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On fixed-width simultaneous confidence intervals for multiple comparisons and some related problems

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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On fixed-width simultaneous confidence intervals for multiple comparisons and some related problems

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This thesis considers inferences about the means of several independently and normally distributed populations with a common variance. The first part discusses the constructions of fixed-width simultaneous confidence intervals when the variance is an unknown parameter by using sequential samplings. A set of fixed-width simultaneous confidence intervals is often used to make simultaneous inferences, with a probability that all the inferences made are simultaneously correct being at least $1-\alpha$, the simultaneous confidence level. Certain probabilities of making simultaneously correct inferences are often larger than the confidence level $1-\alpha$. These are considered in the second part of the thesis. The third and final part of the thesis studies the multiple tests corresponding to the simultaneous confidence intervals. Some new power functions are defined and their properties are investigated.

Contents

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Chapter 1

Introduction and notation

1.1 Construction of a fixed-width confidence interval for the mean of a normal population

Suppose that we have a normally distributed population $N(\eta, \sigma^2)$ with unknown mean η and positive variance σ^2, and that independent observations $Y_1, Y_2, \cdots$ can be taken sequentially from the population. We wish to construct a $100(1-\alpha)\%$ confidence interval for η of width $2d$, in the form of $(\bar{Y} - d,\ \bar{Y} + d)$, where $d > 0$ and $0 < \alpha < 1$ are two given constants, and $\bar{Y}$ is the sample mean of a sample taken from the population.

Inferences about η can be made from this confidence interval. For instance, if $\bar{Y} > d$ then we can infer that $\eta > 0$ since the confidence interval $(\bar{Y} - d, \bar{Y} + d)$ is entirely to the right of zero. Similarly, if $\bar{Y} < -d$ then we can infer that $\eta < 0$. The width d determines the sensitivity of this confidence interval in the following sense. If $\eta > 2d$ then the correct inference "$\eta > 0$" will be made from this confidence interval with probability at least $1 - \alpha$, since the confidence interval for η, $(\bar{Y} - d,\ \bar{Y} + d)$, will be entirely to the right of zero with probability at least $1 - \alpha$. Similarly, if $\eta < -2d$, the correct inference "$\eta < 0$" will be made with probability at least $1 - \alpha$ because the confidence interval for η will be entirely to the left of zero with probability at least $1 - \alpha$.

If σ^2 is known then such a confidence interval can be easily constructed in the following way. A random sample of fixed size n is taken from the population and a confidence interval for η is defined to be

$$\left(\bar{Y}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \bar{Y}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right), \tag{1.1}$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of the standard normal distribution and $\bar{Y}_n = (1/n)\sum_{i=1}^{n} Y_i$. In order that the width of this confidence interval, $2z_{\alpha/2}\,\sigma/\sqrt{n}$, is at most $2d$, the sample size n should satisfy $z_{\alpha/2}\,\sigma/\sqrt{n} \le d$, which implies that

$$n \geq n_0 = d^{-2}(z_{\alpha/2})^2\sigma^2.$$

Therefore, if a sample of fixed size n_0 is taken from the population $N(\eta, \sigma^2)$, then the confidence interval in (1.1) will satisfy the requirement. The value of n_0 is the minimum sample size required to achieve our goal when σ^2 is known and is often called the optimal sample size.

If σ^2 is unknown and a sample of fixed size n is taken, then the usual confidence interval for η with confidence level $1 - \alpha$ is given by

$$\left(\bar{Y}_n - t_{\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}},\ \bar{Y}_n + t_{\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}}\right),$$

where $t_{\alpha/2}$ is the upper $\alpha/2$ quantile of the Student t distribution with $n-1$ degrees of freedom, and

$$\hat{\sigma_n}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2.$$

It is clear that the width of this confidence interval is $2t_{\alpha/2}\,\hat{\sigma}_n/\sqrt{n}$, a random number. Dantzig (1940) proved that if the variance σ^2 is unknown then a fixed-width $100(1-\alpha)\%$ confidence interval for η can not be constructed by using a fixed sample size procedure. For unknown σ^2 it is therefore necessary to use a sequential procedure to achieve our goal.

Stein (1945) proposed a 2-stage procedure to achieve our goal. He showed that a fixed-width confidence interval for η can be constructed if sampling is performed in two stages, and the size of the second sample is a random variable that depends on the observed values of the first sample.

Anscombe (1952) suggested a pure sequential procedure which estimates σ^2 at each stage $n \geq m$ by $\hat{\sigma_n}^2$, where $m \geq 2$ is the size of the first sample, and stop sampling when, for the first time, $n \geq d^{-2} z_{\alpha/2}^2 \hat{\sigma_n}^2$, i.e. stop sampling at

$$T = \inf\{n \geq m : \ n \geq d^{-2} z_{\alpha/2}^2 \hat{\sigma_n}^2\}.$$

On stopping sampling, the confidence interval for η is then defined as

$$I(T) = (\bar{Y}_T - d,\ \bar{Y}_T + d). \tag{1.2}$$

First order approximations to the expected sample size $E(T)$ and the confidence level of this procedure were given by Chow and Robbins (1965). Second order approximations to the $E(T)$ and the confidence level can be found in Woodroofe (1977, 1982). In fact Woodroofe considered the following stopping time which is a simple modification to Anscombe's procedure

$$T = \inf\{n \geq m :\ n \geq d^{-2} z_{\alpha/2}^2\, l_n \hat{\sigma_n}^2\},$$

where $\{l_n\}$ is a sequence of constants of the form

$$l_n = 1 + \frac{1}{n} l_0 + o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty.$$

The first part of this thesis is devoted to develop some pure sequential procedures for constructing fixed-width simultaneous confidence intervals for multiple comparisons. We shall not consider two-stage procedures because they often require considerably more observations than the corresponding pure sequential procedures, as pointed out by Cox (1952) and Mukhopadhyay (1983). Possibilities of developing other sequential procedures are discussed in Chapter 6, Directions of Future Research.

1.2 Fixed-width simultaneous confidence intervals for multiple comparisons

Suppose that we have k independently and normally distributed populations, $N(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, k$, with unknown means μ_i and a common unknown positive variance σ^2, and that we can sample sequentially from each population. Let $Y_{i1}, Y_{i2}, Y_{i3}, \cdots$ denote the observations from the i^{th} population, $i = 1, 2, \cdots, k$ and $\bar{Y}_i$ is the sample mean of a sample taken from the i^{th} population. Our goal is to construct a set of simultaneous confidence intervals of fixed length $2d$ and of simultaneous confidence level $1 - \alpha$ for each of the following three sets of parameters:

$$\mu_i, \quad i = 1, 2, \cdots, k,$$

$$\mu_i - \mu_1, \quad i = 2, 3, \cdots, k,$$

$$\mu_i - \mu_j, \quad 1 \leq i \neq j \leq k,$$

where $d > 0$ and $\alpha \in (0, 1)$ are two given constants.

For the first set of parameters $\{\mu_i, \ i = 1, 2, \cdots, k\}$, we wish to construct a set of fixed-width $2d$ simultaneous confidence intervals with a simultaneous confidence level $1 - \alpha$ of the form

$$\mu_i \in (\bar{Y}_i - d, \ \bar{Y}_i + d), \quad i = 1, 2, \cdots, k.$$

This set of simultaneous confidence intervals can be used to make inference about each individual μ_i and keep the overall error rate controlled at level α. For instance, we can infer that $\mu_i > 0$ for each i satisfying $\bar{Y}_i > d$, since the confidence interval for μ_i, $(\bar{Y}_i - d, \ \bar{Y}_i + d)$, is entirely to the right of zero. Similarly, we can infer that $\mu_i < 0$ for each i satisfying $\bar{Y}_i < -d$. The probability that all the inferences made are simultaneously correct is no less than the confidence level $1 - \alpha$. The value of d determines the sensitivity of this set of simultaneous confidence intervals in the following sense. For each μ_i satisfying $\mu_i > 2d$ ($\mu_i < -2d$), the correct inference $\mu_i > 0 (\mu_i < 0)$ will be made simultaneously from this set of confidence intervals with probability at least $1 - \alpha$, since the confidence interval for μ_i, $(\bar{Y}_i - d,\ \bar{Y}_i + d)$, will be entirely to the right (left) of zero.

For the second set of parameters $\{\mu_i - \mu_1,\ i = 2, 3, \cdots, k\}$, we construct a set of fixed-width $2d$ simultaneous confidence intervals with simultaneous confidence level $1 - \alpha$ of the form

$$\mu_i - \mu_1 \in (\bar{Y}_i - \bar{Y}_1 - d,\ \ \bar{Y}_i - \bar{Y}_1 + d), \quad i = 2, 3, \cdots, k.$$

Here, the first population, $N(\mu_1, \sigma^2)$, may be regarded as the control, the other $k - 1$ ($k \geq 2$) populations as treatments, and we are interested in comparing all the treatments with the control in order to find out if any of the treatments differ from the control. Inferences about $\mu_i - \mu_1$ can be made from this set of simultaneous confidence intervals. For instance, if $\bar{Y}_i - \bar{Y}_1 > d$ then we can infer that $\mu_i > \mu_1$, since the confidence interval for $\mu_i - \mu_1$, $(\bar{Y}_i - \bar{Y}_1 - d,\ \ \bar{Y}_i - \bar{Y}_1 + d)$, is entirely to the right of zero. Similarly, we can infer that $\mu_i < \mu_1$ for each i satisfying $\bar{Y}_i - \bar{Y}_1 < -d$. The probability that all the inferences made are simultaneously correct is no less than the confidence level $1 - \alpha$. The sensitivity of this set of simultaneous confidence intervals is determined by the value of d as can be seen from follows. For each treatment μ_i satisfying $\mu_i - \mu_1 > 2d (< -2d)$, the correct inference $\mu_i > (<) \mu_1$ will be made from this set of simultaneous confidence intervals with probability at least $1 - \alpha$, since the confidence interval for $\mu_i - \mu_1$, $(\bar{Y}_i - \bar{Y}_1 - d,\ \bar{Y}_i - \bar{Y}_1 + d)$, will be entirely to the right (left) of zero.

Finally, for the third set of parameters $\{\mu_i - \mu_j,\ 1 \leq i \neq j \leq k\}$, we wish to construct a set of fixed-width $2d$ simultaneous confidence intervals with a simultaneous confidence level $1 - \alpha$ of the form

$$\mu_i - \mu_j \in (\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d),\ 1 \leq i \neq j \leq k.$$

In this case we are interested in all-pairwise comparisons of the k populations. Inferences about $\mu_i - \mu_j$ can be made based on this set of simultaneous confidence intervals. For instance, if $\bar{Y}_i - \bar{Y}_j > d$ then we can infer that $\mu_i > \mu_j$, since the confidence interval for $\mu_i - \mu_j$, $(\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d)$, is entirely to the right of zero. The probability that all the inferences made are simultaneously correct is no less than the confidence level $1 - \alpha$. The value of d determines the sensitivity of this set of simultaneous confidence intervals in the following sense. For each pair of treatments i and j such that $\mu_i - \mu_j > 2d$, the correct inference $\mu_i > \mu_j$ will be made from this set of simultaneous confidence intervals with probability at least $1 - \alpha$, since the confidence interval for $\mu_i - \mu_j$, $(\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d)$, will be entirely to the right of zero.

1.3 Probabilities of making correct inferences simultaneously

Consider case one: inference on $\{\mu_i,\ i = 1, 2, \cdots, k\}$. From Section 1.2 it is clear that inferences based on the set of simultaneous confidence intervals

$$(\bar{Y}_i - d,\ \bar{Y}_i + d), \quad i = 1, \cdots, k$$

has the property that *the probability of making the correct inference* $\mu_i > 0 (\mu_i < 0)$ *simultaneously for each* μ_i *satisfying* $\mu_i > 2d (\mu_i < -2d)$ *is at least* $1 - \alpha$. The question is "what is the exact value of this probability?"

The same question stands for the cases two and three.

For case two, we know that inferences about $\{\mu_i - \mu_1,\ i = 2, 3, \cdots, k\}$ based on the simultaneous confidence intervals

$$(\bar{Y}_i - \bar{Y}_1 - d,\ \bar{Y}_i - \bar{Y}_1 + d), \quad i = 2, \cdots, k$$

have the property that *the probability of making the correct inference* $\mu_i > \mu_1 (\mu_i < \mu_1)$ *simultaneously for each* μ_i *satisfying* $\mu_i - \mu_1 > 2d (\mu_i - \mu_1 < -2d)$ *is no less than* $1 - \alpha$. However we wish to know the exact value of this probability.

For case three, we know that inferences about $\{\mu_i - \mu_j,\ 1 \le i \ne j \le k\}$ based on the following set of simultaneous confidence intervals

$$(\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d), \quad 1 \le i \ne j \le k$$

have the property that *the probability of making the correct inference* $\mu_i - \mu_j > 0 (\mu_i - \mu_j < 0)$ *simultaneously for each pair* (i, j) *satisfying* $\mu_i - \mu_j > 2d (< -2d)$ *is no less than* $1 - \alpha$. The main problem is to find the exact value of this probability.

The second part of this thesis is concerned with the answers to these three questions.

1.4 Powers of some multiple comparison tests

The inferences about $\{\mu_i,\ i = 1, 2, \cdots, k\}$ discussed in Section 1.2 based on the set of simultaneous confidence intervals

$$(\bar{Y}_i - d,\ \bar{Y}_i + d), \quad i = 1, \cdots, k$$

are equivalent to the following multiple test approach. The hypotheses that need to be tested are

$$H_{i\,0} : \mu_i = 0 \quad \text{vs} \quad H_{i\,+} : \mu_i > 0, \quad \text{or} \quad H_{i\,-} : \mu_i < 0, \quad 1 \le i \le k;$$

the null hypothesis $H_{i\,0}$ is rejected if and only if $|\bar{Y}_i| > d$, and if $H_{i\,0}$ is rejected then $H_{i\,+} (H_{i\,-})$ is preferred if $\bar{Y}_i > d$ ($\bar{Y}_i < -d$).

Similarly, inferences about $\{\mu_i - \mu_1,\ i = 2, 3, \cdots, k\}$ based on the set of simultaneous confidence intervals

$$(\bar{Y}_i - \bar{Y}_1 - d,\ \bar{Y}_i - \bar{Y}_1 + d), \quad i = 2, \cdots, k$$

are equivalent to the following multiple test approach. The hypotheses that need to be tested are

$$H_{i\,0} : \mu_i - \mu_1 = 0 \quad \text{vs} \quad H_{i\,+} : \mu_i > \mu_1, \quad \text{or} \quad H_{i\,-} : \mu_i < \mu_1, \quad 2 \le i \le k;$$

the null hypothesis $H_{i\,0}$ is rejected if and only if $|\bar{Y}_i - \bar{Y}_1| > d$, and if $H_{i\,0}$ is rejected then $H_{i\,+} (H_{i\,-})$ is preferred if $\bar{Y}_i - \bar{Y}_1 > d$ ($\bar{Y}_i - \bar{Y}_1 < -d$).

Inferences about $\{\mu_i - \mu_j,\ 1 \le i \ne j \le k\}$ based on the set of simultaneous confidence intervals

$$(\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d), \quad 1 \le i \ne j \le k$$

are equivalent to the following multiple test approach. The hypotheses that need to be tested are

$$H_{ij\,0} : \mu_i - \mu_j = 0 \quad \text{vs} \quad H_{ij\,+} : \mu_i > \mu_j, \quad \text{or} \quad H_{ij\,-} : \mu_i < \mu_j, \quad 1 \le i \ne j \le k;$$

the null hypothesis H_{ij0} is rejected if and only if $|\bar{Y}_i - \bar{Y}_j| > d$, and if H_{ij0} is rejected then H_{ij+} (H_{ij-}) is preferred if $\bar{Y}_i - \bar{Y}_j > d$ ($\bar{Y}_i - \bar{Y}_j < -d$). The third and final part of this thesis studies the powers of these three multiple tests.

1.5 On the chapters to follow

In Chapter 3, we propose pure sequential procedures for constructing fixed-width $2d$ and (nominal) simultaneous level $1-\alpha$ confidence intervals for each of the following three sets of parameters:

$$\mu_i, \quad i = 1, 2, \cdots, k,$$

$$\mu_i - \mu_1, \quad i = 2, 3, \cdots, k,$$

$$\mu_i - \mu_j, \quad 1 \leq i \neq j \leq k,$$

where $d > 0$ and $\alpha \in (0, 1)$ are two given constants. Second order approximations to the expected sample sizes and the confidence levels are derived. Exact calculations of the distributions of the sample sizes and the confidence levels are discussed.

The stopping times of all the three procedures are of the form

$$T_G = \inf\{n \geq m : \ n \geq d^{-2}\gamma l_n \hat{\sigma_n}^2\},$$

where $\gamma > 0$ is a constant and $l_n = 1 + \frac{1}{n} l_0 + o(\frac{1}{n})$. In Chapter 2, we derive second order approximations to $E(T_G)$ and $E\left[H\left(\gamma \frac{T_G}{n_0}\right)\right]$ as $n_0 \to \infty$ where $H(\cdot)$ is a given function and $n_0 = d^{-2}\gamma\sigma^2$. These results are used in Chapter 3 and the rest of the thesis.

Chapter 4 is devoted to the study of the exact probabilities of making correct inferences based on the corresponding set of simultaneous confidence intervals of fixed-width $2d$ and level $1-\alpha$.

In Chapter 5, we study the power properties of the multiple tests discussed in Section 1.4.

Finally, in Chapter 6, directions of future research are discussed.

1.6 Notation

Throughout this thesis we adopt the following notation.

1. i.i.d. — independently identically distributed.

2. $Z_1, Z_2, \cdots$ — i.i.d $N(0,1)$ random variables.

3. $\phi(x)$ — pdf of the standard normal distribution.

4. $\Phi(x)$ — cdf of the standard normal distribution.

5. χ^2_ν — chi-square random variable with ν degrees of freedom.

6. $f_\nu(x)$ — pdf of $\sqrt{\chi^2_\nu/\nu}$.

7. $\Gamma(x)$ — gamma function.

8. $\bar{Y}_i$ — the sample mean of a sample taken from the i^{th} population.

9. $\bar{Y}_{in} = (1/n)\sum_{j=1}^{n} Y_{ij}$.

10. $|m|_k^\alpha$ — the upper α point of the distribution of the random variable

$$|M|_k = \max_{1\leq i\leq k} |Z_i|.$$

11. $|m|_{k,\nu}^\alpha$ — the upper α point of the distribution of the random variable

$$|M|_{k,\nu} = \frac{\max_{1\leq i\leq k} |Z_i|}{\sqrt{\chi^2_\nu/\nu}}.$$

12. $|t|_{k-1}^\alpha$ — the upper α point of the distribution of the random variable

$$|T|_{k-1} = \max_{2\leq i\leq k} \frac{|Z_i - Z_1|}{\sqrt{2}}.$$

13. $|t|_{k-1,\nu}^\alpha$ — the upper α point of the distribution of the random variable

$$|T|_{k-1,\nu} = \max_{2\leq i\leq k} \frac{|Z_i - Z_1|}{\sqrt{2}\sqrt{\chi^2_\nu/\nu}}.$$

14. q_k^α — the upper α point of the distribution of the random variable

$$Q_k = \max_{1 \le i \ne j \le k} (Z_i - Z_j).$$

15. $q_{k,\nu}^\alpha$ — the upper α point of the distribution of the random variable

$$Q_{k,\nu} = \max_{1 \le i \ne j \le k} \frac{Z_i - Z_j}{\sqrt{\chi_\nu^2/\nu}}.$$

16. $\hat{\sigma_n}^2 = \frac{1}{k(n-1)} \sum_{i=1}^k \sum_{j=1}^n (Y_{ij} - \bar{Y}_{in})^2$, $n \ge 2$.

17. $l_n = 1 + \frac{1}{n} l_0 + o(\frac{1}{n})$ as $n \to \infty$.

18. $m(\ge 2)$ — the initial sample size.

19. T — a stopping time.

20. $T_G = \inf\{n \ge m : \ n > d^{-2} \gamma l_n \hat{\sigma_n}^2\}$.

21. $E(T)$ — the expected value of the stopping time T.

22. CL — confidence level.

23.

$$\rho = \frac{k+2}{2k} - \frac{1}{k} \sum_{n=1}^\infty \frac{1}{n} E \max\left(0, \chi_{nk}^2 - 2nk\right).$$

24. $\langle x \rangle$ — the largest integer $\le x$.

25. u.c.i.p. — uniformly continuous in probability.

26. u.i. — uniform integrable.

27. I_A — indicator function of the set A.

28. $C[A]$ — number of elements in a finite set A.

29. $\mu = (\mu_1, \mu_2, \cdots, \mu_k) \in R^k$.

Chapter 2

The asymptotic theory of the pure sequential procedure

The stopping times of sampling used for constructing fixed-width simultaneous confidence intervals for the three sets of parameters are of the form

$$T_G = \inf\{n \geq m : \; n > d^{-2}\gamma l_n \hat{\sigma_n}^2\}, \tag{2.1}$$

where $\gamma > 0$ and $m (\geq 2)$ are given constants, $l_n = 1 + \frac{1}{n} l_0 + o\left(\frac{1}{n}\right)$ as $n \to \infty$, and

$$\hat{\sigma_n}^2 = \frac{1}{k(n-1)} \sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{in})^2, \; n \geq m \tag{2.2}$$

where Y_{ij}, $1 \leq i \leq k$, $j = 1, 2, \cdots$, are independent random variables with $Y_{ij} \sim N(\mu_i, \sigma^2)$ and $\bar{Y}_{in} = (1/n)\sum_{j=1}^{n} Y_{ij}$. The corresponding confidence levels are of the form

$$E\left[H\left(\gamma \frac{T_G}{n_0}\right)\right]$$

where $H(\cdot)$ is a given function and $n_0 = d^{-2}\gamma\sigma^2$.

In this chapter we first give the second order approximations of $E(T_G)$ and $E\left[H\left(\gamma \frac{T_G}{n_0}\right)\right]$. The proofs of these results follow the lines of Woodroofe (1982), but we try to give all the details. These results will be applied many times in the subsequent chapters. The exact calculation of the distribution of T_G is also discussed.

2.1 Second order approximation to $E(T_G)$

First we write T_G in a more manageable form. For fixed $i, 1 \leq i \leq k$, define

$$W_{ir} = \frac{[\sum_{j=1}^{r}(Y_{ij} - Y_{i(r+1)})]^2}{r(r+1)\sigma^2}, \qquad r = 1, 2, \cdots.$$

Then we have

Lemma 2.1

I $W_{i1}, W_{i2}, \cdots$ *are i.i.d.* χ_1^2 *random variables for each* i, $1 \leq i \leq k$.

II For all $n \geq 2$, $\sum_{i=1}^{k}\sum_{j=1}^{n}(Y_{ij} - \bar{Y}_{in})^2 = \sigma^2 \sum_{i=1}^{k}\sum_{r=1}^{n-1} W_{ir} = \sigma^2 \sum_{r=1}^{n-1} U_r$, *where* $U_1, U_2, \cdots$ *are i.i.d. chi-square random variables with* k *degrees of freedom.*

III $W_{i1}, \cdots, W_{i\,n-1}$ *are independent of* $\bar{Y}_{in}$ *for all* $n \geq 2$.

Proof: Define random variables $R_{i1}, \cdots, R_{i\,n-1}$ and $Q_{in}, n = 2, 3, \cdots,$ by $R_i^n = (R_{i1}, \cdots, R_{i\,n-1}, Q_{in})' = AZ_i^n$, where A is the following orthogonal matrix

$$\begin{pmatrix} \frac{-1}{\sqrt{2\times 1}} & \frac{1}{\sqrt{2\times 1}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{\sqrt{3\times 2}} & \frac{-1}{\sqrt{3\times 2}} & \frac{2}{\sqrt{3\times 2}} & 0 & \cdots & 0 & 0 \\ \vdots & & & & & & \\ \frac{-1}{\sqrt{n\times(n-1)}} & \frac{-1}{\sqrt{n\times(n-1)}} & \frac{-1}{\sqrt{n\times(n-1)}} & \frac{-1}{\sqrt{n\times(n-1)}} & \cdots & \frac{-1}{\sqrt{n\times(n-1)}} & \frac{n-1}{\sqrt{n\times(n-1)}} \\ \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \end{pmatrix}$$

and $Z_i^n = (Z_{i1}, \cdots, Z_{in})'$ where $Z_{ir} = (Y_{ir} - \mu_i)/\sigma$, $i = 1, 2, \cdots, k$, are i.i.d. $N(0,1)$ random variables. By noting that

$$Cov\,(R_i^n, R_i^n) = ACov\,(Z_i^n, Z_i^n)\,A' = AA' = I$$

since A is orthogonal, then $R_{i1}, \cdots, R_{i\,n-1}, Q_{in}$ are i.i.d. standard normal distribution random variables. It is also easy to check that

$$W_{ir} = \frac{[\sum_{j=1}^{r}(Y_{ij} - Y_{i\,r+1})]^2}{r(r+1)\sigma^2} = R_{ir}^2, \qquad r = 1, \cdots, n-1$$

and so, $W_{i1}, W_{i2}, \ldots, W_{i\,n-1}$ are i.i.d. χ_1^2 random variables. Since $n \geq 2$ is arbitrary, $W_{i1}, W_{i2}, \cdots$ are i.i.d. χ_1^2 random variables. This proves (I).

To prove (II), we note that for fixed $1 \leq i \leq k$

$$\begin{aligned} Q_{in} &= \sqrt{n}\bar{Z}_{in}, \\ \sum_{r=1}^{n}(Z_{ir} - \bar{Z}_{in})^2 &= \sum_{r=1}^{n} Z_{ir}^2 - n\bar{Z}_{in}^2, \\ \sum_{r=1}^{n-1} W_{ir} + Q_{in}^2 &= \sum_{r=1}^{n} Z_{ir}^2, \end{aligned}$$

and

$$\sum_{r=1}^{n}(Z_{ir} - \bar{Z}_{in})^2 = \frac{1}{\sigma^2}\sum_{r=1}^{n}(Y_{ir} - \bar{Y}_{in})^2.$$

It follows therefore

$$\begin{aligned} \sum_{r=1}^{n}(Y_{ir} - \bar{Y}_{in})^2 &= \sigma^2 \sum_{r=1}^{n}(Z_{ir} - \bar{Z}_{in})^2 \\ &= \sigma^2 \left(\sum_{r=1}^{n} Z_{ir}^2 - n\bar{Z}_{in}^2\right) \\ &= \sigma^2 \left(\sum_{r=1}^{n-1} W_{ir} + Q_{in}^2 - Q_{in}^2\right) \\ &= \sigma^2 \sum_{r=1}^{n-1} W_{ir}, \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^{k}\sum_{r=1}^{n}(Y_{ir} - \bar{Y}_{in})^2 &= \sigma^2 \sum_{i=1}^{k}\sum_{r=1}^{n-1} W_{ir} \\ &= \sigma^2 \sum_{r=1}^{n-1} U_r, \end{aligned} \tag{2.3}$$

where $U_r = \sum_{i=1}^{k} W_{ir} \sim \chi_k^2$.

Property (III) is obvious since

$$\bar{Y}_{in} = (\sigma/\sqrt{n})Q_{in} + \mu_i \quad \text{and} \quad W_{ir} = R_{ir}^2, \quad r = 1, \cdots, n-1.$$

This completes the proof of the lemma.

Applying this lemma to write $\hat{\sigma_n}^2 = \sigma^2 \bar{U}_{n-1}/k$, where $\bar{U}_n = (1/n)\sum_{i=1}^n U_i$, we have

$$\begin{aligned} T_G &= \inf\{n \geq m : \quad n > d^{-2}\gamma l_n \sigma^2 \bar{U}_{n-1}/k\} \\ &= \inf\Big\{n \geq m : \quad \frac{kn}{l_n}\bar{U}_{n-1}^{-1} > d^{-2}\gamma\sigma^2\Big\}. \end{aligned} \tag{2.4}$$

Hence, T_G assumes the form $T_G = t + 1$, where

$$t = \inf\{n \geq m - 1 : Z_n > n_0\}, \tag{2.5}$$

with

$$n_0 = d^{-2}\gamma\sigma^2, \quad Z_n = \left(\frac{n+1}{n l_{n+1}}\right) kn\bar{U}_n^{-1}.$$

Since the distributions of U_r are independent of μ_i and σ^2 , then the distributions of t and T_G depend only on n_0, which in turn depends on the unknown variance σ^2. It is also noteworthy that t and T_G depend on d and σ^2 only through σ/d.

Note that

$$\left(\frac{n+1}{n l_{n+1}}\right) = 1 + \frac{\Delta_n}{n},$$

where $\Delta_n \to 1 - l_0$ as $n \to \infty$. Using Taylor expansion for $1/x$ about k, we have

$$\begin{aligned} Z_n &= \left(1 + \frac{\Delta_n}{n}\right) kn\bar{U}_n^{-1} \\ &= \left(1 + \frac{\Delta_n}{n}\right) kn\Big\{\frac{1}{k} - \frac{1}{k^2}(\bar{U}_n - k) + (L_n)^{-3}(\bar{U}_n - k)^2\Big\} \\ &= n\left(2 - \frac{\bar{U}_n}{k}\right) + kn\left(1 + \frac{\Delta_n}{n}\right)(L_n)^{-3}(\bar{U}_n - k)^2 + \Delta_n\left(2 - \frac{\bar{U}_n}{k}\right) \\ &= S_n + \xi_n, \end{aligned}$$

where L_n is an intermediate point between $\bar{U}_n$ and k,

$$S_n = X_1 + X_2 + \cdots + X_n, \ n \geq 1,$$

with $X_i = 2 - U_i/k$ for $i \geq 1$, and

$$\xi_n = \left(\frac{1}{L_n}\right)^3 kn(\bar{U}_n - k)^2\left(1 + \frac{\Delta_n}{n}\right) + \Delta_n\bar{X}_n, \tag{2.6}$$

with $\bar{X}_n = 2 - \bar{U}_n/k$.

Since $U_1, U_2, \cdots$ are i.i.d. random variables, $\{S_n, n \geq 1\}$ is a random walk with $E(X_1) = 1$, $V(X_1) = 2/k$. Therefore, the stopping time t defined in (2.5) can be written as

$$t = \inf\{n \geq m - 1 : \ S_n + \xi_n > n_0\}.$$

We intend to apply Theorem A.3 in the appendix to get an asymptotic expansion of $E(t)$. Let $A_n = \Omega$ where Ω denotes the sample space, $h_n = 0$, $n \geq 1$, and $V_n = \xi_n$, we need to check conditions (A.3 - A.9) are satisfied.

Lemma 2.2 *For fixed* $1 \leq i \leq k$, *and* $n \geq 2$, $\{T_G = n\}$ *and* $\bar{Y}_{i\,n}$ *are independent.*

Proof: It follows from (2.4) that

$$T_G = \inf\left\{n \geq m : \quad n > d^{-2}\gamma l_n \sigma^2 \bar{U}_{n-1} /k\right\}$$

and so $\{T_G = n\}$ depends only on $W_{i1}, W_{i2}, \cdots, W_{i\,n-1}$, which, by part (III) of Lemma 2.1, are independent of $\bar{Y}_{i\,n}$. This finishes the proof.

Lemma 2.3 $\{\xi_n, n \geq 1\}$ *is slowly changing (see the appendix for definition).*

Proof: It suffices to show that conditions (A.1) and (A.2) in the appendix hold. For (A.2) we use Lemma A.4 to show that $\xi_n/n \to 0$ w.p.1 as $n \to \infty$. Note that

$$\frac{\xi_n}{n} = \left(\frac{1}{L_n}\right)^3 k(\bar{U}_n - k)^2 \left(1 + \frac{\Delta_n}{n}\right) + \frac{1}{n}\Delta_n \bar{X}_n,$$

and

$$\bar{U}_n \to k \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

$$\frac{\Delta_n \bar{X}_n}{n} \to 0 \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

$$\left(1 + \frac{\Delta_n}{n}\right) \to 1 \quad \text{w.p.1} \quad \text{as } n \to \infty,$$

and $L_n \to k$ w.p.1. since L_n is an intermediate point between $\bar{U}_n$ and k. Hence $\xi_n/n \to 0$ w.p.1 as required.

To prove $\{\xi_n\}$ is u.c.i.p., note that

$$\xi_n = \left(\frac{1}{L_n}\right)^3 2k^2 \left(\left(\sum_{i=1}^{n} U_i - kn\right)/\sqrt{2kn}\right)^2 \left(1 + \frac{\Delta_n}{n}\right) + \Delta_n \bar{X}_n.$$

Now by Lemma A.3, $\{\left(\sum_{i=1}^{n} U_i - kn\right)/\sqrt{2kn}, n \geq 1\}$ is u.c.i.p., and by Lemma A.2 we have

$$\left\{\left(\frac{1}{L_n}\right)^3 \left(1 + \frac{\Delta_n}{n}\right), n \geq 1\right\} \text{ is u.c.i.p., since } \left(\frac{1}{L_n}\right)^3 \left(1 + \frac{\Delta_n}{n}\right) \to \frac{1}{k^3} \text{ w.p.1}$$

$$\{\Delta_n \bar{X}_n, n \geq 1\} \quad \text{is u.c.i.p., since} \Delta_n \bar{X}_n \to 1 - l_0 \text{ w.p.1.}$$

It therefore follows from Lemma A.1 that $\{\xi_n, n \geq 1\}$ is u.c.i.p.. This finishes the proof.

Lemma 2.4 $\xi_n \xrightarrow{D} (2/k)\, \chi_1^2 + (1 - l_0)$ *as* $n \to \infty$.

Proof: Note that

$$\xi_n = \left(\frac{1}{L_n}\right)^3 2k^2 \left(\left(\sum_{i=1}^{n} U_i - kn\right)/\sqrt{2kn}\right)^2 \left(1 + \frac{\Delta_n}{n}\right) + \Delta_n \bar{X}_n.$$

and that

$$\left(\left(\sum_{i=1}^{n} U_i - kn\right)/\sqrt{2kn}\right)^2 \xrightarrow{D} \chi_1^2 \quad \text{as } n \to \infty,$$

$$\Delta_n \bar{X}_n \to (1 - l_0) \quad \text{w.p.1 as } n \to \infty,$$

$$\left(\frac{1}{L_n}\right)^3 \left(1 + \frac{\Delta_n}{n}\right) \to \frac{1}{k^3} \text{ w.p.1 as } n \to \infty,$$

from which the lemma follows.

Lemma 2.5 *Let $F_n(\cdot)$ denote the cumulative distribution function of χ_n^2 and*

$$C_n^* = \frac{1}{2^{n/2}\,\Gamma(1 + n/2)}, \qquad C_n = \frac{1}{2^{n/2}\,\Gamma(1 + n/2)} \left[n \left(\frac{\langle n/k \rangle + 1}{l_{\langle n/k \rangle + 1}}\right)\right]^{n/2}, \quad n \geq 1.$$

Then we have the following results:

I $\quad F_n(x) \sim C_n^* x^{n/2}$ *as $x \to 0$, for all $n \geq 1$,* *II* *there exist a constant $b > 1$ such that $C_n \leq b^n n^{n/2}$ for $n \geq 1$,*

III $P\{t = m-1\} \sim C_{k(m-1)}\ n_0^{-k(m-1)/2}$ *as $n_0 \to \infty$, where t is defined in (2.5) and $m \geq 2$.*

Proof: For (I) it suffices to show that, $\lim_{x\to 0} F_n(x)/(C_n^* x^{n/2}) = 1$. Note that

$$F_n(x) = \int_0^x \frac{1}{2^{n/2}\,\Gamma(n/2)} y^{n/2\,-1}\, e^{-y/2}\ dy.$$

By using L'Hospital's rule, it is easy to show that $\lim_{x\to 0} F_n(x)/(C_n^* x^{n/2}) = 1$, as required.

To prove (II), we use Stirling's formula (see Handbook of Mathematical Function ,1965)

$$\Gamma(x+1) = x^{x+1/2}\, e^{-x+\theta/12x}\ \sqrt{2\pi}, \qquad x > 0,\ 0 < \theta < 1.$$

Then

$$\begin{aligned}
C_n &= \frac{\left(n\frac{(n/k)+1}{l_{(n/k)+1}}\right)^{n/2}}{2^{n/2}\left(\frac{n}{2}\right)^{(n+1)/2} e^{-n/2}\, e^{\theta/6n}\sqrt{2\pi}} \\
&< (n+1)^{n/2}\, 2^{n/2}\, e^{n/2}, \quad \forall n > n_1 \\
&= \left(\frac{n+1}{n}\right)^{n/2} (2en)^{n/2} \\
&< b^n n^{n/2},
\end{aligned}$$

where $n_1 \in N$ is such that $l_{(n/k)} > 1/2\ \forall n > n_1$, and $b \geq 4e \geq 2e(n+1)/n > 1$. This finishes the proof of (II).

To prove (III), we have

$$\begin{aligned}
&P\{t = m-1\} \\
&= P\{Z_{m-1} > n_0\} \\
&= P\Big\{\bar{U}_{m-1}^{-1} > \frac{n_0 l_m}{km}\Big\} \\
&= P\Big\{(m-1)\bar{U}_{m-1} < (m-1)\frac{km}{l_m n_0}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= P\Big\{\chi^2_{k(m-1)} < (m-1)\frac{km}{l_m n_0}\Big\} \\
&\sim C^*_{k(m-1)} \left(k(m-1)\frac{m}{l_m n_0}\right)^{\frac{k(m-1)}{2}} \qquad \text{as } n_0 \to \infty \quad \text{(by (I))} \\
&= \frac{1}{2^{\frac{k(m-1)}{2}}\Gamma\left(\frac{k(m-1)}{2}+1\right)} \left(k(m-1)\left(\frac{m}{l_m}\right)\right)^{\frac{k(m-1)}{2}} n_0^{-\frac{k(m-1)}{2}} \\
&= C_{k(m-1)}\, n_0^{-\frac{k(m-1)}{2}}
\end{aligned}
$$

as required. The proof is thus completed.

Lemma 2.6 *For $m > 1+2/k$, $0 < \varepsilon < 1$ and $l_n = 1 + l_0/n + o(1/n)$ as $n \to \infty$,*

$$
P\{t \le \varepsilon N_{n_0}\} = o(1/N_{n_0}) \quad \text{as } n_0 \to \infty ,
$$

where $N_{n_0} = \langle n_0 \rangle$.

Proof: Noting that for sufficiently large n_0, we have $n_0^{3/4} < \varepsilon N_{n_0}$ and $P\{t \le \varepsilon N_{n_0}\}$ can be written as

$$
P\{t \le \varepsilon N_{n_0}\} = P\{t \le n\} + P\Big\{n < t \le n_0^{3/4}\Big\} + P\Big\{n_0^{3/4} < t \le \varepsilon N_{n_0}\Big\}.
$$

For fixed n we have

$$
\begin{aligned}
P\{t \le n\} &= \sum_{j=m-1}^{n} P\{t = j\} \\
&= P\{t = m-1\} + P\{t = m\} + \cdots + P\{t = n\} \\
&\sim C_{k(m-1)}\, n_0^{-k(m-1)/2} \qquad \text{as } n_0 \to \infty,
\end{aligned}
$$

since for $m \le j \le n$ it can be shown in a way similar to Lemma 2.5 part (III) that

$$
P\{t = j\} \le P\{Z_j > n_0\} \le C_{kj} n_0^{-kj/2}
$$

and $C_{kj} n_0^{-kj/2} = o\left(C_{k(m-1)}\, n_0^{-k(m-1)/2}\right)$ as $n_0 \to \infty$.

Now, by part (II) of Lemma 2.5, we have

$$\begin{aligned}
P\{n < t \leq n_0^{3/4}\} &= \sum_{j=n+1}^{\langle n_0^{3/4}\rangle} P\{t=j\} \\
&\leq \sum_{j=n+1}^{\langle n_0^{3/4}\rangle} P\{Z_j > n_0\} \\
&\leq \sum_{j=n+1}^{\langle n_0^{3/4}\rangle} C_{kj} n_0^{-jk/2} \\
&\leq \sum_{j=n+1}^{\langle n_0^{3/4}\rangle} (k^{1/2}b)^{jk} \left(\frac{j}{n_0}\right)^{jk/2} \\
&\leq \sum_{j=n+1}^{\infty} (k^{1/2}b)^{jk} \left(\frac{n_0^{3/4}}{n_0}\right)^{jk/2} \\
&= \sum_{j=n+1}^{\infty} (k^{1/2}b)^{jk} n_0^{-jk/8} .
\end{aligned}$$

This last summation is of a smaller order of magnitude than $n_0^{-k(m-1)/2}$ as $n_0 \to \infty$ for sufficiently large n. For this, it suffices to show that

$$\lim_{n_0\to\infty} \sum_{j=n+1}^{\infty} (k^{1/2}b)^{jk} n_0^{k(m-1-j/4)/2} = 0.$$

Note that

$$\begin{aligned}
\sum_{j=n+1}^{\infty} (k^{1/2}b)^{jk} n_0^{k(m-1-j/4)/2} &= \sum_{j=n+1}^{\infty} (k^{1/2}b)^{jk} n_0^{k(-j+m_0)/8} \qquad \text{where } m_0 = 4(m-1) \\
&= \sum_{j=n+1}^{\infty} \frac{(k^{1/2}b)^{k(j-m_0-1)}}{n_0^{k(j-m_0-1)/8}} \left(\frac{(k^{1/2}b)^{m_0+1}}{n_0^{1/8}}\right)^k \\
&= \left(\frac{(k^{1/2}b)^{m_0+1}}{n_0^{1/8}}\right)^k \sum_{j=n+1}^{\infty} \left(\frac{k^{1/2}b}{n_0^{1/8}}\right)^{k(j-m_0-1)} .
\end{aligned}$$

Now, $n > 4(m-1)$ is sufficient for this last expression to approach zero as $n_0 \to \infty$.

Finally, $n_0^{3/4} < t \leq \varepsilon N_{n_0}$ implies that $Z_j > n_0$, i.e. $\bar{U}_j < k(j+1)/l_{j+1}n_0$, for some $j \in \left(n_0^{\frac{3}{4}},\ \varepsilon N_{n_0}\right]$. For $j \in \left(n_0^{\frac{3}{4}},\ \varepsilon N_{n_0}\right]$ and sufficiently large n_0, we have

$$\begin{aligned}
\frac{j+1}{n_0 l_{j+1}} &= \frac{j+1}{n_0} \times \frac{1}{1+(l_0/(j+1)+o(1/(j+1)))} \\
&= \frac{j+1}{n_0}(1-l_0/(j+1)+o(1/(j+1))) \\
&= \frac{1}{n_0}(j+1-l_0+o(1)) \\
&\leq \frac{1}{n_0}(\varepsilon N_{n_0}+1-l_0+o(1)) \\
&\leq \varepsilon+\frac{1}{n_0}(1-l_0+o(1)) \\
&\leq \delta_0 < 1,
\end{aligned}$$

and

$$\bar{U}_j < k\delta_0 \text{ implies } \bar{U}_j - k < k(\delta_0 - 1) \equiv -\delta < 0,$$

where $\delta_0 \in (0,1)$ is a constant. Thus for sufficiently large n_0

$$\begin{aligned}
P\Big\{n_0^{3/4} < t \leq \varepsilon N_{n_0}\Big\} &\leq P\Big\{\bar{U}_j - k < -\delta,\ \exists j \in \Big(n_0^{3/4},\ \varepsilon N_{n_0}\Big]\Big\} \\
&\leq P\Big\{\max_{j \leq \langle \varepsilon N_{n_0}\rangle} j|\bar{U}_j - k| > \delta n_0^{3/4}\Big\}.
\end{aligned}$$

Now, by Theorems A.2 and A.1 in the appendix, we have

$$\begin{aligned}
&P\Big\{\max_{j \leq \langle \varepsilon N_{n_0}\rangle} j|\bar{U}_j - k| > \delta n_0^{3/4}\Big\} \\
&\leq \frac{1}{\left(\delta n_0^{3/4}\right)^{\alpha}} E\left|\sum_{i=1}^{\langle \varepsilon N_{n_0}\rangle} U_i - k[\varepsilon N_{n_0}]\right|^{\alpha} \\
&= \frac{(2k)^{\alpha/2}\langle \varepsilon N_{n_0}\rangle^{\alpha/2}}{\left(\delta n_0^{3/4}\right)^{\alpha}} E\left|\frac{\sum_{i=1}^{\langle \varepsilon N_{n_0}\rangle} U_i - k\langle \varepsilon N_{n_0}\rangle}{\sqrt{2k\langle \varepsilon N_{n_0}\rangle}}\right|^{\alpha} \\
&< C n_0^{-\frac{\alpha}{4}}, \quad \forall \alpha \geq 2,
\end{aligned}$$

and so $P\Big\{n_0^{3/4} < t \leq \varepsilon N_{n_0}\Big\} = o\left(n_0^{-k(m-1)/2}\right)$ by choosing α to satisfy $k(m-1)/2 < \alpha/4$.

Combining the above three cases, we have in fact proved that

$$P\{t \leq \varepsilon N_{n_0}\} \sim C_{k(m-1)}\ n_0^{-k(m-1)/2} \quad \text{as } n_0 \to \infty.$$

By noting that

$$C_{k(m-1)}\, n_0^{-k(m-1)/2} = o(1/n_0) \quad \text{as } n_0 \to \infty$$

when $m > 1 + 2/k$, therefore

$$P\{t \le \varepsilon N_{n_0}\} = o(1/N_{n_0}) \quad \text{as } n_0 \to \infty.$$

This completes the proof.

Corollary 2.1 *For* $m > 1 + 2/k$

$$P\{t \le n_0/2\} \sim C_{k(m-1)}\, n_0^{-k(m-1)/2} \qquad \textit{as } n_0 \to \infty.$$

Lemma 2.7 *Let* $Y_n = \max_{0\le s\le n} (n+s)\left(\bar{U}_{n+s} - k\right)^2, n \ge 1$, *then* $\{Y_n^2, n \ge 1\}$ *is uniform integrable. (See the appendix for definition).*

Proof: Note that

$$\begin{aligned}
&P\Big\{\max_{0\le s\le n} (n+s)\left(\bar{U}_{n+s} - k\right)^2 > y\Big\} \\
&\le P\Big\{\max_{0\le s\le 2n} s|\bar{U}_s - k| > \sqrt{ny}\Big\} \\
&\le \left(\frac{1}{ny}\right)^{\alpha/2} E|2n\left(\bar{U}_{2n} - k\right)|^{\alpha} \quad \text{for } \alpha > 1 \\
&= (4k)^{\alpha/2}\, y^{-\alpha/2}\, E\left|\frac{\sum_{i=1}^{2n} U_i - 2nk}{\sqrt{4nk}}\right|^{\alpha},
\end{aligned}$$

where the second inequality follows from the Theorem A.2. Applying Theorem A.1 we have

$$\sup_{n\ge 1} E\left|\frac{\sum_{i=1}^{2n} U_i - 2nk}{\sqrt{4nk}}\right|^{\alpha} \le C_0, \quad \alpha > 2,$$

and so

$$P\Big\{\max_{0\le s\le n} (n+s)\left(\bar{U}_{n+s} - k\right)^2 > y\Big\} < Cy^{-\alpha/2}, \quad \alpha > 2,\ n \ge 1.$$

The lemma now follows from Lemma A.6 in the appendix by choosing $\alpha = 6$ say.

Lemma 2.8 *For given $r > 0$,*

$$\left\{ \max_{0\leq s\leq n} \left(\frac{1}{L_{n+s}} \right)^r ,\ n \geq 1 \right\} \quad \textit{is u.i..}$$

Proof: Again we apply Lemma A.6 in the appendix to prove the lemma. By noting that L_n is an intermediate point between $\bar{U}_n$ and k, we have

$$\begin{aligned}
&P\left\{ \max_{0\leq s\leq n} \left(\frac{1}{L_{n+s}} \right)^r > x \right\} \\
&\leq P\left\{ \max_{0\leq s\leq n} \left(\frac{1}{L_{n+s}} \right) > x^{1/r} \right\} \\
&\leq P\left\{ \min_{0\leq s\leq n} L_{n+s} < \frac{1}{z} \right\} \quad \text{where } x^{1/r} = z \\
&\leq P\left\{ \min_{0\leq s\leq n} \bar{U}_{n+s} < \frac{1}{z} \right\} \qquad \text{(for large } x \text{ so that } 1/z < k) \\
&\leq \sum_{s=n}^{2n} P\left\{ \bar{U}_s < \frac{1}{z} \right\}.
\end{aligned}$$

Now,

$$\begin{aligned}
P\left\{ \bar{U}_s < \frac{1}{z} \right\} &= P\left\{ \sum_{j=1}^{s} U_j < \frac{s}{z} \right\} \\
&= P\{ \chi^2_{sk} < \frac{s}{z} \} \\
&= \int_0^{\frac{s}{z}} \frac{1}{2^{\frac{sk}{2}} \Gamma\left(\frac{sk}{2}\right)} y^{\frac{sk}{2}-1} e^{-\frac{y}{2}}\, dy \\
&\leq \frac{1}{2^{\frac{sk}{2}} \Gamma\left(\frac{sk}{2}\right)} \int_0^{\frac{s}{z}} y^{\frac{sk}{2}-1}\, dy \\
&= \frac{1}{2^{\frac{sk}{2}} \Gamma\left(\frac{sk}{2}\right) \frac{sk}{2}} s^{\frac{sk}{2}} \frac{1}{z^{\frac{sk}{2}}} \\
&\leq d^{sk/2} \left(\frac{s}{sk+1} \right)^{\frac{sk}{2}} \frac{1}{z^{\frac{sk}{2}}} \qquad (2.7) \\
&\leq \left(\frac{d}{z} \right)^{sk/2},
\end{aligned}$$

where d is some constant and the inequality (2.7) follows from part (II) of Lemma 2.5. Consequently

$$\sum_{s=n}^{2n} P\{ \bar{U}_s < \frac{1}{z} \} \leq \sum_{i=0}^{n} \left(\frac{d}{z} \right)^{\frac{k(i+n)}{2}}$$

$$= \left(\frac{d}{z}\right)^{\frac{kn}{2}} \sum_{i=0}^{n} \left(\frac{d}{z}\right)^{\frac{ki}{2}}.$$

Since d is a constant, we can choose x sufficiently large such that $d/z < 1$. Then $\sum_{i=0}^{n} (d/z)^{\frac{ki}{2}} \leq \sum_{i=0}^{\infty} (d/z)^{\frac{ki}{2}} \leq C < \infty$, and then

$$P\left\{ \max_{0 \leq s \leq n} \left(\frac{1}{L_{n+s}}\right)^{r} > x \right\} \leq C(d^{-r} x)^{-\frac{2r+1}{2r}}, \quad \forall\ kn \geq 2r+1.$$

Now, $C(d^{-r} x)^{-(2r+1)/2r}$ is integrable with respect to the Lebesgue measure over $(1, \infty)$. Therefore $\{\max_{0 \leq s \leq n} (1/L_{n+s})^{r}, n \geq (2r+1)/k\}$ is u.i. by Lemma A.6. Also it is easy to show that $\max_{0 \leq s \leq n} (1/L_{n+s})^{r}, \forall\ 1 \leq n \leq 2r/k$, is integrable. So $\{\max_{0 \leq s \leq n} (1/L_{n+s})^{r}, n \geq 1\}$ is u.i..

Lemma 2.9

$$\left\{ \max_{0 \leq s \leq n} |\xi_{n+s}|,\ n \geq 1 \right\} \ \textit{is u.i..}$$

Proof: Since

$$\begin{aligned} &\max_{0 \leq s \leq n} |\xi_{n+s}| \\ &\leq \max_{0 \leq s \leq n} \left(\frac{1}{L_{n+s}}\right)^{3} k(n+s)\left(\bar{U}_{n+s} - k\right)^{2}\left(1 + \frac{\Delta_{n+s}}{n+s}\right) \\ &+ \max_{0 \leq s \leq n} |\bar{X}_{n+s} \Delta_{n+s}|, \end{aligned}$$

it suffices to show that both

$$\left\{ \max_{0 \leq s \leq n} \left(\frac{1}{L_{n+s}}\right)^{3} k(n+s)\left(\bar{U}_{n+s} - k\right)^{2}\left(1 + \frac{\Delta_{n+s}}{n+s}\right),\ n \geq 1 \right\},$$

and $\{\max_{0 \leq s \leq n} |\bar{X}_{n+s} \Delta_{n+s}|,\ n \geq 1\}$ are u.i.. The uniform integrability of the first sequence of random variables follows directly from Lemmas 2.7, 2.8 and part (II) of Lemma A.7. To show the uniform integrability of the second sequence of random variables, it suffices to show that $\{\max_{0 \leq s \leq n} |\bar{X}_{n+s}|,\ n \geq 1\}$ is u.i. since

$$\max_{0 \leq s \leq n} |\bar{X}_{n+s} \Delta_{n+s}| \leq C_1 \max_{0 \leq s \leq n} |\bar{X}_{n+s}|.$$

By noting that

$$\begin{aligned}\max_{0\le s\le n} |\bar{X}_{n+s}| &= \max_{0\le s\le n} |\bar{X}_{n+s}+1-1| \\ &\le \max_{0\le s\le n} |\bar{X}_{n+s}-1|+1,\end{aligned}$$

it suffices to show that $\{\max_{0\le s\le n} |\bar{X}_{n+s}-1|\}$ is u.i.. This follows from Lemma A.6 by noting that

$$\begin{aligned}&P\Big\{\max_{0\le s\le n} |\bar{X}_{n+s}-1|>x\Big\} \\ &\le P\Big\{\max_{0\le s\le n} (n+s)|\bar{X}_{n+s}-1|>nx\Big\} \\ &\le P\Big\{\max_{0\le s\le 2n} s|\bar{X}_{s}-1|>nx\Big\} \\ &\le \left(\frac{1}{nx}\right)^{\alpha} E\left(2n|\bar{X}_{2n}-1|\right)^{\alpha} \le Mx^{-\alpha},\end{aligned}$$

where $\alpha>2$ and $M>0$ are constants.

