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UNIVERSITY OF SOUTHAMPTON

FACULTY OF MATHEMATICAL STUDIES

ISOMETRIC AND TOPOLOGICAL FOLDING OF MANIFOLDS

by

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To my family

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

Doctor of Philosophy

ISOMETRIC AND TOPOLOGICAL FOLDING

OF MANIFOLDS

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Local isometries between Riemannian manifolds may be characterised as maps that send geodesic segments to geodesic segments of the same length. Isometric foldings are likewise characterised by such a property, with the difference that we use piecewise geodesic segments instead of geodesic segments. The theory of isometric foldings studies the stratification determined by the folds or singularities, and relates this structure to classical ideas of Hopf degree, volume and covering spaces.

The idea of topological folding is modelled on that of isometric folding, but in the absence of metrical structure the definition is necessarily inductive. Again a stratification by folds is obtained, and a body of theorems concerning neat foldings has been established. These theorems have a strongly algebraic flavour, and are related to certain aspects of graphs on surfaces and of covering space theory in general.

The first three chapters deal with the theory for manifolds of any dimension. In the final chapter, the special case of surfaces is examined in greater detail.

ACKNOWLEDGEMENTS

I wish to express my deepest thanks to my supervisor Professor S.A. Robertson, for his helpful guidance and valuable discussions during my course of study.

My thanks are also due to the Egyptian Government for its financial support and to my husband and my two sons for their encouragement and kind help.

CHAPTER 1

ISOMETRIC FOLDINGS

1. Review

This section gives a brief outline of previous work on isometric foldings. It is based on work by S.A. Robertson (11).

Local isometries between Riemannian manifolds may be characterised as maps that send geodesic segments to geodesic segments of the same length. Isometric foldings are likewise characterised by such a property with the difference that we use piecewise geodesic segments instead of geodesic segments, that is, a map $\phi : M \to N$, where M and N are C^{∞} Riemannian manifolds of dimensions m, n respectively, is said to be an isometric folding of M into N, iff for any piecewise geodesic path $\gamma : J \to M$, the induced path $\phi o \gamma : J \to N$ is piecewise geodesic and of the same length as γ. The set of points of M where ϕ fails to be differentiable is called the set of singularities of the isometric folding ϕ and it is denoted by $\Sigma(\phi)$. This set corresponds to the 'folds' of the map.

We denote the set of all isometric foldings of M into N by $\mathfrak{F}(M,N)$

Examples

(1.). Any local isometry $\phi : M \to N$ is an isometric folding with $\Sigma(\phi) = \emptyset$. In particular, any locally isometric covering map has this property. For instance, let k be any positive integer, and let M and N be the circles $|Z| = kR$ and $|Z| = R$ in the plane $\mathbb{C}$ of complex numbers. Define $\phi : M \to N$ by $\phi(kRe^{i\theta}) = Re^{ik\theta}$. Then ϕ is an isometric folding with no singularities.

(1.2). Let R be the real line R with the standard Riemannian structure. Let $\phi : R \to R$ be given by $\phi(x) = |x|$. Then ϕ is an isometric folding and $\Sigma(\phi)$ is the origin.

(1.3). Let $\phi : R^3 \to R^3$ be given by $\phi(x,y,z) = (|x|, |y|, |z|)$, Then ϕ is an isometric folding of R^3 into itself with respect to the standard flat structure on R^3, and $\Sigma(\phi) = \{(x,y,z) : xyz = 0\}$ is the union of the three coordinate planes.

(1.4). Let $M = N = S^2$, the unit sphere in Euclidean 3-space, and let $\phi : M \to N$ be given by $\phi(x,y,z) = (x,y,|z|)$. Then ϕ is an isometric folding and $\Sigma(\phi)$ is the great circle $x^2 + y^2 = 1$, $Z = 0$. See Figure (1.1) below.

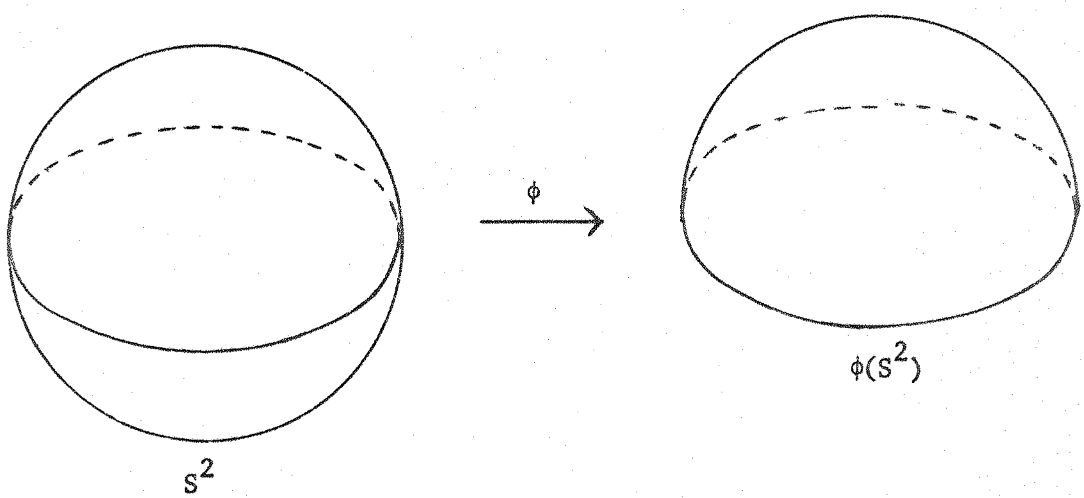


Figure (1.1)

(1.5). Let $M = N = S^2$, as above, and let $\phi : M \to N$ be given by $\phi(x,y,z) = (|x|,|y|,|z|)$. Then ϕ is an isometric folding and $\Sigma(\phi)$ is a graph consisting of the intersection of the three coordinate planes $x = 0$, $y = 0$, $z = 0$, with six vertices $(\pm 1,0,0)$, $(0,\pm 1,0)$ and $(0,0,\pm 1)$, each of valency four and with twelve edges. The image $\phi(S^2)$ is the positive octant. See Figure (1.2), the image is shaded.

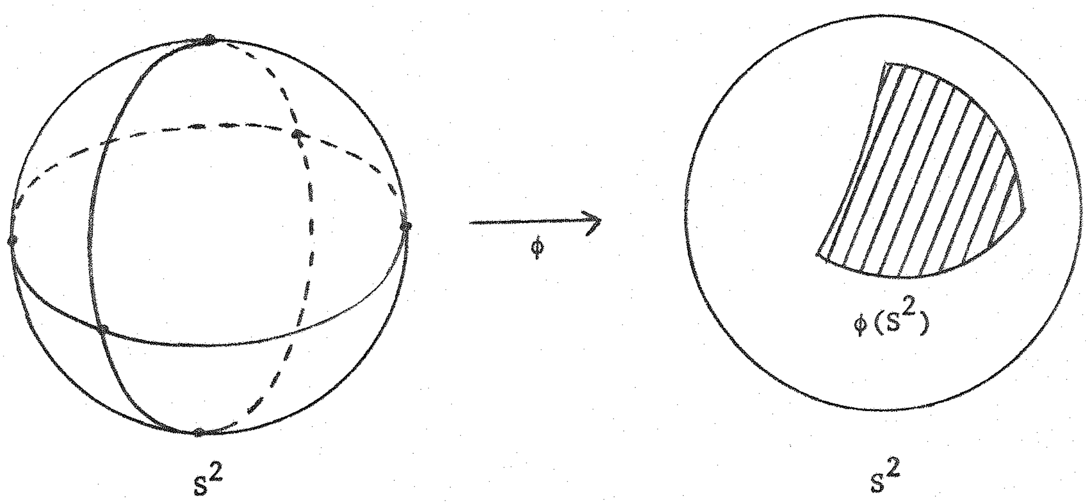


Figure (1.2)

There is no assumption about continuity or differentiability of isometric foldings. However, continuity follows from the definition, since if we denote by d_M and d_N the metrics induced on M and N by their Riemannian structures, we have for all x,y in M, $d_M(x,y) \geqslant d_N(\phi(x),\phi(y))$.

In general an isometric folding need not be differentiable. The local structure of $\Sigma(\phi)$ has been established in detail by Robertson {11}, and this may be used to build up a general picture of $\Sigma(\phi)$ for any isometric folding of M into N, where M and N are complete Riemannian manifolds, as follows:

There is a decomposition of M into mutually disjoint, connected totally geodesic submanifolds which we shall call strata, with the following properties:

(i) Let $\Sigma_k(\phi)$ denote the union of all strata of dimension k. Then $\Sigma(\phi)$ is the union of all $\Sigma_k(\phi)$ for $0 \leqslant k \leqslant m-1$;

(ii) For each stratum S, $\phi|S$ is a locally isometric immersion into N;

(iii) The frontier of each stratum is a union of strata of lower dimension, and in case M is compact, of finitely many such strata;

(iv) The frontier of $\Sigma_k(\phi)$ in M is the union of all the set $\Sigma_\ell(\phi)$ for $0 \leqslant \ell \leqslant k-1$.

Now, for simplicity, we suppose that dim M = dim N = n and that both M and N are oriented and M is compact without boundary. Thus the Hopf degree $\deg\phi$ of ϕ is well defined. This can be calculated locally as follows. Call each n-dimensional stratum S of ϕ positive or negative according as $\phi|S$ orientation-preserving or reversing, and apply the same adjectives to individual points of these strata. Now let y be a point of $\phi(M)$ that is not the image of any singularity of ϕ. Then $\phi^{-1}(y) = \{x_1, \ldots, x_p\}$, where each x_i lies in some n-dimensional stratum, i=1, ..., p. Suppose that of these p points p_+ are positive and $p_- = p - p_+$ are negative. Then $\deg \phi = p_+ - p_-$.

Next. denote by V, V_+, V_- and V_ϕ the n-volume of M, of the positive n-strata, of the negative n-strata, and of $\phi(M)$ respectively. Thus $V = V_+ + V_-$ and it may be shown that $V_+ = V_- + kV_\phi$, $k = \deg \phi$. Note that, if $\deg \phi = 0$ (which happens, for example, if ϕ is not surjective) we have, $V_+ = V_- = \frac{1}{2}V \geqslant V_\phi$.

For surfaces, the local situation is particularly simple. Let $\phi \in \mathfrak{F}(M,N)$, where both M and N are smooth Riemannian 2-manifolds (i.e. surfaces). Then, for each $x \in \Sigma(\phi)$ the singularities of ϕ near x form the images of an even number 2r of geodesic rays emanating from x, making alternate angles $\alpha_1, \beta_1, \ldots, \alpha_r, \beta_r$ where $\sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \beta_r = \pi$.

The set of singularities $\Sigma(\phi)$ of an isometric folding of a smooth Riemannian surface M into another N is a graph on M satisfying the local angle conditions and the area (2-volume) conditions described above. See example (1.5).

Finally we remark that, for any smooth Riemannian manifolds X,Y,Z,W and any isometric foldings $\phi \in \mathfrak{F}(X,Y)$, $\psi \in \mathfrak{F}(Y,Z)$, $\theta \in \mathfrak{F}(X,Z)$ and $\chi \in \mathfrak{F}(Z,W)$, we have

(i) The composite map $\psi \circ \phi \in \mathfrak{F}(X,Z)$;

(ii) $(\phi,\theta) \in \mathfrak{F}(X, Y\times Z)$, and

(iii) $\phi\times\chi \in \mathfrak{F}(X\times Z, Y\times W)$.

It follows that $\mathfrak{F}(M) = \mathfrak{F}(M,M)$ is a semigroup which contains the isometry group $I(M)$ as a subgroup. If M is compact, then for all $\phi \in \mathfrak{F}(M)$, $\deg \phi = 0, \pm 1$. Moreover, $\deg \phi = \pm 1$ iff $\phi \in I(M)$. We may topologise $\mathfrak{F}(M)$ by giving it the compact-open topology. Clearly $\deg \phi$ is constant on each component of this space. An obvious problem is to determine the number of components for each of the values ± 1 of $\deg \phi$. Is there just one component on which $\deg \phi = 0$?

2. Isometric Foldings and Covering Spaces

In this section, we shall use the term manifold to mean a smooth connected Riemannian manifold, unless otherwise stated. Liewise, we suppose that all maps are smooth.

2.1) Invariance

Let M and N be manifolds. Let $p : M \to N$ be a regular locally isometric covering. A covering transformation of p is a homeomorphism $g : M \to M$ such that $pog = p$. We denote by G the group of covering transformations of p. Since p is a regular covering of N, $G \cong \pi_1(N)/p_*\pi_1(M)$, where $p_* : \pi_1(M,x) \to \pi_1(N,P(x))$ is the homeomorphism induced by p.

2.1.1) Definition:- We say that $\phi \in \mathfrak{F}(M)$ is p-invariant iff for all $g \in G$, and all $x \in M, p(\phi(x)) = p(\phi(g.x))$.

We denote the set of all p-invariant isometric foldings of M by $\mathfrak{F}_i(M,p)$.

2.1.2) Example

Let $P_n(R)$ denote real projective n-space, consisting of the equivalence classes $\{x\}$ of points $x \in R^{n+1} \setminus \{o\}$, where x is equivalent to y iff $y = \lambda x$ for some real $\lambda \neq 0$. Define $p : S^n \to P_n(R)$ by $p(x) = \{x\}$. Thus p is the standard double covering.

Consider the isometric folding $\phi \in \mathfrak{F}(S^n)$ given by $\phi(x_1, \ldots, x_{n+1}) = (|x_1|, \ldots, |x_{n+1}|)$. Then $\phi \in \mathfrak{F}_i(S^n,p)$. Here the group G is Z_2 where Z_2 is the group generated by the reflexion, $x \to -x$.

2.1.3) Proposition

For any covering map $p : M \to N$, $\mathfrak{F}_i(M,p)$ is a subsemigroup of $\mathfrak{F}(M)$.

Proof:-

Since $1_M \in \mathfrak{F}_i(M,p)$, hence $\mathfrak{F}_i(M,p) \neq \emptyset$. Let ϕ, $\psi \in \mathfrak{F}_i(M,p)$ be p-invariant isometric foldings of M. Then for all $g \in G$ and all $x \in M$, $p((\phi o \psi)x) = p(\phi(\psi(x)) = p(\phi(\psi(g.x))) = p((\phi o \psi)(g.x))$, so $(\phi o \psi) \in \mathfrak{F}_i(M,p)$.

The next theorem establishes the relation between the set of isometric foldings of a manifold and the set of p-invariant isometric foldings of its universal covering space, where p is its universal covering.

2.1.4) Theorem

Let N be a manifold and $p : M \to N$ its universal covering. Let G be the group of covering transformations of p. Then $\mathfrak{F}(N)$ is isomorphic as a semigroup to $\mathfrak{F}_i(M,p)/G$.

Proof:-

Let $\phi \in \mathfrak{F}_i(M,p)$, and define $\bar{\phi} : N \to N$, by $\bar{\phi}(p(x)) = p(\phi(x))$, for any $x \in M$. Since ϕ is p-invariant, $\bar{\phi}$ is well-defined. For if $p(x) = p(y)$, then $\bar{\phi}(p(y)) = p(\phi(y)) = p(\phi(g.x)) = p(\phi(x)) = \bar{\phi}(p(x))$. The quotient set $\bar{\mathfrak{F}}(N) = \mathfrak{F}_i(M,p)/G$ has a semigroup structure induced by that of $\mathfrak{F}_i(M,p)$. Also since p is a local isometry, $\phi \in \mathfrak{F}(M)$ implies that $\bar{\phi} \in \mathfrak{F}(N)$. Now define a map $F : \bar{\mathfrak{F}}(N) \to \mathfrak{F}(N)$, by $F(G\phi) = \bar{\phi}$. Since $G\phi = G\psi$ iff for all $g \in G$ there is $h \in G$ such that $go\phi = ho\psi$, it follows that if $G\phi = G\psi$, then for all $g \in G$, $\bar{\phi}(p(x)) = p(\phi(g.x)) = p((\phi o g)(x)) = p(((g^{-1} \text{ o } h \text{ o}\psi)\text{o } g)(x)) = p((\psi \text{ o } g)(x)) = p(\psi(x)) = \bar{\psi}(p(x))$.

Hence F is well-defined.

Now, let $\phi, \psi \in \mathfrak{F}_i(M,p)$. Then $F(G\,\phi \text{ o } G\,\psi) = F(G\,\phi\,\psi) = (\overline{\phi \text{ o } \psi})$. But $(\overline{\phi \text{ o } \psi})(p(x)) = p(\phi(\psi(x))) = p(\phi(y))$, where $y = \psi(x)$, hence $(\overline{\phi \text{ o } \psi})(p(x)) = \bar{\phi}(p(y)) = \bar{\phi}(p(\psi(x)) = (\bar{\phi} \text{ o } \bar{\psi})(p(x))$.

Hence F is a homeomorphism.

To prove that F is one-one, let $\phi, \psi \in \mathfrak{F}_i(M,p)$ and suppose that $F(G\phi) = F(G\psi)$. Then according to the definition of F this will imply that $\bar{\phi} = \bar{\psi}$, that is, for all $x \in M$ and all $g \in G$, $\bar{\phi}\,(p(x)) = p(\phi(x)) = \bar{\psi}(p(x)) = p(\psi(x))$. It follows that $\bar{\phi} = \bar{\psi}$ iff for all $x \in M$ and all $g \in G$, $p(\phi(x)) = (p(\psi(x))$. This implies that there is $h \in G$ such that $(h\ o\phi)(x) = \psi(x)$. Hence $G\phi = G\psi$ and F is one-one.

To complete the proof, we have to show that F is onto. For this purpose, let $\beta \in \mathfrak{I}(N)$ and choose $x \in M$, $x_1 \in N$ such that $p(x) = x_1$. Now choose any $y \in M$ such that $p(y) = \beta(x_1) = y_1$, say. Then, if U and V are open neighbourhoods of x in M and x_1 in N such that $p|U$ is a homeomorphism onto V, then there is a unique map $\alpha_U : U \to M$ such that $\alpha_U(x) = y$ and $p o \alpha_U = \beta\ o(p|U)$. See Figure (1.3).

