mid \rho)}{2g} - \frac{(\lambda \mid \lambda)}{2L} \tag{4.3}$$

where $L = L(\lambda)$ and $g = L(\rho)$. In the case of untwisted affine algebras, a tabulation of $s(\Lambda, \lambda)$ can be found in **[KMPS]**. In Table 4.1 we tabulate the modular characteristic of level 2 modules of all affine algebras of rank 2. We denote a modular string function by c^Λ_λ where

$$c^\Lambda_\lambda(\tau) = q^{s(\Lambda,\lambda)} \sigma^\Lambda_\lambda(\tau). \tag{4.4}$$

Table 4.1a : Modular characteristics of level 2 modules of $A_2^{(1)}$.

	(002)	(110)	(020)	(101)	(011)	(200)
(002)	$-\frac{2}{15}$	$-\frac{8}{15}$	0	0	0	0
(110)	$\frac{11}{30}$	$-\frac{1}{30}$	0	0	0	0
(020)	0	0	$-\frac{2}{15}$	$-\frac{8}{15}$	0	0
(101)	0	0	$\frac{11}{30}$	$-\frac{1}{30}$	0	0
(011)	0	0	0	0	$-\frac{1}{30}$	$-\frac{19}{30}$
(200)	0	0	0	0	$\frac{7}{15}$	$-\frac{2}{15}$

Table 4.1b : Modular characteristics of level 2 modules of $C_2^{(1)}$.

	(002)	(020)	(101)	(200)	(011)	(110)
(002)	$-\frac{1}{6}$	$-\frac{17}{30}$	$-\frac{23}{30}$	$-\frac{7}{6}$	0	0
(020)	$\frac{1}{3}$	$-\frac{1}{15}$	$-\frac{4}{15}$	$-\frac{2}{3}$	0	0
(101)	$\frac{7}{12}$	$\frac{11}{60}$	$-\frac{1}{60}$	$-\frac{25}{60}$	0	0
(200)	$\frac{5}{6}$	$\frac{13}{30}$	$\frac{7}{30}$	$-\frac{1}{6}$	0	0
(011)	0	0	0	0	$-\frac{1}{24}$	$-\frac{13}{24}$
(110)	0	0	0	0	$\frac{11}{24}$	$-\frac{1}{24}$

Table 4.1c : Modular characteristics of level 2 modules of $G_2^{(1)}$.

	(002)	(010)	(101)	(200)
(002)	$-\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{7}{12}$
(010)	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{11}{36}$	$\frac{17}{36}$
(101)	$-\frac{19}{36}$	$-\frac{13}{36}$	$-\frac{1}{36}$	$\frac{5}{36}$
(200)	$-\frac{31}{36}$	$-\frac{25}{36}$	$-\frac{13}{36}$	$-\frac{7}{36}$

Table 4.1d : Modular characteristics of level 2 modules of $A_4^{(2)}$.

	(002)	(010)	(100)
(002)	$-\frac{1}{7}$	$\frac{3}{28}$	$\frac{5}{14}$
(010)	$-\frac{2}{7}$	$-\frac{1}{28}$	$\frac{3}{14}$
(100)	$-\frac{4}{7}$	$-\frac{9}{28}$	$-\frac{1}{14}$

Table 4.1e : Modular characteristics of level 2 modules of $D_3^{(2)}$.

	(002)	(010)	(200)	(101)
(002)	$-\frac{5}{24}$	$\frac{7}{24}$	$\frac{19}{24}$	0
(010)	$-\frac{13}{24}$	$-\frac{1}{24}$	$\frac{11}{24}$	0
(100)	$-\frac{29}{24}$	$-\frac{17}{24}$	$-\frac{5}{24}$	0
(101)	0	0	0	$-\frac{1}{24}$

Table 4.1f : Modular characteristics of level 2 modules of $D_4^{(3)}$.

	(010)	(200)
(010)	$-\frac{1}{24}$	$\frac{11}{24}$
(200)	$-\frac{19}{24}$	$-\frac{7}{24}$

A modular function which is holomorphic everywhere (including infinity) is called a modular form. To be precise we need a definition of modular form as follows **[Kac4]**.

Definition 4.1. *Let*

$$\Gamma(n) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \, mod \, n, \; b \equiv c \equiv 0 \, mod \, n\}$$

be the principle congruence subgroup of $SL_2(\mathbb{Z})$. *A function* $f : H \to \mathbb{C}$ *is called a modular form of weight* k *for* Γ *if* f *is holomorphic on* H *and*

$$f(\frac{a\tau + b}{c\tau + d}) = \chi(A)(c\tau + d)^k f(\tau)$$

where the multiplier system χ *satisfies* $\mid \chi(A) \mid = 1$ *for all* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Among the most popular examples of a modular form is the Dedekind η-function

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \quad \text{for } \tau \in H, \tag{4.5}$$

which is a modular form of weight $\frac{1}{2}$ for $\Gamma(1)$. The multiplier system χ is such that $\chi(S) = e^{-\pi i/4}$ and $\chi(T) = e^{\pi i/12}$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generate $\Gamma(1)$. In terms of Euler's function $\phi(q)$, the η-function can be written as

$$\eta(\tau) = q^{\frac{1}{24}} \phi(q) \quad \text{where} \quad \phi(q) = \prod_{j=1}^{\infty} (1 - q^j). \tag{4.6}$$

The relations between modular string functions c_λ^Λ and modular forms can be traced back to the work of Kac and Peterson **[KaP]**. Using the theory of classical theta-functions they obtained the transformation law for string functions of affine algebras of rank r and showed that c_λ^Λ are modular forms of weight $-r/2$. The following theorem and corollary which were proved in the light of modular forms **[KaP]** are very helpful in obtaining explicit form for string functions.

Theorem 4.2. *Let* $\mathcal{G}(A)$ *be an affine Kac-Moody algebra and* c_μ^λ *be a modular string function of a highest weight module* V^λ *of level* L. *Then*

$$det \mid c_\mu^\lambda \mid_{\lambda, \mu \in P^+_{max}} = G(\tau)^{-|P^+_{max}|}$$

where P^+_{max} is the set of maximal dominant weights of level L and

$$G(\tau)=\begin{cases}\eta(\tau)^r & \text{for } X_r^{(1)} \text{ and } A_{2r}^{(2)},\\ \eta(\tau)^{r-1}\eta(2\tau) & \text{for } A_{2r-1}^{(2)},\\ \eta(\tau)\eta(2\tau)^{r-1} & \text{for } D_{r+1}^{(2)},\\ \eta(\tau)^2\eta(2\tau)^2 & \text{for } E_6^{(2)},\\ \eta(\tau)\eta(3\tau) & \text{for } D_4^{(3)},\end{cases}$$

where $X = A, B, C, D, E, F$ or G.

Corollary 4.3. *Let h and g be the Coxeter number and the dual Coxeter number, respectively, as defined in (3.5). Then*

$$\sum_{\lambda\in P^+_{max}} s(\lambda,\lambda) = -\frac{(\bar\rho\mid\bar\rho)}{2g(h_p+1)}\mid P^+_{max}\mid$$

where $h_p = h$ in the case of untwisted algebra and $h_p = g$ in the case of twisted algebra.

For each of the affine algebra we tabulate h, g and $(\bar\rho\mid\bar\rho)$ in Table 4.2. By (1.21) and (3.40) we can see that

$$\Omega^\mu = \sum_\lambda \kappa^\mu_\lambda \sum_\nu \sigma^\lambda_\nu \Omega^\nu ,$$

where λ, μ and ν are all maximal dominant weights in the same affine congruence class. This then implies that

$$\sum_\lambda \kappa^\mu_\lambda \sigma^\lambda_\nu = \delta^\mu_\nu . \tag{4.7}$$

Hence in principle if we could invert the matrix κ^μ_λ then we could obtain the required string functions. We shall call κ^μ_λ an inverse string function. By Theorem 4.2 the determinant of the modular inverse string functions must necessarily be $G(\tau)^{|P^+_{max}|}$.

Let $P^+_{max} = \{\nu_1,\ldots,\nu_n\}$ where $n = \mid P^+_{max}\mid$. Then

$$\begin{aligned}\det\mid c^\lambda_\mu\mid &= \det\mid q^{s(\lambda,\mu)}\sigma^\lambda_\mu\mid_{\lambda,\mu\in P^+_{max}}\\ &= \sum_{\pi\in S_n}\prod_{i=1}^{n} q^{s(\nu_i,\nu_{\pi_i})}\sigma^{\nu_i}_{\nu_{\pi_i}}\\ &= \sum_{\pi\in S_n} q^{\sum s(\nu_i,\nu_{\pi_i})}\prod_{i=1}^{n}\sigma^{\nu_i}_{\nu_{\pi_i}} .\end{aligned}$$

Table 4.2 : Coxeter numbers, dual Coxeter numbers and $(\rho \mid \rho)$

Algebra	h	g	$(\rho \mid \rho)$
$A_r^{(1)}$	$r+1$	$r+1$	$\frac{1}{12}r(r+1)(r+2)$
$B_r^{(1)}$	$2r$	$2r-1$	$\frac{1}{12}r(2r-1)(2r+1)$
$C_r^{(1)}$	$2r$	$r+1$	$\frac{1}{12}r(r+1)(2r+1)$
$D_r^{(1)}$	$2r-2$	$2r-2$	$\frac{1}{6}r(r-1)(2r-1)$
$E_6^{(1)}$	12	12	78
$E_7^{(1)}$	18	18	399/2
$E_8^{(1)}$	30	30	620
$F_4^{(1)}$	12	9	39
$G_2^{(1)}$	6	4	14/3
$A_{2r}^{(2)}$	$2r+1$	$2r+1$	$\frac{1}{12}r(2r-1)(2r+1)$
$A_{2r-1}^{(2)}$	$2r-1$	$2r$	$\frac{1}{6}r(2r+1)(r+1)$
$D_{r+1}^{(2)}$	$r+1$	$2r$	$\frac{1}{6}r(2r-1)(2r+1)$
$E_6^{(2)}$	9	12	78
$D_4^{(3)}$	4	6	14

But

$$s(\nu_i,\nu_{\pi_i}) = \frac{(\nu_i+\rho \mid \nu_i+\rho)}{2(L+g)} - \frac{(\rho \mid \rho)}{2g} - \frac{(\nu_{\pi_i} \mid \nu_{\pi_i})}{2L}$$

and

$$\sum_{i=1}^{n}(\nu_{\pi_i} \mid \nu_{\pi_i}) = \sum_{i=1}^{n}(\nu_i \mid \nu_i)$$

so that

$$\sum_{i=1}^{n} s(\nu_i,\nu_{\pi_i}) = -\frac{g}{2L(L+g)}\sum_i(\nu_i \mid \nu_i) + \frac{1}{(L+g)}\sum_i(\nu_i \mid \rho) - \frac{nL}{2g(L+g)}(\rho \mid \rho)$$

which is independent of the permutation π. Hence $q^{\sum_i s(\nu_i,\nu_i)}$ can be factored out from the expansion of the determinant, i.e.

$$\det \mid c^{\lambda}_{\mu} \mid_{\lambda,\mu\in P^+_{max}} = q^{\sum s(\nu_i,\nu_i)} \det \mid \sigma^{\lambda}_{\mu} \mid_{\lambda,\mu\in P^+_{max}} .$$

However, from Corollary 4.3 and Table 4.2 (modified slightly in the case of $A^{(2)}_{2r}$) we have

$$\mid P^+_{max} \mid^{-1} \sum_{\lambda\in P^+_{max}} s(\lambda,\lambda) = \begin{cases} -r/24 & \text{for } X^{(1)}_r \text{ and } A^{(2)}_{2r}, \\ -(r+1)/24 & \text{for } A^{(2)}_{2r-1}, \\ -(2r-1)/24 & \text{for } D^{(2)}_{r+1}, \\ -1/4 & \text{for } E^{(2)}_6, \\ -1/6 & \text{for } D^{(3)}_4 . \end{cases}$$

It then follows from Theorem 4.2 , (4.6) and (4.7) that

$$det \mid \kappa^{\mu}_{\lambda} \mid = H(q)^{|P^+_{max}|} \tag{4.8}$$

where

$$H(q) = \begin{cases} \phi(q)^r & \text{for } X^{(1)}_r \text{ and } A^{(2)}_{2r}, \\ \phi(q)^{r-1}\phi(q^2) & \text{for } A^{(2)}_{2r-1}, \\ \phi(q)\phi(q^2)^{r-1} & \text{for } D^{(2)}_{r+1}, \\ \phi(q)^2\phi(q^2)^2 & \text{for } E^{(2)}_6, \\ \phi(q)\phi(q^3) & \text{for } D^{(3)}_4 . \end{cases}$$

In the Appendix 4 we have tabulated explicit expressions for some inverse string functions κ^{μ}_{λ}. These functions were expressed as sums. It simplifies things enormously and make inversion easier if these functions are expressed as products. To have some idea of what we are going to do let us invert the inverse string function $\kappa_{(10)}^{(10)} = \sum_n (q^{6n^2-n} - q^{6n^2-5n+1})$ of the algebra $A_1^{(1)}$. Euler's function of (4.6) also has an expansion as a sum given by

$$\phi(q) = \sum_{n\in\mathbb{Z}+} (-1)^n q^{(3n^2+n)/2} = \sum_{n\in\mathbb{Z}} (q^{6n^2-n} - q^{6n^2-5n+1}). \tag{4.9}$$

Thus $\kappa_{(10)}^{(10)} = \phi(q)$. Relation (4.7) then implies that $\kappa_{(10)}^{(10)} \sigma_{(10)}^{(10)} = 1$. Hence

$$\sigma_{(10)}^{(10)} = \phi(q)^{-1} = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)} = \sum_n p_1(n)\, q^n ,$$

where $p_1(n)$ is the partition function. In order to obtain similar results for other inverse string functions one may use the Weyl-Kac denominator identity (1.18) to generalise (4.9). For future reference it is also useful to have a tabulation for the functions $\phi(q)^{-k} = \sum p_k(n)\, q^n$ which can be obtained from **[KMPS]**. The combinatorial interpretation of $p_k(n)$ is the number of distinct partitions of the integer n into integers of k different colours. We tabulate the partition function $p_k(n)$ for $k = 1, \ldots, 6$ and $n = 1, \ldots, 20$ in Appendix 5.

4.2 The Weyl-Kac denominator identity

The Weyl-Kac denominator identity (1.18) takes the form

$$\prod_{\alpha\in\Delta_+} (1 - e^{-\alpha})^{mult\,\alpha} = \sum_{w\in W} \varepsilon(w) e^{w(\rho)-\rho}.$$

By Theorem 3.5 and (3.21) we have for $w \in W$

$$w(\rho) = t_\alpha \bar{w}(\rho) = \bar{w}(\rho) + g\alpha - ((\bar{w}(\rho) \mid \alpha) + \frac{g}{2}(\alpha \mid \alpha))\delta$$

where $\alpha = \sum_{i=1}^r n_i\alpha_i \in M$ and $\bar{w} \in \overline{W}$. Let $\bar{w}(\rho) - \rho = -\sum_{\alpha\in\Phi_w} \alpha = -\sum_{i=1}^r k_i\alpha_i$. Then

$$(\bar{w}(\rho) \mid \alpha) = \sum_{i=1}^r n_i - \sum_{i,j=1}^r k_i n_j A_{ij}$$

$$(\alpha \mid \alpha) = \sum_{i,j=1}^r n_i n_j A_{ij}$$

so that

$$t_\alpha \bar{w}(\rho) - \rho = -\sum_{i=1}^{r}(k_i - gn_i)\alpha_i - (\sum_{i=1}^{r} n_i - \sum_{i,j}^{r} k_i n_j A_{ij} + \frac{g}{2}\sum_{i,j=1}^{r} n_i n_j A_{ij})\delta.$$

Next let $u_i = e^{-\alpha_i}$, $i = 1, \ldots, r$ and $v = e^{-\delta}$. Then

$$\prod_{\alpha \in \Delta+} (1 - e^{-\alpha})^{mult\,\alpha} = \sum_{\alpha \in M} \sum_{\bar{w}} \varepsilon(\bar{w})\, v^{g/2 \sum_{i,j} n_i n_j A_{ij} - \sum_{i,j} k_i n_j A_{ij} + \sum_i n_i} \prod_i u_i^{-gn_i + k_i}. \quad (4.10)$$

To illustrate the method let us apply the denominator identity to the affine algebra $A_2^{(1)}$. The set of positive roots obtained from (3.26), Proposition 3.1 and Proposition 3.2 is

$$\{n\delta \mid n \geq 1\} \cup \{\alpha_1 + n\delta, \alpha_2 + n\delta, \alpha_1 + \alpha_2 + n\delta \mid n \geq 0\}.$$

The real roots have multiplicity 1 but the imaginary roots have multiplicity 2. Hence

$$\begin{aligned}\prod_{\alpha \in \Delta+} (1 - e^{-\alpha})^{mult\,\alpha} &= \prod_{n=1}^{\infty}(1 - e^{-n\delta})^2 \prod_{n=0}^{\infty}(1 - e^{-(\alpha_1 + n\delta)})(1 - e^{-(\alpha_2 + n\delta)})(1 - e^{-(\alpha_1 + \alpha_2 + n\delta)}) \\ &= \prod_{n=1}^{\infty}(1 - v^n)^2 \prod_{n=0}^{\infty}(1 - u_1 v^n)(1 - u_2 v^n)(1 - u_1 u_2 v^n).\end{aligned}$$

On the other hand we can expand $\prod_{\alpha \in \Delta+}(1 - e^{-\alpha})^{mult\,\alpha}$ through (4.10). The Weyl group $\overline{W}$ is given in (2.2) and this gives

$$\begin{aligned}
id(\rho) - \rho = 0 \quad &\Rightarrow \quad \sum_{i,j=1}^{2} k_i n_i A_{ij} = 0 \\
s_1(\rho) - \rho = -\alpha_1 \quad &\Rightarrow \quad \sum_{i,j=1}^{2} k_i n_i A_{ij} = 2n_1 - n_2 \\
s_2(\rho) - \rho = -\alpha_2 \quad &\Rightarrow \quad \sum_{i,j=1}^{2} k_i n_i A_{ij} = -n_1 + 2n_2 \\
s_1 s_2(\rho) - \rho = -2\alpha_1 - \alpha_2 \quad &\Rightarrow \quad \sum_{i,j=1}^{2} k_i n_i A_{ij} = 3n_1 \\
s_2 s_1(\rho) - \rho = -\alpha_1 - 2\alpha_2 \quad &\Rightarrow \quad \sum_{i,j=1}^{2} k_i n_i A_{ij} = 3n_2 \\
s_1 s_2 s_1(\rho) - \rho = -2\alpha_1 - 2\alpha_2 \quad &\Rightarrow \quad \sum_{i,j=1}^{2} k_i n_i A_{ij} = 2n_1 + 2n_2.
\end{aligned}$$

Let $\Gamma = 3\sum_{i,j=1}^{2} n_i n_j A_{ij} = 6(n_1^2 - n_1 n_2 + n_2^2)$ then (4.10) can be expanded to give

$$\begin{aligned} &\sum_{n_1,n_2} v^{\Gamma+n_1+n_2} u_1^{-3n_1} u_2^{-3n_2} - \sum_{n_1,n_2} v^{\Gamma-n_1+2n_2} u_1^{-3n_1+1} u_2^{-3n_2} \\ &- \sum_{n_1,n_2} v^{\Gamma+2n_1-n_2} u_1^{-3n_1} u_2^{-3n_2+1} + \sum_{n_1,n_2} v^{\Gamma-2n_1+n_2} u_1^{-3n_1+2} u_2^{-3n_2+1} \\ &+ \sum_{n_1,n_2} v^{\Gamma+n_1-2n_2} u_1^{-3n_1+1} u_2^{-3n_2+2} - \sum_{n_1,n_2} v^{\Gamma-n_1-n_2} u_1^{-3n_1+2} u_2^{-3n_2+2}, \end{aligned}$$

where n_1 and n_2 are integers. Hence the denominator identity for $A_2^{(1)}$ can now be written down as:

$A_2^{(1)}$:

$$\begin{aligned} &\prod_{n=1}^{\infty} (1-v^n)^2 (1-v^n u_1^{-1})(1-v^{n-1}u_1)(1-v^n u_2^{-1}) \\ &\qquad (1-v^{n-1}u_2)(1-v^n u_1^{-1} u_2^{-1})(1-v^{n-1}u_1 u_2) \\ &= \sum_{n,m} \{ v^{\Gamma+n+m} u_1^{-3n} u_2^{-3m} + v^{\Gamma+n-2m} u_1^{-3n+1} u_2^{-3m+2} + v^{\Gamma-2n+m} u_1^{-3n+2} u_2^{-3m+1} \\ &\qquad - v^{\Gamma-n+2m} u_1^{-3n+1} u_2^{-3m} - v^{\Gamma+2n-m} u_1^{-3n} u_2^{-3m+1} - v^{\Gamma-n-m} u_1^{-3n+2} u_2^{-3m+2} \} \end{aligned} \tag{4.11a}$$

where $\Gamma = 3(n^2 - nm + m^2)$.

In a similar way, the denominator identity expansions that correspond to the other lower rank affine algebras may be expressed in the same form.

$A_1^{(1)}$:

$$\prod_{n=1}^{\infty} (1-v^n)(1-v^n u^{-1})(1-v^{n-1}u) = \sum_{n} \{ v^{n(2n+1)} u^{-2n} - v^{n(2n-1)} u^{-2n+1} \} \tag{4.11b}$$

$A_2^{(2)}$:

$$\begin{aligned} &\prod_{n=1}^{\infty} (1-v^n)(1-v^n u^{-1})(1-v^{n-1}u)(1-v^{2n-1}u^{-2})(1-v^{2n-1}u^2) \\ &= \sum_{n} \{ v^{\frac{n}{2}(3n+1)} u^{-3n} - v^{\frac{n}{2}(3n-1)} u^{-3n+1} \} \end{aligned} \tag{4.11c}$$

$C_2^{(1)}$:

$$\prod_{n=1}^{\infty}\{(1-v^n)^2(1-v^nu_1^{-1})(1-v^{n-1}u_1)(1-v^nu_2^{-1})(1-v^{n-1}u_2)$$
$$(1-v^nu_1^{-1}u_2^{-1})(1-v^{n-1}u_1u_2)(1-v^nu_1^{-2}u_2^{-1})(1-v^{n-1}u_1^2u_2)\}$$

$$\begin{aligned} = \sum_{n,m}\{ & v^{\Gamma+n+m}u_1^{-6n}u_2^{-3m} + w^{\Gamma+3n-2m}u_1^{-6n+1}u_2^{-3m+2} \\ & + v^{\Gamma-3n+2m}u_1^{-6n+3}u_2^{-3m+1} + v^{\Gamma-n-m}u_1^{-6n+4}u_2^{-3m+3} \\ & - v^{\Gamma-n+2m}u_1^{-6n+1}u_2^{-3m} - v^{\Gamma+3n-m}u_1^{-6n}u_2^{-3m+1} \\ & - v^{\Gamma-3n+m}u_1^{-6n+4}u_2^{-3m+2} - v^{\Gamma+n-2m}u_1^{-6n+3}u_2^{-3m+3}\} \end{aligned} \tag{4.11d}$$

where $\Gamma = 3(2n^2 - 2nm + m^2)$.

$G_2^{(1)}$:

$$\prod_{n=1}^{\infty}\{(1-v^n)^2(1-v^nu_1^{-1})(1-v^{n-1}u_1)(1-v^nu_2^{-1})(1-v^{n-1}u_2)$$
$$(1-v^nu_1^{-1}u_2^{-1})(1-v^{n-1}u_1u_2)(1-v^nu_1^{-1}u_2^{-2})(1-v^{n-1}u_1u_2^2)$$
$$(1-v^nu_1^{-1}u_2^{-3})(1-v^{n-1}u_1u_2^3)(1-v^nu_1^{-2}u_2^{-3})(1-v^{n-1}u_1^2u_2^3)\}$$

$$\begin{aligned} = \sum_{n,m}\{ & v^{\Gamma+n+m}u_1^{-4n}u_2^{-12m} + v^{\Gamma+3n-4m}u_1^{-4n+1}u_2^{-12m+4} \\ & + v^{\Gamma-2n+5m}u_1^{-4n+2}u_2^{-12m+1} + v^{\Gamma+2n-5m}u_1^{-4n+4}u_2^{-12m+9} \\ & + v^{\Gamma-3n+4m}u_1^{-4n+5}u_2^{-12m+6} + v^{\Gamma-n-m}u_1^{-4n+6}u_2^{-12m+10} \\ & - v^{\Gamma-n+4m}u_1^{-4n+1}u_2^{-12m} - v^{\Gamma+2n-m}u_1^{-4n}u_2^{-12m+1} \\ & - v^{\Gamma-3n+5m}u_1^{-4n+4}u_2^{-12m+4} - v^{\Gamma+3n-5m}u_1^{-4n+2}u_2^{-12m+6} \\ & - v^{\Gamma-2n+m}u_1^{-4n+6}u_2^{-12m+9} - v^{\Gamma+n-4m}u_1^{-4n+5}u_2^{-12m+10}\} \end{aligned} \tag{4.11e}$$

where $\Gamma = 4(n^2 - 3nm + 3m^2)$.

$A_4^{(2)}$:

$$\prod_{n=1}^{\infty}\{(1-v^n)^2(1-v^{n-1}u_1)(1-v^nu_1^{-1})(1-v^{n-1}u_2)(1-v^nu_2^{-1})$$

$$(1-v^{n-1}u_1u_2)(1-v^nu_1^{-1}u_2^{-1})(1-v^{n-1}u_1u_2^2)(1-v^nu_1^{-1}u_2^{-2})(1-v^{2n-1}u_2^2)$$

$$(1-v^{2n-1}u_2^{-2})(1-v^{2n-1}u_1^2u_2^2)(1-v^{2n-1}u_1^{-2}u_2^{-2})\}$$

$$= \sum_{n,m}\{v^{\frac{1}{2}(\Gamma+2n+m)}u_1^{-5n}u_2^{-5m} + v^{\frac{1}{2}(\Gamma+4n-3m)}u_1^{-5n+1}u_2^{-5m+3}$$

$$+ v^{\frac{1}{2}(\Gamma-4n+3m)}u_1^{-5n+2}u_2^{-5m+1} + v^{\frac{1}{2}(\Gamma-2n-m)}u_1^{-5n+3}u_2^{-5m+4}$$

$$- v^{\frac{1}{2}(\Gamma-2n+3m)}u_1^{-5n+1}u_2^{-5m} - v^{\frac{1}{2}(\Gamma+4n-m)}u_1^{-5n}u_2^{-5m+1}$$

$$- v^{\frac{1}{2}(\Gamma-4n+m)}u_1^{-5n+3}u_2^{-5m+3} - v^{\frac{1}{2}(\Gamma+2n-3m)}u_1^{-5n+2}u_2^{-5m+4}\} \qquad (4.11f)$$

where $\Gamma = 5(2n^2 - 2nm + m^2)$.

$D_3^{(2)}$:

$$\prod_{n=1}^{\infty}\{(1-v^n)(1-v^{2n})(1-v^nu_2^{-1})(1-v^{n-1}u_2)(1-v^nu_1^{-1}u_2^{-1})(1-v^{n-1}u_1u_2)$$

$$(1-v^{2n}u_1^{-1})(1-v^{2n-2}u_1)(1-v^{2n}u_1^{-1}u_2^{-2})(1-v^{2n-2}u_1u_2^2)\}$$

$$= \sum_{n,m}\{v^{\Gamma+2n+m}u_1^{-4n}u_2^{-4m} + v^{\Gamma+4n-3m}u_1^{-4n+1}u_2^{-4m+3}$$

$$+ v^{\Gamma-4n+3m}u_1^{-4n+2}u_2^{-4m+1} + v^{\Gamma-2n-m}u_1^{-4n+3}u_2^{-4m+4}$$

$$- v^{\Gamma-2n+3m}u_1^{-4n+1}u_2^{-4m} - v^{\Gamma+4n-m}u_1^{-4n}u_2^{-4m+1}$$

$$- v^{\Gamma-4n+m}u^{-4n+3}v^{-4m+3} - v^{\Gamma+2n-3m}u^{-4n+2}v^{-4m+4}\} \qquad (4.11g)$$

where $\Gamma = 4(2n^2 - 2nm + m^2)$.

$D_4^{(3)}$:

$$\prod_{n=1}^{\infty}\{(1-v^n)(1-v^{3n})(1-v^nu_1^{-1})(1-v^{n-1}u_1)(1-v^nu_1^{-1}u_2^{-1})(1-v^{n-1}u_1u_2)$$
$$(1-v^nu_1^{-2}u_2^{-1})(1-v^{n-1}u_1^2u_2)(1-v^{3n}u_2^{-1})(1-v^{3n-3}u_2)$$
$$(1-v^{3n}u_1^{-3}u_2^{-1})(1-v^{3n-3}u_1^3u_2)(1-v^{3n}u_1^{-3}u_2^{-2})(1-v^{3n-3}u_1^3u_2^2)\}$$
$$= \sum_{n,m}\{v^{\Gamma+n+3m}u_1^{-6n}u_2^{-6m}+v^{\Gamma+5n-6m}u_1^{-6n+1}u_2^{-6m+2}$$
$$+v^{\Gamma-4n+9m}u_1^{-6n+4}u_2^{-6m+1}+v^{\Gamma+4n-9m}u_1^{-6n+6}u_2^{-6m+5}$$
$$+v^{\Gamma-5n+6m}u_1^{-6n+9}u_2^{-6m+4}+v^{\Gamma-n-3m}u_1^{-6n+10}u_2^{-6m+6}$$
$$-v^{\Gamma-n+6m}u_1^{-6n+1}u_2^{-6m}-v^{\Gamma+4n-3m}u_1^{-6n}u_2^{-6m+1}$$
$$-v^{\Gamma-5n+9m}u_1^{-6n+6}u_2^{-6m+2}-v^{\Gamma+5n-9m}u_1^{-6n+4}u_2^{-6m+4}$$
$$-v^{\Gamma-4n+3m}u_1^{-6n+10}u_2^{-6m+5}+v^{\Gamma+n-6m}u_1^{-6n+9}u_2^{-6m+6}\} \tag{4.11h}$$

where $\Gamma = 6(n^2-3nm+3m^2)$.

In fact (4.11b) is one form of the celebrated Jacobi triple product identity (JTP)

$$\prod_{n=1}^{\infty}(1-v^n)(1-v^nu^{-1})(1-v^{n-1}u)=\sum_n(-1)^nv^{n(n+1)/2}u^{-n}\,.$$

If further we let $v=q^{2k}$ and $u=(-q)^{k+\ell}$ then we obtain another form for the JTP as

$$\prod_{n=1}^{\infty}(1-q^{2kn})(1\pm q^{2kn-k-\ell})(1\pm q^{2kn-k+\ell})=\sum_n(\pm1)^nq^{kn^2+\ell n}\,. \tag{4.12}$$

Specialising to $v=q^r$, $u_1=q^{s_1}$, $u_2=q^{s_2}$ in the respective denominator identities (4.11a - 4.11h), we are able to express the κ_λ^μ that are given in Appendix 4 as sums of products. Specialisation of this form will be denoted by $[r;s_1,s_2]$. A bar represent a negative q specialisation, e.g. $[3;\,1,\overline{1/3}]$ denotes the specialisation $v=q^3$, $u_1=q$ and $u_2=-q^{1/3}$. Also note that the notation $\prod_{\pm a(r)}(1-q^n)$ means $\prod_{n\geq1}(1-q^{rn-a})(1-q^{r(n-1)+a})$.

$A_1^{(1)}$

$$\begin{aligned} [3;\,1] &\quad\Rightarrow\quad \kappa^{(10)}_{(10)} = \phi(q) \\ [2;\,1] &\quad\Rightarrow\quad \kappa^{(11)}_{(11)} = \phi(q)\prod_{1(2)}(1-q^n) \\ [2;\,1/2] &\quad\Rightarrow\quad \kappa^{(20)}_{(20)} + q^{1/2}\kappa^{(02)}_{(20)} = \phi(q^2)\prod_{1(2)}(1-q^{n/2}) \end{aligned} \tag{4.13a}$$

$A_2^{(2)}$

$$\begin{aligned} [4;\,1] &\quad\Rightarrow\quad \kappa^{(01)}_{(01)} = \phi(q) \\ [10;\,1] &\quad\Rightarrow\quad \kappa^{(02)}_{(02)} = \phi(q^{10})\prod_{\pm1(10)}(1-q^n)\prod_{\pm8(20)}(1-q^n) \\ [10;\,2] &\quad\Rightarrow\quad \kappa^{(10)}_{(10)} = \phi(q^{10})\prod_{\pm2(10)}(1-q^n)\prod_{\pm6(20)}(1-q^n) \\ [10;\,3] &\quad\Rightarrow\quad \kappa^{(02)}_{(10)} = -\phi(q^{10})\prod_{\pm3(10)}(1-q^n)\prod_{\pm4(20)}(1-q^n) \\ [10;\,4] &\quad\Rightarrow\quad \kappa^{(10)}_{(02)} = -q\phi(q^{10})\prod_{\pm4(10)}(1-q^n)\prod_{\pm2(20)}(1-q^n) \end{aligned} \tag{4.13b}$$

$A_2^{(1)}$

$$[4;\,1,\,1] \quad\Rightarrow\quad \kappa^{(100)}_{(100)} = \phi(q)^2$$

$$[10;\,4,\,4] \quad\Rightarrow\quad \kappa^{(200)}_{(200)} = \phi(q^2)\phi(q^{10})\prod_{\pm4(10)}(1-q^n)$$

$$[10;\,2,\,2] \quad\Rightarrow\quad \kappa^{(200)}_{(011)} = -q\phi(q^2)\phi(q^{10})\prod_{\pm2(10)}(1-q^n)$$

$[10;\,1,\,2]$ and $[10;\,3,\,3]$

$$\Rightarrow\quad \kappa^{(011)}_{(011)} = \phi(q^{10})^2\Big(\prod_{\pm3,\pm3,\pm4(10)}(1-q^n) - 2q\prod_{\pm1,\pm2,\pm3(10)}(1-q^n)\Big)$$

$[10;\,1,\,1]$ and $[10;\,1,\,3]$

$$\Rightarrow\quad \kappa^{(011)}_{(200)} = -\phi(q^{10})^2\Big(2\prod_{\pm1,\pm3,\pm4(10)}(1-q^n) + q\prod_{\pm1,\pm1,\pm2(10)}(1-q^n)\Big) \tag{4.13c}$$

$C_2^{(1)}$

$$[4;\,1,\,1] \quad \Rightarrow \quad \kappa^{(010)}_{(010)} = \phi(q)^2 \prod (1-q^{2n-1})$$

$$[4;\,1/2,\,1] \quad \Rightarrow \quad \kappa^{(100)}_{(100)} + q^{-1/2}\kappa^{(100)}_{(001)} = \phi(q)\phi(q^2) \prod_{1(2)} (1-q^{n/2})$$

$[10;\,1/2,\,1]$, $[10;\,3/2,\,3]$, $[10;\,7/2,\,7]$ and $[10;\,9/2,\,9]$

$$\Rightarrow \quad \kappa^{(011)}_{(011)} - q^{-1/2}\kappa^{(110)}_{(011)}$$

$$= \phi(q^{10})^2 \prod_{\pm3,\pm4(10)} (1-q^n)\Big(\prod_{\pm3,\pm9(20)} (1-q^{n/2}) + q^{1/2} \prod_{\pm1,\pm7(20)} (1-q^{n/2})\Big)$$
$$+q^{1/2}\phi(q^{10})^2 \prod_{\pm1,\pm2(10)} (1-q^n)\Big(\prod_{\pm7,\pm9(20)} (1-q^{n/2}) + q^{3/2}\phi(q^{10})^2 \prod_{\pm1,\pm3(20)} (1-q^{n/2})\Big)$$

$$[10;\,\bar{2},\,2] \quad \Rightarrow \quad \kappa^{(002)}_{(002)} + q\kappa^{(002)}_{(200)} = \phi(q^4)\phi(q^{10}) \prod_{10(20)} (1-q^n)$$

$$[10;\,\bar{4},\,6] \quad \Rightarrow \quad \kappa^{(002)}_{(020)} = -\phi(q^{20})^2 \prod_{\pm8(20)} (1-q^n) \prod_{\pm4(10)} (1-q^n)$$

$$[10;\,\bar{2},\,8] \quad \Rightarrow \quad \kappa^{(002)}_{(101)} = q\phi(q^{20})^2 \prod_{\pm4(20)} (1-q^n) \prod_{\pm2(10)} (1-q^n)$$

$$[10;\,\bar{1},\,2] \quad \Rightarrow \quad \kappa^{(020)}_{(200)} = -\phi(q^2)\phi(q^{10}) \prod_{\pm1,\pm3(10)} (1+q^n)$$

$$[10;\,\bar{1},\,4] \quad \Rightarrow \quad \kappa^{(020)}_{(020)} = \phi(q^{10})^2 \prod_{\pm1(10)} (1+q^n) \prod_{5(10)} (1+q^n)^2 \prod_{\pm4(10)} (1-q^n)^2$$

$$[10;\,\bar{3},\,2] \quad \Rightarrow \quad \kappa^{(020)}_{(101)} = -\phi(q^{10})^2 \prod_{\pm3(10)} (1+q^n) \prod_{5(10)} (1+q^n)^2 \prod_{\pm2(10)} (1-q^n)^2$$

$[10;\,\bar{0},\,1]$ and $[10;\,\bar{5},\,9]$

$$\Rightarrow \quad \kappa^{(101)}_{(020)} = -q\phi(q^{10})^2 \prod_{\pm4,5,5(10)} (1+q^n) \prod_{\pm1(10)} (1-q^n)^2$$
$$+ 2q^3\phi(q^{20})^2 \prod_{\pm1,\pm2,\pm9(20)} (1-q^n)$$

$[10;\,\bar{2},\,3]$ and $[10;\,\bar{0},\,3]$

$$\Rightarrow \quad \kappa^{(101)}_{(101)} = \phi(q^{10})^2 \prod_{\pm2,5,5(10)} (1+q^n) \prod_{\pm3(10)} (1-q^n)^2$$
$$- 2q\phi(q^{20})^2 \prod_{\pm3,\pm6,\pm7(20)} (1-q^n)$$

$[10;\,\bar{3},\,3]$ and $[10;\,\bar{1},\,7]$

$$\Rightarrow \quad \kappa^{(101)}_{(200)} = -\phi(q^{10})^2 \prod_{\pm3,\pm4(10)} (1+q^n) \prod_{\pm1,\pm3(10)} (1-q^n)$$
$$+ q\phi(q^{10})^2 \prod_{\pm1,\pm2(10)} (1+q^n) \prod_{\pm1,\pm3(10)} (1-q^n)$$

$$(4.13d)$$

$G_2^{(1)}$

$$[5;\,1,\,1/3] \quad\Rightarrow\quad \kappa^{(100)}_{(100)} + q^{1/3}\kappa^{(001)}_{(100)} = \phi(q)\prod_{0,\pm1(5)}(1-q^{\frac{n}{3}})$$

$$[5;\,1,\,2/3] \quad\Rightarrow\quad \kappa^{(001)}_{(001)} + q^{-1/3}\kappa^{(100)}_{(001)} = \phi(q)\prod_{0,\pm2(5)}(1-q^{\frac{n}{3}})$$

$$[3;\,\bar{1},\,1] \quad\Rightarrow\quad \kappa^{(002)}_{(002)} = \phi(q^2)^2\prod_{\pm1(3)}(1+q^n)$$

$$[3;\,1,\,\overline{1/3}] \quad\Rightarrow\quad \kappa^{(002)}_{(010)} + q^{1/3}\kappa^{(002)}_{(101)} = -\phi(q^2)\phi(q^6)\prod_{\pm1,\pm4(9)}(1+q^{n/3})\prod_{\pm4(9)}(1-q^{n/3})$$

$$[3;\,1,\,\overline{4/3}] \quad\Rightarrow\quad \kappa^{(002)}_{(010)} - q^{2/3}\kappa^{(002)}_{(200)} = -\phi(q^2)\phi(q^6)\prod_{\pm2,\pm4(9)}(1+q^{n/3})\prod_{\pm2(9)}(1-q^{n/3})$$

$$[3;\,2,\,\overline{1/3}] \quad\Rightarrow\quad \kappa^{(002)}_{(101)} + q^{1/3}\kappa^{(002)}_{(200)} = -\phi(q^2)\phi(q^6)\prod_{\pm1,\pm2(9)}(1+q^{n/3})\prod_{\pm1(9)}(1-q^{n/3})$$

$$[3;\,1/2,\,1/3] \Rightarrow\quad \kappa^{(101)}_{(101)} + q^{1/3}\kappa^{(101)}_{(200)} + q^{1/2}\kappa^{(002)}_{(101)} + q^{5/6}\kappa^{(002)}_{(200)}$$

$$=\phi(q)\phi(q^3)\prod_{1(2)}(1-q^{n/2})\prod_{3(6)}(1-q^{n/2})\prod_{\pm1(9)}(1-q^{n/3})\prod_{\pm5,\pm7(18)}(1-q^{n/6})$$

$$[3;\,1/2,\,4/3] \Rightarrow\quad \kappa^{(101)}_{(010)} + q^{1/3}\kappa^{(101)}_{(101)} + q^{1/2}\kappa^{(002)}_{(010)} + q^{5/6}\kappa^{(002)}_{(101)}$$

$$= q^{1/3}\phi(q)\phi(q^3)\prod_{1(2)}(1-q^{n/2})\prod_{3(6)}(1-q^{n/2})\prod_{\pm4(9)}(1-q^{n/3})\prod_{\pm1,\pm7(18)}(1-q^{n/6})$$

(4.13e)

$A_4^{(2)}$

$$[6;\,1,\,1] \quad\Rightarrow\quad \kappa^{(001)}_{(001)} = \phi(q)^2$$

$$[14;\,4,\,1] \quad\Rightarrow\quad \kappa^{(002)}_{(002)} = \phi(q^{14})^2\prod_{\pm1,\pm4,\pm5,\pm6(14)}(1-q^n)\prod_{\pm4,\pm12(28)}(1-q^n)$$

$$[14;\,2,\,1] \quad\Rightarrow\quad \kappa^{(002)}_{(010)} = -\phi(q^{14})^2\prod_{\pm1,\pm2,\pm3,\pm4(14)}(1-q^n)\prod_{\pm8,\pm12(28)}(1-q^n)$$

$$[14;\,2,\,3] \quad\Rightarrow\quad \kappa^{(002)}_{(100)} = -\phi(q^{14})^2\prod_{\pm2,\pm3,\pm5,\pm6(14)}(1-q^n)\prod_{\pm4,\pm8(28)}(1-q^n)$$

[14; 3, 1] and [14; 3, 3]

$$\Rightarrow\quad \kappa^{(010)}_{(010)} = \phi(q^{14})^2\prod_{\pm1,\pm3,\pm4,\pm5(14)}(1-q^n)\prod_{\pm6,\pm12(28)}(1-q^n)$$
$$-\,q\phi(q^{14})^2\prod_{\pm3,\pm3,\pm5,\pm6(14)}(1-q^n)\prod_{\pm2,\pm8(28)}(1-q^n)$$

(4.13f)

[14; 1, 2] and [14; 5, 5]

$$\Rightarrow\quad \kappa^{(010)}_{(100)} = -\phi(q^{14})^2\prod_{\pm1,\pm2,\pm3,\pm5(14)}(1-q^n)\prod_{\pm8,\pm10(28)}(1-q^n)$$
$$+\,q\phi(q^{14})^2\prod_{\pm1,\pm4,\pm5,\pm5(14)}(1-q^n)\prod_{\pm4,\pm6(28)}(1-q^n)$$

$$[14;\,1,\,1] \quad \text{and} \quad [14;\,3,\,5]$$

$$\Rightarrow \quad \kappa^{(010)}_{(002)} = -q\phi(q^{14})^2 \prod_{\pm 1,\pm 1,\pm 2,\pm 3(14)} (1-q^n) \prod_{\pm 10,\pm 12(28)} (1-q^n)$$
$$+ q^3\phi(q^{14})^2 \prod_{\pm 1,\pm 3,\pm 5,\pm 6(14)} (1-q^n) \prod_{\pm 2,\pm 4(28)} (1-q^n)$$

$$[14;\,4,\,2] \quad \Rightarrow \quad \kappa^{(100)}_{(002)} = -q\phi(q^2)\phi(q^{14}) \prod_{\pm 2,\pm 6,\pm 8,\pm 10(28)} (1-q^n)$$

$$[14;\,2,\,4] \quad \Rightarrow \quad \kappa^{(100)}_{(010)} = q^2\phi(q^2)\phi(q^{14}) \prod_{\pm 2,\pm 4,\pm 6,\pm 10(28)} (1-q^n)$$

$$[14;\,2,\,2] \quad \Rightarrow \quad \kappa^{(100)}_{(100)} = \phi(q^2)\phi(q^{14}) \prod_{\pm 2,\pm 6,\pm 10,\pm 12(28)} (1-q^n)$$

$D_3^{(2)}$

$$[5;\,2,\,1] \quad \Rightarrow \quad \kappa^{(100)}_{(100)} = \phi(q)\phi(q^2)$$

$$[3;\,\bar{1},\,\bar{1}] \quad \Rightarrow \quad \kappa^{(010)}_{(010)} = \phi(q^2)\phi(q^3)\prod(1+q^{2n-1})(1+q^{6n-3})$$

$$[3;\,\bar{2},\,\bar{2}] \quad \Rightarrow \quad \kappa^{(200)}_{(010)} = -q\phi(q^3)\phi(q^4)\prod(1+q^{6n})$$

$$[3;\,\bar{1},\,2] \quad \Rightarrow \quad \kappa^{(010)}_{(200)} = -\phi(q)\phi(q^{12})\prod(1+q^{2n-1})^2(1+q^{6n}) \tag{4.13g}$$

$$[3;\,1,\,1/2] \quad \Rightarrow \quad \kappa^{(200)}_{(200)} + q^{1/2}\kappa^{(010)}_{(200)} + q\kappa^{(002)}_{(200)}$$
$$= \phi(q)\phi(q^6)\prod(1-q^{(2n-1)/2})(1-q^{(6n-3)/2})$$

$D_4^{(3)}$

$$[7;\,3,\,3] \quad \Rightarrow \quad \kappa^{(100)}_{(100)} = \phi(q)\phi(q^3)$$

$$[4;\,1,\,3/2] \quad \Rightarrow \quad \kappa^{(010)}_{(010)} + q^{-1/2}\kappa^{(200)}_{(010)}$$
$$= \phi(q)\phi(q^3)\prod(1+q^{2n})(1+q^{6n-3})(1-q^{(2n-1)/2})(1-q^{(6n-3)/2})$$

$$[4;\,2,\,3/2] \quad \Rightarrow \quad \kappa^{(200)}_{(200)} + q^{1/2}\kappa^{(010)}_{(200)}$$
$$= \phi(q)\phi(q^3)\prod(1+q^{2n-1})(1+q^{6n})(1-q^{(2n-1)/2})(1-q^{(6n-3)/2}) \tag{4.13h}$$

This complete the determination of all level 1 and level 2 inverse string functions for all rank 1 and 2 affine algebras, although some results are only given implicitly in the form of a linear combination of such functions.

