A Method for Calculating Spectra in Light Nuclei

by

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Abstract

Harmonic oscillator well eigenfunctions, classified according to the group SU_3, are used as a set of zero order approximations to nuclear states. A selection rule is used to put the representations of SU_3 in their order of importance.

A method is developed for calculating energy levels, with these states, by a variational procedure using a central force only.

The appearance of rotational features for even nuclei in the ds-shell is discussed, using only the states of the leading representation, and the low lying energy levels of O^{18}, Ne^{20} and Mg^{24} are calculated. The agreement with the $K = 0$ bands is good but the $K = 0$ and $K = 2$ band spacing in Mg^{24} is too small for any exchange mixture.

The mixing of bands in Mg^{24}, for Rosenfeld exchange, is of the right kind to give almost the correct branching ratio for E2 transitions from the second 2^+ state.

Mixing of representations is small and the changes in the spectrum from the leading representation only slight.

Initial calculations are given on the use of the deformed, harmonic oscillater well eigenfunctions as a set of zero order states. It is found that there is very little difference between the deformation parameters which minimise the deformed harmonic oscillator potential and the Gaussian two body potential.

Contents

Introduction

The states in which the nucleus may exist are given, in the quantum mechanical approach to the nuclear problem, as the eigenfunctions in the Schrödinger equation

$$H\psi = E\psi \tag{1}$$

where H is the nuclear Hamiltonian operator. The energy in the state Ψ is given by the eigenvalue E. The solution of the Schrödinger equation is doubly difficult. In the first place, even if our knowledge of nuclear forces was so complete that we could construct the true Hamiltonian, the resulting many-body problem could not be solved by present mathematical techniques. The task, therefore, is to reduce the insoluble problem to a soluble one while introducing as little error as possible.

In the usual appraoch to such physical problems, models are constructed which approximate to the actual system in some way and are mathematically tractable. Unlike the atomic problem, no single nuclear model has yet been proposed which will predict all the known nuclear properties although the many models already put forward do partially solve the problem.(Mosz.kowski 1957)

The aim today then, is to try to construct a 'unified' model which retains the good points of the old models but rejects their bad ones.

In this thesis, certain ideas connected with an attempt to build up such a unified model are tested. This model was proposed by J.P. Elliott (1958 I & II) and is based on the Independant Particle Model. The significant part about it is that the many-particle wave functions are classified according to the irreducible representations of the unimodular unitary group SU_3. This classification allows states to be associated in a way resembling rotational bands and the construction of the states involves an intrinsic state which may be related to that in the rotational model.

The physical importance of this scheme has already been shown by Elliott. States classified according to the SU_3-group have been favourably compared with those arising from a variational calculation in the nuclear ds-shell using a realistic Hamiltonian.

We shall here develop a method to set up directly the energy matrix spread out by the states of this new model. The method will be applied to nuclei in the nuclear ds-shell and, in particular, to the Mg^{24} nucleus which is treated in greater detail. For these latter reasons, the theory is developed with respect to the ds-shell but it can easily be extended to other shells.

In Chapter 1 a Hamiltonian is defined which is realistic as far as our rather qualitative arguments later on are concerned. The assumptions associated with the model, in the ds-shell, are set out and a brief account of the well-known classification according to the permutation group is recorded.

Chapters 2 and 3 deal with the SU_3-group in some detail. The theory is developed using a simpler set of operators of the group than those suggested by Elliott. The raising and lowering operators in Chapter 3 will be seen to be very important in our method for calculating matrix elements.

The classified, many particle, angular momentum states are given in Chapter 4 as projections out of states classified according to two subgroups of SU_3 - the SU_2 and U_1.

The method for calculating the energy matrix, spread out by these states, is given in Chapter 5.

In Chapter 6 we illustrate the appearance of collective features, as particles are added into the shell, using only the states of the 'leading' representation of SU_3. Approximations to the O^{18}, Ne^{20} and Mg^{24} spectra are shown.

The results of calculations on the mixing of other SU_3-representations, with the 'leading' representation for Mg^{24} are given in Chapter 7.

Some conclusions are drawn after Chapter 7.

At the end of the thesis an addendum has been added. In this we present initial calculations on the use of the eigenfunctions of the axially symmetric, deformed harmonic oscillator as a set of zero order approximations to the true nulcear states. Although these calculations are far from complete, it is felt that at least one of the results so far obtained, is worth recording.

Chapter 1

Assumptions of the Model and the Nuclear Hamiltonian

Assumptions

In the shell model (Elliott and Lane, 1957) the nucleus is assumed to consist of A particles which, in the first approximation (for $A < 40$), move in the central potential of the harmonic oscillator. The nuclear particles all have intrinsic spin $\frac{1}{2}$ and may either be protons or neutrons. The protons and neutrons are assumed to be different states in isotopic spin (charge) space of a hypothetical particle, the nucleon (Rose 1957, p.217). Operators $t_\pm$ and t_o, describing the isotopic spin space, are defined analagously to the operators $s_\pm = \frac{\hbar}{2}\sigma_\pm$ and $s_o = \frac{\hbar}{2}\sigma_\zeta$ of the intrinsic spin space, where $\sigma_{\pm 1}$ and σ_o are the Pauli spin matrices (Schiff 1949). The convention will be used that, for a nucleon state ψ,

$$t_o\psi = \pm \tfrac{1}{2}\psi \qquad \underline{1{\cdot}1}$$

depending on whether ψ is a neutron or proton respectively.

The actual states of the nucleus, in the zero order approximation, are formed by totally antisymmetrising products of A single-particle oscillator states. This ensures that the Pauli Exclusion Principle is obeyed. The low lying energy states are constructed by completely filling the lowest oscillator shells with A-k nucleons and then putting the remaining k-particles in the next (in general unfilled) shell. Thus for $0 < A \leqslant 4$, $4 < A \leqslant 16$ and $16 < A \leqslant 40$, the unfilled shells are the 1s, 1p and (2s, 1d) respectively. In general there are many such low lying states which can be constructed, all being degenerate in energy for the oscillator potential. With such an approximation, the "magic" numbers at 4 and 16 (Elliott and Lane 1957, p. 261) can be associated with the completely filled oscillator orbits 1s and 1p. Indeed, if a single particle spin orbit force is added (see 1.2a below) the magic numbers throughout the whole periodic table can be associated with the closure of shells (Mayer 1949).

As the next approximation to the actual nuclear states, the lowest degenerate oscillator states, which are finite in number, are used to spread out a matrix representation of the realistic Hamiltonian. On diagonalising this energy matrix, the eigenfunctions and eigenvalues correspond to a first order approximation to the states and energy levels respectively. For more than a few particles in the shell, such a program is prohibitive because of the large number of possible degenerate oscillator well states. The problem in the shell model is to reduce this number of states to reasonable proportions. This implies the discovery of possible classification schemes.

We shall here describe a possible classification scheme for the oscillator states with $16 < A < 40$ (the ds-shell) and develop a method for calculating the energy in the states for a realistic Hamiltonian.

The first excited state of O^{16} is about 6Mev above the ground state. It seems reasonable to suppose that the energy differences, less than 6 Mev, between the low-lying excited states of the A-particle nucleus $(16 < A < 40)$ and the ground state, come solely from the interaction between the k-particles in the ds-shell. The first order approximation to the low-lying energies will thus be found by considering the energy matrix spread out by the k-particle degenerate oscillator states.

The nuclear Hamiltonian

Nuclear models do not rely on the detailed form of the Hamiltonian for their construction, but they do stem from some of the general features which this operator is supposed to possess. In the rotational model, for example, the part of the Hamiltonian coupling the intrinsic motion to the collective one is assumed small - the adiabatic assumption. Of course, on testing the validity of a model some realistic Hamiltonian must be used.

We quote then the effective Hamiltonian (Elliott and Lane, 1957, p.336) which is believed to be realistic for a shell model, at least for qualitative arguments, in the region $16 < A < 40$.

The Hamiltonian consists of two parts - the kinetic (T) and potential (V) energies. The explicit form for T is known exactly in the A-particle problem:

$$T = \frac{1}{2m} \sum_{i=1}^{A} [p(i)]^2 \qquad \underline{1.1}$$

where p(i) is the momentum of the i^{th} particle and m the mass of the nucleon (assumed constant).

The potential V is not known exactly because of the uncertainty in the form of the nuclear forces. It is known that the repulsive Coulomb force, existing between protons, is small, in this region, compared with the attractive nuclear forces which must exist to form stable nuclei. Calculations in the p-shell (Inglis 1953, Kurath 1956) have shown that a mass of agreement can be found by using a potential of the form

$$V = \sum_{i<j} V_c(ij) + \xi \sum_{i} (S_i L_i) \qquad \underline{1.2a}$$

where

$$V_o(ij) = (W + MP_{ij}^{r} - HP_{ij}^{\tau} + BP_{ij}^{\sigma}) V_o(r_{ij}) \qquad \underline{1.2b}$$

P_{ij}^{r}, P_{ij}^{τ} and P_{ij}^{σ} are the exchange operators in the ordinary, isotopic spin and intrinsic spin spaces. W,M,H and B are exchange constants, usually normalised such that W + M + B + H = 1. $V_o(r_{ij})$ is the radial shape de~~d~~pending on the internucleon distance r_{ij}.

Experience has shown that the properties of low lying states are reasonably insensitive to the radial shape. Essentially the same results can be obtained by using a Yukawa or Gaussian radial dependence. For mathematical simplicity we shall take

$$V_o(r_{ij}) = - V_o e^{-r_{ij}^2/a^2}$$

where V_o is the (positive) strength and a the range of the force.

The spin orbit force $\sum_i (s_i L_i)$ is introduced to account for the doublet splitting for one particle outside a closed shell e.g. H_e^5 and 0^{17}. Such a term, on its own, cannot arise from a nucleon - nucleon interaction however, it is thought that perhaps a two body vector or tensor force will account for the effects produced.
(See Elliott and Lane 1957, p.336 on this point and for references. Also Feingold 1956.)

In this thesis we concentrate on finding the spectrum produced from a Hamiltonian with central force alone. For those nuclear states which have spin $S = 0$, the matrix elements of the spin-orbit potential are zero and the total angular momentum J is equal to the orbital angular momentum L. In such cases we shall thus be able to compare our predictions for the energy levels directly with those found experimentally.

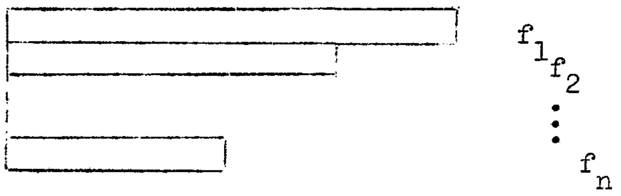
Classification of states according to the permutation group

An irreducible representation of the group of permutations of degree k, S_k, is labelled by a set of numbers $[f] = [f_1 f_2 \dots f_n]$. $[f]$ is a partition of k into n integral parts such that

$$f_1 \geqslant f_2 \cdots \geqslant f_n \quad \text{and} \quad f_1 + f_2 + \dots + f_n = k.$$

(Jahn and van Wieringen (1951))

The set may be represented, diagrammatically, by a Young Tableau with n rows of lengths $f_1 f_2 \dots f_n$ respectively.



A function ψ is said to have symmetry $[f]$ if it belongs to the basis of the representation $[f]$ of S_k. It is usual to describe such a function by $\psi[f]$.

It is well known that totally antisymmetric nuclear states, Ψ, may be formed if orbital functions of symmetry $[f]$ $\{\psi[f]\}$ are combined with charge-spin functions of adjoint symmetry $[\tilde{f}]$ $\{\Gamma[\tilde{f}]\}$ (Bunday 1960, Chapter II). The adjoint representation $[\tilde{f}]$ has the number of rows (columns) in its tableau equal to the number of columns (rows) of the tableau for $[f]$. The antisymmetric state Ψ will be labelled with the $[f]$ of its orbital part.

We notice that since the charge-spin space is of only four dimensions, functions (Γ) in this space cannot be constructed which are antisymmetric in more than four parts. This implies that the representations $[\tilde{f}]$ can have at most four rows i.e. $[f]$ can have at most four columns.

A Hamiltonian whose operators act only in the orbital space (Wigner (W) and Majorana (M) exchanges only), cannot couple states $\Psi[f]$ and $\Psi[f']$ with different $[f]$ labels. This is because functions classified according to S_k, in particular the Γ-states, are orthogonal with respect to the $[f]$-label. Thus, for such a Hamiltonian, the classification according to S_k will certainly reduce the number of zero-order states without loss in accuracy.

Since nuclear forces are attractive, the states of maximum orbital symmetry will be lowest in energy. As we are only interested in the low-lying energy levels, then only those states which have maximum orbital symmetry need be considered.

We shall, in fact, include some Heisenberg (H) and Bartlett (B) exchange in the Hamiltonian but still consider only the states with maximum orbital symmetry. The mixing of states of other symmetries, with the states of maximum orbital symmetry, is expected to be small enough, in a first approximation, to be ignored.

Throughout this thesis the [f]-label will tend to be dropped - in these cases states with maximum orbital symmetry are implied.

Construction of antisymmetric states with maximum orbital symmetry

The k-particle totally antisymmetric states, whose orbital parts have maximum orbital symmetry, may be formed by antisymmetrising between antisymmetric states of four (or less) particles with symmetrical orbital parts.

If a,b,c and d are four different single particle orbital states, a normalised four particle symmetric state will be

$$\psi_{1234}[4] = \sqrt{1/4!}\sum_{P} P(a_1 b_2 c_3 d_4) \equiv (abcd)_{1234} \qquad \underline{1.4a}$$

where $\sum_{P} P$ sums over all permutations of the particle numbers 1,2,3 and 4.

If two particles happen to be in the c-orbital state , we shall have :-

$$\psi_{1234}[4] = \sqrt{2!/4!}\sum_{P}{}' P(a_1 b_2 c_3 c_4) \equiv (abc^2)_{1234} \qquad \underline{1.4b}$$

where $\sum'$ ignores permutations between particles in the same orbital states. The charge-spin functions $\Gamma^{(2T+1)(2S+1)}$ with symmetries [4] [3] and [2] are well known (Jahn and Van Wieringer 1951, Elliott, Hope and Jahn 1953).

We shall write then

$$\overrightarrow{\Gamma}^{11}[4]\ ,\quad \overrightarrow{\Gamma}^{22}[3] \quad \text{and} \quad \overrightarrow{\Gamma}^{13}[2] \quad \text{or} \quad \overrightarrow{\Gamma}^{31}[2] \qquad \underline{1.5}$$

Thus, for example, totally antisymmetric four particle states may be written.

$$\Psi_{1234}[4] = \psi_{1234}[4]\ \Gamma^{11}[\tilde{4}] \qquad \underline{1.6a}$$

Where $\psi_{1234}[4] = (abcd)_{1234}$ etc. We shall adopt the notation

$$\Psi_{1234}^{11}[4] = (abcd)_{1234}^{11} \qquad \text{etc.} \qquad \underline{1.6b}$$

Take now the construction of a k-particle antisymmetric state. If $k = 4p + q$, where $q < 4$, consider

$$\Psi_k^{2T+1,2S+1} = N\sum_P{}'(-)^P P\{\{..\{\Psi_{1234}^{11}[4]\Psi_{5678}^{11}[4]\}^{11}\dots\Psi_{..4p-1,4p}^{11}[4]\}^{11} \times \Psi_{4p+1,..,4p+q}^{2T+1,2S+1}[q]\}^{2T+1,2S+1} \qquad \underline{1.7}$$

where N is a normalisation coefficient. $\sum{}'P$ means the sum over all permutations between the $p + 1$ sets of numbers

$$(1234)(5678)\ \dots\ (..4p + 1,\ 4p)(4p + 1, \dots\ 4p + q)$$

preserving the natural order in the sets. $(-)^{P'} = \pm 1$ depending on whether P is an even or odd permutation. The charge-spin functions are vector coupled (trivially) to a $(2T + 1)(2S + 1)$ state.

The fact that Ψ_k is antisymmetric is clear from the construction.

Because the p-sets of four particles all have symmetry $[4]$ and one set has symmetry $[q]$, Ψ_k cannot have orbital symmetry <u>less</u> than $[44 \ldots 4q]$. The charge-spin space however always prevents Ψ_k having orbital symmetry greater than $[44 \ldots 4q]$. Hence Ψ_k has either orbital symmetry $[44 \ldots 4q]$, i.e. maximum symmetry, or is identically zero.

Further Classification

If the single particle wave functions span a vector space of S-dimensions, the k-particle states classified according to the representation $[f]$ of S_k also spread out an irreducible representation of the group U_s. Here, U_s is the group of all simultaneous unitary transformations among the single particle states. The representations of U_s may then, also be lâbelled with $[f]$. In the nuclear ds-shell, for example, the states $\psi[f]$ spread out the irreducible representation $[f]$ of U_6.

By considering the infinitesimal operators of the group U_s, it can be shown that a state ψ may be classified simultaneously according to the irreducible representations of U_s and its subgroups. Which subgroups we choose for further classification will depend on physical considerations

It has been shown (Elliott 1958 I) that the classification according to the U_3-subgroup and the further reduction to the rotation group R_3, is especially suited for the labelling of states in a degenerate oscillator level. This classification seems to have some physical significance and so we shall use the representation labels of these groups to classify the states $\psi[f]$ further.

In the next chapter we consider the U_3-group in some greater detail and determine the classification of its irreducible representations. The classification of states according to the R_3-group will be left until Chapter 4.

Chapter 2

The U_3 Group

The Group Operators

The oscillator Hamiltonian

$$H_o = (r^2 + b^4p^2)/2b^2 \qquad \underline{2.1}$$

is invariant with respect to the group U_3 described by the nine spherical tensor operators (Elliott 1958 **I** sec. 3).

$$H_o = (r^2 + b^4p^2)/2b^2$$

$$L_q = (r \times p)_q$$

$$Q_q = \sqrt{4\pi/5}\,\{r^2Y_q^2(\theta_r\phi_r) + b^4p^2Y_q^2(\theta_p\phi_p)\}/b^2 \qquad \underline{2.2}$$

Here the L_q are the three infinitesimal rotation operators and the Q_q the five components of a second degree tensor operator. The arguments of the spherical harmonics are the polar angles of space and momentum vectors $\vec{r}$ and $\vec{p}$. In particular

$$Q_o = \{(2z^2 - x^2 - y^2) + b^4(2p_z^2 - p_x^2 - p_y^2)\}/2b^2 \qquad \underline{2.3}$$

The invariance of H_o with respect to U_3 means that it commutes with all the operators of the group. (Jauch and Hill 1940). This implies that the group operators only transform between the degenerate states of H_o. This property can clearly be seen by rewriting the operators in 2.2 in terms of the three creation operators a_i^+ and the three destruction operators a_i of the oscillator quanta in the x,y and z directions where

$$a_x^+ = \sqrt{1/2}\,(x - ib^2 p_x)/b$$

and $$a_x = \sqrt{1/2}\,(x + ib^2 p_x)/b \quad \text{similarly for } y \text{ and } z \qquad \underline{2.4}$$

Thus by defining the operators

$$A_{ij} = (a_i^+ a_j + a_j a_i^+)/2, \; i \text{ and } j = x,\, y \text{ or } z \qquad \underline{2.5}$$

we may deduce the relation (Table I) between these operators A_{ij}, formed from a Cartesian basis, and the spherical tensor operators in 2.2

	Q_{+1}	Q_{-1}	$\sqrt{3}L_{+1}$	$\sqrt{3}L_{-1}$	$\sqrt{3}L_o$	Q_{+2}	Q_{-2}	$\sqrt{2}H_o$	Q_o
$\sqrt{6}A_{xz}$	-1/2	1/2	1/2	1/2					
$\sqrt{6}A_{zx}$	-1/2	+1/2	-1/2	-1/2					
$i\sqrt{6}A_{yz}$	-1/2	-1/2	1/2	-1/2					
$i\sqrt{6}A_{zy}$	-1/2	-1/2	-1/2	+1/2					
$i\sqrt{6}A_{yx}$					$\sqrt{1/2}$	1/2	-1/2		
$i\sqrt{6}A_{xy}$					$-\sqrt{1/2}$	1/2	-1/2		
$\sqrt{6}A_{xx}$						1/2	1/2	$\sqrt{1/3}$	$-\sqrt{1/6}$
$\sqrt{6}A_{yy}$						-1/2	-1/2	$\sqrt{1/3}$	$-\sqrt{1/6}$
$\sqrt{6}A_{zz}$								$\sqrt{1/3}$	$\sqrt{2/3}$

Table I. Transformation Coefficients between the Cartesian operators A_{ij} and the spherical tensor operators.

The many particle operators A_{ij} are defined as the sum of single particle operators $A_{ij}(p)$.

Thus

$$A_{ij} = \sum_{p} A_{ij}(p) \quad \text{summed over all particles} \qquad \underline{2.6}$$

For this reason the commutation properites of the single particle operators will belong also to the many particle operators. Henceforth the symbol A_{ij} will denote the many particle operator unless otherwise stated.

Because the A_{ij} operators are combinations of one destruction with one creation operator, it is clear that they cannot excite a state out of a given degenerate oscillator level. The operators A_{ij} merely shift a quantum from the j-direction into the i-direction.

If s is the number of states in the degenerate oscillator level, then clearly the A_{ij} will be some of the operators of U_s - the set of unitary transformations in s-dimensions.

The Commutation Relations

The commutators of the A_{ij} operators may be deduced from those belonging to the tensor set. (Elliott 1958 I (8)). We shall here find them directly from the simpler relations for the creation and destruction operators.

From the explicit form of the a_i^+ and a_i given in 2.4 we deduce.

$$[a_i^+, a_j^+] = 0 \qquad \underline{2.7a}$$

$$[a_i, a_j] = 0 \qquad \underline{2.7b}$$

$$[a_i^+, a_j] = 0 \quad i \neq j$$

but since $a_x^+ a_x = (x^2 - b^2 - b^4\, \partial^2/\partial x^2)/2b^2$

and $a_x a_x^+ = (x^2 + b^2 - b^4\, \partial^2/\partial x^2)/2b^2$

we may write, in general :-

$$[a_i^+, a_j] = -\delta_{ij} \;. \quad \text{for all } i \text{ and } j \qquad \underline{2.7c}$$

where $\delta_{ij} = 0$ for $i \neq j$

$= 1$ for $i = j$.

The most general commutator for the A_{ij} operators is:

$$[A_{ij}, A_{kl}] = (a_i^+ a_j + a_j a_i^+)(a_k^+ a_l + a_l a_k^+)/4$$

$$- (a_k^+ a_l + a_l a_k^+)(a_i^+ a_j + a_j a_i^+)/4 \qquad \underline{2.8}$$

From equations 2.7 we know that

$$[a_i^+ a_j, a_k^+ a_l] = + \delta_{jk} a_l a_i^+ - \delta_{il} a_j a_k^+$$

$$[a_i^+ a_j, a_l a_k^+] = \delta_{jk} a_l a_i^+ - \delta_{il} a_j a_k^+$$

$$[a_j a_i^+, a_k^+ a_l] = \delta_{jk} a_l a_i^+ - \delta_{il} a_j a_k^+$$

$$[a_j a_i^+, a_l a_k^+] = \delta_{jk} a_l a_i^+ - \delta_{il} a_j a_k^+$$

Thus from 2.8

$$[A_{ij}, A_{kl}] = A_{il} \delta_{jk} - A_{kj} \delta_{il} \qquad 2.9$$

This last relation shows that the set A_{ij} do in fact describe the group U_3.

The commutation relation in 2.9 is very easy to remember and apply. For this reason we prefer to use the set A_{ij}, rather than the tensor set in 2.2, to describe the properties of U_3.

Classification According to the Group U_3

Just as the classification of the group U_S may be described in terms of Young Tableau, with at most s-rows, so may the group U_3 be described by tableau with at most three rows.

The many particle oscillator wave functions may be considered as tensors in the oscillator creation operators. In the p-shell, for example, the single particle wave functions are tensors of degree one

$$\phi = a_i^+ \phi_0 \qquad i = x, y \text{ or } z. \qquad 2.10a$$

where ϕ_0 is the "vacuum" state of no quanta viz. the (1s) state.

In the sd-shell the single particle states are symmetric tensors of degree two

$$\phi = a_i^+ a_j^+ \phi_0 \quad i,j = x,y \text{ or } z \qquad \underline{2.10b}$$

that is, six in number.

The classification of the many particle states with respect to the group U_3 merely describes the tensorial symmetry of the wave functions with respect to the creation operators.

The representations of U_3 are, therefore, described by partitions of n, the numbers of oscillator quanta, into not more than three parts, $[n_1 n_2 n_3]$ where

$$n_1 \geqslant n_2 \geqslant n_3 \qquad \text{and} \qquad n_1 + n_2 + n_3 = n$$

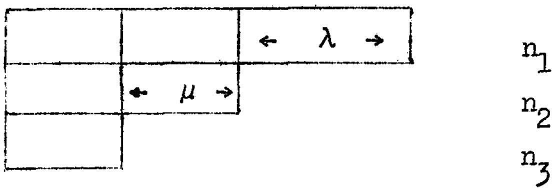
We can define the leading state of the representation [n] as the one having n_1 quanta in the **z-direction and n_2 and n_3** quanta in the x and y directions respectively. This is analagous to saying that, in the rotation group R_3, with angular momentum quantised in the z-direction, the leading state is the one with M = L. The leading state in U_3 has the maximum number of quanta (n_1) in the z direction and the maximum number of quanta in the x direction.where $n_z = n_1$ for the representation $[n_1 n_2 n_3]$. The leading state is then the one with maximum weight (Racah 1951).

The other states of the representation [n] may be generated by some operation with the group operators on the leading state.

The single particle states in the p-shell then spread out the representation [100] of U_3. The two particle states can spread out the representations [200] and [110]. In the latter example the leading state of [200] will have $n_z = 2$ and $n_x = n_y = 0$ and the leading state of [110] will have $n_z = n_x = 1$ and $n_y = 0$ since the single particle in the ds-shell is a symmetric tensor of degree two, it must spread out the representation [200] of U_3.

In physical problems we want to ignore transformations which are simply an overall change of phase. This means dealing with the unimodular unitary group SU_3 obtained by removing the unit infinitesimal operator $H_0 = (A_{xx} + A_{yy} + A_{zz})$. Under this restriction the representations do not reduce further, but those corresponding to tableaux differing only in a number of complete columns become equivalent. Thus the irreducible representations of SU_3 may be labelled by only two numbers $(\lambda\mu)$. These numbers are defined by convention as $\lambda = n_1 - n_2$ and $\mu = n_2 - n_3$. (Elliott 1958 I par. 4.)

The leading state of the $(\lambda\mu)$ representation then has an excess of λ quanta in the z-direction over that in the x.



2.11

The problem of classifying the nuclear states with respect to SU_s and SU_3 involves determining which representions $(\lambda\,\mu)$ occur in a given representation $[f]$. This reduction may be performed, by a chain calculation, in a very simple way using the rules for combining tableau, (Jahn 1950), and the dimension formula,

$$\text{dimension of } [f] \; (D[f]) = \prod_{1\leq i<j\leq s} (f_i - f_j + j - i)/(j-i)$$

for the irreducible representation $[f]$ of U_3 or SU_3 (Weyl 1920 p. 283).

In the ds-shell, for example, each single particle function transforms according to the representation (20) of SU_3. Thus

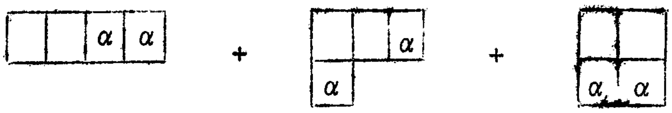
$$[1] \rightarrow (20) \qquad \underline{2.12a}$$

For two particles in the shell, the possible classifications according to SU_S (the permutation symmetry) are given by

$$[1] \times [1] = [2] + [11]$$

Simply by adding <u>one</u> square to the representation $[1]$. The possible representation of SU_3 are found by adding two squares to (20) in a symmetric way.

Thus we get



i.e. $(20) \times (20) = (40) + (21) + (02)$

From the dimension formula

$$D[2] = 21 \qquad D[11] = 15$$

$$\text{and} \quad D(40) = 15 \qquad D(21) = 15 \qquad D(02) = 6$$

The representation (40) of SU_3 means that the four creation operators, describing the two particle state, form totally symmetric tensors of rank four. These tensors can only arise if the two particle states themselves are symmetric. Hence we must have

$$[2] \rightarrow (40) + (02)$$

$$[11] \rightarrow (21)$$

In this way we may build up the complete reduction of any [f] into representations of SU_3 (Elliott 1958 I Table 1).

The Subgroups SU_2 and U_1 of the Group SU_3

The subgroups SU_2 and U_1 do not seem to have any direct physical significance but they will be used later in defining a set of states classified according to the R_3 subgroup. For this reason the classification of states according to SU_2 and U_1 is considered important.

Subgroups of SU_3 may, in general, be found by looking for sets of operators, within the set of SU_3 group operators, which describe a group, i.e. whose commutators are contained within the set (Racah 1951).

The SU_3-group operators can be formed from the U_3 group operators by removing the unit infinitesimal operator. That is by removing H_o or $(A_{xx} + A_{yy} + A_{zz})$
We introduce the operators

$$B_{ij} = A_{ii} - A_{jj} \,.$$

where clearly $B_{ij} = -B_{ji}$, $B_{ii} = 0$ and $B_{ij} + B_{jk} = B_{ik}$
The eight independant operators describing the SU_3 group may thus be written

$$A_{ij},\ i \not\leq j$$

$$B_{ij},\ i,j = x \text{ or } y \text{ or } z. \qquad \underline{2.14}$$

since at most two B_{ij} operators may be written down as being linearly independant.

From the commutation relation in 2.9 we deduce

$$[A_{ij},B_{kl}] = A_{ik}\,\delta_{jk} - A_{kj}\,\delta_{ki} - A_{il}\,\delta_{jl} + A_{lj}\,\delta_{il} \qquad \underline{2.15a}$$

$$[B_{ij},B_{kl}] = 0 \qquad \underline{2.15b}$$

From the commutators in 2.15 with 2.9 we can see that the operators A_{xy}, A_{yx} and B_{xy} form a group, the group SU_2, and all these operators commute with the operator $(B_{zx} + B_{zy})$

which describes the group U_1. Thus the SU_3 group reduces simultaneously to the SU_2 group and the U_1 group i.e.

$$SU_3 \longrightarrow SU_2 \times U_1 \ .$$

Irreducible Representations of the Subgroups SU_2 and U_1

The U_1 group, being of only one dimension, needs but one "number" to label its representations. We shall choose a set of states which diagonalize the operator $(B_{zx} + B_{zy})$ and introduce the quantum number ϵ

Thus

$$(B_{zx} + B_{zy}) \quad \phi(\epsilon) = \epsilon \phi(\epsilon) \qquad \underline{2.16}$$

where we assume now that ϕ implies a normalised state, with permuation symmetry $[f]$, and belonging to the $(\lambda \mu)$ representation of SU_3 (Elliott 1958 II (2)).

The SU_2 group is isomorphic with the R_3 group.

This isomorphism may be seen directly from the operators in 2.14 by defining operators

$$\omega_o = \tfrac{1}{2}B_{xy}, \omega_{+1} = -\sqrt{\tfrac{1}{2}}A_{xy} \quad \text{and} \quad \omega_{-1} = \sqrt{\tfrac{1}{2}}A_{yx} \qquad \underline{2.17}$$

The ω-operators have the commutation relations

$$[\omega_{+1}, \omega_{-1}] = -\omega_o \quad \text{and} \quad [\omega_{\pm 1}, \omega_o] = \mp \omega_{\pm 1} \qquad \underline{2.18}$$

which are just those of the R_3 group.

The classification by the SU_2 group follows in a similar way to the classification by the R_3 group, with a number Λ corresponding to the orbital angular momentum quantum number L.

The Casimir operator of the SU_2 group, corresponding to the operator L^2 of R_3, is

$$\omega^2 = \omega_o^{\,2} - \omega_{+1}\,\omega_{-1} - \omega_{-1}\,\omega_{+1}$$

$$= \tfrac{1}{4}\,B^2_{xy} + \tfrac{1}{2}\,(A_{xy}A_{yx} + A_{yx}A_{xy})$$

We introduce the labelling Λ such that.

$$\omega^2\phi(\epsilon\Lambda) = \Lambda(\Lambda+1)\quad\phi(\epsilon\Lambda)\ \text{(Elliott 1958, II,(7))} \qquad \underline{2.20}$$

The quantum number ϵ is preserved in the operation with ω^2 since $(B_{zx} + B_{zy})$ commutes with all operators of SU_2, in particular with ω^2.

We are free to diagonalize one more operator of SU_2 corresponding to the diagonalization of L_z in R_3. We define a number ν such that

$$B_{xy}\,\phi(\epsilon\Lambda\nu) = 2\ \omega_o\,\phi(\epsilon\Lambda\nu) = \nu\phi(\epsilon\Lambda\nu) \qquad \underline{2.21}$$

Clearly $\frac{1}{2}\nu$ takes on the values $\Lambda, \Lambda-1, \ldots -\Lambda$ by analogy with the R_3 group, that is

$$\nu \text{ takes on the values } 2\Lambda,\ 2\Lambda-2,\ \ldots -2\Lambda. \qquad \underline{2.22}$$

With the exception of the last quantum number 'ν', this classification of states has been given by Elliott (1958, II). In the last stage, Elliott chooses to diagonalize the operator $L_o = i(A_{yx} - A_{xy})$. Thus a set of states $\chi(\epsilon\Lambda K)$ is defined which have the component of angular momentum in the z-direction a good quantum number. The use of the ν-labelling has certain advantages which will be seen in Chapter 4.

So that many of the results obtained by Elliott may be carried over into this paper, the relationship between the sets of states ϕ and χ must be determined. To do this we must first deduce the exact connection between the operators ω_0, $\omega_{\pm 1}$, which we have used to describe SU_2, and the set v_0, $v_\pm$, which Elliott (1958, II) uses to describe the group.

Defining ω_x, ω_y, and ω_z in analogy with the operators L_x,L_y and L_z of R_3 we have:

$$\omega_x = -\sqrt{\tfrac{1}{2}}(\omega_{+1} - \omega_{-1}) = \tfrac{1}{2}(A_{xy} + A_{yx}) = \frac{1}{2i\sqrt{6}}(Q_2 - Q_{-2})$$

$$\omega_y = i\sqrt{\tfrac{1}{2}}(\omega_{+1} + \omega_{-1}) = \tfrac{1}{2}i(-A_{xy} + A_{yx}) = \tfrac{1}{2}L_0$$

$$\omega_z = \omega_0 = \tfrac{1}{2}(A_{xx} - A_{yy}) = \frac{1}{2\sqrt{6}}(Q_2 + Q_{-2})$$

from 2.17 and Table 1.

Elliott describes the SU_2-group by the set of operators

$$v_0 = \tfrac{1}{2}L_0$$

$$\text{and } v_{\pm 1} = \mp\left(\frac{1}{2\sqrt{3}}\right)Q_{\pm 2}.$$

Defining Cartesian components of $\vec{v}$ in the conventional way as above, we deduce:

$$v_{x'} = - \sqrt{\tfrac{1}{2}}(v_{+1} - v_{-1}) = \frac{1}{2\sqrt{6}}(Q_2 + Q_{-2}) \equiv \omega_z .$$

$$v_{y'} = \sqrt{\tfrac{1}{2}}i(v_{+1} + v_{-1}) = \frac{1}{2i\sqrt{6}}(Q_2 - Q_{-2}) \equiv \omega_x$$

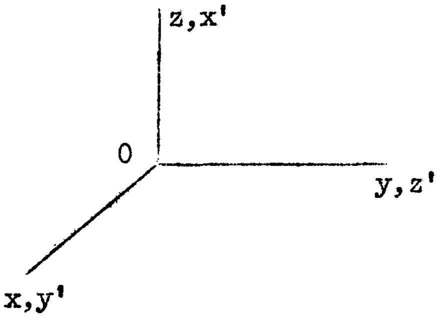
$$v_{z'} = v_0 \qquad = \tfrac{1}{2}L_0 \qquad \equiv \omega_y .$$

The primes have been introduced when discussing $\vec{v}$ since, with these formal operators, the compoents $v_{q'}$ and ω_q were defined quite independently.

Thus, in our language, the set of states $\chi(\Lambda k)$ spread out the representation Λ of SU_2 for which w_y is diagonal. Because of the isomorphism between the SU_2 and R_3-groups, the relationship between the sets $\chi(\Lambda K)$ and $\phi(\Lambda\nu)$ is the same as that between the sets $\psi_y(LM)$ and $\psi_z(LM')$ in this order. Here $\psi_y(LM)\{\psi_z(LM')\}$ belongs to the basis of the representation L of R_3 in which L_y $\{L_z\}$ is diagonal.

We determine now the relationship in the R_3-group.

Consider a set of axes S', which occupy the same positions in space as a set S but where the z'-axis lies along Oy and the x' axis lies along O[illegible].



An Euler angle rotation of $(\pi/2, \pi/2, \pi)$ carries S into S'. Clearly, the set of states $\psi'_{z'}(LM)$, defined in S', which diagonalize $L_{z'}$, are identical with the set $\psi_y(LM)$, defined in S, which diagonalize L_y.

i.e $\psi'_{z'}(LM) \equiv \psi_y(LM)$.

Using the well known transformation properties of angular momentum functions under finite rotations (e.g. Elliott 1958 III, p.29) we have

$$\psi_z(LM) = \sum_{M'} D^L_{MM'}\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)\psi'_{z'}(LM')$$

$$= \sum_{M'} D^L_{MM'}\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi\right)\psi_y(LM')$$

$$\text{i.e} \quad \psi_z(LM) = (-)^M \sum_{M'} d^L_{M'M}(\pi/2)\, \psi_y(LM') \qquad \underline{2.23a}$$

We use the fact that:

$$D^L_{MM'}(\alpha\beta\gamma) = e^{iM\alpha}\, d^L_{MM'}(\beta)\, e^{iM'\gamma}$$

$$\text{and} \quad d^L_{MM'}(\beta) = (-)^{M'-M}\, d^L_{M'M}(\beta)$$

(Elliott 1958 III, p.30)

We may consider the SU_2-group to be a pseudo R_3-group where $v_{x'}, v_{y'}$ and $v_{z'}$ are the angular momentum operators in the S'-frame and ω_x, ω_y and ω_z the operators in the S-frame. Thus we can write immediately, from 2.23a,

$$\phi(\Lambda\nu) = (-i)^{\nu/2} \sum_K d^{\Lambda}_{K/2\ \nu/2}(\pi/2)\ \chi\ (\Lambda K) \tag{2.23b}$$

where it must be remembered that, whereas in the R_3 group we have

$$L_z\ \psi_z(IM) = M\ \psi_z(IM) \quad \text{and} \quad L_y\ \psi_y(IM') = M'\ \psi_y(IM'),$$

in the SU_2-group, because of the definition of the $\phi(\Lambda\nu)$ and $\chi(\Lambda K)$, we have:

$$\omega_z\phi(\Lambda\nu) = (\nu/2)\phi(\Lambda\nu) \qquad \text{and} \qquad \omega_y\chi(\Lambda K) = (K/2)\chi(\Lambda K).$$

We write 2.23b as

$$\phi(\Lambda\nu) = \sum_K b(\Lambda,\nu;K)\ \chi(\Lambda K) \tag{2.24}$$

$$\text{where} \qquad b(\Lambda,\nu:K) = (-i)^{\nu/2}\ d^{\Lambda}_{K/2\ \nu/2}\ (\pi/2) \tag{2.25}$$

We shall see later that we are most interested in the coefficients $b(\Lambda,2\Lambda;K)$. In this case the $d^{\Lambda}_{K/2\ \Lambda}\ (\pi/2)$ simplifies so that we may write :-

$$b(\Lambda,\ 2\Lambda;\ K) = (-i/2)^{\Lambda} \sqrt{\frac{(2\Lambda)!}{(\Lambda+\frac{K}{2})!\,(\Lambda-\frac{K}{2})!}} \tag{2.26}$$

It is useful to know the relation between the numbers ϵ and ν and the oscillator quanta.

