ON A COVARIANT 2+2 FORMULATION OF THE INITIAL VALUE PROBLEM IN GENERAL RELATIVITY

by

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ABSTRACT

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ON A COVARIANT 2+2 FORMULATION OF THE INITIAL VALUE PROBLEM IN GENERAL RELATIVITY

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The initial value problems in general relativity are considered from a geometrical standpoint. First of all a covariant 3+1 formalism is developed for the investigation of non space-like initial value problems. This involves an analysis of the problem of locally imbedding a family of null hypersurfaces in space-time. More precisely, an intrinsic affine connection is constructed on each hypersurface, a rigging is then introduced, and the resulting Gauss-Codazzi equations are derived. The fundamental differences between foliations of space-time into space-like and null hypersurfaces are demonstrated, and the difficulties of applying this work to non space-like initial value problems is discussed. The major part of this thesis is concerned with the development of a covariant 2+2 formalism in which space-time is foliated by space-like 2-surfaces. This foliation is then rigged by a suitably chosen pair of directions in the orthogonal time-like 2-surface elements. The resulting 2+2 break-up of the Einstein vacuum field equations is then used to investigate space-like, characteristic and mixed initial value problems. In each case the so-called conformal 2-structure (essentially the conformal metric of the foliation into space-like 2-surfaces) is identified as explicitly embodying the true gravitational degrees of freedom. By means of the formalism, the geometrical significance of both the physically meaningful initial data and the various possible gauge choices is made clear. Finally, a Lagrangian formulation is included which supports the rôle of the conformal 2-structure as the dynamical variables of the pure gravitational field.

Introduction

The study of initial value problems (or IVP's as we shall refer to them) in relativity is of central importance to the theory as a whole. First of all, IVP's give insight into the analytical structure of the field equations through the investigation of the uniqueness, existence and stability of solutions. Secondly, IVP's provide a means of studying the geometrical structure and physical content of the theory, since they are closely related to the problem of identifying the true dynamical degrees of freedom of the gravitational field. This in turn is of fundamental importance in any investigation of such phenomena as gravitational radiation, and the crucial problem of the quantisation of the gravitational field. Thirdly, IVP's help in the formulation of the problem of actually computing numerical solutions to the field equations. Obtaining such solutions is becoming feasible with the development of high speed computers. One particular application of current interest relates to the discovery of the binary pulsar. This provides one of the first opportunities for accurately testing the theory against observation, since approximate solutions of the two-body problem are involved. In this thesis, we shall be concentrating on the second of the aspects of the IVP's that we have mentioned, although hopefully the techniques developed will be of use in the investigation of the third.

In Chapter I, after some general remarks concerning the nature of the Cauchy problem for a system of partial differential equations, we review one of the earliest formulations of the Cauchy problem in relativity, namely that due to Lichnerowicz(1). In Chapter II we review the covariant, or geometrical, 3+1 formulation of the Cauchy problem, due first to Stachel(2) and later developed by York(3) and others. In particular, we introduce at this stage those geometrical techniques modifications of which are needed in later chapters. In Chapter III we turn our attention to characteristic and mixed IVP's which we shall term collectively as non space-like IVP's, reviewing in particular the work of Sachs[4] and Tamburino and Winicour[5]. In Chapter IV the possibility of a geometrical 3+1 formulation of non space-like IVP's is investigated. The main new result in this chapter concerns the problem of locally imbedding a family of null hypersurfaces in space-time. At the end of this chapter we discuss the major problems which must surround any attempt to pursue this approach further.

The remainder of this thesis deals with a geometrical 2+2 formulation of the IVP's and all the results presented are believed to be new. Chapter V develops the necessary 2+2 formalism, and Chapters VI and VII discuss this formalism as applied to Cauchy and non space-like IVP's respectively. Chapter VIII deals with the derivation of the field equations from a Lagrangian. Finally, some concluding remarks are made, summarising the main results presented in this thesis, and indicating remaining problems and possible future areas of research.

Except where specifically indicated, Greek indices run from 0 to 3. Small Latin indices a,b,c,... run from 0 to 1, and i,j,k,... run from 1 to 3. Capital Latin indices run from 2 to 3. The signature of the space-time metric is assumed to be (+−−−), and a general space-time is denoted by V. Further notation will be introduced where necessary. Our conventions follow Schouten[6] in the main. The notation is adapted from both Schouten and York.[3]

All our considerations are purely local (although hopefully the 2+2 formalism developed will eventually allow non local problems to be investigated) and for simplicity we work only in source-free regions of V, that is in regions where the Einstein vacuum field equations are assumed to hold. Furthermore it is tacitly assumed throughout that the space-time metric can be expanded in a power series about the initial surface(s) since this is a necessary condition for the validity of the various formal integration schemes for the field equations which we discuss. This is undoubtedly a very strong assumption to make, but we leave a more detailed discussion of the significance of the restriction to analytic solutions to the conclusion.

Chapter I. The Cauchy Problem in General Relativity

1.1. Introduction

What precisely is a Cauchy problem? For a determined system of partial differential equations of order n,

$$F_A(y_B, \partial_\alpha y_B, \ldots, \partial^n_{\alpha\beta\ldots\gamma} y_B) = 0 , \qquad (1.1.1)$$

where $x^\alpha, \alpha = 0, \ldots\ldots, m$ are the independent variables, and $y_A, A = 0, \ldots., k$, are the unknown functions, Cauchy's problem consists of giving data on some arbitrary hypersurface $\phi_o : \phi(x^\alpha) = 0$, in the space of independent variables, which are necessary and sufficient to determine a solution in some neighbourhood of ϕ_o.(7) The Cauchy data for (1.1.1) on ϕ_o consist of y_A and their first $n-1$ derivatives in a direction out of ϕ_o. If ϕ_o is such that the Cauchy data determine on ϕ_o, through (1.1.1), the nth derivatives of y_A out of ϕ_o, then ϕ_o is said to be a free surface. In this case, under the assumption of analyticity of both the Cauchy data and of F_A, then the Cauchy-Kowalewski theorem guarantees the existence of a unique analytic solution of (1.1.1) in some neighbourhood of ϕ_o, determined by the Cauchy data. Under less stringent assumptions than analyticity, existence can only be expected if further restrictions on (1.1.1) and ϕ_o are imposed. For example, if (1.1.1) are normal hyperbolic equations of second order, then a necessary condition for existence is that ϕ_o is not only free but space-like.

If ϕ_o is such that the Cauchy data, through (1.1.1), do not determine the nth derivatives of y_A in a direction out of ϕ_o, then (1.1.1) must be restrictions on the Cauchy data. Moreover, even if Cauchy data are given such that these restrictions are satisfied, these data will not determine a unique solution of (1.1.1) in a neighbourhood of ϕ_o, even under the assumption of analyticity. In this case, ϕ_o

is called a characteristic surface, and gives rise to characteristic and mixed IVP's. (In Appendix A, we consider the various types of IVP for the simplest of all second order partial differential equations possessing characteristic surfaces, namely the one-dimensional wave equation. This serves as a useful prototype for illustrating the various different IVP's in relativity.)

1.2. The Structure of the Einstein Vacuum Field Equations

The Einstein field equations for a source-free gravitational field are

$$ {}^4R_{\alpha\beta} = 0 \,, \qquad (1.2.1)$$

where ${}^4R_{\alpha\beta}$ is the Ricci tensor of the normal hyperbolic metric ${}^4g_{\alpha\beta}$ of V. These equations are ten second-order, quasi-linear partial differential equations in the ten unknowns ${}^4g_{\alpha\beta}$. Very naively we might expect that specification of the ten functions ${}^4g_{\alpha\beta}$ and their first time-like derivatives on some space-like hypersurface in V would determine ${}^4g_{\alpha\beta}$ in a neighbourhood of the initial surface. However, it is easy to see why this cannot be the case. Equations (1.2.1) are tensor equations, and are thus invariant under the four-dimensional gauge group of arbitrary coordinate transformations. Thus, suppose we have solution functions ${}^4g_{\alpha\beta}$ such that they and their first time derivatives take on specified values on an initial space-like hypersurface. We can make coordinate transformations which will leave the initial surface and the initial data unchanged, but which will change the functional form of ${}^4g_{\alpha\beta}$ elsewhere. Such transformations yield mathematically distinct solutions for the same initial data, and so no set of initial data can uniquely determine all the components of ${}^4g_{\alpha\beta}$ in a neighbourhood of the initial surface. Of course, the new solutions obtained by such coordinate transformations are not to be regarded as physically distinct, but merely different descriptions of the same physical system because the metric tensor contains not only physical information, but information about the particular coordinates being used. Since less than the full ten components of ${}^4g_{\alpha\beta}$ are required to determine uniquely the physics of a given system, we should expect that the field equations themselves do not determine all the components of ${}^4g_{\alpha\beta}$. However, we are left with the fact that (1.2.1), as they stand, do not form a determined system of equations for ${}^4g_{\alpha\beta}$. In fact, from what we have said so far, it might appear that we have an overdetermined system since we have ten equations in less than ten unknowns. However, the field equations are not independent, they must satisfy the four differential constraints which arise from the contracted Bianchi identities.

$$ {}^4\nabla_\varepsilon {}^4G^\varepsilon{}_\alpha = 0 \ , $$

where ${}^4G_{\alpha\beta}$ is the Einstein tensor of V , and ${}^4\nabla_\varepsilon$ denotes the covariant derivative in V . We shall discuss the rôle of these identities more fully later.

In order to deal with the gauge invariance of the field equations, it is necessary to impose four coordinate, or gauge conditions on ${}^4g_{\alpha\beta}$ in such a way that they single out just one member of each equivalence class of solutions that represents a given physical system. This can be done in a number of ways, not all of which can be formulated in a covariant (and hence geometrically meaningful) manner. The most important example of non covariant coordinate conditions is the use of so-called harmonic coordinates, which are differential constraints on ${}^4g_{\alpha\beta}$. These enable one to deal with a reduced system of field equations which are no longer invariant under arbitrary coordinate transformations, and which have very nice mathematical properties. Specifically, they form a determined system of ten rigorously hyperbolic-normal equations in ${}^4g_{\alpha\beta}$. All of the 'hard' theorems concerning uniqueness, existence and stability of solutions to the field equations, under less stringent assumptions than analyticity, have been proved using these coordinate conditions. We mention here the pioneering work of Choquet-Bruhat(8) on the Cauchy problem, and the more recent work of Müller zum Hagen and Seifert(9) on the non space-like IVP's.

The major drawback of harmonic coordinates is that due to their non covariant nature they are very difficult to interpret geometrically, and we shall not consider them further. On the other hand, we can impose four algebraic conditions on ${}^4g_{\alpha\beta}$ (or on geometrical objects built out of ${}^4g_{\alpha\beta}$), which separate the components of ${}^4g_{\alpha\beta}$ into two groups, one whose evolution is determined by the field equations, and the other to which arbitrary values may be assigned. An approach of this type was first considered by Lichnerowicz(1) and we shall review his results briefly in the next section.

1.3. Lichnerowicz's Formulation of the Cauchy Problem.

Let $\overset{o}{\Sigma} : x^o = 0$ be an initial space-like hypersurface in V . That is, let

$$g^{oo} > 0 \tag{1.3.1}$$

on $\overset{o}{\Sigma}$, and in some neighbourhood of $\overset{o}{\Sigma}$. (We drop the prefix ^{4} on ${}^4g_{\alpha\beta}$, ${}^4R_{\alpha\beta}$, etc., when we are referring to their components in an adapted coordinate system.) The vacuum field equations can then be written in standard form:

$$R_{ij} = -\tfrac{1}{2}\, g^{oo}\, \partial_o^2\, g_{ij} + M_{ij} = 0 \tag{1.3.2}$$

$$G^o{}_o = -\tfrac{1}{2}\, g^{oo}\, M_{oo} + g^{ij}\, M_{ij} = 0 \tag{1.3.3a}$$

$$G^o_i = -g^{oo}\, M_{io} - g^{oj}\, M_{ij} = 0 \tag{1.3.3b}$$

where $M_{\alpha\beta}$ are independent of $\partial_o^2 g_{\alpha\beta}$. We see immediately that the field equations are independent of $\partial_o^2 g_{o\alpha}$. This suggests using up the four coordinate conditions by assigning arbitrary values to $g_{o\alpha}$ everywhere in V (subject to maintaining (1.3.1), of course) and we shall assume that this has been done. Equations (1.3.2) then form a determined system of second order hyperbolic equations in g_{ij}. $\overset{o}{\Sigma}$ is a free surface of the system (1.3.2), i.e. specification of the Cauchy data g_{ij} and $\partial_o g_{ij}$ on $\overset{o}{\Sigma}$ determines $\partial_o^2 g_{ij}$ on $\overset{o}{\Sigma}$ through (1.3.2) and (under the assumption of analyticity) g_{ij} is determined in a neighbourhood of $\overset{o}{\Sigma}$ by means of successive differentiation of (1.3.2) with respect to x^o, and the expansion of g_{ij} in a power series in x^o about $\overset{o}{\Sigma}$. Equations (1.3.2) are termed the evolution equations.

We now turn our attention to equations (1.3.3). These are clearly constraints upon g_{ij}. However, the Bianchi identities can be used to prove a remarkable lemma, namely that if equations (1.3.2) hold everywhere, and (1.3.3) hold on the initial surface $\overset{o}{\Sigma}$ then the latter equations hold everywhere. Hence (1.3.2) act as constraints only upon the Cauchy data.

To summarise the above arguments: in order to determine a solution of the field equations in a neighbourhood of a space-like hypersurface $\overset{o}{\Sigma} : x^o = 0$, the four components $g_{o\alpha}$ must be prescribed arbitrarily in that neighbourhood, subject to maintaining (1.3.1). Then the Cauchy data g_{ij} and $\partial_o g_{ij}$ satisfying the four constraint equations (1.3.3) must be specified on $\overset{o}{\Sigma}$. The evolution equations (1.3.2) then determine g_{ij} in a neighbourhood of $\overset{o}{\Sigma}$, and the Bianchi identities ensure that the constraint equations remain satisfied off $\overset{o}{\Sigma}$.

1.4. Conclusion

The use of a particular coordinate system in the above formulation rather obscures the geometrical significance of both the gauge conditions chosen, and of the Cauchy data. That is to say, the use of coordinate hypersurfaces and partial derivatives destroys the covariance of the approach. Arnowitt, Deser and Misner(10), in their canonical formulation of the dynamics of general relativity, essentially adopt the Lichnerowicz coordinate conditions. They introduce the notion of a 3+1 space + time break up of the metric, into components lying in the three-dimensional hypersurfaces x^o = constant and a one-dimensional time-like direction out of x^o = constant . They interpret the Cauchy data set as being equivalent to giving the first and second fundamental forms (i.e. the intrinsic and extrinsic curvatures) of the initial surface. The coordinate conditions then determine the time development of the initial surface and relate the intrinsic coordinate system of the initial surface to those of successive surfaces x^o = constant. However, although their work is particularly powerful and contains considerable insight, they still make use of particular coordinate systems adapted to a family of space-like hypersurfaces. Stachel(2) was the first to remedy this defficiency by formulating the Cauchy problem in a coordinate independent (covariant, geometrical) way and we review his work, together with the later developments by York(3) and others, in the next chapter.

Chapter II. The Covariant 3 + 1 Formulation of the Cauchy Problem.

2.1 Introduction

In the covariant formulation of the 3 + 1 approach to the Cauchy problem, the essential ideas are to replace the coordinate hypersurfaces x^o = constant by an arbitrary family of space-like hypersurfaces foliating space-time, and to introduce an arbitrary congruence of time-like curves which thread the foliation. The Lie derivative with respect to the tangent vector of this (suitably parametrised) congruence is then used as the natural generalisation of the partial derivative with respect to x^o(11). By means of appropriately constructed projection operators(12) it is then possible to break up space-time objects covariantly into components tangential and orthogonal to the foliation. In particular, a break up of the field equations into those governing the evolution of the Cauchy data, and those which act as constraints on this data, is induced. In what follows we shall lay particular emphasis on the geometrical techniques involved, since generalisations of these will be needed later on.

2.2 Foliations of Space-time by Rigged Space-like Hypersurfaces.

A foliation $\{\Sigma\}$ of V into hypersurfaces ($\{\Sigma\}$ is a foliation of codimension 1) is defined by a closed one-form n, say, with components n_α in an arbitrary space-time basis E^α (reciprocal basis E_α). Since n is closed we have

$$dn = 0 \Longleftrightarrow {}^4\nabla_{[\alpha}n_{\beta]} = 0 \qquad (2.2.1)$$

hence (locally) there exists a scalar function ϕ such that

$$n = d\phi \longleftrightarrow n_\alpha = {}^4\nabla_\alpha \phi \tag{2.2.2}$$

and each hypersurface $\Sigma \in \{\Sigma\}$ arises (locally) as a level surface of ϕ .

An arbitrary basis B_i of vectors tangent to $\{\Sigma\}$ must satisfy

$$< n, B_i > \;=\; n_\alpha B_i^\alpha = 0 \tag{2.2.3}$$

where $<\;\;>$ denotes inner product, and B_i^α are the components of B_i in the general space-time basis. For any vector $\vec{v}$ tangent to $\{\Sigma\}$, we have

$$\vec{v} = v^\alpha E_\alpha \;=\; v^i B_i \tag{2.2.4}$$

where v^α and v^i are the components of $\vec{v}$ with respect to E_α and B_i respectively. From (2.2.3) and (2.2.4), we get

$$v^\alpha = B_i^\alpha v^i, \tag{2.2.5}$$

Hence B_i^α act as connecting quantities of $\{\Sigma\}$. They give the components of $\vec{v}$ referred to E_α in terms of its components in the basis B_i .

We denote by ${}^4g$ the metric of V, and this has components

$${}^4g_{\alpha\beta} = {}^4g(E_\alpha, E_\beta)$$

in the general space-time basis. We define a vector $\vec{n}$ by

$$\vec{n} = {}^4g^{\alpha\beta} n_\alpha E_\beta = n^\alpha E_\alpha \tag{2.2.6}$$

where ${}^4g^{\alpha\beta}$ is the inverse of ${}^4g_{\alpha\beta}$. We shall, in general, follow the convention that vectors and one-forms corresponding in the natural isomorphism determined by ${}^4g$ are denoted by the same kernel letter, the vector being distinguished by the superscript $(\vec{\ })$. From (2.2.6), we see that

$$ {}^4g(\vec{n},B_i) = \langle n,B_i \rangle = 0 , \tag{2.2.7} $$

The necessary and sufficient condition that $\{\Sigma\}$ be a space-like foliation is that $\vec{n}$ is a time-like vector. That is

$$ {}^4g(\vec{n},\vec{n}) = {}^4g^{\alpha\beta}\, n_\alpha n_\beta \stackrel{\text{def}}{=} a^{-2} > 0 , \tag{2.2.8} $$

where a is a strictly positive function, the so-called lapse function. We shall assume that (2.2.8) holds for the remainder of this chapter; that is, that we are dealing with a foliation of space-like hypersurfaces. The unit normal vector to $\{\Sigma\}$ is given by

$$ \vec{u} = a\vec{n} . \tag{2.2.9} $$

A metric g is induced on each member of $\{\Sigma\}$ by the demand that for any vectors $\vec{v}, \vec{w}$ tangent to $\{\Sigma\}$,

$$ g(\vec{v},\vec{w}) = {}^4g(\vec{v},\vec{w}) . $$

The components of g in the basis B_i are given by

$$ g_{ij} = g(B_i,B_j) = {}^4g(B_i,B_j) = {}^4g_{\alpha\beta}B^{\alpha\beta}_{ij} , $$

where $B^{\alpha\beta}_{ij} \equiv B^\alpha_i B^\beta_j$.

The reciprocal basis of forms B^i in $\{\Sigma\}$ have components B^i_α in the general basis. These latter quantities act as connecting quantities for one-forms in $\{\Sigma\}$ in the same way that B^α_i do for vectors in $\{\Sigma\}$ (see equation (2.2.5)). The quantities B^i_α are determined if and only if we choose some vector field $\vec{t}$, defined up to an arbitrary scalar factor, transvecting $\{\Sigma\}$, and set

$$< B^i, \vec{t} > = 0 .$$

The vector $\vec{t}$ defines a rigging of $\{\Sigma\}$; for space-like hypersurfaces the natural rigging, and the one we shall adopt, is the direction orthogonal to $\{\Sigma\}$. With this choice the induced contravariant metric g on $\{\Sigma\}$, with components in the basis B^i given by

$$g^{ij} = {}^4g^{\alpha\beta}B^{ij}_{\alpha\beta} ,$$

is identical to the inverse g^{ij} of the induced metric. This in turn means that the natural isomorphisms, induced by ${}^4g$ and g between vectors and one-forms tangent to $\{\Sigma\}$, are identical.

With the natural choice of rigging, reciprocal bases of V are given by $(B_i,\vec{u})$ and (B^i,u). We can form the quantity

$$B^\alpha_\beta \overset{def}{=} B^\alpha_i \, B^i_\beta = \delta^\alpha_\beta - u^\alpha u_\beta \qquad (2.2.10)$$

The quantities B^α_i, B^i_α and B^α_β act as projection operators into $\{\Sigma\}$ on arbitrary tensors of V . For instance, for any vector $\vec{v}$, the elements

$${}'v^\alpha = B^\alpha_\beta v^\beta$$

are the components of a vector $'\vec{v}$ tangent to $\{\Sigma\}$, in the basis E_α . Similarly,

$${}'v^i = B^i_\alpha v^\alpha$$

are the components of $'\vec{v}$ in the basis B_i . Similar remarks hold for one-forms and higher order tensors. The quantity

$$C^{\alpha}_{\beta} = u^{\alpha}u_{\beta} \qquad (2.2.11)$$

projects tensors orthogonally to $\{\Sigma\}$. Any vector is the sum of its projections tangential and orthogonal to $\{\Sigma\}$. Higher order tensors can also be written as sums of their projections. For example an arbitrary tensor T of type (1,1) can be written in terms of its components as

$$T^{\alpha}{}_{\beta} = B^{\alpha\theta}_{\varepsilon\beta}T^{\varepsilon}{}_{\theta} + B^{\alpha}_{\varepsilon}C^{\theta}_{\beta}T^{\varepsilon}{}_{\theta} + C^{\alpha}_{\varepsilon}B^{\theta}_{\beta}T^{\varepsilon}{}_{\theta} + C^{\alpha\theta}_{\varepsilon\beta}T^{\varepsilon}{}_{\theta} .$$

Using a rather more concise notation, we can write the above as

$$T^{\alpha}{}_{\beta} = \perp T^{\alpha}{}_{\beta} + \perp T^{\alpha}{}_{\hat{u}}u_{\beta} + \perp T^{\hat{u}}{}_{\beta}u^{\alpha} + T^{\hat{u}}{}_{\hat{u}}u^{\alpha}u_{\beta} ,$$

where

$$\perp T^{\alpha}{}_{\hat{u}} \overset{\text{def}}{=} B^{\alpha}_{\varepsilon}\, u^{\theta}T^{\varepsilon}{}_{\theta} ,$$

and the obvious extensions. Then for example, $\perp T^{\alpha}{}_{\hat{u}}$ are the components in E_{α} of a vector tangent to $\{\Sigma\}$.

Although $\vec{u}$ is the unit normal vector to $\{\Sigma\}$, it is not the natural vector with which to propagate geometric objects onto successive members of $\{\Sigma\}$. The natural orthogonal connecting vector of $\{\Sigma\}$ is given by

$$\vec{N} = a\vec{u} \quad \Rightarrow < n,\vec{N} > = 1 \qquad (2.2.12)$$

The orthogonal metrical separation of neighbouring members of $\{\Sigma\}$ parameter distance $\delta\phi$ apart is $a\delta\phi$, where the lapse function, a , is given by equation (2.2.8) or equivalently by

$$a = \left({}^{4}g(\vec{N},\vec{N})\right)^{\frac{1}{2}} .$$

In fact, any vector $\vec{t}$ satisfying

$$< n,\vec{t} > \;= 1 \tag{2.2.13}$$

is a connecting vector of $\{\Sigma\}$. Equation (2.2.13) only defines $\vec{t}$ up to an arbitrary shift vector $\vec{b}$ tangent to $\{\Sigma\}$. That is, any vector

$$\vec{t} = a\vec{u} + \vec{b} = \vec{N} + \vec{b}\,, \quad < n,\vec{b} > = 0 \tag{2.2.14}$$

satisfies (2.2.13).

For a specific choice of shift vector, $\vec{t}$ is tangent to a particular congruence of curves $\mathcal{C}$ threading $\{\Sigma\}$ and fibrating V. These curves set up a one-to-one correspondence between points on the initial surface $\overset{o}{\Sigma}$ and points on any other slice Σ , by identifying points on the same curve of the congruence $\mathcal{C}$. The curves in $\mathcal{C}$ are parametrised by ϕ , and we may write

$$\vec{t} = \frac{\partial}{\partial\phi} \tag{2.2.15}$$

If $\phi = 0$ is the equation of the initial slice $\overset{o}{\Sigma} \in \{\Sigma\}$, the value of a geometric object Ψ_Λ on any other $\Sigma \in \{\Sigma\}$, parameter distance ϕ from $\overset{o}{\Sigma}$ is given by the generalised Taylor expansion

$$\Psi_\Lambda(\phi) = \exp\{\phi \pounds_t\}\Psi_\Lambda(0) \;. \tag{2.2.16}$$

We may think of the foliation and fibration of V described in this section as being generated in a rather different but fully equivalent way. Suppose that in V we are given an arbitrary vector field $\vec{t}$, together with some three-dimensional hypersurface $\overset{o}{\Sigma}$ transvecting $\vec{t}$. The fibration $\mathcal{C}$ is obtained from the integral curves of $\vec{t}$ as before, but the foliation $\{\Sigma\}$ is obtained by Lie dragging the initial surface $\overset{o}{\Sigma}$ with the vector field $\vec{t}$. The one-form n and the vector $\vec{n}$ can then be determined from the foliation. This latter approach is essentially that adopted by Stachel[(2)].

2.3 Projections of the Riemann Tensor and Covariant 3 + 1 Break-up of the Field Equations.

The only non zero projections of the space-time metric are

$$\perp {}^4g_{\alpha\beta} = B^{ij}_{\alpha\beta}\, g_{ij} = g_{\alpha\beta} \,. \tag{2.3.1a}$$

and

$${}^4g_{\hat{u}\hat{u}} = 1 \,. \tag{2.3.1b}$$

Similarly, for the contravariant metric, the non vanishing projections are

$$\perp {}^4g^{\alpha\beta} = B^{\alpha\beta}_{ij}\, g^{ij} = g^{\alpha\beta} \tag{2.3.2a}$$

and

$${}^4g^{\hat{u}\hat{u}} = 1 \,. \tag{2.3.2b}$$

It then follows from (2.3.1) and (2.3.2) that

$$g^{\alpha\varepsilon} g_{\beta\varepsilon} = B^{\alpha}_{\beta} \,.$$

Indices on quantities in $\{\Sigma\}$, written in the general space-time basis, can be raised and lowered by either the full metric of V, or by the induced metric on $\{\Sigma\}$. A covariant derivative ∇ is induced on $\{\Sigma\}$ by projection. We define, for any scalar λ

$$\nabla_\alpha \lambda = \perp {}^4\nabla_\alpha \lambda \tag{2.3.3}$$

and for a vector $\vec{v}$ tangent to $\{\Sigma\}$, we define

$$\nabla_\alpha v^\beta = \perp {}^4\nabla_\alpha v^\beta \tag{2.3.4}$$

Induced covariant derivatives of one-forms and tensors in $\{\Sigma\}$ are defined similarly. It is easy to show that with these definitions

$$\nabla_\gamma g_{\beta\alpha} = 0 , \tag{2.3.5}$$

that is, the induced connection on $\{\Sigma\}$ is the connection of the induced metric. The Riemann tensor of $\{\Sigma\}$ is defined by

$$R_{\delta\gamma\beta}{}^{\alpha} v^\beta = 2\nabla_{[\delta}\nabla_{\gamma]} v^\alpha \; ; \quad R_{\delta\gamma\beta}{}^{\alpha} u^\beta = 0 , \tag{2.3.6}$$

for arbitrary $\vec{v}$ tangent to $\{\Sigma\}$. This reduces to the usual definition of the Riemann tensor when written out in the adapted basis B_i . We define the Ricci tensor and Ricci scalar of $\{\Sigma\}$ by

$$R_{\gamma\beta} = R_{\varepsilon\gamma\beta}{}^{\varepsilon} \; ; \quad R = R_\varepsilon{}^\varepsilon .$$

The second fundamental form of $\{\Sigma\}$, $K_{\alpha\beta}$ is defined by

$$K_{\alpha\beta} = K_{(\alpha\beta)} = - \perp {}^4\nabla_\alpha u_\beta , \tag{2.3.7}$$

from which it is straightforward to show that

$$K_{\alpha\beta} = - \tfrac{1}{2}\pounds_u g_{\alpha\beta} . \tag{2.3.8}$$

$K_{\alpha\beta}$ is also called the extrinsic curvature of $\{\Sigma\}$, and up to a constant scalar factor it is the 'velocity' of $g_{\alpha\beta}$ as defined by Eulerian observers instantaneously at rest in each slice of $\{\Sigma\}$. It is possible to define another extrinsic curvature of $\{\Sigma\}$ by

$$h_{\alpha\beta} = -\tfrac{1}{2}\pounds_N g_{\alpha\beta} = aK_{\alpha\beta} \,, \tag{2.3.9}$$

so $h_{\alpha\beta}$ is the velocity of $g_{\alpha\beta}$ with respect to the natural connecting vector $\vec{N}$ of $\{\Sigma\}$ in the direction of the rigging. It is usual to work with $K_{\alpha\beta}$ rather than $h_{\alpha\beta}$ in the $3 + 1$ formulation of the Cauchy problem, since the former is independent of the behaviour of neighbouring members of $\{\Sigma\}$. That is, specifying $g_{\alpha\beta}$ and $K_{\alpha\beta}$ on one slice Σ characterises uniquely just that slice, as a hypersurface in space-time. If the lapse function a is taken to be a gauge quantity (as is the case in the covariant formulation of Lichnerowicz' approach which we discuss later in this chapter) then it makes no difference whether we work with $h_{\alpha\beta}$ or $K_{\alpha\beta}$, since they only differ by a factor a. However, if other gauge conditions are chosen, such as the maximal slicing condition where a is no longer a gauge variable, it is important to distinguish between $h_{\alpha\beta}$ and $K_{\alpha\beta}$. We shall not pursue this point further here. The reason for introducing $h_{\alpha\beta}$ is that its definition generalises to cases where we are no longer dealing with time-like rigging directions. For example, in Chapter IV we consider null rigging directions for foliations of space-time by null hypersurfaces, in which case there is no unit vector in the direction of the rigging, and we are led naturally to using a rigging vector which is normalised to the one-form defining the foliation. In the space-like case, such a vector is $\vec{N}$, which satisfies the normalising condition (2.2.12). In Chapter V, the situation is even more complicated, since we do not know, a priori, what the metrical properties of the rigging vectors are, and so again it is the generalisation of $h_{\alpha\beta}$ which is appropriate. However, for the remainder of this chapter, we shall follow the standard approach of York, and work with $K_{\alpha\beta}$.

Due to the large number of symmetries of the Riemann tensor ${}^4R_{\delta\gamma\beta\alpha}$ of V, there are only three independent projections. Two of these are given by the well known equations of Gauss and Codazzi for a Riemannian hypersurface imbedded in a Riemannian manifold. They are

$$\perp {}^4R_{\delta\gamma\beta\alpha} = R_{\delta\gamma\beta\alpha} + K_{\delta\beta}K_{\gamma\alpha} - K_{\gamma\beta}K_{\delta\alpha} \qquad \text{(Gauss)} \qquad (2.3.10a)$$

$$\perp {}^4R_{\delta\gamma\beta\hat{u}} = \nabla_\delta K_{\gamma\beta} - \nabla_\gamma K_{\delta\beta} \qquad \text{(Codazzi)} \qquad (2.3.10b)$$

The other independent projection is $\perp {}^4R_{\hat{u}\gamma\beta\hat{u}}$, and this is obtained by considering the definition of $\pounds_u K_{\gamma\beta}$ and using the Ricci identity. One obtains the result

$$\perp {}^4R_{\hat{u}\gamma\beta\hat{u}} = \pounds_u K_{\gamma\beta} + K_\gamma{}^\varepsilon K_{\beta\varepsilon} - a^{-1}\nabla_\gamma\nabla_\beta a \quad . \qquad (2.3.11)$$

Equation (2.3.11) is the only projection involving second time-like derivatives of $g_{\alpha\beta}$. Four of the field equations can be constructed solely from the Gauss-Codazzi equations, and only contain first derivatives of $g_{\alpha\beta}$ out of $\{\Sigma\}$. These four equations are

$$\perp {}^4G^{\alpha\hat{u}} = -\nabla_\varepsilon K^{\alpha\varepsilon} + \nabla^\alpha K = 0 \qquad (2.3.12a)$$

$$2\ {}^4G^{\hat{u}\hat{u}} = K^2 - K_{\varepsilon\theta}K^{\varepsilon\theta} - R = 0 \ . \qquad (2.3.12b)$$

Although we omit the details here, a covariant analysis of the Bianchi identities leads to the usual lemma, namely that (2.3.12) hold automatically everywhere provided they hold on $\overset{o}{\Sigma}$, and the remaining field equations hold everywhere. These latter equations, which do involve (2.3.11) as well as the other projections of ${}^4R_{\delta\gamma\beta}{}^\alpha$, are found to be

$$\perp {}^4R_{\gamma\beta} = \pounds_u K_{\gamma\beta} + 2K_{\gamma\varepsilon}K_\beta{}^\varepsilon - KK_{\gamma\beta} - a^{-1}\nabla_\gamma\nabla_\beta a + R_{\gamma\beta} = 0 \qquad (2.3.13)$$

One integration scheme for equations (2.3.12), (2.3.13) is the direct covariant analogue of Lichnerowicz's treatment: assume without loss of generality, that the initial hypersurface $\overset{o}{\Sigma}$ has equation $\phi = 0$, and specify the lapse function a, and the shift vector $\vec{b}$ quite arbitrarily in V (subject only to $a > 0$). The lapse and shift correspond to the four-dimensional gauge freedom of the theory. Next specify the Cauchy data $g_{\alpha\beta}$ and $K_{\alpha\beta}$ on $\overset{o}{\Sigma}$, subject to the constraints (2.3.12). We can determine $\pounds_t g_{\alpha\beta}$ on $\overset{o}{\Sigma}$ from $K_{\alpha\beta}$, since, by (2.2.14) and (2.3.8)

$$\pounds_t g_{\alpha\beta} = -2aK_{\alpha\beta} + 2\nabla_{(\alpha}b_{\beta)}. \qquad (2.3.14)$$

We now turn our attention to the evolution equations, (2.3.13). These are solved on $\overset{o}{\Sigma}$ for $\pounds_u K_{\alpha\beta}$, and hence $\pounds_t^2 g_{\alpha\beta}$, in terms of the Cauchy data and gauge variables. Successive Lie differentiation with respect to $\vec{t}$ of the evolution equations then allows the determination of $\pounds_t^{(k)} g_{\alpha\beta}$, $k = 2,3,4,\ldots.$ in terms of quantities depending only on the first $k-1$ derivatives of $g_{\alpha\beta}$, and the gauge variables. Then $g_{\alpha\beta}$, in some neighbourhood of $\overset{o}{\Sigma}$, is given by applying equation (2.2.16) to obtain

$$g_{\alpha\beta}(\phi) = \exp\{\phi\pounds_t\}g_{\alpha\beta}(0) \quad . \qquad (2.3.15)$$

The constraint equations now hold automatically off the initial hypersurface by virtue of the Bianchi identities. The metric, in some neighbourhood of $\overset{o}{\Sigma}$, is determined uniquely by the (constrained) initial data, and is given by

$$^4g_{\alpha\beta}(\phi) = \exp\{\phi\pounds_t\}g_{\alpha\beta}(0) + u_\alpha(\phi)u_\beta(\phi), \qquad (2.3.16)$$

where $u_\alpha(\phi)$ is given, using equations (2.2.1), (2.2.9) and (2.2.16), by

$$u_\alpha(\phi) = n_\alpha(0)\exp\{\phi\pounds_t\}a(0) .$$

The geometrical significance of the gauge variables a and $\vec{b}$ has already been discussed in section 2.2, but essentially, they provide a reference system in V, along whose trajectories the field equations are integrated. The initial data (subject to constraints) are the first and second fundamental forms of the initial surface. With the present choice of gauge, specifying $K_{\alpha\beta}$ is, from (2.3.14), equivalent to giving the natural velocity $\pounds_t g_{\alpha\beta}$ of the metric of the initial surface in the direction of $\mathcal{E}$.