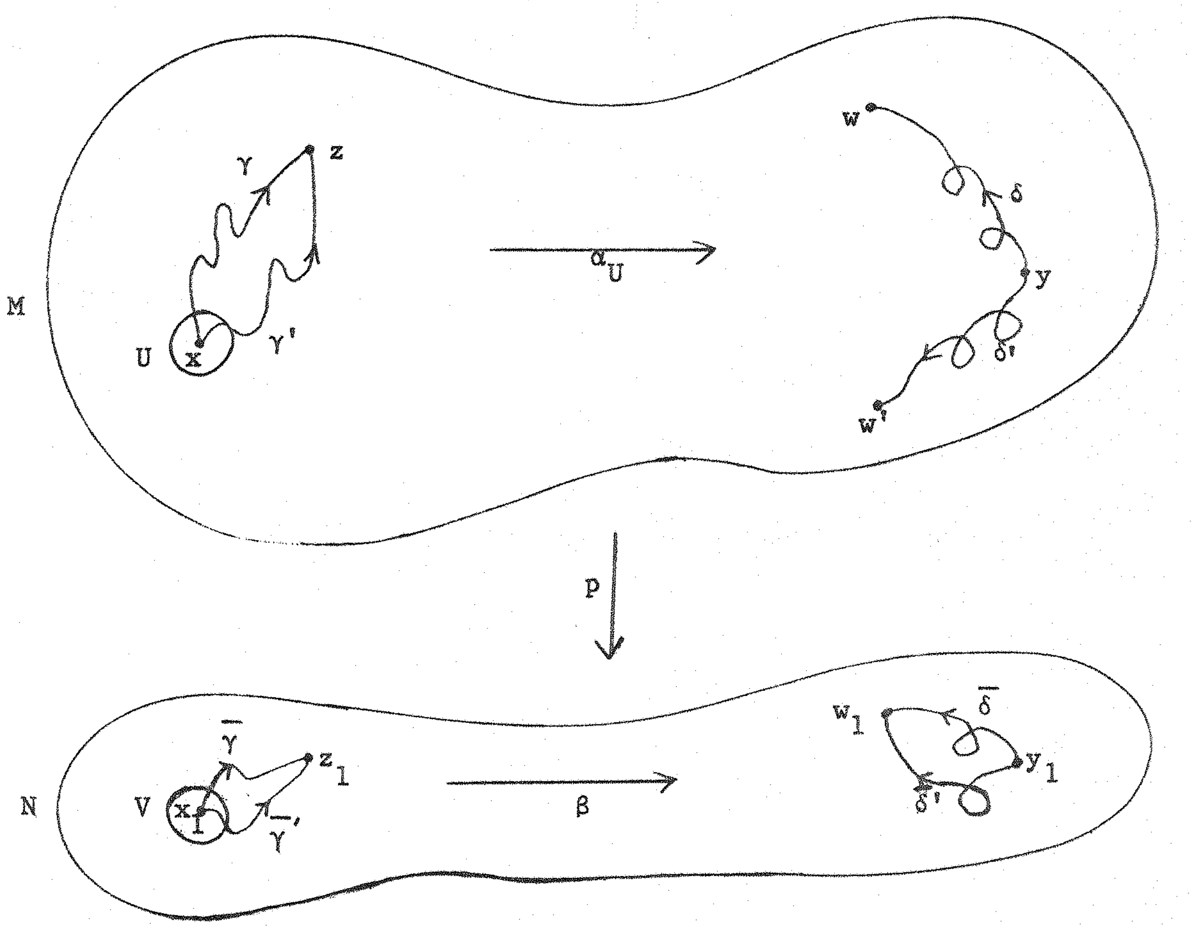


Figure (1.3)

Now, let γ and γ' be any two paths in M beginning at x and having the same end point z and such that $p \circ \gamma = \bar{\gamma}$ and $p \circ \gamma' = \bar{\gamma}'$ are two paths beginning at x_1, with the same end point z_1. Let $\bar{\delta} = \beta o(p \circ \gamma) = \beta \circ \bar{\gamma}$ and $\bar{\delta}' = \beta o(p \circ \gamma') = \beta o\ \bar{\gamma}'$ be two paths at y_1 with the same end point ω_1. Suppose that the unique path liftings δ and δ' of $\bar{\delta}$ and $\bar{\delta}'$

beginning at y have different end points ω and ω' respectively.

Since p is a universal covering of N, M is simply connected, that is every closed path in M is homotopic to a constant, it follows that $\omega = \omega'$.

Thus we can extend α_U to the regions enclosed by the paths and so to the whole of M.

A similar theorem can be obtained in the case of regular covering maps as follows.

2.1.5) Theorem

Suppose that $p : M \to N$ is a regular covering map. Suppose further that, given any points x, $y \in M$ and x_1, $y_1 \in N$ such that $p(x) = x_1$, $p(y) = y_1$ and $\beta(x_1) = y_1$, where $\beta \in \mathfrak{J}(N)$, $(\beta_* \circ p_*)\pi_1(M,x) < p_*\pi_1(M,y)$. Then $\mathfrak{J}(N)$ is isomorphic to $\mathfrak{J}_i(M,p)/G$.

Proof:-

The proof of this theorem is the same as of theorem (2.1.4) except to show that the end points ω and ω' of the paths δ and δ' respectively are the same. This can be proved as follows:

The loop $\ell_1 = \bar{\gamma}^{-1} \circ \bar{\gamma}'$ represents an element $\lambda \varepsilon\ \pi_1(N,x_1)$ and so the loop $\beta\ell_1$ represent the element $\beta(\lambda)$ of $\pi_1(N,y_1)$. But, since β_* carries the image of p_* into that of p_*, there exist a unique map $\alpha : M \to M$, such that $\alpha(x) = y$ and also the element $\beta(\lambda)$ is contained in the subgroup $p_*\pi_1(M,y)$ (7). Hence there exists a loop ℓ_2 at y such that $p\,\ell_2 = \beta\ell_1$, and it follows from the uniqueness of path lifting that $\ell_2(\frac{1}{2}t) = \delta(t)$ and $\ell_2(1 - \frac{1}{2}t) = \delta'(t)$ where $t \in I$ and $\ell_2 : I \to M$. In particular, $\delta(1) = \delta'(1) = \ell_2(\frac{1}{2})$. So δ and δ' have the same end point.

2.2 EQUIVARIANCE

Let M and N be manifolds and let $p : M \to N$ be a regular locally isometric covering. Let G be the group of covering transformations of p as before.

2.2.1) Definition :-

We say that $\phi \in \mathfrak{F}(M)$ is p-equivariant iff for all $g \in G$, the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ g \downarrow & & \downarrow g \\ M & \xrightarrow{\phi} & M \end{array}$$

We denote the set of p-equivariant isometric foldings of M by $\mathfrak{F}_e\,(M,p)$.

2.2.2) Example

Consider the infinite strip $-1 \leq y \leq 1$ in Euclidean plane R^2. Remove from this strip the discs of radius $\varepsilon > o$ and centres (n,o), where n is any integer and $1 > \varepsilon$. Let X denote the remaining closed region, shown in Figure (1.4). Let $\theta : X \to R$ be a function such

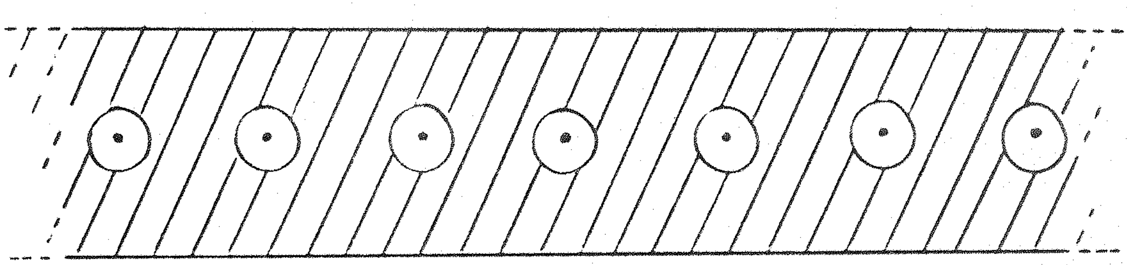


Figure (1.4)

that θ is 0 and has infinite normal derivative on ∂X, and zero derivative in the x-direction whenever $x = m + \frac{1}{2}$, for any integer m. Moreover, let $\theta(x,y) = \theta(x + m, \pm y)$ for all $m \in Z$.

Define a surface M in E^3 by

$$M = \{(x, y, z) : (x, y) \in X,\ z = \pm\theta\,(x, y)\}.$$

The group of integers Z acts freely on M by $m.(x, y, z)=(x+m, y, z)$. Let $N = M/Z$ and let $p : M \to N$ be the covering projection. Then N is homeomorphic to a double torus.

The map $\phi : M \to M$ given by $\phi(x, y, z) =(x, -y, z)$ is an isometric folding which is p-equivariant.

2.2.3) Proposition

For any covering map $p : M \to N$, $\mathfrak{F}_e(M,p)$ is a subsemigroup of $\mathfrak{F}(M)$.

Proof:-

Since $1_M \in \mathfrak{F}_e(M,p)$, $\mathfrak{F}_e(M,p) \neq \emptyset$. Now, let $\phi, \psi \in \mathfrak{F}_e(M,p)$ be p-equivariant isometric foldings of M, that is, for all $g \in G$ the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ g\downarrow & & \downarrow g \\ M & \xrightarrow[\phi]{} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \xrightarrow{\psi} & M \\ g\downarrow & & \downarrow g \\ M & \xrightarrow[\psi]{} & M \end{array}$$

are commutative.

Then for all $g \in G$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi o \psi} & M \\ g\downarrow & & \downarrow g \\ M & \xrightarrow[\psi o \psi]{} & M \end{array}$$

is also commutative. Hence, under this composition of maps $\mathfrak{F}_e(M,p)$ is a semigroup and it is a subsemigroup of $\mathfrak{F}(M)$.

Consider now the special case in which M is the unit sphere S^n. Every isometry of S^n to itself is equivariant with respect to the action of Z_2 on S^n generated by reflexion in 0. We now show that there are no other equivariant isometric foldings of S^n.

2.2.4) Lemma

Let $\phi: S^n \to S^n$ be an isometric folding such that, for all $x \in S^n$, $\phi(-x) = -\phi(x)$. Then ϕ is an isometry.

Proof:-

The result follows directly from the Borsuk-Ulam theorem (13) which can be stated : For $n \geq 1$, there is no continuous map $\phi : S^n \to S^n$ of degree zero such that, for all $x \in S^n$, $\phi(-x) = -\phi(x)$. Since any isometric folding of any compact manifold to itself has degree 1, -1 or 0, and is an isometry in either of the first two cases, the lemma is proved.

We remark that the above lemma may be stated in the form $\mathfrak{F}_e(S^n, \phi) = I(S^n)$, where $\phi(x) = -x$, $x \in S^n$.

2.3) Volume Theorems

If $\phi : M \to N$ is an isometric folding between manifolds M and N of the same dimension, and M is compact, so that the volume Vol M of M is finite, then the volume of the image $\phi(M)$ of ϕ in N cannot exceed VolM itself. The inequality Vol $\phi(M) \leq$ Vol M is an equality iff ϕ is an isometric embedding. If ϕ is k-fold covering of $\phi(M)$, then of course k vol $\phi(M)$ = Vol M.

If ϕ is not an isometric embedding, then the above inequality can be sharpened to 2 Vol $\phi(M) \leq$ Vol M. However, in certain cases Vol $\phi(M)$ is necessarily much smaller. We therefore pose the general question. For a given compact Riemannian manifold, find the infimum e(M) of the ratio Vol M/Vol $\phi(M)$, over all isometric foldings $\phi : M \to M$ of degree zero.

We have succeeded in proving only a few facts about e(M) for particular manifolds M. We cannot say, for example, whether e(M) is always an integer.

2.3.1) Lemma

Let $p : S^n \to P_n(R)$ be the double covering given by $p(x) = \{x\}$. Let $\phi \in \mathfrak{F}_i(S^n, p)$. Then either $\phi \in I(S^n)$, or for all $x, y \in S^n$, $d(\phi(x), \phi(y)) \leq \frac{\pi}{2}$.

Proof:-

Let $x, y \in S^n$, where $x \neq \pm y$. Then there is a unique great circle C such that $x \in C$ and $y \in C$. Hence $-x, -y \in C$, and these points occur on C in the cyclic order x, y, −x, −y.

Let $d(x, y) = \lambda$. Then $d(x, -y) = \pi - \lambda$. Thus $\min(\lambda, \pi-\lambda) \leq \frac{\pi}{2}$.

But $\lambda = d(x,y) \geq d(\phi(x), \phi(y))$ and $\pi - \lambda = d(x, -y) \geq d(\phi(x), \phi(-y))$.

Since for all $x \in S^n$, $p(\phi(x)) = p(\phi(-x))$, we observe that either $\phi(x) = \phi(-x)$ or $-\phi(x) = \phi(-x)$. In the later case, $\phi \in I(S^n)$ (by lemma (2.2.4)). So $\phi(x) = \phi(-x)$. Hence $\pi - \lambda \geq d(\phi(x), \phi(y))$, and so $d(\phi(x)), \phi(y)) \leq \min(\lambda, \pi - \lambda) \leq \frac{\pi}{2}$.

The above lemma allows us to give estimates for the number $e(P_n(R))$, where $P_n(R)$ has its standard Riemannian structure, as explained below.

For any isometric folding $\phi : S^n \to S^n$ such that, for all $x \in S^n$, $\phi(x) = \phi(-x)$, the image $\phi(S^n) = X$ of ϕ is a closed subset of S^n in which the geodesic (great circle arc) distance $d(x, x')$ between any two points $x, x' \in X$ is at most $\pi/2$.

Now consider the family $\mathcal{K}_n$ of all closed subsets of S^n with this property. Denote the supremum of the n-dimensional volume Vol Y over all $Y \in \mathcal{K}_n$ by M_n. Thus $M_1 = \pi/2$. However, we do not know the exact value of M_n for $n > 1$.

We now describe two members of $\mathcal{K}_n$. One is the closed geodesic disc $D_n(\pi/4)$ of radius $\pi/4$, with any centre on S^n. Then $M_n \geq \mathrm{Vol}(D_n(\pi/4)) = \Delta_n$, say. For example, $\Delta_1 = \pi/2$, $\Delta_2 = \pi(2 - \sqrt{2})$. It is tempting to conjecture that $M_n = \Delta_n$.

The second example is the 'Reuleaux' set $R_n = \{x \in S^n;\ x_i \geq 0,\ i=1, \ldots, n+1\}$, see {4} or {10}. Then $R_n \in \mathcal{K}_n$, and $\mathrm{Vol}\ R_n = \mathrm{Vol}\ S^n/2^{n+1} = e_n$, say. Thus $e_1 = e_2 = \pi/2$.

Now the map $\phi : S^n \to S^n$ given by $\phi(x_1, \ldots, x_{n+1}) = (|x_1|, \ldots, |x_{n+1}|)$ is an isometric folding such that $\phi(S^n) = R_n$. It follows that $e(P_n(R)) \leq 2^n$. Thus,

$$\mathrm{Vol}(S^n)/2\, M_n \leq e(P_n(R)) \leq 2^n.$$

Explicit formula for Vol S^n and Vol $D_n(r)$ are derived in the Appendix.

Now, let N be a compact n-manifold and let $\bar{\phi} : N \to N$ be an isometric folding of degree zero. Then, consider the universal covering space M of N with projection map $p : M \to N$, which is local isometry, and induced Riemannian metric. Let G denote the group of covering transformations of p. Then, by theorem (2.1.4), there is an isometric folding $\phi: M \to M$ such that, for all $x \in M$ and all $g \in G$, $p(\phi(x)) = p(\phi(g.x))$. Equivalently, for all $x \in M$ and for all $g \in G$, there is a unique $h \in G$ such that $h \circ \phi(x) = \phi(g.x)$.

In fact, h depends only on g, since G is finite and h varies continuously with x. Thus ϕ in the above sense is p-invariant. So there is a map $f : G \to G$ given by $f(g) = h$. In fact, f is a homomorphism. For if $g, g' \in G$ and $f(g) = h$, $f(g') = h'$, then

$$\phi((g'g).x) = \phi(g'.(g.x)) = h'.\phi(g.x) = h'h.\phi(x).$$

Also, if $\theta = G \to \pi_1 N$ denotes the isomorphism induced by p, then the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ \theta \downarrow & & \downarrow \theta \\ \pi_1 N & \xrightarrow[\bar{\phi}_*]{} & \pi_1 N \end{array}$$

commutes, so $\bar{\phi}_* \circ \theta = \theta \circ f$.

We now consider in more detail the case in which $\bar{\phi}_*$ is trivial. Suppose that $\bar{\phi}_*$ is trivial. Then so is f, and for all $x \in M$, $\phi(G.x) = \phi(x)$. Hence $p|\phi(M)$ is a homeomorphism onto $\bar{\phi}(N)$, and therefore vol $\bar{\phi}(N) = \mathrm{Vol}\,\phi(M)$.

Let F be a fundamental region for G in M. Thus F is a (non empty) closed subset of M such that $\bigcup_{g \in G} g.F = M$ and, for all $x, y \in F$, if $g \in G$ is such that $g(x) = y$, then x and y both lie in the frontier of F.

Now vol F = vol N = (1/k) vol M, where $k = |G|$, is the order of the covering p. Moreover, for all $y \in M$,

$\phi(y) = \phi(g.x)$ for some $x \in F$

$\quad = \phi(x) \in \phi(F)$.

Hence $\phi(M) = \phi(F)$. It follows that $\mathrm{vol}\phi(M) = \mathrm{vol}\phi(F)$, and we conclude that

$$\mathrm{Vol}\ N/\mathrm{Vol}\ \overline{\phi}(N) = \mathrm{Vol}\ F/\mathrm{Vol}\phi(F)\ .$$

We remark that if $\overline{\phi}_*$ is trivial, then each fibre of p is mapped by ϕ to a single point of M. In general, when $\overline{\phi}_*$ need not be trivial, ϕ is p-fibre preserving.

It seems likely that such results on volume can be strengthened considerably. A simple example of what is intended by this remark is obtained by taking N to be a circle S^1 of circumference 2Δ. Thus M is the real line R, and $p : R \to S^1$ may be given by $p(2\Delta t) = \Delta/\pi\ e^{2\pi i t}$.

Suppose that $\phi \in \mathfrak{F}_i(R,p)$ is such that $\overline{\phi}_*\ \pi_1(S^1) = 0$, where $\overline{\phi}$ is the corresponding element of $\mathfrak{F}(S^1)$. Then for all x, $y \in R$, $d(\phi(x),\phi(y)) \leqslant \Delta$. For under these assumptions, for any x, $y \in R$, there is a point $x' = x + 2\Delta m$, for some integer m, such that $d(x',y) \leqslant \Delta$, and since $\phi(x') = \phi(x)$, the result follows.

2.4 Concluding Remarks

It seems difficult to establish any general theorems about the number e(M) for an arbitrary compact manifold M. If $\partial M \neq \emptyset$, then we can also study the number $e(M, \partial M)$, where the isometric foldings concerned lie in the semigroup $\mathfrak{F}(M, \partial M)$. Clearly, $\mathfrak{F}(M, \partial M)$ is a subsemigroup of $\mathfrak{F}(M)$. Also, there is a homeomorphism $\theta : \mathfrak{F}(M, \partial M) \to \mathfrak{F}(\partial M)$ given by $\theta(\phi) = \phi|\ \partial M$.

Even in the case of surfaces, these problems seem quite difficult. For instance, let M be the flat Möbius band represented by the rectangle in Euclidean plane R^2 with vertices $(\pm a, \pm b)$, in which the edges joining $A = (-a, -b)$ to $B = (-a, b)$ and $C = (a, b)$ to $D = (a, -b)$ are identified with the direction reversal as shown in Figure (1.5).

The map $\phi_{\#}$: $M \to M$ induced by the map $\phi: R^2 \to R^2$, given by $\phi(x,y) = (|x|, |y|)$ is such that $\phi \varepsilon \mathfrak{F}(M, \partial M)$ and 4 vol $\phi(M)$ = vol M = 4 ab. Thus $e(M, \partial M) \leq 4$.

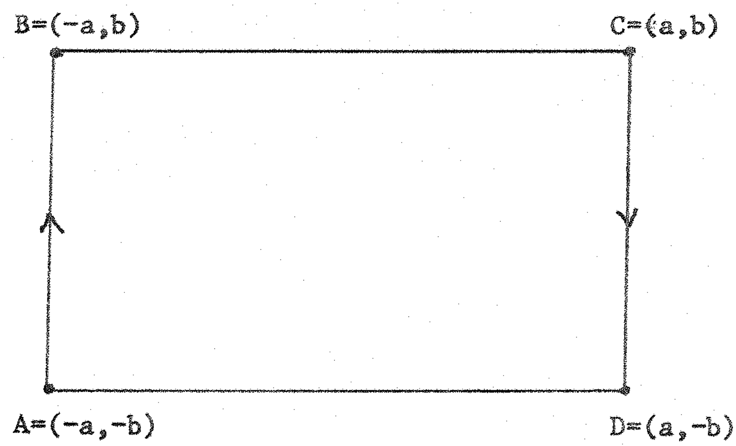


Figure (1.5)

We have failed both to construct any $\psi \varepsilon e(M, \partial M)$ of degree 0 such that vol $\psi(M) > ab$, and to prove that $e(M, \partial M) = 4$.

It is clear that the concept of isometric folding may be extended with only trivial modification to pseudo-riemannian manifolds. There are, however, considerable difficulties in attempting to find analogues of the preceding theorems. The partial results that have been obtained so far are not reported in detail here. A first step is to establish the precise relationship between the isometric foldings of Minkowski (n+1)-space M^{n+1} to itself that keep o fixed and the isometric foldings of the 'positive spheres' $S(\alpha)$ given by

$$x_1^2 + \dots + x_n^2 = x_{n+1}^2 + \alpha, \; x_{n+1} > 0, \; \alpha \neq 0.$$

We can show that the hypersurfaces $S(\alpha)$ carry induced Riemannian metrics (positive definite in case $\alpha > 0$, negative definite in case $\alpha < 0$), and there are natural isomorphisms $\mathfrak{F}(S(\alpha)) \to \mathfrak{F}(S(1)), \mathfrak{F}(S(\beta)) \to \mathfrak{F}(S(-1))$ for $\alpha > 0, \beta < 0$ induced by radial homothety.