4.3 Explicit computation of string functions

Let σ and κ denote the matrices with matrix elements σ_λ^μ and κ_λ^μ, respectively. Then the matrix form for (4.7) is $\sigma = \kappa^{-1}$. Matrices of order less than or equal to 2 can be inverted easily. So whenever $| P_{max} | \leq 2$ we can obtain the string functions directly. For example, consider the task of obtaining all string functions of the level 1 modules of the affine algebra $G_2^{(1)}$. From (4.8) and (4.13e), we have $P_{max} = \{(001),(100)\}$, $\det \kappa = \phi(q)^4$ and

$$\kappa_{(001)}^{(001)} + q^{-1/3}\kappa_{(001)}^{(100)} = \phi(q) \prod_{0,\pm 2(5)} (1-q^{n/3})$$

$$\kappa_{(100)}^{(100)} + q^{1/3}\kappa_{(100)}^{(001)} = \phi(q) \prod_{0,\pm 1(5)} (1-q^{n/3}).$$

Hence

$$\begin{pmatrix} \sigma_{(001)}^{(001)} & \sigma_{(100)}^{(001)} \\ \sigma_{(001)}^{(100)} & \sigma_{(100)}^{(100)} \end{pmatrix} = \frac{1}{\phi(q)^4} \begin{pmatrix} \kappa_{(100)}^{(100)} & -\kappa_{(100)}^{(001)} \\ -\kappa_{(001)}^{(100)} & \kappa_{(001)}^{(001)} \end{pmatrix}$$

so that

$$\sigma_{(100)}^{(100)} - q^{-1/3}\sigma_{(001)}^{(100)} = \phi(q)^{-3} \prod_{0,\pm 2(5)} (1-q^{n/3}) \tag{4.14a}$$

$$\sigma_{(001)}^{(001)} - q^{1/3}\sigma_{(100)}^{(001)} = \phi(q)^{-3} \prod_{0,\pm 1(5)} (1-q^{n/3}). \tag{4.14b}$$

It is also useful to have explicit forms for σ_μ^λ rather than linear combinations of them. By the JTP (4.12)

$$\begin{aligned} &\phi(q^5)\prod(1-q^{5n-1})(1-q^{5n-4}) \\ =&\sum(-1)^n q^{n(5n+3)/2} \\ =&\sum(-1)^{3n}q^{3n(15n+3)/2} + \sum(-1)^{3n+1}q^{(3n+1)(15n+8)/2} + \sum(-1)^{3n+2}q^{(3n+2)(15n+13)/2} \\ =&\sum(-1)^n q^{9n(5n+1)/2} - q^4\sum(-1)^n q^{3n(15n+13)/2} + q^{13}\sum(-1)^n q^{3n(15n+23)/2}. \end{aligned}$$

Hence

$$\begin{aligned} \prod_{0,\pm 1(5)} (1-q^{\frac{n}{3}}) =& \sum(-1)^n q^{n(15n+3)/2} - q^{4/3}\sum(-1)^n q^{n(15n+13)/2} \\ &+ q^{13/3}\sum(-1)^n q^{n(15n+23)/2} \\ =&\phi(q^{15})\Big(\prod_{\pm 6(15)}(1-q^n) - q^{4/3}\prod_{\pm 1(15)}(1-q^n) - q^{1/3}\prod_{\pm 4(15)}(1-q^n)\Big) \end{aligned}$$

This expression and (4.14b) implies that

$$\sigma^{(001)}_{(001)} = \frac{\phi(q^{15})}{\phi(q)^3} \prod_{\pm 6(15)} (1-q^n)$$

$$\sigma^{(001)}_{(100)} = \frac{\phi(q^{15})}{\phi(q)^3} (\prod_{\pm 4(15)} (1-q^n) + q \prod_{\pm 1(15)} (1-q^n)) .$$

Similarly, from the other expression (4.14a), it can be shown that

$$\sigma^{(100)}_{(001)} = \frac{\phi(q^{15})}{\phi(q)^3} \prod_{\pm 3(15)} (1-q^n)$$

$$\sigma^{(100)}_{(100)} = \frac{\phi(q^{15})}{\phi(q)^3} (\prod_{\pm 7(15)} (1-q^n) - q \prod_{\pm 2(15)} (1-q^n)) .$$

Below we give some string functions for the case $| P_{max} | \leq 2$ obtained by inverting expressions from (4.13a - 4.13h). Some of these string functions are expressed as a linear combination of terms. Explicit string functions can be obtained by a similar method to that discussed above. Although cast in slightly different form these results can be compared with those obtained in **[KaP]**. The ones marked * are new results.

$A^{(1)}_1$:

$$\begin{aligned} \sigma^{(20)}_{(20)} - q^{-1/2}\sigma^{(20)}_{(02)} &= \phi(q)^{-1} \prod (1 - q^{(2n-1)/2}) \\ \sigma^{(02)}_{(02)} &= \sigma^{(20)}_{(20)} \\ \sigma^{(20)}_{(02)} &= q\sigma^{(02)}_{(20)} \\ \sigma^{(11)}_{(11)} &= \phi(q)^{-1} \prod (1+q^n) \end{aligned} \qquad (4.15a)$$

$A^{(2)}_2$:

$$\begin{aligned} \sigma^{(02)}_{(02)} &= \phi(q^{10})\phi(q)^{-2} \prod_{\pm 2, \pm 6, \pm 8(20)} (1-q^n) \\ \sigma^{(02)}_{(10)} &= \phi(q^{10})\phi(q)^{-2} \prod_{\pm 3, \pm 4, \pm 7(20)} (1-q^n) \\ \sigma^{(10)}_{(02)} &= q\phi(q^{10})\phi(q)^{-2} \prod_{\pm 2, \pm 4, \pm 6(20)} (1-q^n) \\ \sigma^{(10)}_{(10)} &= \phi(q^{10})\phi(q)^{-2} \prod_{\pm 1, \pm 8, \pm 9(20)} (1-q^n) \end{aligned} \qquad (4.15b)$$

$A_2^{(1)}$:

$$\sigma^{(100)}_{(100)} = \phi(q)^{-2}$$

$$\sigma^{(200)}_{(200)} = \sigma^{(020)}_{(020)} = \sigma^{(002)}_{(002)}$$

$$= \phi(q^{10})^2\phi(q)^{-4}\Big(\prod_{\pm3,\pm3,\pm4(10)}(1-q^n) - 2q\prod_{\pm1,\pm2,\pm3(10)}(1-q^n)\Big)$$

$$\sigma^{(200)}_{(011)} = q\sigma^{(020)}_{(101)} = q\sigma^{(002)}_{(110)}$$

$$= q\phi(q^2)\phi(q^{10})\phi(q)^{-4}\prod_{\pm2(10)}(1-q^n) \tag{4.15c}$$

$$q\sigma^{(011)}_{(200)} = \sigma^{(101)}_{(020)} = \sigma^{(110)}_{(002)}$$

$$= q\phi(q^{10})^2\phi(q)^{-4}\Big(2\prod_{\pm1,\pm3,\pm4(10)}(1-q^n) + q\prod_{\pm1,\pm1,\pm2(10)}(1-q^n)\Big)$$

$$\sigma^{(011)}_{(011)} = \sigma^{(101)}_{(101)} = \sigma^{(110)}_{(110)}$$

$$= \phi(q^2)\phi(q^{10})\phi(q)^{-4}\prod_{\pm4(10)}(1-q^n)$$

$C_2^{(1)}$:

$$* \quad \sigma^{(100)}_{(100)} - q^{-1/2}\sigma^{(100)}_{(001)} = \phi(q)^{-2}\prod(1-q^{(2n-1)/2})$$

$$\sigma^{(100)}_{(100)} = \sigma^{(001)}_{(001)} \tag{4.15d}$$

$$\sigma^{(100)}_{(001)} = q\sigma^{(001)}_{(100)}$$

$$* \quad \sigma^{(010)}_{(010)} = \phi(q)^{-2}\prod(1+q^n)$$

$G_2^{(1)}$:

$$\sigma^{(100)}_{(100)} - q^{-1/3}\sigma^{(100)}_{(001)} = \phi(q)^{-3}\prod_{0,\pm2(5)}(1-q^{\frac{n}{3}})$$

$$\sigma^{(001)}_{(001)} - q^{1/3}\sigma^{(001)}_{(100)} = \phi(q)^{-3}\prod_{0,\pm1(5)}(1-q^{\frac{n}{3}}) \tag{4.15e}$$

$A_4^{(2)}$:

$$\sigma^{(001)}_{(001)} = \phi(q)^{-2} \tag{4.15f}$$

$D_3^{(2)}$:

$$\sigma^{(100)}_{(100)} = \sigma^{(001)}_{(001)} = \phi(q)^{-1}\phi(q^2)^{-1}$$

$$* \quad \sigma^{(101)}_{(101)} = \phi(q)^{-2}\prod(1+q^n) \tag{4.15g}$$

$D_4^{(3)}$:

$$\sigma_{(100)}^{(100)} = \phi(q)^{-1}\phi(q^3)^{-1}$$

$$*\sigma_{(200)}^{(200)} - q^{-1/2}\sigma_{(010)}^{(200)} = \phi(q)^{-1}\phi(q^3)^{-1}\prod(1+q^{2n})(1+q^{6n-3})(1-q^{(2n-1)/2})(1-q^{(6n-3)/2})$$

$$*\sigma_{(010)}^{(010)} - q^{1/2}\sigma_{(200)}^{(010)} = \phi(q)^{-1}\phi(q^3)^{-1}\prod(1+q^{2n+1})(1+q^{6n})(1-q^{(2n-1)/2})(1-q^{(6n-3)/2}) \qquad (4.15h)$$

4.4 Further computation of string functions

For large order matrices it is impractical to invert κ by the method of minors and cofactors because it is quite difficult to simplify combinations of infinite products. Whenever $| P_{max} | \geq 3$ we shall instead resort to directly fitting the weight multiplicities tabulated in **[KMPS]** in the case of untwisted affine algebras or from our program for all low rank affine algebras to various forms of the required weight multiplicity generating functions. Using any algebraic package such as Maple some of the string functions can be fitted quite easily. These are the string functions which consists only of a single infinite product. To illustrate the method let us obtain the string functions of level 2 module of $D_3^{(2)}$. From the numerical values of weight multiplicities we find

$$\begin{aligned}
\sigma_{(010)}^{(002)} &= \phi(q^3)\phi(q^4)\phi(q^{12})\phi(q)^{-2}\phi(q^2)^{-2}\phi(q^6)^{-1} \\
\sigma_{(010)}^{(010)} &= \phi(q^4)^2\phi(q^6)^5\phi(q)^{-1}\phi(q^2)^{-4}\phi(q^3)^{-2}\phi(q^{12})^{-2} \\
\sigma_{(200)}^{(010)} &= \phi(q^{12})^2\phi(q^2)^2\phi(q)^{-3}\phi(q^4)^{-2}\phi(q^6)^{-1} \\
\sigma_{(002)}^{(200)} &= q^2\sigma_{(200)}^{(002)} \\
\sigma_{(010)}^{(200)} &= q\sigma_{(010)}^{(002)} \\
\sigma_{(200)}^{(200)} &= \sigma_{(002)}^{(002)}.
\end{aligned} \qquad (4.16)$$

The modular characteristic of these string functions can be checked to be consistent with that given in Table 4.1e. It then just remain to determine the string functions $\sigma_{(002)}^{(002)}$ and $\sigma_{(200)}^{(002)}$. These remaining string functions cannot be obtained so easily because they may consist of a sum of infinite products. In this case the following proposition **[Kac4]** is very helpful in doing the fitting.

Proposition 4.4. *Let $b_1, b_2, \dots$ be a periodic sequence of integers with period m, such that $b_j = b_{m-j}$ for $j = 1, \dots, m-1$. Set $b = b_1 + b_2 + \dots + b_m$. Then*

$$q^c \prod_{j=1}^{\infty} (1 - q^j)^{b_j}$$

is a modular form (for $\Gamma(n)$, for some n) if and only if the modular characteristic c is given by:

$$c = \frac{bm}{24} - \frac{1}{4m} \sum_{j=1}^{m-1} j(m-j) b_j \,.$$

In particular this proposition implies that $\phi(q^r) \prod_{\pm a(r)} (1 - q^n)$ has modular characteristic $(2a - r)^2/8r$ since $m = r$ and the only non vanishing b_i's are $b_a = b_{m-a} = 1$, $b_m = 1$. The period m in the above proposition can be expected to be the maximun value of k of the form $\prod_{\pm a(k)} (1 - q^n)$ appearing in κ obtained at the end of Section 4.2. With this value of m and modular characteristic Table 4.1a - 4.1f we can generate b_i's that satisfies the Proposition 4.4. There will certainly be an enormous number of different sets of b_i's but it is sometimes the case that by sheer 'good luck' we are able to see how to combine some of them to give the required string functions.

The string functions $\sigma^{(002)}_{(002)}$ and $\sigma^{(002)}_{(200)}$ of $D_3^{(2)}$ generated by our program are

$$\sigma^{(002)}_{(002)} = 1 + q + 5q^2 + 8q^3 + 24q^4 + 39q^5 + 90q^6 + 147q^7 + 297q^8 + 477q^9 + 880q^{10} + 1391q^{11} + 2412q^{12} + \dots$$

$$\sigma^{(002)}_{(200)} = 2 + 3q + 11q^2 + 18q^3 + 47q^4 + 77q^5 + 165q^6 + 268q^7 + 516q^8 + 823q^9 + 1468q^{10} + 2300q^{11} + 3891q^{12} + \dots$$

From the string functions obtained in (4.16) the values of b_i of Proposition 4.4 are in the range of -5 to -1. Another more important observation is that the values for b and b_m are constant for all string functions associated with a given affine algebra. We conjecture that this is also true for other affine algebras and we tabulate these constant for various algebras in Table 4.4.

Table 4.3. : Some parameters arising in fitting string functions of level 2 modules.

Algebra	period m	b	b_m
$A_1^{(1)}$	16	-24	-1
$A_2^{(2)}$	20	-32	-1
$A_2^{(1)}$	10	-32	-2
$C_2^{(1)}$	40	-160	-2
$G_2^{(1)}$	18	-84	-2
$A_4^{(2)}$	28	-96	-2
$D_3^{(2)}$	12	-30	-2
$D_4^{(3)}$	12	-28	-2

Hence on restricting the values of b_i and letting $m = 12$, $b = -30$ and $b_{12} = -2$ we obtained the following possibilities for b_i's in the case of $D_3^{(2)}$.

Modular characteristic= −5/24

	b_1	b_2	b_3	b_4	b_5	b_6
	−4	−2	−1	−1	−4	−4
	−2	−5	−1	−2	−2	−4
	−3	−2	−4	−1	−2	−4
	−3	−3	−1	−4	−1	−4
	−3	−3	−2	−1	−4	−2
	−4	−1	−2	−3	−3	−2
	−2	−3	−5	−1	−2	−2
*	−2	−4	−2	−4	−1	−2
	−3	−1	−5	−3	−1	−2

Modular characteristic= 19/24

	b_1	b_2	b_3	b_4	b_5	b_6
	−3	−2	−1	−1	−5	−4
	−1	−5	−1	−2	−3	−4
	−2	−2	−4	−1	−3	−4
*	−2	−3	−1	−4	−2	−4
	−2	−3	−2	−1	−5	−2
	−3	−1	−2	−3	−4	−2
	−1	−3	−5	−1	−3	−2
*	−1	−4	−2	−4	−2	−2
	−2	−1	−5	−3	−2	−2

By combining line 4 and 8 of the second table we can fit our $D_3^{(2)}$ data to the string function $\sigma_{(200)}^{(002)}$, i.e.

$$\begin{aligned}\sigma_{(200)}^{(002)} &= \phi(q^{12})^{-2} \prod_{\pm 3(12)} (1-q^n)^{-1} \prod_{\pm 1, \pm 5(12)} (1-q^n)^{-2} \prod_{\pm 2(12)} (1-q^n)^{-3} \prod_{\pm 4, 6(12)} (1-q^n)^{-4} \\ &\quad + \phi(q^{12})^{-2} \prod_{\pm 1(12)} (1-q^n)^{-1} \prod_{\pm 3, \pm 5, 6(12)} (1-q^n)^{-2} \prod_{\pm 2, \pm 4(12)} (1-q^n)^{-4} \\ &= \frac{\phi(q^{12})^2}{\phi(q)^2\phi(q^2)^2} \Big(\prod_{\pm 2, \pm 3(12)} (1-q^n) + \prod_{\pm 2, 6, 6(12)} (1-q^n) \Big).\end{aligned}$$

$\sigma_{(002)}^{(002)}$ can be obtained by combining line 8 of the first table and line 4 of the second table,

$$\begin{aligned}\sigma_{(002)}^{(002)} &= \phi(q^{12})^{-2} \prod_{\pm 5(12)} (1-q^n)^{-1} \prod_{\pm 1, \pm 3, 6(12)} (1-q^n)^{-2} \prod_{\pm 2, \pm 4(12)} (1-q^n)^{-4} \\ &\quad - q\phi(q^{12})^{-2} \prod_{\pm 3(12)} (1-q^n)^{-1} \prod_{\pm 1, \pm 5(12)} (1-q^n)^{-2} \prod_{\pm 2(12)} (1-q^n)^{-3} \prod_{\pm 4, 6(12)} (1-q^n)^{-4} \\ &= \frac{\phi(q^{12})^2}{\phi(q)^2\phi(q^2)^2} \Big(\prod_{\pm 5, 6, 6(12)} (1-q^n) - q \prod_{\pm 2, \pm 3(12)} (1-q^n) \Big).\end{aligned}$$

Taken in conjunction with (4.16) these results represent a strikingly simple form for the weight multiplicity generating functions of level 2 modules of $D_3^{(2)}$.

Below we give the string functions for other level 2 modules of the remaining rank 2 affine algebras. It must be admitted that not all of the string functions which consist of sum of infinite products are unambiguously obtained by the method discussed above because of the enormous range of possibilities. But some are obtained instead through the expansion and simplification of the terms arising from minors and cofactors. Further simplification is not out of the question but it would be difficult to pursue this method for higher level cases.

Level 2 (class 0) modules of $C_2^{(1)}$:

$$\sigma^{(200)}_{(200)} = \sigma^{(002)}_{(002)} = \prod_{\pm2,\pm4(20)} (1-q^n) \prod_{\pm12,\pm16(40)} (1-q^n) f_1 - q f_2$$

$$\sigma^{(200)}_{(020)} = q\sigma^{(002)}_{(020)} = \frac{q\phi(q^8)^2\phi(q^{20})^5}{\phi(q)^4\phi(q^4)\phi(q^{10})^2\phi(q^{40})^2} + \frac{q^3\phi(q^4)^5\phi(q^{40})^2}{\phi(q)^4\phi(q^2)^2\phi(q^8)^2\phi(q^{20})}$$

$$\sigma^{(200)}_{(101)} = q\sigma^{(002)}_{(101)} = q\phi(q^2)^2\phi(q^{10})^2\phi(q)^{-5}\phi(q^5)^{-1}$$

$$\sigma^{(200)}_{(002)} = q^2\sigma^{(002)}_{(200)} = q^2 f_2 - q^3 \prod_{\pm6,\pm8(20)} (1-q^n) \prod_{\pm4,\pm8(40)} (1-q^n) f_1$$

$$\sigma^{(020)}_{(002)} = q\sigma^{(020)}_{(200)} = q h_1(q) \prod_{\pm4,\pm16,\pm16(40)} (1-q^n) + q^2 h_2(q) \prod_{\pm4,\pm6,\pm14(40)} (1-q^n)$$

$$\sigma^{(020)}_{(020)} = h_3(q) \prod_{\pm4,\pm16,\pm16(40)} (1-q^n) + q^2 h_4(q) \prod_{\pm4,\pm6,\pm14(40)} (1-q^n)$$

$$\sigma^{(020)}_{(101)} = \phi(q^2)^2\phi(q^5)^2\phi(q)^{-6} \prod_{\pm2,\pm2,\pm3(10)} (1-q^n)$$

$$\sigma^{(101)}_{(002)} = q\sigma^{(101)}_{(200)} = q h_2(q) \prod_{\pm2,\pm12,\pm18(40)} (1-q^n) + q h_1(q) \prod_{\pm8,\pm8,\pm12(40)} (1-q^n)$$

$$\sigma^{(101)}_{(020)} = q h_3(q) \prod_{\pm8,\pm8,\pm12(40)} (1-q^n) + q h_4(q) \prod_{\pm2,\pm12,\pm18(40)} (1-q^n)$$

$$\sigma^{(101)}_{(101)} = \phi(q^2)^2\phi(q^5)^2\phi(q)^{-6} \prod_{\pm1,\pm4,\pm4(10)} (1-q^n)$$

where

$$\begin{aligned} f_1(q) =& \phi(q^{10})^7\phi(q)^{-5}\phi(q^2)^{-3}\phi(q^5)^{-1}\Big(\prod_{\pm3(10)} (1-q^n) \prod_{\pm4(10)} (1-q^n)^4 \prod_{\pm2(20)} (1-q^n) \\ &- q \prod_{\pm1(10)} (1-q^n) \prod_{\pm2(10)} (1-q^n)^4) \prod_{\pm6(20)} (1-q^n)\Big) \\ f_2(q) =& \phi(q^{10})^5\phi(q)^{-5}\phi(q^2)^{-1}\phi(q^5)^{-1}\Big(\prod_{\pm1,\pm2(10)} (1-q^n) \prod_{\pm6,\pm8(20)} (1-q^n) \\ &- q \prod_{\pm3,\pm4(10)} (1-q^n) \prod_{\pm2,\pm4(20)} (1-q^n)\Big) \\ &+ \phi(q^5)\phi(q^{10})\phi(q^{20})^3\phi(q)^{-5}\phi(q^2)^{-1}\phi(q^4)^{-1}\Big(\prod_{\pm4(10)} (1-q^n)^3 \prod_{\pm8(20)} (1-q^n) \\ &- q^2 \prod_{\pm2(10)} (1-q^n)^3 \prod_{\pm4(20)} (1-q^n)\Big) \end{aligned}$$

$$h_1(q) = 2\phi(q^8)^2\phi(q^{20})^2\phi(q)^{-4}\phi(q^4)^{-2}$$

$$h_2(q) = \phi(q^4)^6\phi(q^{10})\phi(q)^{-4}\phi(q^2)^{-3}\phi(q^8)^{-2}$$

$$h_3(q) = \phi(q^4)^4\phi(q^{20})^2\phi(q)^{-4}\phi(q^2)^{-2}\phi(q^8)^{-2}$$

$$h_4(q) = 2\phi(q^8)^2\phi(q^{10})\phi(q)^{-4}\phi(q^2)^{-1}$$

Level 2 (class 1) modules of $C_2^{(1)}$

$$\sigma^{(011)}_{(011)} = \sigma^{(110)}_{(110)} = \phi(q^4)^5\phi(q)^{-5}\phi(q^8)^{-2}$$

$$\sigma^{(110)}_{(011)} = q\sigma^{(011)}_{(110)} = 2q\phi(q^2)^2\phi(q^8)^2\phi(q)^{-5}\phi(q^4)^{-1}$$

Level 2 modules of $G_2^{(1)}$.

$$\sigma^{(002)}_{(002)} = h_4(q)\prod_{\pm 3(9)}(1-q^n)$$

$$\sigma^{(002)}_{(010)} = h_1(q)\prod_{\pm 3(9)}(1-q^n)$$

$$\sigma^{(002)}_{(101)} = h_2(q)\prod_{\pm 3(9)}(1-q^n)$$

$$\sigma^{(002)}_{(200)} = h_3(q)\prod_{\pm 3(9)}(1-q^n)$$

$$\sigma^{(010)}_{(002)} = qh_3(q)\prod_{\pm 2(9)}(1-q^n)$$

$$\sigma^{(010)}_{(010)} = h_1(q)\prod_{\pm 4(9)}(1-q^n) - qh_2(q)\prod_{\pm 1(9)}(1-q^n)$$

$$\sigma^{(010)}_{(101)} = h_1(q)\prod_{\pm 2(9)}(1-q^n)$$

$$\sigma^{(010)}_{(200)} = h_3(q)\prod_{\pm 4(9)}(1-q^n) - h_4(q)\prod_{\pm 1(9)}(1-q^n)$$

$$\sigma^{(101)}_{(002)} = qh_3(q)\prod_{\pm 4(9)}(1-q^n)$$

$$\sigma^{(101)}_{(010)} = qh_1(q)\prod_{\pm 1(9)}(1-q^n) + qh_2(q)\prod_{\pm 2(9)}(1-q^n)$$

$$\sigma^{(101)}_{(101)} = h_1(q)\prod_{\pm 4(9)}(1-q^n)$$

$$\sigma^{(101)}_{(200)} = h_4(q)\prod_{\pm 2(9)}(1-q^n) + qh_3(q)\prod_{\pm 1(9)}(1-q^n)$$

$$\sigma^{(200)}_{(002)} = q^2h_3(q)\prod_{\pm 1(9)}(1-q^n)$$

$$\sigma^{(200)}_{(010)} = qh_2(q)\prod_{\pm 4(9)}(1-q^n) - qh_1(q)\prod_{\pm 2(9)}(1-q^n)$$

$$\sigma^{(200)}_{(101)} = qh_1(q)\prod_{\pm 1(9)}(1-q^n)$$

$$\sigma^{(200)}_{(200)} = h_4(q)\prod_{\pm 4(9)}(1-q^n) - qh_3(q)\prod_{\pm 2(9)}(1-q^n)$$

where

$$h_1(q) = \phi(q^2)^3\phi(q^3)^2\phi(q^9)\phi(q)^{-7}\phi(q^6)^{-1}$$

$$h_2(q) = 2\phi(q^2)^2\phi(q^6)^2\phi(q^9)\phi(q)^{-6}\phi(q^3)^{-1}$$

$$h_3(q) = 3\phi(q^6)^3\phi(q^9)\phi(q)^{-5}\phi(q^2)^{-1}$$

$$h_4(q) = \frac{\phi(q^9)\phi(q^{18})^3}{\phi(q)^5\phi(q^2)}\Big(\prod_{\pm 8(18)}(1-q^n)^3 - q^2\prod_{\pm 4(18)}(1-q^n)^3 + 6q^2\prod_{\pm 2(6)}(1-q^n) - q^4\prod_{\pm 2(18)}(1-q^n)^3\Big)$$

Level 2 modules of $A_4^{(2)}$

$$\sigma_{(002)}^{(002)} = \prod_{\pm 6,\pm 8,\pm 12(28)}(1-q^n)f(q) + q^2\prod_{\pm 4,\pm 4,\pm 10(28)}(1-q^n)f(q)$$

$$\sigma_{(010)}^{(002)} = \phi(q^2)^2\phi(q^7)\phi(q)^{-5}\prod_{\pm 2,\pm 3,\pm 5(14)}(1-q^n)$$

$$\sigma_{(100)}^{(002)} = \prod_{\pm 4(14)}(1-q^n)h_1(q) + \prod_{\pm 6(14)}(1-q^n)h_3(q)$$

$$\sigma_{(002)}^{(010)} = q\prod_{\pm 2,\pm 12,\pm 12(28)}(1-q^n)f(q) + q\prod_{\pm 4,\pm 8,\pm 10(28)}(1-q^n)f(q)$$

$$\sigma_{(010)}^{(010)} = \phi(q^2)^2\phi(q^7)\phi(q)^{-5}\prod_{\pm 1,\pm 5,\pm 6(14)}(1-q^n)$$

$$\sigma_{(100)}^{(010)} = \prod_{\pm 4(14)}(1-q^n)h_2(q) + q\prod_{\pm 2(14)}(1-q^n)h_3(q)$$

$$\sigma_{(002)}^{(100)} = q\prod_{\pm 6,\pm 8,\pm 8(28)}(1-q^n)f(q) - q^3\prod_{\pm 2,\pm 4,\pm 12(28)}(1-q^n)f(q)$$

$$\sigma_{(010)}^{(100)} = \phi(q^2)^2\phi(q^7)\phi(q)^{-5}\prod_{\pm 1,\pm 3,\pm 4(14)}(1-q^n)$$

$$\sigma_{(100)}^{(100)} = \prod_{\pm 6(14)}(1-q^n)h_2(q) - q\prod_{\pm 2(14)}(1-q^n)h_1(q)$$

where

$$f(q) = \phi(q^2)\phi(q^7)\phi(q^{14})\phi(q)^{-5}\prod_{\pm 1,\pm 3,\pm 5(14)}(1-q^n)\prod_{\pm 2,\pm 6,\pm 10(28)}(1-q^n)$$

$$h_1(q) = \phi(q^{14})^2\phi(q)^{-4}\prod_{\pm 1,\pm 3,\pm 4(14)}(1-q^n)\prod_{\pm 4,\pm 6(14)}(1+q^n)$$

$$h_2(q) = \phi(q^{14})^2\phi(q)^{-4}\prod_{\pm 1,\pm 5,\pm 6(14)}(1-q^n)\prod_{\pm 2,\pm 6(14)}(1+q^n)$$

$$h_3(q) = \phi(q^{14})^2\phi(q)^{-4}\prod_{\pm 2,\pm 3,\pm 5(14)}(1-q^n)\prod_{\pm 2,\pm 4(14)}(1+q^n)$$

This complete the level 2 calculation for $C_2^{(1)}$, $G_2^{(1)}$, $A_4^{(2)}$ and $D_3^{(2)}$.

CHAPTER 5

The sets $\{W : \overline{W}\}$ and the actions of their elements

5.1. Specialisation of the Weyl-Kostant-Liu character formula

With reference to Section 1.6., let $\mathcal{G}(A)$ be an affine algebra of rank r with Cartan subalgebra $\mathcal{H}$. Let $U = \{1, 2, \ldots, r\} \subset I = \{0, 1, \ldots, r\}$. Then $\mathcal{G}_U$ is isomorphic to the simple finite-dimensional Lie algebra $\mathcal{G}(\bar{A})$ which we will denoted by $\bar{\mathcal{G}}$. As a consequence of this we will replace all terms in Section 1.6. with a subscript U by corresponding barred symbols. In particular,

$$W(U) = \{w \in W \mid \Phi_w \subset \Delta^+ \backslash \bar{\Delta}^+\}$$

$$\bar{P}^+ = \{\lambda \in \mathcal{H}^* \mid < \lambda, \alpha_i^\vee > \in \mathbb{Z}^+ \quad \text{for } i = 1, \ldots, r\},$$

where W is the affine Weyl group. By Lemma 1.14, $W(U) = \{W : \overline{W}\}$ is the set of right coset representatives of W with respect to the finite Weyl group $\overline{W}$. Then for any $w \in W$, we may write

$$w = \bar{w} w', \tag{5.1}$$

where $\bar{w} \in \bar{W}$ and $w' \in \{W : \overline{W}\}$.

Lemma 3.6 and (3.29) implies that for any $\lambda \in \mathcal{H}^*$ we have

$$\begin{aligned} \lambda &= n\delta + \sum_{i=0}^{r} \lambda_i \Lambda_i \\ &= n\delta + \sum_{i=0}^{r} \lambda_i (\frac{c_i^\vee}{c_0^\vee} \Lambda_0 + \bar{\Lambda}_i) \\ &= \frac{L(\lambda)}{c_0^\vee} \Lambda_0 + n\delta + \sum_{i=1}^{r} \lambda_i \bar{\Lambda}_i \\ &= \frac{L(\lambda)}{c_0^\vee} \Lambda_0 + n\delta + \bar{\lambda}, \end{aligned} \tag{5.2}$$

where $\bar{\lambda} \equiv \sum_{i=1}^{r} \lambda_i \bar{\Lambda}_i$. It should be noted that from (3.30) $\bar{w}(\rho) = (g/c_0^\vee)\Lambda_0 + \bar{w}(\bar{\rho})$ so that

$$\bar{w}(\rho) - \rho = \bar{w}(\bar{\rho}) - \bar{\rho}\,. \tag{5.3}$$

Lemma 5.1. *The denominator D of the Weyl-Kostant-Liu character formula (1.25) can be written as*

$$D = \sum_{w' \in \{W:\overline{W}\}} \varepsilon(w')\, ch\, \overline{V}^{w'(\rho)-\rho} = \prod_{\alpha\in\Delta+\setminus\bar{\Delta}+} (1 - e^{-\alpha})^{mult\,\alpha}$$

Proof First note that the Weyl-Kac denominator identity is given by

$$\sum_{w\in W} \varepsilon(w)\, e^{w(\rho)-\rho} = \prod_{\alpha\in\Delta+} (1 - e^{-\alpha})^{mult\,\alpha}$$

and the original Weyl denominator identity is

$$\sum_{\bar{w}\in\bar{W}} \varepsilon(\bar{w})\, e^{\bar{w}(\bar{\rho})-\bar{\rho}} = \prod_{\alpha\in\bar{\Delta}+} (1 - e^{-\alpha}).$$

Then the above identities together with Weyl character formula (1.19) and (5.3) imply that

$$\begin{aligned}
\sum_{w' \in \{W:\overline{W}\}} \varepsilon(w')\, ch\, \overline{V}^{w'(\rho)-\rho} &= \frac{\sum_{w'\in\{W:\overline{W}\}} \varepsilon(w') \sum_{\bar{w}\in\bar{W}} \varepsilon(\bar{w}) e^{\bar{w}(w'(\rho)-\rho+\bar{\rho})-\bar{\rho}}}{\sum_{\bar{w}\in\bar{W}} \varepsilon(\bar{w}) e^{\bar{w}(\bar{\rho})-\bar{\rho}}} \\
&= \frac{\sum_{w'\in\{W:\overline{W}\}} \sum_{\bar{w}\in\bar{W}} \varepsilon(\bar{w}w') e^{\bar{w}w'(\rho)-\rho}}{\prod_{\alpha\in\bar{\Delta}+}(1 - e^{-\alpha})} \\
&= \frac{\sum_{w\in W} \varepsilon(w) e^{w(\rho)-\rho}}{\prod_{\alpha\in\bar{\Delta}+}(1 - e^{-\alpha})} \\
&= \prod_{\alpha\in\Delta+\setminus\bar{\Delta}+} (1 - e^{-\alpha})^{mult\,\alpha}.
\end{aligned}$$

□

Proposition 5.2. *Let* $D = \sum_{w'\in\{W:\overline{W}\}} \varepsilon(w')\, ch\, \overline{V}^{w'(\rho)-\rho}$. *Then for each infinite series of rank dependent affine algebras we have:*

$$A_r^{(1)} : \quad D = \textstyle\sum_{\xi\in F} (-1)^{|\xi|} q^{|\xi|} \{\bar{\xi};\xi'\}(x)_{r+1}, \tag{5.4a}$$

$$B_r^{(1)} : \quad D = \textstyle\sum_{\alpha\in A} (-1)^{|\alpha|/2} q^{|\alpha|/2} [\alpha](x)_{2r+1}, \tag{5.4b}$$

$$C_r^{(1)} : \quad D = \textstyle\sum_{\gamma\in C} (-1)^{|\gamma|/2} q^{|\gamma|/2} < \gamma > (x)_{2r}, \tag{5.4c}$$

$$D_r^{(1)} : \quad D = \textstyle\sum_{\alpha\in A} (-1)^{|\alpha|/2} q^{|\alpha|/2} [\alpha](x)_{2r}, \tag{5.4d}$$

$$A_{2r-1}^{(2)} : \quad D = \textstyle\sum_{\alpha\in A} (-1)^{|\alpha|/2} q^{|\alpha|/2} < \alpha > (x)_{2r}, \tag{5.4e}$$

$$D_{r+1}^{(2)}: \quad D = \textstyle\sum_{\epsilon \in E} (-1)^{(|\epsilon|+p)/2} q^{|\epsilon|} [\epsilon](x)_{2r+1}, \tag{5.4f}$$

$$A_{2r}^{(2)}: \quad D = \textstyle\sum_{\gamma \in C} (-1)^{|\gamma|/2} q^{|\gamma|/2} [\gamma](x)_{2r+1}. \tag{5.4g}$$

Proof First we need the change of basis from $\{\alpha_0, \alpha_1, \ldots, \alpha_r\}$ to $\{\delta, \epsilon_1, \ldots, \epsilon_r\}$ for each affine algebra and this is given as follows **[Ma]** :

$$\begin{aligned}
A_r^{(1)}: &\quad \alpha_0 = \delta + \epsilon_{r+1} - \epsilon_1,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r).\\
B_r^{(1)}: &\quad \alpha_0 = \delta - \epsilon_1 - \epsilon_2,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r-1),\ \alpha_r = \epsilon_r\\
C_r^{(1)}: &\quad \alpha_0 = \delta - 2\epsilon_1,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r-1),\ \alpha_r = 2\epsilon_r\\
D_r^{(1)}: &\quad \alpha_0 = \delta - \epsilon_1 - \epsilon_2,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r-1),\ \alpha_r = \epsilon_{r-1} + \epsilon_r\\
A_{2r-1}^{(2)}: &\quad \alpha_0 = \delta - \epsilon_1 - \epsilon_2,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r-1),\ \alpha_r = 2\epsilon_r\\
D_{r+1}^{(2)}: &\quad \alpha_0 = \delta - \epsilon_1,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r-1),\ \alpha_r = \epsilon_r\\
A_{2r}^{(2)}: &\quad \alpha_0 = \delta - 2\epsilon_1,\ \alpha_i = \epsilon_i - \epsilon_{i+1}\,(1 \le i \le r-1),\ \alpha_r = \epsilon_r\,.
\end{aligned} \tag{5.5}$$

We will give the proof for the case $A_r^{(1)}$. The proof for the other cases is similar. From Proposition 3.1 and Proposition 3.2 it can be deduced that the positive affine roots of $\Delta^+ \backslash \bar{\Delta}^+$ with multiplicity 1 are $n\delta \pm (\epsilon_i - \epsilon_j)$ for $n > 0$ and $1 \le i < j \le r+1$ and with multiplicity r are $n\delta$ for $n > 0$. Hence by Lemma 5.1 we have

$$\begin{aligned}
D &= \prod_{n=1}^{\infty} (1 - e^{-n\delta})^r \prod_{1 \le i < j \le r+1} (1 - e^{\epsilon_i - \epsilon_j - n\delta})(1 - e^{-\epsilon_i + \epsilon_j - n\delta})\\
&= \prod_{n=1}^{\infty} (1 - q^n)^r \prod_{1 \le i < j \le r+1} (1 - q^n x_i x_j^{-1})(1 - q^n x_i^{-1} x_j)\\
&= \prod_{n=1}^{\infty} \Big(\prod_{1 \le i,j \le r+1} (1 - q^n x_i x_j^{-1}) \Big) / (1 - q^n)\,.
\end{aligned}$$

It then follows from (2.9a) that

$$D = \sum_{\xi \in F} (-1)^{|\xi|} q^{|\xi|} \{\bar{\xi}; \xi'\}(x)_{r+1}$$

where $x_i = e^{\epsilon_i}$ and $q = e^{-\delta}$. □

As emphasised in Section 2.4 if the irreducible characters are not in the standard form for a particular r then we have to apply modification rules.