From the explicit form of the single particle operator $(B_{zx} + B_{zy})$, we can see that the single particle state $\phi_{s.p.}(\epsilon)$ in the oscillator n-shell has the number of quanta in the z-direction (r_z) as a quantum number since :-

$$\epsilon = 2n_z - n_x - n_y \equiv 3n_z - n \qquad \underline{2.27a}$$

Similarly, it can be seen that the state $\phi_{s.p.}(\epsilon\nu)$ has also the difference $(n_x - n_y)$ of the oscillator quanta in the x and y direction a quantum number since

$$\nu = n_x - n_y \qquad \underline{2.27b}$$

Since the many particle operators A_{ij} are linear combinations of the single particle operators, it follows that, for the many particle state $\phi(\epsilon\Lambda\nu)$,

$$\epsilon = 3N_z - N = \sum_i (3n_z(i) - n) = \sum_i \epsilon_i$$

$$\text{and} \quad \nu = N_x - N_y = \sum_i n_x(i) - n_y(i) = \sum_i \nu_i \qquad \underline{2.28}$$

summing over all particle numbers.

In the single particle states the Λ label is unnecessary since it is related to ϵ.

$$\text{i.e. } \Lambda = \tfrac{1}{2}(n_x + n_y) = \tfrac{1}{6}(2N - \epsilon) \qquad \underline{2.29}$$

For the many particle states however, the Λ labelling is not so trivial. It describes the symmetry of the quanta in the xy-plane and thus serves to distinguish states of a representation with the same ϵ and ν labels. The numbers ϵ, Λ and ν form a unique set of labels for a state of the $(\lambda\mu)$ representation.

For the leading state of a representation $(\lambda\mu)$

$$\epsilon = (N_z - N_x) + (N_z - N_y) = 2\lambda + \mu \ (= \bar{\epsilon})$$

$$\text{and } \nu = N_x - N_y = \mu (= \bar{\nu})$$

This then is part of the reduction $SU_3 \rightarrow SU_2 \times U_1$ found in a simple manner. $\bar{\epsilon}$ is the maximum ϵ in the representation and $\bar{\nu}$ is the maximum ν in $\bar{\epsilon}$.

The complete reduction has been studied by Elliott (1958. II), using Young Tableau which illustrate the representation of a unitary group (as in 2.12 for example). He finds the simple results that, within a representation $(\lambda\mu)$,

$$\epsilon = 2\lambda + \mu,\ 2\lambda + \mu - 3,\ \dots,\ -\lambda - 2\mu \qquad 2.30$$

(This can be deduced immediately from 2.28)

For each value of ϵ:-

$$\Lambda = \tfrac{1}{6}|2\lambda-2\mu-\epsilon|,\ \tfrac{1}{6}|2\lambda-2\mu-\epsilon|+1,\ \dots\ ,\min\{\tfrac{1}{6}(2\lambda+4\mu-\epsilon),\tfrac{1}{6}(2\mu+4\lambda+\epsilon)\} \qquad 2.31$$

We notice that for $\epsilon = \bar{\epsilon} = 2\lambda + \mu$, Λ has the single value $\frac{1}{2}\mu$.

It is useful to present the values of 2Λ appearing for each value of ϵ, in a $(\lambda\mu)$ representation, in an array:

$$\begin{array}{lc}
\underline{\epsilon} & \underline{2\Lambda} \\
\bar{\epsilon}=2\lambda+\mu & 2\bar{\Lambda}=\mu \\
\bar{\epsilon}-3 & 2\bar{\Lambda}-1 \quad 2\bar{\Lambda}+1 \\
\bar{\epsilon}-6 & 2\bar{\Lambda}-2 \quad\quad 2\bar{\Lambda} \quad\quad 2\bar{\Lambda}+2 \\
\bar{\epsilon}-9 & 2\bar{\Lambda}-3 \quad\quad 2\bar{\Lambda}-1 \quad 2\bar{\Lambda}+1 \quad\quad 2\bar{\Lambda}+3
\end{array} \qquad \text{etc.} \qquad \underline{2.32}.$$

This is the most general form for the array. In a special case, parts of the array in 2Λ may be missing e.g. the values of 2Λ must always remain positive or zero.

In the next chapter we shall discuss the raising and lowering operators in a $(\lambda\mu)$ representation. These operators will be useful in generating other states of a representation when the leading state is known.

Chapter 3

Raising and Lowering Operators and the Classification of Many Particle States

Raising and Lowering (R and L) Operators in the SU_2-group

The R and L operators in the SU_2-group will be formed from the operators of the group and therefore will not be able to alter the ϵ or Λ labels of a state. From the commutation relation in 2.15 we see

$$[A_{xy}, B_{xy}] = -2A_{xy} .$$

$$[A_{yx}, B_{xy}] = 2A_{yx} .$$

Thus, since $B_{xy}\,\phi(\Lambda\nu) = \nu\,\phi(\Lambda\nu)$

We have $B_{xy}A_{xy}\,\phi(\Lambda\nu) = (A_{xy}B_{xy} + 2A_{xy})\,\phi(\Lambda\nu)$

$$= (\nu + 2)A_{xy}\,\phi(\Lambda\nu)$$

i.e. $A_{xy}\,\phi(\Lambda\nu) = \phi'(\Lambda\,\nu + 2)$ 3.1a

Similarly $A_{yx}\,\phi(\Lambda\nu) = \phi'(\Lambda\nu - 2)$ 3.1b

where throughout the chapter, primed states will be, in general, unnormalised.

Thus the A_{xy} and A_{yx} operators of SU_2 are the R and L operators respectively of the ν-label by two units.

Because of the isomorphism between the R_3 and SU_2 groups, this fact could have been deduced immediately from the well known properties of $L_{\pm 1}$ (Edmonds 1957).

The normalisation coefficient of the ϕ' states may be calculated in the same way as those normalisation coefficients associated with $L_{\pm 1}$.

Thus

$$\omega_{\pm 1}\,\phi(\Lambda\nu) = \mp\sqrt{\tfrac{1}{2}}\sqrt{(\Lambda \mp \tfrac{\nu}{2})(\Lambda \pm \tfrac{\nu}{2} + 1)}\;\phi(\Lambda,\nu \pm 2) \qquad \underline{3.2a}$$

i.e. $$A_{xy}\phi(\Lambda\nu) = \sqrt{(\Lambda - \tfrac{\nu}{2})(\Lambda + \tfrac{\nu}{2} + 1)}\;\phi(\Lambda,\nu + 2) \qquad \underline{3.2b}$$

and $$A_{yx}\phi(\Lambda\nu) = \sqrt{(\Lambda + \tfrac{\nu}{2})(\Lambda - \tfrac{\nu}{2} + 1)}\;\phi(\Lambda,\nu - 2) \qquad \underline{3.2c}$$

The phases have been chosen in accordance with those of Condon and Shortley(1935)

R and L operators in the full SU_3 - group

We wish to find the operators now which raise and lower the ϵ and Λ labels as well as the ν.

Consider the operators A_{xz}, A_{zx}, A_{yz} and A_{zy}. The effect of these operators on the ϵ and ν labels may be deduced immediately from the commutation relations:-

$$[A_{xz}, B_{xy}] = -A_{xy}\,, \quad [A_{xz}, (B_{zx} + B_{zy})] = 3A_{xz}$$

$$[A_{zx}, B_{xy}] = A_{zx}\,, \quad [A_{zx}, (B_{zx} + B_{zy})] = -3A_{zx}$$

$$[A_{yz}, B_{xy}] = A_{yz}\,, \quad [A_{yz}, (B_{zx} + B_{zy})] = 3A_{yz}$$

$$[A_{zy}, B_{xy}] = -A_{zy}\,, \quad [A_{zy}, (B_{zx} + B_{zy})] = -3A_{zy} \qquad \underline{3.3}$$

Thus we find

$$A_{xz}\, \phi(\varepsilon\nu) = \phi'(\varepsilon - 3\,,\ \nu + 1) \qquad \underline{3.4a}$$

$$A_{zx}\, \phi(\varepsilon\nu) = \phi'(\varepsilon + 3\,,\ \nu - 1) \qquad \underline{3.4b}$$

$$A_{yz}\, \phi(\varepsilon\nu) = \phi'(\varepsilon - 3\,,\ \nu - 1) \qquad \underline{3.4c}$$

$$A_{zy}\, \phi(\varepsilon\nu) = \phi'(\varepsilon + 3\,,\ \nu + 1) \qquad \underline{3.4d}$$

These operators do not, in general, preserve the Λ labelling. We should expect some combination of these operators with the A_{xy} and A_{yx} to give states which belong to a definite representation of SU_2.

We shall now derive the explicit form for the operators $0_{\pm}$ where

$$0_{\pm}\, \phi(\varepsilon\Lambda\nu) = \phi'(\varepsilon - 3, \Lambda \pm 1/2, \nu - 1) \qquad \underline{3.5}$$

and then quote the results for the remaining six raising and lowering operators in the ϵ, Λ and ν.

Two operators which decrease ϵ and ν are

$$A_{yz} \quad \text{and} \quad A_{yx} A_{xz} .$$

From the explicit form for ω^2 (2.19) and the commutation relation for the A_{ij} (2.9) we may deduce

$$\omega^2 A_{yx} A_{xz} = A_{yx} A_{xz} \{\omega^2 + (1/2)B_{xy} + 3/4\} + A_{yz}\{ \omega^2 - (1/2)B_{xy} - (1/4)B_{xy}^2$$

$$\omega^2 A_{yz} = A_{yx} A_{xz} \qquad + A_{yz} \{\omega^2 - (1/2)B_{xy} - 1/4\} \qquad 3.6$$

Consider

$$[A_{yx} A_{xz} + \alpha A_{yz}] \quad \phi(\epsilon \Lambda \nu) \qquad 3.7$$

where α is some number.

Then

$$\omega^2 [A_{yx} A_{xz} + \alpha A_{yz}] \phi(\epsilon \Lambda \nu)$$

$$= \{ A_{yx} A_{xz} [\omega^2 + (1/2)B_{xy} + 3/4 + \alpha]$$

$$+ A_{yz} [\omega^2 - (1/2)B_{xy} - (1/4)B_{xy}^2 + \alpha\{\omega^2 - (1/2)B_{xy} - 1/4\}]\} \quad \phi(\epsilon \Lambda \nu)$$

$$3.8a$$

Since $\omega^2 \phi(\epsilon \Lambda \nu) = \Lambda(\Lambda + 1) \phi(\epsilon \Lambda \nu)$ (2.20) and $B_{xy} \phi(\epsilon \Lambda \nu) = \nu \phi(\epsilon \Lambda \nu)$ (2.21), we may rewrite the right hand side of 3.8a as

$$\{ A_{yx} A_{xz} [\Lambda(\Lambda + 1) + \tfrac{1}{2}\nu + 3/4 + \alpha]$$

$$+ A_{yz} [\Lambda(\Lambda + 1) - \tfrac{1}{2}\nu(\tfrac{1}{2}\nu + 1) + \alpha\{\Lambda(\Lambda + 1) - \tfrac{1}{2}\nu - 1/4\}]\} \quad x$$

$$\phi(\epsilon \Lambda \nu) \qquad 3.8b$$

If 3.7 is to be a state with definite ~~ϕ~~ Λ-value, then the ratio of the coefficients of $A_{yx}A_{xz}$ and A_{yz} in 3.8b must be equal to the ratio of the coefficients in 3.7. Two values of a α satisfy the equivalence, they are:

$$\alpha_1 = (\Lambda - \tfrac{1}{2}\nu) \quad \text{and} \quad \alpha_2 = -(\Lambda + \tfrac{1}{2}\nu + 1)$$

substituting the values into 3.8b we deduce the corresponding Λ-values, for the function in 3.7, to be

$$\Lambda_1 = \Lambda + 1/2 \quad \text{and} \quad \Lambda_2 = \Lambda - 1/2.$$

Hence we see

$$[A_{yx}A_{xz} + (\Lambda - \tfrac{1}{2}\nu)A_{yz}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon - 3, \Lambda + 1/2, \nu - 1) \tag{3.9a}$$

$$[A_{yx}A_{xz} - (\Lambda + \tfrac{1}{2}\nu + 1)A_{yz}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon - 3, \Lambda - 1/2, \nu - 1) \tag{3.9b}$$

Similarly we may show.

$$[A_{xy}A_{yz} + (\Lambda + \tfrac{1}{2}\nu)A_{xz}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon - 3, \Lambda + 1/2, \nu + 1) \tag{3.9c}$$

$$[A_{xy}A_{yz} - (\Lambda - \tfrac{1}{2}\nu + 1)A_{xz}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon - 3, \Lambda - 1/2, \nu + 1) \tag{3.9d}$$

$$[A_{zx}A_{xy} + (\Lambda - \tfrac{1}{2}\nu)A_{zy}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon + 3, \Lambda - 1/2, \nu + 1) \tag{3.10a}$$

$$[A_{zx}A_{xy} - (\Lambda + \tfrac{1}{2}\nu + 1)A_{zy}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon + 3, \Lambda + 1/2, \nu + 1) \tag{3.10b}$$

$$[A_{zy}A_{yx} + (\Lambda + \tfrac{1}{2}\nu)A_{zx}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon + 3, \Lambda - 1/2, \nu - 1) \tag{3.10c}$$

$$[A_{zy}A_{yx} - (\Lambda - \tfrac{1}{2}\nu + 1)A_{zx}]\phi(\epsilon\Lambda\nu) = \phi'(\epsilon + 3, \Lambda + 1/2, \nu - 1) \tag{3.10d}$$

The expressions in 3.9 and 3.10 are the most general forms for the raising and lowering operators. Any state $\phi'(\epsilon'\Lambda'\nu')$ can be obtained from a given state $\phi(\epsilon\Lambda\nu)$ by a suitable combination of the operators in 3.9 and 3.10.

A glance at 3.9 and 3.10 will show that the operators are very cumbersome to use as they stand. Powers of operators must be understood since the function of Λ and ν within the operators will change its value after each operation. We are most interested, in this thesis, in generating states $\phi(\epsilon\Lambda\nu)$ from the leading state of a representation $\{\phi(\lambda\mu)\}$.

By observing certain rules on how the state is to be generated the set of R and L operators may be simplified.

States $\phi'(\epsilon\Lambda\nu)$ generated from the leading state $\phi(\lambda\mu)$

We notice that if a state $\phi'(\epsilon\Lambda\nu = 2\Lambda)$ can be generated from the leading state, any state $\phi'(\epsilon\Lambda\nu)$, with the same ϵ and Λ labels, may be generated from it by powers of the operator A_{yx}. Thus we need only use the general R and L operators in 3.9 to generate a state of maximum ν. With these restrictions the operators simplify.

From 3.9d we see that if $\nu = 2\Lambda$ then

$$\phi'(\epsilon - 3, \Lambda - 1/2, 2\Lambda + 1) \equiv 0$$

since the maximum ν in this case is only $(2\Lambda - 1)$.

Thus we deduce

$$A_{xy}A_{yz}\,\phi(\epsilon\Lambda,\ \nu = 2\Lambda) \equiv A_{xz}\,\phi(\epsilon\Lambda\nu = 2\Lambda) \tag{3.11}$$

Substituting this identity into 2.9c we see that

$$A_{xz}\,\phi(\epsilon\Lambda\ \nu = 2\Lambda) = \phi'(\epsilon - 3, \Lambda + 1/2,\ \nu = (2\Lambda + 1)) \tag{3.12a}$$

Defining now

$$E_{+} = A_{xz} \tag{3.12b}$$

any state with maximum ν for a given ϵ can be generated from the leading state by suitable powers of E_{+}. That is

$$E_{+}^{p}\,\phi(\lambda\mu) = \phi'(2\lambda + \mu - 3p,\ 1/2(\mu + p),\ \mu + p) \tag{3.13}$$

since $\phi(\lambda\mu) \equiv \phi(\lambda\mu,\ \epsilon = 2\lambda + \mu,\ \Lambda = \mu/2,\ \nu = \mu)$

By referring to the array in 2.32, which shows explicitly the states of maximum ν for each Λ value within an ϵ, we see that the operators E_{+}^{p} generate the states on the extreme right hand side of the array, from the leading state at the head.

With $\nu = 2\Lambda$, the expression in 3.9b simplifies to

$$[\,A_{yx}A_{xz} - (2\Lambda + 1)A_{yz}\,]\,\phi(\epsilon\Lambda\nu = 2\Lambda) = \phi'(\epsilon - 3, \Lambda - 1/2,\ \nu = (2\Lambda - 1) \tag{3.14}$$

The operator in 3.14 is not in a very convenient form as repeated application requires a change in the coefficient of A_{yz}.

For example, we must write :-

$$[A_{yx}A_{xz} - (2\Lambda)A_{yz}][A_{yx}A_{xz} - (2\Lambda + 1)A_{yz}]\phi(\epsilon\Lambda, 2\Lambda)$$
$$= \phi'(\epsilon - 6, \Lambda - 1, 2\Lambda - 2)$$

The operator in 3.14 however is equivalent to the complete operator form

$$E_- = [A_{yx}A_{xz} - A_{yz}(B_{xy} + 1)] \equiv [A_{yx}A_{xz} - A_{yx}(2\Lambda + 1)] \qquad \underline{3.15}$$

when acting on a state with maximum ν in Λ.

Since E_- will always generate a state with maximum ν in Λ, the complete operator form may always be used. Hence, repeated application of the operator in 3.14 may be written as powers of E_-, e.g.

$$E_-^2\ \phi(\epsilon\Lambda, 2\Lambda) = \phi'(\epsilon - 6, \Lambda - 1, 2\Lambda - 2)$$

Clearly now, any state with maximum ν in Λ can be generated, from the leading state, by a combination of operators $E_-^q E_+^p$. Thus

$$E_-^q E_+^p\ \phi(\lambda\mu) = \phi'(2\lambda + \mu - 3(p + q),\ \tfrac{1}{2}(\mu + p - q),\ \mu + p - q) \qquad \underline{3.16}$$

<u>The leading state generated from a state $\phi(\epsilon\Lambda\,\nu = 2\Lambda)$</u>

We now consider the inverse operation to take a state $\phi(\epsilon\Lambda\,\nu = 2\Lambda)$ back to at least an unnormalised form of the leading state.

First consider the operator which will take a state on the extreme right hand side of the array in 2.32 back to the leading state $\phi'(\lambda\mu)$

From equation 3.10d

$$(A_{zy}A_{yx} - A_{zx})\,\phi(\epsilon\bar{\Lambda}2\bar{\Lambda}) = \phi'(\epsilon + 3, \bar{\Lambda} + 1/2, 2\bar{\Lambda} - 1) \tag{3.17}$$

where $\bar{\Lambda}$ is the maximum Λ in ϵ. Since $\bar{\Lambda}$ is the maximum Λ in ϵ, a glance at the array 2.32 will show that the state $\phi'(\epsilon + 3, \bar{\Lambda} + 1/2, 2\bar{\Lambda} - 1)$ does not exist i.e. is identically zero. Hence

$$A_{zy}A_{yx}\,\phi(\bar{\epsilon}\bar{\Lambda}2\bar{\Lambda}) \equiv A_{zx}\,\phi(\bar{\epsilon}\bar{\Lambda}\,2\bar{\Lambda}) \tag{3.18}$$

Substituting this identity into 2.10c we see that

$$A_{zx}\,\phi(\epsilon\bar{\Lambda}2\bar{\Lambda}) = \phi'(\epsilon + 3, \bar{\Lambda} - 1/2, 2\bar{\Lambda} - 1) \tag{3.19a}$$

Thus by defining

$$F_- = A_{zx} \tag{3.19b}$$

the operator $F_-^{\,p}$ will have exactly the opposite effect to $E_+^{\,p}$. Not only this, $F_-^{\,+} = A_{zx}^{+} = A_{xz} = E_+$. It is for this latter reason that we prefer to operate on $\phi(\lambda\mu)$ with E_+ before E_-.

The inverse operator (F_+) to that in 3.15 arises from that in 3.10b. Thus

$$F_+ = A_{zx}A_{xy} - A_{zy}(B_{xy} + 1) \tag{3.20a}$$

where $F_+ \phi(\epsilon\Lambda, \nu = 2\Lambda) = \phi'(\epsilon + 3, \Lambda + 1/2, \nu = 2\Lambda + 1)$ <u>3.20b</u>

Notice that in this case $F_+^+ \neq E_-$.

The complete inverse operator to $E_-^q E_+^p$ acting on the leading state $\phi(\lambda\mu)$ is $F_-^p F_+^q$, which will take the state $E_-^q E_+^p \phi(\lambda\mu)$ back to an unnormalised form of $\phi(\lambda\mu)$.

<u>Normalisation coefficients for E_- and E_+</u>

Since the inverse operator of E_+ is also the adjoint of E_+, i.e. $E_+^+ = F_- = (E_+)^{-1}$, we are able to determine the normalisation coefficients N_p where

$$E_+^p \phi(\lambda\mu) = N_p \phi\{(2\lambda + \mu - 3p), \ (\tfrac{1}{2}\mu + \tfrac{1}{2}p), \mu + p\} \qquad \underline{3.21}$$

We adapt a method used by Racah (1951).

If $\phi_o = \phi(\lambda\mu)$, we define states ϕ_i such that,

$$E_+ \phi_o = \phi_1' = N_1 \phi_1$$

$$E_+ \phi_1' = \phi_2' = N_2 \phi_2$$

.... etc. where the unprimed ϕ's are normalised.

Introduce the numbers μ_j where $F_- \phi_j' = \mu_j \phi_{j-1}', \mu_o = 0$. <u>3.23</u>

By knowing the μ_j's, it is possible to determine the normalisation coefficient N_p for :-

$$E_+^p \phi_o = E_+ \phi_{p-1}' = \phi_p' \qquad \underline{3.24}$$

But $\phi_p' = N_p \phi_p$ where $(\phi_p | \phi_p) = 1$.

Thus by taking the adjoint of 3.24 with itself, where

$$(\phi'_p)^+ = (E_+\phi'_{p-1})^+ = (\phi'_{p-1})^+ E_+^+ = (\phi'_{p-1})^+ F_- \, , \qquad 3.25$$

we have

$$N_p^* N_p = (\phi'_p \mid \phi'_p) = (\phi'_{p-1}|F_-|\phi'_p)$$

$$= \mu_p(\phi'_{p-1} \mid \phi'_{p-1}) = \mu_p N^*_{p-1} N_{p-1} \qquad 3.26$$

Thus $N_p^* N_p = \mu_p \mu_{p-1} \cdots \mu_1 N_0^* N_0$

If ϕ_0 is normalised, $N_0^* N_0 = 1$

$$\text{i.e.} \quad N_p^* N_p = \prod_{j=1}^{p} \mu_j \qquad 3.27$$

To determine the μ_j, we first derive a recurrence relation.

From 3.23, we have

$$F_-\phi'_{j+1} \equiv \mu_{j+1}\phi'_j \qquad 3.28$$

but $F_-\phi'_{j+1} = F_-E_+\phi'_j$

$= A_{zx}A_{xz}\phi'_j$ from 3.12b and 3.19b

$= [A_{zx}, A_{xz}]\,\phi'_j + A_{xz}A_{zx}\phi'_j$

$= B_{zx}\phi'_j + E_+F_-\phi'_j$

$$= B_{zx}\phi'_j + \mu_j\phi'_j \qquad 3.29$$

Now $B_{zx} = (1/2)\{(B_{zx} + B_{zy}) - B_{xy}\}$

and $(B_{zx} + B_{zy})\phi'_j = (2\lambda + \mu - 3j)\phi'_j$

$B_{xy}\,\phi'_j = (\mu + j)\phi'_j$

from the relations in 3.22, and 3.13.

We find, then, from 3.28, and 3.29,

$$\mu_{j+1}\,\phi_j' = (\mu_j + \lambda - 2j)\phi_j' \quad \text{for any} \quad \phi_j' \tag{3.30a}$$

Hence $\mu_{j+1} = \mu_j + \lambda - 2j.$ $\qquad$ 3.30b

From this recurrence relation and knowing $\mu_0 = 0$ we can deduce :-

$$\mu_j = j(-j + 1 + \lambda) \tag{3.31}$$

Substituting this value in 3.27 we see that the normalisation coefficient in 3.21 is given by

$$N_p^* N_p = \prod_{j=1}^{p} j(-j + 1 + \lambda) \tag{3.32}$$

Notice that in 3.23, F_- is used as the inverse of E_+ whilst in 3.25 we use the fact that $E_+^{+} = F_-$.

As we have already observed, $F_+ \neq E_-^{+}$, and so this approach cannot be used to calculate the normalisation coefficients M_{qp} where

$$E_-^q\,\phi((2\lambda + \mu - 3p),\ \mu/2 + p/2,\ \mu + p)$$

$$= M_{qp}\,\phi((2\lambda + \mu - 3p - 3q),\ \mu/2 + p/2 - q/2,\ = \mu + p - q) \tag{3.33}$$

We therefore have to resort to the straightforward method of calculating explicitly

$$(2\lambda + \mu,\ \mu/2,\ \mu\ |(E_+^{p})^{+}(E_-^{q})^{+}E_-^{q}E_+^{p}|\ 2\lambda + \mu,\ \mu/2,\ \mu)$$

$$= N_p^* N_p M_{qp}^* M_{qp} \quad \text{for each value of } p \text{ and } q\,. \tag{3.34}$$

This may be done using the commutation relation of the A_{ij} and making use of the fact that any creation operator in ϵ, Λ or ν on the leading state $\phi(\lambda\mu)$ gives zero.

For all the normalisation coefficients M_{qp} required in this paper, and calculated in the way indicated in 3.34, we find that they are given by the expression:-

$$M^*_{qp}\, M_{qp} = \prod_{k=1}^{q} k(\mu - k + 1)(\mu + p + 2 - k)(\lambda + \mu + 2 - k) \qquad \underline{3.35}$$

This is verified for all p and q with $p + q = 2$ or 4. Thus we have the result, at least for $p + q = 2$ or 4 and all p with $q = 0$.

$$E_-^{\,q} E_+^{\,p}\, \phi(\lambda\mu) = e^{i\delta}\sqrt{\left\{\prod_{j=1}^{p} j(-j + 1 + \lambda) \prod_{k=1}^{q} k(\mu - k + 1)(\mu + p + 2 - k)\times(\lambda + \mu + 2 - k)\right\}}\times$$
$$\phi(\epsilon = 2\lambda + \mu - 3(p + q), \Lambda = \mu/2 + p/2 - q/2, \nu = \mu + p - q) \qquad \underline{3.36}$$

The phase will be chosen to be $+1$ for convenience i.e. $\delta = 0$. With this convention and that in 3.2, all states of a representation will have real and positive normalisation coefficients if the phase of the leading state is real and positive.

In Chapter 5 we shall use the notation :-

$$E(\lambda\mu, \epsilon\Lambda\nu)$$

to denote the operator which generates a normalised state $\phi((\lambda\mu)\epsilon\Lambda\nu)$, from the leading state $\phi(\lambda\mu)$, in the way indicated in this Chapter. That is, if $\epsilon = 2\lambda + \mu - 3(p + q)$, $\Lambda = 1/2(\mu + p - q)$, $\nu = 2\Lambda - 2r$,

then:-

$$E(\lambda\mu, \epsilon\Lambda\nu)\phi(\lambda\mu) \equiv \frac{A_{yx}^{\ r} \ E_{-}^{\ q} \ E_{+}^{\ p}}{P_{rqp} \ M_{qp} \ N_{p}} \phi(\lambda\mu) = \phi((\lambda\mu)\epsilon\Lambda\nu) \qquad \underline{3.37}$$

where P_{rqp} is the normalisation coefficient derived from

<u>3.2c</u>

<u>Equivalent Operators</u>

We shall find it useful to represent such operators as $A_{yx}^{\ r} \ E_{-}^{\ q} \ E_{+}^{\ p}$, acting on the leading state, as combinations of the operators L_o, $L_{\pm 1}$.

We notice that, from <u>3.1a</u>,

$$A_{xy} \ \phi(\Lambda, \nu = 2\Lambda) \equiv 0 \qquad \underline{3.38}$$

Hence

$$A_{yx} \ \phi(\Lambda, \nu = 2\Lambda) \equiv (A_{yx} - A_{xy})\phi(\Lambda, \nu = 2\Lambda) = - iL_o\phi(\Lambda, \nu = 2\Lambda) \qquad \underline{3.39}$$

(Table 1, Chapter 2)

Since $[A_{yx}, - iL_o] = [A_{yx}, (A_{yx} - A_{xy})] = B_{xy}$

we deduce :-

$$\begin{aligned} A_{yx}^{\ 2} \ \phi(\Lambda, \nu = 2\Lambda) &= A_{yx}(- iL_o)\phi(\Lambda, \nu = 2\Lambda) \\ &= \big((- iL_o)A_{yx} + B_{xy}\big) \ \phi(\Lambda, \nu = 2\Lambda) \\ &= (- L_o^{\ 2} + 2\Lambda) \ \phi(\Lambda, \nu = 2\Lambda). \end{aligned}$$

Similarly for any power A^r_{yx}.

Since $E_-^q E_+^p \phi(\lambda\mu)$ is a state with maximum ν in $\Lambda = (1/2)(\mu + p - q)$, we may replace the A^r_{yx} part of the operator $(A^r_{yx} E_-^q E_+^p)$ immediately in terms of L_o operators and constants.

The operator $E_-^q E_+^p$ is merely a sum of products of the form $A^\alpha_{yx} A^\beta_{yz} A^\gamma_{xz}$. As well as the equivalences in 3.38 and 3.39 we have from 3.4b and 3.4a:

$$A_{zx} \phi(\lambda\mu) \equiv A_{zy} \phi(\lambda\mu) \equiv 0 \qquad \underline{3.40}$$

Hence $A_{xz}\phi(\lambda\mu) \equiv (A_{xz} - A_{zx}) \phi(\lambda\mu) = \sqrt{(1/2)}(L_{+1} + L_{-1}) \phi(\lambda\mu)$ $\qquad$ 3.41a

$$A_{yz}\phi(\lambda\mu) \equiv (A_{yz} - A_{zy}) \phi(\lambda\mu) = - i\sqrt{(1/2)}(L_{+1} - L_{-1}) \phi(\lambda\mu) \qquad \underline{3.41b}$$

Thus from 3.39 and 3.41 we may deduce the equivalences of any product $A^\alpha_{yx} A^\beta_{yx} A^\gamma_{xz}$ and hence $E_-^q E_+^p$, knowing the commutators of the A_{ij} with L_o and $L_{\pm 1}$.

In Tables 3 to 6, given at the end of this Chapter, we quote the expansions and equivalences of certain operators $A_{yx}^r E_-^q E_+^p$. These operators will be of most use to us in the later Chapters.

In Table 3, the expansion of the operators $A_{yx}^r E_-^q E_+^p$ is given in terms of operators of the form $A^\alpha_{yx} A^\beta_{yz} A^\gamma_{xz}$. Table 3a contains only operators which do not change the ε- value i.e. $p + q = 0$. Table 3b contains operators which lower ϵ by 6,$(p + q = 2)$, and Table 3c has operators with $p + q = 4$ i.e. they lower ϵ by 12.

As an example of how Table 3 may be read we quote (from entry 311)

$$A_{yx}^{3}\, E_{-}^{1}\, E_{+}^{1} \equiv A_{yx}^{4}\, A_{xz}^{2} - (\mu + 2)\, A_{yx}^{3}\, A_{yz}\, A_{xz} \qquad \underline{3.42}$$

In Table 4 we list the equivalences of the operators appearing in Tables 3a and 3b using the results of Table 3. These equivalences are written in the general form:-

$$A_{yx}^{r}\, E_{-}^{q}\, E_{+}^{p} \equiv a_1(L_{+1}^{2} + L_{-1}^{2}) + a_2(L_{+1}^{2} - L_{-1}^{2}) + a_3\, L^{2} + a_4 \qquad \underline{3.43a}$$

where the a_i's are functions of L_o only.

Thus, under the entry (002), we have

$$E_{+}^{2} \equiv (1/2)(L_{+1}^{2} + L_{-1}^{2}) - (1/2)L^{2} + ((1/2)L_{o}^{2} + \lambda) \qquad \underline{3.43b}$$

The equivalences of the operators appearing in Table 3c are too cumbersome to be written in one table. We have thus split up the equivalence-reduction into two parts. In Table 5 we list the equivalences

$$A_{yx}^{r}\, E_{-}^{q}\, E_{+}^{p} \equiv b_1\, A_{xz}^{4} + b_2\, A_{yz}\, A_{xz}^{3} + b_3\, A_{yz}^{2}\, A_{xz}^{2}$$

$$+ b_4\, A_{yz}^{3}\, A_{xz} + b_5\, A_{yz}^{4} \qquad \underline{3.44a}$$

where the b_i are functions of L_o only.

Thus, under the entry (113),

$$A_{yx}^{1}\, E_{-}^{1} E_{+}^{3} \equiv - L_{o}^{2}\, A_{xz}^{4} + i(\mu + 4)L_{o}A_{yz}\, A_{xz}^{3} + 0 \qquad \underline{3.44b}$$

In Table 6 the equivalences of all operators $A_{yz}^{m}\, A_{xz}^{n}$ with $m + n = 4$ are listed. These equivalences are of the form.

$$A_{yz}^{m}\, A_{xz}^{n} = d_1 + d_2\, (L_{+1}^{2} + L_{-1}^{2}) + d_3\, (L_{+1}^{2} - L_{-1}^{2}) + d_4\, (L_{+1}^{4} + L_{-1}^{4}) + d_5\, (L_{+1}^{4} - L_{-1}^{4}) \qquad \underline{3.45a}$$

where $d_i = c_{i1}\, L^4 + c_{i2}\, L^2 + c_{i3}$. $\underline{3.45b}$

and the c_{ij}'s are functions of L_o only.

Thus, for the operator A_{xz}^{4},

$$d_2 = \{-1/2\} L^2 + \{(1/2) L_o^{\,2} + 3\,(\lambda - 1)\} \qquad \underline{3.45c}$$

Explicit form for the single particle states

The single particle states in the oscillator n-shell are symmetric tensors of degree n of the creation operators a_i^{+}, that is, they spread out the representation (no) of SU_3. When $\mu = o$, the Λ-number is given directly by the ϵ-number (see 2.31), hence the single particle states are only distinquished by the ϵ and ν labels. In view of the relations in 2.27, we see that the single particle oscillator well functions, which are labelled by ϵ and ν are just those which may also be labelled with n_x, n_y and n_z: that is, the solution of the harmonic oscillator equation in Cartesian coordinates.

The six single particle states in the oscillator $n = 2$ shell are given in Table 2, together with the shorthand notation we shall adopt for them.

Table 2

Total number of quanta	ϵ	2Λ	ν	Shorthand Notation	Explicit form
2	4	0	0	ϕ_0	$\bar{H}_0(\bar{x})\bar{H}_0(\bar{y})\bar{H}_2(\bar{z})e^{-\rho^2/2}$
	1	1	1	ϕ_{+1}	$\bar{H}_1(\bar{x})\bar{H}_0(\bar{y})\bar{H}_1(\bar{z})e^{-\rho^2/2}$
			-1	ϕ_{-1}	$\bar{H}_0(\ )\bar{H}_1(\ )\bar{H}_1(\ )e^{-\rho^2/2}$
	-2	2	2	ϕ_{+2}	$\bar{H}_2(\ )\bar{H}_0(\ )\bar{H}_0(\ )e^{-\rho^2/2}$
			0	ϕ'_0	$\bar{H}_1(\ )\bar{H}_1(\ )\bar{H}_0(\ )e^{-\rho^2/2}$
			2	ϕ_{-2}	$\bar{H}_0(\ )\bar{H}_2(\ )\bar{H}_0(\ )e^{-\rho^2/2}$

In this table $\bar{x}$, $\bar{y}$ and $\bar{z}$ mean $^x/b$, $^y/b$ and $^z/b$ respectively with b the oscillator well parameter and $\rho^2 = (1/b^2)(x^2 + y^2 + z^2)$.

$$\bar{H}_n(\bar{x}) = \sqrt{\frac{1}{b2^n\ n!\ \sqrt{\pi}}}\ H_n(\bar{x}) \qquad \underline{3.46}$$

with $H_n(\bar{x})$ the Hermite Polynomial of degree n.

The phase of the ϕ's has been chosen to tie up with the phases of the raising and lowering operators in 3.2 and 3.36.

The matrix represenetation of the basic operators A_{yx}, A_{yz} and A_{xz}, spread out by the single particle states, have been given in Tables 7. These matrices will be seen to have great use when trying to construct a many particle state of maximum orbital symmetry, $\phi((\lambda\,\mu)\epsilon\Lambda\nu)$, from the leading state $\phi(\lambda\mu)$, using the operators A_{yx}, E_- and E_+.

Tables 7 have to be read downwards. Thus, from Table 7a,

$$A_{xz}\,\phi_o = \sqrt{2}\;\phi_{+1} \qquad 3.47$$

Construction of antisymmetric states, with maximum orbital symmetry, classified according to SU_3, SU_2 and U_1

From the work on raising and lowering operators, states of a $(\lambda\mu)$ representation may be generated from the leading state of the representation. The leading state of the (unique) representation containing the maximum ϵ-value in a given configuration is easy to construct. This 'leading' representation will be the one which maximises $2\lambda + \mu$ (2.30) and it has a leading state which is determined solely from the ϵ and ν values i.e. for this state, the $(\lambda\mu)$ and Λ labels are trivial. The leading state of other representations can be found by orthogonality relations with the states of the leading representation.

The k-particle state of maximum orbital symmetry, which has maximum ϵ and ν values, is formed by putting k particles into the single particle orbits, in the way shown in 1.7, filling those orbits with the greatest value for ϵ and ν first. Thus, in the ds-shell, four particles must be put into the ϕ_0 state before the filling of the ϕ_{+1} etc. Because of the relation in 2.28, the ϵ value of the k-particle state is bound to be a maximum and the ν value will be a maximum for this ϵ.

For $k = 4$, only the ϕ_0 single-particle orbit is filled. Thus $\epsilon = 4 \times 4 = 16$, $\nu = 4 \times 0 = 0$. The only representation $(\lambda\mu)$, which has a state with $\epsilon = 16$ and $\nu = 0$, within $[f] = [4]$, is (80) (Elliott 1958, I, Table 1).

Hence :-

$$\Phi^{11}([4]\,(80),16,0,0) = (\phi_0^4)^{11} \qquad \underline{3.48}$$

using the notation in 1.6b.

When $k = 8$, both the ϕ_0 and ϕ_{+1} orbits are filled. In this case :-

$$\epsilon = (4 \times 4) + (4 \times 1) = 20 \text{ and } \nu = (4 \times 0) + (4 \times 1) = 4.$$

i.e. $(\lambda\mu) = (84)$ (Elliott 1958, I, Table 1).

Hence

$$\Phi^{11}([44]\;(84),20,2,4) = \sqrt{\frac{4!\,4!}{8!}}\sum_P{}' (-)^P\, P\, \left\{(\phi_0^4)^{11}_{1234}(\phi_{+1}^4)^{11}_{5678}\right\}^{11} \qquad \underline{3.49}$$

the coupling of the charge-spin states being trivial.