The next step is to establish whether $\mathfrak{F}(S(\alpha))$ embeds naturally in $\mathfrak{F}_o(M^{n+1})$ where $\mathfrak{F}_o(M^{n+1})$ denotes the semigroup of isometric foldings of M^{n+1} that fix 0. To carry out this step would lead naturally to a theory of isometric foldings for Lorentz manifolds and perhaps to a theory for pseudoriemannian manifolds in general.

CHAPTER 2

TOPOLOGICAL FOLDINGS

The theory described in Chapter 1 made essential use of the Riemannian structure. We now construct a more general theory of a purely topological character. To achieve this, we abandon the definition of isometric folding, which has no obvious analogue in the topological case, and instead we adopt an inductive procedure. We restrict attention to the case in which domain and codomain have the same dimension.

1. Manifolds Without Boundary

We define the following standard subsets of Euclidean n-space E^n for any $n > o$:

$$D^n = \{x \in E^n : |x| \leqslant 1\};$$

$$S^{n-1} = \{y \in E^n : |y| = 1\}.$$

We call D^n and S^{n-1} the unit disc and the unit sphere in Euclidean n-space respectively. Thus $S^{n-1} = \partial D^n$. It follows from the definition that for each $x \in D^n$ with $x \neq 0$, there is a unique real number t and a unique point $y \in S^{n-1}$ such that $x = t\,y$, $o < t \leqslant 1$. Of course for all $y \in S^{n-1}$, $0 = 0y$. See figure (2.1).

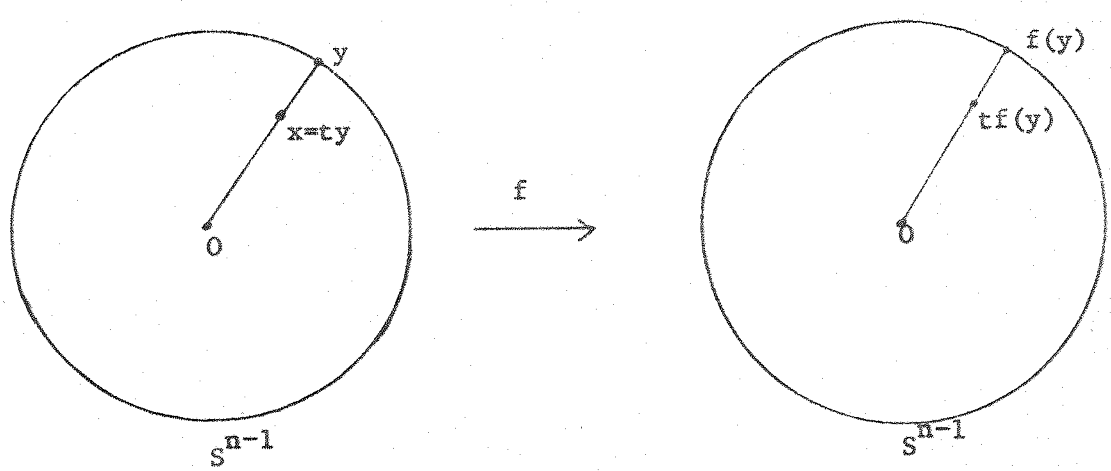


Figure (2.1)

Now suppose that $f : S^{n-1} \to S^{n-1}$ is any map. Then f induces a map $f_* : D^n \to D^n$, given by $f_*(tx) = tf(x)$ where $0 \leq t \leq 1$, $x \in S^{n-1}$ and $f_*(o) = 0$.

By using this construction we can define a topological folding by the following induction. Let M and N be topological manifolds. Where $\dim M = \dim N = n > o$ and $\partial M = \partial N = \emptyset$. For all $x \in M$, a disc chart at x is a homeomorphism $\xi : D^n \to V_x$, where V_x is a neighbourhood of x in M and $\xi(o) = x$. Hence every $x \in M$ has a disc chart.

Now let $\phi : M \to N$ be a continuous map. We say that ϕ is a topological folding of M into N iff, for each $x \in M$, there are disc charts $\xi : D^n \to V_x$ for M at x and $\eta : D^n \to W_y$ for N at $y = \phi(x)$ together with a topological folding $f : S^{n-1} \to S^{n-1}$ such that $\eta \circ f_* = \phi \circ \xi$.

$$\begin{array}{ccc} D^n & \xrightarrow{\;f_*\;} & D^n \\ \xi \big\downarrow & & \big\downarrow \eta \\ V_x & \xrightarrow[\;\phi|V_x\;]{} & W_y \end{array}$$

To complete the definition we say that any map $f : S^o \to S^o$ is a topological folding. Since S^o consists of the two real numbers $1, -1$, there are exactly four topological foldings of S^o to itself. We denote by $\mathfrak{J}(M,N)$ the set of all topological foldings of M into N, and put $\mathfrak{J}(M) = \mathfrak{J}(M,M)$.

If $\phi \in \mathfrak{J}(M,N)$, then $x \in M$ is said to be a singularity of ϕ iff ϕ is not a local homeomorphism at x. The set of all singularities of ϕ is denoted by $\Sigma(\phi)$.

2. Foldings of 1-Manifolds

2.1) Proposition

Let $\phi \in \mathfrak{J}(M,N)$, where M and N are 1-manifolds without boundary. Then $\Sigma(\phi)$ is a discrete subset of M.

Proof:-

Let $x \in M$ and $y = \phi(x)$. Then there are disc charts $\xi : I \to V_x$, $\eta : I \to W_y$ on M and N respectively, and a topological folding $f : S^o \to S^o$ such that $\eta \circ f_* = \phi \circ \xi$, where $I = \{-1, 1\} = D^1$. Now suppose that $x \in \Sigma(\phi)$. Then $f(1) = f(-1) = \pm 1$, say $f(1) = f(-1) = 1$. Then $f_*(t) = |t|$. Hence ϕ is a local homeomorphism on $V_x \setminus \{x\}$. Hence x is an isolated point of $\Sigma(\phi)$, and so $\Sigma(\phi)$ is discrete.

2.2) Corollary

Let $\phi \in \mathfrak{J}(M,N)$. If $M \approx R$, then $\Sigma(\phi)$ is countable. If $M \approx S^1$, then $\Sigma(\phi)$ is finite and $\# \Sigma(\phi)$ is even.

Proof:-

The first statement follows immediately from the proposition. Suppose then that $M \approx S^1$, and let $x \in \Sigma(\phi)$, then there are disc charts $\xi : I \to S^1$, $\eta : I \to S^1$ such that $\xi(o) = x$, $\eta(o) = \phi(x) = y$ and $\phi \circ \xi = \eta o f_*$, where $f_* : I \to I$ is given by $f_*(t) = |t|$. Hence f_* induces orientations on rays $I_-(o < t < 1)$ and $I_+ (-1 < t < o)$ and hence local opposite orientations on $\xi(I_-)$ and $\xi(I_+)$. These local orientations can be chosen so that each region has a unique orientation induced by disc charts. This shows that the singularities of ϕ partition S^1 into arcs in such a way that successive arcs have opposite orientations. Thus the number of arcs is even, and so the number of singularities is also even.

In contrast isometric foldings, topological foldings of S^1 to itself can be of any degree. For example, the power map $\phi_k : S^1 \to S^1$ given by $\phi_k(e^{i\theta}) = e^{ik\theta}$, is a topological folding (without singularities), for any $k \neq 0$.

3. Foldings of Surfaces

Consider now any topological folding $\phi \in \mathcal{J}(M,N)$, where M and N are connected surfaces without boundary. The disc charts provide local models for the set of singularities $\Sigma(\phi)$, as follows. Let $f : S^1 \to S^1$ be a topological folding. Then $\Sigma(f)$ consists of 2k points $p_1, \ldots, p_{2k}$. Hence $\Sigma(f_*)$ consists of the rays joining each p_i to 0. That is $\Sigma(f_*) = \{t\, p_i : 0 \leqslant t \leqslant 1,\ i=1, \ldots, 2k\}$.

It follows that the set $\Sigma(\phi)$ has the structure of a locally finite graph K_ϕ embedded in M, for which every vertex has even valency.

A connected subset of $M \backslash K_\phi$ is called a <u>ϕ-region</u>. We note that the ϕ-regions, together with the edges and vertices of K_ϕ constitute a topological stratification of M.

Any isometric folding of a surface M to another N is an example of a topological folding.

Note also that if M is compact, then K_ϕ is finite and the number of ϕ- regions is finite. Moreover, every ϕ-region is bounded by a closed polygon in K_ϕ.

4. Foldings of Manifolds

From the previous two sections, we can begin to form a picture of how the structure of $\Sigma(\phi)$ may be described, for any $\phi \in \mathcal{J}(M,N)$, where M and N are topological n-manifolds without boundary. We proceed inductively as in the case of isometric foldings, and conclude that $\Sigma(\phi)$ partitions M into disjoint strata that fit together to form a topological stratification S of M. We refer to the r-dimensional strata as <u>r-strata</u>, and to the n-strata as <u>ϕ-regions</u>. This stratification is locally finite and, if M is compact, is finite. Again, any isometric folding is a topological folding.

We observed in chapter 1 that if $\phi \in \mathfrak{J}(X,Y)$ and $\psi \in \mathfrak{J}(Z,W)$, then $\phi \times \psi \in \mathfrak{J}(X\times Z, Y\times W)$. Likewise, if $\phi \in \mathfrak{J}(M,N)$ and $\psi \in \mathfrak{J}(P,Q)$, then $\phi \times \psi \in \mathfrak{J}(M\times P, N\times Q)$. Also, it is easy to check that

$$\Sigma(\phi \times \psi) = (\Sigma(\phi)\times P) \cup (M \times \Sigma(\psi)).$$

For example, let $\phi \in \mathfrak{J}(I)$ and $\psi \in \mathfrak{J}(S^1)$ be the topological foldings given by for all $x \in I$, $\phi(x) = |x|$ and for all $(y, z) \in S^1$, $\psi(y,z) = (y, |z|)$. Then $\phi \times \psi \in \mathfrak{J}(I \times S^1)$. The set $\Sigma(\phi \times \psi)$, and its relation to $\Sigma(\phi)$ and $\Sigma(\psi)$, is indicated in Figure (2.2).

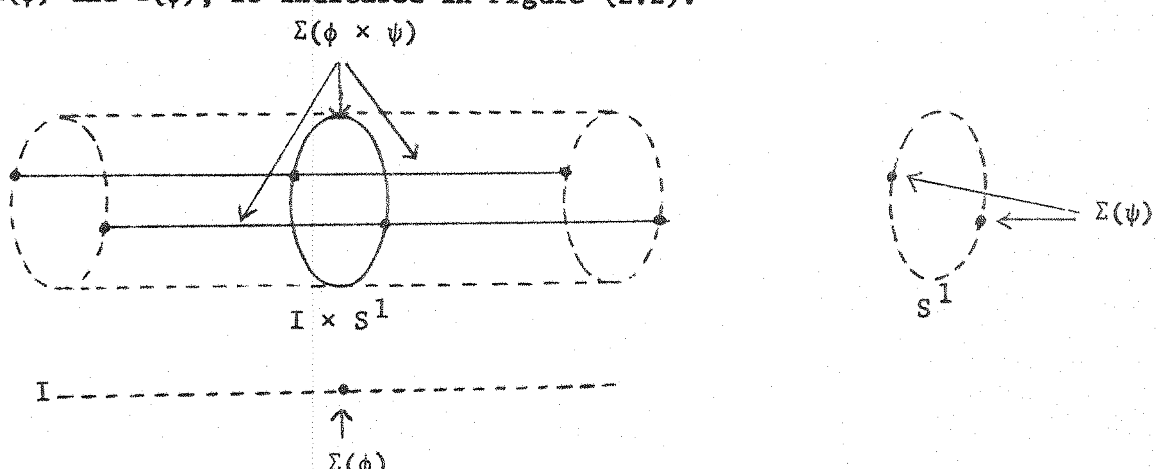


Figure (2.2)

However, the composite of any two topological foldings is not in general a topological folding. We give an example to illustrate the phenomenon.

Let $\phi : S^2 \to S^2$ be given by $\phi(x, y, z) = (x, y, |z|)$. Then $\phi \in \mathfrak{J}(S^2)$, the image of this topological folding being the 'Northern' hemisphere H. Let η be an embedding of the equator $z=0$ of S^2 into S^2, given by $\eta(x,y,o) = (x,y, \varepsilon x \sin \frac{1}{x})$, where $o < \varepsilon < 1, x \neq 0$ and $\eta(0,y,0) = (0,y,0)$. By the Schoenflies theorem, since $\eta : \partial H \to S^2$ is a topological embedding, η extends to a homeomorphism $\bar{\eta} : S^2 \to S^2$. Let

$\psi = \bar{\eta} o \phi$. Then $\psi \in \mathfrak{J}(S^2)$. But $\phi \circ \psi \notin \mathfrak{J}(S^2)$, since $\Sigma(\phi \circ \psi)$ has infinitely many strata.

We observe that for any $\phi \in \mathfrak{J}(M,N)$ and for each stratum $\sigma \in S$, $\phi|\sigma$ is a topological immersion of σ in N. Suppose now that $\psi \in \mathfrak{J}(N,P)$ is a topological folding. Then $\psi o \phi$ will be a topological folding if for each stratum $\sigma \in S$, where S in the topological stratification induced by ϕ on M, $\phi(\sigma)$ is topologically transverse to each stratum of ψ. This condition is not, however, necessary.

5. Manifolds with Boundary

Let M and N be topological manifolds, where dim M = dim N = n > 0, and $\partial M = \partial N \neq \emptyset$. For all $x \in \text{Int } M$ (Int M means interior of M), a disc chart at x can be defined as before. If $x \in \partial M$ a disc chart at x is a homeomorphism $\bar{\xi} = \bar{D}^n \to \bar{V}_x$, where $\bar{V}_x$ is a half disc heighbourhood of x in M, $\bar{D}^n = \{x \in E^n : |x| \leq 1, x_n \geq o\}$ and $\xi(o) = x$. Hence every $x \in M$ has a disc chart.

Now, let $\phi: M \to N$ be a continuous map. We say that ϕ is a <u>topological folding of M into N</u> iff for each $x \in M$, there are disc charts $\xi : D^n \to V_x$ or $\bar{\xi} : \bar{D}^n \to \bar{V}_x$ for M at $x \in \text{Int } M$ or $x \in \partial M$ respectively, and $\eta : D^n \to W_y$ or $\bar{\eta} : \bar{D}^n \to \bar{W}_y$ for N at $y = \phi(x) \in \text{Int } N$ or $y = \phi(x) \in \partial N$, togehter with one of the following topological foldings:

(i) $f : S^{n-1} \to S^{n-1}$ such that $\eta o f_* = \phi o \xi$ ($x \in \text{Int } M$ and $y \in \text{Int } N$);

(ii) $\bar{f} : \bar{S}^{n-1} \to \bar{S}^{n-1}$, where $\bar{S}^{n-1} = \{x \in E^n : |x| = 1, x_n \geq o\}$,
such that $\bar{\eta} \circ \bar{f}_* = \phi o \bar{\xi}$ ($x \in \partial M$ and $y \in \partial N$);

(iii) $f_1 : S^{n-1} \to \bar{S}^{n-1}$ such that $\bar{\eta} o f_{*1} = \phi o \xi$ ($x \in \text{Int } M$ and $y \in \partial N$);

(iv) $f_2 : \bar{S}^{n-1} \to S^{n-1}$ such that $\eta o f_{*2} = \phi o \bar{\xi}$ ($x \in \partial M$ and $y \in \text{Int} N$).

Again we say that any map $f: S^o \to S^o$, $\bar{f} : \bar{S}^o \to \bar{S}^o$, $f_1 : S^o \to \bar{S}^o$ or $f_2 : \bar{S}^o \to S^o$ is a topological folding.

These definitions imply immediately that : If $\phi \in \mathfrak{J}(M,N)$ is a topological folding of M onto N where $\partial M = \partial N \neq \emptyset$, and $\phi(\partial M) \subset \partial N$, then $\phi|\partial M \in \mathfrak{J}(\partial M, \partial N)$.

As before, any such topological folding determines a stratification S on M in which each stratum is a manifold without boundary, and S restricts to a stratification ∂S on ∂M. In constructing this stratification we have considered points in ∂M separately. Thus the set $\Sigma(\phi)$ of singularities of ϕ is a proper subset of the union of the strata of dimension $\leq m-1$. This is becuase the $\phi|\partial M$-regions of ∂M are $(m-1)$-strata in S but ϕ is not singular on these strata.

6. The Graph of a Topological Folding

Let $\phi \in \mathfrak{J}(M,N)$. Then, as we saw in §(2.4) there is a topological stratification S on M by singularities of ϕ. In this section we show that there is a graph Γ_ϕ associated to this stratification in a natural way. In fact the vertices of Γ_ϕ are just the n-strata of S, and its edges are the (n-1)-strata. If $E \in S_{n-1}$, then E lies in the frontiers of exactly two n-strata $\sigma, \sigma' \in S_n$. We then say that E is an edge in Γ_ϕ with end points σ, σ'.

The graph Γ_ϕ can be realised as a graph $\tilde{\Gamma}_\phi$ embedded in M, as follows. For each n-stratum $\sigma \in S_n$, choose any point $\tilde{\sigma} \in \sigma$. If $\sigma, \sigma' \in S_n$ are end-points of $E \in S_{n-1}$, then we can join $\tilde{\sigma}$ to $\tilde{\sigma}'$ by an arc $\tilde{E}$ in M that runs from $\tilde{\sigma}$ through σ and σ' to $\tilde{\sigma}'$, crossing E transversely at a single point. Trivially, the correspondence $\sigma \to \tilde{\sigma}$, $E \to \tilde{E}$ is a graph isomorphism from Γ_ϕ to $\tilde{\Gamma}_\phi$. Figure (2.3) below illustrate this relationship in case n=2.

In this case, the cell complex subdivision of the surface M induced by $\tilde{\Gamma}_\phi$ is the dual of that induced by K_ϕ. These constructions have a greater significance in the case of neat foldings, as we show in the next chapter.

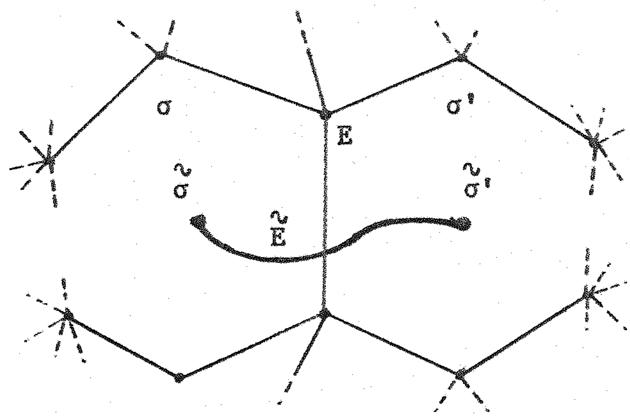


Figure (2.3)

It should be noted that the graph Γ_ϕ may have more than one edge joining a given pair of vertices. For instance, consider the topological folding ϕ of the torus T into itself shown in Figure (2.4) below, induced by the map $\phi : R^3$ given by $\phi(x,y,z) = (x,y,|z|)$. The graph Γ_ϕ has just two vertices but has two edges. See figure (2.4).

