UNIVERSITY OF SOUTHAMPTON

FACULTY OF MATHEMATICAL STUDIES

SINGULARITY THEORY AND GEOMETRY IN THE MOTION OF A TOP.

by

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ABSTRACT

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The aim of this thesis is to examine the spinning top from the point of view of the Smale programme for studying mechanical systems with symmetry. This programme consists of finding the global topological structure of the map $E \times J : TM \to R \times \mathfrak{g}^*$ where E is the total energy of the system, J its momentum mapping, which in our case is just its angular momentum, TM is the phase space and $\mathfrak{g}^*$ is the dual of the Lie algebra of the Lie group G which acts on the configuration space M producing the symmetry.

We are here concerned with examining the nature and configuration of the singularities of this and related maps using the machinery of $\mathcal{K}$ and $\mathcal{A}$ equivalence and of finite determinacy. We are able to interpret various types of motion of the top in terms of singularities and their unfoldings. Of particular importance is the subset of TM corresponding to steady precession whose corresponding geometry in the cotangent bundle we exhibit explicitly.

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CONTENTS

CHAPTER 0

INTRODUCTION

This thesis is devoted to the subject of the spinning top which for our purposes means a rigid body with a point on one of its inertia axes fixed in space and the other two moments of inertia equal. The axis singled out is called the symmetry axis of the top.

The dynamical behaviour of such a system moving under the influence of gravity is well understood and is described, in classical mechanical terms, in such books as Whittaker [20] and Meirovitch [13] and Gray [9]. These descriptions are given in terms of the Euler angles (θ, ϕ, ψ) and their time derivatives which can be briefly described as follows:-

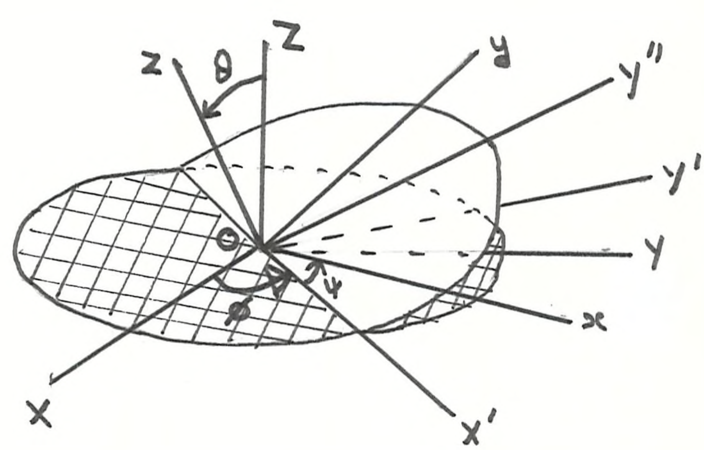


Figure 1.

Let $R(\theta, \phi, \psi)$ denote the rotation of the right handed or orthogonal frame OXYZ into the frame Oxyz (see Fig. 1) such that $R(\theta, \phi, \psi) = r_3 \circ r_2 \circ r_1$ where r_1 is a rotation about OZ of angle ϕ that transforms OX into OX' and OY into OY' (precession), r_2 is a rotation of angle θ around OX' that transforms OZ to Oz and OY' to OY" (nutation) and r_3 is the rotation of angle ψ around Oz (proper rotation). Two sets $(\theta_1, \phi_1, \psi_1)$ and $(\theta_2, \phi_2, \psi_2)$ are called equivalent if $R(\theta_1, \phi_1, \psi_1) = R(\theta_2, \phi_2, \psi_2)$ which, if we restrict θ to $[0, \pi]$, ϕ and ψ to $[0, 2\pi)$, occurs only when $\theta_1 = \theta_2$, $\phi_1 = \phi_2$ and $\psi_1 = \psi_2$.

The Kinetic energy of the top is :-

$$K = \frac{1}{2} A(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} C(\dot{\psi} + \dot{\phi} \cos\theta)^2$$

where C is the moment of inertia of the body about the symmetry axis and A the moment of inertia of the body about any inertia axis perpendicular to the symmetry axis.

The potential energy is

$$V = mga \cos\theta$$

where m is the mass of the body and a is the distance of the centre of gravity from the vertex.

We will also need the expressions for the angular momentum about the axes OZ and Oz. The angular momentum p_ϕ about the axis OZ is

$$p_\phi = A\dot{\phi} \sin^2\theta + C(\dot{\phi} \cos\theta + \dot{\psi})\cos\theta$$

and the angular momentum p_ψ about the axis Oz is

$$p_\psi = C(\dot{\phi} \cos\theta + \dot{\psi})$$

These can all be found on pages 150 and 151 of [13].

In order to simplify the algebra we choose units to make both A and mga equal to one.

When the top spins, with its axis in either the upward or downward vertical position, the Euler angle time derivatives $\dot{\phi}$ and $\dot{\psi}$ are no longer well-defined and so a description of the motion in terms of the Euler angles is difficult. This point is studiously ignored in the classical treatments! However when we come to study the motion of the top with its axis vertically upwards (called the <u>sleeping</u> top) we will introduce a new coordinate chart and recalculate K, V and the angular momenta in terms of this new chart.

Our aim is to examine the spinning top from the point of view of the Smale programme for studying mechanical systems with symmetry. This is described in Smale [17] and also in Abraham and Marsden [1] in full, so we will give only a brief summary here. The description in Smale's paper uses the tangent bundle formulation whilst that in Abraham and Marsden uses the cotangent bundle. We will in fact use both formulations in this thesis but our initial description will follow Smale.

Suppose then we have a classical mechanical system with configuration space a smooth manifold M and tangent bundle TM as the phase space. [In the case of the spinning top, M is SO(3) or equivalently the unit circle tangent bundle of S^2] . The <u>kinetic energy</u> can be thought of as defined by a Riemannian metric on M , so it will be a function $K: TM \to R$ defined by $K(v) = K_x(v, v)$ where $v \in T_x M$ and K_x is an inner product in $T_x M$, smooth in x.

The potential energy is a function $V : M \to R$, normally smooth. The total energy E is the function $E : TM \to R$ given by $E = K + V \circ \pi$, where π is the canonical projection from TM onto M .

If we have a Lie Group G acting on M with action Ψ preserving K and V we define a momentum mapping

$$J : TM \to \mathfrak{g}^*,$$

($\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G) by $J = J_I \circ (2K)^*$ where $(2K)^* : TM \to T^*M$ is the bundle isomorphism defined by $2K$, (twice the Riemannian metric) on M , and $J_I : T^*M \to \mathfrak{g}^*$ is the map which, restricted to each fibre, is the dual of $\alpha_x : \mathfrak{g} \to T_xM$. Here $\alpha_x(X)$ is the value of $\alpha(X)$ at $x \in M$ where $\alpha(X)$ is the vector field on M generated by the 1-parameter transformation group corresponding to $X \in \mathfrak{g}$ that is:-

$$\alpha_x(X) = \frac{d}{dt} \Psi(\exp(tX, x)\Big|_{t=0}$$

So if $\eta \in T^*_x M$ and $X \in \mathfrak{g}$ then J_I is given by :-

$$J_I(\eta)(X) = \eta(\alpha_x(X))$$

In our case G is the torus $T^2 = S^1 \times S^1$.

with action Ψ defined by :-

$$\Psi : G \times M \to M : (\gamma_1, \gamma_2; \theta, \phi, \psi) \to (\theta, \phi + \gamma_1, \psi + \gamma_2)$$

(with addition defined modulo 2π).

We shall show later that in this case the momentum mapping J defined above is just the angular momentum as classically given, that is

$J(v) = (p_\phi, p_\psi) \in R^2$, the Lie algebra of T^2,
where

$$v = (\theta, \phi, \psi; \dot\theta, \dot\phi, \dot\psi) \in TM .$$

The Smale programme consists of the problem of finding the global topological structure of the map $E \times J : TM \to R \times \mathfrak{g}^*$.

This involves at least knowing :-

(i) the topological type of the integral manifolds

$$I_{e,\mu} = (E \times J)^{-1} (e, \mu) .$$

(ii) the bifurcation set $\Sigma_{E\times J}$ of $E\times J$. This is the set of points of $R \times \mathfrak{g}^*$ over which $E\times J$ fails to be locally trivial, in the differentiable sense.

In order to gain information about these, Smale introduces the amended potential, as follows.