All the equations (2.3.3) - (2.3.14) can be rewritten in the basis B_i , by formally replacing indices α,β,γ ... with i,j,k,... and by using the definition that for a tensor $T^{\alpha..}_{..\beta}$ tangent to $\{\Sigma\}$, and any vector $\vec{v}$ in V ,

$$\pounds_v T^{i..}_{..j} = B^{i.\beta}_{\alpha.j}\, \pounds_v T^{\alpha..}_{..\beta} \quad . \qquad (2.3.17)$$

Note that if we demand the basis of vectors B_i be Lie transported along the trajectories of $\vec{t}$, then (2.3.17) holds automatically. Such a basis B_i , together with $\vec{t}$, form what York terms the most general 'computational frames'. Such frames can be related in a natural way to 'Minkowskian observers at spatial infinity'.

2.4 Initial Data and True Gravitational Degrees of Freedom in the 3 + 1 Formulation.

Perhaps the greatest disadvantage of the 3 + 1 formalism is that there is no natural extension of this approach which would enable us to deal with non space-like IVP's, where data is set on (at least one) initial null hypersurface. (This problem is discussed further in Chapter IV). However, even as regards application of the 3 + 1 approach to the Cauchy problem, there remain certain difficulties concerned with the identification of the dynamical degrees of freedom of the gravitational field.

As we have seen, a sufficient set of Cauchy data needed to determine a unique solution of the evolution equations (2.3.13) are the twelve functions

$$\{g_{\alpha\beta}, K_{\alpha\beta}\} \text{ on } \overset{o}{\Sigma} \tag{2.4.1}$$

In the Lagrangian formulation of the 3 + 1 approach, $g_{\alpha\beta}$ are regarded as six configuration coordinates, with $K_{\alpha\beta}$ the corresponding velocities. In classical mechanical systems the velocities become momenta when passing from a Lagrangian to a Hamiltonian formulation. The situation is slightly more complicated in general relativity, however it turns out that the momenta $\pi^{\alpha\beta}$ are closely related to the velocities; they are given by

$$\pi^{\alpha\beta} = \sqrt{-g}\left(K^{\alpha\beta} - g^{\alpha\beta}K\right) \tag{2.4.2}$$

Now the initial data set (2.4.1) (or equivalently $\{g_{\alpha\beta},\pi^{\alpha\beta}\}$) are not freely specifiable on $\overset{o}{\Sigma}$, they must satisfy the four differential constraints (2.3.12). Thus modulo functions of two variables, there are really only eight independent initial data which may be set on the initial hypersurface. The question arises as to where within the constrained data (2.4.1) these independent data reside. There is no unique answer to this question. In principle one may choose any eight of the twelve functions in (2.4.1) as independent data, providing the constraint equations (2.3.12) form a consistent system for the remaining four functions. The approach which has been brought to considerable fruition by York and others is to adopt conformal 3-geometry techniques. In this approach, five of the independent data are taken to be the metric $g_{\alpha\beta}$ of $\overset{o}{\Sigma}$ specified only up to an arbitrary conformal factor, or equivalently, the conformal metric $\tilde{g}_{\alpha\beta}$, where in an adapted basis B_i on $\overset{o}{\Sigma}$,

$$\det(\tilde{g}_{ij}) = -1 \ .$$

The remaining three independent data are taken to be the transverse (covariantly divergence-free) - trace free part of $K_{\alpha\beta}$, and the trace K of $K_{\alpha\beta}$. York then showed that the constraint equations (2.3.12b) and (2.3.12a) become a system of four coupled quasi-linear elliptic partial differential equations for the conformal factor and the longitudinal part of $K_{\alpha\beta}$ respectively. The proof of uniqueness, existence and stability of these equations in the most general case is still the subject of current research, although a number of 'hard' theorems have been obtained. In particular, under the assumption that the trace of $K_{\alpha\beta}$ is constant on $\overset{o}{\Sigma}$, the constraint equations decouple and can be successfully analysed. This condition on K does not restrict the class of solutions to the field equations which may be considered. The eight independent functions which constitute the unconstrained initial data contain four gauge freedoms, three of which correspond to the intrinsic coordinate freedom within $\overset{o}{\Sigma}$, and one which describes how $\overset{o}{\Sigma}$ is imbedded in space-time. This latter gauge freedom is taken as being embodied in K. In particular, the condition

K = constant requires the mean extrinsic curvature of the initial surface to be constant.

The problem of identifying the true dynamical degrees of freedom is closely bound up with that of identifying the freely specifiable initial data. From the above discussion, it is clear that only four pieces of the unconstrained initial data are physically meaningful, and taking two of these data as true dynamical variables, and two as the corresponding velocities or momenta, then this is what is meant by saying that the gravitational field has only two degrees of freedom per space-time point. Thus although we may regard $g_{\alpha\beta}$ as configuration coordinates, they are not the true dynamical variables, since they give six pieces of information at each point, which is four too many. Rather, $g_{\alpha\beta}$ and $\pi^{\alpha\beta}$ are constrained generalised coordinates and momenta respectively, and if one works directly with these quantities then it leads eventually to a constrained Hamiltonian formulation. In the conformal 3-geometry approach, the true dynamical variables are taken on each hypersurface Σ to be the conformal metric $\tilde{g}_{\alpha\beta}$, modulo some choice of basis, with corresponding momenta given by the transverse traceless part of $\pi^{\alpha\beta}$. Thus although in this approach, the configuration coordinate $\sqrt{-g}$ (with conjugate momentum K) is definitely nondynamical, and can be eliminated from the constrained Hamiltonian by solving (2.3.13b), it is still the case that the remaining five configuration coordinates $\tilde{g}_{\alpha\beta}$ contain implicitly the true dynamical variables. That is to say, with the conformal 3-geometry approach one cannot write down explicitly just where within the space-time metric the independent functions representing the dynamical variables reside. Some improvement on this situation can be obtained by the use of a covariantly formulated 'radiation gauge' in which four covariant differential gauge conditions are imposed on $g_{\alpha\beta}$, or rather on the velocities of $g_{\alpha\beta}$. In this case, the lapse and shift cease to be arbitrarily specifiable, but are determined by four of the evolution equations. (This gauge involves what York terms the maximal slicing and minimal shift vector conditions). Nevertheless, since the gauge conditions on $g_{\alpha\beta}$ are differential, it is still not possible to write down in closed form explicit expressions for the generalised coordinates representing the true dynamical degrees of freedom. Whether or not other covariant gauge conditions in the 3+1 approach exist which would enable such a procedure to be carried out is still an open question. This problem with the explicit identification of the dynamical variables leads to certain difficulties, for example, when trying to use the 3+1 formulation to tackle the problem of quantization of the gravitational field. In the canonical quantisation procedure, a necessary first step is to isolate explicitly the two degrees of freedom of the gravitational field.

In the next chapter we shall see how in the classical approaches to the non space-like IVP's one can, at least in a suitable coordinate system, write down closed form expressions for the dynamical variables of the gravitational field.

Chapter III. Characteristic and Mixed Initial Value Problems.

3.1. Introduction.

As a general rule, the greatest insight into a system of partial differential equations comes from the study of the characteristic surfaces of the system. From equation (1.3.1) we see that x^o = constant are characteristic if and only if (1.3.1) do not determine $\partial_o^2 g_{ij}$, and this will be the case if and only if $g^{oo} = 0$. But this latter condition is necessary and sufficient for the surfaces x^o = constant to be null hypersurfaces of the space-time metric, that is surfaces everywhere tangent to the local light cone. Hence the characteristic surfaces of the gravitational field equations are identical to the null surfaces of the metric. It can be deduced in a straightforward manner[(13)] that the bicharacteristics of the system are the null geodesics which necessarily rule any null hypersurface. The physical interpretation of these results is that gravitational and electromagnetic phenomena have the same propagation properties; the wave fronts of both coincide, and the rays along which perturbations in the gravitational and electromagnetic fields spread are the null geodesics ruling the wavefronts. Thus, just from the mere determination of the characteristic surfaces of the Einstein field equations, we obtain an important physical result.

There are a number of physical and geometrical reasons for working with characteristic hypersurfaces. For example, in the study of gravitational radiation, it seems natural to work with a reference system which is adapted to the relevant ray congruence. Characteristic surfaces are bound to be of importance in cosmological problems also, since all information about the universe is received along the past null cone of an observer.

Non space-like IVP's, that is those based on families of null (as opposed to space-like) hypersurfaces, fall into two groups. As we shall see presently, it is not possible to determine uniquely a solution of the field equations by setting data on a single characteristic surface. Additional data must be set on some other initial hypersurface, intersecting the first in a space-like 2-surface. If this other initial surface is itself characteristic, then we are dealing with a characteristic or double-null IVP. If the other initial surface is time-like, then we have a mixed IVP. With regard to the latter, very little is known as to whether such problems are well set[14], although under our assumptions of analyticity there are no more fundamental difficulties in dealing with mixed rather than with characteristic IVP's.

For the remainder of this chapter we shall review some of the main formulations to date of non space-like IVP's. We emphasize here that they have all been based on the introduction of specific coordinate conditions; no successful attempt has yet been made to formulate covariantly either the characteristic or the mixed IVP.

3.2. The Light Cone Gauge.

The classical coordinate-dependent approaches to the non space-like IVP's have been based on the introduction of a coordinate system in which four components of the metric take on specific prescribed values. This is in contradistinction to the space-like case, where the gauge variables can take on more or less arbitrary values. Moreover, in order to isolate the physically meaningful, freely specifiable initial data, the lower dimensional coordinate freedom is used up by imposing (again specific) values on various components of the metric within the initial hypersurfaces. The first choice of four-dimensional coordinate freedom is in fact forced on us, since in order that $x^o \equiv u$ = constant be a family of null hypersurfaces, we must demand that

$$g^{oo} = 0 . \tag{3.2.1a}$$

(As in Chapter I, we drop the prefix ^{4} when referring to components of $^4 g^{\alpha\beta}$, etc., in a particular coordinate system.) Two other coordinate conditions are used up by demanding that x^1 be a parameter along the null geodesics ruling u = constant, in which case

$$g^{oA} = 0 . \tag{3.2.1b}$$

Conditions (3.2.1) use up three of the four available four-dimensional coordinate freedoms, and reduce the space-time metric to the so-called light cone gauge. The IVP in this gauge has recently been studied in a paper by Gambini and Restuccia[15], in the course of which they make various final choices of gauge and discuss the resulting integration schemes for field equations. There does not seem to be any one particular natural choice for the final gauge variable and indeed, although the integration schemes differ in detail for each choice, the overall features are essentially the same. We shall limit ourselves here to a brief review of the double-null problem in the gauge considered originally by Sachs[4] and of the mixed problem as investigated by Tamburino and Winicour[5].

3.3. Sachs' Formulation of the Double-null Initial Value Problem.

In addition to (3.2.1) we demand that $x^1 \equiv v$ = constant be a family of null hypersurfaces, that is

$$g^{11} = 0 . \tag{3.3.1}$$

The lower dimensional gauge freedom is used up by demanding first that u and v respectively are affine parameters along the null rays ruling the initial surfaces $\overset{o}{\Sigma}_o : v = 0$ and $\overset{o}{\Sigma}_1 : u = 0$ respectively, and that the null vectors tangent to the null rays ruling these respective initial surfaces are normalised to unity on the space-like intersection $\overset{o}{S}$ of $\overset{o}{\Sigma}_o$ and $\overset{o}{\Sigma}_1$. This implies that

$$g^{o1} = 1 \quad \text{on} \quad \overset{o}{\Sigma}_o \quad \text{and} \quad \overset{o}{\Sigma}_1 \ ; \quad g^{1A} = 0 \quad \text{on} \quad \overset{o}{\Sigma}_o \ . \qquad (3.3.2a)$$

Secondly, coordinates x^A are chosen on $\overset{o}{S}$ which reflect its conformal flatness; that is, such that

$$g^{AB} = \gamma^{-1}\,\delta^{AB} \quad \text{on} \quad \overset{o}{S} \qquad (3.3.2b)$$

The field equations are split into several groups. Using Sachs' terminology, we write

$$\left.\begin{array}{ll} R_{AB} = 0 \ , & \text{propagating equations} \\ R_{1A} = R_{11} = 0 \ , & \text{hypersurface equations} \end{array}\right\} \text{main equations} \qquad \begin{array}{r}(3.3.3a)\\(3.3.3b)\end{array}$$

$$R_{oA} = R_{oo} = 0 \ , \quad \text{subsidiary conditions} \qquad (3.3.3c)$$

$$R_{o1} = 0 \ , \quad \text{trivial equation.} \qquad (3.3.3d)$$

It turns out from an analysis of the Bianchi identities that providing the expansion of the null rays ruling u = constant is non zero, which we shall assume is the case, the following lemma holds: if the main equations (3.3.3a,b) hold in some sufficiently small region R, bounded from below by $\overset{o}{\Sigma}_o$ and $\overset{o}{\Sigma}_1$, then the trivial equation (3.3.3d) is an algebraic consequence of the main equations in R, and if the subsidiary conditions (3.3.3c) hold on $\overset{o}{\Sigma}_o$, then they hold everywhere in R.

In order to describe the integration scheme for the field equations, it is convenient to split g^{AB} into two parts

$$g^{AB} = \gamma^{-1}\, \tilde{g}^{AB} \,, \quad |\tilde{g}^{AB}| = 1 \,.$$

The subsidiary condition $R_{oo} = 0$, when evaluated on $\overset{o}{\Sigma}_o$ takes the form

$$\partial_o^2\, \gamma = M_{oo}(\gamma, \tilde{g}^{AB}).$$

(In this analysis, $M_{\alpha\beta}$ are known functionals of their arguments).

The above equation determines γ on $\overset{o}{\Sigma}_o$, provided the initial data

$$\tilde{g}^{AB} \text{ on } \overset{o}{\Sigma}_o \,;\; \gamma,\, \partial_o\, \gamma \text{ on } \overset{o}{S}$$

are given. Next the subsidiary conditions $R_{oA} = 0$ imply that on $\overset{o}{\Sigma}_o$

$$\partial_o \partial_1 g^{1A} = M_o{}^A(\partial_1 g^{1B},\, \gamma,\, \tilde{g}^{BC})$$

and this can be solved on $\overset{o}{\Sigma}_o$ for $\partial_1 g^{1A}$ providing the initial data

$$\partial_1 g^{1A} \text{ on } \overset{o}{S}$$

are given. On $\overset{o}{\Sigma}_1$, the hypersurface equation $R_{11} = 0$ has the form

$$\partial_1^2\, \gamma = M_{11}(\gamma, \tilde{g}^{AB})$$

which can be solved for γ providing that the remaining freely specifiable initial data

$$\tilde{g}^{AB} \text{ on } \overset{o}{\Sigma}_1 \,;\; \partial_1\, \gamma \text{ on } \overset{o}{S}$$

are given. At this stage the following are known:

$$g^{o1}(=1),\; g^{1A}(=0),\; g^{AB} \text{ and } \partial_1\, g^{1A} \text{ on } \overset{o}{\Sigma}_o \,;\; g^{o1}(=1) \text{ and } g^{AB} \text{ on } \overset{o}{\Sigma}_1 \,.$$

The above provide necessary and sufficient initial data to integrate the main equations in R . These equations can be written in the form

$$R_{11} = 0 \Rightarrow \partial_1 g^{o1} = M_{11}(g^{AB}) , \quad u \neq 0$$

$$R_{1A} = 0 \Rightarrow \partial_1^2 g^{1A} = M_1{}^A(g^{o1}, g^{1B}, g^{BC})$$

$$R_{AB} = 0 \Rightarrow \partial_1 \partial_o g^{AB} = M^{AB}(g^{o1}, g^{1C}, g^{CD}, \partial_o g^{CD})$$

on each hypersurface u = constant. Formally, we may think of solving the above equations as follows: $R_{11} = 0$ determines g^{o1} as a functional of g^{AB} on each u = constant. Inserting this functional into $R_{1A} = 0$ determines g^{1A} on u = constant, also as a functional of g^{AB} . These functionals are then inserted into $R_{AB} = 0$, and there results an equation for g^{AB} which can be solved (by successive differentiation) in the region R . Insertion of the solution functions g^{AB} into the appropriate functionals then yields g^{o1} and g^{1A} in R .

To summarise: we see that a unique solution to the field equations in a region bounded from below by the null surfaces $\overset{o}{\Sigma}_o$ and $\overset{o}{\Sigma}_1$ is determined by the following initial data:

$$\tilde{g}^{AB} \text{ on } \overset{o}{\Sigma}_o \; ; \; \tilde{g}^{AB} \text{ on } \overset{o}{\Sigma}_1 \; ; \; \gamma, \partial_o\gamma, \partial_1\gamma, \partial_1 g^{1A} \text{ on } \overset{o}{S} .$$

It is to be emphasised that this data is quite arbitrarily specifiable and not subject to differential constraints. Ignoring the lower dimensional data for the moment, we see that essentially the initial data required to determine a solution are two functions of three variables on each of the initial characteristic hypersurfaces. This demonstrates explicitly that the pure gravitational field has two degrees of freedom per space-time point, and that the dynamical variables in the particular coordinate system under consideration are the two independent functions contained in $\tilde{g}^{AB}$, or equivalently its inverse, $\tilde{g}_{AB}$. The dynamical equations can easily be shown to be

$$R_{AB} - \frac{1}{2} g_{AB} g^{EF} R_{EF} = 0 .$$

That these are only two independent equations follows immediately by contraction with g^{AB} .

3.4. The Tamburino-Winicour Formulation of the Mixed Initial Value Problem.

We start again from the light cone gauge (3.2.1), but this time we take the initial surface $\overset{o}{\Sigma}_o$ to be time-like. In fact Tamburino and Winicour take $\overset{o}{\Sigma}_o$ to be a time-like tube, with topology $S^2 \times \mathbb{R}$ which is taken to surround any sources of the external gravitational field. In this case the initial null surface $\overset{o}{\Sigma}_1$ is the future pointing outgoing null surface intersecting $\overset{o}{\Sigma}_o$ in some closed space-like 2-surface $\overset{o}{S}$. The initial data is then set on $\overset{o}{S}$, and the portions of $\overset{o}{\Sigma}_o$ and $\overset{o}{\Sigma}_1$ to the future of $\overset{o}{S}$. The region R in which the solution of the field equations is determined is exterior to $\overset{o}{\Sigma}_o$ and bounded from below by $\overset{o}{\Sigma}_1$.

The final four-dimensional coordinate freedom is used up by demanding that $x^1 \equi r$ be a 'luminosity parameter'[(16)] along the null rays ruling u = constant . That is, we set

$$|g^{AB}| = \left(r^4 f(x^{A}) ^2\right)^{-1} \qquad (3.4.1)$$

where f merely reflects the choice of intrinsic coordinates x^A within $\overset{o}{S}$. In particular, we can always take $f = 1$. The three-dimensional coordinate freedom within $\overset{o}{\Sigma}_o$ is used to demand that the null surfaces u = constant intersect $\overset{o}{\Sigma}_o$ in a family of geodesically parallel (with respect to the inner geometry of $\overset{o}{\Sigma}_o$) space-like 2-surfaces and that

x^A be constant along the associated inner time-like geodesics normal to the 2-surfaces u = constant. In this coordinate system $\overset{o}{\Sigma}_o$ has an equation

$$r = \eta(u,x^A)$$

where η is determined by the requirement that the full metric of space-time reduce to the inner metric of $\overset{o}{\Sigma}_o$ for $r = \eta$. If we write

$$g^{AB} = \gamma^{-1}\,\tilde{g}^{AB} \;, \quad |\tilde{g}^{AB}| = 1$$

and compare this with (3.4.1), we see that in this coordinate system, γ is a known function. The unknown components of the space-time metric in this gauge are g^{o1}, g^{11}, g^{1A} and $\tilde{g}^{AB}$.

The analysis of the field equations and the Bianchi identities follows similar lines to that given for the double-null problem in section 3.3. The only major difference is that the choice of $\overset{o}{\Sigma}_1$ as a time-like surface has the effect of considerably complicating the form of the subsidiary conditions; however a formal iterative scheme for their solution can still be constructed. The initial data required to uniquely determine a solution in R are

$$\tilde{g}^{AB} \text{ on } \overset{o}{\Sigma}_o \;;\; \tilde{g}^{AB} \text{ on } \overset{o}{\Sigma}_1 \;;\; g^{11} \;,\; \partial_1 g^{1A} \text{ on } \overset{o}{S} \;.$$

As with the double-null IVP, we see again that the solution is essentially determined by two functions of three variables given on each of $\overset{o}{\Sigma}_o$ and $\overset{o}{\Sigma}_1$, without constraints. The dynamical variables $\tilde{g}^{AB}$ (or $\tilde{g}_{AB}$) are again propagated by the trace-free part of $R_{AB} = 0$. Tamburino and Winicour in fact consider $\partial_o\, g^{AB}$ as the initial data on $\overset{o}{\Sigma}_o$, which they interpret as the 'news functions' of Bondi et al[17] and Sachs[18]. These functions contain information about the behaviour of sources within $\overset{o}{\Sigma}_o$. Clearly giving $\tilde{g}^{AB}$ or $\partial_o\, \tilde{g}^{AB}$ on $\overset{o}{\Sigma}_o$ is equivalent, for $\tilde{g}^{AB}$ is known on $\overset{o}{S}$, since $\overset{o}{S}$ is conformally flat.

3.5. Conclusion

This brief review of the coordinate dependent formulations of the non space-like IVP's serves to demonstrate the fundamental differences of such problems from the standard Cauchy problem. If we compare the results of this chapter with those of Lichnerowicz reviewed in Chapter I, we note first of all that in the present case the initial data is freely specifiable, and not subject to any differential constraints. Admittedly, the conformal 3-geometry approach identifies the freely specifiable initial data in the 3 + 1 approach to the Cauchy problem, but only by means of a relatively complicated analysis of the constraint equations. No such procedure is required in the approach to the non space-like IVP's discussed in this chapter. Moreover, (in contradistinction to the standard Cauchy problem) the initial data in the present case does not contain any coordinate or gauge freedom, but represents the true physically meaningful quantities defining the gravitational field. This means in turn that for characteristic and mixed IVP's the true gravitational degrees of freedom are, in coordinate dependent terms at least, readily identifiable as $\tilde{g}^{AB}$ or equivalently $\tilde{g}_{AB}$. Such a property, as was indicated at the end of Chapter II, is certainly desirable in any canonical quantisation procedure. However the use of particular coordinate systems, and somewhat ad-hoc choices of coordinate conditions at that, tends to rather obscure the geometrical interpretation of the dynamical variables $\tilde{g}_{AB}$, and various of the other field and gauge variables. The former have been interpreted by most authors as the conformal metric of the null hypersurfaces u = constant. As Sachs[(4)] points out, one only needs, in adapted coordinates, a 2 x 2 matrix to specify the inner metric of a null hypersurface, since such surfaces are degenerate. In adapted coordinates, one can readily invert g_{AB} to obtain g^{AB}.

However in a general coordinate system, the metric of a null hypersurface has no inverse, due to its degeneracy. Thus quite how one interprets g^{AB} geometrically is not clear. In fact, this indicates just one of the difficulties encountered when one tries to formulate non space-like IVP's covariantly using families of null hypersurfaces. One almost immediately runs into several severe problems, due to the fact that null surfaces are not Riemannian manifolds. This problem is discussed in some detail in the next chapter. The problem of covariantly formulating the coordinate conditions adopted in this chapter is resolved in Chapter VII by means of the $2 + 2$ formalism developed in Chapter V.

Chapter IV. On Locally Imbedding a Family of Null Hypersurfaces

4.1. Introduction

A natural approach to covariantly formulating any non space-like initial value problem is to consider a foliation of space-time into a family of null (as opposed to space-like) hypersurfaces, $\{N\}$, and to then attempt to adopt procedures analogous to those used in the covariant 3+1 formulation of the Cauchy problem. The crucial step in this latter formulation is the calculation of expressions for the various projections of ${}^4R_{\delta\gamma\beta\alpha}$ which are dependent only upon the intrinsic affine structure of $\{\Sigma\}$ and the rigging field $\vec{u}$, and not on the affine structure of space-time. (These expressions then allow the straightforward calculation of the projections of the field equations, which can then be used to study the Cauchy problem.) Some of the projections, at least, come from the equations of Gauss and Codazzi for a Riemannian hypersurface in a Riemannian manifold.

In general, Gauss-Codazzi equations arise as integrability conditions for a rigged submanifold with prescribed affine structure, imbedded in an affine manifold of higher dimension, and they invariably take the form of expressions for some of the projections of the Riemann tensor of the imbedding space into the submanifold. In fact, any manifold, considered as a submanifold of some higher dimensional affine space and provided with a rigging, has a unique connection induced on it, dependent only on the connection of the imbedding space and the rigging. In such cases, the Gauss-Codazzi equations are well known and given, for example, by Schouten(12). However, if the submanifold is endowed with some other affine structure, then the Gauss-Codazzi equations must be derived for that particular case.

It turns out, for reasons which will become apparent later in this chapter, that the induced connection in $\{N\}$ is not suitable for applications to the initial value problem, thus we first turn our attention to the problem of choosing a suitable affine connection on $\{N\}$. We then derive the resulting Gauss-Codazzi equations (this involves, as a necessary preliminary, introducing a suitable rigging for $\{N\}$) and using the techniques developed in this derivation, obtain expressions for the remaining projections of $^4R_{\delta\gamma\beta\alpha}$. The Gauss-Codazzi equations can be derived directly as integrability conditions for the imbedding of $\{N\}$, with the particular affine structure under consideration, in V . This has been done elsewhere,[18] but here we shall use a rather simpler method of derivation. This method does not perhaps bring out the significance of the Gauss-Codazzi equations as integrability conditions, but since we are only interested in the final result, namely that of obtaining suitable expressions for the various projections of $^4R_{\delta\gamma\beta\alpha}$ into $\{N\}$, this is no great disadvantage, and we reiterate that the alternative derivation is given in reference 18. We start however by reviewing in the next section some aspects of the intrinsic geometry of a null 3-manifold. This review draws heavily on work by Dautcourt.[19]

4.2. Affine Structure of a Null 3-surface

An n-dimensional null manifold is defined as one upon which a degenerate metric of rank $n-1$ is defined. We shall restrict our attention to $n=3$, and denote a 3-dimensional null manifold by N . We assume without loss of generality that the metric g of N has signature $(0 - -)$. Suppose B_i is an arbitrary basis on N , (reciprocal basis B^i). Then g is of rank 2 if and only if there exists a vector $\vec{k}$, say, such that

$$g(k,B_i) = 0 \Leftrightarrow g_{ij}k^j = 0 \ . \tag{4.2.1}$$

The vector $\vec{k}$ is defined up to scale factor by the above equation, and defines the null directions in N . If N is considered as a hypersurface in V , then the integral curves of $\vec{k}$ are null geodesics in V .

Any vector $\vec{v}$ not parallel to $\vec{k}$ is called space-like, and satisfies

$$g(\vec{v},\vec{v}) < 0 \quad \Longleftrightarrow \vec{v} \neq 0 \; .$$

A one-form v , corresponding to $\vec{v}$, is determined by

$$v = g_{ij} v^i B^j \overset{\text{def}}{=} v_j B^j \; . \tag{4.2.2}$$

Note that any other vector $'\vec{v} = \vec{v} + \lambda\vec{k}$ will define the same one-form v by the above equation. Any pair of linearly independent space-like vectors $\vec{v}$ and $\vec{w}$ (which are also linearly independent of $\vec{k}$) give rise to one-forms v and w which span a unique two-dimensional subspace of one-forms on N . We introduce a one-form n , linearly independent of this subspace and, in particular, use n to fix $\vec{k}$ uniquely by the normalisation condition

$$\langle n,\vec{k}\rangle = 1 \; . \leftrightarrow n_i k^i = 1 \; . \tag{4.2.3}$$

Since g_{ij} is degenerate, it does not have an inverse. However, we may introduce a substitute contravariant metric g^{ij}, defined uniquely if and only if both g_{ij} and n_i are specified. The defining equations are

$$g_{ik} g_{j\ell} g^{k\ell} = g_{ij} \tag{4.2.4a}$$

$$g^{ij} n_j = 0 \; . \tag{4.2.4b}$$

It follows immediately from (4.2.1), (4.2.3) and (4.2.4) that

$$g^{ik} g_{jk} = \delta^i_j - k^i n_j \; . \tag{4.2.5}$$

(Note that strictly speaking, we should use a different kernel letter for g_{ij} and g^{ij} . We do not, since there is no chance of confusion and it avoids unnecessary additional notation.)

Ideally, we should like to construct on N a linear connection which is both torsion-free and metric (as with the case of a Riemannian manifold). Unfortunately, it can be shown that except for a very narrow class of null manifolds, this is impossible.[20] A number of authors have considered the problem of constructing a connection on N , considering variously either the torsion-free property[19,21,22,23] or the metric property[24] to be more fundamental. Of these, references 21, 22 and 23 consider N as a hypersurface in V , and the connections used in these cases in fact depend on quantities extrinsic to N :- a property which seems to have dubious geometrical significance, and certainly renders them useless in applications to the initial value problem. In reference 24, Sokolowski considers a metric connection with torsion, intrinsic to N , but his resulting Gauss-Codazzi equations are valid only for a particular class of null hypersurfaces, and in addition are very complicated.

In what follows, we shall endow N with an affine connection first considered by Dautcourt.[19] The connection is non metric, torsion-free and dependent only upon quantities within N and their internal derivatives. Our motivation for using this connection is purely pragmatic - it enables us to derive a set of Gauss-Codazzi equations valid for any null hypersurface imbedded in space-time, and the resulting equations are suitable for studying non space-like initial value problems. We denote the components of the Dautcourt connection in the basis B_i by Γ^i_{jk} , and the corresponding covariant derivative by ∇ . Then the connection is defined by demanding

$$\nabla_i n_j = \nabla_{[i} n_{j]} \qquad (4.2.6a)$$

$$\nabla_i g_{jk} = 2\, h_{i(j} n_{k)} \qquad (4.2.6b)$$

$$\Gamma^i_{[jk]} = \tfrac{1}{2}\, C^i_{jk} \qquad (4.2.6c)$$

where

$$h_{ij} = h_{(ij)} \overset{\text{def}}{=} -\tfrac{1}{2}\, £_k\, g_{ij} \;. \qquad (4.2.7)$$

This implies

$$h_{ij} k^j = 0 \;, \qquad (4.2.8)$$

and C^i_{jk} are defined by

$$[B_i, B_j] = C^k_{ij}\, B_k \;.$$

Equation (4.2.6c) is merely the standard result expressing the fact that the connection is torsion-free.[25] The structure constants C^i_{jk} contain only information about the particular basis B_i chosen, and in particular they vanish for a coordinate basis. They contain no information about the affine structure of the manifold. In order to calculate a specific expression for Γ^i_{jk} we proceed as follows. From (4.2.6a) we obtain immediately

$$\Gamma^i_{(jk)}\, n_i = \partial_{(j} n_{k)} \;. \qquad (4.2.9)$$

Writing out (4.2.6b) gives

$$\partial_i\, g_{jk} - \Gamma^\ell_{ij} g_{\ell k} - \Gamma^\ell_{ik} g_{j\ell} = h_{ij} n_k + h_{ik} n_j \;.$$

If we cyclicly permute i,j,k in the above equation, and subtract the two resulting equstions, we obtain, remembering that h_{ij} is symmetric and using (4.2.6c)

$$\{i\ j\ k\} + \tfrac{1}{2}\left(g_{\ell k}C^{\ell}_{ij} + g_{\ell j}C^{\ell}_{ik}\right) - g_{i\ell}\Gamma^{\ell}_{(jk)} = n_i h_{jk} \ . \qquad (4.2.10)$$

where

$$\{i\ j\ k\} \stackrel{\text{def}}{=} \tfrac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right) \qquad (4.2.11)$$

is the Christoffel symbol of the first kind. We now contract (4.2.10) with g^{im}, and use (4.2.4b and 5). This yields, on substitution of (4.2.9),

$$\Gamma^{m}_{(jk)} = g^{im}\{i\ j\ k\} + k^{m}\partial_{(j}n_{k)} + \tfrac{1}{2}\left(g^{im}g_{\ell k}C^{\ell}_{ij} + g^{im}g_{\ell j}C^{\ell}_{ik}\right) \qquad (4.2.12)$$

Thus Γ^{i}_{jk} is given explicitly by (4.2.6c) and (4.2.12) .

It is clear from the above definitions that Γ^{i}_{jk} is uniquely defined if and only if g_{ij} and n_i are specified. It is unfortunate that the connection is dependent upon quantities other than g_{ij} , but it is not unexpected. In a three-dimensional Riemannian manifold, the metric connection is specified by the six independent functions contained in the metric. In N , the metric contains only three independent functions, and so it is hardly surprising that further information is needed to fix a connection. This information, for Dautcourt's connection, is supplied by the three independent functions contained in n_i .

The Riemann tensor of Γ^{i}_{jk} is defined in the usual way by

$$R_{\ell kj}{}^{i}v^{j} = 2\,\nabla_{[\ell}\nabla_{k]}v^{i}$$

for any vector $\vec{v}$ in N . In general, calculations involving the affine geometry of N are rather tedious, due to the non-metrical nature of the connection, but a judicious use of the defining equations (4.2.4), (4.2.5) for g^{ij} , (4.2.1), (4.2.3) for k^{i} , and (4.2.6) for the connection, allows us to calculate all the results we need. In particular, it is useful to obtain an expression for $\nabla_i k^{j}$. From (4.2.6b) we get

$$k^j \nabla_i g_{jk} = - g_{jk} \nabla_i k^j = h_{ki} \ .$$

Contracting this with $g^{k\ell}$ gives

$$g^{k\ell} g_{jk} \nabla_i k^j = \left(\delta^\ell_j - k^\ell n_j\right) \nabla_i k^j = - g^{k\ell} h_{ki} \ . \qquad (4.2.13)$$

Next, from (4.2.6a), we get

$$n_j \nabla_i k^j = - k^j \nabla_i n_j = k^j \nabla_j n_i \ .$$

Substituting the above in (4.2.13) then yields

$$\nabla_i k^j = - g^{\ell j} h_{\ell i} + k^j k^\ell \nabla_\ell n_i \ . \qquad (4.2.14)$$

The relationships (4.2.1), (4.2.3-4.2.6) and (4.2.8) are fundamental and most calculations on $\{N\}$ rely heavily on their use. They will be used, for the most part, without reference in the remainder of this chapter, and particularly in section 4.5.

4.3. Foliations of Space-time by Rigged Null Hypersurfaces

We shall follow an analogous procedure to that used in our discussion of a foliation of V by a family of space-like hypersurfaces, given in section 2.2. Although some of the results go through as in section 2.2 there are, as we shall see, some very important differences.

A foliation $\{N\}$ of V into null hypersurfaces is defined by a closed one-form k, say, with components k_α in the arbitrary space-time basis E^α, and which satisfies

$${}^4g^{\alpha\beta} k_\alpha k_\beta = 0 \ . \qquad (4.3.1)$$

Since k is closed, we have, as in section 2.2,

$$dk = 0 \iff {}^4\nabla_{[\alpha}k_{\beta]} = 0 \qquad (4.3.2)$$

and (locally) for some scalar function ϕ

$$k = d\phi \iff k_\alpha = {}^4\nabla_\alpha\phi \ . \qquad (4.3.3)$$

Hence each member of $\{N\}$ arises (locally) as a level surface of ϕ , and from (4.3.1) and (4.3.3)

$${}^4g^{\alpha\beta}\ {}^4\nabla_\alpha\phi\ {}^4\nabla_\beta\phi = 0 \ .$$

An arbitrary basis B_i of vectors tangent to $\{N\}$ must satisfy

$$\langle k, B_i \rangle = k_\alpha B_i^\alpha = 0 \ . \qquad (4.3.4)$$

B_i^α are the components of B_i in the general space-time basis and (c.f. section 2.2) act as connecting quantities of $\{N\}$ for vectors tangent to $\{N\}$.

The vector $\vec{k}$ defined by

$$\vec{k} = {}^4g^{\alpha\beta}k_\alpha E_\beta = k^\alpha E_\alpha \qquad (4.3.5)$$

is a null vector, since

$${}^4g(\vec{k},\vec{k}) = {}^4g^{\alpha\beta}k_\alpha k_\beta = 0 \ ,$$

by (4.3.1). Furthermore $\vec{k}$ is both tangent to $\{N\}$, since

$$\langle k,\vec{k} \rangle = {}^4g^{\alpha\beta}k_\alpha k_\beta = 0$$

and orthogonal to $\{N\}$, since

$${}^4g(\vec{k},B_i) = \langle k,B_i \rangle = 0 \ , \qquad (4.3.6)$$

by (4.3.4).