5.2. The right coset representatives of W with respect to $\overline{W}$ for $A_r^{(1)}$

We have yet to determine the set of right coset representatives $\{W : \overline{W}\}$. Let us work first with the affine algebras $A_r^{(1)}$. Consider the identity (5.4a) obtained in the previous section

$$\sum_{w' \in \{W:\overline{W}\}} \varepsilon(w') ch\, \overline{V}^{w'(\rho)-\rho} = \sum_{\xi \in F} (-1)^{|\xi|} q^{|\xi|} \{\bar{\xi}; \xi'\}(x)_{r+1}, . \tag{5.6}$$

For all a and b such that $0 \leq a \leq r$ and $0 \leq b \leq r$ let

$$w_{\binom{a}{b}} = \begin{cases} s_0 & \text{if } a = b = 0, \\ s_0 s_1 s_2 \cdots s_a & \text{if } 0 < a < r \text{ and } b = 0, \\ s_0 s_r s_{r-1} \cdots s_{r-b+1} & \text{if } a = 0 \text{ and } 0 < b \leq r, \\ s_0 s_1 s_2 \cdots s_a s_r s_{r-1} \cdots s_{r-b+1} & \text{if } 0 < a < r \text{ and } 0 < b \leq r. \end{cases} \tag{5.7}$$

We now compute $w_{\binom{a}{b}}(\rho) - \rho$ for a few cases to see the motivation for introducing these Weyl group elements. For $a + b + 1 \leq r$ the results are given in Table 5.1. From this table we observe that for large r they systematically give a contribution of the required form to (5.6) in the sense that $w_{\binom{a}{b}}(\rho) - \rho \in \bar{P}^+$. If $w' = w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}$ we might expect from these examples that

$$\varepsilon(w') ch\, \overline{V}^{w'(\rho)-\rho} = (-1)^{|\xi|} q^{|\xi|} \{\bar{\xi}; \xi'\}$$

where ξ has partition label $\binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$. However for small r, i.e. when $a_1 + b_1 \geq r$, the right hand side of (5.6) has to be replaced by (2.12a) where modification rules have been taken into consideration. In general the elements of $\{W : \overline{W}\}$ are not in one-to-one correspondence with the partitions $\binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$. Before we arrive at the general result we need the following Lemma which can be proved by direct calculation.

Table 5.1. : Some results from the action $w_{\binom{a}{b}}(\rho) - \rho$

w	$w(\rho) - \rho$	A_r character	depth
s_0	$-\alpha_0$	$\{\overline{1}; 1\}$	1
$s_0 s_1$	$-2\alpha_0 - \alpha_1$	$\{\overline{2}; 1^2\}$	2
$s_0 s_r$	$-2\alpha_0 - \alpha_r$	$\{\overline{1^2}; 2\}$	2
$s_0 s_1 s_2$	$-3\alpha_0 - 2\alpha_1 - \alpha_2$	$\{\overline{3}; 1^3\}$	3
$s_0 s_1 s_r$	$-3\alpha_0 - \alpha_1 - \alpha_r$	$\{\overline{21}; 21\}$	3
$s_0 s_r s_{r-1}$	$-3\alpha_0 - 2\alpha_r - \alpha_{r-1}$	$\{\overline{1^3}; 3\}$	3
$s_0 s_1 s_2 s_3$	$-4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	$\{\overline{4}; 1^4\}$	4
$s_0 s_1 s_2 s_r$	$-4\alpha_0 - 2\alpha_1 - \alpha_2 - \alpha_r$	$\{\overline{31}; 21^2\}$	4
$s_0 s_1 s_r s_0$	$-4\alpha_0 - 2\alpha_1 - 2\alpha_r$	$\{\overline{2^2}; 2^2\}$	4
$s_0 s_1 s_r s_{r-1}$	$-4\alpha_0 - \alpha_1 - 2\alpha_r - \alpha_{r-1}$	$\{\overline{21^2}; 31\}$	4
$s_0 s_r s_{r-1} s_{r-2}$	$-4\alpha_0 - 3\alpha_r - 2\alpha_{r-1} - \alpha_{r-2}$	$\{\overline{1^4}; 4\}$	4

Lemma 5.3. *Let α_i be a simple root and $a+b+1 \le r$. Then*

$$w_{\binom{a}{b}}(\alpha_i) = \begin{cases} \alpha_0+\alpha_1+\alpha_r & i=0, \\ \alpha_{i+1} & 1 \le i \le a-1, \\ -(\alpha_0+\alpha_1+\ldots+\alpha_a) & i=a, \\ \alpha_0+\alpha_1+\ldots+\alpha_{a+1} & i=a+1<r-b, \\ \alpha_i & a+2 \le i \le r-b-1, \\ \alpha_0+\alpha_r+\alpha_{r-1}+\ldots+\alpha_{r-b} & i=r-b>a+1, \\ -(\alpha_0+\alpha_r+\alpha_{r-1}+\ldots+\alpha_{r+1-b}) & i=r+1-b, \\ \alpha_{i-1} & i \ge r+2-b. \end{cases}$$

In the limiting case $i=a+1=r-b$, $w_{\binom{a}{b}}(\alpha_i) = w_{\binom{a}{b}}(\alpha_{a+1}) = \alpha_0+\delta$.

Lemma 5.4. *With the situation as in Lemma 5.3.*

$$i)\quad w_{\binom{a}{b}}(\alpha_0+\alpha_1+\ldots+\alpha_i) = \begin{cases} \alpha_0+\alpha_1+\ldots+\alpha_{i+1}+\alpha_r & 0 \le i \le a-1, \\ \alpha_r & i=a, \\ \alpha_0+\alpha_1+\ldots+\alpha_i+\alpha_r & a+1 \le i \le r-b-1, \\ \alpha_0+\alpha_r+\delta & i=r-b, \\ \alpha_0+\alpha_1+\ldots+\alpha_{i-1}+\alpha_r & r-b+1 \le i < r, \\ \delta & i=r. \end{cases}$$

$$ii)\quad w_{\binom{a}{b}}(\alpha_r+\alpha_{r-1}+\ldots+\alpha_{r-i}) = \begin{cases} \alpha_{r-1}+\alpha_{r-2}+\ldots+\alpha_{r-i-1} & 0 \le i \le b-2, \\ -\alpha_0-\alpha_r & i=b-1, \\ \alpha_{r-1}+\alpha_{r-2}+\ldots+\alpha_{r-b} & b \le i \le r-a-2, \\ -\alpha_r+\delta & i=r-a-1, \\ \alpha_{r-1}+\alpha_{r-2}+\ldots+\alpha_{r-i+1} & r-a \le i < r \\ \delta & i=r. \end{cases}$$

Proof Using Lemma 5.3 and then direct verification for each case.

Proposition 5.5. *Let $a_1+b_1+1 \le r$. The elements of $\{W:\overline{W}\}$ of $A_r^{(1)}$ of length n include all*

$$w_\xi = w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}$$

such that in Frobenius notation ξ is the partition :

$$\xi = \begin{pmatrix} a_1 & a_2 & \dots & a_p \\ b_1 & b_2 & \dots & b_p \end{pmatrix}$$

with $a_1 > a_2 > \dots > a_p \geq 0$, $b_1 > b_2 > \dots > b_p \geq 0$ and $n = \sum_{i=1}^{p}(a_i + b_i + 1)$. The set of all these elements w_ξ will be called the core W_r of $\{W : \overline{W}\}$.

Proof We shall prove this by induction with respect to length n using Proposition 1.16.. Since $\Phi_{s_i} = \{\alpha_i\}$, then the only Φ_{s_i} which is a subset of $\Delta^+ \backslash \bar{\Delta}^+$ is Φ_{s_0}, so that the only element of $\{W : \overline{W}\}$ of length 1 is $s_0 = w_{\binom{0}{0}}$.

Next consider

$$s_0(\alpha_i) = \begin{cases} -\alpha_0 & \text{if } i = 0, \\ \alpha_0 + \alpha_1 \in \Delta^+ \backslash \bar{\Delta}^+ & \text{if } i = 1, \\ \alpha_i & \text{if } i = 2, \dots, r-1, \\ \alpha_0 + \alpha_r \in \Delta^+ \backslash \bar{\Delta}^+ & \text{if } i = r. \end{cases}$$

Then by Proposition 1.16 the elements of $\{W : \overline{W}\}$ of length 2 are $s_0 s_1 = w_{\binom{1}{0}}$ and $s_0 s_r = w_{\binom{0}{1}}$. Hence the Proposition is true for n=1 and 2.

Assume that the proposition is true for n. By hypothesis we have the following interval:

$$0 \leq a_p < a_{p-1} < \dots < a_1 \leq r - b_1 + 1 < r - b_2 + 1 < \dots < r - b_p + 1 \leq r.$$

By Lemma 1.7 and Proposition 1.16 we need to consider only those α_i that satisfies $w(\alpha_i) > 0$ and $w(\alpha_i) \in \Delta^+ \backslash \bar{\Delta}^+$.

If $i = 0$ then

$$w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \dots w_{\binom{a_p}{b_p}}(\alpha_0) = \begin{cases} < 0 & a_p = 0,\ b_p = 0, \\ \alpha_p \notin \Delta^+ \backslash \bar{\Delta}^+ & a_p \neq 0,\ b_p = 0, \\ \alpha_{r+1-p} \notin \Delta^+ \backslash \bar{\Delta}^+ & a_p = 0,\ b_p \neq 0, \\ \sum_{j=0}^{p} \alpha_j + \sum_{j=1}^{p} \alpha_{r+1-j} & a_p \neq 0,\ b_p \neq 0. \end{cases} \tag{5.8a}$$

For $1 \leq i \leq a_p - 1$, we have by Lemma 5.3 and Lemma 5.4

$$w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \dots w_{\binom{a_p}{b_p}}(\alpha_i) = \alpha_{i+p} \tag{5.8b}$$

If $i = a_p$ then $w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}(\alpha_i) < 0$.

If $i = a_1 + 1$ then

$$w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}(\alpha_{a_1+1}) = \begin{cases} \sum_{j=0}^{a_1+1} \alpha_j \in \Delta^+ \backslash \bar{\Delta}^+ & a_1 + b_1 < r - 1, \\ \alpha_0 + \delta \in \Delta^+ \backslash \bar{\Delta}^+ & a_1 + b_1 = r - 1. \end{cases}$$

If $a_1 + 2 \leq i \leq r - b_1 - 1$ then $w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}(\alpha_i) = \alpha_i \notin \Delta^+ \backslash \bar{\Delta}^+$.

If $i = r - b_1$ then

$$w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}(\alpha_{r-b_1}) = \begin{cases} \alpha_0 + \sum_{j=r-b_1}^{r} \alpha_j \in \Delta^+ \backslash \bar{\Delta}^+ & a_1 + b_1 < r - 1, \\ \alpha_0 + \delta \in \Delta^+ \backslash \bar{\Delta}^+ & a_1 + b_1 = r - 1. \end{cases}$$

If $i = r - b_p + 1$ then $w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}(\alpha_i) < 0$.

For $r - b_p + 2 \leq i \leq r$, we have

$$w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}(\alpha_i) = \alpha_{i-p}. \tag{5.8c}$$

We are then left with the following values of i to be considered.

$$a_p + 1 \leq i \leq a_1, \quad \text{and} \quad r - b_1 + 1 \leq i \leq r - b_p.$$

Let us partition the integer interval $a_p < i \leq a_1$ that is $(a_p, a_1]$ into

$$(a_p\,, a_{p-1}] \cup (a_{p-1}\,, a_{p-2}] \cup \ldots \cup (a_2\,, a_1]$$

and the integer interval $r - b_1 + 1 \leq i < r - b_p + 1$ into

$$[r - b_1 + 1\,, r - b_2 + 1) \cup [r - b_2 + 1\,, r - b_3 + 1) \cup \ldots \cup [r - b_{p-1} + 1\,, r - b_p + 1).$$

Consider a case $a_k < i \leq a_{k-1}$. If $a_{k-1} = a_k + 1$ then $i = a_k + 1$ only.

$$\begin{aligned} w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}}(\alpha_{a_k+1}) &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}}(\alpha_{a_k+1}) \\ &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}}(\alpha_0 + \alpha_1 + \ldots + \alpha_{a_k+1}) \\ &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}}(\alpha_r) \\ &= \ \vdots \\ &= \ \alpha_{r-k+2} \end{aligned}$$

If $a_{k-1} = a_k + j$, $j > 1$ then $i = a_k + 1, a_k + 2, \ldots, a_k + j$ and

$$w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}} (\alpha_{a_k+1})$$

$$= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} (\alpha_{a_k+1})$$

$$= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} (\alpha_0 + \alpha_1 + \ldots + \alpha_{a_k+1})$$

$$= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}} (\alpha_0 + \alpha_1 + \ldots + \alpha_{a_k+2} + \alpha_r)$$

$$= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-3}}{b_{k-3}}} (\alpha_0 + \alpha_1 + \ldots + \alpha_{a_k+3} + \alpha_r + \alpha_{r-1})$$

$$= \vdots$$

$$= \alpha_0 + \alpha_1 + \ldots + \alpha_{a_k+k} + \alpha_r + \alpha_{r-1} + \alpha_{r-k+2} \in \Delta^+ \backslash \bar{\Delta}^+.$$

While for $t = 2, 3, \ldots, j-1$

$$\begin{aligned} w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}} (\alpha_{a_k+t}) &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} (\alpha_{a_k+t}) \\ &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}} (\alpha_{a_k+t+1}) \\ &= \vdots \\ &= \alpha_{a_k+t+k-1} \end{aligned}$$

and

$$w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}} (\alpha_{a_k+j}) = w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} (\alpha_{a_k+j}) \ < \ 0.$$

Similarly, consider a case $r - b_{k-1} + 1 \leq i < r - b_k + 1$. If $b_{k-1} = b_k + 1$ then $i = r - b_k$ only.

$$\begin{aligned} w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}} (\alpha_{r-b_k}) &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} (\alpha_0 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{r-b_k}) \\ &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}} (\alpha_0 + \alpha_1 + \alpha_r - \alpha_0 - \alpha_r) \\ &= w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}} (\alpha_1) \\ &= \vdots \\ &= \alpha_{k-1} \end{aligned}$$

If $b_{k-1} = b_k + j,\ j > 1$ then $i = r - b_k - j + 1, r - b_k - j + 2, \ldots, r - b_k$ and

$$\begin{aligned}
&w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}}(\alpha_{r-b_k}) \\
=&w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}}(\alpha_0 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{r-b_k}) \\
=&w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}}(\alpha_0 + \alpha_1 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{r-b_k-1}) \\
=&w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-3}}{b_{k-3}}}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{r-b_k-2}) \\
=& \ \vdots \\
=& \ \alpha_0 + \alpha_1 + \ldots + \alpha_{k-1} + \alpha_r + \alpha_{r-1} + \alpha_{r+1-b_k-k} \in \Delta^+ \backslash \bar{\Delta}^+.
\end{aligned}$$

While for $t = 2, 3, \ldots, j-2$

$$\begin{aligned}
w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}}(\alpha_{r-b_k-t}) =&w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}}(\alpha_{r-b_k-t}) \\
=&w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-2}}{b_{k-2}}}(\alpha_{r-b_k-t-1}) \\
=& \ \vdots \\
=& \ \alpha_{r-b_k-t-k+1}
\end{aligned}$$

and

$$w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k}} \ldots w_{\binom{a_p}{b_p}}(\alpha_{r-b_k-j+1}) = w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_{k-1}}{b_{k-1}}}(\alpha_{r-b_k-j+1}) \ < \ 0.$$

By Proposition 1.16, the expression for elements of $\{W : \overline{W}\}$ of length $n+1$ are then

$$\begin{aligned}
w_1 &= w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}} s_0 \quad \text{if } a_p \neq 0 \text{ and } b_p \neq 0, \\
w_2 &= w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}} s_{a_1+1}, \\
w_3 &= w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}} s_{r-b_1}, \\
w_4 &= w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}} s_{a_k+1} \quad \text{for all k such that } \ a_{k-1} - a_k > 1, \\
w_5 &= w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}} s_{r-b_k} \quad \text{for all k such that } \ b_{k-1} - b_k > 1,
\end{aligned}$$

which can also be written as

$$\begin{aligned}
w_1 &= w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}} w_{\binom{0}{0}} \quad \text{if } a_p \neq 0 \text{ and } b_p \neq 0, \\
w_2 &= w_{\binom{a_1+1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}, \\
w_3 &= w_{\binom{a_1}{b_1+1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}},
\end{aligned}$$

$w_4 = w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \cdots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k+1}{b_k}} \cdots w_{\binom{a_p}{b_p}}$ for all k such that $a_{k-1} - a_k > 1$,

$w_5 = w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \cdots w_{\binom{a_{k-1}}{b_{k-1}}} w_{\binom{a_k}{b_k+1}} \cdots w_{\binom{a_p}{b_p}}$ for all k such that $b_{k-1} - b_k > 1$.

This is precisely the required list of elements of length $n+1$ defined by Proposition 5.5. □

Proposition 5.6. *Let $a_1 + b_1 + 1 \leq r$ and $w_\xi \in W_r \subset \{W : \overline{W}\}$ be a core element of length n as given in Proposition 5.5. Let $\xi = \binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$ be a partition of n. Then*

$$\rho - w(\rho) = n\alpha_0 + \sum_{j=1}^{a_1}(n - \sum_{i=1}^{j} \xi_i')\alpha_j + \sum_{j=1}^{b_1}\sum_{i=1}^{j} \xi_{b_1+2-i}\alpha_{r+j-b_1}$$

Proof We shall prove this result by induction on p. Let $\xi = \binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$ and $\lambda = \binom{a_1 a_2 \cdots a_p\, a_{p+1}}{b_1 b_2 \cdots b_p\, b_{p+1}}$ be partitions of n and m, respectively, and let ξ' and λ' be their conjugates respectively. Thus $m = n + a_{p+1} + b_{p+1} + 1$. Then by (2.5)

$$\lambda_k = \begin{cases} \xi_k & \text{for } k = 1, \ldots, p, \\ \xi_{p+1} + a_{p+1} + 1 & \text{for } k = p+1, \\ \xi_k + 1 & \text{for } k = p+2, \ldots, p+1+b_{p+1}, \\ \xi_k & \text{for } k = p+2+b_{p+1}, \ldots, b_1+1, \\ 0 & \text{for } k \geq b_1 + 2, \end{cases} \tag{5.9a}$$

and

$$\lambda_k' = \begin{cases} \xi_k' & \text{for } k = 1, \ldots, p, \\ \xi_{p+1}' + b_{p+1} + 1 & \text{for } k = p+1, \\ \xi_k' + 1 & \text{for } k = p+2, \ldots, p+1+a_{p+1}, \\ \xi_k' & \text{for } k = p+2+a_{p+1}, \ldots, a_1+1, \\ 0 & \text{for } k \geq a_1 + 2. \end{cases} \tag{5.9b}$$

Let $< \Phi_w > = \sum_{\alpha \in \Phi_w} \alpha$ so that by Proposition 1.11

$$\rho - w(\rho) = < \Phi_w > . \tag{5.10}$$

The fact that $w_{\binom{a_1}{b_1}} = s_0 s_1 s_2 \ldots s_{a_1} s_r s_{r-1} \ldots s_{r-b_1+1}$ and (1.12) we then obtain

$$< \Phi_{w_{\binom{a_1}{b_1}}} > = \alpha_0 + s_0(\alpha_1) + s_0 s_1(\alpha_2) + \ldots + s_0 \ldots s_{a_1} s_r \ldots s_{r-b_1+2}(\alpha_{r-b_1+1})$$

$$= \alpha_0 + w_{\binom{0}{0}}(\alpha_1) + w_{\binom{1}{0}}(\alpha_2) \ldots + w_{\binom{a_1-1}{0}}(\alpha_{a_1})$$

$$+ w_{\binom{a_1}{0}}(\alpha_r) + w_{\binom{a_1}{1}}(\alpha_{r-1}) + \ldots + w_{\binom{a_1}{b_1-1}}(\alpha_{r-b_1+1})$$

$$= \alpha_0 + (\alpha_0 + \alpha_1) + \ldots + (\alpha_0 + \alpha_1 + \ldots + \alpha_{a_1})$$

$$+ (\alpha_0 + \alpha_r) + (\alpha_0 + \alpha_r + \alpha_{r-1}) + \ldots + (\alpha_0 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{r-b_1+1})$$

$$= (a_1 + b_1 + 1)\alpha_0 + a_1\alpha_1 + (a_1 - 1)\alpha_2 + \ldots + \alpha_{a_1}$$

$$+ b_1\alpha_r + (b_1 - 1)\alpha_{r-1} + \ldots + \alpha_{r-b_1+1}.$$

For $\xi = \binom{a_1}{b_1}$ and $\xi' = \binom{b_1}{a_1}$ the above expression gives

$$< \Phi_{w_{\binom{a_1}{b_1}}} >= (a_1 + b_1 + 1)\alpha_0 + \sum_{j=1}^{a_1}(a_1 + b_1 + 1 - \sum_{i=1}^{j}\xi'_i)\alpha_j + \sum_{j=1}^{b_1}\sum_{i=1}^{j}\xi_{b_1+2-i}\alpha_{r+j-b_1}.$$

Hence Proposition 5.6 is true for $p = 1$. Assume that it is true for p and let $w' = w_\xi w_{\binom{a_{p+1}}{b_{p+1}}}$. By the generalisation of (1.12), $< \Phi_{w'} >=< \Phi_{w_\xi} > + < w_\xi \Phi_{w_{\binom{a_{p+1}}{b_{p+1}}}} >$, so that

$$\Phi_{w_{\binom{a_{p+1}}{b_{p+1}}}} = \{\alpha_0, \alpha_0 + \alpha_1, \ldots, \alpha_0 + \alpha_1 + \ldots + \alpha_{a_{p+1}}\}$$

$$\cup \{\alpha_0 + \alpha_r, \alpha_0 + \alpha_r + \alpha_{r-1}, \ldots, \alpha_0 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{r-b_{p+1}+1}\}.$$

This expression and (5.8a - 5.8c) then imply

$$< w_\xi \Phi_{w_{\binom{a_{p+1}}{b_{p+1}}}} > = w_\xi(\alpha_0) + w_\xi(\alpha_0 + \alpha_1) + \ldots + w_\xi(\alpha_0 + \alpha_r + \alpha_{r-1}, \ldots, \alpha_{r-b_{p+1}+1})$$

$$= (a_{p+1} + b_{p+1} + 1)(\alpha_0 + \alpha_1 + \ldots + \alpha_p + \alpha_{r-p+1} + \alpha_{r-p+2} + \ldots + \alpha_r)$$

$$+ a_{p+1}\alpha_{p+1} + (a_{p+1} - 1)\alpha_{p+2} + \ldots + 2\alpha_{p+a_{p+1}-1} + \alpha_{p+a_{p+1}}$$

$$+ b_{p+1}\alpha_{r-p} + (b_{p+1} - 1)\alpha_{r-p-1} + \ldots + 2\alpha_{r-p-b_{p+1}+2} + \alpha_{r-p-b_{p+1}+1}.$$

However by hypothesis

$$< \Phi_{w_\xi} >= n\alpha_0 + \sum_{j=1}^{a_1}(n - \sum_{i=1}^{j}\xi'_i)\alpha_j + \sum_{j=1}^{b_1}\sum_{i=1}^{j}\xi_{b_1+2-i}\alpha_{r+j-b_1}.$$

The coefficient of each α_k in $< \Phi_{w_\xi} > + < w_\xi \Phi_{w_{\binom{a_{p+1}}{b_{p+1}}}} >$ is then given by the following

$$\begin{cases} n + a_{p+1} + b_{p+1} + 1 & \text{for } k = 0, \\ n + a_{p+1} + b_{p+1} + 1 - \sum_{i=1}^{k} \xi_i' & \text{for } k = 1, \ldots, p, \\ n + a_{p+1} + p + 1 - k - \sum_{i=1}^{k} \xi_i' & \text{for } k = p+1, \ldots, p + a_{p+1}, \\ n - \sum_{i=1}^{k} \xi_i' & \text{for } k = p + 1 + a_{p+1}, \ldots, a_1, \\ 0 & \text{for } k = a_1 + 1, \ldots, r - b_1, \\ \sum_{i=1}^{k+b_1-r} \xi_{b_1+2-i} & \text{for } k = r + 1 - b_1, \ldots, r - p - b_{p+1}, \\ k + p + b_{p+1} - r + \sum_{i=1}^{k+b_1-r} \xi_{b_1+2-i} & \text{for } k = r + 1 - p - b_{p+1}, \ldots, r - p, \\ a_{p+1} + b_{p+1} + 1 + \sum_{i=1}^{k+b_1-r} \xi_{b_1+2-i} & \text{for } k = r + 1 - p, \ldots, r. \end{cases}$$

where $a_1 > p + a_{p+1}$ and $r - b_1 < r - p - b_{p+1}$. On noting that $m = n + a_{p+1} + b_{p+1} + 1$, (5.9a) and (5.9b), the above coefficients of the α_k can be simplified and coincide with the coefficient of α_k in

$$< \Phi_w >= m\alpha_0 + \sum_{j=1}^{a_1} (m - \sum_{i=1}^{j} \lambda_i')\alpha_j + \sum_{j=1}^{b_1} \sum_{i=1}^{j} \lambda_{b_1+2-i}\alpha_{r+j-b_1}.$$

Hence, by induction the proposition is true for all p. □

Now we are in a position to prove our key result regarding the core contribution to (5.6).

Proposition 5.7. *Let $q = e^{-\delta}$ and let $w_\xi \in W_r$ be the core element of $\{W : \overline{W}\}$ defined in Proposition 5.5, then*

$$\varepsilon(w_\xi) ch\, \overline{V}^{w_\xi(\rho)-\rho} = (-1)^{|\xi|} q^{|\xi|} \{\bar{\xi}; \xi'\}$$

where $\xi \in F$ is the partition which in Frobenius notation takes the form $\binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$.

Proof Proposition 5.6 implies that

$$w_\xi(\rho) - \rho = -n\alpha_0 - \sum_{j=1}^{a_1} (n - \sum_{i=1}^{j} \xi_i')\alpha_j - \sum_{j=1}^{b_1} \sum_{i=1}^{j} \xi_{b_1+2-i}\alpha_{r+j-b_1}$$

$$w_\xi(\rho)-\rho = -n(\delta+\epsilon_{r+1}-\epsilon_1)-\sum_{j=1}^{a_1}(n-\sum_{i=1}^{j}\xi'_i)(\epsilon_j-\epsilon_{j+1})$$
$$-\sum_{j=1}^{b_1}\sum_{i=1}^{j}\xi_{b_1+2-i}(\epsilon_{r+j-b_1}-\epsilon_{r+j-b_1+1})$$
$$= -n(\delta+\epsilon_{r+1}-\epsilon_1)-(n-\xi'_1)\epsilon_1-\sum_{j=2}^{a_1}(n-\sum_{i=1}^{j}\xi'_i)\epsilon_j+\sum_{j=1}^{a_1}(n-\sum_{i=1}^{j}\xi'_i)\epsilon_{j+1}$$
$$+(\sum_{i=1}^{b_1}\xi_{b_1+2-i})\epsilon_{r+1}-\sum_{j=1}^{b_1}\sum_{i=1}^{j}\xi_{b_1+2-i}\epsilon_{r+j-b_1}+\sum_{j=1}^{b_1-1}\sum_{i=1}^{j}\xi_{b_1+2-i}\epsilon_{r+j-b_1+1}.$$

With the fact that $n=\sum_{i=1}^{b_1+1}\xi_i=\sum_{i=1}^{a_1+1}\xi'_i$ it can be seen that

$$\begin{aligned} w_\xi(\rho)-\rho &= -n\delta+\xi'_1\epsilon_1+\sum_{j=2}^{a_1}\xi'_j\epsilon_j+\xi'_{a_1+1}\epsilon_{a_1+1} \\ &\quad -\xi_1\epsilon_{r+1}-\sum_{j=2}^{b_1}\xi_{b_1+2-j}\epsilon_{r-b_1+j}-\xi_{b_1+1}\epsilon_{r+1-b_1} \\ &= -\mid\xi\mid\delta+\sum_{i=1}^{a_1+1}\xi'_i\epsilon_i-\sum_{i=1}^{b_1+1}\xi_i\epsilon_{r-i+2}. \end{aligned} \tag{5.11}$$

Since $\varepsilon(w_\xi)=(-1)^{|\xi|}$ we have the result. □

Notice that (5.11) can be written in the form

$$w_\xi(\rho)-\rho=\sum_{(i,j)\in F(\xi)}(-\delta+\epsilon_j-\epsilon_{r-i+2}), \tag{5.12}$$

where the summation is carried out over all (i,j) such that a box lies in the ith row and jth column of the Young diagram $F(\xi)$.

Next we consider the non-core action $w_{\binom{c}{d}}$ where $c+d\geq r$. Again by Proposition 1.11 we have

$$\rho-w_{\binom{c}{d}}(\rho)=\alpha_0+s_0(\alpha_1)+\ldots+s_0s_1\ldots s_cs_r\ldots s_{r-d+2}(\alpha_{r-d+1})$$
$$=\delta+(c+d+1)\alpha_0+\sum_{i=1}^{c}(\alpha_1+\alpha_2+\ldots+\alpha_i)$$
$$+\sum_{i=1}^{r-c-1}(\alpha_r+\alpha_{r-1}+\ldots+\alpha_{r+1-i})+\sum_{i=1}^{c+d-r}(\alpha_r+\alpha_{r-1}+\ldots+\alpha_{c+1-i}).$$

When casting this expression in terms of the $\delta-\epsilon$ basis we obtain

$$\begin{aligned} w_{\binom{c}{d}}(\rho)-\rho &= -(c+d+2)\delta+(d+1)\epsilon_1+\sum_{i=2}^{r-d+1}\epsilon_i-(c+2)\epsilon_{r+1}-\sum_{i=2}^{r-c}\epsilon_{r+2-i} \\ &= -(c+d+2)\delta+\sum_{(i,j)\in F(\mu)}\epsilon_i-\sum_{(i,j)\in F(\nu)}\epsilon_{r+2-i} \end{aligned} \tag{5.13}$$

where $\mu = \binom{d}{r-d}$ and $\nu = \binom{c+1}{r-c-1}$.

Next let $\lambda = \sum_{i=1}^{\ell(\mu)} \mu_i \epsilon_i - \sum_{i=1}^{\ell(\nu)} \nu_i \epsilon_{r+2-i}$ where μ and ν are partitions of the same positive integer. Since each ϵ_i $(i = 1, 2, \ldots, r+1)$ lies in $\overline{\mathcal{H}^*}$ then the level $L(\epsilon_i) = 0$ and hence $L(\lambda) = 0$. For $c + d \geq r$ we can write

$$w_{\binom{c}{d}} = s_0 s_1 \ldots s_c s_r \ldots s_{r-d+2} s_{r-d+1} = t_\theta s_r s_{r-1} \ldots s_{c+2} s_1 s_2 \ldots s_{r-d}\,,$$

where $\theta = \epsilon_1 - \epsilon_{r+1}$ and there is no intersection between the intervals $[1, r-d]$ and $[c+2, r]$. Since each Weyl reflection s_i correspond to a transposition $(i, i+1)$ then the permutation correspond to the Weyl reflection $\hat{w} = s_r s_{r-1} \ldots s_{c+2} s_1 s_2 \ldots s_{r-d}$ is the permutation

$$(r+1 \quad r \quad \ldots \quad c+2)\,(1 \quad 2 \quad \ldots \quad r-d+1)\,.$$

Hence

$$\begin{aligned} \hat{w}(\lambda) = \mu_{r-d+1}\epsilon_1 + \sum_{i=2}^{r-d+1} \mu_{i-1}\epsilon_i + \sum_{i=r-d+2}^{\ell(\mu)} \mu_i \epsilon_i \\ - \nu_{r-c}\epsilon_{r+1} - \sum_{i=2}^{r-c} \nu_{i-1}\epsilon_{r+2-i} - \sum_{i=r-c+1}^{\ell(\nu)} \nu_i \epsilon_{r+2-i}\,, \end{aligned} \tag{5.14a}$$

where the second and fourth summations are considered to be zero if $r - d + 2 > \ell(\mu)$ and $r - c + 1 > \ell(\nu)$ respectively. Then by (3.21)

$$\begin{aligned} w_{\binom{c}{d}}(\lambda) &= t_\theta \hat{w}(\lambda) \\ &= \hat{w}(\lambda) + L(\lambda)\theta - ((\hat{w}(\lambda) \mid \theta) + \frac{1}{2}L(\lambda)(\theta \mid \theta))\delta \\ &= \hat{w}(\lambda) - (\mu_{r-d+1} + \nu_{r-c})\delta\,. \end{aligned} \tag{5.14b}$$

Theorem 5.8. *The general form for the right coset representatives of W with respect to $\overline{W}$ of the affine algebra $A_r^{(1)}$ is*

$$w = w_{\binom{c_t}{d_t}} \ldots w_{\binom{c_2}{d_2}} w_{\binom{c_1}{d_1}} w_{\binom{a_1}{b_1}} w_{\binom{a_2}{b_2}} \ldots w_{\binom{a_p}{b_p}}$$

where

$$r > c_t \geq \ldots \geq c_2 \geq c_1 \geq a_1 > a_2 > \ldots > a_p \geq 0,$$

$$r \geq d_t \geq \ldots \geq d_2 \geq d_1 > b_1 > b_2 > \ldots > b_p \geq 0,$$

with $\quad c_1 + d_1 \geq r \geq a_1 + b_1 + 1\,.$

Proof In term of the core elements we can write $w = w_{\binom{c_t}{d_t}} \ldots w_{\binom{c_2}{d_2}} w_{\binom{c_1}{d_1}} w_\xi$. We shall prove the theorem by showing that there is a one-to-one correspondence between the elements of $\{W : \overline{W}\}$ and the expression in (2.15a), namely

$$\sum_{\substack{\zeta \in F \\ \ell(\zeta)+\ell(\zeta')\leq r+1}} \sum_{s=0}^{\infty} \sum_{\substack{k,\bar{k},m_1+\overline{m}_1 \geq r+3 \\ \zeta_1' < m_1 \leq r+1,\, \zeta_1 < \overline{m}_1 \leq r+1}} (-1)^{|\zeta|+m+\bar{m}}\, q^{|\zeta|+n+\bar{n}-s} \{\overline{\nu^s}; \mu^s\}$$

where $k = (m_1, \ldots, m_s)$ with $m_1 \leq \ldots \leq m_s$, $\bar{k} = (\bar{m}_1, \ldots, \bar{m}_s)$ with $\bar{m}_1 \leq \ldots \leq \bar{m}_s$, $m = \sum_{i=1}^s m_i$, $\bar{m} = \sum_{i=1}^s \overline{m}_i$, $n = \sum_{i=1}^s n_i$ and $\bar{n} = \sum_{i=1}^s \overline{n}_i$.

First we note that there is a one-to-one correspondence of labels with the following identification:

$$\overline{m}_i \longleftrightarrow c_i + 2$$

$$m_i \longleftrightarrow d_i + 1$$

$$s \longleftrightarrow t$$

$$\zeta \longleftrightarrow \xi .$$

It just remain to show that that for our particular w we have

$$\varepsilon(w) ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{|\zeta|+m+\bar{m}}\, q^{|\zeta|+n+\bar{n}-s} \{\overline{\nu^s}; \mu^s\}$$

Now by (5.12)

$$\begin{aligned} w_\xi(\rho) &= \rho + \sum_{(i,j)\in F(\xi)} (-\delta + \epsilon_j - \epsilon_{r+2-i}) \\ &= \rho - \mid \xi \mid \delta + \mu^0 - \nu^0, \end{aligned}$$

where $\mu^0 = \xi' = \sum_{(ij)\in F(\xi)} \epsilon_j$ and $\nu^0 = -\bar{\xi} = \sum_{(ij)\in F(\xi)} \epsilon_{r+2-i}$. Furthermore by (5.13) and (5.14)

$$\begin{aligned} w_{\binom{c_1}{d_1}} w_\xi(\rho) = \rho - \mid \xi \mid \delta - (c_1 + d_1 + 2)\delta + \sum_{(i,j)\in F\binom{d_1}{r-d_1}} \epsilon_i - \sum_{(i,j)\in F\binom{c_1+1}{r-c_1-1}} \epsilon_{r+2-i} \\ - (\mu^0_{r-d_1+1} + \nu^0_{r-c_1})\delta + \hat{w}(\mu^0 - \nu^0), \end{aligned}$$

where $\hat{w}(\mu^0 - \nu^0)$ can be computed from (5.14a). Next let

$$\mu^1 = \hat{w}(\mu^0) + \sum_{(i,j)\in F\binom{d_1}{r-d_1}} \epsilon_i$$

$$= (d_1 + 1 + \mu^0_{r-d_1+1})\epsilon_1 + \sum_{i=2}^{r-d_1+1} (\mu^0_{i-1} + 1)\epsilon_i + \sum_{i=r-d_1+2}^{\ell(\mu^0)} \mu^0_i \epsilon_i$$

$$\nu^1 = \hat{w}(\nu^0) + \sum_{(i,j)\in F\binom{c_1+1}{r-c_1-1}} \epsilon_{r+2-i}$$

$$= (c_1 + 2 + \nu^0_{r-c_1})\epsilon_{r+1} + \sum_{i=2}^{r-c_1} (\nu^0_{i-1} + 1)\epsilon_{r+2-i} + \sum_{j=r-c_1+1}^{\ell(\nu^0)} \nu^0_i \epsilon_{r+2-i}$$

$$n_1 = d_1 + 1 + \mu^0_{r+1-d_1} = \mu^1_1$$

$$\overline{n_1} = c_1 + 2 + \nu^0_{r-c_1} = \nu^1_1,$$

then

$$w_{\binom{c_1}{d_1}} w_\xi(\rho) = \rho - (\mid \xi \mid + n_1 + \bar{n}_1 - 1)\delta + \mu^1 - \nu^1.$$

In general μ^i and ν^i are defined recursively as in (2.14b) and (2.14c) respectively, $n_i = d_i + 1 + \mu^{i-1}_{r+1-d_i}$ and $\overline{n_i} = c_i + 2 + \nu^{i-1}_{r-c_i}$. Continuing the procedure iteratively we obtain

$$w(\rho) = w_{\binom{c_t}{d_t}} \dots w_{\binom{c_2}{d_2}} w_{\binom{c_1}{d_1}} w_\xi(\rho)$$

$$= w_{\binom{c_t}{d_t}} \dots w_{\binom{c_2}{d_2}}(- \mid \xi \mid \delta + \rho - (n_1 + \bar{n}_1 - 1)\delta + \mu^1 - \nu^1)$$

$$= \rho - (\mid \xi \mid + \sum_{i=1}^{t} n_i + \sum_{i=1}^{t} \overline{n_i} - t)\delta + \mu^t - \nu^t .$$

Hence

$$w(\rho) - \rho = -(\mid \zeta \mid + n + \bar{n} - s)\delta + \mu^s - \nu^s. \tag{5.15}$$

The parity of w is

$$(-1)^{t+c_1+\dots+c_t+d_1+\dots+d_t+|\xi|}$$

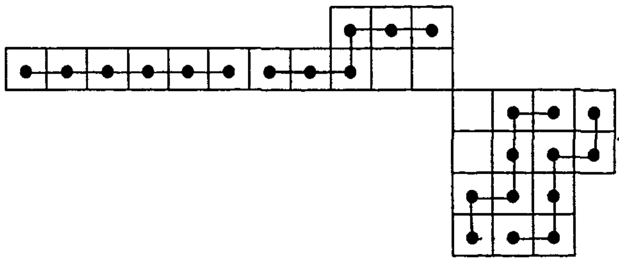
$$=(-1)^{s+(\bar{m}_1-2)+\dots(\bar{m}_s-2)+(m_1-1)+\dots+(m_s-1)+|\zeta|}$$

$$=(-1)^{|\zeta|+m+\bar{m}} .$$

and hence the Theorem is proved. □

Since there is a correspondence between the Weyl group element $w = w_{\binom{c_t}{d_t}} \dots w_{\binom{c_2}{d_2}} w_{\binom{c_1}{d_1}} w_\xi$ with that of (2.15a) then the action of $w \in \{W : \overline{W}\}$ on ρ can be obtained diagrammatically, i.e. $w(\rho) - \rho$ can be obtained from $F(\bar{\xi}; \xi')$

by adding t pairs of boundary strips of length $r+1$. For example, let us note the result of computing $w(\rho)-\rho$ with $w = w_{\binom{4}{2}}w_{\binom{3}{2}}w_{\binom{1}{0}}$ for the affine algebra $A_5^{(1)}$. First note that $w_{\binom{1}{0}} = w_{\xi} = s_0s_1$ is a core element and contribute the Young diagram $F(\overline{\xi};\xi') = F(\overline{2};1^2)$. $w_{\binom{3}{2}} = s_0s_1s_2s_3s_5s_4$ is a non-core element and its action amounts to adding a pair of boundary strips of length $r+1=6$ each extending over 5 and 3 columns respectively. Similarly the action $w_{\binom{4}{2}} = s_0s_1s_2s_3s_5s_4$ amounts to adding a pair of boundary strips each extending over 6 and 3 columns respectively. Hence we obtain the following Young diagram $F(\overline{\nu^2};\mu^2)$:

$F(\overline{\nu^2};\mu^2) =$ .

so that from (5.15) this gives

$$(s_0s_1s_2s_3s_4s_5s_4)(s_0s_1s_2s_3s_5s_4)(s_0s_1)(\rho)-\rho$$

$$= -(2+(3+4)+(5+11)-2)\delta + 4\epsilon_1 + 4\epsilon_2 + 3\epsilon_3 + 3\epsilon_4 - 11\epsilon_6 - 3\epsilon_5$$

$$= -23\delta + 15\epsilon_1 + 15\epsilon_2 + 14\epsilon_3 + 14\epsilon_4 + 8\epsilon_5\,.$$

5.3. The right coset representatives of W with respect to $\overline{W}$ for $X_{N(r)}^{(k)}$

All the results of the previous section for $A_r^{(1)} \supset A_r$ may be extended in very much the same way to more general cases $X_{N(r)}^{(k)} \supset Y_r$. For the other infinite series of rank dependent affine algebras we will be content in this thesis with stating conjectures on the elements of the right coset representatives $\{W : \overline{W}\}$. We suspect that they can all be proved in the same way as in the case of $A_r^{(1)}$. All our results are based on an extensive computer assisted study of $w(\rho)-\rho$ for various w. This has allowed us to identify all $w \in \{W : \overline{W}\}$ with some confidence. The resulting elements are then used to calculate $w(\lambda+\rho)-\rho$.

