أعوذ بالله من الشيطان الرجيم

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THE UNIVERSITY SOUTHAMPTON

KINEMATICS AND SYMMETRY

By

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ABSTRACT

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This thesis is concerned with the study of Kinematics and Symmetry. It begins with an examination of motions in a general metric space X, and gives a complete discussion of the equivalence problem. A symmetry of a motion μ in X is therefore a self-equivalence. The symmetry group $\mathrm{Sym}\,\mu$ of μ and its periodic subgroup $P(\mu)$ are investigated and it is found that $P(\mu)$ is a subgroup of the centre of $\mathrm{Sym}\,\mu$. The symmetry group of individual trajectories of μ is shown to be closed in $I_*(X) \times R$ (where $I_*(X)$ is the identity component of the isometry group $I(X)$) and is isomorphic to $\{0\}$, Z or R. Some special types of symmetries including group motion, where the path μ is a homomorphism, are examined.

Special attention is given to smooth motions in a smooth connected Riemannian n-manifold X. In this context, the centrode $C(\mu)$ of μ is of great interest, each instantaneous axis $C_t(\mu)$ of μ being a totally geodesic submanifold of X of even codimension. The centrode $C(\mu)$ is a 1-parameter family of such axes.

The rest of the thesis is devoted to the case where X is Euclidean n-space E^n. The structure of $I_*(X) = E_+(n)$ is exploited to exhibit more properties of the group $\mathrm{Sym}\,\mu$ (in particular, where μ is translational or spherical). Group motions are studied in the low dimensions $n = 1,2$ and 3. A complete discussion is presented for the symmetry groups that can occur in plane motion.

The study of Kinematics in E^1 is reduced to the study of real-valued continuous functions of a real variable. In particular, stable smooth motions correspond to stable Morse functions $f : R \to R$. The symmetry properties and the classification of smooth stable motions are studied in some detail.

TO MY PARENTS.

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CONTENTS

INTRODUCTION

Kinematics investigates motion in a space X, without discussing the underlying physics. In particular in Kinematics no account is taken of the forces that generate motion. Thus, Kinematics is defined here to be the study of the geometry of the space $\mathcal{M}(X)$ of all motions in X.

This thesis will be devoted to the study of Kinematics and Symmetry in a general metric space X, with particular attention to smooth motions in a smooth connected Riemannian n-manifold.

We begin in chapter 1 by examining motions in an arbitrary metric space X. A criterion for equivalence of motions is given in section 1.3.2. A symmetry of a motion μ is measured by the group $\mathrm{Aut}\,\mu_*$ of automorphisms or self-equivalences of μ. The group $\mathrm{Aut}\,\mu_*$ contains a subgroup $\mathrm{Aut}_*\mu_*$ which preserves every orbit of μ. This subgroup measures the periodic behaviour of μ. We show that the group $P(\mu)$ of all periodicities of μ is a subgroup of the centre of the group $\mathrm{Sym}\,\mu$ of all symmetries of μ. We prove in sections 1.6.6 and 1.6.7 that the group of symmetries of individual trajectories of μ is closed in $I_*(X) \times R$ (where $I_*(X)$ is the identity component of the group $I(X)$ of isometries of X) and is isomorphic to $\{0\}$, Z or R. We end this chapter by examining some special types of symmetries and the special type of group motion (where the path μ is a homomorphism).

Chapter 2 considers the special case where X is a smooth n-manifold with a smooth Riemannian structure. Thus we restrict attention to 'smooth motions', and observe that this term covers a wider range of phenomena than the term 'motion' in Differential Geometry. Following Kobayashi, we prove that an instantaneous axis is a totally geodesic submanifold of even codimension. The relation between the symmetry group $\mathrm{Sym}\,\mu$ and the symmetry group $S(C(\mu))$ of the centrode $C(\mu)$ is given at the end of this chapter.

In chapter 3 we consider Kinematics in the Euclidean n-space E^n as an example of a smooth connected flat Riemannian n-manifold. Thus the group $I_*(X)$ is $E_+(n)$. We give a few examples to show how large the possible symmetry groups can be. Group motions are investigated in detail when $n = 1,2$ and 3. We prove that every 1-dimensional subgroup of $E_+(3)$ is conjugate to a spiral subgroup, a circle subgroup or a translational subgroup.

Chapter 4 deals with Kinematics in E^2. The special structure of $E_+(2)$ helps us to explore more properties of translational and rotational plane motions. Also we use this to give a complete discussion of the kinds of symmetry groups that can occur. As a special case, we show that there is an isomorphism between Sym μ and $S(C(\mu))$.

The final chapter is concerned with the study of Kinematics in E^1. We prove that this can be reduced to the study of real-valued continuous functions of a real variable. We show that stable smooth motions correspond to stable Morse functions $f : R \to R$ and so the classification of such motions can be reduced to the classification of stable Morse functions. The symmetry properties of stable smooth motions are given in some detail.

CHAPTER 1

MOTION IN A METRIC SPACE

1.1 The metric category

Let d_X and d_Y be metrics on sets X and Y respectively; the pairs (X, d_X) and (Y, d_Y) are then metric spaces. An isometry from (X, d_X) to (Y, d_Y) is an injective map $f : X \to Y$, such that for all $x, x' \in X$,

$$d_X(x, x') = d_Y(y, y'),$$

where $y = f(x)$, $y' = f(x')$.

Let us denote the set of isometries from (X, d_X) to (Y, d_Y) by $I(X, Y)$, suppressing the symbols for the metrics. Trivially if (Z, d_Z) is a third metric space, then for any $f \in I(X, Y)$ and any $g \in I(Y, Z)$, the composite map $g \circ f$ is an isometry from X to Z, i.e., $g \circ f \in I(X, Z)$ and so there is a law of composition

$$I(X, Y) \times I(Y, Z) \to I(X, Z)$$

satisfying:

(i) $h \circ (g \circ f) = (h \circ g) \circ f$,
for arbitrary $f \in I(X, Y)$, $g \in I(Y, Z)$ and $h \in I(Z, W)$.

(ii) for each Y, $1_Y \circ f = f$, $g \circ 1_Y = g$,
for arbitrary $f \in I(X, Y)$, $g \in I(Y, Z)$.

Hence there is a category K, the metric category of isometries between metric spaces.

For any metric space (X, d_X) the group $\mathrm{Aut}_K(X, d_X) = I(X)$ is called the isometry group (or group of isometries) of (X, d_X). We can topologise $I(X, Y)$ by giving X and Y their metric topologies and $I(X, Y)$ the corresponding compact-open topology (see for example [10] p. 46). In particular $I(X)$ is a topological group.

We denote the identity component of $I(X)$ by $I_*(X)$, and the set of invertible isometries from X to Y by $\tilde{I}(X, Y)$.

1.2 Motion in a metric space

1.2.1 Definition:

A motion in a metric space (X, d_X) is a continuous path

$$\mu: R \to I_*(X)$$

such that $\mu(0) = 1_X$.

Denote the set of all motions in (X, d_X) by $\mathcal{M}(X)$. Again $\mathcal{M}(X)$ can be topologised by the compact-open topology. In fact $\mathcal{M}(X)$ is a topological group, with respect to the operation $\circ$ given for any μ, $\nu \in \mathcal{M}(X)$ by

$$(\mu \circ \nu)(t) = \mu(t) \circ \nu(t).$$

1.2.2 Definition:

Let $\mu \in \mathcal{M}(X)$. Then the μ-trajectory of a point $x \in X$, is the path

$$\gamma_x : R \to X$$

given by

$$\gamma_x(t) = \mu(t)(x), \qquad \text{for all } t \in R.$$

1.2.3 Definition:

Let $\mu \in \mathcal{M}(X)$. Then the μ-orbit δ_x of $x \in X$ is the path

$$\delta_x : R \to R \times X$$

given by

$$\delta_x(t) = (t, \mu(t)(x)), \qquad \text{for all } t \in R.$$

Intuitively, we can picture two copies of X, one of which is 'fixed', which coincide at 'time' $t = 0$, and we think of the second copy as 'moving' over the first in such a way that $x \in X$ will reach position $\mu(t)(x)$ after time $t > 0$, and was at position $\mu(t)(x)$ at time $t < 0$.

1.3 Induced action of a motion

Each motion μ determines an action μ_* of the additive group of reals on $R \times X$ given by

$$\mu_*(s, (t, \mu(t)(x))) = (s + t, \mu(s + t)(x)).$$

Trivially,

$$\mu_*(0, (t, \mu(t)(x))) = (t, \mu(t)(x))$$

and for all $s, s' \in R$, and all $x \in X$;

$$\mu_*(s', \mu_*(s, (t, \mu(t)(x)))) = (s' + s + t, \mu(s' + s + t)(x))$$
$$= \mu_*(s' + s, (t, \mu(t)(x)))$$

so μ_* is a well-defined group action of R on $R \times X$ (Fig. 1).

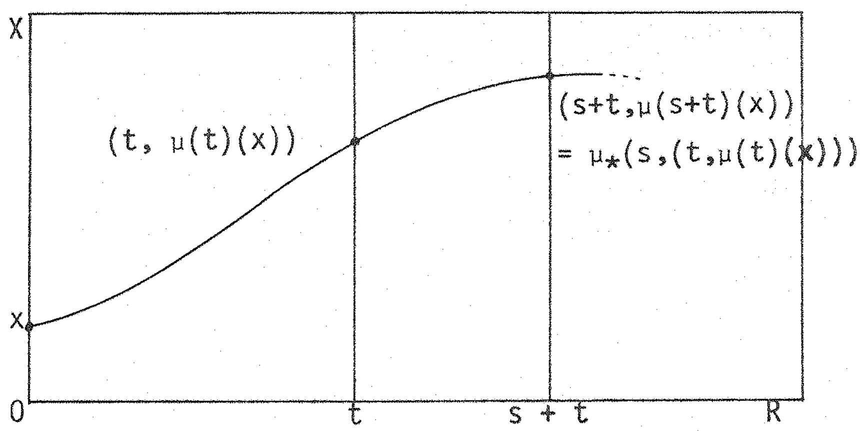


Fig. 1

Recall that if G, H are groups and A, B are sets, a group action $p: G \times A \to A$ on A is said to be <u>equivalent</u> to a group action $q: H \times B \to B$ on B iff there is an isomorphism $\theta: G \to H$ and a bijection $f: A \to B$ such that the diagram

$$\begin{array}{ccc} G \times A & \xrightarrow{\ p\ } & A \\ {\scriptstyle \theta \times f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times B & \xrightarrow[\ q\ ]{} & B \end{array}$$

commutes.

Accordingly, for any two motions $\mu \in \mathcal{M}(X)$ and $\nu \in \mathcal{M}(Y)$, we say that μ_* is <u>equivalent</u> to ν_*, written $\mu_* \overset{\sim}{\approx} \nu_*$, iff there is an automorphism $\theta : R \to R$ and an invertible isometry $\alpha \times \phi : R \times X \to R \times Y$ such that the following diagram commutes.

$$\begin{array}{ccc} R \times (R \times X) & \xrightarrow{\ \mu_*\ } & R \times X \\ {\scriptstyle \theta \times (\alpha \times \phi)}\downarrow & & \downarrow{\scriptstyle \alpha \times \phi} \\ R \times (R \times Y) & \xrightarrow[\ \nu_*\ ]{} & R \times Y \end{array}$$

Thus with an abuse of notation $\alpha(t) = t + \alpha$, for some $\alpha \in R$, and $\phi \in \tilde{I}(X, Y)$. In fact, it is convenient to impose the simplifying condition that $\theta = 1_R$.

We use this idea to define equivalence of the motions μ and ν themselves.

1.3.1 Definition:

Two motions $\mu \in \mathcal{M}(X)$ and $\nu \in \mathcal{M}(Y)$ are <u>equivalent</u> written $\mu \equiv \nu$, iff $\mu_* \approx \nu_*$, and in the above notation (ϕ, α) is called an <u>equivalence</u> from μ_* to ν_*.

The following proposition gives a criterion for the equivalence of motions in metric spaces.

1.3.2 <u>Proposition</u>:

Let $\mu \in \mathcal{M}(X)$, $\nu \in \mathcal{M}(Y)$. Then $\mu \equiv \nu$ iff there exist $\phi, \psi \in \tilde{I}(X,Y)$ and $\alpha \in R$, such that for all $x \in X$, and all $t \in R$,

(1) ... $$\phi(\mu(t)(x)) = \nu(t + \alpha)(\psi(x)).$$

Proof:

Let $\mu \equiv \nu$. Then the diagram

$$\begin{array}{ccc} R \times (R \times X) & \xrightarrow{\mu_*} & R \times X \\ {\scriptstyle 1_R \times (\alpha \times \phi)} \downarrow & & \downarrow {\scriptstyle \alpha \times \phi} \\ R \times (R \times Y) & \xrightarrow[\nu_*]{} & R \times Y \end{array}$$

commutes for some $\alpha \in R$ and some $\phi \in \tilde{I}(X, Y)$. Thus for all $x \in X$, and all $s, t \in R$,

(2) ... $$(s + t + \alpha, \phi(\mu(s + t)(x))) = \nu_*(s,(t + \alpha, \phi(\mu(t)(x)))).$$

Now let $t = 0$, and observe that

$$\phi(x) = \nu(\alpha)(y),$$

for some unique $y \in Y$, since both ϕ and $\nu(\alpha)$ are surjective isometries. Define $\psi : X \to Y$ by $\psi(x) = y$. Thus $\psi(x) = \nu(\alpha)^{-1}(\phi(x))$. Hence $\psi \in \tilde{I}(X, Y)$. From (2), for all $s \in R$,

$$\begin{aligned}(s + \alpha, \phi(\mu(s)(x))) &= \nu_*(s, (\alpha, \phi(x))) \\ &= \nu_*(s, (\alpha, \nu(\alpha)(y))) \\ &= \nu_*(s, (\alpha, \nu(\alpha)(\psi(x)))) \\ &= (s + \alpha, \nu(s + \alpha)(\psi(x))).\end{aligned}$$

Hence for all $s \in R$,

$$\phi(\mu(s)(x)) = \nu(s + \alpha)(\psi(x)).$$

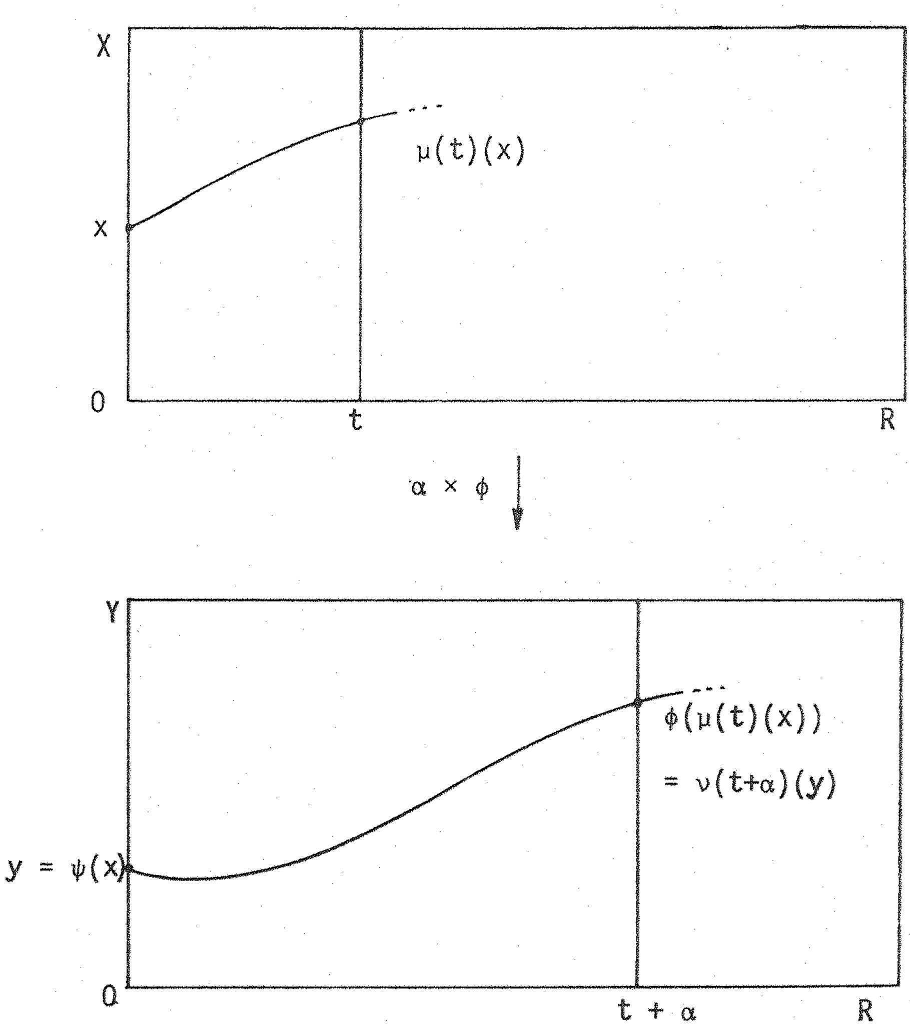


Fig. 2

Conversely, suppose that there are $\phi, \psi \in \tilde{I}(X, Y)$ and $\alpha \in R$, such that equation (1) holds true. Consider the following diagram

$$\begin{array}{ccc} (s, (t, \mu(t)(x))) & \xmapsto{\ \mu_* \ } & (s + t, \mu(s + t)(x)) \\ {\scriptstyle 1_R \times (\alpha \times \phi)} \downarrow & & \downarrow {\scriptstyle \alpha \times \phi} \\ (s, (t+\alpha, \nu(t+\alpha)(\psi(x)))) & \xmapsto[\ \nu_* \ ]{} & (s+t+\alpha, \nu(s+t+\alpha)(\psi(x))) \end{array} .$$

Using (1), we see that the diagram commutes and thus the actions μ_* and ν_* are equivalent. This implies by definition that $\mu \equiv \nu$.

1.4 Symmetry of motion

The reader may notice that we could have defined a category of group actions and hence a category of motions in which our notion of equivalence corresponds to isomorphism. In this spirit, symmetry of motion is measured by the group of automorphisms or self-equivalences of a motion.

Thus if $\mu \in \mathcal{M}(X)$, a symmetry of μ_* is an equivalence $(\phi, \alpha) \in \tilde{I}(X) \times R$ of μ with itself. In fact, with later application in mind, we prefer to restrict ϕ to be an element of $I_*(X)$. The set of all such symmetries (ϕ, α) is the subgroup Aut μ_* of $I_*(X) \times R$. Trivially, Aut μ_* is isomorphic to a subgroup Sym μ of $I_*(X) \times I_*(X) \times R$ under the correspondence $(\phi, \alpha) \to (\phi, \psi, \alpha)$ discussed above. We denote this isomorphism by

$$\tilde{\mu} : \text{Aut } \mu_* \to \text{Sym } \mu.$$

We call Sym μ the symmetry group of the motion μ, and refer to its elements (ϕ, ψ, α) as symmetries of μ.