A metric g is induced on each member of {N} by demanding that for any vectors $\vec{v}$ and $\vec{w}$ tangent to {N}

$$g(\vec{v},\vec{w}) = {}^4g(\vec{v},\vec{w}) \quad . \tag{4.3.7}$$

The components of g in the basis B_i are given by

$$g_{ij} = g(B_i,B_j) = {}^4g(B_i,B_j) = {}^4g_{\alpha\beta}B^{\alpha\beta}_{ij} \tag{4.3.8}$$

Since $\vec{k}$ is tangent to {N}, we have by (4.3.6) and (4.3.7),

$$g(\vec{k},B_i) = {}^4g(\vec{k},B_i) = 0 \quad \text{<->} \quad g_{ij}k^j = 0 \tag{4.3.9}$$

Also, since $\vec{k}$ is null, and ${}^4g(\vec{k},B_i) = 0$, any other linearly independent vectors in {N} must be space-like, since a null vector can never be orthogonal to a time-like vector, and is orthogonal to another null vector if and only if it is parallel to it. From the above argument and equation (4.3.9), we see that each $N \in \{N\}$ is intrinsically a null 3-manifold, as described in section 4.2, by virtue of the metric induced upon it. Note that now, however, we have a normalisation of $\vec{k}$ already determined by equation (4.3.5). In particular, we can show that for this normalisation

$$k^\alpha \, {}^4\nabla_\alpha k^\beta = 0 \ ,$$

since

$$k^\alpha \, {}^4\nabla_\alpha k^\beta = {}^4g^{\gamma\beta}k^\alpha \, {}^4\nabla_\alpha k_\gamma = {}^4g^{\gamma\beta}k^\alpha \, {}^4\nabla_\gamma k_\alpha = \tfrac{1}{2} \, {}^4g^{\gamma\beta} \, {}^4\nabla_\gamma\left(k_\alpha k^\alpha\right) = 0 \ ,$$

by (4.3.5), (4.3.2) and (4.3.1). Hence $\vec{k}$ is not only tangent to null geodesics, but also affinely parametrised.

In order to fix B^i_α, the components of the reciprocal basis B^i in the general basis E^α, we must (c.f. section 2.2) define a rigging of {N}. In the case of a space-like foliation {Σ}, it is the requirement that the correspondence between vectors and forms in $\{\Sigma\}$ defined by the induced metric g be the same as that defined by the full metric ${}^4g$ that leads to the idea of a natural rigging direction for $\{\Sigma\}$. However, since the induced metric on $\{N\}$ is degenerate, we see from the discussion in section 4.2 that we cannot follow an analogous procedure for $\{N\}$. Indeed, the natural rigging for $\{\Sigma\}$ is the direction orthogonal to $\{\Sigma\}$, but the direction orthogonal to $\{N\}$ is, as we have seen, tangent to $\{N\}$ and hence does not provide a rigging. For the moment, we shall assume that we have chosen some arbitrary direction out of $\{N\}$ defined by a vector field $\vec{n}$, and normalised by the requirement that

$$\langle k,\vec{n}\rangle = 1 . \tag{4.3.10}$$

Then B^i_α is determined by the requirement that

$$\langle B^i,\vec{n}\rangle = B^i_\alpha n^\alpha = 0 . \tag{4.3.11}$$

As for space-like foliations, B^i_α are connecting quantities for one-forms in $\{N\}$. The "contravariant metric" $*g$ induced on $\{N\}$ by the requirement that for any one-forms v, w in $\{N\}$

$${}^4g^{\alpha\beta}v_\alpha w_\beta = *g^{ij}v_i w_j ,$$

has components

$$*g^{ij} = {}^4g^{\alpha\beta}B^{ij}_{\alpha\beta} \tag{4.3.12}$$

in the basis B_i, and hence is dependent upon $\vec{n}$.

Since $(B_i,\vec{n})$, (B^i,k) are reciprocal bases of V, as seen from (4.3.4), (4.3.10) and (4.3.11), we can form the quantities

$$B^\alpha_\beta \overset{\text{def}}{=} B^\alpha_i B^i_\beta = \delta^\alpha_\beta - n^\alpha k_\beta . \tag{4.3.13}$$

The quantities B^α_i, B^i_α and B^α_β act as projection operators, projecting tensors of V into {N} (c.f. section 2.2). Note that technically, we should write $B^{\alpha}{}_{\beta}$ for B^{α}_{β}, since

$$B^{\alpha}{}_{\beta} \neq B_{\beta}{}^{\alpha} = \delta^{\alpha}_{\beta} - n_{\beta}k^{\alpha} .$$

However, we shall always understand that $B^{\alpha}_{\beta} \equiv B^{\alpha}{}_{\beta}$. Similarly, we define

$$C^{\alpha}_{\beta} = n^{\alpha}k_{\beta} . \tag{4.3.14}$$

C^{α}_{β} projects covariant quantities normally to {N}, and contravariant quantities in the rigging direction (which we emphasise is not normal to {N}).

The rigging field $\vec{n}$ induces a one-form $'n$ on {N}, where $'n$ is the projection of n into {N}. Note that $'n$ can never be zero, since

$$'n_{\alpha} = B^{\beta}_{\alpha}n_{\beta} = n_{\alpha} - n^{\beta}n_{\beta}k_{\alpha} \tag{4.3.15}$$

by (4.3.13). Hence

$$'n = 0 \Rightarrow \vec{n} = \lambda\vec{k}$$

for some (possibly zero) λ. However, by assumption $\vec{n}$ is not parallel to $\vec{k}$, and non zero. Moreover, since from (4.3.10)

$$\langle k,\vec{n}\rangle = \langle n,\vec{k}\rangle = \langle 'n,\vec{k}\rangle = 1 ,$$

we can use g_{ij} and $'n_i$ to determine a substitute contravariant metric g^{ij}, as described in section 4.2, using equation (4.2.4). We shall demand that this substitute contravariant metric be identical to the induced contravariant metric $*g^{ij}$ given in (4.3.12). Now

$$*g^{ij}\,'n_i = B^{ij}_{\alpha\beta}\,{}^4g^{\alpha\beta}\,'n_i = B^{j}_{\beta}\,{}^4g^{\alpha\beta}\,'n_{\alpha} ,$$

from the definition of $*g^{ij}$. Using (4.3.15), the above becomes

$$*g^{ij}{}'n_i = B^j_\beta \, {}^4g^{\alpha\beta}\left(n_\alpha - n^\varepsilon n_\varepsilon k_\alpha\right) = -\, k^j n^\varepsilon n_\varepsilon \; . \tag{4.3.16a}$$

Again from the definitions of g_{ij} and $*g^{ij}$, we can show easily that

$$g_{ij} g_{k\ell} {*g}^{j\ell} = g_{ik} \; . \tag{4.3.16b}$$

Comparison of (4.3.16) with (4.2.4) shows that

$$*g^{ij} = g^{ij} \iff n^\varepsilon n_\varepsilon = {}^4g(\vec{n},\vec{n}) = 0 \; .$$

That is, the induced and substitute contravariant metrics of $\{N\}$ are identical if and only if the vector $\vec{n}$ defining the rigging is a null vector in V. We shall assume that $\vec{n}$ is null, in which case (4.3.15) immediately yields

$$'n_\alpha = B^\beta_\alpha n_\beta = n_\alpha \; . \tag{4.3.17}$$

that is, n lies in $\{N\}$.

Now, the condition ${}^4g(\vec{n},\vec{n}) = 0$, and equation (4.3.10) are not sufficient to completely determine $\vec{n}$. There remains the freedom of the subgroup of the null rotations[26] about $\vec{k}$ which leave $\vec{k}$ fixed. If we define a complex null vector $\vec{m}$, such that $(\vec{k},\vec{n},\vec{m})$, (k,n,m) form a quasi-orthonormal null basis spanning V, where

$$<k,\vec{n}> = -<\bar{m},\vec{m}> = 1 \tag{4.3.18}$$

(the bar denotes complex conjugate) are the only non zero inner products between the reciprocal bases, then the transformations leaving $\vec{k}$ fixed and preserving (4.3.18) are

$$\vec{k} \to \vec{k} \tag{4.3.19a}$$

$$\vec{m} \to e^{iC}\left(\vec{m} - \bar{B}\,\vec{k}\right) \tag{4.3.19b}$$

$$\vec{n} \to \vec{n} - B\vec{m} - \bar{B}\vec{\bar{m}} + B\bar{B}\,\vec{k} \tag{4.3.19c}$$

where C is a real, and B a complex function. In order to fix $\vec{n}$ completely, some other ad hoc conditions must be imposed on $\vec{n}$. For example, we could demand $n^{\alpha}\,{}^4\nabla_{\alpha}k^{\beta} \propto k^{\beta}$, the gauge used by Sachs[27]. However, we shall make no such restrictions on $\vec{n}$ at this stage, since it is by no means clear just which particular $\vec{n}$ is in any sense the 'best' choice.

4.4. Extrinsic Geometry of $\{N\}$.

In both this section and the next we shall, for convenience, work in the adapted basis B_i, as opposed to the general basis E_{α}. The analysis can be reworked in the latter basis quite straightforwardly (to obtain results in the same form as those presented in Chapter II), but in the first instance the analysis, which is in any case complicated by the fact that $\{N\}$ are non Riemannian manifolds, is far clearer in the adapted basis.

As with a rigged space-like foliation, any tensor in V can be written as the sum of its projections into $\{N\}$ and in the direction of the rigging. In particular, the metric tensor is decomposed as follows:

$$ {}^4g_{\alpha\beta} = B^{\mu\nu}_{\alpha\beta}\,{}^4g_{\mu\nu} + 2B^{\ \mu}_{(\alpha}C^{\ \nu}_{\beta)}\,{}^4g_{\mu\nu} + C^{\mu\nu}_{\alpha\beta}\,{}^4g_{\mu\nu} \quad . $$

Using (4.3.9) and (4.3.13), we obtain

$$ B^{\mu\nu}_{\alpha\beta}\,{}^4g_{\mu\nu} = B^{ij}_{\alpha\beta}g_{ij} \; . $$

From (4.3.14) and (4.3.17) we get

$$ 2B^{\ \mu}_{(\alpha}C^{\ \nu}_{\beta)}\,{}^4g_{\mu\nu} = 2B^{\ \mu}_{(\alpha}k_{\beta)}n^{\nu}\,{}^4g_{\mu\nu} = 2k_{(\alpha}n_{\beta)} \; , $$

and since $\vec{n}$ is a null vector, we have from (4.3.14)

$$ C^{\mu\nu}_{\alpha\beta}\,{}^4g_{\mu\nu} = k_{\alpha}k_{\beta}n^{\mu}n^{\nu}\,{}^4g_{\mu\nu} = 0 \; . $$

Collecting the above results together gives

$$ {}^4g_{\alpha\beta} = B^{ij}_{\alpha\beta}g_{ij} + 2k_{(\alpha}n_{\beta)} \quad . \tag{4.4.1a} $$

In an entirely similar manner we obtain

$$ {}^4g^{\alpha\beta} = B^{\alpha\beta}_{ij}g^{ij} + 2k^{(\alpha}n^{\beta)} \quad . \tag{4.4.1b} $$

We next define a number of tensor fields induced on $\{N\}$, namely

$$ h_{ij} = h_{(ij)} = - B^{\alpha\beta}_{ij}\, {}^4\nabla_\alpha k_\beta \tag{4.4.2a} $$

$$ L_{ij} = - B^{\alpha\beta}_{ij}\, {}^4\nabla_\alpha n_\beta \tag{4.4.2b} $$

$$ {}^*L_i{}^j = - B^{\alpha j}_{i\beta}\, {}^4\nabla_\alpha n^\beta \quad . \tag{4.4.2c} $$

We can show by straightforward manipulations involving the results of the previous section that first of all

$$ h_{ij} = - \tfrac{1}{2}\pounds_k g_{ij} \tag{4.4.3a} $$

and so the definition of h_{ij} given in (4.2.7) agrees with that given above, and in addition

$$ \ell_{ij} \overset{\text{def}}{=} L_{(ij)} = - \tfrac{1}{2}\pounds_n g_{ij} \quad , \tag{4.4.3b} $$

$$ {}^*L_i{}^j = g^{jk}L_{ik} \quad . \tag{4.4.3c} $$

For example, let us prove (4.4.3b). By definition

$$ \ell_{ij} = - B^{\alpha\beta}_{ij}\, {}^4\nabla_{(\alpha}n_{\beta)} $$

Hence

$$ \ell_{ij} = - \tfrac{1}{2}B^{\alpha\beta}_{ij}\, \pounds_n\, {}^4g_{\alpha\beta} $$

Substituting (4.4.1a) in the above gives

$$ \ell_{ij} = - \tfrac{1}{2}B^{\alpha\beta}_{ij}\, \pounds_n\left(B^{k\ell}_{\alpha\beta}\, g_{k\ell} + 2k_{(\alpha}n_{\beta)}\right) \quad . $$

But from (4.3.2), (4.3.4) and (4.3.10),

$$B^{\alpha}_{i}k_{\alpha} = \pounds_{n}k_{\alpha} = 0 ,$$

so

$$\ell_{ij} = -\tfrac{1}{2}B^{\alpha\beta}_{ij}\,\pounds_{n}B^{k\ell}_{\alpha\beta}g_{k\ell} \stackrel{def}{=} -\tfrac{1}{2}\,\pounds_{n}g_{ij}$$

(c.f. equation (2.3.17) for the above definition of $\pounds_{n}g_{ij}$). We shall refer to ℓ_{ij} as the extrinsic curvature of {N} . It is the quantity for {N} which is analogous to the extrinsic curvature $h_{\alpha\beta}$ of {Σ} defined in equation (2.3.8).

A covariant derivative $'\nabla$ is induced on each member of {N} by projection. For any scalar λ we define

$$'\nabla_{i}\lambda = B^{\alpha}_{i}\,{}^{4}\nabla_{\alpha}\lambda \tag{4.4.4a}$$

and for a vector $\vec{v}$ tangent to {N} , $'\nabla$ is defined by

$$'\nabla_{i}v^{j} = B^{\alpha j}_{i\beta}\,{}^{4}\nabla_{\alpha}v^{\beta} . \tag{4.4.4b}$$

Similar formulae hold for one forms and tensors of higher order on {N} . From the definition of $'\nabla$ and equation (4.4.2b), it follows immediately that

$$'\nabla_{i}n_{j} = -L_{ij} . \tag{4.4.5a}$$

It is also quite straightforward to show that

$$'\nabla_{i}g_{jk} = 2\,h_{i(j}n_{k)} , \tag{4.4.5b}$$

since from (4.4.1a) and (4.4.2a),

$$\begin{aligned} '\nabla_{i}g_{jk} &\stackrel{def}{=} B^{\alpha\beta\gamma}_{ijk}\,{}^{4}\nabla_{\alpha}{}^{4}g_{\beta\gamma} \\ &= -2\,B^{\alpha\beta\gamma}_{ijk}\,{}^{4}\nabla_{\alpha}\left(k_{(\beta}n_{\gamma)}\right) \\ &= -2\,n_{(k}B^{\alpha\ \beta}_{|i|j)}\,{}^{4}\nabla_{\alpha}k_{\beta} \\ &= 2\,h_{i(j}n_{k)} . \end{aligned}$$

From the definition (4.4.4), it is clear that the induced connection whose components in the basis B_i we shall denote by $'\Gamma^i_{jk}$, is torsion-free. From (4.4.5b), we see that $'\Gamma^i_{jk}$ is non metric, as we should expect from our discussion in section 4.2. However comparison of (4.4.5a) with (4.2.6a) shows that the induced connection is different from Dautcourt's connection. Moreover, from (4.4.5a) we see that $'\Gamma^i_{jk}$ has the rather odd and dissatisfactory property that it depends upon the extrinsic curvature of $\{N\}$. (This is in stark contrast to the induced connection on a space-like foliation $\{\Sigma\}$, which depends only upon the induced metric.) This in turn means that the Riemann tensor of the induced connection, defined by

$$'R_{\ell kj}{}^{i} v^j = 2\ '\nabla_{[\ell}\, '\nabla_{k]} v^i \tag{4.4.6}$$

for any $\vec{v}$ tangent to $\{N\}$, is also implicitly dependent upon the extrinsic curvature and its derivatives within $\{N\}$. Thus clearly the induced connection $'\Gamma^i_{jk}$ is not suitable for the analysis of initial value problems, since eventually we wish to obtain expressions for the projections of the field equations which isolate explicitly the terms containing derivatives of g_{ij} out of $\{N\}$.

As discussed in section 4.2, Dautcourt's connection Γ^i_{jk} is dependent only upon g_{ij} and n_i, and their derivatives within $\{N\}$. Since Γ^i_{jk} and $'\Gamma^i_{jk}$ are both torsion-free, they must differ by some tensor $A_{jk}{}^{i} = A_{(jk)}{}^{i}$ on $\{N\}$. That is,

$$'\Gamma^i_{[jk]} = \Gamma^i_{[jk]} = \tfrac{1}{2} C^i_{jk} \tag{4.4.7a}$$

and

$$'\Gamma^i_{jk} - \Gamma^i_{jk} = A_{jk}{}^{i} = A_{(jk)}{}^{i} \quad . \tag{4.4.7b}$$

From (4.4.7), we obtain

$$\nabla_i n_j - {}'\nabla_i n_j = A_{ij}{}^{k} n_k \ . \tag{4.4.8}$$

First of all, we note that by antisymmetrising on i and j in the above equation we get, using (4.2.6a) and (4.4.5a)

$$\omega_{ij} \stackrel{\text{def}}{=} L_{[ij]} = -\nabla_i n_j \ . \qquad (4.4.9)$$

Next, we symmetrise on i and j in (4.4.8) and use (4.2.6a), (4.4.3b) and (4.4.5a) to obtain

$$\ell_{ij} = A_{ij}{}^{k} n_k \ . \qquad (4.4.10)$$

Subtracting equation (4.4.5b) from (4.2.6b), and again using (4.4.7) yields

$$\nabla_i g_{jk} - {}'\nabla_i g_{jk} = A_{ij}{}^{\ell} g_{\ell k} + A_{ik}{}^{\ell} g_{j\ell} = 0 \ .$$

Cyclicly permuting the indices in the above equation and subtracting the resulting two permutations leads to

$$A_{ij}{}^{\ell} g_{k\ell} = 0 \ .$$

Contracting this with $g^{\ell m}$ gives

$$A_{ij}{}^{m} - k^{m} n_k A_{ij}{}^{k} = 0 \ ,$$

and substituting (4.4.10) into the above gives

$$A_{jk}{}^{i} = \ell_{jk} k^{i} \ . \qquad (4.4.11)$$

Using a standard result relating the Riemann tensors of two affine connections differing by a tensor, we first of all obtain

$${}'R_{\ell kj}{}^{i} = R_{\ell kj}{}^{i} + 2\nabla_{[\ell} A_{k]j}{}^{i} - 2A_{[\ell|j|}{}^{m} A_{k]m}{}^{i} \ .$$

Next we substitute the expression for $A_{ij}{}^{k}$ given in (4.4.11) into the above. Then after some straightforward manipulation, and the use of equation (4.2.14), we obtain

$$'R_{\ell kj}{}^{i} = R_{\ell kj}{}^{i} + 2\nabla_{[\ell}{}^{\ell}{}_{k]j}\, k^{i} - 2\ell_{[\ell|j|}{}^{k^{m}}L_{k]m}k^{i}$$

$$+ 2g^{im}\, \ell_{[\ell|j|}{}^{h}{}_{k]m} \, . \qquad (4.4.12)$$

Equation (4.4.12) expresses the Riemann tensor of the induced connection explicitly in terms of the extrinsic curvature of $\{N\}$ and quantities intrinsic to $\{N\}$.

4.5 Projections of the Riemann Tensor

We shall denote by $\{N\}^{I}$ and $\{N\}^{D}$ respectively the rigged foliation $\{N\}$ endowed with the induced and Dautcourt's connection. The equations of Gauss and Codazzi for a rigged hypersurface in an affine space endowed with the induced affine structure are well known.[12] For $\{N\}^{I}$ in V , the equation of Gauss is

$$B^{\delta\gamma\beta i}_{\ell kj\alpha}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha} = {}'R_{\ell kj}{}^{i} - 2g^{im}\, L_{[\ell|m|}{}^{h}{}_{k]j} \, . \qquad (4.5.1)$$

This equation is derived as a set of integrability conditions for the imbedding of $\{N\}^{I}$ in V . As far as we are concerned, however, its significance is that it gives an expression for the projection of ${}^{4}R_{\delta\gamma\beta}{}^{\alpha}$ into $\{N\}$. The Codazzi equations arise as further integrability conditions, but again their significance here is that they are expressions for the projections $B^{\delta\gamma\beta}_{\ell kj}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha}k_{\alpha}$ and $B^{\delta\gamma i}_{\ell k\alpha}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha}\, n^{\beta}$. By standard algebraic manipulations, it is quite straightforward to show that

$$B^{\delta\gamma\beta}_{\ell kj}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha}\, k_{\alpha} = g_{jn}k^{m}B^{\delta\gamma\beta n}_{\ell km\alpha}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha} , \qquad (4.5.2a)$$

$$B^{\delta\gamma i}_{\ell k\alpha}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha}\, n^{\beta} = - g^{in}n_{m}B^{\delta\gamma\beta m}_{\ell kn\alpha}\ {}^{4}R_{\delta\gamma\beta}{}^{\alpha} \, . \qquad (4.5.2b)$$

Thus for the special case of a rigged $\{N\}$ in V , the Codazzi equations are algebraic consequences of the Gauss equation.

Now let us consider the Riemann tensor of V in its completely covariant form. It has only three independent projections, due to its symmetries. They are $B^{\delta\gamma\beta\alpha}_{\ell kji}\,{}^4R_{\delta\gamma\beta\alpha}$, $B^{\delta\gamma\beta}_{\ell kj}\,{}^4R_{\delta\gamma\beta\alpha}n^\alpha$ and $B^{\gamma\beta}_{kj}\,{}^4R_{\delta\gamma\beta\alpha}n^\delta n^\alpha$. Again, by standard algebraic manipulation we can show easily that

$$B^{\delta\gamma\beta\alpha}_{\ell kji}\,{}^4R_{\delta\gamma\beta\alpha} = \left(\delta^m_j g_{in} - n_i k^m g_{jn}\right) B^{\delta\gamma\beta n}_{\ell km\alpha}\,{}^4R_{\delta\gamma\beta}{}^{\alpha}, \tag{4.5.3a}$$

$$B^{\delta\gamma\beta}_{\ell kj}\,{}^4R_{\delta\gamma\beta\alpha}n^\alpha = n_m B^{\delta\gamma\beta m}_{\ell kj\alpha}\,{}^4R_{\delta\gamma\beta}{}^{\alpha}. \tag{4.5.3b}$$

Hence two of the independent projections of ${}^4R_{\delta\gamma\beta\alpha}$ can be obtained from the Gauss equation (4.5.1). Substituting this latter equation into (4.5.3) yields

$$B^{\delta\gamma\beta\alpha}_{\ell kji}\,{}^4R_{\delta\gamma\beta\alpha} = g_{in}\,{}'R_{\ell kj}{}^{n} - 2L_{[\ell|i|}{}^{h}{}_{k]j} + 2n_i L_{[\ell|m|}{}^{h}{}_{k]j}k^m$$

$$- n_i g_{jn}\,{}'R_{\ell km}{}^{n}k^m, \tag{4.5.4a}$$

$$B^{\delta\gamma\beta}_{\ell kj}\,{}^4R_{\delta\gamma\beta\alpha}n^\alpha = {}'R_{\ell kj}{}^{m}n_m. \tag{4.5.4b}$$

In analogy to the Gauss-Codazzi equations for $\{\Sigma\}$, given in equations (2.3.10), we shall refer to (4.5.4a and b) respectively as the equations of Gauss and Codazzi for $\{N\}^I$.

It is possible to derive from first principles the equivalent equation to (4.5.1) for $\{N\}^D$, as integrability conditions for the imbedding of $\{N\}^D$ in V. This is done in reference 18. However, it is much simpler to obtain this equation by substituting the expression for ${}'R_{\ell kj}{}^{i}$ in terms of $R_{\ell kj}{}^{i}$ given in (4.4.12), into (4.5.1). Of course, equations (4.5.2) and (4.5.3) still hold, and so we can obtain the equations of Gauss and Codazzi for $\{N\}^D$ merely by substituting (4.4.12) into (4.5.4). We first perform this operation for (4.5.4a), and obtain after some straightforward algebra the following intermediate expression:

$$B^{\delta\gamma\beta\alpha}_{\ell kji}\ {}^4R_{\delta\gamma\beta\alpha} = g_{in}R_{\ell kj}{}^{n} + 2\ell_{[\ell|j|}h_{k]i} + 2h_{[\ell|j|}L_{k]i}$$

$$- 2n_i h_{[\ell|j|}\omega_{k]m}k^m - n_i g_{jn}R_{\ell km}{}^{n}k^m \ . \qquad (4.5.5)$$

The above equation can be put into a slightly neater form by obtaining a suitable expression for the last term on the right hand side. We can show that

$$- n_i g_{jn}R_{\ell km}{}^{n}k^m = 2n_i\nabla_{[\ell}h_{k]j} + 2n_i h_{[\ell|j|}\omega_{k]m}k^m \ , \qquad (4.5.6)$$

since

$$\tfrac{1}{2}g_{jn}R_{\ell km}{}^{n}k^m = g_{jn}\nabla_{[\ell}\nabla_{k]}k^n$$

$$= \nabla_{[\ell}\left(g_{|jn|}\nabla_{k]}k^n\right) - \left(\nabla_{[\ell}g_{|jn|}\right)\left(\nabla_{k]}k^n\right) \ .$$

Using in addition to the fundamental equations in section 4.2 the expression (4.2.14) for $\nabla_k k^n$, the second term of the above equation becomes

$$- \nabla_{[\ell}\left(k^n\nabla_{k]}g_{jn}\right) - \left(h_{[\ell|j}{}^{n}{}_{n|}\right)\left(\omega_{k]m}k^m k^n\right)$$

$$= - \nabla_{[\ell}h_{k]j} - h_{[\ell|j|}\omega_{k]m}k^m \ ,$$

from which (4.5.6) follows immediately. Substituting (4.5.6) into (4.5.5) yields

$$B^{\delta\gamma\beta\alpha}_{\ell kji}\ {}^4R_{\delta\gamma\beta\alpha} = g_{in}R_{\ell kj}{}^{n} + 2\ell_{[\ell|j|}h_{k]i} + 2h_{[\ell|j|}L_{k]i}$$

$$+ 2n_i \nabla_{[\ell}h_{k]j} \ . \qquad (4.5.7a)$$

Next, we substitute (4.4.12) into (4.5.4b) to obtain immediately the equation

$$B^{\delta\gamma\beta}_{\ell kj}\ {}^4R_{\delta\gamma\beta\alpha}n^{\alpha} = 2\nabla_{[\ell}L_{k]j} - 2\ell_{[\ell|j|}L_{k]m}k^{m} \qquad (4.5.7b)$$

We refer to equations (4.5.7a and b) as the equations of Gauss and Codazzi for $\{N\}^D$. They are the true analogies to the Gauss-Codazzi equations (2.3.10) for $\{\Sigma\}$. Firstly, they are the projections of ${}^4R_{\delta\gamma\beta\alpha}$ four times into $\{N\}$, and three times into $\{N\}$, once in the rigging direction. Secondly, the resulting expressions are explicitly in terms of the extrinsic curvature, ℓ_{ij}, and quantities that can be built out of g_{ij}, n_i and their internal derivatives only.

The only other independent projection of ${}^4R_{\delta\gamma\beta\alpha}$ comes from considering the definition of $\pounds_n \ell_{kj}$. Eventually we obtain

$$B^{\gamma\beta}_{kj}\ {}^4R_{\delta\gamma\beta\alpha}n^{\delta}n^{\alpha} = \pounds_n \ell_{kj} - \ell_{kj}\, k^m a_m + g^{mn}L_{km}L_{jn} + \nabla_{(k}a_{j)} \ , \qquad (4.5.8)$$

where

$$a_i \overset{\text{def}}{=} \pounds_n n_i \ . \qquad (4.5.9)$$

Equation (4.5.8) is derived as follows. We first note that $\pounds_n B^{\alpha}_{\beta}=0$. Hence, from the definition of ℓ_{kj}, we obtain immediately

$$\pounds_n \ell_{kj} = - B^{\gamma\beta}_{kj}\pounds_n\ {}^4\nabla_{(\gamma}n_{\beta)} \ .$$

But from the definition of the Lie derivative, we get

$$\pounds_n\ {}^4\nabla_{\gamma}n_{\beta} = n^{\delta}\ {}^4\nabla_{\delta}\ {}^4\nabla_{\gamma}n_{\beta} + \left({}^4\nabla_{\gamma}n^{\delta}\right)\left({}^4\nabla_{\delta}n_{\beta}\right) + \left({}^4\nabla_{\beta}n^{\delta}\right)\left({}^4\nabla_{\gamma}n_{\delta}\right)$$

Using the Ricci identity, this becomes

$$n^{\delta}\,{}^4\nabla_{\gamma}\,{}^4\nabla_{\delta}n_{\beta} - {}^4R_{\delta\gamma\beta\alpha}n^{\delta}n^{\alpha} + \left({}^4\nabla_{\gamma}n^{\delta}\right)\left({}^4\nabla_{\delta}n_{\beta}\right) + \left({}^4\nabla_{\beta}n^{\delta}\right)\left({}^4\nabla_{\gamma}n_{\delta}\right)$$

and again from the definition of the Lie derivative , this becomes

$${}^4\nabla_{\gamma}\pounds_{n}n_{\beta} - {}^4R_{\delta\gamma\beta\alpha}n^{\delta}n^{\alpha} + \left({}^4\nabla_{\beta}n^{\delta}\right)\left({}^4\nabla_{\gamma}n_{\delta}\right) .$$

Symmetrising on γ and β , and projecting into $\{N\}$ then yields

$$-\pounds_{n}\ell_{kj} = {}'\nabla_{(k}a_{j)} - B^{\gamma\beta}_{kj}\,{}^4R_{\delta\gamma\beta\alpha}n^{\delta}n^{\alpha} + L_{(k|m|}\overset{*}{L}_{j)}{}^{m} . \qquad (4.5.10)$$

But from (4.4.7), we have

$${}'\nabla_{(k}a_{j)} = \nabla_{(k}a_{j)} - \ell_{kj}k^{m}a_{m} . \qquad (4.5.11)$$

Equation (4.5.8) follows immediately on substitution of (4.5.11) into (4.5.10).

4.6. Conclusion. Possible Applications to the Non Space-like Initial Value Problem.

From equations (4.5.7) and (4.5.8) it is a straightforward process to calculate the independent projections $B^{\gamma\beta}_{kj}\,{}^4R_{\gamma\beta}$, $B^{\gamma}_{k}\,{}^4R_{\gamma\beta}n^{\beta}$ and ${}^4R_{\gamma\beta}n^{\gamma}n^{\beta}$ of the Ricci tensor of V , and equating these projections to zero gives the covariant 3+1 break up of the field equations with respect to $\{N\}$. It is not hard to see that the projections into $\{N\}$ are the six main equations (c.f. equations (3.3.3)) while the remaining projections are the subsidiary conditions and the trivial equation. The expressions obtained for the projections of the Ricci tensor are, however, quite long and without further analysis not particularly informative. We do not reproduce them here, but they can be found in reference 18. In order to use them to investigate any non space-like initial value problem, it is first necessary to choose a particular null vector $\vec{n}$ rigging {N} . (Remember $\vec{n}$ is only defined at this stage up to the subgroup of the null rotations about $\vec{k}$ given in equations (4.3.19).) Depending on the particular choice of $\vec{n}$, the field equations take on quite different forms. Even after a choice of $\vec{n}$ has been made there still remains, in particular, the problem of extracting the dynamical equations from the main equations. The best way to do this is to project the field equations again into a family of space-like 2-surfaces foliating each member of {N} . Such a family is naturally provided by choosing $\vec{n}$ so that the corresponding one-form n is hypersurface orthogonal, that is proportional to a closed one form $\tilde{n}$, say. (Note that there is insufficient freedom in the equations (4.3.19) to demand that n be equal to a closed one-form.) The two families of null hypersurfaces defined by k and $\tilde{n}$ then intersect in a (two parameter) family of space-like 2-surfaces. In fact, this procedure has been investigated by the author, and work in this direction indicates that it is possible, using this approach, to formulate covariantly a characteristic initial value problem. The difficulties and drawbacks to this approach are, however, manifold. Since {N} are non Riemannian, one has to deal with the problem of imbedding a family of (Riemannian) space-like 2-surfaces in a non Riemannian manifold - essentially the reverse of the problem dealt with in this chapter. One is also restricted with this method to considering only Sachs' double-null problem, since one has naturally introduced two families of null hypersurfaces. If one wishes to impose different conditions on $\vec{n}$, either to investigate other gauge conditions for the double-null problem, or to analyse the mixed problem, then n will not be hypersurface orthogonal. It will instead define anholonomic null 3-surface elements, or anholonomic space-like 2-surface elements within $\{N\}$, and this leads to still further complications.

The fact that one is led to projecting again into space-like 2-surfaces raises the question as to whether it would not be better to project straightaway into a family of (suitably defined) space-like 2-surfaces foliating V , without first projecting into $\{N\}$. In fact this turns out to be a very fruitful approach, to which we devote the remainder of this thesis. Although the 3+1 approach does not seem to be the best way of analysing non space-like initial value problems, it is felt that the work in this chapter is of interest in its own right, since it leads to some deeper insight into the geometry of null hypersurfaces.

Chapter V. The '2+2' Formalism.

5.1 Introduction.

In a recent paper, d'Inverno and Stachel[28] suggested that the so-called 'conformal 2-structure' - essentially the conformal metric of a family of space-like 2-surfaces - might be considered as the dynamical variables of the gravitational field. Using coordinate dependent techniques they show that this prescription works, formally at least, in the characteristic, mixed, and space-like initial value problems. Their work motivates the introduction, in this chapter, of a covariant 2+2 formalism, in which space-time is considered as being foliated by space-like 2-surfaces, rigged by a pair of vector fields which are linearly independent and normal to the foliation, and which thus span time-like 2-surfaces (in general anholonomic) orthogonal to the foliation. By a suitable choice of these rigging fields, subfamilies of the space-like 2-surfaces can be regarded as foliating hypersurfaces in space-time which may be either time-like, space-like or null. This formalism allows us to deal conveniently with all three types of initial value problem, namely Cauchy, characteristic and mixed. In the last two cases, it is because we are working directly with the geometry of Riemannian 2-manifolds that the problems encountered in the last chapter, which arise from the degeneracy of the intrinsic geometry of a null manifold, are essentially bypassed. In the remainder of this chapter we analyse the 2+2 formalism in detail, before going on to consider in the subsequent three chapters the application of this formalism to the various initial value problems.

5.2 Foliations of Space-time by Rigged Space-like 2-surfaces[12].

A foliation $\{S\}$ of V into 2-surfaces ($\{S\}$ is a foliation of codimension 2) is defined by a pair of closed one-forms, n^a, say $(a = 0,1)$ with components in a general space-time basis n^a_α. Now, n^o and n^1 define respectively foliations of V, $\{\Sigma_1\}$ and $\{\Sigma_o\}$, into hypersurfaces. Each $S\varepsilon\{S\}$ can be thought of as the intersection of some $\Sigma_o\varepsilon\{\Sigma_o\}$ and $\Sigma_1\varepsilon\{\Sigma_1\}$. Since n^a are closed, we have

$$dn^a = 0 \longleftrightarrow {}^4\nabla_{[\alpha}n^a_{\beta]} = 0 \tag{5.2.1}$$

hence (locally) there exist scalar functions ϕ^a such that

$$n^a = d\phi^a \longleftrightarrow n^a_\alpha = {}^4\nabla_\alpha\phi^a , \tag{5.2.2}$$

and $\{\Sigma_o\}$, $\{\Sigma_1\}$ arise (locally) as the level surfaces of ϕ^1 and ϕ^o. Each $\Sigma_{(a)}\varepsilon\{\Sigma_{(a)}\}$ is itself foliated by a subset of $\{S\}$, and we denote these subsets by $\{S\}_{(a)}$ (where parentheses around an index indicate that we are referring to a fixed value of that index). The remarks made so far in this section are illustrated in figure 1.

An arbitrary basis of vectors B_A tangent to $\{S\}$ must satisfy

$$< n^a, B_A > = n^a_\alpha B^\alpha_A = 0 \tag{5.2.3}$$

The quantities B^α_A are the components of B_A in the general space-time basis, and act as connecting quantities for vectors tangent to $\{S\}$ (cf. section 2.2).