Definition 5.9.

(i) For $0 \leq a \leq r$ let

$$w_{<a>} = \begin{cases} s_0 & \text{if } a = 0, \\ s_0 s_1 \ldots s_a & \text{if } a \neq 0. \end{cases} \tag{5.16}$$

A general expression is given by $w_{<a_1>} w_{<a_2>} \ldots w_{<a_p>}$.

(ii) For $1 \leq a \leq r$ let

$$w^{(0)}_{[a]} = \begin{cases} s_0 & \text{if } a = 1, \\ s_0 s_2 s_3 \ldots s_a & \text{if } a \neq 1, \end{cases}$$

$$w^{(1)}_{[a]} = \begin{cases} s_1 & \text{if } a = 1, \\ s_1 s_2 s_3 \ldots s_a & \text{if } a \neq 1. \end{cases} \tag{5.17}$$

A general expression is then given by $w_{[a_1]} w_{[a_2]} \ldots w_{[a_i]} \ldots w_{[a_p]}$ with $w_{[a_i]} = w^{(0)}_{[a_i]}$ for i odd, and $w_{[a_i]} = w^{(1)}_{[a_i]}$ for i even. Thus the Weyl reflections for each sequence begin alternately with s_0 and s_1.

Again before giving a general result let us compute some terms for the denominator of the Weyl-Kostant-Liu character formula. Consider first the case when $a_1 \leq r - 1$. In Table 5.2a, 5.2b and 5.2c, respectively, we compute for a few cases $w(\rho) - \rho$ for the representatives affine algebras $B^{(1)}_r$, $D^{(2)}_{r+1}$ and $C^{(1)}_r$.

Table 5.2a : Some results arising from $w_{[a]}(\rho) - \rho$ for $B_r^{(1)}$

w	$w(\rho) - \rho$	B_r character	depth
s_0	$-\alpha_0$	$[1^2]$	1
s_0s_2	$-2\alpha_0 - \alpha_2$	$[21^2]$	2
$s_0s_2s_1$	$-3\alpha_0 - \alpha_1 - 2\alpha_2$	$[2^3]$	3
$s_0s_2s_3$	$-3\alpha_0 - 2\alpha_2 - \alpha_3$	$[31^3]$	3
$s_0s_2s_3s_1$	$-4\alpha_0 - \alpha_1 - 3\alpha_2 - \alpha_3$	$[32^21]$	4
$s_0s_2s_3s_4$	$-4\alpha_0 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	$[41^4]$	4
$s_0s_2s_3s_4s_5$	$-5\alpha_0 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$	$[51^5]$	5
$s_0s_2s_3s_4s_1$	$-5\alpha_0 - \alpha_1 - 4\alpha_2 - 2\alpha_3 - \alpha_4$	$[42^21^2]$	5
$s_0s_2s_3s_1s_2$	$-5\alpha_0 - 2\alpha_1 - 4\alpha_2 - 2\alpha_3$	$[3^22^2]$	5

Table 5.2b : Some results arising from $w_{<a>}(\rho) - \rho$ for $D_{r+1}^{(2)}$

w	$w(\rho) - \rho$	B_r character	depth
s_0	$-\alpha_0$	$[1]$	1
s_0s_1	$-3\alpha_0 - \alpha_1$	$[21]$	3
$s_0s_1s_0$	$-4\alpha_0 - 2\alpha_1$	$[2^2]$	4
$s_0s_1s_2$	$-5\alpha_0 - 2\alpha_1 - \alpha_2$	$[31^2]$	5

Table 5.2c : Some results arising from $w_{<a>}(\rho) - \rho$ for $C_r^{(1)}$

w	$w(\rho) - \rho$	C_r character	depth
s_0	$-\alpha_0$	$< 2 >$	1
$s_0 s_1$	$-2\alpha_0 - \alpha_1$	$< 31 >$	2
$s_0 s_1 s_2$	$-3\alpha_0 - 2\alpha_1 - \alpha_2$	$< 41^2 >$	3
$s_0 s_1 s_0$	$-3\alpha_0 - 3\alpha_1$	$< 3^2 >$	3
$s_0 s_1 s_2 s_3$	$-4\alpha_0 - 3\alpha_1 - 2\alpha_2 - \alpha_3$	$< 51^3 >$	4
$s_0 s_1 s_2 s_0$	$-4\alpha_0 - 4\alpha_1 - \alpha_2$	$< 431 >$	4
$s_0 s_1 s_2 s_3 s_4$	$-5\alpha_0 - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$	$< 61^4 >$	5
$s_0 s_1 s_2 s_3 s_0$	$-5\alpha_0 - 5\alpha_1 - 2\alpha_2 - \alpha_3$	$< 531^2 >$	5
$s_0 s_1 s_2 s_0 s_1$	$-5\alpha_0 - 6\alpha_1 - 2\alpha_2$	$< 4^2 2 >$	5

With Proposition 5.2 in mind we make the following conjectures on the core elements of the right cosets $\{W : \overline{W}\}$ generalising Propositions 5.5 and 5.7 which apply to $A_r^{(1)}$.

Conjecture 5.10. *Let $a_1 \leq r-1$ in the case of affine algebras $B_r^{(1)}$ and $A_{2r-1}^{(2)}$ and $a_1 \leq r-3$ in the case of $D_r^{(1)}$. Core elements of $\{W : \overline{W}\}$ for the algebra $B_r^{(1)}$, $D_r^{(1)}$ and $A_{2r-1}^{(2)}$ of length n are given by*

$$w_\alpha = w_{[a_1]} w_{[a_2]} \dots w_{[a_p]}$$

where $a_1 > a_2 > \dots > a_p \geq 0$ and $n = \sum_{i=1}^p a_i$. These elements are such that

$$\varepsilon(w_\alpha) ch\, \overline{V}^{w_\alpha(\rho)-\rho} = \begin{cases} (-1)^{|\alpha|/2} q^{|\alpha|/2} [\alpha] & \text{for } B_r^{(1)} \text{ or } D_r^{(1)} \\ (-1)^{|\alpha|/2} q^{|\alpha|/2} < \alpha > & \text{for } A_{2r-1}^{(2)} \end{cases}$$

where $\alpha \in A$ *is a partition of the form* $\binom{a_1-1\, a_2-1 \cdots a_p-1}{a_1 \quad a_2 \quad \cdots a_p}$.

Conjecture 5.11. *Let* $a_1 \leq r-1$. *Core elements of* $\{W : \overline{W}\}$ *for the algebra* $C_r^{(1)}$ *and* $A_{2r}^{(2)}$ *of length n are given by*

$$w_\gamma = w_{<a_1>} w_{<a_2>} \dots w_{<a_p>}$$

where $a_1 > a_2 > \dots > a_p \geq 0$ *and* $n = p + \sum_{i=1}^{p} a_i$. *These elements are such that*

$$\varepsilon(w_\gamma) ch\, \overline{V}^{w_\gamma(\rho)-\rho} = \begin{cases} (-1)^{|\gamma|/2} q^{|\gamma|/2} < \gamma > & \text{for } C_r^{(1)} \\ (-1)^{|\gamma|/2} q^{|\gamma|/2} [\gamma] & \text{for } A_{2r}^{(2)} \end{cases}$$

where $\gamma \in C$ *is a partition of the form* $\binom{a_1+1\, a_2+1 \cdots a_p+1}{a_1 \quad a_2 \quad \cdots a_p}$.

Conjecture 5.12. *Let* $a_1 \leq r-1$. *Core elements of* $\{W : \overline{W}\}$ *for the algebra* $D_{r+1}^{(2)}$ *of length n are given by*

$$w_\epsilon = w_{<a_1>} w_{<a_2>} \dots w_{<a_p>}$$

where $a_1 > a_2 > \dots > a_p \geq 0$ *and* $n = p + \sum_{i=1}^{p} a_i$. *These elements are such that*

$$\varepsilon(w_\epsilon) ch\, \overline{V}^{w_\epsilon(\rho)-\rho} = (-1)^{(|\epsilon|+p)/2} q^{|\epsilon|} [\epsilon]$$

where $\epsilon \in E$ *is a partition of the form* $\binom{a_1\, a_2 \cdots a_p}{a_1\, a_2 \cdots a_p}$.

It should be emphasised that thanks to Proposition 5.2 and the fact that the $w(\rho) - \rho \in P^+$ if and only if $w \in \{W : \overline{W}\}$, the only aspect of these Conjectures requiring proof is the precise form of w_α, w_γ and w_ϵ. Next we make further conjectures for arbitrary elements of $\{W : \overline{W}\}$ analogous to Theorem 5.8 in the case of $A_r^{(1)}$.

Conjecture 5.13. *The general form of the right coset representatives of W with respect to* $\overline{W}$ *of the affine algebra* $C_r^{(1)}$ *is*

$$w_{<b_t>} w_{<b_{t-1}>} \dots w_{<b_1>} w_{<a_1>} w_{<a_2>} \dots w_{<a_p>}$$

with $2r-1 \geq b_t \geq \dots \geq b_1 > a_1 > \dots > a_p$ *and for* $b \geq r$,

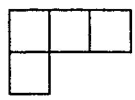
$$w_{<b>} = s_0 s_1 \dots s_{r-1} s_r s_{r-1} \dots s_{2r-b}.$$

Further let $w = w_{<b_t>} w_{<b_{t-1}>} \dots w_{<b_1>} w_\gamma$. *Then*

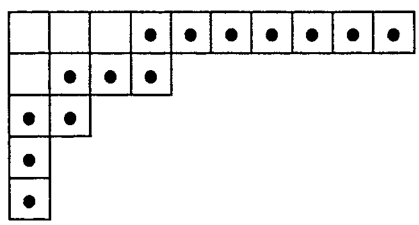
$$\varepsilon(w)\, ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{|\gamma|/2+m} q^{|\gamma|/2+n-t} < \lambda^{(k)} >$$

where all the variables are as described in Proposition 2.1 with the t-tuple given by $k = (b_1 + 3, b_2 + 3, \dots, b_t + 3)$.

For illustration let us note the result of computing $w_{<7>} w_{<1>}(\rho) - \rho$ for the affine algebra $C_6^{(1)}$. First note that $w_{<1>} = s_0 s_1 = w_\gamma$ is a core element and contributes the Young diagram $F(\gamma) = F(31)$



and $w_{<7>} = s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_5$ is a non-core element and contributes an additional boundary strip of length 14 extending over 10 columns.



Hence

$$w_{<7>} w_{<1>}(\rho) - \rho = -11\delta + 10\epsilon_1 + 4\epsilon_2 + 2\epsilon_3 + \epsilon_4 + \epsilon_5.$$

Conjecture 5.14. *The general form of the right coset representatives of* W *with respect to* $\overline{W}$ *of the affine algebra* $A_{2r}^{(2)}$ *is*

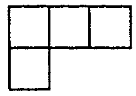
$$w_{<b_t>} w_{<b_{t-1}>} \dots w_{<b_1>} w_{<a_1>} w_{<a_2>} \dots w_{<a_p>},$$

where all the terms are as in Conjecture 5.13. Then

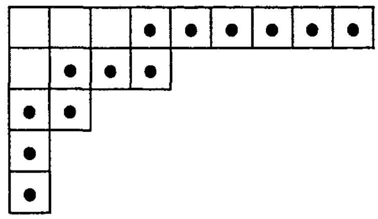
$$\varepsilon(w)\, ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{|\gamma|/2+m-t} q^{|\gamma|/2+n-t} [\lambda^{(k)}],$$

where all the variables are as described in Proposition 2.1 with the t-tuple given by $k = (b_1 + 2, b_2 + 2, \dots, b_t + 2)$.

For illustration let us note the result of computing $w_{<7>}w_{<1>}(\rho) - \rho$ for the affine algebra $A^{(2)}_{12}$. As before $w_{<1>} = s_0 s_1 = w_\gamma$ is a core element and contributes the Young diagram $F(\gamma) = F(31)$



and $w_{<7>} = s_0s_1s_2s_3s_4s_5s_6s_5$ is a non-core element and its action amounts to adding a boundary strip of length 13 extending over 9 columns .



Hence

$$w_{<7>}w_{<1>}(\rho) - \rho = -10\delta + 9\epsilon_1 + 4\epsilon_2 + 2\epsilon_3 + \epsilon_4 + \epsilon_5.$$

Conjecture 5.15. *The general form of the right coset representatives of W with respect to $\overline{W}$ of the affine algebra $D^{(2)}_{r+1}$ is*

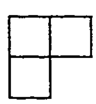
$$w_{<b_t>}w_{<b_{t-1}>} \cdots w_{<b_1>}w_{<a_1>}w_{<a_2>} \cdots w_{<a_p>},$$

where all the terms are as in Conjecture 5.13. Then

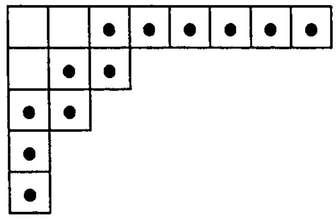
$$\varepsilon(w)\, ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{(|\epsilon|+p)/2+m} q^{|\epsilon|+2n-t}[\lambda^{(k)}],$$

where all the variables are as described in Proposition 2.1 with the t-tuple given by $k = (b_1+1, b_2+1, \ldots, b_t+1)$.

For illustration let us note the result of computing $w_{<7>}w_{<1>}(\rho) - \rho$ for the affine algebra $D^{(2)}_7$. As before $w_{<1>} = s_0 s_1 = w_\gamma$ is a core element and contribute the Young diagram $F(\epsilon) = F(21)$



and $w_{<7>} = s_0s_1s_2s_3s_4s_5s_6s_5$ is a non-core element and its action amounts to adding a boundary strip of length 12 extending over 8 columns.



Hence

$$w_{<7>}w_{<1>}(\rho) - \rho = -18\delta + 8\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4 + \epsilon_5.$$

Conjecture 5.16. *The general form of the right coset representatives of W with respect to $\overline{W}$ of the affine algebra $B_r^{(1)}$ is*

$$w_{[b_t]}w_{[b_{t-1}]} \cdots w_{[b_1]}w_{[a_1]}w_{[a_2]} \cdots w_{[a_p]}$$

such that $2r-1 \geq b_t \geq \ldots \geq b_1 \geq r > a_1 > \ldots > a_p$,

$$w_{[b]}^{(0)} = \begin{cases} s_0s_2\ldots s_{r-1}s_rs_{r-1}\ldots s_{2r-b} & \text{if } b \neq 2r-1, \\ s_0s_2\ldots s_{r-1}s_rs_{r-1}\ldots s_2s_0 & \text{if } b = 2r-1, \end{cases}$$

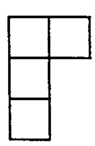
$$w_{[b]}^{(1)} = \begin{cases} s_1s_2\ldots s_{r-1}s_rs_{r-1}\ldots s_{2r-b} & \text{if } b \neq 2r-1, \\ s_1s_2\ldots s_{r-1}s_rs_{r-1}\ldots s_2s_1 & \text{if } b = 2r-1. \end{cases}$$

Further let $w = w_{[b_t]}w_{[b_{t-1}]} \cdots w_{[b_1]}w_\alpha$. *Then*

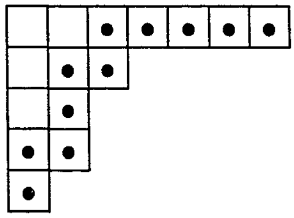
$$\varepsilon(w)\, ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{|\alpha|/2+m}q^{|\alpha|/2+n}[\lambda^{(k)}],$$

where all the variables are as described in Proposition 2.1 with the t-tuple given by $k = (b_1, b_2, \ldots, b_t)$.

For illustration let us note the result of computing $w_{[7]}^{(0)}w_{[2]}^{(1)}(\rho) - \rho$ for the affine algebra $B_6^{(1)}$. As before $w_{[2]}^{(0)} = s_0s_2 = w_\alpha$ is a core element and contributes the Young diagram $F(\alpha) = F(21^2)$



and $w_{[7]}^{(0)} = s_0 s_2 s_3 s_4 s_5 s_6 s_5$ is a non-core element and its action amounts to adding a boundary strip of length 11 extending over 7 columns.



Hence

$$w_{[7]}^{(0)} w_{[2]}^{(1)}(\rho) - \rho = s_0 s_2 s_3 s_4 s_5 s_6 s_5 \, s_1 s_2(\rho) - \rho$$

$$= -9\delta + 7\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + \epsilon_5.$$

Conjecture 5.17. *The general form of the right coset representatives of W with respect to $\overline{W}$ of the affine algebra $A_{2r-1}^{(2)}$ is*

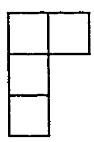
$$w_{[b_t]} w_{[b_{t-1}]} \cdots w_{[b_1]} w_{[a_1]} w_{[a_2]} \cdots w_{[a_p]}$$

where all the terms are as in Conjecture 5.16. Then

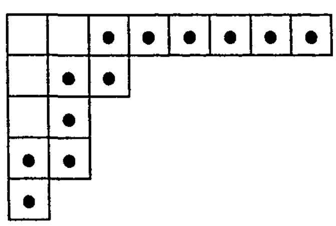
$$\varepsilon(w)\, ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{|\alpha|/2+m-s} q^{|\alpha|/2+n} < \lambda^{(k)} >$$

where all the variables are as described in Proposition 2.1 with the t-tuple given by $k = (b_1+1, b_2+1, \ldots, b_t+1)$.

For illustration let us note the result of computing $w_{[7]}^{(0)} w_{[2]}^{(1)}(\rho) - \rho$ for the affine algebra $A_{11}^{(2)}$. As before $w_{[2]}^{(0)} = s_0 s_2 = w_\alpha$ is a core element and contributes the Young diagram $F(\alpha) = F(21^2)$



and $w_{[7]}^{(0)} = s_0 s_2 s_3 s_4 s_5 s_6 s_5$ is a non-core element and its action amounts to adding a boundary strip of length 12 extending over 8 columns.



Hence

$$w^{(0)}_{[7]}w^{(1)}_{[2]}(\rho) - \rho = s_0s_2s_3s_4s_5s_6s_5\ s_1s_2(\rho) - \rho$$

$$= -10\delta + 8\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + \epsilon_5.$$

Finally, in the case of $D^{(1)}_r$ we have to introduce a slightly different form for the elements of $\{W : \overline{W}\}$. If $(\lambda_1, \ldots, \lambda_{r-1}, \lambda_r)$ is a partition label for a dominant weight of a D_r module then from Table 2.5 we observe that λ_{r-1} is always positive but the range for λ_r is $-\lambda_{r-1} \leq \lambda_r \leq \lambda_{r-1}$. Hence it is possible for λ_r to have negative values. For example, in the case of $D^{(1)}_5$, we obtain

$$s_0s_2s_3s_4(\rho) - \rho = -4\delta + 4\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5$$

$$s_0s_2s_3s_5(\rho) - \rho = -4\delta + 4\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 - \epsilon_5$$

$$s_0s_2s_3s_4s_5(\rho) - \rho = -5\delta + 5\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4$$

$$s_0s_2s_3s_5s_1s_2s_3s_4(\rho) - \rho = -9\delta + 5\epsilon_1 + 5\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 - 2\epsilon_5$$

$$s_0s_2s_3s_5s_1s_2s_3s_5(\rho) - \rho = -7\delta + 3\epsilon_1 + 5\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 - 2\epsilon_5$$

Hence all these Weyl reflections, except the last one, are valid elements of $\{W : \overline{W}\}$. With these examples and from further computations we make the following conjecture on the elements of $\{W : \overline{W}\}$.

Conjecture 5.18. *The general form of the right coset representatives of W with respect to $\overline{W}$ of the affine algebra $D^{(1)}_r$ is*

$$w_{[y_t]}w_{[y_{t-1}]}\ \cdots\ w_{[y_1]}w_{[x_s]}\ \cdots\ w_{[x_1]}w_{[a_1]}w_{[a_2]}\ \cdots\ w_{[a_p]}$$

such that $2r-1 \geq y_t \geq \ldots \geq y_1 > r \geq x_i \neq x_{i+1} \geq r-1 > a_1 > \ldots > a_p$,

$$w^{(0)}_{[x]} = \begin{cases} s_0s_2\ldots s_{r-2}s_{r-1} & \text{if } x = r-1, \\ s_0s_2\ldots s_{r-2}s_r & \text{if } x = r, \end{cases}$$

$$w^{(1)}_{[x]} = \begin{cases} s_1s_2\ldots s_{r-2}s_{r-1} & \text{if } x = r-1, \\ s_1s_2\ldots s_{r-2}s_r & \text{if } x = r, \end{cases}$$

$$w_{[y]}^{(0)} = \begin{cases} s_0 s_2 \ldots s_{r-2} s_{r-1} s_r & \text{if } y = r+1, \\ s_0 s_2 \ldots s_{r-1} s_r s_{r-2} s_{r-3} \ldots s_{2r-y} & \text{if } r+2 \le y \le 2r-2, \\ s_0 s_2 \ldots s_{r-1} s_r s_{r-2} s_{r-3} \ldots s_2 s_0 & \text{if } y = 2r-1, \end{cases}$$

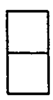
$$w_{[y]}^{(1)} = \begin{cases} s_1 s_2 \ldots s_{r-2} s_{r-1} s_r & \text{if } y = r+1, \\ s_1 s_2 \ldots s_{r-1} s_r s_{r-2} s_{r-3} \ldots s_{2r-y} & \text{if } r+2 \le y \le 2r-2, \\ s_1 s_2 \ldots s_{r-1} s_r s_{r-2} s_{r-3} \ldots s_2 s_1 & \text{if } y = 2r-1. \end{cases}$$

Further let $w = w_{[b_t]} w_{[b_{t-1}]} \ldots w_{[b_1]} w_\alpha$ where $b_i = x_i$ or y_i. Then

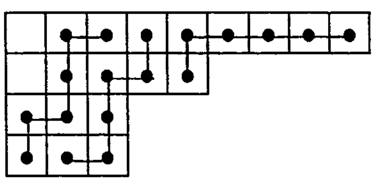
$$\varepsilon(w)\, ch\, \overline{V}^{w(\rho)-\rho} = (-1)^{|\alpha|/2+m} q^{|\alpha|/2+n} [\lambda^{(k)}]$$

where all the variables are as described in Proposition 2.1 with the t-tuple given by $k = (b_1 - 1, b_2 - 1, \ldots, b_t - 1)$ except when $b_i = r-1$ the boundary strips extend over $r-1$ columns as in the case of $b_i = r$. Further if w contains the Weyl reflection $w_{[r]}^{(0)}$ then the coefficient of ϵ_r is negative.

For illustration let us note the result of computing $w(\rho) - \rho$ with $w = w_{[6]}^{(0)} w_{[4]}^{(1)} w_{[3]}^{(0)} w_{[1]}^{(1)}$ for the affine algebra $D_4^{(1)}$. First note that $w_{[1]}^{(0)} = s_0 = w_\alpha$ is a core element and contributes the Young diagram $F(\alpha) = F(1^2)$

 .

$w_{[3]}^{(0)} = s_0 s_2 s_3$ is a non-core element and its action amounts to adding a boundary strip of length 6 extending over 3 columns. Similarly the action of $w_{[4]}^{(0)} = s_0 s_2 s_4$ and $w_{[6]}^{(0)} = s_0 s_2 s_3 s_4 s_2$, respectively, amount to adding boundary strips extending over 3 columns and 5 columns.



Since w does not contains the term $w^{(0)}_{[4]}$ then the coefficient of ϵ_4 is positive. Hence

$$\begin{aligned} w^{(0)}_{[6]}w^{(1)}_{[4]}w^{(0)}_{[3]}w^{(1)}_{[1]}(\rho) - \rho =& s_0s_2s_3s_4s_2\, s_1s_2s_4\, s_0s_2s_3\, s_1(\rho) - \rho \\ =& -17\delta + 9\epsilon_1 + 5\epsilon_2 + 3\epsilon_3 + 3\epsilon_4. \end{aligned}$$

5.4. The actions of the right coset representatives on λ

The numerator of the Weyl-Kostant-Liu character formula involves evaluating expressions of the form $w(\lambda+\rho)-\rho$. We thus need a generalisation of Proposition 1.11, i.e. a general formula to evaluate $w(\lambda)-\lambda$. In this situation we need the following Lemma which appears in one of the exercises in the text by Kac **[Kac4]**.

Lemma 5.19. *Let $w = s_{i_1}\dots s_{i_t}$ be a reduced expression of $w \in W$ and $\beta \in \Phi_w$. Then the sequence $\beta, s_{i_1}(\beta), s_{i_2}s_{i_1}(\beta), \dots$ contains a unique simple root, say $\alpha_{j(\beta)}$, and for $\lambda \in \mathcal{H}^*$*

$$\lambda - w(\lambda) = \sum_{\beta\in\Phi_w} < \lambda, \alpha^\vee_{j(\beta)} > \beta. \tag{5.18}$$

Proof Since $\beta \in \Delta^+$ and $w^{-1}(\beta) < 0$ then at a certain stage, say s_{i_j}, in the sequence of w^{-1} we must have $s_{i_j}, \dots, s_{i_2}s_{i_1}(\beta) < 0$ but $\alpha_{j(\beta)} = s_{i_{j-1}}, \dots, s_{i_2}s_{i_1}(\beta) > 0$. Then $s_{i_j}(\alpha_{j(\beta)}) < 0$. By Lemma 1.5, the fundamental reflection s_i permutes the positive roots other than α_i. Thus $\alpha_{j(\beta)} = \alpha_{i_j}$ which is a simple root.

Suppose that there exist another simple root α_{i_k} in the sequence. Then

$$\begin{aligned} \alpha_{i_k} &= s_{i_{k-1}}\dots s_{i_j}s_{i_{j-1}}\dots s_{i_2}s_{i_1}(\beta) \\ &= s_{i_{k-1}}\dots s_{i_j}(\alpha_{i_j}) > 0. \end{aligned}$$

But $s_{i_{k-1}}\dots s_{i_j}$ is a reduced form so that by Lemma 1.7(b) $s_{i_{k-1}}\dots s_{i_j}(\alpha_{i_j}) < 0$ which is a contradiction. Hence $\alpha_{j(\beta)}$ is unique.

The second part (5.18) of the Lemma can be proved in the same way as in the proof of Proposition 1.11. □

As before let us concentrate first on the case of the affine algebra $A_r^{(1)}$. In this section we will always assume that a weight λ has a Dynkin label $(\lambda_0, \lambda_1, \dots, \lambda_r)$. Let

$$\beta_j = \begin{cases} \alpha_0 & \text{for } j = 0, \\ s_0 s_1 \dots s_{j-1}(\alpha_j) & \text{for } j = 1, \dots, a, \\ s_0 s_1 \dots s_a(\alpha_r) & \text{for } j = r, \\ s_0 s_1 \dots s_a s_r \dots (\alpha_j) & \text{for } j = r-1, \dots, r-b+1. \end{cases} \tag{5.19}$$

Then by (1.12) we have

$$\Phi_{w_{\binom{a}{b}}} = \{\beta_0, \beta_1, \dots, \beta_a, \beta_r, \beta_{r-1}, \dots, \beta_{r-b+1}\}.$$

It can be easily checked that $s_{i_j}(\beta_j) < 0$ so that $\alpha_{j(\beta)}$ of Lemma 5.19 can be taken as α_j for each β_j. If $a + b + 1 \leq r$ then by (1.12)

$$\begin{aligned} \lambda - w_{\binom{a}{b}}(\lambda) &= \sum_{\beta_j \in \Phi_w} < \lambda, \alpha_j > \beta_j \\ &= \lambda_0\alpha_0 + \lambda_1(\alpha_0 + \alpha_1) + \dots + \lambda_a(\alpha_0 + \dots + \alpha_a) \\ &\quad + \lambda_r(\alpha_0 + \alpha_r) + \dots + \lambda_{r-b+1}(\alpha_0 + \alpha_r + \dots + \alpha_{r-b+1}) \\ &= (\sum_{i=0}^{a} \lambda_i + \sum_{i=1}^{b} \lambda_{r+1-i})\alpha_0 + \sum_{i=1}^{a}(\sum_{j=i}^{a} \lambda_j)\alpha_i + \sum_{i=1}^{b}(\sum_{j=i}^{b} \lambda_{r+1-j})\alpha_{r+1-i}. \end{aligned} \tag{5.20}$$

In the $\delta - \epsilon$ basis this reduces to

$$\begin{aligned} w_{\binom{a}{b}}(\lambda) - \lambda = &-(\sum_{j=0}^{a} \lambda_j + \sum_{j=1}^{b} \lambda_{r+1-j})\delta + (\lambda_0 + \sum_{j=1}^{b} \lambda_{r+1-j})\epsilon_1 \\ &+ \sum_{i=2}^{a+1} \lambda_{i-1}\epsilon_i - (\sum_{j=0}^{a} \lambda_j)\epsilon_{r+1} - \sum_{i=1}^{b} \lambda_{r+1-i}\epsilon_{r+1-i}. \end{aligned} \tag{5.21}$$

A generalisation of (5.11) and (5.21) for the action of a core element of $\{W : \overline{W}\}$ takes the following form.

Proposition 5.20. *Let $w_\xi = w_{\binom{a_1}{b_1}} \dots w_{\binom{a_p}{b_p}}$ be a core element of $\{W : \overline{W}\}$. Then the action*

$$\begin{aligned} w_\xi(\lambda) - \lambda = &-(p\lambda_0 + \sum_{j=1}^{p}\sum_{i=1}^{a_j} \lambda_i + \sum_{j=1}^{p}\sum_{i=1}^{b_j} \lambda_{r+1-i})\delta \\ &+ \sum_{i=1}^{p}(\sum_{j=0}^{i-1} \lambda_j + \sum_{j=1}^{b_i} \lambda_{r+1-j})\epsilon_i + \sum_{i=p+1}^{a_1+1}\sum_{j=1}^{\xi_i'} \lambda_{i-j}\epsilon_i \\ &- \sum_{i=1}^{p}(\sum_{j=0}^{a_i} \lambda_j + \sum_{j=1}^{i-1} \lambda_{r+1-j})\epsilon_{r+2-i} - \sum_{i=p+1}^{b_1+1}\sum_{j=1}^{\xi_i} \lambda_{r+1-i+j}\epsilon_{r+2-i}, \end{aligned} \tag{5.22}$$

or in terms of the Young diagram $F(\xi)$

$$w_\xi(\lambda) - \lambda = \sum_{(i,j)\in F(\xi)} (-\lambda_{\eta_{ij}}\delta + \lambda_{\eta_{ij}}\epsilon_j - \lambda_{\eta_{ij}}\epsilon_{r+2-i}), \tag{5.23}$$

where

$$\eta_{ij} = \begin{cases} j-i & \text{if } i \le j, \\ r+1-i+j & \text{if } i > j. \end{cases} \tag{5.24}$$

Proof We shall prove this important result by induction on p. When $p=1$, then (5.21) is the required special case of (5.22) and it is easy to see that the action $w_{\binom{a_1}{b_1}}(\lambda) - \lambda$ can also be written in the form

$$w_{\binom{a_1}{b_1}}(\lambda) - \lambda = \sum_{(i,j)\in F\binom{a_1}{b_1}} (-\lambda_{\eta_{ij}}\delta + \lambda_{\eta_{ij}}\epsilon_j - \lambda_{\eta_{ij}}\epsilon_{r+2-i}),$$

in agreement with (5.23). Hence the Proposition is true when $p=1$. Now let $w_\mu = w_\xi w_{\binom{a_{p+1}}{b_{p+1}}} = w_{\binom{a_1}{b_1}} \dots w_{\binom{a_p}{b_p}} w_{\binom{a_{p+1}}{b_{p+1}}}$. Then from (5.20)

$$\begin{aligned} w_{\binom{a_{p+1}}{b_{p+1}}}(\lambda) - \lambda = &-(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i})\alpha_0 - \sum_{i=1}^{a_{p+1}} (\sum_{j=i}^{a_{p+1}} \lambda_j)\alpha_i \\ &- \sum_{i=r+1-b_{p+1}}^{r} (\sum_{j=r+1-i}^{b_{p+1}} \lambda_{r+1-j})\alpha_i, \end{aligned}$$

so that

$$\begin{aligned} w_\xi w_{\binom{a_{p+1}}{b_{p+1}}}(\lambda) - w_\xi(\lambda) = &-(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}) w_\xi(\alpha_0) - \sum_{i=1}^{a_{p+1}} (\sum_{j=i}^{a_{p+1}} \lambda_j) w_\xi(\alpha_i) \\ &- \sum_{i=r+1-b_{p+1}}^{r} (\sum_{j=r+1-i}^{b_{p+1}} \lambda_{r+1-j}) w_\xi(\alpha_i). \end{aligned}$$

Then from (5.8a - 5.8c) we have:

$$\begin{aligned} w_\mu(\lambda) - w_\xi(\lambda) = &-(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i})(\delta - \alpha_{p+1} - \dots - \alpha_{r-p}) \\ &- \sum_{i=1}^{a_{p+1}} (\sum_{j=i}^{a_{p+1}} \lambda_j)(\alpha_{i+p}) - \sum_{i=r+1-b_{p+1}}^{r} (\sum_{j=r+1-i}^{b_{p+1}} \lambda_{r+1-j})(\alpha_{i-p}) \\ = &-(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i})\delta + (\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i})(\epsilon_{p+1} - \epsilon_{r-p+1}) \\ &+ \sum_{i=1}^{a_{p+1}} (\sum_{j=i}^{a_{p+1}} \lambda_j)(\epsilon_{i+p+1} - \epsilon_{i+p}) + \sum_{i=r+1-b_{p+1}}^{r} (\sum_{j=r+1-i}^{b_{p+1}} \lambda_{r+1-j})(\epsilon_{i-p+1} - \epsilon_{i-p}) \end{aligned}$$

$$w_\mu(\lambda) - w_\xi(\lambda) = -\Big(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}\Big)\delta + \sum_{i=0}^{a_{p+1}} \lambda_i \epsilon_{p+1+i} + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}\epsilon_{p+1}$$
$$- \sum_{i=0}^{a_{p+1}} \lambda_i \epsilon_{r-p+1} - \sum_{i=r+1-b_{p+1}}^{r} \lambda_i \epsilon_{i-p} \, .$$

However using the hypothesis to write down $w_\xi(\lambda) - \lambda$ we have:

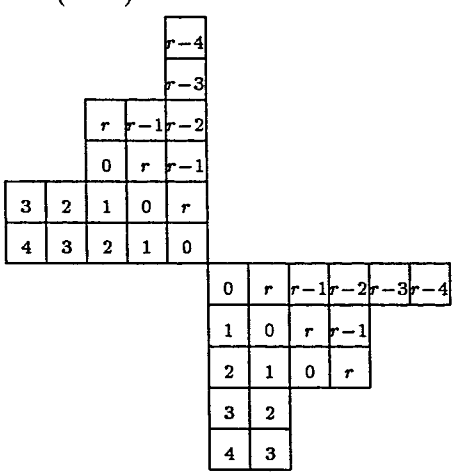
$$w_\mu(\lambda) - \lambda = w_\xi(\lambda) - \lambda - \Big(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}\Big)\delta + \sum_{i=0}^{a_{p+1}} \lambda_i \epsilon_{p+1+i} + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}\epsilon_{p+1}$$
$$- \sum_{i=0}^{a_{p+1}} \lambda_i \epsilon_{r-p+1} - \sum_{i=r+1-b_{p+1}}^{r} \lambda_i \epsilon_{i-p}$$
$$= \sum_{(i,j)\in F(\xi)} (-\lambda_{\eta_{ij}}\delta + \lambda_{\eta_{ij}}\epsilon_j - \lambda_{\eta_{ij}}\epsilon_{r+2-i}) - \Big(\sum_{i=0}^{a_{p+1}} \lambda_i + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}\Big)\delta$$
$$+ \sum_{i=0}^{a_{p+1}} \lambda_i \epsilon_{p+1+i} + \sum_{i=1}^{b_{p+1}} \lambda_{r+1-i}\epsilon_{p+1} - \sum_{i=0}^{a_{p+1}} \lambda_i \epsilon_{r-p+1} - \sum_{i=r+1-b_{p+1}}^{r} \lambda_i \epsilon_{i-p} \, .$$

This can be expanded to show that the coefficients of ϵ_i coincide with the coefficients of ϵ_i in

$$\sum_{(i,j)\in F(\mu)} (-\lambda_{\eta_{ij}}\delta + \lambda_{\eta_{ij}}\epsilon_j - \lambda_{\eta_{ij}}\epsilon_{r+2-i}) \, ,$$

with η as in (5.24). □

The remarkably succinct formulation of (5.23) in terms of Young diagrams lends itself to a simple diagrammatic method for computing $w_\xi(\lambda) - \lambda$. By way of illustration, consider the case of $w_\xi = w_{\binom{4}{5}} w_{\binom{3}{2}} w_{\binom{0}{1}}$ so that $\xi = \binom{430}{521} = (5^2 3^2 1^2)$. The relevant composite Young diagram and the appropriate numbering of its boxes by η_{ij} in accordance with (5.23) and (5.24) take the form:



The depth of $w_\xi(\lambda) - \lambda$ is obtained by adding the contributions λ_η specified by the entries η appearing in each box of $F(\overline{\xi})$ (or equivalently $F(\xi'))$, as displayed above. Similarly the coefficient of ϵ_i is obtained by adding (resp. subtracting) all the contributions λ_η that appear in the corresponding rows of $F(\xi')$ (resp. $F(\overline{\xi})$). Thus for this example the coefficient of $-\delta$ is

$$3\lambda_0 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 + \lambda_{r-4} + \lambda_{r-3} + \lambda_{r-2} + 2\lambda_{r-1} + 3\lambda_r + 3\lambda_{r+1},$$

and the dependence on ϵ_i for $1 \le i \le r+1$ is given by:

$$\begin{aligned}&(\lambda_0 + \lambda_r + \lambda_{r-1} + \lambda_{r-2} + \lambda_{r-3} + \lambda_{r-4})\epsilon_1 + (\lambda_1 + \lambda_0 + \lambda_r + \lambda_{r-1})\epsilon_2\\ &+ (\lambda_2 + \lambda_1 + \lambda_0 + \lambda_r)\epsilon_3 + (\lambda_3 + \lambda_2)\epsilon_4 + (\lambda_4 + \lambda_3)\epsilon_5\\ &- \lambda_{r-4}\epsilon_{r-4} - \lambda_{r-3}\epsilon_{r-3} - (\lambda_r + \lambda_{r-1} + \lambda_{r-2})\epsilon_{r-2}\\ &- (\lambda_0 + \lambda_r + \lambda_{r-1})\epsilon_{r-1} - (\lambda_3 + \lambda_2 + \lambda_1 + \lambda_0 + \lambda_r)\epsilon_r\\ &- (\lambda_4 + \lambda_3 + \lambda_2 + \lambda_1 + \lambda_0)\epsilon_{r+1}.\end{aligned}$$

The above expression is valid for $r \ge 10$. But for the case $r < 10$ we have to apply the modification rule to $F(\overline{\xi};\xi')$ and identify η by filling the remaining boxes with entries taken modulo $(r+1)$ as we will describe below.