Theorem 2.1 *For $m>1+2/k,\ k\ge 1$, then*

$$E(T_G)=n_0+\rho+l_0-\frac{2}{k}+o(1)\quad as\ n_0\to\infty,$$

where $n_0=d^{-2}\gamma\sigma^2$ and

$$\rho=\frac{k+2}{2k}-\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{n}E\max\left(0,\chi^2_{nk}-2nk\right).$$

Proof: Since $T_G=t+1$, it suffices to show that

$$E(t)=n_0+\rho+l_0-1-\frac{2}{k}+o(1)\ \text{ as } n_0\to\infty.$$

For this, we use Theorem A.3 in the appendix. We show that all the conditions (A.3-A.9) hold. Let $A_n=\Omega$, $h_n=0$, and $V_n=\xi_n$, $n\ge 1$. Then (A.3-A.5) are obviously true. Now, (A.8) is true since

$$\xi_n \xrightarrow{D} \frac{2}{k}\chi_1^2+(1-l_0)\quad \text{as } n\to\infty\quad \text{(by Lemma 2.4)}.$$

(A.6) is true by Lemma 2.9, and (A.9) is true by Lemma 2.6. Next, we show that (A.7) holds, i.e.

$$\sum_{n=1}^{\infty} P\{\xi_n \le -n\varepsilon\} < \infty \text{ for some } 0 < \varepsilon < 1. \tag{2.8}$$

By noting that $\bar{X}_n = 2 - \bar{U}_n/k$, we have

$$\begin{aligned}
&P\{\xi_n \le -n\varepsilon\} \\
&= P\left\{\left(\frac{1}{L_n}\right)^3 kn(\bar{U}_n - k)^2\left(1 + \frac{\Delta_n}{n}\right) + \Delta_n\bar{X}_n \le -n\varepsilon\right\} \\
&\le P\{\Delta_n\bar{X}_n \le -n\varepsilon\} \\
&= P\left\{n\varepsilon + 2\Delta_n \le \Delta_n\frac{\bar{U}_n}{k}\right\}.
\end{aligned}$$

Since $\lim_{n\to\infty} \Delta_n = 1 - l_0$, there are constants C_0 and C_1 such that $-C_0 < 2\Delta_n$, $n\varepsilon - C_0 > 0$, and $|\Delta_n| < C_1$. Therefore for sufficiently large n

$$\begin{aligned}
&P\left\{n\varepsilon + 2\Delta_n \le \Delta_n\frac{\bar{U}_n}{k}\right\} \\
&\le P\{k(n\varepsilon - C_0) < \Delta_n\bar{U}_n\} \\
&\le P\{k(n\varepsilon - C_0) < |\Delta_n|\bar{U}_n\} \\
&= P\left\{\frac{k(n\varepsilon - C_0)}{|\Delta_n|} < \bar{U}_n\right\} \\
&\le P\left\{\frac{k(n\varepsilon - C_0)}{C_1} < \bar{U}_n\right\}.
\end{aligned}$$

Now,

$$\begin{aligned}
P\left\{\bar{U}_n > \frac{k(n\varepsilon - C_0)}{C_1}\right\} &\le knP\left\{\chi_1^2 > \frac{k(n\varepsilon - C_0)}{C_1}\right\} \\
&= knP\left\{Z > \sqrt{\frac{k(n\varepsilon - C_0)}{C_1}}\right\} \\
&\le \frac{kn\sqrt{C_1}}{\sqrt{k(n\varepsilon - C_0)}} e^{-k(n\varepsilon - C_0)/2C_1} \\
&= A\frac{kn}{\sqrt{k(n\varepsilon - C_0)}} e^{-kn\varepsilon_1},
\end{aligned}$$

where Z denotes a standard normal random variable, $A = \sqrt{C_1}e^{kC_0/2C_1}$, $\varepsilon_1 = \varepsilon/2C_1$, and the second inequality follows from the well known inequality

$$\int_x^\infty e^{-u^2/2}\ du < \frac{1}{x}e^{-x^2/2}, \qquad \text{for all } x > 0.$$

Therefore (2.8) holds. By Lemma 2.3, $\xi_n, n \geq 1$ are slowly changing. We have therefore shown all the assumptions of Theorem A.3 hold, and so

$$E(T_G) = n_0 + \rho + l_0 - \frac{2}{k} + o(1) \ \text{ as } n_0 \to \infty.$$

2.2 Second order approximation to $E\left[H\left(\gamma\frac{T_G}{n_0}\right)\right]$

In this section we derive a second order expansion of $E\left[H\left(\gamma\frac{T_G}{n_0}\right)\right]$. First, we establish some properties of t which will be used later.

Lemma 2.10 $\int_{t>2n_0} t^2\, dP \to 0$ *as* $n_0 \to \infty$.

Proof: Denote $Y = t^2$, $k_{n_0} = 2n_0^2$, then

$$\begin{aligned}
\int_{t>2n_0} t^2\, dP &= \int_{t^2>4n_0^2} t^2\, dP \\
&= \int_{Y>2k_{n_0}} Y\, dP \\
&\le 2\int_{Y>2k_{n_0}} (Y-k_{n_0})\, dP \qquad (\text{since } Y>2k_{n_0} \Rightarrow Y<2(Y-k_{n_0})) \\
&\le 2\int_{Y>k_{n_0}} (Y-k_{n_0})\, dP \qquad (\text{since } \{Y>2k_{n_0}\}\subset\{Y>k_{n_0}\}) \\
&= 2\sum_{n=\langle k_{n_0}+1\rangle}^{\infty} \int_{\{Y=n\}} (Y-k_{n_0})\, dP \\
&= 2\sum_{n=\langle k_{n_0}+1\rangle}^{\infty} (n-k_{n_0})P\{Y=n\} \\
&= 2\sum_{n=\langle k_{n_0}\rangle}^{\infty} P\{Y>n\} \\
&= 2\sum_{n=\langle k_{n_0}\rangle}^{\infty} P\{t>\sqrt{n}\} \\
&\le 2\sum_{n=\langle k_{n_0}\rangle}^{\infty} P\{t>\langle\sqrt{n}\rangle\}.
\end{aligned}$$

Let $0<\varepsilon<1-1/\sqrt{2}$, $\delta>0$ be so small that $\varepsilon+\delta<1-1/\sqrt{2}$, and $H_{n_0} = n_0/(1-(\varepsilon+\delta))$. Since

$$k_{n_0} = 2n_0^2 > \left(\frac{n_0}{1-(\varepsilon+\delta)}\right)^2 = H_{n_0}^2,$$

we have

$$\begin{aligned}
\sum_{n=\langle k_{n_0}\rangle}^{\infty} P\{t>\langle\sqrt{n}\rangle\} &\le \sum_{n=\langle H_{n_0}^2\rangle}^{\infty} P\{t>\langle\sqrt{n}\rangle\} \\
&\le \sum_{r=\langle H_{n_0}\rangle}^{\infty} 3rP\{t>r\}.
\end{aligned}$$

Note that

$$t > r \ \text{ implies } S_r + \xi_r \le n_0,$$

and so

$$P\{t > r\} \le P\{S_r + \xi_r \le n_0\}.$$

For $r \ge \langle H_{n_0} \rangle$, we have $r \ge n_0/(1-(\varepsilon+\delta))$ and so $n_0 - r \le -r(\varepsilon+\delta)$. Consequently we have for $r \ge \langle H_{n_0} \rangle$

$$\begin{aligned} P\{t > r\} &\le P\{S_r + \xi_r - r \le n_0 - r\}, \\ &\le P\{S_r + \xi_r - r \le -r\varepsilon - r\delta\} \\ &\le P\{S_r - r + r\delta \le 0\} + P\{\xi_r \le -r\varepsilon\}, \end{aligned}$$

which is independent of $n_0 \ge 0$. From the proof of Theorem 2.1, we have $\sum_{r=1}^{\infty} rP\{\xi_r \le -r\varepsilon\} < \infty$. Also note

$$\begin{aligned} rP\{S_r - r + r\delta \le 0\} &= rP\Big\{2r - \frac{1}{k}\sum_{i=1}^{r} U_i - r \le -r\delta\Big\} \\ &= rP\Big\{\frac{1}{k}\sum_{i=1}^{r} U_i - r \ge r\delta\Big\} \\ &\le rP\Big\{\Big|\frac{1}{k}\sum_{i=1}^{r} U_i - r\Big| \ge r\delta\Big\} \\ &\le \frac{1}{r^5\delta^6} E\Big|\frac{1}{k}\sum_{i=1}^{r} U_i - r\Big|^6 \qquad \text{(by Markov's inequality)} \\ &= \frac{2^3}{r^2\delta^6 k^3} E\Big|\frac{\sum_{i=1}^{r} U_i - kr}{\sqrt{2kr}}\Big|^6 \\ &\le \frac{C}{r^2}, \qquad \text{(by Theorem A.1)} \end{aligned}$$

and so $\sum_{r=1}^{\infty} rP\{S_r - r + r\delta \le 0\} < \infty$. We therefore have

$$\begin{aligned} \int_{t > 2n_0} t^2 \, dP &\le 6 \sum_{r=\langle H_{n_0} \rangle}^{\infty} rP\{t > r\} \\ &\le 6 \sum_{r=\langle H_{n_0} \rangle}^{\infty} rP\{\xi_r \le -r\varepsilon\} + 6 \sum_{r=\langle H_{n_0} \rangle}^{\infty} rP\{S_r - r + r\delta \le 0\} \\ &= o(1) \quad \text{as } n_0 \to \infty. \end{aligned}$$

This finishes the proof.

Lemma 2.11 *Let $t^* = (t - n_0)/\sqrt{n_0}$, then*

$$\int_{t>2n_0} t^{*2}\, dP \to 0 \quad as\ n_0 \to \infty\ .$$

Proof: Note that

$$\begin{aligned}
\int_{t>2n_0} t^{*2}\, dP &= \frac{1}{n_0}\int_{t>2n_0} t^2\, dP - 2\int_{t>2n_0} t\, dP + \int_{t>2n_0} n_0\, dP \\
&\leq \frac{1}{n_0}\int_{t>2n_0} t^2\, dP + \int_{t>2n_0} n_0\, dP \\
&\leq \frac{1}{n_0}\int_{t>2n_0} t^2\, dP + \frac{1}{2}\int_{t>2n_0} t\, dP \\
&\leq \frac{1}{n_0}\int_{t>2n_0} t^2\, dP + \frac{1}{2}\int_{t>2n_0} t^2\, dP,
\end{aligned}$$

from which the lemma follows by using Lemma 2.10.

Lemma 2.12 *If $m > 1 + 2/k$, then*

$$\int_{t\leq n_0/2} t^{*2}\, dP \to 0 \quad as\ n_0 \to \infty.$$

Proof: Note that

$$\begin{aligned}
\int_{t\leq n_0/2} t^{*2}\, dP &\leq \int_{t\leq n_0/2} \left(\frac{n_0/2 + n_0}{\sqrt{n_0}}\right)^2 dP \\
&= \frac{9}{4} n_0 \int_{t\leq n_0/2} dP \\
&= \frac{9}{4} n_0 P\{t \leq \frac{n_0}{2}\} \\
&\sim C_{k(m-1)}\, n_0^{1-k(m-1)/2}\ , \qquad \text{(by Corollary 2.1)}
\end{aligned}$$

which goes to zero as $n_0 \to \infty$ for $m > 1 + 2/k$.

Corollary 2.2 *If $m > 1 + 2/k$, then*

$$\int_{T_G\leq n_0/2} \frac{(T_G - n_0)^2}{n_0}\, dP \to 0 \quad as\ n_0 \to \infty.$$

Corollary 2.3 *If $m > 1+2/k$, then $\left\{t^{*2} I_{t\leq n_0/2}\ ,\ n_0 > 2\right\}$ and $\left\{t^{*2} I_{t>2n_0}\ ,\ n_0 > 1\right\}$ are u.i..*

Proof: Note that $E\left|t^{*2} I_{t\le n_0/2}\right| < \infty$ since $\int_{t\le n_0/2} t^{*2}\, dP \to 0$ as $n_0 \to \infty$ by Lemma 2.12. The u.i. of $\left\{t^{*2} I_{t\le n_0/2}\ ,\ n_0 > 2\right\}$ now follows directly from Lemma A.14 by letting $X = 0$ and $p = 1$. A similar argument shows that $\left\{t^{*2} I_{t>2n_0}\ ,\ n_0 > 1\right\}$ is u.i..

Lemma 2.13 *If $m > 1 + 2/k$, then $\{t^{*2}, n_0 \ge 1\}$ is u.i..*

Proof: t^{*2} can be written as

$$t^{*2} = t^{*2} I_{\{t\le n_0/2\}} + t^{*2} I_{\{n_0/2<t\le 2n_0\}} + t^{*2} I_{\{t>2n_0\}}\ .$$

By Corollary 2.3, $\left\{t^{*2} I_{\{t\le n_0/2\}}\right\}$ and $\left\{t^{*2} I_{\{t>2n_0\}}\right\}$ are u.i.. So it remains to show $\left\{t^{*2} I_{\{n_0/2<t\le 2n_0\}}\right\}$ is u.i.. By using Lemma A.6, it suffices to show that there is a function J for which $xJ(x)$ is integrable with respect to Lebesgue measure over $(1, \infty)$, and

$$P\left\{\frac{n_0}{2} < t \le 2n_0,\ |t^*| > x\right\} \le J(x)\,.$$

Note that

$$\begin{aligned} &P\left\{\frac{n_0}{2} < t \le 2n_0,\ |t^*| > x\right\} \\ &\le P\left\{t > \frac{n_0}{2},\ t^* < -x\} + P\{t \le 2n_0,\ t^* > x\right\}, \end{aligned}$$

and we shall consider these two probabilities separately.

For the first probability, since $t > n_0/2$ and $t^* < -x$ imply that $x < \sqrt{n_0}/2$, then

$$P\left\{t > \frac{n_0}{2},\ t^* < -x\right\} = 0 \quad \text{for } x \ge \tfrac{\sqrt{n_0}}{2}.$$

For $1 \le x < \sqrt{n_0}/2$, $t > n_0/2$ and $t^* < -x$, i.e. $n_0/2 < t < n_0 - x\sqrt{n_0}$, we have $Z_j > n_0$ for some $j \in (n_0/2, n_0 - \sqrt{n_0}x]$, i.e. $\bar{U}_j < k(j+1)/(n_0 l_{j+1})$ for some $j \in (n_0/2, n_0 - \sqrt{n_0}x]$. For sufficiently large x and n_0, and $j \in (n_0/2, n_0 - \sqrt{n_0}x]$, we have

$$\frac{j+1}{n_0 l_{j+1}} = \frac{j+1}{n_0} \times \frac{1}{1 + (l_0/(j+1) + o(1/(j+1)))}$$

$$
\begin{aligned}
&= \frac{j+1}{n_0}(1 - l_0/(j+1) + o(1/(j+1))) \\
&= \frac{1}{n_0}(j + 1 - l_0 + o(1)) \\
&\leq \frac{1}{n_0}(n_0 + 1 - x\sqrt{n_0} - l_0 + o(1)) \\
&= 1 - \frac{x}{\sqrt{n_0}} + \frac{1}{n_0}(1 - l_0 + o(1)) \\
&\leq 1 - \frac{x}{2\sqrt{n_0}}.
\end{aligned}
$$

Also note that

$$
\bar{U}_j - k < -\frac{kx}{2\sqrt{n_0}} \Rightarrow |\bar{U}_j - k| > \frac{kx}{2\sqrt{n_0}} \Rightarrow j|\bar{U}_j - k| > \frac{kx\sqrt{n_0}}{4},
$$

for $j > n_0/2$. Consequently

$$
\begin{aligned}
&P\left\{t > \frac{n_0}{2},\ t^* < -x\right\} \\
&= P\left\{\frac{n_0}{2} < t < n_0 - \sqrt{n_0}x\right\} \\
&\leq P\left\{\bar{U}_j < \frac{k(j+1)}{n_0 l_{j+1}} \quad \text{for some } j \in (n_0/2, n_0 - \sqrt{n_0}x]\right\} \\
&\leq P\left\{\bar{U}_j < k\left(1 - \frac{x}{2\sqrt{n_0}}\right) \quad \text{for some } j \in (n_0/2, n_0 - \sqrt{n_0}x]\right\} \\
&\leq P\left\{j|\bar{U}_j - k| > \frac{kx\sqrt{n_0}}{4} \ \text{ for some } j \in (n_0/2, n_0 - \sqrt{n_0}x]\right\} \\
&\leq P\left\{\max_{j \leq n_0} j|\bar{U}_j - k| > \frac{kx\sqrt{n_0}}{4}\right\} \\
&\leq \frac{4^4}{x^4 n_0^2}\int n_0^4|\bar{U}_{n_0} - k|^4\, dP \quad \text{(by Theorem A.2)} \\
&\leq \frac{k^2 4^5}{x^4}\int\left(\frac{\sum_{i=1}^{n_0} U_i - kn_0}{\sqrt{2kn_0}}\right)^4 dP \\
&\leq Cx^{-4} \quad \text{(by Theorem A.1)}
\end{aligned}
$$

where C is a constant.

Next, we show that $P\{t \leq 2n_0,\ t^* > x\} \leq Cx^{-4}$ for sufficiently large x and n_0. Since $t \leq 2n_0$ and $t^* > x$ imply that $x \leq \sqrt{n_0}$, and so

$$
P\{t \leq 2n_0,\ t^* > x\} = 0 \quad \text{for } x > \sqrt{n_0}.
$$

For $1 \leq x \leq \sqrt{n_0}$, $t \leq 2n_0$ and $t^* > x$ we have $n_0 + x\sqrt{n_0} < t \leq 2n_0$ and so $Z_j < n_0 \; \forall j < n_0 + x\sqrt{n_0}$. This implies that for $j \in \left(n_0 + x\sqrt{n_0}/2, n_0 + x\sqrt{n_0}\right)$, and for sufficiently large x and n_0, we have

$$\begin{aligned} \bar{U}_j &> \frac{k(j+1)}{n_0 l_{j+1}} \\ &= \frac{k}{n_0}(j+1-l_0+o(1)) \\ &> \frac{k}{n_0}\left(n_0+1+\frac{x\sqrt{n_0}}{2}-l_0+o(1)\right) \\ &> k+\frac{kx}{2\sqrt{n_0}}-\frac{kx}{4\sqrt{n_0}} = k+\frac{kx}{4\sqrt{n_0}}. \end{aligned}$$

Also note that

$$|\bar{U}_j - k| > \frac{kx}{4\sqrt{n_0}} \Rightarrow j|\bar{U}_j - k| > \frac{kx\sqrt{n_0}}{4},$$

for $j \in \left(n_0 + x\sqrt{n_0}/2, n_0 + x\sqrt{n_0}\right)$. Therefore for sufficiently large x and n_0

$$\begin{aligned} P\{t < 2n_0, t^* > x\} &= P\{n_0 + x\sqrt{n_0} < t \leq 2n_0\} \\ &\leq P\left\{\max_{j \leq 2n_0} j|\bar{U}_j - k| > \frac{kx\sqrt{n_0}}{4}\right\}, \end{aligned}$$

and a similar argument as above shows that $P\{t < 2n_0, t^* > x\} < Cx^{-4}$. Now if we let $J(x) = Cx^{-4}$ then $xJ(x) = Cx^{-3}$ is integrable with respect to Lebesgue measure over $(1, \infty)$. This finishes the proof.

Corollary 2.4 *If $m > 1 + 2/k$, then*

$$\left\{\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)^2, \; n_0 \geq 1\right\} \quad \textit{is u.i..}$$

Proof: The corollary follows by noting that

$$\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)^2 = \left(\frac{t - n_0}{\sqrt{n_0}}\right)^2 + \frac{1}{n_0} + \frac{2}{\sqrt{n_0}}\left(\frac{t - n_0}{\sqrt{n_0}}\right),$$

and the facts that $\{t^{*2}, n_0 \geq 1\}$ is u.i. by Lemma 2.13, $\{(t - n_0)/\sqrt{n_0}\}$ is u.i. by part (I) of Lemma A.7 and $1/n_0$ is bounded.

Lemma 2.14 *Suppose that*

(i) $K(x)$ *is a real valued continuous function and* $|K(x)| \leq Cx^{-\alpha}$ *for constants* $C > 0, \alpha > 0$ *and all* $x > 0$,

(ii) $m > (2/k)(\alpha + 1) + 1$,

(iii) W *is a positive random variable such that*

$$\min\left\{\theta, \frac{T_G}{n_0}\theta\left(1 + \frac{C_1}{T_G}\right)^2\right\} \leq W \leq \max\left\{\theta, \frac{T_G}{n_0}\theta\left(1 + \frac{C_1}{T_G}\right)^2\right\} \qquad w.p.1$$

where $C_1 \geq 0$ *and* $\theta > 0$ *are constants. Then, we have*

$$E\left\{K(W)\left(\frac{T_G}{n_0}\left(1 + \frac{C_1}{T_G}\right)^2 - 1\right)^2\right\} \to \frac{2}{kn_0}K(\theta) + o\left(\frac{1}{n_0}\right) \qquad as\ n_0 \to \infty.$$

Proof: Let

$$\begin{aligned} U &= K(W)\left(\frac{T_G}{\sqrt{n_0}}\left(1 + \frac{C_1}{T_G}\right)^2 - \sqrt{n_0}\right)^2 \\ &= K(W)\left[\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)^2 + \frac{C_1^2}{n_0}\left(\frac{C_1}{T_G} + 2\right)^2 + \frac{2C_1}{\sqrt{n_0}}\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)\left(\frac{C_1}{T_G} + 2\right)\right] \end{aligned}$$

and

$$V = K(W)\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)^2.$$

First we shall show $E(V) \to (2/k)K(\theta)$ as $n_0 \to \infty$. Noting that W is an intermediate value between $\frac{T_G}{n_0}\theta\left(1 + \frac{C_1}{T_G}\right)^2$ and θ, and $\frac{T_G}{n_0}\left(1 + \frac{C_1}{T_G}\right)^2 \to 1$ w.p.1 as $n_0 \to \infty$ by Lemma A.8, so, $W \to \theta$ w.p.1 as $n_0 \to \infty$ and $K(W) \to K(\theta)$ w.p.1 as $n_0 \to \infty$. By Lemma A.11, $\left((T_G - n_0)/\sqrt{n_0}\right)^2 \xrightarrow{D} (2/k)\chi_1^2$. Then the asymptotic distribution of V is $(2/k)K(\theta)\chi_1^2$.

Now, let $A = \{\frac{T_G}{n_0} > \frac{1}{2}\}$, on the event A, $T_G > n_0/2$ and so

$$\frac{T_G}{n_0}\theta > \frac{\theta}{2}.$$

So, on event A, $\theta/2 < W$ and $|K(W)| \leq CW^{-\alpha} \leq C_0\theta^{-\alpha}$, i.e. $K(W)$ is bounded on A, $|K(W)| \leq M$ say, where M is a constant. Hence $\{VI_A\}$ is u.i. since

$$|VI_A| \leq M\left((T_G - n_0)/\sqrt{n_0}\right)^2,$$

and $\{((T_G - n_0)/\sqrt{n_0})^2\}$ is u.i. by Corollary 2.4. Also noting that $VI_A \xrightarrow{D} (2/k)K(\theta)\chi_1^2$, we have

$$\begin{aligned}\lim_{n_0\to\infty} E(VI_A) &= E\left[\frac{2}{k}K(\theta)\chi_1^2\right] \\ &= \frac{2}{k}K(\theta)\end{aligned}$$

by Lemma A.12.

Next, we show that the expectation of V on A^c goes to zero as $n_0 \to \infty$. For this we note that on event A^c and for sufficient large n_0, we have

$$|K(W)| \leq CW^{-\alpha},$$

$$\begin{aligned}\frac{T_G}{n_0} < \frac{1}{2} \;&\Rightarrow\; -1 < \frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2 - 1 < 0 \\ &\Rightarrow\; \left|\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2 - 1\right| = 1 - \frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2,\end{aligned}$$

$$\begin{aligned}&|\theta - W| < \theta\left|\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2 - 1\right| = \theta\left(1 - \frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2\right) \\ &\Rightarrow W > \theta\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2,\end{aligned}$$

$$0 < \frac{T_G}{n_0} < \frac{1}{2} \Rightarrow \left(\frac{T_G}{n_0} - 1\right)^2 < 1.$$

Then

$$\begin{aligned}E(VI_{A^c}) &= E\Big[K(W)\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)^2 I_{A^c}\Big] \\ &\leq C_0\int_{A^c}\left(\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2\right)^{-\alpha} n_0\left(\frac{T_G}{n_0} - 1\right)^2 dP, \\ &\leq C\int_{A^c} n_0^{\alpha+1}\, dP \\ &= Cn_0^{\alpha+1}P(T_G \leq n_0/2) \\ &= CC_{k(m-1)}\, n_0^{\alpha+1} n_0^{-k(m-1)/2}\quad, \quad \text{(by Corollary 2.1)}\end{aligned}$$

which goes to zero as $n_0 \to \infty$, since $m > (2/k)(\alpha+1)+1$. Combining the two cases we have shown that $E(V) \to (2/k)K(\theta)$ as $n_0 \to \infty$. Similarly, we can show

$$E\left[K(W)\frac{C_1^2}{n_0}\left(\frac{C_1}{T_G}+2\right)^2\right] = O\left(\frac{1}{n_0}\right)$$

and

$$E\left[K(W)\frac{2C_1}{\sqrt{n_0}}\left(\frac{T_G - n_0}{\sqrt{n_0}}\right)\left(\frac{C_1}{T_G}+2\right)\right] = O\left(\frac{1}{n_0^{1/2}}\right).$$

Therefore

$$\frac{1}{n_0}E(U) \to \frac{2}{kn_0}K(\theta) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.$$

This completes the proof.

Lemma 2.15 *For T_G define in (2.1) and $m > 1 + 2/k$, we have*

$$E\left(\frac{1}{T_G}\right) = \frac{1}{n_0} + o\left(\frac{1}{n_0}\right) \quad \textit{as } n_0 \to \infty.$$

Proof: Let $A = \{\frac{T_G}{n_0} > \frac{1}{2}\}$, then

$$\begin{aligned} \frac{1}{T_G} &= \frac{1}{n_0}\left(\frac{n_0}{T_G}\right) \\ &= \frac{1}{n_0}\left[\left(\frac{n_0}{T_G}I_A\right) + \left(\frac{n_0}{T_G}I_{A^c}\right)\right], \end{aligned}$$

and

$$\begin{aligned} E\left(\frac{1}{T_G}\right) &= \frac{1}{n_0}E\left(\frac{n_0}{T_G}\right) \\ &= \frac{1}{n_0}\left[E\left(\frac{n_0}{T_G}I_A\right) + E\left(\frac{n_0}{T_G}I_{A^c}\right)\right]. \end{aligned}$$

Note that by Lemma A.8 $\frac{n_0}{T_G} \to 1$ *w.p.*1 as $n_0 \to \infty$, also on event A, $\{\frac{n_0}{T_G}\}$ is u.i. so, by Lemma A.12

$$E\left(\frac{n_0}{T_G}I_A\right) \to 1 \quad \text{as } n_0 \to \infty.$$

Also note

$$\begin{aligned} E\left(\frac{n_0}{T_G}I_{A^c}\right) &\le n_0 \int_{A^c} dP \\ &= n_0 P\left(T_G \le \frac{n_0}{2}\right) \\ &= o(1) \quad \text{(by Corollary 2.1)}. \end{aligned}$$

It follows therefore

$$E\left(\frac{1}{T_G}\right) = \frac{1}{n_0} + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.$$

Theorem 2.2 *Suppose that $H(x)$ is a real valued function of $x > 0$ such that $H''(x)$ is a continuous function and $|H''(x)| \le Cx^{-\beta}$, where $C > 0$ and $\beta > 0$ are constants. If $m > (2/k)(\beta+1)+1$, then*

$$E\left[H\left(\frac{T_G}{n_0}\theta\left(1+\frac{C_1}{T_G}\right)^2\right)\right]$$
$$= H(\theta) + \frac{\theta}{n_0}H'(\theta)\left(\rho + l_0 - \frac{2}{k} + 2C_1\right) + \frac{1}{kn_0}\theta^2 H''(\theta) + o\left(\frac{1}{n_0}\right)$$

where $C_1 \ge 0$ and $\theta > 0$ are constants.

Proof: We expand $H(\cdot)$ in a Taylor series about θ to get

$$E\left[H\left(\frac{T_G}{n_0}\theta\left(1+\frac{C_1}{T_G}\right)^2\right)\right]$$
$$= E\left[H(\theta) + H'(\theta)\left(\frac{T_G}{n_0}\theta\left(1+\frac{C_1}{T_G}\right)^2 - \theta\right) + \frac{1}{2}H''(W)\left(\frac{T_G}{n_0}\theta\left(1+\frac{C_1}{T_G}\right)^2 - \theta\right)^2\right],$$

where

$$|\theta - W| < \left|\frac{T_G}{n_0}\theta\left(1+\frac{C_1}{T_G}\right)^2 - \theta\right|.$$

So

$$\begin{aligned} E\left[H\left(\frac{T_G}{n_0}\theta\left(1+\frac{C_1}{T_G}\right)^2\right)\right] &= H(\theta) + \frac{\theta}{n_0}H'(\theta)E\left((T_G - n_0) + 2C_1 + \frac{C_1^2}{T_G}\right) \\ &\quad + \frac{\theta^2}{2}E\left[H''(W)\left(\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2 - 1\right)^2\right] \\ &= H(\theta) + \frac{\theta}{n_0}H'(\theta)(\rho + l_0 - \frac{2}{k} + 2C_1) \\ &\quad + \frac{\theta^2}{2}E\left[H''(W)\left(\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2 - 1\right)^2\right] + \frac{1}{n_0}o(1), \end{aligned}$$

since $(1/n_0)E(1/T_G) = o(1/n_0)$ as $n_0 \to \infty$ by Lemma 2.15 and $E(T_G - n_0) = \rho + l_0 - 2/k + o(1)$ by Theorem 2.1. By Lemma 2.14 we have

$$E\left[H''(W)\left(\frac{T_G}{n_0}\left(1+\frac{C_1}{T_G}\right)^2 - 1\right)^2\right] \to \frac{2}{kn_0}H''(\theta) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty$$

and so the result follows.

Theorem 2.3 *Suppose that $H(x)$ is a real valued function of $x > 0$ such that $H''(x)$ is a continuous function and $|H'(x)| \leq A_1 x^{-\alpha}$ and $|H''(x)| \leq A_2 x^{-\alpha}$, where A_1, A_2 and α are positive constants. If $m > (1/k)(\alpha + 5) + 1$, then*

$$E\left[H\left(\left(\frac{T_G}{n_0}\right)^{1/2}\left(C_0 - C_1\left(1 + \frac{C_2}{T_G}\right)\right)\right)\right]$$
$$= H(C_0 - C_1) - \frac{1}{n_0}H'(C_0 - C_1)\left\{C_1C_2 - \frac{1}{2}(C_0 - C_1)\left(\rho + l_0 - \frac{2}{k}\right)\right.$$
$$\left. + \frac{1}{4k}(C_0 - C_1)\right\} + \frac{1}{4kn_0}(C_0 - C_1)^2 H''(C_0 - C_1) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty,$$

where $C_0 > C_1$ and C_2 are given positive constants.

Proof: Let $M(x) = H(a\sqrt{x})$. Expanding $M(x)$ about 1 gives

$$M(x) = M(1) + (x-1)M'(1) + \frac{1}{2}(x-1)^2 M''(V),$$

where V is an intermediate value between x and 1. Let $a = C_0 - C_1\left(1 + \frac{C_2}{T_G}\right)$ and $x = \frac{T_G}{n_0}$ and since $M(1) = H(a)$ and $M'(1) = (a/2)H'(a)$, we have

$$E\left[H\left(\left(\frac{T_G}{n_0}\right)^{1/2}\left(C_0 - C_1\left(1 + \frac{C_2}{T_G}\right)\right)\right)\right] = E\left[H\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\right]$$
$$+ \frac{1}{2}E\left[\left(\frac{T_G}{n_0} - 1\right)\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)H'\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\right]$$
$$+ \frac{1}{2}E\left[\left(\frac{T_G}{n_0} - 1\right)^2 M''(V)\right]. \tag{2.9}$$

Now, we find the first expectation on the right hand side of (2.9). For this we expand $H\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)$ in a Taylor series about $(C_0 - C_1)$ to get

$$E\left[H\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\right]$$
$$= E\left[H(C_0 - C_1) - \frac{C_1C_2}{T_G}H'(C_0 - C_1) + \frac{(C_1C_2)^2}{2T_G^2}H''(W_1)\right],$$

where W_1 is an intermediate value between $C_0 - C_1$ and $C_0 - C_1 - \frac{C_1C_2}{T_G}$. By Lemma 2.15 we have

$$E\left[\frac{C_1C_2}{T_G}H'(C_0 - C_1)\right] = \frac{C_1C_2}{n_0}H'(C_0 - C_1) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.$$

By an argument similar to that used in the proof of Lemma 2.14, we can show

$$E\left|H''(W_1)\frac{1}{T_G^2}\right| = o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.$$

So, we have

$$E\Big[H\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\Big] = H\left(C_0 - C_1\right) - \frac{C_1C_2}{n_0}H'\left(C_0 - C_1\right) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.$$

Next, we evaluate the second expectation on the right hand side of (2.9):

$$\begin{aligned}
&E\Big[\frac{1}{2}\left(\frac{T_G}{n_0} - 1\right)\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)H'\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\Big] \\
&= E\Big[\frac{1}{2}\left(\frac{T_G}{n_0} - 1\right)\left(C_0 - C_1\right)H'\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\Big] \\
&-E\Big[\frac{1}{2}\left(\frac{T_G}{n_0} - 1\right)\frac{C_1C_2}{T_G}H'\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right)\Big] \\
&= E1 - E2. \qquad (2.10)
\end{aligned}$$

We have

$$\begin{aligned}
E1 &= E\Big[\frac{1}{2}\left(\frac{T_G}{n_0} - 1\right)\left(C_0 - C_1\right)H'\left(C_0 - C_1\right)\Big] \\
&\quad +E\Big[\frac{1}{2}\left(\frac{T_G}{n_0} - 1\right)\left(C_0 - C_1\right)\left(H'\left(C_0 - C_1 - \frac{C_1C_2}{T_G}\right) - H'\left(C_0 - C_1\right)\right)\Big] \\
&= \frac{1}{2n_0}\left(C_0 - C_1\right)H'\left(C_0 - C_1\right)\left(\rho + l_0 - \frac{2}{k}\right) \\
&\quad -E\Big[\frac{1}{2}\left(\frac{T_G}{n_0} - 1\right)\left(C_0 - C_1\right)H''(W_2)\frac{C_1C_2}{T_G}\Big],
\end{aligned}$$

where W_2 is an intermediate value between $C_0 - C_1$ and $C_0 - C_1 - \frac{C_1C_2}{T_G}$. Same as before, we can show

$$E\Big[\left(\frac{T_G}{n_0} - 1\right)\frac{1}{T_G}\Big] = o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty$$

and so

$$E1 = \frac{1}{2n_0}\left(C_0 - C_1\right)H'\left(C_0 - C_1\right)\left(\rho + l_0 - \frac{2}{k}\right) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty$$

A similar argument establishes that

$$
\begin{aligned}
E2 &= E\Big[\frac{1}{2}\left(\frac{T_G}{n_0}-1\right)\frac{C_1C_2}{T_G}H'\left(C_0-C_1-\frac{C_1C_2}{T_G}\right)\Big] \\
&= E\Big[\frac{1}{2}\left(\frac{T_G}{n_0}-1\right)\frac{C_1C_2}{T_G}\Big(H'\left(C_0-C_1-\frac{C_1C_2}{T_G}\right)-H'\left(C_0-C_1\right)\Big)\Big] \\
&\quad +E\Big[\frac{1}{2}\left(\frac{T_G}{n_0}-1\right)\frac{C_1C_2}{T_G}H'\left(C_0-C_1\right)\Big] \\
&= o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.
\end{aligned}
$$

Finally, the third expectation on the right hand side of (2.9) is given by

$$
\begin{aligned}
&E\Big[\frac{1}{2}M''(V)\left(\frac{T_G}{n_0}-1\right)^2\Big] \\
&= \frac{1}{8n_0}E\Big[\left(\frac{T_G-n_0}{\sqrt{n_0}}\right)^2\Big\{-\frac{C_0-C_1-C_1C_2/T_G}{V^{3/2}}H'\left(\left(C_0-C_1-\frac{C_1C_2}{T_G}\right)\sqrt{V}\right) \\
&+\frac{\left(C_0-C_1-C_1C_2/T_G\right)^2}{V}H''\left(\left(C_0-C_1-\frac{C_1C_2}{T_G}\right)\sqrt{V}\right)\Big\}\Big].
\end{aligned}
$$

By an argument similar to that used in the proof of Lemma 2.14, we can show

$$
\begin{aligned}
&E\Big[\frac{1}{2}M''(V)\left(\frac{T_G}{n_0}-1\right)^2\Big] \\
&= \frac{(C_0-C_1)}{4kn_0}\Big(-H'\left(C_0-C_1\right)+\left(C_0-C_1\right)H''\left(C_0-C_1\right)\Big)+o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty.
\end{aligned}
$$

Putting these together gets the theorem.

2.3 Exact calculations of $E(T_G)$ and $E\left[H\left(\gamma\frac{T_G}{n_0}\right)\right]$

In this section, we evaluate the exact distribution of the t in (2.11) for small and moderate values of n_0, by using a recursive method. Such a recursive computing method was used, for example, by Armitage *et al.* (1969), McPherson and Armitage (1971) and by Jennison and Turnbull (1991). We set $l_n = 1 + l_0/n$.

From (2.4) and (2.5), we have that $T_G = t + 1$ where

$$\begin{aligned} t &= \inf\left\{n \geq m-1 \ : \ U_1 + U_2 + \cdots + U_n \leq \frac{kn(n+1)d^2}{\gamma l_{n+1}\sigma^2}\right\} \\ &= \inf\left\{n \geq m-1 \ : \ U_1 + U_2 + \cdots + U_n \leq \frac{kn(n+1)}{n_0 l_{n+1}}\right\} \\ &= \inf\{n \geq m_0 \ : S_n \leq C_n\}, \end{aligned} \tag{2.11}$$

where $m_0 = m - 1$, $S_n = U_1 + U_2 + \cdots + U_n$, $U_1, U_2, \ldots$ are independent χ_k^2 random variables, and

$$C_n = \frac{kn(n+1)}{n_0\left(1 + \frac{l_0}{n+1}\right)}.$$

If we define

$$R_{m_0}(x) = f_{\chi^2_{k\,m_0}}(x), \tag{2.12}$$

where $f_{\chi^2_\nu}(\cdot)$ denotes a pdf of the χ^2_ν and

$$R_n(x) = \frac{d}{dx}P\{S_{m_0} > C_{m_0}, \cdots, S_{n-1} > C_{n-1}, S_n \leq x\}, \quad n \geq m_0 + 1, \tag{2.13}$$

then we have the following result.

Lemma 2.16 *For $n \geq m_0$*

$$R_{n+1}(x) = \int_{C_n}^{x} R_n(y) f_{\chi^2_k}(x - y)\, dy. \tag{2.14}$$

Proof: By the definitions of $R_n(x)$, we have

$$\begin{aligned} R_{n+1}(x) &= P\{S_{m_0} > C_{m_0}, \cdots, S_{n-1} > C_{n-1}, S_n > C_n, S_{n+1} = x\} \\ &= \int_{C_n}^{\infty} P\{S_{m_0} > C_{m_0}, \cdots, S_{n-1} > C_{n-1}, S_n = y\} \times \end{aligned}$$

$$\begin{aligned}
& P\{S_{n+1} = x | S_{m_0} > C_{m_0}, \cdots, S_{n-1} > C_{n-1}, S_n = y\}\ dy \\
= & \int_{C_n}^{\infty} R_n(y) P\{y + U_{n+1} = x\}\ dy \\
= & \int_{C_n}^{\infty} R_n(y) P\{U_{n+1} = x - y\}\ dy \\
= & \int_{C_n}^{x} R_n(y) f_{\chi_k^2}(x - y)\ dy,
\end{aligned}$$

as required.

Note that $P\{t > m_0 - 1\} = 1$, and

$$P\{t > n+1\} = \int_{C_{n+1}}^{\infty} R_{n+1}(y)\ dy, \qquad n \geq m_0 - 1, \tag{2.15}$$

since $\{t > n+1\} = \{S_{m_0} > C_{m_0}, \cdots, S_{n+1} > C_{n+1}\}$. So

$$\begin{aligned}
E(T_G) &= 1 + E(t) \\
&= 1 + \sum_{n=m_0}^{\infty} nP(t = n) \\
&= 1 + \sum_{n=m_0}^{\infty} n[P(t > n-1) - P(t > n)].
\end{aligned} \tag{2.16}$$

$R_{n+1}(x)$ can thus be calculated recursively. The basic method is to evaluate the right hand side of (2.14) at points on a grid of width h, i.e. for $x = C_n, C_n + h, C_n + 2h, \cdots, C_n + lh$, where l is chosen such that $R_{n+1}(C_n + lh)$ is sufficiently small (here we choose l for which $R_{n+1}(C_n + lh) \leq 5 \times 10^{-6}$); $R_{n+1}(x)$ is approximated by linear interpolation for $x \in [C_n, C_n + lh]$, and approximated by zero for $x > C_n + lh$. From (2.15), $P\{t > n+1\}$ is then approximated by $\int_{C_{n+1}}^{C_{n+1}+lh} R_{n+1}(y)\ dy$. This recursive calculation stops at some r_0 such that $P\{t > r_0\}$ is sufficiently small. From (2.16) the $E(T_G)$ can thus be calculated by a finite summations which sum from $n = m_0$ until $n = r_0$.

Chapter 3

The constructions of fixed-width confidence intervales for multiple comparisons

3.1 Fixed-width simultaneous confidence intervals for the means of several independent normal populations

3.1.1 Introduction

Suppose we have k independently and normally distributed populations $N(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, k$ with unknown $\mu_i, -\infty < \mu_i < \infty$, and a common unknown positive variance σ^2. Assume we can sample sequentially from each population and that $Y_{i1}, Y_{i2}, Y_{i3}, \cdots$ denote the observations from the i^{th} population, $i = 1, 2, \cdots, k$. In this section we construct a set of fixed-width $2d$ simultaneous confidence intervals for the means μ_i of the form

$$\mu_i \in \left(\bar{Y}_i - d, \quad \bar{Y}_i + d\right), \quad i = 1, 2, \cdots, k$$

with a (nominal) confidence level $1 - \alpha$, where $\bar{Y}_i$ is the sample mean of a sample taken from the i^{th} population, and $d > 0$ and $0 < \alpha < 1$ are two given constants.

Let $Z_1, Z_2, \cdots, Z_k$ be i.i.d. $N(0,1)$ random variables, and let χ^2_ν be a chi-square random variable with ν degrees of freedom which is independent of $Z_1, Z_2, \cdots, Z_k$. The distribution of

$$|M|_{k,\nu} = \frac{\max_{1 \le i \le k} |Z_i|}{\sqrt{\chi^2_\nu / \nu}}$$

is called the studentised maximum modulus distribution with parameters k and ν. If $\nu = \infty$ then $\chi^2_\infty/\infty = 1$ and hence the distribution of $|M|_{k,\infty}$ is the same as

$$|M|_k = \max_{1 \le i \le k} |Z_i|.$$

Let $|m|^\alpha_{k,\nu}$ denote the upper α point of the studentised maximum modulus distribution with parameters k and ν, i.e.

$$P\{|M|_{k,\nu} \le |m|^{\alpha}_{k,\nu}\} = 1 - \alpha.$$

Values of $|m|^{\alpha}_{k,\nu}$ for some combinations of α, k and ν can be found in Hahn and Hendrickson (1971).

Suppose a sample of fixed size n is taken from each of the k populations and let $\hat{\sigma}^2_n$ be the pooled sample variance given by

$$\hat{\sigma}^2_n = \frac{1}{k(n-1)} \sum_{i=1}^{k} \sum_{j=1}^{n} (Y_{ij} - \bar{Y}_{in})^2, \; n \ge 2$$

then

$$\max_{1 \le i \le k} \left\{ \frac{\sqrt{n}|\bar{Y}_{in} - \mu_i|}{\hat{\sigma}_n} \right\}$$

has a studentised maximum modulus distribution with parameters k and $\nu = k(n-1)$. Therefore

$$P\left(\left| \frac{\sqrt{n}\left(\bar{Y}_{1n} - \mu_1\right)}{\hat{\sigma}_n} \right| < |m|^{\alpha}_{k,\nu}, \cdots, \left| \frac{\sqrt{n}\left(\bar{Y}_{kn} - \mu_k\right)}{\hat{\sigma}_n} \right| < |m|^{\alpha}_{k,\nu} \right) = 1 - \alpha$$

which can be written as

$$P\left\{ \bar{Y}_{in} - |m|^{\alpha}_{k,\nu} \frac{\hat{\sigma}_n}{\sqrt{n}} < \mu_i < \bar{Y}_{in} + |m|^{\alpha}_{k,\nu} \frac{\hat{\sigma}_n}{\sqrt{n}}, \; 1 \le i \le k \right\} = 1 - \alpha.$$

A set of simultaneous confidence intervals for μ_i with confidence level $1 - \alpha$ is therefore given by

$$\mu_i \in \left(\bar{Y}_{in} - |m|^{\alpha}_{k,\nu} \frac{\hat{\sigma}_n}{\sqrt{n}}, \; \bar{Y}_{in} + |m|^{\alpha}_{k,\nu} \frac{\hat{\sigma}_n}{\sqrt{n}} \right), \quad i = 1, 2, \cdots, k. \tag{3.1}$$

This set of confidence intervals was proposed by Tukey (1952b, 1953).

As we can see, the length of these confidence intervals, $2|m|^{\alpha}_{k,\nu} \hat{\sigma}_n / \sqrt{n}$, is a random number since σ^2 is unknown and so $\nu < \infty$. In fact, when σ^2 is unknown, it is necessary to use a sequential procedure to construct a set of fixed-width $2d$ simultaneous confidence intervals for the means μ_i of the form

$$\mu_i \in \left(\bar{Y}_i - d, \; \bar{Y}_i + d \right), \quad i = 1, 2, \cdots, k.$$

A two-stage procedure based on Stein's (1945) result was proposed by Healy (1956). Here we propose a pure sequential procedure. To appreciate the definition of this pure sequential procedure, we first look at the construction of a set of fixed-width $2d$ simultaneous confidence intervals for the means μ_i when σ^2 is assumed to be a known constant.

Had σ^2 been known, the set of $1-\alpha$ level confidence intervals in (3.1) becomes

$$\mu_i \in \left(\bar{Y}_{in} - |m|_k^\alpha \frac{\sigma}{\sqrt{n}},\ \ \bar{Y}_{in} + |m|_k^\alpha \frac{\sigma}{\sqrt{n}}\right), \quad i = 1, 2, \cdots, k.$$

In order that the width of these confidence intervals is at most $2d$, the sample size n from each of the k populations should satisfy $|m|_k^\alpha \sigma / \sqrt{n} \leq d$, which implies that

$$n \geq d^{-2} (|m|_k^\alpha)^2 \sigma^2. \tag{3.2}$$

That is, when σ^2 is known, we take a sample of size n from each of the k populations where n satisfies (3.2), and then construct a set of simultaneous confidence intervals for the μ_i as

$$\mu_i \in \left(\bar{Y}_{in} - d,\ \ \bar{Y}_{in} + d\right), \quad i = 1, 2, \cdots, k.$$

This set of confidence intervals has width $2d$ and confidence level at least $1-\alpha$.

Now consider our problem in which σ^2 is unknown and so the right side of (3.2) can not be calculated explicitly. A reasonable sample size formula would be similar to (3.2) but with σ^2 replaced by some estimate. Precisely, we take the same number of observations, n, from each of the k populations, starting with m, increasing by one at a time, until

$$T = \inf\{n \geq m:\ n \geq d^{-2} \left(|m|_k^\alpha\right)^2 l_n \hat{\sigma_n}^2\}, \tag{3.3}$$

where $m \geq 2$ is the initial sample size from each population and $l_n = 1 + \frac{1}{n} l_0 + o(\frac{1}{n})$ as $n \to \infty$. On stopping sampling the set of simultaneous confidence intervals for μ_i is defined as

$$\mu_i \in I_i(T) = (\bar{Y}_{iT} - d,\ \bar{Y}_{iT} + d), \quad i = 1, 2, \cdots, k.$$

Next we show that the confidence level of this set of confidence intervals is approximately equal $1 - \alpha$.

3.1.2 Second order approximations to the expected sample size and the confidence level

As the stopping time T defined in (3.3) is of the same form as the stopping time in (2.1) with $\gamma = (|m|_k^\alpha)^2$, the following theorem follows directly from Theorem 2.1.

Theorem 3.1 *For $k \geq 1$ and $m > 1 + 2/k$, we have*

$$E(T) = a + \rho + l_0 - \frac{2}{k} + o(1) \quad as\ a \to \infty,$$

where $a = d^{-2}(|m|_k^\alpha)^2\sigma^2$.

It is noteworthy that a is the right side of (3.2), which can be regarded as the optimal sample size had σ^2 been known. Form Theorem 3.1 the difference between the expected sample size of the pure sequential procedure and the optimal sample size a is about $\rho + l_0 - \frac{2}{k}$, a constant, at least for large a.

In order to deriving the second order approximation to the confidence level, we need the following lemmas.

Lemma 3.1 *For given $a > 0$,*

$$P\{\mu_1 \in I_1(T), \cdots, \mu_k \in I_k(T)\} = E\left[\Psi^k\left((|m|_k^\alpha)^2 \frac{T}{a}\right)\right],$$

where $\Psi(x) = 2\Phi(\sqrt{x}) - 1$ and $a = d^{-2}(|m|_k^\alpha)^2\sigma^2$.

Proof: We have

$$\begin{aligned}
&P\{\mu_1 \in I_1(T), \cdots, \mu_k \in I_k(T)\} \\
&= P\{\bar{Y}_{1T} - d < \mu_1 < \bar{Y}_{1T} + d, \cdots, \bar{Y}_{kT} - d < \mu_k < \bar{Y}_{kT} + d\} \\
&= \sum_{n=m}^{\infty} P\{\bar{Y}_{1T} - d < \mu_1 < \bar{Y}_{1T} + d, \cdots, \bar{Y}_{kT} - d < \mu_k < \bar{Y}_{kT} + d | T = n\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\{\bar{Y}_{1n} - d < \mu_1 < \bar{Y}_{1n} + d, \cdots, \bar{Y}_{kn} - d < \mu_k < \bar{Y}_{kn} + d | T = n\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\{\bar{Y}_{1n} - d < \mu_1 < \bar{Y}_{1n} + d, \cdots, \bar{Y}_{kn} - d < \mu_k < \bar{Y}_{kn} + d\} P\{T = n\},
\end{aligned}$$

where the last equation follows from Lemma 2.2. The lemma now follows by noting that

$$\begin{aligned}
&P\{\bar{Y}_{1n} - d < \mu_1 < \bar{Y}_{1n} + d, \cdots, \bar{Y}_{kn} - d < \mu_k < \bar{Y}_{kn} + d\} \\
&= P\{|\bar{Y}_{1n} - \mu_1| < d, \cdots, |\bar{Y}_{kn} - \mu_k| < d\} \\
&= \left[P\left\{|Z| < \frac{d\sqrt{n}}{\sigma}\right\}\right]^k \\
&= \left[2\Phi\left(\frac{d\sqrt{n}}{\sigma}\right) - 1\right]^k \\
&= \Psi^k\left(\frac{nd^2}{\sigma^2}\right) \\
&= \Psi^k\left((|m|_k^\alpha)^2 \frac{n}{a}\right)
\end{aligned}$$

where Z is a standard normal random variable.

Lemma 3.2 *Let* $\Psi(x) = 2\Phi(\sqrt{x}) - 1$ *and* $h(x) = \Psi^k(x)$. *Then*

I $\Psi''(x)$ *is an increasing function of* $x \in (0, \infty)$.

II *There is a constant* C *for which* $|\Psi''(x)| \leq Cx^{-3/2}$ *for all* $x > 0$.

III *There is a constant* C *for which* $(\Psi'(x))^2 \leq Cx^{-1}$ *for all* $x > 0$.

IV *There is a constant* C *for which* $|h''(x)| \leq Cx^{(k-4)/2}$ *for all* $x > 0$.

Proof: We have

$$\begin{aligned}
\Psi'(x) &= \frac{1}{\sqrt{2\pi x}} e^{-x/2}, \\
\Psi''(x) &= -\frac{1}{2\sqrt{2\pi x}} e^{-x/2}\left(\frac{1}{x} + 1\right), \\
\Psi'''(x) &= \frac{1}{4\sqrt{2\pi x}} e^{-x/2}\left(\left(\frac{1}{x} + 1\right)^2 + \frac{2}{x^2}\right) > 0,
\end{aligned}$$

from which the results I, II and III follow directly.

To prove (IV), we note that for $x > 0$

$$\begin{aligned}
\Psi(x) &= 2\Phi(\sqrt{x}) - 1 \\
&= 2\int_0^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz \\
&< Bx^{1/2}
\end{aligned}$$

where B is a constant and so

$$\begin{aligned}|h''(x)| &= \left|k(k-1)\Psi^{k-2}(x)\left(\Psi'(x)\right)^2 + k\Psi^{k-1}(x)\Psi''(x)\right| \\ &\leq \left|k(k-1)\Psi^{k-2}(x)\left(\Psi'(x)\right)^2\right| + \left|k\Psi^{k-1}(x)\Psi''(x)\right| \\ &\leq A_1 x^{(k-2)/2}\, x^{-1} + A_2 x^{(k-1)/2}\, x^{-3/2} \\ &= C x^{(k-4)/2}\,,\end{aligned}$$

where A_1, A_2 and C are constants, as required.

Now we are ready to give the second order approximation to the confidence level.

Theorem 3.2 *Suppose that $m > 1$ if $k \geq 4$ and $m > 1 + (6-k)/k$ if $k = 2, 3$, then*

$$\begin{aligned}&P\{\mu_1 \in I_1(T), \cdots, \mu_k \in I_k(T)\} \\ &= 1 - \alpha + \frac{1}{a}\Big[\left(|m|_k^\alpha\right)^2 h'\left(\left(|m|_k^\alpha\right)^2\right)\left(\rho + l_0 - \frac{2}{k}\right) \\ &+ \frac{1}{k}\left(|m|_k^\alpha\right)^4 h''\left(\left(|m|_k^\alpha\right)^2\right)\Big] + o\left(\frac{1}{a}\right),\end{aligned}$$

where $h(x) = \Psi^k(x)$, and $\Psi(x) = 2\Phi(\sqrt{x}) - 1$.

Proof: It follows immediately from Lemma 3.1, part IV of Lemma 3.2 and Theorem 2.2 with $\theta = \left(|m|_k^\alpha\right)^2$, $C_1 = 0$ and $n_0 = a$.

3.1.3 Calculations of the approximate values of the expected sample size and the confidence level

In this subsection we calculate the approximate values of the $E(T)$ and CL. First, we calculate the values of

$$\rho = \rho(k) = \frac{k+2}{2k} - \frac{1}{k}\sum_{n=1}^{\infty} \frac{1}{n} E \max\left(0, \chi^2_{nk} - 2nk\right),$$

which is required in Theorem 3.1, for $k = 1(1)\ 20$. By noting that

$$\begin{aligned}
E \max\left(0, \chi^2_{nk} - 2nk\right) &= \int_{\chi^2_{nk} > 2nk} (\chi^2_{nk} - 2nk)\, dP \\
&= \int_{2nk}^{\infty} \frac{1}{2^{nk/2}\,\Gamma(nk/2)} x^{nk/2}\, e^{-x/2}\ dx \\
&\quad -2nk \int_{2nk}^{\infty} \frac{1}{2^{nk/2}\,\Gamma(nk/2)} x^{nk/2-1}\, e^{-x/2}\ dx \\
&= nk \int_{2nk}^{\infty} \frac{1}{2^{1+nk/2}\,\Gamma(1+nk/2)} x^{nk/2}\, e^{-x/2}\ dx \\
&\quad -2nk \int_{2nk}^{\infty} \frac{1}{2^{nk/2}\,\Gamma(nk/2)} x^{nk/2-1}\, e^{-x/2}\ dx
\end{aligned}$$

and using $Q(c, x)$ to denote the incomplete gamma function $\int_x^{\infty} t^{c-1}\, e^{-t}\ dt/\Gamma(c)$, then

$$\rho = \frac{k+2}{2k} - \sum_{n=1}^{\infty}\left[Q\left(\frac{nk+2}{2}, nk\right) - 2Q\left(\frac{nk}{2}, nk\right)\right]. \tag{3.4}$$

The values of $\rho(k)$ for $k = 1(1)\ 20$ are given in Table 3.1. These are calculated from (3.4) by using the NAG routine S14BAF for the incomplete gamma function $Q(\cdot,\cdot)$ and keeping only those terms having magnitude $\geq 10^{-10}$ in the sum.