Since $\mathcal{M}(X)$ is a group, for any motion $\mu \in \mathcal{M}(X)$ there is an inverse motion μ^{-1}, with

$$(\mu^{-1})^{-1} = \mu.$$

The relationship between $\text{Sym}\,\mu$ and $\text{Sym}\,\mu^{-1}$ is given by the following proposition.

1.4.1 Proposition:

Let T be the automorphism of $I_*(X) \times I_*(X) \times R$ given by $T(\phi, \psi, \alpha) = (\psi, \phi, \alpha)$. Then for all $\mu \in \mathcal{M}(X)$, $T(\text{Sym}\,\mu) = \text{Sym}\,\mu^{-1}$.

Proof:

Let $(\phi, \psi, \alpha) \in \text{Sym}\,\mu$, i.e., for all $x \in X$ and all $t \in R$, $\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x))$.

This implies that for all $t \in R$

$$\phi \circ \mu(t) = \mu(t + \alpha) \circ \psi.$$

Hence

$$(\phi \circ \mu(t))^{-1} = (\mu(t + \alpha) \circ \psi)^{-1},$$

and so

$$\mu^{-1}(t) \circ \phi^{-1} = \psi^{-1} \circ \mu^{-1}(t + \alpha).$$

Thus

$$\psi \circ \mu^{-1}(t) = \mu^{-1}(t + \alpha) \circ \phi.$$

Hence for all $x \in X$, and all $t \in R$,

$$\psi(\mu(t)^{-1}(x)) = \mu^{-1}(t + \alpha)(\phi(x)),$$

and therefore $(\psi, \phi, \alpha) \in \text{Sym}\,\mu^{-1}$.

1.4.2 Corollary:

For any motion μ, Sym μ is isomorphic to Sym μ^{-1}.

1.4.3 Proposition:

Let $(\phi, \psi, \alpha) \in I_*(X) \times I_*(X) \times R$, and $\mu \in \mathcal{M}(X)$.

Then $(\phi, \psi, \alpha) \in \text{Sym } \mu \cap \text{ Sym } \mu^{-1} \Longrightarrow \theta^2 = 1_X$

where $\theta = \psi^{-1} \circ \phi$.

Proof:

Suppose that $(\phi, \psi, \alpha) \in \text{Sym } \mu \cap \text{ Sym } \mu^{-1}$.

Then

$$(\phi, \psi, \alpha) \in \text{Sym } \mu \Longleftrightarrow \phi \circ \mu(t) = \mu(t + \alpha) \circ \psi, \quad \text{for all } t \in R,$$

$$\Longleftrightarrow \phi \circ \mu(t) \circ \psi^{-1} = \mu(t + \alpha),$$

and

$$(\phi, \psi, \alpha) \in \text{Sym } \mu^{-1} \Longleftrightarrow \phi \circ \mu^{-1}(t) = \mu^{-1}(t + \alpha) \circ \psi, \quad \text{for all } t \in R,$$
$$\Longleftrightarrow \psi \circ \mu(t) \circ \phi^{-1} = \mu(t + \alpha).$$

Hence, for all $t \in R$,

$$\phi \circ \mu(t) \circ \psi^{-1} = \psi \circ \mu(t) \circ \phi^{-1},$$

that is, for all $t \in R$,

$$(\psi^{-1} \circ \phi) \circ \mu(t) = \mu(t) \circ (\psi^{-1} \circ \phi)^{-1}.$$

In particular for $t = 0$, $\mu(0) = 1_X$ and we get

$$\theta = \theta^{-1}$$

and so

$$\theta^2 = 1_X.$$

$$(\alpha \times \phi)(t, \mu(t)(x)) = \mu_*(\alpha, (t, \mu(t)(x)))$$
$$= (t + \alpha, \mu(t + \alpha)(x)),$$

or

$$(t + \alpha, \phi(\mu(t)(x))) = (t + \alpha, \mu(t + \alpha)(x)).$$

Hence

$$(t + \alpha, \mu(t + \alpha)(\psi(x))) = (t + \alpha, \mu(t + \alpha)(x))$$

for $\quad (\phi, \alpha) \in \text{Aut}\ \mu_*.$

It follows that, for all $x \in X$, and all $t \in R$,

$$\mu(t + \alpha)(\psi(x)) = \mu(t + \alpha)(x).$$

Thus, for all $x \in X$,

$$\psi(x) = x,$$

and so

$$\psi = 1_X.$$

We refer to any element of Sym μ of the form $(\phi, 1_X, \alpha)$ as a <u>periodicity</u> of μ. Let $P(\mu)$ be the set of all periodicities of $\mu \in \mathcal{M}(X)$. Then $P(\mu)$ is a subgroup of Sym μ, and we call this the <u>periodic group</u> of μ, or the group of periodicities of μ. If $(\phi, 1_X, \alpha) \in P(\mu)$, then α is called a <u>period</u> of μ.

1.5.2 Corollary:

$(\phi, \alpha) \in \text{Aut}_* \mu_*$ preserves every orbit in $R \times X \Leftrightarrow (\phi, 1_X, \alpha)$ preserves every trajectory in X.

1.5.3 Proposition:

$P(\mu)$ is a subgroup of the centre of Sym μ.

1.5 Periodicity

In general an automorphism of a group action $p : G \times A \to A$, will not preserve individual orbits. So we may consider the subgroup $\mathrm{Aut}_* p$ of $\mathrm{Aut}\ p$ consisting of all elements $(f, \theta) \in \mathrm{Aut}\ p$ that preserve every orbit in the sense that for each $a \in A$, there exists $g \in G$ such that,

$$f(a) = g.a,$$

i.e., $f(a)$ belongs to the orbit of a.

In case $p = \mu_*$ and so $G = R$, for some motion μ, the group $\mathrm{Aut}_* p$ measures the periodic behaviour of μ.

1.5.1 Proposition:

Let $\mu \in \mathcal{M}(X)$, $(\phi, \alpha) \in I_*(X) \times R$. Then $(\phi, \alpha) \in \mathrm{Aut}_* \mu_*$ iff $\tilde{\mu}(\phi, \alpha) = (\phi, 1_X, \alpha)$.

Proof:

Let $(\phi, 1_X, \alpha) \in \mathrm{Sym}\ \mu$. Then for all $x \in X$, and all $t \in R$,

$$(3) \ldots \qquad \phi(\mu(t)(x)) = \mu(t + \alpha)(x).$$

Let $\delta_x(t) = (t, \mu(t)(x))$, be a μ-orbit. Then

$$\begin{aligned}(\alpha \times \phi)(t, \mu(t)(x)) &= (t + \alpha, \phi(\mu(t)(x))) \\ &= (t + \alpha, \mu(t + \alpha)(x))\end{aligned}$$

and so

$$(\alpha \times \phi)(\delta_x(t)) = \delta_x(t + \alpha).$$

Hence $(\phi, \alpha) \in \mathrm{Aut}_* \mu_*$.

Conversely let $(\phi, \alpha) \in \mathrm{Aut}_* \mu_*$. Then (ϕ, α) preserves every μ-orbit δ_x, that is for all $x \in X$, and all $t \in R$,

Proof:

Let $k = (\phi, \psi, \alpha) \in \text{Sym}\ \mu$, and

$\ell = (\theta, 1_X, \beta) \in P(\mu)$.

Then

$$k^{-1} \circ \ell \circ k = (\phi,\psi,\alpha)^{-1}(\theta, 1_X, \beta)(\phi, \psi, \alpha)$$

$$= (\phi^{-1},\psi^{-1}, -\alpha)(\theta \circ \phi, \psi, \beta + \alpha)$$

$$= (\phi^{-1} \circ \theta \circ \phi, 1_X, \beta),$$ and we have

$$(\phi^{-1} \circ \theta \circ \phi)(\mu(t)(x)) = (\phi^{-1} \circ \theta)(\mu(t + \alpha)(\psi(x))$$

$$= (\phi^{-1})(\mu(t + \alpha + \beta)(\psi(x))$$

$$= \mu(t + \alpha + \beta - \alpha)(\psi^{-1}(\psi(x))) = \mu(t + \beta)(x).$$

Hence

$$(\phi^{-1} \circ \theta \circ \phi) \in P(\mu),$$ and so

$$P(\mu) \lhd \text{Sym}\ \mu.$$

Now $$(\phi^{-1} \circ \theta \circ \phi)(\mu(t)(x)) = \mu(t + \beta)(x)$$

$$= \theta(\mu(t)(x))$$

for all $x \in X$, and all $t \in R$.

In particular, for $t = 0$,

$$\phi^{-1} \circ \theta \circ \phi(x) = \theta(x) \qquad \text{for all } x \in X,$$

$$\Longrightarrow \quad \phi^{-1} \circ \theta \circ \phi = \theta$$

$$\Longrightarrow \quad k^{-1} \circ \ell \circ k = \ell \qquad \text{for all } \ell \in P(\mu), \text{ and all}$$

$k \in \text{Sym}\ \mu$. This proves that $P(\mu)$ is a subgroup of the centre subgroup of Sym μ.

Note that if α and β are periods of μ, then so is $n\alpha + m\beta$, for any integers n and m. We note also that $P(\mu)$ is an abelian group since for all $t \in R$,

$$(\phi_1 \circ \phi_2) \circ \mu(t) = \mu(t + \alpha_1 + \alpha_2)$$
$$= \mu(t + \alpha_2 + \alpha_1)$$
$$= (\phi_2 \circ \phi_1) \circ \mu(t)$$

for all $(\phi_1, 1_X, \alpha_1)$, $(\phi_2, 1_X, \alpha_2) \in P(\mu)$.

1.5.4 Remark

Let Ω denote the set of trajectories of μ. Then $\mathrm{Sym}\,\mu$ acts on Ω by

$$(\phi, \psi, \alpha).\gamma_x = \gamma_y ,$$

where $y = \psi(x)$. The subgroup of $\mathrm{Sym}\,\mu$ that fixes each $\gamma \in \Omega$ is the normal subgroup $P(\mu)$. We denote the quotient group $\mathrm{Sym}\,\mu/P(\mu)$ by $Q(\mu)$. Note that $Q(\mu)$ may be identified with the image of $\mathrm{Sym}\,\mu$ in $I_*(X)$ under projection to the second factor. We therefore regard $Q(\mu)$ as a subgroup of $I_*(X)$. $Q(\mu)$ acts on Ω by

$$\psi.\gamma_x = \gamma_y,$$

where $y = \psi(x)$. We denote the quotient set $\Omega/Q(\mu)$ by Ω_*.

1.6 Symmetry of individual trajectories

1.6.1 Definition:

Let $f : R \to X$, be a path in a metric space (X, d_X). A *symmetry* of f is a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & X \\ \alpha \downarrow & & \downarrow \phi \\ R & \xrightarrow{f} & X \end{array}$$

so that, for all $t \in R$,

$$\phi(f(t)) = f(t + \alpha),$$

where $\alpha(t) = t + \alpha$, for some $\alpha \varepsilon R$,
and $\phi \varepsilon I_*(X)$ is an isometry.

We denote such a symmetry by (ϕ, α). Let $S(f)$ be the set of all such symmetries of the path f.

1.6.2 Proposition:

$S(f)$ is a subgroup of the group $H = I_*(X) \times R$.

Proof:

Suppose that (ϕ, α), $(\psi, \beta) \varepsilon H$. Composition in H is given by

$$(\phi, \alpha)(\psi, \beta) = (\phi \circ \psi, \alpha + \beta),$$

and

$$(\psi, \beta)^{-1} = (\psi^{-1}, -\beta).$$

Let (ϕ, α), $(\psi, \beta) \varepsilon S(f)$. Then for all $s, t \varepsilon R$,

$$\phi(f(t)) = f(t + \alpha),$$

and

$$\psi(f(s)) = f(s + \beta).$$

Hence

$$f(s) = \psi^{-1}(f(s + \beta)).$$

Set $t = s + \beta$,

then

$$f(t - \beta) = \psi^{-1}(f(t)).$$

Thus

$$(\psi^{-1}, -\beta) \varepsilon S(f).$$

Also $$(\phi \circ \psi)(f(t)) = \phi(f(t + \beta))$$
$$= f(t + \beta + \alpha).$$

Hence

$$(\phi \circ \psi, \alpha + \beta) = (\phi, \alpha)(\psi, \beta) \in S(f),$$

and therefore $S(f)$ is a subgroup of H.

1.6.3 Theorem:

Let $f : R \to X$, be a path in a metric space (X, d_X), and suppose that $(\phi, \alpha) \in S(f)$. Let h be defined by $h(t) = f(\rho(t))$, where $\rho(t) = at + b$ for all $t \in R$, $(a \neq 0,\ a, b \in R)$, is a similarity of R. Then h is a path in (X, d_X) with $(\phi, \frac{\alpha}{a}) \in S(h)$.

Proof:

Since $(\phi, \alpha) \in S(f)$,

$$\phi(f(t)) = f(t + \alpha) \qquad \text{for all } t \in R.$$

Set $$\frac{t - b}{a} = s, \qquad a \neq 0,$$

then

$$h(s) = f(\rho(s)) = f(as + b) = f(t),$$

$$\phi(h(s)) = \phi(f(t)) = f(t + \alpha)$$
$$= f(as + b + \alpha)$$
$$= f(a(s + \tfrac{\alpha}{a}) + b)$$
$$= f(\rho(s + \tfrac{\alpha}{a})).$$

Hence for all $s \in R$,

$$\phi(h(s)) = h(s + \tfrac{\alpha}{a}),$$

and so h is a path for which $(\phi, \frac{\alpha}{a}) \in S(h)$.

1.6.4 Similarity between metric spaces

The notion of similarity defined above for the real numbers may be defined between any metric spaces (X, d_X) and (Y, d_Y) as a map $\theta : X \to Y$ such that, for some $p > 0$ and all $x, x' \in X$,

$$p\, d_X(x, x') = d_Y(\theta(x), \theta(x')).$$

The number p is called the <u>modulus</u> $|\theta|$ of θ. If $\sigma : Y \to Z$ is another similarity with modulus q, where (Z, d_Z) is a third metric space, then

$$\sigma \circ \theta : X \to Z$$

is another similarity and

$$|\sigma \circ \theta| = |\sigma||\theta| = qp.$$

Trivially any similarity θ is injective, and is an isometry iff $|\theta| = 1$.

We restrict attention to invertible (i.e., surjective) similarities, and note that the set of all such similarities of X to itself is a group $\Sigma(X)$ under composition. The map

$$m : \Sigma(X) \to R_+$$

into the multiplicative group of positive reals, given by

$$m(\theta) = |\theta|$$

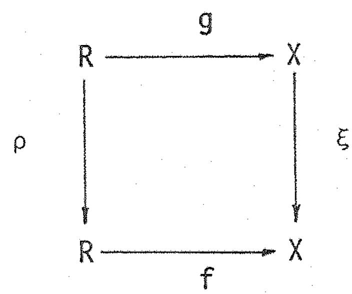
is a homomorphism with kernel $I(X)$.

1.6.5 Theorem:

Let f be a path in a metric space (X, d_X). Let $\rho \in \Sigma(R)$ and $\xi \in \Sigma(X)$. Let $g = \xi^{-1} \circ f \circ \rho$. Then $S(f)$ and $S(g)$ are conjugate subgroups of $\Sigma(X) \times \Sigma(R)$. In fact $S(g) = (\xi, \rho)^{-1}(S(f))(\xi, \rho)$.

Proof:

We have for all $t \in R$, $(\xi \circ g)(t) = (f \circ \rho)(t)$, that is the diagram



commutes. Then for all $(\phi, \alpha) \in S(f)$ and all $t \in R$,

$$\phi((\xi \circ g)(t)) = \phi(f(\rho(t)))$$

but from theorem 1.6.4,

$$\phi((\xi \circ g)(t)) = f(\rho(t + \tfrac{\alpha}{a})), \qquad a \neq 0$$

$$= (\xi \circ g)(t + \tfrac{\alpha}{a})$$

and so

$$(\xi^{-1} \circ \phi \circ \xi \circ g)(t) = g(t + \tfrac{\alpha}{a}).$$

Let

$$\xi^{-1} \circ \phi \circ \xi = \psi, \qquad \beta = \frac{\alpha}{a}.$$

Then we have, for all $t \in R$,

$$\psi(g(t)) = g(t + \beta).$$

Hence

$$(\psi, \beta) \in S(g).$$

Now for each $(\phi, \alpha) \in S(f)$, there corresponds,

$$(\xi, \rho)^{-1}(\phi, \alpha)(\xi, \rho) = (\xi^{-1} \circ \phi \circ \xi, \rho^{-1} \circ \alpha \circ \rho)$$

$$= (\psi, \beta) \in S(g),$$

since $\rho(t) = at + b = s, \qquad a \neq 0$,

$$\rho^{-1}(s) = \frac{s - b}{a}$$

and so

$$\begin{aligned}(\rho^{-1} \circ \alpha \circ \rho)(t) &= (\rho^{-1} \circ \alpha)(at + b) \\ &= \rho^{-1}(at + b + \alpha) \\ &= (at + b + \alpha - b)/a \\ &= t + \frac{\alpha}{a} \\ &= \beta(t).\end{aligned}$$

Hence

$$S(g) = (\xi, \rho)^{-1}(S(f))(\xi, \rho).$$

1.6.6 Lemma:

$S(f)$ is closed in $I_*(X) \times R$.

Proof:

Let $<(\phi_n, \alpha_n)>$, $n = 1, 2, \ldots,$ be a convergent sequence in $S(f)$, with limit point (ϕ, α). Then for all $t \in R$, we have a system of equations of the form

$$\phi_n(f(t)) = f(t + \alpha_n), \qquad n = 1,2,\ldots \ .$$

Since the path f is continuous, $f(t + \alpha_n)$ converges to $f(t + \alpha)$ as $n \to \infty$ and the following equality holds true,

$$\lim_{n\to\infty} \phi_n(f(t)) = \lim_{n\to\infty} f(t + \alpha_n)$$

that is,

$$\phi(f(t)) = f(t + \alpha).$$

Hence

$$(\phi, \alpha) \in S(f),$$

and therefore $S(f)$ is closed in $I_*(X) \times R$.

In the same way, we can show that $\mathrm{Sym}\ \mu$ is closed in $I_*(X) \times I_*(X) \times R$ and $\mathrm{Aut}\ \mu_*$ is closed in $I_*(X) \times R$.

1.6.7 Theorem:

$$S(f) \simeq \{0\},\ R \ \text{ or } \ Z,$$

where $\simeq$ means group isomorphism.

Proof:

Consider the projection $p_2 : I_*(X) \times R$ to the second factor, and the inclusion map

$$\iota : S(f) \to I_*(X) \times R.$$

Then $p_2 \circ \iota$ is a monomorphism of $S(f)$ into R; since $(p_2 \circ \iota)(\phi, \alpha) = (p_2 \circ \iota)(\psi, \beta) \Longleftrightarrow \alpha = \beta$, and for all $t \in R$ $\phi(f(t)) = \psi(f(t))$. Thus $S(f)$ is isomorphic to a closed subgroup of R.