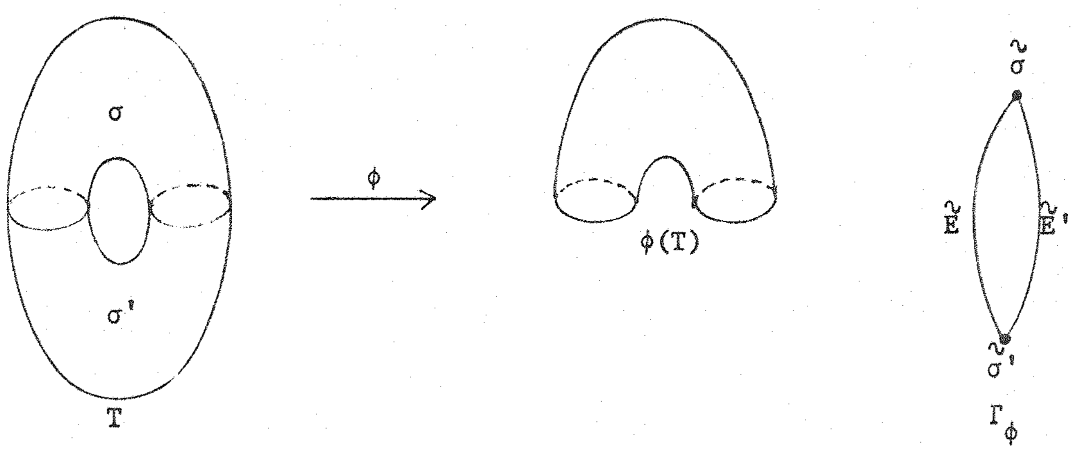


Figure (2.4)

CHAPTER 3

NEAT FOLDINGS

We have seen that any topological folding $\phi : M \to N$ determines a topological stratification S on M. However, no such stratification is induced on N itself. In this chapter, we consider a special class of foldings $\phi : M \to N$ for which N does have a stratification related to the folding ϕ.

1. Definitions and Examples

Let $\phi : M \to N$ be a topological folding and let S be the topological stratification on M whose starta are the singularity manifolds of ϕ. We denote the union of the strata of codimension j by Σ_j. The set of i-dimensional strata in S denoted by S_i.

We say that ϕ is a neat folding iff there is a topological stratification S' on N such that Σ'_o consists of the single n-stratum Int N and for each i-stratum $\sigma \in S_i$, $\phi(\sigma) \in S'_i$, $i = 0, \ldots, n$. It will be noticed that for any neat folding $\phi : M \to N$, $\phi(\partial M) \subset \partial N$. In fact $\phi(\partial M \cup M_{n-1}) = \partial N$, where $M_k = \bigcup_{j \leqslant k} S_j$ and $M_n = M$.

We denote the set of all neat foldings from M to N by $\mathcal{N}(M,N)$. For any neat folding $\phi \in \mathcal{N}(M,N)$, the number of ϕ- regions of M is called the index of ϕ and the number $\# \phi^{-1}(y)$ of points in the inverse image of any $y \in$ Int N is called the order of ϕ. A neat folding of order r is called a neat r-folding. The order of any ϕ-region A is the number of points in $\phi^{-1}(y) \cap A$ for any $y \in$ Int N. Thus if ϕ has index k and its regions are $A_1, \ldots A_k$ of orders $\alpha_1, \ldots, \alpha_k$, then the order of ϕ is $\alpha = \alpha_1 + \ldots + \alpha_k$.

We denote the order of $\phi \in \mathcal{N}(M,N)$ by $w(\phi)$ and its index by $i(\phi)$.

Examples

(1.1). Let $p : M \to N$ be any covering map. Then p is a neat folding without singularities. Also, the composite of any covering map $p : M \to N$ and any neat folding $\phi : N \to Q$ is a neat folding, and if the orders of the covering map and the neat folding are s and r respectively, then the order of

$\psi = \phi o p$ is rs.

(1.2) Let $M = R^2$, and let N be the submanifold of R^2 given by the inequalities $x \geq 0$, $y \geq 0$. Define $\phi : M \to N$ by $\phi(x,y) = (|x| , |y|)$. Then ϕ is a neat folding in which the strata of N are the sets given by:

i) $0 < x < \infty$, $y > 0$;

ii) $x = 0$, $y > 0$;

iii) $y = 0$, $x > 0$;

iv) $x = 0$, $y = 0$.

See Figure (3.1).

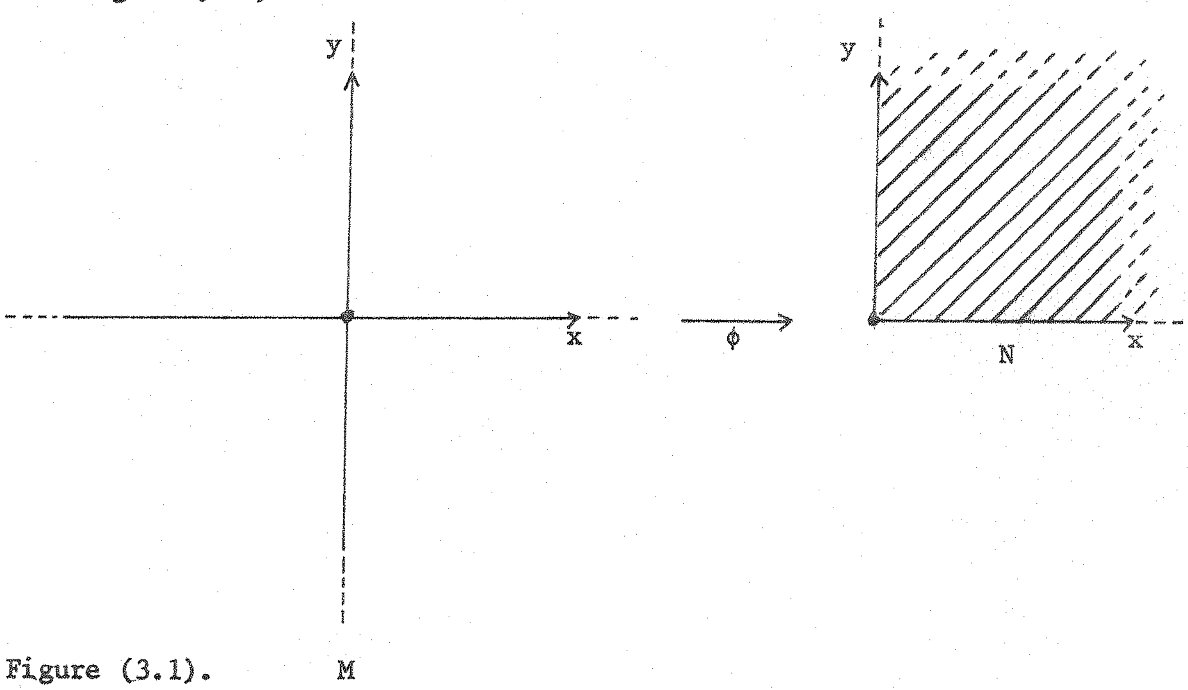


Figure (3.1).

The stratification S on M has only one 0-stratum $\{(0,0)\}$, and has four 1-strata consisting of the four half-axes obtained by removing the origin from the coordinate axes. There are four 2-strata consisting of the open quadrants into which the axes divide R^2.

The set $\Sigma(\phi)$ of singularities of ϕ is the union of the two coordinate axes.

(1.3) Let $M = D^2 = \{(x,y) \in R^2 : x^2 + y^2 \leqslant 1\}$ be the closed unit disc in the plane R^2 with centre (0,0). Let N be the submanifold of D^2 given by the inequalities $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ and $(x,y) \in D^2$. Define $\phi : M \to N$ by $\phi(x,y) = (|x|, |y|)$. Then ϕ is a neat folding. The stratification S of M induced by ϕ consists of five 0-strata, eight 1-strata (four open line segments and four open circular arcs), and four open disc quadrants, as indicated in Figure (3.2).

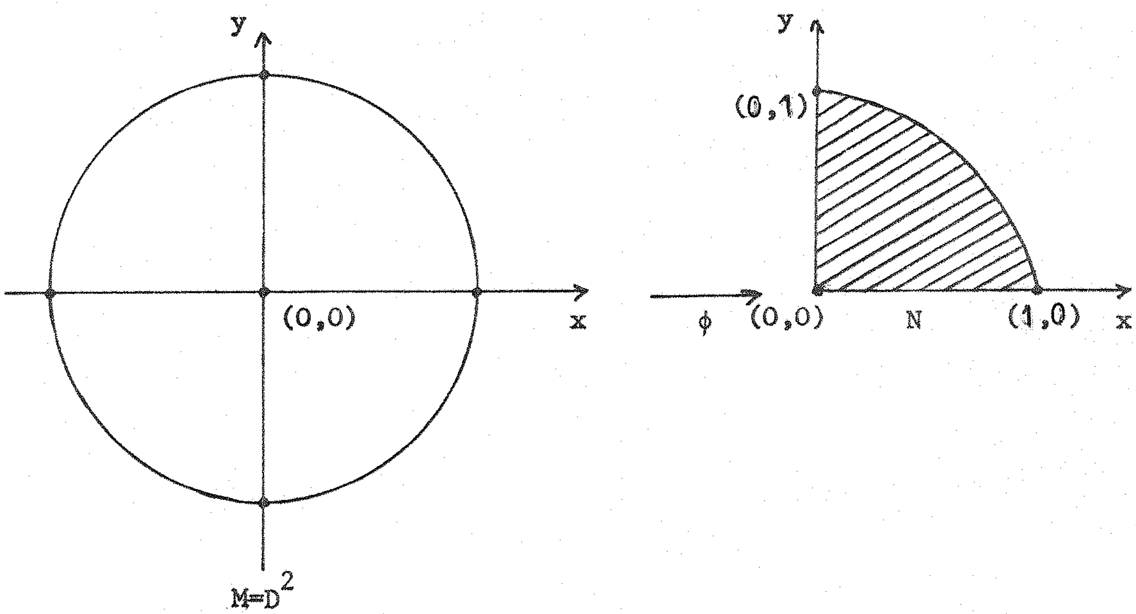


Figure (3.2)

The corresponding stratification S' of N is made up three 0-strata, three 1-strata (two open line-segments and open circular arc), and a single 2-stratum.

Notice that the four circular arcs of ∂M are 1-strata of S but do not lie in the set $\Sigma(\phi)$ of singularities. Thus $\Sigma(\phi)$ consists of the points $(x,0)$ for $-1 \leqslant x \leqslant 1$ and $(0,y)$ for $-1 \leqslant y \leqslant 1$.

(1.4) Let M be the closed half-space $\{(x,y) \in R^2 : y \geqslant 0\}$, and let $N = \{(x,y) \in R^2 : y \geqslant 0,\ 0 \leqslant x \leqslant \frac{1}{2}\}$. Define $\phi : M \to N$ by $\phi(x,y) = (f(x),y)$ where $f(x) = \min_{n \in Z} |x-n|$. Then ϕ is a neat folding in which the strata of N are the sets given by:

i) $0 < x < \frac{1}{2}$, $y > 0$;

ii) $x = 0$, $y > 0$;

iii) $x = \frac{1}{2}$, $y > 0$;

iv) $x = 0$, $y = 0$;

v) $x = \frac{1}{2}$, $y = 0$;

vi) $0 < x < \frac{1}{2}$, $y = 0$.

The set $\Sigma(\phi)$ of singularities of ϕ and the boundary strata of M is composed of:

(i) $-\infty < x < \infty$, $y = 0$;

(ii) $x = n$, $n = 0, \pm\frac{1}{2}, \pm 1, \ldots, \pm\frac{k}{2}, \ldots$, $y > 0$.

See Figure (3.3).

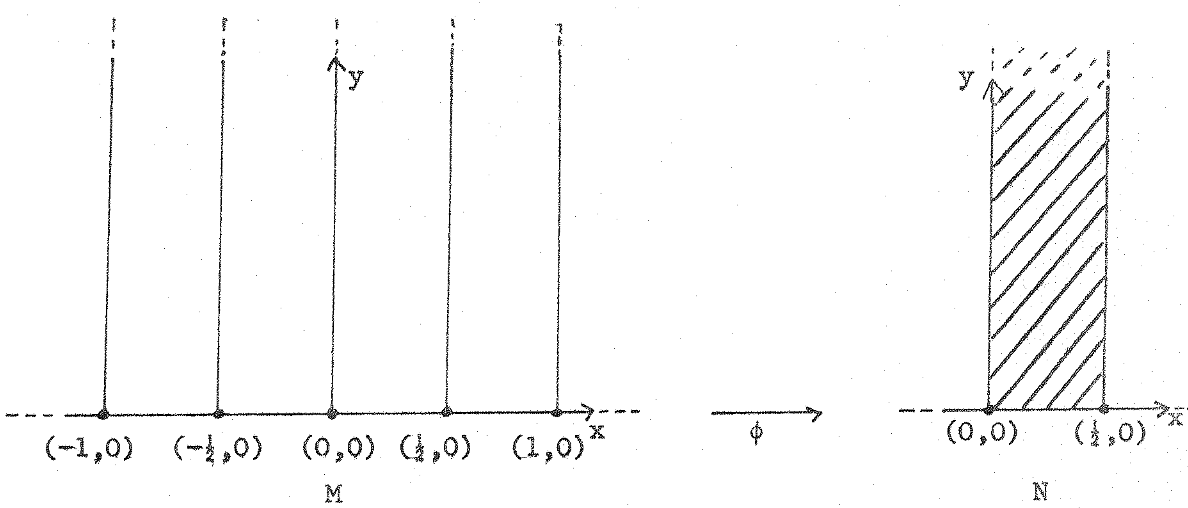


Figure (3.3)

(1.5) Let $M = R^2$, $N = \{(x,y) \in R^2 : 0 \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}\}$.
Let $\phi : M \to N$ be given by $\phi(x,y) = (f(x), f(y))$, where the map f is defined as in example (1.4). Then ϕ is a neat folding from M to N and the strata S' of N consists of four 0-strata (the points (0,0), (0,½), (½,0) and (½,½)), four 1-strata (the four open segments $x = 0$, $x = \frac{1}{2}$ where $0 < y < \frac{1}{2}$ and $y = 0$, $y = \frac{1}{2}$ where $0 < x < \frac{1}{2}$), a single 2-stratum $0 < x < \frac{1}{2}$, $0 < y < \frac{1}{2}$. See Figure (3.4). The set of singularities

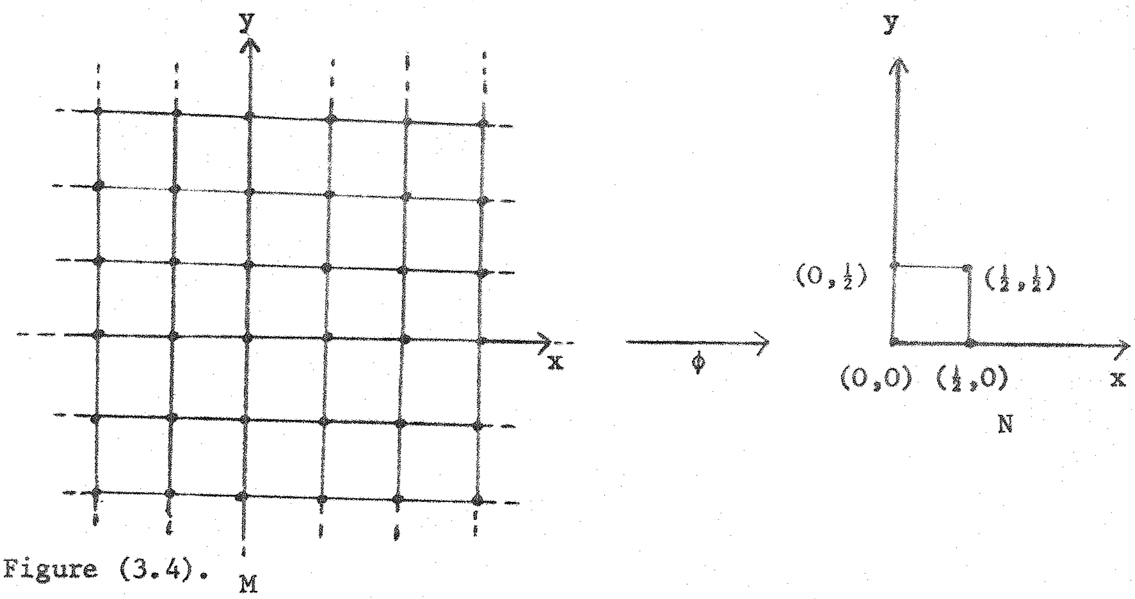


Figure (3.4).

of ϕ is composed of :

(i) $x = n$, $n = 0, \pm\frac{1}{2}, \ldots, \pm\frac{k}{2}, \ldots,$ and $-\infty < y < \infty$

(ii) $-\infty < x < \infty$ and $y = n$, $n = 0, \pm\frac{1}{2}, \pm 1, \ldots, \pm\frac{k}{2}, \ldots$.

(1.6). Let $M = T^2$ be a torus obtained from the square $Q = \{(x,y) \in R^2 : -1 \leq x \leq 1, -1 \leq y \leq 1$ by identifying opposite sides, so that the points $(1,y)$ and $(-1,y)$ are to be identified for $-1 \leq y \leq 1$ and the points $(x,1)$ and $(x,-1)$ are to be identified for $-1 \leq x \leq 1$. See Figure (3.5).

Let $N = \{(x,y) \in R^2 : 0 \leq x \leq 1, 0 \leq y \leq 1$. Define a map $\phi : M \to N$ by $\phi(x,y) = (|x|,|y|)$. Then ϕ is a neat folding. The stratification S of M consists of four 0-strata, eight 1-strata (open line-segments) and four 2-strata.

The corresponding stratification S' on N is made up four 0-strata, four 1-strata and a single 2 stratum.

It should be noted that the eight 1-strata of $S\varepsilon\partial M$ do not lie in $\Sigma(\phi)$. Thus $\Sigma(\phi)$ consists of the points (x,o) for $-1 \leq x \leq 1$ and (o,y) for $-1 \leq y \leq 1$.

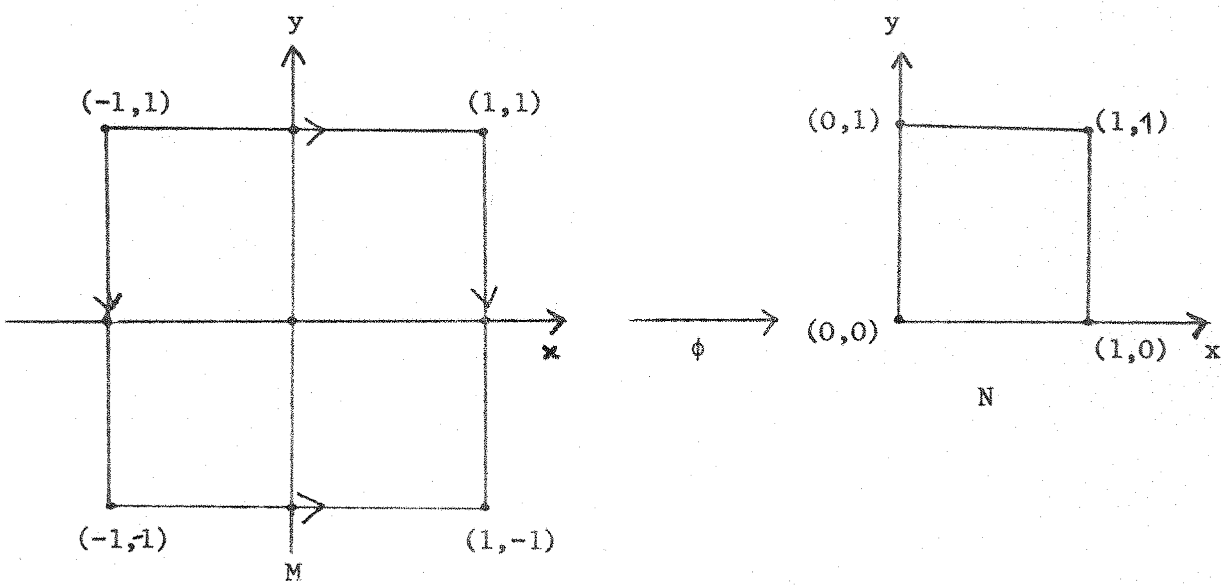


Figure (3.5).

(1.7). Let N be a double torus consisting of a sphere with two handles embedded in R^3 with a plane of symmetry as shown in Figure (3.6). Let C be a generator of one of these handles, see Figure (3.6). Cut N along C, so that N becomes a surface with two boundary curves C' and C".