We define vectors $\vec{n}^a$ by

$$\vec{n}^a = {}^4g^{\alpha\beta}n^a_\alpha E_\beta \tag{5.2.4}$$

Hence, from (5.2.3)

$$ {}^4g(\vec{n}^a, B_A) = \langle n^a, B_A \rangle = 0 \; . \tag{5.2.5}$$

Since $\vec{n}^a$ are orthogonal to B_A, the necessary and sufficient condition that {S} should be space-like is that $\vec{n}^a$ should span a time-like 2-surface element T at each point of V. We shall assume that this is the case, and we denote the totality of these elements by {T}. Note that $\vec{n}^a$ are not, in general, closed under the Lie bracket operation, so the elements {T} do not define a foliation of V ; they are rather anholonomic time-like 2-surface elements, or fields of non integrable time-like 2-planes. Now $\vec{n}^a$ and n^a are bases of vectors and one-forms in {T}, but they are not reciprocal bases. The reciprocal basis to n^a is $'\vec{n}_a$ and is defined by

$$ '\vec{n}_a = \eta_{ab}\vec{n}^b \tag{5.2.6a}$$

$$ \langle n^a, '\vec{n}_b \rangle = \delta^a_b \tag{5.2.6b}$$

where η_{ab} is some 2×2 matrix of (in general non constant) scalars. From (5.2.6a) we have immediately that

$$ 'n_a = \eta_{ab}\, n^b . $$

Then (5.2.6b) and the above yield

$$ \eta_{cb} = \eta_{ca} \langle n^a, '\vec{n}_b \rangle = \langle 'n_c, '\vec{n}_b \rangle = {}^4g('\vec{n}_c, '\vec{n}_b), \tag{5.2.7}$$

hence η_{cb} is a symmetric matrix. We define its inverse by η^{cb}, and thus

$$\eta^{ac}\eta_{bc} = \delta^a_b \ . \tag{5.2.8}$$

From (5.2.6a) and (5.2.8) we then get

$$\eta^{ca}\,'\vec{n}_a = \vec{n}^c \tag{5.2.9}$$

Hence there is a natural isomorphism between $'\vec{n}_a$ and $\vec{n}^a$, and also $'n_a$ and n^a, with the raising and lowering of the indices $a,b,\ldots.$ defined by η^{ab} and η_{ab} respectively. Thus we may write $'n_a = n_a$ and $'\vec{n}_a = \vec{n}_a$, and the vectors $\vec{n}_a$ and one forms n^a form a dyad basis of $\{T\}$. For any tensor $X^{\alpha\ldots}_{\ldots\beta}$ in $\{T\}$, we define its dyad components by

$$X^{a\ldots}_{\ldots b} = n^{a\ldots}_{\alpha\ldots}\, n^{\beta}_{b}\, X^{\alpha\ldots}_{\ldots\beta} \ ,$$

and then it is easy to show from the above arguments that

$$'X_{a\ldots}^{\ \ \ldots b} \stackrel{\text{def}}{=} \eta_{ac\ldots}\eta^{bd}X^{c\ldots}_{\ldots d} = X_{a\ldots}^{\ \ \ldots b} \ . \tag{5.2.10}$$

Thus we may raise and lower dyad indices of the dyad components of tensors in $\{T\}$ with η^{ab} and η_{ab} respectively.

Metrics g and $'g$ are induced in $\{S\}$ and $\{T\}$ respectively by the demand that for vectors $\vec{v}$ and $\vec{w}$ tangent to $\{S\}$ and $'\vec{v}$ and $'\vec{w}$ tangent to $\{T\}$,

$$g(\vec{v},\vec{w}) = {}^4g(\vec{v},\vec{w}) \ ,$$

$$'g('\vec{v},'\vec{w}) = {}^4g('\vec{v},'\vec{w}) \ .$$

Then

$$g_{AB} \stackrel{\text{def}}{=} g(B_A,B_B) = {}^4g(B_A,B_B) = {}^4g_{\alpha\beta}B^{\alpha\beta}_{AB} \tag{5.2.11a}$$

are the components of g in the basis B_A, and

$$'g_{ab} = 'g(\vec{n}_a,\vec{n}_b) = {}^4g(\vec{n}_a,\vec{n}_b) = {}^4g_{\alpha\beta}n^\alpha_a n^\beta_b \qquad (5.2.11b)$$

are the dyad components of $'g$. Comparison of (5.2.7) and (5.2.11b) gives

$$\eta_{ab} = 'g_{ab} , \qquad (5.2.12)$$

and so η_{ab} are the dyad components of the induced metric in {T} .

The natural rigging of {S} is defined by any two linearly independent directions orthogonal to {S} , and in particular, the two directions defined by $\vec{n}_a$ constitute an especially convenient pair of rigging directions. The reciprocal basis of forms B^A in {S} is required to satisfy

$$< B^A, \vec{n}_a > = B^A_\alpha \, n^\alpha_a = 0 , \qquad (5.2.13)$$

and this fixes the components B^A_α of B^A . The vectors $(B_A, \vec{n}_a)$ and one-forms (B^A, n^a) constitute reciprocal bases of V . Hence we can form the quantity

$$B^\alpha_\beta \overset{\text{def}}{=} B^\alpha_A \, B^A_\beta = \delta^\alpha_\beta - n^\alpha_a \, n^a_\beta \quad . \qquad (5.2.14)$$

The quantities B^α_β , B^α_A and B^A_α act as projection operators, projecting arbitrary tensors in V into {S} . For example, for any vector $\vec{v}$ the elements

$$'v^\alpha = B^\alpha_\beta \, v^\beta$$

are the components of a vector $'\vec{v}$ in the general basis E_α , and

$$'v^A = B^A_\alpha \, v^\alpha \qquad (5.2.15)$$

are the components of $'\vec{v}$ in the basis B_A. Similar remarks hold for one-forms and higher order tensors (c.f. section 2.2). The quantity

$$C^{\alpha}_{\beta} \overset{\text{def}}{=} n^{\alpha}_{a}\, n^{a}_{\beta} \tag{5.2.16}$$

projects tensors into {T} and for example, for a vector $\vec{v}$, the elements

$$''v^{\alpha} = C^{\alpha}_{\beta}\, v^{\beta}$$

are the components of a vector $''\vec{v}$ tangent to {T}. The elements

$$''v^{a} = n^{a}_{\alpha}\, v^{\alpha} \tag{5.2.17}$$

are the dyad components of $''\vec{v}$.

It is perhaps worth emphasising at this stage that $'v^A$ (defined by (5.2.15)) are the components of a vector $'\vec{v}$ tangent to {S}, and $'v^A$ transform in the usual way under arbitrary changes of basis in {S}, $B_A \to \lambda_A{}^B B_B$. However, $''v^a$, (as defined by (5.2.17)) are a pair of scalars, $''v^o$, $''v^1$; the dyad index a has no tensorial character. (In the terminology of Schouten[6] the dyad indices are 'dead' indices.) This arises from the fact that $\vec{n}_a$ and n^a are required to be fixed reciprocal bases of {T}, determined uniquely by the demand that n^a be closed one-forms, defining particular foliations $\{\Sigma_a\}$ of V.

Any vector in V is the sum of its projections into {S} and {T}. Higher order tensors can also be written as sums of their projections. For example, an arbitrary tensor X of type (1,1) can be written in terms of its projections as

$$X^{\alpha}{}_{\beta} = B^{\alpha\theta}_{\epsilon\beta}\, X^{\epsilon}{}_{\theta} + B^{\alpha}_{\epsilon}\, C^{\theta}_{\beta}\, X^{\epsilon}{}_{\theta} + C^{\alpha}_{\epsilon}\, B^{\theta}_{\beta}\, X^{\epsilon}{}_{\theta} + C^{\alpha\theta}_{\epsilon\beta}\, X^{\epsilon}{}_{\theta} \quad .$$

This can also be written as follows:

$$X^{\alpha}{}_{\beta} = \perp X^{\alpha}{}_{\beta} + \perp X^{\alpha}{}_{b} n^{b}_{\beta} + \perp X^{a}{}_{\beta} n^{\alpha}_{a} + X^{a}{}_{b} n^{\alpha}_{a}\, n^{b}_{\beta}$$

where, for example

$$\perp X^{\alpha}{}_{b} \overset{def}{=} B^{\alpha}_{\epsilon}\, n^{\beta}_{b}\, X^{\epsilon}{}_{\beta}$$

are the components in E_{α} of a pair of vectors (labelled by the dyad index a) tangent to {S} .

The inverse metrics induced in {S} and {T} have components

$$g^{AB} = B^{AB}_{\alpha\beta}\, {}^{4}g^{\alpha\beta} \tag{5.2.18a}$$

in the basis B_{A}, and dyad components

$$'g^{ab} = n^{a}_{\alpha}\, n^{b}_{\beta}\, {}^{4}g^{\alpha\beta} \tag{5.2.18b}$$

respectively. It is straightforward to show that

$$g^{AC} g_{CB} = \delta^{A}_{B} \tag{5.2.19a}$$

since, from (5.2.11a), (5.2.14) and (5.2.18a)

$$g^{AC} g_{CB} = B^{AC}_{\alpha\gamma}\, {}^{4}g^{\alpha\gamma}\, B^{\epsilon\beta}_{CB}\, {}^{4}g_{\epsilon\beta}$$

$$= B^{A\beta}_{\alpha B}\left(\delta^{\epsilon}_{\gamma} - n^{\epsilon}_{e}\, n^{e}_{\gamma}\right) {}^{4}g^{\alpha\gamma}\, {}^{4}g_{\epsilon\beta}$$

and this gives, using (5.2.3)

$$g^{AC} g_{CB} = B^{A}_{\epsilon}\, B^{\epsilon}_{B} = \delta^{A}_{B} \; .$$

We also have

$$'g^{ab} = \eta^{ab} \iff 'g^{ac}\,'g_{bc} = \delta^a_b \qquad (5.2.19b)$$

since from (5.2.6a), (5.2.11b) and (5.2.12),

$$'g^{ab} = n^a_\alpha n^b_\beta\, {}^4g^{\alpha\beta} = \eta^{ac}\eta^{bd} n^\alpha_c n^\beta_d\, {}^4g_{\alpha\beta} = \eta^{ab}$$

Hence the respective induced inverse metrics in $\{S\}$ and $\{T\}$ are the inverses of the metrics induced in $\{S\}$ and $\{T\}$. This justifies the use of the same kernel letter for the induced inverse metrics in $\{S\}$ and $\{T\}$ respectively. It also means that the natural isomorphisms induced by ${}^4g$ and g between vectors and one-forms tangent to $\{S\}$ are identical. Similar remarks apply to ${}^4g$, $'g$ and vectors and forms tangent to $\{T\}$.

Equations (5.2.2), (5.2.5) and (5.2.6) imply that $\vec{n}_{(a)}$ is tangent to $\{\Sigma_{(a)}\}$, and also that it is the natural orthogonal connecting vector of neighbouring slices of $\{S\}_{(a)}$. The element $\eta_{(a)(a)}$ of η_{ab} defines the lapse function of $\{S\}_{(a)}$; the orthogonal metrical separation of nearby members of $\{S\}_{(a)}$, parameter distance $\delta\phi^{(a)}$ apart, is given by

$$\left({}^4g(\vec{n}_{(a)}\vec{n}_{(a)})\right)^{\frac{1}{2}}\delta\phi^{(a)} = \left|\eta_{(a)(a)}\right|^{\frac{1}{2}}\delta\phi^{(a)} \quad \text{(not summed over } a). \qquad (5.2.20)$$

Then in an analogous manner to that discussed following equation (2.2.12) where we defined the lapse function of the foliation $\{\Sigma\}$ in the 3+1 formalism, we define $\left|\eta_{(a)(a)}\right|^{\frac{1}{2}}$ as the lapse function of $\{S\}_{(a)}$. The above remarks are illustrated in figure 2. The elements η^{oo}, η^{11} of η^{ab} determine the metrical properties of $\{\Sigma_1\}$ and $\{\Sigma_o\}$ respectively, since

$$\eta^{(a)(a)} = {}^4g^{\alpha\beta}\eta^{(a)}_{\alpha}n^{(a)}_{\beta} . \qquad (5.2.21)$$

The orthogonal connecting vectors $\vec{n}_a$ of $\{S\}_a$ do not in general commute, and we define

$$[n_o, n_1]^{\alpha} = \pounds_{n_o} n_1^{\alpha} = -2\,\bar{\Omega}^{\alpha}. \qquad (5.2.22a)$$

Then, since from (5.2.1) and (5.2.6)

$$n^a_{\alpha}\pounds_{n_o} n_1^{\alpha} = -\, n_1^{\alpha}\pounds_{n_o} n^a_{\alpha} = 0 ,$$

we have

$$\bar{\Omega}^{\alpha} = \perp\bar{\Omega}^{\alpha} , \qquad (5.2.22b)$$

that is the commutator of $\vec{n}_a$ is a vector tangent to $\{S\}$. The vanishing of $\bar{\Omega}^{\alpha}$ is the necessary and sufficient condition for $\{T\}$ to be holonomic.

Any vectors $\vec{e}_a$ satisfying

$$< n^a, \vec{e}_b > = \delta^a_b \qquad (5.2.23)$$

are connecting vectors of $\{S\}_a$. Equation (5.2.23) only defines $\vec{e}_{(a)}$ up to an arbitrary shift vector $\vec{b}_{(a)}$ tangent to $\{S\}$. That is, any vectors

$$\vec{e}_a = \vec{n}_a + \vec{b}_a \quad , \quad < n^b, \vec{b}_a > = 0 \qquad (5.2.24)$$

satisfy (5.2.23) (see figure 2). We shall restrict our attention to those $\vec{b}_a$ for which the resulting $\vec{e}_a$ commute. By virtue of (5.2.22), such vectors always exist, but they are not unique. For a given choice of shift vector $\vec{b}_{(a)}$, the resulting $\vec{e}_{(a)}$ is tangent to a congruence of curves $\mathscr{C}_{(a)}$ which lies in $\{\Sigma_{(a)}\}$, threads $\{S\}_{(a)}$ and fibrates V. The curves of $\mathscr{C}_{(a)}$ are parametrised by $\phi^{(a)}$, and we may write

$$\vec{e}_a = \frac{\partial}{\partial \phi^a} , \tag{5.2.25}$$

by virtue of (5.2.2), (5.2.23) and the demand that $\vec{e}_a$ commute. The curves in the congruences $\mathscr{C}_a$ set up a one-to-one correspondence between points on an initial 2-surface $\overset{o}{S}$, and any other $S \in \{S\}$ as follows: if we start at some point $\overset{o}{P}$ on $\overset{o}{S}$, then travelling parameter distances ϕ^o and ϕ^1 from $\overset{o}{P}$ along curves of $\mathscr{C}_o$ and $\mathscr{C}_1$ respectively, in any order, we always arrive at the same point P, say, on a particular 2-surface S . The vectors $\vec{e}_a$ tangent to $\mathscr{C}_a$ are thus the natural ones with which to propagate quantities through V. If, without loss of generality, we assume that $\phi^o = \phi^1 = 0$ is the equation of $\overset{o}{S}$, then the value of some geometric object Ψ_Λ (indices suppressed) on a general $S\epsilon\{S\}$ is given by the generalised Taylor expansion

$$\Psi_\Lambda \ (\phi^o, \phi^1) = \exp\left\{\phi^a \pounds_{e_a}\right\} \Psi_\Lambda(0,0) \tag{5.2.26}$$

(c.f. equation 2.2.16).

The foliation and fibration of V described in this section may be thought of as being generated in a rather different, but fully equivalent way. Suppose that we are given, in V, a pair of commuting vector fields $\vec{e}_a$, and a two-dimensional, space-like cross-section $\overset{o}{S}$ of $\vec{e}_a$. The fibrations $\mathscr{C}_a$ of V are just the integral curves of $\vec{e}_a$ (as before), but the foliation {S} is obtained by Lie dragging $\overset{o}{S}$ with $\vec{e}_a$ so as to fill up V. Since $\vec{e}_a$ commute, the order of dragging is immaterial. The one-forms n^a and the vectors $\vec{n}_a$ can then be determined from {S} .

5.3 Intrinsic Geometry of {S} and {T}.

The only two non vanishing projections of $^4g_{\alpha\beta}$ are

$$B^{\mu\nu}_{\alpha\beta} \, {}^4g_{\mu\nu} = B^{AB}_{\alpha\beta} \, g_{AB} = g_{\alpha\beta} \qquad (5.3.1a)$$

and

$$C^{\mu\nu}_{\alpha\beta} \, {}^4g_{\mu\nu} = n^a_\alpha n^b_\beta \eta_{ab} = {}'g_{\alpha\beta} \, . \qquad (5.3.1b)$$

It follows immediately from (5.2.3) that the other projections of $^4g_{\alpha\beta}$ vanish. Similarly, the non vanishing projections of $^4g^{\alpha\beta}$ are

$$B^{\alpha\beta}_{\mu\nu} \, {}^4g^{\mu\nu} = B^{\alpha\beta}_{AB} \, g^{AB} = g^{\alpha\beta} \qquad (5.3.2a)$$

and

$$C^{\alpha\beta}_{\mu\nu} \, {}^4g^{\mu\nu} = n^\alpha_a n^\beta_b \eta^{ab} = {}'g^{\alpha\beta} \qquad (5.3.2b)$$

From (5.2.14), (5.2.16), (5.3.1) and (5.3.2) it follows that

$$g^{\alpha\gamma} g_{\beta\gamma} = B^\alpha_\beta \qquad (5.3.3a)$$

$${}'g^{\alpha\gamma} \, {}'g_{\beta\gamma} = C^\alpha_\beta \qquad (5.3.3b)$$

From (5.3.1 and 2) it follows that indices on quantities in {S} or {T} can be raised or lowered by either the induced metrics in {S} or {T} respectively, or by the full space-time metric. The results (5.2.1-24) and (5.3.1-3) will be used repeatedly and without reference in the remainder of this chapter.

Covariant derivatives ∇ and $'\nabla$ in {S} and {T} respectively are induced by projection. For any scalar λ , we define

$$\nabla_\alpha \lambda = B^\varepsilon_\alpha \, {}^4\nabla_\varepsilon \lambda \qquad (5.3.4a)$$

$${}'\nabla_\alpha \lambda = C^\varepsilon_\alpha \, {}^4\nabla_\varepsilon \lambda \qquad (5.3.4b)$$

and for vectors $\vec{v}$ and $\vec{w}$ tangent to {S} and {T} respectively, we define

$$\nabla_\alpha v^\beta = B^{\varepsilon\beta}_{\alpha\theta}\, {}^4\nabla_\varepsilon v^\theta \tag{5.3.5a}$$

$${}'\nabla_\alpha w^\beta = C^{\varepsilon\beta}_{\alpha\theta}\, {}^4\nabla_\varepsilon w^\theta \,. \tag{5.3.5b}$$

From the above, it is easy to show that

$$\nabla_\gamma g_{\beta\alpha} = {}'\nabla_\gamma {}'g_{\beta\alpha} = 0 \,, \tag{5.3.6}$$

that is, the induced connections in {S} and {T} are the connections of the respective induced metrics. Next, we define

$$n^\gamma_c n^a_\beta\, {}^4\nabla_\gamma n^\beta_b = -n^\gamma_c n^\beta_b\, {}^4\nabla_\gamma n^a_\beta = \Gamma^a_{cb} = \Gamma^a_{(cb)} \tag{5.3.7}$$

With the above definition, it follows that for any $\vec{w}$ tangent to {T}

$$n^\gamma_c n^a_\alpha {}'\nabla_\gamma w^\alpha = \pounds_{n_c} w^a + \Gamma^a_{cb} w^b \overset{\text{def}}{=} {}'\nabla_c w^a \,. \tag{5.3.8}$$

Hence from (5.3.7 and 8), we obtain

$$n^\gamma_c n^\beta_b n^\alpha_a\, {}'\nabla_\gamma {}'g_{\beta\alpha} = {}'\nabla_c \eta_{ba} = \pounds_{n_c} \eta_{ba} - \Gamma^e_{cb}\eta_{ea} - \Gamma^e_{ca}\eta_{be} = 0 \,,$$

and this yields, by cyclicly permuting the indices c,b,a in the above equation, and subtracting the resulting two equations,

$$\Gamma^a_{cb} = \tfrac{1}{2}\eta^{ad}\left(\pounds_{n_c}\eta_{db} + \pounds_{n_b}\eta_{dc} - \pounds_{n_d}\eta_{cb}\right) . \tag{5.3.9}$$

The set of scalars Γ^a_{bc} are just the dyad components of the induced connection in {T} .

The Riemann tensor of {S} is defined by

$$R_{\delta\gamma\beta}{}^{\alpha} v^\beta = 2\nabla_{[\delta}\nabla_{\gamma]} v^\alpha \;;\quad R_{\delta\gamma\beta}{}^{\alpha} n^\beta_b = 0 \,, \tag{5.3.10}$$

for any $\vec{v}$ tangent to $\{S\}$. This amounts to the usual definition when written in the basis B_A. Since $\{T\}$ is in general anholonomic, its Riemann tensor has a rather more complicated definition. It is given by

$$'R_{\delta\gamma\beta}{}^{\alpha}w^{\beta} = 2'\nabla_{[\delta}{}'\nabla_{\gamma]}w^{\alpha} + 2n^{\alpha}_{a}\Omega_{\delta\gamma}{}^{\varepsilon}\nabla_{\varepsilon}w^{a};\ B^{\varepsilon}_{\beta}{}'R_{\delta\gamma\varepsilon}{}^{\alpha} = 0, \qquad (5.3.11)$$

for any $\vec{w}$ tangent to $\{T\}$. The quantity

$$\Omega_{\delta\gamma}{}^{\alpha} = \Omega_{[\delta\gamma]}{}^{\alpha} \overset{\text{def}}{=} B^{\alpha}_{\varepsilon}n^{d}_{\delta}n^{c}_{\gamma}\,\Omega_{dc}{}^{\varepsilon}, \qquad (5.3.12a)$$

$$\Omega_{dc}{}^{\alpha} \overset{\text{def}}{=} -\tfrac{1}{2}\pounds_{n_d}n^{\alpha}_{c} \qquad (5.3.12b)$$

is the anholonomic object of $\{T\}$, which vanishes if and only if $\{T\}$ is holonomic. $\Omega_{dc}{}^{\alpha}$ has only one independent dyad component, $\Omega_{01}{}^{\alpha}$, and comparison of (5.3.12b) with (5.2.22) shows that

$$\Omega_{01}{}^{\alpha} = \bar{\Omega}^{\alpha} \qquad (5.3.13)$$

The extra term in the definition of $'R_{\delta\gamma\beta}{}^{\alpha}$, involving the anholonomic object, is required to keep the expression linear algebraic in w^{α}. We now take dyad components of (5.3.11) and use (5.3.8) and (5.3.12) to obtain

$$'R_{dcb}{}^{a}w^{b} = 2'\nabla_{[d}{}'\nabla_{c]}w^{a} + 2\Omega_{dc}{}^{\varepsilon}\nabla_{\varepsilon}w^{a}. \qquad (5.3.14)$$

Now, from (5.3.8), we can show that

$$2'\nabla_{[d}{}'\nabla_{c]}w^{a} = [\pounds_{n_d}\pounds_{n_c} - \pounds_{n_c}\pounds_{n_d}]w^{a} + \left(\pounds_{n_d}\Gamma^{a}_{cb} - \pounds_{n_c}\Gamma^{a}_{db} + \Gamma^{a}_{de}\Gamma^{e}_{cb} - \Gamma^{a}_{ce}\Gamma^{e}_{db}\right)w^{b} \qquad (5.3.15)$$

From (5.3.12b) and the commutation law for Lie derivatives, it follows immediately that

$$\left[\pounds_{n_d}\pounds_{n_c} - \pounds_{n_c}\pounds_{n_d}\right] w^a = -\pounds_{\Omega_{dc}} w^a = -\Omega_{dc}{}^{\varepsilon}\nabla_{\varepsilon} w^a \ . \tag{5.3.16}$$

Substituting (5.3.16) into (5.3.15) and comparing the resulting equation with (5.3.14) yields

$$'R_{dcb}{}^{a} = \pounds_{n_d}\Gamma^a_{cb} - \pounds_{n_c}\Gamma^a_{db} + \Gamma^a_{de}\Gamma^e_{cb} - \Gamma^a_{ce}\Gamma^e_{db} \ . \tag{5.3.17}$$

5.4 Extrinsic Geometry of {S} and {T} .

We start by considering what Schouten[(6)] terms the 'curvature tensors of valence 3' of {S} and {T} . These are defined respectively by

$$H_{\delta\gamma}{}^{\alpha} = B^{\lambda\mu}_{\delta\gamma}{}^4\nabla_{\lambda} B^{\alpha}_{\mu} = -B^{\lambda\mu}_{\delta\gamma}{}^4\nabla_{\lambda} C^{\alpha}_{\mu} \tag{5.4.1}$$

$$L_{\delta\gamma}{}^{\alpha} = C^{\lambda\mu}_{\delta\gamma}{}^4\nabla_{\lambda} C^{\alpha}_{\mu} = B^{\alpha}_{\mu} C^{\lambda}_{\delta}{}^4\nabla_{\lambda} C^{\mu}_{\gamma} \ . \tag{5.4.2}$$

From the above definitions, the following identity is obtained immediately:

$$^4\nabla_{\delta} B^{\alpha}_{\gamma} = -\,^4\nabla_{\delta} C^{\alpha}_{\gamma} = H_{\delta\gamma}{}^{\alpha} + H_{\delta}{}^{\alpha}{}_{\gamma} - L_{\delta\gamma}{}^{\alpha} - L_{\delta}{}^{\alpha}{}_{\gamma} \ . \tag{5.4.3}$$

Lowering the contravariant index in the above equation yields

$$^4\nabla_{\delta} g_{\gamma\beta} = -\,^4\nabla_{\delta}\,'g_{\gamma\beta} = 2H_{\delta(\gamma\beta)} - 2L_{\delta(\gamma\beta)} . \tag{5.4.4}$$

From (5.4.1 and 2), we see that the contravariant indices in $H_{\delta\gamma}{}^{\alpha}$ and $L_{\delta\gamma}{}^{\alpha}$ lie in {T} and {S} respectively, and the covariant indices in {S} and {T} . Taking the dyad components of (5.4.1), we get

$$H_{\delta\gamma}{}^{a} = -B^{\lambda\mu}_{\delta\gamma}{}^4\nabla_{\lambda} n^a_{\mu} \ , \tag{5.4.5a}$$

hence

$$H_{[\delta\gamma]}{}^{a} = 0 \qquad (5.4.5b)$$

and

$$h_{\delta\gamma a} \overset{\text{def}}{=} H_{(\delta\gamma)a} = -\tfrac{1}{2}B^{\lambda\mu}_{\delta\gamma}\pounds_{n_a} g_{\lambda\mu} \ . \qquad (5.4.5c)$$

Note that the Lie derivative with respect to $\vec{n}_a$ of a covariant quantity tangent to {S} is not itself necessarily tangent to {S}. For example, for a one-form v in {S} ,

$$n_a^{\varepsilon}\pounds_{n_b} v_{\varepsilon} = -v_{\varepsilon}\pounds_{n_b} n_a^{\varepsilon} = 2\Omega_{ba}{}^{\varepsilon} v_{\varepsilon} \quad ,$$

hence there will in general be a non zero projection of $\pounds_{n_b} v_{\varepsilon}$ into {T} . However, since for the corresponding vector $\vec{v}$,

$$n_{\varepsilon}^{a}\pounds_{n_b} v^{\varepsilon} = -v^{\varepsilon}\pounds_{n_b} n_{\varepsilon}^{a} = 0 \ ,$$

the Lie derivative with respect to $\vec{n}_a$ of a contravariant quantity tangent to {S} remains tangent to {S} . Hence from (5.4.5c), we see that we may write

$$h^{\delta\gamma}{}_{a} = \tfrac{1}{2}\pounds_{n_a} g^{\delta\gamma} \quad . \qquad (5.4.5d)$$

We shall refer to $h_{\delta\gamma a}$ as the extrinsic curvatures of {S} . They are the analagous quantities for {S} to the extrinsic curvature $h_{\delta\gamma}$ of {Σ} defined in equation (2.3.8). Note that {S} , since it is a foliation of codimension 2 , has two extrinsic curvatures. Each $S\varepsilon\{S\}$ arises as the intersection of some $\Sigma_o\varepsilon\{\Sigma_o\}$ and $\Sigma_1\varepsilon\{\Sigma_1\}$, and $h_{\alpha\beta o}$, $h_{\alpha\beta 1}$ are 'velocities' of $g_{\alpha\beta}$ in the direction orthogonal to {S} , but tangent to $\{\Sigma_o\}$ and $\{\Sigma_1\}$ respectively. Put another way, any $S\varepsilon\{S\}$ can be thought of as a hypersurface in some $\Sigma_o\varepsilon\{\Sigma_o\}$ and has a corresponding extrinsic curvature $h_{\alpha\beta o}$, or as a hypersurface in some $\Sigma_1\varepsilon\{\Sigma_1\}$, with corresponding extrinsic curvature $h_{\alpha\beta 1}$.

Taking dyad components of (5.4.2), we get

$$L_{dc}{}^{\alpha} = B^{\alpha}_{\mu} n^{\lambda}_{d} {}^{4}\nabla_{\lambda} n^{\mu}_{c} \tag{5.4.6a}$$

This yields immediately, using (5.3.12), that

$$L_{[dc]}{}^{\alpha} = -\Omega_{dc}{}^{\alpha}. \tag{5.4.6b}$$

It can also be shown from (5.4.6a), that

$$\ell_{dc\alpha} \overset{def}{=} L_{(dc)\alpha} = -\tfrac{1}{2}\nabla_{\alpha}\eta_{dc} , \tag{5.4.6c}$$

which we demonstrate as follows. From (5.4.6a), we have immediately that

$$L_{d}{}^{c}{}_{\alpha} = B^{\mu}_{\alpha} n^{\lambda}_{d} {}^{4}\nabla_{\lambda} n^{c}_{\mu} = B^{\mu}_{\alpha} n^{\lambda}_{d} {}^{4}\nabla_{\mu} n^{c}_{\lambda} = -B^{\mu}_{\alpha} n^{c}_{\lambda} {}^{4}\nabla_{\mu} n^{\lambda}_{d} . \tag{5.4.7}$$

From the last of (5.4.7), we get

$$L_{dc\alpha} = -B^{\mu}_{\alpha} n_{\lambda c} {}^{4}\nabla_{\mu} n^{\lambda}_{d} .$$

Symmetrising on d and c in the above gives

$$\ell_{dc\alpha} = -B^{\mu}_{\alpha} n_{\lambda(c} {}^{4}\nabla_{|\mu|} n^{\lambda}_{d)} = -\tfrac{1}{2} B^{\mu}_{\alpha} {}^{4}\nabla_{\mu}\left(n_{\lambda c} n^{\lambda}_{d}\right) = -\tfrac{1}{2}\nabla_{\alpha}\eta_{cd} ,$$

which is the desired result.

We can decompose ${}^{4}\nabla_{\gamma} n^{a}_{\beta}$ and ${}^{4}\nabla_{\gamma} n^{\alpha}_{b}$ in terms of their projections into {S} and {T} . Collecting together the results (5.3.7), and (5.4.1,2, and 5) yields

$${}^{4}\nabla_{\gamma} n^{a}_{\beta} = -h_{\gamma\beta}{}^{a} + 2L_{b}{}^{a}{}_{(\gamma} n^{b}_{\beta)} - \Gamma^{a}_{cb} n^{c}_{\gamma} n^{b}_{\beta} \tag{5.4.8a}$$

$${}^{4}\nabla_{\gamma} n^{\alpha}_{b} = -h_{\gamma}{}^{\alpha}{}_{b} - L_{b}{}^{a}{}_{\gamma} n^{\alpha}_{a} + L_{cb}{}^{\alpha} n^{c}_{\gamma} + \Gamma^{a}_{cb} n^{\alpha}_{a} n^{c}_{\gamma} . \tag{5.4.8b}$$

We can now decompose the covariant derivative of any tensor in V in a convenient way. Suppose we have a tensor with every index lying either in {S} or in {T} (an arbitrary tensor can always be expressed as a sum of such tensors). For example, suppose that

$$X_{\gamma\beta} = B^{\mu}_{\gamma} n^{b}_{\beta} X_{\mu b} \ .$$

Most of the projections of $^{4}\nabla_{\delta} X_{\gamma\beta}$ are either zero, or follow immediately from (5.4.3), and do not involve derivatives of $X_{\gamma\beta}$. For example,

$$B^{\lambda\mu\nu}_{\delta\gamma\beta} {}^{4}\nabla_{\lambda} X_{\mu\nu} = -B^{\lambda\mu}_{\delta\gamma} X_{\mu\nu} {}^{4}\nabla_{\lambda} B^{\nu}_{\beta} = - X_{\gamma\nu} H_{\delta\beta}{}^{\nu} \ .$$

There are only two projections of $^{4}\nabla_{\delta} X_{\gamma\beta}$ which do involve derivatives of $X_{\gamma\beta}$. The first is

$$B^{\lambda\mu}_{\delta\gamma} n^{\nu}_{b} {}^{4}\nabla_{\lambda} X_{\mu\nu} = \nabla_{\delta} X_{\gamma b} + X_{\gamma e} L_{b}{}^{e}{}_{\delta} \ , \qquad (5.4.9a)$$

where this relation is proved as follows. From the definition of $X_{\gamma\beta}$,

$$B^{\lambda\mu}_{\delta\gamma} n^{\nu}_{b} {}^{4}\nabla_{\lambda} X_{\mu\nu} = B^{\lambda\mu}_{\delta\gamma} n^{\nu}_{b} {}^{4}\nabla_{\lambda} \left(X_{\mu e} n^{e}_{\nu} \right) \ .$$

Expanding the right hand side gives

$$B^{\lambda\mu}_{\delta\gamma} {}^{4}\nabla_{\lambda} X_{\mu b} + X_{\gamma e} B^{\lambda}_{\delta} n^{\nu}_{b} {}^{4}\nabla_{\lambda} n^{e}_{\nu} \ .$$

The definition of ∇, and equation (5.4.8a) then yield the desired result. The only other projection involving a derivative of $X_{\gamma\beta}$ is

$$n^{\lambda}_{d} B^{\mu}_{\gamma} n^{\nu}_{b} {}^{4}\nabla_{\lambda} X_{\mu\nu} = {}'\nabla_{d} X_{\gamma b} + X_{\varepsilon b} h_{\gamma}{}^{\varepsilon}{}_{d} \ , \qquad (5.4.10a)$$

where

$${}'\nabla_{d} X_{\gamma b} \stackrel{def}{=} B^{\mu}_{\gamma} \pounds_{n_d} X_{\mu b} - \Gamma^{e}_{db} X_{\gamma e} \ . \qquad (5.4.11a)$$

To prove (5.4.10a) we first note that, again from the definition of $X_{\gamma\beta}$,

$$n^{\lambda}_{d} B^{\mu}_{\gamma} n^{\nu}_{b} {}^{4}\nabla_{\lambda} X_{\mu\nu} = n^{\lambda}_{d} B^{\mu}_{\gamma} {}^{4}\nabla_{\lambda} X_{\mu b} + X_{\gamma e} n^{\lambda}_{d} n^{\nu}_{b} {}^{4}\nabla_{\lambda} n^{e}_{\nu} .$$

Using the definition of the Lie derivative, the right hand side becomes

$$B^{\mu}_{\gamma}\left(\pounds_{n_d} X_{\mu b} - X_{\varepsilon b} {}^{4}\nabla_{\mu} n^{\varepsilon}_{d}\right) + X_{\gamma e} n^{\lambda}_{d} n^{\nu}_{b} {}^{4}\nabla_{\lambda} n^{e}_{\nu}$$

and using (5.4.8a), this reduces to (5.4.10a), remembering the definition (5.4.11a).

Similar results hold for $X^{\gamma\beta}$, and we obtain

$$B^{\lambda\gamma}_{\delta\mu} n^{b}_{\nu} {}^{4}\nabla_{\lambda} X^{\mu\nu} = \nabla_{\delta} X^{\gamma b} - X^{\gamma e} L_{e}{}^{b}{}_{\delta} , \qquad (5.4.9b)$$

$$n^{\lambda}_{d} B^{\gamma}_{\mu} n^{b}_{\nu} {}^{4}\nabla_{\lambda} X^{\mu\nu} = {}'\nabla_{d} X^{\gamma b} - X^{\varepsilon b} h_{\varepsilon}{}^{\gamma}{}_{d}, \qquad (5.4.10b)$$

where

$${}'\nabla_{d} X^{\gamma b} \overset{\text{def}}{=} \pounds_{n_d} X^{\gamma b} + \Gamma^{b}_{de} X^{\gamma e} , \qquad (5.4.11b)$$

these two projections of ${}^{4}\nabla_{\delta} X^{\gamma\beta}$ being the only ones which involve derivatives of $X^{\gamma\beta}$. Equations (5.4.11) extend the definition of ${}'\nabla_{a}$ to quantities with both dyad indices, and tensor indices lying in {S} . From this definition we see that ${}'\nabla_{a}$ "ignores" these latter indices. Equations (5.4.8) to (5.4.10) help us to calculate in a concise way the projections of the Riemann tensor of V into {S} and {T} .