From Theorem 3.2 it can be seen that the value of l_0 can be chosen to satisfy

$$(|m|_k^{\alpha})^2\, h'\left((|m|_k^{\alpha})^2\right)\left(\rho + l_0 - \frac{2}{k}\right) + \frac{1}{k}(|m|_k^{\alpha})^4\, h''\left((|m|_k^{\alpha})^2\right) = 0$$

so that the CL is equal to $1 - \alpha + o(1/a)$. This $l_0 = l_0(k, \alpha)$ is given by

$$l_0 = \frac{1}{k}\left[2 - \frac{(|m|_k^{\alpha})^2\, h''\left((|m|_k^{\alpha})^2\right)}{h'\left((|m|_k^{\alpha})^2\right)}\right] - \rho, \tag{3.5}$$

Table 3.1: $\rho = \rho(k)$

k	ρ	k	ρ	k	ρ
1	0.817	8	0.608	15	0. 564
2	0.745	9	0.598	16	0.560
3	0.701	10	0.590	17	0.557
4	0.671	11	0.583	18	0.554
5	0.649	12	0.577	19	0.551
6	0.632	13	0.572	20	0.549
7	0.618	14	0.568		

where $h'(x) = k\Psi^{k-1}(x)\Psi'(x)$, $h''(x) = k(k-1)\Psi^{k-2}(x)(\Psi'(x))^2 + k\Psi^{k-1}(x)\Psi''(x)$ and $\Psi(x) = 2\Phi(\sqrt{x}) - 1$. In order to calculate $l_0(k, \alpha)$ I have calculated the values of $|m|_k^\alpha$ for $\alpha = 0.1, 0.05, 0.01$ and $k = 1(1)\ 20$ and they are given in Table 3.2. The value of $l_0(k, \alpha)$ can be easily calculated from (3.5) and the results for $\alpha = 0.1, 0.05, 0.01$ and $k = 1(1)\ 20$ are given in Table 3.3.

From Theorem 2.1, the approximate value of $E(T)$ is

$$a + \rho + l_0 - 2/k.$$

This approximate value corresponding to $l_0 = l_0(k, \alpha)$, is given in Tables 3.4 and 3.5.

Table 3.2: $|m|_k^\alpha$

$k \setminus \alpha$	0.1	0.05	0.01
1	1.645	1.960	2.576
2	1.948	2.236	2.806
3	2.114	2.387	2.934
4	2.226	2.490	3.022
5	2.310	2.568	3.089
6	2.378	2.631	3.142
7	2.433	2.682	3.187
8	2.481	2.727	3.225
9	2.522	2.765	3.259
10	2.559	2.799	3.289
11	2.592	2.830	3.315
12	2.622	2.857	3.340
13	2.649	2.883	3.362
14	2.673	2.906	3.382
15	2.696	2.927	3.401
16	2.718	2.947	3.419
17	2.738	2.966	3.435
18	2.756	2.983	3.451
19	2.774	3.000	3.465
20	2.791	3.016	3.479

Table 3.3: $l_0 = l_0(k, \alpha)$

$k \setminus \alpha$	0.1	0.05	0.01
1	3.0356	3.874	5.463
2	1.393	1.717	2.462
3	0.814	1.044	1.556
4	0.515	0.695	1.085
5	0.332	0.479	0.796
6	0.207	0.333	0.599
7	0.117	0.227	0.457
8	0.049	0.146	0.349
9	-0.004	0.082	0.263
10	-0.048	0.031	0.195
11	-0.083	-0.011	0.138
12	-0.114	-0.046	0.090
13	-0.139	-0.077	0.050
14	-0.162	-0.103	0.015
15	-0.181	-0.126	-0.015
16	-0.198	-0.147	-0.042
17	-0.213	-0.165	-0.066
18	-0.227	-0.181	-0.087
19	-0.239	-0.195	-0.107
20	-0.251	-0.208	-0.124

3.1.4 Exact calculations of the expected sample size and the confidence level

In this subsection, we evaluate, by using a recursive method, the exact distribution of T and hence the exact values of $E(T)$ and CL. Let $t = T - 1$. Then from the Lemma 2.16 and the argument after Lemma 2.16, we have

$$R_{n+1}(x) = \int_{C_n}^{x} R_n(y) f_{\chi_k^2}(x-y)\, dy \tag{3.6}$$

and

$$P\{t > n+1\} = \int_{C_{n+1}}^{\infty} R_{n+1}(y)\, dy, \qquad n \geq m_0 - 1, \tag{3.7}$$

where

$$C_n = \frac{kn(n+1)}{a\left(1 + \frac{l_0}{n+1}\right)}$$

and the value of l_0 is given in Table 3.3. Consequently

$$E(T) = 1 + \sum_{n=m_0}^{\infty} n[P(t > n-1) - P(t > n)] \tag{3.8}$$

and

$$\begin{aligned} CL &= E\left[h\left((|m|_k^\alpha)^2 \frac{T}{a}\right)\right] \\ &= \sum_{n=m_0}^{\infty} P(t=n) h\left((|m|_k^\alpha)^2 \frac{n+1}{a}\right) \\ &= \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)] h\left((|m|_k^\alpha)^2 \frac{n+1}{a}\right), \end{aligned} \tag{3.9}$$

where $h(x) = \Psi^k(x)$ and $\Psi(x) = 2\Phi(\sqrt{x}) - 1$. Now the functions $R_{n+1}(\cdot)$ and thus $E(T)$ and CL, can be calculated in the way discussed after Lemma 2.16.

The results of this calculation are given in Subsection 3.1.5 and were based on a grid of equal width $h = 0.1$. Calculations based on $h = 0.2$ and $h = 0.05$ gave values of the CL differing at the most in the fourth decimal place from those based on $h = 0.1$. Simulations on $E(T)$ and CL were also carried out based on 6,000 experiments and some of the results are given in Table 3.6.

3.1.5 Some comparisons

In this subsection, we compare the second order approximations with the exact calculations of the $E(T)$ and the CL. Throughout, the value of l_0 is given by $l_0 = l_0(k, \alpha)$. From these comparisons we can see when the second order approximations are reasonably accurate.

Firstly, we look at the confidence level CL. The confidence level is equal to $1 - \alpha$ (nominal level) plus an error term of order $o(1/a)$ as $a \to \infty$ and so the approximate is $1 - \alpha$. The true value of the confidence level, however, depends on a, i.e. $CL = CL(a)$. For $m = 2, k = 3, 7, 10$, and $1 - \alpha = 90\%, 99\%$, the exact calculation results of $CL(a)$ at $a = 5(5)60$ are linearly plotted in Figure 1. Figure 2 gives the similar plots for $m = 10$ and $a = 15(5)60$. From Figures 1 and 2 it can be seen that $CL(a)$ is generally closer to the nominal level $1 - \alpha$ for: (i) larger a; (ii) larger k; (iii) larger nominal level $1 - \alpha$; (iv) larger initial sample size m.

Next, we look at the expected sample size $E(T)$. When a is large, the approximation to $E(T)$ is $a + \rho + l_0 - 2/k$. For $m = 2$, $k = 3$, 7, 10, and $1 - \alpha = 90\%$, 99%, Table 3.4 contains the exact values of $E(T)$ calculated using the recursive method and the approximate values of $E(T)$ at $a = 5(5)60$. Table 3.5 contains the similar results for $m = 10$ and $a = 15(5)60$. From Tables 3.4 and 3.5 it can be seen that the approximate value of $E(T)$ are generally closer to the value of $E(T)$ for: (i) large a; (ii) large k; (iii) large initial sample size m. The exact calculations of the $E(T)$ and the CL become quite computationally intensive for $a \geq 60$. However, when $a \geq 60$ the approximations are very good, as can be seen from the results given in this subsection. So approximated results can be used in this case.

Generally, the values of k, α and d are given. However, we don't know the value of σ^2. In most situations we know a range in which σ^2 falls in from the *prior* knowledge. Consequently, we know the range for $a = \sigma^2(|m|_k^\alpha)^2/d^2$.

From this we can find the confidence level either by

(1) if a is large, using the approximation, which is just the nominal level $1 - \alpha$, or

(2) calculating $CL(a)$ for all the a in that range.

In particular, if we are free to choose the initial sample size m then we can bring the true confidence level closer to the nominal level by choosing a suitable value of m.

Table 3.4: *Comparisons between the exact and approximate values of $E(T)$ for $m = 2$ and given values of k, α and a*

$\alpha = 0.1$

a	$k = 3$ Exact	$k = 3$ Appro.	$k = 7$ Exact	$k = 7$ Appro.	$k = 10$ Exact	$k = 10$ Appro.
5	5.3	5.8	5.1	5.4	5.1	5.3
10	10.0	10.8	10.7	10.4	10.3	10.3
15	15.1	15.8	15.4	15.4	15.3	15.3
20	20.2	20.8	20.4	20.4	20.3	20.3
25	25.3	25.8	25.4	25.4	25.3	25.3
30	30.3	30.8	30.4	30.4	30.3	30.3
35	35.4	35.8	35.4	35.4	35.3	35.3
40	40.4	40.8	40.4	40.4	40.3	40.3
45	45.4	45.8	45.4	45.4	45.3	45.3
50	50.5	50.8	50.4	50.4	50.3	50.3
55	55.5	55.8	55.4	55.4	55.3	55.3
60	60.5	60.8	60.4	60.4	60.3	60.3

Table 3.4: *Comparisons between the exact and approximate values of $E(T)$ for $m = 2$ and given values of k, α and a*

$\alpha = 0.01$

	$k = 3$		$k = 7$		$k = 10$	
a	Exact	Appro.	Exact	Appro.	Exact	Appro.
5	5.9	6.6	5.5	5.8	5.4	5.6
10	10.8	11.6	10.6	10.8	10.5	10.6
15	15.9	16.6	15.7	15.8	15.5	15.6
20	21.0	21.6	20.7	20.8	20.6	20.6
25	26.1	26.6	25.7	25.8	25.6	25.6
30	31.2	31.6	30.7	30.8	30.6	30.6
35	36.2	36.6	35.7	35.8	35.6	35.6
40	41.2	41.6	40.7	40.8	40.6	40.6
45	46.2	46.6	45.7	45.8	45.6	45.6
50	51.3	51.6	50.7	50.8	50.6	50.6
55	56.3	56.6	55.7	55.8	55.6	55.6
60	61.3	61.6	60.7	60.8	60.6	60.6

Table 3.5: *Comparisons between the exact and approximate values of $E(T)$ for $m = 10$ and given values of k, α and a*

$\alpha = 0.1$

	$k = 3$		$k = 7$		$k = 10$	
a	Exact	Appro.	Exact	Appro.	Exact	Appro.
15	15.8	15.8	15.4	15.4	15.3	15.3
20	20.7	20.8	20.4	20.4	20.3	20.3
25	25.7	25.8	25.4	25.4	25.3	25.3
30	30.7	30.8	30.4	30.4	30.3	30.3
35	35.8	35.8	35.4	35.4	35.3	35.3
40	40.8	40.8	40.4	40.4	40.3	40.3
45	45.8	45.8	45.4	45.4	45.3	45.3
50	50.8	50.8	50.4	50.4	50.3	50.3
55	55.8	55.8	55.4	55.4	55.3	55.3
60	60.8	60.8	60.4	60.4	60.3	60.3

Table 3.5: *Comparisons between the exact and approximate values of $E(T)$ for $m = 10$ and given values of k, α and a*

$\alpha = 0.01$

a	$k = 3$		$k = 7$		$k = 10$	
	Exact	Appro.	Exact	Appro.	Exact	Appro.
15	16.4	16.6	15.7	15.8	15.6	15.6
20	21.4	21.6	20.7	20.8	20.6	20.6
25	26.5	26.6	25.8	25.8	25.6	25.6
30	31.5	31.6	30.8	30.8	30.6	30.6
35	36.5	36.6	35.8	35.8	35.6	35.6
40	41.5	41.6	40.8	40.8	40.6	40.6
45	46.5	46.6	45.8	45.8	45.6	45.6
50	51.5	51.6	50.8	50.8	50.6	50.6
55	56.5	56.6	55.8	55.8	55.6	55.6
60	61.5	61.6	60.8	60.8	60.6	60.6

Table 3.6: *Comparisons between the exact and simulated values of $E(T)$ and CL for $m = 10$ and $k = 10$ and given values of α and a*

$\alpha = 0.1$

	CL		$E(T)$	
a	Exact	Simul.	Exact	Simul.
15	0.898	0.902	15.3	15.3
20	0.899	0.897	20.3	20.4
25	0.899	0.892	25.3	25.3
30	0.899	0.901	30.3	30.3
35	0.899	0.898	35.3	35.3
40	0.900	0.898	40.3	40.3
45	0.900	0.893	45.3	45.3
50	0.900	0.904	50.3	50.3
55	0.900	0.898	55.3	55.3
60	0.900	0.896	60.3	60.3

Table 3.6: *Comparisons between the exact and simulated values of $E(T)$ and CL for $m = 10$ and $k = 10$ and given values of α and a*

$\alpha = 0.01$

a	CL		$E(T)$	
	Exact	Simul.	Exact	Simul.
15	0.990	0.990	15.6	15.6
20	0.990	0.989	20.6	20.6
25	0.990	0.991	25.6	25.5
30	0.990	0.989	30.6	30.6
35	0.990	0.991	35.6	35.5
40	0.990	0.990	40.6	40.5
45	0.990	0.990	45.6	45.5
50	0.990	0.991	50.6	50.6
55	0.980	0.990	55.6	55.6
60	0.980	0.991	60.6	60.6

Figure 1. The exact confidence level

as a function of $a = a(\sigma)$ for $m = 2$.

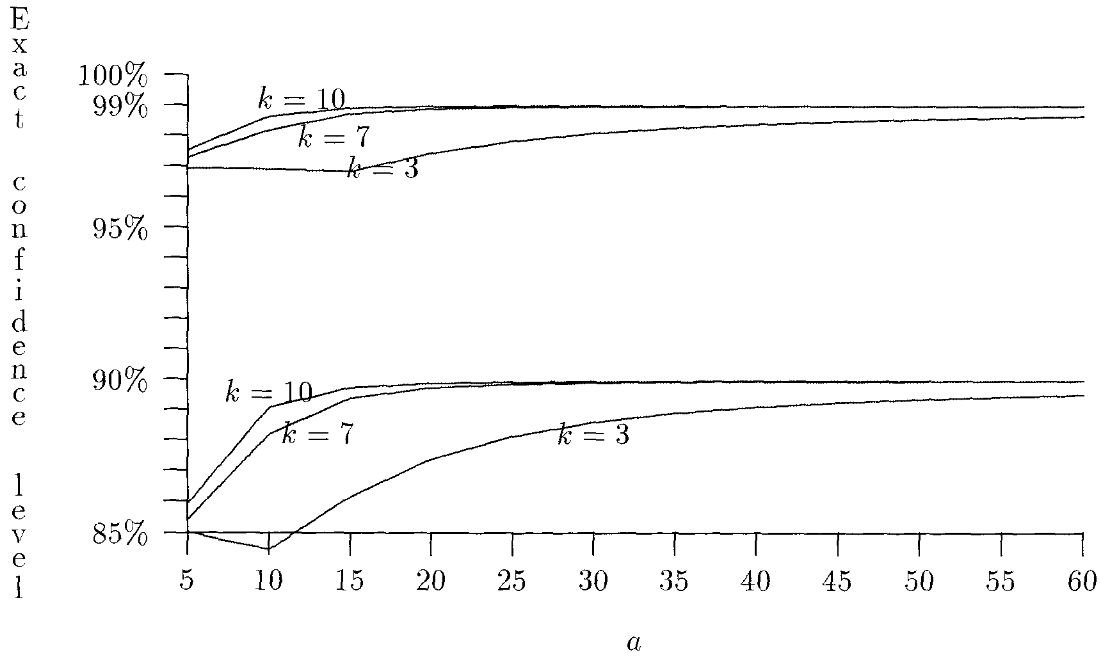
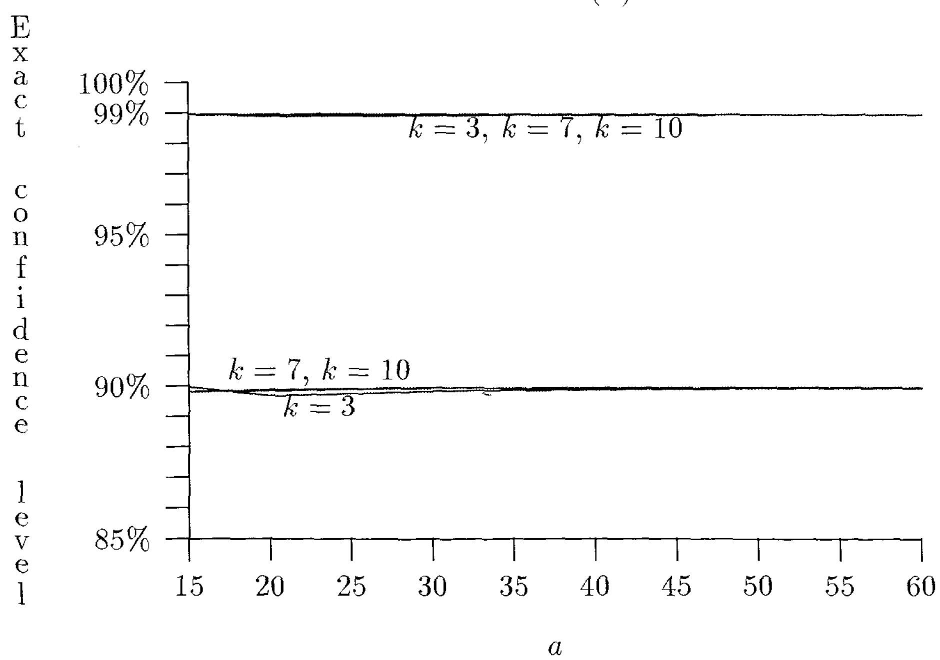


Figure 2. The exact confidence level

as a function of $a = a(\sigma)$ for $m = 10$.



3.2 Fixed-width simultaneous confidence intervals for comparing several treatments with a control

3.2.1 Introduction

Suppose we have k independently and normally distributed populations $N(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, k$ with unknown μ_i, $-\infty < \mu_i < \infty$, and a common unknown positive variance σ^2 and that we can sample sequentially from each population. In this section we construct a set of fixed-width $2d$ simultaneous confidence intervals for

$$\mu_i - \mu_1 \in \left(\bar{Y}_i - \bar{Y}_1 - d,\ \bar{Y}_i - \bar{Y}_1 + d\right), \quad i = 2, 3, \cdots, k,$$

with a (nominal) confidence level $1 - \alpha$, where $d > 0$ and $0 < \alpha < 1$ are two given constants and $\bar{Y}_i$ is the sample mean of a sample taken from the i^{th} population. The first population, $N(\mu_1, \sigma^2)$, may be regarded as a **control** and the other $k-1$ $(k \geq 2)$ populations as **treatments**. This set of confidence intervals can therefore be used to compare the treatments with the control.

Let $|T|_{k-1,\nu}$ denote the random variable

$$|T|_{k-1,\nu} = \max_{2 \leq i \leq k} \frac{|Z_i - Z_1|}{\sqrt{2}\sqrt{\chi^2_\nu/\nu}},$$

where $Z_1, Z_2, \cdots, Z_k$ are i.i.d. random variables and χ^2_ν is independent of $Z_1, Z_2, \cdots, Z_k$. Suppose that $|t|^\alpha_{k-1,\nu}$ is the upper α point of the distribution of $|T|_{k-1,\nu}$. The value of $|t|^\alpha_{k-1,\nu}$ for some combinations of $k-1$, ν and α can be found in Bechhofer and Dunnett (1988). If $\nu = \infty$ we have

$$|T|_{k-1,\infty} \equiv |T|_{k-1} = \max_{2 \leq i \leq k} \frac{|Z_i - Z_1|}{\sqrt{2}}.$$

Suppose a sample of fixed size n is taken from each of the k populations

$N(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, k$. Let $\hat{\sigma}_n^2$ be the pooled sample variance. Then

$$\max_{2 \le i \le k} \left\{ \frac{\sqrt{n}|\bar{Y}_{in} - \bar{Y}_{1n} - (\mu_i - \mu_1)|}{\hat{\sigma}_n \sqrt{2}} \right\}$$

has the same distribution as $|T|_{k-1,\nu}$ with $\nu = k(n-1)$ and so

$$P\left(\frac{\sqrt{n}|\bar{Y}_{in} - \bar{Y}_{1n} - (\mu_i - \mu_1)|}{\hat{\sigma}_n \sqrt{2}} < |t|_{k-1,\nu}^{\alpha}\ ,\ i = 2, 3, \cdots, k \right) = 1 - \alpha.$$

This can be written as

$$P\left\{ \bar{Y}_{in} - \bar{Y}_{1n} - |t|_{k-1,\nu}^{\alpha}\ \hat{\sigma}_n \sqrt{2/n} < \mu_i - \mu_1 < \bar{Y}_{in} - \bar{Y}_{1n} + |t|_{k-1,\nu}^{\alpha}\ \hat{\sigma}_n \sqrt{2/n},\ 2 \le i \le k \right\} = 1 - \alpha.$$

A set of simultaneous confidence intervals for the $\mu_i - \mu_1$ with confidence level $1 - \alpha$ is thus given by

$$\mu_i - \mu_1 \in \left(\bar{Y}_{in} - \bar{Y}_{1n} - |t|_{k-1,\nu}^{\alpha} \frac{\sqrt{2}\hat{\sigma}_n}{\sqrt{n}}, \bar{Y}_{in} - \bar{Y}_{1n} + |t|_{k-1,\nu}^{\alpha} \frac{\sqrt{2}\hat{\sigma}_n}{\sqrt{n}} \right), 2 \le i \le k. \tag{3.10}$$

This set of confidence intervals was proposed by Dunnett (1955, 1964).

As can be seen, the length of these confidence intervals is $2|t|_{k-1,\nu}^{\alpha}\ \hat{\sigma}_n \sqrt{2/n}$, which is a random number. As a matter of fact, in order to construct a set of fixed-width $2d$ and $(1 - \alpha)$-level simultaneous confidence intervals for the $\mu_i - \mu_1$ when σ^2 is unknown, it is necessary to use a sequential procedure. A two-stage procedure based on Stein's (1945) result was proposed by Tong (1969). Here we suggest a pure sequential procedure. To see the motivation behind the definition of this pure sequential procedure. Let us first look at the construction of a set of fixed-width $2d$ simultaneous confidence intervals for the $\mu_i - \mu_1$ when σ^2 is assumed to be a known constant.

Had σ^2 been known, the set of $1 - \alpha$ level confidence intervals in (3.10) becomes

$$\mu_i - \mu_1 \in \left(\bar{Y}_{in} - \bar{Y}_{1n} - |t|_{k-1}^{\alpha} \frac{\sqrt{2}\sigma}{\sqrt{n}}, \bar{Y}_{in} - \bar{Y}_{1n} + |t|_{k-1}^{\alpha} \frac{\sqrt{2}\sigma}{\sqrt{n}} \right), 2 \le i \le k.$$

In order that the width of these confidence intervals is at most $2d$, the sample size n from each of the k populations should satisfy $\sqrt{2}|t|_{k-1}^{\alpha}\ \sigma/\sqrt{n} \le d$, which implies that

$$n \geq 2d^{-2}(|t|_{k-1}^{\alpha})^2\sigma^2. \tag{3.11}$$

That is, when σ^2 is known, we take a sample of size n from each of the k populations where n satisfies (3.11), and then construct a set of simultaneous confidence intervals for the $\mu_i - \mu_1$ as

$$\mu_i - \mu_1 \in (\bar{Y}_{in} - \bar{Y}_{1n} - d,\ \bar{Y}_{in} - \bar{Y}_{1n} + d), \quad 2 \leq i \leq k.$$

This set of confidence intervals has width $2d$ and confidence level at least $1-\alpha$.

Now consider our problem in which σ^2 is unknown and so the right side of (3.11) can not be calculated explicitly. A reasonable sample size formula would be similar to (3.11) but with σ^2 replaced by some estimate. Precisely, we take the same number of observations, n, from each of the k populations, starting with m, increasing by one at a time, until

$$T = \inf\{n \geq m :\ n \geq 2d^{-2}\left(|t|_{k-1}^{\alpha}\right)^2 l_n\hat{\sigma_n}^2\}, \tag{3.12}$$

where $m \geq 2$ is the initial sample size from each population and $l_n = 1 + \frac{1}{n}l_0 + o(\frac{1}{n})$ as $n \to \infty$. On stopping sampling a set of simultaneous confidence intervals for $\mu_i - \mu_1$ is defined as

$$\mu_i - \mu_1 \in I_i(T) = (\bar{Y}_{iT} - \bar{Y}_{1T} - d,\ \bar{Y}_{iT} - \bar{Y}_{1T} + d), \quad 2 \leq i \leq k.$$

Next we show that the confidence level of this set of confidence intervals is approximately equal $1-\alpha$.

3.2.2 Second order approximations to the expected sample size and the confidence level

Applying the general results of Chapter 2, we can find the second order approximations to the expected sample size $E(T)$ and confidence level CL. By noting that the stopping time T in (3.12) is of the same form as the stopping time defined in (2.1) with $\gamma = 2(|t|_{k-1}^{\alpha})^2$, and so the following theorem follows directly from Theorem 2.1.

Theorem 3.3 *for $k \geq 1$ and $m > 1 + 2/k$, we have*

$$E(T) = b + \rho + l_0 - \frac{2}{k} + o(1) \quad as\ b \to \infty,$$

where $b = 2d^{-2}(|t|_{k-1}^{\alpha})^2\sigma^2$.

Note that b is the right side of (3.11), which can be regarded as the optimal sample size had σ^2 been known. From Theorem 3.3, the difference between the expected sample size of the pure sequential procedure and the optimal sample size b is about $\rho + l_0 - \frac{2}{k}$, a constant, at least for large b.

To obtain the second order approximation to the confidence level we first prove the following two lemmas.

Lemma 3.3 *For given $b > 0$*

$$P\{\mu_i - \mu_1 \in I_i(T),\ 2 \leq i \leq k\} = E\left[H\left((|t|_{k-1}^{\alpha})^2\frac{T}{b}\right)\right],$$

where $H(x) = P\{\max_{2\leq i\leq k} |Z_i - Z_1| \leq \sqrt{2x}\}$.

Proof: We have

$$\begin{aligned}
&P\{\mu_i - \mu_1 \in I_i(T),\ 2 \leq i \leq k\} \\
&= P\{\bar{Y}_{iT} - \bar{Y}_{1T} - d < \mu_i - \mu_1 < \bar{Y}_{iT} - \bar{Y}_{1T} + d,\ 2 \leq i \leq k\} \\
&= \sum_{n=m}^{\infty} P\{\bar{Y}_{2T} - \bar{Y}_{1T} - d < \mu_2 - \mu_1 < \bar{Y}_{2T} - \bar{Y}_{1T} + d, \cdots,
\end{aligned}$$

$$
\begin{aligned}
&\bar{Y}_{kT} - \bar{Y}_{1T} - d < \mu_k - \mu_1 < \bar{Y}_{kT} - \bar{Y}_{1T} + d | T = n\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\{\bar{Y}_{2n} - \bar{Y}_{1n} - d < \mu_2 - \mu_1 < \bar{Y}_{2n} - \bar{Y}_{1n} + d, \cdots, \\
&\quad \bar{Y}_{kn} - \bar{Y}_{1n} - d < \mu_k - \mu_1 < \bar{Y}_{kn} - \bar{Y}_{1n} + d | T = n\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\{\bar{Y}_{2n} - \bar{Y}_{1n} - d < \mu_2 - \mu_1 < \bar{Y}_{2n} - \bar{Y}_{1n} + d, \cdots, \\
&\quad \bar{Y}_{kn} - \bar{Y}_{1n} - d < \mu_k - \mu_1 < \bar{Y}_{kn} - \bar{Y}_{1n} + d\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\{|\bar{Y}_{in} - \bar{Y}_{1n} - (\mu_i - \mu_1)| < d, \;\; 2 \le i \le k\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\{\max_{2 \le i \le k} |Z_i - Z_1| < \frac{|t|_{k-1}^{\alpha} \sqrt{2n}}{\sqrt{b}}\} P\{T = n\} \\
&= E\left[H\left((|t|_{k-1}^{\alpha})^2 \frac{T}{b}\right)\right],
\end{aligned}
$$

as required.

Lemma 3.4 *Let* $H(x) = P\{\max_{2 \le i \le k} |Z_i - Z_1| \le \sqrt{2x}\}$, *and* $C_0 > 0$ *is a given constant. Then, for* $0 < x < C_0$, $|H''(x)| < Cx^{(k-5)/2}$ *where* C *is a constant.*

Proof: Let $g(x) = H(x^2)$, then

$$H(x) = g(x^{1/2})$$

$$H'(x) = \frac{1}{2} x^{-1/2} g'(x^{1/2}) \tag{3.13}$$

$$H''(x) = \frac{1}{4}\left[x^{-1} g''(x^{1/2}) - x^{-3/2} g'(x^{1/2})\right]. \tag{3.14}$$

Let $h(x, y) = \Phi(y + \sqrt{2}x) - \Phi(y - \sqrt{2}x)$, then

$$
\begin{aligned}
g(x) &= P\{|Z_i - Z_1| < \sqrt{2}x, \;\; 2 \le i \le k\} \\
&= \int_{-\infty}^{\infty} \phi(y) P\{|Z_i - Z_1| < \sqrt{2}x, \; 2 \le i \le k \big| Z_1 = y\} \, dy \\
&= \int_{-\infty}^{\infty} \phi(y) P\{|Z_i - y| < \sqrt{2}x, \; 2 \le i \le k\} \, dy \\
&= \int_{-\infty}^{\infty} \phi(y) \left(P\{|Z_i - y| < \sqrt{2}x\}\right)^{k-1} \, dy \\
&= \int_{-\infty}^{\infty} \phi(y) \left[\Phi(y + \sqrt{2}x) - \Phi(y - \sqrt{2}x)\right]^{k-1} \, dy,
\end{aligned}
$$

$$g'(x) = \sqrt{2}(k-1)\int_{-\infty}^{\infty} \phi(y)\left(\phi(y+\sqrt{2}x)+\phi(y-\sqrt{2}x)\right)(h(x,y))^{k-2}\ dy, \tag{3.15}$$

$$\begin{aligned} g''(x) &= 2(k-1)\int_{-\infty}^{\infty} \phi(y)\,(h(x,y))^{k-3} \times & (3.16)\\ &\quad \Big\{\Big[-(y+\sqrt{2}x)\phi(y+\sqrt{2}x)+(y-\sqrt{2}x)\phi(y-\sqrt{2}x)\Big]h(x,y)\\ &\quad +(k-2)\Big[\phi(y+\sqrt{2}x)+\phi(y-\sqrt{2}x)\Big]^2\Big\}\ dy. & (3.17)\end{aligned}$$

By noting that

$$h(x,y) = \int_{y-\sqrt{2}x}^{y+\sqrt{2}x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\ dz < C_1 x$$

where C_1 is a constant, we have that

$$\begin{aligned} g'(x) &= \sqrt{2}(k-1)\int_{-\infty}^{\infty} \phi(y)\left(\phi(y+\sqrt{2}x)+\phi(y-\sqrt{2}x)\right)(h(x,y))^{k-2}\ dy\\ &< Mx^{k-2}\int_{-\infty}^{\infty}\phi(y)\ dy\\ &= Mx^{k-2} \end{aligned} \tag{3.18}$$

where M is a constant, and that

$$\begin{aligned} &|g''(x)|\\ \le\ & D_1 x^{k-2}\int_{-\infty}^{\infty}\phi(y)\left\{|-(y+\sqrt{2}x)\phi(y+\sqrt{2}x)+(y-\sqrt{2}x)\phi(y-\sqrt{2}x)|\right\}\ dy\\ &+Lx^{k-3}\\ \le\ & D_2 x^{k-2}\int_{-\infty}^{\infty}\phi(y)\left\{|y|\phi(y+\sqrt{2}x)+\sqrt{2}x\phi(\sqrt{2}x+y)+|y|\phi(y-\sqrt{2}x)\right\}\ dy\\ &+D_2 x^{k-2}\int_{-\infty}^{\infty}\phi(y)\sqrt{2}x\phi(y-\sqrt{2}x)\ dy + Lx^{k-3}\\ \le\ & D_3 x^{k-2}\left(\int_0^{\infty} y\phi(y)\ dy+\sqrt{2}x\int_{-\infty}^{\infty}\phi(y)\ dy\right) + Lx^{k-3}\\ \le\ & Dx^{k-2}(A+Bx)+Lx^{k-3}\\ \le\ & L_1 x^{k-3},\quad \text{for } 0<x<C_0 \end{aligned} \tag{3.19}$$

where $D_1, D_2, D_3, D, A, A_1, B, B_1, L$, and L_1 are constants. It now follows from (3.18), (3.19) and (3.14) that

$$|H''(x)| \ \le\ \frac{1}{4}\left(x^{-1}\,|g''(x^{1/2})| + x^{-3/2}\,|g'(x^{1/2})|\right)$$

$$\leq \ Cx^{(k-5)/2} \, .$$

The proof is thus completed.

The following theorem gives the second order approximation to the confidence level and follows directly from Lemma 3.3 and Theorem 2.2 with $\theta = \left(|t|_{k-1}^{\alpha}\right)^2$, $C_1 = 0$, $\beta = (5-k)/2$ and $n_0 = b$.

Theorem 3.4 *Suppose that $m > 1$ if $k \geq 5$, and $m > 1 + (7-k)/k$ if $k = 2, 3, 4$, then*

$$\begin{aligned}
&P\{\mu_2 - \mu_1 \in I_1(T), \cdots, \mu_k - \mu_1 \in I_{k-1}(T)\} \\
&= 1 - \alpha + \frac{1}{b}\Big[\left(|t|_{k-1}^{\alpha}\right)^2 H'\left(\left(|t|_{k-1}^{\alpha}\right)^2\right)\left(\rho + l_0 - \frac{2}{k}\right) \\
&+ \frac{1}{k}\left(|t|_{k-1}^{\alpha}\right)^4 H''\left(\left(|t|_{k-1}^{\alpha}\right)^2\right)\Big] + o\left(\frac{1}{b}\right),
\end{aligned}$$

where $H(x) = P\{\max_{2 \leq i \leq k} |Z_i - Z_1| \leq \sqrt{2x}\}$.

3.2.3 Calculations of the approximate values of the expected sample size and the confidence level

From Theorem 3.4 it can be seen that the value of l_0 can be chosen to satisfy

$$\left(|t|_{k-1}^{\alpha}\right)^2 H'\left(\left(|t|_{k-1}^{\alpha}\right)^2\right)\left(\rho + l_0 - \frac{2}{k}\right) + \frac{1}{k}\left(|t|_{k-1}^{\alpha}\right)^4 H''\left(\left(|t|_{k-1}^{\alpha}\right)^2\right) = 0$$

so that the CL is equal to $1 - \alpha + o(1/b)$. This $l_0 = l_0(k, \alpha)$ is given by

$$l_0 = \frac{1}{k}\left[2 - \frac{\left(|t|_{k-1}^{\alpha}\right)^2 H''\left(\left(|t|_{k-1}^{\alpha}\right)^2\right)}{H'\left(\left(|t|_{k-1}^{\alpha}\right)^2\right)}\right] - \rho, \tag{3.20}$$

where the functions $H'(\cdot)$, and $H''(\cdot)$ are given in (3.13), and (3.14). In order to calculate $l_0(k, \alpha)$, I have calculated the values of $|t|_{k-1}^{\alpha}$ for $\alpha = 0.1, 0.05, 0.01$ and $k = 2(1)\,20$ and they are given in Table 3.7. The value of $l_0(k, \alpha)$ can now be calculated from (3.20) and the results for $\alpha = 0.1, 0.05, 0.01$ and $k = 2(1)\,20$ are given in Table 3.8.

From Theorem 3.3, the approximate value of $E(T)$ is

$$b + \rho + l_0 - 2/k.$$

This approximate value, corresponding to $l_0 = l_0(k, \alpha)$, is given in Tables 3.9 and 3.10.

Table 3.7: $|t|_{k-1}^{\alpha}$

$k-1 \setminus \alpha$	0.1	0.05	0.01
1	1.645	1.960	2.574
2	1.916	2.213	2.794
3	2.062	2.350	2.916
4	2.160	2.442	2.990
5	2.233	2.511	3.062
6	2.292	2.567	3.111
7	2.340	2.613	3.150
8	2.381	2.652	3.189
9	2.417	2.686	3.219
10	2.448	2.716	3.248
11	2.476	2.743	3.272
12	2.501	2.767	3.292
13	2.525	2.789	3.316
14	2.546	2.810	3.331
15	2.566	2.828	3.351
16	2.583	2.846	3.365
17	2.600	2.862	3.380
18	2.615	2.877	3.394
19	2.631	2.892	3.409

Table 3.8: $l_0 = l_0(k, \alpha)$

$k-1 \setminus \alpha$	0.1	0.05	0.01
1	1.182	1.466	2.162
2	0.697	0.906	1.402
3	0.425	0.590	0.977
4	0.255	0.391	0.708
5	0.138	0.253	0.524
6	0.053	0.154	0.389
7	0.012	0.078	0.285
8	-0.063	0.019	0.206
9	-0.104	-0.029	0.140
10	-0.137	-0.069	0.087
11	-0.165	-0.103	0.042
12	-0.189	-0.131	0.002
13	-0.210	-0.155	-0.030
14	-0.228	-0.176	-0.056
15	-0.243	-0.195	-0.085
16	-0.257	-0.211	-0.108
17	-0.270	-0.226	-0.128
18	-0.281	-0.240	-0.146
19	-0.291	-0.251	-0.162

3.2.4 Exact calculations of the expected sample size and the confidence level

Let $t = T - 1$, then from Lemma 2.16 and the argument after Lemma 2.16, we have

$$R_{n+1}(x) = \int_{C_n}^{x} R_n(y) f_{\chi_k^2}(x-y)\, dy \tag{3.21}$$

and

$$P\{t > n+1\} = \int_{C_{n+1}}^{\infty} R_{n+1}(y)\, dy, \qquad n \geq m_0 - 1, \tag{3.22}$$

where

$$C_n = \frac{kn(n+1)}{b\left(1 + \frac{l_0}{n+1}\right)}$$

and the value of l_0 is given in the Table 3.8. Consequently

$$E(T) = 1 + \sum_{n=m_0}^{\infty} n[P(t > n-1) - P(t > n)] \tag{3.23}$$

and

$$\begin{aligned} CL &= E\left[H\left(\left(|t|_{k-1}^{\alpha}\right)^2 \frac{T}{b}\right)\right] \\ &= \sum_{n=m_0}^{\infty} P(t=n) H\left(\left(|t|_{k-1}^{\alpha}\right)^2 \frac{n+1}{b}\right) \\ &= \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)] H\left(\left(|t|_{k-1}^{\alpha}\right)^2 \frac{n+1}{b}\right), \end{aligned} \tag{3.24}$$

where $H(x) = P\{\max_{2 \leq i \leq k} |Z_i - Z_1| \leq \sqrt{2x}\}$. The functions $R_{n+1}(\cdot)$ and, thus $E(T)$ and CL, can be calculated.

In Subsection 3.2.5, we give the results of this calculation which are based on a grid of equal width $h = 0.1$. We also use grids based on $h = 0.2$ and $h = 0.05$ to find the values of CL, we find some difference in the fourth decimal place from those based on $h = 0.1$. We simulate the $E(T)$ and CL based on 6,000 experiments and some of the results are given in Table 3.11.

3.2.5 Some comparisons

The aim of this subsection is to compare the second order approximations with the exact calculations of the $E(T)$ and the CL. Throughout, the value of l_0 is given by $l_0 = l_0(k, \alpha)$. From these comparisons we can see when the second order approximations are reasonably accurate.

Firstly, we look at the confidence level CL. The confidence level is equal to $1 - \alpha$ (nominal level) plus an error term of order $o(1/b)$ as $b \to \infty$ and so the approximate is $1 - \alpha$. The true value of the confidence level, however, depends on b, i.e. $CL = CL(b)$. For $m = 2, k - 1 = 3, 7, 10$, and $1 - \alpha = 90\%, 99\%$, the exact calculation results of $CL(b)$ at $b = 5(5)60$ are linearly plotted in Figure 3. Figure 4 gives the similar plots for $m = 10$ and $b = 15(5)60$. From Figures 3 and 4 it can be seen that $CL(b)$ is generally closer to the nominal level $1 - \alpha$ for: (i) larger b; (ii) larger k; (iii) larger nominal level $1 - \alpha$; (iv) larger initial sample size m.

Next, we look at the expected sample size $E(T)$. When b is large, the approximation to $E(T)$ is $b + \rho + l_0 - 2/k$. For $m = 2$, $k - 1 = 3$, 7, 10, and $1 - \alpha = 90\%$, 99%, Table 3.9 contains the exact values of $E(T)$ calculated using the recursive method and the approximate values of $E(T)$ at $b = 5(5)60$. Table 3.10 contains the similar results for $m = 10$ and $b = 15(5)60$. From Table 3.9 and 3.10 it can be seen that the approximate value of $E(T)$ are generally closer to the value of $E(T)$ for: (i) large b; (ii) large k; (iii) large initial sample size m. The exact calculations of the $E(T)$ and the CL become quite computationally intensive for $b \geq 60$. However, when $b \geq 60$ the approximations are very good, as can be seen from the results given in this subsection. So approximated results can be used in this case.

Table 3.9: *Comparisons between the exact and approximate values of $E(T)$ for $m = 2$ and given values of k, α and b*

$\alpha = 0.1$

b	$k = 4$		$k = 8$		$k = 11$	
	Exact	Appro.	Exact	Appro.	Exact	Appro.
5	5.1	5.6	5.1	5.3	5.1	5.3
10	10.1	10.6	10.2	10.3	10.2	10.3
15	15.2	15.6	15.3	15.3	15.2	15.3
20	20.3	20.6	20.3	20.3	20.2	20.3
25	25.3	25.6	25.3	25.3	25.2	25.3
30	30.4	30.6	30.3	30.3	30.2	30.3
35	35.4	35.6	35.3	35.3	35.2	35.3
40	40.4	40.6	40.3	40.3	40.2	40.3
45	45.4	45.6	45.3	45.3	45.2	45.3
50	50.5	50.6	50.3	50.3	50.2	50.3
55	55.5	55.6	55.3	55.3	55.2	55.3
60	60.5	60.6	60.3	60.3	60.2	60.3

Table 3.9: *Comparisons between the exact and approximate values of $E(T)$ for $m = 2$ and given values of k, α and b*

$\alpha = 0.01$

b	$k = 4$		$k = 8$		$k = 11$	
	Exact	Appro.	Exact	Appro.	Exact	Appro.
5	5.6	6.1	5.4	5 .6	5.3	5.5
10	10.7	11.1	10.5	10.6	10.4	10.5
15	15.8	16.1	15.6	15.6	15.5	15.5
20	20.9	21.1	20.6	20.6	20.5	20.5
25	25.9	26.1	25.6	25.6	25.5	25.5
30	31.0	31.1	30.6	30.6	30.5	30.5
35	36.0	36.1	35.6	35.6	35.5	35.5
40	41.0	41.1	40.6	40.6	40.5	40.5
45	46.0	46.1	45.6	45.6	45.5	45.5
50	51.0	51.1	50.6	50.6	50.5	50.5
55	56.0	56.1	55.6	55.6	55.5	55.5
60	61.0	61.1	60.6	60.6	60.5	60.5

Table 3.10: *Comparisons between the exact and approximate values of $E(T)$ for $m = 10$ and given values of k, α and b*

$\alpha = 0.1$

	$k = 4$		$k = 8$		$k = 11$	
b	Exact	Appro.	Exact	Appro.	Exact	Appro.
15	15.6	15.6	15.3	15.3	15.2	15.3
20	20.5	20.6	20.3	20.3	20.2	20.3
25	25.5	25.6	25.3	25.3	25.3	25.3
30	30.5	30.6	30.3	30.3	30.3	30.3
35	35.6	35.6	35.3	35.3	35.3	35.3
40	40.6	40.6	40.3	40.3	40.2	40.3
45	45.6	45.6	45.3	45.3	45.2	45.3
50	50.6	50.6	50.3	50.3	50.2	50.3
55	55.6	55.6	55.3	55.3	55.2	55.3
60	60.6	60.6	60.3	60.3	60.2	60.3

Table 3.10: *Comparisons between the exact and approximate values of $E(T)$ for $m = 10$ and given values of k, α and b*

$\alpha = 0.01$

b	$k = 4$ Exact	$k = 4$ Appro.	$k = 8$ Exact	$k = 8$ Appro.	$k = 11$ Exact	$k = 11$ Appro.
15	16.1	16.1	15.6	15.6	15.5	15.5
20	21.1	21.1	20.6	20.6	20.5	20.5
25	26.1	26.1	25.6	25.6	25.5	25.5
30	31.1	31.1	30.6	30.6	30.5	30.5
35	36.1	36.1	35.6	35.6	35.5	35.5
40	41.1	41.1	40.6	40.6	40.5	40.5
45	46.1	46.1	45.6	45.6	45.5	45.5
50	51.1	51.1	50.6	50.6	50.5	50.5
55	56.1	56.1	55.6	55.6	55.5	55.5
60	61.1	61.1	60.6	60.6	60.5	60.5

Table 3.11: *Comparisons between the exact and simulated values of $E(T)$ and CL for $m = 10$ and $k = 11$ and given values of α and b*

$\alpha = 0.1$

	CL		$E(T)$	
b	Exact	Simul.	Exact	Simul.
15	0.899	0.899	15.2	15.2
20	0.899	0.894	20.2	20.3
25	0.899	0.898	25.3	25.2
30	0.900	0.890	30.3	30.3
35	0.900	0.896	35.3	35.2
40	0.900	0.897	40.2	40.3
45	0.900	0.904	45.2	45.3
50	0.900	0.899	50.2	50.3
55	0.900	0.899	55.2	55.3
60	0.900	0.899	60.2	60.3

Table 3.11: *Comparisons between the exact and simulated values of $E(T)$ and CL for $m = 10$ and $k = 11$ and given values of α and b*

$\alpha = 0.01$

b	CL		$E(T)$	
	Exact	Simul.	Exact	Simul.
15	0.990	0.990	15.5	15.5
20	0.990	0.991	20.5	20.6
25	0.990	0.990	25.5	25.5
30	0.990	0.989	30.5	30.5
35	0.990	0.991	35.5	35.5
40	0.990	0.989	40.5	40.5
45	0.990	0.990	45.5	45.6
50	0.990	0.989	50.5	50.6
55	0.990	0.990	55.5	55.5
60	0.990	0.991	60.5	60.6

Figure 3. The exact confidence level

as a function of $b = b(\sigma)$ for $m = 2$.

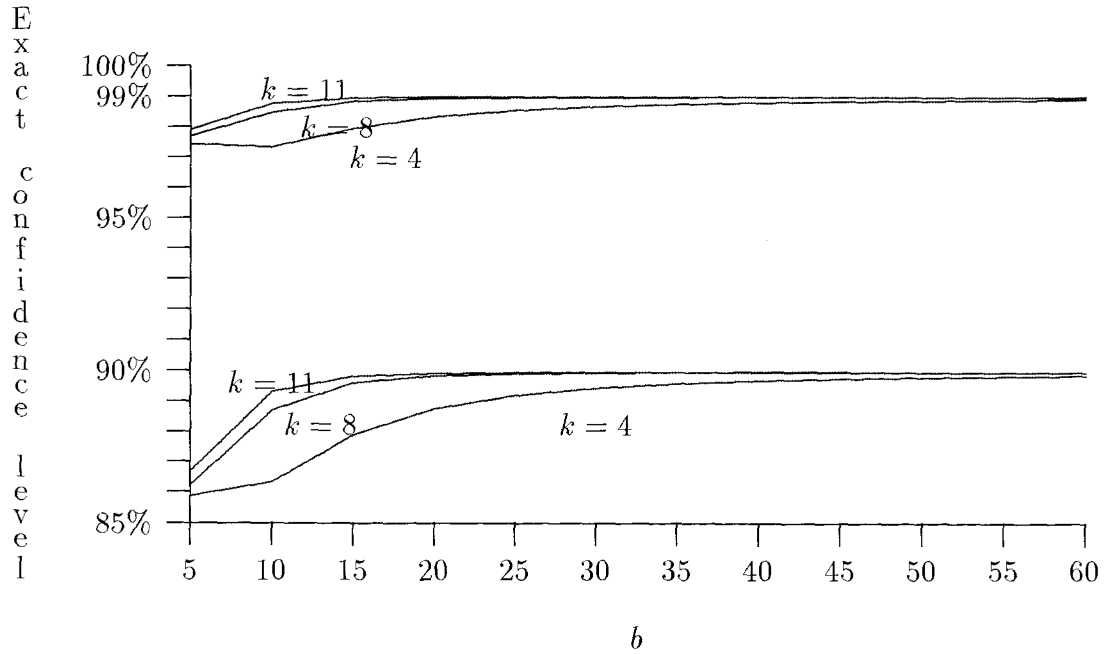
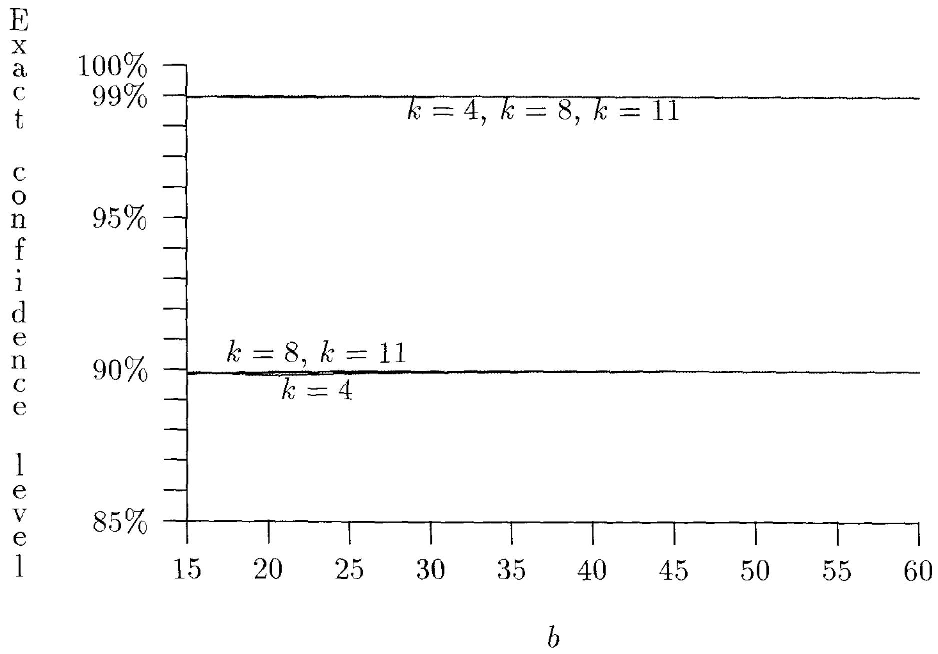


Figure 4. The exact confidence level

as a function of $b = b(\sigma)$ for $m = 10$.



3.3 Fixed-width simultaneous confidence intervals for all-pairwise comparisons of the means of several independent normal populations

3.3.1 Introduction

Suppose we have k independent, normally distributed populations $N(\mu_i, \sigma^2)$, $i = 1, 2, \cdots, k$ with unknown $\mu_i, -\infty < \mu_i < \infty$, and a common unknown positive variance σ^2. Assume we can sample sequentially from each population and that $Y_{i1}, Y_{i2}, Y_{i3}, \cdots$ denote the observations from the i^{th} population, $i = 1, 2, \cdots, k$. In this section we construct a set of fixed-width $2d$ simultaneous confidence intervals for all-pairwise differences $\mu_i - \mu_j$ of the form

$$\mu_i - \mu_j \in (\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d), \quad 1 \leq i \neq j \leq k$$

with a (nominal) confidence level $1 - \alpha$, where $\bar{Y}_i$ is the sample mean of a sample taken from the i^{ih} population, and $d > 0$ and $0 < \alpha < 1$ are two given constants.

Suppose $Z_1, Z_2, \cdots, Z_k$ are i.i.d. $N(0,1)$ random variables, and χ^2_ν is independent of $Z_1, Z_2, \cdots, Z_k$. Let $Q_{k,\nu}$ denote the random variable

$$Q_{k,\nu} = \max_{1 \leq i \neq j \leq k} \frac{Z_i - Z_j}{\sqrt{\chi^2_\nu / \nu}}.$$

The distribution of $Q_{k,\nu}$ is called the studentised range distribution with parameters k and ν. If $\nu = \infty$ then $\chi^2_\infty / \infty = 1$ and hence the distribution of $Q_{k,\infty}$ is the same as

$$Q_k = \max_{1 \leq i \neq j \leq k} (Z_i - Z_j).$$

Suppose that $q^\alpha_{k,\nu}$ is the upper α point of the studentised range distribution with parameters k and ν. The value of $q_{k,\nu}^{\alpha}$ for some combinations of k, α and ν can be found in Harter (1969).

Suppose a sample of size n is taken from each of the k populations $N(\mu_i, \sigma^2)$, $1 \le i \le k$ and $\hat{\sigma}_n^2$ is the pooled sample variance. Then

$$\max_{1\le i\ne j\le k} \left\{ \frac{\sqrt{n}\left((\bar{Y}_{in} - \mu_i) - (\bar{Y}_{jn} - \mu_j)\right)}{\hat{\sigma}_n} \right\}$$

has the same distribution as $Q_{k,\nu}$ with $\nu = k(n-1)$ and so

$$P\left(|\bar{Y}_{in} - \bar{Y}_{jn} - (\mu_i - \mu_j)| \le q_{k,\nu}^{\alpha} \frac{\hat{\sigma}_n}{\sqrt{n}},\ 1 \le i \ne j \le k\right) = 1 - \alpha.$$

This can be written as

$$P\left(\bar{Y}_{in} - \bar{Y}_{jn} - q_{k,\nu}^{\alpha} \frac{\hat{\sigma}_n}{\sqrt{n}} \le \mu_i - \mu_j \le \bar{Y}_{in} - \bar{Y}_{jn} + q_{k,\nu}^{\alpha} \frac{\hat{\sigma}_n}{\sqrt{n}},\ 1 \le i \ne j \le k\right) = 1-\alpha.$$

A set of simultaneous confidence intervals for the $\mu_i - \mu_j$ with confidence level $1-\alpha$ is thus given by

$$\mu_i - \mu_j \in \left(\bar{Y}_{in} - \bar{Y}_{jn} - q_{k,\nu}^{\alpha} \frac{\hat{\sigma}_n}{\sqrt{n}},\ \bar{Y}_{in} - \bar{Y}_{jn} + q_{k,\nu}^{\alpha} \frac{\hat{\sigma}_n}{\sqrt{n}}\right), \quad 1 \le i \ne j \le k. \tag{3.25}$$

This set of confidence intervals was proposed by Tukey (1952a, 1953).

The length of these confidence intervals is $2q_{k,\nu}^{\alpha}\hat{\sigma}_n/\sqrt{n}$, which is a random number. In order to construct a set of fixed-width $2d$ and $1-\alpha$ level simultaneous confidence intervals for all-pairwise differences $\mu_i - \mu_j$ when σ^2 is unknown, it is necessary to use a sequential procedure. A two-stage procedure based on Stein's (1945) result was proposed by Hochberg and Lachenbruch (1976). Here we look at a pure sequential procedure, which was proposed by Liu (1995a). To motivate the definition of this pure sequential procedures, let us first look at the construction of a set of fixed-width $2d$ and $1-\alpha$ level simultaneous confidence intervals for the $\mu_i - \mu_j$ when σ^2 is assumed to be a known constant.

Had σ^2 been known, the set of $1-\alpha$ level confidence intervals in (3.25) becomes

$$\mu_i - \mu_j \in \left(\bar{Y}_{in} - \bar{Y}_{jn} - q_k^\alpha \frac{\sigma}{\sqrt{n}},\ \bar{Y}_{in} - \bar{Y}_{jn} + q_k^\alpha \frac{\sigma}{\sqrt{n}}\right), \quad 1 \le i \ne j \le k.$$

In order that the width of these confidence intervals is at most $2d$, the sample size n from each of the k populations should satisfy $q_k^\alpha \sigma/\sqrt{n} \le d$, which implies that

$$n \ge d^{-2}(q_k^\alpha)^2\sigma^2. \tag{3.26}$$

That is, when σ^2 is known, we take a sample of size n from each of the k populations where n satisfies (3.26), and then construct a set of simultaneous confidence intervals for the $\mu_i - \mu_j$ as

$$\mu_i - \mu_j \in (\bar{Y}_{in} - \bar{Y}_{jn} - d,\ \bar{Y}_{in} - \bar{Y}_{jn} + d), \quad 1 \le i \ne j \le k.$$

This set of confidence intervals has width $2d$ and confidence level at least $1-\alpha$.