Take two copies $N_+ = N \times \{1\}$ and $N_- = N \times \{-1\}$ of this surface, and a fourth surface M by identifying $(c,1) \in C' \times \{1\}$ with $(c,-1) \in C'' \times \{-1\}$ and $(c,1) \in C'' \times \{1\}$ with $(c,-1) \in C' \times \{-1\}$. Then M is a closed surface of genus 3, and if we write $\{x,j\}$ for the point of M obtained from $(x,j) \in N_+ \cup N_-$, then there is a map $p : M \to N$ given by $p\{x,j\} = x$. This is a 2-fold covering map.

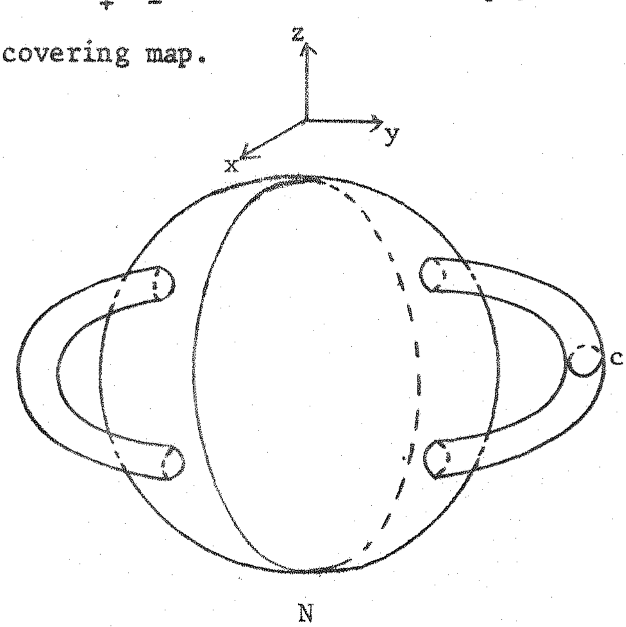


Figure (3.6).

Let us now compose p with a neat folding $\phi : N \to M$ given by $\phi(x,y,z) = (x, |y|, z)$. Then $\psi = p o \phi$ is a neat 4-folding with stratification S on M consisting of no 0-strata, two 1-strata (two simple closed curves) and three 2-strata. The neat folding ψ has index 3 and the regions have orders 1, 1 and 2.

The corresponding stratifications S' on $\phi(N)$ consists of no 0-strata, one 1-stratum and one 2-stratum. The set $\Sigma(\phi)$ of singularities consists of the two 1-strata of M.

For any $\phi \in \mathcal{N}(M,N)$, and for any $\sigma \in S, \phi|\sigma$ is a covering of $\sigma' = \phi(\sigma)$. From now on we denote the ϕ-regions by the letters A,B, ... We say that a neat folding $\phi \in \mathcal{N}(M,N)$ is <u>simple</u> or is a <u>simple folding</u> iff $\phi|A$ is a homeomorphism onto Int N, for each ϕ-region A of M, that is to say $\phi \in \mathcal{N}(M,N)$ is simple iff each ϕ-region is of order 1.

We denote <u>the set of all simple foldings from M to N by $\mathcal{S}(M,N)$</u>.

Each of examples (1.2), (1.3), (1.4), (1.5) and (1.6) is a simple folding, but examples (1.1) and 1.7) are not.

Note that a covering map is simple iff it is a homeomorphism. For any $\phi \in \mathcal{S}(M,N)$, $w(\phi) = i(\phi)$.

(1.8) <u>Lemma</u>

<u>Let $p : M \to N$ be a covering. Let $\phi : N \to P$ be a neat folding such that $\psi = \phi o p$ is simple. Then ϕ is simple.</u>

<u>Proof:-</u>

Let A be an ψ-region. Then $\psi|A$ is a homeomorphism onto Int P. Now $\psi = \phi o p$, and if we put $p(A) = A'$, then $\phi(A') = (\phi o p)(A) = \psi(A) = \text{Int } P$. Hence A' is a union of ϕ-regions. But A' is connected, since A is connected. Hence A' is a ϕ-region. Since $\psi|A$ is a homeomorphism, it follows that both $p|A$ and $\phi|A'$ are homeomorphisms. In particular, ϕ is simple.

(1.9) Corollary

If $\psi = \phi o p \varepsilon \mathcal{S}(M,P)$ as above, and p is of order r, then $i(\psi) = w(\psi) = rw(\phi) = ri(\phi)$.

We observe that example (1.7) gives a neat folding ψ that is formed by composing a covering map with a simple folding, although ψ itself is not simple. By removing one of the ψ-regions of order 1 from M, we obtain a surface M' of genus 2 with boundary and $\psi|M'$ cannot be expressed in the form $\theta o q$ for a covering map q and a simple folding θ.

We have shown in § (2.4) that the composite of topological foldings is not in general a topological folding. This is still true for neat foldings. We give a simple example of two neat foldings that do compose to give a third.

(1.10) Example

Embed a torus $T = S^1 \times S^1$ in R^3 in such a way that the set $X = \{(x,y,z) \varepsilon T : x \leqslant 0\}$ is homeomorphic to D^2. Let $M' = cl(T \setminus X)$, and let M be union of M' with its reflexion in x = 0. Then M is homeomorphic to a double torus, and we can choose cartesian coordinates so that M is invariant under reflexion in any of the coordinate planes. Thus the map $\phi : M \to M$ given by $\phi(x,y,z) = (|x|, |y|, |z|)$ is a simple 8-folding of M onto $N = \phi(M) \subset M$, where N is homeomorphic to D^2 and has a stratification determined by five vertices (and edges) on its boundary. So we can represent N as the disc D^2 with vertices $e^{(2\pi ik)/5}$, k = 0, 1, 2, 3, 4, and consider the simple 10-folding $\psi : N \to P \subset N$, where P is the sector $\{e^{i\theta} : 0 \leqslant \theta \leqslant \pi/5\}$ in D^2 and $\Psi(e^{(\pi k/5+\lambda)i}) = e^{\lambda i}$ for k even and $e^{(\pi/5-\lambda)i}$ for k odd. Thus P carries the stratification of a triangle Δ, and $\psi o \phi : M \to P$ is a simple 80-folding.

We remark that if $\phi \varepsilon \mathcal{N}(M,N)$ and $\psi \varepsilon \mathcal{N}(P,Q)$, then $\phi \times \psi \ \varepsilon \mathcal{N}(M \times P, N \times Q)$.

Also $\Sigma(\phi \times \psi) = (\Sigma(\phi) \times P) \cup (M \times \Sigma(\psi))$. See §(2.4).

2. Neat 2-Foldings

Let H be a topological hypersurface in an n-manifold M, where $\partial M = \emptyset$. Suppose that H is tamely embedded, so that H has a collar V with the structure of 1-disc bundle over H, and $\partial V = H'$ is a double-covering of H.

Let $M' = M \setminus \text{Int } V$. Then $\partial M' = H'$, and there is a continuous map $f : M' \to M$ such that $f|\text{Int } M'$ is a homeomorphism onto $M \setminus H$, and $f|H'$ is the above double covering of H.

We may apply these remarks to the case of a 2-folding $\phi : M \to N$ where M and N are n-manifolds and $\partial M = \emptyset$. Then it is implicit in the definition of topological folding that $\Sigma(\phi)$ is a hypersurface H of M which is tamely embedded, since $\Sigma(\phi)$ has only (n-1)-strata.

Now suppose that ϕ is neat. Then $\phi|H$ is a homeomorphism onto ∂N, and continuing the use of the above notation, we note that $p = \phi o f$ is a 2-fold covering of N. In particular, $f|H'$ is a 2-fold covering of ∂N.

Conversely, we can construct, for any n-manifold N with boundary a neat 2-folding $\psi : M \to N$ as follows. Let $p : W \to N$ be a 2-fold covering (W need not be connected even if N is connected), and define an equivalence relation $\sim$ on W by $x \sim y$ iff $x, y \in \text{Int } W$ and $x = y$ or $x, y \in \partial W$ and $p(x) = p(y)$. Then $M = W \backslash \sim$ is a topological n-manifold without boundary, and there is a unique neat 2-folding $\psi : M \to N$ such that, if $\theta : W \to M$ denotes the quotient map, then $\psi o \theta = p$.

To illustrate this construction, we consider a couple of examples.

(2.1) If $\phi : M \to D^n$ is a neat 2-folding, then M is homeomorphic to S^n.

(2.2) Let $\phi : M \to N$ be a neat 2-folding, where N is a connected surface. If N is a Möbius band, then M is a Klein bottle a torus. If N is an annulus, then M is a torus or a Klein bottle.

3. The Graph of a Neat Folding.

We showed in §(2.6) that to each folding $\phi \varepsilon \mathcal{J}(M,N)$ there is associated a certain graph Γ_ϕ. We now show that if $\phi \varepsilon \mathcal{N}(M,N)$, then Γ_ϕ has the following special features.

(a) Edge-Colouring: The (n-1)-strata of N form a countable set, and we can label them $N_0, N_1, \ldots, N_r, \ldots$, regarding the indices i 'colours'. Each edge of Γ_ϕ is mapped by ϕ to one of these. We may then give Γ_ϕ an edge-colouring by assigning to each edge E the colour i of its image $\phi(E) = N_i$.

(b) Sources and Sinks : It will be noticed that if N is orientable, then any orientation of N induces an orientation for each n-stratum of M. If A and B are regions with a common (n-1)-stratum in their frontiers, then A and B are given opposite orientations by this process. It follows that each edge of the graph Γ_ϕ may be oriented in such a way that every vertex is either a source or a sink (where a vertex ν is a source if all the oriented edges with ν as a vertex begin at ν, and is a sink if all the edges end at ν). See Figure (3.7).

source sink

Figure (3.7).

For such a graph, every circuit has an even number of edges (and hence of vertices).

(c) Regularity: If $\phi \in \mathcal{S}(M,N)$, so that every ϕ-region of M is mapped homeomorphically by ϕ to Int N, then the graph Γ_ϕ is regular. This follows immediately from the fact that the (n-1)-strata in the frontier of each region are in one-one correspondence under ϕ with those of N. It is also worth observing that every colour i occurs exactly once in the set of coloured edges at each vertex of Γ_ϕ. Consequently, the valency of each vertex of Γ_ϕ is the cardinality of the set of (n-1)-strata of N, that is to say, of the set of colours.

We say that ϕ is a Cayley-folding iff Γ_ϕ is a Cayley colour graph.

4. Balanced Foldings.

Let $\phi \in \mathcal{S}(M,N)$. Then for any ϕ-regions A and B there is a homeomorphism $\phi_{AB} : A \to B$ given by $\phi_{AB}(a) = b$ iff $\phi(a) = \phi(b)$, where $a \in A$ and $b \in B$. We can always extend ϕ_{AB} to a homeomorphism $\overline{\phi}_{AB} : \overline{A} \to \overline{B}$, but there need not exist an extension to any open neighbourhood of $\overline{A}$. For instance, consider the following two examples.

4.1) Example

Let $M = \{(x,y) \in R^2 : -2 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 1\}$ be a square in the plane R^2. Let $N = \{(x,y) \in R^2 : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ and define a map $\phi : R^2 \to N$, by $\phi(x,y) = (2f(x), 2f(y))$, where $f(x) = \min_{n \in Z} \left|\frac{x}{2} - n\right|$. Then $\phi|M$ is a simple folding from M to N which maps each ϕ-region of M homeomorphically onto Int N.

Let A, B be the ϕ-regions given by $0 < x < 1$, $-1 < y < 0$ and $0 < x < 1$, $-2 < y < -1$ respectively (see Figure (3.8)). Then there is a homeomorphism $\phi_{AB} : A \to B$, given by $\phi_{AB}(x,y) = (x',y')$ iff $\phi(x,y) = \phi(x',y')$, where $(x,y) \in A$ and $(x',y') \in B$. This homeomorphism has an extension to a homeomorphism $\overline{\phi}_{AB} : \overline{A} \to \overline{B}$ given by $\overline{\phi}_{AB}(x,y) = (x',y')$ iff $\phi(x,y) = \phi(x',y')$ where $(x,y) \varepsilon \overline{A}$ and $(x', y') \varepsilon \overline{B}$. Now consider any open neighbourhoods $\tilde{A}$, $\tilde{B}$ of $\overline{A}$, $\overline{B}$ respectively.

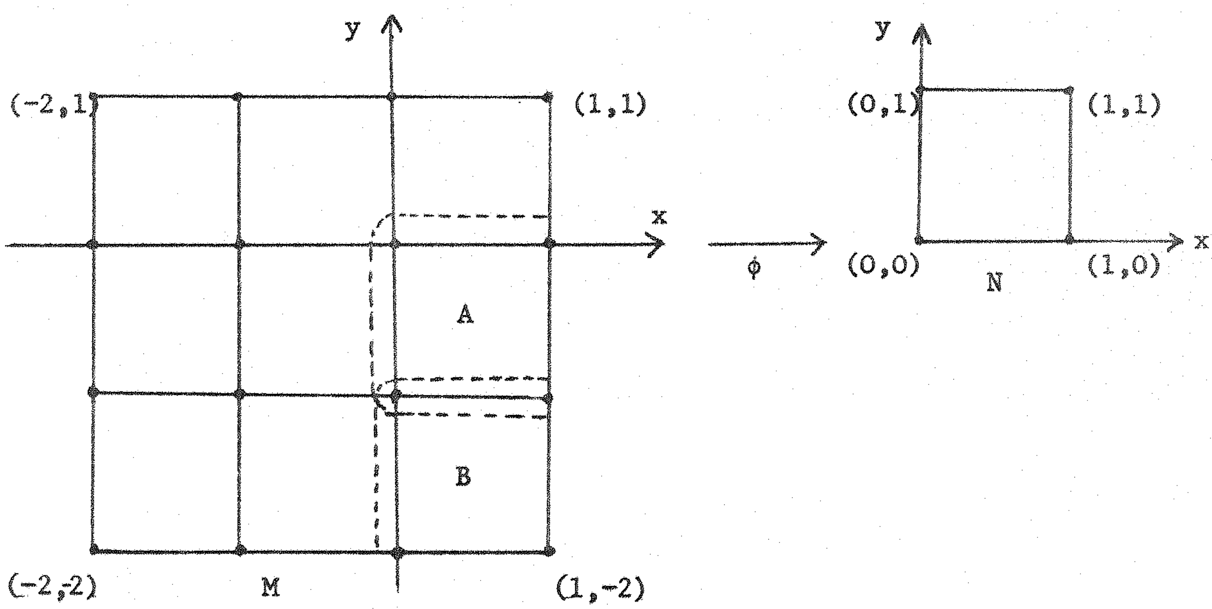


Figure (3.8).

We see that there is no extension of $\overline{\phi}_{AB}$ to a homeomorphism of $\tilde{\phi}_{AB} : \tilde{A} \to \tilde{B}$. This is because three edges of A are interior to M, while only two edges of B have this property.

4.2 <u>Example.</u>

Let M be the unit sphere in Euclidean 3-space, that is, $M = \{\underline{x} \varepsilon E^3 : ||\underline{x}|| = 1\}$. Then M can be partitioned by a triangulation whose vertices are given by $U_k = (\cos\theta_k, \sin\theta_k, 0)$, $V_k = (\cos\theta_k, 0, \sin\theta_k)$, $W_k = (0, \cos\theta_k, \sin\theta_k)$, where $\theta_k = 2\pi/k$, $k = 1, \ldots, 8S$, $S \geq 2$, together with the vertices $(\pm\alpha, \pm\alpha, \pm\alpha)$, where $\alpha = 1/\sqrt{3}$.

There is an essentially unique neat folding $\phi: S^2 \to$ defined by mapping the vertices $(\pm\alpha, \pm\alpha, \pm\alpha)$ to (α,α,α) and the vertices U_k, V_k, W_k to U_1 or V_6 according as k is even or odd. For instance, consider the case $s = 2$. In this case we have a sphere with the triangulation shown in Figure (3.9).

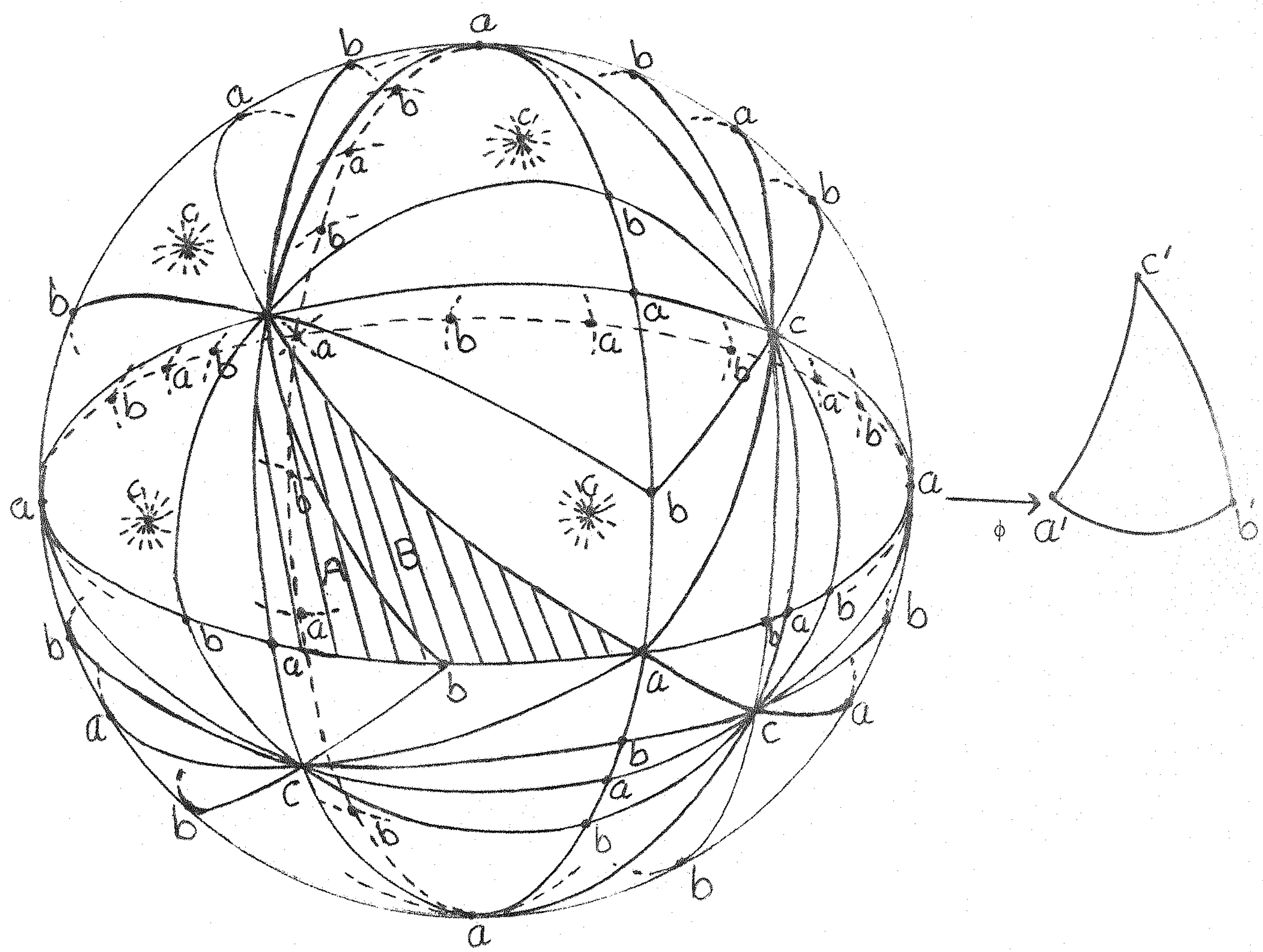


Figure (3.9)

By following the same process explained in example (4.1) it can be checked that a homeomorphism $\phi_{AB} : A \to B$ (where A and B are the ϕ-regions shaded in Figure (3.9)) can not be extended to a homeomorphism of any neighbourhoods $\tilde{A}$, $\tilde{B}$ of $\overline{A}$, $\overline{B}$ respectively. This is because the valencies of the vertices of the ϕ-region A are 12, 4, 4 while those of B are 12, 8, 4.

We say that ϕ is <u>balanced</u> if such an extension exists, for all ϕ-regions A and B. We denote the <u>set of all balanced foldings from M to N</u> by $\mathcal{B}(M,N)$.