Other states $\Phi((\lambda\mu)\varepsilon\Lambda\nu)$ of the leading representation are found by applying the appropriate operator $A^{r}_{yx}\,E^{q}_{-}\,E^{p}_{+}$ to $\Phi(\lambda\mu)$. In practice we expand the operator into a sum over operators $A^{\alpha}_{yx}\,A^{\beta}_{yz}\,A^{\gamma}_{xz}$ (see Table 3) and then proceed to find $\Phi((\lambda\mu)\varepsilon\Lambda\nu)$ by using the known relations of the operators A_{yx}, A_{yz} and A_{xz} on the single particle states (see Table 7).

It is convenient to represent the states in terms of Slater determinants for these operations to be carried out. For the nuclei which we shall consider later {i.e. k = 2,3,4 and 8} , these determinants may be written down immediately e.g.

$$\Phi([4](80)16\ 00) = \sqrt{1/4!}\begin{vmatrix} \overset{++}{\phi}_0(1) & \overset{+-}{\phi}_0(1) & \overset{-+}{\phi}_0(1) & \overset{--}{\phi}_0(1) \\ \overset{++}{\phi}_0(2) & & & \vdots \\ \vdots & & & \vdots \\ \overset{++}{\phi}_0(4) & & --- & \overset{--}{\phi}(4) \end{vmatrix}$$

where $\overset{+-}{\phi}_0(1)$ means that number 'one' particle is in the orbital state ϕ_0 and has isotopic spin projection +1/2 (neutron) and intrinsic spin -1/2.

We shall write the normalised determinant

$$|\phi_0\,\phi_0\,\phi_0\,\phi_0|^{(1)}$$

the order of the charge spin functions being understood.

The bracketed number will give the number of such determinants which sum to give a maximum symmetric orbital state.

Thus, the totally antisymmetric state $|\alpha\,\alpha\,\alpha\,\beta|^{(4)}$, with orbital symmetry $[f] = [4]$, is given explicitly as

$$|\alpha\,\alpha\,\alpha\,\beta|^{(4)} = |\alpha\,\alpha\,\alpha\,\beta| + |\alpha\,\alpha\,\beta\,\alpha| + |\alpha\,\beta\,\alpha\,\alpha| + |\beta\,\alpha\,\alpha\,\alpha| \qquad \underline{3.51}$$

Hence, for example,

$$\Phi^{11}((80)\ 10,\ 1,\ 2) = \frac{E_{\pm}^{2}}{4\sqrt{7}}\ \Phi^{11}(80)$$ where the normalisation is deduced from <u>3.3b</u>

$$\equiv \frac{A_{xz}^{2}}{4\sqrt{7}}\ \Phi^{11}(80) \quad \text{(Table 3)}$$

But :-

$$A_{xz}^{2}\ |\phi_0\,\phi_0\,\phi_0\,\phi_0|^{(1)} = A_{xz}\sqrt{2}\,|\phi_0\,\phi_0\,\phi_0\,\phi_{+1}|^{(4)}$$

$$= \sqrt{2}\ \{2\sqrt{2}\,|\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}|^{(6)} + \sqrt{2}\,|\phi_0\,\phi_0\,\phi_0\,\phi_{+2}|^{(4)}\}$$

$$= 4\sqrt{6}\ (\phi_0^2\,\phi_{+1}^2)^{11} + 2\sqrt{4}(\phi_0^3\,\phi_{+2})^{11}.$$

Thus we deduce :-

$$\Phi^{11}((80)\ 10,\ 1,2) = \sqrt{1/7}\{\sqrt{6}(\phi_0^2\,\phi_{+1}^2)^{11} + (\phi_0^3\,\phi_{+2})^{11}\}$$

the obvious normalisation of which provides a check on the calculations above. In the $k = 8$ particle problem we shall have :-

$$\Phi^{11}(84) = |\phi_0\,\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}|^{(1)} \qquad \underline{3.52}$$

Thus, for example

$$\Phi^{11}((84)\ 14,\ 3,\ 6) = \frac{E_+^2}{4\sqrt{7}}\,\Phi^{11}(84) \qquad \underline{3.36}$$

$$= \frac{A_{xz}^2}{4\sqrt{7}}\,\Phi^{11}(84) \qquad \text{(Table 3).} \qquad \underline{3.53}$$

But

$$A_{xz}^2\,|\phi_0\,\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}|^{(1)} = A_{xz}\,\{\sqrt{2}\,|\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\phi_{+1}\phi_{+1}\phi_{+1}\phi_{+1}|^{(4)}$$

$$+ \sqrt{2}\,|\phi_0\,\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+2}\,|^{(4)}\}.$$

The determinants in the first term all have two columns with the same orbital and charge-spin states and so are identically zero. Thus $A_{xz}^2\,\phi(84) = \sqrt{2}\{\sqrt{2}\,|\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+2}|^{(4)}$

$$+\ 2\sqrt{2}\,|\phi_0\,\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}\,\phi_{+2}\,\phi_{+2}|^{(6)}\} \qquad \underline{3.54}$$

there being only four non-zero determinants of

$$|\phi_0\,\phi_0\,\phi_0\,\phi_0\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+2}|^{(4)} \equiv -\,|\phi_0\,\phi_0\,\phi_0\,\phi_{+2}\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}\,\phi_{+1}|^{(4)}. \qquad \underline{3.55}$$

Hence we deduce :-

$$\Phi^{11}((84)\ 14,\ 3,\ 6) \equiv \sqrt{1/7}\,\{-\,(\phi_0^3\phi_{+2})^{11}(\phi_{+1}^4)^{11} + \sqrt{6}(\phi_0^4)^{11}(\phi_{+1}^2\phi_{+2}^{\,2})^{11}\}^{11} \qquad \underline{3.56}$$

Such states as $(\phi_0^2\,\phi_{+1}^2)^{11}$ in $\Phi^{11}((80),\ 10,\ 12)$, and $\{(\phi_0^3\,\phi_{+2})^{11}_{,}\ (\phi_{+1}^4)^{11}\}^{11}$ in $\Phi^{11}((84)\ 14,\ 3,\ 6)$, we shall call configurational states. These states have $[f], \varepsilon$ and ν as quantum numbers but not $(\lambda\mu)$ or Λ. The configurational states will be written $\Phi_i(\varepsilon\,\nu)$ where i is some arbitrary label.

In general we have

$$\Phi((\lambda\mu)\,\varepsilon\Lambda\nu) = \sum_i p_i((\lambda\mu)\varepsilon\Lambda\nu)\quad \Phi_i(\varepsilon\,\nu) \qquad \underline{3.57}$$

Casimir Operators of the U_3-group

We shall define a Casimir Operator to be any operator of the group which commutes with all operators of the group.

It can be seen that the operators

$$C_1 = 1/2 \sum_{i,j} A_{ij}\, A_{ji} \qquad \underline{3.57a}$$

and

$$C_2 = \sum_{i,j,k} A_{ij}\, A_{jk}\, A_{ki} \qquad \underline{3.57b}$$

are linearly independent and commute with all operators A_{pq} of U_3.

The explicit form for the operator $\tilde{C}_1$, of the group-SU_3, obtained from C_1 by removing the unit operator $(A_{xx} + A_{yy} + A_{zz})$ is:-

$$\begin{aligned}\tilde{C}_1 = 1/6\,[B^2_{xy} + B^2_{zx} + B^2_{zy}] &+ 1/2\,[A_{xy}\,A_{yx} + A_{yx}\,A_{xy}] \\ &+ 1/2\,[A_{yz}\,A_{zy} + A_{zy}\,A_{yz}] \\ &+ 1/2\,[A_{zx}\,A_{xz} + A_{xz}\,A_{zx}]\end{aligned} \qquad \underline{3.58}$$

This operator corresponds to the Casimir Operator formed from the tensor prescription (Racah 1951 p. 30)

$$\tilde{C}_1 = (1/12)\,[(Q.Q) + 3(L.L)] \qquad \underline{3.59}$$

where (.) denotes a scalar product.

States of a particular representation $(\lambda\mu)$ of SU_3, diagonalize the Casimir Operators. We could equally well have defined the numbers λ and μ as being associated with the eigenvalues of $\tilde{C}_1$ and $\tilde{C}_2$ just as L is associated with the eigenvalue of L^2 for the R_3-group.

The eigenvalues of the $\tilde{C}$'s are the same no matter on which state of a representation they are operating. We thus choose the most convenient state - the leading state - to determine the values of $\langle\tilde{C}_1\rangle$ in a representation.

Writing <u>3.58</u> in the more convenient form

$$\tilde{C}_1 = (1/12)\,[(B_{zx} + B_{zy})^2 + 3\,B_{xy}^2] + (1/2)[(B_{zx} + B_{zy}) + B_{xy}]$$

$$+ A_{yx}A_{xy} + A_{yz}A_{zy} + A_{xz}A_{zx} \qquad \underline{3.60}$$

we deduce, since $A_{xy}\,\phi(\lambda\mu) = A_{zy}\,\phi(\lambda\mu) = A_{zx}\,\phi(\lambda\mu) \equiv 0$

(see <u>3.38</u> and <u>3.40</u>)

$$\tilde{C}_1\,\phi(\lambda\mu) = (1/3)\,[\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu]\,\phi(\lambda\mu)$$

(<u>2.16</u> and <u>2.21</u>)

$$= \langle\tilde{C}_1\rangle_{\lambda\mu}\;\phi(\lambda\mu) \qquad \underline{3.61}$$

The eigenvalue of $\tilde{C}_2$ in a $(\lambda\mu)$ representation may be found in a similar way but, as yet, no physical interpretation has been given to this number.

(See Chapter 6, for the use of the operator $\tilde{C}_1$)

Table 3. Expansion of some operators $A_{yx}^{r} E_{-}^{q} E_{+}^{p}$ acting on $\phi(\lambda\mu)$ (for example see 3.42)

Table 3a $p + q = 0$

r q p	1	A_{yx}^{2}	A_{yx}^{4}
0 0 0	1		
2 0 0		1	
4 0 0			1

Table 3b $p + q = 2$

r q p	A_{xz}^{2}	$A_{yx}^{2} A_{xz}^{2}$	$A_{yx}^{4} A_{xz}^{2}$	$A_{yx} A_{yz} A_{xz}$	$A_{yx}^{3} A_{yz} A_{xz}$	A_{yz}^{2}	$A_{yx}^{2} A_{yz}^{2}$
0 0 2	1						
2 0 2		1					
4 0 2			1				
1 1 1		1		$-(\mu + 2)$			
3 1 1			1		$-(\mu + 2)$		
0 2 0		1		$-2(\mu + 1)$		$\mu(\mu+1)$	
2 2 0			1		$-2(\mu + 1)$		$\mu(\mu+1)$

Table 3c. $p + q = 4$

(r q p)	A^4_{xz}	$A^2_{yx}A^4_{xz}$	$A^4_{yx}A^4_{xz}$	$A_{yx}A_{yz}A^3_{xz}$	$A^3_{yx}A_{yz}A^3_{xz}$	$A^2_{yz}A^2_{xz}$	$A^2_{yx}A^2_{yz}A^2_{xz}$	$A_{yx}A^3_{yz}A_{xz}$	A^4_{yz}
0 0 4	1								
2 0 4		1							
4 0 4			1						
1 1 3		1		$-(\mu+4)$					
3 1 3			1		$-(\mu+4)$				
0 2 2		1		$-2(\mu+3)$		$(\mu+2)(\mu+3)$			
2 2 2			1		$-2(\mu+3)$		$(\mu+2)(\mu+3)$		
1 3 1			1		$-3(\mu+2)$		$3(\mu+1)(\mu+2)$	$-\mu(\mu+1)(\mu+2)$	
0 4 0			1		$-4(\mu+1)$		$6(\mu)(\mu-1)$	$-4\mu(\mu-1)(\mu+1)$	$(\mu+1)\mu(\mu-1)(\mu-2)$

Table 4. (for an example see 3.43)

r q p	L_o^6	L_o^4	L_o^3	L_o^2	L_o	1	
2 0 0				-1		μ	1
4 0 0		1		$-2(3\mu-4)$		$3\mu(\mu-2)$	1
0 0 2						$\frac{1}{2}$	$(L_{+1}^2+L_{-1}^2)$
						$-\frac{1}{2}$	L^2
				$\frac{1}{2}$		λ	1
2 0 2				$-\frac{1}{2}$		$\frac{1}{2}(\mu+2)$	$(L_{+1}^2+L_{-1}^2)$
				$\frac{1}{2}$		$-\frac{1}{2}(\mu+2)$	L^2
		$-\frac{1}{2}$		$\frac{1}{2}(\mu+2-2\lambda)$		$\lambda(\mu+2)$	1
4 0 2		$\frac{1}{2}$		$-(3\mu+2)$		$\frac{3}{2}\mu(\mu+2)$	$(L_{+1}^2+L_{-1}^2)$
		$-\frac{1}{2}$		$(3\mu+2)$		$-\frac{3}{2}\mu(\mu+2)$	L^2
	$\frac{1}{2}$	$(\lambda-3\mu-2)$		$\frac{1}{2}(3\mu^2+6\mu-12\lambda\mu-8\lambda)$		$3\lambda\mu(\mu+2)$	1
1 1 1				$-\frac{1}{2}$			$(L_{+1}^2+L_{-1}^2)$
					$\frac{1}{2}(\mu+2)$		$(L_{+1}^2+L_{-1}^2)$
				$\frac{1}{2}$			L^2
		$-\frac{1}{2}$		$-\frac{1}{2}(2\lambda+\mu+2)$			1

Table 4 (contd.)

r q p	L_o^6	L_o^4	L_o^3	L_o^2	L_o	1	
3 1 1		$\frac{1}{2}$		$-\frac{1}{2}(3\mu-2)$			$(L_{+1}^2+L_{-1}^2)$
			$-\frac{1}{2}(\mu+2)$		$\frac{1}{2}(\mu+2)(3\mu-2)$		$(L_{+1}^2-L_{-1}^2)$
		$-\frac{1}{2}$		$\frac{1}{2}(3\mu-2)$			L^2
	$\frac{1}{2}$	$(\lambda-\mu+2)$		$-\frac{1}{2}(2\lambda+\mu+2)(3\mu-2)$			1
0 2 0				$-\frac{1}{2}$		$-\frac{1}{2}\mu(\mu+2)$	$(L_{+1}^2+L_{-1}^2)$
					$(\mu+1)$		$(L_{+1}^2-L_{-1}^2)$
				$\frac{1}{2}$		$-\frac{1}{2}\mu^2$	L^2
		$-\frac{1}{2}$		$-\frac{1}{2}(2\lambda+2\mu-\mu^2+2)$		$\mu^2(\lambda+\mu+1)$	1
2 2 0		$\frac{1}{2}$		$\frac{1}{2}(\mu^2+\mu+2)$		$-\frac{1}{2}\mu(\mu-2)(\mu+2)$	$(L_{+1}^2+L_{-1}^2)$
			$-(\mu+1)$		$(\mu+1)(\mu-2)$		$(L_{+1}^2-L_{-1}^2)$
		$-\frac{1}{2}$		$\frac{1}{2}(\mu-1)(\mu+2)$		$-\frac{1}{2}\mu^2(\mu-2)$	L^2
	$\frac{1}{2}$	$\frac{1}{2}(2\lambda+\mu+4-\mu^2)$		$\frac{1}{2}\Big[(-5\mu^2+4)$ $-(2\lambda+\mu)(\mu-1)(\mu+2)\Big]$		$\mu^2(\mu-2)(\lambda+\mu+1)$	1

Table 5 (For an example see 3.44)

r q p	L_o^4	L_o^3	L_o^2	L_o	1	
0 0 4					1	A_{xz}^4
2 0 4			-1		$(\mu+4)$	A_{xz}^4
4 0 4	1		$-2(3\mu+8)$		$3(\mu+4)(\mu+2)$	A_{xz}^4
1 1 3			-1			A_{xz}^4
				$i(\mu+4)$		$A_{yz}\ A_{xz}^3$
3 1 3	1		$-(3\mu+4)$			A_{xz}^4
		$-i(\mu+4)$		$i(\mu+4)(3\mu+4)$		$A_{yz}\ A_{xz}^3$
0 2 2			-1		$-(\mu+2)$	A_{xz}^4
				$i2(\mu+3)$		$A_{yz}\ A_{xz}^3$
					$(\mu+2)(\mu+3)$	$A_{yz}^2\ A_{xz}^2$

Table 5 (contd.)

r q p	L_o^4	L_o^3	L_o^2	L_o	1	
2 2 2	1		2		$-\mu(\mu+2)$	A_{xz}^4
		$-i2(\mu+3)$		$i2\mu(\mu+3)$		$A_{yz}\ A_{xz}^3$
			$-(\mu+2)(\mu+3)$		$\mu(\mu+2)(\mu+3)$	$A_{yz}^2\ A_{xz}^2$
1 3 1	1		$(3\mu+2)$			A_{xz}^4
		$-i3(\mu+2)$		$-i3\mu(\mu+2)$		$A_{yz}\ A_{xz}^3$
			$-3(\mu+1)(\mu+2)$			$A_{yz}^2\ A_{xz}^2$
				$i\mu(\mu+1)(\mu+2)$		$A_{yz}^3\ A_{xz}$
0 4 0	1		$(6\mu-4)$		$3\mu(\mu-2)$	A_{xz}^4
		$-i4(\mu+1)$		$-i4(\mu+1)(3\mu-4)$		$A_{yz}\ A_{xz}^3$
			$-6\mu(\mu+1)$		$-6\mu(\mu+1)(\mu-2)$	$A_{yz}^2\ A_{xz}^2$
				$i4\mu(\mu-1)(\mu+1)$		$A_{yz}^3\ A_{xz}$
					$(\mu+1)\mu(\mu-1)(\mu-2)$	A_{yz}^4

Table 6 (For an example see 3.45)

	L^4	L^2	1	
A^4_{xz}	$\frac{3}{8}$	$-\frac{3}{4}L_o^2 - \frac{3}{4}(4\lambda-5)$	$\frac{3}{8}L_o^4+\frac{3}{8}(8\lambda-9)L_o^2+3\lambda(\lambda-2)$	1
		$-\frac{1}{2}$	$\frac{1}{2}L_o^2 + 3(\lambda-1)$	$L^2_{+1} + L^2_{-1}$
			$-L_o$	$L^2_{+1} - L^2_{-1}$
			$\frac{1}{4}$	$L^4_{+1} + L^4_{-1}$
$A_{yz}\,A^3_{xz}$		$-\frac{3}{4}\,iL_o$	$\frac{3}{4}iL_o^3 + \frac{3}{4}(2\lambda-1)iL_o$	1
			$\frac{5}{4}iL_o$	$L^2_{+1} + L^2_{-1}$
		$\frac{1}{4}i$	$-\frac{1}{4}iL_o^2-\frac{3}{2}\lambda i$	$(L^2_{+1} - L^2_{-1})$
			$-\frac{1}{4}i$	$L^4_{+1} - L^4_{-1}$
$A^2_{yz}\,A^2_{xz}$	$\frac{1}{8}$	$-\frac{1}{4}L_o^2-\lambda-\frac{\mu}{2}+\frac{5}{4}$	$\frac{1}{8}L_o^4+(\lambda+\frac{\mu}{2}-\frac{13}{8})L_o^2+\lambda(\lambda+\mu-2)$	1
			$\frac{1}{2}(\mu-4)$	$L^2_{+1} + L^2_{-1}$
			L_o	$L^2_{+1} - L^2_{-1}$
			$-\frac{1}{4}$	$L^4_{+1} + L^4_{-1}$

Table 6 (Cont.)

	L^4	L^2	1	
$A^3_{yz}\ A_{xz}$		$-\frac{3}{4}iL_o$	$\frac{3}{4}iL_o^3 + \frac{3}{4}(2\lambda + 2\mu - 3)iL_o$	1
			$-\frac{1}{4}iL_o$	$L^2_{+1} + L^2_{-1}$
		$\frac{1}{4}i$	$-\frac{1}{4}iL_o^2 - \frac{3}{2}(\lambda + \mu - 2)i$	$L^2_{+1} - L^2_{-1}$
			$\frac{1}{4}i$	$L^4_{+1} - L^4_{-1}$
A^4_{yz}	$\frac{3}{8}$	$-\frac{3}{4}L_o^2 - \frac{3}{4}(4\lambda+4\mu-5)$	$\frac{3}{8}L_o^4 + \frac{3}{8}(8\lambda+8\mu-9)L_o^2 + 3(\lambda+\mu)(\lambda+\mu-2)$	1
		$\frac{1}{2}$	$-\frac{1}{2}L_o^2 - 3(\lambda+\mu-1)$	$L^2_{+1} + L^2_{-1}$
			L_o	$L^2_{+1} - L^2_{-1}$
			$\frac{1}{4}$	$L^4_{+1} - L^4_{-1}$

Table 7. Matrix representation of A_{xz}, A_{yz} and A_{yx}.
(For example see 3.47)

7a.

A_{xz}	ϕ_0	ϕ_{+1}	ϕ_{-1}	ϕ_{+2}	ϕ_0'	ϕ_{-2}
ϕ_0						
ϕ_{+1}	$\sqrt{2}$					
ϕ_{-1}						
ϕ_{+2}		$\sqrt{2}$				
ϕ_0'			1			
ϕ_{-2}						

7b.

A_{yz}	ϕ_0	ϕ_{+1}	ϕ_{-1}	ϕ_{+2}	ϕ_0'	ϕ_{-2}
ϕ_0						
ϕ_{+1}						
ϕ_{-1}	$\sqrt{2}$					
ϕ_{+2}						
ϕ_0'		1				
ϕ_{-2}			$\sqrt{2}$			

7c.

A_{yx}	ϕ_0	ϕ_{+1}	ϕ_{-1}	ϕ_{+2}	ϕ_0'	ϕ_{-2}
ϕ_0						
ϕ_{+1}						
ϕ_{-1}		1				
ϕ_{+2}						
ϕ_0'				$\sqrt{2}$		
ϕ_{-2}					$\sqrt{2}$	

Chapter 4

An Integral Form for the Wave Functions

The nuclear Hamiltonian, with central force alone, is invariant with respect to rotations of the orbital co-ordinates in three dimensions. The true eigenfunctions therefore, spread out irreducible representations of the group of rotations in three dimensions (R_3) i.e. they have the orbital angular momentum (L) a good quantum number.

From the operators of the U_3-group (2.2) it can be seen that R_3 is a subgroup of SU_3. Orbital states may therefore be classified according to SU_3 and R_3 in the same way that we have states ϕ classified according to SU_3, SU_2 and U_1. The sets of states classified in the two different ways span the same space and hence a relation exists of the form;-

$$\phi([f]\,(\lambda\mu)\epsilon\Lambda\nu) = \sum_{\alpha\, L'K'} a(\lambda\mu\,\epsilon\Lambda\nu, \alpha L'K')\,\psi([f]\,(\lambda\mu)\alpha L'K') \qquad \underline{4.1}$$

where both ϕ and ψ are normalised.

The a's are numerical coefficients and the ψ's are the states classifed according to R_3 with the operator L_0 diagonal. States with the same L'K' labels, in a representation $(\lambda\mu)$, are distinguished by an arbitrary labelling α.

We define the α-label in the expansion of the leading state i.e.

$$\phi([f]\ (\lambda\mu)) = \sum_{L'K'} a(\lambda\mu, K'L')\ \psi([f](\lambda\mu)K'L'K') \qquad \underline{4.2}$$

Other states ψ may be found by suitable operations with the R_3-group operators. Thus :-

$$L^p_{\pm 1}\ \psi(K'L'K') = \psi'(K'L'K'\pm p)$$

In <u>4.2</u> the ψ-states are defined to within a phase depending on the phase of ϕ and the definition of the a-coefficients. The ψ-states will be defined to be exactly those defined by Elliott (1958, I,(24)) in the equation :-

$$\chi([f]\ (\lambda\mu)K') = \sum_{L'} o(\lambda\mu\ K'L')\ \psi([f](\lambda\mu)K'\ L'\ K') \qquad \underline{4.3}$$

where

$\chi([f]\ (\lambda\mu)K') \equiv \chi([f](\lambda\mu)\ \epsilon = 2\lambda + \mu,\ \Lambda = \mu/2,\ K')$ and the $o(\lambda\mu\ K'L')$ are positive for non-negative K. Since the relation between the ϕ's and the χ's is known (<u>2.24</u>) we have:-

$$\phi([f]\ (\lambda\mu)) = \sum_{K'} b(\mu/2\ \mu, K')\ \chi([f]\ (\lambda\mu)K')$$

$$= \sum_{K'L'} b(\mu/2\ \mu, K')\ o(\lambda\mu\ \ K'L')\ \psi([f]\ (\lambda\mu)\ K'\ L'\ K') \qquad \underline{4.4}$$

$$\equiv \sum_{K'L'} a(\lambda\mu\ \ K'\ L')\ \psi([f]\ (\lambda\mu)\ K'\ L'\ K')$$

from <u>4.2</u>

We deduce

$$a(\lambda\mu\ \ K'\ L') = b(\mu/2\ \mu,\ K')\, c(\lambda\mu\ \ K'\ L') \qquad 4.5a$$

In order that $\psi(K\ L\ M) = \psi(-K\ L\ M)$, the coefficients $c(\lambda\mu\ \ -K\ L) = (-)^{L+\lambda+\mu}\, c(\lambda\mu\ K\ L)$ (Elliott 1958 II sec. 6)
Since $b(\mu/2,\mu,K) = b(\mu/2,\mu - K)$ (2.24), we find

$$a(\lambda\mu\ -K\ L) = (-)^{L+\lambda+\mu}\, a(\lambda\mu\ K\ L) \qquad 4.5b$$

The relation in 4.2 is true in any frame of reference, in particular in a frame rotated through Euler angles α, β, γ $(=\Omega)$ (Edmunds, 1957, p.7) Thus

$$\psi_\Omega([f](\lambda\mu)) = \sum_{L'K'} a(\lambda\mu\ K'\ L')\, \psi_\Omega([f]\ (\lambda\mu)\ K'\ L'\ K') \qquad 4.6$$

where the states are now defined relative to the rotated frame Ω.

With the relationships

$$\psi_\Omega(K'L'K') = \sum_{K''} \left(\mathcal{D}^{L'}_{K''K'}(\Omega)\right)^{*} \psi(K'L'K'') \qquad 4.7a$$

(Elliott 1958, III p.28)

and

$$\int \mathcal{D}^{L}_{MK}(\Omega) \left(\mathcal{D}^{L'}_{K''K'}\right)^{*} d\Omega = \frac{\delta(LL')\ \delta(MK'')\ \delta(KK')}{(2L+1)} \qquad 4.7b$$

(Edmunds 1957, p.62)

we find, by multiplying 4.6 by $\mathcal{D}^{L}_{MK}(\Omega)$ and integrating over the angles Ω, after substituting 4.7a for ψ_Ω,

$$\int \phi_\Omega([f](\lambda\mu))\, \mathcal{D}^{L}_{MK}(\Omega) \quad d\Omega = \frac{a(\lambda\mu\ K\ L)}{(2L+1)}\, \psi([f](\lambda\mu)K\ L\ M) \qquad \underline{4.8}$$

where

$$\int d\Omega \text{ means } \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma$$

Equation 4.8 is very similar to the integral form of the ψ-states given in terms of the χ-states (Elliott 1958 II (21). Elliott has shown that by using all the states $\chi_\Omega([f](\lambda\mu)K)$ with maximum ϵ in a representation all states $\psi([f](\lambda\mu)KLM)$ may be generated. Since the state $\phi_\Omega([f](\lambda\mu))$ may be expanded in terms of all the states $\chi_\Omega([f](\lambda\mu)K)$, because none of the b-coefficients can ever be zero, it follows immediately that the integral in 4.8 must generate all states of a representation.

By using the χ-functions as 'intrinsic' states in the integral, it was found that a different state is needed for each 'band' K. With the ϕ-function however, only one intrinsic state need be defined for each representation and that is the leading state.

We may express the representations (L) of R_3 appearing in the $(\lambda\mu)$ representation of SU_3 in the simple way (Elliott 1958, I, (24)):-

$$K = \min(\lambda\mu),\ \min(\lambda\mu) - 2,\ \ldots,\ 0 \text{ or } 1 \qquad \underline{4.9a}$$

while, for each value of K,

$$L = K, K + 1, \ldots , K + \max (\lambda\mu) \qquad \underline{4.9b}$$

except when $K = 0$ when

$$L = \max (\lambda\mu), \max (\lambda\mu) - 2, \ldots , 0 \text{ or } 1 \qquad \underline{4.9c}$$

These relations may be deduced from the explicit calculations of the $c(\lambda\mu\ KL)$, defined in 4.3, or the $a(\lambda\mu\ KL)$, defined in 4.2.

The appearance of a band-like structure for the ψ-states is clear from the relations in 4.9. Each band is labelled with a K-number. Within each band, states with definite orbital angular momentum appear in the way that angular momentum states appear in a rotational band. In this classification scheme however, the bands are cut off at some upper limit while the rotational bands extend to infinity.

In this analysis, the K-label in the ψ-states is not a proper quantum number in the sense that it describes a classification with respect to some group. States with different K-labels are not necessarily orthogonal, in fact, it is possible for one state to have two different K-labels.

The K-label will help to distinguish different states with the same angular momentum.

The expansion coefficients $a(\lambda\mu\, KL)$

The $a(\lambda\mu\ KL)$ may be derived explicitly from the relation in 4.5a. The calculation of the $b(\mu/2,\mu\, K)$ is straightforward using the closed expression in 2.26. Recurrence relations derived by Elliott (1958, II, sec. 5), enable the $c(\lambda\mu\, KL)$ to be calculated without too much difficulty.

A more direct method for calculating the a-coefficients has recently been found (J.P. Elliott private communication).

Since the a-coefficients depend only on the group properties of the representation $(\lambda\mu)$ of SU_3, the $[f]$ label for the states ψ & ϕ will not enter into the problem. This has already been indicated by the labelling of the coefficients. We shall therefore, omit the $[f]$ labels in the following proof.

On forming the product:-

$$\Big(\phi(\lambda\mu)\ \Big|\ \psi\,((\lambda\mu),\,K\,L\,K')\Big)\ a(\lambda\mu.K\,L) \qquad 4.10$$

and expanding the $\phi([f]\,(\lambda\mu))$ using 4.2 we find:-

$$\Big(\phi(\lambda\mu)\ \Big|\ \psi((\lambda\mu)K\,L\,K')\Big)\ a(\lambda\mu\,K\,L)$$

$$= \sum_{K''L''} a^*(\lambda\mu\,K''\,L'')\ \Big(\psi((\lambda\mu),K''\,L''\,K'')\Big|\,\psi((\lambda\mu)KLK')\Big)\ a(\lambda\mu\,.KL)$$

$$= a^*(\lambda\mu\,K'L)\ a(\lambda\mu\,KL)\ \Big(\psi((\lambda\mu)K'LK')\ \Big|\ \psi((\lambda\mu)KLK')\Big) \qquad 4.11$$

The last relation coming from the orthogonality of the ψ -states with respect to the proper-quantum numbers representing the angular momentum (L) and its projection.

The a-coefficients are given directly by putting $K' = K$ in <u>4.11</u>. Thus

$$a^2(\lambda\mu\ K\ L) = \Big(\phi(\lambda\mu)\ \Big|\ \psi((\lambda\mu)K'LK)\Big)\ a(\lambda\mu\ K\ L) \qquad \underline{4.12}$$

Knowing the a-coefficients from <u>4.12</u>, the overlaps of the ψ -states may be deduced directly from <u>4.11</u>.

The problem reduces to the calculation of such functions shown in <u>4.10</u>.

Defining the rotation operator $\mathcal{R}(\Omega)$ by $\mathcal{R}(\Omega)\phi = \phi_\Omega$ we have, from <u>4.10</u> and <u>4.8</u> :-

$$\Big(\phi(\lambda\mu)\ \Big|\ \psi((\lambda\mu)KLK')\Big)\ a(\lambda\mu\ KL) = (2L+1)\int \mathcal{D}^{L}_{K'K}(\Omega)\Big(\phi(\lambda\mu)\ \Big|\mathcal{R}(\Omega)\Big|\ \phi(\lambda\mu)\Big)d\Omega \qquad \underline{4.13}$$

We appeal to a particularly simple explicit form for the $\phi(\lambda\mu)$ in terms of a product of one quantum, normalised states with symmetry $(\lambda\mu)$. There will, of course, be a number of such functions corresponding to the ways of ordering the numbers $1, 2, \ldots (2\mu + \lambda)$ in the tableau of $(\lambda\mu)$. Since one such arrangement can be obtained from another by a permutation, the overlap $\Big(\phi(\lambda\mu)\ \Big|\ \phi_\Omega(\lambda\mu)\Big)$ in <u>4.13</u> will be independant of which one we use. We therefore choose the most convenient.

A state $\phi(\lambda 0)$ made up from single particle states $\phi(i)$ may be written:-

$$\phi(\lambda 0) = \phi_z(1)\phi_z(2) \dots \phi_z(\lambda) \qquad \underline{4.15a}$$

where $\phi_z(i)$ is a single particle state with one quanta in the z-direction and zero quanta in the xy-plane i.e. $\phi_z \sim z$

The state $\phi(\lambda\mu)$ has less than full symmetry and so a representative state may be written :-

$$\phi(\lambda\mu) = \sqrt{\tfrac{1}{2}}\{\phi_z(1)\phi_x(2) - \phi_x(1)\phi_z(2)\}\sqrt{\tfrac{1}{2}}\{\phi_z(3)\phi_x(4) - \phi_x(3)\phi_z(4)\} \ldots$$

$$\ldots\sqrt{\tfrac{1}{2}}\{\phi_z(2\mu-1)\,\phi_x(2\mu) - \phi_x(2\mu-1)\phi_z(2\mu)\}\phi_z(2\mu+1)\,\ldots\,\phi_z(\lambda+2\mu) \qquad \underline{4.15b}$$

Now $\phi_\Omega(\lambda\mu)$ will be the product of all rotated single particle functions and hence $\left(\phi(\lambda\mu)\middle|R(\Omega)\middle|\phi(\lambda\mu)\right)$ reduces to a product of such matrix elements as:

$$\left(\phi_z\,\middle|\,(\phi_z)_\Omega\right) = \cos\beta \qquad \underline{4.16a}$$

and

$$\left(\sqrt{\tfrac{1}{2}}(\phi_z(1)\phi_x(2) - \phi_x(1)\phi_z(2))\middle|\sqrt{\tfrac{1}{2}}(\phi_z(1)\phi_x(2) - \phi_x(1)\phi_z(2))_\Omega\right)$$

$$\equiv (\cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma) \qquad \underline{4.16b}$$

which follows from the explicit form for the ϕ's.

Hence we may write:-

$$\left(\phi(\lambda\mu)\,\middle|\,\phi_\Omega(\lambda\mu)\right) = (\cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma)^\mu(\cos\beta)^\lambda \qquad \underline{4.17}$$

Substituting <u>4.17</u> in <u>4.13</u> we find:-

$$\left(\phi(\lambda\mu)\middle|\psi((\lambda\mu)K\,L\,K')\right) a(\lambda\mu\,KL) = (2L+1)\int \mathcal{D}^L_{K'K}(\Omega)\ \times$$

$$(\cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma)^\mu \times (\cos\beta)^\lambda\,d\Omega \qquad \underline{4.18}$$

Since $\mathcal{D}^L_{K'K}(\Omega) = e^{iK'\alpha}\,e^{iK\gamma}\,d^L_{K'K}(\beta)$ (Edmunds 1957, 4.1)

and

$$(\cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma)^\mu = \sum_{n=0}^{\mu}\frac{\mu!(-)^n}{n!(\mu-n)!}\times$$

$$(\sin\alpha\sin\gamma\cos\beta)^n(\cos\alpha\cos\gamma)^{\mu-n} \qquad \underline{4.19}$$

by a binomial expansion,

we may rewrite 4.18 such that:-

$$\left(\phi(\lambda\mu)\middle|\psi((\lambda\mu)KLK')\right) a(\lambda\mu\ KL) = \frac{(2L+1)}{8\pi^2} \sum_{n=0}^{\mu} \frac{\mu!(-)^n}{n!(\mu-n)!} \times$$

$$A_n(\mu K')\, B_n(\mu K)\, C_n(\lambda LK'K) \qquad \underline{4.20}$$

where :-

$$A_n(\mu K') = \int_0^{2\pi} d\alpha\, e^{iK'\alpha} \sin^n\alpha \cos^{\mu-n}(\alpha) \qquad \underline{4.21a}$$

$$B_n(\mu K) = \int_0^{2\pi} d\gamma\, e^{iK\gamma} \sin^n\gamma \cos^{\mu-n}(\gamma) \qquad \underline{4.21b}$$

$$C_n(\lambda LK'K) = \int_{-1}^{+1} d(\cos\beta)\, d^L_{K'K}(\beta) \cos^{\lambda+n}(\beta) \qquad \underline{4.21c}$$

Thus, by substituting 4.20 into 4.11, a closed expression for the a-coefficients and the overlaps may be found.

The functions A_n and B_n are of the same form and the integrals may be performed in general. It can be shown that:-

$$A_n(\mu K') = \frac{2\pi n!(\mu-n)!}{(i)^n 2^\mu} \sum_p \frac{(-)^{n-p}}{p!(n-p)!\left(\frac{\mu-K'}{2}-p\right)!\left(\frac{\mu+K'}{2}-n+p\right)!}$$

$$\underline{4.22a}$$

where p is summed from $\max\left[0, \frac{1}{2}(2n-\mu-K')\right]$ to $\min\left[n, \frac{\mu-K'}{2}\right]$.

In the actual evaluation of A_n it has been found useful to use the expression:-

$$A_n(\mu K') = \frac{2\pi(\mu - n)!(-)^n}{(i)^n 2^\mu m!\ell!} \left((1 + D_y)^\ell (1 + D_x)^m (y - x)^n\right)_{x=y=0} \qquad \underline{4.22b}$$

where $\ell = \frac{\mu + K'}{2}$, $m = \frac{\mu - K'}{2}$ and $D_y = \frac{\partial}{\partial y}$, $D_x = \frac{\partial}{\partial x}$.

It can easily be seen that the expansion of 4.22b yields the summation in 422a.

No general expression has yet been found for the C_n functions but it is possible to evaluate them in special cases.

When $K = K' = 0$ (implying μ even), $d^L_{oo}(\beta) = P_L(\cos\beta)$ (Edmunds 1957. p.59).

Thus $C_n(\lambda L00) = \int_{-1}^{+1} P_L(x)\, x^{\lambda+n}\, dx.$ 4.23

Expanding $x^{\lambda+n}$ in terms of Legendre functions we find:-

$$x^{\lambda+n} = \sum_{L'} \frac{(2L' + 1)2^{L'}(\lambda + n)!\{(\lambda + n + L')/2\}!}{(n+ \lambda + L' + 1)!\ \{(\lambda + n - L')/2\}!} P_{L'}(x) \qquad \underline{4.24}$$

where L' is such that $(\lambda + n + L')$ is even. (Whittaker & Watson 1927, p.310).

Thus, from the orthogonality relations for the Legendre Polynomials, we have:-

$$C_n(\lambda L00) = \frac{2^{L+1}(\lambda + n)!\{(\lambda + n + L)/2\}!}{(\lambda + n + L + 1)!\{(\lambda + n - L)/2\}!}$$

if $\lambda + n + L$ is even

$= 0$ otherwise. 4.25

In these particular cases the A and B coefficients simplify to:-

$$A_n(\mu 0) = B_n(\mu 0) = \frac{(n-1)!!\;(\mu-n-1)!!}{\mu!!}\,2\pi \qquad \text{for } n \text{ and } \mu \text{ even}$$

$$= 0 \text{ otherwise} \qquad \underline{4.26}$$

where $\mu!! \equiv \mu(\mu-2)(\mu-4)\ \ldots\ 2.$ etc.