Now consider our problem in which σ^2 is unknown and so the right side of (3.26) can not be calculated explicitly. A reasonable sample size formula would be similar to (3.26) but with σ^2 replaced by some estimate. Precisely, we take the same number of observations, n, from each of the k populations, starting with m, increasing by one at a time, until

$$T = \inf\{n \ge m :\ n > d^{-2}\,(q_k^\alpha)^2\, l_n \hat{\sigma_n}^2\}. \tag{3.27}$$

where $m \ge 2$ is the initial sample size from each population and $l_n = 1 + \frac{1}{n}l_0 + o(\frac{1}{n})$ as $n \to \infty$. On stopping sampling the set of simultaneous confidence intervals for $\mu_i - \mu_j$ is defined as

$$\mu_i - \mu_j \in I_{ij}(T) = (\bar{Y}_{iT} - \bar{Y}_{jT} - d,\ \bar{Y}_{iT} - \bar{Y}_{jT} + d), \quad 1 \le i \ne j \le k.$$

Next we show that the confidence level of this set of confidence intervals is approximately equal $1-\alpha$.

3.3.2 Second order approximations to the expected sample size and the confidence level

In this subsection, we use the results of Chapter 2 to find the second order approximations to the expected sample size $E(T)$ and confidence level CL. As we can see the stopping time T in (3.27) is of the same form as the stopping time defined in (2.1) with $\gamma = (q_k^\alpha)^2$, and so the following theorem follows directly from Theorem 2.1.

Theorem 3.5 *For $k \geq 1$ and $m > 1 + 2/k$, we have*

$$E(T) = c + \rho + l_0 - \frac{2}{k} + o(1) \quad as\ c \to \infty,$$

where $c = d^{-2}(q_k^\alpha)^2\sigma^2$.

The value of c, given on the right side of (3.26), can be regarded as the optimal sample size had σ^2 been known. From Theorem 3.5, at least for large c, the difference between the expected sample size of the pure sequential procedure and the optimal sample size c is about $\rho + l_0 - \frac{2}{k}$, a constant.

Now we derive the second order approximation to the confidence level. For this, we require the following lemmas.

Lemma 3.5 *For given $c > 0$,*

$$CL = P\Big\{\max_{1\leq i\neq j\leq k} |\bar{Y}_{iT} - \mu_i - \bar{Y}_{jT} + \mu_j| \leq d\Big\} = E\left[H\left((q_k^\alpha)^2\frac{T}{c}\right)\right],$$

where $H(x) = P\{\max_{1\leq i\neq j\leq k} |Z_i - Z_j| \leq \sqrt{x}\}$.

Proof: We have

$$\begin{aligned}
&P\Big\{\max_{1\leq i\neq j\leq k} |\bar{Y}_{iT} - \mu_i - \bar{Y}_{jT} + \mu_j| \leq d\Big\} \\
&= \sum_{n=m}^{\infty} P\Big\{\max_{1\leq i\neq j\leq k} |\bar{Y}_{iT} - \mu_i - \bar{Y}_{jT} + \mu_j| \leq d \Big| T = n\Big\} P\{T = n\} \\
&= \sum_{n=m}^{\infty} P\Big\{\max_{1\leq i\neq j\leq k} |\bar{Y}_{in} - \mu_i - \bar{Y}_{jn} + \mu_j| \leq d \Big| T = n\Big\} P\{T = n\}
\end{aligned}$$

$$= \sum_{n=m}^{\infty} P\Big\{\max_{1\le i\neq j\le k} |\bar{Y}_{in} - \mu_i - \bar{Y}_{jn} + \mu_j| \le d\Big\} P\{T = n\}$$
$$= \sum_{n=m}^{\infty} P\Big\{\max_{1\le i\neq j\le k} |Z_i - Z_j| < \frac{d\sqrt{n}}{\sigma}\Big\} P\{T = n\}$$
$$= E\left[H\left((q_k^\alpha)^2 \frac{T}{c}\right)\right],$$

where $c = d^{-2}(q_k^\alpha)^2\sigma^2$, as required.

Lemma 3.6 *Let $H(x) = P\{\max_{1\le i\neq j\le k} |Z_i - Z_j| \le \sqrt{x}\}$, and $C_0 > 0$ is a given constant. Then, for $0 < x < C_0$, $|H''(x)| < Cx^{(k-5)/2}$ where C is a constant.*

Proof: Let $g(x) = H(x^2)$, then

$$H(x) = g(x^{1/2}),$$
$$H'(x) = \frac{1}{2}x^{-1/2}\, g'(x^{1/2}), \tag{3.28}$$
$$H''(x) = \frac{1}{4}\left[x^{-1} g''(x^{1/2}) - x^{-3/2} g'(x^{1/2})\right]. \tag{3.29}$$

Let $h(x, y) = \Phi(y) - \Phi(y - x)$, then

$$\begin{aligned} g(x) &= P\Big\{\max_{1\le i\neq j\le k} |Z_i - Z_j| \le x\Big\} \\ &= kP\{Z_1 - x < Z_2 < Z_1, Z_1 - x < Z_3 < Z_1, \cdots, Z_1 - x < Z_k < Z_1\} \\ &= k\int_{-\infty}^{\infty} \phi(y) P\{y - x < Z_2 < y, \cdots, y - x < Z_k < y \Big| Z_1 = y\}\, dy \\ &= k\int_{-\infty}^{\infty} \phi(y)\Big[P\{y - x < Z_2 < y\}\Big]^{k-1}\, dy \\ &= k\int_{-\infty}^{\infty} \phi(y)\Big[\Phi(y) - \Phi(y - x)\Big]^{k-1}\, dy, \end{aligned}$$

$$g'(x) = k(k-1)\int_{-\infty}^{\infty} \phi(y)\phi(y - x)\Big[\Phi(y) - \Phi(y - x)\Big]^{k-2}\, dy, \tag{3.30}$$

$$\begin{aligned} g''(x) &= k(k-1)(k-2)\int_{-\infty}^{\infty} \phi(y)\left(\phi(y - x)\right)^2 \Big[\Phi(y) - \Phi(y - x)\Big]^{k-3}\, dy \\ &+ k(k-1)\int_{-\infty}^{\infty} (y - x)\phi(y)\phi(y - x)\Big[\Phi(y) - \Phi(y - x)\Big]^{k-2}\, dy. \end{aligned} \tag{3.31}$$

By noting that

$$h(x,y) = \int_{y-x}^{y} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\ dz < C_1 x$$

where C_1 is a constant, we have

$$g^{'}(x) < Ax^{k-2} \int_{-\infty}^{\infty} \phi(y)\ dy = Ax^{k-2}\,, \tag{3.32}$$

and

$$\begin{aligned} |g''(x)| &< D_1 x^{k-3} \int_{-\infty}^{\infty} \phi(y)\ dy + D_2 x^{k-2} \int_{-\infty}^{\infty} \phi(y)\ dy \\ &+ D_3 x^{k-1} \int_{-\infty}^{\infty} \phi(y)\ dy \\ &\le Dx^{k-3}\,, \end{aligned} \tag{3.33}$$

where D_1, D_2 D_3, and D are constants. It now follow from (3.32), (3.33) and (3.29) that

$$\begin{aligned} |H''(x)| &\le \frac{1}{4}\left[x^{-1}\,|g''(x^{1/2})| + x^{-3/2}\,|g'(x^{1/2})|\right] \\ &\le Cx^{(k-5)/2}\,. \end{aligned}$$

This finishes the proof.

By using Theorem 2.2 with $\theta = (q_k^{\alpha})^2$, $C_1 = 0$, $\beta = (5-k)/2$ and $n_0 = c$, and using Lemma 3.6, we have the following theorem.

Theorem 3.6 *Suppose that $l_n = 1 + l_0/n + o(1/n)$ as $n \to \infty$, and $m > 1$ if $k \ge 5$ and $m > 1 + (7-k)/k$ if $k = 2,3,4$, then*

$$\begin{aligned} &P\Big\{\mu_i - \mu_j \in I_{ij}(T),\quad 1 \le i \ne j \le k\Big\} \\ &= 1 - \alpha + \frac{1}{c}\Big[(q_k^{\alpha})^2\, H'\left((q_k^{\alpha})^2\right)\left(\rho + l_0 - \frac{2}{k}\right) \\ &+ \frac{1}{k}\,(q_k^{\alpha})^4\, H''\left((q_k^{\alpha})^2\right)\Big] + o\left(\frac{1}{c}\right), \end{aligned}$$

where $H(x) = P\{\max_{1\le i\ne j\le k}\ |Z_i - Z_j| \le \sqrt{x}\}$ and $c = d^{-2}(q_k^{\alpha})^2\sigma^2$.

3.3.3 Calculations of the approximate values of the expected sample size and the confidence level

From Theorem 3.6 it can be seen that the value of l_0 can be chosen to satisfy

$$(q_k^\alpha)^2 H'\left((q_k^\alpha)^2\right)\left(\rho + l_0 - \frac{2}{k}\right) + \frac{1}{k}(q_k^\alpha)^4 H''\left((q_k^\alpha)^2\right) = 0$$

so that the CL is equal to $1 - \alpha + o(1/c)$. This l_0 is given by

$$l_0 = \frac{1}{k}\left[2 - \frac{(|q|_k^\alpha)^2 H''\left((|q|_k^\alpha)^2\right)}{H'\left((|q|_k^\alpha)^2\right)}\right] - \rho, \tag{3.34}$$

where the functions $H'(\cdot)$ and $H''(\cdot)$ are given in (3.28) and (3.29). In order to calculate $l_0(k, \alpha)$, I have calculated the values of q_k^α for $\alpha = 0.1, 0.05, 0.01$ and $k = 2(1)\,20$ and they are given in Table 3.12. The values of $\rho = \rho(k)$ have already been given in Table 3.1. The value of $l_0(k, \alpha)$ can now be calculated from (3.34) and the results for $\alpha = 0.1, 0.05, 0.01$ and $k = 2(1)\,20$ are given in Table 3.13.

From Theorem 3.5, the approximate value of $E(T)$ is

$$c + \rho + l_0 - 2/k.$$

This approximate value corresponding to $l_0 = l_0(k, \alpha)$ is given in Tables 3.14 and 3.15.

Table 3.12: $|q|_k^\alpha$

$k \setminus \alpha$	0.1	0.05	0.01
2	2.326	2.771	3.644
3	2.902	3.314	4.120
4	3.240	3.633	4.405
5	3.478	3.857	4.605
6	3.661	4.030	4.756
7	3.808	4.170	4.886
8	3.931	4.286	4.986
9	4.037	4.386	5.076
10	4.129	4.474	5.156
11	4.211	4.551	5.226
12	4.284	4.621	5.291
13	4.351	4.685	5.346
14	4.411	4.743	5.401
15	4.468	4.796	5.446
16	4.519	4.846	5.496
17	4.568	4.891	5.536
18	4.612	4.933	5.576
19	4.654	4.973	5.611
20	4.694	5.011	5.646

Table 3.13: $l_0 = l_0(k, \alpha)$

$k \setminus \alpha$	0.1	0.05	0.01
2	1.181	1.465	2.165
3	0.732	0.960	1.486
4	0.480	0.661	1.093
5	0.319	0.481	0.835
6	0.207	0.348	0.652
7	0.125	0.251	0.520
8	0.061	0.174	0.413
9	0.010	0.114	0.330
10	-0.031	0.064	0.262
11	-0.066	0.022	0.205
12	-0.095	-0.013	0.157
13	-0.120	-0.043	0.114
14	-0.142	-0.0693	0.079
15	-0.161	-0.093	0.046
16	-0.178	-0.113	0.019
17	-0.193	-0.132	-0.006
18	-0.207	-0.148	-0.029
19	-0.220	-0.163	-0.044
20	-0.231	-0.177	-0.068

3.3.4 Exact calculations of the expected sample size and the confidence level

In this subsection, we evaluate, by using the recursive method discussed in Section 2.3, the exact distribution of T and hence the exact values of $E(T)$ and CL. In this case, we have

$$R_{n+1}(x) = \int_{C_n}^{x} R_n(y) f_{\chi_k^2}(x-y)\, dy, \quad n \geq m_0 \tag{3.35}$$

and

$$P\{t > n+1\} = \int_{C_{n+1}}^{\infty} R_{n+1}(y)\, dy, \qquad n \geq m_0 - 1, \tag{3.36}$$

where

$$C_n = \frac{kn(n+1)}{c\left(1 + \frac{l_0}{n+1}\right)}$$

and the value of l_0 is given in Table 3.13. Consequently

$$E(T) = 1 + \sum_{n=m_0}^{\infty} n[P(t > n-1) - P(t > n)] \tag{3.37}$$

and

$$\begin{aligned} CL &= E\left[H\left((|q|_k^{\alpha})^2 \frac{T}{c}\right)\right] \\ &= \sum_{n=m_0}^{\infty} P(t=n) H\left((|q|_k^{\alpha})^2 \frac{n+1}{c}\right) \\ &= \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)] H\left((|q|_k^{\alpha})^2 \frac{n+1}{c}\right), \end{aligned} \tag{3.38}$$

where $H(x) = P\{\max_{1 \leq i \neq j \leq k} |Z_i - Z_j| \leq \sqrt{x}\}$. The results of calculation are given in Subsection 3.3.5 and were based on a grid with $h = 0.1$. Calculations based on $h = 0.2$ and $h = 0.05$ gave values of the CL differing at the most in the fourth decimal place from those based on $h = 0.1$. Simulations on $E(T)$ and CL were also carried out based on 6,000 experiments and some of the results are given in the Table 3.16.

3.3.5 Some comparisons

In this subsection, the second order approximations and the exact calculations of the $E(T)$ and the CL are compared, from which we can judge when the second order approximations are reasonably accurate.

Firstly, we look at the confidence level CL. The confidence level is equal to $1-\alpha$ (nominal level) plus an error term of order $o(1/c)$ as $c \to \infty$ and so the approximate is $1-\alpha$. The true value of the confidence level, however, depends on c, i.e. $CL = CL(c)$. For $m = 2, k = 3, 7, 10$, and $1-\alpha = 90\%, 99\%$, the exact calculation results of $CL(c)$ at $c = 5(5)60$ are linearly plotted in Figure 5. Figure 6 gives the similar plots for $m = 10$ and $c = 15(5)60$. From Figures 5 and 6 it can be seen that $CL(c)$ is generally closer to the nominal level $1-\alpha$ for: (i) larger c; (ii) larger k; (iii) larger nominal level $1-\alpha$; (iv) larger initial sample size m.

Next consider the expected sample size $E(T)$. By Theorem 3.5, we know that for large c, $E(T) = E_c(T) = c + \rho + l_0 - 2/k$ as $c \to \infty$. Table 3.14 contains the values of $E(T)$ calculated using the recursive method and the approximation formula at $c = 5(5)\ 60$, for $m = 2$, $k = 3,\ 7,\ 10$, and $1-\alpha = 90\%,\ 99\%$. Similar results are given in Table 3.15 for $m = 10$ and $c = 15(5)\ 60$. We note from Table 3.14 and 3.15 that the approximate value of $E(T)$ is generally closer to the value of $E(T)$ for: (i) large c; (ii) large k; (iii) large m. The exact calculations of the $E(T)$ and the CL become quite computationally intensive for $c \geq 60$. However, when $c \geq 60$ the approximations are very good, as can be seen from the results given in this subsection. So approximated results can be used in this case.

Table 3.14: *Comparisons between the exact and approximate values of $E(T)$ for $m = 2$ and given values of k, α and c*

$\alpha = 0.1$

	$k = 3$		$k = 7$		$k = 10$	
c	Exact	Appro.	Exact	Appro.	Exact	Appro.
5	5.2	6.0	5.8	6.1	6.0	6.1
10	9.9	11.0	10.9	11.1	11.0	11.1
15	15.0	16.0	16.0	16.1	16.1	16.1
20	20.1	21.0	21.0	21.1	21.1	21.1
25	25.2	26.0	26.0	26.1	26.1	26.1
30	30.2	31.0	31.0	31.1	31.1	31.1
35	35.3	36.0	36.0	36.1	36.1	36.1
40	40.3	41.0	41.0	41.1	41.1	41.1
45	45.3	46.0	46.0	46.1	46.1	46.1
50	50.4	51.0	51.0	51.1	51.1	51.1
55	55.4	56.0	56.0	56.1	56.1	56.1
60	60.4	61.0	61.0	61.1	61.1	61.1

Table 3.14: *Comparisons between the exact and approximate values of $E(T)$ for $m = 2$ and given values of k, α and c*

$\alpha = 0.01$

c	$k = 3$ Exact	$k = 3$ Appro.	$k = 7$ Exact	$k = 7$ Appro.	$k = 10$ Exact	$k = 10$ Appro.
5	6.0	6.5	5.6	5.8	5.5	5.6
10	11.0	11.5	10.7	10.8	10.6	10.6
15	16.0	16.5	15.8	15.8	15.6	15.6
20	21.0	21.5	20.8	20.8	20.6	20.6
25	26.0	26.5	25.8	25.8	25.6	25.6
30	31.1	31.5	30.8	30.8	30.6	30.6
35	36.1	36.5	35.8	35.8	35.6	35.6
40	41.1	41.5	40.8	40.8	40.6	40.6
45	46.2	46.5	45.8	45.8	45.6	45.6
50	51.2	51.5	50.8	50.8	50.6	50.6
55	56.2	56.5	55.8	55.8	55.6	55.6
60	61.2	61.5	60.8	60.8	60.6	60.6

Table 3.15: *Comparisons between the exact and approximate values of $E(T)$ for $m = 10$ and given values of k, α and c*

$\alpha = 0.1$

c	$k = 3$		$k = 7$		$k = 10$	
	Exact	Appro.	Exact	Appro.	Exact	Appro.
15	15.7	15.7	16.0	16.0	16.1	16.1
20	20.6	20.7	21.0	21.0	21.1	21.1
25	25.6	25.7	26.0	26.0	26.1	26.1
30	30.7	35.7	31.0	31.0	31.1	31.1
35	35.7	35.7	36.0	36.0	36.1	36.1
40	40.7	40.7	41.0	41.0	41.1	41.1
45	45.7	45.7	46.0	46.0	46.1	46.1
50	50.7	50.7	51.0	51.0	51.1	51.1
55	55.7	55.7	56.0	61.0	56.1	56.1
60	60.7	60.7	61.0	61.0	61.1	61.1

Table 3.15: *Comparisons between the exact and approximate values of $E(T)$ for $m = 10$ and given values of k, α and c*

$\alpha = 0.01$

c	$k = 3$		$k = 7$		$k = 10$	
	Exact	Appro.	Exact	Appro.	Exact	Appro.
15	16.4	16.5	15.8	15.8	15.6	15.6
20	21.4	21.5	20.8	20.8	20.6	20.6
25	26.4	26.5	25.8	25.8	25.6	25.6
30	31.4	31.5	30.8	30.8	30.6	30.6
35	36.4	36.5	35.8	35.8	35.6	35.6
40	41.4	41.5	40.8	40.8	40.6	40.6
45	46.4	46.5	45.8	45.8	45.6	45.6
50	51.4	51.5	50.8	50.8	50.6	50.6
55	56.4	56.5	55.8	55.8	55.6	55.6
60	61.4	61.5	60.8	60.8	60.6	60.6

Table 3.16: *Comparisons between the exact and simulated values of $E(T)$ and CL for $m = 10$ and $k = 10$ and given values of α and c*

$\alpha = 0.1$

c	CL Simul.	CL Exact	$E(T)$ Simul.	$E(T)$ Exact
15	0.904	0.916	15.4	15.4
20	0.894	0.913	20.4	20.4
25	0.897	0.910	25.3	25.4
30	0.899	0.909	30.3	30.4
35	0.903	0.908	35.3	35.4
40	0.897	0.907	40.3	40.4
45	0.892	0.906	45.3	45.4
50	0.896	0.905	50.4	50.4
55	0.905	0.905	55.4	55.4
60	0.899	0.904	60.4	60.4

Table 3.16: *Comparisons between the exact and simulated values of $E(T)$ and CL for $m = 10$ and $k = 10$ and given values of α and c*

$\alpha = 0.01$

	CL		$E(T)$	
c	Simul.	Exact	Simul.	Exact
15	1.000	0.989	15.7	15.6
20	1.000	0.990	20.7	20.6
25	1.000	0.990	25.6	25.6
30	1.000	0.990	30.6	30.6
35	1.000	0.990	35.6	35.6
40	1.000	0.990	40.6	40.6
45	1.000	0.990	45.6	45.6
50	1.000	0.990	50.6	50.6
55	1.000	0.990	55.6	55.6
60	1.000	0.990	60.6	60.6

Figure 5. The exact confidence level

as a function of $c = c(\sigma)$ for $m = 2$.

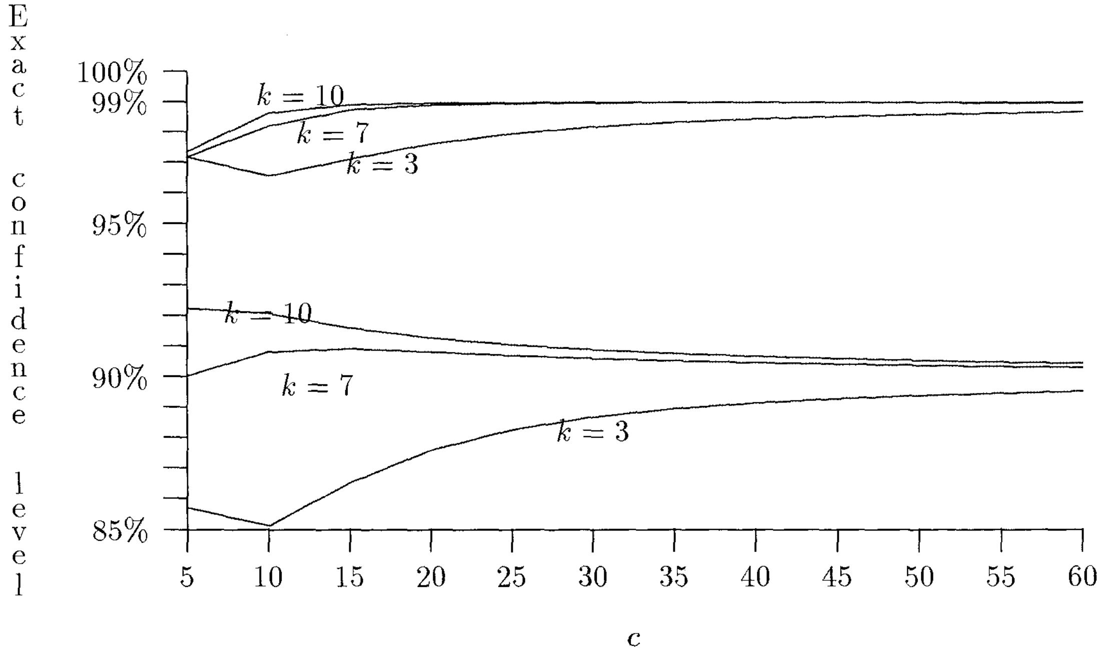
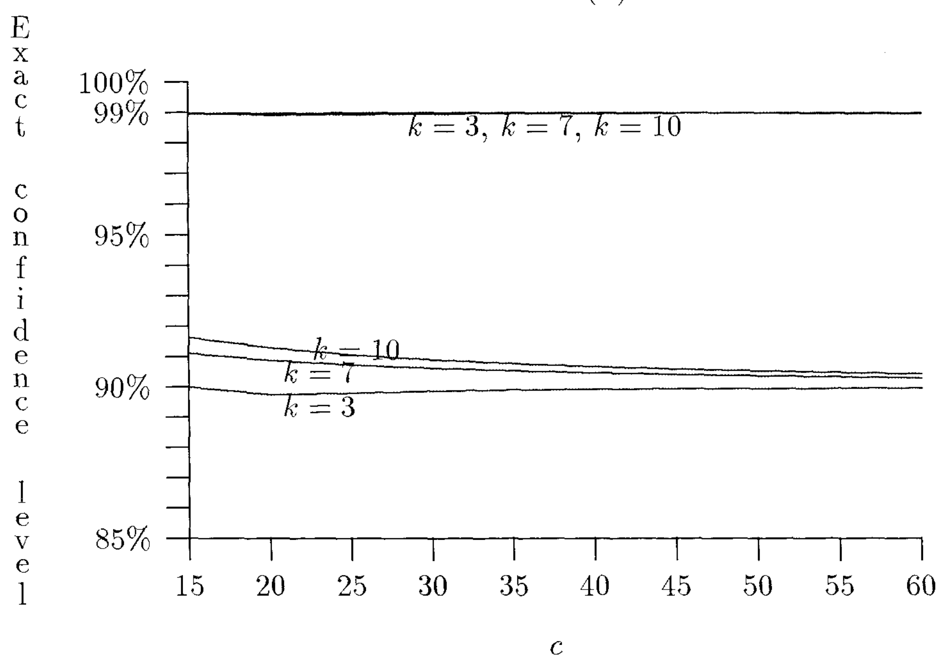


Figure 6. The exact confidence level

as a function of $c = c(\sigma)$ for $m = 10$.



Chapter 4

The exact probabilities of making correct inferences

4.1 The exact probability of making correct inference about the means of several independent normal populations

4.1.1 Introduction

Suppose that we have the following set of $2d$-width and $(1-\alpha)$-level simultaneous confidence intervals for the μ_i's

$$P\{\mu_i \in (\bar{Y}_i - d,\ \bar{Y}_i + d), \quad i = 1, 2, \cdots, k\} = 1 - \alpha.$$

As has already been pointed out in Section 3.1, simultaneous inference about each μ_i can be made from this set of confidence intervals. For example, we can infer that $\mu_i > 0 (\mu_i < 0)$ if $\bar{Y}_i - d > 0 (\bar{Y}_i + d < 0)$. Furthermore, the probability of making correct inferences, either $\mu_i > 0$ or $\mu_i < 0$, for every μ_i satisfying $|\mu_i| \geq 2d$, is at least $1-\alpha$, the confidence level. The problem that we want to study in this section is " what is the exact value of this probability?" More precisely, we want to investigate the following probability

$$P\{\text{making correct inferences, either } \mu_i > 0 \text{ or } \mu_i < 0, \text{ for each } \mu_i \text{ satisfying } |\mu_i| \geq 2d\}.$$

Let

$$\Omega_U(d) = \{i : \mu_i \geq 2d\} \quad \text{and} \quad \Omega_L(d) = \{j : \mu_j \leq -2d\}.$$

The above probability is then equal to

$$P\{\text{making correct inferences } \mu_i > 0 \text{ for each } i \in \Omega_U(d) \text{ and}$$
$$\text{making correct inferences } \mu_j < 0 \text{ for each } j \in \Omega_L(d)\}.$$

This probability is of course dependent on the true value of $\mu = (\mu_1, \mu_2, \cdots, \mu_k) \in R^k$, and let it be denoted by $\beta(\mu, d)$. For obvious reason, we impose $\beta(\mu, d) = 1$

if for a given value of μ and d both the sets $\Omega_U(d)$ and $\Omega_L(d)$ are empty. We wish to assess

$$\beta(d) = \min_{\mu \in R^k} \beta(\mu, d)$$

in this section. As one should expect, $\beta(d)$ must be no less than $1 - \alpha$.

Two different situations will be considered. In Subsection 4.1.2 we consider the known variance case in which the set of confidence intervals for μ_i is given in (4.1). In Subsection 4.1.3 we consider the unknown variance case in which the set of confidence intervals for μ_i is constructed by using the pure sequential procedure of Section 3.1.

4.1.2 When the variance is known

In this subsection we suppose that σ^2 is known. Take a random sample of fixed size $a = \left(|m|_k^\alpha\right)^2 \sigma^2 d^{-2}$ from each of the k populations and construct the following set of simultaneous confidence intervals for μ_i

$$\mu_i \in \left(\bar{Y}_{ia} - d,\ \bar{Y}_{ia} + d\right), \quad i = 1, 2, \cdots, k. \tag{4.1}$$

From Section 3.1.1, we know that this set of confidence intervals has exact level $1 - \alpha$. In order to compute the exact value of $\beta(d)$, we require

Theorem 4.1 *For* $a > 0$

$$\beta(d) = \Phi^k\left(\frac{\sqrt{a}d}{\sigma}\right) = \Phi^k\left(|m|_k^\alpha\right). \tag{4.2}$$

Proof: By definition we have

$$\begin{aligned}
\beta(\mu, d) &= P\{\bar{Y}_{ia} > d\ \forall i \in \Omega_U(d) \text{ and } \bar{Y}_{ja} < -d\ \forall j \in \Omega_L(d)\} \\
&= \prod_{i \in \Omega_U(d)} P\{\bar{Y}_{ia} > d\} \prod_{j \in \Omega_L(d)} P\{\bar{Y}_{ja} < -d\} \\
&= \prod_{i \in \Omega_U(d)} \Phi\left(-\frac{\sqrt{a}(d - \mu_i)}{\sigma}\right) \prod_{j \in \Omega_L(d)} \Phi\left(-\frac{\sqrt{a}(d + \mu_j)}{\sigma}\right).
\end{aligned}$$

From this, it is clear that $\beta(\mu, d)$ attains its minimum at $\mu^*(d) = (\mu_1^*, \cdots, \mu_k^*)$, where each μ_i^* is equal to either $2d$ or $-2d$. Consequently

$$\beta(d) = \Phi^k\left(\frac{\sqrt{a}d}{\sigma}\right) = \Phi^k\left(|m|_k^\alpha\right).$$

This finishes the proof.

It is interesting to note that the value of $\beta(d)$ depends only on α and k, but not on d and σ^2. This is because of the way in which we set the sample size $a = \left(|m|_k^\alpha\right)^2 \sigma^2 d^{-2}$. Table 4.1 presents the values of $\beta(d)$ for $k = 2(1)20$ and $\alpha = 0.1, 0.05, 0.01$. It can see that the value of the $\beta(d) = \Phi^k\left(|m|_k^\alpha\right)$ is very stable to the value of k, and is strictly large than $1 - \alpha$, the confidence level. In fact it is close to $1 - \alpha/2$, as it is a sort of one saided probability.

Therefore, if $\alpha = 0.10$ say, we can claim that, with probability at least 0.95, rather than $1 - \alpha = 0.90$, correct inference, based on the set of confidence intervals in (4.1), will be made for each μ_i satisfying $|\mu_i| \geq 2d$.

Table 4.1: $\Phi^k\left(|m|_k^\alpha\right)$

$k \setminus 1-\alpha$	0.90	0.95	0.99
2	0.949	0.975	0.995
3	0.949	0.975	0.995
4	0.949	0.975	0.995
5	0.949	0.975	0.995
6	0.949	0.975	0.995
7	0.949	0.975	0.995
8	0.949	0.975	0.995
9	0.949	0.975	0.995
10	0.949	0.975	0.995
11	0.949	0.975	0.995
12	0.949	0.975	0.995
13	0.949	0.975	0.995
14	0.949	0.975	0.995
15	0.949	0.975	0.995
16	0.949	0.975	0.995
17	0.949	0.975	0.995
18	0.949	0.975	0.995
19	0.949	0.975	0.995
20	0.949	0.975	0.995

4.1.3 When the variance is unknown

In this subsection, we suppose σ^2 is unknown and consider inferences based on the set of confidence intervals

$$\mu_i \in (\bar{Y}_{iT} - d,\ \bar{Y}_{iT} + d), \quad 1 \le i \le k$$

constructed by using the pure sequential procedure given in Subsection 3.1.1, in which the stopping time T is given by

$$T = \inf\{n \ge m:\ n \ge d^{-2}(|m|_k^\alpha)^2 l_n(\hat{\sigma_n})^2\}.$$

We know that, for each treatment satisfying $\mu_i \ge 2d(\le -2d)$, the correct inference $\mu_i > 0(< 0)$ will be made from this set of simultaneous confidence intervals with a probability of at least $1 - \alpha + o(d^2)$, since the confidence level of this set of confidence intervals is equal to $1 - \alpha + o(d^2)$. We wish to assess

$$\beta_U(d) = \min_{\mu \in R^k} \beta_U(\mu, d),$$

where

$$\beta_U(\mu, d) = P\{\bar{Y}_{iT} > d\ \forall i \in \Omega_U(d),\ \ \bar{Y}_{jT} < -d\ \forall j \in \Omega_L(d)\}. \tag{4.3}$$

In particular, we define $\beta_U(\mu, d) = 1$ if all treatments satisfy $|\mu_i| < 2d$. First we have

Lemma 4.1

$$\beta_U(d) = E\left[\Psi^k\left((|m|_k^\alpha)^2 \frac{T}{a}\right)\right],$$

where $\Psi(x) = \Phi(\sqrt{x})$ *and* $a = (|m|_k^\alpha)^2\, d^{-2} \sigma^2$.

Proof: By definition we have

$$\begin{aligned} \beta_U(\mu, d) &= P\{\bar{Y}_{iT} > d\ \forall i \in \Omega_U(d),\ \bar{Y}_{jT} < -d\ \forall j \in \Omega_L(d)\} \\ &= \sum_{n=m}^{\infty} P\{\bar{Y}_{in} > d\ \forall i \in \Omega_U(d),\ \bar{Y}_{jn} < -d\ \forall j \in \Omega_L(d)\} P\{T = n\}. \end{aligned}$$

From Theorem 4.1 we know that for each n, the minimum value of

$$P\{\bar{Y}_{in} > d\ \forall i \in \Omega_U(d),\ \bar{Y}_{jn} < -d\ \forall j \in \Omega_L(d)\}$$

over $\mu \in R^k$ is attained at $\mu^*(d) = (2d, \cdots, 2d)$ say, and given by $\Psi^k\left(\frac{nd^2}{\sigma^2}\right)$. So the minimum of $\beta_U(\mu, d)$ over $\mu \in R^k$ is given by

$$\begin{aligned}\beta_U(d) &= \sum_{n=m}^{\infty} \Psi^k\left(\frac{(|m|_k^\alpha)^2 n}{a}\right) P\{T = n\} \\ &= E\left[\Psi^k\left((|m|_k^\alpha)^2 \frac{T}{a}\right)\right].\end{aligned}$$

This completes the proof.

An argument similar to the proof of Lemma 3.2 establishes

Lemma 4.2 *Let $H(x) = \Psi^k(x)$ and $\Psi(x) = \Phi(\sqrt{x})$. Then, there is a constant C for which $|H''(x)| \leq Cx^{(k-4)/2}$ for $x > 0$.*

The following theorem, which follows directly from Theorem 2.2, gives the second order approximation to $\beta_U(d)$.

Theorem 4.2 *Let $H(x) = \Psi^k(x)$ and $\Psi(x) = \Phi(\sqrt{x})$, and suppose $m > 1$ if $k \geq 4$ and $m > 1 + (6-k)/k$ if $k = 2, 3$. Then*

$$\begin{aligned}\beta_U(d) &= \Phi^k\left(|m|_k^\alpha\right) + \frac{1}{a}\Big[(|m|_k^\alpha)^2 H'\left((|m|_k^\alpha)^2\right)\left(\rho + l_0 - \frac{2}{k}\right) \\ &\quad + \frac{1}{k}(|m|_k^\alpha)^4 H''\left((|m|_k^\alpha)^2\right)\Big] + o\left(\frac{1}{a}\right),\end{aligned}$$

where $a = d^{-2}(|m|_k^\alpha)^2 \sigma^2$.

The exact value of $\beta_U(d)$ can be calculated by using a recursive method similar to that discussed in Subsection 3.1.4 since the stopping time is the same as before and

$$\begin{aligned}\beta_U(d) &= E\left[\Psi^k\left((|m|_k^\alpha)^2 \frac{T}{a}\right)\right] \\ &= \sum_{n=m_0}^{\infty} P(t = n)\Psi^k\left((|m|_k^\alpha)^2 \frac{n+1}{a}\right) \\ &= \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)]\Psi^k\left((|m|_k^\alpha)^2 \frac{n+1}{a}\right),\end{aligned}$$

where $\Psi(x) = \Phi(\sqrt{x})$ and $m_0 = m - 1$. Simulation to estimate $\beta_U(d)$, based on 6,000 experiments, was also carried out.

For $k = 5, 7, 10$ and $1 - \alpha = 0.90, 0.99$, Tables 4.2 and 4.3 give the exact, simulated and second order approximate values of $\beta_U(d)$ at $a = 5(5)60$ and $a = 15(5)60$. For $m = 3, 10, k = 5, 7, 10$, and $1 - \alpha = 90\%, 99\%$, the exact calculation results and approximations of $\beta_U(d)$ at $a = 5(5)\ 60$ and $a = 15(5)60$ are linearly plotted in Figures 7-12. From these tables and figures it can be seen that the exact values and the second order approximations of the $\beta_U(d)$ are generally closer together for: (i) larger a; (ii) larger k; (iii) larger initial sample size m. For larger k, the exact values and approximations of the $\beta_U(d)$ are almost 0.05 larger than $1 - \alpha$ when $\alpha = 0.1$. When $\alpha = 0.01$ the exact values and approximations of the $\beta_U(d)$ are almost 0.005 larger than $1 - \alpha$.

Table 4.2: *Comparisons between the exact, approximate*

and simulation results of $\beta_U(d)$

for $m = 3$ *and given values of* k, $1 - \alpha$ *and* a

$1 - \alpha = 0.90$

a	$k = 5$			$k = 7$			$k = 10$		
	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
5	0.942	0.948	0.942	0.940	0.948	0.932	0.945	0.948	0.931
10	0.939	0.949	0.938	0.942	0.949	0.945	0.947	0.949	0.947
15	0.945	0.949	0.946	0.946	0.949	0.945	0.948	0.949	0.949
20	0.947	0.949	0.946	0.948	0.949	0.953	0.948	0.949	0.949
25	0.948	0.949	0.945	0.948	0.949	0.948	0.948	0.949	0.944
30	0.948	0.949	0.949	0.948	0.949	0.950	0.948	0.949	0.951
35	0.948	0.949	0.949	0.948	0.949	0.946	0.948	0.949	0.946
40	0.948	0.949	0.949	0.948	0.949	0.948	0.948	0.949	0.949
45	0.948	0.949	0.951	0.948	0.949	0.949	0.948	0.949	0.946
50	0.948	0.949	0.948	0.948	0.949	0.950	0.948	0.949	0.953
55	0.948	0.949	0.951	0.949	0.949	0.948	0.948	0.949	0.947
60	0.949	0.949	0.950	0.949	0.949	0.950	0.948	0.949	0.950

Table 4.2: *Comparisons between the exact, approximate and simulation results of* $\beta_U(d)$

for $m = 10$ *and given values of* k, $1-\alpha$ *and* a

$1 - \alpha = 0.90$

a	$k=5$			$k=7$			$k=10$		
	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
15	0.946	0.949	0.952	0.951	0.948	0.953	0.949	0.948	0.949
20	0.945	0.949	0.951	0.946	0.948	0.948	0.949	0.948	0.949
25	0.946	0.949	0.944	0.946	0.948	0.949	0.949	0.948	0.949
30	0.946	0.949	0.946	0.948	0.948	0.949	0.949	0.948	0.949
35	0.946	0.949	0.951	0.949	0.948	0.954	0.949	0.948	0.949
40	0.945	0.949	0.953	0.947	0.949	0.950	0.949	0.949	0.949
45	0.945	0.949	0.949	0.949	0.949	0.949	0.949	0.949	0.949
50	0.945	0.949	0.949	0.949	0.949	0.948	0.949	0.949	0.949
55	0.945	0.949	0.949	0.949	0.949	0.952	0.949	0.949	0.949
60	0.945	0.949	0.952	0.949	0.949	0.946	0.949	0.949	0.949

Table 4.3: *Comparisons between the exact, approximate and simulation results of* $\beta_U(d)$ *for* $m = 3$ *and given values of* k, $1-\alpha$ *and* a

$1 - \alpha = 0.99$

a	$k=5$			$k=7$			$k=10$		
	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
5	0.993	0.995	0.991	0.993	0.995	0.985	0.993	0.995	0.983
10	0.992	0.995	0.991	0.993	0.995	0.988	0.994	0.995	0.993
15	0.994	0.995	0.994	0.994	0.995	0.993	0.995	0.995	0.996
20	0.994	0.995	0.994	0.995	0.995	0.995	0.995	0.995	0.994
25	0.995	0.995	0.994	0.995	0.995	0.994	0.995	0.995	0.995
30	0.995	0.995	0.995	0.995	0.995	0.993	0.995	0.995	0.994
35	0.995	0.995	0.995	0.995	0.995	0.996	0.995	0.995	0.995
40	0.995	0.995	0.996	0.995	0.995	0.995	0.995	0.995	0.994
45	0.995	0.995	0.994	0.995	0.995	0.996	0.995	0.995	0.994
50	0.995	0.995	0.996	0.995	0.995	0.994	0.995	0.995	0.995
55	0.995	0.995	0.996	0.995	0.995	0.996	0.995	0.995	0.995
60	0.995	0.995	0.996	0.995	0.995	0.995	0.995	0.995	0.995

Table 4.3: *Comparisons between the exact, approximate and simulation results of* $\beta_U(d)$

for $m = 10$ *and given values of* k, $1-\alpha$ *and* a

$1 - \alpha = 0.99$

	$k=5$			$k=7$			$k=10$		
a	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
15	0.995	0.995	0.996	0.994	0.995	0.996	0.995	0.995	0.995
20	0.995	0.995	0.996	0.995	0.995	0.995	0.995	0.995	0.995
25	0.995	0.995	0.996	0.996	0.995	0.993	0.995	0.995	0.995
30	0.995	0.995	0.995	0.995	0.995	0.993	0.995	0.995	0.995
35	0.995	0.995	0.996	0.996	0.995	0.995	0.995	0.995	0.995
40	0.995	0.995	0.996	0.995	0.995	0.995	0.995	0.995	0.995
45	0.995	0.995	0.995	0.996	0.995	0.995	0.995	0.995	0.995
50	0.995	0.995	0.995	0.996	0.995	0.995	0.995	0.995	0.995
55	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995
60	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995

Figure 7. The exact (-) and approximate ($\cdots$) values of $\beta_U(d)$ and $1-\alpha$ (- -) as a function of $a = a(\sigma)$ for $m = 3$, $k = 5$ and $\alpha = 0.10,\ 0.01$.

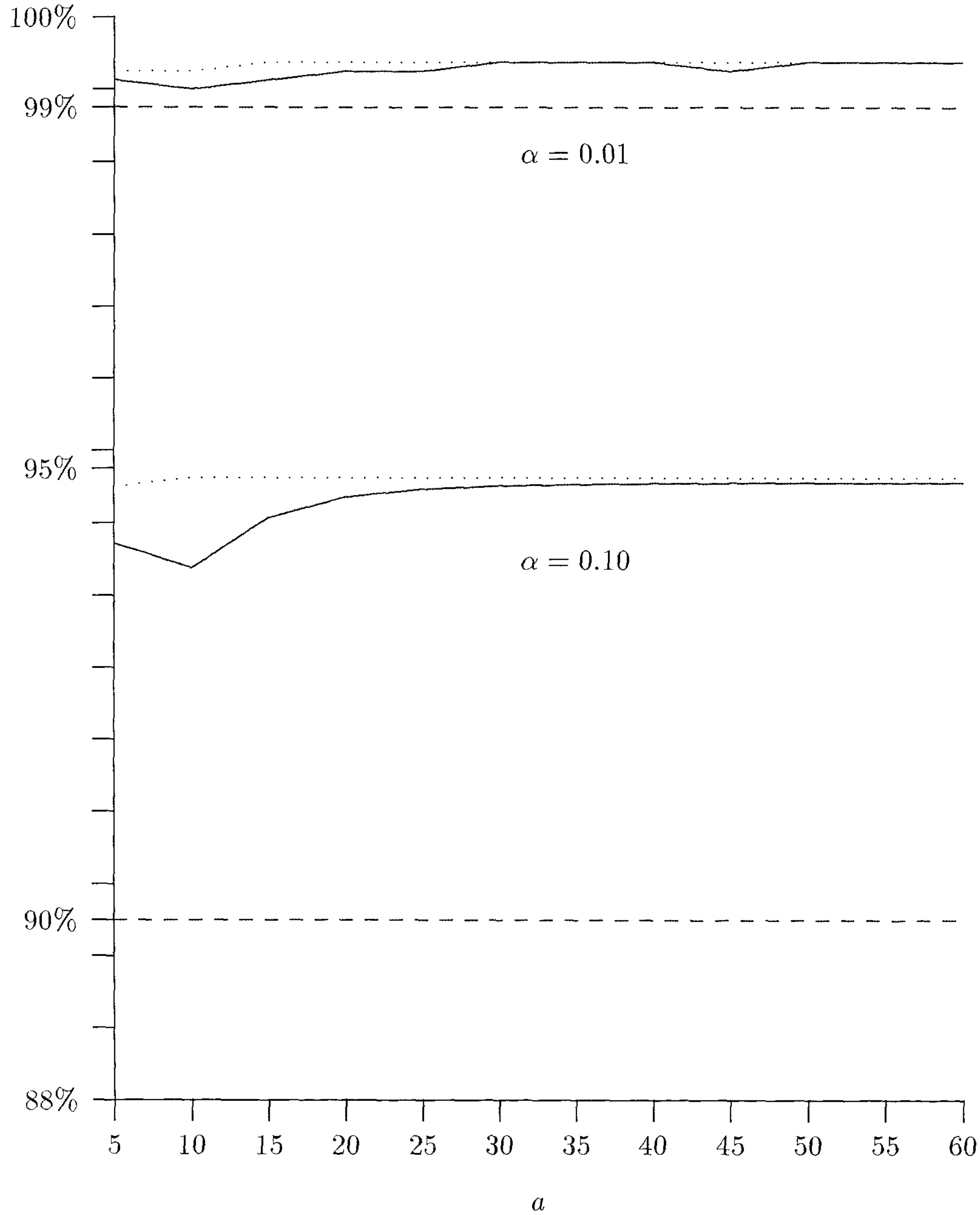


Figure 8. The exact (-) and approximate ($\cdots$) values of $\beta_U(d)$ and $1-\alpha$ (- -) as a function of $a = a(\sigma)$ for $m = 10$, $k = 5$ and $\alpha = 0.10,\ 0.01$

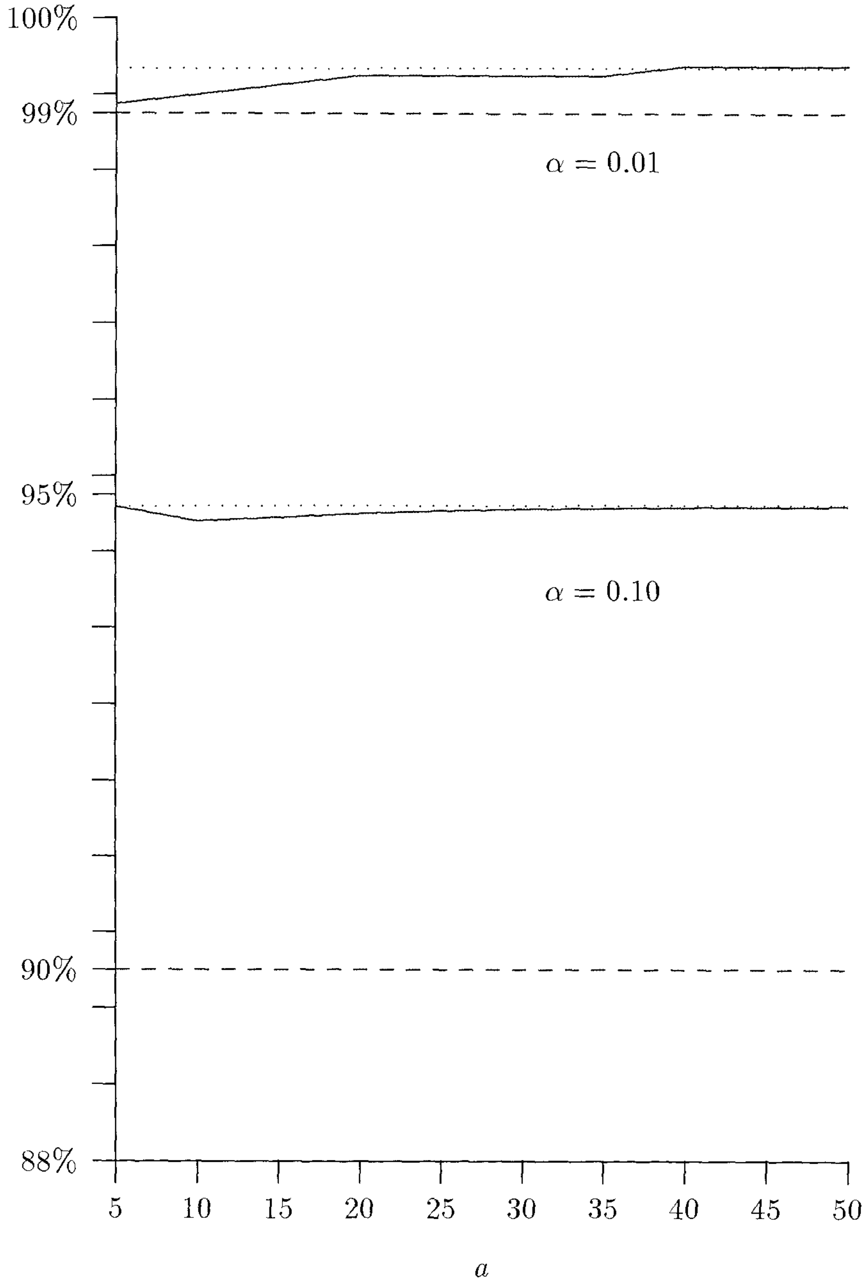


Figure 9. The exact (-) and approximate $(\cdots)$ values of $\beta_U(d)$ and $1-\alpha$ (- -) as a function of $a = a(\sigma)$ for $m = 3,\ k = 7$ and $\alpha = 0.10,\ 0.01$.

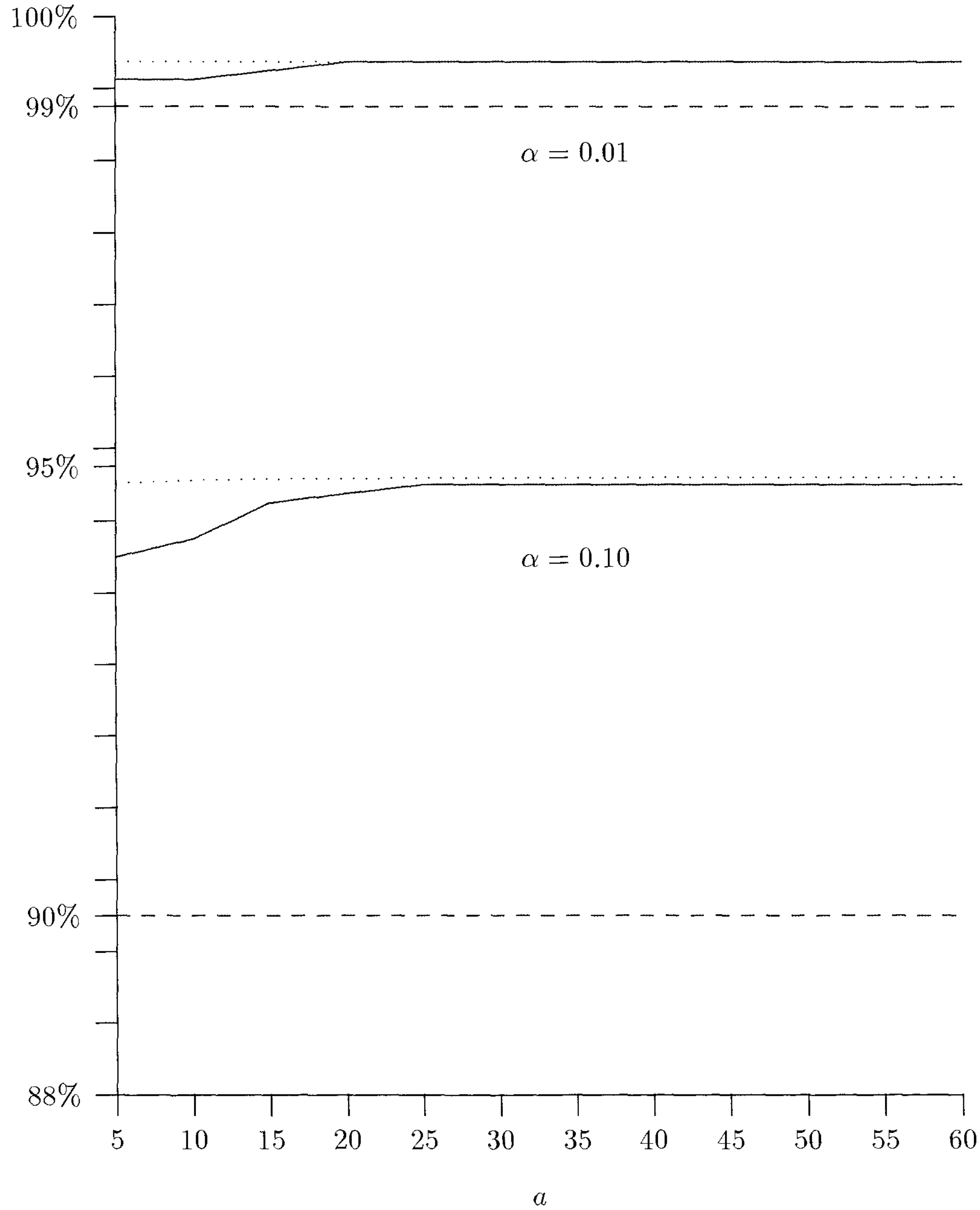


Figure 10. The exact (-) and approximate ($\cdots$) values of $\beta_U(d)$ and $1-\alpha$ (- -) as a function of $a = a(\sigma)$ for $m = 10,\ k = 7$ and $\alpha = 0.10,\ 0.01$.

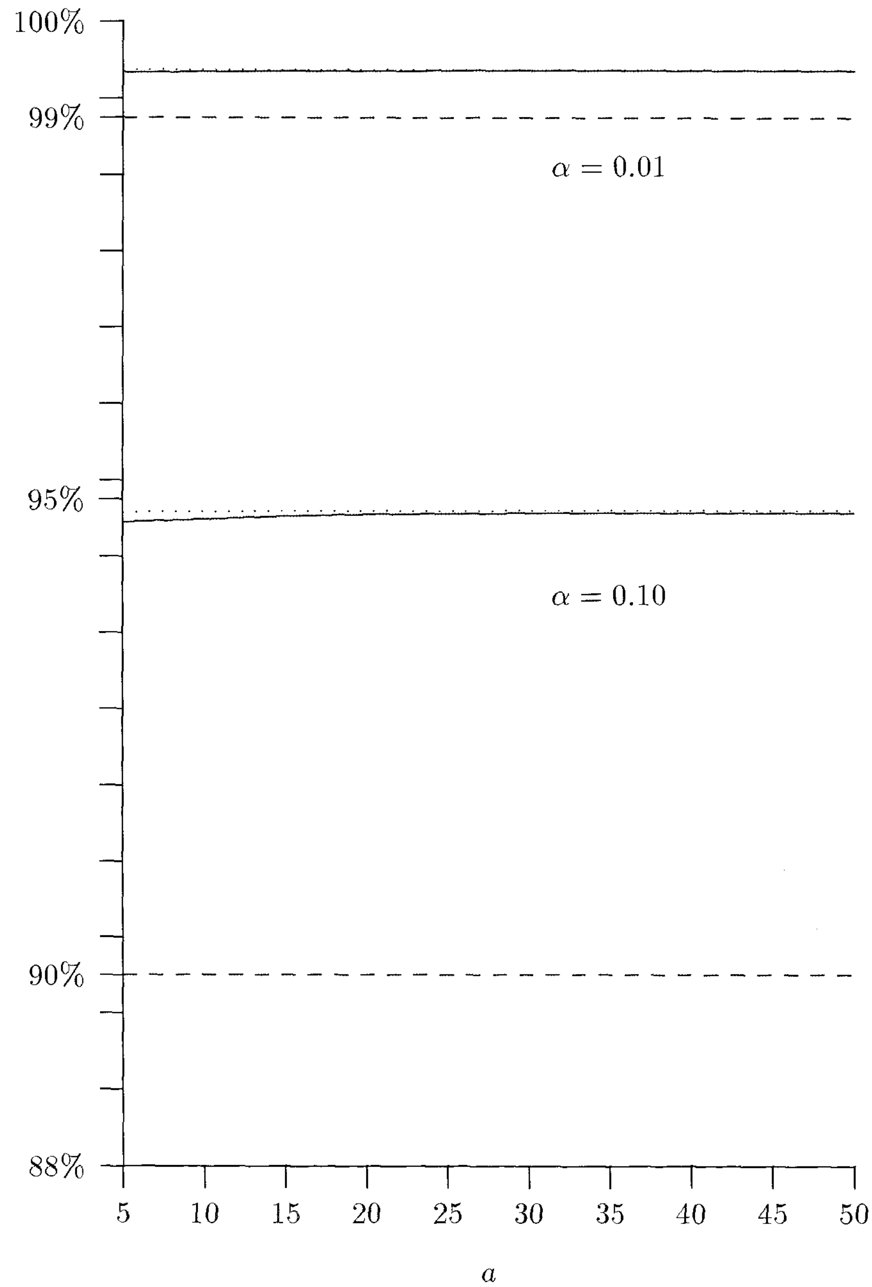


Figure 11. The exact (-) and approximate (···) values of $\beta_U(d)$ and $1-\alpha$ (- -) as a function of $a = a(\sigma)$ for $m = 3$, $k = 10$ and $\alpha = 0.10,\ 0.01$.