4.3) Example

Let M be the unit sphere in Euclidean 3-space. Consider a triangulation whose vertices are given by $U_k = (\cos\theta_k, \sin\theta_k, 0)$, $V_k = (\cos\theta_k, 0, \sin\theta_k)$, $W_k = (0, \cos\theta_k, \sin\theta_k)$ where $\theta_k = 2\pi/k$, $\theta = 1, \ldots, 8$, together with the vertices $(\pm\alpha, \pm\alpha, \pm\alpha)$ where $\alpha = 1/\sqrt{3}$. This triangulation partitions the sphere into 48 triangles and an essentially unique simple 48-folding $\phi : S^2 \circlearrowleft$ is defined again by mapping the vertices $(\pm\alpha, \pm\alpha, \pm\alpha)$ to (α,α,α) and the vertices U_k, V_k, W_k to U_1 or V_6 according as k is even or odd. See Figure (3.10).

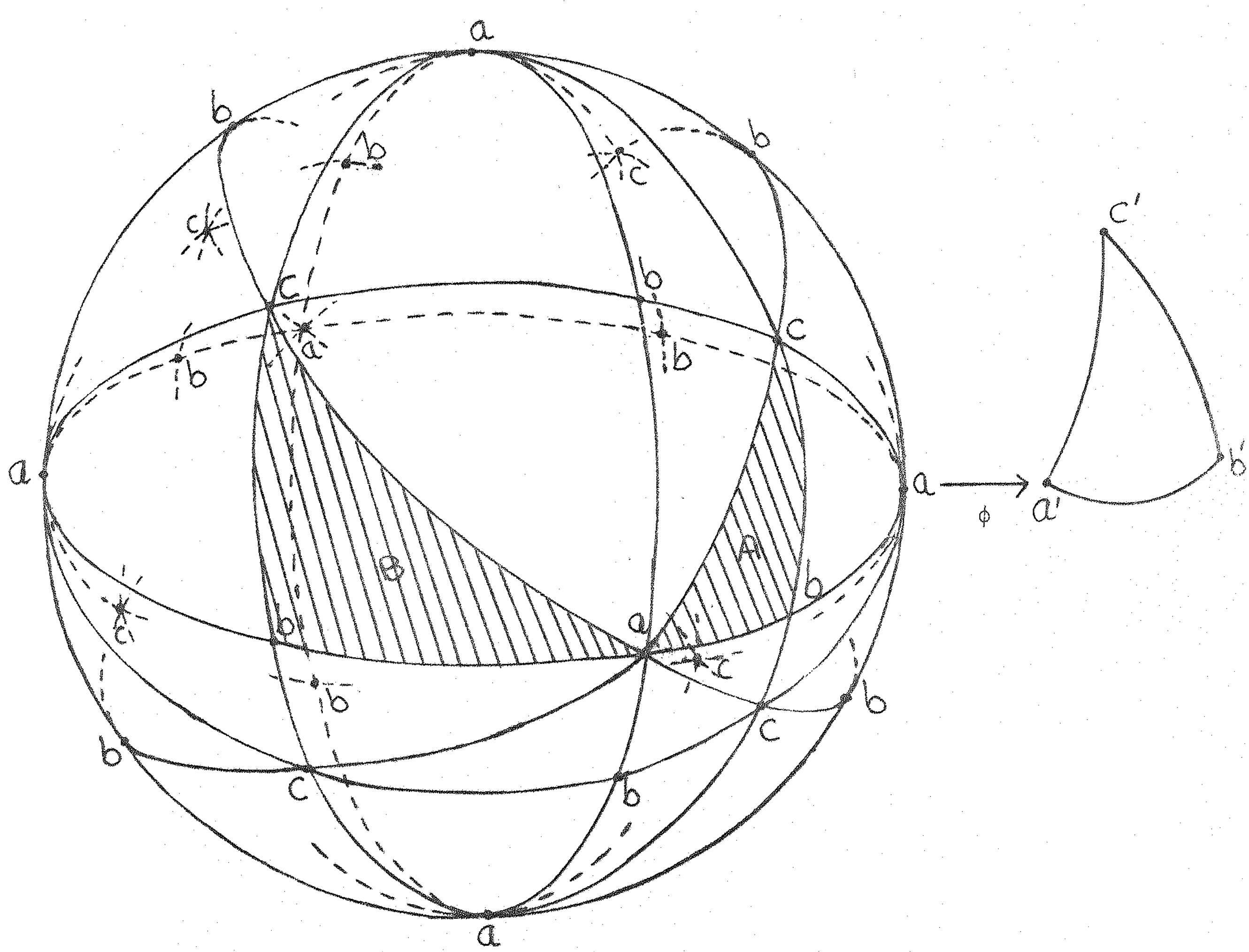


Figure (3.10)

If we consider any ϕ-regions A, B of M, for example the ones shaded in the figure, then there is a homeomorphism $\phi_{AB} : A \to B$, given by $\phi_{AB}(x,y,z) = (x', y', z')$ iff $\phi(x,y,z) = \phi(x', y', z')$, where $(x,y,z) \in A$ and $(x', y', z') \in B$. This homeomorphism has an extension $\overline{\phi}_{AB} : \overline{A} \to \overline{B}$

defined in the same way. Now consider any open neighbourhoods $\tilde{A}$, $\tilde{B}$ of $\overline{A}$, $\overline{B}$ respectively. A homeomorphism $\tilde{\phi}_{AB} : \tilde{A} \to \tilde{B}$ can be defined because the valencies of the vertices of A and B are 6, 8, 4. It follows that ϕ is balanced.

5. Cayley Graphs and Cayley Foldings

The properties of the graph Γ_ϕ that we have already discussed, in section 3, suggest that in certain cases the graph Γ_ϕ may be a Cayley colour graph. We now show that this is indeed the case, for a large class of balanced foldings.

Note first, that, for any map $\phi : M \to N$, we can associate a group $G(\phi)$ namely the group of all homeomorphisms $h : M \to M$ such that $\phi o h = \phi$. In case ϕ is a neat folding, we may ask whether the induced action of $G(\phi)$ on the stratification S of M is transitive on the set of ϕ-regions. In particular, we ask whether there is a subgroup $H(\phi)$ of $G(\phi)$ that acts 1-transitively on the set of ϕ-regions.

In general, this is not true. For instance, consider example (1.7). There are three ψ-regions two of order one and one of order two. Hence no subgroup $H(\psi)$ of $G(\psi)$ act 1-transitively on the ψ-regions in this case. However the following theorem gives us the conditions under which the group $H(\phi)$, where $\phi : M \to N$ is a neat folding, may act 1-transitively on the set of ϕ-regions.

5.1) Theorem

Let $\phi : M \to N$ be a balanced folding, and M be simply connected. Then there is a subgroup $H(\phi)$ of $G(\phi)$ that acts 1-transitively on the set of ϕ-regions. Moreover Γ_ϕ is a Cayley colour graph of the group $H(\phi)$.

Proof:-

Let $\phi \in \mathfrak{F}(M,N)$ be a balanced folding. Let A,B be ϕ-regions. Then $\phi_{AB} : A \to B$ extends to a homeomorphism $\tilde{\phi}_{AB} : \tilde{A} \to \tilde{B}$, where $\tilde{A}$ and $\tilde{B}$ are open neighbourhoods of A and B respectively. Let C be a ϕ-region such that $C \neq A$ and $C \cap \tilde{A} \neq \emptyset$. Let $\tilde{\phi}_{AB}(C) \subset D$. Then there are open neighbourhoods $\tilde{C}$ and $\tilde{D}$ of C and D such that ϕ_{CD} extends to a homeomorphism $\tilde{\phi}_{CD}: \tilde{C} \to \tilde{D}$, where $\tilde{\phi}_{CD}$ and $\tilde{\phi}_{AB}$ agree on $\tilde{A} \cap \tilde{C}$. Iterate this procedure to extend ϕ_{AB} to a map $\Phi_{AB}: M \to M$. The existence and uniqueness of the extension are guaranteed by the fact that M is 1-connected. For, let $\alpha = \alpha_1 \alpha_2 \ldots \alpha_k$ be an edge path in S*, the dual graph of S, starting at $a \in A_1$ and joining equivalent points $a = a_1, a_2, \ldots, a_{k+1} = x$, where $a_i \in A_i$ (by equivalent points we mean that they have the same image under ϕ). Let $\alpha' = \alpha_1' \alpha_2' \ldots \alpha_k'$ be another edge path in S* starting at $a \in A_1$ and joining equivalent points $a = a_1', a_2', \ldots, a_{k+1}' = x$, where $a_i' \in A_i'$. Let $\beta = \beta_1 \beta_2 \ldots \beta_k$ and $\beta' = \beta_1' \beta_2' \ldots \beta_k'$ be the images of α and α' under Φ_{AB} with vertices $b = b_1, b_2, \ldots, b_{k+1} = y$ and $b = b_1', b_2', \ldots, b_{k+1}' = y'$ respectively.

Since M is 1-connected, there is a subgraph Γ' of Γ_ϕ that spans a disc and whose boundary is made up of the edge paths α and α' from a to x. Then Φ_{AB} maps Γ' onto a subgraph Γ'' of Γ_ϕ, in which α and α' are mapped to β and β', both of which must have the same end point. Then $y = y'$ and it follows that Φ_{AB} is well-defined.

Now, to prove that Φ_{AB} is onto, let $y \in M$ a nonsingular point. Then y belongs to an m-stratum Y. Let $B_1, B_2, \ldots, B_{k+1} = Y$, be a sequence of m-strata such that B_j, B_{j+1} are contigous, $j = 1, 2, \ldots, k$. The sequence $B_1, B_2, \ldots, B_{k+1}$ of m-strata is the image under Φ_{AB} of a unique sequence $A_1, A_2, \ldots, A_{k+1} = X$ of m-strata such that A_j, A_{j+1} are contiguous, $j = 1, 2, \ldots, k$ and each $\phi_{A_iB_i}: A_i \to B_i$ extends to a homeomorphism $\tilde{\phi}_{A_iB_i}: \tilde{A}_i \to \tilde{B}_i$ where $\tilde{\phi}_{A_iB_i}$ and $\tilde{\phi}_{A_{i+1}B_{i+1}}$ agree on $\tilde{A}_i \cap \tilde{A}_{i+1}$. Hence Φ_{AB} is onto.

We have now shown that Φ_{AB} is a local homemorphism of the simply-connected manifold M onto itself. In fact, Φ_{AB} is a covering map. Thus Φ_{AB} is a homeomorphism.

The set of all such homeomorphisms is the required group $H(\phi)$, which by its construction acts 1-transitively on the set of ϕ-regions.

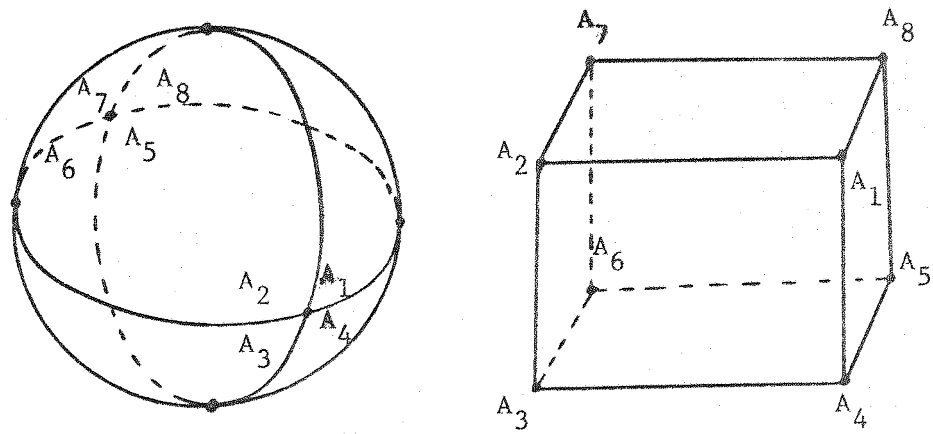
The relationship of $H(\phi)$ to the graph Γ_ϕ is as follows. Choose some ϕ-region A. Thus A is a vertex of Γ_ϕ. Identify any other vertex (ϕ-region) B of Γ_ϕ with the unique element Φ_{AB} of $H(\phi)$ such that $\Phi_{AB}(A) = B$.

It follows trivially that the graph Γ_ϕ is a Cayley colour graph of $H(\phi)$, with generators $\phi_B = \phi_{AB}$, where B runs through the set of m-strata $B \neq A$ having an (m-1)-stratum in its common frontier with A.

Note that for surfaces M,N and any $\phi \varepsilon \mathcal{N}(M,N)$ the singularity sets Σ_1 and Σ_2 form the edges and vertices of a graph K_ϕ. If ϕ is balanced, then the valencies of the vertices are invariant under any of the extended homeomorphisms $\tilde{\phi}_{AB}$. In particular, if $\phi \varepsilon \mathcal{B}(M,N)$ be such that K_ϕ is a regular graph embedded in M, then $\phi \varepsilon \mathcal{R}(M,N)$. Moreover, if M is simply connected, then $H(\phi)$ will act 1-transitively on the set of ϕ-regions and Γ_ϕ will be a Cayley colour graph of the group $H(\phi)$.

5.2 Example

Let $M = S^2 = \{\underline{x} \varepsilon E^3 : ||\underline{x}|| = 1\}$, be the unit sphere in Euclidean 3-space. Let $\phi : M \to M$ be given by $\phi(x,y,z) = (|x|, |y|, |z|)$. Then ϕ is a simple folding and the graph K_ϕ is a regular graph of valency 4, with 6 vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$, twelve edges and eight regions. The image of ϕ is the positive octant $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$. See Figure (3.11)(a) below.



(a) S^2 with the graph K_ϕ

(b) The graph Γ_ϕ is a Cayley colour graph.

Figure (3.11)

Since K_ϕ is a regular graph, it follows that ϕ is a balanced folding and the graph Γ_ϕ , which is a Cayley colour graph, has the form given by Figure (3.11) (b). Hence $H(\phi)$ is isomorphic to $Z_2 \times Z_2 \times Z_2$ and it acts 1-transitively on the set of eight regions $A_1, A_2, \ldots, A_8$.

We now explore the relationship between balanced foldings and covering maps.

5.3) Theorem.

Let $\phi \in \mathcal{N}(M,N)$ and let $p : \tilde{M} \to M$ be the universal covering. Suppose that $\tilde{\phi} = \phi o p \in \mathcal{B}(\tilde{M},N)$ and that $G(p) \triangleleft H(\tilde{\phi})$. Then there is a subgroup $H(\phi)$ of $G(\phi)$, isomorphic to $H(\tilde{\phi})/G(p)$, acting 1-transitively on the set of ϕ-regions.

Proof:-

We first construct the group $H(\phi)$. Let $\tilde{h} \in H(\tilde{\phi})$. We now show that $\tilde{h}$ covers a (unique) homeomorphism $h : M \to M$, that is $hop = po\tilde{h}$. Let $a \in M$, and let $\tilde{a} \in p^{-1}(a)$. Put $b = p(\tilde{b})$, where $\tilde{b} = \tilde{h}(\tilde{a})$. The point b is independent of the choice of $\tilde{a} \in p^{-1}(a)$. For if $p(\tilde{c}) = a$, and $d = p(\tilde{d})$ where $\tilde{d} = \tilde{h}(\tilde{c})$, then there is an element $g \in G(p)$ such that $g(\tilde{a}) = \tilde{c}$. Consider $g' = \tilde{h} \circ g \circ \tilde{h}^{-1}$. Then $g'(\tilde{b}) = \tilde{d}$. Since $G(p) \triangleleft H(\tilde{\phi})$, $g' \in G(p)$. Thus $b = p(\tilde{b}) = p(\tilde{d}) = d$. Define $h : M \to M$ by $h(a) = b$. Then h is a homeomorphism of M, and, trivially, the set $H(\phi) = \{h : \tilde{h} \in H(\tilde{\phi})\}$ is a subgroup of $G(\phi)$ isomorphic to $H(\tilde{\phi})/G(p)$. Thus there is an epimorphism $\theta : H(\tilde{\phi}) \to H(\phi)$ given by $\theta(\tilde{h}) = h$.

Secondly, we show that $H(\phi)$ acts 1-transitively on the set of ϕ-regions. By lemma (1.8) in this chapter, $\phi \in \mathcal{B}(M,N)$. Let A,B be ϕ-regions. Then there are $\tilde{\phi}$-regions $\tilde{A}$ and $\tilde{B}$ such that $p(\tilde{A}) = A$ and $p(\tilde{B}) = B$. Let $\tilde{h}$ be the unique element of $H(\tilde{\phi})$ such that $\tilde{h}(\tilde{A}) = \tilde{B}$ and let $h = \theta(\tilde{h})$. Then $h(A) = B$, and there is only one such element of $H(\phi)$.

Remark: If $p : \tilde{M} \to M$ is a covering map, and $\tilde{\phi} = \phi op$, where $\phi \in \mathcal{N}(M,N)$, then $\tilde{\phi} \in \mathcal{B}(\tilde{M},N)$ implies that $\phi \in \mathcal{B}(M,N)$.

5.4 Example

Let $M = P_n(R)$, and let N be the n-simplex $\{t \in R^{n+1} : \sum_{i=1}^{n+1} t_i = 1, 0 \leq t_i \leq 1\}$. Define $\phi : M \to N$ by $\phi(\{x\}) = (|x_1|, \ldots, |x_{n+1}|)/||x||$. Then $\tilde{M}$ may be identified with S^n, and $p : \tilde{M} \to M$ is given by $p(x) = \{x\}$. In this case $G(p) \simeq Z_2$ is generated by the map $g : S^n \to S^n$, $g(x) = -x$, and $H(\tilde{\phi}) \simeq (Z_2)^{n+1}$ is generated by the reflexions $g_i : R^{n+1} \to R^{n+1}$, $g_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_{n+1})$ and $\tilde{\phi}(x) = (\phi\ op)(x) = (|x_1|, \ldots, |x_{n+1}|)/||x||$ as above.

5.5) Theorem

Let $\tilde{\phi}$, ϕ be as in theorem (5.3) such that $G(p) \lhd H(\tilde{\phi})$. Let $\gamma : P \to M$ be a regular covering. Then $H(\psi)$, where $\psi = \phi o \gamma$, acts 1-transitively on the set of ψ-regions of P.

Proof:-

Since $\tilde{M}$ is simply-connected, for any other covering map $\gamma : P \to M$ there exists a universal covering map $h : \tilde{M} \to P$ such that $\gamma o h = p$.

Now $G(p) \simeq \pi_1(M)$ and $G(h) \simeq \pi_1(P)$. Since $\gamma : P \to M$ is regular $\gamma_* \pi_1(P,y) \lhd \pi_1(M,x)$, where $\gamma(y) = x$. There are isomorphisms $f : G(p) \to \pi_1(M)$ and $g : G(h) \to \pi_1(P)$ such that the following diagram is commutative.

$$\begin{array}{ccc} G(h) & \xrightarrow{\ g\ } & \pi_1(P) \\ \alpha \downarrow & & \downarrow \gamma_* \\ G(p) & \xrightarrow[\ f\ ]{} & \pi_1(M) \end{array}$$

It follows from elementary group theory that, since $\pi_1(P)$ is embedded in $\pi_1(M)$ as a normal subgroup, then $G(h)$ is embedded by α in $G(p)$ as a normal subgroup. But $G(p) \lhd H(\tilde{\phi})$ by assumption. Hence $G(h) \lhd H(\tilde{\phi})$ and theorem (5.3) can be applied for ψ, yielding that $G(\psi) = H(\tilde{\phi})/G(h)$ acts 1-transitively on the set of ψ-regions of P.