We now show that every non-discrete subgroup of R is dense in R. If G is a subgroup of R which is not discrete, then for every $\varepsilon > 0$ there is a point $x \neq 0$ of G which belongs to the interval $[-\varepsilon, +\varepsilon]$; since all integral multiples of x belong to G, every interval of length $\varepsilon > 0$ contains an element of G, and therefore G is dense in R. Hence every closed subgroup of the additive group R other than R itself is discrete.

Let G be a closed subgroup of R, such that $G \neq \{0\}$, R. G is discrete and the relation $-G = G$ implies that the set

$$H = \{y \in G: \ y > 0\}$$

is non-empty.

If $b \in H$, then $[0, b] \cap G$ is compact and discrete and is therefore finite. Let $\lambda \in H$, $\lambda = \inf H$, so $\lambda \geqslant 0$. If $\lambda = 0$, then $G = R$ which is a contradiction, so $\lambda > 0$. For every $x \in G$ put $m = \left[\frac{x}{\lambda}\right]$, the integer part of $\frac{x}{\lambda}$, then we have $x - \lambda m \in G$ and $0 \leqslant x - \lambda m < \lambda$. By the definition of λ it follows that $x - \lambda m = 0$ and therefore

$$G = \{m\lambda : m \in Z\}, \qquad G \simeq Z.$$

Thus every closed subgroup of R other than $\{0\}$, R is a discrete group of the form λZ where $\lambda > 0$, and hence

$$S(f) \simeq \{0\}, \ R \ \text{or} \ Z.$$

Let $Per(\mu)$ denote the set of all periods of a motion μ. Then $Per(\mu)$ is a subgroup of R, isomorphic to the group $P(\mu)$ under the projection

$$I_*(X) \times I_*(X) \times R \to R$$

to the third factor.

The proof of the next theorem is similar to that of theorem 1.6.7, and is omitted.

1.6.8 Theorem:

$$Per(\mu) \simeq \{0\}, \ R \ \text{or} \ Z.$$

1.6.9 Definition:

Let $Per(\mu) \simeq Z$. Since $Per(\mu)$ is a subgroup of R, it has a unique +ve generator called the <u>primitive period</u> of μ.

1.6.10 Proposition:

Let γ_x be the trajectory of a motion $\mu \in \mathcal{M}(X)$ through $x \in X$. Then the map

$$\pi : P(\mu) \to S(\gamma_x)$$

given by, $\pi(\phi, 1_X, \alpha) = (\phi, \alpha)$ is a monomorphism.

Proof:

Suppose that $\mu \in \mathcal{M}(X)$, and let $(\phi, 1_X, \alpha) \in P(\mu)$. Then for all $x \in X$, and all $t \in R$,

(4) ... $$\phi(\mu(t)(x)) = \mu(t + \alpha)(x).$$

Since

$$\phi(\mu(t)(x)) = \phi(\gamma_x(t)),$$

we conclude that

$$\phi(\gamma_x(t)) = \gamma_x(t + \alpha).$$

Hence

$$(\phi, \alpha) \in S(\gamma_x).$$

The result now follows immediately.

1.6.11 Remark

Conversely let $(\phi, \alpha) \in I_*(X) \times R$, be such that, (ϕ, α) is a symmetry of each trajectory of a motion μ, then $(\phi, 1_X, \alpha) \in P(\mu)$.

1.7 Some particular types of symmetry

Let $\mu \in \mathcal{M}(X)$. Then $\text{Sym}\,\mu$ is a subgroup of $I_*(X) \times I_*(X) \times R$. Consider the projections p_i, $i = 1,2,3$ of $\text{Sym}\,\mu$ into the i^{th} factor of $I_*(X) \times I_*(X) \times R$, $i = 1,2,3$. Then we have the following results.

1.7.1 Proposition:

If p_3 is trivial, then $p_2 = p_1$.

Proof:

Let $p_3 = 0$. Then any element of $\mathrm{Sym}\,\mu$ is of the form, $(\phi, \psi, 0)$ and we have for all $x \in X$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t)\psi(x).$$

In particular, for $t = 0$, $\mu(0) = 1_X$, and we obtain, for all $x \in X$,

$$\phi(x) = \psi(x).$$

Hence

$$\phi = \psi.$$

1.7.2 Proposition:

If p_2 and p_3 are trivial, then so is p_1.

Proof:

Let p_2 and p_3 be trivial. Then the elements of $\mathrm{Sym}\,\mu$ are then of the form $(\phi, 1_X, 0)$, and we have for all $x \in X$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t)(x).$$

In particular, for $t = 0$, $\mu(0) = 1_X$, and we get for all $x \in X$, $\phi(x) = x$.

Hence

$$\phi = 1_X.$$

It is of interest to consider the various possibilities that can arise when at least one of the projections p_1, p_2, p_3 is trivial. Interesting phenomena are found also when $p_1 = p_2$.

The following table shows all such possible cases. However, using propositions 1.7.1 and 1.7.2, these can be reduced to only five cases; since cases (2), (3) and (7) are equivalent, as are cases (3) and (9).

Case	p_1	p_2	p_3	type
(1)	ϕ	1_X	α	A
(2)	ϕ	1_X	0	(7)
(3)	ϕ	ψ	0	(9)
(4)	1_X	ψ	α	B
(5)	1_X	ψ	0	(7)
(6)	1_X	1_X	α	(1)
(7)	1_X	1_X	0	C
(8)	ϕ	ϕ	α	D
(9)	ϕ	ϕ	0	E

1.7.3 Type A: p_2 is trivial.

Elements of Sym μ, are then of the form $(\phi, 1_X, \alpha)$ and we have for all $x \in X$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(x),$$

and μ is therefore periodic, with period α .

Example: Cycloidal plane motion (see 4.4, example (8)).

1.7.4 Type B: p_1 is trivial, where p_2, p_3 are non-trivial.

In this case, the orbit of any $x \in X$ is "parallel" to that through $y = \psi(x)$, and one may be mapped to the other by translation $(t, z) \to (t + \alpha, z)$. It follows that the trajectory through y coincides with that through x as a subset of X, with a time delay α.

Clearly μ is of type A iff μ^{-1} is of type B. Thus an example of type B is the inverse cycloidal plane motion (rolling of a line on a circle).

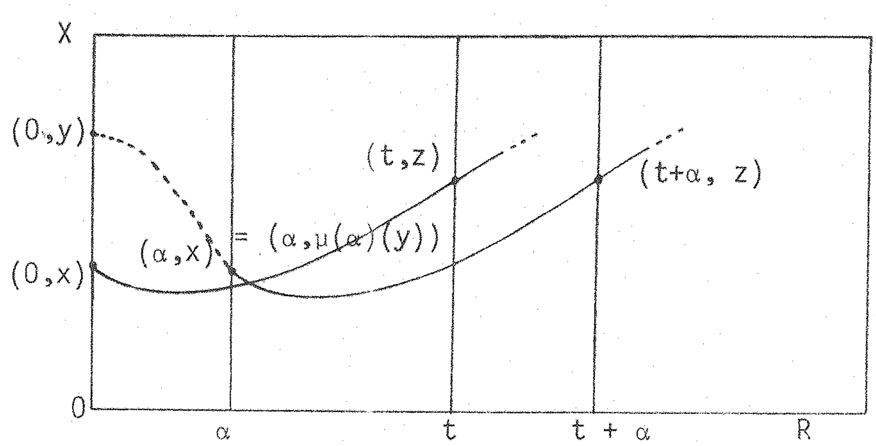


Fig. 3

1.7.5 Type C: p_i is trivial for all $i = 1,2,3$.

In this case Sym μ is trivial.

1.7.6 Types D and E: $p_1 = p_2 \neq 1_X$.

In this case, every element of Sym μ is of the form (ϕ, ϕ, α) and for such ϕ we have for all $x \in X$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\phi(x)).$$

This means that for any $x \in X$, the trajectory γ_x through x is 'parallel' to that through $\phi(x)$, with a time delay α.

The case $\alpha = 0$ is mentioned separately for completeness, in view of case 3 in the table.

We discuss translational motion in a normed linear space to illustrate some aspects of type E.

Let V be a normed linear space, and let d be given by

$$d(x, y) = \|x - y\|.$$

Let $f \in I_*(V)$, and suppose that $f(0) = a$. Consider the isometry $g \in I_*(V)$ given by

$$g(x) = f(x) - a, \quad \text{for all } x \in V.$$

Then

$$g(0) = 0.$$

So every element of $I_*(V)$ can be expressed uniquely in the form (g, a), where $g \in SO(V)$, and $SO(V)$ denotes the identity component of the orthogonal group of isometries of V that fix 0.

Conversely for any $g \in SO(V)$ and any $a \in V$, the transformation

$$f : V \to V,$$

given by

$$f(x) = g(x) + a, \quad \text{for all } x \in V,$$

is an element of $I_*(V)$, for, let $x, y \in V$, then

$$\begin{aligned} d(f(x), f(y)) &= \|f(x) - f(y)\| \\ &= \|g(x) - g(y)\| = d(x, y). \end{aligned}$$

In Chapter 3, we discuss in more detail the particular case in which V is the n-dimensional Euclidean space E^n. In fact $I_*(V)$ is a semi-direct product of $SO(V)$ with V itself. Thus $\mu \varepsilon \mathcal{M}(V)$ may be written

$$\mu(t) = (g(t), a(t)), \qquad t \varepsilon R,$$

for some paths

$$g : R \to SO(V),$$

and

$$a : R \to V.$$

We say that μ is <u>translational</u> iff for all $t \varepsilon R$, $g(t) = 1_V$, and <u>rotational</u> iff for all $t \varepsilon R$, $a(t) = 0$.

For any $p \varepsilon V$, the isometry $(1_V, p)$ of V is denoted by τ_p. Thus for all $x \varepsilon V$,

$$\tau_p(x) = x + p.$$

1.7.8 Theorem:

Let V be a normed linear space, and let $\mu \varepsilon \mathcal{M}(V)$ be translational. Then there is a monomorphism $\sigma : V \to \mathrm{Sym}\,\mu$, given by $\sigma(p) = (\tau_p, \tau_p, 0)$, $p \varepsilon V$.

Proof:

Let $\mu \varepsilon \mathcal{M}(V)$ be translational. Then

$$\mu(t) = (1_V, a(t))$$

for some path $a : R \to V$, and so for all $x \varepsilon V$, and all $t \varepsilon R$,

$$\mu(t)(x) = x + a(t).$$

Thus for any $\phi \varepsilon I_*(V)$, $(\phi, \phi, 0) \varepsilon \mathrm{Sym}\,\mu$ iff for all $x \varepsilon V$, and all $t \in R$,

$$\phi(x + a(t)) = \phi(x) + a(t).$$

In particular, for any $p \in V$, $(\tau_p, \tau_p, 0) \in \text{Sym}\, \mu$ iff

$$(x + a(t)) + p = (x + p) + a(t),$$

which proves the theorem.

1.8 Group motions

An obvious special type of motion $\mu : R \to I_*(X)$ occurs when the path μ is a homomorphism. The motion μ is then called a group motion (in the geometrical literature, a group of motions). Thus if $\mathfrak{G}(X)$ denotes the set of group motions in X, then $\mu \in \mathfrak{G}(X)$ iff $\mu : R \to I_*(X)$ is such that for all $s, t \in R$,

$$\mu(s + t) = \mu(s) \circ \mu(t).$$

Group motions have special symmetry properties as follows. Let $\mu \in \mathfrak{G}(X)$. Then $(\phi, \psi, \alpha) \in \text{Sym}\, \mu$ iff for all $x \in X$, and all $t \in R$,

$$\begin{aligned} \phi(\mu(t)(x)) &= \mu(t + \alpha)(\psi(x)) \\ &= \mu(t) \circ \mu(\alpha)(\psi(x)) \\ &= \mu(\alpha) \circ \mu(t)(\psi(x)), \end{aligned}$$

where, as before, $\psi = \mu^{-1}(\alpha) \circ \phi.$

Since $(\phi, \phi, 0) \in \text{Sym}\, \mu$ iff

$$\phi \circ \mu = \mu \circ \phi,$$

Sym μ contains the image in $I_*(X) \times I_*(X) \times 0$ of the centralizer of $\mu(R)$ in $I_*(X)$ under the homomorphism

$$\phi \mapsto (\phi, \phi, 0).$$

Moreover, if $(\phi, \phi, 0) \in \mathrm{Sym}\,\mu$, $\mu \in \mathcal{G}(X)$, and $\alpha \in R$, then $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$, where $\psi = \mu^{-1}(\alpha) \circ \phi$.

CHAPTER 2

MOTION IN A RIEMANNIAN MANIFOLD

The concept of motion that has been developed in the previous chapter assumes an especially interesting form in the special case where X is a connected Riemannian manifold and the metric d_X is given by a Riemannian structure g on X.

It is important to emphasise at the outset that in the literature of Differential Geometry, the term 'motion' is used in somewhat different, but related, senses. We shall explain this in the course of the discussion below.

The term 'smooth' will be used as a synonym for 'C^∞'. For simplicity we use the term 'n-manifold' to mean an n-dimensional connected manifold without boundary.

2.1 Smooth motions

Let X be a smooth n-manifold with smooth complete Riemannian structure g. Thus g is a smooth Riemannian tensor field on X. Let d_X be the associated metric. Then the concept of motion in (X, d_X) is well-defined according to the theory of Chapter 1. However, in order to take advantage of the fact that X has the structure of a smooth manifold, it is natural to restrict attention to smooth motions.

Recall that the group $I(X)$ of isometries of a smooth n-manifold is a Lie group of dimension $k \leq \frac{1}{2}n(n + 1)$. It follows that $I_*(X)$ is a connected Lie group, also of dimension k.

A <u>smooth motion</u> in X is a smooth path

$$\mu : R \to I_*(X)$$

such that $\mu(0) = 1_X$. The set of smooth motions in X is a subgroup $\mathcal{R}(X)$ of the group $\mathcal{M}(X)$ of all motions of X. Of course $\mathcal{R}(X)$ need not be closed in $\mathcal{M}(X)$, since a sequence μ_i, $i = 1, 2, \ldots$ of smooth paths in $I_*(X)$ need not have a smooth limit (see Figure 4).

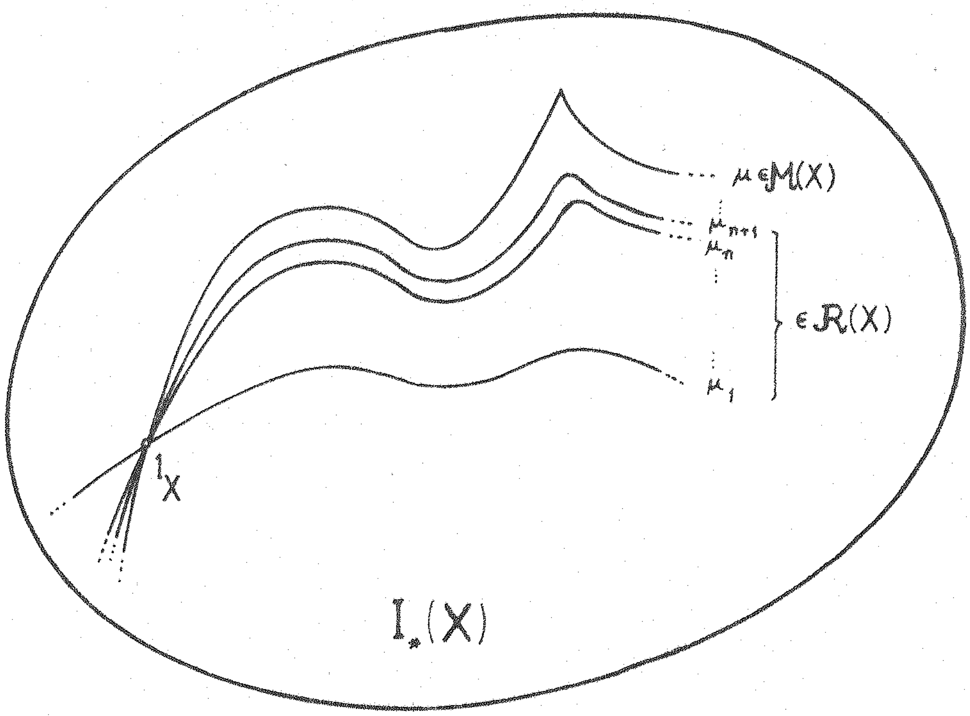


Fig 4

2.2 Velocity vector fields and Killing vector fields

Let $\mu \in \mathcal{R}(X)$. Then the path μ has a well-defined tangent vector $\dot{\mu}(t) = d\mu(t)/dt$ for each $t \in R$. We observe that $\dot{\mu}(t)$ determines a smooth vector field on X, defined by

$$v_t(\mu)(\mu(t)(x)) = \dot{\gamma}_x(t),$$

where γ_x is the μ-trajectory of a point $x \in X$. Thus if $\mathcal{V}(X)$ denotes the set of all smooth vector fields on X, then there is a map

$$V : \mathcal{R}(X) \times R \to \mathcal{V}(X)$$

given by $V(\mu, t) = v_t(\mu)$. We say that $v_t(\mu)$ is the velocity vector field of μ at time t.

The set $\mathcal{V}(X)$ has the structure of a Lie algebra under pointwise addition and the Lie bracket multiplication. This algebra contains a subalgebra $\mathcal{K}(X)$ of Killing vector fields. These are defined as follows. Let $v \in \mathcal{V}(X)$, and let $L_v g$ denote the Lie derivative of the Riemannian tensor field g along v. Then v is said to be a Killing vector field [9] iff,

(1) ... $$L_v g = 0.$$

Equation (1) is sometimes called the Killing-equation. In local coordinates $x^1, \ldots, x^n$, it takes the form,

(2) ... $$v^\sigma \frac{\partial g_{\lambda\tau}}{\partial x^\sigma} + g_{\rho\tau} \frac{\partial v^\rho}{\partial x^\lambda} + g_{\lambda\rho} \frac{\partial v^\rho}{\partial x^\tau} = 0,$$

where the symbols have their standard meanings and the summation convention applies.

Let $\mathcal{K}(X)$ denote the set of all Killing vector fields defined in this way. If $v, w \in \mathcal{K}(X)$, then

$$\begin{aligned} L_{[v,w]} g &= L_v L_w g - L_w L_v g \\ &= L_v 0 - L_w 0 = 0. \end{aligned}$$

Thus $[v,w] \in \mathcal{K}(X)$. Likewise, $v + w \in \mathcal{K}(X)$ and $\mathcal{K}(X)$ is a Lie subalgebra of $\mathcal{V}(X)$.

Although $\mathcal{V}(X)$ is not of finite dimension, dim $\mathcal{K}(X)$ is finite.