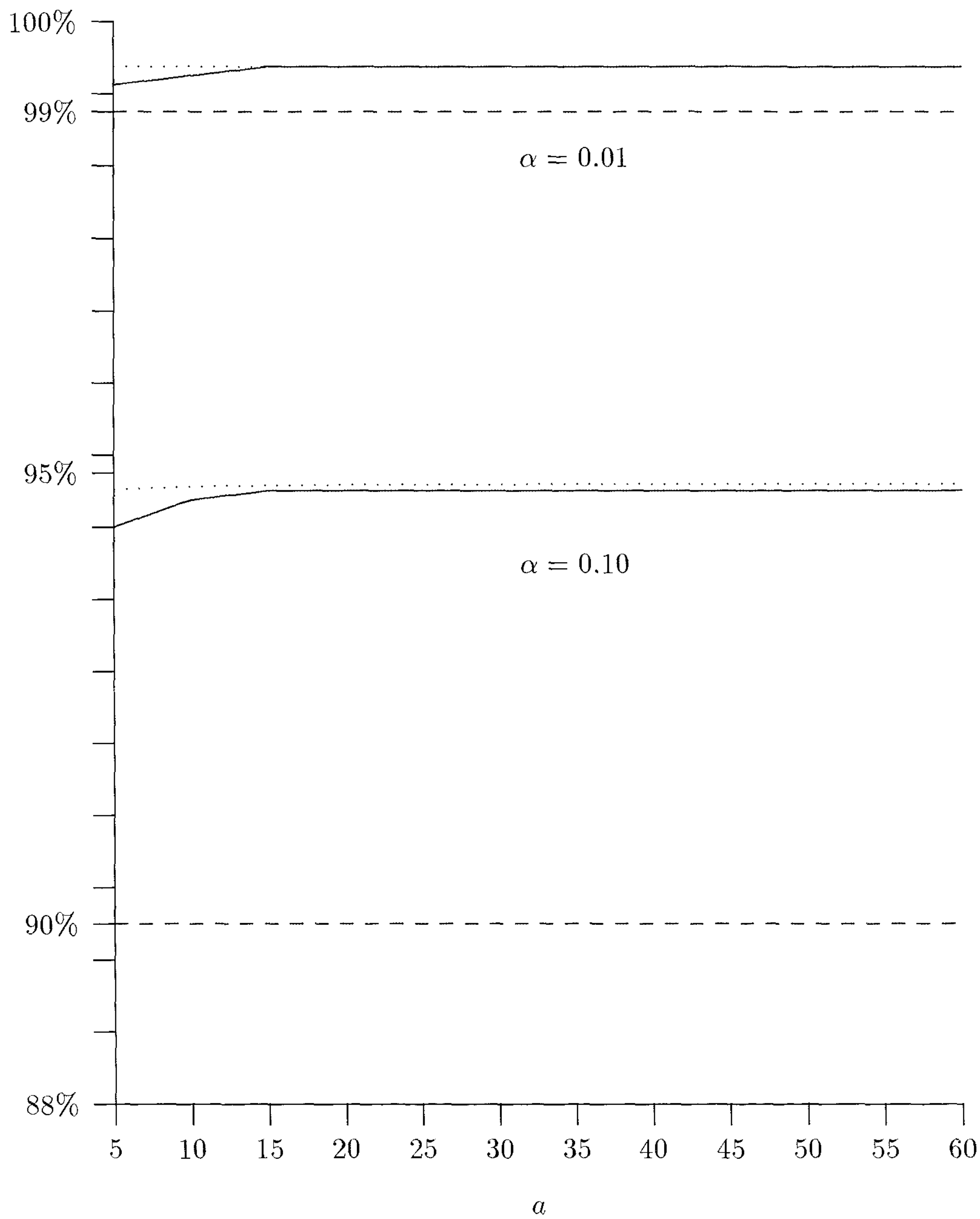
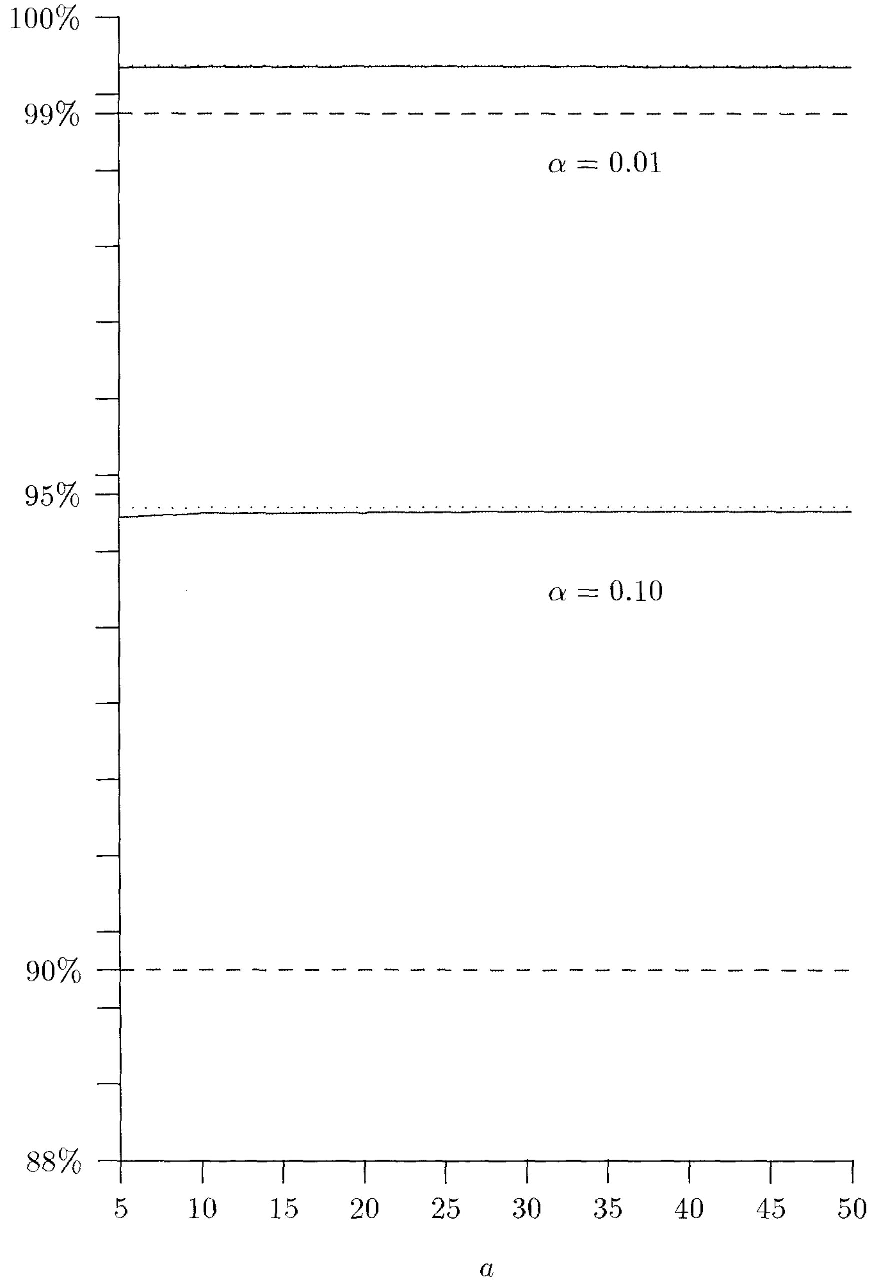


Figure 12. The exact (-) and approximate ($\cdots$) values of $\beta_U(d)$ and $1-\alpha$ (- -) as a function of $a = a(\sigma)$ for $m = 10$, $k = 10$ and $\alpha = 0.10,\ 0.01$



4.2 The exact probability of making correct inference for comparing several treatments with a control

4.2.1 Introduction

Suppose the first population, $N(\mu_1, \sigma^2)$, is the control, the other $k-1 (k \geq 2)$ populations are treatments, and the set of $2d$-width and $(1-\alpha)$-level simultaneous confidence intervals for the $\mu_i - \mu_1$ is given by

$$P\{\mu_i - \mu_1 \in \left(\bar{Y}_i - \bar{Y}_1 - d,\ \bar{Y}_i - \bar{Y}_1 + d\right), \quad i = 2, \cdots, k\} = 1 - \alpha.$$

Based on this set of confidence intervals, simultaneous inference about each $\mu_i - \mu_1$ can be made. For example, if $\bar{Y}_i - \bar{Y}_1 - d > 0 (\bar{Y}_i - \bar{Y}_1 + d < 0)$ we can infer that $\mu_i - \mu_1 > 0 (\mu_i - \mu_1 < 0)$. Furthermore, the probability of making correct inferences, either $\mu_i - \mu_1 > 0$ or $\mu_i - \mu_1 < 0$, for every μ_i satisfying $|\mu_i - \mu_1| \geq 2d$, is at least $1 - \alpha$, the confidence level. The purpose of this section is to study the exact value of this probability.

Let $\beta^*(\mu, d)$ be the probability of making correct inferences, either $\mu_i - \mu_1 > 0$ or $\mu_i - \mu_1 < 0$, for each μ_i satisfying $|\mu_i - \mu_1| \geq 2d$, and

$$\Omega_U^*(d) = \{i : \mu_i - \mu_1 > 2d\} \quad \text{and} \quad \Omega_L^*(d) = \{j : \mu_j - \mu_1 < -2d\}.$$

Then $\beta^*(\mu, d)$ is equal to

$$P\{\text{making correct inferences } \mu_i - \mu_1 > 0 \text{ for each } i \in \Omega_U^*(d) \text{ and making correct inferences } \mu_j - \mu_1 < 0 \text{ for each } j \in \Omega_L^*(d)\}.$$

In particular, we impose $\beta^*(\mu, d) = 1$ if for a given value of μ both the sets $\Omega_U^*(d)$ and $\Omega_L^*(d)$ are empty. Let the minimum value of $\beta^*(\mu, d)$ over $\mu \in R^k$ be

$$\beta^*(d) = \min_{\mu \in R^k} \beta^*(\mu, d).$$

As one should expect, $\beta^*(d)$ must be no less than $1-\alpha$, but we want to assess the exact value of $\beta^*(d)$.

Two different cases, known and unknown variance, will be considered separately. In Subsection 4.2.2 we consider the known variance case in which the set of confidence intervals for $\mu_i - \mu_1$ is given in (4.4). In Subsection 4.2.3 we consider the unknown variance case in which the set of confidence intervals for $\mu_i - \mu_1$ is constructed by using the pure sequential procedure of Section 3.2.

4.2.2 When the variance is known

Let σ^2 be known, we draw a random sample of fixed size $b = 2\left(|t|_{k-1}^{\alpha}\right)^2 \sigma^2 d^{-2}$ from each of the k populations and construct the following set of simultaneous confidence intervals

$$\mu_i - \mu_1 \in \left(\bar{Y}_{ib} - \bar{Y}_{1b} - d,\ \bar{Y}_{ib} - \bar{Y}_{1b} + d\right), \quad 2 \le i \le k. \tag{4.4}$$

It is known from Subsection 3.2.1 that this set of confidence intervals has simultaneous level $1 - \alpha$. In order to compute the exact value of $\beta^*(d)$, we need the following theorem.

Theorem 4.3 *Let $k \ge 3$, $p = \langle (k+1)/2 \rangle$ and $\mu^*(d) = (0, 2d, \cdots, 2d, -2d, \cdots, -2d) \in R^k$ where the first component is zero, the last $k-p$ components are $-2d$ and the rest $p-1$ components are $2d$. Then*

$$\begin{aligned}
\beta^*(d) &= \beta^*(\mu^*(d), d) \\
&= \int_{-\infty}^{\infty} \phi(x) \Phi^{p-1}\left(-x + \sqrt{2}|t|_{k-1}^{\alpha}\right) \Phi^{k-p}\left(x + \sqrt{2}|t|_{k-1}^{\alpha}\right) dx.
\end{aligned}$$

Proof: From the definition of $\beta^*(\mu, d)$, we have

$$\begin{aligned}
&\beta^*(\mu, d) \\
&= P\{\bar{Y}_{ib} - \bar{Y}_{1b} > d\ \forall i \in \Omega_U^*(d),\ \ \bar{Y}_{jb} - \bar{Y}_{1b} < -d\ \forall j \in \Omega_L^*(d)\} \\
&= P\Big\{\frac{\sqrt{b}(\bar{Y}_{ib} - \mu_i)}{\sigma} - \frac{\sqrt{b}(\bar{Y}_{1b} - \mu_1)}{\sigma} > \frac{\sqrt{b}(d - (\mu_i - \mu_1))}{\sigma}\ \ \forall i \in \Omega_U^*(d), \\
&\qquad \frac{\sqrt{b}(\bar{Y}_{jb} - \mu_j)}{\sigma} - \frac{\sqrt{b}(\bar{Y}_{1b} - \mu_1)}{\sigma} < \frac{\sqrt{b}(-d - (\mu_j - \mu_1))}{\sigma}\ \ \forall j \in \Omega_L^*(d)\Big\} \\
&= P\Big\{Z_i - Z_1 > \frac{\sqrt{b}(d - (\mu_i - \mu_1))}{\sigma}\ \ \forall i \in \Omega_U^*(d), \\
&\qquad Z_j - Z_1 < \frac{\sqrt{b}(-d - (\mu_j - \mu_1))}{\sigma}\ \ \forall j \in \Omega_L^*(d)\Big\} \\
&= \int_{-\infty}^{\infty} \phi(x) P\Big\{Z_i > x + \frac{\sqrt{b}(d - (\mu_i - \mu_1))}{\sigma}\ \ \forall i \in \Omega_U^*(d), \\
&\qquad Z_j < x + \frac{\sqrt{b}(-d - (\mu_j - \mu_1))}{\sigma}\ \ \forall j \in \Omega_L^*(d)\Big\}\, dx
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \phi(x) \prod_{i \in \Omega_U^*(d)} \Phi\left(-x - \frac{\sqrt{b}(d-(\mu_i-\mu_1))}{\sigma}\right) \times$$
$$\prod_{j \in \Omega_L^*(d)} \Phi\left(x + \frac{\sqrt{b}(-d-(\mu_j-\mu_1))}{\sigma}\right) \, dx.$$

It is clear that

$$\prod_{i \in \Omega_U^*(d)} \Phi\left(-x - \frac{\sqrt{b}(d-(\mu_i-\mu_1))}{\sigma}\right) \prod_{j \in \Omega_L^*(d)} \Phi\left(x + \frac{\sqrt{b}(-d-(\mu_j-\mu_1))}{\sigma}\right)$$

attains its minimum value over $\mu \in R^k$ when $\mu_i - \mu_1 = 2d$ for all $i \in \Omega_U^*(d)$, $\mu_j - \mu_1 = -2d$ for all $j \in \Omega_L^*(d)$ and $C[\Omega_U^*(d)] + C[\Omega_L^*(d)] = k-1$. Without loss of generality, let $\mu_i - \mu_1 = 2d$ for $2 \le i \le l$ and $\mu_j - \mu_1 = -2d$ for $l < j \le k$. Now, let

$$M(l) = \int_{-\infty}^{\infty} \phi(x)\Phi^{l-1}\left(-x + \sqrt{2}|t|_{k-1}^{\alpha}\right)\Phi^{k-l}\left(x + \sqrt{2}|t|_{k-1}^{\alpha}\right) \, dx.$$

It is easy to show that $M(l) \ge M(l+1)$ for $2 \le l < p$ by using the inequality

$$ab\left(a^{s-2(r+1)} + b^{s-2(r+1)}\right) \le a^{s-2r} + b^{s-2r}, \quad \text{for } a, b \in R^+ \text{ and } s \ge 2r+1.$$

Also, it is clear that $M(l) = M(k+1-l)$. So $M(l)$ is minimized over $2 \le l \le k$ at $l = p = \langle (k+1)/2 \rangle$. Consequently

$$\begin{aligned}
\beta^*(d) &= \beta^*(\mu^*(d), d) \\
&= \int_{-\infty}^{\infty} \phi(x)\Phi^{p-1}\left(-x + \frac{\sqrt{b}d}{\sigma}\right)\Phi^{k-p}\left(x + \frac{\sqrt{b}d}{\sigma}\right) \, dx. \\
&= \int_{-\infty}^{\infty} \phi(x)\Phi^{p-1}\left(-x + \sqrt{2}|t|_{k-1}^{\alpha}\right)\Phi^{k-p}\left(x + \sqrt{2}|t|_{k-1}^{\alpha}\right) \, dx.
\end{aligned}$$

This finishes the proof.

It is interesting to note that the value of $\beta^*(d)$ depends only on α and k, but not on d and σ^2. This is because of the way in which we set the sample size $b = 2\left(|t|_{k-1}^{\alpha}\right)^2 \sigma^2 d^{-2}$.

Table 4.4 shows the values of $\beta^*(d)$ for $k-1 = 2(1)20$ and $\alpha = 0.1, 0.05, 0.01$. As we can see, the values of the $\beta^*(d)$ are always larger than $1-\alpha$. For example, if $\alpha = 0.10$ and $k = 4$, we can claim that, with probability 0.945, rather than $1 - \alpha = 0.90$, correct inference, based on the set of confidence intervals in (4.4), will be made for each $\mu_i - \mu_1$ satisfying $|\mu_i - \mu_1| \geq 2d$.

Table 4.4: $\beta^*(d) = \min_{\mu \in R^k} \ \beta^*(\mu, d)$

$k-1 \setminus 1-\alpha$	0.90	0.95	0.99
2	0.945	0.973	0.995
3	0.945	0.973	0.995
4	0.943	0.972	0.995
5	0.943	0.972	0.995
6	0.942	0.972	0.995
7	0.942	0.972	0.995
8	0.942	0.972	0.995
9	0.941	0.972	0.995
10	0.941	0.972	0.995
11	0.941	0.971	0.995
12	0.941	0.971	0.995
13	0.940	0.971	0.995
14	0.940	0.971	0.995
15	0.940	0.971	0.995
16	0.940	0.971	0.995
17	0.940	0.971	0.995
18	0.940	0.971	0.995
19	0.940	0.971	0.995
20	0.939	0.971	0.995

4.2.3 When the variance is unknown

Suppose that σ^2 is unknown, we consider inferences based on the set of confidence intervals

$$\mu_i - \mu_1 \in (\bar{Y}_{iT} - \bar{Y}_{1T} - d,\ \bar{Y}_{iT} - \bar{Y}_{1T} + d), \quad 2 \le i \le k,$$

where the stopping time T is given in Subsection 3.2.1 by

$$T = \inf\{n \ge m :\ n \ge 2d^{-2}(|t|_{k-1}^{\alpha})^2 l_n(\hat{\sigma_n})^2\}.$$

We know that, for each treatment satisfying $\mu_i - \mu_1 \ge 2d(\le -2d)$, the correct inference $\mu_i - \mu_1 > 0(< 0)$ will be made from this set of simultaneous confidence intervals with a probability of at least $1 - \alpha + o(d^2)$, since the confidence level of this set of confidence intervals is equal to $1 - \alpha + o(d^2)$. We wish to assess

$$\beta_U^*(d) = \min_{\mu \in R^k} \beta_U^*(\mu, d),$$

where

$$\beta_U^*(\mu, d) = P\{\bar{Y}_{iT} - \bar{Y}_{1T} > d\ \forall i \in \Omega_U^*(d),\ \ \bar{Y}_{jT} - \bar{Y}_{1T} < -d\ \forall j \in \Omega_L^*(d)\}. \quad (4.5)$$

In particular, we define $\beta_U^*(\mu, d) = 1$ if all the treatments satisfy $|\mu_i - \mu_1| < 2d$. For this we need the following lemma, which can be proved in a way similarly to Lemma 4.1.

Lemma 4.3 *For $k \ge 3$*

$$\beta_U^*(d) = E\left[H\left(\left(|t|_{k-1}^{\alpha}\right)^2 \frac{T}{b}\right)\right],$$

where

$$H(x) = P\left\{\frac{Z_i - Z_1}{\sqrt{2}} > -\sqrt{x}, 2 \le i \le p, \frac{Z_j - Z_1}{\sqrt{2}} < \sqrt{x}, p < j \le k\right\}, \quad (4.6)$$

$b = 2\left(\sigma|t|_{k-1}^{\alpha}/d\right)^2$ *and* $p = \langle (k+1)/2 \rangle$.

Lemma 4.4 *Let $k \geq 3$, $p = \langle (k+1)/2 \rangle$ and*

$$H(x) = P\Big\{ \frac{Z_i - Z_1}{\sqrt{2}} > -\sqrt{x}, 2 \leq i \leq p, \quad \frac{Z_j - Z_1}{\sqrt{2}} < \sqrt{x}, p+1 \leq j \leq k \Big\}.$$

Then for $0 < x < x_0$, we have $|H''(x)| < Cx^{(k-6)/2}$ *, where C is a constant.*

Proof: Let $g(x) = H(x^2)$, then

$$H(x) = g(x^{1/2}),$$
$$H'(x) = \frac{1}{2} x^{-1/2}\, g'(x^{1/2}),$$
$$H''(x) = \frac{1}{4}\left[x^{-1} g''(x^{1/2}) - x^{-3/2}\, g'(x^{1/2}) \right], \tag{4.7}$$

and

$$\begin{aligned} & g(x) \\ &= P\{Z_i - Z_1 > -x\sqrt{2}, 2 \leq i \leq p, \quad Z_j - Z_1 < x\sqrt{2}, p+1 \leq j \leq k\} \\ &= \int_{-\infty}^{\infty} \phi(y) P\{Z_i - y > -x\sqrt{2}, \\ &\quad 2 \leq i \leq p, \quad Z_j - y < x\sqrt{2}, p+1 \leq j \leq k \Big| Z_1 = y\}\, dy \\ &= \int_{-\infty}^{\infty} \phi(y) P\{Z_i - y > -x\sqrt{2}, 2 \leq i \leq p, \quad Z_j - y < x\sqrt{2}, p+1 \leq j \leq k\}\, dy \\ &= \int_{-\infty}^{\infty} \phi(y) \Phi^{p-1}(x\sqrt{2} - y) \Phi^{k-p}(x\sqrt{2} + y)\, dy, \end{aligned}$$

$$\begin{aligned} g'(x) \;&=\; \sqrt{2} \int_{-\infty}^{\infty} \phi(y) \Big((p-1)\phi(x\sqrt{2} - y) \Phi^{p-2}(x\sqrt{2} - y) \Phi^{k-p}(x\sqrt{2} + y) \\ &\quad + (k-p)\phi(y + x\sqrt{2}) \Phi^{p-1}(x\sqrt{2} - y) \Phi^{k-p-1}(x\sqrt{2} + y) \Big)\, dy. \end{aligned}$$

First, observe that

$$\begin{aligned} g'(x) \;&\leq\; \int_{-\infty}^{\infty} B_1 \phi(y) \Phi^{p-2}(x\sqrt{2} - y) \Phi^{k-p-1}(x\sqrt{2} + y)\, dy \\ &\leq\; \int_{-\infty}^{\infty} B_2 \phi(y) (\Phi(x\sqrt{2} - y) + \Phi(x\sqrt{2} + y))^{k-3}\, dy \\ &\leq\; \int_{-\infty}^{\infty} B_3 \phi(y) x^{k-3}\, dy \\ &\leq\; B x^{k-3}, \end{aligned} \tag{4.8}$$

where B, B_1, B_2, and B_3 are constants. Next, we have

$$
\begin{aligned}
&g''(x) = \\
&2\int_{-\infty}^{\infty}\phi(y)\Big((p-1)(y-x\sqrt{2})\phi(x\sqrt{2}-y)\Phi^{p-2}(x\sqrt{2}-y)\Phi^{k-p}(x\sqrt{2}+y) \\
&\quad +(p-1)(p-2)\phi^2(x\sqrt{2}-y)\Phi^{p-3}(x\sqrt{2}-y)\Phi^{k-p}(x\sqrt{2}+y) \\
&+(p-1)(k-p)\phi(x\sqrt{2}-y)\Phi^{p-2}(x\sqrt{2}-y)\phi(x\sqrt{2}+y)\Phi^{k-p-1}(x\sqrt{2}+y) \\
&-(k-p)(x\sqrt{2}+y)\phi(x\sqrt{2}+y)\Phi^{p-1}(x\sqrt{2}-y)\Phi^{k-p-1}(x\sqrt{2}+y) \\
&+(k-p)(p-1)\phi(x\sqrt{2}+y)\phi(x\sqrt{2}-y)\Phi^{p-2}(x\sqrt{2}-y)\Phi^{k-p-1}(x\sqrt{2}+y) \\
&+(k-p)(k-p-1)\phi^2(x\sqrt{2}+y)\Phi^{p-1}(x\sqrt{2}-y)\Phi^{k-p-2}(x\sqrt{2}+y)\Big)\,dy
\end{aligned}
$$

and so

$$
\begin{aligned}
&|g''(x)| \\
&\le A_1\int_{-\infty}^{\infty}\phi(y)\Big[\Big(|(y-x\sqrt{2})|\phi(x\sqrt{2}-y) \\
&+|(y+x\sqrt{2})|\phi(y+x\sqrt{2})\Big)\Phi^{p-2}(x\sqrt{2}-y)\Phi^{k-p-1}(x\sqrt{2}+y) \\
&+A_2\Big(\Big(\phi^2(x\sqrt{2}-y)+\phi(y+x\sqrt{2})\Big)\,\Phi^{p-3}(x\sqrt{2}-y)\Phi^{k-p-1}(x\sqrt{2}+y)\Big) \\
&+A_3\Big(\Big(\phi(x\sqrt{2}-y)\phi(x\sqrt{2}+y)+\phi^2(y+x\sqrt{2})\Big)\times \\
&\qquad \Phi^{p-2}(x\sqrt{2}-y)\Phi^{k-p-2}(x\sqrt{2}+y)\Big)\Big]\,dy \\
&\le D_1x^{k-3}\int_{-\infty}^{\infty}\phi(y)\Big(|y|\phi(x\sqrt{2}-y)+\sqrt{2}x\phi(x\sqrt{2}-y)+|y|\phi(y+x\sqrt{2}) \\
&\qquad +\sqrt{2}x\phi(x\sqrt{2}+y)\Big)\,dy + D_2x^{k-4} \\
&\le D_1x^{k-3}(A+Bx)+D_2x^{k-4} \\
&\le Dx^{k-4} \qquad\qquad (4.9)
\end{aligned}
$$

where D_1, D_2, A, B, and D are constants. By substituting (4.8) and (4.9) in to (4.7), we get

$$
\begin{aligned}
|H''(x)| &\le \frac{1}{4}\left(x^{-1}\,|g''(x^{1/2})| + x^{-3/2}\,|g'(x^{1/2})|\right) \\
&\le Cx^{(k-6)/2}
\end{aligned}
$$

and the proof is thus completed.

Now, by using Theorem 2.2 with $\theta = \left(|t|^{\alpha}_{k-1}\right)^2$, $n_0 = b$, $C_1 = 0$, $\beta = (6-k)/2$ and Lemma 4.4 we have following second order approximation to the $\beta^*_U(d)$.

Theorem 4.4 *For $H(x)$ defined in (4.6), and suppose $m > 1$ if $k \geq 6$ and $m > 1 + (8-k)/k$ if $k = 3, 4, 5$. Then*

$$\begin{aligned}\beta^*_U(d) &= \beta^*(d) + \frac{1}{b}\Big[\left(|t|^{\alpha}_{k-1}\right)^2 H'\left(\left(|t|^{\alpha}_{k-1}\right)^2\right)\left(\rho + l_0 - \frac{2}{k}\right) \\ &\quad + \frac{1}{k}\left(|t|^{\alpha}_{k-1}\right)^4 H''\left(\left(|t|^{\alpha}_{k-1}\right)^2\right)\Big] + o\left(\frac{1}{b}\right),\end{aligned}$$

where $b = 2d^{-2}\left(|t|^{\alpha}_{k-1}\right)^2 \sigma^2$.

The exact value of $\beta^*_U(d)$ can be calculated by using the recursive method discussed in Subsection 3.2.4 since the stopping time is the same as before and

$$\begin{aligned}\beta^*_U(d) &= E\left[H\left(\left(|t|^{\alpha}_{k-1}\right)^2 \frac{T}{b}\right)\right] \\ &= \sum_{n=m_0}^{\infty} P(t=n) H\left(\left(|t|^{\alpha}_{k-1}\right)^2 \frac{n+1}{b}\right) \\ &= \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)] H\left(\left(|t|^{\alpha}_{k-1}\right)^2 \frac{n+1}{b}\right),\end{aligned}$$

where $H(x)$ is defined in (4.6) and $m_0 = m - 1$. Simulation to estimate $\beta^*_U(d)$, based on 6,000 experiments, was also carried out.

For $k = 5, 7, 10$ and $1 - \alpha = 0.90, 0.99$, Tables 4.5 and 4.6 give the exact, simulated and second order approximate values of $\beta^*_U(d)$ at $b = 5(5)60$ and $b = 15(5)60$. For $m = 2, 10, k = 5, 7, 10$, and $1 - \alpha = 90\%, 99\%$, the exact calculation results and approximations of $\beta^*_U(d)$ at $b = 5(5)\ 60$ and $b = 15(5)60$ are linearly plotted in Figures 13-18.

Table 4.5: *Comparisons between the exact, approximate and simulation results of* $\beta_U^*(d)$ *for* $m = 2$ *and given values of* k, $1 - \alpha$ *and* b

$1 - \alpha = 0.90$

	$k = 5$			$k = 7$			$k = 10$		
b	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
5	0.915	0.949	0.908	0.915	0.952	0.920	0.916	0.956	0.907
10	0.923	0.946	0.924	0.930	0.947	0.931	0.935	0.948	0.929
15	0.933	0.945	0.939	0.938	0.945	0.940	0.939	0.946	0.929
20	0.938	0.944	0.938	0.940	0.945	0.942	0.940	0.945	0.935
25	0.940	0.944	0.938	0.941	0.944	0.945	0.941	0.944	0.935
30	0.941	0.944	0.945	0.941	0.944	0.943	0.941	0.943	0.938
35	0.941	0.944	0.938	0.942	0.943	0.946	0.941	0.943	0.935
40	0.942	0.944	0.945	0.942	0.943	0.942	0.941	0.943	0.935
45	0.942	0.944	0.941	0.942	0.943	0.943	0.941	0.943	0.936
50	0.942	0.944	0.950	0.942	0.943	0.941	0.941	0.942	0.936
55	0.942	0.944	0.942	0.942	0.943	0.942	0.941	0.942	0.934
60	0.942	0.943	0.941	0.942	0.943	0.942	0.941	0.942	0.935

Table 4.5: *Comparisons between the exact, approximate and simulation results of* $\beta_U^*(d)$

for $m = 10$ *and given values of* k, $1-\alpha$ *and* b

$1 - \alpha = 0.90$

b	$k=5$			$k=7$			$k=10$		
	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
15	0.942	0.945	0.944	0.941	0.945	0.941	0.940	0.946	0.939
20	0.942	0.944	0.939	0.941	0.945	0.944	0.941	0.945	0.933
25	0.942	0.944	0.949	0.942	0.944	0.945	0.941	0.944	0.936
30	0.943	0.944	0.942	0.942	0.944	0.944	0.941	0.943	0.938
35	0.943	0.944	0.938	0.942	0.943	0.938	0.941	0.943	0.935
40	0.943	0.944	0.944	0.942	0.943	0.939	0.941	0.943	0.941
45	0.943	0.944	0.945	0.942	0.943	0.945	0.941	0.943	0.934
50	0.943	0.944	0.943	0.942	0.943	0.941	0.941	0.942	0.934
55	0.943	0.944	0.942	0.942	0.943	0.940	0.941	0.942	0.933
60	0.943	0.943	0.943	0.942	0.943	0.942	0.941	0.942	0.940

Table 4.6: *Comparisons between the exact, approximate*

and simulation results of $\beta_U^*(d)$

for $m = 2$ *and given values of* k, $1-\alpha$ *and* b

$1 - \alpha = 0.99$

b	$k=5$			$k=7$			$k=10$		
	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
5	0.985	0.996	0.987	0.986	0.997	0.986	0.987	0.997	0.986
10	0.987	0.995	0.985	0.990	0.996	0.990	0.993	0.996	0.990
15	0.990	0.995	0.990	0.993	0.995	0.992	0.994	0.996	0.992
20	0.992	0.995	0.994	0.994	0.995	0.994	0.994	0.996	0.994
25	0.993	0.995	0.994	0.994	0.995	0.994	0.994	0.995	0.994
30	0.993	0.995	0.993	0.994	0.995	0.994	0.994	0.995	0.994
35	0.994	0.995	0.993	0.994	0.995	0.995	0.994	0.995	0.995
40	0.994	0.995	0.993	0.994	0.995	0.995	0.994	0.995	0.995
45	0.994	0.995	0.993	0.994	0.995	0.994	0.994	0.995	0.994
50	0.994	0.995	0.994	0.994	0.995	0.995	0.994	0.995	0.995
55	0.994	0.995	0.995	0.994	0.995	0.995	0.994	0.995	0.995
60	0.994	0.995	0.996	0.994	0.995	0.995	0.994	0.995	0.995

Table 4.6: *Comparisons between the exact, approximate and simulation results of* $\beta_U^*(d)$

for $m = 10$ *and given values of* k, $1 - \alpha$ *and* a

$1 - \alpha = 0.99$

b	$k = 5$			$k = 7$			$k = 10$		
	Exact	Appro.	Simul.	Exact	Appro.	Simul.	Exact	Appro.	Simul.
15	0.994	0.995	0.995	0.994	0.995	0.996	0.994	0.996	0.994
20	0.994	0.995	0.997	0.994	0.995	0.995	0.994	0.996	0.993
25	0.994	0.995	0.996	0.995	0.995	0.994	0.995	0.995	0.995
30	0.994	0.995	0.994	0.995	0.995	0.995	0.995	0.995	0.993
35	0.994	0.995	0.993	0.995	0.995	0.996	0.995	0.995	0.993
40	0.994	0.995	0.993	0.995	0.995	0.995	0.995	0.995	0.995
45	0.995	0.995	0.993	0.995	0.995	0.996	0.995	0.995	0.995
50	0.995	0.995	0.994	0.995	0.995	0.994	0.995	0.995	0.995
55	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.995
60	0.995	0.995	0.994	0.995	0.995	0.995	0.995	0.995	0.993

Figure 13. The exact (-) and approximate ($\cdots$) values of $\beta_U^*(d)$ as a function of $b = b(\sigma)$ and $1 - \alpha$ (- -) for $m = 2$, $k = 5$ and $\alpha = 0.10,\ 0.01$.

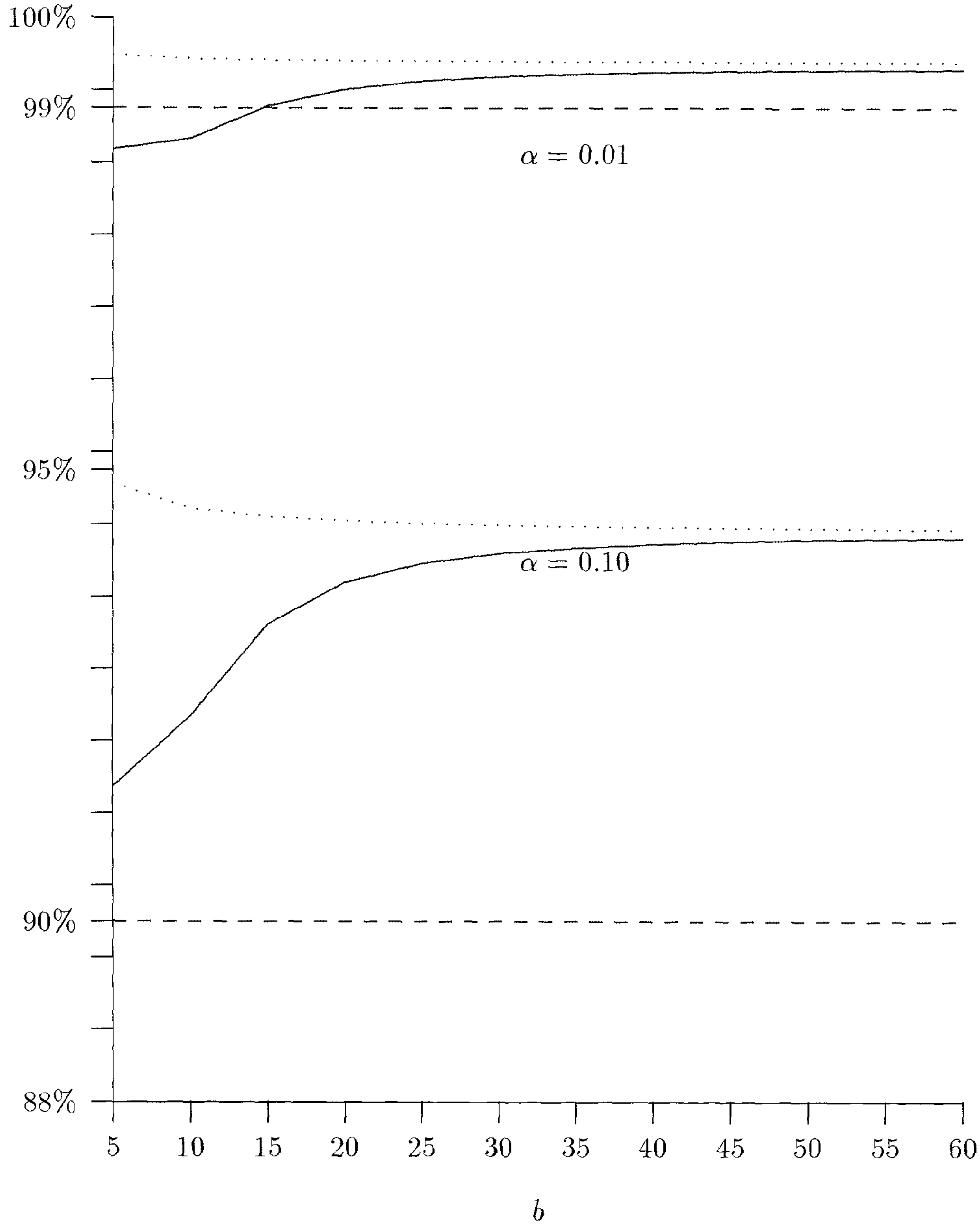


Figure 14. The exact (-) and approximate ($\cdots$) values of $\beta_U^*(d)$ as a function of $b = b(\sigma)$ and $1 - \alpha$ (- -) for $m = 10$, $k = 5$ and $\alpha = 0.10,\ 0.01$.

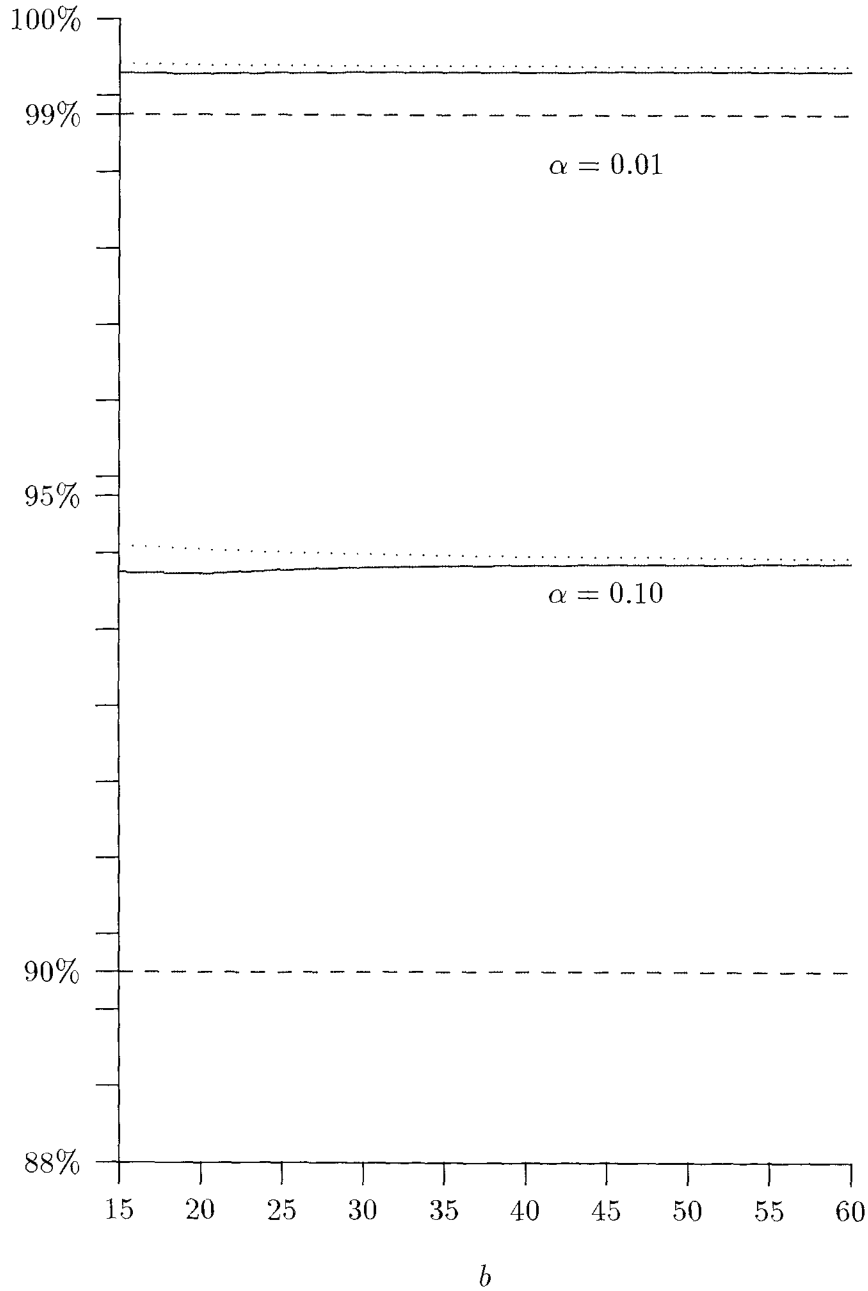


Figure 15. The exact (-) and approximate ($\cdots$) values of $\beta_U^*(d)$ as a function of $b = b(\sigma)$ and $1 - \alpha$ (- -) for $m = 2$, $k = 7$ and $\alpha = 0.10,\ 0.01$.

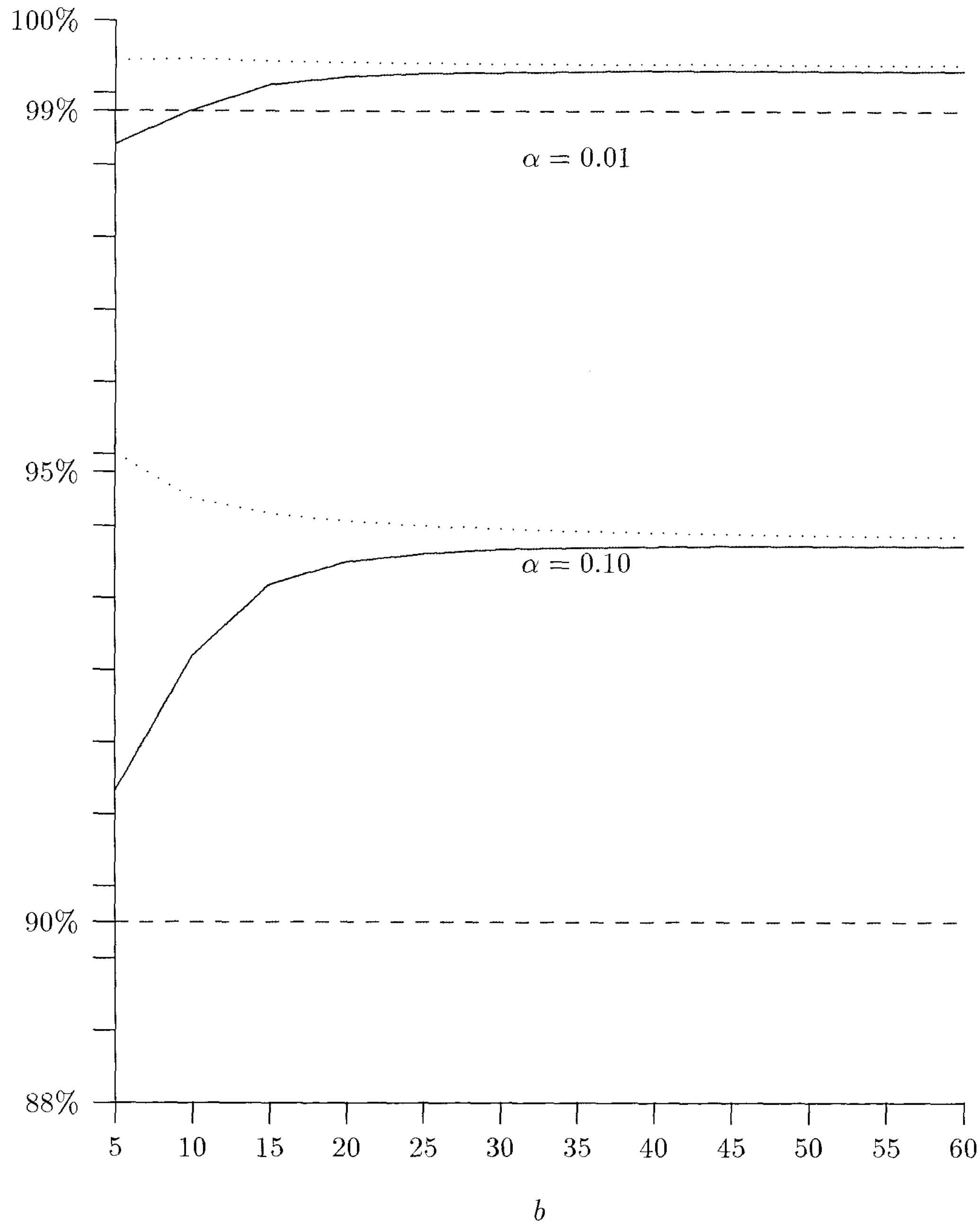


Figure 16. The exact (-) and approximate (···) values of $\beta_U^*(d)$ as a function of $b = b(\sigma)$ and $1 - \alpha$ (- -) for $m = 10$, $k = 7$ and $\alpha = 0.10,\ 0.01$.

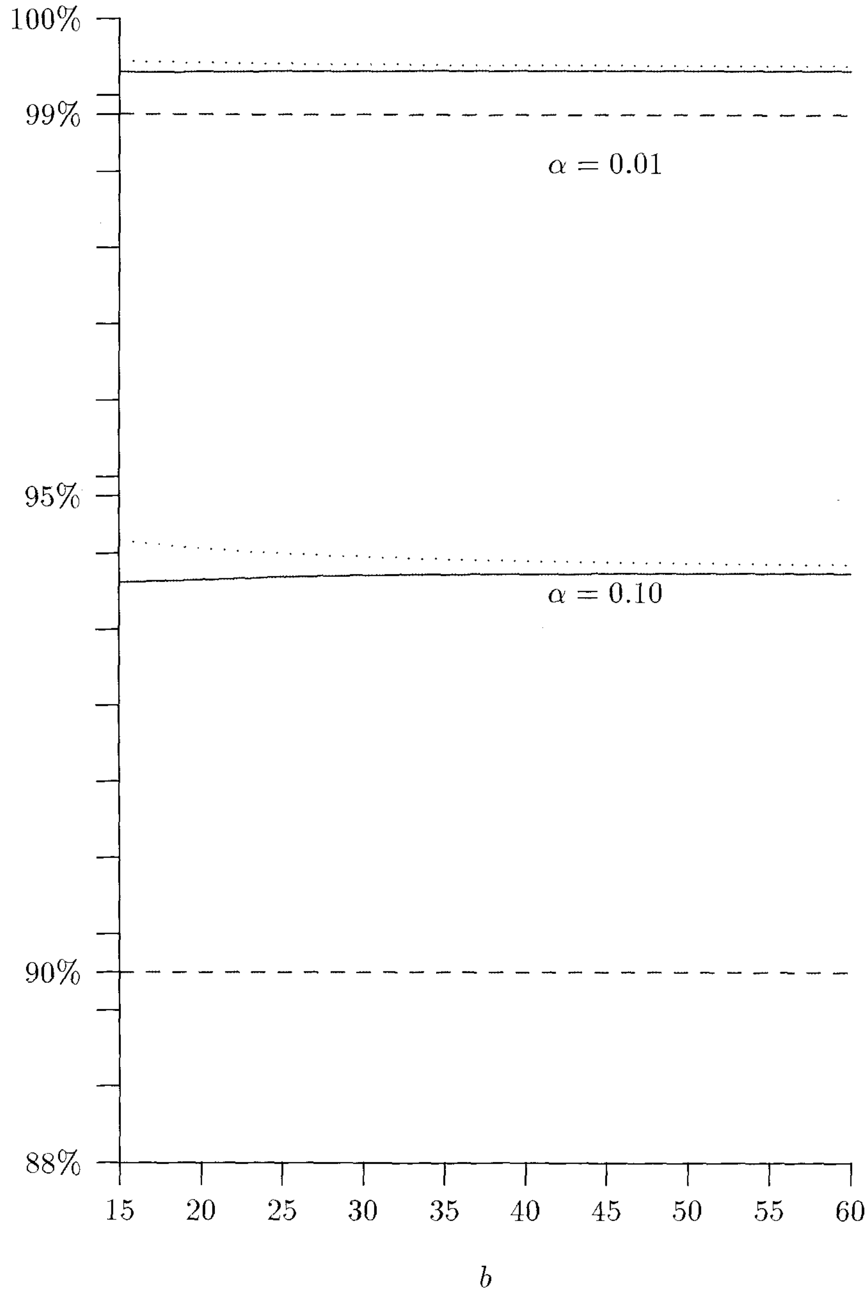


Figure 17. The exact (-) and approximate ($\cdots$) values of $\beta_U^*(d)$ as a function of $b = b(\sigma)$ and $1 - \alpha$ (- -) for $m = 2$, $k = 10$ and $\alpha = 0.10,\ 0.01$.

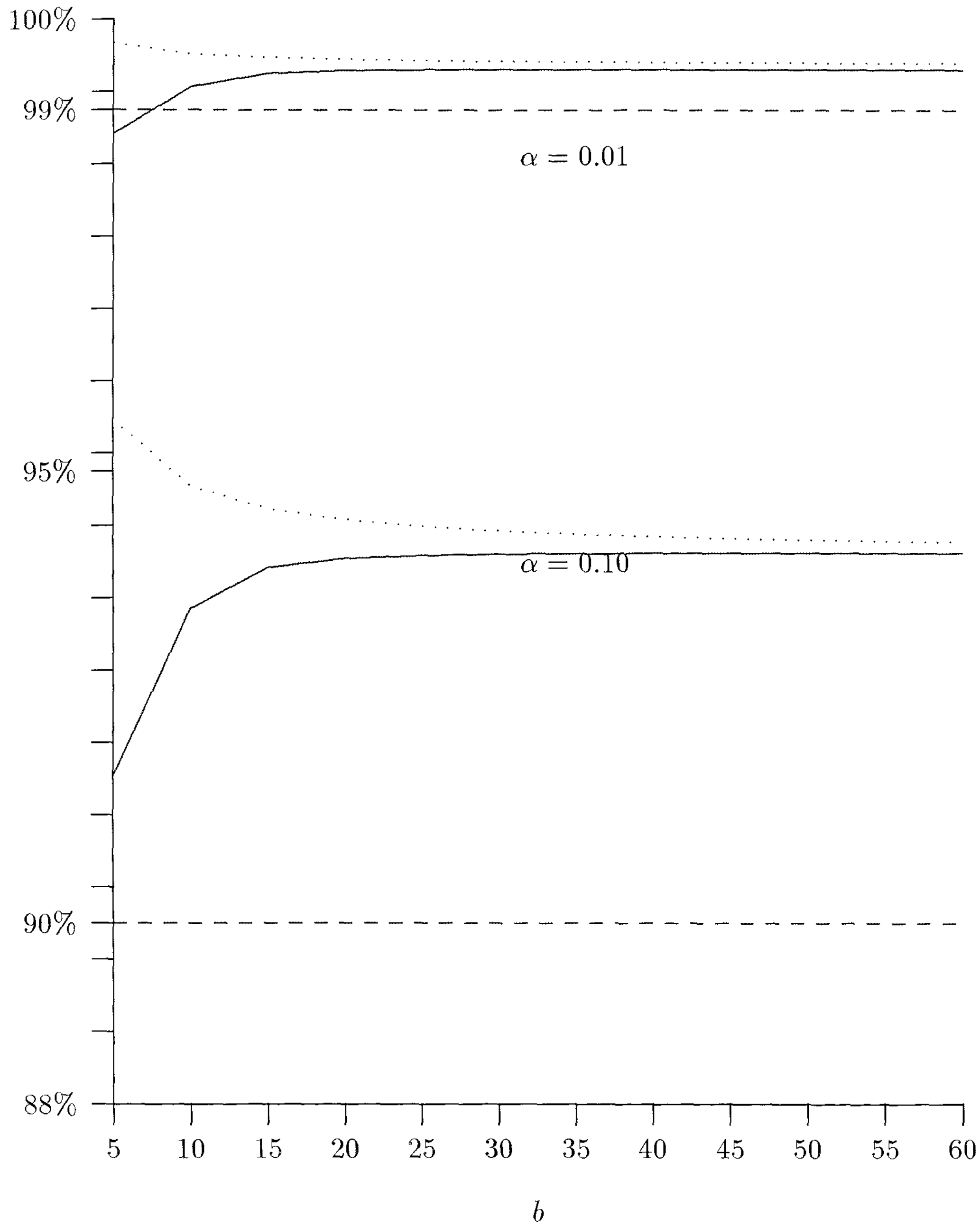
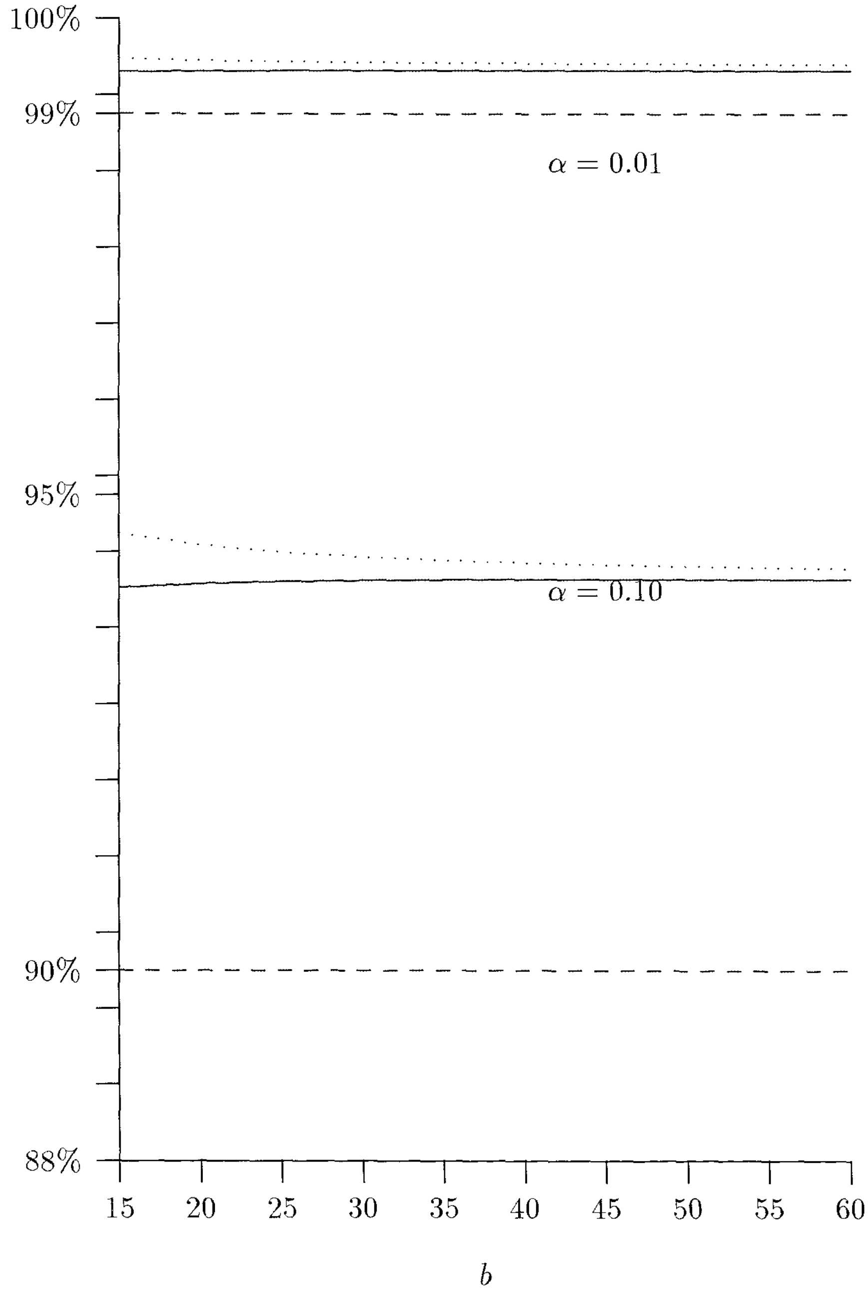


Figure 18. The exact (-) and approximate ($\cdots$) values of $\beta_U^*(d)$ as a function of $b = b(\sigma)$ and $1 - \alpha$ (- -) for $m = 10$, $k = 10$ and $\alpha = 0.10,\ 0.01$.