5.5 The Alternating Quantity $\bar{\varepsilon}_{ab}$

We introduce a pair of antisymmetric quantities $\bar{\varepsilon}_{ab}, \bar{\varepsilon}^{ab}$ defined by

$$\bar{\varepsilon}_{o1} = -\bar{\varepsilon}^{o1} = 1 \qquad (5.5.1)$$

Next, we define

$$\eta = \left(-\det(\eta_{ab})\right)^{\frac{1}{2}} \qquad (5.5.2a)$$

$$\bar{\eta}_{ab} = \eta^{-1}\eta_{ab} \, , \; \bar{\eta}^{ab} = \eta\eta^{ab} \; . \qquad (5.5.2b)$$

From these definitions we obtain immediately the following results:

$$-\tfrac{1}{2}\bar{\varepsilon}^{ac}\bar{\varepsilon}^{bd}\eta_{ab}\eta_{cd} = \eta^{2} \qquad (5.5.3a)$$

$$-\tfrac{1}{2}\bar{\varepsilon}_{ac}\bar{\varepsilon}_{bd}\eta^{ab}\eta^{cd} = \eta^{-2} \qquad (5.5.3b)$$

$$\bar{\varepsilon}^{ab} = \eta^{2}\eta^{ac}\eta^{bd}\bar{\varepsilon}_{cd} \qquad (5.5.4a)$$

$$\bar{\varepsilon}_{ab} = \eta^{-2}\eta_{ac}\eta_{bd}\bar{\varepsilon}^{cd} \; . \qquad (5.5.4b)$$

From (5.5.3 and 4), we see that we must lower and raise indices on $\bar{\varepsilon}^{ab}$ and $\bar{\varepsilon}_{ab}$ with $\bar{\eta}_{ab}$ and its inverse, $\bar{\eta}^{ab}$ respectively. A further relationship between η^{ab} and $\bar{\varepsilon}^{ab}$ is

$$\eta^{ad}\eta^{bc} - \eta^{ac}\eta^{bd} = \eta^{-2}\bar{\varepsilon}^{ab}\bar{\varepsilon}^{cd} \; . \qquad (5.5.5)$$

Now, from (5.5.2a), we have

$$'\nabla_{a}\eta = \tfrac{1}{2}\eta^{bc}\,'\nabla_{a}\eta_{bc} = 0 \; , \qquad (5.5.6)$$

that is, η behaves like a scalar density of weight $+1$ with respect to the operator $'\nabla_{a}$. Then it is clear from (5.5.3) that $\bar{\varepsilon}_{ab}$ and $\bar{\varepsilon}^{ab}$ must behave like densities of weight -1 and $+1$ respectively, with respect to $'\nabla_a$. It then follows immediately that

$$'\nabla_a \bar{\varepsilon}^{bc} = '\nabla_a \bar{\varepsilon}_{bc} = 0 \; . \qquad (5.5.7)$$

From the above discussion we see that, comparing (5.2.22) and (5.3.12b), we may write

$$\Omega_{dc}{}^{\alpha} = \bar{\varepsilon}_{dc} \bar{\Omega}^{\alpha} \; , \qquad (5.5.8a)$$

and we then have that

$$'\nabla_a \bar{\Omega}^{\alpha} = \pounds_{n_a} \bar{\Omega}^{\alpha} - \Gamma^e_{ae} \bar{\Omega}^{\alpha} \qquad (5.5.8b)$$

that is, $\bar{\Omega}^{\alpha}$ behaves like a scalar density of weight $+1$ with respect to $'\nabla_a$. We can also prove some useful results about $'R_{dcba}$. From its definition in (5.3.17), $'R_{dcba}$ is clearly antisymmetric in its first pair of indices. Thus we may write

$$'R_{dcba} = \bar{\varepsilon}_{dc} \bar{\varepsilon}_{ba} \, 'R_{0101} + 'R_{dc(ba)} \qquad (5.5.9)$$

It is possible to show that

$$'R_{dc(ba)} = 2\bar{\varepsilon}_{dc} \bar{\Omega}^{\varepsilon} \ell_{ba\varepsilon} \qquad (5.5.10)$$

as follows: from (5.3.17), we obtain directly that

$$'R_{dcba} = \pounds_{n_d} \Gamma_{cba} - \pounds_{n_c} \Gamma_{dba} + \Gamma_{ace} \Gamma^e_{db} - \Gamma_{ade} \Gamma^e_{cb} \qquad (5.5.11)$$

where, from (5.3.9)

$$\Gamma_{cba} \overset{\mathrm{def}}{=} \eta_{ae} \Gamma^e_{cb} = \tfrac{1}{2}\left(\pounds_{n_c} \eta_{ba} + \pounds_{n_b} \eta_{ca} - \pounds_{n_a} \eta_{cb}\right) \qquad (5.5.12)$$

Symmetrising on a,b in (5.5.11) and using (5.5.12) yields

$$'R_{dc(ba)} = \tfrac{1}{2}\left[\pounds_{n_d}\pounds_{n_c} - \pounds_{n_c}\pounds_{n_d}\right]\eta_{ab} \ .$$

The equation (5.5.10) then follows from the definitions (5.3.12b) and (5.4.6c), and the commutation law for Lie derivatives. Equations (5.5.9 and 10) yield immediately the following:

$$'R_{cb} \stackrel{def}{=} {'R_{ecb}}^{e} = \eta^{-2}R_{olol} + 2\bar{\varepsilon}_{ec}\ell^{e}{}_{b\varepsilon}\bar{\Omega}^{\varepsilon}$$

$$'R \stackrel{def}{=} {'R_{e}}^{e} = 2\eta^{-2}R_{olol} \ . \qquad (5.5.13)$$

Hence

$$'G_{cb} \stackrel{def}{=} {'R_{cb}} - \tfrac{1}{2}\eta_{cb}{'R} = 2\bar{\varepsilon}_{ec}\ell^{e}{}_{b\varepsilon}\bar{\Omega}^{\varepsilon} \ . \qquad (5.5.14)$$

The quantities $'R_{cb}$, $'R$ and $'G_{cb}$ are the dyad components of the Ricci tensor and scalar, and Einstein tensor of {T} respectively. As we would expect, the Einstein tensor vanishes if and only if the Riemannian 2-surface elements {T} are holonomic.

Finally in this section, we note that by virtue of (5.5.5), the dyad components of any tensor $T^{\alpha\beta}$ in {T} obey the following identity:

$$T^{ab} - \eta^{ab}T_{e}{}^{e} = \left(\eta^{ac}\eta^{bd} - \eta^{ab}\eta^{cd}\right)T_{cd} = \eta^{-2}\bar{\varepsilon}^{ad}\bar{\varepsilon}^{bc}T_{cd} . \qquad (5.5.15)$$

5.6 Conformal 2-structure.

The concept of conformal 2-structure is due originally to d'Inverno and Stachel, and an extensive discussion of it can be found in reference 28. They in fact only consider in detail the case of 2-surfaces foliating a 3-dimensional manifold. However, as they indicate, it is straightforward to extend their invariant definition to the case of 2-surfaces foliating a 4-dimensional manifold. We make such an extension here. We start by removing a conformal factor γ from the induced metric of $\{S\}$, and we denote the resulting metric by $\tilde{g}_{\alpha\beta}$. That is, we define

$$g_{\alpha\beta} = \gamma\tilde{g}_{\alpha\beta} \ , \ g^{\alpha\beta} = \gamma^{-1}\tilde{g}^{\alpha\beta} \tag{5.6.1}$$

and then the conformal extrinsic curvatures of $\{S\}$ are defined by

$$\tilde{h}^{\gamma\beta}{}_{a} = \tfrac{1}{2}\pounds_{n_a}\tilde{g}^{\gamma\beta} \ . \tag{5.6.2}$$

We now look for a conformal factor such that the trace of each of the resulting conformal extrinsic curvatures vanishes, that is, such that

$$\tilde{h}_{\varepsilon}{}^{\varepsilon}{}_{a} \overset{\text{def}}{=} \tilde{g}_{\gamma\beta}\tilde{h}^{\gamma\beta}{}_{a} = 0 \ , \tag{5.6.3}$$

(Note that indices on quantitities with a superscript $(\tilde{\ })$ are raised and lowered by the conformal metric.) Now, we have

$${}^{4}g^{\gamma\beta}\pounds_{n_a}{}^{4}g_{\gamma\beta} = g^{\gamma\beta}\pounds_{n_a}g_{\gamma\beta} + {}'g^{\gamma\beta}\pounds_{n_a}{}'g_{\gamma\beta} \ .$$

Using (5.5.2) and (5.6.1-3), the above equation can be written as

$$(\sqrt{-{}^{4}g})^{-1}\,\pounds_{n_a}\sqrt{-{}^{4}g} = \gamma^{-1}\pounds_{n_a}\gamma - \tilde{h}_{\varepsilon}{}^{\varepsilon}{}_{a} + \eta^{-1}\pounds_{n_a}\eta \ , \tag{5.6.4}$$

where

$${}^{4}g \overset{\text{def}}{=} \det({}^{4}g_{\alpha\beta}) \ .$$

From (5.6.4), we see that (5.6.3) is satisfied if and only if

$$\pounds_{n_a}\left(\eta\gamma(\sqrt{-{}^{4}g})^{-1}\right) = 0 \ . \tag{5.6.5}$$

Since $\vec{n}_a$ do not commute, the above equations have integrability conditions which must be satisfied, and in general, the only solution of (5.6.5) (unique up to a constant scalar factor) is

$$\gamma = \eta^{-1}\sqrt{-{}^4g} \quad . \tag{5.6.6}$$

We shall assume for the remainder of this thesis that γ is given by the above equation, and we denote the resulting conformal metric $\tilde{g}_{\alpha\beta}$ as the conformal 2-structure. We see that $\tilde{g}_{\alpha\beta}$ has only two independent components, by virtue of (5.6.3) . In fact γ^2 is just the determinant of the induced metric of $\{S\}$, when the latter is written in terms of the basis B_A . Note that γ has the same tensorial character as $\sqrt{-{}^4g}$, namely it is a scalar density of weight + 1 in V.

We may now decompose the extrinsic curvatures into their trace and trace-free parts. We get

$$h_{\gamma\beta a} = \gamma\tilde{h}_{\gamma\beta a} + \tfrac{1}{2}g_{\gamma\beta}h_a \tag{5.6.7}$$

where

$$h_a \overset{\text{def}}{=} h_{\varepsilon}{}^{\varepsilon}{}_a = -\gamma^{-1}\pounds_{n_a}\gamma \; . \tag{5.6.8}$$

We define the operator $'\nabla_a$ on γ by

$$'\nabla_a\gamma = \pounds_{n_a}\gamma \tag{5.6.9}$$

and this in turn defines the operation of ∇_a on all quantities superscripted by $(\tilde{\ })$. For example

$$'\nabla_a\tilde{g}^{\gamma\beta} = \pounds_{n_a}\tilde{g}^{\gamma\beta} = 2\tilde{h}^{\gamma\beta}{}_a,$$

and so on in an obvious way.

5.7 Projections of the Riemann Tensor.

On account of the many symmetries of the Riemann tensor of V, it is not difficult to see that it has only five independent projections into $\{S\}$ and $\{T\}$. Four of these are given by the well known equations of Gauss and Codazzi for (possibly anholonomic) Riemannian submanifolds of a Riemann space. In the present case, they reduce to

$$B^{\lambda\mu\nu\pi}_{\delta\gamma\beta\alpha}\,{}^4R_{\lambda\mu\nu\pi} = R_{\delta\gamma\beta\alpha} + 2H_{[\delta|\beta|}{}^{\varepsilon}H_{\gamma]\alpha\varepsilon} \qquad \text{(Gauss)},$$

$$B^{\lambda\mu\nu}_{\delta\gamma\beta}C^{\pi}_{\alpha}\,{}^4R_{\lambda\mu\nu\pi} = 2B^{\lambda\mu}_{[\delta\gamma]}{}^{\nu}_{\beta}C^{\pi}_{\alpha}\,{}^4\nabla_{\lambda}H_{\mu\nu\pi} \qquad \text{(Codazzi)},$$

for $\{S\}$, and

$$C^{\lambda\mu\nu\pi}_{\delta\gamma\beta\alpha}\,{}^4R_{\lambda\mu\nu\pi} = {}'R_{\delta\gamma\beta\alpha} + 2L_{[\delta|\beta|}{}^{\varepsilon}L_{\gamma]\alpha\varepsilon} - 2\Omega_{\delta\gamma}{}^{\varepsilon}L_{\beta\alpha\varepsilon} \qquad \text{(Gauss)},$$

$$C^{\lambda\mu\nu}_{\delta\gamma\beta}B^{\pi}_{\alpha}\,{}^4R_{\lambda\mu\nu\pi} = 2C^{\lambda\mu}_{[\delta\gamma]}{}^{\nu}_{\beta}B^{\pi}_{\alpha}\,{}^4\nabla_{\lambda}L_{\mu\nu\pi} - 2\Omega_{\delta\gamma}{}^{\varepsilon}H_{\varepsilon\alpha\beta} \qquad \text{(Codazzi)},$$

for $\{T\}$. We now take dyad components of the above four equations, and use (5.4.9 and 10) to expand the first terms on the right hand side of the respective Codazzi equations. We obtain, respectively

$$\perp^4R_{\delta\gamma\beta\alpha} = R_{\delta\gamma\beta\alpha} + 2h_{[\delta|\beta|}{}^{e}h_{\gamma]\alpha e} \qquad (5.7.1)$$

$$\perp^4R_{\delta\gamma\beta a} = 2\nabla_{[\delta}h_{\gamma]\beta a} + 2L_{a}{}^{e}{}_{[\delta}h_{\gamma]\beta e} \qquad (5.7.2)$$

$${}^4R_{dcba} = {}'R_{dcba} + 2L_{[d|b|}{}^{\varepsilon}L_{c]a\varepsilon} - 2\Omega_{dc}{}^{\varepsilon}L_{ba\varepsilon} \qquad (5.7.3)$$

$$\perp^4R_{dcb\alpha} = 2'\nabla_{[d}L_{c]b\alpha} + 2h_{\alpha}{}^{\varepsilon}{}_{[d}L_{c]b\varepsilon} - 2\Omega_{dc}{}^{\varepsilon}h_{\varepsilon\alpha b}, \qquad (5.7.4)$$

where, on the left hand sides of the above equations, we have used the abbreviated notation, described in section 5.2, that

$$B^{\lambda\mu\nu}_{\delta\gamma\beta} n^{\pi}_{a} {}^4R_{\lambda\mu\nu\pi} \overset{\text{def}}{=} \perp {}^4R_{\delta\gamma\beta a}$$

and so on. The remaining independent projection of the Riemann tensor is $\perp {}^4R_{d\gamma\beta a}$, and we obtain a suitable expression for this as follows: we start by considering the Ricci identity, applied to $'g_{\alpha\beta}$. This gives immediately

$$-2{}^4\nabla_{[\lambda}{}^4\nabla_{\mu]}{}'g_{\nu\pi} = {}^4R_{\lambda\mu\nu}{}^{\varepsilon}{}'g_{\varepsilon\pi} + {}^4R_{\lambda\mu\pi}{}^{\varepsilon}{}'g_{\varepsilon\nu}$$

$$= {}^4R_{\lambda\mu\nu\varepsilon}C^{\varepsilon}_{\pi} + {}^4R_{\lambda\mu\pi\varepsilon}C^{\varepsilon}_{\nu} .$$

Next, we project this quantity twice into $\{S\}$, and twice into $\{T\}$. Taking dyad components, this yields

$$\perp {}^4R_{d\gamma\beta a} = -2n^{\lambda}_{d}B^{\mu\nu}_{\gamma\beta}n^{\pi}_{a}{}^4\nabla_{[\lambda}{}^4\nabla_{\mu]}{}'g_{\nu\pi} .$$

The above expression for $\perp {}^4R_{d\gamma\beta a}$ becomes, using the identity (5.4.4)

$$\perp {}^4R_{d\gamma\beta a} = 2n^{\lambda}_{d}B^{\mu\nu}_{\gamma\beta}n^{\pi}_{a}{}^4\nabla_{[\lambda}\left(H_{\mu]\nu\pi} - L_{\mu]\nu\pi}\right) .$$

Finally, we use the identity (5.4.3), and equations (5.4.9 and 10) in the above, and after some straight forward algebraic manipulation we get

$$\perp {}^4R_{d\gamma\beta a} = {}'\nabla_{d}h_{\gamma\beta a} + h_{\gamma\varepsilon a}h_{\beta}{}^{\varepsilon}{}_{d} + \nabla_{\gamma}L_{da\beta} + L_{de\beta}L_{a}{}^{e}{}_{\gamma} \qquad (5.7.5)$$

5.8 Projections of the Einstein Tensor.

The Ricci tensor of V is defined by

$$^4R^{\gamma\beta} = {}^4R^{\varepsilon\gamma\beta}{}_{\varepsilon} \qquad (5.8.1)$$

and the Einstein tensor by

$$ {}^4G^{\gamma\beta} = {}^4R^{\gamma\beta} - \tfrac{1}{2}g^{\gamma\beta}\,{}^4R \tag{5.8.2} $$

where

$$ {}^4R = {}^4R_{\varepsilon}{}^{\varepsilon} \tag{5.8.3} $$

is the Ricci scalar. We split the projections into $\{S\}$ of ${}^4R^{\gamma\beta}$ and ${}^4G^{\gamma\beta}$ into their respective trace and trace-free parts, defined by

$$ \perp{}^4R = g_{\gamma\beta}\perp{}^4R^{\gamma\beta} \tag{5.8.4a} $$

$$ \perp{}^4G = g_{\gamma\beta}\perp{}^4G^{\gamma\beta} \tag{5.8.4b} $$

$$ \gamma^{-1}\perp{}^4\tilde{R}^{\gamma\beta} = \perp{}^4R^{\gamma\beta} - \tfrac{1}{2}g^{\gamma\beta}\perp{}^4R\ , \tag{5.8.5a} $$

$$ \gamma^{-1}\perp{}^4\tilde{G}^{\gamma\beta} = \perp{}^4G^{\gamma\beta} - \tfrac{1}{2}g^{\gamma\beta}\perp{}^4G\ . \tag{5.8.5b} $$

Equations (5.8.2-5) imply the following relationships between the projections of the Ricci and Einstein tensors:

$$ \perp{}^4\tilde{G}^{\gamma\beta} = \perp{}^4\tilde{R}^{\gamma\beta} \tag{5.8.6a} $$

$$ \perp{}^4G = -{}^4R_e{}^e \tag{5.8.6b} $$

$$ \perp{}^4G^{c\beta} = \perp{}^4R^{c\beta} \tag{5.8.6c} $$

$$ {}^4G^{cb} = {}^4R^{cb} - \tfrac{1}{2}\eta^{cb}\left(\perp{}^4R + {}^4R_e{}^e\right) \tag{5.8.6d} $$

$$ {}^4G_e{}^e = -\perp{}^4R\ . \tag{5.8.6e} $$

Using equations (5.8.1-6), we can write down the projections of the Einstein tensor in terms of the projections of the Riemann tensor. We obtain

$$\gamma^{-1}\perp{}^4\tilde{G}^{\gamma\beta} = \perp{}^4R^{e\gamma\beta}{}_{e} + \perp{}^4R^{\varepsilon\gamma\beta}{}_{\varepsilon} - \tfrac{1}{2}g^{\gamma\beta}\left(\perp{}^4R^{e\varepsilon}{}_{\varepsilon e} + \perp{}^4R^{\varepsilon\theta}{}_{\theta\varepsilon}\right)$$

$$-\perp{}^4G = {}^4R^{ef}{}_{fe} + \perp{}^4R^{e\varepsilon}{}_{\varepsilon e}$$

$$\perp{}^4G^{c\beta} = \perp{}^4R^{ce}{}_{e}{}^{\beta} + \perp{}^4R^{\beta\varepsilon}{}_{\varepsilon}{}^{c}$$

$${}^4G^{cb} = {}^4R^{ecb}{}_{e} + \perp{}^4R^{c\varepsilon}{}_{\varepsilon}{}^{b} - \tfrac{1}{2}\eta^{cb}\left({}^4R^{ef}{}_{fe} + \perp{}^4R^{\varepsilon\theta}{}_{\theta\varepsilon} + 2\perp{}^4R^{e\varepsilon}{}_{\varepsilon e}\right)$$

$$-{}^4G_{e}{}^{e} = \perp{}^4R^{e\varepsilon}{}_{\varepsilon e} + \perp{}^4R^{\varepsilon\theta}{}_{\theta\varepsilon} \,,$$

It is straightforward to calculate all the necessary contractions of (5.7.1-5) needed in order to calculate explicit expressions for the projections of ${}^4G^{\alpha\beta}$. It must be remembered that ∇_α and the raising and lowering of dyad indices do not commute, neither does $'\nabla_a$ and the raising and lowering of tensor indices in $\{S\}$. The following formulae are easily established for any $\vec{v}$ and $\vec{w}$ tangent to $\{S\}$ and $\{T\}$ respectively:

$$g^{\alpha\beta}\,'\nabla_a v_\alpha = {}'\nabla_a v^\beta - 2v^\alpha h_\alpha{}^\beta{}_a$$

$$\eta^{ab}\nabla_\alpha w_a = \nabla_\alpha w^b - 2w^a \ell_a{}^b{}_\alpha \quad .$$

Using these results, together with the decomposition of $L_{dc\alpha}$ into its symmetric and antisymmetric parts, given by (5.4.6) we get

$$\perp {}^4R^{\varepsilon\gamma\beta}{}_{\varepsilon} = R^{\gamma\beta} + h^{\gamma\varepsilon e}h^{\beta}{}_{\varepsilon e} - h^{\gamma\beta e}h_{e}$$

$$\perp {}^4R^{\varepsilon\theta}{}_{\theta\varepsilon} = R + h^{\varepsilon\theta e}h_{\varepsilon\theta e} - h^{e}h_{e}$$

$$\perp {}^4R^{\beta\varepsilon}{}_{\varepsilon}{}^{c} = \nabla^{\beta}h^{c} - \nabla^{\varepsilon}h^{\beta}{}_{\varepsilon}{}^{c} - h_{e}\ell^{ec\beta} + h^{\beta}{}_{\varepsilon e}\ell^{ec\varepsilon} - h_{e}\Omega^{ce\beta} + h^{\beta}{}_{\varepsilon e}\Omega^{ce\varepsilon}$$

$$\perp {}^4R^{e\gamma\beta}{}_{e} = {}'\nabla_{e}h^{\gamma\beta e} - 3h^{\gamma\varepsilon e}h^{\beta}{}_{\varepsilon e} + \nabla^{\gamma}\ell^{\beta} - \ell_{ef}{}^{\gamma}\ell^{ef\beta} + \Omega_{ef}{}^{\gamma}\Omega^{ef\beta}$$

$$\perp {}^4R^{e\varepsilon}{}_{\varepsilon e} = {}'\nabla_{e}h^{e} - h^{\varepsilon\theta e}h_{\varepsilon\theta e} + \nabla_{\varepsilon}\ell^{\varepsilon} - \ell_{ef\varepsilon}\ell^{ef\varepsilon} + \Omega_{ef\varepsilon}\Omega^{ef\varepsilon}$$

$$\perp {}^4R^{c\varepsilon}{}_{\varepsilon}{}^{b} = \nabla_{\varepsilon}\ell^{cb\varepsilon} - 3\ell^{ce\varepsilon}\ell^{b}{}_{e\varepsilon} + {}'\nabla^{(c}h^{b)} - h^{\varepsilon\theta c}h_{\varepsilon\theta}{}^{b} + \Omega^{ce\varepsilon}\Omega^{b}{}_{e\varepsilon} + 2\Omega^{e(b}{}_{\varepsilon}\ell^{c)}{}_{e}{}^{\varepsilon}$$

$$\perp {}^4R^{ce}{}_{e}{}^{\beta} = {}'\nabla^{c}\ell^{\beta} - {}'\nabla_{e}\ell^{ce\beta} - h^{\beta}{}_{\varepsilon}{}^{c}\ell^{\varepsilon} + \ell^{ce\varepsilon}h^{\beta}{}_{\varepsilon e} + {}'\nabla_{e}\Omega^{ce\beta} - 3h^{\beta}{}_{\varepsilon e}\Omega^{ce\varepsilon}$$

$${}^4R^{ecb}{}_{e} = {}'R^{(cb)} + \ell^{ce\varepsilon}\ell^{b}{}_{e\varepsilon} - \ell^{cb\varepsilon}\ell_{\varepsilon} - 3\Omega^{ec\varepsilon}\Omega_{e}{}^{b}{}_{\varepsilon} - 2\Omega^{e(c}{}_{\varepsilon}\ell^{b)}{}_{e}{}^{\varepsilon}$$

$${}^4R^{ef}{}_{fe} = {}'R + \ell^{ef\varepsilon}\ell_{ef\varepsilon} - \ell^{\varepsilon}\ell_{\varepsilon} - 3\Omega^{ef\varepsilon}\Omega_{ef\varepsilon} .$$

To obtain the projections of ${}^4G^{\alpha\beta}$ in final form, we use the expression for $\Omega_{dc}{}^{\alpha}$ given by (5.5.8a) and that for ${}'R^{cb}$ given by (5.5.14). In addition, we decompose $h_{\gamma\beta a}$ using (5.6.7), and apply (5.5.15) to ${}'\nabla^{(c}h^{b)}$ in the calculation of ${}^4G^{cb}$. Then we obtain, by straightforward algebraic manipulation,

$$\perp {}^4\tilde{G}^{\gamma\beta} = {}'\nabla_{e}\tilde{h}^{\gamma\beta e} - 2\tilde{h}^{\gamma\varepsilon e}\tilde{h}^{\beta}{}_{\varepsilon e} - \tilde{h}^{\gamma\beta e}h_{e} + \tilde{T}\left(\nabla^{\gamma}\ell^{\beta} - \ell^{ef\gamma}\ell_{ef}{}^{\beta} - 2\eta^{-2}\bar{\Omega}^{\gamma}\bar{\Omega}^{\beta}\right) \qquad (5.8.7)$$

$$-\perp {}^4G = {}'R + {}'\nabla_{e}h^{e} - \tfrac{1}{2}h_{e}h^{e} - \tilde{h}^{\varepsilon\theta e}\tilde{h}_{\varepsilon\theta e} + \nabla_{\varepsilon}\ell^{\varepsilon} - \ell_{\varepsilon}\ell^{\varepsilon} + 4\eta^{-2}\bar{\Omega}^{\varepsilon}\bar{\Omega}_{\varepsilon} \qquad (5.8.8)$$

$$\perp{}^4G^{c\beta} = \eta^{-2}\bar{\varepsilon}^{ce}\left('\nabla_e\bar{\Omega}^{\beta} - 2(\tilde{h}^{\beta}{}_{\varepsilon e} + \delta^{\beta}_{\varepsilon}h_e)\bar{\Omega}^{\varepsilon}\right) - \nabla^{\varepsilon}\tilde{h}^{\beta}{}_{\varepsilon}{}^{c} + \tfrac{1}{2}\nabla^{\beta}h^{c}$$

$$+ 2\tilde{h}^{\beta}{}_{\varepsilon e}\ell^{ce\varepsilon} - \tilde{h}^{\beta}{}_{\varepsilon}{}^{c}\ell^{\varepsilon} - \tfrac{1}{2}h^{c}\ell^{\beta} - '\nabla_e\ell^{ce\beta} + '\nabla^{c}\ell^{\beta} \qquad (5.8.9)$$

$${}^4G^{cb} = \eta^{-2}\bar{\varepsilon}^{ce}\bar{\varepsilon}^{bf}\,'\nabla_{(e}h_{f)} + 2\eta^{-2}\bar{\varepsilon}^{e(c}\ell^{b)}{}_{e}{}^{\varepsilon}\bar{\Omega}_{\varepsilon} - \tfrac{1}{2}h^{c}h^{b} - \tilde{h}^{\varepsilon\theta c}\tilde{h}_{\varepsilon\theta}{}^{b} + \nabla_{\varepsilon}\ell^{cb\varepsilon}$$

$$-2\ell^{ce\varepsilon}\ell^{b}{}_{e\varepsilon} - \ell^{cb\varepsilon}\ell_{\varepsilon} + \eta^{cb}\Big(\tfrac{1}{2}\tilde{h}^{\varepsilon\theta e}\tilde{h}_{\varepsilon\theta e} + \tfrac{3}{4}h^{e}h_{e} - \nabla_{\varepsilon}\ell^{\varepsilon} + \tfrac{1}{2}\ell^{ef\varepsilon}\ell_{ef\varepsilon}$$

$$+ \tfrac{1}{2}\ell^{\varepsilon}\ell_{\varepsilon} + \eta^{-2}\bar{\Omega}^{\varepsilon}\bar{\Omega}_{\varepsilon} - \tfrac{1}{2}R\Big) \qquad (5.8.10)$$

$$-{}^4G_e{}^{e} = '\nabla_e h^{e} - h^{e}h_e + \nabla_{\varepsilon}\ell^{\varepsilon} - \ell^{ef\varepsilon}\ell_{ef\varepsilon} - 2\eta^{-2}\bar{\Omega}^{\varepsilon}\bar{\Omega}_{\varepsilon} + R \;, \qquad (5.8.11)$$

where in (5.8.7), we have used the notation

$$\tilde{T}\left(X^{\gamma\beta}\right) \overset{\text{def}}{=} \gamma X^{(\gamma\beta)} - \tfrac{1}{2}\tilde{g}^{\gamma\beta}X_{\varepsilon}{}^{\varepsilon} \;, \qquad \text{for any } X^{\gamma\beta} \text{ in } \{S\} \,.$$

5.9 The Bianchi Identities

The Bianchi identities ${}^4\nabla_{\varepsilon}{}^4G^{\alpha\varepsilon} = 0$ can be decomposed in terms of the projections of ${}^4G^{\alpha\beta}$. First of all we consider the projection of the identities into $\{S\}$. We get

$$0 = B^{\alpha}_{\pi}{}^4\nabla_{\varepsilon}{}^4G^{\pi\varepsilon} = B^{\alpha\varepsilon}_{\pi\mu}\,{}^4\nabla_{\varepsilon}{}^4G^{\pi\mu} + B^{\alpha}_{\pi}C^{\varepsilon}_{\mu}\,{}^4\nabla_{\varepsilon}{}^4G^{\pi\mu}$$

$$= B^{\alpha\varepsilon}_{\pi\mu}{}^4\nabla_{\varepsilon}\left(\perp{}^4G^{\pi\mu} + n^{\pi}_{e}\perp{}^4G^{e\mu} + n^{\mu}_{e}\perp{}^4G^{e\pi}\right) + B^{\alpha}_{\pi}C^{\varepsilon}_{\mu}{}^4\nabla_{\varepsilon}\left(\perp{}^4G^{\pi\mu} + n^{\mu}_{e}\perp{}^4G^{e\pi} + n^{\mu}_{e}n^{\pi}_{f}{}^4G^{ef}\right) \,.$$

Application of (5.4.3), (5.4.8-10) to the above then results in

$$B^{\alpha}_{\pi}\ {}^4\nabla_{\varepsilon}\ {}^4G^{\pi\varepsilon} = {}'\nabla_e \perp {}^4G^{e\alpha} - \left(2h^{\alpha}{}_{\varepsilon e} + \delta^{\alpha}_{\varepsilon} h_e\right) \perp {}^4G^{e\varepsilon} + \nabla_{\varepsilon} \perp {}^4G^{\alpha\varepsilon} - \ell_{\varepsilon} \perp {}^4G^{\alpha\varepsilon} + \ell_{ef}{}^{\alpha}\, {}^4G^{ef} = 0 \ . \tag{5.9.1}$$

Similarly, we get for the projection into $\{T\}$

$$0 = C^{\alpha}_{\pi}{}^4\nabla_{\varepsilon}\ {}^4G^{\pi\varepsilon} = C^{\alpha}_{\pi}B^{\varepsilon}_{\mu}{}^4\nabla_{\varepsilon}\left(\perp {}^4G^{\pi\mu} + n^{\pi}_{e} \perp {}^4G^{e\mu} + n^{\pi}_{e} n^{\mu}_{f}\, {}^4G^{ef}\right)$$

$$+ C^{\alpha\varepsilon}_{\pi\mu}{}^4\nabla_{\varepsilon}\left(n^{\pi}_{e} \perp {}^4G^{e\mu} + n^{\mu}_{e} \perp {}^4G^{e\pi} + n^{\pi}_{e} n^{\mu}_{f}\, {}^4G^{ef}\right)$$

which yields, on taking the dyad components

$$n^{a}_{\pi}{}^4\nabla_{\varepsilon}\ {}^4G^{\pi\varepsilon} = {}'\nabla_e\, {}^4G^{ea} - h_e\, {}^4G^{ea} + \nabla_{\varepsilon} \perp {}^4G^{a\varepsilon} - \left(2L_e{}^{a}{}_{\varepsilon} + \delta^{a}_{e} \ell_{\varepsilon}\right) \perp {}^4G^{e\varepsilon} + h_{\varepsilon\theta}{}^{a} \perp {}^4G^{\varepsilon\theta} = 0 \ . \tag{5.9.2}$$

5.10 Conclusion

In the next two chapters, we shall examine the following problem. Given a solution to the vacuum field equations ${}^4G_{\alpha\beta} = 0$ in V, together with a particular foliation $\{S\}$ and fibrations $\mathcal{C}_a$, then what data, on an initial 2-surface $\overset{o}{S}$ and either or both of the hypersurfaces $\overset{o}{\Sigma}_a$ intersecting in $\overset{o}{S}$, are necessary and sufficient to determine that solution, in some neighbourhood of $\overset{o}{S}$. In this section, however, we shall make some preliminary remarks about the nature of this problem, and introduce some notation, which will simplify the subsequent discussions in Chapters VI and VII.

The metric of V can be written, using equations (5.2.24), (5.3.2) and (5.6.1), as

$$ {}^4g^{\alpha\beta} = \gamma^{-1}\tilde{g}^{\alpha\beta} + \eta^{ab}\left(e_a^\alpha - b_a^\alpha\right)\left(e_b^\beta - b_b^\beta\right) = {}^4g^{\alpha\beta}(\gamma, \tilde{g}^{\alpha\beta}, \eta^{ab}, b_a^\alpha). $$

Hence to determine a solution, we must find the ten independent components $\{\gamma, \tilde{g}^{\alpha\beta}, \eta^{ab}, b_a^\alpha\}$ of ${}^4g^{\alpha\beta}$. We may choose four of these components arbitrarily (subject to maintaining the correct signature of ${}^4g^{\alpha\beta}$), since the field equations are invariant under a four-dimensional coordinate gauge group. This gauge choice essentially gives us freedom to specify some (though not all) of the metrical properties of, and relations between the foliation $\{S\}$ and the fibrations $\mathcal{C}_a$. For example, if we demand that one of the foliations $\{\Sigma_a\}, \{\Sigma_1\}$ say, be a family of null hypersurfaces, then we must, by (5.2.21) impose the gauge condition

$$ \eta^{oo} = 0 . \tag{5.10.1} $$

The imposition of this gauge, which allows us to study characteristic and mixed IVP's, makes the ensuing analysis fundamentally different from the analysis of space-like IVP's, in which case we demand that $\{\Sigma_1\}$ be space-like and $\{\Sigma_o\}$ time-like. That is, by (5.2.21)

$$ \eta^{oo} > 0, \ \eta^{11} < 0 . \tag{5.10.2} $$

Having chosen either (5.10.1) or (5.10.2), the remaining analysis falls, roughly speaking, into three parts: firstly, the investigation of the rôle of the Bianchi identities; secondly, the choice of the (remaining) gauge quantities and thirdly, the construction of a formal integration scheme. The scheme then indicates how the field equations propagate the six field variables into some region of V in the neighbourhood of an initial $\overset{o}{S}\epsilon\{S\}$. This leads naturally to the identification of the freely specifiable initial data, and hence the dynamical variables. Since, as we have said, the analysis is very different under the alternative choices (5.10.2) and (5.10.1), we shall consider them separately in Chapters VI and VII respectively. First, we introduce some notation which will facilitate the ensuing discussion.

In what follows, $\overset{o}{S}$ denotes an initial space-like 2-surface, and $\overset{o}{\Sigma}_a$ denote the initial hypersurfaces emanating from $\overset{o}{S}$. In addition, we denote the 'kth neighbouring hypersurface' to $\overset{o}{\Sigma}_a$ by $\overset{k}{\Sigma}_a$, and the subsets of {S} foliating $\overset{k}{\Sigma}_a$ by $\overset{k}{\{S\}}_a$. Finally, we denote the 2-surface defined by the intersection of $\overset{i}{\Sigma}_o$ and $\overset{j}{\Sigma}_1$ by $\overset{i,j}{S}$. Then for example the 'jth neighbouring 2-surface' to $\overset{o}{S}$ within $\overset{o}{\Sigma}_o$ is $\overset{o,j}{S}$. When we refer to knowing some field variable Φ_Λ , say, on $\overset{i_o,j_o}{S}$, what we mean is that we know

$$\pounds^{(j)}_{e_o}\pounds^{(i)}_{e_1}\Phi_\Lambda\Big|_{\overset{o}{S}} , \quad i = 0,1,\ldots i_o, \quad j = 0,1,\ldots j_o .$$

The value of Φ_Λ on any S in some neighbourhood of $\overset{o}{S}$ is then determined by the generalised Taylor expansion given in equation (5.2.26).

Chapter VI. The 2+2 Formulation of the Cauchy Problem.

In this chapter, we shall assume that (5.10.2) holds, that is, that $\{\Sigma_1\}$ are space-like hypersurfaces, and $\{\Sigma_o\}$ time-like. We group the field equations ${}^4G^{\alpha\beta}=0$ as follows, using essentially the terminology of Bondi[(17)]:

$$
\left.\begin{array}{ll}
\perp {}^4G^{\alpha\beta} = 0 & \text{dynamical equations} \\
\left.\begin{array}{l}
\perp {}^4G = 0 \\
{}^4G^{11} = 0 \\
\perp {}^4G^{1\alpha} = 0
\end{array}\right\} & \text{constraint equations}
\end{array}\right\} \text{main equations}
$$

$$
\left.\begin{array}{l}
{}^4G^{oa} = 0 \\
\perp {}^4G^{o\alpha} = 0
\end{array}\right\} \text{subsidiary conditions.}
$$

First of all we analyse the Bianchi identities. If we assume that the main equations hold everywhere, and that on some arbitrary member of $\{\Sigma_1\}$ the subsidiary conditions hold also, then on this hypersurface the Bianchi identities (5.9.1) and (5.9.2) imply respectively that

$$\pounds_{e_o} \perp {}^4G^{o\alpha} = 0 \tag{6.1a}$$

and

$$\pounds_{e_o} {}^4G^{oa} = 0 \, , \tag{6.1b}$$

by virtue of (5.3.8) and (5.2.24). Equations (6.1) then lead immediately to the following lemma: the subsidiary equations are satisfied everywhere if they are satisfied on the initial hypersurface $\overset{o}{\Sigma}_1$ and the main equations hold everywhere.