By Lemma 5.19 and (5.19) it is not difficult to show that for $c + d \ge r$

$$\begin{aligned}\lambda - w_{(\substack{c\\d})}(\lambda) = (\sum_{i=0}^{r}\lambda_i)\alpha_0 + \lambda_{c+1}\delta + \sum_{i=1}^{c}\lambda_i(\alpha_1 + \alpha_2 + \ldots + \alpha_i)\\ + \sum_{i=1}^{r-c-1}\lambda_{r+1-i}(\alpha_r + \alpha_{r-1} + \ldots + \alpha_{r+1-i})\\ + \sum_{i=1}^{c+d-r}\lambda_{c+1-i}(\alpha_0 + \alpha_r + \alpha_{r-1} + \ldots + \alpha_{c+2-i}),\end{aligned}$$

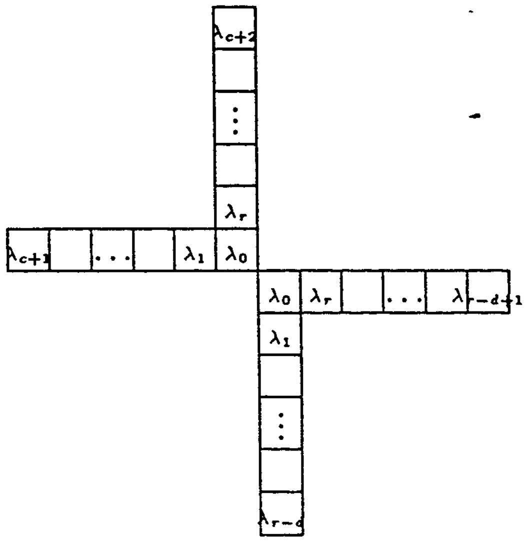
where the third and fourth summations are considered to be zero if $r - c - 1 \le 0$ and $c + d - r \le 0$ respectively. In term of the $\delta - \epsilon$ basis,

$$\begin{aligned}w_{(\substack{c\\d})}(\lambda) - \lambda = (\sum_{j=r-d+1}^{r}\lambda_j + \sum_{j=0}^{c+1}\lambda_j)\delta + (\lambda_0 + \sum_{j=r-d+1}^{r}\lambda_j)\epsilon_1 + \sum_{i=1}^{r-d}\lambda_i\,\epsilon_{i+1}\\ - (\sum_{j=0}^{c+1}\lambda_j)\epsilon_{r+1} - \sum_{i=1}^{r-c-1}\lambda_{r+1-i}\,\epsilon_{r+1-i}\end{aligned}$$

In the light of (5.13) and (5.23), the above expression can be written as

$$w_{({}^{c}_{d})}(\lambda) - \lambda = -\Big(\sum_{j=r-d+1}^{r} \lambda_j + \sum_{j=0}^{c+1} \lambda_j\Big)\delta + \sum_{(i,j)\in F(\mu)} \lambda_{\eta_{ji}}\epsilon_i - \sum_{(i,j)\in F(\nu)} \lambda_{\eta_{ij}}\epsilon_{r+2-i} \qquad (5.25)$$

where $\mu = ({}^{d}_{r-d})$ and $\nu = ({}^{c+1}_{r-c-1})$. Diagrammatically the contributions of $\lambda_{\eta_{ji}}$ and $\lambda_{\eta_{ij}}$ are specified by:



Next let

$$\begin{aligned}\gamma &= \sum_{(i,j)\in F(\mu)} \lambda_{\eta_{ji}}\epsilon_i - \sum_{(i,j)\in F(\nu)} \lambda_{\eta_{ij}}\epsilon_{r+2-i} \\ &= \sum_{i=1}^{\ell(\mu)}\sum_{j=1}^{\mu_i} \lambda_{\eta_{ji}}\epsilon_i - \sum_{i=1}^{\ell(\nu)}\sum_{j=1}^{\nu_i} \lambda_{\eta_{ij}}\epsilon_{r+2-i}\end{aligned}$$

where μ and ν are partition of the same integer. Comparing with (5.14) we can make the following correspondence

$$\mu_i \longleftrightarrow \sum_{j=1}^{\mu_i} \lambda_{\eta_{ji}}$$

$$\nu_i \longleftrightarrow \sum_{j=1}^{\nu_i} \lambda_{\eta_{ij}}\,,$$

and these implies that

$$w_{({}^{c}_{d})}(\gamma) = \hat{w}(\gamma) - \Big(\sum_{j=1}^{\mu_{r-d+1}} \lambda_{\eta_{j,r-d+1}} + \sum_{j=1}^{\nu_{r-c}} \lambda_{\eta_{r-c,j}}\Big)\delta$$

where

$$\begin{aligned}\hat{w}(\gamma) = &\sum_{j=1}^{\mu_{r-d+1}} \lambda_{\eta_{j,r-d+1}}\epsilon_1 + \sum_{i=2}^{r-d+1}\sum_{j=1}^{\mu_{i-1}} \lambda_{\eta_{j,i-1}}\epsilon_i + \sum_{i=r-d+2}^{\ell(\mu)}\sum_{j=1}^{\mu_i} \lambda_{\eta_{ji}}\epsilon_i \\ &- \sum_{j=1}^{\nu_{r-c}} \lambda_{\eta_{r-c,j}}\epsilon_{r+1} - \sum_{i=2}^{r-c}\sum_{j=1}^{\nu_{i-1}} \lambda_{\eta_{i-1,j}}\epsilon_{r+2-i} - \sum_{i=r-c+1}^{\ell(\nu)}\sum_{j=1}^{\nu_i} \lambda_{\eta_{ij}}\epsilon_{r+2-i}.\end{aligned} \qquad (5.26)$$

Noting that by (5.23)

$$\begin{aligned} w_\xi(\lambda) &= \lambda - (\sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}})\delta + \sum_{(ij)\in F(\xi')} \lambda_{\eta_{ji}}\epsilon_i - \sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}}\epsilon_{r+2-i} \\ &= \lambda - (\sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}})\delta + \mu^0(\lambda) - \nu^0(\lambda) \end{aligned}$$

with $\mu^0(\lambda) = \sum_{(ij)\in F(\xi')} \lambda_{\eta_{ji}}\epsilon_i$ and $\nu^0(\lambda) = \sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}}\epsilon_{r+2-i}$. Further by (5.25)

$$\begin{aligned} w_{\binom{c_1}{d_1}} w_\xi(\lambda) = \lambda - (&\sum_{i=r-d_1+1}^{r} \lambda_i + \sum_{i=0}^{c_1+1} \lambda_i)\delta + \sum_{(i,j)\in F(\binom{d_1}{r-d_1})} \lambda_{\eta_{ji}}\epsilon_i \\ &- \sum_{(i,j)\in F(\binom{c_1+1}{r-c_1-1})} \lambda_{\eta_{ij}}\epsilon_{r+2-i} - (\sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}})\delta \\ &- (\sum_{j=1}^{\mu^0_{r-d_1+1}} \lambda_{\eta_{j,r-d_1+1}} + \sum_{j=1}^{\nu^0_{r-c_1}} \lambda_{\eta_{r-c_1,j}})\delta + \hat{w}(\mu^0(\lambda) - \nu^0(\lambda))], \end{aligned} \tag{5.27}$$

where $\hat{w}(\mu^0(\lambda) - \nu^0(\lambda))$ can be obtained from (5.26). As in the case of (5.24) $\mu^0(\lambda) - \nu^0(\lambda)$ can be computed by filling the composite Young diagram $F(\overline{\xi};\xi')$ with corresponding entries $\lambda_{\eta_{ij}}$. It then follows that

$$\begin{aligned} w_{\binom{c_1}{d_1}} w_\xi(\lambda) = \lambda - (&\sum_{i=r-d_1+1}^{r} \lambda_i + \sum_{i=0}^{c_1+1} \lambda_i + \sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}} + \sum_{j=1}^{\mu^0_{r-d_1+1}} \lambda_{\eta_{j,r-d_1+1}} + \sum_{j=1}^{\nu^0_{r-c_1}} \lambda_{\eta_{r-c_1,j}})\delta \\ &+ \mu^1(\lambda) - \nu^1(\lambda) \end{aligned}$$

where

$$\begin{aligned} \mu^1(\lambda) - \nu^1(\lambda) &= \sum_{(i,j)\in F(\binom{d_1}{r-d_1})} \lambda_{\eta_{ji}}\epsilon_i - \sum_{(i,j)\in F(\binom{c_1+1}{r-c_1-1})} \lambda_{\eta_{ij}}\epsilon_{r+2-i} + \hat{w}(\mu^0(\lambda) - \nu^0(\lambda)) \\ &= (\lambda_0 + \sum_{j=r-d_1+1}^{r} \lambda_j)\epsilon_1 + \sum_{i=1}^{r-d_1} \lambda_i\epsilon_{i+1} - (\sum_{j=0}^{c_1+1} \lambda_j)\epsilon_{r+1} - \sum_{i=1}^{r-c_1-1} \lambda_{r+1-i}\epsilon_{r+1-i} \\ &+ \sum_{j=1}^{\mu^0_{r-d_1+1}} \lambda_{\eta_{j,r-d_1+1}}\epsilon_1 + \sum_{i=2}^{r-d_1+1}\sum_{j=1}^{\mu^0_{i-1}} \lambda_{\eta_{j,i-1}}\epsilon_i + \sum_{i=r-d_1+2}^{\ell(\mu^0)}\sum_{j=1}^{\mu^0_i} \lambda_{\eta_{ji}}\epsilon_i \\ &- \sum_{j=1}^{\nu^0_{r-c_1}} \lambda_{\eta_{r-c_1,j}}\epsilon_{r+1} - \sum_{i=2}^{r-c_1}\sum_{j=1}^{\nu^0_{i-1}} \lambda_{\eta_{i-1,j}}\epsilon_{r+2-i} - \sum_{i=r-c_1+1}^{\ell(\nu^0)}\sum_{j=1}^{\nu^0_i} \lambda_{\eta_{ij}}\epsilon_{r+2-i}, \end{aligned} \tag{5.28}$$

where the summations $\sum_{i=r-d_1+2}^{\ell(\mu^0)}\sum_{j=1}^{\mu^0_i}$ and $\sum_{i=r-c_1+1}^{\ell(\nu^0)}\sum_{j=1}^{\nu^0_i}$ are considered to be zero if $r - d_1 + 2 > \ell(\mu^0)$ and $r - c_1 + 1 > \ell(\nu^0)$ respectively.

All the subscripts η of λ necessarily lie in the range $0, 1, \ldots, r$. Without loss of generality we may take these subscripts η modulo $(r+1)$. With this convention it follows from (5.24) that

$$\lambda_{\eta_{i,j}} = \lambda_{j-i} \quad \text{for all } i,j. \tag{5.29}$$

Then

$$\begin{aligned}
\mu^1(\lambda) - \nu^1(\lambda) = &(\sum_{j=0}^{d_1} \lambda_{-j})\epsilon_1 + \sum_{i=1}^{r-d_1} \lambda_i \epsilon_{i+1} - (\sum_{j=0}^{c_1+1} \lambda_j)\epsilon_{r+1} - \sum_{i=1}^{r-c_1-1} \lambda_{-i}\epsilon_{r+1-i} \\
&+ \sum_{j=1}^{\mu^0_{r-d_1+1}} \lambda_{-d_1-j}\epsilon_1 + \sum_{i=2}^{r-d_1+1} \sum_{j=1}^{\mu^0_{i-1}} \lambda_{i-j-1}\epsilon_i + \sum_{i=r-d_1+2}^{\ell(\mu^0)} \sum_{j=1}^{\mu^0_i} \lambda_{i-j}\epsilon_i \\
&- \sum_{j=1}^{\nu^0_{r-c_1}} \lambda_{c_1+1+j}\epsilon_{r+1} - \sum_{i=2}^{r-c_1} \sum_{j=1}^{\nu^0_{i-1}} \lambda_{j-i+1}\epsilon_{r+2-i} - \sum_{i=r-c_1+1}^{\ell(\nu^0)} \sum_{j=1}^{\nu^0_i} \lambda_{j-i}\epsilon_{r+2-i} \\
= &(\sum_{j=0}^{d_1+\mu^0_{r-d_1+1}} \lambda_{-j})\epsilon_1 + \sum_{i=2}^{r-d_1+1} \sum_{j=0}^{\mu^0_{i-1}} \lambda_{i-j-1}\, \epsilon_i + \sum_{i=r-d_1+2}^{\ell(\mu^0)} \sum_{j=1}^{\mu^0_i} \lambda_{i-j}\epsilon_i \\
&- (\sum_{j=0}^{c_1+1+\nu^0_{r-c_1}} \lambda_j)\epsilon_{r+1} - \sum_{i=2}^{r-c_1} \sum_{j=0}^{\nu^0_{i-1}} \lambda_{j-i+1}\epsilon_{r+1-i} - \sum_{i=r-c_1+1}^{\ell(\nu^0)} \sum_{j=1}^{\nu^0_i} \lambda_{j-i}\epsilon_{r+2-i}.
\end{aligned}$$

Let $F(\mu^1)$ and $F(\nu^1)$ be the Young diagrams that can be obtained from $F(\mu^0)$ and $F(\nu^0)$ respectively by adding strips of length $(r+1)$ as in (2.12a). Then $\mu^1(\lambda)$ can be obtained diagrammatically by filling the i^{th}-row of boxes of $F(\mu^1)$ from left to right with the sequence

$$\lambda_{i-1}\,,\ \lambda_{i-2}\,,\ \ldots\,,\ \lambda_{i-\mu^1_i}$$

where

$$\mu^1_i = \begin{cases} d_1 + 1 + \mu^0_{r+1-d_1} & i = 1, \\ \mu^0_{i-1} + 1 & i = 2, \ldots, r+1-d_1, \\ \mu^0_i & i = r+2-d_1, \ldots, \ell(\mu^0) \end{cases}$$

in accordance with (2.14b). Similarly $\nu^1(\lambda)$ can be obtained diagrammatically by filling the i^{th}-row of boxes of $F(\nu^1)$ from right to left with the sequence

$$\lambda_{-i+1}\,,\ \lambda_{-i+2}\,,\ \ldots\,,\ \lambda_{-i+\nu^1_i},$$

where

$$\nu^1_i = \begin{cases} c_1 + 2 + \nu^0_{r-c_1} & i = 1, \\ \nu^0_{i-1} + 1 & i = 2, \ldots, r-c_1, \\ \nu^0_i & i = r+1-c_1, \ldots, \ell(\nu^0). \end{cases}$$

It should be noted that the entries in each added strip are then precisely $\lambda_0, \lambda_1, \ldots, \lambda_r$.

In general $\mu^i(\lambda)$ may be obtained by filling the i^{th} row of $F(\mu^i)$ from left to right with the sequence

$$\lambda_{i-1}\,,\ \lambda_{i-2}\,,\ \ldots\,,\ \lambda_1\,,\ \overline{\lambda_0\,,\ \lambda_r\,,\ \lambda_{r-1}\,,\ \ldots\,,\ \lambda_1} \tag{5.30a}$$

and $\nu^i(\lambda)$ may be obtained by filling the i^{th} row of $F(\nu^i)$ from right to left with the sequence

$$\lambda_{r+2-i}\,,\ \lambda_{r+3-i}\,,\ \ldots\,,\ \lambda_r\,,\ \overline{\lambda_0\,,\ \lambda_1\,,\ \ldots\,,\ \lambda_r} \tag{5.30b}$$

where the overline sequence may be repeated as necessary. Hence we may write (5.28) as

$$\mu^1(\lambda) - \nu^1(\lambda) = \sum_{(i,j)\in F(\mu^1)} \lambda_{\eta_{ji}} \epsilon_i - \sum_{(i,j)\in F(\nu^1)} \lambda_{\eta_{ij}} \epsilon_{r+2-i} \tag{5.31}$$

where $\eta_{ji} = i - j$ and all entries are to be taken modulo $(r+1)$ so as to lie in the range $0, 1, \ldots, r$.

Let $w = w_{\binom{c_q}{d_q}} \ldots w_{\binom{c_1}{d_1}} w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_p}{b_p}}$ be an element of $\{W : \overline{W}\}$ as in Theorem 5.8 with core term $w_\xi = w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_p}{b_p}}$ and such that $\xi = \binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$. Then (5.28), (5.29) and (5.31) implies

$$\begin{aligned} w(\lambda) &= w_{\binom{c_q}{d_q}} \ldots w_{\binom{c_2}{d_2}}[\lambda - (\sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}} - \lambda_0 + \lambda_0 + \sum_{j=1}^{d_1} \lambda_{r+1-j} + \sum_{j=1}^{\mu^0_{r-d_1+1}} \lambda_{r-d_1+1-j})\delta \\ &\quad - (\sum_{j=0}^{c_1+1} \lambda_j + \sum_{j=1}^{\nu^0_{r-c_1}} \lambda_{j+c+1})\delta + \mu^1(\lambda) - \nu^1(\lambda)] \\ &= w_{\binom{c_q}{d_q}} \ldots w_{\binom{c_2}{d_2}}[\lambda - (-\lambda_0 + \sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}} + \mu^1_1(\lambda) + \nu^1_1(\lambda))\delta \\ &\quad + \sum_{(i,j)\in F(\mu^1)} \lambda_{\eta_{ji}} \epsilon_i - \sum_{(i,j)\in F(\nu^1)} \lambda_{\eta_{ij}} \epsilon_{r+2-i}] \end{aligned}$$

where $\mu^1_1(\lambda)$ and $\nu^1_1(\lambda)$ are the coefficients of ϵ_1 and $-\epsilon_{r+1}$, respectively in

$$\sum_{(i,j)\in F(\mu^1)} \lambda_{\eta_{ji}} \epsilon_i - \sum_{(i,j)\in F(\nu^1)} \lambda_{\eta_{ij}} \epsilon_{r+2-i}.$$

Proceeding iteratively,

$$w(\lambda) = \lambda - (-q\lambda_0 + \sum_{(ij)\in F(\xi)} \lambda_{\eta_{ij}} + \sum_{t=1}^{q} \sum_{(1,j)\in F(\mu^t)} \lambda_{\eta_{j,1}} + \sum_{t=1}^{q} \sum_{(1,j)\in F(\nu^t)} \lambda_{\eta_{1,j}})\delta$$
$$+ \sum_{(i,j)\in F(\mu^q)} \lambda_{\eta_{ji}}\epsilon_i - \sum_{(i,j)\in F(\nu^q)} \lambda_{\eta_{ij}}\epsilon_{r+2-i})$$

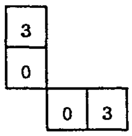
where $F(\mu^t)$ and $F(\nu^t)$ are defined in terms of $F(\mu^{t-1})$ and $F(\nu^{t-1})$, respectively, by adding strips of length $(r+1)$. These results are can all be summarised in the following theorem.

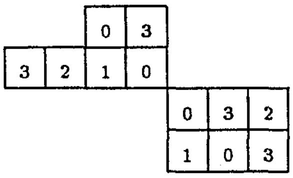
Theorem 5.21. *For affine algebra $A_r^{(1)}$, let $w = w_{\binom{c_q}{d_q}} \dots w_{\binom{c_1}{d_1}} w_{\binom{a_1}{b_1}} \dots w_{\binom{a_p}{b_p}}$ as in Theorem 5.8 and $\xi = \binom{a_1 a_2 \cdots a_p}{b_1 b_2 \cdots b_p}$. Let $F(\mu^t)$ (resp. $F(\overline{\nu^t})$) be the Young diagram obtained by adding t boundary strips each of length $r+1$ to ξ' (resp. $\overline{\xi}$) and covering $d_1+1, d_2+1, \dots, d_t+1$ (resp. $c_1+2, c_2+2, \dots, c_t+2$) columns consecutively. Let $\xi'(\lambda)$ correspond to filling the i^{th} row of boxes of $F(\xi')$ with the sequence in (5.30a). Similarly let $\mu^t(\lambda)$ (resp. $\nu^t(\lambda)$) correspond to filling the i^{th} row of boxes of $F(\mu^t)$ (resp. $F(\overline{\nu^t})$) with the sequence in (5.30a) (resp. (5.30b)). Then*

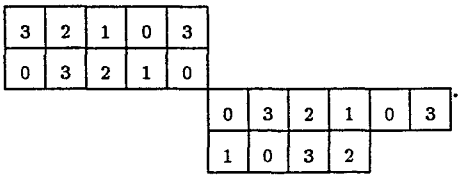
$$w(\lambda) = \lambda - (\xi'(\lambda) + \sum_{t=1}^{q} \mu_1^t(\lambda) + \sum_{t=1}^{q} \nu_1^t(\lambda) - q\lambda_0)\delta + \mu^q(\lambda) - \nu^q(\lambda)\,.$$

where $\mu_1^t(\lambda)$ and $\nu_1^t(\lambda)$ are the coefficients of ϵ_1 and $-\epsilon_{r+1}$, respectively, in $\mu^t(\lambda) - \nu^t(\lambda)$.

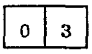
It should be noted that the specific case of this corresponding to $w(\rho) - \rho$ may be recovered directly by setting $\lambda_0 = \lambda_1 = \dots = \lambda_r = 1$ so that the shape of $F(\nu^t; \mu^t)$ is sufficient to define $w(\rho) - \rho$. To illustrate Theorem 5.21 let us note the result of computing $w(\lambda) - \lambda$ where $w = w_{\binom{1}{3}} w_{\binom{1}{2}} w_{\binom{0}{1}}$ in the case of $A_3^{(1)}$. Here $\xi = \binom{0}{1}$ and $q = 2$. First we obtain the Young diagrams $F(\mu^2)$ (resp. $F(\overline{\nu^2})$) by adding to $F\binom{1}{0}$ (resp. $F\binom{0}{1}$) 2 boundary strips each of length $r+1 = 4$. Then we fill the boxes of the composite Young diagram $F(\overline{\nu^2}; \mu^2)$ with the sequence of λ_i as described in (5.30a) and (5.30b). This will gives $\mu^2(\lambda) - \nu^2(\lambda)$.

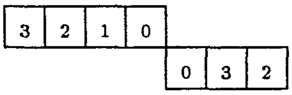
$$F(\overline{\nu^0}; \mu^0) = F(\overline{\xi}; \xi') =$$


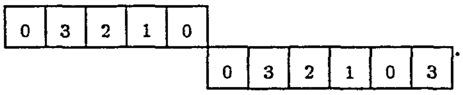
$$F(\overline{\nu^1}; \mu^1) =$$


$$F(\overline{\nu^2}; \mu^2) =$$


The contribution to δ comes from the following:

$$-2\lambda_0 +$$


$$+$$


$$+$$


Hence

$$\begin{aligned}
&(s_0 s_1 s_3 s_2)^2 s_0 s_3(\lambda) - \lambda \\
= &-(-2\lambda_0 + (\lambda_0 + \lambda_3) + (\lambda_3 + \lambda_2 + \lambda_1 + \lambda_0) + (\lambda_0 + \lambda_3 + \lambda_2) \\
&\quad + (\lambda_0 + \lambda_3 + \lambda_2 + \lambda_1 + \lambda_0) + (\lambda_0 + \lambda_3 + \lambda_2 + \lambda_1 + \lambda_0 + \lambda_3))\delta \\
&+ (2\lambda_0 + \lambda_1 + \lambda_2 + 2\lambda_3)\epsilon_1 + (\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)\epsilon_2 \\
&- (2\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)\epsilon_4 - (\lambda_0 + \lambda_1 + \lambda_2 + 2\lambda_3)\epsilon_3 \\
= &-(3L + 2\lambda_0 + \lambda_2 + 3\lambda_3)\delta + (L + \lambda_0 + \lambda_3)\epsilon_1 + L\epsilon_2 - (L + \lambda_3)\epsilon_3 - (L + \lambda_0)\epsilon_4
\end{aligned}$$

where $L = \sum_{i=0}^{3} \lambda_i$ is the level of λ.

5.5. Conjectures on the actions of the right coset representatives on λ

For the other affine algebras we give the following conjectures on the form of $w(\lambda)-\lambda$ which have been arrived at.

Conjecture 5.22. *For $C_r^{(1)}$, let $w = w_{<b_q>}\cdots w_{<b_1>}w_{<a_1>}\cdots w_{<a_p>}$ as in Conjecture 5.13 and $\gamma = \begin{pmatrix} a_1+1 & a_2+1 & \cdots & a_p+1 \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$. Let $F(\mu^t)$ be the Young diagram obtained by adding t boundary strips each of length $2r+2$ to γ and covering $b_1+3, b_2+3, \ldots, b_t+3$ columns consecutively. Let $\mu^t(\lambda)$ (resp. $\gamma(\lambda)$) correspond to filling the i^{th} row of boxes of $F(\mu^t)$ (resp. $F(\gamma)$) with the sequence*

$$\lambda_{i-1}, \lambda_{i-2}, \ldots, \lambda_2, \lambda_1, \lambda_0, \overline{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r, \lambda_r, \lambda_{r-1}, \ldots, \lambda_1, \lambda_0}$$

where the overlined sequence may be repeated as necessary. Then

$$w(\lambda) - \lambda = -\left(\frac{1}{2}\gamma(\lambda) + \sum_{t=1}^{q} \mu_1^t(\lambda) - q\lambda_0\right)\delta + \mu^q(\lambda).$$

To illustrate this, let compute $w_{<3>}^2 w_{<1>}(\lambda) - \lambda$ of $C_3^{(1)}$. In Table 5.3 we have written down the sequences as described in Conjecture 5.22 when $r = 3$.

Table 5.3 : The sequences for computing $w(\lambda) - \lambda$ in the case of $C_3^{(1)}$

0	0	1	2	3	3	2	1	0	0	1	2	3	3	...
1	0	0	1	2	3	3	2	1	0	0	1	2	3	...
2	1	0	0	1	2	3	3	2	1	0	0	1	2	...

On superimposing the Young diagrams $F(\mu^0)$, $F(\mu^1)$ and $F(\mu^2)$, respectively, on the top left hand corner of Table 5.3 we obtain

$F(\mu^0) = F(\gamma) =$

0	0	1
1		

,

$F(\mu^1) =$

0	0	1	2	3	3
1	0	0	1		
2	1				

,

$$F(\mu^2) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ \hline 1 & 0 & 0 & 1 & 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-7} 2 & 1 & 0 & 0 & 1 & \multicolumn{3}{c}{} \\ \cline{1-5} \end{array} .$$

The contribution to δ comes from the following:

$$-2\lambda_0 + \boxed{0 \mid 1} + \boxed{0 \mid 0 \mid 1 \mid 2 \mid 3 \mid 3} + \boxed{0 \mid 0 \mid 1 \mid 2 \mid 3 \mid 3 \mid 2 \mid 1}.$$

This then implies

$$(s_0s_1s_2s_3)^2 s_0s_1(\lambda) - \lambda = -(3\lambda_0 + 4\lambda_1 + 3\lambda_2 + 4\lambda_3)\delta + (2\lambda_0 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3)\epsilon_1$$
$$+ (2\lambda_0 + 2\lambda_1 + \lambda_2 + 2\lambda_3)\epsilon_2 + (2\lambda_0 + 2\lambda_1 + \lambda_2)\epsilon_3.$$

Conjecture 5.23. *For $A_{2r}^{(2)}$, let $w = w_{<b_q>} \dots w_{<b_1>} w_{<a_1>} \dots w_{<a_p>}$ as in Conjecture 5.14 and $\gamma = \begin{pmatrix} a_1+1 & a_2+1 & \cdots & a_p+1 \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$. Let $F(\mu^t)$ be the Young diagram obtained by adding t boundary strips each of length $2r+1$ to γ and covering $b_1+2, b_2+2, \dots, b_t+2$ columns consecutively. Let $\mu^t(\lambda)$ (resp. $\gamma(\lambda)$) correspond to filling the i^{th} row of boxes of $F(\mu^t)$ (resp. $F(\gamma)$) with the sequence*

$$\lambda_{i-1}, \lambda_{i-2}, \dots, \lambda_2, \lambda_1, \lambda_0, \overline{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \dots, \lambda_1, \lambda_0}$$

where the overlined sequence may be repeated as necessary. Then

$$w(\lambda) - \lambda = -(\frac{1}{2}\gamma(\lambda) + \sum_{t=1}^{q} \mu_1^t(\lambda) - q\lambda_0)\delta + \mu^q(\lambda).$$

To illustrate this, let compute $w_{<3>}^2 w_{<1>}(\lambda) - \lambda$ of $A_6^{(2)}$. In Table 5.4 we have written down the sequences as described in Conjecture 5.23 when $r = 3$.

Table 5.4 : The sequences for computing $w(\lambda) - \lambda$ in the case of $A_6^{(2)}$

0	0	1	2	3	2	1	0	0	1	2	3	2	1	...
1	0	0	1	2	3	2	1	0	0	1	2	3	2	...
2	1	0	0	1	2	3	2	1	0	0	1	2	3	...

On superimposing the Young diagrams $F(\mu^0)$, $F(\mu^1)$ and $F(\mu^2)$, respectively, on the top left hand corner of Table 5.4 we obtain

$$F(\mu^0) = F(\gamma) = \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array},$$

$$F(\mu^1) = \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 1 & \multicolumn{3}{c}{} \\ \cline{1-2}\end{array},$$

$$F(\mu^2) = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 3 & 2 & 1 \\ \hline 1 & 0 & 0 & 1 & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-6} 2 & 1 & 0 & 0 & 1 & \multicolumn{2}{c}{} \\ \cline{1-5}\end{array}.$$

The contribution to δ comes from the following:

$$-2\lambda_0 + \begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array} + \begin{array}{|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 0 & 1 & 2 & 3 & 2 & 1 \\ \hline\end{array}.$$

This then implies

$$(s_0 s_1 s_2 s_3)^2 s_0 s_1(\lambda) - \lambda = -(3\lambda_0 + 4\lambda_1 + 3\lambda_2 + 2\lambda_3)\delta + (2\lambda_0 + 2\lambda_1 + 2\lambda_2 + \lambda_3)\epsilon_1$$
$$+ (2\lambda_0 + 2\lambda_1 + \lambda_2 + \lambda_3)\epsilon_2 + (2\lambda_0 + 2\lambda_1 + \lambda_2)\epsilon_3.$$

Conjecture 5.24. *For $D^{(2)}_{r+1}$, let $w = w_{<b_q>} \cdots w_{<b_1>} w_{<a_1>} \cdots w_{<a_p>}$ as in Conjecture 5.15 and $\epsilon = \binom{a_1\, a_2 \cdots a_p}{a_1\, a_2 \cdots a_p}$. Let $F(\mu^t)$ be the Young diagram obtained by adding t boundary strips each of length $2r$ to ϵ and covering $b_1+1, b_2+1, \ldots, b_t+1$ columns consecutively. Let $\mu^t(\lambda)$ (resp. $\epsilon(\lambda)$) correspond to filling the i^{th} row of boxes of $F(\mu^t)$ (resp. $F(\epsilon)$) with the sequence*

$$\lambda_{i-1}, \lambda_{i-2}, \ldots, \lambda_1, \overline{\lambda_0, \lambda_1, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \ldots, \lambda_1}$$

where the overlined sequence may be repeated as necessary. Then

$$w(\lambda) - \lambda = -(\epsilon(\lambda) + 2\sum_{t=1}^{q} \mu_1^t(\lambda) - q\lambda_0)\delta + \mu^q(\lambda).$$

To illustrate this, let compute $w^2_{<3>} w_{<1>}(\lambda) - \lambda$ of $D_4^{(2)}$. In Table 5.5 we have written down the sequences as described in Conjecture 5.24 when $r = 3$.

Table 5.5 : The sequences for computing $w(\lambda) - \lambda$ in the case of $D_4^{(2)}$

0	1	2	3	2	1	0	1	2	3	2	1	0	1	...
1	0	1	2	3	2	1	0	1	2	3	2	1	0	...
2	1	0	1	2	3	2	1	0	1	2	3	2	1	...

On superimposing the Young diagrams $F(\mu^0)$, $F(\mu^1)$ and $F(\mu^2)$, respectively, on the top left hand corner of Table 5.5 we obtain

$$F(\mu^0) = F(\epsilon) = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

$$F(\mu^1) = \begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & \\ \cline{1-3} 2 & 1 & & \\ \cline{1-2}\end{array}$$

$$F(\mu^2) = \begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 2 & 1 \\ \hline 1 & 0 & 1 & 2 & 3 & \\ \cline{1-5} 2 & 1 & 0 & 1 & & \\ \cline{1-4}\end{array}.$$

The contribution to δ comes from the following:

$$-2\lambda_0 + \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array} + 2(\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline\end{array}) + 2(\begin{array}{|c|c|c|c|c|c|}\hline 0 & 1 & 2 & 3 & 2 & 1 \\ \hline\end{array}).$$

This then implies

$$(s_0s_1s_2s_3)^2s_0s_1(\lambda) - \lambda = -(3\lambda_0 + 8\lambda_1 + 6\lambda_2 + 4\lambda_3)\delta + (\lambda_0 + 2\lambda_1 + 2\lambda_2 + \lambda_3)\epsilon_1$$
$$+ (\lambda_0 + 2\lambda_1 + \lambda_2 + \lambda_3)\epsilon_2 + (\lambda_0 + 2\lambda_1 + \lambda_2)\epsilon_3.$$

Conjecture 5.25. *For $B_r^{(1)}$, let $w = w_{[b_q]} \dots w_{[b_1]} w_{[a_1]} \dots w_{[a_p]}$ as in Conjecture 5.16 and $\alpha = \left({a_1-1 \atop a_1} {a_2-1 \atop a_2} {\cdots \atop \cdots} {a_p-1 \atop a_p} \right)$. Let $F(\mu^t)$ be the Young diagram obtained by adding t boundary strips each of length $2r-1$ to α and covering $b_1, b_2, \dots, b_t$ columns consecutively. Let $\mu^t(\lambda)$ (resp. $\alpha(\lambda)$) correspond to filling the i^{th} row of boxes of $F(\mu^t)$ (resp. $F(\alpha)$) with the following sequence:*

if $p+q$ even

$$\lambda_{i-1}, \lambda_{i-2}, \dots, \lambda_1, \overline{\lambda_0, \lambda_2, \lambda_3, \dots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \dots, \lambda_1}$$

and if $p+q$ odd

$$\begin{cases} \lambda_0, \lambda_2, \lambda_3, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \ldots, \lambda_2, \lambda_0, \overline{\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \ldots, \lambda_2, \lambda_0} & i=1, \\ \lambda_0, \overline{\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \ldots, \lambda_2, \lambda_0} & i=2, \\ \lambda_{i-1}, \ldots, \lambda_2, \lambda_0, \overline{\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r, \lambda_{r-1}, \ldots, \lambda_2, \lambda_0} & i>2. \end{cases}$$

Further suppose that the element $w_{[b_i]}$ begins with the fundamental reflection s_k ($k = 0,1$). Let $\hat{\mu}_1^t(\lambda)$ be obtained from $\mu_1^t(\lambda)$ by replacing the first entry with λ_k but retaining the rest of the entries. Then

$$w(\lambda) - \lambda == -(\frac{1}{2}\alpha(\lambda) + \sum_{t=1}^{q} \hat{\mu}_1^t(\lambda))\delta + \mu^q(\lambda).$$

To illustrate this let us note the result of computing $w(\lambda) - \lambda$ of $B_4^{(1)}$ for a few cases. In Table 5.6 we have written down the sequences as described in Conjecture 5.25 when $r = 4$.

Table 5.6 : The sequences for computing $w(\lambda) - \lambda$ in the case of $B_4^{(1)}$

If $p+q$ is even

0	2	3	4	3	2	1	0	2	3	4	3	2	1	...
1	0	2	3	4	3	2	1	0	2	3	4	3	2	...
2	1	0	2	3	4	3	2	1	0	2	3	4	3	...
3	2	1	0	2	3	4	3	2	1	0	2	3	4	...

If $p+q$ is odd

0	2	3	4	3	2	0	1	2	3	4	3	2	0	...
0	1	2	3	4	3	2	0	1	2	3	4	3	2	...
2	0	1	2	3	4	3	2	0	1	2	3	4	3	...
3	2	0	1	2	3	4	3	2	0	1	2	3	4	...

Let $w = w_{[3]}^{(0)} w_{[2]}^{(1)} w_{[1]}^{(0)} = s_0 s_2 s_3\, s_1 s_2\, s_0$. This is a core element with p odd. On superimposing the Young diagram $F(\alpha)$ on the top left hand corner of Table 5.6 we obtain

$$\begin{array}{|c|c|c|}\hline 0 & 2 & 3 \\ \hline 0 & 1 & 2 \\ \hline 2 & 0 & 1 \\ \hline 3 & 2 & 0 \\ \hline \end{array}$$

Then

$$w(\lambda) - \lambda = -(2\lambda_0 + \lambda_1 + 2\lambda_2 + \lambda_3)\delta + (\lambda_0 + \lambda_2 + \lambda_3)\epsilon_1$$

$$+ (\lambda_0 + \lambda_1 + \lambda_2)\epsilon_2 + (\lambda_0 + \lambda_1 + \lambda_2)\epsilon_3 + (\lambda_0 + \lambda_2 + \lambda_3)\epsilon_4.$$

Next let $w = w^{(0)}_{[5]} w^{(1)}_{[5]} w^{(0)}_{[3]} w^{(1)}_{[1]} = s_0 s_2 s_3 s_4 s_3\, s_1 s_2 s_3 s_4 s_3\, s_0 s_2 s_3\, s_1$ where $p = 2$ and $q = 2$. On superimposing the Young diagrams $F(\mu^0)$, $F(\mu^1)$ and $F(\mu^2)$, respectively, on the top left hand corner of Table 5.6 we obtain

$$F(\mu^0) = F(\alpha) = \begin{array}{|c|c|c|}\hline 0 & 2 & 3 \\ \hline 1 & 0 \\ \cline{1-2} 2 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

$$F(\mu^1) = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 1 \\ \hline 1 & 0 & 2 & 3 \\ \cline{1-4} 2 & 1 & 0 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

$$F(\mu^2) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 2 & 3 & 4 & 3 & 2 & 1 \\ \hline 2 & 1 & 0 & 2 & 3 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array}.$$

The depth comes from the following diagrams

$$\frac{1}{2}\left(\begin{array}{|c|c|c|}\hline 0 & 2 & 3 \\ \hline 1 & 0 \\ \cline{1-2} 2 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}\right) + \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}.$$

Then

$$w(\lambda) - \lambda = -(3\lambda_0 + 4\lambda_1 + 5\lambda_2 + 5\lambda_3 + 2\lambda_4)\delta + (2\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4)\epsilon_1$$

$$+ (\lambda_0 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4)\epsilon_2 + (\lambda_0 + \lambda_1 + 2\lambda_2 + \lambda_3)\epsilon_3 + \lambda_3\epsilon_4.$$

Finally let $w = w^{(0)}_{[5]} w^{(1)}_{[5]} w^{(0)}_{[5]} w^{(1)}_{[3]} w^{(0)}_{[1]} = s_0 s_2 s_3 s_4 s_3 s_1 s_2 s_3 s_4 s_3 s_0 s_2 s_3 s_4 s_3 s_1 s_2 s_3 s_0$ where $p = 2$ and $q = 3$. On superimposing the Young diagrams $F(\mu^0)$, $F(\mu^1)$, $F(\mu^2)$ and

$F(\mu^3)$, respectively, on the top left hand corner of Table 5.6 we obtain

$$F(\mu^0) = F(\alpha) = \begin{array}{|c|c|c|}\hline 0 & 2 & 3 \\ \hline 1 & 0 & \\ \cline{1-2} 2 & 1 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array}$$

$$F(\mu^1) = \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 0 \\ \hline 0 & 1 & 2 & 3 & & & \\ \cline{1-4} 2 & 0 & 1 & & & & \\ \cline{1-3} 3 & & & & & & \\ \cline{1-1} \end{array}$$

$$F(\mu^2) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 0 & 1 \\ \hline 0 & 1 & 2 & 3 & 4 & 3 & 2 & 0 \\ \hline 2 & 0 & 1 & 2 & 3 & & & \\ \cline{1-5} 3 & & & & & & & \\ \cline{1-1} \end{array}$$

$$F(\mu^3) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & 3 & 4 & 3 & 2 & 0 & 1 & \\ \cline{1-9} 2 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 0 & \\ \cline{1-9} 3 & & & & & & & & & \\ \cline{1-1} \end{array}.$$

The depth comes from the following diagrams

$$\frac{1}{2}\left(\begin{array}{|c|c|c|}\hline 0 & 2 & 3 \\ \hline 1 & 0 & \\ \cline{1-2} 2 & 1 & \\ \cline{1-2} 3 & & \\ \cline{1-1} \end{array}\right) + \begin{array}{|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 0 \\ \hline \end{array}$$

$$+ \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 3 & 2 & 0 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 0 & 2 & 3 & 4 & 3 & 2 & 0 & 1 & 2 & 3 \\ \hline \end{array}.$$

Then

$$\begin{aligned} w(\lambda) - \lambda = & - (6\lambda_0 + 4\lambda_1 + 8\lambda_2 + 8\lambda_3 + 3\lambda_4)\delta \\ & + (2\lambda_0 + \lambda_1 + 3\lambda_2 + 3\lambda_3 + \lambda_4)\epsilon_1 + (2\lambda_0 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4)\epsilon_2 \\ & + (2\lambda_0 + \lambda_1 + 3\lambda_2 + 2\lambda_3 + \lambda_4)\epsilon_3 + \lambda_3\epsilon_4. \end{aligned}$$

Conjecture 5.26. *For $A^{(2)}_{2r-1}$, let $w = w_{[b_q]} \dots w_{[b_1]} w_{[a_1]} \dots w_{[a_p]}$ as in Conjecture 5.17 and $\alpha = \begin{pmatrix} a_1-1 & a_2-1 & \cdots & a_p-1 \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$. Let $F(\mu^t)$ be the Young diagram obtained by adding t boundary strips each of length $2r$ to α and covering $b_1+1, b_2+1, \dots, b_t+1$ columns consecutively. Let $\mu^t(\lambda)$ (resp. $\alpha(\lambda)$) correspond to filling the i^{th} row of boxes of $F(\mu^t)$ (resp. $F(\alpha)$) with the following sequence:*

if $p+q$ even

$$\lambda_{i-1}, \lambda_{i-2}, \dots, \lambda_1, \overline{\lambda_0, \lambda_2, \lambda_3, \dots, \lambda_{r-1}, \lambda_r, \lambda_r, \lambda_{r-1}, \dots, \lambda_1}$$

and if $p+q$ odd

$$\begin{cases} \lambda_0, \lambda_2, \dots, \lambda_{r-1}, \lambda_r, \lambda_r, \lambda_{r-1}, \dots, \lambda_2, \lambda_0, \overline{\lambda_1, \dots, \lambda_r, \lambda_r, \lambda_{r-1}, \dots, \lambda_2, \lambda_0} & i=1, \\ \lambda_0, \overline{\lambda_1, \dots, \lambda_{r-1}, \lambda_r, \lambda_r, \lambda_{r-1}, \dots, \lambda_2, \lambda_0} & i=2, \\ \lambda_{i-1}, \dots, \lambda_2, \lambda_0, \overline{\lambda_1, \dots, \lambda_{r-1}, \lambda_r, \lambda_r, \lambda_{r-1}, \dots, \lambda_2, \lambda_0} & i>2. \end{cases}$$

Then

$$w(\lambda) - \lambda == -(\frac{1}{2}\alpha(\lambda) + \sum_{t=1}^{q} \hat{\mu}_1^t(\lambda))\delta + \mu^q(\lambda)$$

where $\hat{\mu}_1^t(\lambda)$ are as in Conjecture 5.25.