Let

$$\Lambda = \left\{ x\varepsilon M : J_x = J\Big|_{T_xM} : T_xM \to \mathfrak{g}^* \text{ is not surjective} \right\}$$

Then Λ is closed and G-invariant. For $x \in M \setminus \Lambda$ and $\mu \in \mathfrak{g}^*$ we define $\alpha_\mu(x) \in T_xM$ by the conditions

(a) $\alpha_\mu(x) \in J_x^{-1}(\mu)$

and (b) $K(\alpha_\mu(x)) = \inf\limits_{\alpha \in J_x^{-1}(\mu)} K(\alpha)$

and then define the amended potential $V_\mu : M\setminus\Lambda \to R$ by

$$V_\mu(x) = V(x) + K(\alpha_\mu(x)) .$$

A particularly important subset of $\Sigma_{E\times J}$ is the set $\Sigma'_{E\times J}$ of singular values of $E\times J$. This is the image of the set of singular points of $E\times J$. The reason for the importance of this subset is given in the following two theorems taken from [1]:

<u>Theorem 1.1</u>

Singular points of $E\times J$ on $J^{-1}(\mu)$ correspond to singular points of V_μ. ∎

<u>Theorem 1.2</u>

Singular points of V_μ are in one to one correspondence, using the diffeomorphism induced by α_μ, with <u>relative equilibria</u>. ∎

The relative equilibria are defined by considering the reduced phase space $J^{-1}(\mu)/G_\mu$, where G_μ is the isotropy subgroup of G at μ for the adjoint action of G on $\mathfrak{g}^*$. On this reduced phase space the energy mapping canonically induces a flow with a corresponding "reduced" energy mapping. A relative equilibrium is a singular point for this reduced mapping. (This definition is given on page 306 of [1]). Intuitively a relative equilibrium is a point where the system could be considered to be at rest except for the motion due to the action of the symmetry group.

We can also construct the reduced amended potential:

$$\tilde{V}_\mu : M/G_\mu \to R$$

and them make use of :

<u>Theorem 1.3</u>

The nondegenerate maxima or minima of $\tilde{V}_\mu$ give stable relative equilibria. ∎

The three theorems quoted can be found on pages 348, 354 and 360 of [1].

In terms of the top, the relative equilibria give the points at which the motion of steady precession takes place and so using Theorem 1.3 the stability of steady precession can be immediately deduced (see Corollary 4.13).

Although the Smale programme concentrates on determining the topological type of the level sets of $E \times J$ we shall, rather, be concerned with examining the nature of the singularities of this map. Since $\mathfrak{g}^* = R^{2*} \cong R^2$ we write $J = (J_1, J_2)$ and look at $E \times J$: $TM \to R^3$. For this we will mainly consider the S^1 action on M that gives rotation about the vertical axis as we can trivially factor out by the other part of the action that gives rotation about the axis of the top.

We shall first examine this map in full generality, for which we will need a different coordinate chart on M from the Euler angles. We shall show that locally in a neighbourhood of the position corresponding to a sleeping top the mapping $E \times J$ is equivalent, with respect to the action of the symmetry group, to a polynomial mapping and in fact J can, in a certain sense, be made equivalent to a quadratic mapping. (see Theorem 2.12).

Next in the context of steady precession we shall investigate the singularities of $\Phi = E \times J_1 \big|_{\dot{\theta}=0}$ and Φ_{θ_o} which is Φ restricted to a constant value of θ. The reason for investigating Φ is that the singularities of $E \times J_1$ occur when $\dot{\theta} = 0$. We shall find that the singularities of Φ all correspond to the motion of steady precession (see the remark preceeding Proposition 3.3) and that all the singularities of Φ_{θ_o} are fold points (see Proposition 3.2) while for Φ we have fold points and one standard Whitney cusp point if the moments of inertia satisfy a particular condition (Theorem 3.6). A description of the inverse image of the singular set for the standard Whitney cusp map from R^3 to R^2 is given and the position of the tangent space to the steady precession manifold at the cusp point is determined for this standard picture.

We then turn our attention to the amended potential. Following the Smale programme we analyse the general motion of the axis of the top. We do this by examining the path the axis would trace out on the surface of a sphere. We shall find (Theorems 4.7 and 4.8) that the stable motion of the axis, that is the axis oscillating between two circles and meeting the upper circle in loops or waves, can be expressed as a universal unfolding of the unstable case where the axis of the top meets the upper bounding circle in cusps, this unfolding being parametrised by the initial conditions.

Then using the amended potential, we investigate steady precession showing that for fixed values of J_1 and J_2 there is only one possible angle of inclination of the axis of the top from the vertical at which steady precession can take place, and that this motion is a stable relative equilibrium. This is illustrated by a certain mapping of $\mathfrak{g}^* = R^2$ into the parameter space of a section of a swallowtail catastrophe and generalises the local version of Arnol'd [3].

Finally, we analyse the case of the sleeping top by seeing what happens to the steady precession surface near the points corresponding to sleeping. We find (Theorem 4.17) that as we approach the sleeping position the steady precession surface is bounded by the line corresponding to the stable sleeping top. By adding in the line that corresponds to the unstable sleeping top we are able to construct a global picture of the steady precession surface in the cotangent bundle, a picture that is reminiscent of a motorway bridge!

CHAPTER 1

SUMMARY OF SINGULARITY THEORY USED

We take the following definition from page 37 of [5] .

Definition 1.1

The jet-space $J^k(n,p)$ is the set of all mappings $f : R^n \to R^p$ each of whose components is a polynomial of degree less than or equal to k in the standard coordinates $x_1,\dots,x_n$ in R^n . Elements of $J^k(n,p)$ are called k-jets.

Suppose that $f : R^n \to R^p$ is a smooth map. Then we can expand f around some point a in R^n by constructing the Taylor series in terms of the standard bases for R^n and R^p . If we then delete all terms with degree greater than k we obtain a k-jet which we denote $j^k f(a)$ and call the k-jet of f at a. This defines the mapping $j^k f : R^n \to J^k(n,p)$ called the k-jet extension of f .

Although the definition given here is couched in terms of local coordinates, a coordinate free definition is easy to give as for instance in [4] where $J^k(n,p)$ is defined as

$$J^k(n,p) = R^p \times \mathrm{Hom}(R^n,R^p) \times \mathrm{Hom}_s^2(R^n,R^p)\times\dots\times\mathrm{Hom}_s^k(R^n,R^p)$$

where $\mathrm{Hom}_s^j(R^n,R^p)$ is the space of symmetric j-multilinear mappings from j copies of R^n to R^p .

The next definition and theorems are taken from [6].

Definition 1.2

Given a point a in R^n , and a mapping $f : R^n \to R^p$

then $Df(a)$ is a unique linear mapping of $T_a R^n \to T_{f(a)} R^p$.
Define the <u>rank of f</u> as the rank of $Df(a)$ and the <u>corank of f</u> as $\min(n,p)$ - rank f.

Let $S_r = \{f \in J^1(n,p) \; ; \; \text{corank} \;\; f = r\}$

<u>Theorem 1.4</u>

The set S_r is a submanifold of $J^1(n,p)$ of codimension $r(|n-p| + r)$. ∎

We let $S_r(f) = (j^1 f)^{-1}(S_r)$. The sets $S_r(f)$ are not necessarily submanifolds of R^n but using the Thom Transversality Theorem one can prove:-

<u>Theorem 1.5</u>

The set of f for which $j^1 f$ is transverse to S_r for all r is a residual set with respect to the Whitney topology. For such an f the set $S_r(f)$ is a submanifold of R^n with codimension equal to the codimension of S_r. ∎

Such maps are called <u>1-generic.</u>

Now suppose $f : R^n \to R^p$ is 1-generic. Denote by $S_{r,s}(f)$ the set of points where the map $f : S_r(f) \to R^p$ drops rank by s. We will now show how to construct $S_{r,s}$ in $J^2(n,p)$ such that

$$x \in S_{r,s}(f) \iff j^2 f(x) \in S_{r,s} .$$

The method is that given on pages 149-155 of [6]. We first of all construct the <u>intrinsic derivative</u> of a smooth map $f : R^n \to R^p$.