The one-form defining the foliation $\{\Sigma_1\}$ is n^o, as follows from equation (5.2.2), and the lapse function, a, of $\{\Sigma_1\}$ is defined by (c.f. equation (2.2.8))

$$\eta^{oo} = {}^4g^{\alpha\beta}n^o_\alpha n^o_\beta = a^{-2} . \tag{6.2}$$

Thus η^{oo} determines the time development of the foliation $\{\Sigma_1\}$ from the initial hypersurface $\overset{o}{\Sigma}_1$, since (c.f. section 2.2, equation (2.2.12)ff.) the orthogonal metrical separation of two members of $\{\Sigma_1\}$ parameter distance $\delta\phi^o$ apart, is given by

$$\left({}^4g(\vec{N},\vec{N})\right)^{\frac{1}{2}}\delta\phi^o = a\delta\phi^o ,$$

where $\vec{N}$ is the orthogonal connecting vector of $\{\Sigma_1\}$, defined by

$$\vec{N} = a^2\vec{n}^o = (\eta^{oo})^{-1}\vec{n}^o . \tag{6.3}$$

Now, any vector $\vec{t}$ satisfying

$$\langle n^o,\vec{t}\rangle = 1 \tag{6.4}$$

is a connecting vector of $\{\Sigma_1\}$, and in particular, the vector $\vec{e}_o$ satisfies (6.4). From (5.2.9), (5.2.24) and (6.3) we have

$$\vec{e}_o = \vec{N} + \vec{b} , \tag{6.5a}$$

where

$$\vec{b} = -(\eta^{oo})^{-1}\eta^{o1}\vec{n}_1 + \vec{b}_o \Rightarrow \langle n^o,\vec{b}\rangle = 0 . \tag{6.5b}$$

In particular, if $\eta^{o1}=0$ and $\vec{b}_o=0$, then $\vec{e}_o$ and $\vec{N}$ coincide and the trajectories $\mathcal{C}_o$ of $\vec{e}_o$ are orthogonal to $\{\Sigma_1\}$. However, arbitrary choices of $\vec{b}_o$ and η^{o1} (for a specific value of η^{oo}) give rise to corresponding shifts within {S} and in the direction of $\vec{n}_1$ respectively. These two shifts combine to give the arbitrary shift $\vec{b}$ tangent to $\{\Sigma_1\}$ defined by equation (6.5b). Conversely, for an arbitrary choice of $\vec{b}$, we can always find values of η^{o1} and $\vec{b}_o$ for which (6.5b) holds. For $\vec{b} \neq 0$, the curves $\mathcal{C}_o$ are tilted relative to $\{\Sigma_1\}$, and a specific choice of $\vec{b}$ sets up a particular correspondence between points on different members of $\{\Sigma_1\}$, by identifying points on the same curves of $\mathcal{C}_o$.

For the remainder of this chapter we shall regard the set

$$\{\eta^{oo}, \eta^{o1}, b^{\alpha}_{o}\} \tag{6.6}$$

as arbitrarily specifiable in V, that is, as representing the four-dimensional gauge freedom.

There are some additional three-dimensional choices of gauge within $\overset{o}{\Sigma}_1$ which we shall take as corresponding to the freedom to specify the development of $\{\overset{o}{S}\}_1$ from $\overset{o}{S}$ and the correspondence between points on different members of $\{\overset{o}{S}\}_1$. These are governed respectively by the lapse $|\eta_{11}|^{\frac{1}{2}}$, and shift $\vec{b}_1$, of $\{\overset{o}{S}\}_1$, which we shall thus allow to take arbitrary values within $\overset{o}{\Sigma}_1$. There remains one final choice of gauge on $\overset{o}{\Sigma}_1$, which corresponds to the specification $\overset{o}{\Sigma}_1$ as a hypersurface in V. One possible choice is $\pounds_{n_o}\eta_{11}$, and to interpret the geometrical significance of this quantity we proceed as follows. We first construct a projection operator ${}^3B^{\alpha}_{\beta}$ defined by

$${}^3B^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} - u^{\alpha}u_{\beta} \tag{6.7}$$

where, from (6.2) it follows that

$$\vec{u} \overset{\text{def}}{=} (\eta^{oo})^{-\frac{1}{2}} \vec{n}^{o} \tag{6.8}$$

is the unit time-like vector orthogonal to $\{\Sigma_1\}$. Thus ${}^3B^{\alpha}_{\beta}$ is identical to the projection operator defined by equation (2.2.10). In particular the induced covariant derivative ${}^3\nabla$ on $\{\Sigma_1\}$ is Riemannian, that is

$$ {}^{3}\nabla_{\gamma}\ {}^{3}g_{\beta\alpha} \overset{\text{def}}{=} {}^{3}B^{\lambda\mu\nu}_{\gamma\beta\alpha}\ {}^{4}\nabla_{\lambda}\ {}^{4}g_{\mu\nu} = 0 $$

where ${}^{3}g_{\alpha\beta}$ is the induced metric on $\{\Sigma_1\}$. Now, from (5.4.8b) and (6.7 and 8) we obtain

$$ n_1^{\alpha}\,{}^{3}\nabla_{\alpha} n_1^{\beta} \overset{\text{def}}{=} {}^{3}B^{\beta}_{\epsilon} n_1^{\alpha}\,{}^{4}\nabla_{\alpha} n_1^{\epsilon} = L_{11}{}^{\beta} + (\Gamma^{1}_{11} - \eta^{o1}\Gamma^{o}_{11}) n_1^{\beta} . $$

Hence the trajectories of $\vec{n}_1$ are geodesics with respect to the inner geometry of $\{\Sigma_1\}$ if and only if

$$ L_{11}{}^{\beta} \overset{\text{def}}{=} -\tfrac{1}{2}\,\nabla^{\beta}\eta_{11} = 0 . \tag{6.9} $$

From (5.4.8b), we find

$$ n_1^{\alpha}\,{}^{4}\nabla_{\alpha} n_1^{\beta} = L_{11}{}^{\beta} + \Gamma^{1}_{11} n_1^{\beta} + \Gamma^{o}_{11} n_o^{\beta} , $$

from which we see that the trajectories of $\vec{n}_1$, will be geodesics in V if and only if, in addition to (6.9), the equation

$$ \Gamma^{o}_{11} \overset{\text{def}}{=} \tfrac{1}{2}\,\eta^{oo}(2\pounds_{n_1}\eta_{o1} - \pounds_{n_o}\eta_{11}) + \tfrac{1}{2}\eta^{o1}\pounds_{n_1}\eta_{11} = 0 $$

holds. Rearranging the above equation yields

$$ \pounds_{n_o}\eta_{11} = \eta^{o1}(\eta^{oo})^{-1}\pounds_{n_1}\eta_{11} + 2\pounds_{n_1}\eta_{o1} . \tag{6.10} $$

Now, within $\overset{o}{\Sigma}_1$, the shifts $\vec{b}_a$ and all the components of η_{ab} are gauge quantities. Let us suppose that some choice of these quantities has been made. It is then always possible to choose $\pounds_{n_o}\eta_{11}$ so that (6.10) holds, in which case if the trajectories of $\vec{n}_1$ are geodesics in $\overset{o}{\Sigma}_1$, they are geodesics in V also. One may thus regard $\pounds_{n_o}\eta_{11}$ as measuring by how much the trajectories of $\vec{n}_1$ fail to be geodesics in V, given that they are geodesics in $\overset{o}{\Sigma}_1$.

Finally, since any Riemannian 2-surface is conformally flat, $\tilde{g}^{\alpha\beta}$ on $\overset{o}{S}$ represents two-dimensional gauge freedom, corresponding to an initial choice of basis on $\overset{o}{S}$. Hence in summary, the lower two- and three-dimensional gauge freedom is

$$\{b_1^{\alpha}\ ,\ \eta_{11} \text{ and } \pounds_{n_o}\eta_{11} \text{ on } \overset{o}{\Sigma}_1\ ;\ \tilde{g}^{\alpha\beta} \text{ on } \overset{o}{S}\}\ . \tag{6.11}$$

All of the gauge variables are completely arbitrarily specifiable, subject only, of course, to maintaining equation (5.10.2). In this gauge the field variables, that is those components of $^4g^{\alpha\beta}$ to be determined by the six main equations, are the set of quantities

$$\{\eta_{11}, b_1^{\alpha}, \gamma, \tilde{g}^{\alpha\beta}\}\ . \tag{6.12}$$

We now use (5.8.7-11) to write the field equations in the following form

$$\perp{}^4\tilde{G}^{\gamma\beta} = 0 \Rightarrow {}'\nabla_e\tilde{h}^{\gamma\beta e} = 2\tilde{h}^{\gamma\varepsilon e}\tilde{h}^{\beta}{}_{\varepsilon e} + \tilde{h}^{\gamma\beta e}h_e - \tilde{T}\left(\nabla^{\gamma}\ell^{\beta} - \ell^{ef\gamma}\ell_{ef}{}^{\beta} - 2\eta^{-2}\bar{\Omega}^{\gamma}\bar{\Omega}^{\beta}\right)$$

$$\perp{}^4G - {}^4G_e{}^e = 0 \Rightarrow {}'R = \tilde{h}^{\varepsilon\theta e}\tilde{h}_{\varepsilon\theta e} - \tfrac{1}{2}h^e h_e - \ell^{ef\varepsilon}\ell_{ef\varepsilon} + \ell^{\varepsilon}\ell_{\varepsilon} - 6\eta^{-2}\bar{\Omega}^{\varepsilon}\bar{\Omega}_{\varepsilon} + R$$

$$\perp{}^4G^{c\beta} = 0 \Rightarrow \bar{\varepsilon}^{ce}\,{}'\nabla_e\bar{\Omega}^{\beta} = 2\bar{\varepsilon}^{ce}\left(\tilde{h}^{\beta}{}_{\varepsilon e} + \delta^{\beta}_{\varepsilon}h_e\right)\bar{\Omega}^{\varepsilon} + \eta^2\left[\nabla^{\varepsilon}\tilde{h}^{\beta}{}_{\varepsilon}{}^{c} - \tfrac{1}{2}\nabla^{\beta}h^c - 2\tilde{h}^{\beta}{}_{\varepsilon e}\ell^{ce\varepsilon}\right.$$

$$\left. + \tilde{h}^{\beta}{}_{\varepsilon}{}^{c}\ell^{\varepsilon} + \tfrac{1}{2}h^c\ell^{\beta} + {}'\nabla_e\ell^{ce\beta} - {}'\nabla^c\ell^{\beta}\right)$$

$${}^4G^{cb} = 0 \Rightarrow \bar{\varepsilon}^{ce}\bar{\varepsilon}^{bf}\,{}'\nabla_{(e}h_{f)} = +\ \bar{\varepsilon}^{(c|e|}\ell^{b)}{}_{e\varepsilon}\bar{\Omega}^{\varepsilon} + \eta^2\left(\tfrac{1}{2}h^c h^b + \tilde{h}^{\varepsilon\theta c}\tilde{h}_{\varepsilon\theta}{}^{b} - \nabla_{\varepsilon}\ell^{cb\varepsilon}\right.$$

$$+ 2\ell^{ce\varepsilon}\ell^{b}{}_{e\varepsilon} + \ell^{cb\varepsilon}\ell_{\varepsilon} - \eta^{cb}(\tfrac{1}{2}\tilde{h}^{\varepsilon\theta e}\tilde{h}_{\varepsilon\theta e} + \tfrac{3}{4}h^e h_e - \nabla_{\varepsilon}\ell^{\varepsilon} + \tfrac{1}{2}\ell^{ef\varepsilon}\ell_{ef\varepsilon}$$

$$\left. + \tfrac{1}{2}\ell^{\varepsilon}\ell_{\varepsilon} + \eta^{-2}\bar{\Omega}^{\varepsilon}\bar{\Omega}_{\varepsilon} - \tfrac{1}{2}R)\right)\ .$$

Remembering the defining equations (5.3.8) and (5.2.24) of $'\nabla_a$ and $\vec{e}_a$ respectively, the following can be deduced immediately.

First of all no second derivatives with respect to $\vec{e}_a$ (that is derivatives out of $\{S\}$) of any of the field variables occur on the right hand side of any of the above equations. Secondly, we see that no second derivatives with respect to $\vec{e}_o$ ('time' derivatives) occur in the subsidiary conditions. In fact the leading terms (i.e. those involving second extrinsic derivatives) of the four subsidiary conditions ${}^4G^{oo} = 0$, ${}^4G^{o1} = 0$, ${\perp}{}^4G^{o\beta} = 0$ are, from (5.6.8), (5.2.22a) and (5.2.24)

$$-\pounds_{n_1} h_1 \equiv \gamma^{-1} \pounds^2_{e_1} \gamma + \ldots ,$$

$$-\left(\tfrac{1}{2}\pounds_{n_o} h_1 + \tfrac{1}{2}\pounds_{n_1} h_o\right) \equiv \gamma^{-1} \pounds_{e_1} \pounds_{e_o} \gamma + \ldots ,$$

$$\pounds_{n_1} \bar{\Omega}^{\alpha} \equiv \tfrac{1}{2}\pounds_{e_1} \pounds_{e_o} b^{\alpha}_1 + \ldots ,$$

respectively. Now, let us suppose that in addition to some choice of gauge variables, the following initial data are given:

$$\tilde{h}^{\alpha\beta}{}_o,\ \tilde{h}^{\alpha\beta}{}_1 \text{ on } \overset{o}{\Sigma}_1 ;\ \gamma ,\ h_o ,\ h_1 \text{ and } \bar{\Omega}^{\alpha} \text{ on } \overset{o}{S} . \qquad (6.13)$$

Then on $\overset{o}{S}$ all the field variables and their first extrinsic derivatives are known. Thus we can solve the subsidiary equations on $\overset{o}{S}$ for $\pounds^2_{e_1}\gamma$, $\pounds_{e_1}\pounds_{e_o}\gamma$ and $\pounds_{e_1}\pounds_{e_o} b^{\alpha}_1$, which is equivalent to knowing $\pounds_{e_1}\gamma$, $\pounds_{e_o}\gamma$ and $\pounds_{e_o} b^{\alpha}_1$ respectively on $\overset{1,o}{S}$. This then allows us to solve the subsidiary conditions again on $\overset{1,o}{S}$. Repeating the above procedure on all successive $\overset{i,o}{S}$, $i = 2,3,\ldots.$ we build up a knowledge of γ, $\pounds_{e_o}\gamma$ and $\pounds_{e_o} b^{\alpha}_1$ on $\overset{o}{\Sigma}_1$. Assuming that we have solved the subsidiary conditions on $\overset{o}{\Sigma}_1$, we now know all the field variables and their first 'time' derivatives on $\overset{o}{\Sigma}_1$. By virtue of the Bianchi identities, the subsidiary conditions hold automatically on all other members of $\{\Sigma_1\}$. We now turn our attention to the main equations.

The main equations all contain second 'time' derivatives of the field variables. The leading terms of ${}^4G^{11} = 0$, $\perp{}^4G - {}^4G_e{}^e = 0$, $\perp{}^4G^{1\alpha} = 0$ and $\perp{}^4\tilde{G}^{\alpha\beta} = 0$ are, from (5.6.8)

$$\pounds_{\underset{o}{n}} h_o \equiv -\gamma^{-1}\pounds^2_{\underset{o}{e}}\gamma + \ldots,$$

from (5.22a)

$$\pounds_{\underset{o}{n}} \bar{\Omega}^\alpha \equiv -\tfrac{1}{2}\pounds^2_{\underset{o}{e}} b^\alpha_1 + \ldots,$$

from (5.3.9), (5.3.17) and (5.5.13)

$$'R \equiv \eta^{-2}\pounds^2_{\underset{o}{e}}\eta_{11} + \ldots,$$

and from (5.6.2)

$$\pounds_{n_e} \tilde{h}^{\gamma\beta e} \equiv \tfrac{1}{2}\eta^{oo}\pounds^2_{\underset{o}{e}}\tilde{g}^{\alpha\beta} + \ldots$$

respectively. At this stage we have sufficient initial data to solve the main equations for the second 'time' derivatives of all the field variables on $\overset{o}{\Sigma}_1$. Solving the main equations on $\overset{o}{\Sigma}_1$ then allows us to solve them again on $\overset{1}{\Sigma}_1$ and then on all subsequent $\overset{k}{\Sigma}_1$, k = 2,3,... . In this way we build up a solution of the field equations in some neighbourhood of $\overset{o}{S}$. It is easy to see that the initial data given in (6.13) is necessary and sufficient to determine the solution. Giving $\tilde{h}^{\alpha\beta}{}_o$ and $\tilde{h}^{\alpha\beta}{}_1$ on $\overset{o}{\Sigma}_1$ is fully equivalent to giving the conformal 2-structure and its first 'time' derivative on $\overset{o}{\Sigma}_1$. On $\overset{o}{S}$, the quantities γ, h_1 and h_o then determine the complete intrinsic and extrinsic geometry of $\overset{o}{S}$. Finally giving $\bar{\Omega}^\alpha$ on $\overset{o}{S}$ is equivalent to specifying the first 'time' derivative of the shift vector b^α_1.

In the 2+2 approach to the Cauchy problem discussed in this chapter, we have been able to formulate covariantly algebraic gauge conditions on the space-time metric which eliminate not only the four-dimensional gauge freedom, but also the lower dimensional gauge freedom on the initial hypersurface $\overset{o}{\Sigma}_1$. This in turn enables us to identify precisely and in closed form the physically meaningful initial data which must be specified on an initial space-like hypersurface in order to determine a unique analytic solution to the vacuum field equations. Modulo functions of two variables, this data consists of four functions of three variables at each point of $\overset{o}{\Sigma}_1$. Two of these four functions, namely the conformal 2-structure, we interpret as generalised coordinates, and the other two, namely the first 'time' derivatives of the conformal 2-structure, as 'velocities'. The propagating equations of $\tilde{g}^{\alpha\beta}$, namely $\perp^4\tilde{G}^{\alpha\beta} = 0$, are, from (5.8.7) quite clearly normal hyperbolic in $\tilde{g}^{\alpha\beta}$. Comparison with the usual results for dynamical systems described by second order hyperbolic equations (see Appendix A) shows that, as has long been known, the gravitational field possesses two degrees of freedom per space-time point. Furthermore, however, we see that in the 2+2 approach to the Cauchy problem, the dynamical variables may be regarded as being embodied explicitly and covariantly in the conformal 2-structure. This contrasts strongly with the 3+1 approach where, as discussed in section 2.4, no such explicit isolation of the dynamical variables has yet proved possible.

Chapter VII. The 2+2 Formulation of the Characteristic and Mixed Initial Value Problems.

7.1 The Bianchi Identities and Reduction of the Field Equations to the Generalised Light Cone Gauge.

In this chapter, we shall assume that (5.10.1) holds, that is, that $\{\Sigma_1\}$ is a foliation of V into null hypersurfaces. Under the specific gauge choice (5.10.1), several dyad components of various quantities vanish identically. We obtain immediately that

$$\eta^{oo} = 0 \Rightarrow \eta_{11} = 0 \quad \text{and} \quad \eta_{o1} = (\eta^{o1})^{-1} = \eta \ . \qquad (7.1.1a)$$

Substituting the above into (5.3.9) and (5.4.6c) yields

$$\Gamma^{o}_{1a} = 0 \quad \text{and} \quad \ell^{oo}{}_{\alpha} = \ell_{11\alpha} = 0 \qquad (7.1.1b)$$

respectively.

In order to analyse the Bianchi identities, we first group the field equations in a particular way. As in the previous chapter, we use the terminology (essentially) of Bondi and write

$$\left.\begin{array}{ll} \underline{\perp}{}^4\tilde{G}^{\alpha\beta} = 0 & \text{dynamical equations} \\ \left.\begin{array}{l} \underline{\perp}{}^4G^{o\alpha} = 0 \\ {}^4G^{oa} = 0 \end{array}\right\} & \text{constraint equations} \end{array}\right\} \text{main equations}$$

$$\left.\begin{array}{l} {}^4G^{11} = 0 \\ \underline{\perp}{}^4G^{1\alpha} = 0 \end{array}\right\} \quad \text{subsidiary conditions}$$

$$\underline{\perp}{}^4G = 0 \qquad \text{trivial equation.}$$

Next, we assume that the main equations vanish everywhere in V .
Then, substituting (7.1.1) into (5.9.2) gives first of all

$$n^{o}_{\mu}{}^{4}\nabla_{\nu}{}^{4}G^{\mu\nu} \equiv \tfrac{1}{2}\eta h_{1}\perp{}^{4}G = 0 \qquad (7.1.2)$$

Now $\eta \neq 0$, otherwise ${}^{4}g^{\alpha\beta}$ is degenerate. The expansion[26] of the null rays ruling $\{\Sigma_1\}$ vanishes if and only if

$${}^{4}\nabla_{\alpha}n^{o\alpha} = 0 , \qquad (7.1.3)$$

since $\vec{n}^{o}$ is an affinely parametrised vector tangent to the null rays. But from (7.1.1a), we have

$${}^{4}\nabla_{\alpha}n^{o\alpha} = {}^{4}\nabla_{\alpha}\left(\eta^{-1}n^{\alpha}_{1}\right) .$$

Expanding the right hand side of the above, we obtain

$${}^{4}\nabla_{\alpha}n^{o\alpha} = \eta^{-1}\left(\tfrac{1}{2}{}^{4}g^{\alpha\beta}\pounds_{n_1}{}^{4}g_{\alpha\beta} - \pounds_{n_1}\ln\eta\right) . \qquad (7.1.4)$$

However, from (5.6.4 and 8), we have

$$\tfrac{1}{2}{}^{4}g^{\alpha\beta}\pounds_{n_1}{}^{4}g_{\alpha\beta} = (\sqrt{-{}^{4}g})^{-1}\,\pounds_{n_1}\sqrt{-{}^{4}g} = -h_1 + \pounds_{n_1}\ln\eta . \qquad (7.1.5)$$

substituting (7.1.5) into (7.1.4), we get

$${}^{4}\nabla_{\alpha}n^{o\alpha} = -\eta^{-1}h_1 .$$

Hence

$$h_1 \neq 0 \qquad (7.1.6)$$

is a necessary and sufficient condition for the expansion of the null rays ruling $\{\Sigma_1\}$ to be non zero. We shall assume for the remainder of this chapter that we are working in a region where (7.1.6) holds. In that case, equation (7.1.2) implies immediately that

$$\perp {}^4G = 0 . \qquad (7.1.7)$$

Equation (5.9.1) now becomes

$$\pounds_{n_1} \perp {}^4G^{1\alpha} - \left(2h_{\varepsilon}{}^{\alpha}{}_1 + (h_1 - \pounds_{n_1} \ell n\eta)\delta^{\alpha}_{\varepsilon}\right) \perp {}^4G^{1\varepsilon} = 0 \qquad (7.1.8)$$

and so if $\perp {}^4G^{1\alpha}$ vanishes on any cross-section of $\{\Sigma_1\}$ then it vanishes everywhere. Under the assumption that $\perp {}^4G^{1\alpha}$ does indeed vanish everywhere, equation (5.9.2) gives

$$\pounds_{n_1} {}^4G^{11} + \left(2\pounds_{n_1} \ln\eta - h_1\right) {}^4G^{11} = 0 . \qquad (7.1.9)$$

Hence ${}^4G^{11}$ vanishes everywhere if it vanishes on some cross-section of $\{\Sigma_1\}$. Collecting the results of (7.1.7-9) together, we obtain the usual lemma: the trivial equation is an algebraic consequence of the main equations; the subsidiary conditions hold everywhere if they hold on some hypersurface transvecting $\{\Sigma_1\}$ and the main equations hold everywhere. In practice, we solve the subsidiary conditions on $\overset{o}{\Sigma}_o$.

The condition (5.10.1) imposes one specific choice of gauge on ${}^4g^{\alpha\beta}$, and its significance, as stated in section 5.10, is that it ensures that $\{\Sigma_1\}$ is a family of null hypersurfaces. Equivalently, since

$$\eta^{oo} = 0 \iff \eta_{11} = 0$$

this gauge choice may be regarded as choosing the lapse of the foliations $\{S\}_1$ of each $\Sigma_1 \in \{\Sigma_1\}$ to be zero (see figure 2). The exact covariant analogue of the light cone gauge (c.f. section 3.2) would be to choose, in addition, the shift $\vec{b}_1$ of $\{S\}_1$ to be zero, in which case the trajectories $\mathcal{C}_1$ of $\vec{e}_1$ would be the null geodesics ruling $\{\Sigma_1\}$ (arbitrarily parametrised at this stage). In general, however, we shall regard $\vec{b}_1$ as being arbitrarily specifiable. If $\vec{b}_1$ is non zero, then the surfaces $S\epsilon\{S\}_1$ are tilted relative to the curves of $\mathcal{C}_1$ (see figure 2). We shall term the set of gauge quantities

$$\{\eta_{11} = 0,\ b^{\alpha}_{1}\} \tag{7.1.10}$$

the generalised light cone (GLC) gauge. This gauge leaves one further four-dimensional gauge freedom, and we shall consider various possible choices for this remaining gauge quantity later in the chapter.

In the GLC, the main equations take on a remarkably simple form. Substituting (7.1.1) into equations (5.8.7–11) yields, in particular, for the main equations

$$\eta^{2}\,{}^{4}G^{oo} \equiv \pounds_{n_1} h_1 - \tfrac{1}{2}(h_1)^2 - h_1 \pounds_{n_1} \ell n\eta - \tilde{h}^{\epsilon\theta}{}_{1}\tilde{h}_{\epsilon\theta 1} = 0$$

$$-\eta^{2}\,{}^{4}G^{o1} \equiv \pounds_{n_1} h_o - \tfrac{1}{2}\eta^{-1} h_1 \pounds_{n_1}\eta_{oo} - h_o h_1 + \tfrac{1}{4}\eta^{-1}\eta_{oo}(h_1)^2 - \tfrac{1}{2}\eta^{-1}\eta_{oo}\tilde{h}^{\epsilon\theta}{}_{1}\tilde{h}_{\epsilon\theta 1} + \nabla_{\epsilon}\bar{\Omega}^{\epsilon}$$
$$- \eta^{-2}\bar{\Omega}^{\epsilon}\bar{\Omega}_{\epsilon} - \tfrac{1}{2}\nabla^2\eta + \tfrac{1}{4}\eta^{-1}(\nabla_{\epsilon}\eta)(\nabla^{\epsilon}\eta) + \tfrac{1}{2}\eta R = 0$$

$$-\eta^{2}\perp{}^{4}G^{o\alpha} \equiv \pounds_{n_1}\bar{\Omega}^{\alpha} - 2\left(\tilde{h}^{\alpha}{}_{\epsilon 1} + \delta^{\alpha}_{\epsilon}(h_1 + \tfrac{1}{2}\pounds_{n_1} \ln\eta)\right)\bar{\Omega}^{\epsilon} + \eta\nabla^{\epsilon}\tilde{h}^{\alpha}{}_{\epsilon 1} - \tfrac{1}{2}\eta^{2}\nabla^{\alpha}\left(\eta^{-1}h_1\right)$$
$$+ \tfrac{1}{2}\eta\nabla^{\alpha}\pounds_{n_1} \ln\eta = 0$$

$$\tfrac{1}{2}\eta\perp{}^{4}\tilde{G}^{\alpha\beta} \equiv \pounds_{n_1}\tilde{h}^{\alpha\beta}{}_{o} - \tfrac{1}{2}\pounds_{n_1}\left(\eta^{-1}\eta_{oo}\tilde{h}^{\alpha\beta}{}_{1}\right) - 2\tilde{h}^{\alpha}{}_{\epsilon(o}\tilde{h}^{\beta\epsilon}{}_{1)} + \eta^{-1}\eta_{oo}\tilde{h}^{\alpha\epsilon}{}_{1}\tilde{h}^{\beta}{}_{\epsilon 1} - \tilde{h}^{\alpha\beta}{}_{(o}h_{1)}$$
$$+ \eta^{-1}\eta_{oo}\tilde{h}^{\alpha\beta}{}_{1}h_1 + \tilde{T}\left(\nabla^{\alpha}\bar{\Omega}^{\beta} - \eta^{-1}\bar{\Omega}^{\alpha}\bar{\Omega}^{\beta} - \tfrac{1}{2}\nabla^{\alpha}\nabla^{\beta}\eta + \tfrac{1}{4}\nabla^{\alpha}\eta\nabla^{\beta}\eta\right) = 0\ .$$

Note that in the above equations, we have written out explicitly the dyad components of the various quantities involved. Unfortunately, no great simplification of the subsidiary conditions occurs in the GLC gauge. To proceed further, we must consider separately the characteristic and mixed IVP's, in which case the initial surface $\overset{o}{\Sigma}_o$ is null or time-like respectively. We consider the characteristic IVP first. In this case the subsidiary conditions, evaluated on $\overset{o}{\Sigma}_o$, take on a much simpler form than in the case when $\overset{o}{\Sigma}_o$ is considered to be a time-like hypersurface.

7.2 The Characteristic Initial Value Problem.

The initial surface $\overset{o}{\Sigma}_o$ is a null hypersurface if and only if

$$\eta^{11} = 0 \iff \eta_{oo} = 0 \quad \text{on} \quad \overset{o}{\Sigma}_o \,. \qquad (7.2.1a)$$

This leads immediately to the following conditions which hold on $\overset{o}{\Sigma}_o$ only:

$$\ell^{11}{}_{\alpha} = \ell_{oo\alpha} = 0 \text{ and } \Gamma^{1}_{oo} = 0 \,. \qquad (7.2.1b)$$

Substituting (7.1.1) and (7.2.1) into the expressions for the subsidiary conditions ${}^4G^{11} = 0$ and $\perp{}^4G^{1\alpha} = 0$, given by equations (5.8.9 and 10), yields the following on $\overset{o}{\Sigma}_o$:

$$\eta^2\,{}^4G^{11} = \pounds_{n_o} h_o - \tfrac{1}{2}(h_o)^2 - Uh_o - \tilde{h}^{\varepsilon\theta}{}_o\tilde{h}_{\varepsilon\theta o} = 0$$

$$\eta^2\perp{}^4G^{1\alpha} = \pounds_{n_o}\bar{\Omega}^{\alpha} - 2\bar{\Omega}^{\varepsilon}\left(\tilde{h}^{\alpha}{}_{\varepsilon o} + \delta^{\alpha}_{\varepsilon}(h_o + \tfrac{1}{2}\pounds_{n_o}\ln\eta)\right) + \tfrac{1}{2}\eta\nabla^{\alpha}\pounds_{n_o}\ln\eta - \eta\nabla^{\varepsilon}\tilde{h}^{\alpha}{}_{\varepsilon o}$$
$$+ \tfrac{1}{2}\eta^2\nabla^{\alpha}\left(\eta^{-1}h_o\right) - \eta\nabla^{\alpha}U = 0 \,,$$

where

$$U \overset{\text{def}}{\equiv} \Gamma^{o}_{oo} = \pounds_{n_o}\ell n\eta - \tfrac{1}{2}\eta^{-1}\pounds_{n_1}\eta_{oo} \,. \qquad (7.2.2)$$

There are a number of ways in which the remaining four-dimensional gauge freedom can be used up. We shall consider just two possibilities here.

<u>Case 1:</u> We allow η to be freely specifiable everywhere, subject, of course, to $\eta \neq 0$. The geometrical interpretation of η is as follows. The foliation $\{\Sigma_1\}$ is a family of null hypersurfaces, defined by the one-form n^o. The trajectories of $\vec{n}^o$ are thus the congruence of null geodesics ruling $\{\Sigma_1\}$, and these trajectories are parametrised by an affine parameter r, say, determined up to an additive function, constant on each null geodesic[26]. We may fix r uniquely by further demanding, without loss of generality, that $r = 0$ on $\overset{o}{\Sigma}_o$. Under an arbitrary reparametrisation of these trajectories,

$$r \rightarrow r' = f(r)$$

the tangent vector of the reparametrised curves $'\vec{n}^o$, say, is given by

$$'\vec{n}^o = \lambda \vec{n}^o \; ; \quad \lambda^{-1} = \frac{df}{dr} \, .$$

Now, the vector $\vec{n}_1$ is also tangent to the null geodesics ruling $\{\Sigma_1\}$ and in fact from (7.1.1a) we see that

$$\vec{n}_1 = \eta \vec{n}^o$$

Hence we see that a choice of η is equivalent to a choice of parametrisation r' of the null geodesics ruling $\{\Sigma_1\}$, that is, of the trajectories of $\vec{n}_1$. Now, each $\Sigma_1 \epsilon \{\Sigma_1\}$ is foliated by a family of 2-surfaces $\{S\}_1$, and we define the initial member of $\{S\}_1$ to be the intersection of Σ_1 with $\overset{o}{\Sigma}_o$. Since the lapse function $|\eta_{11}|^{\frac{1}{2}}$ of $\{S\}_1$ is zero, we see that this function does not determine the development of $\{S\}_1$ from its initial member. The lapse merely tells us that the orthogonal metrical separation of members of $\{S\}_1$ is zero. However, since $\vec{n}_1$ is a connecting vector of $\{S\}_1$ we may regard $r' =$ constant as the equation of each of the members of $\{S\}_1$, considered as hypersurfaces in Σ_1, and without loss of generality, we may fix the origin of r' by demanding that $r' = 0$ be the equation of the initial member of $\{S\}_1$. Hence it is the parametrisation of the trajectories of $\vec{n}_1$ which determines the development of $\{S\}_1$ and thus from the above discussion we see that we may interpret η as the quantity which determines the development of each $\{S\}_1$ from its initial member. In particular,

$$\eta = 1 \Leftrightarrow \vec{n}_1 = \vec{n}^o \Leftrightarrow r = r'$$

and also, since from (5.4.8b) and (7.1.1) we can show easily that

$$n_1^\alpha \, {}^4\nabla_\alpha n_1^\beta = n_1^\beta \pounds_{n_1} \ell n \eta \ ,$$

we see that any η satisfying $\pounds_{n_1} \ell n \eta = 0$ implies that r' is an affine parameter, related to r by

$$r' = Ar \ ; \ \pounds_{n_1} A = 0 \ .$$

<u>Case 2.</u> We allow η_{oo} to be freely specifiable everywhere, subject of course to its vanishing on $\overset{o}{\Sigma}_o$. If η_{oo} is chosen to be non zero, then, since $|\eta_{oo}|^{\frac{1}{2}}$ is the lapse function of the foliations $\{S\}_o$ of each $\Sigma_o \epsilon \{\Sigma_o\}$ we see that η_{oo} determines the development

of each $\{S\}_0$ from its initial member, which we take to be the intersection of Σ_0 with $\overset{o}{\Sigma}_1$. Now, since

$$\eta_{oo} = \eta^{-1}\eta^{11} ,$$

we may equally well regard η^{11} as the gauge quantity; this latter quantity determines the metrical properties of $\{\Sigma_1\}$. In particular if we choose

$$\eta^{11} = 0 \iff \eta_{oo} = 0 \text{ everywhere,}$$

then this implies that $\{\Sigma_0\}$ is a foliation into null hypersurfaces.