5.6) Example

Let $M = S^1 \times S^1$, $\tilde{M} = R^2$, and let $p : \tilde{M} \to M$ be given by $p(x,y) = (e^{2\pi ix}, e^{2\pi iy})$. Let $N = S^1 \times I$ where $I = [0,1]$ and let $\phi : M \to N$ be given by $\phi(a,b) = (a,c)$, and if $b = e^{2\pi iy} = \cos 2\pi y + i \sin 2\pi y$, then $\cos c = |\cos 2\pi y|$, $\sin c = |\sin 2\pi y|$. Let $P = R \times S^1$. So that $h : R^2 \to P$ is given by $h(x,y) = (x, e^{2\pi iy})$ and $\gamma : P \to M$ be given by $\gamma(x, e^{2\pi iy}) = (e^{2\pi ix}, e^{2\pi iy})$. Thus $G(p) \simeq Z \times Z$ is generated by the translations $(x,y) \to (x+1,y)$, $(x,y) \to (x,y+1)$, while $H(\tilde{\phi}) \simeq Z \times Z$, generated by $(x,y) \to (x+1,y)$ and $(x,y) \to (x,y+\frac{1}{2})$. The quotient group $H(\phi) \simeq H(\tilde{\phi})/G(p) \simeq Z_2$. Also $G(h) \simeq Z$ and it is generated by the translations $(x,y) \to (x,y+1)$. Finally, $H(\psi)$ is isomorphic to $Z \times Z_2$ and is generated by $(x, e^{2\pi iy}) \to (x+1, e^{2\pi iy})$ and $(x, e^{2\pi iy}) \to (x, e^{-2\pi iy})$.

6. Uniform Foldings

In this section we consider a class of neat foldings a little more general than simple foldings. Suppose that $\phi : M \to N$ is a neat folding. Then for any ϕ-regions A and B the maps $\phi_A = \phi|A$ and $\phi_B = \phi|B$ are coverings of Int N. If, for any such A and B, the coverings ϕ_A and ϕ_B are isomorphic, that is to say there is a homeomorphism $\theta : A \to B$ such that $\phi_B o \theta = \phi_A$, then ϕ is said to be uniform folding. It follows that all the coverings $\phi|A$ in a uniform folding are of the same order. If this order is finite, say k, then ϕ is said to be k-uniform.

Note that 1-uniform foldings are just simple foldings.

We remark that if ϕ is a k-uniform folding of index j and order r, then $r = kj$.

We have observed previously that a covering map is a neat folding. Also, if $p : \tilde{M} \to M$ is a covering and $\phi : M \to N$ is a neat folding, then $\tilde{\phi} = \phi o p$ is a neat folding. It is natural then to consider whether $\tilde{\phi}$ is uniform if ϕ is uniform. In general, this will not be so. However, the following remarks may help to clarify the position.

Let $\phi : M \to N$ be a uniform folding, and let A and B be ϕ-regions. Then there is homeomorphism $\theta : A \to B$ such that $\phi_B o \theta = \phi_A$. Now suppose that $p : \tilde{M} \to M$ is a covering, and let $\tilde{A}$, $\tilde{B}$ be connected components of

$p^{-1}(A))$ and $p^{-1}(B)$ respectively, and let $p_A = p|\tilde{A}$ and $p_B = p|\tilde{B}$. Thus p_A and p_B are covering maps. In order that $\tilde{\phi} = \phi o p$ be a uniform folding, we require that for all such $\tilde{A}$ and $\tilde{B}$ the covering maps $\tilde{\phi}_A = \phi_A \circ p_A$ and $\tilde{\phi}_B = \phi_B \circ p_B$ be equivalent. We therefore seek conditions that guarantee such an equivalence.

Consider base points $a \varepsilon A$, $b \varepsilon B$, $\tilde{a} \varepsilon \tilde{A}$, $\tilde{b} \varepsilon \tilde{B}$ such that $\theta(a) = b$, $p(\tilde{a}) = a$, $p(\tilde{b}) = b$. Let $G = p_{A_*}(\pi_1(\tilde{A},\tilde{a})) < \pi_1(A,a)$, $H = p_{B_*}(\pi_1(\tilde{B},\tilde{b})) < \pi_1(B,b.)$ Suppose that $\theta_* : \pi_1(A,a) \to \pi_1(B,b)$ maps G isomorphically onto H. Then there is a unique hoeomorphism $\tilde{\theta} : \tilde{A} \to \tilde{B}$ such that $\tilde{\theta}(\tilde{a}) = \tilde{b}$, and $p_B \circ \tilde{\theta} = \theta o p_A$. See Figure (3.12) below. It follows that $\tilde{\phi}$ is uniform in this case.

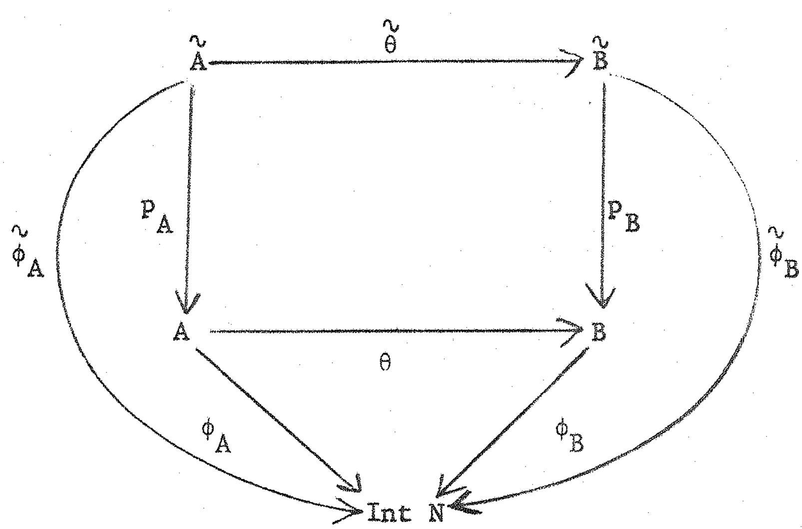


Figure (3.12)

CHAPTER 4

FOLDINGS OF SURFACES

As the theory has built up, we have noted at many points that surfaces are particularly interesting because of the fact that the set of singularities of any folding ϕ of a surface M forms a graph K_ϕ on M.

In this chapter we examine a few simple aspects of the relationships between the topology of M and the structure of K_ϕ

1. General Considerations

Consider a neat folding $\phi : M \to N$ where M and N are surfaces. To avoid too many complications, let us suppose that M is compact, connected and with empty boundary, and let N be connected. Thus the boundary of N is composed of finitely many closed curves.

Since M is compact, ϕ is of finite order $k \geq 1$, and the graph $K = K_\phi$ is a finite graph. Let K divide M into ϕ-regions $A_1, A_2, \ldots, A_n$, say, and let $\phi|A_j$ be a covering map of order k_j. Thus $k = \sum_{j=1}^{n} k_j$.

1.1 Proposition

If $\partial N \neq \emptyset$, then k is even.

Proof:-

Suppose $\partial N \neq \emptyset$. Then there is at least one component C of ∂N. Suppose that K has a vertex ν such that $\phi(\nu) = w \in C$. Then $\phi^{-1}(w)$ is a subset of the set of vertices of K, say $\phi^{-1}(w) = \{\nu_1, \ldots, \nu_h\}$. Now each ν_s, $s = 1, \ldots, h$ has even valency $2Z_s$, say. Hence $k = \sum_{s=1}^{h} 2Z_s$ is even.

Suppose on the other hand that K has no vertices. Since $\partial N \neq \emptyset$, K consists of closed curves. Choose any point $w \in C$, and let

$\phi^{-1}(w) = \{\nu_1, \ldots, \nu_h\}$ as above. It follows that $k = 2h$, and so again k is even.

Remark: If $\partial N = \emptyset$ then k can be odd, since ϕ is then just a covering map.

2. Euler Numbers

Let $\phi : M \to N$ be a neat folding, as above. We can triangulate N by a simplicial complex T_N such that every vertex of the ϕ-stratification of ∂N is a vertex of T_N. Let T_M be the triangulation of M induced by ϕ.

Consider the regions $A_1, \ldots, A_n$ and their closures $B_1, \ldots, B_n$. Then for $j = 1, 2, \ldots, n, \phi|B_j$ is a k_j - fold covering of N. Thus $e(B_j) = k_j\, e(N)$, where $e(X)$ is the Euler number of X. If we now calculate the Euler number $e(M)$ of M using the triangulation T_M, then we can compare $e(M)$ with $\sum_{j=1}^{n} e(B_j) = \sum_{j=1}^{n} k_j\, e(N) = k\, e(N)$. We note that for each vertex of K with valency ν exactly ν vertices have been counted in the calculation of the Euler number $k\, e\,(N)$ of the disjoint union of $B_1, \ldots, B_n$. Likewise, every edge of K appears twice in these calculations. Figure (4.1) which shows the neighbourhood of a vertex with valency 4, may help to clarify these remarks.

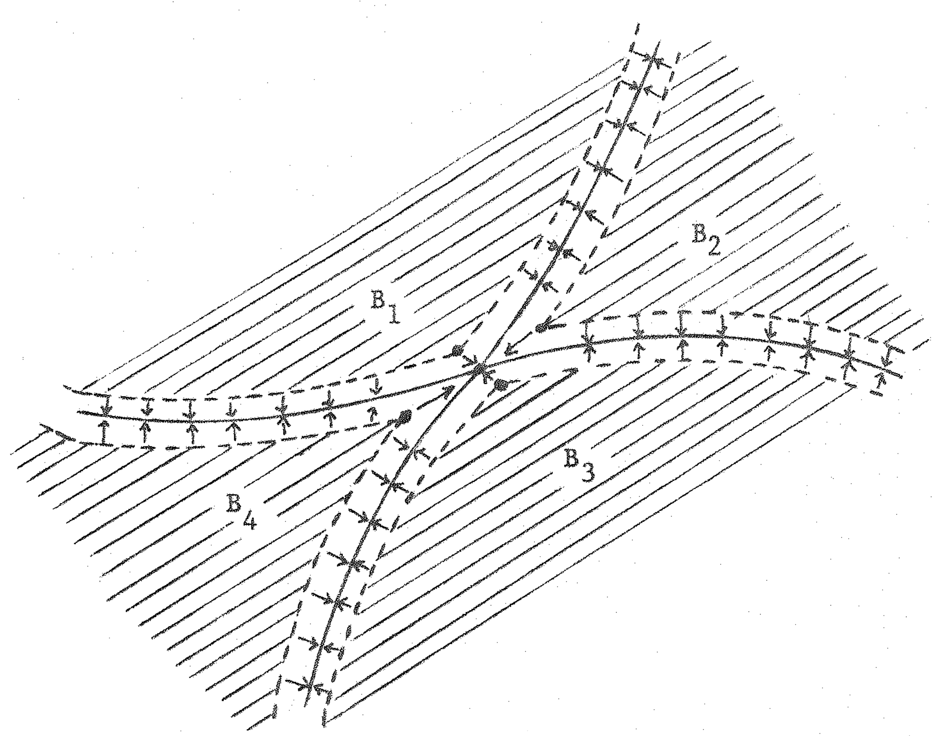


Figure (4.1)

Thus to obtain e(M) from $\sum_{j=1}^{h} e(B_j)$ we must subtract $\nu-1$ for each vertex of K (of valency ν) and add the number of edges of K. The first of these is V-pk, where V is the number of vertices of K, and p is the number of vertices of ∂N. The second is equal to $\frac{1}{2}pk$. We conclude that

$$e(M) = k(e(N)) + V - \tfrac{1}{2}pk. \qquad (4.1)$$

It may be worth observing that the number of closed curves (without vertices) in ∂N does not influence this relation.

3. Neat Foldings Over a Disc

We now study the case in which N is the disc D^2. In this case e(N) =1 and each ϕ-region A is itself homeomorphic to D^2. It follows that $\phi|A$ is a homeomorphism, and so ϕ is simple. Equation (4.1) now reduces to

$$2e(M) = k(2-p) + 2V \qquad (4.2)$$

Notice that if $N = D^2$ has no 0-strata, then p = V = 0, k = 2 and M is homeomorphic to the 2-sphere S^2. Thus for a neat folding over a disc, with no 0-strata, the graph K consists of a single simple closed curve, and ϕ is a 2-folding of S^2.

4. Balanced Foldings Over a Disc

Equations (4.1) and (4.2) can be refined slightly when ϕ is balanced. In this case, if we label the vertices of the disc D^2 as $V_1, \ldots, V_p$, then each vertex in the set $\phi^{-1}(V_i)$ has the same valency $2q_i$, i = 1, ..., p.

It follows that $\phi^{-1}(V_i)$ contains $k/2q_i$ elements. Thus the number of vertices of K_ϕ is

$$V = (k/2)\sum_{i=1}^{p} 1/q_i \ . \qquad (4.3)$$

Hence for a balanced folding over a disc, (4.2) may be reduced to

$$2e(M) = k\{(2-p) + \sum_{i=1}^{p} 1/q_i\} \ . \qquad (4.4)$$

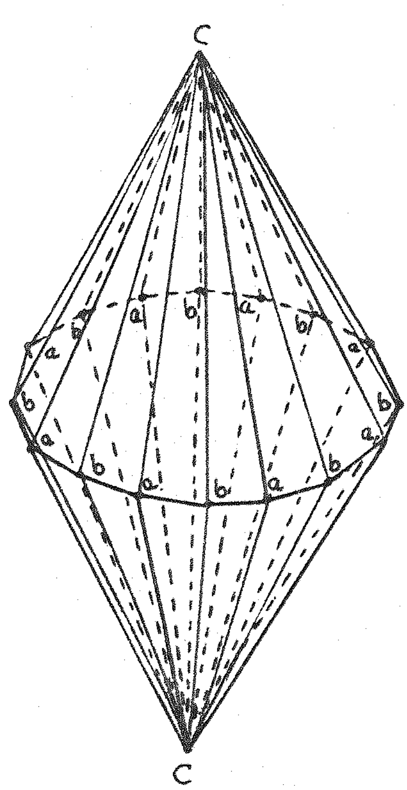
Certain cases of relation (4.4) are of special interest. For instance, let p = 3, so that M is triangulated by K, and (4.4) becomes

$$2e(M) = k\{\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_2} - 1\} . \qquad (4.5)$$

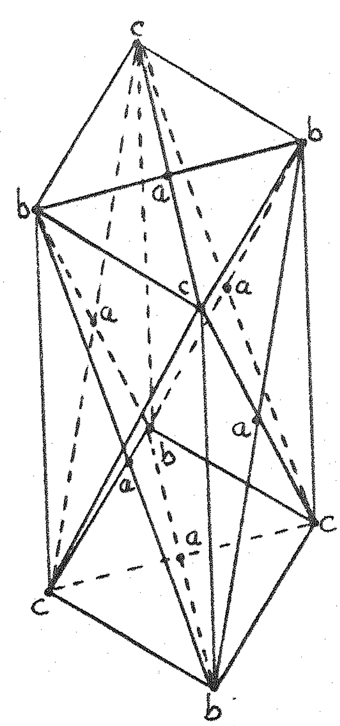
Thus if M is a sphere, then $\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}\right) > 1$ and $k \geqslant 4$. The only possibilities are listed in the following table

q_1	q_2	q_3	k	$H(\phi)$
2	2	n n>1	4n	D_{2n}
2	3	3	24	O
2	3	4	48	$\overline{O}$
2	3	5	120	$\overline{I}$

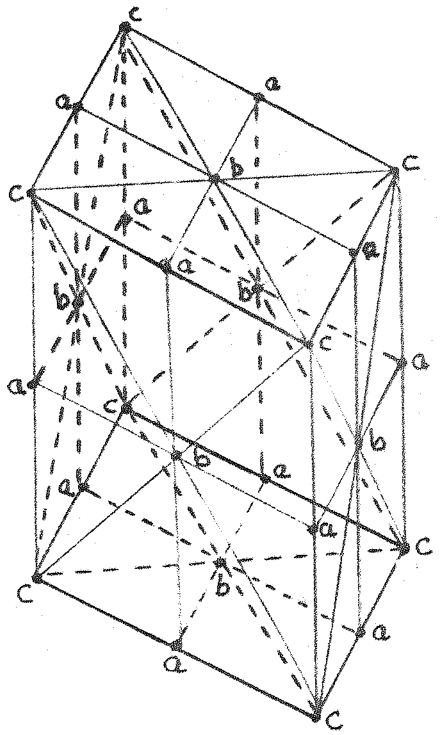
The group $H(\phi)$ associated with ϕ according to theorem (5.1) in chapter 3 is shown in column 5, and the corresponding triangulation of S^2 are shown in Figure (4.2) (i), (ii), (iii) and (iv). Note that in Figure (4.2) (iv) we have drawn only one side. The vertices are labelled in such a way that vertices with the same image under ϕ are labelled alike.



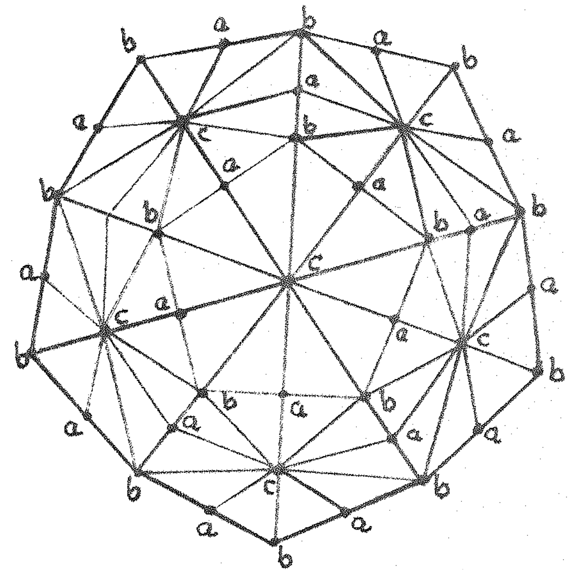
(i) (2, 2, 8)



(ii) (2, 3, 3)



(iii) (2, 3, 4)



(iv) (2, 3, 5)

Figure (4.2)

5. Regular Folding Over a Disc

For any surfaces M and N, we say that $\phi \varepsilon \mathcal{S}(M,N)$ is regular iff the graph $K = K_\phi$ of the set of singularities of ϕ is a regular graph. We concentrate on the case $N = D^2$ and denote the set of all regular foldings of M onto D^2 by $\mathfrak{D}(M)$. For each non-negative integer p, we examine the set $\mathfrak{D}_p(M)$ of regular foldings of M over a disc D^2 for which the stratification of D^2 has p vertices. Thus (for $p \geqslant 3$ at least), we study the problem of folding regular subdivisions of a surface M into topological p-sided polygons that correspond to some neat foldings.

As we have already observed, $\mathfrak{D}(M) \subset \mathcal{B}(M,D^2)$, and Γ_ϕ is a Cayley colour graph for a group $H(\phi)$ acting 1-transitively on the ϕ-regions, for each $\phi \varepsilon \mathfrak{D}(M)$.

Suppose then that M is a closed connected surface with Euler number e, and let $\phi \varepsilon \mathfrak{D}_p(M)$ be such that the graph K has E edges, and V vertices. Then there are k ϕ-regions, where

$$k - E + V = e.$$

Since ϕ is regular, each vertex has valency 2s for some positive integer s. Also k = 2m for some positive integer m. Thus,

$$V = pk/2s. \tag{4.6}$$

Also, since K is regular, 2sV = 2E, and so

$$E = pk/2 \tag{4.7}$$

Equation (4.4) now reduces to

$$2e(M) = k\{(2-p) + \frac{p}{s}\} \tag{4.8}$$

Note that $k \geq 2s$. It is convenient to denote by $\mathfrak{D}_p^k(M)$ the set of all $\phi \varepsilon \mathfrak{D}_p(M)$ with k ϕ-regions. We are only interested in foldings up to equivalence, where ϕ, $\psi \varepsilon \mathfrak{D}(M)$ are equivalent iff for some homeomorphisms $f : M \to M$ and $g : D^2 \to D^2$, $\psi o f = g \text{ o } \phi$, and when we refer to a folding we mean the equivalence class of that folding. Thus to say that $\mathfrak{D}(M)$ contains only one element means that all $\phi \varepsilon \mathfrak{D}(M)$ are equivalent.

From equation (4.8) we see that if p = 0 then k = e = 2, and so $M = S^2$. In this case K is a single closed curve, and ϕ is represented by the map $\phi(x,y,z) = (x,y,|z|)$. Thus $\mathfrak{D}_o^k(M) \neq \emptyset$ iff $k = 2$ and $M = S^2$, and $\mathfrak{D}_0^2(S^2)$ has only one element.