Given an $f : R^n \to R^p$ we have $Df : R^n \to Hom(R^n, R^p)$. For $x \in S_r(f)$ we calculate $D(Df)_x \in Hom\big(R^n, Hom(R^n, R^p)\big)$. We then restrict to the kernel K_x of Df_x and project to the cokernel L_x of Df_x to get the intrinsic derivative

$$(d\ Df)_x \in Hom\big(R^n, Hom(K_x, L_x)\big)$$

This will be determined by the 2-jet of f at x . By restricting the intrinsic derivative on R^n to K_x we induce a symmetric mapping

$$\delta^2 f_x \in Hom(K_x, Hom(K_x, L_x)) = Hom^2_s(K_x, L_x)$$

Now if we let $S_r^{(2)}$ be the preimage in $J^2(n,p)$ of S_r under the projection $J^2(n,p) \to J^1(n,p)$, we can summarise the above by saying:-

Let x belong to $S_r(f)$ and $\sigma \in S_r^{(2)}$. There exists an $f : R^n \to R^p$ such that $j^2 f(x) = \sigma$ and $j^1 f(x) \in S_r$. Then we calculate a symmetric mapping $\delta^2 f_x \in Hom\big(K_x, Hom(K_x, L_x)\big)$ depending only on σ . So we have a mapping

$$\mathcal{N} : S_r^{(2)} \to Hom\big(K, Hom(K,L)\big)$$

where $K = \bigcup_{\sigma \in S_r} K_x$ and $L = \bigcup_{\sigma \in S_r} L_x$. (Since fixing a σ in S_r fixes a K_x and L_x in $T_x R^n$ and $T_y R^p$ respectively). Consider the set of maps in $Hom\big(K, Hom(K, L)\big)$ of corank s . The pullback of this set by $\mathcal{N}$ in $S_r^{(2)}$ is the set we will denote by $S_{r,s}$. Then it can be proved, see [6], that

Theorem 1.6

(1) $S_{r,s}$ is a submanifold of $S_r^{(2)}$ of codimension

$$\frac{\ell}{2}\,k(k+1) - \frac{\ell}{2}\,(k-s)\,(k-s+1) - s(k-s)$$

where $k = r + \max(n-p,\, 0)$ and $\ell = p - n + k$.

(2) $x \in S_{r,s}(f) \iff j^2 f(x) \in S_{r,s}$ ■

We can extend all the above definitions to smooth mappings $f : X \to Y$ where X and Y are smooth manifolds of n , respectively p , dimensions in the obvious manner by choosing local coordinates in X and Y .

In fact in this paper we will be mainly concerned with the case when $n = 3$ and $p = 2$ or $n = p = 2$. In both these cases the codimension calculations show that only S_1 and S_{11} type singularities can occur for 1-generic maps. Complete classifications of these singularities in terms of local coordinates exist, see [19] from whence comes :

<u>Theorem 1.7</u> (Whitney).

Let X and Y be 2-dimensional manifolds and let $f : X \to Y$ be 1-generic. Then if $a \in S_1(f)$ and $b = f(a)$ either

(a) $T_a S_1(f) \oplus \mathrm{Ker}\ Df(a) = T_a X$
in which case a is called a <u>fold point</u> and one can choose a system of coordinates (x_1, x_2) centred at a and (y_1, y_2) centred at b such that $f : (x_1, x_2) \mapsto (x_1, x_2^2)$.

or

(b) $T_a\, S_1(f) = \mathrm{Ker}\ \ Df(a)$
in which case the situation is more complex. ■

In order to explain this last case it is easier to take the more general form of the above theorem due to B. Morin [14].

Let S_{1k} stand for S_1 if $k = 1$ and $S_{1,1}$ if $k = 2$.

<u>Theorem 1.8</u> (Morin)

If $f : X \to Y$ with $\dim X \geqslant \dim Y$ satisfies the condition that $j^k f$ is transverse to S_{1k} and if a is in $S_{1k}(f)$, then there exists a coordinate system $x_{1,\ldots,}x_n$ centred at a and a coordinate system $y_{1,\ldots,}y_p$ centred at $f(a)$ such that f has the form

$$f : (x_1,\ldots,x_n) \mapsto \left(x_1,\ldots,x_{p-1}, \pm x_p^2 \pm \ldots \pm x_{n-1}^2 \pm x_n^{k+1} + \sum_{j=1}^{k-1} x_j x_n^j\right)$$

This theorem gives the equal dimensional case by putting $n = p$, and is a specific example of the more general result proved in [14].

Letting $(n, p) = (3, 2)$ or $(4, 2)$ and $k = 2$ will give us the standard Whitney cusp maps from R^3 to R^2 or R^4 to R^2 respectively, that is :

$$f(x_1, x_2, x_3) = (x_1, x_2^2 + x_3^3 + x_1x_3)$$

or

$$f(x_1, x_2, x_3, x_4) = (x_1, x_2^2 + x_3^2 + x_4^3 + x_1x_4)$$

We are going to look at the local behaviour of maps and define a notion of local equivalence of mappings. In order to do this we need to use germs of mappings. From [5] page 33 we take :

<u>Definition 1.3</u>

Consider the set of smooth maps from $X \to Y$, where X and

Y are smooth manifolds. Given a point $x \in X$ we look at the set of all such maps whose domain U is a neighbourhood of x in X. If $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ are two such maps we say $f_1 \sim f_2$ if there exists a neighbourhood U of x in X depending on f_1 and f_2 such that

$$f_1|U = f_2|U .$$

This defines an equivalence relation on these maps. The equivalence classes are called <u>germs</u> of maps at x. If f_1 and f_2 represent the same germ then $f_1(x) = f_2(x) = y$ so we use the notation

$$f : (X, x) \to (Y, y) \text{ for the germ}$$

Now suppose that Γ is a compact group acting orthogonally on R^n and R^p. We say that a germ $f : (R^n, 0) \to (R^p, 0)$ is <u>Γ-equivariant</u> if $f(\gamma x) = \gamma f(x)$ for all γ in Γ. A function germ $h : (R^n, 0) \to R$ is called Γ-invariant if $h(\gamma x) = h(x)$ for all γ in Γ.

Let $\mathcal{E}^\Gamma(n)$ be the ring of Γ-invariant function germs on R^n and $\mathcal{E}^\Gamma(p)$ the ring of Γ-invariant function germs on R^p. Now define the free $\mathcal{E}^\Gamma(n)$-module $\mathcal{E}^\Gamma(n, p)$ as the set of Γ-equivariant map germs from R^n to R^p.

We will now give definitions of equivalence of germs taking into account the action of Γ. Putting Γ equal to the trivial group consisting of the identity only enables the more common definitions to be seen.

<u>Definition 1.4</u>

(a) Two map germs f and $g : (R^n, 0) \to (R^p, 0)$ are

<u>Γ-right-left equivalent</u> or <u>Γ-$\mathcal{A}$ equivalent</u> if there exist diffeomorphism germs $h : (R^n, 0) \to (R^n, 0)$ and $k : (R^p, 0) \to (R^p, 0)$ such that

$$\text{(i)} \quad k\big(f(x)\big) = g\big(h(x)\big) \text{ for all } x \text{ in } R^n$$
$$\text{(ii)} \quad h(\gamma x) = \gamma h(x) \text{ for all } x \text{ in } R^n \text{ and for all } \gamma \text{ in } \Gamma$$
$$\text{(iii)} \quad k(\gamma y) = \gamma k(y) \text{ for all } y \text{ in } R^p \text{ and for all } \gamma \text{ in } \Gamma$$

(b) Two maps germs f and $g : (R^n, 0) \to (R^p, 0)$ are <u>Γ-contact equivalent</u> or <u>Γ-$\mathcal{K}$ equivalent</u> if there exist a diffeomorphism germ $h : (R^n, 0) \to (R^n, 0)$ and $L \in C^\infty\big(R^n, GL(R^p)\big)$ such that

$$\text{(i)} \quad f\big(h(x)\big) = L(x)\, g(x) \text{ for all } x \text{ in } R^n$$
$$\text{(ii)} \quad h\big(\gamma(x)\big) = \gamma h(x) \text{ for all } x \text{ in } R^n \text{ and for all } \gamma \text{ in } \Gamma$$
$$\text{(iii)} \quad \gamma^{-1} L(\gamma x)\gamma = L(x) \text{ for all } x \text{ in } R^n \text{ and for all } \gamma \text{ in } \Gamma$$

The definition of Γ-$\mathcal{K}$ equivalence can be found on page 1-1 of [8] and that of Γ-$\mathcal{A}$ equivalence is closely modelled on it. Γ-$\mathcal{A}$ equivalence would seem the most natural form of equivalence to use and we shall try to work with it as much as possible. Contact equivalence arises more naturally in considering algebraic varieties defined as zero sets in algebraic geometry but its importance for us lies in the fact that it yields algebraic conditions for equivalence that are easy to check, unlike $\mathcal{A}$-equivalence. Of course Γ-$\mathcal{A}$ equivalence implies Γ-$\mathcal{K}$ equivalence but the converse is false. We will use the notation Γ-$\mathcal{F}$-equivalence when we do not need to specify whether it is contact or right-left equivalence under discussion.