In fact $\mathbb{K}(X)$ is isomorphic to the Lie algebra $\mathcal{I}(X)$ of $I(X)$ and hence

$$\dim \mathbb{K}(X) \leq \tfrac{1}{2}n(n+1).$$

Thus if θ is an element of the Lie algebra $\mathcal{I}(X)$, then there is a unique 1-parameter subgroup G of $I_*(X)$ generated by $\exp \theta$. In fact, G may be regarded as a smooth motion μ in the sense of Section 2.1. The velocity vector field v of μ at $t = 0$ is then a Killing vector field. It is a standard exercise to show that this correspondence $\theta \to v$ is an isomorphism of Lie algebras from $\mathcal{I}(X)$ to $\mathbb{K}(X)$.

In Differential Geometry, a <u>motion</u> in X is a 1-parameter group G of isometries of X, and an <u>infinitesimal motion</u> in X is an element of $\mathcal{I}(X)$ or of $\mathbb{K}(X)$. Our term 'smooth motion' covers a wider range of phenomena, since not every smooth path $\mu : R \to I_*(X)$ with $\mu(0) = 1_X$ is a 1-parameter subgroup of $I_*(X)$, and to each smooth motion μ in our sense there is associated not just a single Killing vector field but a 1-parameter family of such fields.

We remark that the group $\mathcal{R}(X)$ is not, in general, a (finite-dimensional) Lie group.

2.3 Centrodes

In classical Kinematics, attention is confined to motions whose local structure is identical to that of smooth motions in our sense. In particular, we are able to define the classical ideas of <u>instantaneous axis</u>, and the twin notions of <u>fixed</u> and <u>moving</u> centrodes for smooth motions, as follows.

Let $\mu \in \mathcal{R}(X)$. The set of all points of X, where the velocity vector field $v_t(\mu)$ vanishes is called the <u>instantaneous axis</u> $C_t(\mu)$ of the motion μ at time $t \in R$. The union of all instantaneous axes of μ, is called the <u>centrode</u> $C(\mu)$ of the motion μ. Thus

$$C(\mu) = \bigcup_{t \in R} C_t(\mu).$$

In classical Kinematics, the centrode $C(\mu)$ is called the <u>fixed centrode</u> $C_F(\mu)$ of μ. Likewise the <u>moving centrode</u> $C_M(\mu)$ of μ is the (fixed) centrode of the inverse motion μ^{-1}. Thus

$$C_M(\mu) = C_F(\mu^{-1}).$$

The classical description of μ is to picture two 'copies' X_F and X_M of X, one of which is 'fixed'. The fixed centrode $C_F(\mu)$ is regarded as a subset of X_F, and the moving centrode $C_M(\mu)$ as a subset of X_M. Then X_M moves over X_F by 'rolling' $C_M(\mu)$ along $C_F(\mu)$ and/or 'sliding' $C_M(\mu)$ over $C_F(\mu)$ along an instantaneous axis at which they touch. For example in cycloidal plane motion, $X = E^2$, $C_F(\mu)$ is a straight line L, and $C_M(\mu)$ is a circle C. The moving plane travels over the fixed plane by rolling C along L.

2.4 Lie group actions

In order to analyse the structure of the centrode of a motion, we consider some general aspects of Lie group actions.

Consider a smooth action

$$\alpha : G \times X \to X$$

given by $\alpha(g, x) = g \cdot x$, where G is a Lie group. Then for each $g \in G$, there corresponds a diffeomorphism $\phi(g)$ of X so we have a homomorphism

$$\phi : G \to \text{Diff } X$$

given by $\phi(g)(x) = g \cdot x$, for all $x \varepsilon X$, where Diff X denotes the group of diffeomorphisms of X. Thus for all $g, h \varepsilon G$,

(3) ... $$\phi(g)\phi(h) = \phi(g \circ h).$$

Let T denote the tangent functor. Then TX is the total space of the tangent bundle of X, and thus α induces a map

$$\alpha_* : G \times TX \to TX$$

given for all $g \varepsilon G$, and all $v \varepsilon TX$ by

$$\alpha_*(g, v) = T(\phi(g))(v).$$

We prove that α_* is a group action, as follows.

(i) For all $g, h \varepsilon G$, and all $v \varepsilon TX$,

$$\begin{aligned} \alpha_*(g, \alpha_*(h, v)) &= T(\phi(g))(\alpha_*(h, v)) \\ &= T(\phi(g))(T(\phi(h))(v)) \\ &= T(\phi(g)\phi(h))(v) \\ &= T(\phi(g \circ h))(v), \quad \text{from (3)} \\ &= \alpha_*(g \circ h, v). \end{aligned}$$

(ii) For all $h \varepsilon G$, and all $v \varepsilon TX$,

$$\begin{aligned} \alpha_*(1_G, \alpha_*(h, v)) &= \alpha_*(1_G \circ h, v) \\ &= \alpha_*(h, v). \end{aligned}$$

We consider in particular the case $G = R$, so we have a smooth action of the additive group R as a Lie transformation group on X.

Let $t \in R$, and $y \in X$ be such that,

$$\alpha(t, y) = y.$$

Then for all $v \in T_yX$,

$$\alpha_*(t, v) = w \in T_yX,$$

and hence α_* defines a linear automorphism $\eta : T_yX \circlearrowleft$ given by $\eta(v) = w$.

2.5 Structure of instantaneous axes

The essential content of the following theorem is included in the work of Kobayashi [11, 12]. However we give a modified version to fit the present context for the sake of completeness.

2.5.1 Theorem:

Let $\mu \in \mathcal{R}(X)$. Then each instantaneous axis $C_t(\mu)$ is a totally geodesic submanifold of even codimension in X.

Proof:

Let σ_t be the local 1-parameter group of motions whose velocity vector field is v_t. Then,

$$C_t(\mu) = \{y \in X : v_t(\mu)(y) = 0\}.$$

Suppose $y \in C_t(\mu)$, and let $W \subset T_yX$ be the fixed point set of the automorphism $\eta : T_yX \circlearrowleft$. Thus W is a linear subspace of T_yX. Let U^* be a neighbourhood of the origin in T_yX such that the exponential mapping $\exp_y : U^* \to X$ is an embedding. Let $U = \exp_y(U^*)$ and assume

U is a convex neighbourhood (in the sense that any two points of U can be joined by a unique minimizing geodesic). Then,

$$(U \cap C_t(\mu)) = \exp_y(U^* \cap W).$$

This shows that a neighbourhood $U \cap C_t(\mu)$ of y in $C_t(\mu)$ consists of submanifolds of X.

Now if a point $a \in C_t(\mu)$ is sufficiently close to y so that they can be joined by a unique minimizing geodesic τ, then the map $\exp_y^{-1}(\tau)$ is a straight line in T_yX through the origin. Hence this straight line belongs to the linear subspace W of T_yX, and so is fixed by the automorphism η. It follows that each point of τ is fixed. Hence $C_t(\mu)$ is totally geodesic.

Let us regard T_yX as a Riemannian manifold, with flat Riemannian structure induced by its inner product. Then there is a unique Killing vector field w_t on T_yX such that $T\exp_y$ maps $w_t|U^*$ to $v_t|U$. Since $w_t(0) = 0$, it is an element of the Lie algebra of the orthogonal group of T_yX as an Euclidean inner product space. Thus if we choose suitable orthonormal basis $e_1, \ldots, e_n$ for T_yX, the vector field w_t, which we regard as a linear endomorphism of T_yX, is given by a skew-symmetric $n \times n$ matrix B of the form,

$$B = \left[\begin{array}{ccc|ccccc} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ \hline & & & 0 & a_1 & & & \\ & & & -a_1 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & a_k \\ & & & & & & -a_k & 0 \end{array}\right], \quad a_i \neq 0.$$

$$\underbrace{\qquad\qquad}_{n-2k} \quad \underbrace{\qquad\qquad}_{2k}$$

Let T' be the linear subspace of T_yX generated by $e_1, \ldots, e_{n-2k}$. Then T' is the tangent space to the components of $C_t(\mu)$ at y. Hence,

$$\dim C_t(\mu) = \dim T' = n - 2k.$$

Thus codimension of $C_t(\mu)$ is even.

Following Kobayashi, one can deduce from the form of the matrix B that the normal bundle to each component of $C_t(\mu)$ is orientable. Hence if X is orientable so is each component of $C_t(\mu)$.

For examples and discussion of symmetry phenomena, we specialise the Euclidean case, which is the subject of the remaining chapters.

However, we can observe that for any smooth motion $\mu \in \mathcal{R}(X)$, a symmetry of the centrode $C(\mu)$ may be defined as a pair $(\phi, \alpha) \in I_*(X) \times R$ such that for all $t \in R$,

$$(4) \ldots \qquad \phi C_t(\mu) = C_{t+\alpha}(\mu).$$

The set of all symmetries of $C(\mu)$ is a subgroup $S(C(\mu))$ of $I_*(X) \times R$. We now construct a monomorphism

$$\rho_F : \text{Sym}\,\mu \to S(C_F(\mu))$$

as follows. Consider the trajectory γ_x of μ through $x \in X$. Then the velocity vector field v_t is given by

$$v_t(\mu)(\mu(t)(x)) = \dot{\gamma}_x(t).$$

Thus if $(\phi, \psi, \alpha) \in \text{Sym}\,\mu$, so that for all $x \in X$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)),$$

that is,

$$\phi(\gamma_x(t)) = \gamma_y(t + \alpha),$$

where $y = \psi(x)$, then,

$$T\phi(\dot{\gamma}_x(t)) = \dot{\gamma}_y(t + \alpha),$$

from which equation (4) follows immediately.

Consider now the inverse motion μ^{-1} of the motion μ. Then $(\psi, \phi, \alpha) \in \text{Sym}\, \mu^{-1}$, and hence there is another monomorphism

$$\rho_M : \text{Sym}\, \mu \to S(C_M(\mu))$$

given by $\rho_M(\phi, \psi, \alpha) = (\psi, \alpha) = (\mu^{-1}(\alpha)(\phi), \alpha)$.

It should be noted, however, that $C_F(\mu)$ and $C_M(\mu)$ may have symmetries that do not arise in this way. For more details see Chapter 4.

CHAPTER 3

EUCLIDEAN KINEMATICS

3.1 The Euclidean group

We now consider Euclidean n-space E^n as an example of a smooth, connected n-dimensional Riemannian manifold. The Riemannian tensor field g being given at every $x \in E^2$ by the identity matrix I_n with respect to the standard coordinate system. Thus E^n is flat and the metric d is given by

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}.$$

The isometry.group $I(E^n)$ is known as the <u>Euclidean group</u> and its dimension is the maximum possible, $\dim I(E^n) = \frac{1}{2}n(n + 1)$. It is customary to denote $I(E^n)$ by $E(n)$.

Euclidean space E^{n+1} contains the n-dimensional sphere $S^n = \{x \in E^{n+1} : \|x\| = 1\}$ as a smooth submanifold to which we may assign the induced Riemannian structure. Thus S^n is a smooth Riemannian n-manifold of constant curvature 1, and its isometry group $I(S^n)$ is the <u>orthogonal group</u> $O(n + 1)$. Again, $\dim O(n + 1) = \frac{1}{2}n(n + 1)$ is maximal.

We now have two examples of Riemannian manifolds with the highest possible degrees of symmetry in the compact and noncompact cases. In fact, motions in S^n can be studied as restrictions to S^n of motions in E^{n+1} that fix the origin.

The structure of $E(n)$ is that of a semidirect product of the orthogonal group $O(n)$ with the group T^n of translations of E^n. We can identify the former with the group of all real $n \times n$ matrices A

such that $AA^t = I_n$, (where A^t is the transpose of A) and the latter with the real linear space R^n itself under the following identifications.

If $A \in O(n)$, and $x \in E^n$, then A acts on x by

$$A \cdot x = y,$$

where $y_i = \sum_{j=1}^{n} a_{ij}x_j$, and a_{ij} is the $(i, j)^{th}$ element of A. If $a \in R^n$, then a acts on x by

$$a \cdot x = x + a.$$

As a set, E(n) may be identified with the cartesian product $O(n) \times R^n$, with group multiplication given by

$$(A, a)(B, b) = (AB, A \cdot b + a),$$

and

$$(A, a)^{-1} = (A^{-1}, -A^{-1} \cdot a).$$

There is a short exact sequence

$$1 \rightarrow R^n \xrightarrow{\iota} E(n) \xrightarrow{\pi} O(n) \rightarrow 1$$

where $\iota(a) = (I_n, a)$, $\pi(A, a) = A$, which splits $A \mapsto (A, 0)$. The action of E(n) on E^n may now be written as

$$(A, a) \cdot x = A \cdot x + a.$$

As we have remarked above, O(n) fixed $0 \in E^n$ and S^{n-1} is among its orbits. The group R^n of translations acts 1-transitively on E^n, since for all $x, y \in E^n$

$$a \cdot x = x + a = y \text{ iff } a = y - x.$$

Of course R^n is a normal subgroup of $E(n)$, with $E(n)/R^n \approx O(n)$, and the isotropy subgroup of any $x \in E^n$ is conjugate to $O(n)$.

We are concerned with the path-component $I_*(E^n) = E_+(n)$ of the identity element $(I_n, 0)$. This subgroup $E_+(n)$ of $E(n)$ consists of all $(A, a) \in O(n) \times R^n$ such that $A \in SO(n)$, the group of all orthogonal matrices A with $\det A = 1$. Thus $I_*(S^n) = SO(n+1)$.

We observe that if $\mu \in \mathcal{R}(E^n) = \mathcal{R}(n)$ is given by

$$\mu(t) = (A(t), a(t)),$$

then,

$$\mu^{-1}(t) = (A^{-1}(t), -A^{-1}(t)\cdot a(t)).$$

3.2 Euclidean motion

Since E^n is a smooth connected Riemannian n-manifold, we can apply the results of Chapter 2. Firstly we observe that the totally geodesic submanifolds of E^n are the affine subspaces of E^n, and so the instantaneous axis $C_t(\mu)$ of any motion $\mu \in \mathcal{R}(n)$ is an affine subspace (or affine plane) of dimension $n - 2k$ for some integer k, $0 \leqslant 2k \leqslant n$. So the centrode $C(\mu)$ is a 1-parameter family of such affine planes.

In particular, if $n = 1$, any instantaneous axis is either the whole line $E^1 = R$ or is empty. If $n = 2$, then an instantaneous axis is either the whole of E^2 or is a singleton or is empty. If $n = 3$, then an instantaneous axis is either E^3, a line, or empty. Thus we see that for motion in Euclidean 3-space, the centrode is a ruled surface. The motion consists of rolling and sliding of $C_M(\mu)$ on $C_F(\mu)$ along the generator $C_t(\mu)$. In the plane only 'rolling' occurs.

3.3 Spherical motion

A motion μ on S^n may be regarded as the restriction of a motion μ' in E^{n+1}, where

$$\mu'(t) = (\mu(t), 0), \quad \mu(t) \in SO(n + 1) \quad \text{and} \quad \mu'(t) \in E_+(n + 1).$$

Any instantaneous axis of such a motion μ is a great sphere of even co-dimension in S^n, formed by the intersection with S^n of a linear subspace of E^{n+1}.

If $n = 2$, then the instantaneous axes are pairs of antipodal points, and the centrode of the corresponding Euclidean motion μ' is a cone with vertex 0. Thus μ' may be pictured as the rolling (without sliding) of one such cone on another.

3.4 Translational motion

In contrast to the spherical motions considered above, we can define a translational motion to be a Euclidean motion $\mu \in \mathcal{R}(n)$ of the form

$$\mu(t) = (I_n, a(t)), \qquad t \in R.$$

Then the velocity vector field $v_t(\mu)$ is given by

$$v_t(\mu)(x) = \dot{a}(t), \qquad \text{for all} \quad x \in E^n.$$

Thus $v_t(\mu)$ is constant, and so each instantaneous axis is either E^n itself or is empty.

3.5 Symmetry groups

We have seen that for any motion μ in a metric space (X, d_X), the symmetry group $\mathrm{Sym}\,\mu$ of μ is a subgroup of $I_*(X) \times I_*(X) \times R$.

In case X is a Euclidean n-space, the group $I_*(X)$ is $E_+(n)$ and we can say a little more about the structure of $\mathrm{Sym}\,\mu$.

Recall first that $\mathrm{Sym}\,\mu$ is isomorphic to a subgroup $S(\mu)$ of $I_*(X) \times R$, and there is an embedding of $S(\mu)$ into $I_*(X) \times I_*(X) \times R$ sending $(\phi, \alpha) \in S(\mu)$ to $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$.

Suppose then that $\mu \in \mathcal{R}(n)$, and let $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$. Identifying $E_+(n)$ with $SO(n) \times R^n$ as above, we can write,

$$\phi = (A, a), \quad \psi = (B, b) \text{ and for all } t \in R$$

$$\mu(t) = (P(t), p(t)),$$

where $A, B, P(t) \in SO(n)$ and $a, b, p(t) \in R^n$. Thus since for all $t \in R$,

$$\phi(\mu(t)) = \mu(t + \alpha)(\psi)$$

we have

$$(A, a)(P(t), p(t)) = (P(t + \alpha), p(t + \alpha))(B, b).$$

In particular, for $t = 0$, we obtain

$$(A, a) = (P(\alpha), p(\alpha))(B, b).$$

Hence

(1) ... $$(B, b) = (P^{-1}(\alpha)A, P^{-1}(\alpha)\cdot(a - p(\alpha))),$$

and so

$$B = P^{-1}(\alpha)A, \qquad b = P^{-1}(\alpha)\cdot(a - p(\alpha)).$$

It is natural to consider the question of which subgroups of $E_+(n) \times R$ may occur as symmetry groups $S(\mu)$, these being embedded in $E_+(n) \times E_+(n) \times R$ according to equation (1) above as Sym μ.

We have not investigated this question in any detail. However we give a few examples to indicate how large these groups can be.

We can gain some insight by examining the possible symmetry groups for translational motion.

Suppose that $\mu \in \mathcal{R}(n)$ is a translational motion, given as in Section 3.4 by $\mu(t) = (I_n, a(t))$, $t \in R$. Thus the behaviour of μ depends entirely on the path

$$a : R \to R^n.$$

We discuss three special cases.

(i) Suppose that for all $t \in R$,

$$a(t) = 0.$$

Hence for all $x \in E^n$,

$$\mu(t)(x) = x.$$

Let $(\phi, \alpha) \in E_+(n) \times R$, and consider the following diagram

(2) ...

$$\begin{array}{ccc} (s, (t, \mu(t)(x))) & \xrightarrow{\ \mu_* \ } & (s + t, \mu(s + t)(x)) \\ \downarrow {\scriptstyle 1_R \times (\alpha \times \phi)} & & \downarrow {\scriptstyle \alpha \times \phi} \\ (s, (t+\alpha, \phi(\mu(t)(x)))) & \xrightarrow[\ \mu_* \ ]{} & (s+t+\alpha, \phi(\mu(s+t)(x))) \end{array}$$

This diagram commutes iff $(\phi, \alpha) \in S(\mu)$, and this is so iff

(3) ... $$\phi(\mu(t)(x)) = \mu(t + \alpha)(y),$$

for some $y = \psi(x)$, $\psi \in E_+(n)$.