To illustrate this, let compute $w = w_{[5]}^{(0)} w_{[5]}^{(1)} w_{[3]}^{(0)} w_{[1]}^{(1)}(\lambda) - \lambda$ of $A_7^{(2)}$. In Table 5.7 we have written down the sequences as described in Conjecture 5.26 when $r=4$.

Table 5.7 : The sequences for computing $w(\lambda) - \lambda$ in the case of $A_7^{(2)}$

If $p+q$ is even

0	2	3	4	4	3	2	1	0	2	3	4	4	3	...
1	0	2	3	4	4	3	2	1	0	2	3	4	4	...
2	1	0	2	3	4	4	3	2	1	0	2	3	4	...
3	2	1	0	2	3	4	4	3	2	1	0	2	3	...

If $p+q$ is odd

0	2	3	4	4	3	2	0	1	2	3	4	4	3	...
0	1	2	3	4	4	3	2	0	1	2	3	4	4	...
2	0	1	2	3	4	4	3	2	0	1	2	3	4	...
3	2	0	1	2	3	4	4	3	2	0	1	2	3	...

On superimposing the Young diagrams $F(\mu^0)$, $F(\mu^1)$ and $F(\mu^2)$, respectively, on the top left hand corner of Table 5.7 we obtain

$$F(\mu^0) = F(\alpha) = \begin{array}{|c|c|c|} \hline 0 & 2 & 3 \\ \hline 1 & 0 \\ \cline{1-2} 2 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

$$F(\mu^1) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\ \hline 1 & 0 & 2 & 3 \\ \cline{1-4} 2 & 1 & 0 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

$$F(\mu^2) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 0 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 0 \\ \hline 1 & 0 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\ \hline 2 & 1 & 0 & 2 & 3 \\ \cline{1-5} 3 \\ \cline{1-1} \end{array}.$$

The depth comes from the following diagrams

$$\frac{1}{2}\left(\begin{array}{|c|c|c|} \hline 0 & 2 & 3 \\ \hline 1 & 0 \\ \cline{1-2} 2 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}\right) + \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 4 & 3 & 2 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 0 & 2 & 3 & 4 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}.$$

Then

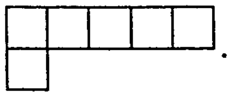
$$w(\lambda) - \lambda = -(3\lambda_0 + 4\lambda_1 + 5\lambda_2 + 5\lambda_3 + 4\lambda_4)\delta + (2\lambda_0 + \lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4)\epsilon_1$$
$$+ (\lambda_0 + 2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 2\lambda_4)\epsilon_2 + (\lambda_0 + \lambda_1 + 2\lambda_2 + \lambda_3)\epsilon_3 + \lambda_3\epsilon_4.$$

For the rank dependent series of affine algebras we are finally left to determine the action $w(\lambda) - \lambda$ for $D_r^{(1)}$. As has been noted in obtaining Conjecture 5.18 we have found it is necessary to introduce a slightly different form for the elements of $\{W : \overline{W}\}$. This create further difficulties, in determining the action $w(\lambda) - \lambda$ diagrammatically. We have yet to resolve these problem. To illustrate these difficulties let us compute $w_{[6]}^{(0)}(\lambda) - \lambda$ in the case of $D_4^{(1)}$.

As has been noted in the example following Conjecture 5.18, $w_{[6]}^{(0)} = s_0 s_2 s_3 s_4 s_2$ is a valid non-core element of $\{W : \overline{W}\}$ since

$$w_{[6]}^{(0)}(\rho) - \rho = -5\delta + 5\epsilon_1 + \epsilon_2\,.$$

The Young diagram associated with this Weyl element and action is



However

$$\begin{aligned} w^{(0)}_{[6]}(\lambda) - \lambda =& -(\lambda_0 + 2\lambda_2 + \lambda_3 + \lambda_4)\alpha_0 - (2\lambda_2 + \lambda_3 + \lambda_4)\alpha_2 \\ & - (\lambda_2 + \lambda_3)\alpha_3 - (\lambda_2 + \lambda_4)\alpha_4 \\ =& -(\lambda_0 + 2\lambda_2 + \lambda_3 + \lambda_4)\delta + (\lambda_0 + 2\lambda_2 + \lambda_3 + \lambda_4)\epsilon_1 \\ & + \lambda_0\epsilon_2 + (\lambda_3 - \lambda_4)\epsilon_4 \,. \end{aligned}$$

Since there is a gap with the coefficient of ϵ_3 being zero, there is no way that we can represent the action $w^{(0)}_{[6]}(\lambda) - \lambda$ by filling a continuous boundary strip with Dynkin components of λ. There is also a term in ϵ_4 whose coefficient is zero if $\lambda = \rho$, but may be positive, negative or zero for other λ.

However, it should be noted that, although $w^{(0)}_{[6]}(\lambda) - \lambda \notin P^+$ for some λ, but Lemma 1.13 implies that $w^{(0)}_{[6]}(\lambda + \rho) - \rho$ is still a dominant weight if λ itself is dominant. In this particular example, we have

$$\begin{aligned} w^{(0)}_{[6]}(\lambda + \rho) - \rho =& L(\lambda)\Lambda_0 - (\lambda_0 + 2\lambda_2 + \lambda_3 + \lambda_4 + 5)\delta \\ & + (\lambda_0 + \lambda_1 + 3\lambda_2 + \frac{3}{2}\lambda_3 + \frac{3}{2}\lambda_4 + 5)\epsilon_1 \\ & + (\lambda_0 + \lambda_2 + \frac{1}{2}\lambda_3 + \frac{1}{2}\lambda_4 + 1)\epsilon_2 \\ & + (\frac{1}{2}\lambda_3 + \frac{1}{2}\lambda_4)\epsilon_3 + (\frac{1}{2}\lambda_3 - \frac{1}{2}\lambda_4)\epsilon_4 \,, \end{aligned}$$

which is dominant for all non-negative $\lambda_0, \lambda_1, \ldots, \lambda_r$.

CHAPTER 6

Branching Rules

6.1 Basic theory

A Lie subalgebra $\mathcal{G}'$ of the Lie algebra $\mathcal{G}$ is a subvectorspace which itself is a Lie algebra. A subalgebra $\mathcal{G}'$ is called a regular subalgebra if the roots of $\mathcal{G}'$ are contained in the root system of $\mathcal{G}$. Otherwise $\mathcal{G}'$ is called a special subalgebra. The problem of classifying the maximal semisimple subalgebras of simple finite-dimensional Lie algebras has been dealt with in the article of Dynkin **[D]**.

An embedding of a subalgebra $\mathcal{G}'$ into a Lie algebra $\mathcal{G}$ is a mapping f of $\mathcal{G}'$ into $\mathcal{G}$. Given an embedding $f : \mathcal{G}' \to \mathcal{G}$ and an irreducible representation $\psi(\mathcal{G})$, the representation $\psi(\mathcal{G})$ becomes a representation $\psi(f(\mathcal{G}'))$ of $\mathcal{G}'$ which can be either reducible or irreducible. If $\psi(f(\mathcal{G}'))$ is reducible then the decomposition **[McP]**

$$\psi(\mathcal{G}) \supset \psi(f(\mathcal{G}')) = \psi_1(f(\mathcal{G}')) \oplus \psi_2(f(\mathcal{G}')) \oplus \ldots \tag{6.1}$$

is called the branching rule of $\mathcal{G}$ with respect to the subalgebra $\mathcal{G}'$. The multiplicity of occurrence of the irreducible representations $\psi_i(f(\mathcal{G}'))$ in the decomposition (6.1) are called the branching rule multiplicities and they are necessarily non-negative integers. The same subalgebra $\mathcal{G}'$ can often be embedded in a given algebra $\mathcal{G}$ in different ways with different branching rules. The embedding $f : \mathcal{G}' \to \mathcal{G}$ induces a projection between the weight spaces of $\mathcal{G}$ and of $\mathcal{G}'$.

Correspondingly, the restriction of the characters $ch\, V^\lambda$, of $\mathcal{G}$ to $\mathcal{G}'$, induces a mapping of the form

$$ch\, V^\lambda \to \sum_{\mu'} b^\lambda_{\mu'}\, ch\, V^{\mu'} \tag{6.2}$$

where care has to be taken in defining consistently the mapping from the weight space of $\mathcal{G}$ to that of $\mathcal{G}'$. The coeffients $b^{\lambda}_{\mu'}$ are the branching rule multiplicities of each irreducible constituent $V^{\mu'}$. If $\mathcal{G}'$ is a regular subalgebra then the Dynkin labels of the weights of $\mathcal{G}'$-module under the projection are just the Dynkin labels given in the usual way by

$$\mu'_i = < \mu, \alpha'^{\vee}_i > . \tag{6.3}$$

The problem of obtaining branching rules for representations of simple finite-dimensional Lie algebras restricted to Lie subalgebras has been treated by various methods. Extensive tables of branching rules for simple finite-dimensional Lie algebras have already been given by McKay and Patera **[McP]**. An obvious method for obtaining the branching rule (6.2) is to proceed in three stages. First we find the weights of $\mathcal{G}$-modules. Then the weights are transformed into the weights of the subalgebra $\mathcal{G}'$. Finally these weights are sorted out into the weights of $\mathcal{G}'$-modules.

In order to make use of the orbit-character and character-orbit expansions given in Chapter 3 and 4 in obtaining affine branching rules we describe first the method discussed by Patera and Sharp **[PS]** in the framework of simple finite-dimensional Lie alegebras. This technique also works in the affine algebra case **[B]**. The method as described in **[PS]** consists of three steps:

(B1) Express the irreducible $\mathcal{G}$-character in terms of $\mathcal{G}$-orbits;

(B2) Decompose $\mathcal{G}$-orbits to $\mathcal{G}'$-orbits; (6.4)

(B3) Express the $\mathcal{G}'$-orbits in term of irreducible $\mathcal{G}$-characters.

Step B1 requires the weight multiplicities of dominant weights which can be obtained, for example, directly from the tabulation of **[BMP]**. Step B3 just amounts to inverting weight multiplicity matrices which also can be done easily. The only problem lies in decomposing the $\mathcal{G}$-orbit into $\mathcal{G}'$-orbits. However if the projection of the weights are known then the decomposition of $\mathcal{G}$-orbits into $\mathcal{G}'$-orbits are obtained merely by retaining the weights of $\mathcal{G}'$ modules which have all components non-negative, i.e. are $\mathcal{G}'$-dominant.

For illustration let us consider an embedding of $A_2 \oplus u_1$ in A_3 where u_1 is the abelian Lie algebra of dimension 1. The representation theory of u_1 is quite trivial. The embedding is such that the simple roots of A_2 may be taken to be:

$$\alpha_1' \rightarrow \alpha_1 + \alpha_2$$

$$\alpha_2' \rightarrow \alpha_3$$

where α_1, α_2 and α_3 are the simple roots of A_3. Then an A_3 weight $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ becomes an A_2 weight (λ_1', λ_2') where

$$\lambda_1' = <\lambda, \alpha_1'^{\vee}> = <\lambda, \alpha_1 + \alpha_2> = \lambda_1 + \lambda_2$$

$$\lambda_2' = <\lambda, \alpha_2'^{\vee}> = <\lambda, \alpha_3> = \lambda_3 .$$

In order to obtain the label for u_1 which necessarily takes the form $k_1\lambda_1 + k_2\lambda_2 + k_3\lambda_3$ where k_1, k_2 and k_3 are constants to be determined, consider the Weyl orbit of $(1,0,0)$,

$$\{\,(1,0,0), (-1,1,0), (0,-1,1), (0,0,-1)\,\}.$$

As an $A_2 \oplus u_1$ weight these become

$$\{\,(1,0;k_1), (0,0;-k_1+k_2), (-1,1;-k_2+k_3), (0,-1;-k_3)\,\}\,.$$

However the weights $(1,0)$, $(-1,1)$ and $(0,-1)$ form the Weyl orbit of $(1,0)$ so that

$$k_1 \;=\; -k_2 + k_3 \;=\; -k_3\,.$$

If we further fix the scale by letting the u_1 label of A_2 Weyl orbits $(1,0)$ and $(0,0)$ differ by unity then we obtain the following projection for the weights of A_3 to the weights of $A_2 \oplus u_1$

$$(\lambda_1, \lambda_2, \lambda_3) \rightarrow (\lambda_1 + \lambda_2, \lambda_3\,;\, (\lambda_1 - 2\lambda_2 - \lambda_3)/4)\,.$$

In particular the projected weights of the $(1,0,0)$ orbit becomes

$$\{\,(1,0;\frac{1}{4}),(0,0;-\frac{3}{4}),(-1,1;\frac{1}{4}),(0,-1;\frac{1}{4})\,\}$$

and on retaining the components of A_2 which are non-negative we obtained the decomposition

$$\Omega^{(1,0,0)} \longrightarrow \Omega'^{(1,0;1/4)} + \Omega'^{(0,0;-3/4)}.$$

In a similar way we obtain

$$\Omega^{(0,1,0)} \longrightarrow \Omega'^{(0,1;1/2)} + \Omega'^{(1,0;-1/2)}$$

$$\Omega^{(0,0,1)} \longrightarrow \Omega'^{(0,0;3/4)} + \Omega'^{(0,1;-1/4)}$$

$$\Omega^{(1,0,1)} \longrightarrow \Omega'^{(1,0;1)} + \Omega'^{(1,1;0)} + \Omega'^{(0,1;-1)}$$

$$\Omega^{(0,2,0)} \longrightarrow \Omega'^{(0,2;1)} + \Omega'^{(2,0;-1)}$$

$$\Omega^{(2,1,0)} \longrightarrow \Omega'^{(2,1;1)} + \Omega'^{(3,0;0)} + \Omega'^{(1,0;-2)}$$

Then from the orbit multiplicities table **[BMP]**, we have

$$\begin{aligned}
ch\,V^{(2,1,0)} &= \Omega^{(2,1,0)} + \Omega^{(0,2,0)} + 2\Omega^{(1,0,1)} + 3\Omega^{(0,0,0)} \\
&\longrightarrow \{\Omega'^{(2,1;1)} + \Omega'^{(3,0;0)} + \Omega'^{(1,0;-2)}\} + \{\Omega'^{(0,2;1)} + \Omega'^{(2,0;-1)}\} \\
&\quad + 2\{\Omega'^{(1,0;1)} + \Omega'^{(1,1;0)} + \Omega'^{(0,1;-1)}\} + 3\Omega'^{(0,0;0)} \\
&= \Omega'^{(2,1;1)} + \Omega'^{(0,2;1)} + 2\Omega'^{(1,0;1)} + \Omega'^{(3,0;0)} + 2\Omega'^{(1,1;0)} + 3\Omega'^{(0,0;0)} \\
&\quad + \{\Omega'^{(2,0;-1)} + 2\Omega'^{(0,1;-1)}\} + \Omega'^{(1,0;-2)} \\
&= ch\,V^{(2,1;1)} + ch\,V^{(3,0;0)} + ch\,V^{(1,1;0)} + ch\,V^{(2,0;-1)} + ch\,V^{(0,1;-1)} + ch\,V^{(1,0;-2)}
\end{aligned}$$

We see that in this particular example the $\mathcal{G}$-module decomposes into a finite number of $\mathcal{G}'$-modules. The same is true for all finite dimensional modules of simple finite-dimensional Lie algebras. However for affine algebras this is no longer the case and in general $\mathcal{G}'$-modules may appear with infinite multiplicity in an affine $\mathcal{G}$-module.

6.2. Simple finite-dimensional Lie subalgebras of affine algebras and weight multiplicity polynomials.

The simplest subalgebras of a given affine algebra $\mathcal{G}(A)$ are those whose Dynkin diagram may be obtained from the Dynkin diagram of $\mathcal{G}(A)$ by dropping one node, say the i^{th} node. The resulting diagram is that of a semisimple finite-dimensional Lie algebra $\overline{\mathcal{G}}_i$. Although there already exist extensive tables of branching rules $\mathcal{G} \supset \overline{\mathcal{G}}_i$ of these regular embeddings **[KMPS]**, the computation has been done case by case, one rank at a time. Rather than dropping an arbitrary node we consider here the more specific case of dropping the zeroth node from the Dynkin diagram of $\mathcal{G}(A)$. Then the resulting simple finite-dimensional Lie algebra is $\mathcal{G}(\bar{A})$ or just $\overline{\mathcal{G}}$.

From (5.2) we can write

$$w(\lambda+\rho)-\rho = \frac{L(\lambda)}{c_0^\vee}\Lambda_0 - d_w(\lambda+\rho)\delta + \overline{w(\lambda+\rho)-\rho}\,. \tag{6.5}$$

Then the numerator of the Weyl-Kostant-Liu character formula (1.25) can be written as

$$\begin{aligned} N^\lambda &= \sum_{w\in\{W:\overline{W}\}} \varepsilon(w)\, ch\, \overline{V}^{w(\lambda+\rho)-\rho} \\ &= \sum_{w\in\{W:\overline{W}\}} \varepsilon(w)(e^{(L(\lambda)/c_0^\vee)\Lambda_0 - d_w(\lambda+\rho)\delta} \frac{\sum_{\bar{w}\in\overline{W}} \varepsilon(\bar{w}) e^{\bar{w}(\overline{w(\lambda+\rho)-\rho}+\bar{\rho})-\bar{\rho}}}{\sum_{\bar{w}\in\overline{W}} \varepsilon(\bar{w}) e^{\bar{w}(\bar{\rho})-\bar{\rho}}} \\ &= e^{(L(\lambda)/c_0^\vee)\Lambda_0} \sum_{w\in\{W:\overline{W}\}} \varepsilon(w)\, q^{d_w(\lambda+\rho)}\, ch\, \overline{V}^{\overline{w(\lambda+\rho)-\rho}}, \end{aligned} \tag{6.6}$$

where $q = e^{-\delta}$. In a similar way the denominator of the Weyl-Kostant-Liu character formula (1.25) can be written as

$$D = \sum_{w\in\{W:\bar{W}\}} \varepsilon(w)\, q^{d_w(\rho)}\, ch\, \overline{V}^{\overline{w(\rho)-\rho}}\,. \tag{6.7}$$

In the following, the computations are done independently of the rank r of the affine algebras by assuming that r is sufficiently large for no modifications to be required. Rank dependent calculations can be taken care of by the use of modifications rules as discussed in Chapter 5. For each affine algebra the denominator can be computed easily from Proposition 5.2. Let us denote the denominators for affine algebras $A_r^{(1)}$, $B_r^{(1)}$, $C_r^{(1)}$ and $D_{r+1}^{(2)}$ respectively by K_q, A_q, C_q and E_q. Then up to depth 4 we obtain

$$\begin{aligned}
K_q =&\{0\} - q\{\bar{1};1\} + q^2(\{\bar{2};1^2\} + \{\bar{1}^2;2\}) - q^3(\{\bar{3};1^3\} + \{\bar{2}\bar{1};21\} + \{\bar{1}^3;3\}) \\
&+ q^4(\{\bar{4};1^4\} + \{\bar{3}\bar{1};21^2\} + \{\bar{2}^2;2^2\} + \{\bar{1}^4;4\}) + \dots \\
A_q =&[0] - q[1^2] + q^2[21^2] - q^3([31^3] + [2^3]) + q^4([41^4] + [32^21]) + \dots \\
C_q =& <0> - q <2> + q^2 <31> - q^3(<41^2> + <3^2>) \\
&+ q^4(<51^3> + <431>) + \dots \\
E_q =&[0] - q[1] + q^3[21] - q^4[2^2] + \dots
\end{aligned} \tag{6.8}$$

The inverse of K_q for example can be calculated as follows. Let $K_q^{-1} = k_0 + qk_1 + q^2k_2 + \dots$. Then $K_q^{-1} \times K_q = \{0\}$ and on comparing the coefficients of various powers of q we obtain

$$\begin{aligned}
k_0 =&\{0\}, \\
k_1 =&k_0 \times \{\bar{1};1\} \\
=&\{\bar{1};1\}, \\
k_2 =&k_1 \times \{\bar{1};1\} - k_0 \times (\{\bar{2};1^2\} + \{\bar{1}^2;2\}) \\
=&\{\bar{2};2\} + \{\bar{1}^2;1^2\} + 2\{\bar{1};1\} + \{0\}.
\end{aligned}$$

The above tensor products and others like them may be carried out with the help of SCHUR software [W]. Similar computations can be done for other affine algebras. The results take the form:

$$
\begin{aligned}
K_q^{-1} = & \{0\} \\
& + q\{\bar{1};1\} \\
& + q^2(\{\bar{2};2\} + \{\bar{1}^2;1^2\} + 2\{\bar{1};1\} + \{0\}) \\
& + q^3(\{\bar{3};3\} + \{\bar{2}\bar{1};21\} + \{\bar{1}^3;1^3\} + 2\{\bar{2};2\} \\
& \quad + 2\{\bar{2};1^2\} + 2\{\bar{1}^2;2\} + 2\{\bar{1}^2;1^2\} + 5\{\bar{1};1\} + 2\{0\}) \\
& + q^4(\{\bar{4};4\} + \{\bar{3}\bar{1};31\} + \{\bar{2}^2;2^2\} + \{\bar{2}\bar{1}^2;21^2\} + \{\bar{1}^4;1^4\} + 2\{\bar{3};3\} \\
& \quad + 2\{\bar{3};21\} + 2\{\bar{2}\bar{1};3\} + 4\{\bar{2}\bar{1};21\} + 2\{\bar{2}\bar{1};1^3\} + 2\{\bar{1}^3;21\} + 2\{\bar{1}^3;1^3\} \\
& \quad + 8\{\bar{2};2\} + 5\{\bar{2};1^2\} + 5\{\bar{1}^2;2\} + 8\{\bar{1}^2;1^2\} + 12\{\bar{1};1\} + 5\{0\}) \\
& + \dots
\end{aligned} \tag{6.9a}
$$

$$
\begin{aligned}
A_q^{-1} = & [0] \\
& + q[1^2] \\
& + q^2([2^2] + [1^4] + [2] + [1^2] + [0]) \\
& + q^3([3^2] + [2^2 1^2] + [1^6] + [31] + [2^2] + 2[21^2] + [1^4] + [2] + 4[1^2] + [0]) \\
& + q^4([4^2] + [3^2 1^2] + [2^4] + [2^2 1^4] + [1^8] + [42] + [3^2] + 2[321] + [31^3] \\
& \quad + [2^3] + 2[2^2 1^2] + 2[21^4] + [1^6] + [4] + 2[31] + 6[2^2] + 5[21^2] + 5[1^4] \\
& \quad + 5[2] + 6[1^2] + 4[0]) \\
& + \dots
\end{aligned} \tag{6.9b}
$$

$$
\begin{aligned}
C_q^{-1} = &< 0 > \\
&+ q < 2 > \\
&+ q^2(< 4 > + < 2^2 > + < 2 > + < 1^2 > + < 0 >) \\
&+ q^3(< 6 > + < 42 > + < 2^3 > + < 4 > +2 < 31 > + < 2^2 > \\
&\quad + < 21^2 > +4 < 2 > + < 1^2 > + < 0 >) \\
&+ q^4(< 8 > + < 62 > + < 4^2 > + < 42^2 > + < 2^4 > + < 6 > \\
&\quad + 2 < 51 > +2 < 42 > + < 41^2 > + < 3^2 > +2 < 321 > + < 2^3 > \\
&\quad + < 2^2 1^2 > +5 < 4 > +5 < 31 > +6 < 2^2 > +2 < 21^2 > + < 1^4 > \\
&\quad + 6 < 2 > +5 < 1^2 > +4 < 0 >) \\
&+ \cdots
\end{aligned}
\tag{6.9c}
$$

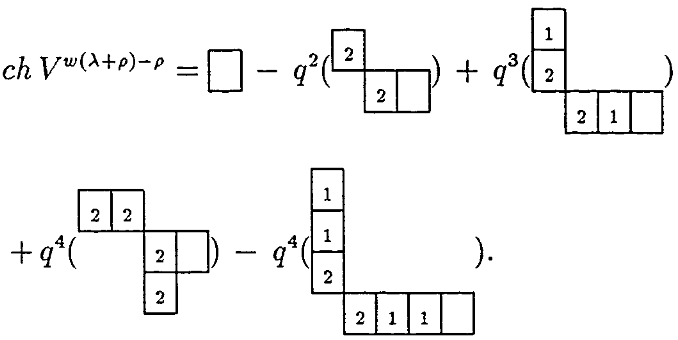
$$
\begin{aligned}
E_q^{-1} = &[0] \\
&+ q[1] \\
&+ q^2([2] + [1^2] + [0]) \\
&+ q^3([3] + [21] + [1^3] + 3[1]) \\
&+ q^4([4] + [31] + [2^2] + [21^2] + [1^4] + 4[2] + 4[1^2] + 3[0]) \quad . \\
&+ \cdots
\end{aligned}
\tag{6.9d}
$$

For the numerator N^λ of (6.6), we make use of the Young diagram method to compute $w(\lambda+\rho)-\rho$ by noting that

$$
\begin{aligned}
w(\lambda+\rho)-\rho &= w(\lambda+\rho)-(\lambda+\rho)+\lambda \\
&= w(\mu)-\mu+\lambda
\end{aligned}
$$

where $\mu = \lambda + \rho$. First we compute $w(\mu) - \mu$ by the Young diagrammatic method as in the example following Proposition 5.20 and to each Young diagram term we add the boxes that correspond to λ. By way of illustration let us compute the numerator when the highest weight representation of $A_r^{(1)}$ is $\lambda = \Lambda_0 + \Lambda_1 = (1,1,0,\ldots)$ so that $\lambda = 2\Lambda_0 + \epsilon_1$ and $\lambda + \rho = \mu = (2,2,1,1,\ldots,1)$. Let $w = w_\xi = w_{\binom{a_1}{b_1}} \ldots w_{\binom{a_p}{b_p}}$ where ξ is the partition $\binom{a_1 \ldots a_p}{b_1 \ldots b_p}$. First we list all the Young diagrams that correspond to ξ' and fill the boxes with the appropriate numerical values of μ_η where η is given in (5.24) Next we annex to these Young diagrams the empty boxes that correspond to $\bar{\lambda} = \epsilon_1$ and this will determine $ch\, V^{w(\lambda+\rho)-\rho}$. Then up to depth 4 we obtain

$$\sum_{w \in \{W:\overline{W}\}} \varepsilon(w) ch\, V^{w(\lambda+\rho)-\rho} =$$



Algebraically these come about through applying id, s_0, $s_0 s_r$, $s_0 s_1$ and $s_0 s_r s_{r-1}$ which are the the Weyl core elements of Proposition 5.5. The empty boxes denote the contribution from $\bar{\lambda}$ in the ϵ basis. Every empty box will contribute 1 unit while the contribution of the other boxes is according to the numerical values of their entries. Hence the expansion for the numerator can be written as follows:

$$\sum_{w \in \{W:\overline{W}\}} \varepsilon(w) ch\, V^{w(\Lambda_0+\Lambda_1+\rho)-\rho} = e^{2\Lambda_0}(\{1\} - q^2\{\bar{2};3\} + q^3\{\overline{21};4\}$$
$$+ q^4(\{\bar{4};32\} - \{\overline{21^2};5\}) + \ldots).$$

The tensor product of the above numerator expression with K_q^{-1} of (6.9a) then gives the expression for $ch\, V^{\Lambda_0+\Lambda_1}$ as

$$
\begin{aligned}
ch\ V^{\Lambda_0+\Lambda_1} =& e^{2\Lambda_0}(\{1\} + q(\{\bar{1};2\} + \{\bar{1};1^2\} + \{1\}) \\
& + q^2(\{\bar{2};21\} + \{\overline{1^2};21\} + 3\{\bar{1};2\} + \{\overline{1^2};1^3\} + 3\{\bar{1};1^2\} + 3\{1\}) \\
& + q^3(\{\bar{2};3\} + \{\overline{1^2};3\} + \{\overline{21};2^2\} + \{\overline{21};21^2\} + \{\overline{1^3};21^2\} \\
& \quad + 4\{\bar{2};21\} + 5\{\overline{1^2};21\} + 8\{\bar{1};2\} + \{\overline{1^3};1^4\} + 2\{\bar{2};1^3\} \\
& \quad + 3\{\overline{1^2};1^3\} + 9\{\bar{1};1^2\} + 7\{1\}) \\
& + q^4(2\{\overline{21};31\} + \{\overline{1^3};31\} + 4\{\bar{2};3\} + 4\{\overline{1^2};3\} + \{\overline{2^2};2^21\} \\
& \quad + \{\overline{21^2};2^21\} + \{\bar{3};2^2\} + 4\{\overline{21};2^2\} + 2\{\overline{1^3};2^2\} + \{\overline{21^2};21^3\} \\
& \quad + \{\overline{1^4};21^3\} + \{\bar{3};21^2\} + 6\{\overline{21};21^2\} + 5\{\overline{1^3};21^2\} + 14\{\bar{2};21\} \\
& \quad + 17\{\overline{1^2};21\} + 21\{\bar{1};2\} + \{\overline{1^4};1^5\} + 2\{\overline{21};1^4\} + 3\{\overline{1^3};1^4\} \\
& \quad + 7\{\bar{2};1^3\} + 12\{\overline{1^2};1^3\} + 24\{\bar{1};1^2\} + 16\{1\} \\
& + \ldots).
\end{aligned}
$$

This expression for the character of $\Lambda_0 + \Lambda_1$ defines a branching rule of the affine algebras $A_r^{(1)}$ to the simple finite-dimensional algebra A_r down to depth 4 since $\{\bar{\nu};\mu\}$ is to be interpreted as the A_r character $ch\,\overline{V}^{\{\bar{\nu};\mu\}}$. In contrast to the other methods discussed elsewhere [**KMPS**], the branching rule has been obtained without the need to compute weight multiplicities or Weyl orbits and is done independently of the rank of the affine algebras. Below are some character expressions up to depth 4 of affine algebras that we have computed using the algorithm discussed above.

$A_r^{(1)} \supset A_r$

$$
\begin{aligned}
ch\ V^{\Lambda_0} =& e^{\Lambda_0}(\{0\} + q\{\overline{1};1\} \\
& + q^2(\{\overline{1^2};1^2\} + 2\{\overline{1};1\} + \{0\}) \\
& + q^3(\{\overline{1^3};1^3\} + \{\overline{1^2};2\} + \{\overline{2};1^2\} + 2\{\overline{1^2};1^2\} + 4\{\overline{1};1\} + 2\{0\}) \\
& + q^4(\{\overline{1^4};1^4\} + \{\overline{21};1^3\} + \{\overline{1^3};21\} + 2\{\overline{1^3};1^3\} + \{\overline{2};2\} \\
& \quad + 2\{\overline{1^2};2\} + 2\{\overline{2};1^2\} + 6\{\overline{1^2};1^2\} + 8\{\overline{1};1\} + 4\{0\}) \\
& + \ldots)
\end{aligned}
$$

$$
\begin{aligned}
ch\ V^{\Lambda_1} =& e^{\Lambda_0}(\{1\} + q(\{\overline{1};1^2\} + \{1\}) \\
& + q^2(\{\overline{1};2\} + \{\overline{1^2};1^3\} + 2\{\overline{1};1^2\} + 2\{1\}) \\
& + q^3(\{\overline{1^2};21\} + 2\{\overline{1};2\} + \{\overline{1^3};1^4\} + \{\overline{2};1^3\} + 2\{\overline{1^2};1^3\} + 5\{\overline{1};1^2\} + 4\{1\}) \\
& + q^4(\{\overline{1^3};21^2\} + \{\overline{2};21\} + 3\{\overline{1^2};21\} + 5\{\overline{1};2\} + \{\overline{1^4};1^5\} \\
& \quad + \{\overline{21};1^4\} + 2\{\overline{1^3};1^4\} + 2\{\overline{2};1^3\} + 6\{\overline{1^2};1^3\} + 10\{\overline{1};1^2\} + 8\{1\}) \\
& + \ldots)
\end{aligned}
$$

$$
\begin{aligned}
ch\ V^{2\Lambda_0} =& e^{2\Lambda_0}(\{0\} + q\{\overline{1};1\} + q^2(\{\overline{2};2\} + \{\overline{1^2};1^2\} + 2\{\overline{1};1\} + \{0\}) \\
& + q^3(\{\overline{21};21\} + \{\overline{1^3};1^3\} + 2\{\overline{2};2\} + 2\{\overline{2};1^2\} + 2\{\overline{1^2};2\} \\
& \quad + 2\{\overline{1^2};1^2\} + 5\{\overline{1};1\} + 2\{0\}) \\
& + q^4(\{\overline{2^2};2^2\} + \{\overline{21^2};21^2\} + \{\overline{1^4};1^4\} + \{\overline{3};21\} + \{\overline{21};3\} \\
& \quad + 4\{\overline{21};21\} + 2\{\overline{21};1^3\} + 2\{\overline{1^3};21\} + 2\{\overline{1^3};1^3\} + 7\{\overline{2};2\} \\
& \quad + 5\{\overline{2};1^2\} + 5\{\overline{1^2};2\} + 8\{\overline{1^2};1^2\} + 12\{\overline{1};1\} + 5\{0\}) \\
& + \ldots)
\end{aligned}
$$

$B_r^{(1)} \supset B_r$

$$
\begin{aligned}
ch\ V^{\Lambda_0} =& e^{\Lambda_0}([0] + q[1^2] + q^2([1^4] + [2] + [1^2] + [0]) \\
& + q^3([1^6] + [21^2] + [1^4] + [2] + 3[1^2] + [0]) \\
& + q^4([1^8] + [21^4] + [1^6] + [2^2] + 2[21^2] \\
& \quad + 3[1^4] + 3[2] + 4[1^2] + 3[0]) + \ldots) \\
ch\ V^{\Lambda_1} =& e^{\Lambda_0}([1] + q([1^3] + [1]) + q^2([21] + [1^5] + [1^3] + 2[1]) \\
& + q^3([21^3] + 2[21] + [1^7] + [1^5] + 3[1^3] + 3[1] \\
& + q^4([3] + [2^21] + [21^5] + 2[21^3] + 4[21] + [1^9] + [1^7] \\
& \quad + 3[1^5] + 5[1^3] + 6[1]) + \ldots) \\
ch\ V^{\Lambda_0+\Lambda_1} =& e^{2\Lambda_0}([1] + q([1^3] + [21] + [1]) \\
& + q^2([3] + [2^21] + [21^3] + 3[21] + [1^5] + 2[1^3] + 3[1]) \\
& + q^3([32] + 2[31^2] + 2[3] + [2^31] + [2^21^3] + 3[2^21] + [21^5] + 4[21^3] \\
& \quad + 8[21] + [1^7] + 2[1^5] + 7[1^3] + 6[1]) \\
& + q^4([41] + [32^2] + 2[321^2] + 4[32] + 2[31^4] + 6[31^2] + 6[3] + [2^41] \\
& \quad + [2^31^3] + 3[2^31] + [2^21^5] + 4[2^21^3] + 11[2^21] + [21^7] + 4[21^5] \\
& \quad + 13[21^3] + 20[21] + [1^9] + 2[1^7] + 8[1^5] + 15[1^3] + 14[1]) + \ldots) \\
ch\ V^{2\Lambda_0} =& e^{2\Lambda_0}([0] + q[1^2] \\
& + q^2([2^2] + [1^4] + [2] + [1^2] + [0]) \\
& + q^3([2^21^2] + [1^6] + [31] + [2^2] + 2[21^2] + [1^4] + [2] + 4[1^2] + [0]) \\
& + q^4([4] + [321] + [31^3] + 2[31] + [2^4] + [2^3] + [2^21^4] \\
& \quad + 2[2^21^2] + 5[2^2] + 2[21^4] + 5[21^2] + 5[2] + [1^8] + [1^6] \\
& \quad + 5[1^4] + 6[1^2] + 4[0]) + \ldots)
\end{aligned}
$$

$C_r^{(1)} \supset C_r$

$$ch\ V^{\Lambda_0} = e^{\Lambda_0}(<0> + q<2>$$
$$+ q^2(<2^2> + <2> + <1^2> + <0>)$$
$$+ q^3(<2^3> + <31> + <2^2> + <21^2>$$
$$+ 3<2> + <1^2> + <0>)$$
$$+ q^4(<2^4> + <321> + <2^3> + <2^21^2>$$
$$+ <4> + 2<31> + 4<2^2> + 2<21^2>$$
$$+ <1^4> + 4<2> + 4<1^2> + 3<0>)$$
$$+ \ldots)$$

$$ch\ V^{\Lambda_1} = e^{\Lambda_0}(<1> + q(<21> + <1>)$$
$$+ q^2(<2^21> + <3> + 2<21> + <1^3> + 2<1>)$$
$$+ q^3(<32> + <31^2> + 2<3> + <2^31> + 2<2^21>$$
$$+ <21^3> + 5<21> + 2<1^3> + 4<1>)$$
$$+ q^4(<41> + <32^2> + <321^2> + 3<32> + 3<31^2>$$
$$+ 4<3> + <2^41> + 2<2^31> + <2^21^3> + 6<2^21>$$
$$+ 3<21^3> + 11<21> + <1^5> + 5<1^3> + 8<1>)$$
$$+ \ldots)$$

$$
\begin{aligned}
ch\ V^{\Lambda_0+\Lambda_1} = e^{2\Lambda_0}(&<1> + q(<3> + <21> + <1>) \\
&+ q^2(<41> + <32> + <2^21> + 2<3> + 3<21> \\
&\quad + <1^3> + 3<1>) \\
&+ q^3(<5> + <43> + <421> + 3<41> + <32^2> \\
&\quad + 4<32> + 3<31^2> + 6<3> + <2^31> + 3<2^21> \\
&\quad + <21^3> + 9<21> + 2<1^3> + 6<1>) \\
&+ q^4(2<52> + <51^2> + 3<5> + <4^21> + <432> \\
&\quad + 3<43> + <42^21> + 5<421> + <41^3> + 11<41> \\
&\quad + 3<3^21> + <32^3> + 4<32^2> + 3<321^2> + 14<32> \\
&\quad + 10<31^2> + 14<3> + <2^41> + 3<2^31> + <2^21^3> \\
&\quad + 12<2^21> + 4<21^3> + 23<21> + <1^5> \\
&\quad + 8<1^3> + 14<1>) + \ldots)
\end{aligned}
$$

$$
\begin{aligned}
ch\ V^{2\Lambda_0} = e^{2\Lambda_0}(&<0> + q<2> \\
&+ q^2(<4> + <2^2> + <2> + <1^2> + <0>) \\
&+ q^3(<42> + <2^3> + <4> + 2<31> + <2^2> \\
&\quad + <21^2> + 4<2> + <1^2> + <0>) \\
&+ q^4(<51> + <4^2> + <42^2> + 2<42> + <41^2> \\
&\quad + 4<4> + <3^2> + 2<321> + 5<31> + <2^4> \\
&\quad + <2^3> + <2^21^2> + 6<2^2> + 2<21^2> \\
&\quad + 6<2> + <1^4> + 5<1^2> + 4<0>) + \ldots)
\end{aligned}
$$

$D^{(2)}_{r+1} \supset B_r$

$$
\begin{aligned}
ch\ V^{\Lambda_0} =& e^{\Lambda_0}([0] + q[1] + q^2([1^2] + [0]) \\
& + q^3([1^3] + 2[1]) \\
& + q^4([1^4] + [2] + 2[1^2] + 2[0]) + \ldots) \\
ch\ V^{\Lambda_1} =& e^{\Lambda_0}([1] + q([1^2] + [0]) + q^2([21] + [1^3] + 2[1]) \\
& + q^3([21^2] + 2[2] + [1^4] + 3[1^2] + 2[0]) \\
& + q^4([3] + [2^2 1] + [21^3] + 4[21] + [1^5] + 4[1^3] + 6[1]) + \ldots) \\
ch\ V^{2\Lambda_0} =& e^{2\Lambda_0}([0] + q[1] + q^2([2] + [1^2] + [0]) \\
& + q^3([21] + [1^3] + 3[1]) \\
& + q^4([2^2] + [21^2] + 3[2] + [1^4] + 4[1^2] + 3[0]) + \ldots) \\
ch\ V^{\Lambda_0+\Lambda_1} =& e^{2\Lambda_0}([1] + q([2] + [1^2] + [0]) + q^2(2[21] + [1^3] + 3[1]) \\
& + q^3([31] + [2^2] + 2[21^2] + 4[2] + [1^4] + 5[1^2] + 3[0]) \\
& + q^4([32] + [31^2] + 3[3] + 2[2^2 1] + 2[21^3] + 9[21] \\
& + [1^5] + 6[1^3] + 10[1]) + \ldots).
\end{aligned}
$$

For sufficiently large r, the branching rule of representations of $A^{(2)}_{2r}$ restricted to B_r is the same as that of $C^{(1)}_r$ to C_r. While the branching rule of representations of $D^{(1)}_r$ restricted to D_r and $A^{(2)}_{2r-1}$ restricted to C_r is the same as that of $B^{(1)}_r$ to B_r.