We will use the notation $\mathcal{m}^\Gamma(n)$ to represent the maximal ideal of the ring $\mathcal{E}^\Gamma(n)$ and also let

$$\mathcal{m}^\Gamma_k(n) = \{g \in \mathcal{E}^\Gamma(n);\ \text{the } (k-1) \text{ jet of } g \text{ is zero}\}$$

and

$$\mathfrak{m}_k^{\Gamma}(n,p) = \{\xi \in \mathcal{E}^{\Gamma}(n,p);\ \text{the } (k-1) \text{ jet of } \xi \text{ is zero}\}$$

Consider now all map germs $f : (R \times R^n, 0) \to (R^p, 0)$ such that $f(t, x)$ is Γ-$\mathcal{F}$-equivalent to $f(0, x)$ where the equivalence varies smoothly with t. If we denote $f(t, x)$ by $f_t(x)$ then we can say that we have a curve $t \to f_t$ where f_t is Γ-$\mathcal{F}$ equivalent to f_0 with the equivalence varying smoothly with t. Using this idea we make the following definition :-

Definition 1.5

The Γ-$\mathcal{F}$ tangent space $T_{\mathcal{F}}f$ of a germ $f : (R^n, 0) \to (R^p, 0)$ is the totality of derivatives $\left.\frac{\partial f_t}{\partial t}\right|_{t=0}$ of all curves $t \to f_t$, where f_t is as given above and f_0 is f.

We can now prove:

Proposition 1.9

Let $f : (R^n, 0) \to (R^p, 0)$ be a map germ,

a) if $\mathcal{F} = \mathcal{A}$ then $T_{\mathcal{A}} f$ is given by

$$T_{\mathcal{A}} f = \Big\{Df(x)\ a(x) + b\big(f(x)\big)\Big\}$$

for all $a \in \mathfrak{m}_1^{\Gamma}(n, n)$ and $b \in \mathfrak{m}_1^{\Gamma}(p, p)$

b) If $\mathcal{F} = \mathcal{K}$ then $T_{\mathcal{K}}f$ is given by

$$T_{\mathcal{K}}f = \Big\{Df(x)\ a(x) + L(x)\ f(x)\Big\}$$

for all $a \in \mathfrak{m}_1^{\Gamma}(n, n)$ and $L(x)$ being a (possibly singular) $p \times p$ matrix satisfying the condition :

$$\gamma^{-1} L(\gamma x)\gamma = L(x) \quad \text{for all } \gamma \in \Gamma .$$

Proof

(a) We assumed that the equivalence varies smoothly with t . So if f_t is Γ-$\mathcal{A}$ equivalent to f then there exists an h_t : $(R^n, 0) \to (R^n, 0)$ and $k_t : (R^p, 0) \to (R^p, 0)$ both diffeomorphisms such that

$$f_t = k_t \circ f \circ h_t$$

where if $f_o = f$ both k_o and h_o are the identity mappings. By the Chain Rule :

$$\frac{\partial f_t}{\partial t} = \frac{\partial k_t}{\partial t}(fh_t) + Dk_t(fh_t)\, Df(h_t)\, \frac{\partial h_t}{\partial t} \; .$$

Putting $t = 0$ gives

$$\left.\frac{\partial f_t}{\partial t}\right|_{t=0} = \left.\frac{\partial k_t}{\partial t}\right|_{t=0} \circ f + Df\left(\left.\frac{\partial h_t}{\partial t}\right|_{t=0}\right),$$

So

$$\left.\frac{\partial f_t}{\partial t}\right|_{t=0} \in \left\{b\big(f(x)\big) + Df(x)\; a(x)\right\}$$

i.e. $T_{\mathcal{A}} f \subset \left\{Df(x)\; a(x) + b\big(f(x)\big)\right\}$

To prove the inclusion the other way we need to be able to express an arbitrary Γ-equivariant map $H : R^n \to R^n$ as $\left.\frac{\partial h_t}{\partial t}\right|_{t=0}$ where h_t is a Γ-equivariant diffeomorphism and h_o is the identity on R^n, and similarly an arbitrary Γ-equivariant map $K : R^p \to R^p$ as $\left.\frac{\partial k_t}{\partial t}\right|_{t=0}$ with k_t a Γ-equivariant diffeomorphism and k_o the identity on R^p .

To do this let $h_t = id + tH$ and $k_t = id + tK$.
This completes the proof.

(b) This is proved similarly and the proof can be found in [7] for Γ the identity and sketched in [8] for the more general case. ■

We now use these ideas to make the following definitions (adapted from [8]) :

Definition 1.6

A germ $f : (R^n, 0) \to (R^p, 0)$ is said to be Γ-$\mathcal{F}$- k determined if whenever g in $\mathcal{E}^\Gamma(n, p)$ has the same k-jet as f then f is Γ-$\mathcal{F}$ equivalent to g .

By an argument similar to that used in the proof of part (a) of Proposition 1.9 it is easy to see that if, for all positive integer values of k ,

$$\mathfrak{m}^\Gamma_{k+1}(n, p) \not\subset T_{\mathcal{A}} f$$

then f is not Γ-$\mathcal{A}$ finitely determined.

Both necessary and sufficient conditions for contact k-determinary are given by Golubitsky and Schaeffer in [7]. However when looking at Γ- contact k determinacy the situation is made slightly more complicated because in general.

$$\mathfrak{m}^\Gamma(n)\,\mathfrak{m}^\Gamma_k(n,p) \neq \mathfrak{m}^\Gamma_{k+1}(n,p)$$

In order to prove that f is Γ-$\mathcal{K}$- k determined we want to show that if $f_t = f + th$ where $h \in \mathfrak{m}^\Gamma_{k+1}(n, p)$ then $h \in T_{\mathcal{K}} f_t$.
We need the following lemma :

Lemma 1.10

If (1) $h \in \mathcal{M}^{\Gamma}(n)\, T_{\mathcal{K}} f$

and (2) $Dh(x)\, a(x) \in \mathcal{M}^{\Gamma}(n)\, T_{\mathcal{K}} f$

then $T_{\mathcal{K}} f \subset T_{\mathcal{K}} f_t$.

Proof

Each component of an element of $T_{\mathcal{K}} f_t$ (that is a p-vector) is of the form

$$\sum_{j=1}^{p} \ell_{ij}(x)\, f_{t_j}(x) + \sum_{k=1}^{n} \frac{\partial f_t}{\partial x_k} a_k(x)$$

$$= \sum_{j=1}^{p} \ell_{ij}(x) \left[f_j(x) + t h_j(x) \right] + \sum_{k=1}^{n} \left(\frac{\partial f_i}{\partial x_k} + t \frac{\partial h_i}{\partial x_k} \right) a_k(x)$$

But each $\ell_{ij}\, h_j$ and each $\frac{\partial h_i}{\partial x_k} a_k$ belong to $\mathcal{M}^{\Gamma}(n)\, T_{\mathcal{K}} f$. Hence Nakayama's lemma (see page 102 of [5]) implies that

$$T_{\mathcal{K}} f \subset T_{\mathcal{K}} f_t \ . \qquad \blacksquare$$

Since if h belongs to $\mathcal{M}^{\Gamma}_{k+1}(n, p)$ then so does $Dh(x)\, a(x)$ we deduce :

Corollary 1.11

A sufficient condition for $f : (R^n, 0) \to (R^p, 0)$ to be $\Gamma - \mathcal{K} - k$ determined is

$$\mathcal{M}^{\Gamma}_{k+1}(n, p) \subset \mathcal{M}^{\Gamma}(n)\, T_{\mathcal{K}} f \ . \qquad \blacksquare$$

Note that a generalisation of the condition given by Golubitsky and Schaeffer as Theorem 2.8 in [7] to the case of $\Gamma - \mathcal{K}$ determinacy would give a stronger condition than that in Corollary 1.11 and hence one that is harder to verify. In fact in the particular example we will be examining in Chapter 2 the generalisation of Golubitsky and Schaeffer's condition will not hold whereas Corollary 1.11 will be satisfied.

Although we will not need this, it is straightforward to prove that a necessary and sufficient condition for f to be $\Gamma - \mathcal{K} - k$ determined is given by :

(1) $\mathfrak{m}^{\Gamma}_{k+1}(n, p) \subset T_{\mathcal{K}}(f + h) + \mathfrak{m}^{\Gamma}_{k+2}(n, p)$

for every $h \in \mathfrak{m}^{\Gamma}_{k+1}(n, p)$ and

(2) f is $\Gamma - \mathcal{K} - (k + 1)$ determined.

Definition 1.7

Let $f : (R^n, 0) \to (R^p, 0)$ be a map germ. The $\Gamma\mathcal{F}$- codimension of f is the dimension of $\mathcal{E}^{\Gamma}(n,p)/T_{\mathcal{F}}f$ over R. If the $\Gamma\mathcal{F}$-codimension of f is zero then f is said to be $\Gamma\mathcal{F}$ infinitesimally stable.