We can, in this case, using 4.12, write down the general expression:-

$$a^2(\lambda\mu 0L) = \frac{(2L+1)\,2^{L-\mu}\,\mu!}{(\mu/2)!\,(\mu/2)!}\sum_{n=0}^{\mu}{}' \frac{(n-1)!!\,(\mu-n-1)!!}{n!!\,(\mu-n)!!} \times$$

$$\frac{(\lambda+n)!\,\{(\lambda+n+L)/2\}!}{(\lambda+n+L+1)!\,\{(\lambda+n-L)/2\}!} \qquad \underline{4.27}$$

where Σ' sums over <u>even</u> n only.

The form of the $c_n(\lambda LK'K)$ coefficients in two more important cases are given below for K and K' $\leqslant 2$

$$c_n(\lambda 220) = \frac{\sqrt{6}}{(\lambda+n+1)\,(\lambda+n+3)}\left(\frac{1-(-)^{\lambda+n+1}}{2}\right) \qquad \underline{4.28a}$$

$$c_n(\lambda 222) = \frac{(\lambda+n+2)}{(\lambda+n+1)(\lambda+n+3)}\left(\frac{1-(-)^{\lambda+n+1}}{2}\right) \qquad \underline{4.28b}$$

$$+\left(\frac{1}{(\lambda+n+2)}\right)\left(\frac{1-(-)^{\lambda+n}}{2}\right)$$

Using 4.22, 4.27 and 4.28, all the a-coefficients required in this thesis may be calculated.

Chapter 5

Energy Matrix Spread Out by the States $\Psi([f](\lambda\mu)KLM)$

The general method for calculating matrix elements.

We now develop the method for determining the energy in totally antisymmetric states $\Psi([f](\lambda\mu)KLM)$ of the realistic Hamiltonian discussed in Chapter 1. The states Ψ have orbital functions defined in the last chapter and, for the moment the charge-spin labels have been suppressed. Since the Hamiltonian is scalar in the charge and spin spaces, it follows that the matrix elements of H between states Ψ with different charge-spin labels must be zero.

The Hamiltonian with central potential alone is spherically symmetric. This means that the operator H will have the same form no matter in which frame of reference it is seen.

Thus

$$H\Psi([f](\overline{\lambda\mu})KLM) = \frac{(2L+1)}{a(\overline{\lambda\mu},KL)} \int \mathcal{D}^{L}_{MK}(\Omega) H \, \Phi_{\Omega}([f](\overline{\lambda\mu})) \, d\Omega$$

(from 4.8)

$$\equiv \frac{(2L+1)}{a(\overline{\lambda\mu},KL)} \int \mathcal{D}^{L}_{MK}(\Omega) \, H_{\Omega} \, \Phi_{\Omega}([f](\overline{\lambda\mu}) \, d\Omega \qquad 5.1$$

where H_{Ω} is the Hamiltonian seen in the intrinsic frame. With the assumption that $[f]$ is a good quantum number, we may just consider the states of maximum orbital symmetry when calculating the energy of low lying energy levels (Chapter 1).

We have then

$$H_\Omega \, \Phi_\Omega(\overline{\lambda\mu}) = \sum_{\lambda\mu\epsilon\Lambda\nu} ((\lambda\mu)\epsilon\Lambda\nu \,|\, H_\Omega \,|\, \overline{\lambda\mu}) \, \Phi_\Omega((\lambda\mu)\epsilon\Lambda\nu) \qquad 5.2$$

where $((\lambda\mu)\epsilon\Lambda\nu|H_\Omega|\lambda\bar{\mu})$ is the matrix element coupling the state $\Phi_\Omega((\lambda\mu)\epsilon\Lambda\nu)$ to the leading state of the $(\overline{\lambda\mu})$ - representation in the matrix representation of H_Ω, spread out by the states $\Phi_\Omega(\lambda\mu\epsilon\Lambda\nu)$, in the intrinsic frame.

From 3.37 we may write :-

$$E(\lambda\mu\epsilon\Lambda\nu) \quad \Phi_\Omega(\lambda\mu) = \Phi_\Omega((\lambda\mu)\epsilon\Lambda\nu) \qquad 5.3$$

Thus

$$H_\Omega \, \Phi_\Omega(\overline{\lambda\mu}) = \sum_{\lambda\mu\epsilon\Lambda\nu} ((\lambda\mu)\epsilon\Lambda\nu|H_\Omega|\overline{\lambda\mu}) E(\lambda\mu,\epsilon\Lambda\nu) \, \Phi_\Omega(\lambda\mu) \qquad 5.4$$

Expanding $\Phi_\Omega(\lambda\mu)$ in terms of $\Psi_\Omega((\lambda\mu)K'L'K')$, using 4.2, we have :-

$$\Phi_\Omega(\lambda\mu) = \sum_{K'L'} a(\lambda\mu K'L') \; \Psi_\Omega((\lambda\mu)K'L'K') \qquad 5.5$$

Since in 5.4 the operator E acts on the leading state of a representation, it may be replaced by the equivalent operator $\tilde{E}$. From 5.4 and 5.5 we see that the $\tilde{E}$ operators will now act directly on the states Ψ_Ω. Because the $\tilde{E}$ - operators are functions only of the angular momentum operators L_0 and $L_{\pm 1}$, we can deduce the numbers f where:-

$$\tilde{E}(\lambda\mu\varepsilon_\Lambda\nu)\ \Psi_\Omega((\lambda\mu)K'L'K') = \sum_{K''} f(\lambda\mu\varepsilon_\Lambda\nu|L'K'K'')\Psi_\Omega((\lambda\mu)K'L'K'') \qquad \underline{5.6}$$

Notice that the $\tilde{E}$ operators can only change the last label in Ψ_Ω.

Thus, from 5.4, 5.5 and 5.6,

$$H_\Omega\Phi_\Omega(\overline{\lambda\mu}) = \sum_{\lambda\mu\varepsilon_\Lambda\nu}((\lambda\mu)\varepsilon_\Lambda\nu|H_\Omega|\overline{\lambda\mu})\sum_{K'L'}a(\lambda\mu K'L')\sum_{K''}f(\lambda\mu\varepsilon_\Lambda\nu|L'K'K'')\ x$$

$$\Psi_\Omega((\lambda\mu)K'L'K'') \qquad \underline{5.7}$$

Defining the number g by:-

$$g(\overline{\lambda\mu}|\lambda\mu\ L'K'K'') = \sum_{\varepsilon\Lambda\nu}(\lambda\mu\varepsilon_\Lambda\nu|H_\Omega|\overline{\lambda\mu})f(\lambda\mu,\varepsilon_\Lambda\nu|L'K'K'')\ , \qquad \underline{5.8}$$

equation 5.7 reduces to:-

$$H_\Omega\ \Phi_\Omega(\overline{\lambda\mu}) = \sum_{\lambda\mu K'L'K''} a(\lambda\mu K'L')\ g(\overline{\lambda\mu}|\lambda\mu L'K'K'')\Psi_\Omega((\lambda\mu)K'L'K'') \qquad \underline{5.9}$$

Substituting 5.9 into 5.1 we find

$$H\ \psi((\overline{\lambda\mu})KLM) = \frac{(2L+1)}{a(\overline{\lambda\mu}KL)}\int \mathcal{D}^L_{MK}(\Omega) \sum_{\substack{K'L'K''\\ \lambda\mu}} a(\lambda\mu K'L')g(\overline{\lambda\mu}|\lambda\mu L'K'K'')\ x$$

$$\Psi_\Omega((\lambda\mu)K'L'K'')d\Omega \qquad \underline{5.10}$$

Since $\Psi_{\Omega}((\lambda\mu)K'L'K'') = \sum_{K'''} \left(\mathcal{D}^{L'}_{K'''K''}(\Omega) \right)^{*} \Psi((\lambda\mu)K'L'K''')$ 5.11

(Elliott 1958, III, p.28)

we may integrate directly over the $\mathcal{D}$-functions, using 4.7b, to find:-

$$H\Psi((\bar{\lambda}\bar{\mu})KLM) = \sum_{\lambda\mu K'} \frac{a(\lambda\mu K'L)}{a(\overline{\lambda\mu}KL)} \, g(\overline{\lambda\mu}|\lambda\mu LK'K) \; \Psi((\lambda\mu)K'LM) \qquad 5.12$$

From the definition of the phases of the ψ-state in 4.5b we have

$$\Psi((\lambda\mu)|K'|LM) = \Psi((\lambda\mu) - |K'|LM) \qquad 5.13$$

with $a(\lambda\mu - |K'|L) = (-)^{L+\lambda+\mu} a(\lambda\mu|K'|L)$.

Defining now

$$\tilde{g}(\overline{\lambda\mu}|\lambda\mu\ L|K'|K) = g(\overline{\lambda\mu}|\lambda\mu\ L|K'|K) + (-)^{L+\lambda+\mu}\, g(\overline{\lambda\mu}|\lambda\mu\ L -|K'|K), \qquad 5.14a$$

with the exception of $K' = 0$, when $\tilde{g}(\overline{\lambda\mu}|\lambda\mu\ L0K) = g(\overline{\lambda\mu}|\lambda\mu\ L0K)$, 5.14b

we may rewrite 5.12 as:-

$$H\psi((\lambda\mu)KLM) = \sum_{\lambda\mu|K'|} \frac{a(\lambda\mu|K'|L)}{a(\overline{\lambda\mu}\ KL)} \, \tilde{g}(\overline{\lambda\mu}|\lambda\mu L|K'|K) \, \psi((\lambda\mu)|K'|LM) \qquad 5.15$$

We therefore deduce the elements of the energy matrix $[H]$ to be:-

$$\left\langle (\lambda''\mu'')|K''|\,L''M''\,\middle|\,H\,\middle|\,(\overline{\lambda\mu})KLM\right\rangle = \sum_{|K'|} \frac{a(\lambda''\mu''\,|K'|L)}{a(\overline{\lambda\mu}KL)} \times$$

$$\tilde{g}(\overline{\lambda\mu}\,|\lambda''\mu''\,L|K'|K) \times \left\langle (\lambda''\mu'')|K''|LM\,\middle|\,(\lambda''\mu'')\,|K'|\,LM\right\rangle \delta(L''L)\,\delta(M''M), \quad \underline{5.16}$$

the sum over $|K'|$ persisting since the states are not orthogonal with respect to this number.

The eigenvalues of $[H]$ will be the prediction of the energy levels. The eigenfunctions will correspond to the stationary states of these levels. If the classification is a good one, the off-diagonal elements in $[H]$ will be small, that is, the diagonal elements will be a good approximation to the true eigenvalues and the true eigenfunctions will involve little mixing of λ,μ or K.

It is not necessary to diagonalize the matrix $[H]$, with elements defined in 5.16, to deduce the energy levels, even though it has the advantage of being symmetric. The non-symmetric matrix with elements $\tilde{g}(\overline{\lambda\mu}\,|\lambda\mu\;L|K'|K)$ has the same eigenvalues as $[H]$

To prove this last statement, let us first define unnormalised states:-

$$\Psi'((\lambda\mu)K\,LM) = a(\lambda\mu\;K\;L)\,\Psi((\lambda\mu)\;K\;LM) \qquad \underline{5.17}$$

Equation 5.15 may now be written as:-

$$H\;\Psi'((\overline{\lambda\mu})KLM) = \sum_{\lambda\mu|K'|} \tilde{g}(\overline{\lambda\mu}|\;\lambda\mu\;L|K'|K)\;\;\Psi'((\lambda\mu)|K'|LM) \qquad \underline{5.18}$$

The eigenfunctions $\tilde{\Psi}_i(LM)$ of the matrix $[H]$ will be linear combinations of the Ψ' thus :-

$$\tilde{\Psi}_i(LM) = \sum_{\overline{\lambda\mu}|K|} \Psi'((\overline{\lambda\mu})|K|LM) \ \alpha_i((\overline{\lambda\mu})|K|LM) \tag{5.19}$$

where α_i are expansion constants.

If E_i is the eigenvalue of $[H]$ corresponding to $\tilde{\Psi}_i$ then, within the given $[f]$,

$$H\,\tilde{\Psi}_i(LM) = E_i\,\tilde{\Psi}_i(LM) = E_i \sum_{\overline{\lambda\mu}|K|} \Psi'((\overline{\lambda\mu})|K|LM) \ \alpha_i((\overline{\lambda\mu})|K|LM) \tag{5.20}$$

also $\quad H\,\tilde{\Psi}_i(LM) = \sum_{\overline{\lambda\mu}|K|} \alpha_i((\overline{\lambda\mu})|K|LM) \ H\Psi'((\overline{\lambda\mu}|K|LM)$

$$= \sum_{\overline{\lambda\mu}|K|} \ \sum_{\lambda\mu|K'|} \alpha_i(\overline{\lambda\mu}|K|LM) \ \tilde{g}'(\overline{\lambda\mu}|\lambda\mu L|K'|K) \times \Psi'((\lambda\mu)|K'|LM) \tag{5.21}$$

Replacing the left-hand side of 5·20 by the right-hand side of 5·21 and equating coefficients of Ψ' - since these states are independent - we find :-

$$E_i \ \alpha_i((\lambda\mu)|K'|LM) = \sum_{\overline{\lambda\mu}|K|} \tilde{g}(\overline{\lambda\mu}|\lambda\mu L|K'|K) \ \alpha_i \ ((\lambda\mu)|K|LM) \tag{5.22a}$$

or, in matrix notation,

$$E_i \, [\alpha_i] = [\tilde{g}'] \, [\alpha_i] \tag{5.22b}$$

Clearly now the E_i are the eigenvalues of the $[\tilde{g}]$ matrix. The eigenfunctions of $[\tilde{g}]$ give the values of $\alpha_i((\overline{\lambda\mu})|K|LM)$ to within a normalisation factor. If the eigenfunctions $\tilde{\Psi}_i$ of $[H]$ are to be normalised then, from 5.19 and 5.17,

$$(\tilde{\Psi}_i(LM)|\tilde{\Psi}_i(LM)) = 1$$

$$= \sum_{|K||K'|} \alpha_i((\overline{\lambda\mu})|K|LM)\ \alpha_i^*((\lambda\mu)|K'|LM)\ a(\overline{\lambda\mu}|K|L)\ a^*(\overline{\lambda\mu}|K'|L)$$

$$\Big((\overline{\lambda\mu})|K'|LM\Big|(\overline{\lambda\mu})|K|LM\Big) \qquad \underline{5.23}$$

Knowing the a-coefficients and the overlaps in 5.23 we can deduce the unknown normalisation coefficient for $\tilde{\Psi}_i$.

We notice that, in deducing the eigenvalues from 5.22, no knowledge of the a-coefficients or the overlaps $(\Psi'|\Psi)$ need be possessed since $\tilde{g}$ is independent of both.

Since most of the calculation of the matrix elements of $[H]$ go into the $\tilde{g}$ - functions, (see 5.16), a useful check may be put on the results by using the symmetrical property of the matrix for H. Alternatively, the symmetry of $[H]$ will provide a method for calculating one $\tilde{g}$-element from another when the a- coefficients and overlaps are known.

Since

$$\Big((\lambda''\mu'')|K''|\ LM\Big|H\Big|(\overline{\lambda\mu})KLM\Big) \equiv \Big((\overline{\lambda\mu})KLM\Big|H\Big|(\lambda''\mu'')|K''|LM\Big),$$

from 5·16 we find :-

$$\sum_{|K'|} \frac{a(\lambda''\mu''\,|K'|\,L)}{a(\overline{\lambda\mu}\,|K|\,L)}\,\tilde{g}(\overline{\lambda\mu}\,|\,\lambda''\mu''\,L\,|K'|\,K)\Big((\lambda''\mu'')\,|K''|\,LM\,\Big|\,(\lambda''\mu'')\,|K'|\,LM\Big)$$

$$\equiv \sum_{|K'|} \frac{a(\lambda\mu\,|K'|\,L)}{a(\lambda''\mu''|K''|\,L)}\,\tilde{g}(\lambda''\mu''\,|\,\overline{\lambda\mu}\,L\,|K'|\,|K''|)\Big((\overline{\lambda\mu})\,|K|\,LM\,\Big|\,(\overline{\lambda\mu})\,|K'|\,LM\Big) \qquad \underline{5.24}$$

In particular, when L = 0, the only K value will be K = 0. Thus,

$$\frac{a(\lambda''\mu''\,00)}{a(\overline{\lambda\mu}\,00)}\,\tilde{g}(\overline{\lambda\mu}\,|\,\lambda''\mu''\,000) \equiv \frac{a(\overline{\lambda\mu}\,00)}{a(\lambda''\mu''\,00)}\,\tilde{g}(\lambda''\mu''\,|\,\overline{\lambda\mu}\,00) \qquad \underline{5.25}$$

From the selection rule derived later (Table 8), we shall see that, when $(2\bar{\lambda} + \bar{\mu}) - (2\lambda'' + \mu'') = 12$ there can be no mixing of K, that is:

$$\tilde{g}(\overline{\lambda\mu}\,|\,\lambda''\mu''\,L\,|K'|\,K) \equiv 0 \quad \text{unless } |K'| = |K|$$

When $(2\bar{\lambda}+\bar{\mu}) - (2\lambda''+\mu'') = 6$, it is found that:-

$\tilde{g}(\overline{\lambda\mu}\,|\,\lambda''\mu''\,L\,|K'|\,K)$ is small for $|K'| \neq |K|$ compared with the term when $|K'| = |K|$.

If also the overlaps of the states in the $(\overline{\lambda\mu})$ representation are small, then :-

$$\frac{a(\lambda''\mu''\,|K''|\,L)}{a(\overline{\lambda\mu}\,|K|\,L)}\,\tilde{g}(\overline{\lambda\mu}\,|\,\lambda''\mu''\,L\,|K''|\,K\,)$$

$$\simeq \frac{a(\overline{\lambda\mu}\,|K|\,L)}{a(\lambda''\mu''\,|K''|\,L)}\,\tilde{g}(\lambda''\mu''\,|\,\overline{\lambda\mu}\,L\,|K|\,|K''|\,). \qquad \underline{5.26}$$

Of course, if the overlaps in the $(\lambda''\mu'')$ representation are also small, 5·26 will be very accurate.

The relations in 5·25 and 5·26 will be useful later on when we are estimating the mixing of representations in the Mg^{24} nucleus ($k = 8$ particles in the ds-shell)

The determination of the energy levels and stationary states using the states Ψ, merely depends on our ability to calculate the functions $\tilde{g}$. From 5·14 and 5·8 we see that the $\tilde{g}$'s are known if the matrix elements in the intrinsic frame can be calculated and if the constants f can be determined from 5·6.

Matrix elements in the intrinsic frame.

Only the matrix elements coupling a state $\Phi((\lambda\mu)\varepsilon\Lambda\nu)$ to the leading state, $\Phi(\bar{\lambda}\bar{\mu})$, of the $(\bar{\lambda}\bar{\mu})$ - representation have to be calculated (5·8). The construction of $\Phi((\lambda\mu)\varepsilon\Lambda\nu)$ from the leading state $\Phi(\lambda\mu)$ in Chapter 3 involved the introduction of certain configurational states $\Phi_i(\varepsilon\nu)$ (3·57).

$$\text{Thus}\quad \left((\lambda\mu)\varepsilon\Lambda\nu\,\middle|\,H_{\Omega}\middle|\bar{\lambda}\bar{\mu}\right) = \sum_i p_i^{*}(\lambda\mu,\Lambda)\left(i\varepsilon\nu\middle|H_{\Omega}\middle|\bar{\lambda}\bar{\mu}\right) \qquad 5.27$$

Since the $\Phi_i(\varepsilon\nu)$ and $\Phi(\bar{\lambda}\bar{\mu})$ are known explicitly, the calculation now follows the standard process of determining the (k-2, 2) fractional parentage coefficients for the configurational states i.e.

$$\Phi_i(\varepsilon\nu) = \sum_{\mathfrak{F}\,\Phi'} a^{\Phi_i(\varepsilon\nu)}_{\mathfrak{F}\Phi'}\;\mathfrak{F}\,\Phi' \qquad 5.28$$

where $a^{\Phi}_{\tilde{\Phi}\Phi'}$ are the fractional parentage coefficients

$\tilde{\Phi}$ is an antisymmetric state of k-2 particles in Φ and Φ' is an antisymmetric state of 2 particles in Φ and we are summing over all possible states $\tilde{\Phi},\Phi'$.

Thus we may write:

$$(i\varepsilon\ \nu\,|H_{\Omega}|\overline{\lambda\mu}) = \frac{k(k-1)}{2}\sum_{\tilde{\Phi}_1\Phi_1'}\ \sum_{\tilde{\Phi}_2\Phi_2'} a^{\Phi_i(\varepsilon\nu)}_{\tilde{\Phi}_1\Phi_1'}\ a^{\Phi(\overline{\lambda\mu})}_{\tilde{\Phi}_2\Phi_2'} \times$$
$$(\tilde{\Phi}_1\,|\,\tilde{\Phi}_2)\ (\Phi_1'\,|\,H_{ij}|\Phi_2') \qquad \underline{5.29}$$

since H is a two body operator i.e. $H = \sum_{i<j} H_{ij}$.

The $a^{\Phi_i}_{\tilde{\Phi}\,\Phi'}$ may be found by going back to the form for the $\Phi_i(\varepsilon\nu)$ (<u>3·51</u>) and explicitly taking out the last two particles using the known charge-spin fractional parentage reduction by one and two particles (Jahn & van Wieringen 1951, Elliott, Hope and Jahn, 1953) and the orbital fractional parentage coefficients. Since our orbital states are not vector coupled in any way, the orbital coefficients will just be normalisation constants. For example, with the four particle state $(ab\ c\ c)^{11}$, with symmetry [4] (<u>1·4b</u>), we find :-

$$(abcc)^{11}_{1234} = \left(\sqrt{\frac{1}{2}}\,(abc)^{22}_{123}c^{22}_{4} + \sqrt{\frac{1}{4}}\,(acc)^{22}_{123}b^{22}_{4} + \sqrt{\frac{1}{4}}(bcc)^{22}_{123}\ a^{22}_{4}\right)^{11}$$

and

$$(abcc)^{11}_{1234} = \sqrt{\tfrac{1}{2}}\Big(\Big\{\sqrt{\tfrac{1}{6}}(ab)^{13}_{12}(c^2)^{13}_{34} + \sqrt{\tfrac{1}{3}}(ac)^{13}_{13}(cb)^{13}_{34} + \sqrt{\tfrac{1}{3}}(bc)^{13}_{12}(ac)^{13}_{34}$$

$$+ \sqrt{\tfrac{1}{6}}(c^2)^{13}_{12}(ab)^{13}_{34}\Big\} - \Big\{\sqrt{\tfrac{1}{6}}(ab)^{31}_{12}(c^2)^{31}_{34} + \sqrt{\tfrac{1}{3}}(\quad)^{31}(\quad)^{31} + \ldots\ldots\Big\}\Big)^{11}$$

(See also Appendix 2 for an example of the reduction of an eight particle state).

From 5.30 we see that the only coefficients $a^{\Phi_i(\varepsilon\nu)}_{\tilde{\Phi}\Phi'}$ which need be calculated are those where $(\tilde{\Phi}_1 | \tilde{\Phi}_2)$ are non zero. A considerable amount of labour is saved by observing this fact.

For central forces, the two body matrix elements in 5.29 will vanish unless Φ'_1 and Φ'_2 have the same charge-spin labels.

Since the states Φ'^{13} and Φ'^{31} are symmetric in their orbital spaces ($\phi(s)$) while Φ'^{11} and Φ'^{33} are antisymmetric ($\phi(a)$), the only matrix elements of $V_c(ij)$ (defined in 1.2b) which can occur, are:-

$$(\Phi_1'^{13} | V_c(ij) | \Phi_2'^{13}) = - V_o(W + M + H + B)(\phi_1(s) | e^{-r^2 ij/a^2} | \phi_2(s))$$

$$(\Phi_1'^{31} | \, | \Phi_2'^{31}) = - V_o(W + M - H - B)(\phi_1(s) | \, | \phi_2(s))$$

$$(\Phi_1'^{11} | \, | \Phi_2'^{11}) = - V_o(W - M + H - B)(\phi_1(a) | \, | \phi_2(a))$$

$$(\Phi_1'^{33} | \, | \Phi_2'^{33}) = - V_o(W - M - H + B)(\phi_1(a) | \, | \phi_2(a)) \qquad 5.31$$

The two body matrix elements have been evaluated in general, for a Gaussian radial dependence, in Appendix 1.

Selection rules for intrinsic states.

Not all states $\Phi_i(\epsilon\nu)$ will couple to $\Phi(\bar{\lambda}\bar{\mu})$. For the states in the ds-shell, since H_Ω is a two body operator, the greatest change in ϵ which can occur through H_Ω, in going from $\Phi(\bar{\lambda}\bar{\mu})$ to $\Phi_i(\epsilon\nu)$, is $\Delta\epsilon = 12$. From A1·14 we see that the two body matrix elements, $(\Phi_1' \mid V_c(ij) \mid \Phi_2')$, vanish, for a Gaussian potential, unless the sums of the oscillator quanta in the x, y and z directions are all even for the two functions Φ_1' and Φ_2'. The resulting selection rule for the coupling of many particle states to $\Phi(\bar{\lambda}\bar{\mu})$ is given in Table 8.

Table 8. Selection rule for the states $\Phi_i(\epsilon\nu)$ coupling to $\Phi(\bar{\lambda}\bar{\mu})$. $\Delta\epsilon = |2\bar{\lambda} + \bar{\mu} - \epsilon|$ and $\Delta\nu = |\bar{\mu} - \nu|$

$\Delta\epsilon$	$\Delta\nu$
0	0, 4, 8.
6	2, 6.
12	0, 4.

$f(\lambda\mu\epsilon\Delta\nu \mid L'K'K'')$ coefficients.

Because of the selection rule and the relation in 5·8, it is clear that only the f -coefficients, whose ϵ and ν values satisfy one of the relations in Table 8, need be calculated. That is, only a few operators $E(\lambda\mu, \epsilon\Lambda\nu)$ and their equivalences are needed to form the required f-coefficients.

For the coupling of intrinsic states within the leading representation, the operators required and their equivalences have been listed in Tables 3 to 6 (Chapter 3). These tables also contain some of the E's which are necessary for the mixing of representations ($\lambda\mu$) into the leading representation ($\bar{\lambda}\bar{\mu}$) where

$$(2\bar{\lambda} + \bar{\mu}) - (2\lambda + \mu) = 6 \text{ or } 12.$$

The mixing of other representations has not been considered here although the calculations are straightforeward once the expansion of the operators E and their equivalences are known.

In the remaining chapters we shall use the method now developed to calculate the energy in the states of two, three, four and eight particles in the ds-shell and try to fit the results onto the physical picture.

Parameter fitting.

Before attempting any calculations, we must first decide what values of the parameters will give realistic results for the nuclei in the ds-shell. In the remaining part of this chapter, we discuss these same paramters in the p-shell and, from the work of Kurath (1956), deduce approximate, realistic values for them.

In the p-shell, the only orbital matrix elements which occur are

$$((1p)^2\ [2]\ SO \,|\, V_{c(ij)} \,|\, (1p)^2\ [2]\ SO) = L + 2K$$

$$((1p)^2\ [2]\ DM \,|\, V_{c(ij)} \,|\, (1p)^2\ [2]\ DM) = L - K$$

$$((1p)^2 [11]\ PM \,|\, V_{c(ij)} \,|\, (1p)^2 [11]\ PM) = L - 3K \qquad 5.32$$

(Elliott 1952, p.49)

Evaluation of the matrix elements in 5·32 yields the results:-

$$L = F^{(o)} + 4/25 \; F^{(2)}$$
$$K = 3/25 \; F^{(2)} \qquad \ldots \; 5{\cdot}33$$

where

$$F^{(o)} = \int_{0}^{\infty}\int_{0}^{\infty} U^2(\rho_1)\, U^2(\rho_2)\, J_o(r_1,r_2)\rho_1^2\rho_2^2 \; d\rho_1 \; d\rho_2$$
$$F^{(2)} = \int_{0}^{\infty}\int_{0}^{\infty} U^2(\rho_1)\, U^2(\rho_2)\, J_2(r_1,r_2)\rho_1^2\rho_2^2 \; d\rho_1 \; d\rho_2 \qquad 5.34$$
$$V_{12} = + V_o \, e^{-r_{12}^2/a^2} = \sum_{k=o}^{\infty} J_k(r_1,r_2)\, P_k \; (\cos \omega_{12})$$

with P_k a Legendre Polynomial

and $$U(\rho) = \frac{2}{\pi^{1/4}} \sqrt{\frac{2}{3}}\, \rho \, e^{-\rho^2/2}$$

= radial part of the harmonic oscillator wave function in the nuclear p-shell.

$\rho = r_{12}/b$ with b the oscillator well parameter.

(Elliott and Lane 1957, Appendix 8)

The radial integrals in 5.34 have been evaluated by Jahn (I). Hence we can deduce, for a Gaussian potential,

$$L = V_o \frac{1}{4} \; \beta^{3/2}\left[3 - 2\beta + 3\beta^2\right]$$
$$K = V_o \frac{1}{4} \; \beta^{3/2}\left[1 - 2\beta + \beta^2\right], \; \beta = \frac{\alpha^2}{\alpha^2 + 2}, \; \alpha = a/b \qquad 5.35$$

Consistent results may be obtained for the p-shell by taking $^{L}/K = 6{\cdot}8$, and $K = -0.9$ Mev. (Kurath, 1956). This implies that $\alpha^2 = 1{\cdot}2$.

An estimate of the nuclear mean square radius may be made using the formula

$$\left\langle \frac{1}{A} \sum_i r_i^2 \right\rangle = \frac{3}{5} R^2 \quad \text{with } R = A^{1/3}\, 1{\cdot}3 \times 10^{-13} \text{ cms.} \qquad \underline{5{\cdot}36}$$

where centre of mass motion is neglected. The expectation value of $\sum_i r_i^2$ is easily evaluated using the lowest oscillator well wave-functions.

For L_i^6 we find

$$b = 1{\cdot}5 \times 10^{-13} \text{ cms.}$$

Thus, if $\alpha^2 = a^2/b^2 = 1{\cdot}2$, we have $a = 1{\cdot}65 \times 10^{-13}$ cms. $\underline{5{\cdot}37}$

Using Kurath's value for $K = -0{\cdot}9$ and with $\alpha = 1{\cdot}2$, we may estimate the strength of the force to be

$$V_0 = 40 \text{ Mev.} \qquad \ldots \underline{5{\cdot}38}$$

In the cases we have considered only two exchange parameters occur. We write these as:-

$$\left\{(\Gamma^{13}|\bar{\bar{E}}|\Gamma^{13}) + (\Gamma^{31}|\bar{\bar{E}}|\Gamma^{31})\right\}\Big/2 = W + M = X$$

and
$$\left\{(\Gamma^{11}|\bar{\bar{E}}|\Gamma^{11}) + 9(\Gamma^{33}|\bar{\bar{E}}|\Gamma^{33})\right\}\Big/10$$

$$= \left\{10W - 10M - 8H + 8B\right\}\Big/10 = Y \qquad \underline{5{\cdot}39}$$

where $\bar{\bar{E}} = W + M P_{ij}^{r} - H P_{ij}^{\tau} + B P_{ij}^{\sigma}$ and Γ is the charge-spin state of the i and j th particles.

Table 9 indicates the values of X and $^{Y}/X$ for various well-known exchange parameters.

Table 9

Mixture	W	M	B	H	X	$^{Y}/X$
Serber	0.5	0.5	0	0	1	0
Rosenfeld	-0.13	0.93	0.43	-0.26	0.8	-0.6
Inglis	0	0.8	0.2	0	0.8	-0.8

In Mg^{24} the mean square radius formula in 5.36 yields a value of $b = 1.7 \times 10^{-13}$ cms. Hence, with $a = 1.65 \times 10^{-13}$, we shall assume that the ratio $\alpha = {}^{a}/b = 1$ gives realistic results.

All energy levels are plotted in units of $({}^{V_0}/40)$ Mev.

Chapter 6.

Application of the Method and the Appearance of Rotational Features.

Elliott has compared the lowest eigenstates from shell-model calculations, in the beginning of the ds-shell, with those states classified according to the representation of SU_3. (Elliott 1958, I. Elliott and Macfarlane, Private communication). Large overlaps were found and it was seen that the level ordering, by a realistic potential, corresponded closely to that by $\langle \tilde{C}_1 \rangle_{\lambda\mu}$ The representation which maximises $\langle \tilde{C}_1 \rangle_{\lambda\mu}$ was always found to be lowest.

Such a result may be seen, in a very crude way, by making a 'long-range' expansion of the Gaussian potential. That is, assuming the nuclear distances r_{ij} remain small for a long range force, we may write:-

$$\sum_{i<j} e^{-r_{ij}^2/a^2} = \sum_{i<j}(1 - r_{ij}^2/a^2 + r_{ij}^4/a^4 \ldots)$$

- the terms getting successively smaller.

The first two terms in this expansion contribute the same amount of energy to all states within an oscillater shell. The third term consists of functions which contribute the same, or almost the same amount of energy to states within a shell, and a term of the form

$$+\sum_{i<j} r_i^2 \, r_j^2 \, P_2(\cos \theta_{ij})$$

- the quadrupole force appearing with a positive sign.

The latter function may be written in terms of the Casimir operator $\tilde{C}_1$ which appears again with a positive sign.

With these rather general arguments, we are led to make the assumption that, for an attractive force, the states belonging to the representation $(\tilde{\lambda}\tilde{\mu})$, where $\langle \tilde{C}_1 \rangle_{\tilde{\lambda}\tilde{\mu}}$ is a maximum, form the major part of the low lying states of nuclei. How accurate is this assumption will appear when we discuss representation mixing (see Chapter 7).

In this chapter we consider the significance of taking only the representation which maximises $\langle \hat{C}'_1 \rangle_{\lambda\mu}$, for k = 2,3,4 and 8 particles in the ds-shell, while illustrating the method, developed in Chapter 5, for calculating the energy in the states $\Psi((\lambda\mu)KLM)$ from a central potential.

General Remarks

(a) The kinetic energy will not affect the results as we are only concerned with energies above the ground state and our wavefunctions for the k particles come only from the degenerate ds-shell. We therefore ignore this part of the Hamiltonian in all the calculations, and measure energies from the ground state.

(b) The supermultiplet classification for the low lying states of O^{18}, Ne^{20} and Mg^{24} give these muclei a spin $S = 0$, hence the total spin $J = L$ (the orbital angular momentum). The addition of a spin-orbit force in the Hamiltonian, to make it more realistic, will give zero contribution to the energy in these cases. If we take the k(= 2,4, and 8) particle states to have $S = 0$, we shall be able to relate our results, using only a central potential, to the physical pictures of O^{18} Ne^{20} and Mg^{24}.

For the $k = 3$ problem to represent F^{19}, some spin dependant force must be introduced into the Hamiltonian - this will not be done here. The $k = 3$ problem is considered only to show how rotational features appear in the orbital space as particles are added into the shell.

(c) All tables and figures have been collected at the end of the chapter.

$k = 2$ (O^{18})

The leading representation of SU_3 appearing in $[f] = [2]$ is $(\lambda\mu) = (40)$. (Elliott I Table 1). The leading state of (40) for the O^{18} nucleus is

$$\Phi^{31}((40)\ 800) = (\phi_0)^{31} \qquad \ldots\ \underline{6.1}$$

The states within the (40) representation which satisfy the selection rules in Table 8 (Chapter 5) are given in Table 10 in terms of the configurational states ϕ_i $(\varepsilon\bullet)$. The powers (rqp) in the operators $A^r_{yx}\ E_-^{\,q} E_+^{\,p}$ needed to generate these states, are also shown in this table.

Non-zero two body matrix elements coupling the configurational states to (ϕ_o^2) are given in Table 11. These have been evaluated directly from the formulae given in Appendix 1.

The expansion of $\phi((40)800)$ in terms of $\psi((40)KLK)$ yields only a K = 0 band i.e.

$$\phi((40)8\,00) = \sqrt{1/5}\,\psi((40)\,000) + \sqrt{4/7}\;\psi((40)\,020) + \sqrt{8/35}\,\psi((40)\,040) \qquad \ldots\ \underline{6.2}$$

where the coefficients may be deduced from $\underline{4.12}$

With only a K = 0 band, the $\tilde{g}$-elements ($\underline{5.14}$) are equal to the g-elements and we can now deduce from $\underline{5.8}$, Tables 10 and 11 and Tables 4 to 6 (Chapter 3)

$$g(40|40,L00) = V_o\{A_1 + [-\tfrac{1}{6}(\beta+\omega) - \tfrac{11}{48}(3\gamma+\delta+\rho)]\,L(L+1) + [\tfrac{1}{96}(3\gamma+\delta+\rho)]\,L^2(L+1)^2 \qquad \underline{6.3}$$

Where β, ω, γ, δ and ρ are defined in Table 11, and A_1 is independent of L.

In this simple case we find (from $\underline{5.16}$):-

$$\big((40)0LM\big|V_c\big|(40)0LM\big) \equiv g\,(40|40,\,L00) \qquad \ldots\ \underline{6.4}$$

The ground state has L = 0. The energies of other states above the ground state are thus:-

$$E(0LM) = \big((40)\,0LM\,\big|\,V_c\big|(40)\,0LM\big) - \big(40,000\,\big|\,V_c\big|(40)\,000\big)$$
$$= -V_o\Big\{[-\tfrac{1}{6}(\beta+\omega) - \tfrac{11}{48}(3\gamma+\rho+\delta)]\;L(L+1) + [\tfrac{1}{96}(3\gamma+\rho+\delta)]\;L^2(L+1)^2\Big\} \qquad \underline{6.6a}$$

i.e. E(OLM)

$$= + V_o \rho \left\{ \frac{1}{48} (80\,\alpha^4 + 32\,\alpha^2 + 500)\, L(L + 1) - \frac{18}{48}\, L^2 (L + 1)^2 \right\} \qquad 6.6b$$

with α = the ratio of range (a) to well parameter $b (\alpha = {}^a/b)$ and $\rho = \alpha^2/4(2+\alpha^2)^{11/2}$

The ratio E (04M)/E (02M) is plotted in Figure 1 against the parameter α. For a fixed mean square radius (implying b is constant), the variation in α corresponds to varying the range of the force. We see then that for a long range, E (04M)/E (02M) tends to the limit 10/3 - just that for a rotational band. For a realistic value of the range $(\alpha \sim 1)$, E (04M)/E (02M) = 1.7 , which compares favourably with the experimental value of 1.8. (Ajzenberg-Selove & Lauritsen, 1959).It is instructive to note from where the various terms in E (OIM) come and what approximations we can make to them.

If only the states with $\Delta\epsilon = 0$ are considered in the intrinsic frame i.e. $\beta = \omega = \gamma = \delta = \rho = 0$ in 6.6a (see Table 10 & 11) all the levels are degenerate. Including the mixing of states with $\Delta\epsilon \leq 6$ (only $\gamma = \delta = \rho = 0$) it is clear from (6.6a) that we get just a rotational band. Comparing (6.6a) in this approximation, with the energy levels obtained from the rotational model, we have the moment of inertia (I) given by

$$\frac{\hbar^2}{2I} = + V_o(\beta + \omega)/6 \qquad 6.7a$$

Including the intrinsic states with $\Delta\epsilon = 12$, we get a more realistic picture with the moment of inertia decreased to satisfy:-

$$\frac{\hbar^2}{2I} = V_o \left[(\beta + \omega)/6 + 11(3\gamma + \rho + \delta)/48 \right] \qquad \underline{6.7b}$$

In Figure 2 the various approximation to the energy levels have been plotted in units of $(V_o/40)$ Mev for $\alpha = 1$ and Serber Exchange. Figure 6 shows how the energy level spacing varies with α.