4.3 The exact probability of making correct inference for all-pairwise comparisons of the means of several independent normal populations

4.3.1 Introduction

From a set of $2d$-width and $(1-\alpha)$-level simultaneous confidence interval for the $\mu_i - \mu_j$

$$P\Big\{\mu_i - \mu_j \in \Big(\bar{Y}_i - \bar{Y}_j - d,\ \bar{Y}_i - \bar{Y}_j + d\Big),\ \ 1 \le i \ne j \le k\Big\} = 1-\alpha$$

simultaneous inferences about each $\mu_i - \mu_j$ can be made. For example, we can infer that $\mu_i - \mu_j > 0$ if $\bar{Y}_i - \bar{Y}_j - d > 0$. Furthermore, the probability of making correct inferences $\mu_i - \mu_j > 0$, for each pair satisfying $\mu_i - \mu_j \ge 2d$, is at least $1-\alpha$, the confidence level. The question we want to answer in this section is " what is the exact value of this probability?" More precisely, we want to investigate the following probability

$$P\{\text{making correct inferences } \mu_i - \mu_j > 0, \text{ for each pair satisfying } \mu_i - \mu_j \ge 2d\}.$$

Let

$$\Omega_U^{**}(d) = \{(i,j) : \mu_i - \mu_j > 2d\}.$$

The above probability is then equal to

$$P\{\text{making correct inferences } \mu_i - \mu_j > 0 \text{ for each } (i,j) \in \Omega_U^{**}(d)\}.$$

This probability is of course dependent on the true value of $\mu = (\mu_1, \mu_2, \cdots, \mu_k) \in R^k$, and let it be denoted by $\beta^{**}(\mu, d)$. For obvious reason, we impose $\beta^{**}(\mu, d) = 1$ if for given values of μ and d the set $\Omega_U^{**}(d)$ is empty. In this section, we wish to assess

$$\beta^{**}(d) = \min_{\mu \in R^k} \beta^{**}(\mu, d).$$

It is clear that $\beta^{**}(d)$ must be no less than $1-\alpha$, the confidence level. We shall compute the exact value of $\beta^{**}(d)$ when $k = 3$ and a lower bound on $\beta^{**}(d)$ when $k = 4$. Although $\beta^{**}(\mu, d)$ and $\beta^{**}(d)$ are well defined for general $k \geq 2$, to find an explicit formula for $\beta^{**}(d)$ when $k \geq 4$ encounters great difficulty and might be impossible.

Two different situations are considered. In Subsection 4.3.2 we consider the known variance case in which the set of confidence intervals for $\mu_i - \mu_j$ is given in (4.10). In Subsection 4.3.3 we consider the unknown variance case in which the set of confidence intervals for $\mu_i - \mu_j$ is constructed by using the pure sequential procedure of Section 3.3.

The following notation is used. Let $\mu_{[1]} \leq \mu_{[2]} \leq \mu_{[3]} \leq \mu_{[4]}$ denote the ordered values of μ_1, μ_2, μ_3 and μ_4, and let $\bar{Y}_{(i)}$ denote the sample mean from the population with mean $\mu_{[i]}, i = 1, 2, 3, 4$.

4.3.2 When the variance is known

Suppose that σ^2 is known and a random sample of fixed size $c = (q_k^\alpha)^2 \sigma^2 d^{-2}$ is taken from each of the k populations, we construct the following set of simultaneous confidence intervals for $\mu_i - \mu_j$

$$\mu_i - \mu_j \in \left(\bar{Y}_{ic} - \bar{Y}_{jc} - d,\ \bar{Y}_{ic} - \bar{Y}_{jc} + d\right), \quad 1 \le i \ne j \le k. \tag{4.10}$$

It is known from Subsection 3.3.1 that this set of confidence intervals has simultaneous level $1 - \alpha$.

First, when $k = 3$, in order to compute the exact value of $\beta^{**}(d)$, we have the following theorem.

Theorem 4.5 *For given $d > 0, c = d^{-2}(q_3^\alpha)^2 \sigma^2$ and $\mu^*(d) = (0, -2d, 2d) \in R^3$. We have*

$$\begin{aligned} \beta^{**}(d) &= \beta^{**}(\mu^*(d), d) \\ &= \int_{-\infty}^{\infty} \phi(x)\Phi(\frac{d\sqrt{c}}{\sigma} - x)\Phi(\frac{d\sqrt{c}}{\sigma} + x)\, dx \\ &= \int_{-\infty}^{\infty} \phi(x)\Phi(q_3^\alpha - x)\Phi(q_3^\alpha + x)\, dx. \end{aligned}$$

Proof: Dividing the whole space of $\mu = (\mu_1, \mu_2, \mu_3)$, R^3, into five regions as follows:

1. $R_1 = \{\mu_{[3]} - \mu_{[1]} < 2d\}$
2. $R_2 = \{\mu_{[3]} - \mu_{[1]} \ge 2d, \mu_{[3]} - \mu_{[2]} < 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$
3. $R_3 = \{\mu_{[3]} - \mu_{[1]} \ge 2d, \mu_{[3]} - \mu_{[2]} < 2d, \mu_{[2]} - \mu_{[1]} \ge 2d\}$
4. $R_4 = \{\mu_{[3]} - \mu_{[1]} \ge 2d, \mu_{[3]} - \mu_{[2]} \ge 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$
5. $R_5 = \{\mu_{[3]} - \mu_{[2]} \ge 2d, \mu_{[2]} - \mu_{[1]} \ge 2d\}$.

Now consider the function $\beta^{**}(\mu, d)$ for $\mu = (\mu_1, \mu_2, \mu_3)$ in each of the five regions. When $\mu \in R_1$, $\beta^{**}(\mu, d) = 1$ by definition since $|\mu_i - \mu_j| < 2d, \forall 1 \le$

$i < j \leq 3$. So, the minimum value of $\beta^{**}(\mu, d)$ will not be attained at $\mu \in R_1$. When $\mu \in R_2$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(3)} - \bar{Y}_{(1)} > d\}. \tag{4.11}$$

When $\mu \in R_3$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(3)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}. \tag{4.12}$$

When $\mu \in R_4$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(3)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(2)} > d\}. \tag{4.13}$$

When $\mu \in R_5$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(3)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}. \tag{4.14}$$

Now, we compare $\min_{\mu \in R_i} \ \beta^{**}(\mu, d)$ for $i = 2, 3, 4, 5$. When $\mu \in R_5$, we have

$$\begin{aligned}
&\beta^{**}(\mu, d) \\
=\ & P\Big(\frac{\bar{Y}_{(3)} - \bar{Y}_{(2)} - (\mu_{[3]} - \mu_{[2]})}{\sigma/\sqrt{c}} > \frac{d\sqrt{c}}{\sigma} - \frac{(\mu_{[3]} - \mu_{[2]})\sqrt{c}}{\sigma}, \\
& \quad \frac{\bar{Y}_{(2)} - \bar{Y}_{(1)} - (\mu_{[2]} - \mu_{[1]})}{\sigma/\sqrt{c}} > \frac{d\sqrt{c}}{\sigma} - \frac{(\mu_{[2]} - \mu_{[1]})\sqrt{c}}{\sigma}\Big) \\
=\ & P\Big(Z_3 - Z_2 > \frac{d\sqrt{c}}{\sigma} - \frac{(\mu_{[3]} - \mu_{[2]})\sqrt{c}}{\sigma},\ Z_2 - Z_1 > \frac{d\sqrt{c}}{\sigma} - \frac{(\mu_{[2]} - \mu_{[1]})\sqrt{c}}{\sigma}\Big),
\end{aligned}$$

which clearly attains its minimum at

$$\mu_{[1]} = -2d, \quad \mu_{[2]} = 0, \quad \mu_{[3]} = 2d.$$

So

$$\begin{aligned}
A_5 &= \min_{\mu \in R_5} \beta^{**}(\mu, d) \\
&= P(Z_3 - Z_2 > -q_3^{\alpha},\ Z_2 - Z_1 > -q_3^{\alpha}) \\
&= P\Big(X_5 > -\frac{q_3^{\alpha}}{\sqrt{2}},\ Y_5 > -\frac{q_3^{\alpha}}{\sqrt{2}}\Big)
\end{aligned}$$

where (X_5, Y_5) has a bivariate normal distribution with mean $(0,0)$ and a covariance matrix

$$\begin{pmatrix} 1 & , & -1/2 \\ -1/2 & , & 1 \end{pmatrix}.$$

Similarly, when $\mu \in R_4$, we have

$$\begin{aligned} A_4 &= \min_{\mu \in R_4} \beta^{**}(\mu, d) \\ &= P(Z_3 - Z_1 > -q_3^\alpha,\ Z_3 - Z_2 > -q_3^\alpha) \\ &= P\Big(X_4 > -\frac{q_3^\alpha}{\sqrt{2}},\ Y_4 > -\frac{q_3^\alpha}{\sqrt{2}}\Big) \end{aligned}$$

where (X_4, Y_4) has a bivariate normal distribution with mean $(0,0)$ and a covariance matrix

$$\begin{pmatrix} 1 & , & 1/2 \\ 1/2 & , & 1 \end{pmatrix}.$$

When $\mu \in R_3$, we have

$$\begin{aligned} A_3 &= \min_{\mu \in R_3} \beta^{**}(\mu, d) \\ &= P\Big(Z_3 - Z_1 > -\frac{d\sqrt{c}}{\sigma},\ Z_2 - Z_1 > -\frac{d\sqrt{c}}{\sigma}\Big) \\ &= P\Big(X_3 > -\frac{q_3^\alpha}{\sqrt{2}},\ Y_3 > -\frac{q_3^\alpha}{\sqrt{2}}\Big), \end{aligned}$$

where (X_3, Y_3) has a bivariate normal distribution with mean $(0,0)$ and a covariance matrix

$$\begin{pmatrix} 1 & , & 1/2 \\ 1/2 & , & 1 \end{pmatrix}.$$

Now, by Slepians's inequality (Theorem A.4), we have $A_5 \leq A_3 = A_4$. Also, $\min_{\mu \in R_2} \beta^{**}(\mu, d) \geq A_3$ is obvious. So, $\beta^{**}(\mu, d)$ attains its minimum in R_5 at $\mu = (0, -2d, 2d)$, and the minimum is given by

$$\begin{aligned} \beta^{**}(d) &= P(Z_3 - Z_2 > -q_3^\alpha,\ Z_2 - Z_1 > -q_3^\alpha) \\ &= \int_{-\infty}^{\infty} \phi(x) P(Z_3 > -q_3^\alpha + x,\ Z_1 < q_3^\alpha + x)\, dx \\ &= \int_{-\infty}^{\infty} \phi(x) \Phi(q_3^\alpha - x) \Phi(q_3^\alpha + x)\, dx \qquad (4.15) \end{aligned}$$

as required. This completes the proof.

Table 4.7 shows the values of $\beta^{**}(d)$ for $k = 3, \alpha = 0.1, 0.05, 0.01$.

Table 4.7: $\beta^{**}(d)$ *for* $k = 3$

1-α	$\beta^{**}(d)$
0.90	0.960
0.95	0.981
0.99	0.996

Now we consider $k = 4$. The following theorem gives a lower bound, $\beta_L^{**}(d)$, on $\min_{\mu \in R^4} \ \beta^{**}(\mu, d)$.

Theorem 4.6 *For given* $d > 0$ *and* $c = d^{-2}\,(q_4^\alpha)^2\,\sigma^2$, *we have*

$$\beta^{**}(d) \geq \beta_L^{**}(d),$$

where

$$\beta_L^{**}(d) = \int_{-\infty}^{\infty} \int_{-\infty}^{x+2q_4^\alpha} \phi(x)\phi(y)\Phi(q_4^\alpha - x)[\Phi(q_4^\alpha + x) - \Phi(y - q_4^\alpha)]^2 \, dy dx.$$

Proof: Divide the whole space of $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, R^4, into fourteen regions as follows:

1. $R_1 = \{\mu_{[4]} - \mu_{[1]} < 2d\}$

2. $R_2 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} < 2d\}$

3. $R_3 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

4. $R_4 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} \geq 2d\}$

5. $R_5 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} < 2d,$
   $\mu_{[3]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

6. $R_6 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} < 2d,$
$\mu_{[3]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} \geq 2d\}$

7. $R_7 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} < 2d,$
$\mu_{[3]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} < 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

8. $R_8 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} < 2d,$
$\mu_{[3]} - \mu_{[2]} \geq 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

9. $R_9 = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} < 2d,$
$\mu_{[3]} - \mu_{[2]} \geq 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} \geq 2d\}$

10. $R_{10} = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} \geq 2d,$
$\mu_{[3]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

11. $R_{11} = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} \geq 2d,$
$\mu_{[3]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} \geq 2d\}$

12. $R_{12} = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} \geq 2d,$
$\mu_{[3]} - \mu_{[2]} < 2d, \mu_{[3]} - \mu_{[1]} < 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

13. $R_{13} = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} \geq 2d,$
$\mu_{[3]} - \mu_{[2]} \geq 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} < 2d\}$

14. $R_{14} = \{\mu_{[4]} - \mu_{[1]} \geq 2d, \mu_{[4]} - \mu_{[2]} \geq 2d, \mu_{[4]} - \mu_{[3]} \geq 2d,$
$\mu_{[3]} - \mu_{[2]} \geq 2d, \mu_{[3]} - \mu_{[1]} \geq 2d, \mu_{[2]} - \mu_{[1]} \geq 2d\}.$

Now consider the function $\beta^{**}(\mu, d)$ for μ in each of the fourteen regions. When $\mu \in R_1$, $\beta^{**}(\mu, d) = 1$, since $|\mu_i - \mu_j| < 2d, \forall 1 \leq i < j \leq 4$. So the minimum value of $\beta^{**}(\mu, d)$ will not be attained in R_1. Let $B_i = \min_{\mu \in R_i} \beta^{**}(\mu, d)$. When $\mu \in R_2$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d\}$$

and

$$B_2 = P\{Z_4 - Z_1 > -q_4^{\alpha}\}.$$

When $\mu \in R_3$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d\}$$

and

$$B_3 = P\{Z_4 - Z_1 > -q_4^\alpha,\ \ Z_3 - Z_1 > -q_4^\alpha\}.$$

When $\mu \in R_4$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}$$

and

$$B_4 = P\{Z_4 - Z_1 > -q_4^\alpha,\ \ Z_3 - Z_1 > -q_4^\alpha,\ \ Z_2 - Z_1 > -q_4^\alpha\}.$$

When $\mu \in R_5$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d\}$$

and

$$B_5 = P\{Z_4 - Z_1 > -q_4^\alpha,\ \ Z_4 - Z_2 > -q_4^\alpha,\ \ Z_3 - Z_1 > -q_4^\alpha\}.$$

When $\mu \in R_6$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}$$

and

$$B_6 = P\{Z_4 - Z_2 > -q_4^\alpha,\ \ Z_3 - Z_1 > -q_4^\alpha,\ \ Z_2 - Z_1 > -q_4^\alpha\}.$$

When $\mu \in R_7$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(2)} > d\}$$

and

$$B_7 = P\{Z_4 - Z_1 > -q_4^\alpha,\ \ Z_4 - Z_2 > -q_4^\alpha\}.$$

When $\mu \in R_8$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d\}$$

and

$$B_8 = P\{Z_4 - Z_1 > -q_4^{\alpha},\ \ Z_4 - Z_2 > -q_4^{\alpha},\ \ Z_3 - Z_2 > -q_4^{\alpha},\ \ Z_3 - Z_1 > -q_4^{\alpha}\}.$$

When $\mu \in R_9$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}$$

and

$$B_9 = P\{Z_4 - Z_2 > -q_4^{\alpha},\ \ Z_3 - Z_2 > -q_4^{\alpha},\ \ Z_2 - Z_1 > -q_4^{\alpha}\}.$$

When $\mu \in R_{10}$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(3)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d\}$$

and

$$B_{10} = P\{Z_4 - Z_2 > -q_4^{\alpha},\ \ Z_4 - Z_3 > -q_4^{\alpha},\ \ Z_3 - Z_1 > -q_4^{\alpha}\}.$$

When $\mu \in R_{11}$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(3)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}$$

and

$$B_{11} = P\{Z_4 - Z_2 > -q_4^{\alpha},\ \ Z_4 - Z_3 > -q_4^{\alpha},\ \ Z_3 - Z_1 > -q_4^{\alpha},\ \ Z_2 - Z_1 > -q_4^{\alpha}\}.$$

When $\mu \in R_{12}$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(1)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(4)} - \bar{Y}_{(3)} > d\}$$

and

$$B_{12} = P\{Z_4 - Z_1 > -q_4^{\alpha},\ \ Z_4 - Z_2 > -q_4^{\alpha},\ \ Z_4 - Z_3 > -q_4^{\alpha}\}.$$

When $\mu \in R_{13}$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(3)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(1)} > d\}$$

and

$$B_{13} = P\{Z_4 - Z_3 > -q_4^\alpha, \quad Z_3 - Z_2 > -q_4^\alpha, \quad Z_3 - Z_1 > -q_4^\alpha\}.$$

When $\mu \in R_{14}$,

$$\beta^{**}(\mu, d) = P\{\bar{Y}_{(4)} - \bar{Y}_{(3)} > d,\ \bar{Y}_{(3)} - \bar{Y}_{(2)} > d,\ \bar{Y}_{(2)} - \bar{Y}_{(1)} > d\}$$

and

$$B_{14} = P\{Z_4 - Z_3 > -q_4^\alpha, \quad Z_3 - Z_2 > -q_4^\alpha, \quad Z_2 - Z_1 > -q_4^\alpha\}.$$

Now, it is clear that $B_4 \leq B_3 \leq B_2$ and $B_5 \leq B_7$, and so the minimum is among $B_4, B_5, B_6, B_8, B_9, B_{10}, B_{11}, B_{12}, B_{13}$ and B_{14}.

Dividing these B_i's in to two groups, one group contains $B_4, B_5, B_6, B_9, B_{10}$, B_{12}, B_{13} and B_{14}, and the other group contains B_8, B_{11}. Now by using Slepians's inequality it can be shown that B_{14} is the minimum in the first group and that $B_{11} \leq B_8$. Consequently

$$\beta^{**}(d) = \min(B_{11}, B_{14}).$$

It is straightforward to show that

$$\begin{aligned} B_{14} &= \min_{\mu \in R_{14}} \beta^{**}(\mu, d) = \beta^{**}((-2d, 0, 2d, 4d), d) \\ &= P(Z_4 - Z_3 > -q_4^\alpha, Z_3 - Z_2 > -q_4^\alpha,\ Z_2 - Z_1 > -q_4^\alpha) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x)\phi(y) P(Z_4 > -q_4^\alpha + x,\ Z_2 < q_4^\alpha + x,\ Z_2 > -q_4^\alpha + y)\, dy\, dx \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x+2q_4^\alpha} \phi(y)\phi(x)\Phi(q_4^\alpha - x)(\Phi(q_4^\alpha + x) - \Phi(y - q_4^\alpha))\, dy\, dx, \end{aligned}$$

and

$$\begin{aligned} B_{11} &= \min_{\mu \in R_{11}} \beta^{**}(\mu, d) = \beta^{**}((-2d, 0, 0, 2d), d) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x+2q_4^\alpha} \phi(y)\phi(x)[\Phi(q_4^\alpha + x) - \Phi(y - q_4^\alpha)]^2\, dy\, dx, \end{aligned}$$

from which the theorem follows clearly. The proof is thus completed.

From this proof, we see that

$$\beta^{**}(d) = \min(B_{11},\ B_{14}).$$

The numerical calculation shows that for some values of q_4^α, $B_{14} < B_{11}$ and for other values of q_4^α, $B_{14} > B_{11}$. For example, if $q_4^\alpha = 1$ then $B_{14} = 0.369$ and $B_{11} = 0.377$, but if $q_4^\alpha = 3$ then $B_{14} = 0.949$ and $B_{11} = 0.938$. It is therefore most unlikely that an explicit expression of $\beta^{**}(d)$ can be given.

Table 4.8 presents the values of $\beta^{**}(d)$ and $\beta_L^{**}(d)$ for $k = 4, \alpha = 0.1, 0.05, 0.01$. From these, it seems that $\beta_L^{**}(d)$ is a reasonably tight lower bound on $\beta^{**}(d)$. It is interesting to note that both the values of $\beta^{**}(d)$ and $\beta_L^{**}(d)$ depend only on α and k, but not on d and σ^2.

Table 4.8: $\beta^{**}(d)$ *and* $\beta_L^{**}(d)$ *for* $k = 4$

1-α	$\beta^{**}(d)$	$\beta_L^{**}(d)$
0.90	0.959	0.949
0.95	0.981	0.976
0.99	0.996	0.996

4.3.3 When the variance is unknown

In this subsection, we suppose σ^2 is unknown and consider inferences based on the set of confidence intervals

$$\mu_i - \mu_j \in (\bar{Y}_{iT} - \bar{Y}_{jT} - d,\ \bar{Y}_{iT} - \bar{Y}_{jT} + d), \quad 1 \le i \ne j \le k,$$

where the stopping time T is given in Subsection 3.3.1 by

$$T = \inf\{n \ge m :\ n \ge d^{-2}(q_k^\alpha)^2 l_n(\hat{\sigma_n})^2\}.$$

We know that, for each pair (i,j) of treatments satisfying $\mu_i - \mu_j \ge 2d$, the correct inference $\mu_i - \mu_j > 0$ will be made from this set of simultaneous confidence intervals with a probability of at least $1 - \alpha + o(d^2)$, since the confidence level of this set of confidence intervals is equal to $1 - \alpha + o(d^2)$. We wish to assess

$$\beta_U^{**}(d) = \min_{\mu \in R^k} \beta_U^{**}(\mu, d),$$

where

$$\beta_U^{**}(\mu, d) = P\{\bar{Y}_{iT} - \bar{Y}_{jT} > d\ \forall (i,j) \in \Omega_U^{**}(d)\}. \tag{4.16}$$

In particular, we define $\beta_U^{**}(\mu, d) = 1$ if all treatments satisfy $|\mu_i - \mu_j| < 2d$. We again consider only $k = 3$ and $k = 4$.

When $k = 3$, an argument similar to the proof of Lemma 4.1 establishes

Lemma 4.5

$$\beta_U^{**}(d) = E\left[H\left((q_3^\alpha)^2 \frac{T}{c}\right)\right],$$

where

$$H(x) = P\{Z_3 - Z_2 > -\sqrt{x},\ \ Z_2 - Z_1 > -\sqrt{x}\} \tag{4.17}$$

and $c = (\sigma q_3^\alpha / d)^2$.

The following lemma can be proved similarly as before.

Lemma 4.6 *Let* $H(x) = P\{Z_3 - Z_2 > -\sqrt{x},\ \ Z_2 - Z_1 > -\sqrt{x}\}$*, and* $C_0 > 0$ *is a given constant. Then for* $0 < x < C_0,\ \ |H''(x)| < Dx^{-3/2}$*, where* D *is a constant.*

Now Lemma 4.6 and Theorem 2.2 with $\theta = (q_3^\alpha)^2$, $C_1 = 0$ and $n_0 = c$ give the second order approximation to the $\beta_U^{**}(d)$.

Theorem 4.7 *For* $H(x)$ *defined in (4.17) and* $m > 2$ *we have*

$$\begin{aligned}\beta_U^{**}(d) &= \beta^{**}(d) + \frac{1}{c}\Big[(q_3^\alpha)^2 H'\left((q_3^\alpha)^2\right)\left(\rho + l_0 - \frac{2}{3}\right) \\ &\quad + \frac{1}{3}(q_3^\alpha)^4 H''\left((q_3^\alpha)^2\right)\Big] + o\left(\frac{1}{c}\right),\end{aligned}$$

where $c = d^{-2}(q_3^\alpha)^2\sigma^2$.

For $k = 3$ and $1 - \alpha = 0.90, 0.99$, Tables 4.9 gives the values of the second order approximations to $\beta_U^{**}(d)$ at $c = 5(5)60$ and $c = 15(5)60$

Next, when $k = 4$, by using Lemma 4.6, we have the following lemma.

Lemma 4.7

$$\beta^{**}(d) \geq \beta_{LU}^{**}(d) = E\left[H\left((q_4^\alpha)^2\frac{T}{c}\right)\right],$$

where

$$H(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{t+2\sqrt{x}} \phi(t)\phi(r)\Phi(\sqrt{x} - t)[\Phi(\sqrt{x} + t) - \Phi(r - \sqrt{x})]^2\, dr dt \quad (4.18)$$

and $c = (\sigma q_4^\alpha / d)^2$.

Lemma 4.8 *Suppose* $H(x)$ *is given by (4.18) and* $C_0 > 0$ *is a given constant. Then for* $0 < x < C_0,\ \ |H''(x)| < Mx^{-3/2}$*, where* M *is a constant.*

Proof: Let $g(x) = H(x^2)$. Then

$$g(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{t+2x} \phi(t)\phi(r)\Phi(x - t)[\Phi(x + t) - \Phi(r - x)]^2\, dr dt$$

and

$$
\begin{aligned}
g'(x) &= \frac{d}{dx}\int_{-\infty}^{\infty}\int_{-\infty}^{t+2x} \phi(t)\phi(r)\Phi(x-t)[\Phi(x+t)-\Phi(r-x)]^2\, drdt \\
&= \int_{-\infty}^{\infty}\frac{\partial}{\partial x}\int_{-\infty}^{t+2x} \phi(t)\phi(r)\Phi(x-t)[\Phi(x+t)-\Phi(r-x)]^2\, drdt \\
&= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{t+2x}\frac{\partial}{\partial x}K(t,r,x)\, dr + K(t,t+2x,x)\frac{\partial(t+2x)}{\partial x}\right\}\, dt,
\end{aligned}
$$

where

$$K(t,r,x) = \phi(t)\phi(r)\Phi(x-t)[\Phi(x+t)-\Phi(r-x)]^2$$

and so

$$
\begin{aligned}
g'(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{t+2x} \phi(t)\phi(r)\Big\{\phi(x-t)[\Phi(x+t)-\Phi(r-x)]^2 \\
&\quad +2\Phi(x-t)[\Phi(x+t)-\Phi(r-x)]\left(\phi(x+t)+\phi(r-x)\right)\Big\}\, drdt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{t+2x} \phi(t)\phi(r)\Big\{\phi(x-t)W^2 + 2\Phi(x-t)\left(\phi(x+t)+\phi(r-x)\right)W\Big\}\, dr\, dt,
\end{aligned}
$$

where

$$W = W(t,r,x) = \Phi(x+t) - \Phi(r-x).$$

By noting that

$$W = \int_{r-x}^{x+t}\frac{1}{\sqrt{2\pi}}e^{-s^2/2}\ ds \le 2x+t-r,$$

we have

$$
\begin{aligned}
|g'(x)| &\le A_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(t)\phi(r)\left(W^2+W\right)\ drdt \\
&\le A_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(t)\phi(r)W\ drdt \\
&\le A_3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(t)\phi(r)(2x+t-r)\ drdt, \\
&\le Ax, \qquad (4.19)
\end{aligned}
$$

where A_1, A_2, A_3, and A are constants.

Now we find $g''(x)$.

$g''(x)$

$$= \frac{d}{dx}\int_{-\infty}^{\infty}\int_{-\infty}^{t+2x}\phi(t)\phi(r)\Big\{\phi(x-t)[W^2+2\Phi(x-t)\left(\phi(x+t)+\phi(r-x)\right)W]\Big\}\,dr\,dt,$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{t+2x}\frac{\partial}{\partial x}N(t,r,x)\,dr + N(t,t+2x,x)\frac{\partial(t+2x)}{\partial x}\right\}\,dt,$$

where

$$N(t,r,x) = \phi(t)\phi(r)\Big\{\phi(x-t)W^2 + 2\Phi(x-t)\left(\phi(x+t)+\phi(r-x)\right)W\Big\},$$

and so, we have

$$\begin{aligned} g''(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{t+2x}\phi(t)\phi(r)\Big\{-(x-t)\phi(x-t)W^2 \\ &\quad +4\phi(x-t)\left(\phi(x+t)+\phi(r-x)\right)W + 2\Phi(x-t)\left(\phi(x+t)+\phi(r-x)\right)^2 \\ &\quad +2\Phi(x-t)\left(-(x+t)\phi(x+t)+(r-x)\phi(r-x)\right)W\Big\}\,drdt. \end{aligned}$$

Therefore

$$\begin{aligned} &|g''(x)| \\ &\le B_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(t)\phi(r)\Big((x+|t|)W^2+W+D+(2x+|t|+|r|)W\Big)\,drdt \\ &\le B_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(t)\phi(r)\left((3x+1)(2x+t-r)+(2|t|+|r|)(2x+t-r)+D_0\right)\,drdt \\ &\le B_3x^2+B_4x+B_5,\ \text{for } 0<x<C_0, \qquad (4.20) \end{aligned}$$

where B_1, B_2, B_3, B_4, B_5 and D_0 are constants. It now follows from (4.19), (4.20) that

$$\begin{aligned} |H''(x)| &\le \frac{1}{4}\left(x^{-1}\left|g''(x^{1/2})\right| + x^{-3/2}\left|g'(x^{1/2})\right|\right) \\ &\le Mx^{-3/2}, \end{aligned}$$

where M is a constant.

Now Lemma 4.8 and Theorem 2.2 with $\theta = (q_4^\alpha)^2$, $C_1 = 0$ and $n_0 = c$ give the second order approximation to the $\beta_{LU}^{**}(d)$.

Theorem 4.8 *For $H(x)$ defined in (4.18) and $m > 2$, we have*

$$\begin{aligned} \beta_{LU}^{**}(d) &= \beta_L^{**}(d) + \frac{1}{c}\Big[(q_4^\alpha)^2 H'\left((q_4^\alpha)^2\right)\left(\rho + l_0 - \frac{2}{4}\right) \\ &\quad + \frac{1}{4}(q_4^\alpha)^4 H''\left((q_4^\alpha)^2\right)\Big] + o\left(\frac{1}{c}\right), \end{aligned}$$

where $c = d^{-2}\,(q_4^\alpha)^2\,\sigma^2$.

Table 4.10 presents the values of the second order approximation to the $\beta_{LU}^{**}(d)$.

The exact values of $\beta_U^{**}(d)$ when $k = 3$ and $\beta_{LU}^{**}(d)$ when $k = 4$ can be calculated by using the recursive method discussed in Subsection 3.3.4, since the stopping time is the same as before,

$$\beta_U^{**}(d) = \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)] H\left((q_3^\alpha)^2 \frac{n+1}{c}\right)$$

where $H(x)$ is defined by (4.17), and

$$\beta_{LU}^{**}(d) = \sum_{n=m_0}^{\infty} [P(t > n-1) - P(t > n)] H\left((q_4^\alpha)^2 \frac{n+1}{c}\right)$$

where $H(x)$ is defined by (4.18), where $m_0 = m - 1$. Simulations to estimate $\beta_U^{**}(d)$ and $\beta_{LU}^{**}(d)$, based on 6,000 experiments, were also carried out.

For $k = 3, 4$ and $1 - \alpha = 0.90, 0.99$, Tables 4.9 and 4.10 give the exact, simulated and approximate values of $\beta_U^{**}(d)$ and $\beta_{LU}^{**}(d)$ at $c = 5(5)60$ and $c = 15(5)60$.

Table 4.9: *Comparisons between the exact, approximate and simulation results of* $\beta_U^{**}(d)$ *for* $m = 2$ *and given values of* $k = 3$, $1 - \alpha$ *and* c

$1 - \alpha = 0.90$

c	$k = 3$		
	Exact	Appro.	Simul.
5	0.936	0.943	0.939
10	0.931	0.951	0.941
15	0.938	0.954	0.952
20	0.944	0.956	0.953
25	0.948	0.956	0.954
30	0.951	0.957	0.955
35	0.952	0.957	0.956
40	0.954	0.958	0.958
45	0.955	0.958	0.952
50	0.955	0.958	0.958
55	0.956	0.958	0.963
60	0.956	0.958	0.963

Table 4.9: *Comparisons between the exact, approximate and simulation results of* $\beta_U^{**}(d)$

for $m = 10$ *and given values of* $k = 3$, $1 - \alpha$ *and* c

$1 - \alpha = 0.90$

	$k = 3$		
c	Exact	Appro.	Simul.
15	0.959	0.954	0.967
20	0.958	0.956	0.963
25	0.958	0.956	0.961
30	0.959	0.957	0.962
35	0.959	0.957	0.962
40	0.959	0.958	0.962
45	0.959	0.958	0.960
50	0.959	0.958	0.963
55	0.959	0.958	0.965
60	0.959	0.958	0.967

Table 4.9: *Comparisons between the exact, approximate and simulation results of* $\beta_U^{**}(d)$ *for* $m = 2$ *and given values of* $k = 3$, $1 - \alpha$ *and* c

$1 - \alpha = 0.99$

c	$k = 3$		
	Exact	Appro.	Simul.
5	0.989	0.995	0.990
10	0.985	0.995	0.987
15	0.987	0.996	0.991
20	0.989	0.996	0.991
25	0.990	0.996	0.993
30	0.991	0.996	0.994
35	0.992	0.996	0.995
40	0.993	0.996	0.993
45	0.993	0.996	0.995
50	0.994	0.996	0.994
55	0.994	0.996	0.995
60	0.994	0.996	0.996

Table 4.9: *Comparisons between the exact, approximate and simulation results of* $\beta_U^{**}(d)$

for $m = 10$ *and given values of* $k = 3$, $1-\alpha$ *and* c

$1 - \alpha = 0.99$

c	$k = 3$		
	Exact	Appro.	Simul.
15	0.996	0.996	0.998
20	0.996	0.996	0.995
25	0.996	0.996	0.995
30	0.996	0.996	0.997
35	0.996	0.996	0.997
40	0.996	0.996	0.997
45	0.996	0.996	0.997
50	0.996	0.996	0.997
55	0.996	0.996	0.996
60	0.996	0.996	0.997

Table 4.10: *Comparisons between the exact, approximate and simulation results of* $\beta^{**}_{LU}(d)$ *for* $m = 2$ *and given values of* $k = 4$, $1-\alpha$ *and* c

$1 - \alpha = 0.90$

c	$k = 4$		
	Exact	Appro.	Simul.
5	0.942	0.940	0.928
10	0.946	0.944	0.932
15	0.949	0.946	0.947
20	0.949	0.946	0.944
25	0.949	0.947	0.947
30	0.949	0.947	0.951
35	0.949	0.947	0.944
40	0.949	0.947	0.950
45	0.949	0.948	0.950
50	0.949	0.948	0.950
55	0.949	0.948	0.945
60	0.949	0.948	0.949

Table 4.10: *Comparisons between the exact, approximate and simulation results of* $\beta^{**}_{LU}(d)$

for $m = 10$ *and given values of* $k = 4$, $1 - \alpha$ *and* c

$1 - \alpha = 0.90$

c	$k = 4$ Exact	Appro.	Simul.
15	0.949	0.946	0.952
20	0.949	0.946	0.948
25	0.949	0.947	0.951
30	0.949	0.947	0.954
35	0.949	0.947	0.944
40	0.949	0.947	0.953
45	0.949	0.948	0.949
50	0.949	0.948	0.951
55	0.949	0.948	0.945
60	0.949	0.948	0.949

Table 4.10: *Comparisons between the exact, approximate and simulation results of* $\beta^{**}_{LU}(d)$ *for* $m = 2$ *and given values of* $k = 4$*,* $1 - \alpha$ *and* c

$1 - \alpha = 0.99$

c	$k = 4$		
	Exact	Appro.	Simul.
5	0.995	0.995	0.989
10	0.995	0.995	0.988
15	0.995	0.995	0.993
20	0.995	0.995	0.994
25	0.995	0.995	0.995
30	0.995	0.995	0.994
35	0.995	0.995	0.993
40	0.995	0.995	0.994
45	0.995	0.995	0.994
50	0.995	0.995	0.996
55	0.995	0.995	0.995
60	0.995	0.995	0.995

Table 4.10: *Comparisons between the exact, approximate and simulation results of* $\beta^{**}_{LU}(d)$

for $m = 10$ *and given values of* $k = 4$, $1-\alpha$ *and* c

$1 - \alpha = 0.99$

	$k=4$		
c	Exact	Appro.	Simul.
15	0.995	0.995	0.995
20	0.995	0.995	0.996
25	0.995	0.995	0.996
30	0.995	0.995	0.994
35	0.995	0.995	0.994
40	0.995	0.995	0.994
45	0.995	0.995	0.994
50	0.995	0.995	0.996
55	0.995	0.995	0.995
60	0.995	0.995	0.995

Chapter 5

Some power functions of multiple tests

5.1 A power function for testing the means of several independent normal populations

5.1.1 Introduction

Suppose we have k independently and normally distributed populations $N(\mu_i, \sigma^2)$, $1 \leq i \leq k$, with unknown means μ_i and a common positive variance σ^2. We are again interested in making inferences about the μ_i and in particular, we want to test the family of two-sided hypotheses

$$H_{i\,0} : \mu_i = 0 \quad \text{vs} \quad H_{i\,1} : \mu_i \neq 0, \quad 1 \leq i \leq k. \tag{5.1}$$

Assume that $\bar{Y}_{in}$ denotes the sample mean of a sample of fixed size n from the i^{th} population, $1 \leq i \leq k$, and that S^2 is an estimate of σ^2 which is independent of the $\bar{Y}_{in}$ and distributed as a χ^2_ν/ν random variable. If σ^2 is known then $\nu = \infty$, otherwise $0 < \nu < \infty$. It is well known that the family of hypotheses (5.1) can be tested in the following way

$$\text{reject } H_{i\,0} \text{ in favour of } H_{i\,1} \text{ iff } |\frac{\sqrt{n}\bar{Y}_{in}}{S}| \geq |m|^{\alpha}_{k,\nu}, \quad 1 \leq i \leq k, \tag{5.2}$$

and accompany the rejection of any $H_{i\,0}$ by the directional decision that $\mu_i > 0$ if $\bar{Y}_{in} > 0$ and $\mu_i < 0$ if $\bar{Y}_{in} < 0$, where $|m|^{\alpha}_{k,\nu}$ is the upper α point of the distribution of the random variable

$$\frac{\max_{1 \leq i \leq k} |Z_i|}{\sqrt{\chi^2_\nu/\nu}}.$$

This multiple test procedure controls strongly the type I error rate at α (see appendix for definition), since it is actually derived from the following set of simultaneous confidence intervals of level $1 - \alpha$

$$\mu_i \in \left(\bar{Y}_{in} - |m|^{\alpha}_{k,\nu}\frac{S}{\sqrt{n}},\ \bar{Y}_{in} + |m|^{\alpha}_{k,\nu}\frac{S}{\sqrt{n}}\right), \quad i = 1, 2, \cdots, k.$$

To assess the sensitivity of this test procedure, we want to calculate the probability that this test will detect, with a correct directional decision, each treatment whose mean μ_i is significantly different from zero in terms of $|\mu_i| \geq d\sigma$, where $d > 0$ is a given constant. For this we define a power function $\gamma(\mu, d)$ to be

$$P\{\text{all false } H_{i\,0} \text{ with } |\mu_i| \geq d\sigma \text{ are rejected with correct directional decisions}\} \tag{5.3}$$

and, in particular, $\gamma(\mu, d) = 1$ if all the treatments satisfy $|\mu_i| < d\sigma$. The sensitivity of this multiple test procedure can then be measured by $\gamma(d) = \min_{\mu \in R^k} \gamma(\mu, d)$. The problem that we want to investigate is how large the sample size n should be if we require test (5.2) has the sensitivity $\gamma(d) = \gamma$ for preassigned values of $d > 0$ and $0 < \gamma < 1$. This is treated in Subsection 5.1.2.

Note that, in the definition of power function $\gamma(\mu, d)$ in (5.3), the departure of the μ_i from the origin, $|\mu_i|$, is measured in unit of σ. It certainly makes sense to define a power function, $\hat{\gamma}(\mu, d)$, to be

$$P\{\text{all false } H_{i\,0} \text{ with } |\mu_i| \geq d \text{ are rejected with correct directional decisions}\}$$

and, in particular, $\hat{\gamma}(\mu, d) = 1$ if all the treatments satisfy $|\mu_i| < d$. The sensitivity of a test of (5.1) can be measured by the quantity $\hat{\gamma}(d) = \min_{\mu \in R^k} \hat{\gamma}(\mu, d)$. Now assume σ^2 is an unknown parameter and we wish to design a test of (5.1) such that this test has type I error rate α and sensitivity $\hat{\gamma}(d) = \gamma$, for given values of α, d and γ. For this it is necessary to use a sequential sampling scheme. In Subsection 5.1.3 we discuss a pure sequential procedure.

5.1.2 A fixed sample size procedure

In order to determine the sample size n so that test (5.2) has $\gamma(d) = \gamma$ for given values of k, ν, $d > 0$ and $0 < \gamma < 1$, we first find a configuration of the population means μ at which the power function $\gamma(\mu, d)$ attains its minimum. We have the following result. The proof is similar to that of Theorem 4.1 and omitted.

Theorem 5.1 *Let $k \geq 2$, $p = \langle k/2 \rangle$ and $\mu^*(d) = (d\sigma, \cdots, d\sigma, -d\sigma, \cdots, -d\sigma)$ which has the first p components equal to $d\sigma$ and the last $k - p$ components equal to $-d\sigma$. Then*

$$\gamma(d) = \gamma(\mu^*(d), d) = \int_0^\infty \Phi^k(d\sqrt{n} - s|m|_{k,\nu}^{\alpha}) f_\nu(s)\, ds$$

where $f_\nu(x)$ denotes a pdf of the $\sqrt{\chi_\nu^2/\nu}$.

Notice that, if the variance σ^2 is known then

$$\gamma(d) = \min_{\mu \in R^k} \gamma(\mu, d) = \Phi^k \left(d\sqrt{n} - |m|_k^\alpha \right).$$

For given values of k, ν, α and γ, Tables 5.1 and 5.2 give the values of $d\sqrt{n}$ such that $\gamma(d) = \gamma$.

Table 5.1: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma(d) = \gamma$*

for $\alpha = 0.05$ and $\gamma = 0.90$

	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
ν=10	4.411	4.869	5.189	5.434	5.632	5.798	5.941	6.066	6.177
ν=12	4.309	4.744	5.047	5.278	5.466	5.622	5.757	5.875	5.980
ν=14	4.239	4.659	4.950	5.172	5.351	5.501	5.630	5.743	5.843
ν=16	4.188	4.596	4.879	5.094	5.268	5.413	5.538	5.647	5.743
ν=18	4.149	4.549	4.825	5.035	5.204	5.346	5.467	5.573	5.667
ν=20	4.119	4.512	4.783	4.989	5.154	5.293	5.412	5.515	5.608
ν=30	4.031	4.405	4.660	4.854	5.010	5.139	5.250	5.347	5.433
ν=40	3.989	4.353	4.602	4.789	4.940	5.105	5.172	5.266	5.349
ν=60	3.948	4.303	4.544	4.726	4.872	4.993	5.097	5.187	5.267
ν=120	3.908	4.254	4.488	4.665	4.806	4.923	5.023	5.110	5.186
$\nu=\infty$	3.869	4.206	4.434	4.605	4.742	4.855	4.951	5.034	5.108

Table 5.1: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma(d) = \gamma$ *for* $\alpha = 0.05$ *and* $\gamma = 0.90$

	$k = 11$	$k = 12$	$k = 13$	$k = 14$	$k = 15$	$k = 16$	$k = 17$	$k = 18$	$k = 19$	$k = 20$
ν=10	6.278	6.368	6.452	6.528	6.560	6.666	6.728	6.786	6.841	6.893
ν=12	6.074	6.160	6.238	6.310	6.377	6.440	6.498	6.553	6.605	6.654
ν=14	5.933	6.015	6.090	6.159	6.223	6.283	6.339	6.391	6.441	6.488
ν=16	5.831	5.910	5.982	6.049	6.111	6.168	6.222	6.273	6.321	6.366
ν=18	5.752	5.829	5.899	5.964	6.024	6.080	6.133	6.182	6.229	6.272
ν=20	5.720	5.795	5.865	5.929	5.989	6.044	6.096	6.144	6.190	6.233
ν=30	5.510	5.580	5.644	5.703	5.758	5.809	5.856	5.901	5.943	5.983
ν=40	5.423	5.491	5.552	5.609	5.661	5.710	5.756	5.799	5.839	5.877
ν=60	5.338	5.403	5.462	5.517	5.567	5.614	5.658	5.699	5.738	5.774
ν=120	5.255	5.318	5.374	5.427	5.475	5.520	5.562	5.601	5.638	5.673
ν=∞	5.174	5.234	5.289	5.339	5.385	5.428	5.468	5.506	5.541	5.575

Table 5.1: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma(d) = \gamma$ *for* $\alpha = 0.05$ *and* $\gamma = 0.95$

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
ν=10	4.799	5.251	5.569	5.814	6.013	6.180	6.324	6.450	6.563
ν=12	4.683	5.109	5.408	5.637	5.823	5.980	6.114	6.232	6.338
ν=14	4.603	5.012	5.230	5.517	5.694	5.842	5.971	6.083	6.183
ν=16	4.546	4.942	5.218	5.429	5.600	5.743	5.866	5.974	6.070
ν=18	4.503	4.889	5.158	5.363	5.529	5.668	5.787	5.892	5.985
ν=20	4.469	4.848	5.110	5.311	5.473	5.608	5.725	5.827	5.918
ν=30	4.371	4.728	4.974	5.161	5.312	5.437	5.545	5.640	5.723
ν=40	4.324	4.671	4.909	5.089	5.234	5.355	5.459	5.550	5.630
ν=60	4.278	4.615	4.846	5.020	5.160	5.276	5.376	5.463	5.540
ν=120	4.234	4.561	4.784	4.953	5.087	5.199	5.295	5.378	5.452
$\nu=\infty$	4.191	4.509	4.725	4.887	5.017	5.125	5.217	5.297	5.368

Table 5.1: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma(d) = \gamma$

for $\alpha = 0.05$ *and* $\gamma = 0.95$

	$k=11$	$k=12$	$k=13$	$k=14$	$k=15$	$k=16$	$k=17$	$k=18$	$k=19$	$k=20$
ν=10	6.664	6.757	6.841	6.919	6.992	7.059	7.122	7.182	7.238	7.291
ν=12	6.432	6.518	6.597	6.670	6.738	6.801	6.860	6.916	6.968	7.018
ν=14	6.273	6.355	6.430	6.499	6.563	6.623	6.679	6.732	6.782	6.829
ν=16	6.157	6.236	6.308	6.374	6.436	6.493	6.547	6.598	6.646	6.691
ν=18	6.069	6.145	6.215	6.279	6.339	6.394	6.447	6.496	6.542	6.586
ν=20	5.999	6.074	6.142	6.204	6.262	6.317	6.367	6.415	6.460	6.503
ν=30	5.799	5.867	5.930	5.987	6.041	6.090	6.137	6.181	6.222	6.261
ν=40	5.702	5.768	5.828	5.883	5.934	5.982	6.026	6.068	6.107	6.145
ν=60	5.609	5.672	5.729	5.782	5.831	5.876	5.918	5.958	5.996	6.031
ν=120	5.518	5.578	5.633	5.683	5.730	5.773	5.814	5.852	5.888	5.922
$\nu=\infty$	5.431	5.488	5.540	5.589	5.633	5.674	5.713	5.749	5.783	5.815

Table 5.1: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma(d) = \gamma$ for $\alpha = 0.05$ and $\gamma = 0.99$*

	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
ν=10	5.541	5.986	6.303	6.548	6.749	6.918	7.065	7.194	7.309
ν=12	5.340	5.813	6.105	6.332	6.517	6.673	6.807	6.926	7.032
ν=14	5.302	5.695	5.972	6.186	6.359	6.506	6.633	6.744	6.844
ν=16	5.233	5.611	5.876	6.080	6.246	6.386	6.507	6.613	6.708
ν=18	5.181	5.547	5.803	6.001	6.161	6.296	6.412	6.514	6.605
ν=20	5.140	5.497	5.747	5.939	6.094	6.225	6.338	6.437	6.525
ν=30	5.023	5.355	5.586	5.762	5.904	6.024	6.126	6.216	6.297
ν=40	4.967	5.288	5.509	5.678	5.814	5.928	6.026	6.112	6.188
ν=60	4.914	5.223	5.436	5.598	5.728	5.837	5.930	6.011	6.084
ν=120	4.862	5.160	5.365	5.520	5.645	5.749	5.837	5.915	5.984
$\nu=\infty$	4.811	5.100	5.297	5.446	5.565	5.664	5.749	5.823	5.889

Table 5.1: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma(d) = \gamma$

for $\alpha = 0.05$ *and* $\gamma = 0.99$

	$k = 11$	$k = 12$	$k = 13$	$k = 14$	$k = 15$	$k = 16$	$k = 17$	$k = 18$	$k = 19$	$k = 20$
ν=10	7.413	7.508	7.595	7.675	7.750	7.820	7.885	7.947	8.005	8.061
ν=12	7.128	7.215	7.295	7.369	7.438	7.502	7.562	7.619	7.672	7.723
ν=14	6.934	7.015	7.091	7.160	7.224	7.285	7.341	7.394	7.445	7.492
ν=16	6.793	6.871	6.943	7.009	7.070	7.127	7.181	7.232	7.280	7.325
ν=18	6.687	6.762	6.831	6.894	6.953	7.008	7.060	7.109	7.154	7.198
ν=20	6.605	6.677	6.744	6.805	6.862	6.915	6.965	7.012	7.057	7.099
ν=30	6.368	6.434	6.494	6.549	6.601	6.649	6.693	6.736	6.776	6.813
ν=40	6.256	6.319	6.376	6.428	6.477	6.522	6.565	6.605	6.643	6.678
ν=60	6.149	6.208	6.262	6.312	6.358	6.401	6.441	6.478	6.515	6.548
ν=120	6.046	6.102	6.153	6.201	6.244	6.285	6.323	6.358	6.392	6.424
∞	5.947	6.000	6.049	6.094	6.135	6.174	6.210	6.243	6.276	6.305

Table 5.2: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma(d) = \gamma$ for $\alpha = 0.01$ and $\gamma = 0.95$*

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
ν=10	5.961	6.453	6.801	7.069	7.288	7.472	7.631	7.770	7.895
ν=12	5.718	6.170	6.487	6.732	6.931	7.099	7.243	7.370	7.484
ν=14	5.556	5.980	6.278	6.507	6.693	6.849	6.984	7.103	7.209
ν=16	5.440	5.845	6.129	6.347	6.523	6.672	6.800	6.912	7.012
ν=18	5.354	5.745	6.017	6.227	6.396	6.539	6.661	6.769	6.865
ν=20	5.287	5.680	5.931	6.133	6.297	6.435	6.554	6.658	6.750
ν=30	5.096	5.445	5.686	5.879	6.018	6.142	6.249	6.342	6.425
ν=40	5.007	5.341	5.571	5.746	5.887	6.005	6.106	6.194	6.273
ν=60	4.921	5.241	5.461	5.628	5.762	5.874	5.969	6.053	6.128
ν=120	4.839	5.146	5.356	5.515	5.643	5.749	5.839	5.919	5.989
$\nu=\infty$	4.761	5.055	5.256	5.408	5.522	5.630	5.716	5.791	5.857

Table 5.2: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma(d) = \gamma$

for $\alpha = 0.01$ *and* $\gamma = 0.95$

	$k=11$	$k=12$	$k=13$	$k=14$	$k=15$	$k=16$	$k=17$	$k=18$	$k=19$	$k=20$
ν=10	8.007	8.109	8.203	8.289	8.370	8.445	8.515	8.581	8.643	8.703
ν=12	7.586	7.679	7.764	7.843	7.916	7.984	8.048	8.108	8.165	8.219
ν=14	7.304	7.391	7.470	7.543	7.612	7.675	7.735	7.791	7.844	7.894
ν=16	7.102	7.184	7.259	7.329	7.394	7.454	7.510	7.563	7.613	7.661
ν=18	6.951	7.030	7.102	7.168	7.230	7.288	7.342	7.392	7.440	7.486
ν=20	6.834	6.979	7.101	7.043	7.103	7.158	7.210	7.259	7.306	7.349
ν=30	6.500	6.568	6.630	6.688	6.741	6.791	6.837	6.881	6.922	6.961
ν=40	6.344	6.408	6.467	6.521	6.571	6.618	6.662	6.703	6.742	6.778
ν=60	6.194	6.255	6.310	6.361	6.409	6.453	6.494	6.532	6.569	6.603
ν=120	6.052	6.109	6.161	6.210	6.254	6.295	6.334	6.370	6.404	6.437
$\nu=\infty$	5.917	5.970	6.020	6.065	6.107	6.146	6.182	6.216	6.248	6.279

Table 5.2: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma(d) = \gamma$*

for $\alpha = 0.01$ and $\gamma = 0.99$

	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
ν=10	6.804	7.298	7.651	7.925	8.150	8.340	8.504	8.649	8.779
ν=12	6.509	6.955	7.271	7.517	7.718	7.888	8.035	8.165	8.281
ν=14	6.314	6.727	7.020	7.247	7.432	7.588	7.724	7.843	7.950
ν=16	6.176	6.566	6.842	7.055	7.229	7.376	7.503	7.615	7.714
ν=18	6.073	6.446	6.709	6.913	7.078	7.218	7.338	7.444	7.539
ν=20	5.994	6.368	6.607	6.803	6.961	7.095	7.211	7.313	7.404
ν=30	5.769	6.093	6.320	6.493	6.634	6.752	6.854	6.943	7.023
ν=40	5.665	5.973	6.187	6.351	6.483	6.594	6.700	6.773	6.848
ν=60	5.566	5.858	6.061	6.215	6.340	6.444	6.533	6.612	6.682
ν=120	5.471	5.749	5.941	6.087	6.204	6.302	6.386	6.460	6.525
$\nu=\infty$	5.381	5.646	5.828	5.966	6.077	6.169	6.248	6.317	6.378

Table 5.2: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma(d) = \gamma$*

for $\alpha = 0.01$ and $\gamma = 0.99$

	$k = 11$	$k = 12$	$k = 13$	$k = 14$	$k = 15$	$k = 16$	$k = 17$	$k = 18$	$k = 19$	$k = 20$
ν=10	8.896	9.003	9.101	9.192	9.277	9.355	9.429	9.499	9.565	9.627
ν=12	8.386	8.481	8.569	8.650	8.726	8.796	8.863	8.925	8.984	9.040
ν=14	8.046	8.134	8.215	8.289	8.359	8.424	8.485	8.542	8.596	8.647
ν=16	7.805	7.887	7.962	8.032	8.097	8.158	8.215	8.269	8.420	8.368
ν=18	7.625	7.703	7.775	7.841	7.902	7.960	8.014	8.065	8.113	8.159
ν=20	7.486	7.764	7.629	7.693	7.752	7.807	7.859	7.907	7.953	7.997
ν=30	7.095	7.161	7.221	7.276	7.327	7.376	7.421	7.463	7.503	7.541
ν=40	6.915	6.976	7.032	7.084	7.131	7.176	7.218	7.258	7.295	7.330
ν=60	6.744	6.801	6.853	6.902	6.946	6.988	7.027	7.064	7.098	7.131
ν=120	6.584	6.637	6.686	6.731	6.772	6.811	6.847	6.881	6.913	6.944
ν=∞	6.433	6.483	6.528	6.570	6.609	6.646	6.678	6.710	6.741	6.769

5.1.3 A pure sequential procedure

In this subsection, σ^2 is assumed to be an unknown parameter. We want to design a test of the family of hypotheses (5.1) which has, at least approximately, type I error rate α and power $\hat{\gamma}(d) = \gamma$, where $0 < \alpha < 1,\ 0 < \gamma < 1$ and $d > 0$ are prefixed constants. To motivate the definition of a pure sequential procedure, we first look at the known σ^2 case which is covered in the last subsection.

had σ^2 been known, we would take a sample of size n_0 from each of the k populations and test the family of hypotheses (5.1) by:

$$\text{reject } H_{i\,0} \text{ in favour of } H_{i\,1} \text{ iff } |\bar{Y}_{in_0}| > \frac{\sigma |m|_k^\alpha}{\sqrt{n_0}}, \quad 1 \le i \le k,$$

and accompany the rejection of any $H_{i\,0}$ by the directional decision that $\mu_i > 0$ if $\bar{Y}_{in_0} > 0$ and $\mu_i < 0$ if $\bar{Y}_{in_0} < 0$, where n_0 satisfies

$$\Phi^k\left(\frac{d\sqrt{n_0}}{\sigma} - |m|_k^\alpha\right) = \gamma.$$

This last equation gives

$$n_0 = \sigma^2 d^{-2}\left(|m|_k^\alpha + \Phi^{-1}\left(\gamma^{1/k}\right)\right)^2 \tag{5.4}$$

and so the test can be rewritten as

$$\text{reject } H_{i\,0} \text{ in favour of } H_{i\,1} \text{ iff } |\bar{Y}_{in_0}| > \frac{d|m|_k^\alpha}{|m|_k^\alpha + \Phi^{-1}\left(\gamma^{1/k}\right)}, \quad 1 \le i \le k,$$

and accompany the rejection of any $H_{i\,0}$ by the directional decision that $\mu_i > 0$ if $\bar{Y}_{in_0} > 0$ and $\mu_i < 0$ if $\bar{Y}_{in_0} < 0$.