We also wish to consider unfoldings for germs in $\mathcal{E}(n, p)$ without the group action.

The following definitions and results are taken from Martinet's paper [11].

Definition 1.8

Let $f_0 : (R^n, 0) \to (R^p, 0)$ be a map germ. All germs of the form

$$F : (R^q \times R^n, 0) \to (R^q \times R^p, 0)$$
$$(u, x) \mapsto \big(u, f(u, x)\big)$$

where

$$f(0, x) = f_0(x)$$

are called <u>q-parameter unfoldings of f_0.</u>

<u>Definition 1.9</u>

Two q-parameter unfoldings of the same germ f_0, F_1 and F_2 are <u>isomorphic</u> if there exist local diffeomorphisms

$$H : (R^q \times R^n, 0) \to (R^q \times R^n, 0) \quad \text{and} \quad K : (R^q \times R^p, 0) \to (R^q \times R^p, 0)$$

which are q-parameter unfoldings of the identity of R^n , respectively R^p, such that :

$$F_2 = K \circ F_1 \circ H$$

More generally F_1 and F_2 are said to be <u>equivalent</u> if there exists a local diffeomorphism $g : (R^q, 0) \to (R^q, 0)$ such that F_2 is isomorphic to g^*F_1 where g^*F_1 the pull back of F_1 by g is defined by :

$$g^*F_1(u, x) = \big(u, f_1\big(g(u), x\big)\big) \quad .$$

The pull back of an unfolding can be defined for any map $g : (R^t, 0) \to (R^q, 0)$ in a similar way.

<u>Definition 1.10</u>

An unfolding of f_0 is <u>trivial</u> if it is isomorphic to the constant unfolding $(u, x) \mapsto \big(u, f_0(x)\big)$.

An unfolding F of f_0 is called versal if any other unfolding F' of f_0 is isomorphic to the pull back of F by a suitable map of the parameter space of F' into that of F.

A germ is called stable if all its unfoldings are trivial.

From these definitions Martinet proves the following theorems:

Theorem 1.12

A q-parameter unfolding F of a germ $f_0 : (R^n, 0) \to (R^p, 0)$ is versal if and only if the initial speeds $\dot{F}_i \in \mathcal{E}(n, p)$ with $i = 1,\ldots,q$ span a real vector sub-space $R\{\dot{F}_1,\ldots,\dot{F}_q\}$ of $\mathcal{E}(n, p)$ such that

$$\mathcal{E}(n, p) = T_{\mathcal{A}} f_0 + R\{\dot{F}_1,\ldots,\dot{F}_q\}$$

where $\dot{F}_i = \frac{\partial f}{\partial u_i}(0, x)$ ■

Corollary 1.13

A germ that is $\mathcal{A}$ infinitesimally stable is stable. ■

Corollary 1.14

All c-parameter versal unfoldings of a germ f_0 with $\mathcal{A}$-codimension c are equivalent; they are called universal unfoldings of f_0. ■

Corollary 1.15

All q-parameter versal unfoldings of f_0, where $q > c$, are equivalent to a constant unfolding of $q - c$ parameters of a universal unfolding of f_0. ■

All the definitions and theorems in this section stem from the work of Mather [12].

CHAPTER 2

NEW COORDINATES FOR THE CONFIGURATION SPACE

Although it is traditional and convenient to use the Euler angles θ, ϕ and ψ as a coordinate chart for the configuration space of a spinning top, yet when $\theta = 0$, that is when the axis of symmetry of the top coincides with the vertical, ϕ and ψ become indistinguishable from each other, hence these angles no longer form a valid coordinate chart for the configuration space.

Thus in order to study the behaviour of a top near this position, as for instance in examining the phenomenon of the sleeping top (when it spins with its axis of symmetry vertical), it is necessary to use a different set of coordinates. The following system, is well defined, at least while the angle between the vertical and the top's symmetry axis is no greater than $\frac{\pi}{2}$, and so is an adequate coordinate chart for the configuration space in a neighbourhood of $\theta = 0$.

Starting from an orthogonal set of axes OXYZ, fixed in space, a rotation θ_1 is made about the axis OX bringing the system into the position $O\xi'\eta'\zeta'$. Then a second rotation θ_2 is made about $O\eta'$ giving the position $O\xi\eta\zeta$ and finally a third rotation θ_3 is made about $O\zeta$ moving the axis into coincidence with a set of axes Oxyz fixed in the top and thus moving in space. This construction gives a coordinate chart around the identity element in the Lie group SO(3), with which we can identify our configuration space.

Thus the coordinates $(\theta_1, \theta_2, \theta_3)$ specify the position of the top relative to the axes OXYZ fixed in space. By writing down the three rotation matrices and multiplying them, we can calculate

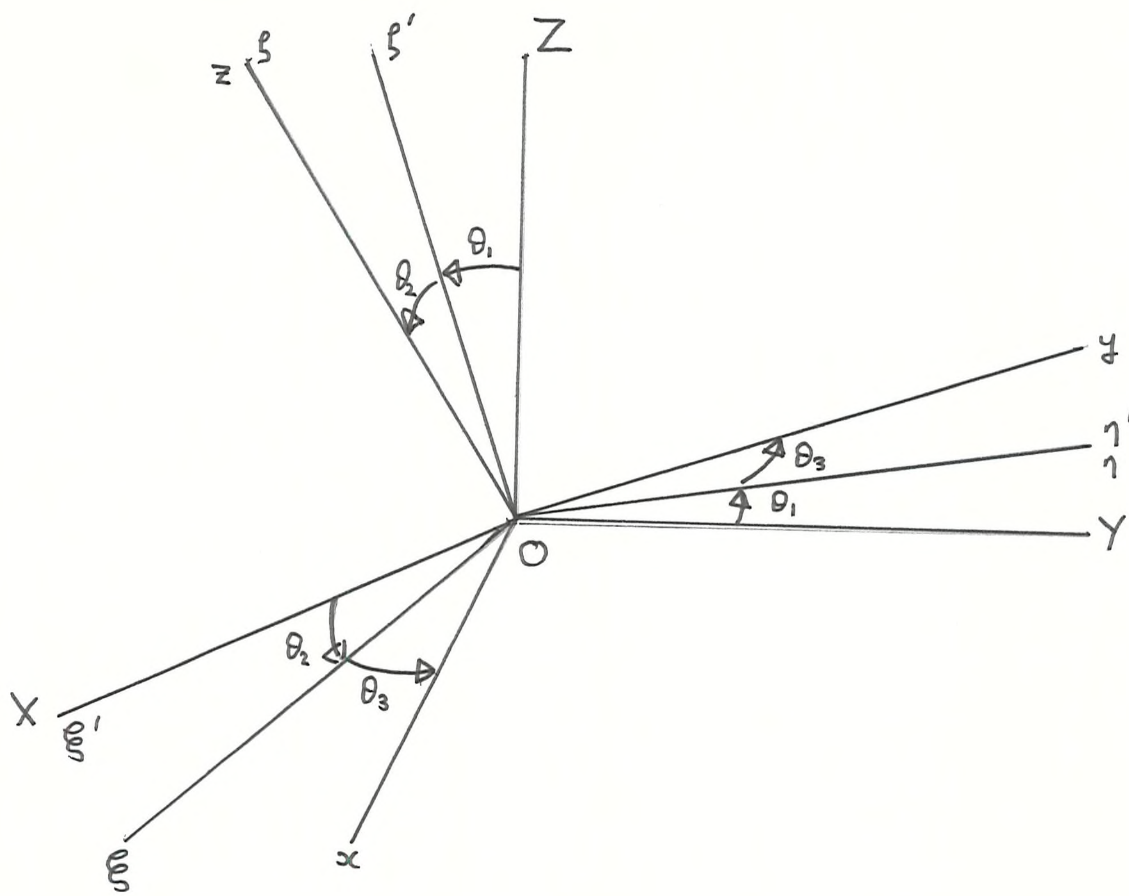


Figure 2.

the relationship between the two sets of orthogonal axes OXYZ and Oxyz as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c\theta_2 c\theta_3 & c\theta_1 s\theta_3 + s\theta_1 s\theta_2 c\theta_3 & s\theta_1 s\theta_3 - c\theta_1 s\theta_2 c\theta_3 \\ -c\theta_2 s\theta_3 & c\theta_1 c\theta_3 - s\theta_1 s\theta_2 s\theta_3 & s\theta_1 c\theta_3 + c\theta_1 s\theta_2 s\theta_3 \\ s\theta_2 & -s\theta_1 c\theta_2 & c\theta_1 c\theta_2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad (1)$$

where the abbreviations $s\theta_i = \sin\theta_i$ and $c\theta_i = \cos\theta_i$ are used to save space. This calculation is carried out in full on page 105 of [13].