<u>k = 3</u>

The leading SU_3 representation in $[f] = [3]$ is $(\lambda\mu) = (60)$. The leading state for the representation is :-

$$\Phi^{22}(60) \equiv \Phi^{22}((60), 12, 00) = (\phi_o{}^3)^{22} \qquad \ldots \underline{6.8}$$

The states within the (60) representation, which satisfy the selection rules in Table 8 (Chapter 5), are given in Table 12. The matrix elements in the intrinsic frame which are required are listed in Table 13.

The expansion of $\phi(60)$, in terms of the $\psi(KLK)$ -states, yields only a $K = 0$ band,

$$\phi(60) = \sqrt{1/7}\,\psi(000) + \sqrt{10/21}\,\psi(020) + \sqrt{24/77}\,\psi(040) + \sqrt{16/231}\,\psi(060) \quad \underline{6.9}$$

As in the $k = 2$ case, we deduce that, for energies above the ground state $(L = 0)$:-

$$\begin{aligned} E(0LM) &= g\,(60|60\ L00) - g(60|60\ 000) \\ &= -V_o \left\{ \left[-(1/5)(\beta + \omega) - \frac{19}{240}(3\gamma + \delta + \rho) \right] L(L+1) \right. \qquad \ldots \underline{6.10a} \\ &\qquad \left. + \left[(1/480)(3\gamma + \delta + \rho) \right] L^2(L+1)^2 \right\} \end{aligned}$$

i.e. $$E(0LM) = +V_0\, p\left\{(40\,\alpha^4 + 16\,\alpha^2 + 109)(1/20)L(L+1) - (3/40)L^2(L+1)^2\right\} \qquad \ldots\ \underline{6.10b}.$$

The ratio E (04M)/E(02M) has been plotted against α in Figure 1. Again we notice that, for a long range force, the ratio tends to the 10/3 limit. By analysing 6.10a it can be seen that it is the $\Delta\epsilon = 6$ states which give the rotational spectrum and the $\Delta\epsilon = 12$ states which introduce the term $L^2(L + 1)^2$.

k = 4 (Ne^{20})

Table 14 shows all the SU_3 representations within the representation [f] = [4] for SU_6. On this table also is shown the reduction $SU_3 \rightarrow SU_2 \times U_1$ for those states with $\Delta\epsilon \leq 12$. Here $\Delta\epsilon$ is the difference in ϵ from that of the leading state of the leading representation..

The leading SU_3 representation is seen to be $(\lambda\mu) = (80)$, and the leading state is:-

$$\Phi^{11}(80) \equiv \Phi^{11}((80), 16, 00) = (\phi_0{}^4)^{11} \qquad \ldots\ \underline{6.11}$$

The explicit form for the other states of the representation, satisfying the selection rules in Table 8 (Chapter 5), are given in Table 15 together with the powers (r q p) in the operators $A_{yx}{}^r E_-{}^q E_+{}^p$ required to generate the state.

The required energy matrix elements in the intrinsic frame are listed in Table 16.

Expanding $\phi(80)$ in terms of $\psi(KLK)$ we find, from 4.12,

$$\phi(80) = \sqrt{1/9}\,\psi(000) + \sqrt{40/99}\,\psi(020) + \sqrt{48/143}\,\psi(040)$$

$$+ \sqrt{64/495}\,\psi(060) + \sqrt{128/6435}\,\psi(080) \qquad \underline{6.12}$$

Again only a $K = 0$ band is predicted.

In this case,

$$\begin{aligned} E(0LM) &= g(80|80\ L00) - g(80|80\ 000) \\ &= +V_o\{[(3/14)(B + \omega) + (27/560)(3\gamma + \delta + \rho)]\,L(L+1) \\ &\quad -[(1/32.35)(3\gamma + \delta + \rho)]\,L^2(L+1)^2 \qquad \underline{6.13a} \\ &= +V_o p\{(3/140)(100\,\alpha^4 + 40\,\alpha^2 + 211)\,L(L+1) \\ &\quad - 9/280\,L^2(L+1)^2\} \qquad \underline{6.13b} \end{aligned}$$

The ratio $E(04M)/E(02M)$ has been drawn in Figure 1 for various values of the parameter α. For a long range force we have a rotational spectrum whilst for a realistic range ($\alpha \sim 1$) the ratio compares favourably with the value 3.05 from the experimental spectrum (Azjenberg - Selove and Lauritsen, 1959).

The approximations to the energy levels, as first the $\Delta\epsilon = 6$ and then the $\Delta\epsilon = 12$ terms are included in the energy calculation in the intrinsic frame, are shown in Figure 3 in units of $(V_o/40)$ Mev. Figure 7 shows how the energy level spacings vary with α.

Summary for k = 2, 3 and 4 particle problems.

Already it can be seen that a rotational spectrum appears in this scheme for realistic value of $\alpha(\sim 1)$ as particles are added into the shell (Figure 1). This result is directly related to the fact that the contribution of the $\Delta\epsilon = 12$ intrinsic states to the energy grows relatively smaller with the greater number of particles. For more particles we should expect a good estimate to the spectrum by ignoring the $\Delta\epsilon = 12$ terms (see k = 8 particle problem).

The absolute energies compare quite well with the experimental data. A value of $V_o \sim 54$ Mev. will fit the k = 2 results to the O^{18} spectrum whilst a value $V_o \sim 60$ Mev. will fit the k = 4 results to Ne^{20}. It is encouraging to see that these strengths are about the same and near $V_o = 40$ Mev. which we have estimated to be the correct value from p-shell data (Chapter 5).

<u>k = 8 (Mg^{24})</u>

Because of the four dimensional charge spin space the maximum orbital symmetry is $[f] = [44]$.

The representations of SU_3 contained in $[f] = [44]$ of SU_6, which have ϵ-values differing by less than 12 from the leading state of the leading representation, are shown in Table 17.

The leading representation is seen to be $(\lambda\mu) = (84)$. With the notation of <u>3.50</u>, the leading state of (84) (from <u>3.53</u>) is:-

$$\Phi^{11}(84) \equiv \Phi^{11}((84)\ 20,\ 2,\ 4) = (\phi_0^{\,4})(\phi_{+1}^{\,4})^{11} \qquad \ldots\ \underline{6.14}$$

From the two body matrix elements deduced in (<u>A 1.5</u>) the configurational states, $\Phi_i(\epsilon\nu)$, coupling directly to the state $\Phi(84)$ can be written down immediately. These are the states $|1) \ldots |15)$ listed in Table 18. The coefficients $p_i(84\epsilon\Lambda\nu)$ of the $\Phi_i(\epsilon\,\nu)$ in the expansion of states $\Phi((84)\epsilon\Lambda\nu)$ (see 3.56) are given in Table 19. The $p_i(84\,\epsilon\Lambda\nu)$ may be calculated independently of each other hence only those belonging to configurational states which couple directly to the leading state need be known. This saves a great amount of labour.

The matrix elements $(\Phi_i(\epsilon\nu)|V_c|\,\Phi(84))$ can immediately be deduced (using A 1.5) once the fractional parentage coefficient are known for the reduction of the eight particle states by two particles (see, for example, Appendix 2).

Since the two particle states can now be either symmetric or antisymmetric, the form of the matrix elements will depend on the exchange mixture to a much larger extent than in the simpler cases already considered.

The elements $\left(\Phi_i(\epsilon\nu)\middle|V_{c'}\middle|\,\Phi(84)\right)$ in Table 20 have been given in two parts - the contribution (S and A) from the symmetric and antisymmetric pairs of particles respectively. The full element will then be:-

$$\left(\Phi_i(\epsilon\nu)\middle|V_c'\middle|\,\Phi(84)\right) = X\left(\,|V_c'|\,\right)_S + Y\left(|V_c'|\,\right)_A \qquad \ldots\ 6.15.$$

with X and Y defined in 5.38,

$$V_c' = -V_o \sum_{i<j} \exp\left(-r_{ij}^2/a^2\right) \text{ and}$$

$\left(\Phi(84)\middle|V_c'\middle|\,\Phi_i(\epsilon\nu)\right)_S \equiv$ Contribution from symmetric pairs of particles

$\left(\Phi(84) \middle| V'_c \middle| \Phi_i(\epsilon \nu) \right)_A \equiv$ Contribution from antisymmetric pairs of particles.

Using 5.8, we may now construct $g(84|84\ L'\, K'\, K'')$ and hence deduce $\tilde{g}(84|84\ L'\ |K'|\ K')$ from 5.14.

The expansion of $\Phi(84)$ (as in 4.2) shows that there are $K = 0$, 2 and 4 bands. Thus the $\tilde{g}$ elements, in this case, are not trivially equal to the g-elements for all values of K.

To simplify the analysis we divide the $\tilde{g}$ elements into four parts. First we define $\tilde{g}_S$ and $\tilde{g}_A$ to be the contribution to $\tilde{g}$ from the two body matrix elements of the symmetric and antisymmetric pair of particles respectively. We then define $\tilde{g}'_S$ and $\tilde{g}''_S$ to be the contribution to $\tilde{g}_S$ from the intrinsic terms with $\Delta\epsilon \leqslant 6$ and $\Delta\epsilon = 12$ respectively. A similar definition will stand for $\tilde{g}_A$.

Thus we may write:-

$$\tilde{g} = \tilde{g}'_S + \tilde{g}''_S + \tilde{g}'_A + \tilde{g}''_A \qquad \ldots\ 6.16$$

For a Serber force $\tilde{g} = \tilde{g}'_S + \tilde{g}''_S$ since Y in 6.15 is zero.

The expression for $\tilde{g}$ in terms of the parameter α is very complicated in this case and no attempt has been made to write it down in general. It is possible to derive the general form for $\tilde{g}_{S\text{ or }A}$ however. Ignoring, for the moment, the $\Delta\epsilon = 12$ intrinsic states, we have, from 5.4 and Table 4 (Chapter 3) :-

$$
\begin{aligned}
V_c\ \Phi(84) \equiv \Big[& h_1 + h_2 A_{yx}^2 + h_3 E_+^2 + h_4 A_{yx}^2 E_+^2 + h_5 A_{yx}^4 E_+^2 \\
& + h_6 A_{yx} E_- E_+ + h_7 A_{yx}^3 E_- E_+ + h_8 E_-^2 + h_9 A_{yx}^2 E_-^2 \Big]\ \Phi(84) \qquad \underline{6.17}
\end{aligned}
$$

$$
\begin{aligned}
= \Big[& \Big\{ (1/2)(h_5 + h_7 + h_9) L_o^4 - \tfrac{1}{2}(h_4 + 28h_5 + h_6 + 10h_7 + h_8 - 22h_9) L_o^2 \\
& + 1/2(h_3 + 6h_4 + 72h_5 - 24h_8 - 48h_9) \Big\} (L_{+1}^2 + L_{-1}^2) \\
& + \Big\{ -(3h_7 + 5h_9) L_o^2 + (3h_6 + 30h_7 + 5h_8 + 10h_9) L_o \Big\} (L_{+1}^2 - L_{-1}^2) \\
& + \Big\{ -\tfrac{1}{2}(h_5 + h_7 + h_9) L_o^4 + \tfrac{1}{2}(h_4 + 28h_5 + h_6 + 10h_7 + h_8 + 18h_9) L_o^2 \\
& \quad - \tfrac{1}{2}(h_3 + 6h_4 + 72h_5 + 16h_8 + 32h_9) \Big\} L^2 \\
& + \Big\{ (1/2)(h_5 + h_7 + h_9) L_o^6 - \tfrac{1}{2}(h_4 + 12h_5 + h_6 - 12h_7 + h_8 - 8h_9 \qquad) L_o^4 \\
& + \tfrac{1}{2}(-2h_2 + h_3 - 10h_4 - 376h_5 - 22h_6 - 220h_7 - 10h_8 - 436h_9) L_o^2 \\
& + (h_1 + 4h_2 + 8h_3 + 48h_4 + 576h_5 + 208h_8 + 416h_9) \Big\} \Big]\ \overset{\Phi}{\cancel{\phi}}(84) \qquad \underline{6.18a}
\end{aligned}
$$

Denoting the operator in the R.H.S. of <u>6.18a</u> by G' we have:

$$
V_c \Phi(84) = G'\ \Phi(84) \qquad \underline{6.18b}
$$

The h_i are defined in 6.17 as the matrix elements coupling an intrinsic state of the (84) representation to the leading state, divided by the normalization factor $(N_p M_{qp} P_{rqp})$ of that particular intrinsic state (3.37)

From 5.6 to 5.8 we deduce

$$G' \psi((84)\, K'L'K') = \sum_{K''} g'(84|84L'K'K'')\, \psi((84\, K'L'K'') \qquad 6.19$$

Thus, by knowing the form of the h_i's in 6.17, we can immediately deduce the g' and hence $\tilde{g}'$.

Table 21 lists the form of the h_i's in terms of α. In Tables 22 the functions of the h_i's in 6.18 have been evaluated for several values of the parameter α. Table 22a lists the contribution to the functions from the symmetric pairs of particles and Table 22b shows the contribution from the antisymmetric pairs.

The $\tilde{g}'_s$ and $\tilde{g}'_A$ matrices may now be deduced and are shown in Tables 23 and 24 for $\alpha = 1$. The $\tilde{g}''_s$ matrices, are found to be the same for all values of the parameter α within a factor of

$$p = \alpha^3/4(2+\alpha^2)^{11/2} ,$$

and are shown in Table 25. The $\tilde{g}''_A$ matrices are identically zero for all values of α.

The energy levels, deduced from a diagonalisation of the $\tilde{g}$ - matrices for various values of α, have been drawn in Figures 8, 9 and 10, as functions of α for three different exchange mixtures.

The form of the $\tilde{g}$ - matrices will now be examined, for a Serber force, after making various approximations. First we ignore all intrinsic states except those with $\Delta\varepsilon = 0$; secondly we include those states with $\Delta\varepsilon \leq 6$; finally we take into account all the intrinsic states. (Figure 4).

In the first approximation then, the g matrices are given by the G' operator, from 6.19, where all the h_i's in 6.18, except h_1 and h_2, are zero. In this case, since there are no operators to mix bands $g \equiv \tilde{g}$. We see that all levels with the same K label are degenerate but the different bands are split by an amount proportional to $-h_2K^2$. Since h_2 is negative (Table 21), the K = 0 band is lowest.

The next approximation has all the h_i's in 6.17 non-zero. Ignoring the mixing of K we see that again the g-matrices $\equiv$ $\tilde{g}$-matrices and, from 5.15, the $\tilde{g}$-elements give the energy levels directly. Since the K-band mixing will come from the $L^2_{\pm 1}$ terms in 6.18, it can be seen that the energy levels within a band are just of the form

$$A + B\,L\,(L+1).$$

The mixing of the K-bands at this stage depends on the coefficients of the $L^2_{\pm 1}$ terms. From Tables 22 we see that these coefficients are very small. This is directly responsible for the small off-diagonal matrix elements in Tables 23.

We note here that the most important contribution to the energy, from the terms with $\Delta\epsilon = 6$, comes from the state with maximum ν.

The $\tilde{g}_S$-matrices, resulting from the inclusion of all the intrinsic states, can be found by adding the matrices $\tilde{g}'_S$ to $\tilde{g}''_S$ from Tables 24 and 26 (see 6.16).

A first approximation to the energy levels is found from the diagonal elements of $\tilde{g}_S$. Since the diagonal elements of $\tilde{g}''_S$ are very small, the diagonal elements of $\tilde{g}_S$ are almost equal to those of $\tilde{g}'_S$.

The off-diagonal elements of $\tilde{g}''_S$ are of the same order of magnitude as the same elements for $\tilde{g}'_S$. It is, therefore, not consistent to calculate the K-band mixing when the intrinsic states with $\Delta\epsilon = 12$ are ignored. This approximation will still give a good approximation to the energy levels however as the K-band mixing is always very small. For a Rosenfeld - type exchange the mixing of K is found to be even smaller.

In Figures 4 and 5 the spectra for the Serber and Rosenfeld type exchanges (with $\alpha = 1$) are drawn for these various approximations.

From these diagrams we can see that the error in ignoring the K-band mixing and the $\Delta\epsilon = 12$ terms is not great. We have, then, almost pure rotational bands described by the K-label.

The actual states with L = 2 for the levels deduced in the Serber and Rosenfeld-type exchange cases have been calculated, for $\alpha = 1$, from <u>5.23</u>.

In this calculation we use the facts that :-

$$((84)22M\,|\,(84)02M)^2 = 3/19.137 \qquad \underline{6.20}$$

$$a^2(84\ 22) = 137/2.9.11.13. \quad \text{and} \quad a^2(84\ 02) = 19/11.13. \qquad \underline{6.21}$$

(See <u>4.11</u> and <u>4.12</u>)

Thus for a Serber exchange.

$$\psi_1(L = 2,M) = + \cdot 99097\ \psi((84)02M) + \cdot 10462\ \psi((84)22M)$$

$$\psi_2(2,M) = - \cdot 13965\ \psi((84)02M) + \cdot 99493\ \psi((84)22M) \qquad \underline{6.22}$$

and for Rosenfeld exchange

$$\psi_1(2M) = + \cdot 99650\ \psi((84)02M) + \cdot 05613\ \psi((84)22M)$$

$$\psi_2(2M) = - \cdot 09110\ \psi((84)02M) + \cdot 9990\ \psi((84)22M) \qquad \underline{6.23}$$

This classification scheme is slightly better for a Rosenfeld-type exchange than for a Serber.

<u>Fitting the experimental data</u>.

The results of all our calculations so far are illustrated in Figures 6 to 10. The energy levels are shown as functions of the parameter α and are plotted in units of $V_o/40$ Mev. To make direct comparisons, the experimentally deduced spectra have also been drawn on these figures in units of $(V_o/40)$ Mev for various values of the strength V_o.

We find that the best consistent fit is made, to the O^{18}, Ne^{20} and Mg^{24} spectra, for a range $a = 1.85 \times 10^{-13}$ cms. This is a slightly larger value than that deduced from p-shell data (Chapter 5).

From the mean square radius formula in 5.36 we find the results:-

O^{18} : b = ~~.97~~ 1·71 $\times 10^{-13}$ cms
Ne^{20}: b = ~~.96~~ 1·73 $\times 10^{-13}$ cms
Mg^{24}: b = ~~.93~~ 1·78 $\times 10^{-13}$ cms

We thus make the comparison of our results, with experiment, for values of the parameter α $(={}^a\!/b)$ = 1.08, 1.07 and 1.04 (i.e. about 1) for O^{18}, Ne^{20} and Mg^{24} respectively.

A value for the strength V_o is deduced in each case by fitting the energy of the firstexcited state exactly to the experimental one. The values of V_o chosen in the different cases for the various exchanges are shown below. The intermediate exchange has been chosen to see how the Mg^{24} spectrum changes as ${}^Y\!/X$ changes from 0 (Serber) through -0.2 (Intermediate) to -0.6 (Rosenfeld)

Exchange	X	Y	V_o		
			O^{18}	Ne^{20}	Mg^{24}
Serber	1	0	54	60	60
Intermediate	0.8	-0.16	90	75	78
Rosenfeld	0.8	-0.48	90	75	110

The calculations on the O^{18} and Ne^{20} problems were only for a Serber-type exchange but, since the exchange contribution is merely an overall multiplicative factor of (W + M - H - B) and (W + M) in these respective cases, the results for any other exchange mixture may easily be deduced. It is clear that the same results can be obtained from any exchange by taking different strengths.

We notice that the strengths required, with the Serber and intermediate exchanges, are roughly consistent to within 15 Mev. The apparent inconsistency in the Rosenfeld case is not so serious as it would at first seem. From Figure ~~8~~ 10, we see that a slight increase of just 0.15 $V_o/40$ Mev of the first excited state (coming from representation mixing maybe) would imply a strength of only V_o = 80 Mev.

The actual spectra we have deduced are compared with the experimental ones in Figure 11 for Mg^{24}. The fitting of the lowest K = 0 bands of O^{18} and Ne^{20} is good.

The lowest K = 0 band of Mg^{24} is given quite well for any exchange. The K = 2 band is consistently too low although a better result is obtained by using a Rosenfeld exchange instead of a Serber. The relative spacing of the L = 2 and 3 levels of the K = 2 band are then given correctly and the L = 4 level is only .6 Mev from the experimental level at 6 Mev. One further piece of information which favours this model is in the calculation of the branching ratio from the second 2^+ level in Mg^{24}.

The lifetime for an E2 transition of energy E from a level J' to a level J is given by:-

$$\tau = \hbar/\Gamma \qquad \text{with}$$

$$\Gamma(J' \to J) = \frac{4\pi E^5}{75\,\hbar^5\,c^5} \times \frac{(J'||O^{(2)}||J)^2}{(2J'+1)} \qquad \underline{6.24a}$$

where the reduced matrix element is defined in:-

$$(J'M'\,|\,O_q^{(2)}\,|\,JM) = (J2Mq\,|\,J'M')\,\frac{(J'\,||O^{(2)}||\,J)}{(2J'+1)^{1/2}}$$

and $O_q^{(2)}$ is the quadrupole moment operator:-

(Blatt and Weisskopf, 1952).

$$O_q^{(2)} = e\sum_p r_p^2\;Y_q^2(p) \equiv \frac{e}{2}\sum_i (1 - \tau_z(i))\,r_i^2\;Y_q^2(i)\,. \qquad \underline{6.24b}$$

p sums over all protons and i sums over all particles. τ_z is the isotopic spin operator such that $\langle\tau_z(i)\rangle = \pm 1$ depending on whether the i^{th} particle is a neutron or proton respectively. e is the charge on the proton.

Thus the branching ratio :-

$$\frac{\Gamma(\Psi_2(2M) \to \Psi(00))}{\Gamma(\Psi_2(2M) \to \Psi_1(2M))} = \left(\frac{E_2^* - E_0}{E_2^* - E_2}\right)^5 \left(\frac{(\Psi_2(2)||O^2||\;\Psi(0))}{(\Psi_2(2)||O^2||\;\Psi_1(2))}\right)^2$$

where E_o, E_2 and E_2^* are the energies of the first L = 0 ($\Psi(00)$) the first L = 2 ($\Psi_1(2M)$), and second L = 2 ($\Psi_2(2M)$) states respectively.

Putting in the experimental values (Batchelor et al, 1960) for the branching ratio (~2.8) and for the energies, we deduce:-

$$R = \left(\frac{(\Psi_2(2)|| \, O^{(2)} \, || \, \Psi(0))}{(\Psi_2(2)|| \, O^{(2)} \, || \Psi_1(2))} \right)^2 = 0{\cdot}4 \qquad \underline{6.25}$$

We now calculate a value for R using the wave functions deduced from this model with band mixing.

Although the second degree tensor operators Q_q, of the U_3-group operators (2.2), contain the momentum operators, the symmetry between the momenta and the coordinates in a oscillater well means that, <u>within an oscillator well</u>:-

$$Q_q \equiv \sqrt{4\pi/5} \sum_i r_i^2 \; Y_q^2(i)/b^2 \qquad \underline{6.26}$$

Thus, Q_q is equivalent to a quadrupole mass operator within a shell, with sum over all particles. In a nucleus with T = 0, the contribution from the term $\tau_z(i)$ in 6.25 is zero and so, in this case,

$$O_q^2 \equiv eb^2/8 \sqrt{5/\pi} \;\; Q_q \qquad \underline{6.27}$$

Thus the ratio R for the Mg^{24} nucleus is equivalent to :-

$$R = \frac{(\Psi_2(2M) \, || \, Q \, || \, \Psi(00) \,)^2}{(\Psi_2(2M) \, || \, Q \, || \, \Psi_1(2M) \,)^2}$$

If $\Psi_1(2,M) = \alpha_1 \Psi(02M) + \beta_1 \Psi(22M)$

and $\Psi_2(2,M) = \alpha_2 \Psi(02M) + \beta_2 \Psi(22M)$

then

$$R = \frac{1}{\alpha_1^2} \left\{ \frac{(\alpha_2/\beta_2)(02||Q||00) + (22||Q||00)}{(\alpha_2/\beta_2)(02)||Q||02) + (\beta_1/\alpha_1)(22||Q||22) + \left(1 + \frac{\beta_1\alpha_2}{\alpha_1\beta_2}\right)(22||Q||02)} \right\}^2$$

Elliott (1958, II (45)) has derived the result:- <u>6.29</u>

$$Q_0 \Psi((\lambda\mu) KLM) = \sum_{L'} \frac{(2L + 1)}{(2L' + 1)} \frac{(L2M0|L'M')}{c(\lambda\mu K,L)} \Big[(L2K0|L'K)\, c(\lambda\mu\, KL')$$

$$\times \left\{\mu + 2\lambda + \tfrac{1}{2}(L'(L' + 1) + 6 - L\,(L + 1))\right\} \Psi((\lambda\mu)KL'M)$$

$$+ \sum_{\pm} (L2K{\pm}2|\, L'K{\pm}2)((3/2)(\mu \mp K)\ (\mu \pm K + 2))^{1/2}$$

$$\times \quad c(\lambda\mu\ K \pm 2,\ L')\ \psi((\lambda\mu)\ K \pm 2,\ L'M)\Big]. \tag{6.30}$$

From which we may deduce, by replacing the c-coefficients with the b and a-coefficients (<u>4.5a</u>) and with the explicit form for the b's (<u>2.26</u>):-

$$\left((\lambda\mu)K'L'\,||Q||\,(\lambda\mu)KL\right) = \frac{(2L + 1)}{(2L' + 1)}^{1/2} \left[\frac{a(\lambda\mu\ KL')}{a(\lambda\mu\ KL)}\right] \quad \times$$

$$\left\{ 2\lambda + \mu + \left(L'(L' + 1) + 6 - L(L + 1)\right)/2 \right\} (L2K0|\,L'K) \left((\lambda\mu)K'L'\,\middle|\,(\lambda\mu)KL'\right)$$

$$+ \sum_{\pm} (L2K \pm 2|L'\ K{\pm}2)\sqrt{3/2}(\mu \pm K + 2)\ \frac{a(\lambda\mu,K \pm 2,\ L')}{a(\lambda\mu,\ KL)} \quad \times$$

$$\left((\lambda\mu)K'L'\,\middle|\,(\lambda\mu)\ K \pm 2,\ L'\right) \tag{6.31}$$

For pure K-bands, $\alpha_2 = \beta_1 = 0$ and, from 6.29 and 6.31 we find $R = 0.79$.

For the states deduced using a Serber exchange, $R = 0.12$ and for a Rosenfeld, $R = 0.23$.

Thus we see that the ratio R is very sensitive to the mixing of bands. The change in this mixing, on going from a Serber exchange to a Rosenfeld, doubles the small value for R deduced in the Serber case, to get within .17 of the given experimental value.

Adjustment to the nuclear Hamiltonian

Up to now the 'realistic' potential has consisted only of two-body operators between particles outside the O^{16} closed shell. Such a potential will not give any s - d spacing in O^{17} which has just one particle in the ds-shell. Introducing a one body potential of the form.

$$x \sum_i L_i^2 \qquad 6.33$$

into the Hamiltonian, the correct order of spacing in O^{17} can be achieved by putting $x = + .2$ Mev. Here L_i is the orbital angular mementum operator for the i^{th} particle.

We now ask: What effect does such a potential have in the $k = 8$ particle case?

Since $\sum_i L_i^2$ is a spherically symmetric operator, the general method of Chapter 5 can be used to calculate the matrix representation of the potential by the states $\Psi((\tilde{\lambda}\tilde{\mu})KLM)$.

The calculation of the required elements of the matrix of $\sum_i L_i^2$ in the intrinsic frame follows the usual procedure. First, the single particle elements must be determined. Secondly, the fractional parentage coefficients of the reduction of the many particle state by one particle must be found. This latter problem is solved in a similar way to the reduction of the many particle states by two particles (See Appendix 2). The former problem entails finding matrix elements of the form,

$$\left(\phi((20)\epsilon\Lambda\nu)\right| L^2 \left| \phi((20)\epsilon'\Lambda'\nu')\right) \qquad \underline{6.34}$$

where the ϕ's are single particle states with two oscillator quanta. (The Λ labelling is really trivial for these states from 2.31).

$$\text{Now} \quad \phi((20)\epsilon\Lambda\nu) = E((20)\epsilon\Lambda\nu)\ \phi((20)400)$$

$$\equiv E((20)\epsilon\Lambda\nu)\phi_0 \qquad \underline{6.35}$$

$$\text{also} \quad \phi((20)400) = \sqrt{1/3}\ \Psi((20)000) + \sqrt{2/3}\ \Psi((20)020)$$

$$\equiv \sqrt{1/3}\ \Psi_0 + \sqrt{2/3}\ \Psi_2 \qquad \underline{6.36}$$

(4.11 and 4.12)

$$\text{Thus} \quad (\phi((20)\epsilon\Lambda\nu)|\ L^2\ |\phi((20)\epsilon'\Lambda'\nu'))$$

$$= (\sqrt{1/3}\ \Psi_0 + \sqrt{2/3}\ \Psi_2\ |\ \tilde{E}^+((20)\epsilon\Lambda\nu)\ L^2 \tilde{E}((20)\epsilon'\Lambda'\nu')| \times$$

$$|\sqrt{1/3}\ \Psi_0 + \sqrt{2/3}\ \Psi_2) \qquad \underline{6.37}$$

In the k = 8 particle problem, the only non-zero orbital matrix elements required are:-

$$(\phi_o|L^2|\phi_o) = 4$$
$$(\phi_o|L^2|\phi_{+2}) = -2$$
$$(\phi_o|L^2|\phi_{-2}) = -2$$
$$(\phi_{+1}|L^2|\phi_{+1}) = +6 \qquad \underline{6.38}$$

In this case, the only configurational states which will couple to the leading state $\phi(84)$ are:

$$(\phi_o^4)(\phi_{+1}^4)^{11} \quad , \quad (\phi_o^3\,\phi_{+2})(\phi_{+1}^4)^{11} \quad \text{and} \quad (\phi_o^3\,\phi_{-2})(\phi_{+1}^4)^{11} \qquad \underline{6.39}$$

We see that only states with $\Delta\epsilon \leqslant 6$ will couple to $\Phi(84)$ by this operator.

A similar expression to that in 6.18, with $V_c = \sum_i L_i^2$, yields:-

$h'_1 = 40$
$h'_2 = 0$
$h'_3 = 0.143$
$h'_4 = 0.005$
$h'_5 = 0$
$h'_6 = 0.003$
$h'_7 = 0$
$h'_8 = -0.004$
$h'_9 = 0$ ... 6.40

Neglecting band mixing, the increase in the energy of the first excited state above the ground state is given by:-

$$x\left\{(020|\sum_i L_i^2|020) - (000|\sum_i L_i^2|000)\right\} ,$$

which, from 6.18 and 6.40, is found to give:

$$- x (.330) = - .066 \text{ Mev.}$$

Thus, this change in the potential tends to decrease the energy of the first excited state above the ground state although by a negligible amount. Small changes are found in the other spacings also.

In the next chapter, we consider the accuracy of the assumption concerning the smallness of the representation mixing.

ϵ	Λ	ν	Configurational States $\phi_i(\epsilon\nu)$	Operator r q p
8	0	0	(ϕ_0^2)	0 0 0
2	1	2	$\sqrt{1/6}\,[\sqrt{2}(\phi_0\phi_{+2}) + 2(\phi^2_{+1})]$	0 0 2
		-2	$\sqrt{1/6}\,[\sqrt{2}(\phi_0\phi_{-2}) + 2(\phi^2_{-1})]$	2 0 2
-4	2	4	(ϕ^2_{+2})	0 0 4
		0	$\sqrt{1/6}\,[\sqrt{2}(\phi_{+2}\phi_{-2}) + 2(\phi'^2_0)]$	2 0 4
		-4	(ϕ^2_{-2})	4 0 4

Table 10. States of the (40) representation k=2

$\phi_i(\epsilon\nu)$	$(\phi(40)\,\vert\,\exp(-r^2_{12}/a^2)\,\vert\,\phi_i(\epsilon\nu))$	Shorthand Notation
(ϕ_0^2)	$p(4\alpha^8 + 16\alpha^6 + 60\alpha^4 + 56\alpha^2 + 41)$	$\bar{\alpha}$
$(\phi_0\phi_{+2})$	$\sqrt{2}p(\,2\alpha^4 - 4\alpha^2 - 1\,)$	$\sqrt{2}\beta$
$(\phi_0\phi_{-2})$	$\sqrt{2}p(\,2\alpha^4 - 4\alpha^2 - 1\,)$	$\sqrt{2}\beta$
(ϕ^2_{+1})	$2p(\,4\alpha^4 + 4\alpha^2 + 7\,)$	ω
(ϕ^2_{-1})	$2p(\,4\alpha^4 + 4\alpha^2 + 7\,)$	ω
(ϕ^2_{+2})	$9p$	γ
(ϕ^2_{-2})	$9p$	γ
(ϕ'^2_0)	$6p$	ρ
$(\phi_{+2}\phi_{-2})$	$\sqrt{2}\ 3p$	$\sqrt{2}\delta$

Table 11. Two-body matrix elements for k = 2

$$p = \alpha^3/\{4(2+\alpha^2)^{11/2}\}$$

ϵ	Λ	ν	Configurational States $\phi_i(\epsilon\nu)$	Operator		
				r	q	p
12	0	0	(ϕ_0^3)	0	0	0
6	1	2	$\sqrt{1/5}\,[(\phi_{+2}\phi_0^2) + 2(\phi_0\phi_{+1}^2)]$	0	0	2
		-2	$\sqrt{1/5}\,[(\phi_{-2}\phi_0^2) + 2(\phi_0\phi_{-1}^2)]$	2	0	2
0	2	4	$\sqrt{1/5}\,[(\phi_0\phi_{+2}^2) + 2(\phi_{+1}^2\phi_{+2})]$	0	0	4
		0	$\sqrt{1/30}\,[\sqrt{2}(\phi_0\phi_{+2}\phi_{-2})+2(\phi_0\phi_0'^2)+2(\phi_{+1}^2\phi_{-2})+$ $2(\phi_{-1}^2\phi_{+2})+4(\phi_{+1}\phi_{-1}\phi_0')]$	2	0	4
		-4	$\sqrt{1/5}\,[(\phi_0\phi_{-2}^2) + 2(\phi_{-1}^2\phi_{-2})]$	4	0	4

Table 12. States of the (60) representation. k = 3

$\phi_i(\epsilon\nu)$	$(\phi(60)\,\vert V_c'/V_0\vert\,\phi_i(\epsilon\nu))$
(ϕ_0^3)	$3\bar{\alpha}$
$(\phi_0\phi_{+1}^2)$	$\sqrt{3}\,\omega$
$(\phi_0^2\phi_{+2})$	$2\sqrt{3}\,\beta$
$(\phi_0^2\phi_{-2})$	$2\sqrt{3}\,\omega$
$(\phi_0\phi_{-1}^2)$	$\sqrt{3}\,\omega$
$(\phi_0\phi_{+2}^2)$	$\sqrt{3}\,\gamma$
$(\phi_0\phi_0'^2)$	$\sqrt{3}\,\rho$
$(\phi_0\phi_{+2}\phi_{-2})$	$\sqrt{6}\,\delta$
$(\phi_0\phi_{-2}^2)$	$\sqrt{3}\,\gamma$

$$V_c' = V_0 \sum_{i<j} \exp\left[\frac{-r_{ij}^2}{a^2}\right]$$

Table 13. Matrix elements for k = 3.
(See Table 11 for definition of Greek symbols)

$(\lambda\mu)$	ϵ	2Λ
(80)	16	0
	13	1
	10	2
	7	3
	4_+	4
(42)	10	2
	7	1 3
	4_+	0 2 4
(04)	4_+	4
(20)	4_+	0

Table 14. Classification of states with orbital symmetry [4] showing only those which couple to $\phi(80)$ by the Gaussian Potential.

ϵ	2Λ	ν	Configurational states $\phi_i(\epsilon\nu)$	Operator r	 q	 p
16	0	0	(ϕ_0^4)	0	0	0
10	2	2	$\sqrt{1/7}\left[(\phi_0^3\phi_{+2}) + \sqrt{6}(\phi_0^2\phi_{+1}^2)\right]$	0	0	2
		-2	$\sqrt{1/7}\left[(\phi_0^3\phi_{-2}) + \sqrt{6}(\phi_0^2\phi_{-1}^2)\right]$	2	0	2
4	4	4	$\sqrt{1/70}\left[\sqrt{6}(\phi_0^2\phi_{+2}^2)+4\sqrt{3}(\phi_0\phi_{+1}^2\phi_{+2})+4(\phi_{+1}^4)\right]$	0	0	4
		0	$\sqrt{1/35}\left[(\phi_0^2\phi_{+2}\phi_{-2})+\sqrt{2}(\phi_0^2\phi_0'^2)+2(\phi_0\phi_{+1}^2\phi_{-2}) +\right.$ $\left.2(\phi_0\phi_{-1}^2\phi_{+2})+4(\phi_0\phi_{+1}\phi_{-1}\phi_0')+2\sqrt{2}(\phi_{+1}^2\phi_{-1}^2)\right]$	2	0	4
		-4	$\sqrt{1/70}\left[\sqrt{6}(\phi_0^2\phi_{-2}^2)+4\sqrt{3}(\phi_0\phi_{-1}^2\phi_{-2})+4(\phi_{-1}^4)\right]$	4	0	4

Table 15. States of the (80) representation. k = 4.

$\phi_i(\epsilon\nu)$	$(\phi(80)\|V'_c/V_o\|\phi_i(\epsilon\nu))$
(ϕ_o^4)	$6\bar{\alpha}$
$(\phi_o^2\phi_{+1}^2)$	$\sqrt{6}\,\omega$
$(\phi_o^3\phi_{+2})$	6β
$(\phi_o^2\phi_{-1}^2)$	$\sqrt{6}\,\omega$
$(\phi_o^3\phi_{-2})$	6β
$(\phi_o^2\phi_{+2}^2)$	$\sqrt{6}\,\gamma$
$(\phi_o^2\phi_o'^2)$	$\sqrt{6}\,\rho$
$(\phi_{+2}\phi_{-2}\phi_o^2)$	$2\sqrt{3}\,\delta$
$(\phi_o^2\phi_{-2}^2)$	$\sqrt{6}\,\gamma$

$$V'_c = V_o \sum_{i<j} \exp\left[\frac{-r_{ij}^2}{a^2}\right]$$

Table 16. Matrix elements for k = 4.
(See Table 11 for definition of Greek symbols)

$\langle \tilde{C}_1 \rangle$	$(\lambda\mu)$	ϵ	2Λ
148	(84)	20	4
		17	3 5
		14	2 4 6
		11	1 3 5 7
		8_{+}	0 2 4 6 8
109	(73)	17	3
		14	2 4
		11	1 3 5
		8_{+}	0 2 4 6
106	(46)	14	6
		11	5 7
		8_{+}	4 6 8
100	(81)	17	1
		14	0 2
		11	1 3
		8_{+}	2 4
88	(54)	14	4
		11	3 5
		8_{+}	2 4 6
88	(08)	8_{+}	8
76	2x(62)	14	2
		11	1 3
		8_{+}	0 2 4
73	(35)	11	5
		8_{+}	4 6
58	(43)	11	3
		8_{+}	2 4
49	2x(51)	11	1
		8_{+}	0 2
46	2x(24)	8_{+}	4
34	(32)	8_{+}	2
28	2x(40)	8	0

Table 17. Classification of states with orbital symmetry [44].

Table 18. Definition of configurational states for $k = 8, [f] = [44]$.