In the present example, $\mu(t)(x) = x$ for all $x \in E^n$, and all $t \in R$. In particular for $t = 0$, we obtain from (3)

$$\phi(x) = y.$$

Thus $(\phi, \alpha) \in S(\mu)$ for any $(\phi, \alpha) \in E_+(n) \times R$, since if we take $y = \phi(x)$, then for all $s, t, \alpha \in R$,

$$\begin{aligned} \mu_*(s, (t + \alpha, \mu(t + \alpha)(y))) &= (s + t + \alpha, \mu(s + t + \alpha)(y)) \\ &= (s + t + \alpha, y) \\ &= (s + t + \alpha, \phi(x)). \end{aligned}$$

Thus $S(\mu) = E_+(n) \times R$.

(ii) Let $a(t)$ be a linear function of t. That is, there exists $v \in R^n$ such that for all $t \in R$,

$$a(t) = tv.$$

Diagram (2) may be written

$$\begin{array}{ccc} (s, (t, x+tv)) & \xrightarrow{\mu_*} & (s+t, x + tv + sv) \\ {\scriptstyle 1_R \times (\alpha \times \phi)} \downarrow & & \downarrow {\scriptstyle \alpha \times \phi} \\ (s, (t+\alpha, \phi(x + tv))) & \xrightarrow[\mu_*]{} & (s+t+\alpha, \phi(x + tv + sv)) \end{array}$$

Thus $(\phi, \alpha) \in S(\mu)$ iff for some $\psi \in E_+(n)$, and for all $x \in E^n$, and all $t \in R$,

$$\phi(x + tv) = y + tv + \alpha v,$$

where $y = \psi(x)$, and

$$y + tv + sv + \alpha v = \phi(x + tv + sv).$$

That is for all $x \in E^n$, and all $t, s \in R$,

$$(4) \ldots \qquad \phi(x + tv) + sv = \phi(x + tv + sv).$$

Now let $\phi = (B, b) \in E_+(n)$, so (4) can be written

$$B \cdot x + tB \cdot v + b + sv = B \cdot x + tB \cdot v + sB \cdot v + b$$

which reduces to

$$sv = sB \cdot v \quad \text{for all } s \in R,$$

i.e., $$B \cdot v = v.$$

There is no loss of generality in taking $v = e_n$, where $e_1, \ldots, e_n$ is the standard basis for R^n. Then $S(\mu)$ consists of all $((B, b), \alpha)$, where $\alpha \in R$, $b \in R^n$ and

$$B = \begin{bmatrix} \tilde{B} & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B} \in SO(n - 1).$$

(iii) Suppose there exists $\omega \in R$ such that $\omega > 0$, and for all $t \in R$,

$$a(t + \omega) = a(t).$$

Then diagram (2) may be written

$$\begin{array}{ccc}
(s,(t, x+a(t))) & \xrightarrow{\ \mu_*\ } & (s+t, x + a(s+t)) \\
{\scriptstyle 1_R \times (\alpha \times \phi)} \downarrow & & \downarrow {\scriptstyle \alpha \times \phi} \\
(s,(t+\alpha,\phi(x+a(t)))) & \xrightarrow[\ \mu_*\ ]{} & (s+t+\alpha,\phi(x+a(t+s)))
\end{array} .$$

Thus $(\phi, \alpha) \in S(\mu)$ iff for some $\psi \in E_+(n)$, and for all $x \in E^n$ and all $t \in R$,

$$(5) \ldots \qquad \phi(x + a(t)) = y + a(t + \alpha),$$

where $y = \psi(x)$, and

$$(6) \ldots \qquad y + a(s + t + \alpha) = \phi(x + a(s + t)).$$

Let $\alpha = m\omega$, for some $m \in Z$. Then

$$a(s + t + \alpha) = a(s + t),$$

and (5) becomes

$$\phi(x + a(t)) = y + a(t).$$

Hence (6) may be written

$$(7) \ldots \qquad \phi(x + a(t)) - a(t) + a(s + t) = \phi(x + a(s + t)).$$

Now let $\phi = (B, b) \in E_+(n)$. Hence (7) may be written

$$B\cdot(x + a(t)) + b - a(t) + a(s + t) = B\cdot(x + a(s + t)) + b$$

which reduces to

$$(B - I_n)(a(t) - a(s + t)) = 0.$$

If we now choose B to be any matrix in $SO(n)$ having $a(s)$ as an eigenvector, for all $s \in R$, then $((B, b), m\omega) \in S(\mu)$ for any $b \in R^n$ and any $m \in Z$. Let V be the linear subspace of R^n generated by the vectors $a(s)$, $s \in R$, and let $\dim V = k$. Then B is conjugate to a matrix of the form

$$P = \begin{bmatrix} \tilde{B} & 0 \\ 0 & I_k \end{bmatrix},$$

where $\tilde{B} \in SO(n - k)$. It follows that $S(\mu)$ contains a subgroup of the form

$$G \times Z < E_+(n) \times R,$$

where $G = \{(P, b) \in SO(n) \times R^n\}$, and P is of the above form. In particular, if $n = 2$, then $P = I_2$ or $a(t) = 0$ for all $t \in R$.

Note that case (i) is obtained by taking $k = 0$, when $V = \{0\}$ and $P = B$.

3.6 Group motion in low dimensions

In looking for interesting examples of motions in any metric space, it is worthwhile considering the homomorphism

$$\mu : R \to I_*(X)$$

some of whose properties were discussed in Section 1.8.

In the case of Euclidean n-space E^n, many such homomorphisms exist, and the set $\mathfrak{G}(n) = \mathfrak{G}(E^n)$ is difficult to describe. For $n \leq 3$, however, the situation is fairly simple. We concentrate our discussion of Euclidean Kinematics in low dimensions, therefore, by determining the sets $\mathfrak{G}(1)$, $\mathfrak{G}(2)$ and $\mathfrak{G}(3)$.

Trivially, $\mathfrak{G}(1) \approx \mathrm{Hom}(R, R)$ is R itself. In the case of $\mathfrak{G}(2)$, every $\mu \in \mathfrak{G}(2)$ is of the form

$$\mu(t) = (A(t), a(t)),$$

where,

$$A(t) = \begin{bmatrix} \cos \lambda t & -\sin \lambda t \\ \sin \lambda t & \cos \lambda t \end{bmatrix}, \text{ for some } \lambda \in R.$$

Thus

$$\mu(t + s) = \mu(t)\cdot\mu(s)$$

iff, for all $t, s \in R$, $x \in E^2$,

$$\begin{aligned}(A(t), a(t))(A(s), a(s))(x) &= A(t)\cdot(A(s)\cdot x + a(s)) + a(t) \\ &= A(t)A(s)\cdot x + A(t)\cdot a(s) + a(t) \\ &= A(t + s)\cdot x + a(t + s).\end{aligned}$$

That is, iff for all $s, t \in R$,

$$A(t)A(s) = A(t + s),$$

and

$$A(t)\cdot a(s) + a(t) = a(t + s).$$

Since $A \in SO(2)$, $A(t)A(s) = A(t + s)$ and hence $\mu(t + s) = \mu(t)\cdot\mu(s)$ iff $A(t)\cdot a(s) = a(s)$. Thus for each pair $t, s \in R$, either

$$A(t) = I_2 \quad \text{or} \quad a(s) = 0.$$

Hence either for some $\lambda \in R$,

$$\mu(t) = (A(t), 0),$$

or, for some $v \in R^2$,

$$\mu(t) = (I_2, tv).$$

Next consider the epimorphism

$$\pi : E_+(n) \to SO(n),$$

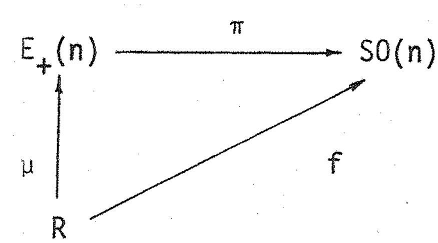
given by the projection to the first factor in the set $E_+(n) = SO(n) \times R^n$, by $\pi(A, a) = A$. Then any homomorphism

$$\mu : R \to E_+(n)$$

determines a homomorphism

$$f = \pi \circ \mu : R \to SO(n),$$

that is, the diagram



commutes. Thus $f(R)$ is a connected abelian subgroup of $SO(n)$.

For $n = 3$, we show that $f(R)$ is a circle subgroup of $SO(3)$ as follows.

The Lie algebra $so(3)$ of $SO(3)$ is 3-dimensional and may be identified with the algebra of all skew-symmetric 3×3 real matrices A, where

$$[A, B] = AB - BA.$$

A basis for $so(3)$ is given by

$$v_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We regard v_1, v_2 and v_3 as tangent vectors at the identity of $SO(3)$ to the paths

$$h_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \quad h_2(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix} \text{ and}$$

$$h_3(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{respectively.}$$

Let $v \in so(3)$ and consider the map

$$f_v : R \to SO(3),$$

given by $f_v(t) = \exp(tv)$. Suppose $a_1v_1 + a_2v_2 + a_3v_3 = v$. Then $f_v(t)$ is a smooth path in $SO(3)$ whose tangent vector at the origin is v. Furthermore one finds from the properties of the exponential map

$$f_v(t + s) = \exp(t + s)v = \exp(tv)\exp(sv)$$

$$= f_v(t).f_v(s), \quad \text{for all } s, t \in R.$$

Hence the set $\{\exp(tv) : t \in R\}$ is a 1-dimensional subgroup of $SO(3)$ with tangent vector at the origin equal to v, and one finds that

$$f_{v_i}(t) = \exp(tv_i) = h_i(t), \qquad i = 1,2,3.$$

In particular, $f_{v_1}(t) = h_1(t)$, and $h_1(t)$ is isomorphic to the circle group.

Now let G, H be any 1-dimensional subgroups of $SO(3)$. Then G is conjugate to H and hence every 1-dimensional subgroup G of $SO(3)$, $(G \neq 1)$ is isomorphic to the circle group. To show this we consider the following.

Let $G = \{\exp tv : t \in R\}$ and $H = \{\exp tw : t \in R\}$ be 1-dimensional subgroups of SO(n), where $v = [a_{ij}]$, $w = [b_{ij}]$ are skew-symmetric $n \times n$ real matrices, and $v, w \in so(n)$ are tangent vectors to G, H respectively at the identity of SO(n). The eigenvalues λ_i, ρ_i of v, w are the roots of the characteristic equations

$$|\lambda I_n - v| = \lambda^n + s_1\lambda^{n-1} + \quad \ldots + s_{n-1}\lambda + (-1)^n|\det v| = 0,$$

$$|\rho I_n - w| = \rho^n + r_1\rho^{n-1} + \quad \ldots . + r_{n-1}\rho + (-1)^n|\det w| = 0,$$

where $s_m(r_m)$, $(m = 1, 2, \ldots, n-1)$ is $(-1)^m$ times the sum of all the m-square principal minors of v(w).

Since v, w are skew-symmetric, we observe that for n=3,

$$\lambda^3 + (a_1^2 + a_2^2 + a_3^2)\lambda = 0, \qquad \rho^3 + (b_1^2 + b_2^2 + b_3^2)\rho = 0.$$

Set $(\sum_1^3 a_i^2)^{\frac{1}{2}} = \|v\|$ and $(\sum_1^3 b_i^2)^{\frac{1}{2}} = \|w\|$. Then the eigenvalues of v and w are $0, \pm\|v\|$ and $0, \pm\|w\|$, and they depend only on the norm of the corresponding tangent vectors v, w. Hence there exists $k \in R$, $k \neq 0$ such that v, kw have the same eigenvalues.

Since the eigenvalues of v, kw are all different, v, kw can be diagonalised and they are similar to the same diagonal matrix. Thus v is similar to kw, i.e., there exists $A \in SO(3)$ such that

$$v = A^{-1}\, kw\, A.$$

This implies that for all $t \in R$,

$$\begin{aligned} \exp tv &= \exp A^{-1}\, ktw\, A \\ &= I + A^{-1}(ktw)A + (A^{-1}\, ktwA)^2/2! + \ldots \\ &= A^{-1}(I + (ktw) + (ktw)^2/2! + \ldots)A \\ &= A^{-1}(\exp ktw)A. \end{aligned}$$

Hence G is conjugate to H.

We remark that in case $n > 3$ this is no longer true. The following example shows that there exist 1-dimensional subgroups of $SO(n)$, $n > 3$ that are $\neq S^1$.

3.6.1 Example:

Suppose $n = 4$ and let

$$A(\lambda_i t) = \begin{bmatrix} \cos \lambda_i t & -\sin \lambda_i t \\ \sin \lambda_i t & \cos \lambda_i t \end{bmatrix} \quad \varepsilon \quad SO(2),$$

where $\lambda_i \varepsilon R$, $i = 1,2$. Let G, H be 1-dimensional subgroups of $SO(4)$ given by

$$G(t) = \left[\begin{array}{c:c} A(\lambda_1 t) & 0 \\ \hdashline 0 & A(\lambda_2 t) \end{array}\right], \qquad H(t) = \left[\begin{array}{c:c} A(\lambda_1 t) & 0 \\ \hdashline 0 & I_2 \end{array}\right].$$

We note that $H(t) \simeq S^1$, where $G(t) \neq S^1$ in general. $G(t)$ is periodic and hence is closed, so $G(t) \simeq S^1$ iff $A(\lambda_1 t)$, $A(\lambda_2 t)$ have the same period $\Leftrightarrow$ λ_1/λ_2 is rational. Note that $G(t)$ is not conjugate to $H(t)$ since their corresponding eigenvalues are $e^{i\lambda_1 t}$, $e^{-i\lambda_1 t}$, $e^{i\lambda_2 t}$, $e^{-i\lambda_2 t}$ and $e^{i\lambda_1 t}$, $e^{-i\lambda_1 t}$, 1, 1 respectively, and they are different.

Now consider the homomorphism $f : R \to SO(3)$. Suppose that $f(R) \neq$ identity of $SO(3)$, then $f(R) \simeq S^1$. We can choose orthogonal cartesian coordinates for E^3 so that

$$f(t) = \begin{bmatrix} \cos \lambda t & -\sin \lambda t & 0 \\ \sin \lambda t & \cos \lambda t & 0 \\ 0 & 0 & 1 \end{bmatrix} = A(t) = \begin{bmatrix} \tilde{A}(t) & 0 \\ 0 & 1 \end{bmatrix}$$

and $A(t)$ is then a rotation about z-axis. Note that

(8) ... $$A(s)A(t) = A(s + t).$$

Let $a(t) = (a_1(t), a_2(t), a_3(t)) \varepsilon R^3$, and let $\tilde{a}(t) = (a_1(t), a_2(t))$. Then there is a homomorphism

$$\tilde{\mu} : R \to E_+(2)$$

given by $\tilde{\mu}(t) = (\tilde{A}(t), \tilde{a}(t))$, and for all $x \varepsilon E$ and all $t \varepsilon R$, $\tilde{\mu}(t)(x) = \tilde{A}(t) \cdot x + \tilde{a}(t)$.

Suppose $\tilde{A}(t) \neq I_2$, then there exists a unique point $q \varepsilon E^2$ such that for all $t \varepsilon R$,

$$\tilde{\mu}(t)(q) = \tilde{A}(t) \cdot q + \tilde{a}(t) = q$$

$$\Leftrightarrow \qquad q = (I_2 - \tilde{A}(t))^{-1} . \tilde{a}(t)$$

and $\tilde{\mu}(t)$ is a rotation about the point q.

If we move the origin to the point $q \varepsilon E^2$, (q is conjugate by a translation $(I_2 - \tilde{A}(t))^{-1} . \tilde{a}(t)$ orthogonal to z-axis) then for all $s, t \varepsilon R$,

(9) ... $$A(t) \cdot d(s) = d(s), \quad \text{where } d(s) = \begin{bmatrix} 0 \\ 0 \\ b(s) \end{bmatrix} .$$

Consider $\mu(t) = (A(t), d(t))$, $\mu(s) = (A(s), d(s))$ then

$$\begin{aligned} \mu(t) \cdot \mu(s) &= (A(t), d(t))(A(s), d(s)) \\ &= (A(t)A(s), A(t) \cdot d(s) + d(t)) \\ &= \mu(t + s) = (A(t + s), d(t + s)) \quad \text{iff} \end{aligned}$$

(i) $$A(t)A(s) = A(t + s) \quad \text{and}$$

(ii) $$A(t) \cdot d(s) + d(t) = d(t + s).$$

We have already observed that (i) is satisfied, and condition (ii) reduces to

$$d(t + s) = d(t) + d(s), \quad \text{by} \quad (9).$$

Hence d is a homomorphism, and μ is a 'screw motion' along the translated axis z_* through q. Thus in general $\mu(t) = (A(t), a(t))$ is a screw motion along a line z_* parallel to z.

We can summarise what we have proved in the following theorem and lemmas which describe the set $\mathfrak{G}(3)$.

Consider the subgroups $G(\lambda, \kappa)$ of $E_+(3)$ of the form

$$\mu(t) = (A(t), a(t)),$$

where

$$A(t) = \begin{bmatrix} \cos \lambda t & -\sin \lambda t & 0 \\ \sin \lambda t & \cos \lambda t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $$a(t) = (0, 0, t\kappa), \qquad \lambda, \kappa \in R.$$

3.6.2 Theorem:

Every 1-dimensional subgroup of $E_+(3)$ is conjugate to $G(\lambda, \kappa)$ for some unique pair (λ, κ), $\kappa \geq 0$.

We call $G(\lambda, \kappa)$ a <u>spiral</u> subgroup if $\lambda \neq 0$, $\kappa > 0$, a <u>circle</u> subgroup if $\lambda > 0$, $\kappa = 0$, and a <u>translational</u> subgroup if $\lambda = 0$, $\kappa > 0$.

3.6.3 Lemma:

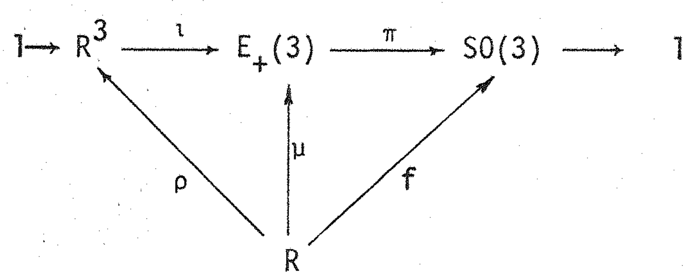
Every 1-dimensional subgroup of R^3 is conjugate to a translational subgroup.

3.6.4 Lemma:

Every 1-dimensional subgroup of SO(3) is conjugate to a circle subgroup.

3.6.5 Lemma:

For every homomorphism $\mu : R \to E_+(3)$ there is a homomorphism $\rho : R \to R^3$, such that the following diagram



commutes.

In higher dimensions, one has to investigate the geometry of ruled submanifolds of E^n. There has been considerable recent work on such problems. See, for example, [4, 8, 14]

CHAPTER 4

KINEMATICS IN THE EUCLIDEAN PLANE

Our aim in this chapter is to exploit the relatively straightforward structure of $E_+(2)$ to give a fairly complete discussion of the kinds of symmetry that occur in plane motion.