In general we can write

$$ch\, V^\lambda = e^{L(\lambda)\Lambda_0/c_0^\vee} \sum_{n=0}^{\infty} \sum_{\bar\mu \in \bar P+} b^\lambda_{\bar\mu} ch\, \overline{V}^{\bar\mu}\, q^n \tag{6.10}$$

where the sum is over the set $\bar P^+$ of dominant weights $\bar\mu$ of $\mathcal{G}(\bar A)$. Then

$$ch\, V^\lambda = e^{L(\lambda)\Lambda_0/c_0^\vee} \sum_{n=0}^{\infty} \sum_{\bar\mu \in \bar P+} \sum_{\bar\nu \in \bar P} b^\lambda_{\bar\mu} (dim\, V^{\bar\mu}_{\bar\nu})\, e^{\bar\nu} q^n\,. \tag{6.11}$$

Alternatively $ch\, V^\lambda = \sum_{\nu \in \mathcal{H}^*} (dim\, V^\lambda_\nu)\, e^\nu$ where $dim\, V^\lambda_\nu = 0$ if ν is not a weight of the highest weight module V^λ. As has been discussed in Chapter 4, each weight

$\nu = \bar{\nu} - n\delta + (L(\nu)/c_0^\vee)\Lambda_0$ appears in a string so that we may write

$$ch\, V^\lambda = \sum_{n=0}^{\infty} \sum_{\bar{\nu} \in \bar{P}} (dim\, V_\nu^\lambda)\, e^{\bar{\nu}} q^n e^{L(\lambda)\Lambda_0/c_0^\vee} . \tag{6.12}$$

On comparing this expression with that of (6.11) we obtain

$$dim\, V_\nu^\lambda \;=\; \sum_{\bar{\mu} \in \bar{P}+} b_{\bar{\mu}}^\lambda (dim\, V_{\bar{\nu}}^{\bar{\mu}}) . \tag{6.13}$$

In term of the weight multiplicity generating function or string function σ_ν^λ we may write

$$\sigma_\nu^\lambda \;=\; \sum_{n=0}^{\infty} \sum_{\bar{\mu} \in \bar{P}+} b_{\bar{\nu}}^\lambda \,(dim\, V_{\bar{\nu}}^{\bar{\mu}})\, q^n . \tag{6.14}$$

Tabulation of $dim\, V_{\bar{\nu}}^{\bar{\mu}}$ in terms of the rank of the algebras can be obtained from the work of **[KiP]** and **[BBL]** whereby it was established that the weight multiplicities of dominant weights of finite-dimensional modules of the classical series of simple finite-dimensional Lie algebras are polynomials in the rank of the algebra. It then follows that the weight multiplicities of the highest weight modules of the rank dependent series of affine algebras are necessarily polynomials in the rank of the algebra.

It is well known **[Kac4]** that the string functions $\sigma_{\Lambda_i}^{\Lambda_i}$ for level 1 modules of the affine algebras $A_r^{(1)}$ and $D_r^{(1)}$ and the string function $\sigma_{\Lambda_r}^{\Lambda_r}$ of $A_{2r}^{(2)}$ are all given by $\phi(q)^{-r}$. But

$$\begin{aligned}\phi(q)^{-r} &= \prod_{i>0} (1-q^i)^{-r} \\ &= 1 + rq + (\frac{1}{2}r^2 + \frac{3}{2}r)q^2 + (\frac{1}{6}r^3 + \frac{3}{2}r^2 + \frac{4}{3}r)q^3 + \dots \\ &\quad + (\frac{1}{24}r^4 + \frac{3}{4}r^3 + \frac{59}{24}r^2 + \frac{7}{4}r)q^4 + \dots\end{aligned}$$

This illustrates the polynomial rank dependence of the weight multiplicities with the degree of the polynomial given by the depth of the weights.

In the case of other affine modules, using the weight multiplicity polynomials of the simple finite-dimensional Lie algebras tabulated in **[KiP]** or **[BBL]** and (6.14), we find from our branching rule results:

$A_r^{(1)}$

$$\sigma^{\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1} = 1 + 2rq + \frac{r}{2}(5r+3)q^2 + \frac{r}{3}(7r^2+9r+8)q^3 + \frac{r}{12}(21r^3+44r^2+87r+16)q^4 + \dots$$

$$\sigma^{\Lambda_0+\Lambda_1}_{\Lambda_2+\Lambda_r} = 2q + (5r-3)q^2 + (7r^2-7r+8)q^3 + (7r^3-9r^2+28r-20)q^4 + \dots$$

$$\sigma^{2\Lambda_0}_{2\Lambda_0} = 1 + rq + r(r+2)q^2 + \frac{r}{6}(5r^2+15r+10)q^3 + \frac{r}{12}(7r^3+30r^2+53r+30)q^4 + \dots$$

$$\sigma^{2\Lambda_0}_{\Lambda_1+\Lambda_r} = q + 2rq^2 + \frac{1}{2}(5r^2+r+2)q^3 + \frac{1}{6}(14r^3+6r^2+51r-18)q^4 + \dots$$

$$\sigma^{2\Lambda_0}_{\Lambda_2+\Lambda_{r-1}} = 2q^2 + (5r-4)q^3 + (7r^2-11r+15)q^4 + \dots$$

$B_r^{(1)}$

$$\sigma^{\Lambda_0}_{\Lambda_0} = \sigma^{\Lambda_1}_{\Lambda_1} = 1 + rq + \frac{1}{2}(r^2+3r+2)q^2 + \frac{1}{6}(r^3+9r^2+14r+6)q^3 + \frac{1}{24}(r^4+18r^3+71r^2+102r+48)q^4 + \dots$$

$$\sigma^{\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1} = 1 + (3r-1)q + (5r^2-2r+3)q^2 + \frac{1}{6}(35r^3-12r^2+85r-36)q^3 + \frac{1}{12}(63r^4-10r^3+375r^2-248r+132)q^4 + \dots$$

$$\sigma^{\Lambda_0+\Lambda_1}_{2\Lambda_1} = q\sigma^{\Lambda_0+\Lambda_1}_{2\Lambda_0} = q + 3rq^2 + r(5r+2)q^3 + \frac{r}{6}(35r^2+33r+22)q^4 + \dots + \frac{r}{4}(21r^3+34r^2+57r+8)q^5 + \dots$$

$$\sigma^{2\Lambda_0}_{2\Lambda_0} = \sigma^{2\Lambda_1}_{2\Lambda_1} = 1 + rq + \frac{r}{2}(3r+3)q^2 + \frac{r}{3}(5r^2+6r+7)q^3 + \frac{r}{24}(35r^3+66r^2+169r+42)q^4 + \dots$$

$$\sigma^{2\Lambda_0}_{\Lambda_0+\Lambda_1} = q + (3r-1)q^2 + (5r^2-2r+2)q^3 + \frac{1}{6}(35r^3-12r^2+61r-24)q^4 + \dots$$

$$\sigma^{2\Lambda_0}_{2\Lambda_1} = rq^2 + \frac{r}{2}(3r+1)q^3 + \frac{r}{3}(5r^2+6r+7)q^4 + \dots$$

$C_r^{(1)}$

$$\sigma_{\Lambda_0}^{\Lambda_0} = 1 + rq + r(r+1)q^2 + \frac{r}{6}(5r^2+6r+7)q^3 + \frac{r}{12}(7r^3+12r^2+29r+12)q^4 + \dots$$

$$\sigma_{\Lambda_1}^{\Lambda_1} = 1 + (2r-1)q + \frac{1}{2}(5r^2-3r+2)q^2 + \frac{1}{6}(14r^3-9r^2+25r-12)q^3 + \frac{1}{4}(7r^4-4r^3+31r^2-26r+12)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_1}^{\Lambda_0+\Lambda_1} = 1 + (3r-1)q + (7r^2-5r+2)q^2 + (14r^3-23r^2+28r-11)q^3 + \frac{1}{4}(99r^4-304r^3+655r^2-638r+244)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_3}^{\Lambda_0+\Lambda_1} = 3q + (14r-15)q^2 + (42r^2-108r+99)q^3 + (99r^3-454r^2+953r-762)q^4 + \dots$$

$$\sigma_{2\Lambda_0}^{2\Lambda_0} = 1 + rq + \frac{r}{2}(3r+3)q^2 + \frac{r}{3}(7r^2+3r+5)q^3 + \frac{r}{6}(21r^3-10r^2+54r-5)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_2}^{2\Lambda_0} = q + (3r-1)q^2 + (7r^2-8r+6)q^3 + (14r^3-36r^2+62r-37)q^4 + \dots$$

$$\sigma_{2\Lambda_1}^{2\Lambda_0} = q + 2rq^2 + (4r^2-r+1)q^3 + \frac{1}{3}(22r^3-30r^2+47r-18)q^4 + \dots$$

$D_r^{(1)}$

$$\sigma_{\Lambda_0+\Lambda_1}^{\Lambda_0+\Lambda_1} = 1 + (3r-2)q + (5r^2-5r+2)q^2 + \frac{1}{6}(35r^3-42r^2+55r-30)q^3 + \frac{1}{12}(63r^4-80r^3+249r^2-280r+108)q^4 + \dots$$

$$\sigma_{\Lambda_3}^{\Lambda_0+\Lambda_1} = 3q + (10r-15)q^2 + \frac{1}{2}(35r^2-99r+108)q^3 + \frac{1}{2}(42r^3-169r^2+389r-360)q^4 + \dots$$

$$\sigma_{2\Lambda_0}^{2\Lambda_0} = 1 + rq + \frac{1}{2}(3r^2+r)q^2 + \frac{1}{6}(10r^3+3r^2-r+6)q^3 + \frac{1}{24}(35r^4+26r^3+37r^2-2r+24)q^4 + \dots$$

$$\sigma_{\Lambda_2}^{2\Lambda_0} = q + (3r-3)q^2 + (5r^2-9r+7)q^3 + \frac{1}{6}(35r^3-87r^2+160r-120)q^4 + \dots$$

$D_{r+1}^{(2)}$

$$\sigma_{\Lambda_1}^{\Lambda_1} = 1 + q + 3rq^2 + (4r+1)q^3 + (5r^2+2r+3)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_1}^{\Lambda_0+\Lambda_1} = 1 + 2q + 5rq^2 + (12r-2)q^3 + (15r^2-r+6)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_1}^{3\Lambda_0} = 1 + (2r+1)q + (5r+1)q^2 + (6r^2+5r+2)q^3 + (15r^2+7r+2)q^4 + \dots$$

$$\sigma_{2\Lambda_0}^{2\Lambda_0} = 1 + q + (2r+1)q^2 + (3r+2)q^3 + (3r^2+5r+2)q^4 + \dots$$

$$\sigma_{\Lambda_1}^{2\Lambda_0} = q + 2q^2 + (3r+1)q^3 + (6r+1)q^4 + \dots$$

$$\sigma_{2\Lambda_0}^{2\Lambda_0} = 2q^2 + 3q^3 + (6r-1)q^4 + \dots$$

$A_{2r}^{(2)}$

$$\sigma_{\Lambda_0}^{\Lambda_0} = 1 + (r-1)q + (r^2+r)q^2 + \frac{1}{6}(5r^3+9r^2-8r-6)q^3 + \frac{r}{12}(7r^3+30r^2+59r+24)q^4 + \dots$$

$$\sigma_{\Lambda_1}^{\Lambda_1} = 1 + 2rq + \frac{1}{2}(5r^2+3r)q^2 + \frac{1}{3}(7r^3+9r^2+11r-3)q^3 + \frac{1}{12}(21r^4+44r^3+117r^2-26r+24)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_1}^{\Lambda_0+\Lambda_1} = 1 + 3rq + (7r^2+r)q^2 + (14r^3-3r^2+11r-3)q^3 + \dots$$

$$\sigma_{2\Lambda_0}^{\Lambda_0+\Lambda_1} = 1 + 3rq + r(7r+2)q^2 + r(14r^2+2r+4)q^3 + \dots$$

$$\sigma_{2\Lambda_0}^{2\Lambda_0} = 1 + rq + \frac{1}{2}r(3r+5)q^2 + \frac{1}{3}r(7r^2+12r+5)q^3 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_1}^{2\Lambda_0} = q + 3rq^2 + (7r^2-r+1)q^3 + \dots$$

$$\sigma_{2\Lambda_1}^{2\Lambda_0} = q + 2rq^2 + r(4r+2)q^3 + \dots$$

$A_{2r-1}^{(2)}$

$$\sigma_{\Lambda_0}^{\Lambda_0} = \sigma_{\Lambda_1}^{\Lambda_1} = 1 + (r-1)q + \frac{1}{2}(r^2+r)q^2 + \frac{1}{6}(r^3+6r^2-r-6)q^3 + \frac{1}{24}(r^4+14r^3+23r^2-14r+24)q^4 + \dots$$

$$\sigma_{\Lambda_0+\Lambda_1}^{\Lambda_0+\Lambda_1} = 1 + (3r-3)q + (5r^2-9r+6)q^2 + \frac{1}{6}(35r^3-87r^2+130r-78)q^3 + \frac{1}{4}(21r^4-64r^3+161r^2-202r+100)q^4 + \dots$$

These results are a significant generalisation of those obtained previously for $A_r^{(1)}$ **[BKM2]**.

6.3 Self embedding

Although not possible in the finite-dimensional case, it is a remarkable fact that an affine algebra may be embedded in itself. Indeed this can be done in a number of distinct ways. The simplest way is to define the following transformation of the roots

$$\begin{aligned} \alpha'_0 &\longrightarrow \alpha_0 + \delta \\ \alpha'_i &\longrightarrow \alpha_i \quad i = 1, \ldots, r. \end{aligned} \tag{6.15}$$

It can then be seen that the GCM $A'_{ij} =< \alpha'_i, \alpha'^{\vee}_j >$ coincides with the affine GCM $A_{ij} =< \alpha_i, \alpha^{\vee}_j >$. The weights of the $\mathcal{G}$ modules of level L and depth d are transformed to weights of $\mathcal{G}' = \mathcal{G}$ modules of level $2L$ and depth $d/2$. This type of self embedding is possible for all highest weight modules of affine algebras except $A^{(2)}_{2r}$. In the case of $A^{(2)}_{2r}$, by (6.3), (3.6) and (3.8), the transformation (6.15) would give:

$$\begin{aligned} \lambda'_0 &=< \lambda, \alpha'^{\vee}_0 >=< \lambda, 2\alpha^{\vee}_0 + \alpha^{\vee}_1 + \ldots + \alpha^{\vee}_{r-1} + \frac{1}{2}\alpha^{\vee}_r > \\ &= 2\lambda_0 + \lambda_1 + \ldots + \lambda_{r-1} + \frac{1}{2}\lambda_r \\ \lambda'_i &=< \lambda, \alpha'^{\vee}_i >= \lambda_i \quad \text{for} \quad i = 1, \ldots, r. \end{aligned}$$

Since the weight label must be integer, we see that unless the r^{th} Dynkin component of the highest weight is even then the projection (6.15) does not define an embedding $A^{(2)}_{2r} \supset A^{(2)}_{2r}$.

In the case of the self embeddings $A^{(1)}_1 \supset A^{(1)}_1$ some branching rules have been computed by Hussin, King, Leng and Patera **[HKLP]**. However most of their results are given numerically. Here we undertake the compution of branching rules analytically by obtaining the branching rule multiplicity generating functions for level 1 modules to level 2 modules using the algorithm discussed in (6.4).

For illustration let us consider the branching rule for $D^{(2)}_3 \supset D^{(2)}_3$. The transformation (6.15) implies that the weights are projected as follows:

$$(\lambda_0, \lambda_1, \lambda_2)_d \longrightarrow (2\lambda_0 + 2\lambda_1 + \lambda_2, \lambda_1, \lambda_2)_{\frac{d}{2}}. \tag{6.16}$$

From the orbit-weight generating function given in (3.36f) and the projection (6.16) we obtain the following decomposition of level 1 orbits of $D_3^{(2)}$ on retaining the weights that have all their components non negative and are thus dominant:

$$\Omega^{(001)_d} \longrightarrow \Omega'^{(101)_{d/2}}$$

$$\Omega^{(100)_d} \longrightarrow \Omega'^{(200)_{d/2}} + \Omega'^{(010)_{(d+1)/2}} + \Omega'^{(002)_{(d+2)/2}}.$$

In term of generating functions we can write

$$\begin{aligned} ch\, V^{(001)} &= \sigma^{(001)}_{(001)}(q)\Omega^{(001)} \\ &\rightarrow \sigma^{(001)}_{(001)}(q^{1/2})\Omega'^{(101)_0} \\ &= \sigma^{(001)}_{(001)}(q^{1/2})\kappa^{(101)}_{(101)}(q)ch\, V'^{(101)}. \end{aligned}$$

On substituting the string functions given in (4.15g) and the inverse string functions given in (4.13g) we obtain the branching rule multiplicity generating function for $b^{(001)}_{(101)}$ as

$$b^{(001)}_{(101)} = \frac{1}{\phi(q^{1/2})\phi(q)}\frac{\phi(q)^3}{\phi(q^2)} = \prod(1+q^{(2n-1)/2}).$$

In a similar way,

$$\begin{aligned} ch\, V^{(100)} &= \sigma^{(100)}_{(100)}(q)\Omega^{(100)} \\ &\rightarrow \sigma^{(100)}_{(100)}(q^{1/2})(\Omega'^{(200)_0} + \Omega'^{(010)_{1/2}} + \Omega'^{(002)_1}) \\ &= \sigma^{(100)}_{(100)}(q^{1/2})(\kappa^{(200)}_{(002)}(q)ch\, V'^{(002)} + \kappa^{(200)}_{(010)}(q)ch\, V'^{(010)} + \kappa^{(200)}_{(200)}(q)ch\, V'^{(200)}) \\ &\quad + q^{1/2}\sigma^{(100)}_{(100)}(q^{1/2})(\kappa^{(010)}_{(002)}(q)ch\, V'^{(002)} + \kappa^{(010)}_{(010)}(q)ch\, V'^{(010)} + \kappa^{(010)}_{(200)}(q)ch\, V'^{(200)}) \\ &\quad + q\sigma^{(100)}_{(100)}(q^{1/2})(\kappa^{(002)}_{(002)}(q)ch\, V'^{(002)} + \kappa^{(002)}_{(010)}(q)ch\, V'^{(010)} + \kappa^{(002)}_{(200)}(q)ch\, V'^{(200)}). \end{aligned}$$

Then

$$\begin{aligned} b^{(100)}_{(200)} &= \sigma^{(100)}_{(100)}(q^{1/2})(\kappa^{(200)}_{(200)} + q\kappa^{(002)}_{(200)} + q^{1/2}\kappa^{(010)}_{(200)}) \\ &= \frac{1}{\phi(q^{1/2})\phi(q)}\phi(q)\phi(q^6)\prod(1-q^{(2n-1)/2})(1-q^{(6n-3)/2}) \\ &= \phi(q^6)\phi(q)^{-1}\prod(1-q^{(6n-3)/2}) \end{aligned}$$

$$b^{(100)}_{(002)} = \sigma^{(100)}_{(100)}(q^{1/2})(\kappa^{(200)}_{(002)} + q\kappa^{(002)}_{(002)} + q^{1/2}\kappa^{(010)}_{(002)}) = qb^{(100)}_{(200)}$$

$$b^{(100)}_{(010)} = \sigma^{(100)}_{(100)}(q^{1/2})(\kappa^{(200)}_{(010)} + q\kappa^{(002)}_{(010)} + q^{1/2}\kappa^{(010)}_{(010)})$$
$$= q^{1/2}\phi(q^2)\phi(q^3)\phi(q^{1/2})^{-1}\phi(q)^{-1}(\prod(1+q^{2n-1})(1+q^{6n-3})$$
$$- 2q^{1/2}\prod(1+q^{2n})(1+q^{6n}))$$

Below we give branching rules for the self embedding of affine algebras $A_1^{(1)}$, $A_2^{(1)}$, $C_2^{(1)}$, $G_2^{(1)}$ and $D_4^{(3)}$ defined in each case by (6.15). Because we could not find ways of simplifying them, some of them look quite 'ugly'. Those marked * have been obtained previously in **[HKLP]**.

$A_1^{(1)} \supset A_1^{(1)}$.

Weight projection: $(\lambda_0, \lambda_1)_d \to (2\lambda_0 + \lambda_1, \lambda_1)_{\frac{d}{2}}$.

$$*b^{(10)}_{(02)} = q^{1/2}\prod(1+q^n)$$
$$*b^{(10)}_{(20)} = \prod(1+q^n)$$
$$b^{(01)}_{(11)} = \prod(1+q^{(2n-1)/2})$$

$A_2^{(1)} \supset A_2^{(1)}$.

Weight projection: $(\lambda_0, \lambda_1, \lambda_2)_d \to (2\lambda_0 + \lambda_1 + \lambda_2, \lambda_1, \lambda_2)_{\frac{d}{2}}$.

$$b^{(100)}_{(011)} = q^{1/2}\phi(q)^{-2}\phi(q^{10})^2\prod(1+q^{n/2})^2(\prod_{\pm3,\pm3,\pm4(10)}(1-q^n)$$
$$- 2q\prod_{\pm1,\pm2,\pm3(10)}(1-q^n) - q^{1/2}\prod_{\pm2,\pm2,\pm4(10)}(1-q^n))$$
$$b^{(100)}_{(200)} = \phi(q)^{-2}\phi(q^{10})^2\prod(1+q^{n/2})^2(\prod_{\pm2,\pm4,\pm4(10)}(1-q^n)$$
$$- 2q^{1/2}\prod_{\pm1,\pm3,\pm4(10)}(1-q^n) - q^{3/2}\prod_{\pm1,\pm1,\pm2(10)}(1-q^n))$$

$C_2^{(1)} \quad \supset \quad C_2^{(1)}$

Weight projection: $(\lambda_0, \lambda_1, \lambda_2)_d \to (2\lambda_0 + \lambda_1 + \lambda_2, \lambda_1, \lambda_2)_{\frac{d}{2}}$.

$$\begin{aligned}
b^{(100)}_{(002)} &= q b^{(100)}_{(200)} \\
&= q\phi(q^{1/2})^{-2}\phi(q)^{-1}\phi(q^4)\phi(q^{10})(\phi(q^4) \prod_{\pm 3(8)} (1+q^{n/2}) \prod_{10(20)} (1-q^n) \\
&\quad - q^{1/2}\phi(q^2) \prod_{\pm 3(8)} (1+q^{n/2}) \prod_{\pm 1, \pm 3(10)} (1+q^n) \\
&\quad - q^{1/2}\phi(q^{10}) \prod_{\pm 1(8)} (1+q^{n/2}) \prod_{\pm 1, \pm 3(10)} (1-q^n) \prod_{\pm 3, \pm 4(10)} (1+q^n) \\
&\quad + q^{3/2}\phi(q^{10}) \prod_{\pm 1(8)} (1+q^{n/2}) \prod_{\pm 1, \pm 3(10)} (1-q^n) \prod_{\pm 1, \pm 2(10)} (1+q^n)
\end{aligned}$$

$$\begin{aligned}
b^{(100)}_{(020)} &= \frac{q^{1/2}\phi(q^4)\phi(q^{10})^2}{\phi(q^{1/2})^2\phi(q)} \prod_{5(10)} (1+q^n)^2 (\prod_{\pm 3(8)} (1+q^{n/2}) \prod_{\pm 1(10)} (1+q^n) \prod_{\pm 4(10)} (1-q^n)^2 \\
&\quad - q \prod_{\pm 1(8)} (1+q^{n/2}) \prod_{\pm 4(10)} (1+q^n) \prod_{\pm 1(10)} (1-q^n)^2) \\
&\quad + \frac{2q\phi(q^4)\phi(q^{20})^2}{\phi(q^{1/2})^2\phi(q)} (- \prod_{\pm 3(8)} (1+q^{n/2}) \prod_{\pm 4, \pm 6, \pm 8(20)} (1-q^n) \\
&\quad + q^{5/2} \prod_{\pm 1(8)} (1+q^{n/2}) \prod_{\pm 1, \pm 2, \pm 9(10)} (1-q^n))
\end{aligned}$$

$$\begin{aligned}
b^{(100)}_{(101)} &= \frac{q^{1/2}\phi(q^4)\phi(q^{10})^2}{\phi(q^{1/2})^2\phi(q)} \prod_{5(10)} (1+q^n)^2 (\prod_{\pm 1(8)} (1+q^{n/2}) \prod_{\pm 2(10)} (1+q^n) \prod_{\pm 3(10)} (1-q^n)^2 \\
&\quad - \prod_{\pm 3(8)} (1+q^{n/2}) \prod_{\pm 3(10)} (1+q^n) \prod_{\pm 2(10)} (1-q^n)^2) \\
&\quad + \frac{2q\phi(q^4)\phi(q^{20})^2}{\phi(q^{1/2})^2\phi(q)} (-q^{1/2} \prod_{\pm 1(8)} (1+q^{n/2}) \prod_{\pm 3, \pm 6, \pm 7(20)} (1-q^n) \\
&\quad + q \prod_{\pm 3(8)} (1+q^{n/2}) \prod_{\pm 2, \pm 4, \pm 8(20)} (1-q^n)
\end{aligned}$$

$$\begin{aligned}
b^{(010)}_{(011)} &= q^{1/2} b^{(010)}_{(110)} \\
&= q^{1/2}\phi(q^{10})^2 \prod_{\pm 3, \pm 4(10)} (1-q^n) (\prod_{\pm 3, \pm 9(20)} (1+q^{n/2}) + q^{1/2} \prod_{\pm 1, \pm 7(20)} (1+q^{n/2})) \\
&\quad + q\phi(q^{10})^2 \prod_{\pm 1, \pm 2(10)} (1-q^n) (\prod_{\pm 7, \pm 9(20)} (1+q^{n/2}) + q^{3/2} \prod_{\pm 1, \pm 3(20)} (1+q^{n/2}))
\end{aligned}$$

$G_2^{(1)} \quad \supset \quad G_2^{(1)}$

Weight projection: $(\lambda_0,\lambda_1,\lambda_2)_d \to (2\lambda_0+2\lambda_1+\lambda_2,\lambda_1,\lambda_2)_{\frac{d}{2}}$.

$$b^{(100)}_{(002)} = q^{1/2}\phi(q^{15/2})\phi(q^{1/2})^{-1}\prod_{\pm1(6)}(1-q^{n/2})\prod_{3(15)}(1-q^{n/2})$$

$$b^{(100)}_{(010)}+q^{1/3}b^{(100)}_{(101)} = q^{1/3}\prod_{\pm3(15)}(1-q^{n/2})f_2(q) + (q^{1/2}\prod_{\pm7(15)}(1-q^{n/2}) - q\prod_{\pm2(15)}(1-q^{n/2}))f_1(q)$$

$$b^{(100)}_{(101)}+q^{1/3}b^{(100)}_{(200)} = q^{1/6}\phi(q^{15/2})^{-1}\prod_{0,\pm2(5)}(1+q^{n/6})f_1(q) + (-q^{1/6}\prod_{\pm7(15)}(1-q^{n/2}) + q^{2/3}\prod_{\pm2(15)}(1-q^{n/2}))f_2(q)$$

$$b^{(001)}_{(002)} = q^{1/2}\phi(q^{15/2})\phi(q^{1/2})^{-1}\prod_{\pm1(6)}(1-q^{n/2})\prod_{6(15)}(1-q^{n/2})$$

$$b^{(001)}_{(010)}+q^{1/3}b^{(001)}_{(101)} = q^{1/3}\prod_{\pm6(15)}(1-q^{n/2})f_2(q) + (q^{1/2}\prod_{\pm4(15)}(1-q^{n/2}) + q\prod_{\pm1(15)}(1-q^{n/2}))f_1(q)$$

$$b^{(001)}_{(101)}+q^{1/3}b^{(001)}_{(200)} = \phi(q^{15/2})^{-1}\prod_{0,\pm1(5)}(1+q^{n/6})f_1(q) + (-q^{1/6}\prod_{\pm4(15)}(1-q^{n/2}) - q^{2/3}\prod_{\pm1(15)}(1-q^{n/2}))f_2(q)+$$

where

$$f_1(q) = \phi(q^{1/2})^{-2}\phi(q^{3/2})\phi(q^{15/2})\prod_{\pm1(9)}(1-q^{n/3})\prod_{\pm5,\pm7(18)}(1-q^{n/6})$$

$$f_2(q) = \phi(q^{1/2})^{-2}\phi(q^{3/2})\phi(q^{15/2})\prod_{\pm4(9)}(1-q^{n/3})\prod_{\pm1,\pm7(18)}(1-q^{n/6})$$

$D_4^{(3)} \quad \supset \quad D_4^{(3)}$

Weight projection: $(\lambda_0,\lambda_1,\lambda_2)_d \to (2\lambda_0+2\lambda_1+3\lambda_2,\lambda_1,\lambda_2)_{\frac{d}{2}}$.

$$b^{(100)}_{(200)} = \prod(1+q^{2n-1})(1+q^{6n})$$

$$b^{(100)}_{(010)} = q^{1/2}\prod(1+q^{2n})(1+q^{6n-3})$$

6.4 Other affine algebra to affine algebra branching rules

In the case of affine algebras of rank 2 most of the maximal equal rank affine subalgebras have been identified by Begin and Sharp [**BS1**]. As before the branching rules multiplicity generating functions are expressed in terms of the string functions and inverse string functions obtained in Chapter 4. For reason of simplicity we shall consider only a few cases. Others can be obtained in a similar fashion.

For illustration let us consider the embedding $C_2^{(1)} \supset A_1^{(1)} \oplus u_1$. The transformation of the roots have been given in [**BS1**]. Here we shall give the projection of the weights only. It takes the form:

$$\{\lambda_0, \lambda_1\, \lambda_2\}_d \longrightarrow \{2\lambda_0 + \lambda_1, \lambda_1 + 2\lambda_2\, ; \lambda_1\}_d.$$

Then from the orbit-weight generating function given in (3.36d) we obtain

$$\begin{aligned}
\Omega^{(100)_d} &\longrightarrow \sum_{n\in\mathbb{Z}}(\Omega^{(20;4n)_{d+2n^2}} + \Omega^{(02;4n-2)_{d+2n^2-2n+1}}) \\
\Omega^{(010)_d} &\longrightarrow \sum_{n\in\mathbb{Z}}(\Omega^{(11;4n+1)_{d+2n^2+n}} + \Omega^{(11;4n-1)_{d+2n^2-n}}) \\
\Omega^{(001)_d} &\longrightarrow \sum_{n\in\mathbb{Z}}(\Omega^{(20;4n+2)_{d+2n^2+2n}} + \Omega^{(02;4n)_{d+2n^2}}).
\end{aligned}$$

For the highest weight representation (100) we then have

$$\begin{aligned}
&ch\, V^{(100)} \\
&= \sigma^{(100)}_{(001)}\Omega^{(001)} + \sigma^{(100)}_{(100)}\Omega^{(100)} \\
&\longrightarrow \sigma^{(100)}_{(001)}(\sum_{n\in\mathbb{Z}}(\Omega^{(20;4n+2)_{d+2n^2+2n}} + \Omega^{(02;4n)_{d+2n^2}}) \\
&\qquad + \sigma^{(100)}_{(100)}\sum_{n\in\mathbb{Z}}(\Omega^{(20;4n)_{d+2n^2}} + \Omega^{(02;4n-2)_{d+2n^2-2n+1}}) \\
&= \sum_{n\in\mathbb{Z}}\sigma^{(100)}_{(001)}(\kappa^{(20)}_{(02)}\, ch\, V^{(02;4n+2)_{2n^2+2n}} + \kappa^{(20)}_{(20)}\, ch\, V^{(20;4n+2)_{2n^2+2n}}) \\
&\qquad + \sum_{n\in\mathbb{Z}}\sigma^{(100)}_{(001)}(\kappa^{(02)}_{(02)}\, ch\, V^{(02;4n)_{2n^2}} + \kappa^{(02)}_{(20)}\, ch\, V^{(20;4n)_{2n^2}}) \\
&\qquad + \sum_{n\in\mathbb{Z}}\sigma^{(100)}_{(100)}(\kappa^{(20)}_{(02)}\, ch\, V^{(02;4n)_{2n^2}} + \kappa^{(20)}_{(20)}\, ch\, V^{(20;4n)_{2n^2}}) \\
&\qquad + \sum_{n\in\mathbb{Z}}\sigma^{(100)}_{(100)}(\kappa^{(02)}_{(02)}\, ch\, V^{(02;4n-2)_{2n^2-2n+1}} + \kappa^{(02)}_{(20)}\, ch\, V^{(20;4n-2)_{2n^2-2n+1}}).
\end{aligned}$$

This implies that for $n \in \mathbb{Z}$, the branching rules are

$$b^{(100)}_{(20;4n)} = q^{2n^2}(\sigma^{(100)}_{(001)}\kappa^{(02)}_{(20)} + \sigma^{(100)}_{(100)}\kappa^{(20)}_{(20)})$$

$$b^{(100)}_{(20;4n+2)} = q^{2n^2+2n}(\sigma^{(100)}_{(001)}\kappa^{(20)}_{(20)} + q\sigma^{(100)}_{(100)}\kappa^{(02)}_{(20)})$$

$$b^{(100)}_{(02;4n)} = q^{2n^2}(\sigma^{(100)}_{(001)}\kappa^{(02)}_{(02)} + \sigma^{(100)}_{(100)}\kappa^{(20)}_{(02)})$$

$$= q^{2n^2}(\sigma^{(100)}_{(001)}\kappa^{(20)}_{(20)} + q\sigma^{(100)}_{(100)}\kappa^{(02)}_{(20)})$$

$$b^{(100)}_{(02;4n+2)} = q^{2n^2+2n}(\sigma^{(100)}_{(001)}\kappa^{(20)}_{(02)} + q\sigma^{(100)}_{(100)}\kappa^{(02)}_{(02)})$$

$$= q^{2n^2+2n+1}(\sigma^{(100)}_{(001)}\kappa^{(02)}_{(20)} + q\sigma^{(100)}_{(100)}\kappa^{(20)}_{(20)})$$

However from (4.13a) $\kappa^{(20)}_{(20)} + q^{1/2}\kappa^{(02)}_{(20)} = \phi(q^2)\prod(1 - q^{(2n-1)/2})$ and from (4.15d) $\sigma^{(100)}_{(100)} + q^{-1/2}\sigma^{(100)}_{(001)} = \phi(q)^{-2}\prod(1 + q^{(2n-1)/2})$. Then

$$\phi(q)^{-1} = (\sigma^{(100)}_{(100)} + q^{-1/2}\sigma^{(100)}_{(001)})(\kappa^{(20)}_{(20)} + q^{1/2}\kappa^{(02)}_{(20)})$$

$$= (\sigma^{(100)}_{(100)}\kappa^{(20)}_{(20)} + \sigma^{(100)}_{(001)}\kappa^{(02)}_{(20)}) + q^{-1/2}(\sigma^{(100)}_{(001)}\kappa^{(20)}_{(20)} + q\sigma^{(100)}_{(100)}\kappa^{(02)}_{(20)}).$$

Hence

$$b^{(100)}_{(20;4n)} = q^{2n^2}\phi(q)^{-1}$$

$$b^{(100)}_{(20;4n+2)} = 0$$

$$b^{(100)}_{(02;4n)} = 0$$

$$b^{(100)}_{(02;4n+2)} = q^{2n^2+2n+1}\phi(q)^{-1}.$$

Similarly, for highest weight representation (010) we have

$$ch\, V^{(010)} = \sigma^{(010)}_{(010)}\Omega^{(010)}$$

$$\to \sum_{n\in\mathbb{Z}} \sigma^{(010)}_{(010)}\Omega^{(11;2n-1)_{n(n-1)/2}}$$

$$= \sum_{n\in\mathbb{Z}} \sigma^{(010)}_{(010)}\kappa^{(11)}_{(11)}\, ch\, V^{(11;2n-1)_{n(n-1)/2}}.$$

Hence on substituting $\sigma^{(010)}_{(010)}$ and $\kappa^{(11)}_{(11)}$ from (4.15d) and (4.13a) respectively, we obtain

$$b^{(010)}_{(11;2n-1)} = q^{n(n-1)/2}\phi(q)^{-1}.$$

Below we give some branching rule multiplicity generating functions for the affine subalgebras of affine algebras identified in [**BS1**]. The branching rule multiplicities marked * can be inferred from those of [**BS1**], while others are new.

$A_2^{(1)} \quad \supset \quad A_1^{(1)} \oplus u_1$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \longrightarrow (\lambda_0, \lambda_1 + \lambda_2\, ; \frac{1}{3}(\lambda_1 - \lambda_2))_d$

$$* \, b^{(100)}_{(10;2n)} = q^{3n^2} \phi(q)^{-1}$$

$$* \, b^{(100)}_{(01;2n+1)} = q^{3n^2+3n+1} \phi(q)^{-1}$$

$C_2^{(1)} \quad \supset \quad A_1^{(1)} \oplus A_1$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \longrightarrow (\lambda_0, \lambda_1 + \lambda_2\, ; \lambda_2)_d.$

$$b^{(100)}_{(01;2n+1)} = (q^{n^2+n+1} - q^{n^2+3n+3}) \phi(q^8) \phi(q)^{-2} \prod_{\pm 2, \pm 3, \pm 5(16)} (1 - q^n)$$

$$b^{(100)}_{(10;2n)} = (q^{n^2} - q^{(n+1)^2}) \phi(q^8) \phi(q)^{-2} \prod_{\pm 1, \pm 6, \pm 7(16)} (1 - q^n)$$

$$* \quad b^{(010)}_{(01;2n)} = (q^{n^2} - q^{(n+1)^2}) \phi(q^2) \phi(q)^{-2}$$

$$* \quad b^{(010)}_{(10;2n+1)} = (q^{n(n+1)} - q^{(n+1)(n+2)}) \phi(q^2) \phi(q)^{-2}$$

$D_3^{(2)} \quad \supset \quad A_1^{(1)} \oplus A_1$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \rightarrow (\lambda_0 + \lambda_1 + \lambda_2, \lambda_1\, ; \lambda_1 + \lambda_2)_{\frac{d}{2}}.$

$$b^{(001)}_{(01;2n)} = (q^{m^2} - q^{(m+1)^2}) \phi(q)^{-1}$$

$$b^{(001)}_{(10;2n+1)} = (q^{n(n+1)} - q^{(n+1)(n+2)}) \phi(q)^{-1}$$

$$* \quad b^{(100)}_{(01;2n+1)} = (q^{n^2+n+1/2} - q^{n^2+3n+5/2}) \phi(q)^{-1}$$

$$* \quad b^{(100)}_{(10;2n)} = (q^{n^2} - q^{(n+1)^2}) \phi(q)^{-1}$$

$$b^{(101)}_{(02;2n+1)} = (q^{(n^2+n+1)/2} - q^{(n^2+3n+4)/2}) \phi(q^{1/2})^{-2} \phi(q^2)$$

$$b^{(101)}_{(11;2n)} = (q^{n^2/2} - q^{(n+1)^2/2}) \phi(q^{1/2})^{-3} \phi(q^2)^{-1} \phi(q)^3$$

$$b^{(101)}_{(20;2n+1)} = (q^{(n^2+n)/2} - q^{(n^2+3n+3)/2}) \phi(q^{1/2})^{-2} \phi(q^2)$$

$D_3^{(2)} \quad \supset \quad A_1^{(1)} \oplus u_1$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \longrightarrow (\lambda_0, 2\lambda_1 + \lambda_2\, ; \lambda_2)_d.$

$$b^{(001)}_{(01;2n+1)} = q^{n(n+1)} \phi(q^2)^{-1}$$

$$* \quad b^{(100)}_{(10;2n)} = q^{n^2} \phi(q^2)^{-1}$$

$$b^{(101)}_{(11;2n+1)} = q^{n(n+1)/2} \phi(q)^{-1}$$

$A_4^{(2)} \quad \supset \quad A_1^{(1)} \oplus A_1$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \longrightarrow (2\lambda_0 + \lambda_1 + \lambda_2, \lambda_1\, ; \lambda_1 + \lambda_2)_d.$

$$* \quad b^{(001)}_{(01;2n)} = (q^{n^2} - q^{(n+1)^2})\phi(q)^{-1}$$

$$* \quad b^{(001)}_{(10;2n+1)} = (q^{n(n+1)} - q^{(n+1)(n+2)})\phi(q)^{-1}$$

$A_4^{(2)} \quad \supset \quad A_2^{(2)} \oplus u_1$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \longrightarrow (\lambda_0, 2\lambda_1 + \lambda_2\, ; \lambda_2)_d.$

$$* \quad b^{(001)}_{(01;2n+1)} = q^{n(n+1)/2}\phi(q)^{-1}$$

$G_2^{(1)} \quad \supset \quad A_2^{(1)}$

Weight projection: $(\lambda_0, \lambda_1\, \lambda_2)_d \rightarrow (\lambda_0, \lambda_1, \lambda_1 + \lambda_2)_d.$

$$b^{(001)}_{(100)} = \phi(q)^{-1}\phi(q^{15})\Big(\prod_{\pm 4(15)} (1-q^n) + q \prod_{\pm 1(15)} (1-q^n)\Big)$$

$$b^{(100)}_{(100)} = \phi(q)^{-1}\phi(q^{15})\Big(\prod_{\pm 7(15)} (1-q^n) - q \prod_{\pm 2(15)} (1-q^n)\Big)$$

$$b^{(001)}_{(001)} = b^{(001)}_{(010)} = \phi(q)^{-1}\phi(q^{15}) \prod_{\pm 6(15)} (1-q^n)$$

$$b^{(100)}_{(001)} = b^{(100)}_{(010)} = q\phi(q)^{-1}\phi(q^{15}) \prod_{\pm 3(15)} (1-q^n)$$

CHAPTER 7

Conclusion

In this thesis we have presented two methods of computing weight multiplicities of highest weight modules of affine Kac-Moody algebras. The first method depends on reorganising the Weyl-Kac character formula and on making use of the fact that the affine Weyl group is a semidirect product of a translation group and a finite Weyl group. This allowed us to obtain analytic expressions for orbit sum to irreducible character expansions for low level and low rank affine algebras. These expansions were further simplified by specialising the Weyl-Kac denominator identity before being inverted to obtain weight multiplicity generating functions. These analytic functions were later used to obtain analytic branching rule multiplicities for the embedding of one affine algebra in another or in itself.