$(2T+1, 2S+1) = (1,1)$

$$|1) = (\phi_0^4)(\phi_{+1}^4)$$

$$|2) = (\phi_0^4)(\phi_{+1}^2\phi_{-1}^2)$$

$$|3) = (\phi_0^4)(\phi_{+1}^2\phi_{+2}^2)$$

$$|4) = (\phi_0^3\phi_{+2})(\phi_{+1}^4)$$

$$|5) = (\phi_0^4)(\phi_{+1}^2\phi_0'^2)$$

$$|6) = (\phi_0^4)(\phi_{+1}^2\phi_{+2}\phi_{-2})$$

$$|7) = (\phi_0^3\phi_{-2})(\phi_{+1}^4)$$

$$|8) = (\phi_0^2\phi_{-1}^2)(\phi_{+1}^4)$$

$$|9) = \sqrt{2/3}\left[(\phi_0^3\phi_{-1})(\phi_{+1}^3\phi_0') + (\phi_0^3\phi_0')(\phi_{+1}^3\phi_{-1})\right]$$

$$|10) = \sqrt{2/5}\left[(\phi_0^3\phi_{-1})(\phi_{+1}^3\phi_0') - (\phi_0^3\phi_0')(\phi_{+1}^3\phi_{-1})\right]$$

$$|11) = (\phi_0^4)(\phi_{+1}^2\phi_{-2}^2)$$

$$|12) = (\phi_0^2\phi_{+2}^2)(\phi_{+1}^4)$$

$$|13) = (\phi_0^2\phi_0'^2)(\phi_{+1}^4)$$

$$|14) = (\phi_0^2\phi_{+2}\phi_{-2})(\phi_{+1}^4)$$

$$|15) = (\phi_0^2\phi_{-2}^2)(\phi_{+1}^4)$$

$$|16) = (\phi_0^4)(\phi_{-1}^2\phi_{+2}^2)$$

$$|17) = (\phi_0^4)(\phi_{+1}\phi_{-1}\phi_{+2}\phi_0')$$

$$|18) = (\phi_0^3\phi_{+2})(\phi_{+1}^2\phi_{-2}^2)$$

$$|19) = \sqrt{1/5}\left[(\phi_0^3\phi_{+2})(\phi_{+1}^2\phi_{-1}^2) + \sqrt{6}(\phi_0^3\phi_{-1})(\phi_{-1}\phi_{+1}^2\phi_{+2})\right]$$

$$|20) = (\phi_0^4)(\phi_{+2}^4)$$

$$|21) = (\phi_0^3\phi_{+1})(\phi_{+1}\phi_{+2}^3)$$

$$|22) = (\phi_0^4)(\phi_{+2}^3\phi_{-2})$$

$$|23) = (\phi_0^4)(\phi_{+2}^2\phi_0'^2)$$

Table 19.

States $\phi((84)\epsilon\Lambda\nu)$ in terms of configurational states $\phi_i(\epsilon\nu)$.

ϵ	Λ	ν	$\sum_i p_i(84,\epsilon\Lambda\nu)\phi_i(\epsilon\nu)$
20	2	4	$\vert 1)$
		0	$\vert 2)$
14	3	6	$\sqrt{6/7}\,\vert 3)-\sqrt{1/7}\,\vert 4)$
		2	$\sqrt{8/35}\,\vert 5)+\sqrt{4/35}\,\vert 6)-\sqrt{1/105}\,\vert 7)-\sqrt{1/35}\,\vert 9)+\sqrt{1/21}\,\vert 10)+\sqrt{2/35}\,\vert 16)+\sqrt{16/35}\,\vert 17)-\sqrt{2/35}\,\vert 18)$
		-2	$\sqrt{2/35}\,\vert 11)+\dots$
	2	2	$-\sqrt{1/26}\,\vert 5)-\sqrt{1/52}\,\vert 6)-\sqrt{1/156}\,\vert 7)-\sqrt{4/13}\,\vert 9)-\sqrt{5/39}\,\vert 10)+\sqrt{1/26}\,\vert 16)+\sqrt{1/52}\,\vert 17)+\sqrt{1/104}\,\vert 18)-\sqrt{45/104}\,\vert 19)$
		-2	$-\sqrt{1/26}\,\vert 11)+\dots$
	1	2	$\sqrt{4/195}\,\vert 5)-\sqrt{3/130}\,\vert 6)+\sqrt{9/130}\,\vert 7)+\sqrt{20/39}\,\vert 8)+\sqrt{8/195}\,\vert 9)+\sqrt{2/13}\,\vert 10)+\sqrt{1/195}\,\vert 16)-\sqrt{1/39}\,\vert 17)+\sqrt{3/260}\,\vert 18)-\sqrt{25/156}\,\vert 19)$
		-2	$\sqrt{1/195}\,\vert 11)+\dots$
8	4	8	$\sqrt{3/35}\,\vert 12)+\sqrt{24/35}\,\vert 21)+\sqrt{8/35}\,\vert 20)$
		4	$\sqrt{3/245}\,\vert 13)+\sqrt{3/490}\,\vert 14)+\dots$
		0	$\sqrt{3/2450}\,\vert 15)+\dots$
	3	4	$\sqrt{1/91}\,\vert 13)+\sqrt{1/182}\,\vert 14)+\dots$
		0	$\sqrt{3/910}\,\vert 15)+\dots$
	2	4	$\sqrt{2/637}\,\vert 13)-\sqrt{25/2548}\,\vert 14)+\dots$
		0	$-\sqrt{1/7644}\,\vert 15)+\dots$
	1	0	$-\sqrt{27/2860}\,\vert 15)+\dots$
	0	0	$\sqrt{128/10725}\,\vert 15)+\dots$

Table 20.

Matrix elements for k = 8.

Coupling to $\phi(84)$.

$\phi_i(\epsilon\nu)$	$(\phi(84)\|V'_c\|\phi_i(\epsilon\nu))_S/V_o p$					$(\phi(84)\|V'_c\|\phi_i(\epsilon\nu))_A/V_o p$				
	α^8	α^6	α^4	α^2	1		α^8	α^6	α^4	α^2
$\|1)$	72	288	900	936	630		40	160	340	360
$\|2)$			$4\sqrt{6}$	$8\sqrt{6}$	$12\sqrt{6}$					
$\|3)$			$8\sqrt{6}$	$8\sqrt{6}$	$14\sqrt{6}$					
$\|4)$			42	-12	6				-30	-60
$\|5)$			$4\sqrt{6}$	$8\sqrt{6}$	$12\sqrt{6}$					
$\|6)$				$-8\sqrt{3}$	$-4\sqrt{3}$					
$\|7)$			18	-36	-18				10	20
$\|8)$			$8\sqrt{6}$	$8\sqrt{6}$	$14\sqrt{6}$					
$\|9)$			$8\sqrt{3}$	$8\sqrt{3}$	$8\sqrt{3}$					
$\|10)$									$8\sqrt{5}$	$16\sqrt{5}$
$\|11)$					$6\sqrt{6}$					
$\|12)$					$9\sqrt{6}$					
$\|13)$					$6\sqrt{6}$					
$\|14)$					$6\sqrt{3}$					
$\|15)$					$9\sqrt{6}$					

$$V'_c = V_o \sum_{i<j} \exp\left[\frac{-r_{ij}^2}{a^2}\right]$$

$$p = \frac{\alpha^3}{4(2+\alpha^2)^{11/2}}$$

$(\ \|\ \|\ )_S$ = contribution from Symmetric pairs of particles.

$(\ \|\ \|\ )_A$ = " " Antisymmetric " " "

Table 21.

Expansion of h_i coefficients in terms of α, divided into the contribution from the symmetric and antisymmetric pairs of particles. Units of $(-V_o p)$ Mev.

h_i	Symmetic					Antisymmetric			
	α^8	α^6	α^4	α^2	1	α^8	α^6	α^4	α^2
h_1	72	288	900	936	630	40	160	340	360
h_2			2	4	6				
h_3			3/14	15/7	39/14			15/14	15/7
h_4			1/140	1/14	19/140			1/28	1/14
h_5					1/280				
h_6			$\frac{-23}{208}$	$\frac{-7}{104}$	$\frac{-23}{208}$.			$\frac{-15}{208}$	$\frac{-15}{104}$
h_7					-1/208				
h_8			$\frac{131}{1560}$	$\frac{7}{156}$	$\frac{53}{520}$			$\frac{11}{312}$	$\frac{11}{156}$
h_9					1/520				

Table 22a.

α	0.5	0.7	0.8	1.0	1.2	1.4
$\frac{1}{2}(h_5+h_7+h_9)$	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
$\frac{1}{2}(h_4+28h_5+h_6+10h_7+h_8-22h_9)$	0.0738	0.0783	0.0800	0.0831	0.0834	0.0790
$\frac{1}{2}(h_3+6h_4+72h_5-24h_8-48h_9)$	0.7912	0.8144	0.7775	0.5275	-0.0876	-1.2533
$-(3h_7+5h_9)$	0.0048	0.0048	0.0048	0.0048	0.0048	0.0048
$(3h_6+30h_7+5h_8+10h_9)$	0.0640	0.0850	0.1033	0.1634	0.2680	0.4355
$\frac{1}{2}(h_4+28h_5+h_6+10h_7+h_8+18h_9)$	0.1123	0.1165	0.1185	0.1216	0.1219	0.1174
$\frac{1}{2}(h_3+6h_4+72h_5+16h_8+32h_9)$	2.7747	3.2359	3.7728	4.1551	5.2198	6.8027
$\frac{1}{2}(h_4+12h_5+h_6-12h_7+h_8-8h_9)$	0.1116	0.1158	0.1178	0.1209	0.1212	0.1167
$\frac{1}{2}(2h_2-h_3+10h_4+376h_5+22h_6+220h_7+10h_8+436h_9)$	5.9038	6.7696	7.3770	9.0424	11.4765	14.9183
$(h_1+4h_2+8h_3+48h_4+576h_5+208h_8+416h_9)$	1015+ 0.0844	1448+ 0.7226	1802+ 0.7653	2976+ 0.2854	5213+ 0.6622	9425+ 0.6414

Evaluation of the symmetric contribution to the functions of h_i in 6.19. Units of $(-V_o p)$Mev.

Table 22b.

α	0.5	0.7	0.8	1.0	1.2	1.4
$\frac{1}{2}(h_5+h_7+h_9)$	0	0	0	0	0	0
$-\frac{1}{2}(h_4+28h_5+h_6+10h_7+h_8-22h_9)$	0.0003	0.0007	0.0010	0.0017	0.0028	0.0044
$\frac{1}{2}(h_3+6h_4+72h_5-24h_8-48h_9)$	0.1236	0.2682	0.3713	0.6593	1.0887	1.7058
$-(3h_7+5h_9)$	0	0	0	0	0	0
$-(3h_6+30h_7+5h_8+10h_9)$	0.0225	0.0489	0.0677	0.1202	0.1985	0.3110
$-\frac{1}{2}(h_4+28h_5+h_6+10h_7+h_8+18h_9)$	0.0003	0.0007	0.0010	0.0017	0.0028	0.0044
$\frac{1}{2}(h_3+6h_4+72h_5+16h_8+32h_9)$	0.5203	1.1285	1.5627	2.7747	4.5816	7.1788
$-\frac{1}{2}(h_4+12h_5+h_6-12h_7+h_8-8h_9)$	0.0003	0.0007	0.0010	0.0017	0.0028	0.0044
$-\frac{1}{2}(2h_2-h_3+10h_4+376h_5+22h_6+220h_7+10h_8+436h_9)$	0.5479	1.1885	1.6459	2.9224	4.8255	7.5608
$(h_1+4h_2+8h_3+48h_4+576h_5+208h_8+416h_9)$	104+ 0.6920	300+ 0.6609	448+ 0.0863	952+ 0.8571	1960+ 0.4513	3948+ 0.5380

Evaluation of the antisymmetric contribution to the functions of h_i in 6.19. Units of $(-V_o p)$Mev.

Table 23. $\tilde{g}'_S(84|84\ LKK')$ (relative to $\tilde{g}'_S(84|84\ 000)$)

L = 2

K' \ K	0	2
0	0.7457	-0.0615
2	-0.0330	1.5837

L = 3

K' \ K	0
0	2.2608

L = 4

K' \ K	0	2	4
0	2.4856	-0.2383	0
2	-0.1278	3.1634	0.0208
4	0	-0.0312	5.7493

Table 24. $\tilde{g}'_A(84|84\ LKK')$ (relative to $\tilde{g}'_A(84|84\ 000)$)

L = 2

K' \ K	0	2
0	0.3964	-0.0769
2	-0.0248	0.1184

L = 3

K' \ K	0
0	0.5157

L = 4

K' \ K	0	2	4
0	1.3213	-0.2979	0
2	-0.0962	1.0455	-0.1142
4	0	-0.0260	0.2097

Table 25. $\tilde{g}''_S(84|84\ LKK')$ (relative to $\tilde{g}'_S(84|84\ 000)$)

L = 2

K' \ K	0	2
0	0.1313	-0.0679
2	-0.0360	0.0580

L = 3

K' \ K	0
0	0.0505

L = 4

K' \ K	0	2	4
0	0.3181	-0.1431	-0.0452
2	-0.0785	0.1878	0.0032
4	-0.0102	-0.0306	0.0836

Figure 1

Ratio of the energies E(04M) and E(02M) for k = 2,3&4.

Experimental values for O^{18} and Ne^{20} are indicated by the broken lines.

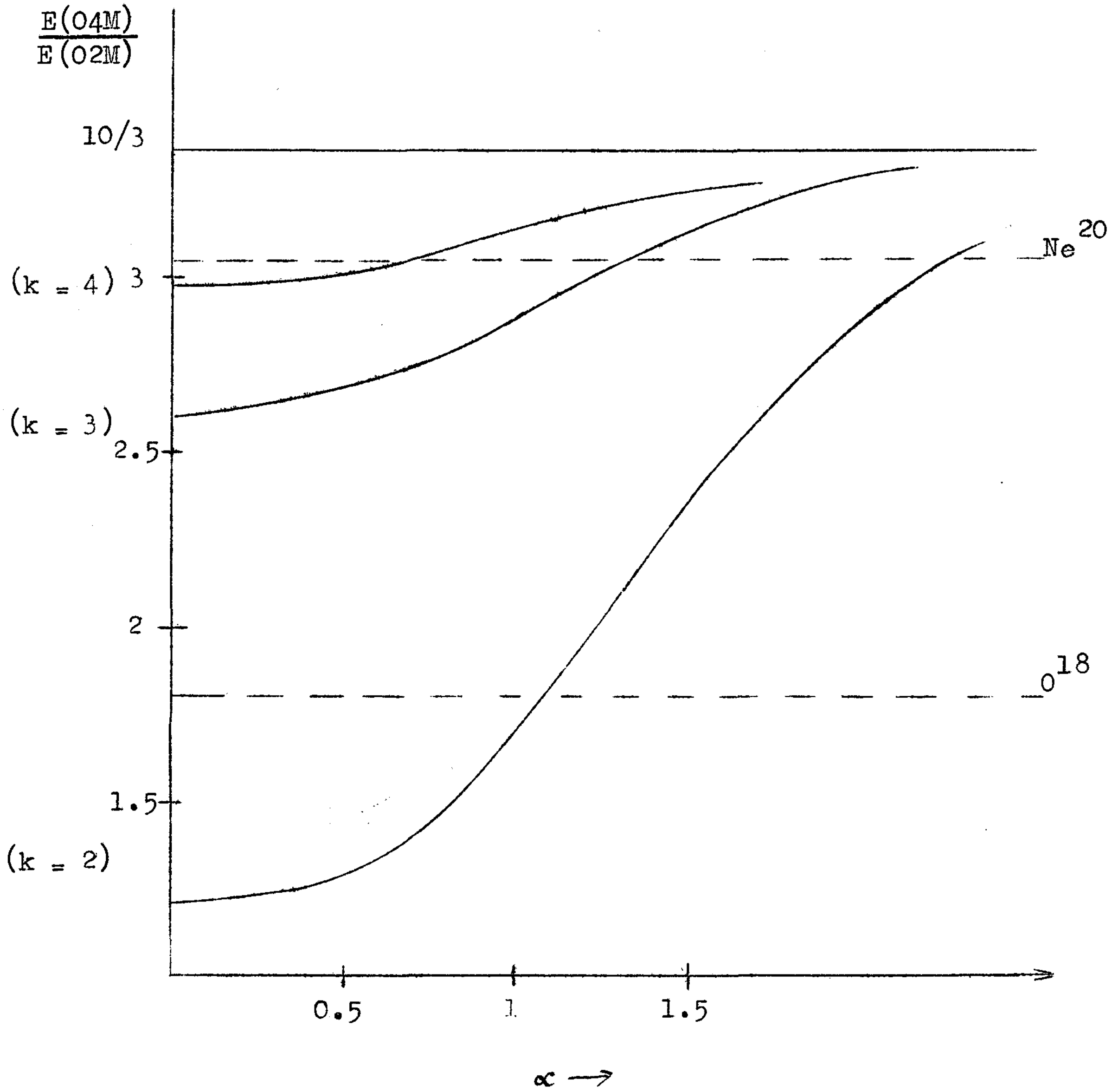


Figure 2

Approximations to the O^{18} spectrum. (Units of $V_o/40$ Mev)

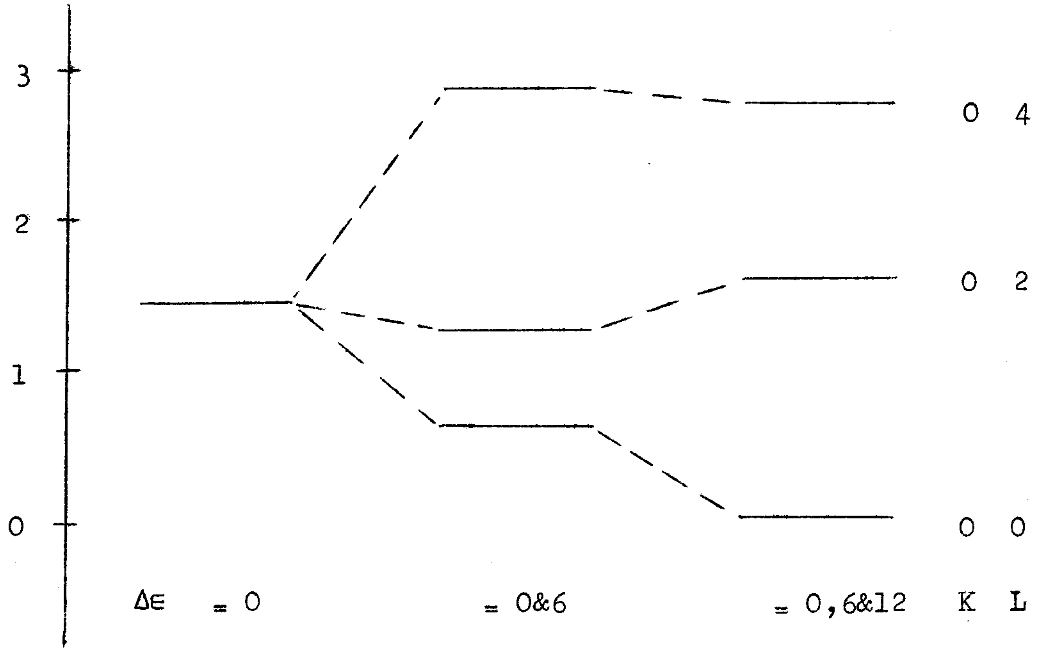


Figure 3

Approximations to the Ne^{20} spectrum. (Units of $V_o/40$ Mev)

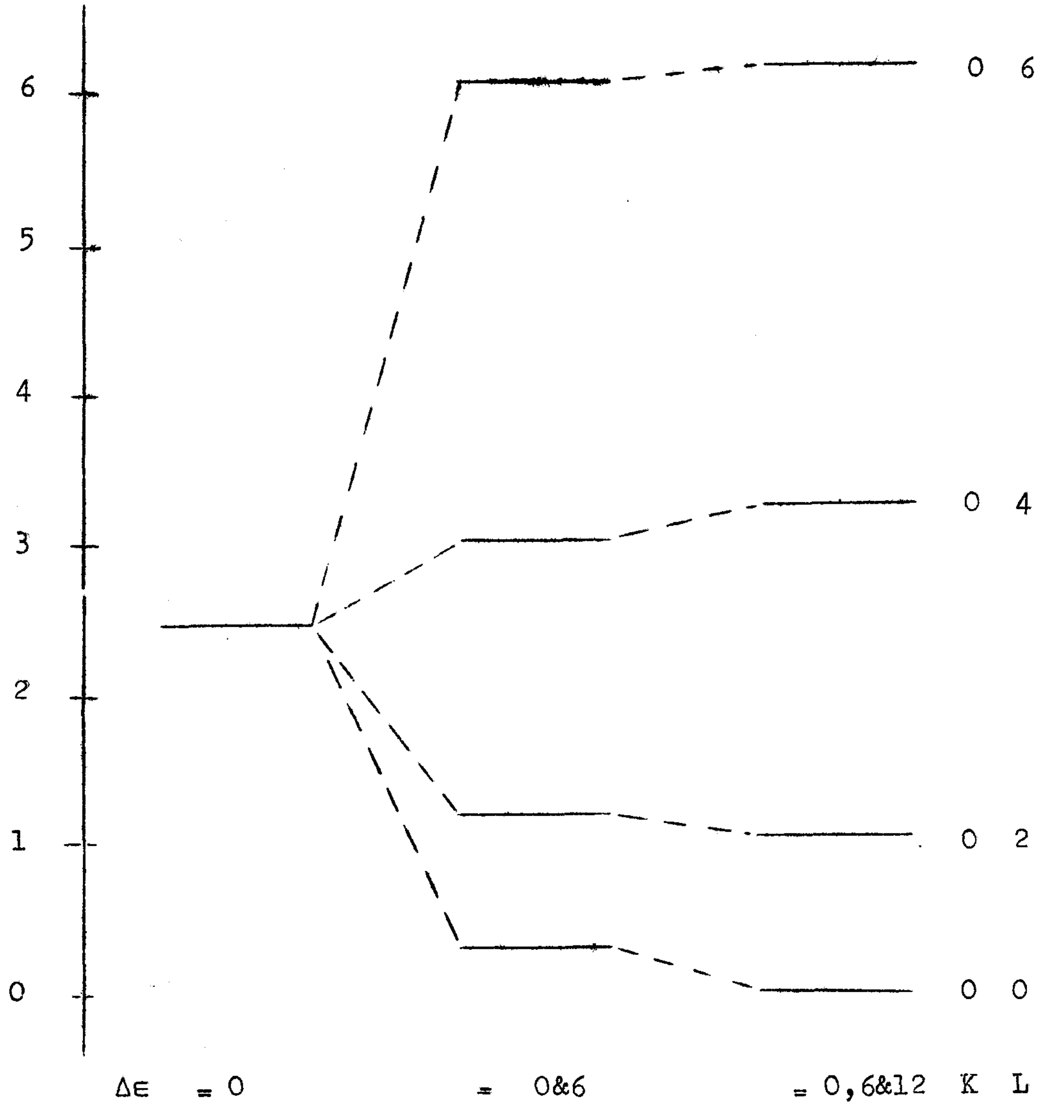


Figure 4

Approximations to the Mg^{24} spectrum - Serber-type exchange.

(Units of $V_o/40$ Mev)

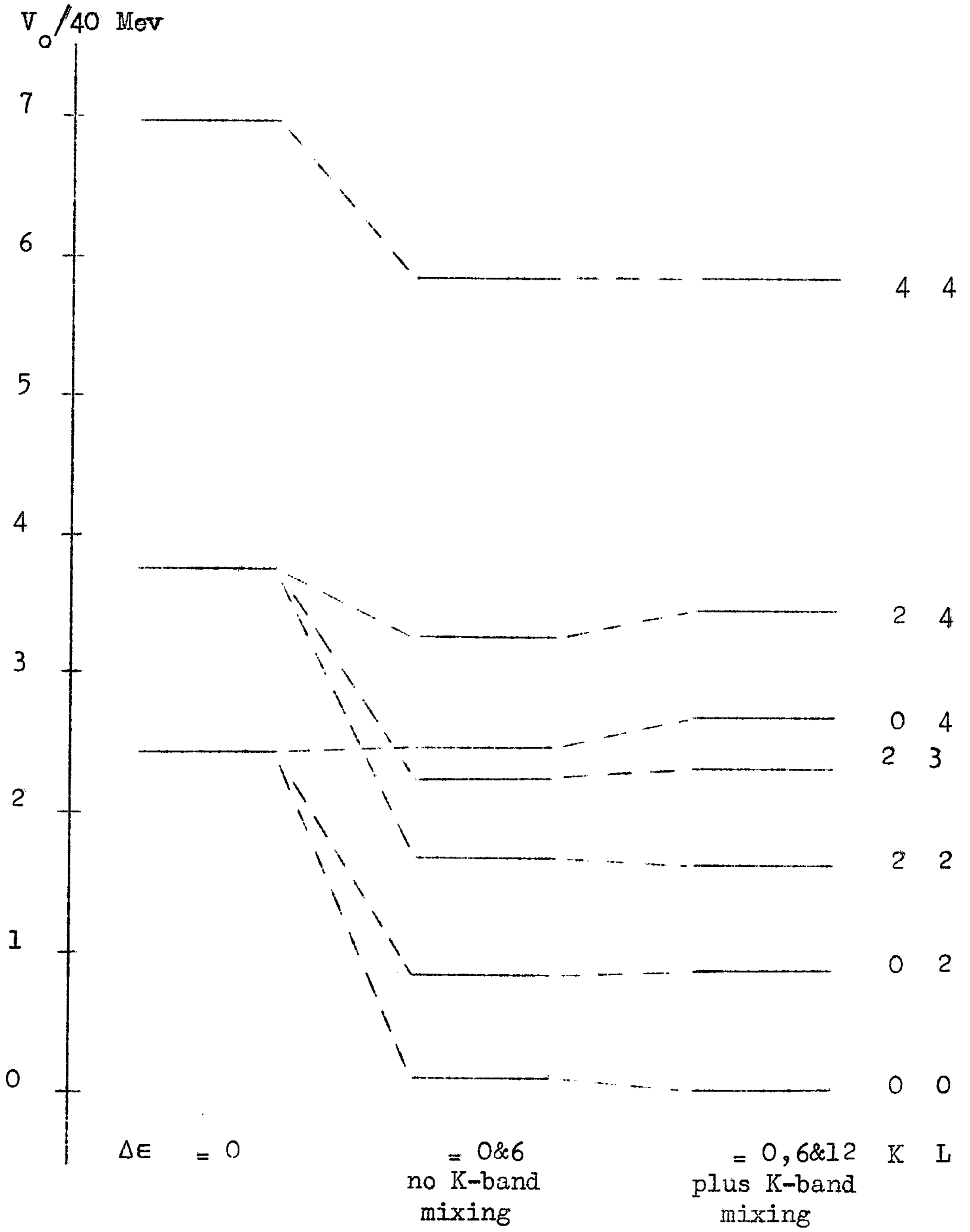


Figure 5

Approximations to the Mg^{24} spectrum - Rosenfeld-type exchange.

(Units of $V_o/40$ Mev)

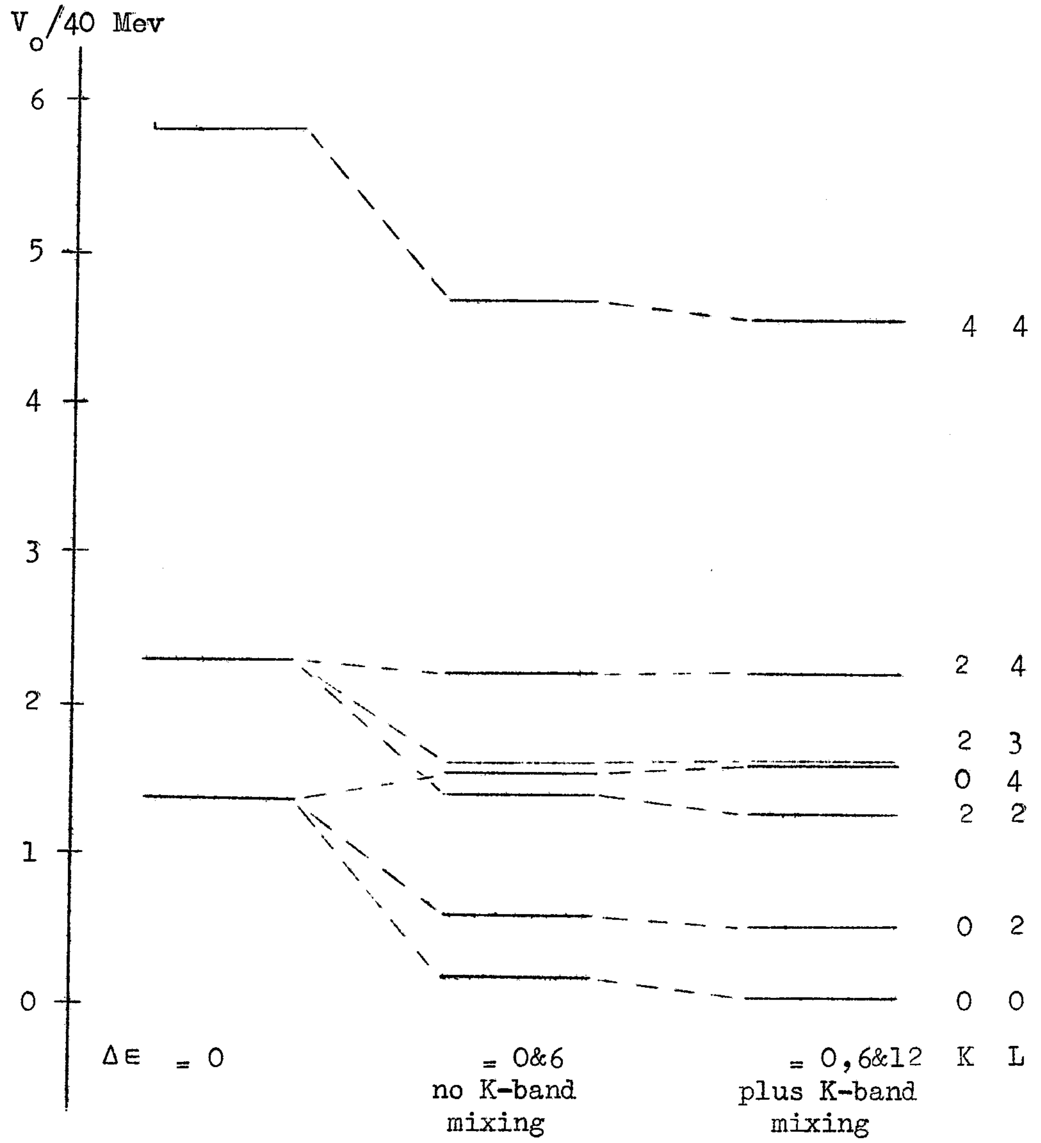


Figure 6

Energy levels for 0^{18}. (Units of $V_o/40$ Mev)

(a) Theory: shown as a variation with α for a Serber exchange.

(b) Experiment: shown for varying V_o.

(Azjenberg-Selove and Lauritsen, 1959.)

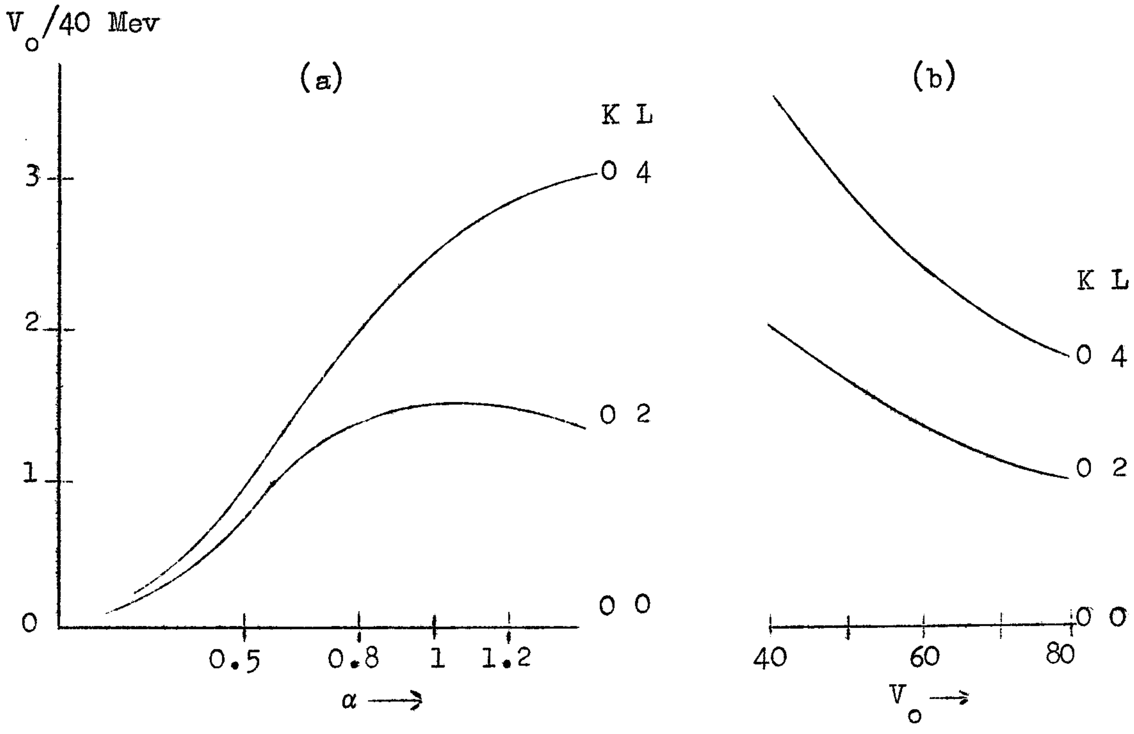


Figure 7

Energy levels for Ne^{20}. (Units of $V_o/40$ Mev)

(a) Theory: shown as a variation with α for a Serber exchange.

(b) Experiment: shown for varying V_o.

(Azjenberg-Selove and Lauritsen, 1959.)

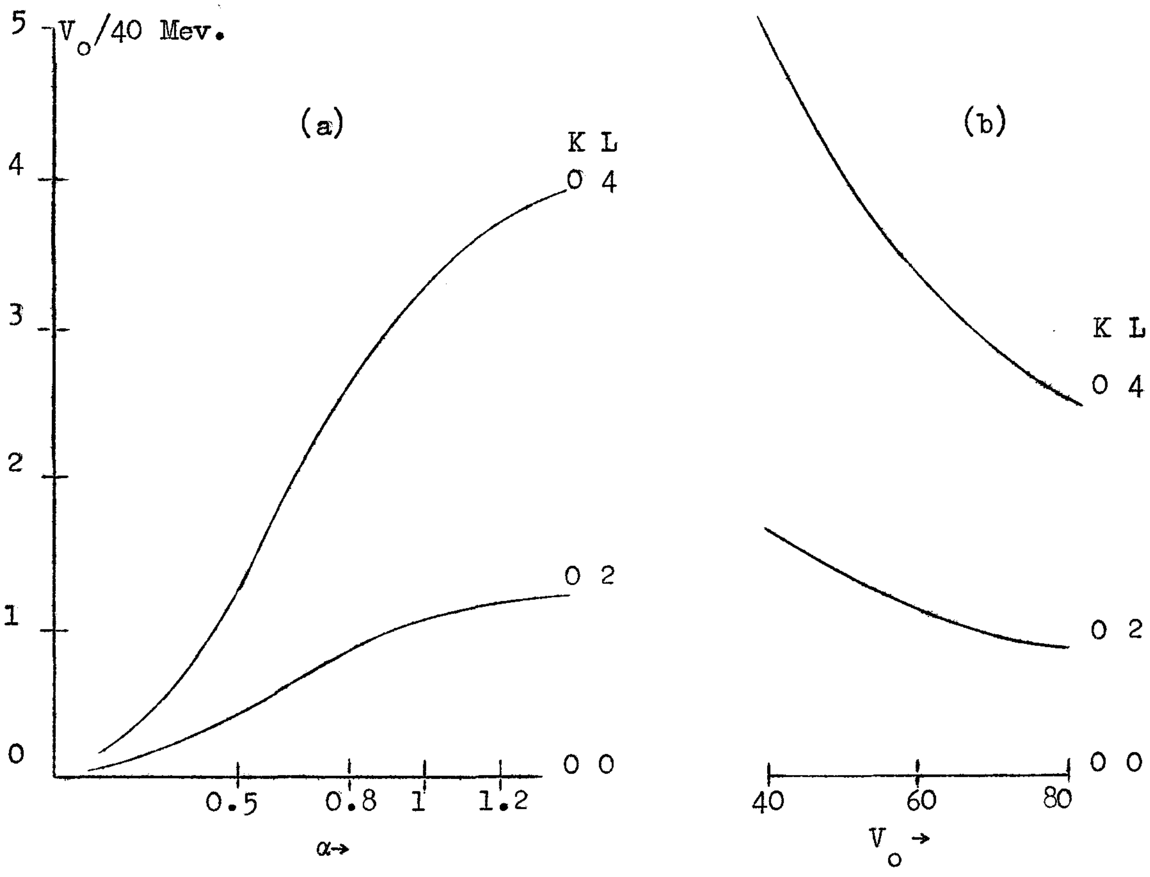


Figure 8

Energy levels for Mg^{24}. (Units of $V_o/40$ Mev)

(a) Theory: shown as a variation with α for a Serber exchange.

(b) Experiment: shown for varying V_o. (Endt and Braams, 1957)

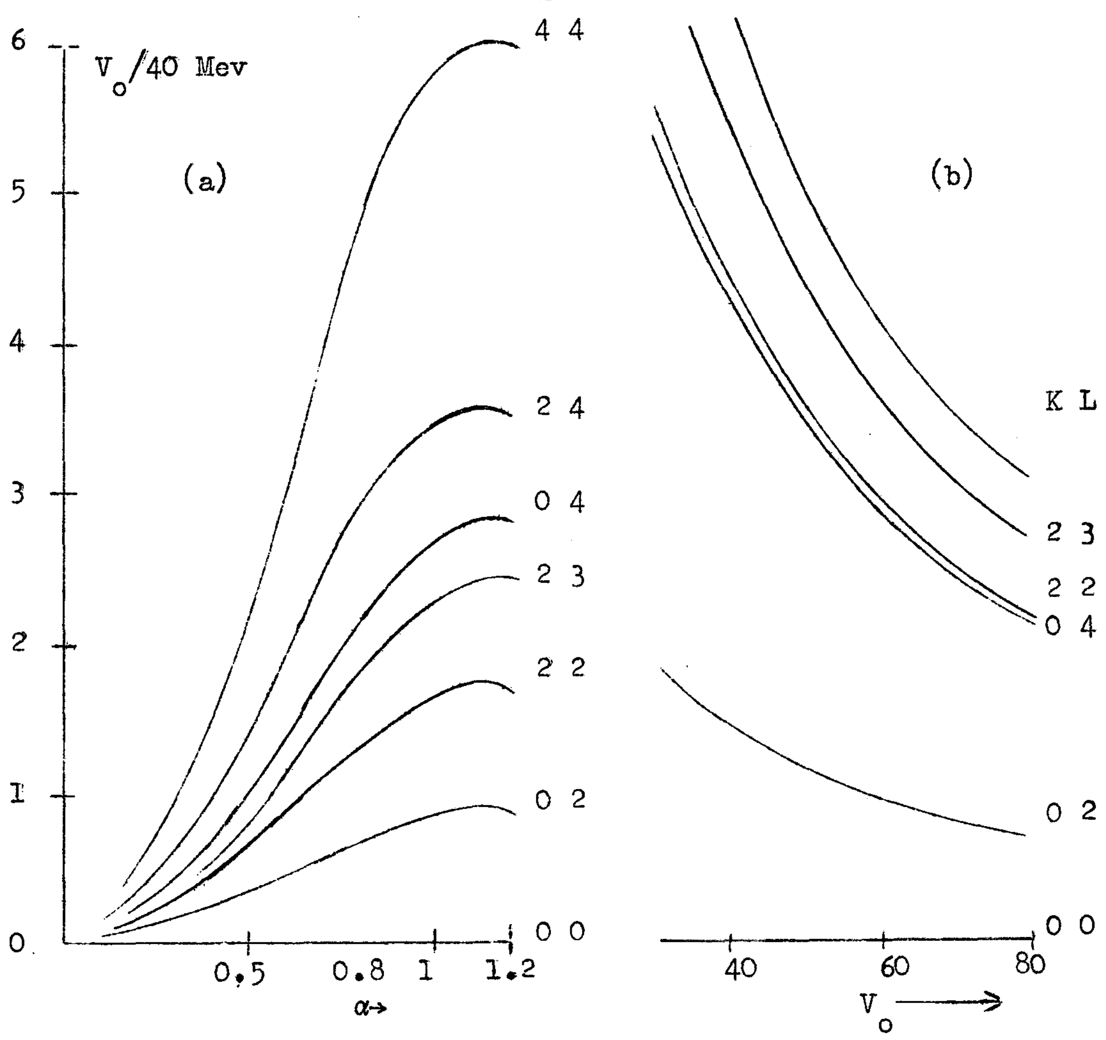


Figure 9

Energy levels for Mg^{24}. (units of V_o/40 Mev)

(a) Theory: shown as a variation with α for an Intermediate exchange.