Based on these observations, we can now define a sequential procedure for the situation of unknown σ^2 that is assumed in this subsection. Take a sample of size m from each of the k populations, then take one observation from each populations at a time until

$$T = \inf\{n \ge m :\ n \ge (1 + \xi_1/n) d^{-2} C^2 \hat{\sigma_n}^2\},$$

where $0 < C = |m|_k^\alpha + \Phi^{-1}(\gamma^{1/k})$ and ξ_1 is a given constant whose value will be determined later. On stopping sampling,

$$\text{reject } H_{i\,0} \text{ in favour of } H_{i\,1} \text{ iff } |\bar{Y}_{iT}| > \frac{d|m|_k^\alpha}{C}\left(1+\frac{\eta_1}{T}\right), \quad 1 \le i \le k,$$

and accompany the rejection of any $H_{i\,0}$ by the directional decision that $\mu_i > 0$ if $\bar{Y}_{iT} > 0$ and $\mu_i < 0$ if $\bar{Y}_{iT} < 0$, where η_1 is a given constant whose value is given below.

Note that the stopping time T uses formula (5.4) adaptively by replacing σ^2 with $\hat{\sigma}_n^2$ to check whether enough observations have already been drawn, and the test mimics the test for the known σ^2 situation. Next we show that this procedure has the required properties, at least for large n_0.

First, we show that this procedure controls strongly the type I error rate at α, at least for large n_0. For this, it is sufficient to show that

$$CL = P\left\{|\bar{Y}_{iT} - \mu_i| < \frac{d|m|_k^\alpha}{C}\left(1+\frac{\eta_1}{T}\right), \quad 1 \le i \le k\right\}$$

is equal to $1-\alpha+o(1)$ as $n_0 \to \infty$. By noting that

$$CL = E\left[H\left(\frac{T}{n_0}\left(1+\frac{\eta_1}{T}\right)^2\right)\right]$$

where

$$H(x) = \left(2\Phi(|m|_k^\alpha\sqrt{x}) - 1\right)^k,$$

it therefore follows from Theorem 2.2 with $\theta = 1$ and $C_1 = \eta_1$ that

$$\begin{aligned} CL &= 1-\alpha+\frac{1}{n_0}H'(1)\left(\rho+\xi_1-\frac{2}{k}+2\eta_1\right)+\frac{1}{kn_0}H''(1)+o\left(\frac{1}{n_0}\right) \quad (5.5) \\ &= 1-\alpha+o(1) \quad \text{as } n_0 \to \infty. \end{aligned}$$

Next, we find the second order approximation to the value of $\hat{\gamma}(d)$ of this procedure. Let

$$\Omega_U(d) = \{i : \mu_i \ge d\} \quad \text{and} \quad \Omega_L(d) = \{j : \mu_j \le -d\}.$$

From the definition, we have

$$\begin{aligned}
\hat{\gamma}(d) &= \min_{\mu \in R^k} P\{\text{all false } H_{i0} \text{ with } |\mu_i| \geq d \\
&\qquad \text{are rejected with correct directional decisions}\} \\
&= \min_{\mu \in R^k} P\Big\{\bar{Y}_{iT} > \frac{d|m|_k^\alpha}{C}\left(1+\frac{\eta_1}{T}\right) \ \forall i \in \Omega_U(d), \\
&\qquad \bar{Y}_{jT} < -\frac{d|m|_k^\alpha}{C}\left(1+\frac{\eta_1}{T}\right) \ \forall j \in \Omega_L(d)\Big\} \\
&= \min_{\mu \in R^k} \sum_{n=m}^{\infty} P\Big\{\bar{Y}_{in} > \frac{d|m|_k^\alpha}{C}\left(1+\frac{\eta_1}{n}\right) \ \forall i \in \Omega_U(d), \\
&\qquad \bar{Y}_{jn} < -\frac{d|m|_k^\alpha}{C}\left(1+\frac{\eta_1}{n}\right) \ \forall j \in \Omega_L(d)\Big\} P\{T=n\} \\
&= \min_{\mu \in R^k} \sum_{n=m}^{\infty} P\Big\{Z_i > \frac{d|m|_k^\alpha\sqrt{n}}{C\sigma}\left(1+\frac{\eta_1}{n}\right) - \frac{\mu_i\sqrt{n}}{\sigma} \ \forall i \in \Omega_U(d), \\
&\qquad Z_j < -\frac{d|m|_k^\alpha\sqrt{n}}{C\sigma}\left(1+\frac{\eta_1}{n}\right) - \frac{\mu_j\sqrt{n}}{\sigma} \ \forall j \in \Omega_L(d)\Big\} P\{T=n\} \\
&= \sum_{n=m}^{\infty} \Phi^k\left(\frac{d\sqrt{n}}{\sigma} - \frac{d|m|_k^\alpha\sqrt{n}}{C\sigma}\left(1+\frac{\eta_1}{n}\right)\right) P\{T=n\} \\
&= \sum_{n=m}^{\infty} \Phi^k\left(\left(C - |m|_k^\alpha\left(1+\frac{\eta_1}{n}\right)\right)\frac{\sqrt{n}}{\sqrt{n_0}}\right) P\{T=n\} \\
&= E\left[\Phi^k\left\{\left(C - |m|_k^\alpha\left(1+\frac{\eta_1}{T}\right)\right)\frac{\sqrt{T}}{\sqrt{n_0}}\right\}\right].
\end{aligned}$$

It therefore follows from Theorem 2.3 with $H(x) = \Phi^k(x), C_0 = C,\ C_1 = |m|_k^\alpha$ and $C_2 = \eta_1$ that

$$\begin{aligned}
&\hat{\gamma}(d) = \gamma - \frac{\gamma^{(k-1)/k}}{n_0}\phi\left(\Phi^{-1}\left(\gamma^{1/k}\right)\right) \times \\
&\left\{k|m|_k^\alpha\eta_1 - \frac{k\Phi^{-1}\left(\gamma^{1/k}\right)}{2}\left(\rho+\xi_1-\frac{2}{k}\right) - \frac{\Phi^{-1}\left(\gamma^{1/k}\right)}{4} \times \right. \\
&\left.\left((k-1)\Phi^{-1}\left(\gamma^{1/k}\right)\frac{\phi\left(\Phi^{-1}\left(\gamma^{1/k}\right)\right)}{\gamma^{1/k}} - 1 - \left(\Phi^{-1}\left(\gamma^{1/k}\right)\right)^2\right)\right\} + o\left(\frac{1}{n_0}\right) \quad (5.6)
\end{aligned}$$

From (5.5) and (5.6), we set the values of ξ_1 and η_1 satisfying simultaneously

$$\xi_1 + 2\eta_1 = -\rho + \frac{2}{k} - \frac{H''(1)}{kH'(1)},$$

$$
\begin{aligned}
&|m|_k^\alpha \eta_1 - \frac{\Phi^{-1}\left(\gamma^{1/k}\right)}{2}\left(\rho + \xi_1 - \frac{2}{k}\right) \\
&= \frac{\Phi^{-1}\left(\gamma^{1/k}\right)}{4k}\left((k-1)\Phi^{-1}\left(\gamma^{1/k}\right)\frac{\phi\left(\Phi^{-1}\left(\gamma^{1/k}\right)\right)}{\gamma^{1/k}} - 1 - \left(\Phi^{-1}\left(\gamma^{1/k}\right)\right)^2\right),
\end{aligned}
$$

so that the procedure has type I error rate $\alpha + o(1/n_0)$ and power $\hat{\gamma}(d) = \gamma + o(1/n_0)$ as $n_0 \to \infty$.

Table 5.3 presents the values of ξ_1 and η_1 for given values of α, γ and k.

The expected sample size from each population of this sequential procedure is given by

$$E(T) = n_0 + \rho + \xi_1 - \frac{2}{k} + o(1) \quad \text{as } n_0 \to \infty,$$

which follows directly from Theorem 2.1. A simulation exercise has been carried out to assess the performance of this procedure for small and moderate values of n_0. Table 5.4 shows the values of $\Phi^{-1}\left(\gamma^{1/k}\right)$ and Table 5.5 presents the simulated and approximate values of $E(T)$. Table 5.6 shows the simulation results of (1$-$ type I error rate) and $\hat{\gamma}(d) = \gamma$ for $m = 10, k = 3, 10$ and $\alpha = 0.1, 0.05$.

Table 5.3: *Values of ξ_1 and η_1*

for $\alpha = 0.05$ and given values of γ and k

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
k	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1
2	1.269	0.225	1.144	0.287	0.986	0.366	0.733	0.492
3	0.653	0.196	0.571	0.237	0.463	0.291	0.289	0.378
4	0.354	0.171	0.291	0.202	0.208	0.244	0.073	0.311
5	0.178	0.151	0.126	0.177	0.058	0.211	-0.052	0.266
6	0.062	0.135	0.018	0.158	-0.040	0.187	-0.133	0.233
7	-0.019	0.123	-0.058	0.143	-0.109	0.168	-0.190	0.209
8	-0.080	0.113	-0.115	0.130	-0.160	0.153	-0.232	0.189
9	-0.127	0.105	-0.158	0.120	-0.199	0.141	-0.264	0.173
10	-0.164	0.098	-0.193	0.112	-0.230	0.131	-0.289	0.160
11	-0.194	0.092	-0.221	0.105	-0.255	0.122	-0.309	0.149
12	-0.219	0.086	-0.244	0.099	-0.276	0.115	-0.326	0.140
13	-0.240	0.082	-0.264	0.093	-0.294	0.108	-0.340	0.131
14	-0.259	0.077	-0.280	0.088	-0.309	0.102	-0.352	0.124
15	-0.274	0.074	-0.295	0.084	-0.321	0.097	-0.362	0.118
16	-0.288	0.070	-0.308	0.080	-0.333	0.093	-0.371	0.112
17	-0.300	0.067	-0.319	0.077	-0.342	0.089	-0.379	0.107
18	-0.311	0.065	-0.328	0.074	-0.351	0.085	-0.386	0.102
19	-0.320	0.062	-0.337	0.071	-0.359	0.082	-0.392	0.098
20	-0.329	0.060	-0.345	0.068	-0.366	0.078	-0.397	0.094

Table 5.3: *Values of ξ_1 and η_1*

for $\alpha = 0.1$ and given values of γ and k

k	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1
2	0.987	0.203	0.872	0.261	0.723	0.335	0.477	0.458
3	0.462	0.176	0.385	0.214	0.282	0.266	0.112	0.351
4	0.210	0.153	0.150	0.183	0.070	0.223	-0.062	0.289
5	0.062	0.135	0.012	0.160	-0.054	0.193	-0.162	0.247
6	-0.035	0.121	-0.078	0.143	-0.134	0.171	-0.226	0.217
7	-0.103	0.110	-0.141	0.129	-0.191	0.154	-0.271	0.194
8	-0.154	0.101	-0.187	0.118	-0.232	0.141	-0.303	0.176
9	-0.193	0.094	-0.223	0.109	-0.264	0.130	-0.328	0.161
10	-0.223	0.088	-0.252	0.102	-0.288	0.120	-0.347	0.149
11	-0.248	0.082	-0.275	0.095	-0.309	0.112	-0.362	0.139
12	-0.269	0.078	-0.294	0.090	-0.325	0.106	-0.375	0.130
13	-0.287	0.073	-0.310	0.085	-0.339	0.100	-0.385	0.123
14	-0.302	0.070	-0.323	0.080	-0.351	0.094	-0.394	0.116
15	-0.314	0.066	-0.335	0.077	-0.361	0.090	-0.401	0.110
16	-0.326	0.063	-0.345	0.073	-0.370	0.086	-0.408	0.105
17	-0.336	0.061	-0.354	0.070	-0.378	0.082	-0.414	0.100
18	-0.344	0.058	-0.362	0.067	-0.384	0.078	-0.419	0.096
19	-0.352	0.056	-0.369	0.065	-0.391	0.075	-0.423	0.092
20	-0.359	0.054	-0.375	0.062	-0.396	0.072	-0.427	0.088

Table 5.4: *Values of* $\Phi^{-1}\left(\gamma^{1/k}\right)$

for given values of γ *and* k

	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
k =2	0.754	0.981	1.250	1.632
$k = 3$	1.009	1.215	1.463	1.818
$k = 4$	1.176	1.370	1.605	1.943
$k = 5$	1.298	1.484	1.710	2.036
$k = 6$	1.394	1.574	1.793	2.111
$k = 7$	1.473	1.648	1.861	2.172
$k = 8$	1.539	1.710	1.919	2.224
$k = 9$	1.597	1.764	1.969	2.269
$k = 10$	1.647	1.811	2.013	2.309
$k = 11$	1.691	1.854	2.052	2.344
$k = 12$	1.732	1.891	2.087	2.376
$k = 13$	1.768	1.926	2.120	2.406
$k = 14$	1.801	1.957	2.149	2.433
$k = 15$	1.832	1.986	2.176	2.457
$k = 16$	1.860	2.013	2.202	2.480
$k = 17$	1.887	2.038	2.225	2.502
$k = 18$	1.911	2.062	2.247	2.522
$k = 19$	1.934	2.084	2.268	2.541
$k = 20$	1.956	2.104	2.287	2.559

Table 5.5: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 3$, $\alpha = 0.1$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.6	15.5	15.5	15.4	15.4	15.3	15.3	15.1
20	20.4	20.5	20.3	20.4	20.2	20.3	20.0	20.1
25	25.3	25.5	25.2	25.4	25.1	25.3	25.0	25.1
30	30.3	30.5	30.2	30.4	30.1	30.3	30.0	30.1
35	35.4	35.5	35.3	35.4	35.2	35.3	35.0	35.1
40	40.4	40.5	40.3	40.4	40.2	40.3	40.0	40.1
45	45.4	45.5	45.3	45.4	45.2	45.3	45.0	45.1
50	50.4	50.5	50.3	50.4	50.2	50.3	50.1	50.1
55	55.6	55.5	55.5	55.4	55.4	55.3	55.2	55.1
60	60.6	60.5	60.5	60.4	60.4	60.3	60.2	60.1

Table 5.5: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 10$, $\alpha = 0.1$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.2	15.2	15.1	15.1	15.1	15.1	15.0	15.0
20	20.2	20.2	20.1	20.1	20.1	20.1	20.0	20.0
25	25.1	25.2	25.1	25.1	25.1	25.1	25.0	25.0
30	30.1	30.2	30.1	30.1	30.1	30.1	30.0	30.0
35	35.1	35.2	35.0	35.1	35.0	35.1	35.0	35.0
40	40.1	40.2	40.1	40.1	40.1	40.1	40.0	40.0
45	45.1	45.2	45.1	45.1	45.1	45.1	45.0	45.0
50	50.2	50.2	50.1	50.1	50.1	50.1	50.0	50.0
55	55.2	55.2	55.1	55.1	55.1	55.1	55.0	55.0
60	60.2	60.2	60.1	60.1	60.1	60.1	60.0	60.0

Table 5.5: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 3$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.8	15.7	15.5	15.6	15.5	15.5	15.3	15.3
20	20.6	20.7	20.5	20.6	20.4	20.5	20.2	20.3
25	25.5	25.7	25.4	25.6	25.3	25.5	25.1	25.3
30	30.5	30.7	30.4	30.6	30.3	30.5	30.2	30.3
35	35.6	35.7	35.5	35.6	35.4	35.5	35.2	35.3
40	40.5	40.7	40.5	40.6	40.4	40.5	40.2	40.3
45	45.6	45.7	45.5	45.6	45.4	45.5	45.2	45.3
50	50.6	50.7	50.5	50.6	50.4	50.5	50.2	50.3
55	55.8	55.7	55.7	55.6	55.6	55.5	55.4	55.3
60	60.8	60.7	60.7	60.6	60.6	60.5	60.4	60.3

Table 5.5: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 10$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.2	15.2	15.2	15.2	15.1	15.2	15.1	15.1
20	20.2	20.2	20.2	20.2	20.2	20.2	20.1	20.1
25	25.2	25.2	25.2	25.2	25.1	25.2	25.1	25.1
30	30.2	30.2	30.2	30.2	30.1	30.2	30.1	30.1
35	35.2	35.2	35.1	35.2	35.1	35.2	35.0	35.1
40	40.2	40.2	40.2	40.2	40.1	40.2	40.1	40.1
45	45.2	45.2	45.2	45.2	45.1	45.2	45.1	45.1
50	50.2	50.2	50.2	50.2	50.2	50.2	50.1	50.1
55	55.3	55.2	55.2	55.2	55.2	55.2	55.1	55.1
60	60.3	60.2	60.2	60.2	60.2	60.2	60.1	60.1

Table 5.6: *Simulation values of $\alpha^c = (1-$ type I error rate) and $\hat{\gamma}(d)$ for $m = 10$, $k = 3$, $\alpha = 0.1$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c
15	0.598	0.904	0.696	0.906	0.783	0.907	0.879	0.904
20	0.591	0.897	0.695	0.897	0.784	0.894	0.884	0.895
25	0.584	0.894	0.692	0.893	0.787	0.895	0.892	0.901
30	0.592	0.899	0.686	0.896	0.787	0.894	0.888	0.888
35	0.596	0.898	0.690	0.900	0.789	0.897	0.898	0.896
40	0.587	0.898	0.687	0.900	0.786	0.902	0.894	0.902
45	0.591	0.897	0.699	0.902	0.793	0.903	0.902	0.904
50	0.606	0.903	0.698	0.896	0.795	0.900	0.888	0.899
55	0.590	0.891	0.701	0.901	0.798	0.901	0.897	0.900
60	0.600	0.907	0.689	0.900	0.802	0.903	0.899	0.900

Table 5.6: *Simulation values of* $\alpha^c = (1-$ *type I error rate) and* $\hat{\gamma}(d)$ *for* $m = 10$, $k = 10$, $\alpha = 0.1$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c
15	0.592	0.897	0.688	0.904	0.783	0.904	0.888	0.921
20	0.593	0.899	0.693	0.896	0.789	0.901	0.882	0.917
25	0.574	0.899	0.689	0.899	0.797	0.902	0.891	0.905
30	0.595	0.903	0.686	0.896	0.793	0.900	0.894	0.904
35	0.598	0.905	0.702	0.904	0.793	0.904	0.897	0.908
40	0.599	0.902	0.700	0.902	0.794	0.895	0.900	0.913
45	0.603	0.898	0.697	0.901	0.805	0.901	0.899	0.903
50	0.601	0.899	0.705	0.907	0.796	0.904	0.896	0.903
55	0.594	0.901	0.703	0.900	0.796	0.895	0.897	0.907
60	0.598	0.898	0.707	0.899	0.801	0.897	0.905	0.904

Table 5.6: *Simulation values of $\alpha^c = (1-$ type I error rate) and $\hat{\gamma}(d)$ for $m = 10$, $k = 3$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c
15	0.588	0.952	0.696	0.951	0.787	0.953	0.880	0.952
20	0.591	0.947	0.698	0.947	0.789	0.945	0.882	0.947
25	0.592	0.949	0.684	0.944	0.788	0.947	0.886	0.950
30	0.594	0.946	0.694	0.948	0.793	0.949	0.891	0.947
35	0.593	0.952	0.693	0.953	0.794	0.947	0.894	0.947
40	0.595	0.953	0.692	0.949	0.785	0.949	0.889	0.953
45	0.585	0.946	0.692	0.951	0.793	0.945	0.896	0.950
50	0.593	0.948	0.703	0.954	0.809	0.953	0.892	0.949
55	0.599	0.953	0.693	0.952	0.796	0.946	0.892	0.950
60	0.604	0.949	0.695	0.952	0.806	0.952	0.900	0.947

Table 5.6: *Simulation values of* $\alpha^c = (1-$ *type I error rate) and* $\hat{\gamma}(d)$ *for* $m = 10$, $k = 10$, $\alpha = 0.05$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c	$\hat{\gamma}(d)$	α^c
15	0.597	0.953	0.685	0.947	0.787	0.950	0.886	0.950
20	0.595	0.951	0.690	0.951	0.787	0.947	0.888	0.948
25	0.589	0.952	0.689	0.952	0.785	0.951	0.894	0.954
30	0.595	0.947	0.699	0.952	0.789	0.951	0.890	0.948
35	0.600	0.945	0.701	0.947	0.796	0.952	0.895	0.951
40	0.590	0.953	0.694	0.954	0.794	0.950	0.891	0.948
45	0.587	0.945	0.711	0.949	0.801	0.947	0.900	0.951
50	0.598	0.948	0.695	0.949	0.789	0.950	0.891	0.951
55	0.602	0.950	0.699	0.953	0.798	0.951	0.892	0.946
60	0.591	0.947	0.697	0.944	0.803	0.950	0.902	0.947

5.2 A power function for comparing several treatments with a control

5.2.1 Introduction

Suppose we have k independently and normally distributed populations $N(\mu_i, \sigma^2)$, $1 \le i \le k$, with unknown means μ_i and a common positive variance σ^2. Assume that the first population, $N(\mu_1, \sigma^2)$, is the control, and that the other $k-1 (k \ge 2)$ populations are treatments. We are interested in making inferences about $\mu_i - \mu_1$ and, in particular, testing the family of two-sided hypotheses

$$H_{i\,0} : \mu_i - \mu_1 = 0 \quad \text{vs} \quad H_{i\,1} : \mu_i - \mu_1 \neq 0, \quad 2 \le i \le k. \tag{5.7}$$

Assume that $\bar{Y}_{in}$ denotes the sample mean of a sample of fixed size n from the i^{th} population, $1 \le i \le k$, and that S^2 is an estimate of σ^2 which is independent of the $\bar{Y}_{in}$ and distributed as a χ^2_ν/ν random variable. If σ^2 is known then $\nu = \infty$, otherwise $0 < \nu < \infty$. Then it is well known that the family of hypotheses (5.7) can be tested in the following way

$$\text{reject } H_{i\,0} \text{ in favour of } H_{i\,1} \text{ iff } \frac{\sqrt{n}|\bar{Y}_{in} - \bar{Y}_{1n}|}{S\sqrt{2}} \ge |t|^{\alpha}_{k-1,\nu}, \quad 2 \le i \le k, \tag{5.8}$$

and accompany the rejection of any $H_{i\,0}$ by the directional decision that $\mu_i - \mu_1 > 0$ if $\bar{Y}_{in} - \bar{Y}_{1n} > 0$ and $\mu_i - \mu_1 < 0$ if $\bar{Y}_{in} - \bar{Y}_{1n} < 0$, where $|t|^{\alpha}_{k-1,\nu}$ is the upper α point of the distribution of the random variable

$$|T|_{k-1,\nu} = \max_{2 \le i \le k} \frac{|Z_i - Z_1|}{\sqrt{2}\sqrt{\chi^2_\nu/\nu}}.$$

This multiple test procedure controls strongly the type I error rate at α, since it is actually derived from the following set of simultaneous confidence intervals of level $1-\alpha$

$$\mu_i - \mu_1 \in \left(\bar{Y}_{in} - \bar{Y}_{1n} - |t|^{\alpha}_{k-1,\nu} \frac{S\sqrt{2}}{\sqrt{n}},\ \bar{Y}_{in} - \bar{Y}_{1n} + |t|^{\alpha}_{k-1,\nu} \frac{S\sqrt{2}}{\sqrt{n}} \right), \quad i = 2, \cdots, k.$$

To assess the sensitivity of this test procedure, we calculate the probability that this test will detect, with a correct directional decision, each treatment whose mean μ_i is significantly different from μ_1 in terms of $|\mu_i - \mu_1| \geq d\sigma$, where $d > 0$ is a given constant. For this we define a power function $\gamma^*(\mu, d)$ to be

$$P\{\text{all false } H_{i\,0} \text{ with } |\mu_i - \mu_1| \geq d\sigma \text{ are rejected with correct directional decisions}\} \tag{5.9}$$

and, in particular, $\gamma^*(\mu, d) = 1$ if all the treatments satisfy $|\mu_i - \mu_1| < d\sigma$. The sensitivity of this multiple comparisons procedure can then be measured by $\gamma^*(d) = \min_{\mu \in R^k} \gamma^*(\mu, d)$. We shall investigate that how large the sample size n should be if we require test (5.8) has the sensitivity $\gamma^*(d) = \gamma$ for preassigned values of $d > 0$ and $0 < \gamma < 1$. We consider this in Subsection 5.2.2.

In the definition of the power function $\gamma^*(\mu, d)$ in (5.9), the departure of the μ_i from μ_1 is measured in unit of σ. It certainly makes sense to define a power function, $\hat{\gamma}^*(\mu, d)$, to be

$$P\{\text{all false } H_{i\,0} \text{ with } |\mu_i - \mu_1| \geq d \text{ are rejected with correct directional decisions}\}$$

and, in particular, $\hat{\gamma}^*(\mu, d) = 1$ if all the treatments satisfy $|\mu_i - \mu_1| < d$. The sensitivity of a test of (5.7) can be measured by the quantity $\hat{\gamma}^*(d) = \min_{\mu \in R^k} \hat{\gamma}^*(\mu, d)$. Now assume σ^2 is unknown and we wish to design a test of (5.7) such that this test has type I error rate α and sensitivity $\hat{\gamma}^*(d) = \gamma$, for given values of α, d and γ. For this it is necessary to use a sequential sampling scheme. In Subsection 5.2.3 we discuss a pure sequential procedure.

5.2.2 A fixed sample size procedure

In this subsection, we determine the sample size n so that test (5.8) has $\gamma^*(d) = \gamma$ for given values of $k,\ \nu,\ d > 0$ and $0 < \gamma < 1$. For this, we have the following theorem, whose proof is similar to that of Theorem 4.3.

Theorem 5.2 *Let $k \geq 3,\ \ p = \langle (k+1)/2 \rangle$ and $\mu^*(d) = (0, d\sigma, \cdots, d\sigma, -d\sigma, \cdots, -d\sigma) \in R^k$ which has the first component equal to zero, the last $k-p$ components equal to $-d\sigma$ and the rest $p-1$ components equal to $d\sigma$. Then*

$$\begin{aligned}\gamma^*(d) &= \gamma^*(\mu^*(d), d) \\ &= \int_0^\infty \int_{-\infty}^\infty \Phi^{p-1}\left(d\sqrt{n} - s|t|_{k-1,\nu}^{\alpha}\sqrt{2} - x\right) \times \\ &\qquad \Phi^{k-p}\left(d\sqrt{n} - s|t|_{k-1,\nu}^{\alpha}\sqrt{2} + x\right)\phi(x) f_\nu(s)\, dxds, \qquad (5.10)\end{aligned}$$

where $f_\nu(x)$ denotes a pdf of the $\sqrt{\chi^2_\nu/\nu}$.

Notice that, if σ^2 is known then

$$\begin{aligned}\gamma^*(d) &= \int_{-\infty}^\infty \Phi^{p-1}\left(d\sqrt{n} - |t|_{k-1}^{\alpha}\sqrt{2} - x\right) \times \\ &\qquad \Phi^{k-p}\left(d\sqrt{n} - |t|_{k-1}^{\alpha}\sqrt{2} + x\right)\phi(x)\, dx.\end{aligned}$$

For given values of k, ν, α and γ, Tables 5.7 and 5.8 give the value of $d\sqrt{n}$ such that $\gamma^*(d) = \gamma$.

Table 5.7: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma^*(d) = \gamma$*

for $\alpha = 0.05$ and $\gamma = 0.90$

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
ν=10	6.216	6.761	7.158	7.442	7.680	7.870	8.037	8.179	8.307
ν=12	6.072	6.592	6.970	7.241	7.466	7.646	7.805	7.940	8.062
ν=14	5.973	6.476	6.841	7.102	7.320	7.493	7.646	7.776	7.893
ν=16	5.902	6.392	6.748	7.002	7.213	7.382	7.531	7.657	7.771
ν=18	5.848	6.329	6.677	6.925	7.132	7.298	7.443	7.566	7.677
ν=20	5.805	6.279	6.621	6.866	7.069	7.231	7.374	7.495	7.604
ν=30	5.682	6.134	6.460	6.692	6.885	7.039	7.174	7.288	7.391
ν=40	5.622	6.065	6.382	6.609	6.796	6.946	7.078	7.189	7.289
ν=60	5.565	5.997	6.307	6.528	6.711	6.856	6.984	7.092	7.190
ν=120	5.509	5.932	6.234	6.450	6.627	6.770	6.894	6.999	7.094
$\nu=\infty$	5.454	5.868	6.163	6.373	6.546	6.685	6.806	6.908	7.000

Table 5.7: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^*(d) = \gamma$

for $\alpha = 0.05$ *and* $\gamma = 0.90$

	$k=12$	$k=13$	$k=14$	$k=15$	$k=16$	$k=17$	$k=18$	$k=19$	$k=20$
ν=10	8.419	8.523	8.616	8.703	8.781	8.856	8.924	8.989	9.050
ν=12	8.168	8.267	8.355	8.437	8.512	8.583	8.647	8.709	8.766
ν=14	7.996	8.091	8.175	8.254	8.327	8.394	8.457	8.516	8.571
ν=16	7.870	7.962	8.044	8.121	8.191	8.257	8.318	8.376	8.429
ν=18	7.775	7.865	7.945	8.020	8.088	8.153	8.212	8.268	8.321
ν=20	7.700	7.788	7.867	7.940	8.007	8.071	8.129	8.184	8.235
ν=30	7.482	7.565	7.640	7.709	7.773	7.832	7.887	7.939	7.7987
ν=40	7.377	7.458	7.531	7.598	7.660	7.717	7.771	7.821	7.868
ν=60	7.275	7.354	7.424	7.490	7.550	7.606	7.658	7.707	7.752
ν=120	7.177	7.253	7.321	7.385	7.443	7.498	7.548	7.595	7.640
$\nu=\infty$	7.081	7.155	7.222	7.284	7.340	7.393	7.442	7.488	7.531

Table 5.7: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma^*(d) = \gamma$ for $\alpha = 0.05$ and $\gamma = 0.95$*

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
ν=10	6.745	7.290	7.686	7.973	8.211	8.404	8.573	8.716	8.847
ν=12	6.583	7.100	7.473	7.744	7.969	8.151	8.310	8.446	8.568
ν=14	6.472	6.971	7.329	7.589	7.805	7.979	8.131	8.261	8.378
ν=16	6.393	6.877	7.224	7.477	7.685	7.854	8.001	8.127	8.241
ν=18	6.332	6.806	7.145	7.392	7.595	7.759	7.903	8.026	8.136
ν=20	6.285	6.750	7.083	7.325	7.524	7.686	7.826	7.946	8.055
ν=30	6.149	6.591	6.905	7.133	7.321	7.473	7.605	7.718	7.820
ν=40	6.084	6.515	6.820	7.042	7.224	7.371	7.499	7.609	7.708
ν=60	6.021	6.441	6.738	6.953	7.130	7.273	7.398	7.504	7.599
ν=120	5.960	6.369	6.659	6.868	7.040	7.178	7.299	7.402	7.494
$\nu=\infty$	5.900	6.300	6.582	6.785	6.952	7.087	7.204	7.304	7.393

Table 5.7: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma^*(d) = \gamma$ for $\alpha = 0.05$ and $\gamma = 0.95$*

	$k=12$	$k=13$	$k=14$	$k=15$	$k=16$	$k=17$	$k=18$	$k=19$	$k=20$
ν=10	8.961	9.066	9.161	9.249	9.330	9.405	9.475	9.542	9.604
ν=12	8.676	8.775	8.864	8.948	9.023	9.095	9.161	9.223	9.282
ν=14	8.482	8.577	8.662	8.741	8.814	8.882	8.945	9.005	9.061
ν=16	8.340	8.432	8.515	8.592	8.662	8.728	8.789	8.847	8.901
ν=18	8.234	8.323	8.404	8.478	8.547	8.611	8.670	8.727	8.779
ν=20	8.150	8.238	8.316	8.389	8.456	8.520	8.578	8.633	8.684
ν=30	7.909	7.991	8.065	8.134	8.196	8.255	8.310	8.361	8.409
ν=40	7.794	7.873	7.945	8.011	8.072	8.129	8.182	8.232	8.278
ν=60	7.683	7.760	7.829	7.893	7.952	8.007	8.058	8.106	8.151
ν=120	7.575	7.650	7.717	7.779	7.836	7.889	7.939	7.985	8.029
ν=∞	7.472	7.544	7.609	7.670	7.724	7.776	7.823	7.869	7.910

Table 5.8: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^*(d) = \gamma$

for $\alpha = 0.01$ *and* $\gamma = 0.90$

	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$	$k = 11$
ν=10	7.808	8.397	8.830	9.140	9.400	9.608	9.792	9.947	10.089
ν=12	7.503	8.052	8.455	8.744	8.985	9.178	9.349	9.493	9.624
ν=14	7.299	7.821	8.204	8.477	8.706	8.889	9.051	9.188	9.312
ν=16	7.152	7.656	8.023	8.287	8.506	8.682	8.838	8.969	9.088
ν=18	7.042	7.531	7.888	8.143	8.356	8.527	8.677	8.804	8.920
ν=20	6.957	7.434	7.783	8.032	8.240	8.406	8.552	8.676	8.789
ν=30	6.712	7.159	7.483	7.714	7.906	8.060	8.196	8.311	8.415
ν=40	6.597	7.029	7.341	7.564	7.749	7.898	8.028	8.138	8.238
ν=60	6.486	6.904	7.205	7.420	7.598	7.741	7.866	7.972	8.068
ν=120	6.380	6.784	7.074	7.282	7.453	7.591	7.712	7.814	7.905
ν=∞	6.277	6.669	6.949	7.149	7.315	7.447	7.563	7.662	7.750

Table 5.8: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^*(d) = \gamma$

for $\alpha = 0.01$ *and* $\gamma = 0.90$

	$k = 12$	$k = 13$	$k = 14$	$k = 15$	$k = 16$	$k = 17$	$k = 18$	$k = 19$	$k = 20$
ν=10	10.212	10.327	10.429	10.525	10.612	10.694	10.770	10.842	10.909
ν=12	9.739	9.845	9.940	10.029	10.110	10.186	10.256	10.323	10.385
ν=14	9.421	9.521	9.611	9.695	9.772	9.844	9.911	9.974	10.032
ν=16	9.192	9.289	9.375	9.456	9.529	9.599	9.662	9.723	9.974
ν=18	9.021	9.114	9.198	9.276	9.347	9.414	9.476	9.534	9.589
ν=20	8.887	8.978	9.060	9.136	9.205	9.270	9.330	9.388	9.441
ν=30	8.506	8.590	8.665	8.735	8.799	8.859	8.915	8.967	9.016
ν=40	8.326	8.406	8.478	8.546	8.607	8.665	8.718	8.769	8.816
ν=60	8.152	8.230	8.299	8.364	8.423	8.478	8.529	8.578	8.623
ν=120	7.987	8.061	8.128	8.189	8.246	8.299	8.348	8.395	8.438
$\nu=\infty$	7.828	7.899	7.963	8.023	8.077	8.128	8.175	8.220	8.261

Table 5.8: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^*(d) = \gamma$

for $\alpha = 0.01$ *and* $\gamma = 0.95$

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
ν=10	8.410	9.007	9.442	9.759	10.022	10.235	10.423	10.583	10.727
ν=12	8.069	8.620	9.022	9.313	9.556	9.752	9.925	10.072	10.205
ν=14	7.841	8.362	8.741	9.016	9.245	9.429	9.592	9.730	9.855
ν=16	7.679	8.178	8.541	8.804	9.022	9.199	9.354	9.486	9.606
ν=18	7.557	8.040	8.391	8.645	8.856	9.026	9.176	9.303	9.419
ν=20	7.463	7.934	8.274	8.521	8.727	8.892	9.038	9.161	9.274
ν=30	7.195	7.631	7.945	8.172	8.361	8.513	8.646	8.759	8.862
ν=40	7.069	7.489	7.791	8.009	8.189	8.335	8.463	8.571	8.669
ν=60	6.949	7.354	7.643	7.853	8.026	8.165	8.287	8.391	8.485
ν=120	6.834	7.224	7.502	7.704	7.869	8.003	8.121	8.221	8.310
$\nu=\infty$	6.723	7.101	7.368	7.562	7.721	7.849	7.961	8.057	8.143

Table 5.8: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^*(d) = \gamma$

for $\alpha = 0.01$ *and* $\gamma = 0.95$

	$k=12$	$k=13$	$k=14$	$k=15$	$k=16$	$k=17$	$k=18$	$k=19$	$k=20$
ν=10	10.854	10.972	11.077	11.175	11.265	11.350	11.428	11.502	11.571
ν=12	10.322	10.430	10.526	10.617	10.699	10.777	10.849	10.917	10.981
ν=14	9.965	10.067	10.158	10.243	10.321	10.394	10.462	10.526	10.585
ν=16	9.711	9.808	9.895	9.976	10.050	10.120	10.184	10.246	10.303
ν=18	9.520	9.613	9.697	9.776	9.847	9.915	9.977	10.036	10.091
ν=20	9.372	9.463	9.544	9.620	9.690	9.755	9.816	9.873	9.926
ν=30	8.952	9.035	9.109	9.179	9.243	9.302	9.358	9.410	9.459
ν=40	8.755	8.835	8.906	8.972	9.033	9.090	9.142	9.193	9.239
ν=60	8.568	8.644	8.712	8.775	8.833	8.888	8.938	8.986	9.030
ν=120	8.389	8.461	8.527	8.587	8.643	8.695	8.743	8.789	8.831
$\nu=\infty$	8.219	8.288	8.350	8.408	8.461	8.511	8.557	8.601	8.641

Table 5.8: *Values of the parameter $d\sqrt{n}$ satisfying $\gamma^*(d) = \gamma$*

for $\alpha = 0.01$ and $\gamma = 0.99$

	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$
ν=10	9.574	10.184	10.627	10.955	11.226	11.448	11.643	11.810	11.961
ν=12	9.164	9.719	10.120	10.417	10.663	10.865	11.041	11.192	11.329
ν=14	8.892	9.411	9.784	10.061	10.290	10.478	10.641	10.782	10.910
ν=16	8.700	9.193	9.547	9.809	10.026	10.203	10.359	10.492	10.612
ν=18	8.556	9.031	9.370	9.622	9.829	9.999	10.148	10.276	10.391
ν=20	8.446	8.905	9.233	9.477	9.678	9.842	9.985	10.109	10.220
ν=30	8.134	8.553	8.850	9.070	9.252	9.399	9.529	9.640	9.740
ν=40	7.989	8.389	8.672	8.882	9.054	9.195	9.318	9.423	9.517
ν=60	7.851	8.234	8.503	8.704	8.867	9.000	9.117	9.218	9.307
ν=120	7.720	8.086	8.343	8.534	8.689	8.817	8.927	9.023	9.108
$\nu=\infty$	7.594	7.946	8.191	8.373	8.522	8.643	8.748	8.838	8.919

Table 5.8: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^*(d) = \gamma$

for $\alpha = 0.01$ *and* $\gamma = 0.99$

	$k=12$	$k=13$	$k=14$	$k=15$	$k=16$	$k=17$	$k=18$	$k=19$	$k=20$
ν=10	12.094	12.217	12.328	12.431	12.526	12.615	12.697	12.775	12.848
ν=12	11.450	11.561	11.662	11.755	11.841	11.922	11.996	12.067	12.133
ν=14	11.022	11.125	11.219	11.306	11.386	11.461	11.530	11.596	11.658
ν=16	10.719	10.816	10.905	10.987	11.063	11.133	11.199	11.262	11.320
ν=18	10.493	10.586	10.671	10.750	10.822	10.890	10.953	11.012	11.068
ν=20	10.318	10.409	10.490	10.566	10.636	10.701	10.762	10.820	10.873
ν=30	9.828	9.910	9.983	10.051	10.114	10.172	10.227	10.278	10.326
ν=40	9.601	9.678	9.747	9.812	9.872	9.927	9.979	10.027	10.073
ν=60	9.387	9.459	9.525	9.586	9.642	9.695	9.744	9.790	9.833
ν=120	9.183	9.252	9.315	9.373	9.426	9.476	9.522	9.566	9.607
ν=∞	8.991	9.057	9.116	9.171	9.221	9.269	9.313	9.354	9.393

5.2.3 A pure sequential procedure

Let σ^2 be an unknown parameter. We want to design a test of the family of hypotheses (5.7) which has, at least approximately, the type I error rate α and power $\hat{\gamma}^*(d) = \gamma$, where $0 < \alpha < 1$, $0 < \gamma < 1$ and $d > 0$ are prefixed constants. To motivate the definition of a pure sequential procedure, we first look at the known σ^2 case which is covered in the last subsection.

had σ^2 been known, we would take a sample of size n_0 from each of the k populations and test the family of hypotheses (5.7) by:

$$\text{reject } H_{i0} \text{ in favour of } H_{i1} \text{ iff } |\bar{Y}_{in_0} - \bar{Y}_{1n_0}| > \frac{\sigma |t|_{k-1}^{\alpha} \sqrt{2}}{\sqrt{n_0}}, \quad 2 \le i \le k,$$

and accompany the rejection of any H_{i0} by the directional decision that $\mu_i - \mu_1 > 0$ if $\bar{Y}_{in_0} - \bar{Y}_{1n_0} > 0$ and $\mu_i - \mu_1 < 0$ if $\bar{Y}_{in_0} - \bar{Y}_{1n_0} < 0$, where n_0 satisfies

$$\int_{-\infty}^{\infty} \Phi^{p-1}\left(\frac{d\sqrt{n_0}}{\sigma} - |t|_{k-1}^{\alpha}\sqrt{2} - x\right)\Phi^{k-p}\left(\frac{d\sqrt{n_0}}{\sigma} - |t|_{k-1}^{\alpha}\sqrt{2} + x\right)\phi(x)\,dx = \gamma \tag{5.11}$$

where $p = \langle (k+1)/2 \rangle$. Denote

$$t_\gamma = \frac{d\sqrt{n_0}}{\sigma} - |t|_{k-1}^{\alpha}\sqrt{2}, \tag{5.12}$$

which can be solved from equation (5.11). Then sample size n_0 is given by

$$n_0 = \sigma^2 d^{-2}\left(t_\gamma + |t|_{k-1}^{\alpha}\sqrt{2}\right)^2 \tag{5.13}$$

and so the test can be rewritten as

$$\text{reject } H_{i0} \text{ in favour of } H_{i1} \text{ iff } |\bar{Y}_{in_0} - \bar{Y}_{1n_0}| > \frac{d|t|_{k-1}^{\alpha}\sqrt{2}}{t_\gamma + |t|_{k-1}^{\alpha}\sqrt{2}}, \quad 2 \le i \le k,$$

and accompany the rejection of any H_{i0} by the directional decision that $\mu_i - \mu_1 > 0$ if $\bar{Y}_{in_0} - \bar{Y}_{1n_0} > 0$ and $\mu_i - \mu_1 < 0$ if $\bar{Y}_{in_0} - \bar{Y}_{1n_0} < 0$.

Based on these observations, we can now define a sequential procedure for the situation of unknown σ^2 that is assumed in this subsection. Take a sample of size m from each of the k populations, then take one observation from each population at a time until

$$T = \inf\{n \geq m : \; n \geq (1 + \xi_1/n) d^{-2} C^2 \hat{\sigma}_n{}^2\},$$

where $0 < C = t_\gamma + |t|_{k-1}^{\alpha} \sqrt{2}$ and ξ_1 is a given constant whose value will be determined later. On stopping sampling

$$\text{reject } H_{i\,0} \text{ in favour of } H_{i\,1} \text{ iff } |\bar{Y}_{iT} - \bar{Y}_{1T}| > \frac{d|t|_{k-1}^{\alpha}\sqrt{2}}{C}\left(1 + \frac{\eta_1}{T}\right), \quad 2 \leq i \leq k,$$

and accompany the rejection of any $H_{i\,0}$ by the directional decision that $\mu_i - \mu_1 > 0$ if $\bar{Y}_{iT} - \bar{Y}_{1T} > 0$ and $\mu_i - \mu_1 < 0$ if $\bar{Y}_{iT} - \bar{Y}_{1T} < 0$, where η_1 is a given constant whose value is given below.

Note that the stopping time T uses formula (5.13) adaptively by replacing σ^2 with $\hat{\sigma}_n^2$ to check whether enough observations have already been drawn, and the test mimics the test for the known σ^2 situation. Next we show that this procedure has the required properties, at least for large n_0.

First, we show that this procedure controls strongly the type I error rate at α, at least for large n_0. For this, it is sufficient to show that

$$CL = P\left\{|\bar{Y}_{iT} - \bar{Y}_{1T} - (\mu_i - \mu_1)| < \frac{d|t|_{k-1}^{\alpha}\sqrt{2}}{C}\left(1 + \frac{\eta_1}{T}\right), \quad 2 \leq i \leq k\right\}$$

is equal to $1 - \alpha + o(1)$ as $n_0 \to \infty$. By noting that

$$CL = E\left[H\left(\frac{T}{n_0}\left(1 + \frac{\eta_1}{T}\right)^2\right)\right]$$

where

$$H(x) = P\left(\max_{2 \leq i \leq k} |Z_i - Z_1| \leq |t|_{k-1}^{\alpha}\sqrt{2}\sqrt{x}\right),$$

it therefore follows from Theorem 2.2 with $\theta = 1$ and $C_1 = \eta_1$ that

$$\begin{aligned} CL &= 1 - \alpha + \frac{H'(1)}{n_0}\left(\rho + \xi_1 - \frac{2}{k} + 2\eta_1\right) + \frac{H''(1)}{kn_0} + o\left(\frac{1}{n_0}\right) \qquad (5.14) \\ &= 1 - \alpha + o(1) \quad \text{as } n_0 \to \infty. \end{aligned}$$

Next, we find the second order approximation to the value of $\hat{\gamma}^*(d)$ of this procedure. Let

$$\Omega_U^*(d) = \{i : \mu_i - \mu_1 \geq d\} \quad \text{and} \quad \Omega_L^*(d) = \{j : \mu_j - \mu_1 \leq -d\}.$$

From the definition and Theorem 4.3, we have

$$\begin{aligned}
\hat{\gamma}^*(d) &= \min_{\mu \in R^k} P\{\text{all false } H_{i0} \text{ with } |\mu_i - \mu_1| \geq d \\
&\qquad \text{are rejected with correct directional decisions}\} \\
&= \min_{\mu \in R^k} P\Big\{\bar{Y}_{iT} - \bar{Y}_{1T} > \frac{d|t|_{k-1}^{\alpha}\sqrt{2}}{C}\left(1 + \frac{\eta_1}{T}\right) \ \forall i \in \Omega_U^*(d), \\
&\qquad \bar{Y}_{jT} - \bar{Y}_{1T} < -\frac{d|t|_{k-1}^{\alpha}\sqrt{2}}{C}\left(1 + \frac{\eta_1}{T}\right) \ \forall j \in \Omega_L^*(d)\Big\} \\
&= E\left[G\left\{\left(C - |t|_{k-1}^{\alpha}\sqrt{2}\left(1 + \frac{\eta_1}{T}\right)\right)\frac{\sqrt{T}}{\sqrt{n_0}}\right\}\right],
\end{aligned}$$

where

$$G(x) = P\{Z_i - Z_1 > -x,\ 2 \leq i \leq p,\ Z_i - Z_1 < x,\ p + 1 \leq i \leq k\}$$

and $p = \langle (k+1)/2 \rangle$. It therefore follows from Theorem 2.3 with $H(x) = G(x)$, $C_0 = C$, $C_1 = |t|_{k-1}^{\alpha}\sqrt{2}$ and $C_2 = \eta_1$ that

$$\begin{aligned}
\hat{\gamma}^*(d) &= \gamma - \frac{1}{n_0}G'(t_\gamma)\left(\eta_1|t|_{k-1}^{\alpha}\sqrt{2} - \frac{t_\gamma}{2}\left(\rho + \xi_1 - \frac{2}{k}\right) + \frac{t_\gamma}{4k}\right) \\
&\quad + \frac{1}{4kn_0}t_\gamma^2 G''(t_\gamma) + o\left(\frac{1}{n_0}\right). \qquad (5.15)
\end{aligned}$$

Note that

$$\begin{aligned}
G'(t_\gamma) &= \int_{-\infty}^{\infty} \phi(y)\Big\{(p-1)\phi(t_\gamma - y)\Phi^{p-2}(t_\gamma - y)\Phi^{k-p}(t_\gamma + y) \\
&\quad + (k-p)\phi(t_\gamma + y)\Phi^{p-1}(t_\gamma - y)\Phi^{k-p-1}(t_\gamma + y)\Big\}\, dy
\end{aligned}$$

and

$$G''(t_\gamma) =$$

$$\int_{-\infty}^{\infty} \phi(y)\Big((p-1)(y - t_\gamma)\phi(t_\gamma - y)\Phi^{p-2}(t_\gamma - y)\Phi^{k-p}(t_\gamma + y)$$

$$+(p-1)(p-2)\phi^2(t_\gamma - y)\Phi^{p-3}(t_\gamma - y)\Phi^{k-p}(t_\gamma + y)$$
$$+(p-1)(k-p)\phi(t_\gamma - y)\Phi^{p-2}(t_\gamma - y)\phi(t_\gamma + y)\Phi^{k-p-1}(t_\gamma + y)$$
$$-(k-p)(t_\gamma + y)\phi(t_\gamma + y)\Phi^{p-1}(t_\gamma - y)\Phi^{k-p-1}(t_\gamma + y)$$
$$+(k-p)(p-1)\phi(t_\gamma + y)\phi(t_\gamma - y)\Phi^{p-2}(t_\gamma - y)\Phi^{k-p-1}(t_\gamma + y)$$
$$+(k-p)(k-p-1)\phi^2(t_\gamma + y)\Phi^{p-1}(t_\gamma - y)\Phi^{k-p-2}(t_\gamma + y)\Big)\,dy.$$

From (5.14) and (5.15), we set the values of ξ_1 and η_1 satisfying simultaneously

$$\xi_1 + 2\eta_1 = -\rho + \frac{2}{k} - \frac{H''(1)}{kH'(1)},$$
$$2kt_\gamma\left(\rho + \xi_1 - \frac{2}{k}\right)G'(t_\gamma) = (4k\eta_1|t|_{k-1}^{\alpha}\sqrt{2} + t_\gamma)G'(t_\gamma) - t_\gamma^2 G''(t_\gamma),$$

so that the procedure has type I error rate $\alpha + o(1/n_0)$ and power $\hat{\gamma}^*(d) = \gamma + o(1/n_0)$ as $n_0 \to \infty$.

Table 5.9 presents the values of ξ_1 and η_1 for given values of α, γ and k.

By Theorem 2.1, the expected sample size from each population is given by

$$E(T) = n_0 + \rho + \xi_1 - \frac{2}{k} + o(1) \;\text{ as } n_0 \to \infty\;.$$

A simulation exercise has been carried out to assess the performance of this procedure for small and moderate values of n_0. Table 5.10 shows the values of t_γ for $k = 2(1)20$ and Table 5.11 presents the simulated and approximate values of $E(T)$. For $m = 10, k = 3, 10$ and $\alpha = 0.1, 0.05$, Table 5.12 shows the simulation results of (1− type I error rate) and $\hat{\gamma}^*(d)$.