We can now calculate the energy of the top in these new coordinates.

The system of axes $O\xi\eta\zeta$ has angular velocity components $\dot{\theta}_1 \cos\theta_2$ about $O\xi$, $\dot{\theta}_2$ about $O\eta$ and $\dot{\theta}_1 \sin\theta_2$ about $O\zeta$.

The top is moving relative to the system $O\xi\eta\zeta$ with an angular velocity $\dot{\theta}_3$ about $O\zeta$ so we can express the angular velocity of the top in terms of components $(\omega_\xi, \omega_\eta, \omega_\zeta)$ along system $O\xi\eta\zeta$ by $\omega_\xi = \dot{\theta}_1 \cos\theta_2$, $\omega_\eta = \dot{\theta}_2$ and $\omega_\zeta = \dot{\theta}_1 \sin\theta_2 + \dot{\theta}_3$.

Using the units mentioned in Chapter 1, we can express the rotational kinetic energy as :

$$K = \frac{1}{2}\left(\omega_\xi^2 + \omega_\eta^2\right) + \frac{1}{2} C\, \omega_\zeta^2$$

Using the XY plane as reference the potential energy is

$$V = \cos\theta_1 \cos\theta_2 ,$$

so the energy is given by :

$$E = K + V = \frac{1}{2}\left(\dot{\theta}^2_1 \cos^2\theta_2 + \dot{\theta}_2^2\right) + \frac{1}{2} C\left(\dot{\theta}_1 \sin\theta_2 + \dot{\theta}_3\right)^2 + \cos\theta_1 \cos\theta_2 .$$

We can also calculate the angular momentum of the top about the axis OZ. The angular momentum about the axis $O\xi$ is $\dot{\theta}_1 \cos\theta_2$, that about the axis $O\eta$ is $\dot{\theta}_2$ and about the axis $O\zeta$ is $C(\dot{\theta}_1 \sin\theta_2 + \dot{\theta}_3)$. Now the angle between the axis OZ and $O\xi$ has cosine equal to $-\cos\theta_1 \sin\theta_2$, that between OZ and $O\eta$ has cosine equal to $\sin\theta_1$, and that between OZ and $O\zeta$ has cosine equal to $\cos\theta_1 \cos\theta_2$, thus the angular momentum J_1 about OZ is given by

$$J_1 = -\dot{\theta}_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 + \dot{\theta}_2 \sin\theta_1 + C(\dot{\theta}_1 \sin\theta_2 + \dot{\theta}_3) \cos\theta_1 \cos\theta_2 .$$

These calculations are similar to those done for an artillery shell on page 255 of [13].

The Effect of Rotational Symmetry

Here we prove that the top forms a mechanical system with symmetry as defined in Chapter 0. Intuitively this is so because the values of the angles ϕ and ψ do not affect the energy or the angular momentum.

Let M be the unit circle tangent bundle of S^2 with the Riemmanian metric $K : TM \times TM \to R$ given by

$$K(v, w) = \frac{1}{2} (v_1 \; v_2 \; v_3) \begin{pmatrix} \cos^2\theta_2 + C\sin^2\theta_2 & 0 & C\sin\theta_2 \\ 0 & 1 & 0 \\ C\sin\theta_2 & 0 & C \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

or in the Euler coordinate chart,

$$K(v, w) = \frac{1}{2} (v_1 \; v_2 \; v_3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta + C\cos^2\theta & C\cos\theta \\ 0 & C\cos\theta & C \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

The symmetry group action is that of the torus T^2 acting on M as $S^1 \times S^1$, that is rotation about the vertical axis and rotation about the symmetry axis of the top, referred to as ψ in Chapter 1.

So we have the action $\Psi : T^2 \times M \to M$ defined by

$$\Psi(\gamma_1, \gamma_2; \theta, \phi, \psi) = (\theta, \phi + \gamma_1, \psi + \gamma_2)$$

In a neighbourhood of $\theta = 0$, where we must use $(\theta_1, \theta_2, \theta_3)$ to coordinatise M the T^2 action has a more complicated expression which we shall need later.

Proposition 2.1

The action of T^2 on M for $(\theta_1, \theta_2, \theta_3)$ in a toral neighbourhood given by θ_1 and θ_2 each belonging to a neighbourhood of 0 and θ_3 being any point in S^1, is a smooth mapping $\Psi : T^2 \times M \to M$ defined by

$\Psi(\gamma_1, \gamma_2; \theta_1, \theta_2, \theta_3) = (\theta_1', \theta_2', \theta_3')$ where the θ_i' are given by :-

$$\tan\theta_1' = \tan\theta_1 \cos\gamma_1 - \tan\theta_2 \sec\theta_1 \sin\gamma_1$$

$$\sin\theta_2' = \sin\theta_2 \cos\gamma_1 + \sin\theta_1 \cos\theta_2 \sin\gamma_1$$

$$\tan\theta_3' = \frac{\cos\theta_1 \sin\gamma_1 \cos(\theta_3+\gamma_2) + \sin(\theta_3+\gamma_2)(\cos\theta_2 \cos\gamma_1 - \sin\theta_1 \sin\theta_2 \sin\gamma_1)}{\cos\theta_2 \cos\gamma_1 \cos(\theta_3+\gamma_2) - \sin\gamma_1 \cos\theta_1 \sin(\theta_3+\gamma_2) - \sin\gamma_1 \sin\theta_1 \sin\theta_2 \cos(\theta_3+\gamma_2)}$$

Proof

In order to see this we use the matrix (1) constructed above which transforms coordinates relative to the fixed axes into coordinates relative to the rotating axes, in terms of the angles $\theta_1, \theta_2, \theta_3$.

Of course if we performed the rotations corresponding to the Euler angles we would obtain another matrix which performs exactly the same function, but gives the transformations in terms of θ, ϕ and ψ. This matrix is :-

$$\begin{pmatrix} c\phi\ c\psi - s\phi\ c\theta\ s\psi & s\phi\ c\psi + c\theta\ c\phi\ s\psi & s\theta\ s\psi \\ -c\phi\ s\psi - s\phi\ c\theta\ c\psi & -s\phi\ s\psi + c\theta\ c\phi\ c\psi & s\theta\ c\psi \\ s\phi\ s\theta & -c\phi\ s\theta & c\theta \end{pmatrix} \qquad (2)$$

where $c\theta = \cos\theta$, $c\phi = \cos\phi$, $c\psi = \cos\psi$ etc.

Because both matrices take the triple (X, Y, Z) to the same triple (x, y, z) corresponding entries must be equal.

In particular

$$\sin\theta_2 = \sin\phi \sin\theta \tag{3}$$

and, combining expressions for $\sin\theta_1$ and $\cos\theta_1$,

$$\tan\theta_1 = \cos\phi \tan\theta \tag{4}$$

and similarly

$$\tan\theta_3 = \frac{\tan\psi + \tan\phi \cos\theta}{1 - \tan\phi \tan\psi \cos\theta} \tag{5}$$

Now, in terms of the Euler angles, the T^2 action just increases ϕ and ψ and leaves θ unchanged. Using this together with equations 3, 4 and 5 gives the expressions for the T^2 action in terms of θ_1, θ_2 and θ_3 that appear above. $\blacksquare$

Next we check that we do have a mechanical system with symmetry in the case of the spinning top.

Proposition 2.2

M, K, V and G as given above do form a mechanical system with symmetry in the sense of Smale.

Proof

We need to show that K and V are invariant under the T^2-action given in Proposition 2.1. In order to facilitate the calculations we will use the Euler angle coordinate chart.

For V this means that $V \circ \Psi_\gamma = V$ where $\Psi_\gamma : M \to M$ is given by :

$$\Psi_\gamma(\theta, \phi, \psi) = \Psi(\gamma; \theta, \phi, \psi)$$

for γ in T^2.

Going back to Chapter 0 we can say that

$$V(\theta, \phi, \psi) = \cos\theta$$

As Ψ leaves θ unchanged, $\cos\theta$ also is unchanged so V is invariant under the T^2 action.

For K we have to prove that

$$K(T\Psi_\gamma v, T\Psi_\gamma w) = K(v, w)$$

Since the T^2 action takes $(\dot\theta, \dot\phi, \dot\psi,)$ at $x \in M$ to $(\dot\theta, \dot\phi, \dot\psi)$ at $\Psi_\gamma x$ this follows immediately. ■

We now carry out the construction given in Smale's paper [17] and outlined in Chapter 0 to calculate the angular momentum mapping J, showing that the final result agrees with the angular momentum as calculated by classical mechanical methods.