(b) Experiment: shown for varying V_o. (Endt and Braams, 1957)

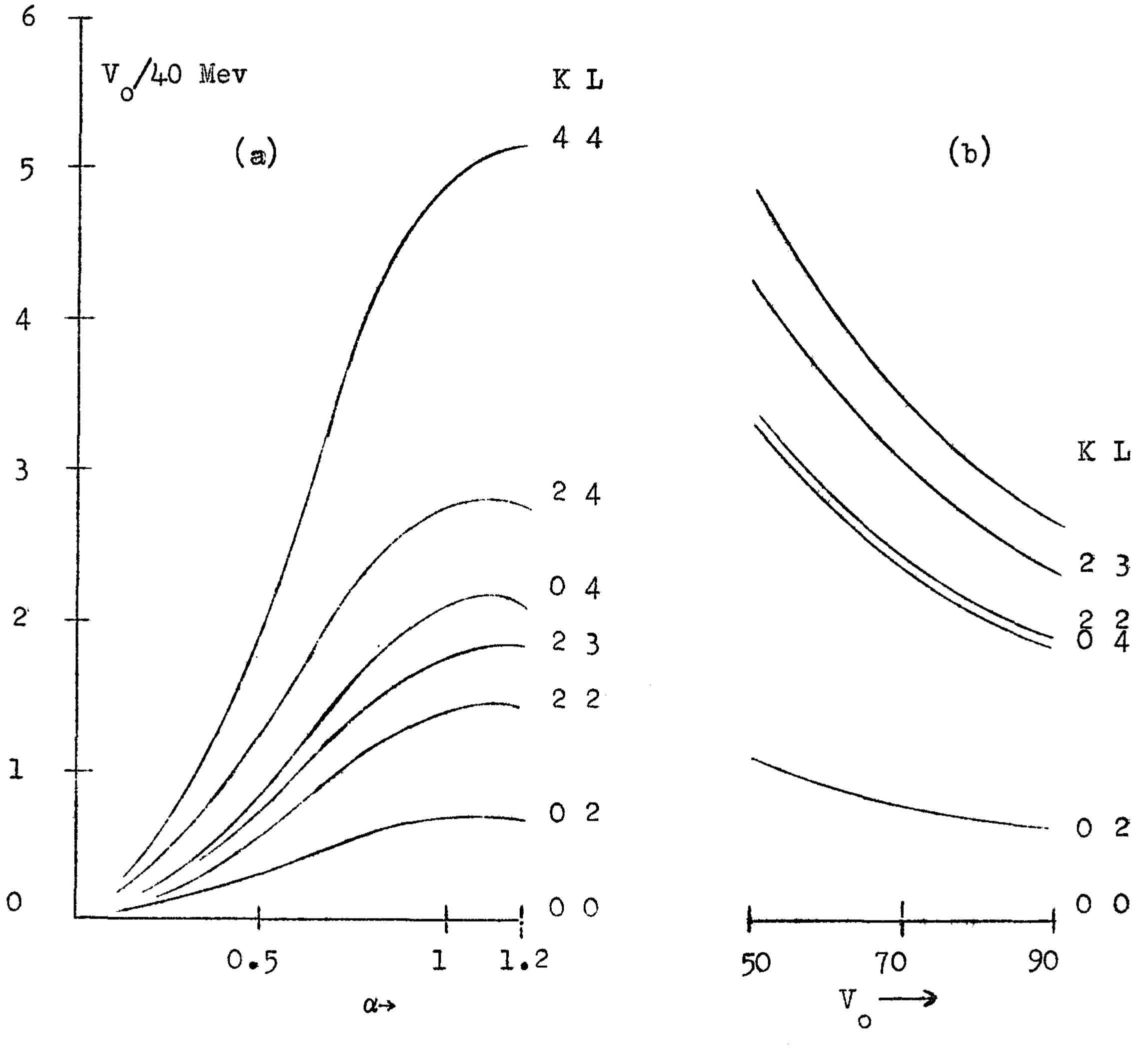


Figure 10

Energy levels for Mg^{24}. (Units of $V_o/40$ Mev)

(a) Theory: shown as a variation with α for a Rosenfeld exchange.

(b) Experiment: shown for varying V_o. (Endt and Braams, 1957)

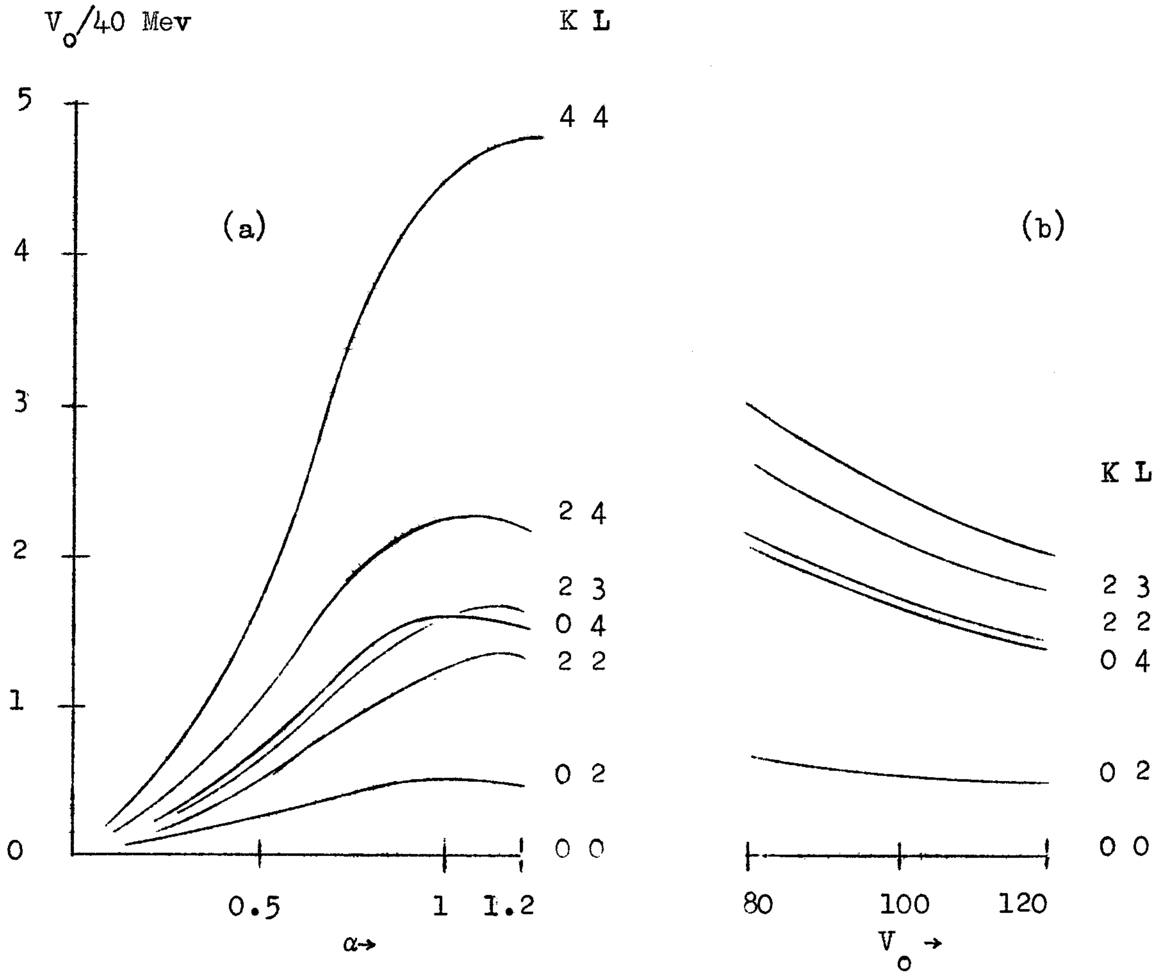
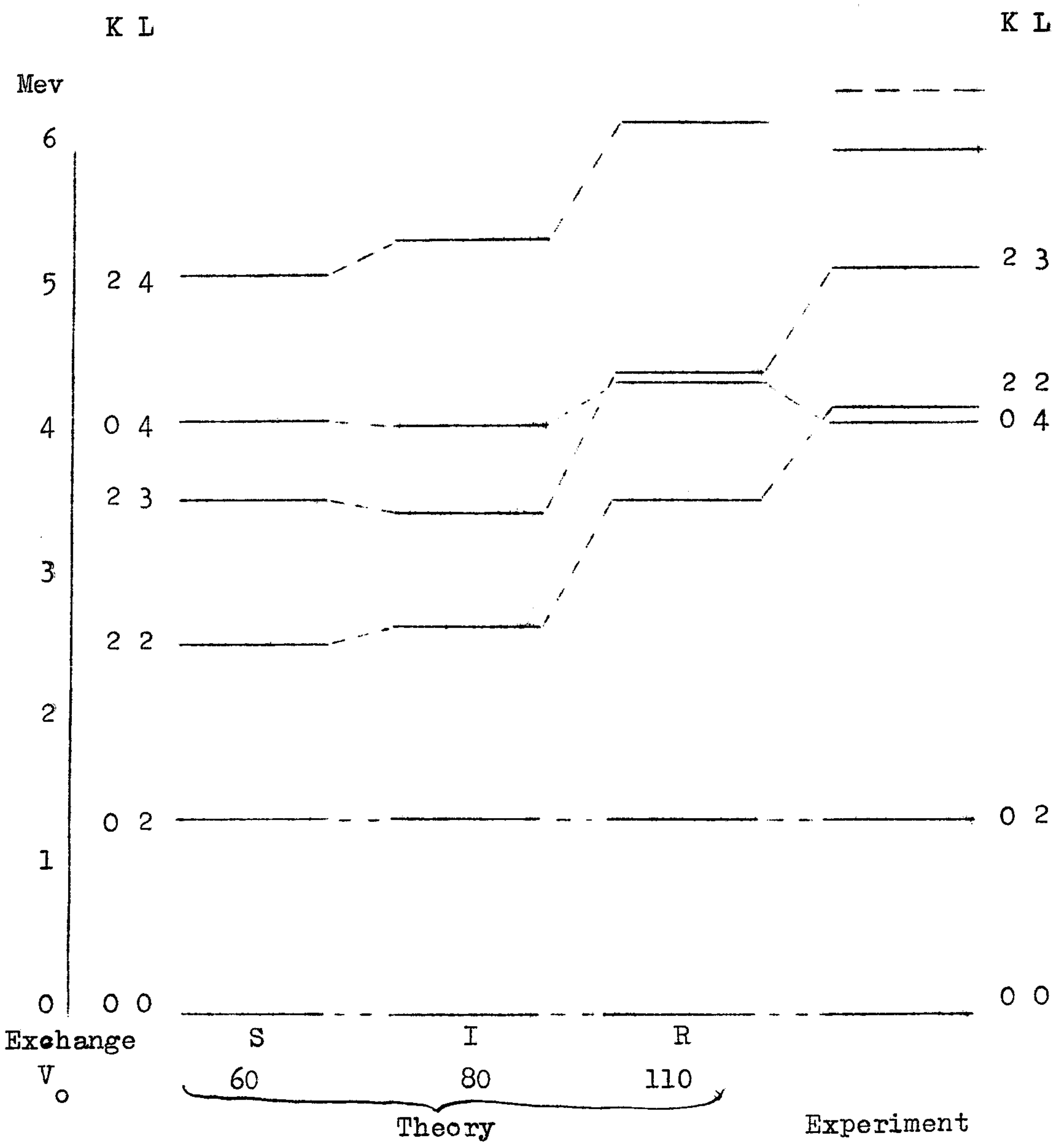


Figure 11

Energy levels for Mg^{24} for three exchange mixtures:- Serber (S), Intermediate (I) and Rosenfeld (R).



Chapter 7

Representation Mixing For The Eight Particle Problem

In this chapter we begin to discuss the magnitude and effect of the mixing of other representations with the (84) representation.

The mixing of different K-bands, from different representations, is expected to be small in the same way that we have seen the mixing of bands within the (84)-representation to be small. Thus, when considering the mixing of states of different representations into the lowest-lying states of $(\lambda\mu) = (84)$, we shall here, as a first step, only consider these representations containing $K = 0$ bands with even L. From Table 17 and 4.9 these are seen to be

$$(84), \quad (46), \quad (08), \quad 2x(62), \quad (24), \quad 2x(40) \qquad 7.1$$

From the general remarks on the Casimir operator $\tilde{C}_1$ in Chapter 6, we may suppose that the next most important representation to (84) is (46) - this having the next highest value for $\langle \tilde{C}_1 \rangle_{\lambda\mu}$ (Table 17)

The mixing of the (46)-representation

The leading intrinsic state of the (46)-representation must be orthogonal to $\Phi^{11}((84), 14, 3, 6)$. We find then, from Table 19,

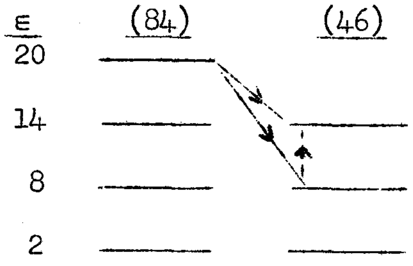
$$\Phi^{11}((46)\ 14, 3, 6) = \sqrt{1/7}(\phi_0^4)(\phi_{+1}^2\ \phi_{+2}^2)^{11} + \sqrt{6/7}(\phi_0^3\ \phi_{+2})(\phi_{+1}^4)^{11} \qquad \underline{7.2}$$

The $\tilde{g}$-matrix elements, which couple the K = 0 bands of the (84) and (46) representations, are of the form:-

$$\tilde{g}(84|46, L\ 0\ 0) \quad \text{and} \quad \tilde{g}(46|84, L\ 0\ 0) \qquad \underline{7.3}$$

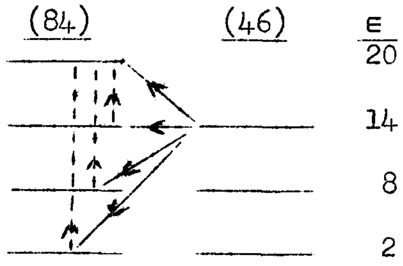
We illustrate the difference between these two elements with the help of the diagrams shown below.

Diagram 1



etc.

Diagram 2



etc.

In the diagrams, the horizontal lines represent the different ε-values, at a distance $\Delta\epsilon = 6$ apart, with the maximum ε (= 20) highest.

Diagram 1 indicates the steps required to build up $\tilde{g}(84|46, L\ 0\ 0)$. The unbroken arrowed line represents the coupling of the state $\Phi((46), \epsilon\ \nu)$ to $\Phi(84)$ by the two body potential. The broken arrowed line represents the equivalent operator $\tilde{E}((46)\epsilon_{\Lambda\nu})$ where :-

$$E((46)\epsilon_{\Lambda\nu})\ \Phi(46) = \Phi((46)\epsilon_{\Lambda\nu})$$

Diagram 2 indicates the steps required to build up $\tilde{g}(46|84, L\ 0\ 0)$.

Remembering that for each ϵ-value there may be several Λ & ν numbers, it can be seen that the $\tilde{g}(84|46\ L\ 0\ 0)$-element is much easier to build up than the $\tilde{g}(46|84\ L\ 0\ 0)$.

The $\tilde{g}(46|84, L\ 0\ 0)$ - elements can be deduced exactly from the $\tilde{g}(84|46, L\ 0\ 0)$ using the set of equations in 5.24, however, to estimate the (46) representation mixing, only the $\tilde{g}(84|46\ L\ 0\ 0)$ and $\tilde{g}(84|46\ L\ 2\ 2)$ off diagonal elements have been calculated exactly. The elements $\tilde{g}(46|84\ L\ 0\ 0)$ and $\tilde{g}(46|84\ L\ 2\ 2)$ are estimated using the relation in 5.26. We take as our justification for doing this the smallness of the overlaps in the (84) representation for $L = 0, 2$ & 4 and the smallness of the terms $\tilde{g}(84|46\ L\ K\ K')_{K \neq K'}$ compared with $\tilde{g}(84|46\ L\ K\ K)$. This latter result arises from the fact that $\tilde{g}(84|46\ L\ K\ K')_{K \neq K'}$ only comes from the coupling of the states $\Phi((46), 8_{\Lambda\nu})$ to $\Phi(84)$ by the potential, whereas $\tilde{g}(84|40\ L\ K\ K)$ comes from the coupling of states $\Phi((40)\ 14_{\Lambda\nu})$ and $\Phi((40)\ 8_{\Lambda\nu})$ to $\Phi(84)$. The smallness of the matrix elements in the former case together with the small mormalisation coefficients of the operators $E((46), 8\Lambda\nu)$, compared with the same functions in the latter case, justify over assumptions. We note here that, for the representations $(\lambda\mu)$ such that $(20-2\lambda-\mu) = 12$, the elements $\tilde{g}(84|\lambda\mu\ L\ K\ K')_{K \not\leq K'}$ are identically zero. For these cases then, there can be no mixing of bands with the mixing of representations. For the representation $(\lambda\mu)$ such that $(20-2\lambda-\mu) = 6$, e.g. $(\lambda\mu) = (46)$, the mixing of bands will be small and, as we have already stated, may be ignored.

The configurational states with $\epsilon = 14$ and 8, which will couple to $\Phi(84)$ by a two body potential, have been deduced in the last Chapter. The intrinsic states of the (46) representation which contain these configurational states are given in Table 26.

We find that, for $\alpha = 1$,

$$\tilde{g}_S(84|46\ L00) = \sqrt{6/7}\{1{\cdot}594 + {\cdot}048\ L(L+1)\}\ \left(\frac{V_0}{40}\text{ Mev.}\right) \qquad \underline{7.4a}$$

$$\tilde{g}_A(84|46\ L00) = -\sqrt{6/7}\{2{\cdot}571\}\ \left(\frac{V_0}{40}\text{ Mev.}\right) \qquad \underline{7.4b}$$

$$\tilde{g}_S(84|46\ L22) = \sqrt{6/7}\{1{\cdot}223 + {\cdot}039\ L(L+1)\}\ \left(\frac{V_0}{40}\text{ Mev.}\right) \qquad \underline{7.5a}$$

$$\tilde{g}_A(84|46\ I22) = -\sqrt{6/7}\{2{\cdot}286\}\ \left(\frac{V_0}{40}\text{ Mev.}\right) \qquad \underline{7.5b}$$

The elements $\tilde{g}(46|84\ L00)$ may be calculated from 7.4 for L = 0, 2 and 4 using 5.26 with :-

$a^2(84\ 00) = 5/11.13 \qquad a^2(46\ 00) = 10/3.7.11$

$a^2(84\ 02) = 19/11.13 \qquad a^2(46\ 02) = 5.5.17/3.7.11.13$

$a^2(84\ 04) = 3.807/8.11.13.17 \qquad a^2(46\ 04) = 149.5/7.8.11.13$

Similarly we may deduce the element $\tilde{g}(46|84\ L22)$ from 7.5 with :-

$a^2(84\ 22) = 137/2.9.11.13$, $a^2(46\ 22) = 5.5.17/2.3.7.11.13$

$a^2(84\ 23) = 4.7/3.11.15$, $a^2(46\ 23) = 5/2.3.13.$

$a^2(84\ 24) = 223/4.5.13.17$, $a^2(46\ 24) = 3.5/4.7.11.$

The exact form of the element $\tilde{g}(46|46\ LKK)$ should be deduced in the same way as the $\tilde{g}(84|84\ LKK)$ of the last chapter. However, to estimate the order of magnitude of the representation mixing, it is only necessary to deduce the order of magnitude of the spacings between the terms $\tilde{g}(46|46\ LKK)$ and $\tilde{g}(84|84\ LKK)$.

In building up $\tilde{g}(84|84\ LKK)$, we saw that the intrinsic states, differing in their ε-values by more than 12 from the leading state, $\Phi(84)$, contribute a negligible amount to the energy. The states differing in their ε-values by 6 from that of $\Phi(84)$ contribute only a small amount of energy compared with the $\Delta\epsilon = 0$ terms. Of the states with $\Delta\epsilon = 6$, the most important one had maximum ν. Thus, in estimating $\tilde{g}(46|46\ 000)$, the only intrinsic states which have been considered are:

$$\Phi((46)\ 14, 3, 6)$$

$$\Phi((46)\ 14, 3, 2)$$

$$\Phi((46)\ 14, 3, -2)$$

$$\Phi((46)\ 8, 4, 8) \qquad \underline{7.6}$$

The configurational states of these intrinsic states, which couple to $\Phi(46)$ via a two body potential, are included in Table 26. The non-zero matrix elements, $(\Phi_i(\epsilon\nu)|V_o|\Phi(46))$, are listed in Table 27.

We find the differences,

$$D_S = \tilde{g}(46|46\ 000) - \tilde{g}(84|84\ 000) = 8\cdot30\ (\frac{V_o}{40}\ \text{Mev.})$$

for the Serber-type exchange

$$D_R = \tilde{g}(46|46000) - \tilde{g}(84|84\ 000) = 6\cdot04\ (\frac{V_o}{40}\ \text{Mev.})$$

for the Rosenfeld-type exchange $\underline{7.7}$

Incidentally, since the $\tilde{g}(\lambda\mu|\lambda\mu\ 000)$ is the first approximation to the lowest state of the $(\lambda\mu)$ representation, 7.7 shows that the (84) representation is indeed lower than the (46) in energy. This then partially justifies our assumption at the beginning of Chapter 6.

In the estimation of the mixing of the other states $\Psi((46)KLM)$, with K = 0 and 2 we have assumed:-

$$\tilde{g}(46|46\ LKK) - \tilde{g}(84|84\ LKK) = D_{S \text{ or } R}$$

Thus, we have assumed that the error in the spacing of levels in the (46) representation is small compared with the spacing of the (84) and (46) representation (i.e. the difference in energy in the states $\Psi((84)000)$ and $\Psi((46)000)$).

An estimation of the difference:

$$\tilde{g}(46|46\ 200) - \tilde{g}(84|84\ 200) = D'$$

using only the states in 7.6, yields the results:-

$$D'_S = 8{\cdot}11\ (\tfrac{V_o}{40}\ \text{Mev.})\ \text{ for the Serber-type exchange}$$

$$\text{and }\ D'_R = 5{\cdot}88\ (\tfrac{V_o}{40}\ \text{Mev.})\ \text{ for Rosenfeld-type exchange} \qquad \underline{7.8}$$

These figures must be taken as an underestimation of the true differences and so we may conclude that the use of the D's in 7.7, for all spacings, is probably as accurate as we require for these rather qualitative arguments.

On setting up the various matrices :-

$$\begin{bmatrix} \tilde{g}(84|84\ LKK) & \tilde{g}(46|84\ LKK) \\ \tilde{g}(84|46\ LKK) & \tilde{g}(46|46\ LKK) \end{bmatrix} \qquad \underline{7.9}$$

and diagonalising, we deduce the results shown in Figures 12a and 13a for the Serber and Rosenfeld exchanges. In these figures we are only comparing the energy spacings not the absolute energies. The amount of (84) representation states in the lowest eigenstates is indicated as a percentage.

We see that the mixing of the representations is small and that the spectrum still possesses the main features of the spectrum of the (84) representation.

Discussion on representation mixing

Since the $\langle \tilde{C}_1 \rangle_{\lambda\mu}$ of the other representations in 7.1 are even smaller than $\langle \tilde{C}_1 \rangle_{46}$, we might expect the percentage of mixing of these representations to be very small.

The slight changes in the spectrum, from that of the (84) representation, due to the (46) representation, have very little significance until the effect of the mixing of other representations has been considered.

We notice that the energy of the first excited state of the (84) representation is decreased slightly in the mixing of the (46) representation. Referring back to Figures 8 to 10, this implies that a larger value of V_0 must be chosen to fit the experimental value. An estimate of the effect of the (08) representation mixing has been made, however, in the same way as that for the (46) with $\alpha = 1$ (Figures 12b and 13b). We see that in this case the first excited state of the (84) representation is *increased* in energy. Both the (46) and (08) representations increase the $K = 0$ and $K = 2$ band spacing - a pleasing result since this spacing was found to be too small in the (84) representation.

It is perhaps unwise, at this state, to compare these few results of representation mixing directly with experiment until the effects of other representations has been examined. We notice that, although $\langle \tilde{C}_1 \rangle_{62}$ is smaller than either $\langle \tilde{C}_1 \rangle_{46}$ or $\langle \tilde{C}_1 \rangle_{08}$, the fact that there are two (62) representations may have a noticeable effect.

The representations with odd K, which were not included in *7.1*, must also be examined, although the smallness of the mixing of bands indicates that they will again have little effect on mixing with the (84) representation.

Table 26.

States of the (46) representation in terms of configurational states which couple to $\Phi(84)$ and $\Phi(46)$ with the Gaussian potential.

ϵ	Λ	ν	$\sum_i p_i(46,\epsilon\Lambda\nu)\Phi_i(\epsilon\nu)$
14	3	6	$\sqrt{1/7}\,\vert 3)+\sqrt{6/7}\,\vert 4)$
		2	$\sqrt{1/105}\,[2\vert 5)+\sqrt{2}\,\vert 6)+\sqrt{6}\,\vert 7)+\vert 16)+2\sqrt{2}\,\vert 17)+6\vert 18)+3\sqrt{2}\,\vert 9)-\sqrt{30}\,\vert 10)]$
		-2	$\sqrt{1/105}\,[\vert 11)+\dots]$
8	4	8	$\sqrt{1/42}\,[2\sqrt{6}\,\vert 20)-\sqrt{2}\,\vert 21)-4\vert 12)]$
		4	$1/21\,[-2\sqrt{3}\,\vert 14)-2\sqrt{6}\,\vert 13)+\dots]$
		0	$(1/7)\sqrt{1/15}\,[-2\vert 15)+\dots]$
	3	4	$(1/3)\sqrt{1/77}\,[-2\vert 14)-2\vert 13)+\dots]$
		0	$\sqrt{1/1155}\,[-2\vert 15)+\dots]$
	2	4	$(1/105)\sqrt{1/11}\,[59\sqrt{2}\,\vert 14)-8\vert 13)+\dots]$
		0	$(1/35)\sqrt{1/33}\,[17\sqrt{2}\,\vert 15)+\dots]$

Table 27.

Matrix elements for k = 8.

Coupling to $\Phi(46)$.

$\phi_i(\varepsilon\nu)$	B	$(\bar{\Phi}(46)\vert V'_c\vert\bar{\phi}_i(\varepsilon\nu))_S/V_o p$					$(\bar{\Phi}(46)\vert V'_c\vert\bar{\phi}_i(\varepsilon\nu))_A/V_o p$			
		α^8	α^6	α^4	α^2	1	α^8	α^6	α^4	α^2
3)	$\sqrt{1/7}$	72	288	846	826	561	20	160	310	300
4)	$\sqrt{6/7}$	72	288	796	832	559	40	160	340	360
5)	$\sqrt{1/7}$			8	8	14				
6)	$\sqrt{2/7}$			9	6	6			5	10
7)	$\sqrt{6/7}$			9	6	12			5	10
8)	$-\sqrt{1/7}$				24	12				
9)	$\sqrt{2/7}$			12	12	12				
10)	$-\sqrt{30/7}$								4	8
11)	$\sqrt{1/7}$					9				
12)	$\sqrt{1/7}$			130	-80	49			-45	-105
16)	$\sqrt{1/7}$			4	8	12				
17)	$\sqrt{2/7}$			8	8	8				
18)	$\sqrt{1/7}$			24	50	73				
19)	$-\sqrt{1/35}$				22	11				
20)	$\sqrt{6/7}$			8	8	14				
21)	$-\sqrt{2/7}$			29	2	35			-15	-30

e.g. $(\ \vert\ \vert\ )_{S/A} = B(\ a_8\alpha^8 + a_6\alpha^6 + \dots\ )V_o p$

$$V'_c = V_o \sum_{i<j} \exp\left[\frac{-r_{ij}^2}{a^2}\right] \qquad p = \frac{\alpha^2}{4(2+\alpha^2)^{11/2}}$$

Mixing of representations:-

Figure 12. -for a Serber exchange. ($\alpha=1$)

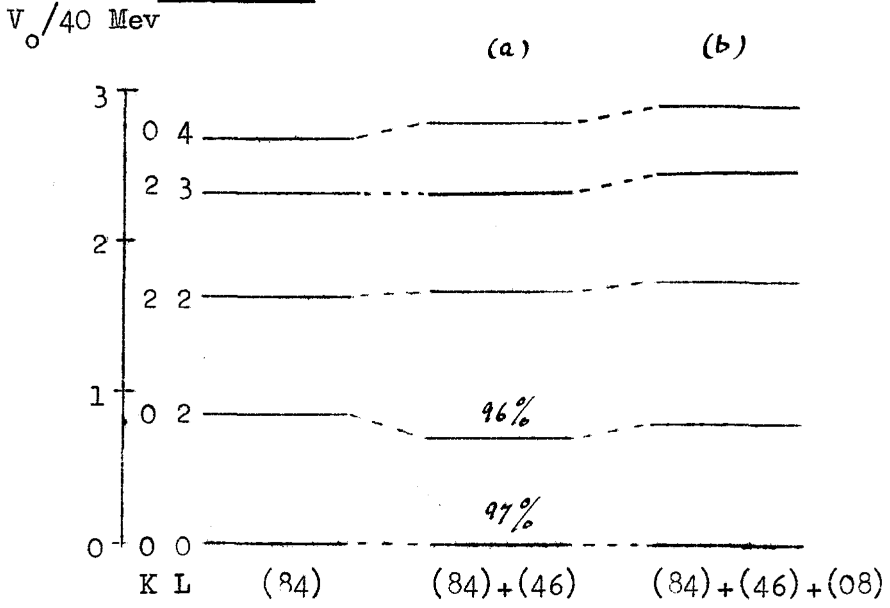
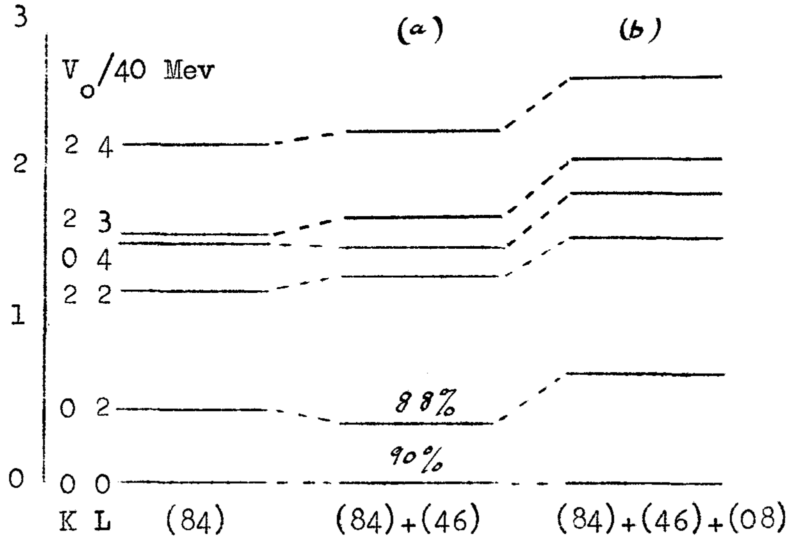


Figure 13 -for a Rosenfeld exchange. ($\alpha=1$)



Summary and Conclusions

The method developed here, for calculating the spectra of light nuclei, is sufficiently streamlined to make a number of such calculations possible for a large number of particles in the shell.

Taking only the states from the leading SU_3 representation $\widetilde{(\lambda\mu)}$, the only labour in the calculations came in the expansion of the intrinsic states $\Phi(\widetilde{(\lambda\mu)}\epsilon\Lambda\nu)$, in terms of configurational states, and the coupling of the configurational states to $\Phi\widetilde{(\lambda\mu)}$ by the potential. Because of the simple form of the intrinsic states, both of these calculations are straight forward but, sometimes, a little tedious.

Analysis has shown that, in the calculations without representation mixing, the intrinsic states of the leading representation with $\Delta\epsilon = 0$ give a spectrum in which the states with different K-labels are separated in energy but where the states with the same K-label are degenerate. A rotational spectrum within a K-band is formed by the inclusion of the $\Delta\epsilon = 6$ intrinsic states and the differences from rotational spectra come from the addition of the $\Delta\epsilon = 12$ states. We have seen that the contribution from the $\Delta\epsilon = 12$ states grows relatively smaller as particles are added into the shell - thus producing the well known rotational features away from closed shells. The comparison of our results with experiment for the spectra of O^{18}, Ne^{20} and Mg^{24}, using only the leading SU_3 representation, depends on the choice of the range and strength of the central force. The results with the values of the parameters chosen in Chapter 6 are encouraging. In general, the $K = 0$ bands can be fitted close to the experimental ones, for any of the exchanges we have used, but the spacing between the $K = 0$ and $K = 2$ bands in Mg^{24} is too small, the least error coming with a Rosenfeld exchange. The spacing of the first and second states in the $K = 2$ band is found to be almost correct for any exchange.

The strengths chosen in each case are roughly consistent for Serber and Intermediate exchanges but the strength, with a Rosenfeld exchange, for Mg^{24} is perhaps too large compared with those for O^{18}, and Ne^{20}.

With the small band mixing in the Mg^{24} case, deduced with Rosenfeld exchange, and the correct energy spacings, we are able to calculate the branching ratio of the E2 transitions from the second 2^+ state to be very close to the experimental value.

The introduction of the term $\Sigma_i L_i^2$ into the Hamiltonian was found to have little effect on the Mg^{24} spectra.

The initial calculations on representation mixing indicate that these are probably small. The (46) and (08) representations change the spectrum, from that of the (84)-representation, only slightly and the increases in the $K = 0$ and $K = 2$ spacing are also small. The actual effect of the representations with odd λ and/or μ will have to be examined although, since these do not contain $K = 0$ bands with even L, the changes in the lowest states from that of the (84)-representation is expected to be small. It is difficult to see, in view of this, how any of these representations alone can increase the $K = 0$ and $K = 2$ band spacing appreciably.

Work is now in progress at Southampton on the introduction of a spin-orbit force into the potential so that complete calculations on all nuclei in the ds-shell may be made.

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Appendix 1

Two Body Matrix Elements of the Gaussian Potential

The spherical harmonic oscillator equation in three dimensions, for a single particle, can be written:

$$-\frac{1}{2}\frac{\hbar^2}{m}\nabla^2 U + \frac{1}{2}\frac{\hbar^2}{b^4 m} r^2 U = \frac{\hbar^2}{b^2 m} EU \qquad \text{A1.1}$$

The solution of this equation in Cartesian co-ordinates is:

$$U(n_x n_y n_z) = \bar{\bar{H}}_{n_x}(\bar{x})\, \bar{\bar{H}}_{n_y}(\bar{y})\, \bar{\bar{H}}_{n_z}(\bar{z})\, e^{-(\bar{x}^2 + \bar{y}^2 + \bar{z}^2)/2} \qquad \text{A1.2a}$$

where $\bar{\bar{H}}_m(\bar{x}) = \sqrt{1/(2^m\, m!\, \sqrt{\pi}\, b)}\; H_m(x/b)$, $\bar{x} = x/b$ etc.

with $H_m(x/b)$ a Hermite polynomial of degree m.

(Shiff 1955)

The eigenvalue, corresponding to this eigenfunction, gives

$$E = (n_x + n_y + n_z + 3/2) \qquad \text{A1.3}$$

Using the set of k-particle oscillator well functions to describe the nuclear states, we have found it necessary to evaluate such integrals as

$$M = (U_1(n_{x1}n_{y1}n_{z1})U_2(n_{x2}n_{y2}n_{z2})\,|\,V_{ij}\,|\,U_1(n_{x3}n_{y3}n_{z3})U_2(n_{x4}n_{y4}n_{z4})) \qquad \text{A1.4}$$

Where the i^{th} particle is in the state $U_i(n_x n_y n_z)$

The two body potential V_{ij} has been taken to be

$e^{-r_{ij}^2/a^2}$ – the Gaussian potential apart from the strength parameter V_0.

We shall show that

$$M = \sqrt{[n_x]!\ [n_y]!\ [n_z]!} \quad \frac{\alpha^3}{(\alpha^2 + 2)^{3/2}} \quad x$$

$$f(n_{x1}n_{x2}n_{x3}n_{x4})\ f(n_{y1}n_{y2}n_{y3}n_{y4})\ f(n_{z1}n_{z2}n_{z3}n_{z4}) \qquad \underline{\text{A1.5}}$$

where $\alpha = a/b$, $[n_x]! \equiv n_{x1}!\ n_{x2}!\ n_{x3}!\ n_{x4}!$ etc.

$$\text{and} \quad f(n_1n_2n_3n_4) = \sum_{P=\bar{n}_{13}}^{n_{13}} \ \sum_{Q=\bar{n}_{24}}^{n_{24}} (-)^{P-Q}\{2(P+Q)\}!\ (1/\alpha^2 + 2)^{P+Q} \ x$$

$$1 \Big/ \Big[(P + \bar{n}_{13})!(P - \bar{n}_{13})!(Q + \bar{n}_{24})!(Q - \bar{n}_{24})!(n_{13} - P)! \ x$$

$$(n_{24} - Q)!(P + Q)!\ 2^{P+Q}\Big]$$

with $(n_1 + n_2 + n_3 + n_4)$ an even integer

and $f(n_1n_2n_3n_4) = 0$ otherwise $\qquad \underline{\text{A1.6}}$

Here, $n_{13} = \dfrac{n_1 + n_3}{2}$ and $\bar{n}_{13} = \dfrac{n_1 - n_3}{2}$ etc.

The integrals in <u>A1.4</u> reduce to a product of three in x,y and z spaces, of the form :-

$$I_x = \frac{1}{b^2\pi} \sqrt{\frac{1}{2^{n_1+n_2+n_3+n_4} \quad n_1!n_2!n_3!n_4!}}\ F_x \qquad \underline{\text{A1.7a}}$$

(Here $n_1 \equiv n_{x_1}$ etc. for convenience in writing)

$$\text{where} \quad F_x = \int\!\!\int_{x_1,x_2=-\infty}^{+\infty} H_{n_1}(\bar{x}_1)H_{n_2}(\bar{x}_2)H_{n_3}(\bar{x}_1)H_{n_4}(\bar{x}_2)e^{-(\bar{x}_1^{\,2}+\bar{x}_2^{\,2})}e^{-\frac{(x_1-x_2)^2}{a^2}}\,dx_1dx_2$$

$\underline{\text{A1.7b}}$

Putting $R_x = x_1 + x_2$ and $P_x = x_1 - x_2$, we find

$$F = 1/2 \int\int_{R_x P_x = -\infty}^{+\infty} H_{n_1}(\bar{x}_1) H_{n_2}(\bar{x}_2) H_{n_3}(\bar{x}_1) H_{n_4}(\bar{x}_2) e^{-c^2 R_x^2 - d^2 P_x^2} \, dR_x . dP_x . \qquad \text{A1.8a}$$

where $c^2 = 1/2b^2$ and $d^2 = (1/2b^2 + 1/a^2)$ A1.8b

The problem in evaluating the integral in A1.8a reduces to representing the product of Hermite polynomials, in the integrand, in terms of functions of R_x and P_x.