4.1 Translational motion

We saw in Chapter 2 that the centrode $C(\mu) = \bigcup_{t \in R} C_t(\mu)$ of a smooth motion $\mu \in \mathcal{R}(X)$ is a 1-parameter family of totally geodesic submanifolds $C_t(\mu)$ of X of even codimension. Thus in case $X = E^2$, $C_t(\mu)$ is empty, or a singleton, or is E^2 itself.

Recall that a motion $\mu \in \mathcal{M}(n)$ is said to be translational iff $\mu(t) = (I_n, a(t))$, for some path $a : R \to R^n$.

4.1.1 Theorem:

Let $\mu \in \mathcal{R}(2)$. Then μ is translational iff, for all $t \in R$, $C_t(\mu) = \varnothing$ or $C_t(\mu) = E^2$.

Proof:

Let μ be translational motion. Then for all $t \in R$,

$$\mu(t) = (I_2, a(t)),$$

where $a: R \to R^2$ is a smooth map.

Hence for all $x \in E^2$, the velocity vector field v at $t \in R$ is given by

$$v_t(\mu)(\mu(t)(x)) = \dot{a}(t) = \frac{da(t)}{dt}.$$

Thus $v_t(\mu)$ is constant for all $x \in E^2$. If $\dot{a}(t) \neq 0$ then $C_t(\mu) = \emptyset$, likewise if $\dot{a}(t) = 0$, then $C_t(\mu) = E^2$.

Conversely, let $\mu \in \mathcal{R}(2)$ and suppose that

$$\mu(t) = (A(t), a(t)),$$

where $A : R \to SO(2)$ and $a : R \to R^2$, are smooth maps. Let $x \in E^2$, then

$$\mu(t)(x) = A(t)x + a(t), \qquad t \in R$$
$$= y.$$

Hence

$$v_t(\mu)(y) = \dot{A}(t)x + \dot{a}(t).$$

Thus $y \in C_t(\mu)$ iff

$$0 = \dot{A}(t)x + \dot{a}(t),$$

that is iff

$$\dot{A}(t)A^{-1}(t)(y - a(t)) + \dot{a}(t) = 0.$$

Now suppose that μ is not translational. Then $A(s) \neq I_2$, for some $s \in R$, and $\dot{A}(t) \neq 0$ for some $t \in R$. For such t, consider the matrix

$$P(t) = \dot{A}(t)A^{-1}(t).$$

Since the matrix $A(t)$ is orthogonal, the matrix $P(t)$ is skew-symmetric. Thus $P(t)$ can be written

$$P(t) = \begin{bmatrix} 0 & \lambda(t) \\ -\lambda(t) & 0 \end{bmatrix},$$

where $\lambda(t) = -\dot{\theta}(t)$, and

$$A(t) = \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}.$$

Hence $\dot{A}(t) \neq 0 \iff \dot{\theta}(t) \neq 0 \iff \lambda(t) \neq 0$

$\iff P(t) \neq 0 \iff C_t(\mu) = \{y\}.$

Thus if μ is not translational, then for some $t \in R$, $C_t(\mu)$ is a singleton.

This theorem is false for motions in E^n, $n > 2$. For example, let $\mu \in \mathcal{R}(n)$ be given by

$$\mu(t) = (A(t), a(t)),$$

where

$$A(t) = \left[\begin{array}{cc:c} \cos t & -\sin t & \\ \sin t & \cos t & 0 \\ \hdashline & 0 & I_{n-2} \end{array}\right], \quad a(t) = (0,0,\ldots,0,t).$$

Then μ is not translational, but $C_t(\mu) = \varnothing$ for all $t \in R$.

4.2 Rotational motion

A <u>spherical motion</u> or <u>rotational motion</u> in E^n is a motion μ of the form

$$\mu(t) = (A(t), 0),$$

where $A(t) \in SO(n)$. In case $n = 2$, such a motion has instantaneous axis $C_t(\mu) = E^2$ or a singleton.

Now we can identify $SO(2)$ with the circle group S^1 of complex numbers of unit modulus. Thus $z \in S^1$ iff $z = e^{i\theta}$, for some $\theta \in R$. We can then regard a rotational motion as a path $\mu : R \to E_+(2)$ of the form

$$\mu(t) = (e^{i\theta(t)}, 0),$$

where $\theta : R \to R$ is some continuous function, such that $\theta(0) = 0$.

Recall that $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$ iff for all $t \in R$,

(1) ... $$\phi \circ \mu(t) = \mu(t + \alpha) \circ \psi$$

Let $\phi = (e^{ip}, a)$ and $\psi = (e^{iq}, b)$. Then

$$\begin{aligned} \psi &= \mu^{-1}(\alpha) \circ \phi \\ &= (e^{-i\theta(\alpha)}, 0)(e^{ip}, a) \\ &= (e^{i(p-\theta(\alpha))}, e^{-i\theta(\alpha)}a). \end{aligned}$$

So we can take

(2) ... $$q = p - \theta(\alpha) \qquad \text{and,}$$

(3) ... $$b = e^{-i\theta(\alpha)}a.$$

Equation (1) then becomes

$$(e^{ip},a)(e^{i\theta(t)},0) = (e^{i\theta(t+\alpha)}, 0)(e^{i(p-\theta(\alpha))}, e^{-i\theta(\alpha)}a),$$

that is, for all $t \in R$,

$$(e^{i(p+\theta(t))}, a) = (e^{i(\theta(t+\alpha)+p-\theta(\alpha))}, e^{i(\theta(t+\alpha)-\theta(\alpha))}a)$$

which reduces to

$$\theta(t + \alpha) = \theta(t) + \theta(\alpha), \qquad \text{for all } t \in R,$$

and either $a = 0$ or $\theta(t + \alpha) = \theta(\alpha)$.
But $\theta(t + \alpha) = \theta(\alpha)$ iff for all $t \in R$, $\theta(t) = \theta(\alpha)$. Since $\theta(0) = 0$, we then have $\theta(t) = 0$ for all $t \in R$. In this case μ is the trivial motion.

Thus if μ is non-trivial, then $a = b = 0$. It follows that for any non-trivial rotational plane motion $\mu = (e^{i\theta(t)}, 0)$, every element of Sym μ is of the form

$$((e^{ip}, 0), (e^{i(p-\theta(\alpha))}, 0), \alpha).$$

Conversely if $\theta(t + \alpha) = \theta(t) + \theta(\alpha)$ for all $t \in R$, then

$$((e^{ip}, 0), (e^{i(p-\theta(\alpha))}, 0) \alpha) \in \text{Sym } \mu \quad \text{for any } p \in R.$$

4.3 Rhythmic functions

It seems that such functions θ have not been discussed in standard texts, so we propose the following.

Recall that a function $\phi : R \to R$ is <u>periodic</u> with <u>period</u> $\alpha \neq 0$, or is <u>α-periodic</u>, iff for all $x \in R$,

$$\phi(x + \alpha) = \phi(x).$$

Also a function $\psi : R \to R$ is <u>linear</u> with slope $k \in R$, iff for all $x \in R$,

$$\psi(x) = kx.$$

We say that a function $f : R \to R$ is rhythmic with beat $\alpha \neq 0$ and pitch $f(\alpha)$, or α-rhythmic iff for all $x \in R$,

$$f(x + \alpha) = f(x) + f(\alpha).$$

Thus any α-periodic function that vanishes at 0, is α-rhythmic with beat $\alpha \neq 0$ and pitch 0. Also any linear function is rhythmic with beat α, for any $\alpha \in R$, $\alpha \neq 0$.

Let P_α denote the set of all α-periodic functions that vanish at 0, and let L denote the set of all linear functions. We denote the set of all α-rhythmic functions by $\mathcal{R}_\alpha$.

4.3.1 Theorem:

There is a bijection $\beta : \mathcal{R}_\alpha \to P_\alpha \times L$ given by $\beta(f) = (\phi, \psi)$, where

$$\phi(x) = f(x) - \frac{f(\alpha)}{\alpha} x,$$

and
$$\psi(x) = \frac{f(\alpha)}{\alpha} x.$$

The inverse of β is given by $\beta^{-1}(\phi, \psi) = \phi + \psi$.

Proof:

Let $f \in \mathcal{R}_\alpha$. Thus for any $x \in R$,

$$f(x + \alpha) = f(x) + f(\alpha).$$

Hence, if ϕ and ψ are as stated, then ψ is linear, and for all $x \in R$,

$$\phi(x + \alpha) = f(x + \alpha) - \frac{f(\alpha)}{\alpha}(x + \alpha)$$

$$= f(x) + f(\alpha) - \frac{f(\alpha)}{\alpha}x - f(\alpha)$$

$$= f(x) - \frac{f(\alpha)}{\alpha}x$$

$$= \phi(x).$$

Thus ϕ is α-periodic.

Conversely let ϕ be α-periodic with $\phi(0) = 0$, and let ψ be linear with slope k. Let $f = \phi + \psi$. Then for all $x \varepsilon R$,

$$f(x + \alpha) = \phi(x + \alpha) + \psi(x + \alpha)$$

$$= \phi(x) + k(x + \alpha)$$

$$= (\phi(x) + kx) + k\alpha$$

$$= f(x) + \phi(\alpha) + k\alpha,$$

since, $0 = \phi(0) = \phi(\alpha)$.

Thus

$$f(x + \alpha) = f(x) + f(\alpha).$$

Note that

$$f(\alpha) = k\alpha.$$

Finally, we observe that

$$\beta(\phi + \psi) = (\phi, \psi).$$

For let $\beta(\phi + \psi) = (\phi', \psi') \varepsilon P_\alpha \times L$. Thus

$$\phi'(x) = f(x) - \frac{f(\alpha)}{\alpha}x$$

$$= \phi(x) + kx - \frac{(\phi(\alpha) + k\alpha)}{\alpha}x$$

$$= \phi(x).$$

Hence

$$\phi' = \phi \quad \text{and} \quad \psi' = \psi.$$

4.3.2 Corollary:

If f is rhythmic, then $f(0) = 0$.

Proof:

Let $f \in \mathcal{R}_\alpha$. Then $f = \phi + \psi$, where $\phi \in P_\alpha$, $\psi \in L$.

Hence

$$f(0) = \phi(0) + \psi(0)$$
$$= \phi(0) = 0.$$

4.3.3 Remarks

(1) The above argument shows that for any $\alpha \neq 0$ and any $\beta \in R$, we may construct a rhythmic function f with beat α and pitch β as follows. Let $\phi \in P_\alpha$ and let $\psi \in L$ have a slope $\frac{\beta}{\alpha}$. Then

$$f = \phi + \psi$$

has the required properties.

(2) Let $f \in \mathcal{R}_\alpha$, with $f(\alpha) \neq 0$. Then

$$f(t + n\alpha) = f(t) + nf(\alpha), \qquad n \in Z.$$

Thus

$$\lim_{x \to \pm\infty} f(x) = \pm\infty.$$

(3) Let $f \in \mathcal{R}_\alpha$. Then for all $n \in Z$,

$$f(0) = f(n\alpha + (-n\alpha)) = f(n\alpha) + f(-n\alpha) = 0.$$

4.3.4 Example

Let $\phi(x) = \sin x$, $\psi(x) = x$, then

$$f(x) = \sin x + x.$$

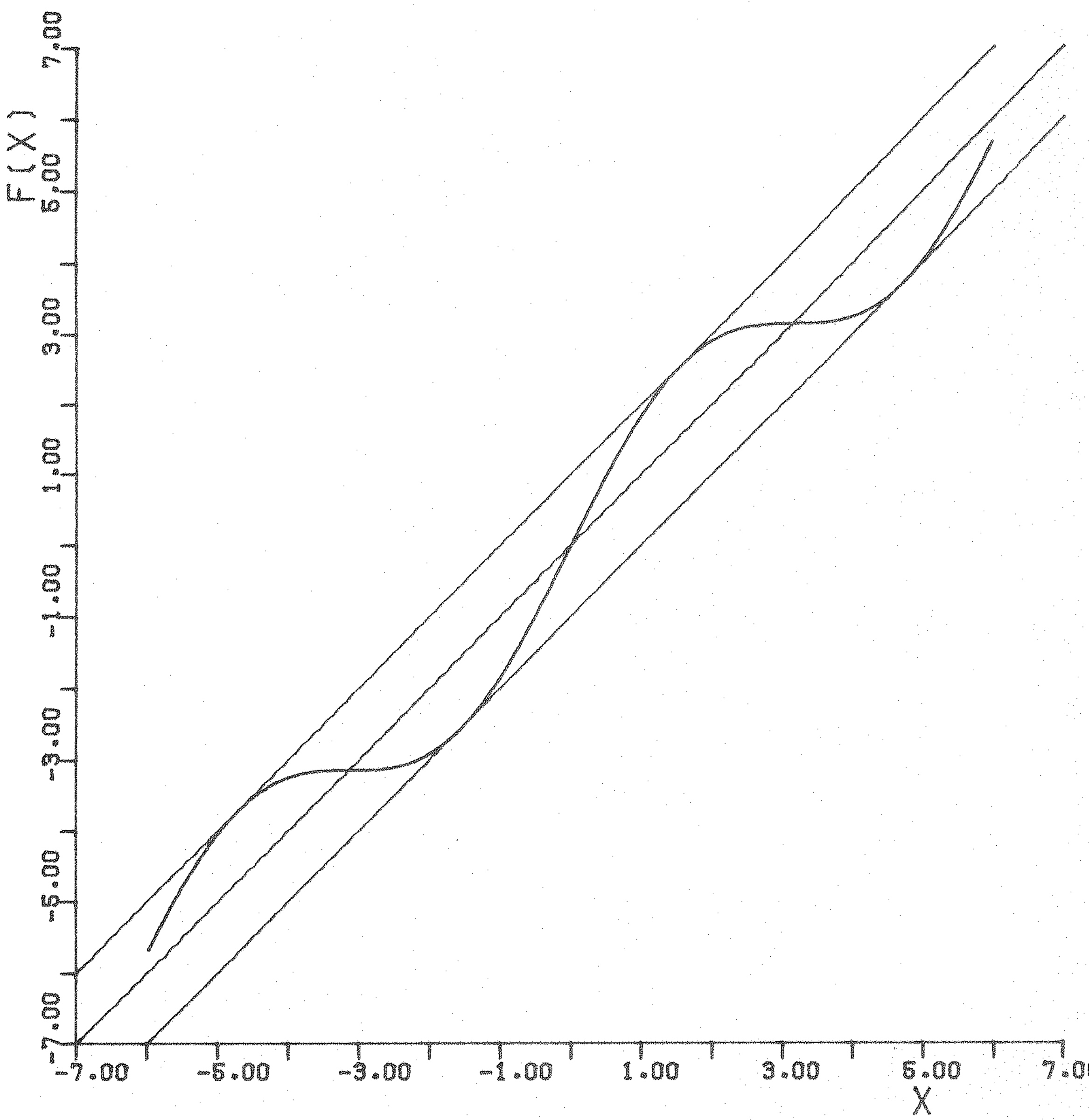


Fig. 5

4.4 Symmetry of Centrodes

We constructed in 2.5 a monomorphism

$$\rho_F : \mathrm{Sym}\,\mu \to S(C_F(\mu)),$$

for a motion μ in a Riemannian manifold.

We should expect that 'in general', ρ_F will be an isomorphism. This is easy to see for motions in E^2 as follows.

Let $\mu \in \mathcal{R}(2)$, and let $C_F(\mu) = C$, $C_M(\mu) = \tilde{C}$. Then the relation between $C, \tilde{C}$ is given by

$$(4) \ldots \qquad C(t) = \mu(t)\tilde{C}(t), \qquad \text{for all} \quad t \in R.$$

Let $H(\mu) = \{(\phi,\psi,\alpha) \in E_+(2) \times E_+(2) \times R : \text{ for all } t \in R,$

$$(\phi(C(t)),\ \psi(\tilde{C}(t))) = (C(t+\alpha),\ \tilde{C}(t+\alpha))\}.$$

Consider the monomorphism

$$\sigma : \mathrm{Sym}\,\mu \to H(\mu),$$

and suppose that $(\phi, \psi, \alpha) \in H(\mu)$. Then for all $t \in R$,

$$(5) \ldots \qquad \phi(C(t)) = C(t+\alpha) \qquad \text{and,}$$

$$(6) \ldots \qquad \psi(\tilde{C}(t)) = \tilde{C}(t+\alpha).$$

From (4), (5), we get

$$C(t) = \mu(t)\tilde{C}(t) = \phi^{-1}(C(t+\alpha)).$$

That is

$$\phi(\mu(t)(\tilde{C}(t))) = \mu(t+\alpha)(\tilde{C}(t+\alpha)).$$

From (6) we obtain

$$\phi(\mu(t)(\tilde{C}(t))) = \mu(t + \alpha)(\psi(\tilde{C}(t))), \quad \text{for all} \quad t \in R.$$

This implies that, for all $t \in R$,

$$\phi \circ \mu(t) = \mu(t + \alpha) \circ \psi$$

and hence $(\phi, \psi, \alpha) \in \text{Sym}\, \mu$ iff there are two distinct points in each of the families $C(t)$ and $\tilde{C}(t)$, $t \in R$. Now this is so iff the motion μ is neither rotational nor translational.

If μ is translational, then

$$C = \tilde{C} = \varnothing,$$

and $\text{Sym}\, \mu$ is isomorphic to a subgroup of $E_+(2) \times R$, but trivially

$$H(\mu) \simeq E_+(2) \times E_+(2) \times R.$$

Likewise if μ is rotational and non-trivial, then

$$C = \tilde{C} = \{x\} = \{0\}, \quad \text{say, and}$$

in this case $\text{Sym}\, \mu$ is isomorphic to a subgroup of $SO(2) \times R$, but $H(\mu) \simeq SO(2) \times SO(2) \times R$.

4.5 Symmetry groups of plane motions

The following examples of plane motions are designed to exhibit the instances of every possible type of symmetry group. However, I cannot prove that this list is exhaustive.

Consider a motion $\mu \in \mathcal{R}(2)$. Let γ be a μ-trajectory. Then there exists a function θ unique up to an additive constant such that any other trajectory δ of μ is given by

$$\delta(t) = \gamma(t) + \zeta_\delta \, e^{i(\theta(t) + k_\delta)},$$

for all $t \in R$, where $\theta(0) = 0$. Suppose γ is such that there is no other μ-trajectory with a bigger symmetry group. Since $S(\gamma) \simeq \{0\}$, Z or R (see section 1.6, theorem 1.6.7), we have the following cases;

(1) $S(\gamma) \simeq R$, and γ is therefore:

(i) a fixed point path;

or (ii) a circular path with uniform velocity;

or (iii) a straight line path with uniform velocity.

(2) $S(\gamma) \simeq Z$, and γ is therefore:

(iv) a closed path;

or (v) is not closed but bounded;

or (vi) is not closed and is unbounded.

We can choose γ according to the priorities (i), (ii), ..., (vi), given above. That is if there is a fixed point path, then we choose γ to be this path, and if there is no such path but there is a circular path, then we choose γ to be this circular path and so on.