There remain certain lower dimensional gauge choices on each of the initial hypersurfaces $\overset{o}{\Sigma}_{(a)}$. In each case we shall take these choices as corresponding to the freedom to specify the development of $\{\overset{o}{S}\}_{(a)}$ from $\overset{o}{S}$, and the correspondence between points on different members of $\{\overset{o}{S}\}_{(a)}$. (In contradistinction to to the space-like case, there is no freedom to specify $\overset{o}{\Sigma}_a$ as hypersurfaces in V. Once $\overset{o}{S}$ is chosen, $\overset{o}{\Sigma}_a$ are uniquely determined since, as is well known, there are precisely two null hypersurfaces intersecting in any given space-like 2-surface.) The latter freedom is governed by the shift vector $\vec{b}_{(a)}$. Since $\vec{b}_1$ has already been designated a four-dimensional gauge quantity, this does not lead to any additional gauge choice in $\overset{o}{\Sigma}_1$. however, we shall regard the shift $\vec{b}_0$ of $\{\overset{o}{S}\}_0$ as being freely specifiable on $\overset{o}{\Sigma}_0$. The development of $\{\overset{o}{S}\}_1$ from $\overset{o}{S}$ is governed by η, as has been discussed above. Hence in case 1, there is again no extra gauge choice, however in case 2, we shall regard η as being freely specifiable on $\overset{o}{\Sigma}_1$. Since $\overset{o}{\Sigma}_0$ is null, the development of $\{\overset{o}{S}\}_0$

from $\overset{o}{S}$ is determined by the parametrisation u', say, of the trajectories of $\vec{n}_o$ on $\overset{o}{\Sigma}_o$, which are of course, null geodesics ruling $\overset{o}{\Sigma}_o$. From (5.8.4b) and (7.2.1) it follows that

$$n_o^\alpha \, {}^4\nabla_\alpha n_o^\beta = U n_o^\beta$$

where U is defined by equation (7.2.2). A given choice of U specifies uniquely the parameter u'. For example, choosing $U = 0$ implies that $\vec{n}_o$ is affinely parametrised, and so u' is determined up to the transformation

$$u' \rightarrow Au'; \quad \pounds_{n_o} A = 0, \tag{7.2.3}$$

(we assume without loss of generality that $u' = 0$ is the equation of $\overset{o}{S}$, for any parameter u'). Under the transformation (7.2.3), then

$$\vec{n}_o \rightarrow A^{-1}\vec{n}_o \; .$$

The function η on $\overset{o}{S}$ (already designated as a gauge variable on $\overset{o}{S}$) fixes A, since the former governs the normalisation of $\vec{n}_o$ to $\vec{n}_1$. Once A is determined on $\overset{o}{S}$, then it is known everywhere in $\overset{o}{\Sigma}_o$, since $\pounds_{n_o} A = 0$. If U is chosen to be non zero on $\overset{o}{\Sigma}_o$, then the parameter is determined in a rather more complicated way, but is still uniquely determined by U on $\overset{o}{\Sigma}_o$ and η on $\overset{o}{S}$. In case 1, giving U is equivalent to giving $\pounds_{n_1} \eta_{oo}$, since η is already a known function. Similarly, in case 2, once U is specified on $\overset{o}{\Sigma}_o$, then η is known everywhere on $\overset{o}{\Sigma}_o$, once it is known on $\overset{o}{S}$.

In both cases $\tilde{g}^{\alpha\beta}$ on $\overset{o}{S}$ is two-dimensional gauge freedom reflecting the conformal flatness of $\overset{o}{S}$. Hence in addition to the

GLC gauge conditions given in equation (7.1.10) and the condition (7.2.1a), we choose in case 1 the set

$$\{\eta \text{ in } V \; ; \; \pounds_{n_1}\eta_{oo}, b^{\alpha}_{o} \text{ on } \overset{o}{\Sigma}_{o} ; \; \tilde{g}^{\alpha\beta} \text{ on } \overset{o}{S}\}$$

and in case 2 the set

$$\{\eta_{oo} \text{ in } V \; ; \; \eta \text{ on } \overset{o}{\Sigma}_{1} ; \; \eta, \; b^{\alpha}_{o} \text{ on } \overset{o}{\Sigma}_{o} ; \; \tilde{g}^{\alpha\beta} \text{ on } \overset{o}{S}\} .$$

as freely specifiable gauge quantities.

The alternative gauges considered here are generalisations of well known gauges in which, using coordinate dependent approaches the characteristic IVP has been solved. In particular if we choose the actual covariant analogue of the light cone gauge, namely

$$b^{\alpha}_{1} = 0 ,$$

and in addition, in case 1

$$\eta = 1 \text{ in } V; \; \pounds_{n_1}\eta_{oo} = 0, \; b^{\alpha}_{o} = 0 \text{ on } \overset{o}{\Sigma}_{o}$$

and in case 2

$$\eta_{oo} = 0 \text{ in } V; \; \eta = 1 \text{ on } \overset{o}{\Sigma}_{1} ; \; \eta = 1, \; b^{\alpha}_{o} = 0 \text{ on } \overset{o}{\Sigma}_{1}$$

we obtain the exact covariant analogues of the Newman-Penrose[(29)] or Robinson - Trautman[(30)] gauge, and the Sachs[(4)] gauge respectively. The characteristic IVP in the former gauge has recently been solved using coordinate-dependent techniques by Gambini and Restuccia[(15)]. The coordinate-dependent version of the latter gauge has been discussed briefly in section 3.3 , and a full discussion can be found in reference 4. In fact the generalisations of these respective gauges which we consider here do not affect the resulting integration schemes greatly, and these schemes follow essentially their coordinate-dependent counterparts. In addition to some particular choice of gauge quantities, the following initial data are required, in each case, to determine a solution:

$$\tilde{h}^{\alpha\beta}{}_{o} \text{ on } \overset{o}{\Sigma}_{o};\ \tilde{h}^{\alpha\beta}{}_{1} \text{ on } \overset{o}{\Sigma}_{1};\ \gamma\ ,\ h_{o},\ h_{1},\ \text{and } \bar{\Omega}^{\alpha} \text{ on } \overset{o}{S}\ . \qquad (7.2.4)$$

The data $\tilde{h}^{\alpha\beta}{}_{o}$ and $\tilde{h}^{\alpha\beta}{}_{1}$ are set respectively on the portions of $\overset{o}{\Sigma}_{o}$ and $\overset{o}{\Sigma}_{1}$ to the future of $\overset{o}{S}$, and the resulting region of integration of the field equations is some (sufficiently small) region R bounded from below by $\overset{o}{\Sigma}_{o}$ $\overset{o}{\Sigma}_{1}$ and their space-like intersection $\overset{o}{S}$.

We consider first the integration scheme for case 2, the generalised Sachs gauge. We start with the subsidiary conditions, and in particular the equation ${}^{4}G^{11} = 0$. This is solved on ${}^{o}\overset{o}{S}{}^{o}$ for $\pounds_{n_o} h_{o}$, and hence $\pounds^{2}_{e_o}\gamma$, in terms of the initial data . ${}^{4}G^{11} = 0$ can then be solved on successive ${}^{o}\overset{o}{S}{}^{i}$, i=1,2,3,... for $\pounds^{2}_{e_o}\gamma$, and in this way the equation is solved for γ on $\overset{o}{\Sigma}_{o}$, by using the generalised Taylor expansion (5.2.26). We now have sufficient data to solve $\perp{}^{4}G^{o\alpha} = 0$ on $\overset{o}{\Sigma}_{o}$ in a similar fashion for $\bar{\Omega}^{\alpha}$. We now turn our attention to the main equations. Solution of the subsidiary conditions on $\overset{o}{\Sigma}_{o}$, together with the initial data specified in (7.2.4) allows the solution of the main equations, in the order ${}^{4}G^{oo} = 0$, $\perp{}^{4}G^{o\alpha} = 0$, ${}^{4}G^{o1} = 0$, $\perp{}^{4}\tilde{G}^{\alpha\beta} = 0$ on successive $\overset{k}{\Sigma}_{1}$, k=0,1,2,... . The first of these determines γ on $\overset{o}{\Sigma}_{1}$, and η thereafter. The remaining equations determine $\bar{\Omega}^{\alpha}$, h_{o} and $\tilde{h}^{\alpha\beta}{}_{o}$ respectively. $\bar{\Omega}^{\alpha}$ determines b^{α}_{o} on any $\overset{k}{\Sigma}_{1}$; h_{o} , $\tilde{h}^{\alpha\beta}{}_{o}$ on $\overset{k}{\Sigma}_{1}$ determine γ and $\tilde{g}^{\alpha\beta}$ on $\overset{k+1}{\Sigma}_{1}$

respectively. The Bianchi identities then ensure that the subsidiary conditions hold automatically everywhere else in the region R of integration.

The integration scheme for case 1, the generalised Newman-Penrose gauge, is more complicated, but the outline is as follows: on $\overset{o}{\Sigma}_o$, the subsidiary conditions ${}^4G^{11} = 0$ and $\perp{}^4G^{1\alpha} = 0$ determine γ and $\bar{\Omega}^\alpha$ respectively. We next solve ${}^4G^{o1} = 0$ on $\overset{o}{\Sigma}_o$ for h_1 (using the relationship $\pounds_{n_1} h_o = \pounds_{n_o} h_1 - 2\nabla_\varepsilon \bar{\Omega}^\varepsilon$) . The integration scheme for the main equations is rather involved. We first solve ${}^4G^{oo} = 0$ on $\overset{o}{\Sigma}_1$ for γ , and then $\perp{}^4G^{o\alpha} = 0$ on $\overset{o}{\Sigma}_1$ for $\bar{\Omega}^\alpha$. We now solve $\perp{}^4\tilde{G}^{\alpha\beta} = 0$ on $\overset{o,o}{S}$ for $\pounds_{n_1} \tilde{h}^{\alpha\beta}{}_o$, which determines $\tilde{g}^{\alpha\beta}$ on $\overset{1,1}{S}$. We can then solve ${}^4G^{oo} = 0$ on $\overset{o,1}{S}$ for $\pounds_{n_1} h_1$, which determines γ on $\overset{2,1}{S}$. Next we can solve ${}^4G^{o1} = 0$ on $\overset{1,o}{S}$ for $\pounds_{n_1} \eta_{oo}$, which determines η_{oo} on $\overset{2,o}{S}$. This allows us to solve $\perp{}^4\tilde{G}^{\alpha\beta} = 0$ again on $\overset{1,o}{S}$. Continuing in this way, we solve $\perp{}^4\tilde{G}^{\alpha\beta} = {}^4G^{o1} = 0$ on $\overset{o}{\Sigma}_1$, and ${}^4G^{oo} = 0$ on $\overset{1}{\Sigma}_1$. We can summarise the above process:

$$\perp{}^4G^{\alpha\beta}=0 \text{ on } \overset{o,i}{S} \rightarrow {}^4G^{oo}=0 \text{ on } \overset{1,i}{S} \rightarrow {}^4G^{o1}=0 \text{ on } \overset{o,i+1}{S} \rightarrow \perp{}^4\tilde{G}^{\alpha\beta} = 0 \text{ on } \overset{o,i+1}{S} \text{ , } i=0,1,\ldots \text{ .}$$

We repeat the integration scheme for the main equations on each successive pair of hypersurfaces $\overset{k}{\Sigma}_1$, $\overset{k+1}{\Sigma}_1$, $k = 1,2,\ldots$, starting by solving $\perp{}^4G^{o\alpha} = 0$ on $\overset{1}{\Sigma}_1$. Again, once the main equations have been solved in some region R, the Bianchi identities ensure that the subsidiary conditions hold in that region.

7.3 The Mixed Initial Value Problem.

We start again from the GLC gauge (equation (7.1.10)). Now, however, we assume that $\overset{o}{\Sigma}_o$ is a time-like hypersurface, say a time-like tube with topology $S^2\times\mathbb{R}$, in which case $\overset{o}{S}$ is a closed space-like 2-surface, and $\overset{o}{\Sigma}_1$ is the outgoing null hypersurface intersecting $\overset{o}{\Sigma}_o$ in $\overset{o}{S}$. (Strictly speaking the topology of $\overset{o}{\Sigma}_o$ is a non local question.) We take as lower dimensional gauge freedom on $\overset{o}{\Sigma}_o$ the set of quantities $\{\eta_{oo} > 0 \ , \ b^\alpha_o\}$. Since $\eta^{11}=\eta^{-2}\eta_{oo}$, we must have $\eta_{oo} > 0$ to ensure that $\overset{o}{\Sigma}_o$ is time-like. As the final four-dimensional gauge freedom, we shall allow γ to be freely specifiable, subject of course to the condition $h_1 \neq 0$ which is necessary to ensure that the Bianchi identities play their usual rôle. Again $\tilde{g}^{\alpha\beta}$ is a gauge quantity on $\overset{o}{S}$. The set of quantities

$$\left\{ \gamma \text{ in } V; \ \eta_{oo} > 0, \ b^\alpha_o \text{ on } \overset{o}{\Sigma}_o; \ \tilde{g}^{\alpha\beta} \text{ on } \overset{o}{S} \right\} \qquad (7.3.1)$$

together with the GLC gauge, are generalisations of the gauge considered by Tamburino and Winicour[5]. The exact covariant analogue of the latter gauge is to choose the actual light cone gauge, $b^\alpha_1 = 0$ in V, and in addition

$$\pounds^2_{n_1} h_1 - \tfrac{1}{2}h^2_1 = 0 \text{ in } V; \quad b^\alpha_o = 0, \ h_o = 0 \text{ on } \overset{o}{\Sigma}_o; \ \nabla_\alpha h_1 = 0 \text{ on } \overset{o}{S} . \qquad (7.3.2)$$

In an adapted coordinate basis $E_\alpha \overset{*}{=} \dfrac{\partial}{\partial x^\alpha}$, where $x^\alpha \overset{*}{=} (u,r,x^A)$ are chosen so that

$$B^\alpha_A \overset{*}{=} \delta^\alpha_A \ , \ e^\alpha_a \overset{*}{=} \delta^\alpha_a \ , \ \tilde{g}^{\alpha\beta} \overset{*}{=} \delta^\alpha_A\delta^\beta_B g^{AB} \quad \text{and} \ |\tilde{g}^{AB}| = 1 \ ,$$

then the conditions imposed on h_o and h_1 in equation (7.3.2) imply that

$$\gamma \overset{*}{=} r^2 \ ,$$

that is, that r is a 'luminosity parameter'[16] along the null rays ruling $\{\Sigma_1\}$. Note that it is more convenient from a covariant point of view to impose conditions on h_o and h_1, which are scalars, than on γ directly, since the latter is a scalar density. In particular, we cannot put $\gamma = (\phi^1)^2$, since ϕ^1 is a scalar function. A specific choice of γ, for which $h_1 \neq 0$, leads to a particular parametrisation of the null rays ruling $\{\Sigma_1\}$, and in particular, uses up the remaining gauge freedom in $\overset{o}{\Sigma}_1$ (c.f. the discussion in the previous section). The lower dimensional gauge conditions on $\overset{o}{\Sigma}_o$ represent the freedom to specify the development of $\{\overset{o}{S}\}_o$ from $\overset{o}{S}$, the correspondence between points on different members of $\{\overset{o}{S}\}_o$, determined by the lapse $(\eta_{oo})^{\frac{1}{2}}$ and shift b^α_o respectively, and finally the freedom to specify $\overset{o}{\Sigma}_o$ as a hypersurface in V. However, we regard γ as being freely specifiable in V, and hence, in particular h_o on $\overset{o}{\Sigma}_o$. h_o is the trace of the extrinsic curvature of $\{\overset{o}{S}\}_o$, the latter regarded as hypersurfaces in $\overset{o}{\Sigma}_o$. Specifying both the lapse $(\eta_{oo})^{\frac{1}{2}}$, and the trace of the extrinsic curvature h_o, of $\{\overset{o}{S}\}_o$ is an overdetermination of the development of $\{\overset{o}{S}\}_o$ from $\overset{o}{S}$. For example choosing

$$\eta_{oo} = 1 , \quad h_o = 0$$

implies that $\{\overset{o}{S}\}_o$ are both geodesically parallel and maximal hypersurfaces in $\overset{o}{\Sigma}_o$. This overdetermination serves to fix $\overset{o}{\Sigma}_o$ as a hypersurface in V. To summarise: the GLC gauge, together with (7.3.1) use up all the available gauge freedom at our disposal.

Apart from some specific choice of gauge, the initial data required to specify a solution of the field equations in this gauge are

$$\tilde{h}^{\alpha\beta}{}_{o} \text{ on } \overset{o}{\Sigma}_{o} \;;\; \tilde{h}^{\alpha\beta}{}_{1} \text{ on } \overset{o}{\Sigma}_{1} ;\; \bar{\Omega}^{\alpha} \text{ and } \eta \text{ on } \overset{o}{S} . \tag{7.3.3}$$

As in the characteristic IVP, the data $\tilde{h}^{\alpha\beta}{}_{o}$ and $\tilde{h}^{\alpha\beta}{}_{1}$ respectively are set on the portions of $\overset{o}{\Sigma}_{1}$ and $\overset{o}{\Sigma}_{o}$ to the future of $\overset{o}{S}$. The resulting region R of intergration is bounded from below by $\overset{o}{\Sigma}_{1}$ and $\overset{o}{S}$, and (regarding $\overset{o}{\Sigma}_{o}$ as a time-like tube) exterior to $\overset{o}{\Sigma}_{o}$. The subsidiary conditions in this gauge even when evaluated on $\overset{o}{\Sigma}_{o}$ have a very complicated explicit form, due to the non vanishing of η^{11} on $\overset{o}{\Sigma}_{o}$. Nevertheless, we may write them formally on $\overset{o}{\Sigma}_{o}$ as

$$^{4}G^{11} = 0 \Rightarrow \pounds_{n_o}\eta = f^{11}\left(\tilde{h}^{\alpha\beta}{}_{o}\,,\ \tilde{h}^{\alpha\beta}{}_{1},\ \eta\,,\ \pounds_{n_1}\eta_{oo},\bar{\Omega}^{\alpha},\ \text{gauge variables}\right)$$

$$\perp{}^{4}G^{1\alpha} = 0 \Rightarrow \pounds_{n_o}\bar{\Omega}^{\alpha} = f^{1\alpha}\left(\tilde{h}^{\alpha\beta}{}_{o}\,,\tilde{h}^{\alpha\beta}{}_{1},\bar{\Omega}^{\alpha},\eta,\pounds_{n_o}\eta,\pounds_{n_1}\eta,\pounds_{n_1}\eta_{oo},\ \text{gauge variables}\right)$$

where f^{11} and $f^{1\alpha}$ are some complicated functionals of their arguments. Explicit expressions for f^{11} and $f^{1\alpha}$ can be obtained from equations (5.8.9 and 10), but they are very long and not particularly informative, so we do not give them here.

The occurrence of derivatives of field variables out of $\overset{o}{\Sigma}_{o}$ in the subsidiary conditions means that they cannot be solved independently of the main equations. The formal integration scheme is as follows: the main equations, in the order $^{4}G^{oo} = 0$, $\perp{}^{4}G^{o\alpha} = 0$, $^{4}G^{o1} = 0$, $\perp{}^{4}\tilde{G}^{\alpha\beta} = 0$ are solved on $\overset{o}{\Sigma}_{1}$ for η, $\bar{\Omega}^{\alpha}$, η_{oo}, $\tilde{h}^{\alpha\beta}{}_{o}$

respectively. The subsidiary equations can then be solved on $\overset{o,o}{S}$, in the order ${}^4G^{11} = 0$, $\perp{}^4G^{1\alpha} = 0$, for $\pounds_{n_o} \eta$ and $\pounds_{n_o} \bar{\Omega}^{\alpha}$ respectively. This enables the main equations to be solved again on $\overset{1}{\Sigma}_1$. In general, solving the main equations on $\overset{k}{\Sigma}_1$ allows the subsidiary conditions to be solved on $\overset{o,k}{S}$, which then allows the main equations to be solved again on $\overset{k+1}{\Sigma}_1$, $k = 0,1,\ldots$.

7.4 Conclusion.

It is easy to see that the initial data given in equations (7.2.4) and (7.3.3) for the characteristic and mixed IVP's respectively are both necessary and sufficient to determine a unique analytic solution. The interpretation of this initial data is as follows. The conformal extrinsic curvatures $\tilde{h}^{\alpha\beta}{}_o$ and $\tilde{h}^{\alpha\beta}{}_1$ on $\overset{o}{\Sigma}_o$ and $\overset{o}{\Sigma}_1$ respectively are entirely equivalent to specifying the conformal 2-structure on the respective initial hypersurfaces. Additionally, lower dimensional data is required on $\overset{o}{S}$. In each case one must give $\bar{\Omega}^{\alpha}$. This is equivalent to specifying $\pounds_{e_1} b^{\alpha}_o$, the one extrinsic derivative of b^{α}_o left undetermined by the gauge conditions. In the characteristic IVP one must also give γ , h_o and h_1 , which determine the entire intrinsic and extrinsic geometry of $\overset{o}{S}$. In the mixed IVP the quantity η must be given. This latter quantity determines the initial normalisation on $\overset{o}{S}$ of the null vector $\vec{n}_1$ to the time-like vector $\vec{n}_o$.

We see that the main physically meaningful initial data required to determine a solution in both the characteristic and mixed

IVP's, are two functions of three variables on each of the two initial hypersurfaces. These two functions constitute the conformal 2-structure, and as in the space-like case, we may interpret $\tilde{g}^{\alpha\beta}$ as generalised coordinates.

Comparison with the usual results for non space-like IVP's of second order hyperbolic equations indicates that the dynamical variables describing the pure gravitational field may be regarded in the 2+2 approach to the characteristic and mixed IVP's as being explicitly embodied in the conformal 2-structure. Comparing the results of this chapter with those of the previous one, we see that in all three types of IVP we have considered, the gravitational degrees of freedom can be cast in a covariant manner in the conformal 2-structure. It is interesting to note the 'doubling' of the initial data required in the Cauchy problem, as opposed to the non space-like IVP's; this behaviour is typical of normal hyperbolic partial differential equations, as is demonstrated in Appendix A.

Chapter VIII. A Covariant Lagrangian 2 + 2 Formulation

We start by assuming that we have a bare manifold with some fixed foliation $\{S\}$ and fibrations $\mathcal{E}_a$ defined by the closed one-forms n^a and commuting vector fields $\vec{e}_a$ respectively, satisfying

$$\langle n^a, \vec{e}_b \rangle = \delta^a_b \quad .$$

We now impose a metric ${}^4g$, with components ${}^4g^{\alpha\beta}$ in an arbitrary coordinate basis $E_\alpha = \frac{\partial}{\partial x^\alpha}$. This metric can be decomposed with respect to $\{S\}$ and $\mathcal{E}_a$ in the way described in section 5.2 , and hence written as

$$ {}^4g^{\alpha\beta} = \gamma^{-1}\,\tilde{g}^{\alpha\beta} + \eta^{ab}\left(e^\alpha_a - b^\alpha_a\right)\left(e^\beta_b - b^\beta_b\right) \qquad (8.1)$$

which is just the same expression as given in section 5.10 . We may denote the set of independent components of ${}^4g^{\alpha\beta}$ by

$$\{\Phi_\Lambda\} = \{\gamma,\ \tilde{g}^{\alpha\beta},\ \eta^{ab},\ b^\alpha_a\} \quad . \qquad (8.2)$$

Note that $\vec{e}_a$ and n^a are defined independently of any metrical considerations, they merely provide a framework in the bare manifold upon which the metric is constructed. We shall consider variations in the metrical components

$$\Phi_\Lambda \rightarrow \Phi_\Lambda + \delta\Phi_\Lambda = {}'\Phi_\Lambda$$

such that ${}'\Phi_\Lambda$ are related to $\vec{e}_a$ and n^a in the same way as the original components Φ_Λ . We define

$$\delta\tilde{g}^{\alpha\beta} = \tilde{\delta}^{\alpha\beta},\ \delta\eta^{ab} = \delta^{ab},\ \delta b^\alpha_a = \xi^\alpha_a \quad . \qquad (8.3)$$

Then, from the above discussion, we see that the following must hold:

$$n^a_\alpha \tilde{\delta}^{\alpha\beta} = 0 \ , \qquad (8.4)$$

hence $\tilde{\delta}^{\alpha\beta}$ is tangent to $\{S\}$;

$$\delta n^\alpha_a = \mathcal{g}^\alpha_a \ , \qquad (8.5)$$

$$n^a_\alpha \mathcal{g}^\alpha_a = 0 \ , \qquad (8.6)$$

hence $\mathcal{g}^\alpha_a$ are vectors tangent to $\{S\}$. In order to calculate $\delta\tilde{g}_{\alpha\beta}$, we first note that

$$0 = \delta\left(\tilde{g}_{\alpha\beta} n^\beta_b\right) = n^\beta_b \, \delta\tilde{g}_{\alpha\beta} + \gamma^{-1} \mathcal{g}_{\alpha b} \ ,$$

from which we obtain

$$n^\beta_b \, \delta\tilde{g}_{\alpha\beta} = -\gamma^{-1} \mathcal{g}_{\alpha b} \ ; \quad n^\alpha_a \, n^\beta_b \, \delta\tilde{g}_{\alpha\beta} = 0 \ . \qquad (8.7)$$

To calculate $B^{\mu\nu}_{\alpha\beta} \, \delta\tilde{g}_{\mu\nu}$, we first obtain an expression for δB^α_β. First of all

$$\delta B^\alpha_\beta = -\delta C^\alpha_\beta = -n^a_\beta \, \mathcal{g}^\alpha_a \ , \qquad (8.8)$$

but in addition we have

$$\delta B^\alpha_\beta = \delta\left(\tilde{g}^{\alpha\gamma} \tilde{g}_{\beta\gamma}\right) = \tilde{g}^{\alpha\gamma} \, \delta\tilde{g}_{\beta\gamma} + \tilde{\delta}^\alpha{}_\beta \ . \qquad (8.9)$$

Then equating (8.8) and (8.9) gives

$$\tilde{g}^{\alpha\gamma} \, \delta\tilde{g}_{\beta\gamma} + \tilde{\delta}^\alpha{}_\beta = -n^a_\beta \, \mathcal{g}^\alpha_a$$

$$\Rightarrow \quad B^\gamma_\alpha \, \delta\tilde{g}_{\beta\gamma} + \tilde{\delta}_{\alpha\beta} = -\gamma^{-1} n^a_\beta \, \mathcal{g}_{\alpha a} \ .$$

$$\Rightarrow \quad B^{\mu\nu}_{\alpha\beta} \, \delta\tilde{g}_{\mu\nu} = -\tilde{\delta}_{\alpha\beta} \ . \qquad (8.10)$$

Collecting together equations (8.7 and 10), we obtain

$$\delta\tilde{g}_{\alpha\beta} = -\tilde{\delta}_{\alpha\beta} - 2\gamma^{-1} n^a_{(\alpha} \mathcal{g}_{\beta)a} \ . \qquad (8.11)$$

Now,

$$\delta\left(\tilde{g}^{\alpha\beta}\ \pounds_{n_a}\ \tilde{g}_{\alpha\beta}\right) = 0\ . \tag{8.12}$$

In order to write out the above equation explicitly, we first note that since variation and partial differentiation commute, we have the operator equivalence

$$\delta\pounds_v \equiv \pounds_{\delta v} + \pounds_v \delta \tag{8.13}$$

for any vector $\vec{v}$. Hence, from (8.12) we obtain, using (8.11 and 13),

$$\tilde{\delta}^{\alpha\beta}\pounds_{n_a}\ \tilde{g}_{\alpha\beta} - \tilde{g}^{\alpha\beta}\ \pounds_{n_a}\ \tilde{\delta}_{\alpha\beta} + \tilde{g}^{\alpha\beta}\ \pounds_{\mathcal{J}_a}\ \tilde{g}_{\alpha\beta} = 0\ . \tag{8.14}$$

But

$$\pounds_{\mathcal{J}_a}\ \tilde{g}_{\alpha\beta} = 2\tilde{T}\left(\nabla_\alpha \mathcal{J}_{\beta a}\right)$$

and so the third term in (8.14) vanishes. Hence from (8.14) we get

$$\pounds_{n_a}\ \tilde{\delta}^\alpha{}_\alpha = 0 \Rightarrow \tilde{\delta}^\alpha{}_\alpha = 0\ . \tag{8.15}$$

From (8.4), (8.6) and (8.15) we see that the set

$$\delta\Phi_\Lambda = \{\delta\gamma,\ \tilde{\delta}^{\alpha\beta},\ \mathcal{J}^\alpha_a,\ \delta^{ab}\}$$

are ten independent variations of Φ_Λ , and from (8.1 and 3) we may write

$$\delta^4 g^{\alpha\beta} = \gamma^{-1}\ \tilde{\delta}^{\alpha\beta} - \tilde{g}^{\alpha\beta}\ \gamma^{-1}\ \delta\gamma + 2\eta^{ab}\ n_a^{(\alpha}\mathcal{J}_b^{\beta)} + n_a^\alpha\ n_b^\beta\ \delta^{ab}\ . \tag{8.16}$$

The action function I is defined by

$$I = \int \mathcal{L}\, d^4x\ ,$$

where

$$\mathcal{L} \stackrel{\text{def}}{=} \sqrt{-{}^4g}\ {}^4R \tag{8.17}$$

is the Lagrangian density. Replacing $\mathcal{L}$ with any $\mathcal{L}^D$ defined by

$$\mathcal{L}^D = \mathcal{L} - \sqrt{-{}^4g}\ {}^4\nabla_\varepsilon\ T^\varepsilon\ , \tag{8.18}$$

where T^α is a vector functional of ${}^4g^{\alpha\beta}$ and $\partial_\gamma {}^4g^{\alpha\beta}$, leads to an action function I^D . As defined in (8.18), $\mathcal{L}^D$ has the same tensorial character as $\mathcal{L}$, namely a scalar density. Variation with respect to ${}^4g^{\alpha\beta}$ of I is identical to the variation of I^D , and leads to the same field equations. We obtain

$$\delta I^D = \int \delta\mathcal{L}^D d^4x = \int \left(\mathcal{L}_{\alpha\beta}\ \delta^4g^{\alpha\beta} + \sqrt{-{}^4g}\ {}^4\nabla_\alpha Z^\alpha\right) d^4x\ , \tag{8.19}$$

where Z^α are linear in $\delta^4g^{\alpha\beta}$ and $\partial\gamma\delta^4g^{\alpha\beta}$. Hence

$$\delta I^D = 0 \Rightarrow \mathcal{L}_{\alpha\beta} \equiv \sqrt{-{}^4g}\ {}^4G_{\alpha\beta} = 0\ . \tag{8.20}$$

We would expect independent variations of I^D with respect to the different Φ_Λ to give rise to different subsets of the field equations. In fact, substituting (8.16) in (8.19), and remembering the definition of $\mathcal{L}_{\alpha\beta}$ in (8.20), we see that

$$0 = \delta_{\tilde{g}}\alpha\beta I^D = \int \delta_{\tilde{g}}\alpha\beta\mathcal{L}^D d^4x = \int \left(\sqrt{-{}^4g}\lfloor{}^4G_{\alpha\beta}\tilde{\delta}^{\alpha\beta} + \sqrt{-{}^4g}\ {}^4\nabla_\alpha Z_1^\alpha\right) d^4x \Rightarrow \lfloor{}^4\tilde{G}_{\alpha\beta} = 0 \tag{8.21a}$$

$$0 = \delta_\gamma I^D = \int \delta_\gamma\mathcal{L}^D d^4x = \int \left(-\sqrt{-{}^4g}\lfloor{}^4G\gamma^{-1}\delta\gamma + \sqrt{-{}^4g}\ {}^4\nabla_\alpha Z_2^\alpha\right) d^4x \Rightarrow \lfloor{}^4G = 0 \tag{8.21b}$$

$$0 = \delta_\eta ab I^D = \int \delta_\eta ab\,\mathcal{L}^D d^4x = \int \left(\sqrt{-{}^4g}\ {}^4G_{ab}\delta^{ab} + \sqrt{-{}^4g}\ {}^4\nabla_\alpha Z_3^\alpha\right) d^4x \Rightarrow {}^4G_{ab} = 0 \tag{8.21c}$$

$$0 = \delta_{b}{}^\alpha{}_a I^D = \int \delta_{b}{}^\alpha{}_a\mathcal{L}^D d^4x = \int \left(2\sqrt{-{}^4g}\lfloor{}^4G_{a\alpha}\eta^{ab}\delta_b^\alpha + \sqrt{-{}^4g}{}^4\nabla_\alpha Z_4^\alpha\right) d^4x \Rightarrow \lfloor{}^4G^a{}_\alpha = 0\ . \tag{8.21d}$$

The above results allow us to obtain explicit expressions for the various projections of the Einstein tensor. We first obtain an expression for $\mathcal{L}$ in terms of Φ_Λ and their derivatives. From (5.8.6b,e) we see that

$$^4R = -\left(\perp^4G + {^4G_e}^e\right) ,$$

and hence from (5.8.8 and 11) we get

$$\mathcal{L} = \sqrt{-^4g}\left(2'\nabla_e h^e + 2\nabla_\varepsilon \ell^\varepsilon - \frac{3}{2} h_e h^e - \ell_\varepsilon \ell^\varepsilon - \tilde{h}_{\varepsilon\theta e}\tilde{h}^{\varepsilon\theta e} - \ell_{ef\varepsilon}\ell^{ef\varepsilon} - \Omega_{ef\varepsilon}\Omega^{ef\varepsilon} + {'R} + R\right) . \tag{8.22}$$

We can establish the result

$$'\nabla_e h^e = h_e h^e + {^4\nabla_\varepsilon}\left(n_e^\varepsilon h^e\right) , \tag{8.23}$$

since, from (5.3.5 and 8) we have

$$'\nabla_e h^e = C_\theta^{\varepsilon\,4}\nabla_\varepsilon\left(n_e^\theta h^e\right) = {^4\nabla_\varepsilon}\left(n_e^\varepsilon h^e\right) - h^e n_e^{\theta\,4}\nabla_\varepsilon C_\theta^\varepsilon , \tag{8.24}$$

and from (5.4.8), we obtain immediately that

$$h^e n_e^{\theta\,4}\nabla_\varepsilon C_\theta^\varepsilon = - h^e h_e . \tag{8.25}$$

In an entirely similar manner, we can show that

$$\nabla_\varepsilon \ell^\varepsilon = \ell_\varepsilon \ell^\varepsilon + {^4\nabla_\varepsilon}\ell^\varepsilon . \tag{8.26}$$

Substituting (8.23) and (8.26) into the expression (8.22) for $\mathcal{L}$, we obtain an equivalent Lagrangian density $\mathcal{L}^D$, given by

$$\mathcal{L}^D = \sqrt{-^4g}\left(\frac{1}{2} h_e h^e + \ell_\varepsilon \ell^\varepsilon - \tilde{h}_{\varepsilon\theta e}\tilde{h}^{\varepsilon\theta e} - \ell_{ef\varepsilon}\ell^{ef\varepsilon} - \Omega_{ef\varepsilon}\Omega^{ef\varepsilon} + {'R} + R\right) \tag{8.27}$$

where

$$\mathcal{L}' - \mathcal{L}^{D} = 2\sqrt{-{}^4g}\,{}^4\nabla_{\varepsilon}(\ell^{\varepsilon} + n^{\varepsilon}_{e}h^{e}) \ .$$

From equations (8.21), it is clear that variation of $\mathcal{L}^{D}$, as defined in (8.27), with respect to each member of $\{\Phi_{\Lambda}\}$, should lead to respective subsets of the field equations. We can perform the explicit term by term variation of $\mathcal{L}^{D}$, and this will provide an alternative derivation of the 2 + 2 break up of the field equations. In particular, comparison of the expressions for the field equations obtained by this method with those obtained in Chapter V, equations (5.8.7-10) provides a check on the internal consistency of the formalism we have developed in this thesis. In order to perform the variation of $\mathcal{L}^{D}$, we use the fact that variation and partial differentiation commute, which means that equation (8.13) holds. We then obtain after rather long but straightforward calculations, the following results:

$$\delta\sqrt{-{}^4g} = \delta\left(\eta\gamma\right) = \sqrt{-{}^4g}\,\gamma^{-1}\delta\gamma - \tfrac{1}{2}\sqrt{-{}^4g}\,\delta_{e}{}^{e} \tag{8.28a}$$

$$\delta\left(\tfrac{1}{2}h_{e}h^{e}\right) = \left({}'\nabla_{e}h^{e} - h_{e}h^{e}\right)\gamma^{-1}\delta\gamma + \tfrac{1}{2}h_{e}h_{f}\delta^{ef} - \left(\nabla_{\varepsilon}h^{e} + \ell_{\varepsilon}h^{e}\right)\mathfrak{f}^{\varepsilon}_{e}$$

$$+ {}^4\nabla_{\varepsilon}\left(n^{\varepsilon}_{e}h^{e}\gamma^{-1}\delta\gamma + h^{e}\mathfrak{f}^{\varepsilon}_{e}\right) \tag{8.28b}$$

$$\delta\left(\ell_{\varepsilon}\ell^{\varepsilon}\right) = -\ell_{\varepsilon}\ell^{\varepsilon}\gamma^{-1}\delta\gamma + \tilde{T}\left(\ell_{\varepsilon}\ell_{\theta}\right)\tilde{\delta}^{\varepsilon\theta} + \left(\ell_{e}\ell^{\varepsilon} - \nabla_{\varepsilon}\ell^{\varepsilon}\right)\delta_{e}{}^{e} + {}^4\nabla_{\varepsilon}\left(\ell^{\varepsilon}\delta_{e}{}^{e}\right) \tag{8.28c}$$

$$\delta\left(-\ell_{ef\varepsilon}\ell^{ef\varepsilon}\right) = \ell_{ef\varepsilon}\ell^{ef\varepsilon}\gamma^{-1}\delta\gamma - \tilde{T}\left(\ell_{ef\theta}\ell^{ef}{}_{\varepsilon}\right)\tilde{\delta}^{\varepsilon\theta}$$

$$- \left(\nabla_{\varepsilon}\ell_{ef}{}^{\varepsilon} + \ell_{ef\varepsilon}\ell^{\varepsilon} + 2\ell_{e}{}^{d\varepsilon}\ell_{fd\varepsilon}\right)\delta^{ef} - {}^4\nabla_{\varepsilon}\left(\ell_{ef}{}^{\varepsilon}\delta^{ef}\right) \tag{8.28d}$$

$$\delta\left(-\tilde{h}_{\varepsilon\theta e}\tilde{h}^{\varepsilon\theta e}\right) = \left({}'\nabla_e \tilde{h}_{\varepsilon\theta}{}^e + 2\tilde{h}_{\varepsilon\mu e}\tilde{h}_\theta{}^{\mu e} - \tilde{h}_{\varepsilon\theta e}h^e\right)\tilde{\delta}^{\varepsilon\theta} - \tilde{h}_{\varepsilon\theta e}\tilde{h}^{\varepsilon\theta}{}_f\delta^{ef}$$

$$- 2\left(\nabla_\varepsilon \tilde{h}^\varepsilon{}_\theta{}^e - \ell_\varepsilon \tilde{h}^\varepsilon{}_\theta{}^e\right)\mathcal{S}^\theta_e - {}^4\nabla_\varepsilon\left(n^\varepsilon_e \tilde{h}_{\theta\mu}{}^e\tilde{\delta}^{\theta\mu} - 2\tilde{h}^\varepsilon{}_\theta{}^e\mathcal{S}^\theta_e\right) \quad (8.28e)$$

$$\delta\left(-\Omega_{ef\varepsilon}\Omega^{ef\varepsilon}\right) = -\Omega_{ef\varepsilon}\Omega^{ef\varepsilon}\gamma^{-1}\delta\gamma + \tilde{T}\left(\Omega_{ef\varepsilon}\Omega^{ef}{}_\theta\right)\tilde{\delta}^{\varepsilon\theta} - 2\Omega_e{}^{d\varepsilon}\Omega_{fd\varepsilon}\delta^{ef}$$

$$+ 2\left({}'\nabla_e\Omega^{ef}{}_\varepsilon - \Omega^{ef}{}_\varepsilon h_e\right)\mathcal{S}^\varepsilon_f - 2\,{}^4\nabla_\varepsilon\left(n^\varepsilon_e\Omega^{ef}{}_\theta\mathcal{S}^\theta_f\right) \quad (8.28f)$$

$$\delta' R = \delta' R_{ef}\delta^{ef} + \eta^{ef}\delta' R_{ef} \quad (8.28g)$$

$$\delta R = -R\gamma^{-1}\delta\gamma + g^{\varepsilon\theta}\delta R_{\varepsilon\theta} \quad . \quad (8.28h)$$