Table 5.9: *Values of ξ_1 and η_1*

for $\alpha = 0.05$ and given values of γ and k

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
k	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1
3	0.949	0.138	0.927	0.150	0.920	0.153	0.950	0.138
4	0.606	0.124	0.596	0.129	0.598	0.128	0.628	0.113
5	0.394	0.110	0.391	0.111	0.398	0.108	0.427	0.094
6	0.252	0.098	0.252	0.098	0.260	0.094	0.286	0.081
7	0.150	0.088	0.152	0.087	0.160	0.083	0.184	0.071
8	0.073	0.080	0.075	0.079	0.083	0.075	0.106	0.063
9	0.012	0.073	0.015	0.072	0.023	0.068	0.044	0.057
10	-0.036	0.068	-0.033	0.066	-0.025	0.062	-0.005	0.052
11	-0.076	0.063	-0.072	0.061	-0.065	0.057	-0.047	0.048
12	-0.109	0.059	-0.106	0.057	-0.099	0.053	-0.081	0.045
13	-0.137	0.055	-0.134	0.053	-0.127	0.050	-0.111	0.042
14	-0.162	0.052	-0.158	0.050	-0.152	0.047	-0.136	0.039
15	-0.183	0.049	-0.179	0.047	-0.173	0.044	-0.158	0.037
16	-0.201	0.047	-0.198	0.045	-0.192	0.042	-0.178	0.035
17	-0.218	0.044	-0.214	0.043	-0.208	0.040	-0.195	0.033
18	-0.232	0.042	-0.229	0.041	-0.223	0.038	-0.210	0.031
19	-0.246	0.040	-0.242	0.039	-0.237	0.036	-0.224	0.030
20	-0.257	0.039	-0.254	0.037	-0.249	0.034	-0.237	0.028

Table 5.9: *Values of* ξ_1 *and* η_1

for $\alpha = 0.1$ *and given values of* γ *and* k

k	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1
3	0.713	0.109	0.704	0.113	0.715	0.108	0.768	0.081
4	0.427	0.097	0.427	0.097	0.441	0.090	0.487	0.067
5	0.250	0.086	0.255	0.083	0.271	0.076	0.312	0.055
6	0.131	0.076	0.137	0.073	0.152	0.066	0.189	0.047
7	0.046	0.069	0.052	0.065	0.067	0.058	0.100	0.041
8	-0.019	0.062	-0.012	0.059	0.001	0.052	0.031	0.037
9	-0.070	0.057	-0.063	0.053	-0.050	0.047	-0.023	0.033
10	-0.110	0.052	-0.104	0.049	-0.092	0.043	-0.066	0.031
11	-0.144	0.049	-0.137	0.045	-0.126	0.040	-0.102	0.028
12	-0.171	0.045	-0.165	0.042	-0.155	0.037	-0.133	0.026
13	-0.195	0.043	-0.189	0.040	-0.179	0.035	-0.158	0.024
14	-0.215	0.040	-0.210	0.037	-0.200	0.032	-0.181	0.023
15	-0.233	0.038	-0.227	0.035	-0.219	0.031	-0.200	0.021
16	-0.249	0.036	-0.243	0.033	-0.235	0.029	-0.217	0.020
17	-0.262	0.034	-0.257	0.032	-0.249	0.027	-0.232	0.019
18	-0.275	0.033	-0.270	0.030	-0.262	0.026	-0.246	0.018
19	-0.286	0.031	-0.281	0.029	-0.273	0.025	-0.258	0.017
20	-0.296	0.030	-0.291	0.027	-0.284	0.024	-0.269	0.016

Table 5.10: *Values of t_γ*

for given values of γ and k

k	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
3	1.167	1.456	1.809	2.326
4	1.430	1.707	2.047	2.546
5	1.639	1.903	2.229	2.710
6	1.769	2.027	2.348	2.821
7	1.883	2.136	2.451	2.916
8	1.967	2.217	2.528	2.989
9	2.045	2.292	2.599	3.055
10	2.106	2.351	2.656	3.109
11	2.164	2.407	2.709	3.159
12	2.212	2.453	2.754	3.202
13	2.258	2.497	2.797	3.242
14	2.297	2.536	2.834	3.277
15	2.335	2.572	2.869	3.310
16	2.368	2.604	2.900	3.339
17	2.400	2.635	2.930	3.368
18	2.429	2.663	2.957	3.394
19	2.456	2.690	2.983	3.418
20	2.481	2.714	3.006	3.441
21	2.506	2.738	3.029	3.463

Table 5.11: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 3$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	16.0	16.0	16.0	16.0	16.0	16.0	16.0	16.0
20	20.9	21.0	20.9	21.0	20.9	21.0	20.9	21.0
25	25.9	26.0	25.9	26.0	25.8	26.0	25.9	26.0
30	30.8	31.0	30.8	31.0	30.8	31.0	30.8	31.0
35	36.0	36.0	35.8	36.0	35.8	36.0	35.8	36.0
40	41.0	41.0	40.9	41.0	40.9	41.0	40.9	41.0
45	45.9	46.0	45.9	46.0	45.9	46.0	45.9	46.0
50	50.9	51.0	50.9	51.0	50.9	51.0	50.9	51.0
55	55.9	56.0	56.0	56.0	56.0	56.0	56.0	56.0
60	60.9	61.0	61.0	61.0	61.0	61.0	61.0	61.0

Table 5.11: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 10$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.4	15.4	15.4	15.4	15.4	15.4	15.4	15.4
20	20.4	20.4	20.4	20.4	20.4	20.4	20.4	20.4
25	25.3	25.4	25.3	25.4	25.3	25.4	25.3	25.4
30	30.3	30.4	30.3	30.4	30.4	30.4	30.4	30.4
35	35.3	35.4	35.3	35.4	35.3	35.4	35.3	35.4
40	40.3	40.4	40.3	40.4	40.3	40.4	40.4	40.4
45	45.3	45.4	45.3	45.4	45.4	45.4	45.4	45.4
50	50.3	50.4	50.4	50.4	50.4	50.4	50.4	50.4
55	55.4	55.4	55.4	55.4	55.4	55.4	55.4	55.4
60	60.4	60.4	60.4	60.4	60.4	60.4	60.4	60.4

Table 5.11: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 3$, $\alpha = 0.1$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.8	15.7	15.8	15.7	15.8	15.7	15.9	15.8
20	20.7	20.7	20.7	20.7	20.7	20.7	20.7	20.8
25	25.6	25.7	25.6	25.7	25.6	25.7	25.6	25.8
30	30.6	30.7	30.6	30.7	30.6	30.7	30.7	30.8
35	35.6	35.7	35.6	35.7	35.6	35.7	35.7	35.8
40	40.7	40.7	40.7	40.7	40.7	40.7	40.7	40.8
45	45.7	45.7	45.7	45.7	45.7	45.7	45.7	45.8
50	50.7	50.7	50.7	50.7	50.7	50.7	50.7	50.8
55	58.8	55.7	55.8	55.7	55.8	55.7	55.8	55.8
60	60.8	60.7	60.8	60.7	60.8	60.7	60.9	60.8

Table 5.11: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 10$, $\alpha = 0.1$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.3	15.3	15.3	15.3	15.3	15.3	15.3	15.3
20	20.3	20.3	20.3	20.3	20.3	20.3	20.4	20.3
25	25.3	25.3	25.3	25.3	25.3	25.3	25.3	25.3
30	30.3	30.3	30.3	30.3	30.3	30.3	30.3	30.3
35	35.2	35.3	35.2	35.3	35.3	35.3	35.3	35.3
40	40.3	40.3	40.3	40.3	40.3	40.3	40.3	40.3
45	45.3	45.3	45.3	45.3	45.3	45.3	45.3	45.3
50	50.3	50.3	50.3	50.3	50.3	50.3	50.3	50.3
55	55.3	55.3	55.3	55.3	55.3	55.3	55.4	55.3
60	60.3	60.3	60.3	60.3	60.3	60.3	60.4	60.3

Table 5.12: *Simulation values of $\alpha^c = (1-$ type I error rate) and $\hat{\gamma}^*(d)$ for $m = 10$, $k = 3$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c
15	0.597	0.951	0.697	0.952	0.797	0.952	0.900	0.952
20	0.597	0.950	0.697	0.950	0.804	0.949	0.898	0.951
25	0.599	0.951	0.700	0.949	0.796	0.949	0.899	0.951
30	0.591	0.951	0.691	0.952	0.796	0.951	0.900	0.951
35	0.599	0.948	0.693	0.948	0.798	0.949	0.893	0.948
40	0.583	0.948	0.698	0.951	0.799	0.950	0.897	0.948
45	0.583	0.948	0.694	0.949	0.797	0.948	0.896	0.948
50	0.598	0.949	0.697	0.947	0.795	0.947	0.900	0.949
55	0.599	0.951	0.695	0.952	0.797	0.953	0.900	0.951
60	0.688	0.952	0.702	0.952	0.808	0.956	0.905	0.951

Table 5.12: *Simulation values of* $\alpha^c = (1-$ *type I error rate) and* $\hat{\gamma}^*(d)$

for $m = 10,\ k = 10,\ \alpha = 0.05$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c
15	0.607	0.955	0.708	0.956	0.806	0.953	0.900	0.953
20	0.587	0.948	0.690	0.950	0.799	0.950	0.896	0.952
25	0.601	0.951	0.701	0.952	0.807	0.951	0.905	0.951
30	0.594	0.947	0.698	0.949	0.801	0.953	0.904	0.947
35	0.602	0.956	0.698	0.958	0.799	0.955	0.905	0.953
40	0.603	0.948	0.703	0.950	0.795	0.951	0.891	0.948
45	0.604	0.952	0.698	0.953	0.800	0.951	0.898	0.946
50	0.603	0.953	0.703	0.953	0.799	0.951	0.900	0.950
55	0.592	0.951	0.697	0.951	0.798	0.953	0.898	0.949
60	0.599	0.951	0.696	0.950	0.801	0.951	0.893	0.948

Table 5.12: *Simulation values of* $\alpha^c = (1-$ *type I error rate)* *and* $\hat{\gamma}^*(d)$ *for* $m = 10,\ k = 3,\ \alpha = 0.1$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c
15	0.586	0.902	0.696	0.902	0.800	0.902	0.902	0.904
20	0.588	0.899	0.692	0.901	0.793	0.899	0.900	0.900
25	0.581	0.893	0.694	0.893	0.798	0.893	0.900	0.897
30	0.587	0.897	0.696	0.900	0.801	0.897	0.897	0.899
35	0.594	0.902	0.699	0.903	0.799	0.903	0.895	0.902
40	0.594	0.902	0.697	0.900	0.805	0.902	0.904	0.900
45	0.595	0.904	0.701	0.903	0.801	0.907	0.901	0.900
50	0.586	0.896	0.686	0.896	0.789	0.896	0.901	0.899
55	0.594	0.901	0.697	0.898	0.799	0.902	0.892	0.897
60	0.593	0.909	0.690	0.911	0.796	0.907	0.905	0.912

Table 5.12: *Simulation values of* $\alpha^c = (1-$ *type I error rate) and* $\hat{\gamma}^*(d)$

for $m = 10,\ k = 10,\ \alpha = 0.1$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c	$\hat{\gamma}^*(d)$	α^c
15	0.602	0.904	0.702	0.905	0.795	0.898	0.902	0.899
20	0.585	0.895	0.697	0.900	0.802	0.900	0.900	0.903
25	0.604	0.901	0.709	0.903	0.795	0.905	0.899	0.904
30	0.599	0.899	0.693	0.898	0.800	0.900	0.900	0.899
35	0.601	0.901	0.697	0.904	0.803	0.901	0.900	0.902
40	0.595	0.893	0.696	0.900	0.802	0.898	0.903	0.900
45	0.606	0.892	0.702	0.900	0.798	0.896	0.892	0.891
50	0.601	0.895	0.706	0.899	0.805	0.903	0.906	0.908
55	0.601	0.905	0.702	0.901	0.806	0.904	0.905	0.907
60	0.608	0.902	0.701	0.900	0.798	0.896	0.897	0.902

5.3 A power function for all pairwise comparisons of several treatments

5.3.1 Introduction

Suppose we have k independently and normally distributed populations $N(\mu_i, \sigma^2)$, $1 \leq i \leq k$, with unknown means μ_i and a common positive variance σ^2. We are interested in making inferences about $\mu_i - \mu_j$ and, in particular, we want to test the family of two-sided hypotheses

$$H_{ij0} : \mu_i - \mu_j = 0 \quad \text{vs} \quad H_{ij1} : \mu_i - \mu_j \neq 0, \quad 1 \leq i \neq j \leq k. \tag{5.16}$$

Assume that $\bar{Y}_{in}$ denotes the sample mean of a sample of fixed size n from the i^{th} population, $1 \leq i \leq k$, and that S^2 is an estimate of σ^2 which is independent of the $\bar{Y}_{in}$ and distributed as a χ^2_ν/ν random variable. If σ^2 is known then $\nu = \infty$, otherwise $0 < \nu < \infty$. It is well known that the family of hypotheses (5.16) can be tested in the following way

$$\text{reject } H_{ij0} \text{ in favour of } H_{ij1} \text{ iff } \frac{\sqrt{n}|\bar{Y}_{in} - \bar{Y}_{jn}|}{S} \geq q^\alpha_{k,\nu}, \quad 1 \leq i \neq j \leq k, \tag{5.17}$$

and accompany the rejection of any H_{ij0} by the directional decision that $\mu_i - \mu_j > 0$ if $\bar{Y}_{in} - \bar{Y}_{jn} > 0$, where $q^\alpha_{k,\nu}$ is the upper α point of the distribution of the random variable

$$Q_{k,\nu} = \max_{1 \leq i \neq j \leq k} \frac{Z_i - Z_j}{\sqrt{\chi^2_\nu/\nu}}.$$

This multiple test procedure controls strongly the type I error rate at α, since it is actually derived from the following set of simultaneous confidence intervals of level $1 - \alpha$

$$\mu_i - \mu_j \in \left(\bar{Y}_{in} - \bar{Y}_{jn} - q^\alpha_{k,\nu} \frac{S}{\sqrt{n}}, \ \bar{Y}_{in} - \bar{Y}_{jn} + q^\alpha_{k,\nu} \frac{S}{\sqrt{n}} \right), \quad 1 \leq i \neq j \leq k.$$

To assess the sensitivity of this test procedure, we calculate the probability that this test will detect, with a correct directional decision, each pair (i, j)

of treatments whose means μ_i and μ_j are significantly different in terms of $|\mu_i - \mu_j| \geq d\sigma$, where $d > 0$ is a given constant. For this we define a power function $\gamma^{**}(\mu, d)$ to be

$$P\{\text{all false } H_{ij0} \text{ with } |\mu_i - \mu_j| \geq d\sigma \text{ are rejected with correct directional decisions}\} \tag{5.18}$$

and, in particular, $\gamma^{**}(\mu, d) = 1$ if all pair of the treatments satisfy $|\mu_i - \mu_j| < d\sigma$. The sensitivity of this multiple comparisons procedure can then be measured by $\gamma^{**}(d) = \min_{\mu \in R^k} \gamma^{**}(\mu, d)$. In this section we investigate that how large the sample size n should be if we require test (5.17) has the sensitivity $\gamma^{**}(d) = \gamma$ for preassigned values of $d > 0$ and $0 < \gamma < 1$. This is treated in Subsection 5.3.2 for $k = 3$. When $k = 4$ we find the sample size n necessary to guarantee $\gamma^{**}(d) \geq \gamma$. Although, the power function defined here is suitable for general $k \geq 4$, to find an explicit formula for the minimum of the power function when $k \geq 4$ seems impossible.

Note that, in the definition of the power function $\gamma^{**}(\mu, d)$ in (5.18), the departure of the μ_i from the μ_j, is measured in unit of σ. We may define a power function, $\hat{\gamma}^{**}(\mu, d)$, to be

$$P\{\text{all false } H_{ij0} \text{ with } |\mu_i - \mu_j| \geq d \text{ are rejected with correct directional decisions}\}$$

and, in particular, $\hat{\gamma}^{**}(\mu, d) = 1$ if all pair of the treatments satisfy $|\mu_i - \mu_j| < d$. The sensitivity of a test of (5.16) can be measured by the quantity $\hat{\gamma}^{**}(d) = \min_{\mu \in R^k} \hat{\gamma}^{**}(\mu, d)$. Now assume σ^2 is an unknown parameter and we wish to design a test of (5.16) such that this test has type I error rate α and sensitivity $\hat{\gamma}^{**}(d) = \gamma$, for given values of α, d and γ. For this it is necessary to use a sequential sampling scheme. A pure sequential procedure will be discussed in Subsection 5.3.3.

5.3.2 A fixed sample size procedure

This subsection is devoted to determine the sample size n so that test (5.17) has $\gamma^{**}(d) \geq \gamma$ for given values of ν, $d > 0$, $0 < \gamma < 1$ and $k = 3, 4$.

First, when $k = 3$, we have the following theorem, whose proof is similar to Theorem 4.5.

Theorem 5.3 *Let $k = 3$ and $\mu^*(d) = (0, -d\sigma, d\sigma) \in R^3$, then*

$$\begin{aligned}\gamma^{**}(d) &= \gamma^{**}(\mu^*(d), d) \\ &= \int_0^\infty \int_{-\infty}^\infty \Phi(d\sqrt{n} - sq_{3,\nu}^\alpha - x) \times \\ &\qquad \Phi(d\sqrt{n} - sq_{3,\nu}^\alpha + x)\phi(x) f_\nu(s) dx ds, \end{aligned} \tag{5.19}$$

where $f_\nu(x)$ denotes a pdf of the $\sqrt{\chi_\nu^2/\nu}$.

Notice that, if the variance σ^2 is known then

$$\gamma^{**}(d) = \int_{-\infty}^\infty \Phi(d\sqrt{n} - q_3^\alpha - x)\Phi(d\sqrt{n} - q_3^\alpha + x)\phi(x)dx.$$

Table 5.13 presents the values of $d\sqrt{n}$ for given values of $k = 3, \nu, \alpha$ and γ.

Now, when $k = 4$, we have the following theorem which can be proved in a way similar to Theorem 4.6.

Theorem 5.4 *Let*

$$M = \phi(x)\phi(y)[\Phi(x - sq_{4,\nu}^\alpha + d\sqrt{n}) - \Phi(y + sq_{4,\nu}^\alpha - d\sqrt{n})]^2,$$

$$N = \phi(x)\phi(y)\Phi(-x - sq_{4,\nu}^\alpha + d\sqrt{n})[\Phi(x - sq_{4,\nu}^\alpha + d\sqrt{n}) - \Phi(y + q_{4,\nu}^\alpha s - d\sqrt{n})],$$

$$A = \int_{-\infty}^\infty \int_{-\infty}^{x+2d\sqrt{n}} \int_0^{(x-y)/(2q_{4,\nu}^\alpha)+d\sqrt{n}/q_{4,\nu}^\alpha} M f_\nu(s)\, ds\, dy\, dx,$$

$$B = \int_{-\infty}^\infty \int_{-\infty}^{x+2d\sqrt{n}} \int_0^{(x-y)/(2q_{4,\nu}^\alpha)+d\sqrt{n}/q_{4,\nu}^\alpha} N f_\nu(s)\, ds\, dy\, dx.$$

Then

$$\gamma^{**}(d) = \min(A, B).$$

Table 5.14 presents the values of $d\sqrt{n}$ for which $\gamma^{**}(d) = \gamma$ for given values of ν, α, $k = 4$ and γ.

Table 5.13: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^{**}(d) = \gamma$

for $\alpha = 0.05, k = 3$ *and* $\gamma = 0.95$

ν	$d\sqrt{n}$	ν	$d\sqrt{n}$
10	7.039	30	6.367
12	6.856	40	6.293
14	6.732	60	6.223
16	6.642	120	6.153
18	6.574	∞	6.086
20	6.521		

Table 5.13: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^{**}(d) = \gamma$

for $\alpha = 0.01, k = 3$ *and* $\gamma = 0.95$

ν	$d\sqrt{n}$	ν	$d\sqrt{n}$
10	8.755	30	7.413
12	8.378	40	7.274
14	8.127	60	7.139
16	7.947	120	7.014
18	7.812	∞	6.892
20	7.708		

Table 5.13: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^{**}(d) = \gamma$

for $\alpha = 0.01, k = 3$ *and* $\gamma = 0.99$

ν	$d\sqrt{n}$	ν	$d\sqrt{n}$
10	9.950	30	8.361
12	9.497	40	8.200
14	9.196	60	8.047
16	8.983	120	7.903
18	8.825	∞	7.763
20	8.703		

Table 5.14: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^{**}(d) = \gamma$

for $\alpha = 0.05, k = 4$ *and* $\gamma = 0.95$

ν	$d\sqrt{n}$	ν	$d\sqrt{n}$
10	15.323	30	13.319
12	14.827	40	13.073
14	14.443	60	12.825
16	14.181	120	12.659
18	13.974	∞	12.437
20	13.799		

Table 5.14: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^{**}(d) = \gamma$

for $\alpha = 0.01, k = 4$ *and* $\gamma = 0.95$

ν	$d\sqrt{n}$	ν	$d\sqrt{n}$
10	20.294	30	17.634
12	20.216	40	17.182
14	19.450	60	16.725
16	19.216	120	16.572
18	18.549	∞	16.321
20	18.423		

Table 5.14: *Values of the parameter* $d\sqrt{n}$ *satisfying* $\gamma^{**}(d) = \gamma$

for $\alpha = 0.01, k = 4$ *and* $\gamma = 0.99$

ν	$d\sqrt{n}$	ν	$d\sqrt{n}$
10	29.345	30	22.108
12	27.826	40	21.723
14	25.988	60	20.795
16	24.530	120	19.664
18	24.471	∞	19.431
20	23.385		

5.3.3 A pure sequential procedure

Let σ^2 be an unknown parameter and $k = 3$. In this section we design a test of the family of hypotheses (5.16) which has, at least approximately, type I error rate α and power $\hat{\gamma}^{**}(d) = d$, where $0 < \alpha < 1$, $0 < \gamma < 1$ and $d > 0$ are prefixed constants. To motivate the definition of a pure sequential procedure, we first look at the known σ^2 case which is discussed in the last subsection.

had σ^2 been known, we would take a sample of size n_0 from each of the k populations and test the family of hypotheses (5.16) by:

$$\text{reject } H_{ij0} \text{ in favour of } H_{ij1} \text{ iff } |\bar{Y}_{in_0} - \bar{Y}_{jn_0}| > \frac{\sigma q_3^\alpha}{\sqrt{n_0}}, \quad 1 \leq i \neq j \leq 3,$$

and accompany the rejection of any H_{ij0} by the directional decision that $\mu_i - \mu_j > 0$ if $\bar{Y}_{in_0} - \bar{Y}_{jn_0} > 0$, where n_0 satisfies

$$\int_{-\infty}^{\infty} \Phi\left(\frac{d\sqrt{n_0}}{\sigma} - q_3^\alpha - x\right) \Phi\left(\frac{d\sqrt{n_0}}{\sigma} - q_3^\alpha + x\right) \phi(x)\, dx = \gamma. \tag{5.20}$$

Denote

$$r_\gamma = \frac{d\sqrt{n_0}}{\sigma} - q_3^\alpha, \tag{5.21}$$

which can be solved from the equation (5.20). Then sample size n_0 is given by

$$n_0 = \sigma^2 d^{-2} \left(r_\gamma + q_3^\alpha\right)^2. \tag{5.22}$$

and so the test can be written as

$$\text{reject } H_{ij0} \text{ in favour of } H_{ij1} \text{ iff } |\bar{Y}_{in_0} - \bar{Y}_{jn_0}| > \frac{d q_3^\alpha}{r_\gamma + q_3^\alpha}, \quad 1 \leq i \neq j \leq 3,$$

and accompany the rejection of any H_{ij0} by the directional decision that $\mu_i - \mu_j > 0$ if $\bar{Y}_{in_0} - \bar{Y}_{jn_0} > 0$.

Based on these observations, we can now define a sequential procedure for the situation of unknown σ^2 that is assumed in this subsection. Take a sample of size m from each of the $k = 3$ populations, then take one observation from each population at a time until

$$T = \inf\{n \geq m : \ n \geq (1 + \xi_1/n) d^{-2} C^2 \hat{\sigma}_n{}^2\},$$

where $0 < C = r_\gamma + q_3^\alpha$ and ξ_1 is a given constant whose value will be determined later. On stopping sampling,

reject H_{ij0} in favour of H_{ij1} iff $|\bar{Y}_{iT} - \bar{Y}_{jT}| > \dfrac{dq_3^\alpha}{C}\left(1 + \dfrac{\eta_1}{T}\right)$, $1 \le i \ne j \le 3$,

and accompany the rejection of any H_{ij0} by the directional decision that $\mu_i - \mu_j > 0$ if $\bar{Y}_{iT} - \bar{Y}_{jT} > 0$, where η_1 is a given constant whose value is given below. Next we show that this procedure has the required properties, at least for large n_0.

First, we show that this procedure controls strongly the type I error rate at α, at least for large n_0. For this, it is sufficient to show that

$$CL = P\left\{|\bar{Y}_{iT} - \bar{Y}_{jT} - (\mu_i - \mu_j)| < \frac{dq_3^\alpha}{C}\left(1 + \frac{\eta_1}{T}\right), \quad 1 \le i \ne j \le 3\right\}$$

is equal to $1 - \alpha + o(1)$ as $n_0 \to \infty$. By noting that

$$CL = E\left[H\left(\frac{T}{n_0}\left(1 + \frac{\eta_1}{T}\right)^2\right)\right]$$

where

$$H(x) = P\left\{\max_{1 \le i \ne j \le 3} |Z_i - Z_j| \le q_3^\alpha \sqrt{x}\right\},$$

it therefore follows from Theorem 2.2 with $\theta = 1$ and $C_1 = \eta_1$ that

$$\begin{aligned} CL &= 1 - \alpha + \frac{H'(1)}{n_0}\left(\rho + \xi_1 - \frac{2}{3} + 2\eta_1\right) + \frac{H''(1)}{3n_0} + o\left(\frac{1}{n_0}\right) \qquad (5.23) \\ &= 1 - \alpha + o(1) \quad \text{as } n_0 \to \infty. \end{aligned}$$

Next, we find the second order approximation to the value of $\hat{\gamma}^{**}(d)$ of this procedure. Let

$$\Omega(d) = \{i, j : \mu_i - \mu_j \ge d\}.$$

From the definition and Theorem 4.5 we have

$$\begin{aligned} \hat{\gamma}^{**}(d) &= \min_{\mu \in R^3} P\{\text{all false } H_{ij0} \text{ with} |\mu_i - \mu_j| \ge d \\ &\qquad \text{are rejected with correct directional decisions}\} \\ &= \min_{\mu \in R^3} P\left\{\bar{Y}_{iT} - \bar{Y}_{jT} > \frac{dq_3^\alpha}{C}\left(1 + \frac{\eta_1}{T}\right) \ \forall (i,j) \in \Omega(d)\right\} \\ &= E\left[G\left\{\left(C - q_3^\alpha\left(1 + \frac{\eta_1}{T}\right)\right)\frac{\sqrt{T}}{\sqrt{n_0}}\right\}\right], \end{aligned}$$

where

$$G(x) = P\{Z_3 - Z_2 > -x, \quad Z_2 - Z_1 > -x\}.$$

It therefore follows from Theorem 2.3 with $H(x) = G(x), C_0 = C, C_1 = q_3^\alpha$ and $C_2 = \eta_1$ that

$$\begin{aligned} \hat{\gamma}^{**}(d) &= \gamma - \frac{1}{n_0} G'(r_\gamma)\left(\eta_1 q_3^\alpha - \frac{r_\gamma}{2}\left(\rho + \xi_1 - \frac{2}{3}\right) + \frac{r_\gamma}{12}\right) \\ &\quad + \frac{1}{12 n_0} r_\gamma^2 G''(r_\gamma) + o\left(\frac{1}{n_0}\right) \quad \text{as } n_0 \to \infty. \end{aligned} \tag{5.24}$$

Note that

$$G'(r_\gamma) = \int_{-\infty}^{\infty} \phi(y)\Big\{\phi(r_\gamma - y)\Phi(r_\gamma + y) + \phi(r_\gamma + y)\Phi(r_\gamma - y)\Big\}\, dy$$

and

$$\begin{aligned} G''(r_\gamma) &= \int_{-\infty}^{\infty} \phi(y)\Big\{(y - r_\gamma)\phi(r_\gamma - y)\Phi(r_\gamma + y) \\ &\quad + 2\phi(r_\gamma + y)\phi(r_\gamma - y) - (r_\gamma + y)\phi(r_\gamma + y)\Phi(r_\gamma - y)\Big\}\, dy. \end{aligned}$$

From (5.23) and (5.24), we set the values of ξ_1 and η_1 satisfying simultaneously

$$\xi_1 + 2\eta_1 = -\rho + \frac{2}{3} - \frac{H''(1)}{3H'(1)}$$

$$6 r_\gamma \left(\rho + \xi_1 - \frac{2}{3}\right) G'(r_\gamma) = (12\eta_1 q_3^\alpha + r_\gamma) G'(r_\gamma) - r_\gamma^2 G''(r_\gamma),$$

so that the procedure has the type I error rate $\alpha + o(1/n_0)$ and power $\hat{\gamma}^{**}(d) = \gamma + o(1/n_0)$ as $n_0 \to \infty$.

Table 5.15 presents the values of ξ_1 and η_1 for given values of α, γ and k and Table 5.16 shows the values of r_γ.

The expected sample size from each population of this sequential procedure is given by

$$E(T) = n_0 + \rho + \xi_1 - \frac{2}{3} + o(1) \quad \text{as } n_0 \to \infty,$$

which follows directly from Theorem 2.1. A simulation exercise has been carried out to assess the performance of this procedure for small and moderate values of n_0. Table 5.17 presents the simulated and approximate values of $E(T)$. For $m = 10, k = 3$ and $\alpha = 0.1, 0.05$, Table 5.18 shows the simulation results of (1$-$ type I error rate) and $\hat{\gamma}^{**}(d)$.

Table 5.15: *Values of ξ_1 and η_1*

for $k = 3$ and given values of γ and α

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
α	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1	ξ_1	η_1
0.05	0.128	0.002	0.129	0.001	0.131	0.001	0.132	0.000
0.1	0.127	0.002	0.129	0.001	0.130	0.001	0.131	0.000

Table 5.16: *Values of r_γ*

for $k = 3$ and given values of γ

$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$
1.167	1.456	1.809	2.326

Table 5.17: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 3$, $\alpha = 0.05$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.3	15.2	15.3	15.2	15.3	15.2	15.3	15.2
20	20.1	20.2	20.2	20.2	20.1	20.2	20.1	20.2
25	25.0	25.2	25.0	25.2	25.0	25.2	25.0	25.2
30	30.0	30.2	30.0	30.2	30.0	30.2	30.0	30.2
35	35.0	35.2	35.0	35.2	35.0	35.2	35.0	35.2
40	40.0	40.2	40.0	40.2	40.0	40.2	40.0	40.2
45	45.1	45.2	45.0	45.2	45.0	45.2	45.0	45.2
50	50.1	50.2	50.1	50.2	50.0	50.2	50.0	50.2
55	55.2	55.2	55.2	55.2	55.2	55.2	55.2	55.2
60	60.2	60.2	60.2	60.2	60.2	60.2	60.2	60.2

Table 5.17: *Comparisons between the simulated and approximate values of $E(T)$ for $m = 10$, $k = 3$, $\alpha = 0.1$ and given values of n_0 and γ*

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.	Simul.	Appro.
15	15.3	15.2	15.3	15.2	15.3	15.2	15.3	15.2
20	20.1	20.2	20.1	20.2	20.1	20.2	20.1	20.2
25	25.0	25.2	25.0	25.2	25.0	25.2	25.0	25.2
30	30.0	30.2	30.0	30.2	30.0	30.2	30.0	30.2
35	35.0	35.2	35.0	35.2	35.0	35.2	35.0	35.2
40	40.0	40.2	40.0	40.2	40.0	40.2	40.0	40.2
45	45.1	45.2	45.1	45.2	45.0	45.2	45.0	45.2
50	50.1	50.2	50.1	50.2	50.1	50.2	50.0	50.2
55	55.2	55.2	55.2	55.2	55.2	55.2	55.2	55.2
60	60.2	60.2	60.2	60.2	60.2	60.2	60.2	60.2

Table 5.18: *Simulation values of* $\alpha^c = (1-$ *type I error rate) and* $\gamma^{**}(d)$

for $m = 10$, $k = 3$, $\alpha = 0.1$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}^{**}(d)$	α^c	$\hat{\gamma}^{**}(d)$	α^c	$\hat{\gamma}^{**}(d)$	α^c	$\hat{\gamma}^{**}(d)$	α^c
15	0.607	0.895	0.704	0.895	0.805	0.895	0.907	0.895
20	0.599	0.890	0.699	0.890	0.791	0.889	0.894	0.890
25	0.600	0.888	0.696	0.888	0.801	0.888	0.905	0.889
30	0.592	0.888	0.692	0.886	0.793	0.887	0.894	0.887
35	0.605	0.893	0.703	0.893	0.797	0.893	0.902	0.892
40	0.602	0.896	0.695	0.895	0.798	0.896	0.896	0.896
45	0.611	0.893	0.707	0.893	0.808	0.893	0.901	0.893
50	0.598	0.896	0.698	0.897	0.794	0.897	0.891	0.896
55	0.594	0.902	0.698	0.901	0.795	0.902	0.901	0.903
60	0.609	0.899	0.707	0.897	0.803	0.897	0.901	0.897

Table 5.18: *Simulation values of* $\alpha^c = (1-$ *type I error rate) and* $\gamma^{**}(d)$

for $m = 10$, $k = 3$, $\alpha = 0.05$ *and given values of* n_0 *and* γ

	$\gamma = 0.6$		$\gamma = 0.7$		$\gamma = 0.8$		$\gamma = 0.9$	
n_0	$\hat{\gamma}^{**}(d)$	α^c	$\hat{\gamma}^{**}(d)$	α^c	$\hat{\gamma}^{**}(d)$	α^c	$\hat{\gamma}^{**}(d)$	α^c
15	0.606	0.945	0.704	0.944	0.805	0.945	0.907	0.945
20	0.599	0.941	0.699	0.941	0.791	0.941	0.894	0.941
25	0.601	0.941	0.696	0.941	0.801	0.941	0.905	0.941
30	0.592	0.945	0.692	0.942	0.793	0.942	0.894	0.942
35	0.605	0.945	0.703	0.944	0.797	0.943	0.902	0.943
40	0.603	0.947	0.695	0.947	0.798	0.947	0.896	0.947
45	0.612	0.946	0.706	0.946	0.808	0.946	0.901	0.946
50	0.599	0.949	0.698	0.950	0.794	0.949	0.891	0.949
55	0.594	0.948	0.698	0.948	0.795	0.949	0.901	0.949
60	0.609	0.948	0.707	0.947	0.803	0.947	0.901	0.948

Chapter 6

Directions of future research

In this thesis, we have applied Anscombe-Chow-Robbin's pure sequential sampling scheme to some multiple comparison problems. Two obvious directions of further research are to use different sequential sampling schemes and to consider other problems which require prescript accuracy when some nuisance parameters are involved.

6.1 Other sequential sampling schemes

Hall (1981) proposed a triple stage procedure to construct a fixed-width confidence interval of length $2d$ and (nominal) confidence level $1-\alpha$ for the mean of a normal population, where $d > 0$ and $0 < \alpha < 1$ are two given constants. This triple sampling procedure involves only three sampling operations. By sampling in bulk, a considerable saving in time and money can be achieved. It also requires an average sample size which is comparable to the corresponding Anscombe-Chow-Robbin's (ACR) " one-by-one" sampling scheme. Hall's procedure operates as follows. Let m be the initial sample size. Calculate

$$M = \max\{m, \langle c\lambda\hat{\sigma}_m^2\rangle + 1\},$$

where $\lambda = (z_{\alpha/2}/d)^2$ and $c \in (0,1)$. If $M = m$, we do not take any more sample, otherwise, if $M > m$, we draw a second sample of size $M - m$, and calculate $\hat{\sigma}_M^2$. Now based on M observations we define

$$T = \max\{M, \langle \lambda\hat{\sigma}_M^2 + m_1\rangle + 1\}$$

where $m_1 = (5 - z_{\alpha/2}^2 - c)/2c$, and draw a sample of size $T - M$. Let $\bar{Y}_T$ be the mean of the pooled sample of size T. Then an approximate $(1-\alpha)$-level confidence interval for μ is given by

$$I_T = (\bar{Y}_T - d,\ \bar{Y}_T + d).$$

Hall showed that

$$P\{|\bar{Y}_T - \mu| < d\} = 1 - \alpha + o(d^2),$$

$$E(T) = n_0 + (1 + z_{\alpha/2}^2)/2c + o(1) \quad \text{as } n_0 \to \infty,$$

where $n_0 = \lambda\sigma^2$.

Liu (1995b) generalized Hall's three-stage procedure to the general $k(\geq 3)$-stage procedure.

Hall (1983) proposed another sequential procedure which uses an ACR procedure only to determine a preliminary sample and then jump ahead to obtain the final sample. After taking the initial sample of size m, it takes observations one by one until

$$N_1 = \inf\{n \geq m : \quad n > c\lambda\hat{\sigma}_n^2\},$$

where $\lambda = (z_{\alpha/2}/d)^2$ and $c \in (0,1)$. Then draw a final sample of size

$$M_1 = \max\{N_1, \langle c\lambda\hat{\sigma}_{N_1}^2 + m_2\rangle + 1\},$$

where $m_2 = (5 + z_{\alpha/2}^2)/(2c) + \beta$ for any $\beta > 0$, and a confidence interval for μ is defined as

$$I_{M_1} = (\bar{Y}_{M_1} - d,\ \bar{Y}_{M_1} + d).$$

It has been shown in Hall (1983) that I_{M_1} has a confidence level greater than $(1-\alpha)$ for all sufficiently small d and

$$E(M_1) = n_0 + (1 + z_{\alpha/2}^2)/2c + \beta + o(1) \quad \text{as } n_0 \to \infty,$$

where $n_0 = \lambda\sigma^2$.

In contrast to Hall's (1983) procedure, Liu (1995c) proposed a new procedure which starts with two samples followed by pure sequential sampling. Take a " pilot" sample of size m. Fix c in the range $0 < c < 1$ and take second sample of size $M_1 - m$ where

$$M_1 = \max\{m, \langle c\lambda\hat{\sigma}_m^2\rangle + 1\}.$$

Continue sampling one observation at a time until

$$M_2 = \inf\{n \geq M_1 : \quad n > \lambda l_n\hat{\sigma}_n^2\}.$$

The confidence interval for μ is given by

$$I_{M_2} = (\bar{Y}_{M_2} - d,\ \bar{Y}_{M_2} + d).$$

The motivation behind this new procedure is that when we are far away from the target we can leap forward by taking clusters of observations, and when we are getting closer to the target we should approach with care by taking one observation at a time. The new procedure not only inherits the great efficiency of the ACR procedure in that it has the same large sample property as the ACR procedure, but also has the ability to reduce the number of sampling operations by an arbitrary factor (which is about $1 - c$). Under the assumptions as in Hall (1981), it has been shown that

$$E(M_2) = n_0 + \rho + l_0 - 2 + o(1), \quad \text{as } n_0 \to \infty,$$

$$P\{\mu \in I_{M_2}\} = 1 - \alpha + \frac{1}{n_0}\{z^2\phi'(z^2)(\rho + l_0 - 2) + z^4\phi''(z^2)\} + o(\frac{1}{n_0}),$$

where $\phi(x) = 2\Phi(\sqrt{x}) - 1$ and $n_0 = \lambda\sigma^2$.

All these sequential sampling ideas can be used to replace the pure sequential sampling idea to solve the problems considered in this thesis. It would be interesting to compare the performance of these procedures.

6.2 Other problems

The basic idea behind sequential sampling is to achieve a prescribed accuracy, e.g. fixed-width confidence interval, fixed type I and type II error of a test, when some nuisance parameters are involved, such as the unknown σ^2 when we want to make inference about μ of a normal population $N(\mu, \sigma^2)$. There are many such problems, and most of these problems have been solved only by using the pure sequential sampling scheme and the two-stage sampling method. Applying the new sequential sampling schemes, such as Hall's three-stage scheme and Liu's (1995b) scheme, to solve these problems is certainly worthwhile and requires a lot of research.

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Appendix

Some definitions and theorems in probability theory

Definition A.1 : *A sequence of random variables* $\{\xi_n, n \geq 1\}$ *is said to be* **uniformly continuous in probability** , *abbreviated u.c.i.p., if and only if for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *for which*

$$P\left\{\max_{0\leq k\leq n\delta} |\xi_{n+k} - \xi_n| > \varepsilon\right\} < \varepsilon \text{ for all } n \geq 1. \tag{A.1}$$

Definition A.2 : *A sequence of random variables* $\{\xi_n, n \geq 1\}$ *is said to be* **stochastically bounded** *if and only if for every* $\varepsilon > 0$ *there is a number* $C > 0$ *for which*

$$P\{|\xi_n| > C\} < \varepsilon \quad \text{for all } n \geq 1.$$

Note that, if ξ_n converges in distribution, then $\{\xi_n, n \geq 1\}$ is stochastically bounded.

Lemma A.1 : *If* $\{X_n, n \geq 1\}$ *and* $\{Y_n, n \geq 1\}$ *are u.c.i.p., then so is* $\{X_n + Y_n, n \geq 1\}$. *If in addition* $\{X_n, n \geq 1\}$ *and* $\{Y_n, n \geq 1\}$ *are stochastically bounded, and* f *is any continuous function on* R^2, *then* $\{f(X_n, Y_n), n \geq 1\}$ *is u.c.i.p. (see Woodroofe ,1982, page 10).*

Lemma A.2 : *If* $X_n \to C$ *w.p.1, then* $\{X_n, n \geq 1\}$ *is u.c.i.p..*

Proof: Suppose that $X_n \to C$ w.p.1, then $X_n - C \to 0$ w.p.1. By Lemma A.13 $\sup_{m>n} |X_m - C| \to 0$ in probability as $n \to \infty$. Therefore, for given $\varepsilon > 0$, there exist N_0, such that for all $n \geq N_0$

$$P\Big\{\sup_{m>n} |X_m - C| > \frac{\varepsilon}{2}\Big\} < \frac{\varepsilon}{2}.$$

Now

$$\begin{aligned} P\Big\{\sup_{m>n} |X_m - X_n| > \varepsilon\Big\} &\leq P\Big\{\sup_{m>n} |X_m - C| > \frac{\varepsilon}{2}\Big\} \\ &\quad + P\Big\{\sup_{m>n} |C - X_n| > \frac{\varepsilon}{2}\Big\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Also note

$$\Big\{\omega : \sup_{0\leq k\leq n\delta} |X_{n+k} - X_n| > \varepsilon\Big\} \subseteq \Big\{\omega : \sup_{m>n} |X_m - X_n| > \varepsilon\Big\},$$

and so

$$P\Big\{\sup_{0\leq k\leq n\delta} |X_{n+k} - X_n| > \varepsilon\Big\} < P\Big\{\sup_{m>n} |X_m - X_n| > \varepsilon\Big\} < \varepsilon.$$

Therefore, if $n \geq N_0$, (A.1) is correct for all $\delta > 0$. If $1 \leq n < N_0$, (A.1) is correct for all $\delta < 1/N_0 + 1$ since the probability in (A.1) is zero. So, (A.1) holds for $\delta < 1/N_0 + 1$ and $n \geq 1$.

Lemma A.3 : *If $X_1, X_2, \cdots$ are i.i.d. with finite mean μ and finite positive variance σ^2, then*

$$S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \quad n \geq 1,$$

is u.c.i.p. (see Woodroofe ,1982, page 11).

Theorem A.1 : *(Von Bahr's Theorem) Let $X_1, X_2, \cdots$ be i.i.d. random variables with finite mean μ, finite positive σ^2, and finite α^{th} absolute moment $E|X_1|^\alpha < \infty$, then*

$$E|S_n^*|^\alpha \to 2^{\alpha/2}\frac{\Gamma(1/2 + \alpha/2)}{\sqrt{\pi}} \qquad \alpha > 2,$$

where $S_n^* = (S_n - n\mu)/\sigma\sqrt{n}$, *and* $S_n = X_1 + X_2 + \cdots + X_n$ *(see Woodroofe ,1982, page 12).*

The following theorem follows directly from the submartingale inequality (see Woodroofe ,1982, page 8).

Theorem A.2 : *Let* $X_1, X_2, \cdots$ *be independent random variables for which* $E(X_i) = 0$ *and* $E|X_i|^\alpha < \infty$ *for* $i \geq 1$, *where* $\alpha > 1$. *Then*

$$P\{\max_{k\leq n} |S_k| > y\} \leq \frac{1}{y^\alpha} \int_{\max_{k\leq n} |S_k|>y} |S_n|^\alpha dP$$

for all $y > 1$ *and* $n \geq 1$.

Definition A.3 : *A sequence of random variables* $\{\xi_n, n \geq 1\}$ *is said to be* **slowly changing** *if and only if*

$$\frac{1}{n}\max\{|\xi_1|, |\xi_2|, \cdots, |\xi_n|\} \to 0 \text{ in probability as } n \to \infty, \tag{A.2}$$

and $\{\xi_n, n \geq 1\}$ *is uniform continuous in probability.*

Lemma A.4 : *(A.2) holds if* $\xi_n/n \to 0$ *w.p.1 as* $n \to \infty$.

Proof: Suppose that $\xi_n/n \to 0$ w.p.1 as $n \to \infty$, we want to show (A.2) holds. Note that

$$\begin{aligned}
&\frac{1}{n}\max\{|\xi_1|, |\xi_2|, \cdots, |\xi_n|\} \\
&\leq \frac{1}{n}\max\{|\xi_1|, |\xi_2|, \cdots, |\xi_{j_0-1}|\} + \frac{1}{n}\max\{|\xi_{j_0}|, \cdots, |\xi_n|\} \\
&\leq \frac{1}{n}\max\{|\xi_1|, |\xi_2|, \cdots, |\xi_{j_0-1}|\} + \max\left\{\frac{|\xi_{j_0}|}{j_0}, \frac{|\xi_{j_0+1}|}{j_0+1}, \cdots, \frac{|\xi_n|}{n}\right\}
\end{aligned}$$

for all $1 < j_0 \leq n$, and $(1/n)\max\{|\xi_1|, \cdots, |\xi_{j_0-1}|\} \to 0$ in probability for each fixed j_0 as $n \to \infty$. By Lemma A.13, $\xi_n/n \to 0$ w.p.1 implies that $\max_{n>j} |\xi_n|/n \to 0$ in probability as $j \to \infty$, and so for given $\varepsilon > 0$, there exist j_0, such that

$$P\{\max_{n\geq j_0} |\xi_n|/n > \frac{\varepsilon}{2}\} < \frac{\varepsilon}{2}.$$

Consequently

$$P\Big\{\max\Big\{\frac{|\xi_{j_0}|}{j_0},\frac{|\xi_{j_0+1}|}{j_{0+1}},\cdots,\frac{|\xi_n|}{n}\Big\}>\frac{\varepsilon}{2}\Big\}\le P\Big\{\max_{n\ge j_0}\frac{|\xi_n|}{n}>\frac{\varepsilon}{2}\Big\}<\frac{\varepsilon}{2}$$

and

$$\begin{aligned}
&P\Big\{\frac{1}{n}\max\{|\xi_1|,|\xi_2|,\cdots,|\xi_n|\}>\varepsilon\Big\}\\
&\le P\Big\{\frac{1}{n}\max\{|\xi_1|,|\xi_2|,\cdots,|\xi_{j_0-1}|\}>\frac{\varepsilon}{2}\Big\}+P\Big\{\max\Big\{\frac{|\xi_{j_0}|}{j_0},\frac{|\xi_{j_0+1}|}{j_0+1},\cdots,\frac{|\xi_n|}{n}\Big\}>\frac{\varepsilon}{2}\Big\}\\
&<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \text{as } n\to\infty.
\end{aligned}$$

This completes the proof.

Definition A.4 : *A sequence of random variables $\{X_n,\ n\ge 1\}$ is said to be* **uniform integrable** , *abbreviated u.i., if for every $\varepsilon>0$ there is a $\delta>0$ such that*

$$\sup_{n\ge 1}\int_A |X_n|\ dP<\varepsilon,$$

whenever $P\{A\}<\delta$ and, in addition,

$$\sup_{n\ge 1} E|X_n|<\infty.$$

The following result is well known (see Chow and Teicher ,1978, page 93).

Lemma A.5 : *A sequence of random variables $\{X_n,\ n\ge 1\}$ is u.i. if and only if*

$$\lim_{a\to\infty}\sup_{n\ge 1}\int_{(|X_n|>a)}|X_n|\ dP=0.$$

The next lemma is often useful in establishing the u.i., and taken from Woodroofe (1982).

Lemma A.6 : *Let $\{X_n, n\ge 1\}$ be random variables and*

$$G(x)=\sup_{n\ge 1}P\{|X_n|>x\},\qquad x>0.$$

If $r>0$ and $x^{r-1}G(x)$ is integrable with respect to Lebesgue measure over (A_0,∞) where $A_0>0$ is a given constant, then $\{|X_n|^r, n\ge 1\}$ is u.i..

The next result follows easily from the definition of u.i. and the Cauchy-Schwarz inequality.

Lemma A.7 *:*

I $\{X_n, n \geq 1\}$ *is u.i. if* $\{X_n^2, n \geq 1\}$ *is u.i..*

II $\{X_n Y_n, n \geq 1\}$ *is u.i. if* $\{X_n^2, n \geq 1\}$ *and* $\{Y_n^2, n \geq 1\}$ *are u.i..*

Let $\{\xi_n, n \geq 1\}$ denote random variables for which $(X_1, \xi_1), \cdots, (X_n, \xi_n)$ are independent of $\{X_k, k > n\}$ for every $n \geq 1$, where $X_1, X_2, \cdots$ are i.i.d. random variables with $\mu = E(X_1)$. Let Ω is the sample space, $\Re_0 = \{\phi, \Omega\}$ and $\Re_n = \sigma\{(X_k, \xi_k); k \leq n\}, n \geq 1$. Suppose that there are $\Re_n$ measurable events A_n, $n \geq 1$, constants h_n, $n \geq 1$, and $\Re_n$ measurable random variables V_n, $n \geq 1$, such that

$$\sum_{n=1}^{\infty} P\left(\cup_{k=n}^{\infty} A_k^c\right) < \infty, \tag{A.3}$$

$$\xi_n = h_n + V_n \text{ on } A_n, \; n \geq 1, \tag{A.4}$$

$$\sup_{n \geq 1} \max_{0 \leq k \leq n\delta} |h_{n+k} - h_n| \to 0 \text{ as } \delta \to 0, \tag{A.5}$$

$$\max_{0 \leq k \leq n} |V_{n+k}|, n \geq 1, \text{ are uniformly integrable,} \tag{A.6}$$

$$\sum_{n=1}^{\infty} P\{V_n \leq -n\varepsilon\} < \infty \text{ for some } \varepsilon, \; 0 < \varepsilon < \mu, \tag{A.7}$$

$$V_n \text{ converges in distribution to a random variable } V, \tag{A.8}$$

$$P\{t \leq \varepsilon N_a\} = o\left(\frac{1}{N_a}\right), \text{ as } a \to \infty, \forall \varepsilon > 0, \tag{A.9}$$

where $N_a = \langle a/\mu \rangle$, $a \geq 0$ and t is defined in (A.10).

Let F be the common distribution of i.i.d. random variables X_i with $E(X_1) = \mu$, $0 < \mu < \infty$ and $S_n = X_1 + X_2 + \cdots + X_n, n \geq 1$, denotes the partial sums. Next, let

$$Z_n = S_n + \xi_n, \quad n \geq 1$$

and

$$t = \inf\{n \geq 1 : Z_n > a\}. \tag{A.10}$$

Theorem A.3 : *Suppose that F has a finite positive variance σ^2 and a finite positive mean μ, and also that conditions (A.3- A.9) hold and $V_n, n \geq 1$, are slowly changing. If F is nonarithmetic, then*

$$E(t) = \frac{1}{\mu}\left(a + \rho - h_{N_a} - E(V)\right) + o(1) \ \ \textit{as } a \to \infty,$$

where

$$\rho = \frac{\mu^2 + \sigma^2}{2\mu} - \sum_{k=1}^{\infty} \frac{1}{k} E\left(S_k^-\right),$$

and S_k^- denotes the negative part of S_k. (See Woodroofe ,1982, page 48).

Lemma A.8 : *Let $\xi_n/n \to 0$ w.p.1 as $n \to \infty$, and $t = \inf\{n \geq 1 : Z_n > a\}$, then*

$$\frac{t}{a} \to \frac{1}{\mu} \ \ \textit{w.p.1 as } a \to \infty.$$

(See Woodroofe ,1982, page 42).

Lemma A.9 : *Suppose that $X_1, X_2, \cdots$ are i.i.d. with $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$ and $t/a \to c$, $0 < c < \infty$, in probability as $a \to \infty$, then*

$$S_t^{\#} = \frac{S_t - t\mu}{\sigma\sqrt{ac}} \xrightarrow{D} N(0,1) \ \ \textit{as } a \to \infty.$$

(See Woodroofe ,1982, page 12).

Lemma A.10 : *If $\{X_n,\ n \geq 1\}$ is a sequence of i.i.d. random variables with finite variance. Then $X_n^2/n \to 0$ in probability as $n \to \infty$.*

Proof: Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left\{\frac{X_n^2}{n} > \varepsilon\right\} \\
&= \sum_{n=1}^{\infty} P\{\frac{X_1^2}{\varepsilon} > n\} \\
&\leq E\frac{X_1^2}{\varepsilon} < \infty,
\end{aligned}$$

the last inequality follows from the well known inequality (see Chow and Teicher ,1978, page 89)

$$\sum_{n=1}^{\infty} P\{|X| \geq n^{1/r}\} \leq E|X|^r \leq \sum_{n=0}^{\infty} P\{|X| \geq n^{1/r}\}.$$

Therefore $P\{X_n^2/n > \varepsilon\} \to 0$ as $n \to \infty$ and so $X_n^2/n \xrightarrow{P} 0$ as $n \to \infty$.

Lemma A.11 : *Suppose that F has a finite variance σ^2, and $\xi_n/\sqrt{n} \to 0$ in probability as $n \to \infty$. Then*

$$t^* = \frac{t - N_a}{\sqrt{N_a}} \xrightarrow{D} N(0, \mu^{-2}\sigma^2) \quad \text{as } a \to \infty,$$

where $N_a = \langle a/\mu \rangle$. (See Woodroofe ,1982, page 42).

Lemma A.12 : *If $\{X_n, n \geq 1\}$ is u.i. and X_n converge in distribution to a random variable X, then $E|X| < \infty$ and $E(X_n) \to E(X)$. (See Woodroofe ,1982, page 12).*

Lemma A.13 : *$X_n \to X$ w.p.1 iff $\sup_{j>n} |X_j - X| \xrightarrow{P} 0$. (See Chow and Teicher ,1978, page 66).*

Definition A.5 : *A sequence $\{X_n, n \geq 1\}$ of ℓ_p random variables (i.e. $E|X_n|^p < \infty$) is said to* **converge** *in* **mean** *of* **order** *p (to a random variable X) if $E|X_n - X|^p \to 0$ as $n \to \infty$. This will be denoted by $X_n \xrightarrow{\ell_p} X$.*

Lemma A.14 : *If $X_n, n \geq 1$, are ℓ_p random variables and $X_n \xrightarrow{\ell_p} X$, then $\{|X_n|^p, n \geq 1\}$ is u.i.. (See Chow and Teicher ,1978, page 98).*

Definition A.6 :*Type I error rate is defined as the probability of at least one Type I error.*

Theorem A.4 *(Slepian's inequality). Let $X = (X_1, X_2, \cdots, X_k)'$ be distributed according to $N(0, \Sigma)$, where Σ is a correlation matrix. Let $R = (\rho_{ij})$, $T = (\pi_{ij})$*

be two positive semidefinite correlation matrices. If $\rho_{ij} \geq \pi_{ij}$ *holds for all* i, j*,*

then

$$P_{\Sigma=R}\Big[\bigcap_{i=1}^{k}\{X_i \leq a_i\}\Big] \geq P_{\Sigma=T}\Big[\bigcap_{i=1}^{k}\{X_i \leq a_i\}\Big],$$

and

$$P_{\Sigma=R}\Big[\bigcap_{i=1}^{k}\{X_i \geq a_i\}\Big] \geq P_{\Sigma=T}\Big[\bigcap_{i=1}^{k}\{X_i \geq a_i\}\Big].$$

(See Tong ,1980, pages 10 and 11).