The symmetry properties of μ are closely related to the structure of the set Ω_* (see remark, section 1.5.4).

We now consider the following examples.

(1) Trivial motion

The trajectory γ is a fixed point path and so $S(\gamma) \simeq R$. The motion μ is such that for all $x \in E^2$, and all $t \in R$,

$$\delta_x(t) = \mu(t)(x) = x$$

that is each $x \in E^2$ is at rest. Let $(\phi, \psi, \alpha) \in E_+(2) \times E_+(2) \times R$. Then $(\phi,\psi,\alpha) \in \text{Sym}\,\mu$ iff for all $x \in E^2$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)).$$

That is iff for all $x \varepsilon E^2$,

$$\phi(x) = \mu(\alpha)(\psi(x))$$

$$= \psi(x) \Leftrightarrow \phi = \psi.$$

The elements of $P(\mu)$ are therefore of the form $(1, 1, \alpha)$, where $1 = 1_{E^2}$. Thus we have

$$\text{Sym } \mu \simeq E_+(2) \times R,$$

$$P(\mu) \simeq R,$$

$$Q(\mu) \simeq E_+(2).$$

The action of $Q(\mu)$ on Ω is transitive and Ω_* is a singleton.

(2) Uniform straight line motion

The trajectory γ is a straight line path with uniform velocity and so $S(\gamma) \simeq R$. The motion μ is given by

$$\mu(t) = (I_2, a(t)),$$

where $a(t) = tv$, $v \varepsilon E^2 \setminus 0$.

The trajectory δ_x of any $x \varepsilon E^2$ is then given by

$$\delta_x(t) = \mu(t)(x) = x + tv, \qquad t \varepsilon R,$$

and Ω is therefore a 1-parameter family of parallel straight lines. Let T denote the group of translations in E^2, and let $\phi \varepsilon T$. Then $\phi = (I_2, w)$, where $w \varepsilon E^2 \setminus 0$, and so we have for all $x \varepsilon E^2$, and all $t \varepsilon R$,

$$\phi(\mu(t)(x)) = (x + tv) + w.$$

Let $\alpha \varepsilon R$, then there exists

$$\psi = \mu^{-1}(\alpha) \circ \phi$$

$$= (I_2, -\alpha v)(I_2, w)$$

$$= (I_2, w - \alpha v) \varepsilon T,$$

such that for all $x \varepsilon E^2$, and all $t \varepsilon R$,

$$\mu(t + \alpha)(\psi(x)) = (x + w - \alpha v) + (t + \alpha)v$$

$$= (x + tv) + w$$

$$= \phi(\mu(t)(x)).$$

Thus $(\phi, \psi, \alpha) \varepsilon \text{Sym } \mu$, and so

$$\text{Sym } \mu \simeq T \times R.$$

Now $\psi = 1 \Leftrightarrow w - \alpha v = 0 \Leftrightarrow w = \alpha v$ and so each element of $P(\mu)$ is of the form $((I_2, \alpha v), 1, \alpha)$ and we get

$$P(\mu) \simeq R,$$

and

$$Q(\mu) \simeq T.$$

The action of $Q(\mu)$ on Ω is 1-transitive and Ω_* is a singleton.

(3) Uniform translational circular motion

The trajectory γ is a circular path of uniform velocity and so $S(\gamma) \simeq R$. Suppose that γ is of radius r and centre 0. The motion μ is such that for all $x \varepsilon E^2$, and all $t \varepsilon R$,

$$\delta_x(t) = \mu(t)(x) = re^{ikt} + \zeta_x e^{i(\theta(t)+\beta_x)}, \text{ where } k, \beta_x \varepsilon R,$$

and θ is constant and so δ_x is a circle of the same radius r.

Each element $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$ is of the form

$$((A(k\alpha), b), (I_2, b), \alpha),$$

where $\phi = (A(k\alpha), b) \in E_+(2)$, $\alpha \in R$. Thus

$$\mathrm{Sym}\,\mu \simeq T \times R.$$

Now $\psi = 1 \Leftrightarrow b = 0$, and so each element of $P(\mu)$ is of the form $((A(k\alpha), 0), 1, \alpha)$. Hence

$$P(\mu) \simeq R,$$

and so $$Q(\mu) \simeq T.$$

The action of $Q(\mu)$ on Ω is 1-transitive and Ω_* is a singleton.

(4) Uniform concentric circular motion

The trajectory γ is a fixed point path, say 0, and so $S(\gamma) \simeq R$. The motion μ is such that for all $x \in E^2$, and all $t \in R$,

$$\delta_x(t) = \mu(t)(x) = \zeta_x e^{i(kt+\beta_x)},$$

(where $k, \beta_x \in R$), that is δ_x is a circle of radius $|\zeta_x|$ and centre 0. The motion μ can be given by

$$\mu(t) = (A(kt), 0),$$

where $A \in SO(2)$. Let $\phi \in E_+(2)$ be given by $\phi = (A(\beta), b)$. Then for all $x \in E^2$ and all $t \in R$,

$$\mu(t)(x) = A(kt)x$$

and

$$\begin{aligned}\phi\mu(t)(x) &= A(\beta)A(kt)x + b \\ &= A(kt + \beta)x + b.\end{aligned}$$

Let $\alpha \in R$, then there exists

$$\psi = \mu^{-1}(\alpha) \circ \phi$$

$$= (A^{-1}(k\alpha), 0)(A(\beta), b)$$

$$= (A(\beta - k\alpha), A(-k\alpha)b),$$

such that for all $x \in E^2$, and all $t \in R$,

$$\mu(t + \alpha)(\psi(x)) = A(kt + k\alpha)(A(\beta - k\alpha)x + A(-k\alpha)b)$$

$$= A(kt + \beta)x + A(kt)b$$

$$= \phi(\mu(t)(x)) \quad \text{iff,}$$

$$A(kt + \beta)x + A(kt)b = A(kt + \beta)x + b$$

$\Leftrightarrow \quad (A(kt) - I_2)b = 0 \quad$ for all $t \in R$,

$\Leftrightarrow \quad b = 0$, since $(A(kt) - I_2)$ is non singular for some $t \in R$, which implies that $\phi, \psi \in SO(2)$. That is for arbitrary $\alpha \in R$ and $\phi \in SO(2)$

$$((A(\beta), 0), (A(\beta - k\alpha), 0), \alpha) \in \text{Sym}\ \mu.$$

Hence

$$\text{Sym}\ \mu \simeq SO(2) \times R.$$

Now $\psi = 1 \Leftrightarrow \beta = k\alpha + 2\pi n$, $n \in Z$ and this implies that the elements of $P(\mu)$ are of the form

$$((A(k\alpha + 2\pi n), 0), 1, \alpha),$$

and so

$$P(\mu) \simeq R.$$

Hence

$$Q(\mu) \simeq SO(2).$$

The action of $Q(\mu)$ on Ω is intransitive. We can identify Ω_* with the set of non-negative real numbers. Thus δ_x lies in the class $|\zeta_x|$.

(5) Closed circular arcs motion

The trajectory γ is a fixed point, say O, and so $S(\gamma) \simeq R$. The motion μ is such that for all $x \in E^2$, and all $t \in R$,

$$\delta_x(t) = \mu(t)(x) = \zeta_x e^{i(\theta(t)+\beta_x)},$$

where θ is periodic with period ω. Therefore each $\delta \in \Omega$, $\delta \neq \gamma$, is a closed arc which subtends a fixed angle at O. Let $\phi, \psi \in E_+(2)$ be given by

$$\phi = (e^{ia}, 0), \psi = (e^{ib}, 0).$$

Then

$$\phi(\mu(t)(x)) = \zeta_x e^{i(\theta(t)+\beta_x+a)}.$$

Let $\alpha \in R$, and consider

$$\mu(t + \alpha)(\psi(x)) = \zeta_x e^{i(\theta(t+\alpha)+\beta_x+b)}.$$

Now $(\phi, \psi, \alpha) \in \text{Sym}\,\mu$ iff,

$$\zeta_x e^{i(\theta(t)+\beta_x+a)} = \zeta_x e^{i(\theta(t+\alpha)+\beta_x+b)}$$

$$\Longleftrightarrow \quad \theta(t) + a = \theta(t + \alpha) + b$$

$$\Longleftrightarrow \quad \theta(t + \alpha) = \theta(t) + c, \quad \text{where } c = a - b.$$

Suppose that $c \neq 0$, then we have

$$\theta(t + 2\alpha) = \theta((t + \alpha) + \alpha) = \theta(t + \alpha) + c = \theta(t) + 2c,$$

and so for $n \in Z$

$$\theta(t + n\alpha) = \theta(t) + nc.$$

Thus θ is not bounded, which is a contradiction since θ is periodic. Hence $c = 0$, and so $\alpha = n\omega$. Also $c = 0 \Leftrightarrow a = b \Leftrightarrow \phi = \psi$. Therefore the elements of Sym μ are of the form

$$((e^{ia}, 0), (e^{ia}, 0), n\omega), \quad \text{and so}$$

$$\text{Sym } \mu \simeq SO(2) \times Z.$$

Now $\psi = 1 \Leftrightarrow a = 2n\pi$, $n = Z$, and so each element of $P(\mu)$ is of the form $(1, 1, n\omega)$ and hence

$$P(\mu) \simeq Z.$$

Thus

$$Q(\mu) \simeq SO(2).$$

The action of $Q(\mu)$ on Ω is intransitive and again Ω_* can be identified with the set of non-negative real numbers. Thus δ_X lies in the class $|\zeta_X|$.

(6) Rhythmic circular arcs motion

Same as in example (5) but with θ a rhythmic function. Then by theorem 4.3.1, θ can be given by

$$\theta(t) = K(t) + \lambda(t), \quad \text{for all } t \in R,$$

where K is periodic with $K(0) = 0$, and λ is linear with slope ρ. Thus for all $t \in R$

$$\theta(t) = K(t) + \rho t.$$

Let $\alpha \in R$, and $\phi, \psi \in E_+(2)$ be given by $\phi = (e^{ia}, 0)$, $\psi = (e^{ib}, 0)$. Then $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$ iff for all $x \in E^2$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)).$$

That is iff

$$\zeta_x e^{i(K(t)+\rho t+\beta_x+a)} = \zeta_x e^{i(K(t+\alpha)+\rho(t+\alpha)+\beta_x+b)}$$

$$\Leftrightarrow \quad K(t) + a = K(t + \alpha) + \rho\alpha + b$$

$$\Leftrightarrow \quad K(t + \alpha) = K(t) + c_\alpha, \quad \text{where } c_\alpha = a - b - \rho\alpha.$$

In particular for $t = 0$ we get

$$K(\alpha) = c_\alpha.$$

Thus $(\phi, \psi, \alpha) \in \mathrm{Sym}\,\mu$ iff for all $t \in R$,

$$K(t + \alpha) = K(t) + K(\alpha).$$

Again since K is periodic, $K(\alpha) = 0$, and so $\alpha = n\sigma$ where σ is the beat of θ, $n \in Z$. Also $c_\alpha = K(\alpha) = 0 \Leftrightarrow b = a - \rho\alpha = a - n\rho\sigma$. Hence each element of $\mathrm{Sym}\,\mu$ is of the form

$$((e^{ia}, 0), (e^{i(a - n\rho\sigma)}, 0), n\sigma).$$

Therefore

$$\mathrm{Sym}\,\mu \simeq SO(2) \times Z.$$

Now $\psi = 1 \Leftrightarrow a - n\rho\sigma = 2n\pi \Leftrightarrow a = n\rho\sigma + 2n\pi$, and so the elements of $P(\mu)$ are of the form

$$((e^{in(\rho\sigma + 2\pi)}, 0), 1, n\sigma)$$

and thus

$$P(\mu) \simeq Z.$$

Hence
$$Q(\mu) \simeq SO(2).$$

The action of $Q(\mu)$ on Ω is intransitive, and Ω_* can be identified with the set of non-negative real numbers. Thus δ_x lies in the class $|\zeta_x|$.

(7) Translational motion with closed trajectories

The trajectory γ is closed with $S(\gamma) \simeq Z$. The motion μ is such that for all $x \in E^2$, and all $t \in R$,

$$\delta_x(t) = \mu(t)(x) = \gamma(t) + \zeta_x e^{i(\theta(t)+\beta_x)},$$

where θ is constant. Therefore each $\delta \in \Omega$ is isometric to γ. Let ω be the period of γ, that is for all $t \in R$,

$$\gamma(t + \omega) = \gamma(t).$$

Each element of Sym μ is of the form (ϕ, ψ, α_k), where $\phi = (A_k, a)$,

$$A_k = \begin{bmatrix} \cos\frac{2\pi k}{m} & -\sin\frac{2\pi k}{m} \\ \sin\frac{2\pi k}{m} & \cos\frac{2\pi k}{m} \end{bmatrix}, \quad m > 0, \; m, k \in Z, \text{ and } a \in R^2$$

with $\alpha_k = \dfrac{k\omega}{m}$.

Let x_* be the centre of γ. Then for each k

$$\mu(\alpha)(x) = A_k(x - x_*) + x_* = y,$$

and so

$$\mu^{-1}(\alpha)(y) = x = A_k^{-1}(y - x_*) + x_*.$$

Hence

$$\psi(x) = \mu^{-1}(\alpha) \circ \phi(x) = A_k^{-1}(A_k x + a - x_*) + x_*$$

$$= x + A_k^{-1}(a - x_*) + x_*$$

$$= x + b,$$

where $b = A_k^{-1}(a - x_*) + x_*$, and so $\psi \in T$. Therefore each element $(\phi, \psi, \alpha_k) \in \text{Sym}\,\mu$ is of the form

$$((A_k, a), (I_2, b), \alpha_k)$$

and we have

$$\text{Sym}\,\mu \simeq T \times Z,$$

$$P(\mu) \simeq Z$$

and

$$Q(\mu) \simeq T.$$

The action of $Q(\mu)$ on Ω is 1-transitive and Ω_* is a singleton.

(8) Cycloidal motion

The trajectory γ is a straight line path of uniform velocity and thus $S(\gamma) \simeq R$. The motion μ is such that for all $x \in E^2$, and all $t \in R$

$$\delta_x(t) = \mu(t)(x) = \gamma(t) + \zeta_x e^{i(\theta(t)+\beta_x)},$$

where θ is periodic. The motion μ can be given by

$$\mu(t) = (A^{-1}(t), a(t)),$$

where $A \in SO(2)$ and $a(t) \in R^2$, $a(t) = (t, 0)$. Then for all $x \in E^2$, and all $t \in R$,

$$\mu(t)(x) = A^{-1}(t)x + a(t),$$

$$\mu(\alpha)(x) = A^{-1}(\alpha)x + a(\alpha), \quad \text{for some } \alpha \in R.$$

Let $\phi = (I_2, b) \in T$, then there exists

$$\begin{aligned}\psi &= \mu^{-1}(\alpha) \circ \phi \\ &= (A(\alpha), -A(\alpha).a(\alpha))(I_2, b) \\ &= (A(\alpha), A(\alpha).(b - a(\alpha))),\end{aligned}$$

s.t. $(\phi, \psi, \alpha) \in \text{Sym}\,\mu$ iff, for all $x \in E^2$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)).$$

That is iff

$$\begin{aligned}A^{-1}(t)x + a(t) + b &= A^{-1}(t + \alpha)(A(\alpha)x + A(\alpha)(b - a(\alpha))) + a(t + \alpha) \\ &= A^{-1}(t)x + A^{-1}(t)(b - a(\alpha)) + a(t + \alpha))\end{aligned}$$

$$\Longleftrightarrow \quad b = A^{-1}(t)(b - a(\alpha)) + a(\alpha)$$

$$\Longleftrightarrow \quad (A(t) - I_2)b = (A(t) - I_2)a(\alpha)$$

$$\Longleftrightarrow \quad (A(t) - I_2)(b - a(\alpha)) = 0, \quad \text{for all } t \in R$$

$$\Longleftrightarrow \quad b - a(\alpha) = 0, \quad \text{since } (A(t) - I_2) \text{ is}$$

nonsingular for at least some $t \in R$.

Thus $(\phi, \psi, \alpha) \in \text{Sym}\,\mu \Longleftrightarrow b = a(\alpha)$, and so each element of $\text{Sym}\,\mu$ is of the form

$$((I_2, a(\alpha)), \quad (A(\alpha), 0), \quad \alpha).$$

Thus

$$\text{Sym}\ \mu \simeq R$$

Now $\psi = 1 \Leftrightarrow \alpha = 2n\pi,\ n \in Z$ and so the elements of $P(\mu)$ are of the form $((I_2, a(2n\pi)), 1, 2n\pi)$. Hence

$$P(\mu) \simeq Z,$$

and so

$$Q(\mu) \simeq SO(2).$$

The action of $Q(\mu)$ on Ω is intransitive and Ω_* can be identified with the set of non-negative real numbers. Thus δ_x lies in the class $|\zeta_x|$.

(9) Planetary motion

The trajectory γ is a circular path, and so $S(\gamma) \simeq R$. Let γ have radius r, centre 0 and angular velocity β_1. Let $x = \rho e^{i\lambda}$ and $v = \gamma(0) = re^{i\sigma}$. Then for all $x \in E^2$, and all $t \in R$

$$\delta_x(t) = \mu(t)(x) = \gamma(t) + (x - v)e^{i\beta_2 t}.$$

That is,

$$\mu(t)(x) = r(e^{i(\beta_1 t+\sigma)} - e^{i(\beta_2 t+\sigma)}) + \rho e^{i(\beta_2 t+\lambda)},$$

where $\beta_1, \beta_2, \sigma, \lambda \in R$, and $\beta_1 \neq \beta_2$.

Let $\phi = (e^{ia}, 0)$ and let $\alpha \in R$. Then

$$\mu(\alpha)(x) = r(e^{i(\beta_1\alpha+\sigma)} - e^{i(\beta_2\alpha+\sigma)}) + \rho e^{i(\beta_2\alpha+\lambda)} = z,$$

and so

$$\mu^{-1}(\alpha)(z) = e^{-i\beta_2\alpha}(z - r(e^{i(\beta_1\alpha+\sigma)} - e^{i(\beta_2\alpha+\sigma)})).$$

Consider $\psi \in E_+(2)$ given by

$$\begin{aligned}\psi(x) &= \mu^{-1}(\alpha)(\phi(x)) \\ &= e^{-i\beta_2\alpha}(\rho e^{i(\lambda+a)} - r(e^{i(\beta_1\alpha+\sigma)} - e^{i(\beta_2\alpha+\sigma)})) \\ &= \rho e^{i(\lambda+a - \beta_2\alpha)} - r(e^{i(\beta_1\alpha-\beta_2\alpha+\sigma)} - e^{i\sigma}).\end{aligned}$$

Then $(\phi,\psi,\alpha) \in \text{Sym}\,\mu$ iff for all $x \in E^2$, and all $t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)).$$

That is iff

$$\begin{aligned}&r(e^{i(\beta_1 t+\sigma+a)} - e^{i(\beta_2 t+\sigma+a)}) + \rho e^{i(\beta_2 t+\lambda+a)} \\ &= r(e^{i(\beta_1 t+\beta_1\alpha+\sigma)} - e^{i(\beta_2 t+\beta_2\alpha+\sigma)}) + e^{i(\beta_2 t+\beta_2\alpha)}(\rho e^{i(\lambda+a-\beta_2\alpha)} - \\ &\qquad - r(e^{i(\beta_1\alpha-\beta_2\alpha+\sigma)} - e^{i\sigma})) \\ &= r(e^{i(\beta_1 t+\beta_1\alpha+\sigma)} - e^{i(\beta_2 t+\beta_2\alpha+\sigma)}) + \rho e^{i(\beta_2 t+\lambda+a)}.\end{aligned}$$

Thus $(\phi,\psi,\alpha) \in \text{Sym}\,\mu \iff a = \beta_1\alpha$, and so the elements of $\text{Sym}\,\mu$ are of the form

$$((e^{i\beta_1\alpha}, 0), (e^{i(\beta_1-\beta_2)\alpha}, re^{i\sigma}(1 - e^{i(\beta_1-\beta_2)\alpha})),\alpha).$$

Hence $\qquad \text{Sym}\,\mu \simeq R.$

Now $\psi = 1 \iff \alpha = 2n\pi/(\beta_1 - \beta_2)$, $n \in Z$.