The variations $g^{\varepsilon\theta}\delta R_{\varepsilon\theta}$ and $\eta^{ef}\delta' R_{ef}$ are rather more difficult to calculate, and the procedure is outlined in Appendix B. Eventually, we get

$$\eta^{ef}\delta' R_{ef} = \left(h_e h^e - {}'\nabla_e h^e\right)\delta_f{}^f + \left({}'\nabla_{(e}h_{f)} - h_e h_f\right)\delta^{ef}$$

$$+ 2\left(\ell_\varepsilon h^e - h^f\ell_f{}^e{}_\varepsilon + {}'\nabla^e\ell_\varepsilon - {}'\nabla^f\ell^e{}_{f\varepsilon}\right)\mathcal{S}^\varepsilon_e$$

$$+ \nabla_\varepsilon\left(n^\varepsilon_e({}'\nabla^e\delta_f{}^f - {}'\nabla_f\delta^{ef} + h^e\delta_f{}^f - h_f\delta^{ef} + 2\ell_\theta\mathcal{S}^{\theta e} - 2\ell^{ef}{}_\theta\mathcal{S}^\theta_f)\right) \quad . \quad (8.28i)$$

$$g^{\varepsilon\theta}\delta R_{\varepsilon\theta} = \left(\nabla_\varepsilon\ell^\varepsilon - \ell_\varepsilon\ell^\varepsilon\right)\gamma^{-1}\delta\gamma + \tilde{T}\left(\nabla_\varepsilon\ell_\theta - \ell_\varepsilon\ell_\theta\right)\tilde{\delta}^{\varepsilon\theta}$$

$$- {}^4\nabla_\varepsilon\left(\nabla_\theta(\gamma^{-1}\tilde{\delta}^{\varepsilon\theta}) + \nabla^\varepsilon(\gamma^{-1}\delta\gamma) + \ell^\varepsilon\gamma^{-1}\delta\gamma + \ell_\theta\gamma^{-1}\tilde{\delta}^{\varepsilon\theta}\right) \quad . \quad (8.28j)$$

From equations (8.27) and (8.28) we then obtain the following:

$$\delta_{\tilde{g}\varepsilon\theta}\mathcal{L}^D = \sqrt{-{}^4g}\Big\{{}'\nabla_e\tilde{h}_{\varepsilon\theta}{}^e + 2\tilde{h}_{\varepsilon\mu e}\tilde{h}_\theta{}^{\mu e} - \tilde{h}_{\varepsilon\theta e}h^e + \tilde{T}\Big(\nabla_\varepsilon\ell_\theta - \ell_{ef\varepsilon}\ell^{ef}{}_\theta$$

$$- \Omega_{ef\varepsilon}\Omega^{ef}{}_\theta\Big)\Big\}\tilde{\delta}^{\varepsilon\theta} + \sqrt{-{}^4g}\;{}^4\nabla_\varepsilon Z_1^\varepsilon \qquad (8.29a)$$

$$\delta_\gamma\mathcal{L}^D = \sqrt{-{}^4g}\Big\{\nabla_e h^e - \tfrac{1}{2}h_e h^e + \nabla_\varepsilon\ell^\varepsilon - \ell_\varepsilon\ell^\varepsilon - \tilde{h}_{\varepsilon\theta e}\tilde{h}^{\varepsilon\theta e} - 2\Omega_{ef\varepsilon}\Omega^{ef\varepsilon} + {}'R\Big\}\gamma^{-1}\delta\gamma$$

$$+ \sqrt{-{}^4g}\;{}^4\nabla_\varepsilon Z_2^\varepsilon \qquad (8.29b)$$

$$\delta_{\eta^{ef}}\mathcal{L}^D = \sqrt{-{}^4g}\Big\{{}'\nabla_{(e}h_{f)} - \tilde{h}_{\varepsilon\theta e}\tilde{h}^{\varepsilon\theta}{}_f + \nabla_\varepsilon\ell_{ef}{}^\varepsilon + 2\ell_e{}^{d\varepsilon}\ell_{fd\varepsilon} - \ell_{ef\varepsilon}\ell^\varepsilon$$

$$+ {}'R_{ef} - \eta_{ef}\Big(\tfrac{1}{2}{}'R + \tfrac{1}{2}R + {}'\nabla_d h^d + \nabla_\varepsilon\ell^\varepsilon - \tfrac{3}{4}h_d h^d - \tfrac{1}{2}\ell_\varepsilon\ell^\varepsilon$$

$$- \tfrac{1}{2}\tilde{h}_{\varepsilon\theta d}\tilde{h}^{\varepsilon\theta d} - \tfrac{1}{2}\ell_{dc\varepsilon}\ell^{dc\varepsilon} + \tfrac{1}{2}\Omega_{dc\varepsilon}\Omega^{dc\varepsilon}\Big)\Big\}\delta^{ef} + \sqrt{-{}^4g}\;{}^4\nabla_\varepsilon Z_3^\varepsilon$$

$$(8.29c)$$

$$\delta_{b^\varepsilon_e}\mathcal{L}^D = 2\sqrt{-{}^4g}\Big\{{}'\nabla_f\Omega^{fe}{}_\varepsilon - h_f\Omega^{fe}{}_\varepsilon - \nabla_\theta\tilde{h}^\theta{}_\varepsilon{}^e + \tfrac{1}{2}\nabla_\varepsilon h^e + {}'\nabla^e\ell_\varepsilon - {}'\nabla^f\ell^e{}_{f\varepsilon}$$

$$+ \ell_\theta\tilde{h}^\theta{}_\varepsilon{}^e - \ell_f{}^e{}_\varepsilon h^f + \tfrac{1}{2}h^e\ell_\varepsilon\Big\}\delta b^\varepsilon_e + \sqrt{-{}^4g}\;{}^4\nabla_\varepsilon Z_4^\varepsilon\,. \qquad (8.29d)$$

Comparing (8.29a,b,c,d) with (8.21a,b,c,d) respectively gives us explicit expressions for the various projections of ${}^4G_{\alpha\beta}$. It is a simple matter to check that these expressions are identical to the corresponding ones for the projections of ${}^4G^{\alpha\beta}$ given by (5.8.7-10). Explicit expressions can also be calculated for the Z^α from equations (8.28) and (8.29).

Conclusion

To date most work on the IVP's of general relativity has been focussed on the standard Cauchy problem. Here the 3+1 approach first suggested by Lichnerowicz and extensively developed by others has led to some significant advances. In particular, the use of harmonic coordinates, starting with the fundamental work of Choquet-Bruhat, has allowed strong theorems of existence, uniqueness and stability of the evolution equations to be proved. The introduction of coordinate independent techniques by Stachel, and especially his use of the Lie derivative as the natural covariant analogue of the partial derivative, have done much to aid our geometrical insight into the 3+1 formulation. They allow, for example, a covariant interpretation of the gauge freedom of the theory and of the (constrained) initial data required to solve the evolution equations. The later development by York of these techniques, and his introduction of conformal 3-geometry methods has allowed an identification of the freely specifiable initial data which reduces the constraint equations to a system of four coupled, quasi-linear partial differential equations, for which many 'good' theorems of uniqueness, existence and stability have been proved.

In recent years, Müller zum Hagen and Seifert have used harmonic coordinate conditions to prove that the double-null IVP is well set. While this is an important result from an analytical point of view, it does not lead to any geometrical insight, since harmonic coordinate conditions are inherently non covariant. In fact all work to date on characteristic and mixed IVP's has suffered from the disadvantage of it being couched in rather ad hoc coordinate-dependent form. Part of the purpose of the present work is to remove this limitation.

The main results presented in this thesis rely on the development in Chapter V, of a covariant 2+2 formalism, in which space-time is foliated by a family of space-like 2-surfaces, and all space-time objects are covariantly decomposed into their projections tangential and orthogonal to this foliation. The formalism was applied first to the Cauchy problem, in Chapter VI, and then to the characteristic and mixed IVPs, in Chapter VII. In all three cases the gauge conditions were formulated covariantly, and in particular, in the case of the non space-like IVP's considered, it was shown that certain specialisations of these gauge conditions are covariant analogues of coordinate conditions used by various authors in previous coordinate dependent analyses. Furthermore it was shown, in each case, that the dynamical degrees of freedom of the gravitational field may be explicitly and covariantly embodied in the conformal 2-structure; that is, that the two independent quantities constituting the conformal 2-structure may be regarded as unconstrained generalised coordinates, or true dynamical variables, of the gravitational field. Thus the gravitational degrees of freedom in the 2+2 approach have an immediately clear and local geometrical significance. Moreover it is possible to identify the two field equations $\perp^4G^{\alpha\beta} = 0$ as dynamical equations, that is as equations propagating the dynamical variables off the initial hypersurface(s). In Chapter VIII we obtained the expected result that variation of the action function with respect to the dynamical variables does indeed lead to the dynamical equations.

However, work on the 2+2 formulation is still in its early stages and the fundamental question as to whether the IVP's we have considered are properly set with our choice of initial data is still an open one, although as we have said Müller zum Hagen and Seifert have proved that the problem is well posed in the double-null case. The next task is clearly to determine whether the same is true for the other cases considered in this thesis. Of course the presciption works in the very limited case of analytic solutions, and the probability is that, pathologies aside, it will prove to work in the more realistic case of smooth (C^{∞}) solutions (although there may be difficulties with the mixed IVP, due to problems with the 'coherency' of initial data set on a time-like hypersurface). Once the theoretical existence of non analytic solutions has been established, then the restriction in practice to analytic solutions is not so important (even finite discontinuities in the solution functions can be approximated by sufficiently rapidly varying analytic functions) and the iterative schemes which we have described in this thesis may then well prove of direct use in allowing one to actually compute solutions from given initial data.

Our analysis so far has been purely local, and certainly another important area of investigation is the rôle of boundary conditions in the 2+2 formulation, and in particular a covariant analysis of the original Bondi formulation of the non space-like IVP, where some data is set at future null infinity ($\mathcal{J}^+$). Another interesting question is to consider what happens when the solution admits a Killing vector. In the investigation of this latter problem it is likely that an anholonomic 2+2 break-up, in analogy with the 3+1 anholonomic break-up first considered by Ó Murchadha[(31)] and later by Stachel[(32)], will be of more use than that considered here. In fact such a formalism has been developed by the author[(33)] in which space-time is considered as being fibrated by the trajectories of two commuting vector fields which span a family of time-like 2-surfaces, and 'foliated' by the orthogonal space-like 2-surface elements (these latter 2-surface elements are in general anholonomic and so do not constitute a proper foliation). One then proceeds by demanding that one of the two commuting vector fields is also Killing. This problem is under study.

Perhaps the most immediately interesting question is how the 2+2 approach compares with the 3+1. First of all, of course, we see that the 3+1 formulation only allows an analysis of the standard Cauchy problem. As we demonstrated in Chapter IV, there are serious problems involved in trying to apply a 3+1 approach to non space-like IVP's, and even were one to succeed in such a formulation, it would bear little relation to the 3+1 formulation of the Cauchy problem, since the geometry of space-like and null hypersurfaces is so different. In contrast, one clear advantage of the 2+2 approach is that it provides a unification, at least locally, of all the various types of IVP, since the dynamical variables in each case are the same. Secondly, as we discussed in Chapter II the 3+1 approach has difficulties associated with it when it comes to the problem of identifying the gravitational degrees of freedom. By means of the conformal 3-geometry techniques of York the freely specifiable initial data can be identified, but this contains implicitly within it the gauge freedom associated with the initial surface. In particular the dynamical degrees of freedom are identified as the conformal 3-geometry of a family of space-like hypersurfaces, modulo a choice of basis at each point. It is possible to construct a quantity (specifically a transverse trace-free tensor, the so-called Bach tensor) invariant under conformal transformations and diffeomorphisms ("conformeomorphisms") of the 3-geometry of each slice. Although this invariant characterises uniquely the gravitational degrees of freedom, it depends upon space-like derivatives of the conformal metric, and hence has a non-local interpretation. It is hard to see how the Bach tensor could be interpreted as explicitly embodying the dynamical variables. For example, it is not possible to isolate a subset of the six evolution equations as dynamical equations propagating the Bach tensor. The non local character of the dynamical degrees of freedom in the 3+1 approach is in direct contrast to the situation, already discussed, in the 2+2 formulation.

With regard to the problem of the quantisation of the gravitational field, the 3+1 approach suggests the use of a constrained Hamiltonian formulation, with the five functions contained in the conformal 3-geometry acting as constrained configuration coordinates. However, a necessary condition in at least one approach to quantisation is to have the two unconstrained dynamical variables explicitly isolated, and this suggests further investigation of the 2+2 approach in relation to the quantum problem. Some work in this direction has been done, in particular by Gambini and Restuccia,[(15)] working in the light cone gauge. The formalism developed in this thesis may well prove useful in any future work on a canonical quantisation procedure for the gravitational field.

Appendix A. The One-dimensional Wave Equation.

The one-dimensional scalar wave equation[7] is given, in suitable units, by

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \, . \qquad \text{(A1)}$$

This is the simplest of all systems of normal hyperbolic equations, which possesses one degree of freedom corresponding to the generalised coordinate, or dynamical variable, $\phi(x,t)$. By transforming to new (null) coordinates

$$u = t - x$$
$$v = t + x \, ,$$

it may be shown that the general (d'Alembert) solution of (A1) is given by

$$\phi = f(u) + g(v)$$

where f and g are arbitrary functions. The lines $u = \text{const}$ and $v = \text{const}$ are characteristics of equation (A1). Using the above results, we may write down the general solution of (A1) in terms of the initial data in a two-dimensional region R, for each of the three types of IVP considered in this thesis.

1) The Cauchy Problem: we write the wave equation in terms of the space-like and time-like coordinates x and t respectively, which yields the form of the equation given in (A1), namely

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = 0 \, .$$

The necessary initial data is

$$\phi = \psi(x) \quad \text{on} \quad t = t_o \,, \quad x_o \leq x \leq x_1$$

$$\frac{\partial \phi}{\partial t} = \chi(x) \quad \text{on} \quad t = t_o \quad x_o \leq x \leq x_1 \,.$$

The general solution is then

$$\phi(x,t) = \tfrac{1}{2}\Big(\psi(x+t-t_o) + \psi(x-t+t_o)\Big) + \tfrac{1}{2} \int_{x-t+t_o}^{x+t-t_o} \chi(y)\,dy \,.$$

in the region $(t_o - x_o \leq t - x \leq t_o - x_1 \,,\ t_o + x_o \leq t + x \leq t_o + x_1)$, shown in figure 3.

2) <u>The Characteristic IVP</u>: we write the wave equation in terms of the two null coordinates u and v , which yields

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0 \,.$$

The necessary initial data is

$$\phi = \psi(u) \quad \text{on} \quad v = v_o \,, \ u \geq u_o$$

$$\phi = \chi(v) \quad \text{on} \quad u = u_o \,, \ v \geq v_o \,, \ \chi(v_o) = \psi(u_o)$$

The general solution is then

$$\phi(u,v) = \psi(u) + \chi(v) - \psi(u_o) \,.$$

in the region $(u \geq u_o ,\ v \geq v_o)$, shown in figure 3.

3) <u>The Mixed IVP</u>: we write the wave equation in terms of the null and space-like coordinates u and x respectively, which yields

$$2\,\frac{\partial^2 \phi}{\partial u \partial x} - \frac{\partial^2 \phi}{\partial x^2} = 0 \,.$$

The necessary initial data is

$$\phi = \psi(u) \quad \text{on} \quad x = x_o \; , \; u \geqslant u_o \; .$$

$$\phi = \chi(x) \quad \text{on} \quad u = u_o \; , \quad x \geqslant x_o \; , \; \chi(x_o) = \psi(u_o) \; .$$

The general solution is then

$$\phi(u,x) = \chi\left(x - \tfrac{1}{2}(u-u_o)\right) - \chi\left(x_o - \tfrac{1}{2}(u-u_o)\right) + \psi(u)$$

in the region $(u \geqslant u_o \; , \; x \geqslant x_o)$ shown in figure 3.

Thus we see that apart from the fact that the scalar wave equation possesses only one degree of freedom, the initial data which may be freely specified is entirely analogous to that required for the Einstein vacuum field equations, as considered in the 2+2 formulation.

Appendix B. Calculation of $g^{\alpha\beta}\delta R_{\alpha\beta}$ and $\eta^{ab}\delta' R_{ab}$.

In order to calculate $g^{\alpha\beta}\delta R_{\alpha\beta}$, it is convenient to introduce an arbitrary coordinate basis

$$B_A = \frac{\partial}{\partial' x^A}$$

into $\{S\}$. Since we have already assumed in Chapter VIII that we are working in a general coordinate space-time basis

$$E_\alpha = \frac{\partial}{\partial x^\alpha} \quad ,$$

we see that the connecting quantities B^α_A defined in equation (5.2.3) for arbitrary (and possibly non coordinate) bases E_α , B_A become

$$B^\alpha_A = \frac{\partial x^\alpha}{\partial' x^A} \quad . \qquad \text{(B1)}$$

Equation (B1) shows that B^α_A is independent of any variations in Φ_Λ since it depends only upon some arbitrary choice of coordinate basis $\frac{\partial}{\partial' x^A}$ and $\frac{\partial}{\partial x^\alpha}$ in $\{S\}$ and V respectively. Hence

$$\delta g^{\alpha\beta} = \delta\left(B^{\alpha\beta}_{AB} g^{AB}\right) = B^{\alpha\beta}_{AB}\delta g^{AB} \Rightarrow \delta g^{AB} = B^{AB}_{\alpha\beta}\delta g^{\alpha\beta}$$

and it then follows that

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{AB}\delta R_{AB} \quad .$$

Now R_{AB} has its usual definition in terms of the Christoffel symbols Γ^A_{BC} , namely

$$R_{CB} = \partial_E \Gamma^E_{CB} - \partial_C \Gamma^E_{EB} + \Gamma^E_{EF}\Gamma^F_{CB} - \Gamma^E_{CF}\Gamma^F_{EB} \quad .$$

We can calculate the variation in Γ^A_{CB} , and we obtain

$$\delta\Gamma^{A}_{CB} = \nabla_C\left(g^{AE}\delta g_{BE}\right) + \nabla_B\left(g^{AE}\delta g_{CE}\right) - \nabla_E\left(g^{AE}\delta g_{BC}\right) . \tag{B2}$$

Note in particular that $\delta\Gamma^{A}_{CB}$ is a tensor in $\{S\}$. We then obtain

$$\delta R_{CB} = \nabla_E\delta\Gamma^{E}_{CB} - \nabla_C\delta\Gamma^{E}_{EB} \tag{B3}$$

from which it follows, using (B2) and (B3), that

$$g^{CB}\delta R_{CB} = \nabla_E\nabla^E\left(g_{CB}\delta g^{CB}\right) - \nabla_C\nabla_B\delta g^{CB} . \tag{B4}$$

But we can now write this in terms of the basis of V , and we obtain

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\varepsilon\nabla^\varepsilon\left(g_{\alpha\beta}\delta g^{\alpha\beta}\right) - \nabla_\alpha\nabla_\beta\delta g^{\alpha\beta} \tag{B5}$$

from which (8.28j) follows straightforwardly.

In order to calculate $\eta^{cb}\delta{}'R_{cb}$, we first note that

$$'R_{cb} = \pounds_{n_e}\Gamma^{e}_{cb} - \pounds_{n_c}\Gamma^{e}_{eb} + \Gamma^{e}_{ef}\Gamma^{f}_{cb} - \Gamma^{e}_{cf}\Gamma^{f}_{eb} ,$$

and hence it is dependent both on η_{ab} and b^{α}_{a} . Let us calculate the variation with respect to η^{ab} first. We start by calculating $\delta\Gamma^{a}_{bc}$. From the definition (5.3.9), we can easily show that

$$\delta\Gamma^{a}_{cb} = -\tfrac{1}{2}{}'\nabla_c\delta_b{}^a - \tfrac{1}{2}{}'\nabla_b\delta_c{}^a + \tfrac{1}{2}{}'\nabla^a\delta_{cb} ,$$

hence $\delta\Gamma^{a}_{cb}$ is a 'tensor' with respect to $'\nabla_a$. Then we may write

$$\delta{}'R_{cb} = {}'\nabla_e\delta\Gamma^{e}_{cb} - {}'\nabla_c\delta\Gamma^{e}_{eb} ,$$

and

$$\eta^{cb}\delta{}'R_{cb} = {}'\nabla_c{}'\nabla^c\delta_b{}^b - {}'\nabla_c{}'\nabla_b\delta^{cb} \tag{B6}$$

To calculate the variation with respect to b^{α}_{a} , we first calculate $\mathcal{S}\Gamma^{a}_{cb}$, and obtain

$$\mathcal{S}\Gamma^{a}_{cb} = \tfrac{1}{2}\eta^{ae}\left(\pounds_{\mathcal{S}_c}\eta_{eb} + \pounds_{\mathcal{S}_b}\eta_{ec} - \pounds_{\mathcal{S}_e}\eta_{cb}\right) = -\mathcal{S}^{\varepsilon}_{c}\ell^{a}{}_{b\varepsilon} - \mathcal{S}^{\varepsilon}_{b}\ell^{a}{}_{c\varepsilon} + \mathcal{S}^{\varepsilon a}\ell_{cb\varepsilon} , \tag{B7}$$

and so again $\delta\Gamma^a_{cb}$ is a 'tensor' with respect to $'\nabla_a$. Next we can show that

$$\delta\,'R_{cb} = \pounds_{\delta_e}\Gamma^e_{cb} - \pounds_{\delta_c}\Gamma^e_{eb} + '\nabla_e\delta\Gamma^e_{cb} - '\nabla_c\delta\Gamma^e_{eb} \,. \qquad \text{(B8)}$$

In order to proceed, we need an expression for $\pounds_{\delta_d}\Gamma^a_{cb}$. We use the following result for the commutator of ∇_α and $\pounds_{n_a}$ acting on a scalar,

$$\nabla_\alpha\pounds_{n_a} - \pounds_{n_a}\nabla_\alpha = 2n^e_\alpha\pounds_{\Omega_{ae}} \,.$$

Then

$$\pounds_{\delta_d}\Gamma^a_{cb} = \delta^\alpha_d\nabla_\alpha\Gamma^a_{cb} = 2\delta^\alpha_d\ell^{ae}{}_\alpha\Gamma_{cbe} - \delta^\alpha_d\eta^{ae}\left(\pounds_{n_b}\ell_{ec\alpha} + \pounds_{n_c}\ell_{eb\alpha} - \pounds_{n_e}\ell_{cb\alpha}\right)$$

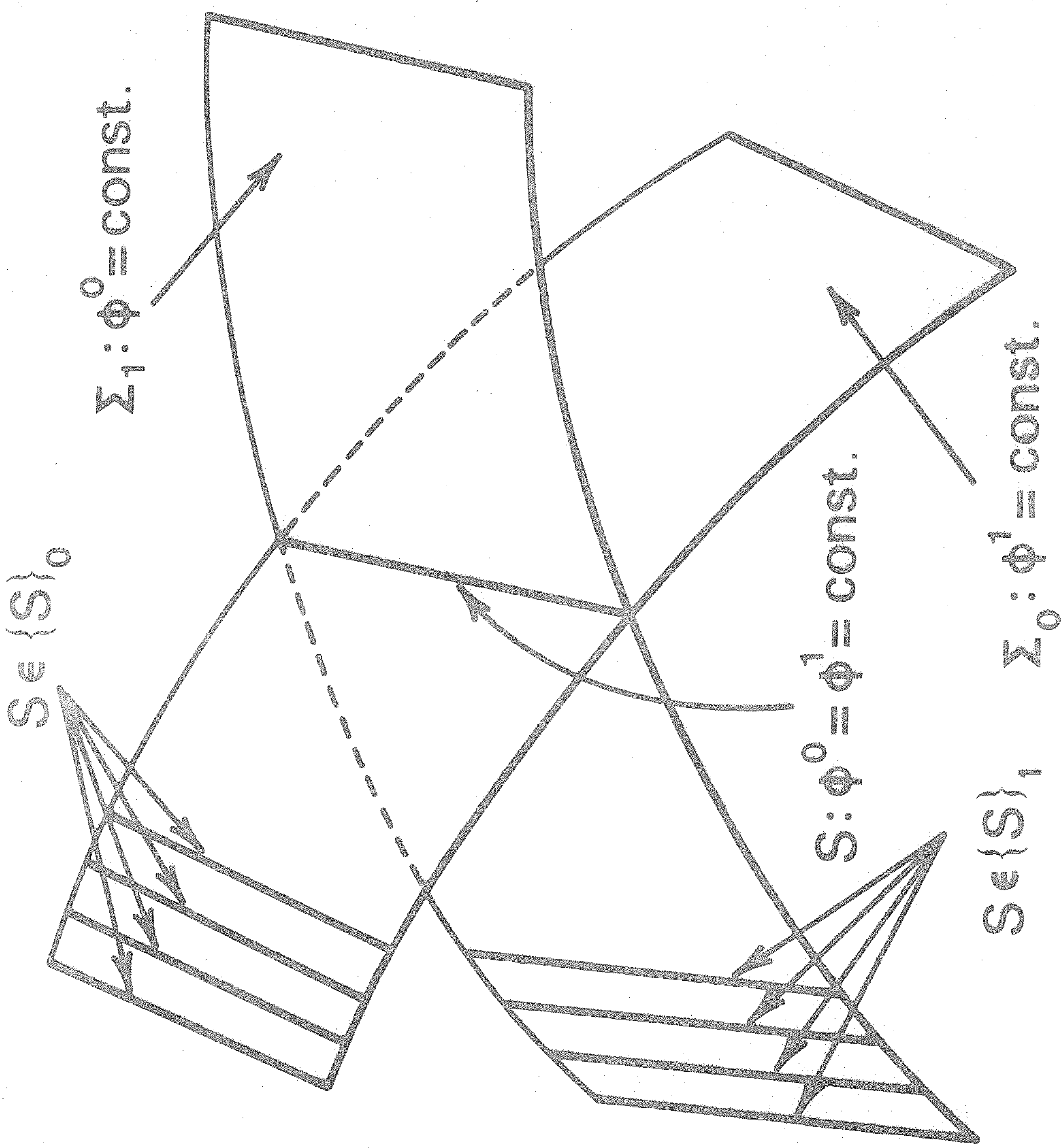
and hence

$$\pounds_{\delta_d}\Gamma^a_{cb} = -\delta^\alpha_d\left('\nabla_b\ell^a{}_{c\alpha} + '\nabla_c\ell^a{}_{b\alpha} - '\nabla^a\ell_{cb\alpha}\right) \,. \qquad \text{(B9)}$$

Substituting (B7) and (B9) into (B8) gives

$$\eta^{cb}\delta R_{cb} = 2\delta^\varepsilon_b\left('\nabla^b\ell_\varepsilon - '\nabla^c\ell^b{}_{c\varepsilon}\right) + 2'\nabla^b\left(\delta^\varepsilon_b\ell_\varepsilon - \delta^\varepsilon_c\ell^c{}_{b\varepsilon}\right). \qquad \text{(B10)}$$

Then (B6) and (B10) lead directly to (8.28i).


$\Sigma_1 : \phi^0 = const.$
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$\Sigma_0 : \phi^1 = const.$
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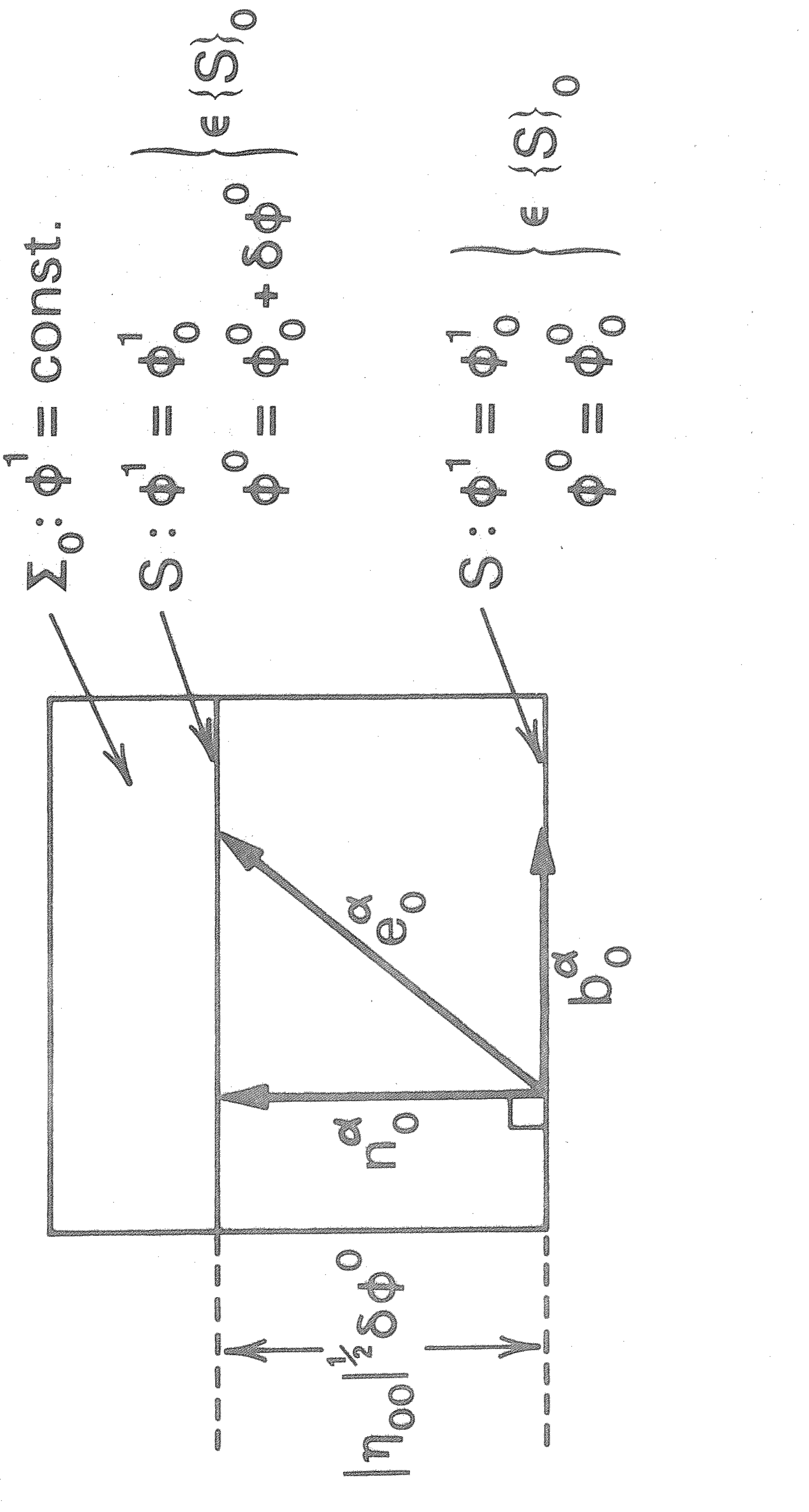
Σ_o: ϕ^1 = const.
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S: $\phi^1 = \phi^1_o$
$\phi^0 = \phi^0_o$
$\in \{S\}_o$
e^α_o
b^α_o
n^α_o
$|n_{oo}|^{\frac{1}{2}}\delta\phi^0$

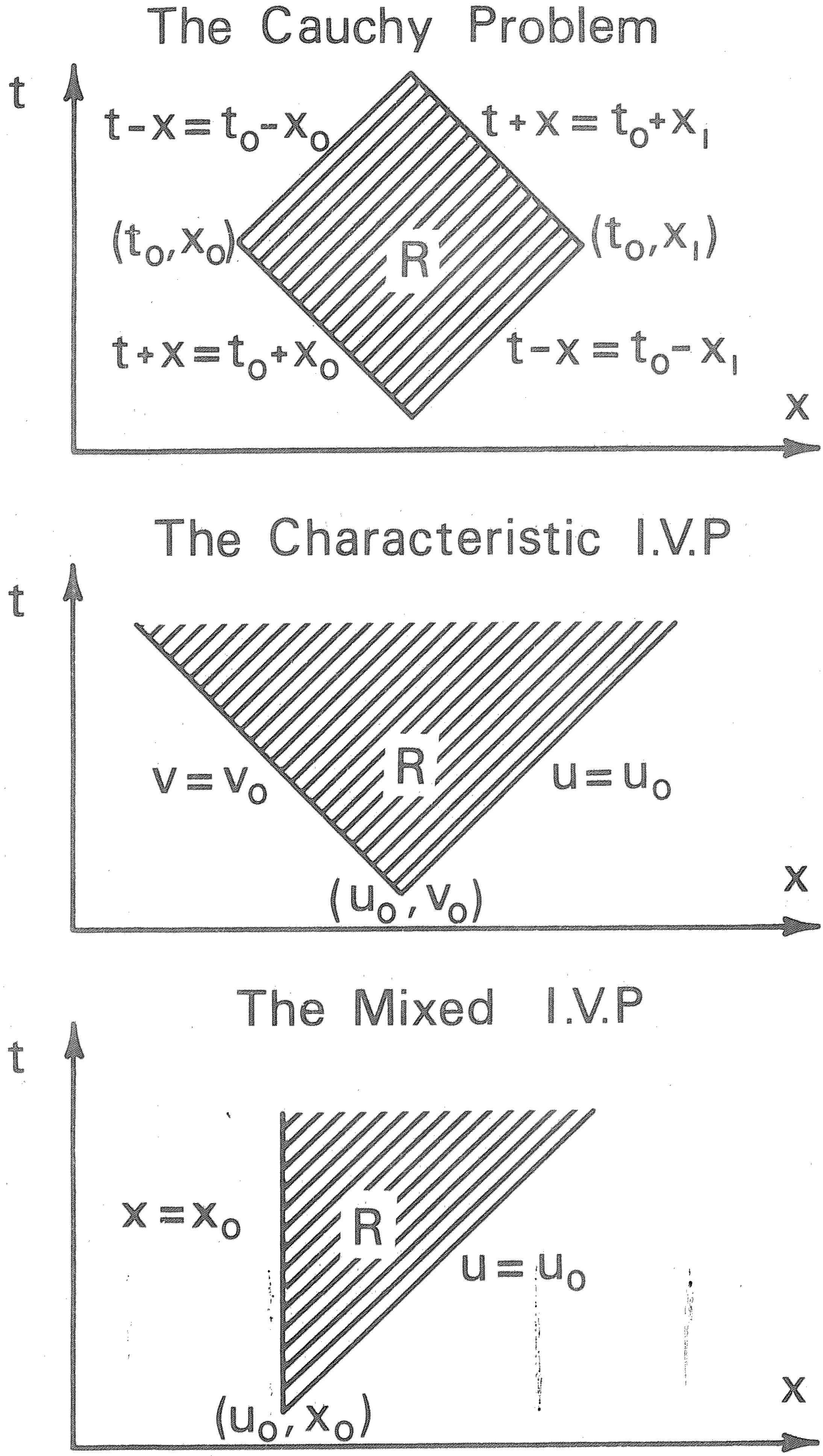

Figure 3. The region of integration R for each of the three types of IVP considered in Appendix A.

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