Likewise, $\mathfrak{D}_1^k(M) = \emptyset$ for all M, and $\mathfrak{D}_2^2(M) = \emptyset$ for all M. Reference to (4.8) also show that $\mathfrak{D}_2^4(M) \neq \emptyset$ iff $M = S^2$, and the only element of $\mathfrak{D}_2^4(S^2)$ is represented by the map $\phi(x,y,z) = (x, |y|, |z|)$. In this case, E = 4, V = 2, and s = 2. See Figure (4.3).

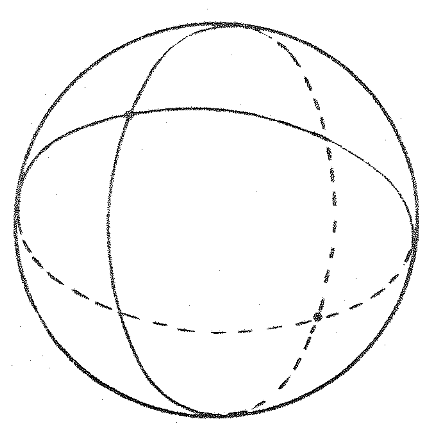


Figure (4.3). $\phi \varepsilon \mathfrak{D}_2^4(S^2)$

Also, $\mathcal{D}_2^6(M) \neq \emptyset$ iff $M = S^2$ and $\mathcal{D}_2^6(S^2)$ contains only one element. A representative graph K is shown in Figure (4.4). It has 6 edges, and 2 vertices, each of valency 6.

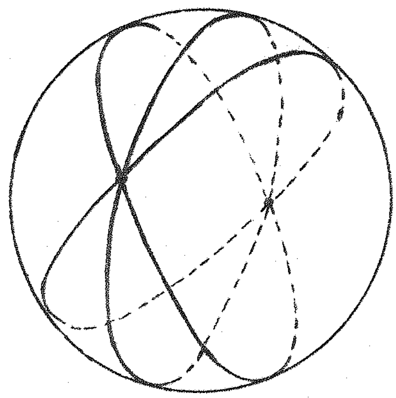


Figure (4.4). $\phi \in \mathcal{D}_2^6(S^2)$

A corresponding map ϕ can be defined as follows. Let $p(\theta,\psi) = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)$ be a point on the sphere where $0 \leq \theta \leq 2\pi, 0 \leq \psi \leq \pi$. Let $\theta = \frac{\ell\pi}{3} + \alpha$ where $0 \leq \alpha \leq \pi/3$. Then ϕ is defined by mapping $P(\theta,\psi)$ to $P(\alpha,\psi)$ or $P(\frac{\pi}{3} - \alpha,\psi)$ if ℓ is even or odd respectively.

In general $\mathcal{D}_2^k(M) \neq \emptyset$ iff $M = S^2$ and $\mathcal{D}_2^k(S^2)$ has only one element. The graph K has k edges, and 2 vertices each of valency k. A representative map for the k-regular foldings of the sphere over a disc with two vertices on its boundary can be defined as above, where θ in this case is given by $\theta = \frac{\ell 2\pi}{n} + \alpha$, $0 \leq \alpha \leq \frac{2\pi}{n}$ and ϕ is mapping $p(\theta,\psi)$ to $p(\alpha,\psi)$ or $p(\frac{2\pi}{n} - \alpha,\psi)$ if ℓ is even or odd respectively.

In the next three sections we study the sets $\mathcal{D}_p^k(M)$ for $p = 3$, 4 and 5. The results give some indication of how a choice of p restricts the topology of M and the value of k, and serve to illustrate the force of the relation (4.8).

6. Regular Foldings Over a Triangle

From a topological point of view, a triangle may be regarded as a disc which has a stratification on the boundary consisting of 3 vertices and 3 edges.

Hence $e = k(3-s)/2s$. We study the set $\mathfrak{D}_3^k(M)$. So if $k=2$, then $e = (3-s)/s$. Hence $e = 2$ and $s = 1$ or $e = 0$ and $s = 3$ are the only solutions. But neither of these can be realised. So $\mathfrak{D}_3^2(M) = \emptyset$ for any compact surface M.

Now if $k = 4$, then $e = 2(3-s)/s$, and the only solution is given by $s = 2$. In this case $e = 1$ and so $M = P_2(R)$. The graph K has 3 vertices and 6 edges, and $\mathfrak{D}_3^4(P_2(R))$ contains only one element represented by the map $\phi(x,y,z) = (|x|, |y|, |z|)$. That is $\mathfrak{D}_3^4(M) \neq \emptyset$ iff $M = P_2(R)$.

If $k = 6$, then $e = 3(3-s)/s$, and the only solution is given by $s = 3$. In this case $e = 0$, $V = 3$, $E = 9$. Such a graph cannot be constructed and hence $\mathfrak{D}_3^6(M) = \emptyset$.

If $k = 8$, then $e = 4(3-s)/s$, and the only solutions are given by $s = 2$, $s = 3$ and $s = 4$. For the first case $e = 2$ and we have a sphere. A representative graph K is shown in Figure (4.5). It has 12 edges, and 6 vertices each of valency 4. The corresponding map is given by $\phi(x,y,z) = (|x|, |y|, |z|)$. The image of the sphere is the positive octant.

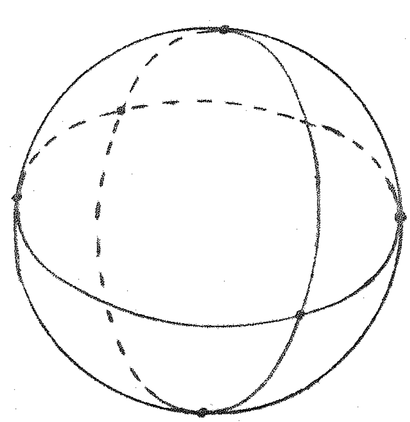


Figure (4.5). $\phi \in \mathfrak{D}_3^8(S^2)$

In the second case (s = 3), e = 0 and we have a torus or a Klein bottle. In both cases the graph has 12 edges, and 4 vertices each of valency 6. See Figure (4.6) below.

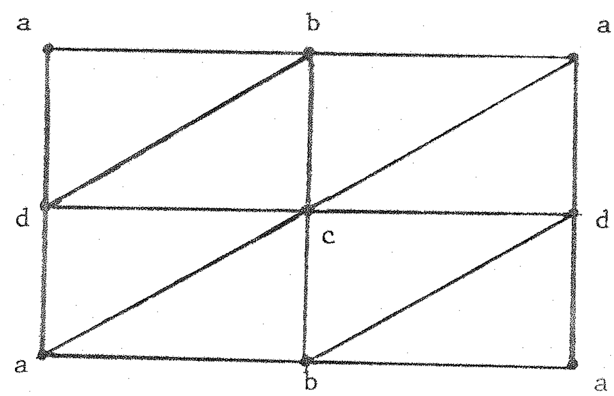


Figure (4.6) $\phi \varepsilon \mathfrak{D}_3^8$(T or K)

A representative map is indicated by the labelling of the vertices of the graph as shown in Figure (4.6).

In the third case (s = 4) we have the closed surface of Euler number e = −1 . Thus M is homeomorphic to $P_2(R) \# P_2(R) \# P_2(R)$. A representative graph K would have V = 3, E = 12 and F = 8, but no such graph exists. Hence $\mathfrak{D}_3^8(M) \neq \emptyset$ iff $M = S^2$ or M = T or M = K.

If k = 10, then e = 5(3−s)/s. The only solutions given by s = 3 and s = 5. In the first case e = 0 and we have a torus or a Klein bottle. The graph in both cases has 15 edges, and 5 vertices each of valency 6. Such a graph does not exists. In the second case e = −2, and a representative graph K would have V = 3, E = 15, F = 10. Again such a graph does not exist. It follows that $\mathfrak{D}_3^{10}(M) = \emptyset$.

Now, we consider the case $k = 12$. In this case $e = 6(3-s)/s$. The only solutions are given by $s = 3$ and $s = 6$. This corresponds to $e = 0$ and $e = -3$ respectively. In the first case a representative graph K has 18 edges and 6 vertices. See Figure (4.7). The corresponding map is indicated by labelling of the vertices of the graph.

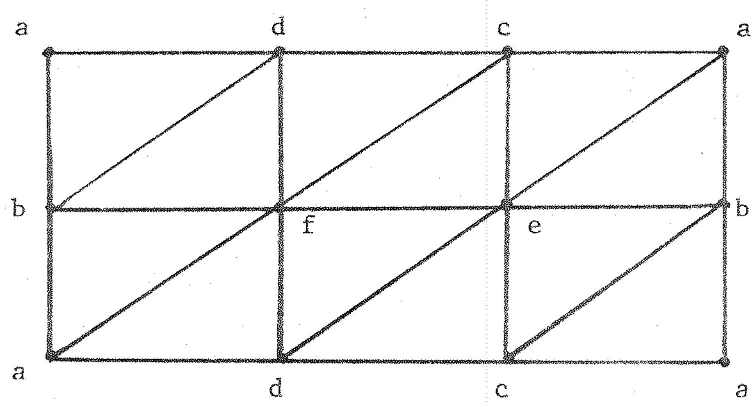


Figure (4.7). $\phi \varepsilon \mathfrak{D}_3^{12}$(T or K).

In the second case $E = 18$, $V = 3$, and $F = 12$. Such a graph does not occur. Hence $\mathfrak{D}_3^{12}(M) \neq \emptyset$ iff $M = T$ or $M = K$.

There is no difficulty in carrying on for $k \geq 14$ to know the sets $\mathfrak{D}_3^k(M)$, $k \geq 14$. Anyhow from the above discussions we can pick out the following results.

1. $\mathfrak{D}_3^k(S^2) \neq \emptyset$ iff $k = 8$,
2. $\mathfrak{D}_3^k(P_2(R)) \neq \emptyset$ iff $k = 4$,
3. $\mathfrak{D}_3^k(T) \neq \emptyset$ iff $k = 4m$, $m = 2, 3, \ldots$,
4. $\mathfrak{D}_3^k(K) \neq \emptyset$ iff $k = 4m$, $m = 2, 3, \ldots$.

It seems likely that no other closed surface can be regularly folded over a triangle.

7. Regular Foldings Over a Square

As in the case of foldings over a triangle, we regard a square as a topological disc whose frontier is stratified into 4 vertices and 4 arcs. Putting $p = 4$ in (4.8), we get $e = k(2-s)/s$. For each $k = 4, 6, 8, \ldots, 2m, \ldots,$ we can calculate e, E, V and s, and attempt to construct a corresponding graph on a surface of Euler number e.

We observe first that $\mathfrak{D}_4^k(S^2) = \emptyset$ for all k, since if $e = 2$, then $s = 2m/(1+m)$ which is not positive for any integer $m \geq 2$.

On the other hand, $\mathfrak{D}_4^k(M)$ has many elements for M a torus or klein bottle. In fact, if $k = 2^{j+2} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where $j \geq 0$ and $p_1, \ldots, p_r$ are distinct odd primes, then there are at least n mutually inequivalent regular foldings of a torus over a square where n is the smallest integer such that $2n \geq (j+1)(\alpha_1 + 1) \ldots (\alpha_r + 1)$. For, let $k = 4rs$. Then we can partition a rectangle into 4rs rectangles as in Figure (4.8), to obtain such a folding.

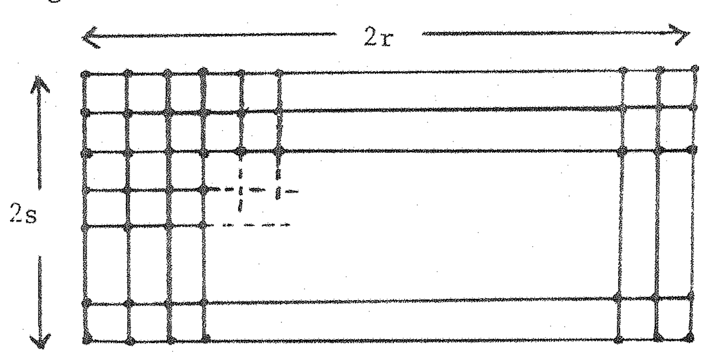


Figure (4.8). The graph K_ϕ of $\mathfrak{D}_4^k$(T or K)

Then the decomposition into 2r columns and 2s rows is equivalent to that into 2s columns and 2r rows when the edges of the rectangle are identified to give a torus.

Note, however that if we identify the vertical ends oppositely to give a Klein bottle, then this symmetry is lost for $r \neq s$, and so even more foldings are obtained in this case.

To illustrate these remarks, let $k = 24 = 2^3.3$. Then $j = r = \alpha_1 = 1$. So we construct two inequivalent foldings of a torus, and four for a Klein bottle. See Figure (4.9).

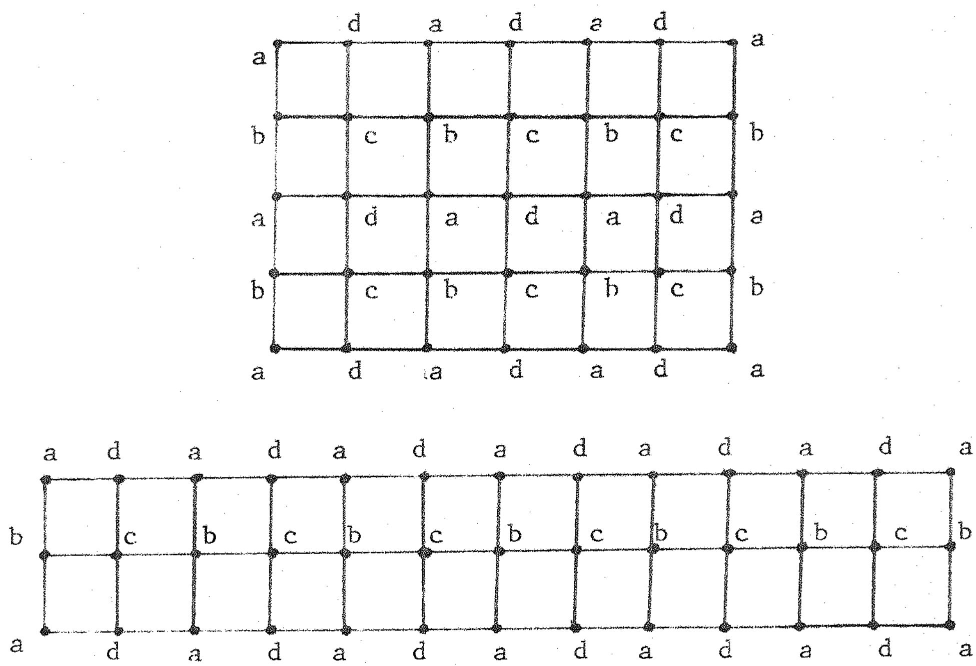


Figure (4.9). The graph K_ϕ, $\phi \varepsilon \mathfrak{D}^{24}_4$ (T)

We note that $\mathfrak{D}^4_4(M) \neq \emptyset$ iff M is a torus or Klein bottle. For $k > 4$, other possibilities may exist, but we have not succeeded in constructing any example. In fact, it may be conjectured that non exists.

8. Regular Foldings Over Polygons

One may continue to explore the possibilities indicated by equation (4.8), for $p = 5, 6, \ldots$. Unfortunately the information so obtained gives no help in deciding whether a regular folding exists with such a specification.

In this context, the use of many regular tessellations of the hyperbolic plane may prove fruitful.

To conclude, we point out that there is a regular 8-folding with valency 4 of the double torus M over a pentagon. This indicated in Figure (4.10), below. We embed M in E^3 in such a way that M is invariant under the group $Z_2 \times Z_2 \times Z_2$ generated by reflexions in the three coordinate planes. Then the familiar map $f : E^3 \to E^3$ given by $f(x, y, z) = (|x|, |y|, |z|)$ restricts to M to give an 8-regular folding $\phi : M \to M$ whose image is a topological disc with pentagonally subdivided rim.

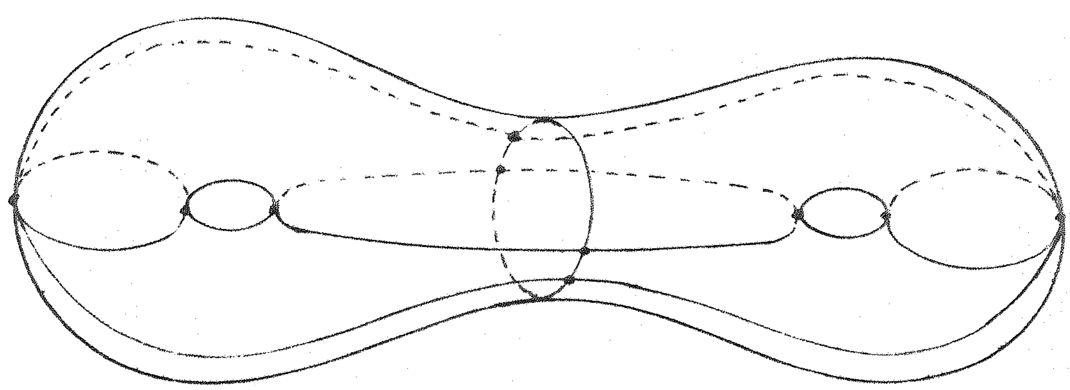


Figure (4.10). K_ϕ, $\phi \in \mathcal{D}^8_5(T)$

APPENDIX

THE VOLUME OF SPHERES AND GEODESIC DISCS

In chapter 1, § (2.3), we discussed certain inequalities involving the n-volume $\Sigma_n = \text{Vol } S^n$ of the unit n-sphere in E^{n+1}. In fact, it is possible to give explicit formulae for Σ_{2n+1} and Σ_{2n}.

It may be shown by straightforward integration (see, for example, Coxeter {3}) that

$$\Sigma_n = 2\,\Gamma\,(\tfrac{1}{2})^{n+1}/\Gamma\,(\tfrac{1}{2}(n+1)) \tag{1}$$

But $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$, so

$$\Sigma_n = 2\pi^{\frac{1}{2}(n+1)}/\Gamma(\tfrac{1}{2}(n+1)). \tag{2}$$

Also, from the recurrence relation

$$\Gamma(m+1) = m\Gamma(m),$$

we can deduce that

$$\Sigma_{n+2} = 2\pi\Sigma_n/(n+1). \tag{3}$$

Since $\Sigma_1 = 2\pi$ and $\Sigma_2 = 4\pi$, we find that

$$\Sigma_{2n+1} = \frac{2}{n!}\,\pi^{n+1}, \tag{4}$$

and

$$\Sigma_{2n} = (2^{2n+1}\,\pi^n)\,\frac{n!}{(2n)!} \tag{5}$$

Now the n-volume $\Sigma_n(R)$ of a sphere of radius R in E^{n+1} is given by

$$\Sigma_n(R) = R^n\,\Sigma_n\,. \tag{6}$$

Thus

$$\Sigma_{2n+1}(R) = \frac{2}{n!}\,\pi^{n+1}\,R^{2n+1}, \tag{7}$$

and

$$\Sigma_{2n}(R) = (2^{2n+1}\,\pi^n)\,\frac{n!}{(2n)!}\,R^{2n}. \tag{8}$$

Now, denote by $D_n(r)$ the volume of a closed geodesic disc of radius r and with any centre on S^n. Then, from (7) and (8) we have,

$$D_{2n+1}(r) = \frac{2\pi^{n+1}}{n!} \int_o^r (\sin\theta)^{2n+1} d\theta, \qquad (9)$$

and

$$D_{2n}(r) = (2^{2n+1}\pi^n) \frac{n!}{(2n)!} \int_o^r (\sin\theta)^{2n} d\theta, \qquad (10)$$

where $R = \sin\theta$, $0 \leq \theta \leq r$. See Figure (1) in the case of S^2.

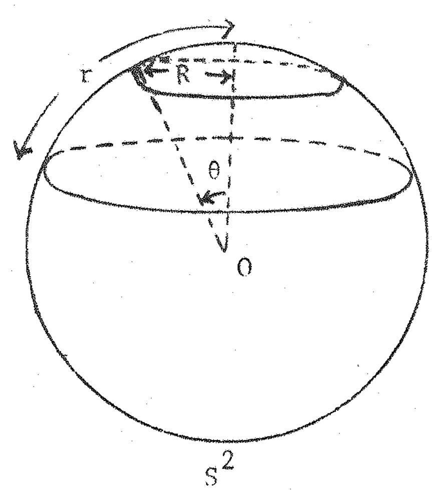


Figure (1)

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