It follows that $P(\mu) \simeq Z$.

Let $y \varepsilon E^2$, then y is a fixed point of ψ iff

$$\psi z = z,$$

i.e.,

$$e^{i(\beta_1-\beta_2)} z + re^{i\sigma}(1 - e^{i(\beta_1-\beta_2)\alpha}) = z,$$

and so

$$(e^{i(\beta_1-\beta_2)\alpha} - 1)z = re^{i\sigma}(e^{i(\beta_1-\beta_2)\alpha} - 1).$$

Since $\beta_1 \neq \beta_2$ and $e^{i(\beta_1-\beta_2)\alpha} \neq 1$ for some $\alpha \varepsilon R$,

$$z = re^{i\sigma} = v$$

that is the fixed point of ψ is independent on α, and thus

$$Q(\mu) \simeq SO(2).$$

Note that if we put

$$\Phi(\mu) = \{\phi : (\phi, 1, \alpha) \varepsilon P(\mu)\},$$

so that $\Phi(\mu)$ is a subgroup of $E_+(2)$, then for the motion μ that we are now considering, $\Phi(\mu)$ is a subgroup of the circle group $S^1 = SO(2)$, and is the image of non-trivial homomorphism

$$h : Z \to S^1,$$

with $h(n) = e^{i2\pi n\theta}$, where $\theta = \dfrac{1}{1 - \beta_2/\beta_1}$.

It follows that if β_2/β_1 is rational, then $\Phi(\mu) \simeq Z_m$ for some m. If β_2/β_1 is irrational, then $\Phi(\mu) \simeq Z$, but $\Phi(\mu)$ is dense in S^1.

The action of $Q(\mu)$ on Ω is intransitive, and again Ω_* can be identified with the set of non-negative real numbers. Thus δ_x lies in the class $|x - v|$.

(10) Translational motion with non-closed, unbounded trajectories

The trajectory γ is non-closed, unbounded with $S(\gamma) \simeq Z$. The motion μ is such that for all $x \in E^2$, and all $t \in R$,

$$\delta_x(t) = \mu(t)(x) = \gamma(t) + \zeta_x e^{i(\theta(t)+\beta_x)},$$

where θ is constant. Thus each $\delta \in \Omega$ is isometric to γ, and one can show that

$$\text{Sym}\,\mu \simeq T \times Z, \quad P(\mu) \simeq Z \quad \text{and} \quad Q(\mu) \simeq T.$$

The action of $Q(\mu)$ on Ω is 1-transitive and Ω_* is a singleton.

(11) Translational motion with bounded non-closed trajectories

The trajectory γ is non-closed and bounded with $S(\gamma) \simeq Z$. The motion μ is such that for all $x \in E^2$, and all $t \in R$,

$$\delta_x(t) = \mu(t)(x) = \gamma(t) + \zeta_x\, e^{i(\theta(t)+\beta_x)},$$

where θ is constant. Thus each $\delta \in \Omega$ is isometric to γ, and one can show that

$$\text{Sym}\,\mu \simeq T, \quad P(\mu) \text{ is trivial and } Q(\mu) \simeq T.$$

The action of $Q(\mu)$ is 1-transitive and Ω_* is a singleton.

For further study of the Euclidean plane kinematics see [3]. In his paper [7], A. Karger studied the kinematic geometry in homogeneous spaces the group of motions of which are special 3-dimensional Lie-groups of motions. In [3] there is a method leading to the solution of an equivalence problem for all Lie-groups of motions. In addition there is a description of all transitive 1-parametric systems of motions in E^2.

CHAPTER 5

KINEMATICS IN THE EUCLIDEAN LINE

The main theme of this chapter is that the study of Kinematics in the Euclidean line E^1 is essentially the same thing as the study of real-valued continuous functions of a real variable. Thus we can reduce the study of smooth motions in E^1 to the study of smooth functions $f : R \to R$.

Many of the concepts of Kinematics assume a very simple form in this context. In particular, stable smooth motions in E^1 correspond to stable Morse functions $f : R \to R$. Consequently, we can classify such stable motions and discuss their symmetry properties in some detail.

5.1 Structure of $E_+(1)$

The connection between continuous real-valued functions and motions in E^1 depends on the following fact.

5.1.1 Proposition:

The map

$$\theta : E_+(1) \to R$$

given by

$$\theta(f) = f(0)$$

is an isomorphism. Thus for all $f \in E_+(1)$ and for all $x \in R$,

$$f(x) = x + \theta(f).$$

Proof:

Let $f \in E_+(1)$. Then for all $x, y \in R$,

(i) $x < y$ iff $f(x) < f(y)$,

(ii) $|x - y| = |f(x) - f(y)|$.

Now (i) and (ii) are equivalent to the statement that for all $x, y \in R$,

$$x - y = f(x) - f(y).$$

Hence if $f \in E_+(1)$, then for all $x \in R$,

$$x = x - 0 = f(x) - f(0),$$

and so

$$\begin{aligned} f(x) &= x + f(0) \\ &= x + \theta(f). \end{aligned}$$

Trivially, if $f, g \in E_+(1)$, then

$$f = g \iff \theta(f) = \theta(g).$$

Likewise every map f of the form

$$f(x) = x + k$$

is in $E_+(1)$. So θ is a bijection. Further, for all $f, g \in E_+(1)$,

$$\begin{aligned} (g \circ f)(x) &= g(x + \theta(f)) \\ &= (x + \theta(f)) + \theta(g) \\ &= x + \theta(f) + \theta(g) \\ &= x + \theta(g) + \theta(f) \\ &= x + \theta(g \circ f). \end{aligned}$$

This completes the proof of the proposition.

5.2 Smooth motions

Let $\mathcal{R} = \mathcal{R}(1)$ denote the group of smooth motions in E^1, as before, and let $\mathcal{F}$ denote the group of smooth real-valued functions on R, that vanish at 0, under pointwise addition. By proposition 5.1.1, there is an isomorphism

$$i : \mathcal{R} \to \mathcal{F}$$

given by

$$(i(\mu))(t) = \theta(\mu(t)).$$

We now see that for any motion $\mu \in \mathcal{R}$, we can write, for any $t, x \in R$,

$$\mu(t)(x) = x + f(t),$$

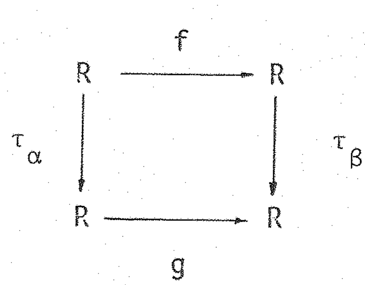
where $f = i(\mu)$.

We give $\mathcal{F}$ the Whitney C^1-topology [5].

5.3 Equivalence

Since the study of motions in E^1 reduces to the study of continuous real-valued functions of a real variable, we expect that equivalence of motions can be expressed as equivalence of functions.

Let $f, g \in \mathcal{F}$. Then f is equivalent to g, written $f \approx g$, iff there exists $(\tau_\alpha, \tau_\beta) \in E_+(1) \times E_+(1)$ such that the diagram



commutes, or for all $x \in R$,

$$g(x + \alpha) = f(x) + \beta,$$

where $\tau_\lambda(x) = x + \lambda$, for all $x \varepsilon R$.

Another way of saying that is to observe that $E_+(1) \times E_+(1)$ acts on $\mathcal{F}$ by

$$(\tau_\alpha, \tau_\beta)\cdot f = \tau_\beta \circ f \circ \tau_\alpha^{-1}.$$

Then $f \overset{\sim}{\sim} g$ iff f and g are in the same orbit under this action. Thus the equivalence classes of smooth functions in $\mathcal{F}$ are the orbits of $E_+(1) \times E_+(1)$.

We now show that equivalence of motions corresponds to equivalence of functions.

5.3.1 Theorem:

Let $\mu, \nu \varepsilon \mathcal{R}$ and $f = i(\mu)$, $g = i(\nu)$. Then $\mu \equiv \nu \Leftrightarrow f \overset{\sim}{\sim} g$.

Proof:

The motions μ, ν are equivalent iff there exists $(\phi,\psi,\alpha) \varepsilon E_+(1) \times E_+(1) \times R$ such that for all $x, t \varepsilon R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)).$$

Now

$$\phi(u) = u + a,$$

$$\psi(v) = v + b,$$

for all $u, v \varepsilon R$ and some $a, b \varepsilon R$. Also for all $s, t, x, y \varepsilon R$,

$$\mu(t)(x) = x + f(t)$$

and

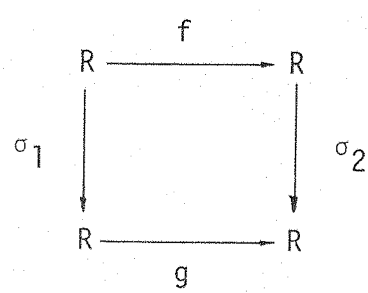
$$\nu(s)(y) = y + g(s).$$

Hence

$$\begin{aligned} \mu \equiv \nu \iff & x + f(t) + a = x + b + g(t + \alpha) \\ \iff & f(t) + (a - b) = g(t + \alpha) \\ \iff & f(t) + \beta = g(t + \alpha), \qquad \beta = a - b, \\ \iff & \tau_\beta \circ f = g \circ \tau_\alpha \\ \iff & f \approx g . \end{aligned}$$

This classification is 'very fine' in the sense that there are 'very many' equivalence classes. A more familiar and coarser, classification is based on diffeomorphisms rather than isometries.

Let $\mathrm{Diff}_+(R)$ denote the group of orientation preserving diffeomorphisms of R. Then we say that $f, g \in \mathcal{F}$ are (diffeomorphism) equivalent, written $f \sim g$, iff there exist $\sigma_1, \sigma_2 \in \mathrm{Diff}_+(R)$ such that the diagram



commutes.

We can therefore introduce a corresponding classification relation $\sim$ on $\mathcal{R}$ by putting $\mu \sim \nu$ iff $i(\mu) \sim i(\nu)$. The equivalence class of f in $\mathcal{F}$ is denoted by $[f]$. Now f is said to be <u>stable</u> iff $[f]$ is open in $\mathcal{F}$. We may therefore say that μ is <u>stable</u> iff $f = i(\mu)$ is stable.

Also there is a well-known characterisation of stable functions as follows.

Let $f \in \mathcal{F}$. Then $x \in R$ is a <u>critical point</u> of f iff

$f'(x) = df(x)/dx = 0$. Such a critical point is non-degenerate iff $f''(x) \neq 0$. Any non-degenerate critical point is isolated in the set Γ_f of all critical points of f. If Γ_f consists entirely of non-degenerate critical points, then f is said to be a Morse function, and so for such a function, Γ_f is countable.

It is well-known (see for example [5, p.87]) that a Morse function f is stable iff it has distinct critical values, i.e., iff for any $x, y \in \Gamma_f$,

$$x \neq y \Rightarrow f(x) \neq f(y).$$

We observe next that if $\mu \in \mathcal{R}$, and $f = i(\mu)$, then the velocity vector field $v_t(\mu)$ is the constant vector field on R whose value at any point is $\dot{f}(t)$. Thus the instantaneous axis at any time is $\emptyset$ if $\dot{f}(t) \neq 0$ and is E^1 itself if $\dot{f}(t) = 0$.

5.4 Classification of stable motions

We have now reduced the classification of stable smooth functions in E^1 to the classification of stable Morse functions $f : R \to R$. This can be done as follows.

Let K be a non-empty discrete countable (written ndc) subset of R. A labelling of K is an injective order-preserving map

$$\xi : K \to Z$$

such that for any $p, q \in K$ with $p > q$,

$$\# K \cap [p, q] = \xi(p) - \xi(q) + 1.$$

Thus a labelling of K enumerates its elements in succession along the real line.

If ξ, ξ' are two labellings of K, then there is a unique integer m such that for all $p \in K$,

$$\xi(p) = \xi'(p) + m.$$

If K, L are two ndc subsets of R, then we say that K is equivalent to L, written $K \cong L$ iff there are bijection $\beta : K \to L$ and labellings ξ, η of K, L such that $\eta \circ \beta = \xi$.

Now let $f : R \to R$, be a stable Morse function, and suppose that its set Γ_f of critical points is non-empty. A labelling ξ of Γ_f is said to be <u>proper</u> iff for some (and hence for every) maximum p of f, $\xi(p)$ is even. Consider also the set

$$\Delta_f = \{f(p) : p \in \Gamma_f\}.$$

A pair (ξ, ξ') where ξ is a proper labelling of Γ_f and ξ' is a labelling of Δ_f is called a <u>labelling of f</u>.

We also must take into account the behaviour of f at $\pm\infty$. Here we assign to f the quadruple $(\lambda, \lambda', \rho, \rho')$, where each component may take the value 1, 0 or -1. Thus we put $\lambda = 1, 0$ or -1 according as $\overline{\lim}_{x\to\infty} f(x)$ is ∞, is finite, or is $-\infty$, and define λ', ρ, ρ' likewise with respect to $\underline{\lim}_{x\to\infty} f(x)$, $\overline{\lim}_{x\to-\infty} f(x)$, $\underline{\lim}_{x\to-\infty} f(x)$.

We call $(\lambda, \lambda', \rho, \rho')$ the <u>type</u> of f. Note that not all of the 81 possibilities can occur. For example, if $\lambda' = 1$, then $\lambda = 1$. This is because $\overline{\lim} \geq \underline{\lim}$. In fact one may readily check that there are 36 types.

Now let $f, g : R \to R$, be stable Morse functions. If (ξ, ξ') and (η, η') are labellings of f, g respectively, then we say that

(ξ, ξ') is equivalent to (η, η') iff there is a bijection $\gamma : \Gamma_f \to \Gamma_g$ mapping

$$\begin{matrix} \text{maxima} \\ \text{minima} \end{matrix} \text{ to } \begin{matrix} \text{maxima} \\ \text{minima} \end{matrix}$$

such that, if

$$\delta : \Delta_f \to \Delta_g$$

given by $\delta(f(p)) = g(\gamma(p))$, then for all $p \in \Gamma_f$,

$$\xi(p) = \eta(\gamma(p)) + m ,$$

$$\xi'(f(p)) = \eta'(\delta(f(p))) + n,$$

where $m, n \in Z$.

One would expect that if $f, g : R \to R$ are two stable Morse functions, then $f \sim g$ iff they are of the same type and either $\Gamma_f = \Gamma_g = \varnothing$ or admit equivalent labellings. We do not pursue this question.

5.5 Symmetry of motion

Let $\mu \in \mathcal{R}$ and let $i(\mu) = f \in \mathcal{F}$. A symmetry of μ is (by definition) an element $(\phi, \psi, \alpha) \in E_+(1) \times E_+(1) \times R$ such that for all $x, t \in R$,

(1) ... $$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x)).$$

Let ϕ, ψ be given by $\phi(x) = x + \beta$, $\psi(y) = y + \sigma$ for all $x, y \in R$, and for some $\beta, \sigma \in R$. Equation (1) then becomes

$$\phi(x + f(t)) = \psi(x) + f(t + \alpha),$$

and so

$$x + f(t) + \beta = x + \sigma + f(t + \alpha),$$

which reduces to

(2) ... $$f(t) + (\beta - \sigma) = f(t + \alpha).$$

In particular, for $t = 0$, $f(0) = 0$ and we get

$$(\beta - \sigma) = f(\alpha).$$

Let $\theta_\alpha \in E_+(1)$, be given by

(3) ... $$\theta_\alpha(x) = x + (\beta - \sigma) = x + f(\alpha)$$

for all $x \in R$. Then (2) becomes

(4)... $$\theta_\alpha(f(t)) = f(t + \alpha), \text{ for all } t \in R,$$

thus $(\theta_\alpha, \alpha) \in S(f)$.

By proposition 1.6.2 $S(f)$ is a subgroup of $E_+(1) \times R$. The inverse element of (θ_α, α) is given by

$$(\theta_\alpha, \alpha)^{-1} = (\theta_\alpha^{-1}, -\alpha)$$

where, for all $t \in R$

$$\begin{aligned}\theta_\alpha^{-1}(f(t)) &= f(t + (-\alpha)) \\ &= f(t) + f(-\alpha) \\ &= f(t) - f(\alpha),\end{aligned}$$

because $f(0) = f(\alpha - \alpha) = f(\alpha) + f(-\alpha) = 0$.

Since each element of $S(f)$ is of the form (θ_α, α), where θ_α is given by (3), $S(f)$ is isomorphic to the projection of $S(f)$ into the second factor of $E_+(1) \times R$.

5.5.1 Proposition:

Let $\mu \in \mathcal{R}$, and let $\phi, \psi \in E_+(1)$ be given by $\phi(x) = x + a$, $\psi(x) = x + b$, $x \in R$. Then for any $\alpha \in R\backslash 0$, $(\phi, \psi, \alpha) \in \text{Sym}\ \mu$ iff $f = i(\mu)$ is rhythmic with beat α and pitch $(a - b)$.

Proof:

Such $(\phi, \psi, \alpha) \in \text{Sym}\ \mu$ iff for all $x, t \in R$,

$$\phi(\mu(t)(x)) = \mu(t + \alpha)(\psi(x))$$

$$\Leftrightarrow \quad \phi(x + f(t)) = x + b + f(t + \alpha)$$

$$\Leftrightarrow \quad x + f(t) + a = x + b + f(t + \alpha)$$

$$\Leftrightarrow \quad f(t) + (a - b) = f(t + \alpha)$$

$$\Leftrightarrow \quad f(t) + f(\alpha) = f(t + \alpha)$$

$$\Leftrightarrow \quad f \text{ is rhythmic.}$$

The classification of stable motions in R whose symmetry group is isomorphic to Z or to R may be pursued in the spirit indicated in section 5.4. In this context, the possibilities are severely limited, but again we regard this as outside the scope of our investigation.

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