In this particular case where $G = T^2$, $\mathcal{G} = R^2$ and $\mathcal{G}^* = \mathrm{Hom}(R^2, R)$ we can write:

$$\begin{aligned} J(v_x)(X) &= J_I\big((2K)^*(v_x)\big)(X) \\ &= \big[(2K)^*(v_x)\big](\alpha_x(X)) \\ &= 2K_x(\alpha_x(X), v_x) \in R . \end{aligned}$$

where $v_x \in T_x M$ and $\alpha_x(X)$ is the instantaneous velocity vector at $x \in M$ corresponding to the choice of $X \in R^2$ and can be thought of as the angular speed of the rotational actions.

<u>Proposition 2.3</u>

Choosing X first as $(1, 0)$ and then as $(0, 1)$ to correspond to the infinitessimal speeds of the group action we have that $J(v_x)(X)$

agrees with the classical calculations for the angular momentum about the vertical axis and the symmetry axis respectively, namely:-

(1) In terms of the Euler coordinate chart $x = (\theta, \phi, \psi)$ we have

$$J(v_x)(1,0) = \dot{\phi}\sin^2\theta + C(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta$$

and $J(v_x)(0,1) = C(\dot{\phi}\cos\theta + \dot{\psi})$

(2) In terms of the $(\theta_1, \theta_2, \theta_3)$ coordinate chart $x = (\theta_1, \theta_2, \theta_3)$ we have

$$J(v_x)(1,0) = -\dot{\theta}_1\cos\theta_1\cos\theta_2\sin\theta_2 + \dot{\theta}_2\sin\theta_1 + C(\dot{\theta}_1\sin\theta_2 + \dot{\theta}_3)\cos\theta_1\cos\theta_2$$

and $J(v_x)(0,1) = C(\dot{\theta}_1 \sin\theta_2 + \dot{\theta}_3)$

<u>Proof</u>

(1) The instantaneous velocity vector $\alpha_x(1, 0)$ is given by $(0, 1, 0)$ and that for $\alpha_x(0, 1)$ by $(0, 0, 1)$ and for v_x we take any vector in $T_x M$ i.e. $(\dot{\theta}, \dot{\phi}, \dot{\psi})$ where these form the natural coordinate chart on TM induced by the Euler chart on M. We obtain:

$$J(\theta, \phi, \psi; \dot{\theta}, \dot{\phi}, \dot{\psi})\ (1,0) = (0\ 1\ 0)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta + C\cos^2\theta & C\cos\theta \\ 0 & C\cos\theta & C \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix}$$

$$= \dot{\phi}\sin^2\theta + C(\dot{\phi}\cos\theta + \dot{\psi})\cos\theta$$

$$J(\theta, \phi, \psi; \dot{\theta}, \dot{\phi}, \dot{\psi})\ (0,1) = (0\ 0\ 1)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta + C\cos^2\theta & C\cos\theta \\ 0 & C\cos\theta & C \end{pmatrix}\begin{pmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{pmatrix}$$

$$= C(\dot{\phi}\cos\theta + \dot{\psi})$$

as obtained in Chapter 0.

(2) In the case of the $(\theta_1, \theta_2, \theta_3)$- chart to calculate $\alpha_x(1, 0)$ we differentiate equations 3, 4 and 5 with respect to ϕ as that gives the direction of rotation for the first factor of T^2, and then substitute from the equality of the two coordinate transformation matrices to express everything in terms of θ_1, θ_2 and θ_3. For the second factor of the T^2 action, that is $\alpha_x(0, 1)$, we just have $(0, 0, 1)$ as before.

Denoting $\alpha_x(1, 0)$ by $(\theta_1', \theta_2', \theta_3')$ we get

From 3 : $$\cos\theta_2\ \theta_2' = \sin\theta\ \cos\phi = \sin\theta_1\ \cos\theta_2$$

So $$\theta_2' = \sin\theta_1\ ;$$

From 4 : $$\sec^2\theta_1\ \theta_1' = \frac{-\sin\theta\ \sin\phi}{\cos\theta} = -\frac{\sin\theta_2}{\cos\theta_1\cos\theta_2}$$

thus $$\theta_1' = -\frac{\cos\theta_1\sin\theta_2}{\cos\theta_2}$$

From 5 :

$$\sec^2\theta_3\theta_3' = \frac{[(c\phi c\psi - s\phi c\theta s\psi)(-s\phi s\psi + c\phi c\theta c\psi) - (c\phi s\psi + s\phi c\theta c\psi)(-s\phi c\psi - c\phi c\theta s\psi)]}{(c\phi\ c\psi\ -\ s\phi\ c\theta\ s\psi)^2}$$

Thus $$\theta_3' = \frac{\cos^2\ \theta_3}{\cos^2\theta_3\cos^2\theta_2}\left[c\theta_1 c\theta_3(c\theta_1 c\theta_3 - s\theta_1 s\theta_2 s\theta_3) - c\theta_2 s\theta_3(-c\theta_1 s\theta_3 - s\theta_1 s\theta_2 c\theta_3)\right]$$

$$= \frac{\cos\theta_1}{\cos\theta_2}$$

So $$\alpha_x(1, 0) = \left(-\frac{\cos\theta_1\ \sin\theta_2}{\cos\theta_2} \quad \sin\theta_1 \quad \frac{\cos\theta_1}{\cos\theta_2}\right)$$

For v_x we take any vector $(\dot\theta_1, \dot\theta_2, \dot\theta_3)$ in $T_x\ M$, and we get

$J(\theta_1, \theta_2, \theta_3; \dot\theta_1, \dot\theta_2, \dot\theta_3)(1, 0)$

$$= \begin{pmatrix} -\dfrac{\cos\theta_1\sin\theta_2}{\cos\theta_2} & \sin\theta_1 & \dfrac{\cos\theta_1}{\cos\theta_2} \end{pmatrix} \begin{pmatrix} \cos^2\theta_2 + C\sin^2\theta_2 & 0 & C\sin\theta_2 \\ 0 & 1 & 0 \\ C\sin\theta_2 & 0 & C \end{pmatrix} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix}$$

$$= -\dot\theta_1\cos\theta_1\cos\theta_2\sin\theta_2 + \dot\theta_2 \sin\theta_1 + C(\dot\theta_1 \sin\theta_2 + \dot\theta_3)\cos\theta_1 \cos\theta_2$$

and

$$J(\theta_1,\theta_2,\theta_3;\dot\theta_1,\dot\theta_2,\dot\theta_3)(0,1) = (0 \quad 0 \quad 1) \begin{pmatrix} \cos^2\theta_2 + C\sin^2\theta_2 & 0 & C\sin\theta_2 \\ 0 & 1 & 0 \\ C\sin\theta_2 & 0 & C \end{pmatrix} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix}$$

$$= C(\dot\theta_1 \sin\theta_2 + \dot\theta_3)$$

as obtained earlier in this chapter. ■

Corollary

The spin defined in terms of the Euler angles by

$$\dot\phi \cos\theta + \dot\psi$$

is given in terms of the $(\theta_1, \theta_2, \theta_3)$ coordinate chart by

$$\dot\theta_3 + \dot\theta_1 \sin\theta_2$$ ■

We now go on to analyse the map $E \times J$ under this T^2 action.

Orthogonal symmetry group action

We have now calculated explicitly the energy-momentum mapping $E \times J : TM \to R \times \mathfrak{g}^*$, which we can regard as a map $E \times \bar{J}$ from TM to $R \times R^2$ given by

$$(E \times \bar{J})(v) = \big(E(v), J(v)(1, 0), J(v)(0, 1)\big)$$
$$= \big(E(v), J_1(v), J_2(v)\big)$$

where

$$E(v) = \frac{1}{2}\left(\dot\theta_1^2 \cos^2\theta_2 + \dot\theta_2^2\right) + \frac{1}{2} C\left(\dot\theta_1 \sin\theta_2 + \dot\theta_3\right)^2 + \cos\theta_1 \cos\theta_2$$

$$J_1(v) = - \dot\theta_1 \cos\theta_1 \cos\theta_2 \sin\theta_2 + \dot\theta_2 \sin\theta_1 + C\left(\dot\theta_1 \sin\theta_2 + \dot\theta_3\right) \cos\theta_1 \cos\theta_2$$

$$J_2(v) = C(\dot\theta_1 \sin\theta_2 + \dot\theta_3)$$

and $$v = \left(\theta_1, \theta_2, \theta_3 ; \dot\theta_1, \dot\theta_2, \dot\theta_3\right)$$

From now on we will abuse notation by referring to $E \times \bar{J}$ as $E \times J$

Notice that as θ_3 does not occur in $E \times J\ (v)$ we can factor TM out by the second part of the $S^1 \times S^1$ action deriving a new domain $T_1M = TM/_{S^1}$ for $E \times J$. This corresponds to the classical treatment of ignorable coordinates.