Although the method itself is of general validity, it seems quite impractical in the case of affine algebras to proceed beyond level 2 and rank 2 as the number of irreducible characters tends to increase rapidly as well as the number of weights in each congruence class. It remains to be seen how the compatibility rules stated by Begin and Sharp **[BS2]** may be used for anything beyond the rank 1 affine algebras. Numerically with the help of computers, some progress could be made but certainly there will be a practical bound because the computations depend on the explicit generation of Weyl group element.

In the second method, the Weyl-Kostant-Liu character formula together with the identification of the set $\{W : \overline{W}\}$ and the Young diagrammatic technique for computing $w(\lambda) - \lambda$ allowed us to expand the irreducible affine characters directly in terms of irreducible characters of simple finite-dimensional Lie algebras. For sufficiently large rank, this computation is independent of the rank of the algebra. Since the weight multiplicities of the simple finite-dimensional Lie algebras are polynomial in the rank, we have thereby established that the weight multiplicities of affine algebras are also polynomial.

In the process of obtaining the action $w(\lambda) - \lambda$ in the ϵ basis, it is a bit of a surprise that the entries in the boxes of the Young diagrams are just the Dynkin labels of the weight λ which are actually components of λ in the fundamantal basis. Another unexpected coincidence is that the core elements of $\{W : \overline{W}\}$ are in such close correspondence with the Frobenius notation for partitions. Both of these factors make the results much easier to express than would be the case without the use of partitions and Young diagrams.

One obvious extension of this work is surely to find proofs of all the conjectures stated for the affine algebras $C_r^{(1)}$, $A_{2r}^{(2)}$, $D_{r+1}^{(2)}$, $B_r^{(1)}$, $A_{2r-1}^{(2)}$ and $D_r^{(1)}$. It is expected that the proofs in the case of $C_r^{(1)}$, $A_{2r}^{(2)}$ and $D_{r+1}^{(2)}$ will be similar to that of $A_r^{(1)}$. Although it might be more difficult, it is also reasonable to expect that the conjectures for cases $B_r^{(1)}$ and $A_{2r-1}^{(2)}$ can also be proved in the near future with a two-step inductive argument taking into account the distinction between $w_{[a]}^{(0)}$ and $w_{[a]}^{(1)}$. The case of $D_r^{(1)}$ is a bit subtle and surely needs some further ingredient especially in obtaining the action $w(\lambda) - \lambda$.

In the thesis we have been most concerned with the determination of $\{W : \overline{W}\}$ for the seven infinite series of rank dependent affine algebras and their restriction to one specific infinite series of rank dependent simple finite-dimensional Lie algebra. It would also be interesting to know what the set $\{W : \hat{W}\}$ looks like where $\hat{W}$ is the Weyl group of the semisimple Lie algebra $\hat{\mathcal{G}}$ obtained from the Dynkin diagram of the affine algebra $\mathcal{G}$ by dropping a node other than the zeroth node. Similarly it would be interesting to know $\{W : \overline{W}\}$ in the case of exceptional affine algebras. Maybe we are not concerned with the computation of weight multiplicities this time, but the possibility of obtaining branching rules is certainly of interest.

The computations so far have been made only for a few representation of the affine algebras and have been carried out only up to depth 4. They are already quite involved. It would be helpful if a program could be written in SCHUR to do similar computations for these and other representations going beyond depth 4. It should be stressed that in computing up to depth 4 the expansions of the inverse D^{-1}, (6.7), have been given in full. To proceed it is only necessary to expand N^{λ}, (6.6), up to terms involving q^4. Since $d_{w(\lambda)}$ is proportional to the level $L(\lambda)$ of λ very few coset elements $w \in \{W : \overline{W}\}$ are required. In fact for $L(\lambda) \geq 4$ it is sufficient to just take $w = id$ in the numerator.

Beyond the context of affine Kac-Moody algebras, it would also be interesting to know the impact of the polynomial nature of the weight multiplicities of affine algebras on the determination of the root multiplicities of hyperbolic Kac-Moody algebras [KM].

Appendix 1 : Generalised Cartan matrices of affine type.

1. GCM for $A_1^{(1)}$

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

2. GCM for $A_r^{(1)}$, $r \geq 2$ is the $(r+1) \times (r+1)$ matrix.

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

In particular, for $r = 2$

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -2 \end{pmatrix}$$

3. GCM for $B_r^{(1)}$, $r \geq 3$ is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

In particular, for $r = 3$

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

4. GCM for $C_r^{(1)}$, $r \geq 2$ is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

In particular, for $r = 2$

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

5. GCM for $D_r^{(1)}$, $r \geq 4$ is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 0 & 2 \end{pmatrix}$$

In particular, for $r = 4$

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

6. GCM for $E_6^{(1)}$ is

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

7. GCM for $E_7^{(1)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

8. GCM for $E_8^{(1)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

9. GCM for $F_4^{(1)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

10. GCM for $G_2^{(1)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$$

11. GCM for $A_2^{(2)}$

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

12. GCM for $A_{2r}^{(2)}$, $r \geq 2$ is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

In particular, for $r = 2$

$$A = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

13. GCM for $A_{2r-1}^{(2)}$, $r \geq 3$ is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}$$

In particular, for $r = 3$

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

14. GCM for $D_{r+1}^{(2)}$, $r \geq 2$ is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -2 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

In particular, for $r = 2$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

15. GCM for $E_6^{(2)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

16. GCM for $D_4^{(3)}$ is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 2 \end{pmatrix}$$

Appendix 2 : The symmetric $\bar{G}$ matrices.

1. For $A_r^{(1)}$, $D_r^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$ the matrix $\bar{G}$ is the same as matrix $\bar{A}^{-1}$

2. For $A_2^{(2)}$, $\bar{G} = (\frac{1}{4})$

3. For $G_2^{(1)}$

$$\bar{G} = \frac{1}{3}\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix}$$

4. For $D_4^{(3)}$

$$\bar{G} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$$

5. For $F_4^{(1)}$

$$\bar{G} = \frac{1}{2}\begin{pmatrix} 4 & 6 & 4 & 2 \\ 6 & 12 & 8 & 4 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{pmatrix}$$

6. For $E_6^{(2)}$

$$\bar{G} = \begin{pmatrix} 2 & 3 & 4 & 2 \\ 3 & 6 & 8 & 4 \\ 4 & 8 & 12 & 6 \\ 2 & 4 & 6 & 4 \end{pmatrix}$$

7. For $B_r^{(1)}$ or $A_{2r}^{(2)}$

$$\bar{G} = \frac{1}{4}\begin{pmatrix} 4 & 4 & 4 & \dots & 4 & 4 & 2 \\ 4 & 8 & 8 & \dots & 8 & 8 & 4 \\ 4 & 8 & 12 & \dots & 12 & 12 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \dots & 4(r-2) & 4(r-2) & 2(r-2) \\ 4 & 8 & 12 & \dots & 4(r-2) & 4(r-1) & 2(r-1) \\ 2 & 4 & 6 & \dots & 2(r-2) & 2(r-1) & r \end{pmatrix}$$

8. For $D_{r+1}^{(2)}$

$$\bar{G} = \frac{1}{2}\begin{pmatrix} 4 & 4 & 4 & \dots & 4 & 4 & 2 \\ 4 & 8 & 8 & \dots & 8 & 8 & 4 \\ 4 & 8 & 12 & \dots & 12 & 12 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \dots & 4(r-2) & 4(r-2) & 2(r-2) \\ 4 & 8 & 12 & \dots & 4(r-2) & 4(r-1) & 2(r-1) \\ 2 & 4 & 6 & \dots & 2(r-2) & 2(r-1) & r \end{pmatrix}$$

9. For $C_r^{(1)}$

$$\bar{G} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 & 2 \\ 1 & 2 & 3 & \dots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & r-2 & r-2 & r-2 \\ 1 & 2 & 3 & \dots & r-2 & r-1 & r-1 \\ 1 & 2 & 3 & \dots & r-2 & r-1 & r \end{pmatrix}$$

10. For $A_{2r-1}^{(2)}$

$$\bar{G} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 & 2 \\ 1 & 2 & 3 & \dots & 3 & 3 & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & r-2 & r-2 & r-2 \\ 1 & 2 & 3 & \dots & r-2 & r-1 & r-1 \\ 1 & 2 & 3 & \dots & r-2 & r-1 & r \end{pmatrix}$$

Appendix 3 : Weight multiplicities of twisted affine algebras of level 2.

$A_2^{(2)}$ - Class 0 - Highest weight (02) and (10)

Depth	(02)		(10)	
	(02)	(10)	(02)	(10)
0	1	1	0	1
1	2	2	1	1
2	4	5	2	3
3	8	9	4	5
4	15	17	8	10
5	26	29	14	16
6	44	50	24	29
7	72	80	40	45
8	115	129	64	74
9	180	199	101	113
10	276	306	156	176
11	416	458	236	261
12	619	682	352	393
13	908	994	519	570
14	1316	1442	754	832
15	1888	2059	1084	1186
16	2682	2923	1544	1691
17	3774	4100	2177	2369
18	5268	5719	3044	3317
19	7296	7898	4224	4578
20	10032	10852	5816	6307

$A_4^{(2)}$ - Class 0 - Highest weight (002)

Depth	(002)	(010)	(100)
0	1	1	2
1	4	5	7
2	14	17	24
3	40	49	64
4	104	126	162
5	248	298	371
6	556	663	816
7	1184	1403	1696
8	2421	2849	3414
9	4776	5589	6623
10	9144	10643	12524
11	17048	19747	23057
12	31055	35810	41582
13	55404	63627	73454
14	97020	110994	127560
15	167040	190431	217861
16	283202	321804	366774
17	473404	536297	608989
18	781124	882383	998800
19	1273440	1434697	1618978
20	2052979	2307165	2596392
21	3275392	3672284	4121772
22	5175012	5789225	6482332
23	8101952	9044581	10104295
24	12575799	14011106	15619824
25	19362520	21531867	23955810
26	29584406	32840234	36468828
27	44876016	49730097	55125988
28	67604838	74796125	82772398

$A_4^{(2)}$ - Class 0 - Highest weight (010)

Depth	(002)	(010)	(100)
0	0	1	1
1	2	4	4
2	8	13	15
3	25	37	42
4	68	94	109
5	168	221	256
6	384	491	571
7	832	1038	1202
8	1720	2108	2442
9	3426	4139	4776
10	6608	7890	9086
11	12397	14657	16822
12	22696	26617	30471
13	40672	47359	54044
14	71488	82732	94169
15	123488	142143	161328
16	209968	240533	272317
17	351894	401391	453260
18	581968	661275	744987
19	950753	1076529	1209974
20	1535664	1733263	1943939
21	2454316	2761993	3091152
22	3883936	4358997	4868861
23	6089647	6817339	7600122
24	9465260	10571599	11764154
25	14591966	16261984	18064744
26	22321992	24825871	27532285
27	33897746	37627706	41662824
28	51120104	56642461	62621070

$A_4^{(2)}$ - Class 0 - Highest weight (100)

Depth	(002)	(010)	(100)
0	0	0	1
1	1	1	2
2	4	4	8
3	12	13	20
4	32	36	53
5	79	89	120
6	180	205	271
7	390	446	564
8	808	925	1154
9	1613	1847	2252
10	3120	3570	4307
11	5872	6708	7980
12	10784	12299	14519
13	19387	22066	25802
14	34184	38824	45126
15	59230	67124	77496
16	101008	114222	131236
17	169770	191559	218976
18	281540	317001	360953
19	461160	518167	587644
20	746752	837368	946542
21	1196350	1338904	1508534
22	1897588	2119697	2381611
23	2981818	3324766	3725400
24	4644496	5169603	5778673
25	7174599	7972279	8890794
26	10996576	12199331	13576397
27	16730180	18531033	20581100
28	25275136	27953657	30988700

$D_3^{(2)}$ - Class 0 - Highest weight (002)

Depth	(002)	(010)	(200)
0	1	1	2
1	1	2	3
2	5	7	11
3	8	13	18
4	24	32	47
5	39	57	77
6	90	119	165
7	147	204	268
8	297	385	516
9	477	638	823
10	880	1125	1468
11	1391	1812	2300
12	2412	3041	3891

$D_3^{(2)}$ - Class 0 - Highest weight (010)

Depth	(002)	(010)	(200)
0	0	1	1
1	1	1	3
2	3	6	7
3	7	9	16
4	16	27	34
5	34	43	67
6	67	101	127
7	127	161	232
8	232	328	412
9	412	520	713
10	713	964	1205
11	1205	1508	1997
12	1997	2623	3255

$D_3^{(2)}$ - Class 0 - Highest weight (200)

Depth	(002)	(010)	(200)
0	0	0	1
1	0	1	1
2	2	2	5
3	3	7	8
4	11	13	24
5	18	32	39
6	47	57	90
7	77	119	147
8	165	204	297
9	268	385	477
10	516	638	880
11	823	1125	1391
12	1468	1812	2412

$D_3^{(2)}$ - Class 1 - Highest weight (101)

Depth	(101)
0	1
1	3
2	8
3	19
4	41
5	83
6	161
7	299
8	538
9	942
10	1610
11	2694
12	4427

$D_4^{(3)}$ - Highest weight (010) and (200)

Depth	(010)		(200)	
	(010)	(200)	(010)	(200)
0	1	1	0	1
1	2	4	1	1
2	5	8	3	5
3	13	17	6	10
4	25	37	15	21
5	49	68	31	42
6	96	125	57	83
7	169	229	110	143
8	296	390	198	263
9	515	658	338	448
10	851	1101	583	749
11	1393	1774	971	1237
12	2261	2832	1569	2012

Appendix 4 : Inverse string functions κ

$A_1^{(1)}$:

Level 1 : $P^+_{max} = \{(10)\} \cup \{(01)\}$

$$\kappa^{(10)}_{(10)} = \kappa^{(01)}_{(01)} = \sum_n \{q^{6n^2-n} - q^{6n^2-5n+1}\}$$

Level 2 : $P^+_{max} = \{(11)\} \cup \{(2,0),(02)\}$

$$\kappa^{(11)}_{(11)} = \sum_n \{q^{4n^2} - q^{4n^2-4n+1}\}$$

$$\kappa^{(20)}_{(20)} = \kappa^{(02)}_{(02)} = \sum_n q^{4n^2-n}$$

$$\kappa^{(20)}_{(02)} = q\kappa^{(02)}_{(20)} = -\sum_n q^{4n^2-5n+2}$$

$A_2^{(2)}$:

Level 1 : $P^+_{max} = \{(01)\}$

$$\kappa^{(01)}_{(01)} = \sum_n (q^{6n^2+n} - q^{6n^2-5n+1})$$

Level 2 : $P^+_{max} = \{(10),(02)\}$

$$\kappa^{(02)}_{(02)} = \sum_n \{q^{15n^2+2n} - q^{15n^2+8n+1}\}$$

$$\kappa^{(02)}_{(10)} = \sum_n \{q^{15n^2+14n+3} - q^{15n^2-4n}\}$$

$$\kappa^{(10)}_{(02)} = \sum_n \{q^{15n^2-13n+3} - q^{15n^2-7n+1}\}$$

$$\kappa^{(10)}_{(10)} = \sum_n \{q^{15n^2-n} - q^{15n^2+11n+2}\}$$

$A_2^{(1)}$:

Level 1 : $P^+_{max} = \{(100)\} \cup \{(010)\} \cup \{(001)\}$

$$\kappa^{(100)}_{(100)} = \kappa^{(010)}_{(010)} = \kappa^{(001)}_{(001)}$$

$$= \sum_{m,n} \{q^{\Gamma-m-n} + 2q^{\Gamma-m-10n+3} - 2q^{\Gamma-7m+2n+1} - q^{\Gamma-7m-7n+4}\}$$

where $\Gamma = 12(m^2 - mn + n^2)$.

Level 2 : $P^+_{max} = \{(200),(011)\} \cup \{(020),(101)\} \cup \{(002),(110)\}$

$$\begin{aligned}\kappa^{(011)}_{(011)} =&\kappa^{(101)}_{(101)} = \kappa^{(110)}_{(110)}\\ =&\sum_{m,n}\{q^{\Gamma+m+n} - 2q^{\Gamma+49m-23n+20} + 2q^{\Gamma+31m+13n+17} - q^{\Gamma+19m+19n+12}\\ &+ 2q^{\Gamma+31m-14n+8} - 2q^{\Gamma+19m-8n+3} - 2q^{\Gamma+7m+34n+16} + 2q^{\Gamma+m+28n+9}\\ &+ 2q^{\Gamma-17m+46n+18} - 2q^{\Gamma-11m+34n+10}\}\end{aligned}$$

$$\begin{aligned}\kappa^{(110)}_{(002)} =&\kappa^{(101)}_{(020)} = q\kappa^{(011)}_{(200)}\\ =&\sum_{m,n}\{q^{\Gamma+13m+13n+6} - 2q^{\Gamma+37m+n+16} + 2q^{\Gamma-11m+43n+17} - q^{\Gamma+7m+7n+2}\\ &+ 2q^{\Gamma-17m+28n+7} - 2q^{\Gamma+7m+16n+5} - 2q^{\Gamma-29m+52n+23} + 2q^{\Gamma+13m+4n+3}\\ &+ 2q^{\Gamma+19m+28n+17} - 2q^{\Gamma+37m-8n+13}\}\end{aligned}$$

$$\begin{aligned}\kappa^{(200)}_{(011)} =&q\kappa^{(002)}_{(110)} = q\kappa^{(020)}_{(101)}\\ =&\sum_{m,n}\{q^{\Gamma+16m+16n+9} - 2q^{\Gamma+4m+22n+7} + 2q^{\Gamma+16m-2n+3} - q^{\Gamma+4m+4n+1}\}\end{aligned}$$

$$\begin{aligned}\kappa^{(200)}_{(200)} =&\kappa^{(020)}_{(020)} = \kappa^{(002)}_{(002)}\\ =&\sum_{m,n}\{q^{\Gamma-2m-2n} - 2q^{\Gamma+22m-14n+4} + 2q^{\Gamma+34m-2n+12} - q^{\Gamma+22m+22n+16}\}\end{aligned}$$

where $\Gamma = 30(m^2 - mn + n^2)$.

$C^{(1)}_2$:

Level 1 : $P^+_{max} = \{(010)\} \cup \{(100),(001)\}$

$$\begin{aligned}\kappa^{(010)}_{(010)} = &\sum_{m,n}\{q^{\Gamma+2m-n} + q^{\Gamma+8m-13n+4} + q^{\Gamma-16m+5n+3} + q^{\Gamma-10m-7n+7}\\ &- q^{\Gamma-10m+5n+1} - q^{\Gamma+8m-7n+1} - q^{\Gamma-16m-n+6} + q^{\Gamma+2m-13n+6}\}\end{aligned}$$

$$\kappa^{(100)}_{(100)} = \kappa^{(001)}_{(001)} = \sum_{m,n}\{q^{\Gamma-m-n} + q^{\Gamma-7m-7n+5} - q^{\Gamma+5m-7n+1} - q^{\Gamma+11m-10n+2}\}$$

$$\kappa^{(100)}_{(001)} = q\kappa^{(001)}_{(001)} = \sum_{m,n}\{q^{\Gamma+11m-13n+4} + q^{\Gamma-19m+5n+5} - q^{\Gamma-7m+5n+1} - q^{\Gamma-m-13n+8}\}$$

where $\Gamma = 12(2m^2 - 2mn + n^2)$.

Level 2 : $P^+_{max} = \{(002),(020),(101),(200)\} \cup \{(011),(110)\}$

$$\kappa^{(110)}_{(110)} = \kappa^{(011)}_{(011)} = \sum_{m,n}\{q^{\Gamma+m-2n} + q^{\Gamma+37m-14n+6} + q^{\Gamma-47m+34n+11}$$
$$+ q^{\Gamma+49m-38n+13} + q^{\Gamma+31m-2n+7} + q^{\Gamma+7m-14n+2} + q^{\Gamma+43m-26n+8}$$
$$+ q^{\Gamma-41m+22n+7} - q^{\Gamma+49m-26n+10} - q^{\Gamma-23m+22n+4} - q^{\Gamma+13m-2n+1}$$
$$- q^{\Gamma+m-14n+3} - q^{\Gamma-41m+34n+10} - q^{\Gamma+67m-38n+19} - q^{\Gamma-17m-2n+3}$$
$$- \sum q^{\Gamma+31m-14n+4}\}$$

$$\kappa^{(110)}_{(011)} = q\kappa^{(011)}_{(110)} = \sum_{m,n}\{q^{\Gamma+31m+n+9} + q^{\Gamma-17m+13n+2} + q^{\Gamma-53m+37n+14}$$
$$+ q^{\Gamma+19m-11n+2} + q^{\Gamma+61m-29n+16} + q^{\Gamma+13m-17n+3} + q^{\Gamma-23m+7n+3}$$
$$+ q^{\Gamma+49m-41n+15} - q^{\Gamma-41m+37n+12} - q^{\Gamma+43m-11n+10} - q^{\Gamma+7m+n+1}$$
$$- q^{\Gamma-29m+13n+4} - q^{\Gamma-11m+7n+1} - q^{\Gamma+73m-41n+23} - q^{\Gamma+37m-29n+8}$$
$$- \sum q^{\Gamma+m-17n+5}\}$$

$$\kappa^{(101)}_{(002)} = q\kappa^{(101)}_{(200)} = \sum_{m,n}\{q^{\Gamma-14m+13n+2} + q^{\Gamma+34m-23n+6} + q^{\Gamma+16m-17n+3}$$
$$+ q^{\Gamma+4m+7n+2} + q^{\Gamma+76m-47n+26} + q^{\Gamma-56m+37n+15} + q^{\Gamma+46m-17n+10}$$
$$+ q^{\Gamma-26m+7n+4} - q^{\Gamma-38m+37n+12} - q^{\Gamma+58m-47n+20} - q^{\Gamma-8m+7n+1}$$
$$- q^{\Gamma+28m-17n+4} - q^{\Gamma+52m-23n+12} - q^{\Gamma-32m+13n+5} - q^{\Gamma+22m+7n+8}$$
$$- \sum q^{\Gamma-2m-17n+6}\}$$

$$\kappa^{(101)}_{(020)} = \sum_{m,n}\{q^{\Gamma+70m-47n+23} + q^{\Gamma-50m+37n+13} + q^{\Gamma+40m-17n+7} + q^{\Gamma-20m+7n+2}$$
$$+ q^{\Gamma-20m+13n+2} + q^{\Gamma+40m-23n+7} + q^{\Gamma+10m-17n+3} + q^{\Gamma+10m+7n+3}$$
$$- q^{\Gamma+46m-23n+9} - q^{\Gamma-26m+13n+3} - q^{\Gamma+16m+7n+5} - q^{\Gamma+4m-17n+4}$$
$$- q^{\Gamma-44m+37n+12} - q^{\Gamma+64m-47n+21} - q^{\Gamma-14m+7n+1} - q^{\Gamma+34m-17n+5}\}$$

$$\kappa^{(101)}_{(101)} = \sum_{m,n}\{q^{\Gamma-2m+n} + q^{\Gamma+22m-11n+2} + q^{\Gamma-32m+31n+8} + q^{\Gamma+52m-41n+15}$$
$$+ q^{\Gamma+28m+n+7} + q^{\Gamma-8m-11n+4} + q^{\Gamma+58m-29n+14} + q^{\Gamma-38m+19n+6}$$
$$- q^{\Gamma+10m-11n+1} - q^{\Gamma+10m+n+1} - q^{\Gamma-20m+19n+3} - q^{\Gamma+40m-11n+8}$$
$$- q^{\Gamma+40m-29n+8} - q^{\Gamma-20m+n+3} - q^{\Gamma+70m-41n+21} - q^{\Gamma-50m+31n+11}\}$$

$$\kappa^{(200)}_{(002)} = q^2\kappa^{(002)}_{(200)} = \sum_{m,n}\{q^{\Gamma+58m-26n+15} + q^{\Gamma+22m-14n+3} - q^{\Gamma+46m-14n+11} - q^{\Gamma+34m-26n+7}\}$$

$$\kappa^{(200)}_{(020)} = q\kappa^{(002)}_{(020)} = \sum_{m,n}\{q^{\Gamma+34m-2n+9} + q^{\Gamma+46m-38n+13} - q^{\Gamma+70m-38n+21} - q^{\Gamma+10m-2n+1}\}$$

$$\kappa^{(200)}_{(101)} = q\kappa^{(002)}_{(101)} = \sum_{m,n}\{q^{\Gamma+10m-14n+2} + q^{\Gamma-50m+34n+12} - q^{\Gamma-38m+34n+10} - q^{\Gamma-2m-14n+4}\}$$

$$\kappa^{(200)}_{(200)} = \kappa^{(002)}_{(002)} = \sum_{m,n}\{q^{\Gamma-2m-2n} + q^{\Gamma-38m+22n+6} - q^{\Gamma-26m+22n+4} - q^{\Gamma-14m-2n+2}\}$$

$$\kappa^{(020)}_{(002)} = q\kappa^{(020)}_{(200)} = \sum_{m,n}\{q^{\Gamma-32m+34n+10} + q^{\Gamma-8m-14n+6} + q^{\Gamma+28m+4n+9} + q^{\Gamma+52m-44n+17} - q^{\Gamma+16m-14n+2} - q^{\Gamma-56m+34n+14} - q^{\Gamma+76m-44n+25} - q^{\Gamma+4m+4n+1}\}$$

$$\kappa^{(020)}_{(020)} = \sum_{m,n}\{q^{\Gamma+4m-2n} + q^{\Gamma-44m+22n+8} + q^{\Gamma+64m-32n+17} + q^{\Gamma+16m-8n+1} - q^{\Gamma-20m+22n+4} - q^{\Gamma-20m-2n+4} - q^{\Gamma+40m-8n+9} - q^{\Gamma+40m-32n+9}\}$$

$$\kappa^{(020)}_{(101)} = \sum_{m,n}\{q^{\Gamma+40m-14n+7} + q^{\Gamma+40m-26n+7} + q^{\Gamma-20m+16n+2} + q^{\Gamma-20m+4n+2} - q^{\Gamma+52m-26n+11} - q^{\Gamma+28m-14n+3} - q^{\Gamma-8m+4n} - q^{\Gamma-32m+16n+4}\}$$

where $\Gamma = 30(2m^2 - 2mn + n^2)$.

$G_2^{(1)}$:

Level 1 : $P^+_{max} = \{(001), (100)\}$

$$\kappa^{(001)}_{(001)} = \sum_{m,n}\{q^{\Gamma-m+3n} + q^{\Gamma+7m-33n+9} + q^{\Gamma-21m+15n+10} - q^{\Gamma-9m+15n+1} - q^{\Gamma+11m-33n+6} - q^{\Gamma-17m+3n+12}\}$$

$$\kappa^{(001)}_{(100)} = \sum_{m,n}\{q^{\Gamma+7m-21n+2} + q^{\Gamma-13m+15n+2} + q^{\Gamma-9m-9n+9} - q^{\Gamma+3m-9n} - q^{\Gamma-17m+15n+5} - q^{\Gamma-m-21n+8}\}$$

$$\kappa^{(100)}_{(001)} = \sum_{m,n}\{q^{\Gamma+11m-25n+3} + q^{\Gamma-17m+23n+4} + q^{\Gamma-9m-13n+13} - q^{\Gamma+7m-13n+1} - q^{\Gamma-21m+23n+7} - q^{\Gamma-m-25n+12}\}$$

$$\kappa^{(100)}_{(100)} = \sum_{m,n}\{q^{\Gamma-m-n} + q^{\Gamma+3m-25n+7} + q^{\Gamma-17m+11n+7} - q^{\Gamma-9m+11n+1} - q^{\Gamma+7m-25n+4} - q^{\Gamma-13m-n+9}\}$$

where $\Gamma = 20m^2 - 60mn + 60n^2$.

Level 2 : $P^+_{max} = \{(002),(010),(101),(200)\}$

$$\kappa^{(101)}_{(002)} = 3\sum_{m,n}\{q^{\Gamma+7m-15n+2} - q^{\Gamma+5m-9n+1}\}$$

$$\kappa^{(010)}_{(200)} = -q^{-1}\kappa^{(101)}_{(010)} = \sum_{m,n}\{q^{\Gamma+m-13n+3} + q^{\Gamma-9m+5n+3} - q^{\Gamma-7m-n+4} - q^{\Gamma-5m+5n}\}$$

$$\kappa^{(010)}_{(010)} = \kappa^{(101)}_{(101)} = \sum_{m,n}\{q^{\Gamma+m-n} + q^{\Gamma+3m-19n+6} - q^{\Gamma-7m+11n+1} - q^{\Gamma+7m-19n+3}\}$$

$$\kappa^{(010)}_{(101)} = \kappa^{(101)}_{(200)} = \sum_{m,n}\{q^{\Gamma+5m-13n+1} + q^{\Gamma-5m-7n+6} - q^{\Gamma-11m+11n+3} - q^{\Gamma+3m-7n}\}$$

$$\kappa^{(002)}_{(002)} = = \sum_{m,n}\{q^{\Gamma-m+3n} + q^{\Gamma-5m-9n+8} - q^{\Gamma-13m+15n+4} - q^{\Gamma+7m-21n+4}\}$$

$$\kappa^{(200)}_{(002)} = \kappa^{(010)}_{(002)} = 0$$

$$\kappa^{(002)}_{(101)} = q^{-1}\kappa^{(200)}_{(010)} = \sum_{m,n}\{q^{\Gamma+3m-17n+4} - q^{\Gamma-5m+7n}\}$$

$$\kappa^{(002)}_{(200)} = q^{-1}\kappa^{(200)}_{(101)} = \sum_{m,n}\{q^{\Gamma-5m-5n+4} - q^{\Gamma-9m+7n+2}\}$$

$$\kappa^{(002)}_{(010)} = -\kappa^{(200)}_{(200)} = \sum_{m,n}\{q^{\Gamma+7m-17n+2} - q^{\Gamma+3m-5n}\}$$

where $\Gamma = 12m^2 - 36mn + 36n^2$.

$A^{(2)}_4$:

Level 1 : $P^+_{max} = \{(001)\}$

$$\kappa^{(001)}_{(001)} = \sum_{m,n}\{q^{\Gamma-m+2n} + q^{\Gamma+9m-13n+3} + q^{\Gamma-21m+7n+4} + q^{\Gamma-11m-8n+7} - q^{\Gamma-11m+7n+1} - q^{\Gamma+9m-8n+1} - q^{\Gamma-21m+2n+6} - q^{\Gamma-m-13n+6}\}$$

where $\Gamma = 30m^2 - 30mn + 15n^2$.

Level 2 : $P^+_{max} = \{(002),(010),(100)\}$

$$\kappa^{(002)}_{(002)} = \sum_{m,n}\{q^{\Gamma-2m+4n} + q^{\Gamma+48m-16n+9} + q^{\Gamma+78m-26n+24} + q^{\Gamma-12m+24n+5}$$
$$- q^{\Gamma+58m-26n+12} - q^{\Gamma-22m+24n+4} - q^{\Gamma+8m+4n+1} - q^{\Gamma+68m-16n+21}\}$$

$$\kappa^{(002)}_{(010)} = \sum_{m,n}\{q^{\Gamma-32m+34n+8} + q^{\Gamma+78m-16n+29} + q^{\Gamma-22m+44n+17} + q^{\Gamma+18m-6n+1}$$
$$- q^{\Gamma+88m-26n+32} - q^{\Gamma-62m+64n+29} - q^{\Gamma-22m+34n+9} - q^{\Gamma+38m-16n+5}\}$$

$$\kappa^{(002)}_{(100)} = \sum_{m,n}\{q^{\Gamma-2m+34n+15} + q^{\Gamma+108m-46n+42} + q^{\Gamma+18m+4n+3} + q^{\Gamma+58m-6n+19}$$
$$- q^{\Gamma+58m+4n+27} - q^{\Gamma+38m-6n+7} - q^{\Gamma+88m-36n+28} - q^{\Gamma-2m+24n+7}\}$$

$$\kappa^{(010)}_{(002)} = \sum_{m,n}\{q^{\Gamma-37m+39n+11} + q^{\Gamma+13m+19n+10} + q^{\Gamma+43m+9n+20} + q^{\Gamma+23m-11n+2}$$
$$- q^{\Gamma+23m+9n+8} - q^{\Gamma-57m+59n+25} - q^{\Gamma-27m+39n+12} - q^{\Gamma+103m-51n+38}\}$$
$$+ q^{\Gamma+33m+4n+10} + q^{\Gamma-57m+54n+21} + q^{\Gamma-27m+44n+16} + q^{\Gamma+93m-46n+31}$$
$$- q^{\Gamma-47m+44n+14} - q^{\Gamma+13m+24n+14} - q^{\Gamma+43m+4n+16} - q^{\Gamma+33m-16n+4}\}$$

$$\kappa^{(010)}_{(010)} = \sum_{m,n}\{q^{\Gamma+3m-n} + q^{\Gamma+113m-51n+46} + q^{\Gamma+13m+9n+4} + q^{\Gamma-17m+29n+7}$$
$$- q^{\Gamma+123m-61n+54} - q^{\Gamma-27m+29n+6} - q^{\Gamma+13m-n+1} - q^{\Gamma+3m+19n+6}\}$$
$$+ q^{\Gamma+73m-36n+19} + q^{\Gamma+43m-16n+7} + q^{\Gamma+83m-26n+28} + q^{\Gamma+53m-6n+16}$$
$$- q^{\Gamma+53m-26n+10} - q^{\Gamma+43m-6n+10}\} - q^{\Gamma+83m-36n+25} - q^{\Gamma+73m-16n+25}\}$$

$$\kappa^{(010)}_{(100)} = \sum_{m,n}\{q^{\Gamma+33m-n+7} + q^{\Gamma+73m-11n+28} + q^{\Gamma-17m+39n+14} + q^{\Gamma+93m-41n+31}$$
$$- q^{\Gamma+93m-31n+34} - q^{\Gamma+3m+29n+13} - q^{\Gamma+53m-n+19} - q^{\Gamma+33m-11n+4}\}$$
$$+ q^{\Gamma-37m+34n+8} + q^{\Gamma+3m+24n+9} + q^{\Gamma+53m+4n+23} + q^{\Gamma+23m-6n+2}$$
$$- q^{\Gamma+23m+4n+5} - q^{\Gamma-67m+64n+29} - q^{\Gamma-17m+34n+10} - q^{\Gamma+103m-46n+38}\}$$

$$\kappa^{(100)}_{(002)} = \sum_{m,n}\{q^{\Gamma+68m-31n+17} + q^{\Gamma-22m+19n+3} + q^{\Gamma+8m+9n+3} + q^{\Gamma+58m-11n+17}$$
$$- q^{\Gamma-12m+9n+1} - q^{\Gamma+48m-11n+11} - q^{\Gamma+78m-31n+23} - q^{\Gamma-2m+19n+5}\}$$

$$\kappa^{(100)}_{(010)} = \sum_{m,n}\{q^{\Gamma+38m-n+10} + q^{\Gamma+8m+19n+8} + q^{\Gamma+118m-61n+50} + q^{\Gamma+88m-41n+28}$$
$$- q^{\Gamma+18m+9n+6} - q^{\Gamma+78m-41n+22} - q^{\Gamma+48m-n+16} - q^{\Gamma+108m-51n+42}\}$$

$$\kappa^{(100)}_{(100)} = \sum_{m,n}\{q^{\Gamma-2m-n} + q^{\Gamma+38m-11n+6} + q^{\Gamma+88m-31n+30} + q^{\Gamma-12m+29n+8}$$
$$- q^{\Gamma+58m-31n+12} - q^{\Gamma-32m+29n+6} - q^{\Gamma+18m-n+2} - q^{\Gamma+68m-11n+24}\}$$

where $\Gamma = 70m^2 - 70mn + 35n^2$.

$D_3^{(2)}$:

Level 1 : $P^+_{max} = \{(100)\} \cup \{(001)\}$

$$\kappa^{(100)}_{(100)} = \kappa^{(001)}_{(1001)} = \sum_{m,n}\{q^{\Gamma-2m-n} + q^{\Gamma+6m-17n+5}$$
$$+ q^{\Gamma-26m+7n+5} + q^{\Gamma-18m-9n+10}$$
$$- q^{\Gamma-18m+7n+2} - q^{\Gamma+6m-9n+1}$$
$$- q^{\Gamma-26m-n+9} - q^{\Gamma-2m-17n+8}\}$$

where $\Gamma = 40m^2 - 40mn + 20n^2$.

Level 2 : $P^+_{max} = \{(002), (010), (200)\}$

$$\kappa^{(010)}_{(002)} = q\kappa^{(010)}_{(200)} = \sum_{m,n}\{q^{\Gamma-22m+7n+6} - q^{\Gamma+10m-9n+2}$$
$$+ q^{\Gamma+10m-13n+4} - q^{\Gamma-22m+3n+8}\}$$

$$\kappa^{(010)}_{(010)} = \sum_{m,n}\{q^{\Gamma+2m-n} - q^{\Gamma-14m+7n+2}$$
$$+ q^{\Gamma-14m-5n+8} - q^{\Gamma+2m-13n+6}\}$$

$$\kappa^{(200)}_{(002)} = q^{-2}\kappa^{(002)}_{(200)} = \sum_{m,n}\{q^{\Gamma-10m-9n+10} - q^{\Gamma-10m+7n+2}\}$$

$$\kappa^{(200)}_{(010)} = q^{-1}\kappa^{(002)}_{(010)} = \sum_{m,n}\{q^{\Gamma-18m+7n+4} - q^{\Gamma-18m-n+8}\}$$

$$\kappa^{(200)}_{(200)} = \kappa^{(002)}_{(002)} = \sum_{m,n}\{q^{\Gamma-2m-n} - q^{\Gamma-2m-9n+4}\}$$

where $\Gamma = 24m^2 - 24mn + 12n^2$.

$D_4^{(3)}$:

Level 1 : $P^+_{max} = \{(100)\}$

$$\kappa^{(100)}_{(100)} = \sum_{m,n}\{q^{\Gamma-m-3n} + q^{\Gamma+23m-57n+7} + q^{\Gamma-31m+33n+7} + q^{\Gamma+17m-75n+21}$$
$$+ q^{\Gamma-37m+15n+21} + q^{\Gamma-13m-39n+28} - q^{\Gamma-13m+15n+1} - q^{\Gamma+17m-39n+3}$$
$$- q^{\Gamma-37m+33n+12} - q^{\Gamma+23m-75n+16} - q^{\Gamma-31m-3n+25} + q^{\Gamma-m-57n+27}\}$$

where $\Gamma = 42m^2 - 126mn + 126n^2$.

Level 2 : $P^+_{max} = \{(010), (200)\}$

$$\kappa^{(010)}_{(010)} = \sum_{m,n}\{q^{\Gamma+2m-3n} + q^{\Gamma-10m-21n+19} - q^{\Gamma-10m+15n+1} - q^{\Gamma+2m-39n+18}$$
$$+ q^{\Gamma-19m+24n+4} + q^{\Gamma+11m-48n+15} - q^{\Gamma-25m+24n+9} - q^{\Gamma+17m-48n+10}\}$$

$$\kappa^{(010)}_{(200)} = \sum_{m,n}\{q^{\Gamma-16m+15n+3} + q^{\Gamma+8m-39n+10} - q^{\Gamma+8m-21n+1} - q^{\Gamma-16m-3n+12}$$
$$+ q^{\Gamma+11m-30n+3} + q^{\Gamma-19m+6n+10} - q^{\Gamma-7m+6n} - q^{\Gamma-m-30n+13}\}$$

$$\kappa^{(200)}_{(010)} = \sum_{m,n}\{q^{\Gamma+17m-39n+6} + q^{\Gamma-25m+15n+14} - q^{\Gamma+11m-21n+2} - q^{\Gamma-19m-3n+18}\}$$

$$\kappa^{(200)}_{(200)} = \sum_{m,n}\{q^{\Gamma-m-3n} + q^{\Gamma-7m-21n+14} - q^{\Gamma-19m+15n+6} - q^{\Gamma+11m-39n+8}\}$$

where $\Gamma = 24m^2 - 72mn + 72n^2$.

Appendix 5 : The values of the partition function p_k.

	p_1	p_2	p_3	p_4	p_5	p_6
0	1	1	1	1	1	1
1	1	2	3	4	5	6
2	2	5	9	14	20	27
3	3	10	22	40	65	98
4	5	20	51	105	190	315
5	7	36	108	252	506	918
6	11	65	221	574	1265	2492
7	15	110	429	1240	2990	6372
8	22	185	810	2580	6765	15525
9	30	300	1479	5180	14725	36280
10	42	481	2640	10108	31027	81816
11	56	752	4599	19208	63505	178794
12	77	1165	7868	35693	126730	380051
13	101	1770	13209	64960	247170	788004
14	135	2665	21843	116090	472295	1597725
15	176	3956	35581	203984	885723	3174210
16	231	5822	57222	353017	1633000	6190182
17	297	8470	90882	602348	2963840	11867310
18	385	12230	142769	1014580	5302075	22395359
19	490	17490	221910	1688400	9358470	41650050
20	627	24842	341649	2778517	16313440	76413078

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