We know, from Bateman 1953, Vol.II. p. 193 - 4,

$$H_{n_1}(\bar{x}_1) H_{n_3}(\bar{x}_1) = \sum_{p=0}^{\min(n_1,n_3)} 2^p \, p! \binom{n_1}{p} \binom{n_3}{p} H_{(n_1+n_3-2p)}(\bar{x}_1) \qquad \text{A1.9}$$

where $\binom{n}{p}$ are the binomial coefficients,

and $$H_{n_1+n_3-2p}(\bar{x}_1) = (1/2)^{(n_1+n_3-2p)/2} \sum_{k=0}^{n_1+n_3-2p} \binom{n_1+n_3-2p}{k} H_k(cR_x) H_{(n_1+n_3-2p-k)}(cP_x) \qquad \text{A1.10}$$

Similar transformations of $H_{n_2}(\bar{x}_2) H_{n_4}(\bar{x}_2)$ produces sums over q and ℓ

Defining now $2n = n_1 + n_2 + n_3 + n_4$, $P = n_{13} - p$, $Q = n_{24} - q$ A1.11

we may write

$$H_{n_1}(\bar{x}_1)H_{n_2}(\bar{x}_2)H_{n_3}(\bar{x}_1)H_{n_4}(\bar{x}_2) =$$

$$\sum_{P=\bar{n}_{13}}^{n_{13}} \sum_{Q=\bar{n}_{24}}^{n_{24}} 2^{n-2P-2Q}\,(n_{13}-P)!\,(n_{24}-Q)! \quad x$$

$$\binom{n_1}{n_{13}-P}\binom{n_2}{n_{24}-Q}\binom{n_3}{n_{13}-P}\binom{n_4}{n_{24}-Q} \quad x$$

$$\sum_{k=0}^{2P} \sum_{\ell=0}^{2Q} \binom{2P}{k}\binom{2Q}{\ell} H_k(cR_x)H_\ell(cR_x)H_{(2P-k)}(cP_x)H_{(2Q-\ell)}(cP_x)(-)^{2Q-\ell}$$

A1.12

The integral in A1.8a now reduces to a product of two integrals:

$$\int_{-\infty}^{+\infty} H_k(cR_x)H_\ell(cR_x)e^{-c^2R_x^{\ 2}}\,dR_x = \delta_{k\ell}\,2^k k!\,\sqrt{\pi}/c \qquad \text{A1.13}$$

and $$\int_{-\infty}^{+\infty} H_{(2P-k)}(cP_x)H_{(2Q-k)}(cP_x)\,e^{-d^2P_x^{\ 2}}\,dP_x$$

$$\sum_{r=0} 2^r\,r!\,\binom{2P-k}{r}\binom{2Q-k}{r}\int_{-\infty}^{+\infty} H_{2(P+Q-k-r)}(cP_x)\quad e^{-d^2P_x^{\ 2}}\,dP_x \qquad \text{A1.14}$$

(Bateman 1953, Vol. II, p.194)

Since k (A1.10), 2P and 2Q (A1.12) and r (A1.14) are all integers, we have that $2(P+Q-k-r)$ is an integer. But the integral in A1.14 vanishes unless $2(P + Q - k - r)$ is an even integer. This implies that $(P + Q)$ is an integer.

From A1.11

$$P + Q = (n_1 + n_2 + n_3 + n_4)/2 - p - q = \text{an integer}$$

Since p (and q) is an integer from A1.9 we deduce that

$$(n_1 + n_2 + n_3 + n_4)/2 = n = \text{an integer for non}$$

zero solutions of A1.14 A1.15

Using the fact that :-

$$\int_{-\infty}^{+\infty} H_{2(P + Q - k - r)}(cP_x) \; e^{-d^2 P_x^2} \, dP_x = \frac{\sqrt{\pi}}{d} \frac{[2(P+Q-k-r)]!}{[P+Q-k-r]!} \left[(c/d)^2 - 1\right]^{P+Q-k-r} \qquad \text{A1.16}$$

we find, from A1.8 to A1.15

$$F = \frac{\pi}{2cd} \sum_{P=\bar{n}_{13}}^{n_{13}} \sum_{Q=\bar{n}_{24}}^{n_{24}} \sum_{k=o} \sum_{r=o} 2^{N-2P-2Q+r+k} (-)^{2Q-k} \quad x$$

$$(n_{13} - P)!(n_{24} - Q)! \; k!r! \binom{n_1}{n_{13}-P} \binom{n_2}{n_{24}-Q} \binom{n_3}{n_{13}-P} \binom{n_4}{n_{24}-Q} \quad x$$

$$\binom{2P}{k} \binom{2Q}{k} \binom{2P-k}{r} \binom{2Q-k}{r} \frac{[2(P + Q - k - r)]!}{(P + Q - k - r)!} \left[(c/d)^2 - 1\right]^{P+Q-k-r} \qquad \text{A1.17}$$

with k summing to $\min\{2Q, 2P\}$

and r " " $\min\{2P-k, 2Q-k\}$

Putting $Q + P - k = K$ and $K - r = R$ we have

$$F = \frac{\pi n_1!n_2!n_3!n_4!}{2cd} \sum_{P=\bar{n}_{13}}^{n_{13}} \sum_{Q=\bar{n}_{24}}^{n_{24}} \frac{2^{N-P-Q}(-)^{Q-P}(2P)!(2Q)!}{} \quad x$$

$$1/(P+\bar{n}_{13})!(P-\bar{n}_{13})!(Q+\bar{n}_{24})!(Q-\bar{n}_{24})!(n_{13}-P)!(n_{24}-Q)! \quad x$$

$$\sum_{R=P-Q}^{P+Q} \sum_{K=R}^{P+Q} \frac{(-)^K (2R)! \left[(c/d)^2-1\right]^R}{R!(Q+P-K)!(K-R)!(P-Q+R)!(Q-P+R)!}$$

<u>A1.18</u>

The sum over K vanishes unless $P + Q - R = 0$.

With $(c/d)^2-1 = {}^{-2}/\alpha^2 + 2$, $\alpha = a/b$ we find:

$$F = \frac{\pi n_1!n_2!n_3!n_4!a^2 2^n}{\alpha(\alpha^2+2)^{1/2}} f(n_1n_2n_3n_4) \qquad \text{A1.19}$$

Where $f(n_1n_2n_3n_4)$ is defined in <u>A1.6</u>.

The result in <u>A1.5</u> now follows immediately from <u>A1.17a</u>.

We notice that the f-functions have the following symmetry relations.

$$\begin{aligned} f(n_1n_2n_3n_4) &= f(n_3n_2n_1n_4) \\ &= f(n_2n_1n_4n_3) \end{aligned}$$

The only f-functions which are needed for the calculation of matrix elements in this thesis are those for which $n_i \leqslant 2$. These have been listed in Table <u>32</u>.

Table 32 f-functions presented in the form.

$$f(n_1 n_2 n_3 n_4) = K(1/\alpha^2+2)^t \left[a_8\alpha^8 + a_6\alpha^6 + a_4\alpha^4 + a_2\alpha^2 + a_o \right]$$

for $(n_1 + n_2 + n_3 + n_4)$ even

= 0 otherwise

n_1	n_2	n_3	n_4	K	t	a_8	a_6	a_4	a_2	a_o
0	0	0	0	1						1
1	1	0	0	1	1					1
1	0	1	0	1	1				1	1
1	1	1	1	1	2			1	2	3
2	0	0	0	-1/2	1					1
2	1	1	0	1/2	2				2	1
2	1	0	1	-1/2	2				1	-1
2	2	0	0	3/4	2					1
2	0	2	0	1/4	2			2	4	3
2	2	1	1	1/4	3			4	4	7
2	1	2	1	1/4	3		2	6	15	7
2	2	2	0	-1/8	3			2	-4	-1
2	2	2	2	1/16	4	4	16	60	56	41

Appendix 2

Fractional Parentage Reduction of the Eight Particle State $(a^4)^{II}\ (b^4)^{II}$ by Two Particles

The state has been defined in 3.51 to be.

$$(a^4)^{11}(b^4)^{11} = \sqrt{4!4!/_{8!}}\ \sum{}^{\prime}(-)^{P}P\left\{(a^4)^{11}_{1234}(b^4)^{11}_{5678}\right\}^{11} \qquad \text{A2.1}$$

where $\sum{}^{\prime}$ sums over all the permutations P between the two sets of numbers (1234) and (5678) perserving the natural order in the sets. The charge spin functions being vector coupled to a singlet-singlet state. Writing the sum over permutations involving 7 and 8 explicitly we find:

$$(a^4)^{11}(b^4)^{11} = \sqrt{4!4!/_{8!}}\ \sum{}^{\prime\prime}(-)^{P}P\left\{(a^4)^{11}_{1234}(b^4)^{11}_{5678} - \right.$$

$$\left.(a^4)^{11}_{1237}(b^4)^{11}_{4568} \overset{-}{\oplus} (a^4)^{11}_{1238}(b^4)^{11}_{4567} + (a^4)^{11}_{1278}(b^4)^{11}_{3456}\right\}^{II} \qquad \text{A2.2}$$

Where now $\sum{}^{\prime\prime}$ sums over all permuation of $\sum{}^{\prime}$ except those involving 7 and 8 i.e. the permutations involve the numbers 12...6 only.

The fractional parentage coefficients (c.f.p) of the reduction of a totally symmetric fourparticle orbital state by one or two particles are trivial since the orbital states are not coupled. The charge spin c.f.p. are well known (Jahn Van Wierengen, 1951. Elliott, Hope and Jahn, 1953).

Thus, for example,

$$(b^4)^{11}_{5678} = \Big(\sqrt{1/2}(b^2)^{13}_{56}(b^2)^{13}_{78} - \sqrt{1/2}(b^2)^{31}_{56}(b^2)^{31}_{78}\Big)^{11} \qquad \underline{\text{A2.3a}}$$

Similarly for $(a^4)^{11}_{1278}$

and $(a^4)^{11}_{1237} = ((a^3)^{22}_{123}\ (a)^{22}_{7})^{11}$ $\underline{\text{A2.3b}}$

Similarly for $(b^4)^{11}_{4568}$, $(a^4)^{11}_{1238}$ and $(b^4)^{11}_{4567}$

Using the recoupling coefficients for the charge-spin spaces, i.e. the normalised Racah U-function defined in:

$$\Psi((j_1j_2)J_{12},j_3JM) = \sum_{J_{23}} U(j_1j_2Jj_2:J_{12}J_{23})\Psi(j_1(j_2j_3)J_{23},JM)$$

(Elliott 1958, III, p.10)

and the Hope χ-function defined in:

$$\Psi((j_1j_2)J_{12},(j_3j_4)J_{34},JM) =$$

$$\sum_{J_{13},J_{24}} \chi((j_1j_2)J_{12},(j_3j_4)J_{34},J \mid (j_1j_3)J_{13},(j_2j_4)J_{24},J) \times \Psi((j_1j_3)J_{13},(j_2j_4)J_{24},JM)$$

with

$$\chi = \left[(2J_{12}+1)(2J_{34}+1)(2J_{13}+1)(2J_{24}+1)\right]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & J_{12} \\ j_3 & j_4 & J_{34} \\ J_{13} & J_{24} & J \end{Bmatrix}$$

$\{\ \}$ being a nine-j symbol.

(Elliott 1958, III, p.15)

we find,

$$\left\{(a^4)^{11}_{1234}(b^4)^{11}_{5678}\right\}^{11} = \left\{\sqrt{1/2}((a^4)^{11}_{1234}(b^2)^{13}_{56})^{13}(b^2)^{13}_{78}\right.$$

$$\left. - \sqrt{1/2}((a^4)^{11}_{1234}(b^2)^{31}_{56})^{31}(b^2)^{31}_{78}\right\}^{11} \qquad \underline{A2.4a}$$

$$\left\{(a^4)^{11}_{1237}(b^4)^{11}_{4568}\right\}^{11} = 1/4\left\{((a^3)^{22}_{123}(b^3)^{22}_{456})^{11}(a_7b_8)^{11} + \right.$$

$$3((a^3)^{22}_{123}(b^3)^{22}_{456})^{33}(a_7b_8)^{33} + \sqrt{3}((a^3)^{22}_{123}(b^3)^{22}_{456})^{13}(a_7b_8)^{13}$$

$$\left. + \sqrt{3}((a^3)^{22}_{123}(b^3)^{22}_{456})^{31}(a_7b_8)^{31}\right\}^{11} \qquad \underline{A2.4b}$$

etc.

Collecting up the terms from A2.3 and A2.4 and putting them back into A2.2 we find, by summing over the permutations between the particle numbers 1 ....6:-

$$(a^4)^{11}(b^4)^{11} = \sqrt{6/28}\left\{\sqrt{1/2}((a^4)^{11}(b^2)^{13})^{13}_{1..6}(b^2)^{13}_{78} - \sqrt{1/2}((a^4)^{11}(b^2)^{31})_{1..6}(b^2)^{31}_{78}\right\}^{11}$$

$$+ \sqrt{6/28}\left\{\sqrt{1/2}((a^2)^{13}(b^4)^{11})^{13}_{1..6}(a^2)^{13}_{78} - \sqrt{1/2}((a^2)^{31}(b^4)^{11})^{31}_{1..6}(a^2)^{31}_{78}\right\}^{11}$$

$$- \sqrt{6/28}\left\{\sqrt{1/2}((a^3)^{22}(b^3)^{22})^{13}_{1..6}(ab)^{13}_{78} + \sqrt{1/2}((a^3)^{22}(b^3)^{22})^{31}_{1..6}(ab)^{31}_{78}\right\}^{11}$$

$$- \sqrt{10/28}\left\{\sqrt{1/10}((a^3)^{22}(b^3)^{22})^{11}_{1..6}(\widetilde{ab})^{11}_{78} + \sqrt{9/10}((a^3)^{22}(b^3)^{22})^{33}_{1..6}(\widetilde{ab})^{33}_{78}\right\}^{11}$$

$$\underline{A2.5}$$

The first six-particle states in <u>A2.5</u> are antisymmetric and are coupled in the charge-spin spaces to antisymmetric two-particle states of the last two particles. The orbital symmetry of the last pair is given by the charge spin functions.

e.g.

$$(a^2)^{13}_{78} \equiv a_7 a_8 \Gamma^{13}_{78}$$

$$(ab)^{13}_{78} \equiv \sqrt{1/2}(a_7 b_8 + a_8 b_7)\Gamma^{13}_{78}$$

$$(\widetilde{ab})^{11}_{78} \equiv \sqrt{1/2}(a_7 b_8 - a_8 b_7)\Gamma^{11}_{78}$$

etc.

Addendum

On the Use of the Eigenfunctions of the Deformed Harmonic Oscillator

Throughout this thesis so far we have been concerned with the energy, in the k-particle states in the ds-shell, of an A-particle nucleus with $16 < A < 40$. The method for calculating energy levels has been designed to bring out certain features of the rotational model. No excitation of the closed shells (- corresponding to the 0^{16} core) has been permitted however. We know that such an excitation must be considered if the correct quadrupole moments are to be predicted. From the known features of the rotational model a deformed shape, which is stable in some sense, is suggested.

In this addendum, we record the results from initial calculations in the p-shell $\{4 < A < 16\}$ for the binding energy, using the many particle eigenfunctions of the axially symmetric deformed oscillator (H_d) as a set of zero order approximations to the nuclear states. The energy in these states, from the Hamiltonian with only central forces (H_c), is now given in terms of two oscillator-well parameters one of which can measure the deformation of the well. The best approximation to the binding energy, with a few of these states, is found by diagonalising the energy matrix and minimising the lowest eigenvalue with respect to the two parameters.

We show that very similar values for the deformation parameter will minimise the energy from either H_d or H_c.

The deformed harmonic oscillator.

We may write the axially symmetric deformed harmonic oscillator as:-

$$H_d = \frac{\hbar^2}{2m} \left[-\nabla^2 + \alpha^4 x^2 + \alpha^4 y^2 + \beta^4 z^2 \right] \qquad \ldots \text{ Ad.1}$$

where, for zero deformation (spherical oscillator), $\alpha = \beta$.

The single particle eigenfunctions of H_d can be expressed in either of two ways:

$$U_{n_x n_y n_z} = \sqrt{\frac{\beta\alpha^2}{2^n n_x! n_y! n_z! \pi^{3/2}}} H_{n_x}(\alpha x) H_{n_y}(\alpha y) H_{n_z}(\beta z) e^{-(\alpha^2 x^2 + \alpha^2 y^2 + \beta^2 z^2)/2} \qquad \underline{\text{Ad.2a}}$$

or

$$Y_{n_\perp m n_z} = \sqrt{\frac{\beta\alpha^2 \{(n_\perp - |m|)/2\}!}{2^{n_z} n_z! \{(n_\perp + |m|)/2\}! \pi^{3/2}}} e^{im\phi} (\alpha\rho)^{|m|} \quad x$$

$$\mathcal{L}^{|m|}_{(n_\perp - |m|)/2} (\alpha^2 \rho^2) \; H_{n_z}(\beta z) \; e^{-(\alpha^2 \rho^2 + \beta^2 z^2)/2} \qquad \underline{\text{Ad.2b}}$$

Here H is a Hermite Polynomial and $\mathcal{L}$ a Laguerre Polynomial (Bateman 1953 III). ρ, ϕ, z are cylindrical polar coordinates and

$$n_\perp = n_x + n_y \qquad\qquad n = n_x + n_y + n_z$$

The eigenvalues of H_d are:

$$E_{n_x n_y n_z} \equiv E_{n_\perp m\, n} = \frac{\hbar}{m}\{\alpha^2(n_x + \tfrac{1}{2}) + \alpha^2(n_y + 1/2) + \beta^2(n_z + 1/2)\} \qquad \underline{\text{Ad.3b}}$$

The functions U and V both spread out irreducible representations of the group SU_2 described by the three operators B_{xy}, A_{xy}, A_{yx} of Chapter 2 $(\alpha = 1/b)$. We may write:

$$U_{n_x n_y n_z} = U(n\Lambda s)$$

$$\text{and} \quad V_{n_\perp m\, n_z} = V(n\Lambda t) \qquad \underline{\text{Ad.4}}$$

where Λ is a representation of SU_2 such that $\Lambda = (n_x + n_y)/2$ and

$$s = (n_x - n_y)/2 \qquad\qquad t = m/2. \qquad \ldots \text{Ad.5}$$

The difference between the U and V functions is, then, just the same as that between the ϕ and χ of Chapter 2. The relations between U and V can immediately be deduced from <u>2.23</u> to be:-

$$U(n\Lambda s) = \sum_t b(\Lambda 2s : 2t)\, V(n\Lambda t)$$

$$\text{where} \quad b(\Lambda, 2s : 2t) = (-i)^s\, d^{\Lambda}_{ts}(\pi/2) \qquad \underline{\text{Ad.6}}$$

We define paramters γ and δ after Nilsson (1955) by the relation :-

$$\alpha^4 = \gamma^4(1 + (2/3)\delta)$$

$$\beta^4 = \gamma^4(1 - (4/3)\delta) \qquad \underline{\text{Ad.7}}$$

where the parameter δ is a measure of the deformation. The $(\text{volume})^{-2}$ of an equipotential ellipsoid in H_d is proportional to:

$$\alpha^4\beta^2 = \gamma^6[1 + (2/3)\delta][1 - (4/3)\delta]^{1/2} = \gamma_o^{\ 6} \text{ (say)}. \qquad \underline{\text{Ad.8}}$$

By varying δ with γ_o constant, we shall be varying the deformation of the potential keeping the volume of each equipotential surface constant.

Since the potential derives from the nuclear density, this is an approximation to keeping the nuclear volume fixed while the shape is varied.

Many particles in the deformed oscillator potential.

The eigenvalue of a many particle state of H_d, at a certain deformation (δ), can be calculated by adding up the energies of its single particle constituents from Ad.3b. The energies, in units of $\gamma_o^{\ 2}\hbar^2/m$, in the lowest configurations for A = 6, 8 and 10 particles, are shown as functions of the parameter δ in Figures 14, 15 and 16.

We notice that for A = 6 and 8, the lowest energy is given for a positive value of the δ -prolate deformation. For A = 10 the lowest energy is given for a negative value of the deformation although there is a state, $(V_{000}{}^{4}\,V_{001}{}^{4}\,V_{1-10}\overset{V_{110}}{)}$, having a minimum energy of almost the same value but with a positive deformation - the transition from a prolate to oblate shape.

Many particles in a two body potential.

The two body central potential is given in 1.2. For these calculations we used the Hu-Massey (1949) values for the strength and range of the force:

$$Vo = 29{\cdot}59 \text{ Mev} \qquad a = 2{\cdot}18 \times 10^{-17} \text{ cms.}$$

and the Rosenfeld exchange constants.

The many particle states χ have been taken in terms of the single particle states in Ad.2b. In this way we can preserve the labelling with the projection of the orbital angular momentum $M = \sum_i m_i$ on the symmetry axis where m_i refers to the i^{th} nucleon. By taking the low lying deformed oscillator states, however, it is not possible to form simply a state which can be labelled with the orbital angular momentum quantum number L. This is a disadvantage since we know that, for a central potential, L is a good quantum number.

For the lowest nuclear states, we again just consider those oscillator well states with maximum orbital symmetry (see Chapter 1 for the general construction).

The calculation of the matrix elements of the potential follows the standard procedure of finding the c.f.p of the reduction of the many particle states by two particles. Since the single particle states will only be vector coupled in their charge-spin spaces, the calculation of the c.f.p. follows in a similar way to that in Appendix 2 for an eight particle state.

The two body matrix elements using the set of single particle states in Ad.2b may be written in terms of those using the set in Ad.2a with the aid of Ad.6. These latter matrix elements can be deduced immediately from the general formulae in Appendix 1. The energy now will be given in terms of two parameters.

$$e = a\alpha \quad \text{and} \quad f = a\beta \qquad \text{Ad.9}$$

Matrix elements of the kinetic energy.

The kinetic energy of a nucleus, relative to the centre of mass, may be written:-

$$T = \frac{1}{2m} \sum_i p(i)^2 - \frac{1}{2mA} P^2$$

$$\equiv \frac{1}{2mA} \sum_{i<j} (p(i) - p(j))^2 \qquad \text{Ad.10}$$

where $\vec{P} = \sum_i \vec{p}(i)$ is the momentum of the centre of mass and $\vec{p}(i)$ is the momentum of the i^{th} nucleon.

For oscillator well states ψ, with total quanta N, we have the equivalence between momenta and coordinates:-

$$\left(\psi'(N) \;\middle|\; \sum_{i<j} (p(i) - p(j))^2 \;\middle|\; \psi(N)\right)$$

$$\equiv (\psi'(N) \mid \sum_{i<j} \{\alpha^4 x_{ij}^2 + \alpha^4 y_{ij}^2 + \beta^4 z_{ij}^2\} \mid \psi(N)) \qquad \text{Ad.11a}$$

with $x_{ij} = x(i) - x(j)$ etc.

The matrix elements of the kinetic energy may be deduced from those of the Gaussian potential (Appendix 1) for:

$$\sum_{i<j} e^{-x_{ij}^2/a^2} = \sum_{i<j} \{1 - \frac{1}{a^2} x_{ij}^2 + O(\frac{1}{a^4})\}$$

Ignoring $O(\frac{1}{a})^4$ we can deduce:

$$\frac{1}{a^2} (\psi' | \sum_{i<j} x_{ij}^2 | \psi) = \frac{A(A-1)}{2} \delta(\psi\psi')$$

$$-(\psi' | \sum_{i<j}{}' e^{x_{ij}^2/a^2} | \psi)_{O(\frac{1}{a^4})} = 0 \qquad \underline{\text{Ad.11b}}$$

Similarly for $\sum_{i<j} y_{ij}^2$ and $\sum_{i<j}{}' z_{ij}^2$

The kinetic energy matrix elements are of the form

$$(\psi' | \frac{1}{2mA} \sum_{i<j} (p(i) - p(j))^2 | \psi)$$

$$= \frac{1}{2mA} (\tilde{A}\alpha^2 + \tilde{B}\beta^2) \qquad \underline{\text{Ad.12}}$$

where $\tilde{A}$ and $\tilde{B}$ are constants.

Parameters

For the purpose of minimising the energy matrix spread out by the states χ, we choose parameters (functions of e and f) such that the mean square radius,

$$\text{M.S.R.} = \frac{1}{A} \sum_i (r_i - R)^2 = \frac{1}{A^2} \sum_{i<j} (r_i - r_j)^2 \qquad \underline{\text{Ad.13}}$$

remains a constant

From the form of the kinetic energy matrix elements we deduce:

$$\text{M.S.R.} = \frac{1}{A^2}\{\tilde{A}/\alpha^2 + \tilde{B}/\beta^2\} =$$

$$\frac{a^2}{A^2}\{\tilde{A}/e^2 + \tilde{B}/f^2\} \qquad \text{Ad.14}$$

Writing $1/e^2 = p + y\varepsilon$ and $1/f^2 = p + x\varepsilon$ Ad.15

where for the moment x, y are arbitrary, we have

$$\frac{a^2}{A^2}\{(\tilde{A} + \tilde{B})p + (\tilde{A}y + \tilde{B}x)\varepsilon\} = \text{M.S.R.} \qquad \text{Ad.16}$$

We choose the ratio of x to y such that $\tilde{A}y + \tilde{B}x = 0$. In this case,

$$\text{M.S.R.} = \frac{a^2}{A^2}\{\tilde{A} + \tilde{B}\}p \qquad \text{Ad.17}$$

p is then proportional to the M.S.R.

The actual values for x, y are unimportant. A change in their values will just change the definition of the parameter ε. Notice that the ε also measures the deformation of the oscillator well since, when $\varepsilon = 0$, $e=f$ i.e. $\alpha=\beta$ (Ad.15, Ad.9). It is possible to relate the parameters p, ε to the Nilsson parameter δ.

On the one hand we have :-

$$\frac{e^4}{f^4} = \frac{a^4\alpha^4}{a^4\beta^4} = \frac{(1 + (2/3)\delta)}{(1 - (4/3)\delta)} = S^2 \text{ (say)} \qquad \text{Ad.18}$$

On the other hand

$$\frac{e^4}{f^4} = \left(\frac{p + x\varepsilon}{p + y\varepsilon}\right)^2 \equiv S^2 \qquad \text{Ad.19}$$

Thus we have

$$\delta = \frac{3}{2}\left(\frac{S^2 - 1}{1 + 2S^2}\right) \quad \text{where} \quad S = \left(\frac{p + x\epsilon}{p + y\epsilon}\right) \qquad \underline{\text{Ad.20}}$$

Although the calculations have been done using the p, ε parameters and the energy, in the many particle states of the deformed oscillator, minimised with respect to these parameters, the results have been given for convenience in terms of the mean square radius (deduced from ρ in <u>Ad.17</u>) and Nilsson's δ (deduced from p and ε in Ad.20).

We now discuss the results of the calculations for the A = 6, 8 and 10 particle nuclei corresponding to L_i^6, B_e^8 and B^{10} respectively.

<u>The L_i^6 Problem.</u>

The lowest six particle states in the deformed oscillator potential with M = 0, N = 2 $\{ N = \sum_i (n_x(i) + n_y(i) + n_z(i))\}$ and orbital symmetry [42] are:-

$$\chi^{13}_{002} = \Big\{\big((V_{000}^{4})^{11}\,(V_{001}^{2})^{13}\Big)^{13}$$

$$\chi^{13}_{200} = \Big((V_{000}^{4})^{11}\,(V_{110}V_{1-10})^{13}\Big)^{13} \qquad \ldots \; \underline{\text{Ad.21}}$$

where the many particle states are of the form

$$\chi^{2T+1,\,2S+1}_{N_\perp M N_z}$$

The energy in these two states, for H_c, was evaluated for several values of the parameters p and ε. Near the minimum energy, small changes in p were found not to change the energy appreciably. The value of p which actually gives the minimum leads to a mean square radius 9.5×10^{-26} cms^2.

The variation of the energy with, δ, at this value for p, is shown by the continuous curves (a) and (b) in Figure 17.

In this one case, the mixing of the two states was considered. The minimum of the lowest eigenvalue was found for the same value of p which gave a minimum without mixing. The variation of the eigenvalues with δ is shown by the ~~broken~~ curves (c) and (d) in Figure 17.

We notice that the mixing of states does not contribute appreciably to the energy at that value for δ which minimises the energy. The δ which minimises the energy is seen to be positive and to have about the same value as that which minimises H_d when A = 6 (Fig. 14).

The B_e^8 problem.

In this case there are three states of maximum orbital symmetry with lowest oscillator quanta ($N = 4$) and angular momentum projection $M = 0$.

$$\chi_{004}^{11} = \left((V_{000}^{4})^{11} \; (V_{001}^{4})^{11} \right)^{11}$$

$$\chi^{11}_{202} = ((V_{000}^{4})^{11}\ (V_{001}^{2}\ V_{110}\ V_{1-10})^{11})^{11}$$

$$\chi^{11}_{400} = ((V_{000}^{4})^{11}\ (V_{110}^{2}\ V_{1-10}^{2})^{11})^{11} \qquad \ldots \text{Ad.22}$$

We have only performed the variational calculation on the first of these states.

The minimum energy was found for a mean square radius of 10.5×10^{-26} cms^2 - a slight increase over that for L_i^6. Again only slight changes in the energy were detected for small changes of p around the minimum.

The variation of the energy with δ in the state χ^{11}_{004} is shown in Figure 18.

The minimum energy is given for a value $\delta \sim 0.57$ again comparing favourably with that value which minimises H_d (Figure 15)

The B^{10} problem.

This nuclus is in the middle of the p-shell. We notice that the states

$$\chi^{13}_{204} = ((V_{000}^{4})^{11}\ (V_{001}^{4})^{11}\ (V_{110}\ V_{1-10})^{13})^{13}$$

$$\chi^{13}_{600} = ((V_{000}^{4})^{11}\ (V_{110}^{3}\ V_{1-10})^{11}\ (V_{1-10})^{13})^{13} \qquad \ldots \text{Ad.23a}$$

have near enough the same minimum energies in H_d but for different values for δ. The former favours a positive deformation and the latter a negative one. (Figure 16) The energy in the state χ^{13}_{204} with H_c has been found to be minimised for a mean square radius value of 15.2 x 10^{-26} cms^2 - again an increase over that for L_i^6 and B_e^8.

The variation of the energy in this state with δ, for this M.S.R., is shown in Figure 19. Notice again that the δ which minimises the energy, is the same one that minimises the energy with H_d.

Although time has not permitted the calculation of the energy in χ^{13}_{600}, it is expected to be minimised for a value of δ which minimises the energy in the state from H_d.

Three states, of the form

$$\chi^{13}_{402} = (V_{000}^{4}\, V_{001}^{2}\, V_{110}^{2}\, V_{1-10}^{2})^{13} \qquad \ldots \underline{\text{Ad.23b}}$$

may be constructed which have a minimum energy with H_d for $\delta \sim 0$. It would be interesting to see if the mixing of the states in Ad.23a and Ad.23b produce a lowest eigenvalue which is U-shaped and centred about $\delta = 0$ when the variation with δ is considered. The B^{10} nucleus could then be thought of as having an unstable spherical shape.

Conclusions.

We have here only begun to discuss the use of deformed oscillator well states however, with the calculations already done, we can begin to get a picture of the part the deformation plays in energy calculations. It is seen that a gain in binding energy is achieved by going away from spherical symmetry to axial symmetry with that deformation which minimises the energy for H_d. For L_i^6 a gain of 0.85 Mev is obtained over the S-state for zero deformation (i.e. the lowest eigenstate at $\delta = 0$).

The mixing of all higher states must make the lowest state one with good angular momentum the main single part of which is the lowest deformed oscillator configuration.

The $\chi_{N_1 M N_z}$, of course, form a complete set of states for any value of the deformation parameter and the true states of a k-particle nucleus can be expanded in terms of any of these sets. Thus by mixing in <u>all</u> states of a set, and diagonalising the resulting energy matrix, the lowest eigenvalue, when plotted against the deformation parameter, would be a horizontal line. (This is mathematically correct but cannot be checked in practise since the complete sets are infinite). It appears, however, that by expanding the true nuclear eigenfunction in terms of the complete set of states which minimise the deformed oscillator energy, a better approximation to the binding energy is obtained in a fewer number of terms than for any other value of the deformation parameter. This approximation though will not, in general, give a state with good angular momentum. A study of the variation of $\langle L^2 \rangle$ with deformation will have to be carried out.

Lowest energy levels for A-particle states (M=0) in the deformed harmonic oscillator potential.

Figure 14. A = 6

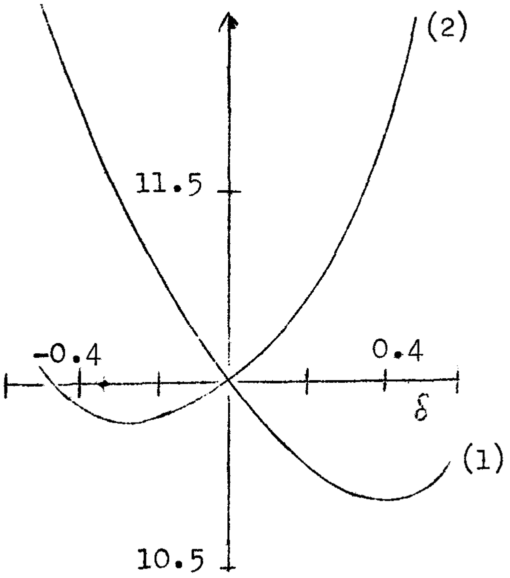
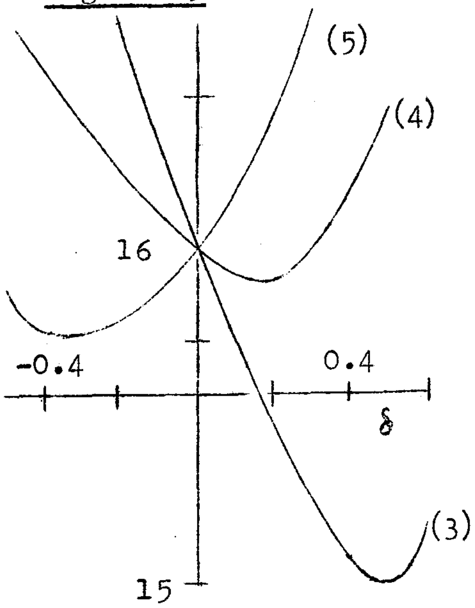
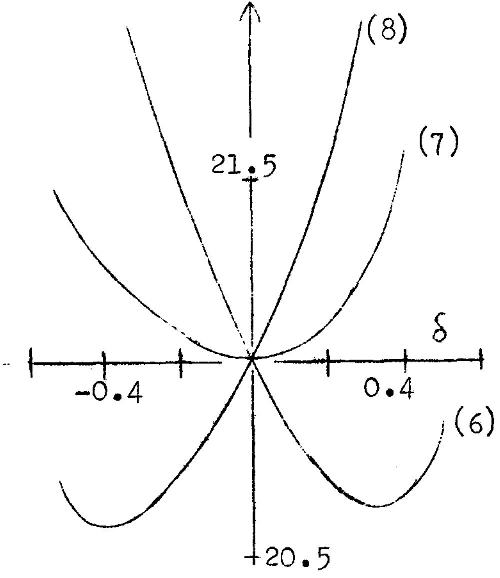


Figure 15. A = 8



Units of $\frac{\gamma_o^2 \hbar^2}{m}$

Figure 16. A = 10



States

(1) = $V^4_{ooo} V^2_{ool}$

(2) = $V^4_{ooo} V_{llo} V_{l-lo}$

(3) = $V^4_{ooo} V^4_{ool}$

(4) = $V^4_{ooo} V^2_{ool} V^2_{l-lo}$

(5) = $V^4_{ooo} V^2_{llo} V^2_{l-lo}$

(6) = $V^4_{ooo} V^4_{ool} V_{llo} V_{l-lo}$

(7) = $V^4_{ooo} V^2_{ool} V^2_{llo} V^2_{l-lo}$

(8) = $V^4_{ooo} V^3_{llo} V^3_{l-lo}$

Energy in the lowest A-particle ,deformed oscillator eigenfunctions from H_c.

Figure 17. A = 6

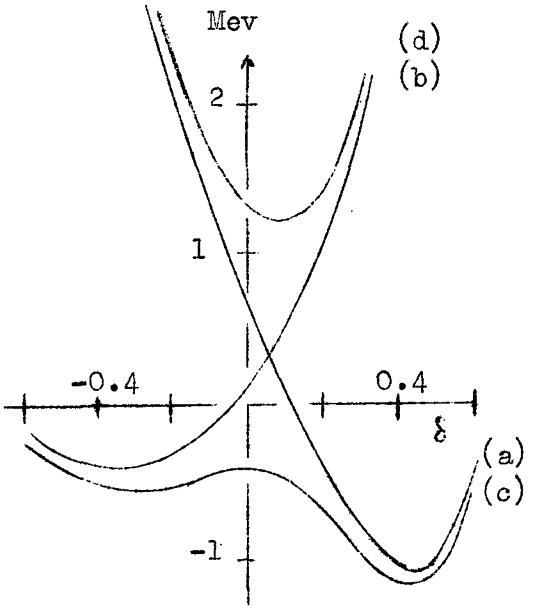


Figure 18. A = 8

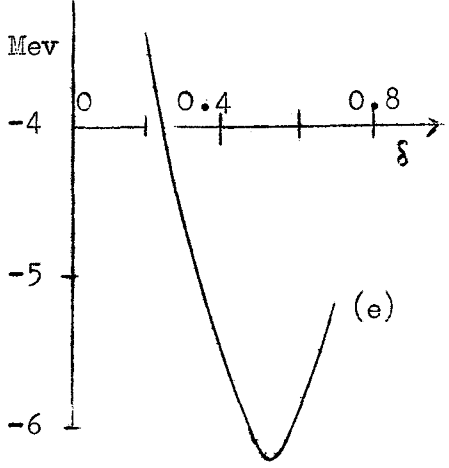
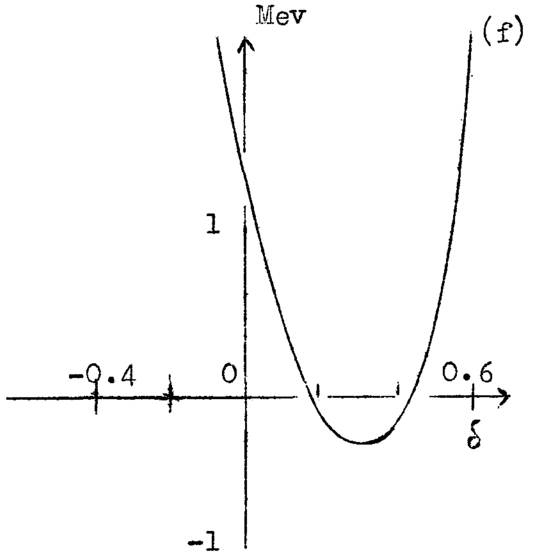


Figure 19. A = 10



States

(a) = χ^{13}_{002}

(b) = χ^{13}_{200}

(c) = lowest eigenvalue from (a)&(b)

(d) = highest " " " "

(e) = χ^{13}_{004}

(f) = χ^{13}_{204}