The remaining S^1 action which we will denote Ψ' on the five dimensional space T_1M has an invariant manifold I given by

$$I = \left\{(\theta_1, \theta_2 ; \dot\theta_1, \dot\theta_2, \dot\theta_3) ; \theta_1 = \theta_2 = 0 \right\} .$$

In order to apply methods of singularity theory to the analysis of $E \times J$ near I we need to find new coordinates with respect to which the S^1 action is orthogonal.

Accordingly, define new coordinates for T_1M by :

$$x_1 = \sin\theta_1 \cos\theta_2$$
$$x_2 = \sin\theta_2$$
$$x_3 = \cos\theta_1 \cos\theta_2 \dot\theta_1 - \sin\theta_1 \sin\theta_2 \dot\theta_2$$
$$x_4 = \cos\theta_2 \dot\theta_2$$
$$x_5 = \dot\theta_3 + \dot\theta_1 \sin\theta_2$$

This change of coordinates defines a diffeomorphism near $(0, 0, 0, 0, 0)$ since in fact at that point the Jacobian of the transformation is the identity.

Note that x_3 and x_4 are the time derivatives of x_1 and x_2, and x_5 is the spin.

Proposition 2.4

With the new coordinate chart given above, the S^1-action $\Psi' : S^1 \times T_1 M \to T_1 M$ is given by

$$\Psi'(\gamma; x_1, x_2, x_3, x_4, x_5) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 & 0 \\ 0 & 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

i.e. the S^1 action is orthogonal.

Proof

It is easier to prove this by expressing x_1, x_2, x_3, x_4, x_5 in terms of the Euler angles.

Then

$$x_1 = \cos\phi \sin\theta$$
$$x_2 = \sin\phi \sin\theta$$
$$x_3 = -\sin\phi \sin\theta\, \dot{\phi} + \cos\phi \cos\theta\, \dot{\theta}$$
$$x_4 = \cos\phi \sin\theta\, \dot{\phi} + \sin\phi \cos\theta\, \dot{\theta}$$

To evaluate the effect of Ψ'_γ on the first four coordinates, replace ϕ by $\phi + \gamma$ and expand. The results are as given.

In terms of the Euler angles, $x_5 = \dot{\phi}\cos\theta + \dot{\psi}$ and as Ψ'_{γ} leaves θ, $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$ unchanged the proposition is proved. $\blacksquare$

We now need to express $E \times J$ in terms of these new coordinates.

<u>Proposition 2.5</u>

The energy-momentum mapping, in the new coordinates is given by :-

$$E \times J(x_1, x_2, x_3, x_4, x_5) = \left(E(x), J_1(x), J_2(x)\right) \quad \text{where}$$

$$E(x) = \tfrac{1}{2}(1-x_1^2-x_2^2)^{-1}\left[(x_3^2+x_4^2) - (x_1x_4-x_2x_3)^2\right] + \tfrac{1}{2}C\,x_5^2 + (1-x_1^2-x_2^2)^{\frac{1}{2}}$$

$$J_1(x) = x_1x_4 - x_2x_3 + Cx_5(1-x_1^2-x_2^2)^{\frac{1}{2}}$$

$$J_2(x) = Cx_5$$

<u>Proof</u>

This follows from the expression for $E \times J$ given at the beginning of this section by substituting :-

$$\dot{\theta}_2 = \frac{x_4}{(1-x_2^2)^{\frac{1}{2}}}$$

$$\dot{\theta}_1 = \left(x_3 + \frac{x_1x_2x_4}{1-x_2^2}\right)(1-x_1^2-x_2^2)^{-\frac{1}{2}}$$

$$\dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2 = x_5$$

$$\cos\theta_1 \cos\theta_2 = (1-x_1^2-x_2^2)^{1/2}$$

$$\sin\theta_1 = \frac{x_1}{(1-x_2^2)^{\frac{1}{2}}}$$

$$\cos\theta_2 = (1-x_2^2)^{\frac{1}{2}}$$

$\blacksquare$

To understand the nature of the singularity of $E \times J$ at the origin we examine whether $E \times J$ is k-determined, for some k, under the S^1 symmetry. In order to do this we first calculate the tangent space to $E \times J$ as given in Proposition 1.9.

<u>The S^1-tangent space to $E \times J$ at $0 \in R^5$</u>

From now on whenever we use Γ we will mean S^1.

To simplify the calculations we will change coordinates in the codomain and work with a mapping F which is Γ-$\mathcal{A}$ equivalent to $E \times J$, namely

$$F(x) = \left(E(x) - \frac{1}{2C}\left[J_2(x)\right]^2 , J_1(x) - J_2(x), J_2(x)\right)$$

and denote the k-jet of F by F_k.

To begin we need some information about the ring of invariant functions $\mathcal{E}^\Gamma(5)$.

The ring of polynomials in $\mathcal{E}^\Gamma(5)$ is generated by the following invariant polynomials.:

$$\sigma_1 = x_1^2 + x_2^2$$

$$\sigma_2 = x_3^2 + x_4^2$$

$$\sigma_3 = x_1x_4 - x_2x_3$$

$$\sigma_4 = x_1x_3 + x_2x_4$$

$$\sigma_5 = x_5$$

It is clear that these polynomials are invariant under the S^1-action. The fact that they generate the polynomial ring can best be seen by using complex coordinates to replace the first four x-coordinates, that is letting

$$z = x_1 + i\,x_2 \quad \text{and} \quad w = x_3 + i\,x_4 .$$

Then any polynomial in $x_1,\ldots,x_5$ can be written as

$$f(z, \bar{z}, w, \bar{w}, x_5) = \sum (a_{jk\ell mn} + i\, b_{jk\ell mn})\, z^j (\bar{z})^k w^\ell (\bar{w})^m x_5^n$$

where $a_{jk\ell mn}$ and $b_{jk\ell mn}$ are real.

The action of S^1 is given by multiplication by $e^{i\theta}$ hence if f is Γ- invariant then we can write

$$f(e^{i\theta} z, e^{-i\theta} \bar{z}, e^{i\theta} w, e^{-i\theta} \bar{w}, x_5) = f(z, \bar{z}, w, \bar{w}, x_5)$$

as the action on the fifth coordinate is trivial. Substituting into the expression for f given above we deduce that

$$j + \ell = k + m$$

Hence f can be written as

$$f(z, \bar{z}, w, \bar{w}, x_5) = \sum (a_{jk\ell mn} + ib_{jk\ell mn})(z\bar{z})^j \, (w\bar{w})^m (\bar{z}w)^{k-j} \, x_5^n$$

if $k > j$, or as

$$f(z, \bar{z}, w, \bar{w}, x_5) = \sum (a_{jk\ell mn} + i\, b_{jk\ell mn})(z\bar{z})^k (w\bar{w})^\ell (z\bar{w})^{j-k} \, x_5^n$$

if $k \leqslant j$

and since $z\bar{z} = \sigma_1$, $w\bar{w} = \sigma_2$, $z\bar{w} = \sigma_4 - i\sigma_3$, $\bar{z}w = \sigma_4 + i\sigma_3$

and $x_5 = \sigma_5$, the σ's generate the polynomial ring.

Note that we have the relationship

$$\sigma_1 \, \sigma_2 = \sigma_3^2 + \sigma_4^2$$

which we will use in the determinacy calculations below.

In order to calculate $T_{\mathcal{F}}F$ we need the generators of the module $\mathcal{E}^{\Gamma}(5, 5)$ over $\mathcal{E}^{\Gamma}(5)$, from which we can deduce the generators of $\mathfrak{m}_1^{\Gamma}(5, 5)$ over $\mathcal{E}^{\Gamma}(5)$, as $\{DF(x)\ a(x)\ ;\ a(x) \in \mathfrak{m}_1^{\Gamma}(5, 5)\}$ is one summand of $T_{\mathcal{F}}F$ for both $\mathcal{F} = \mathcal{A}$ and $\mathcal{F} = \mathcal{K}$. Since the S^1 action is trivial on the last coordinate all we need to calculate are the generators of $\mathcal{E}^{\Gamma}(4, 4)$ over $\mathcal{E}^{\Gamma}(4)$ and then use the following lemma.

Lemma 2.6

If a compact group Γ acts orthogonally on $R^n \times R$ such that its action on the last coordinate is trivial, and if the $\mathcal{E}^{\Gamma}(n)$ - module $\mathcal{E}^{\Gamma}(n, n)$ (considering the action of Γ restricted to R^n) is generated by $\gamma_1, \ldots, \gamma_s$ then $\mathcal{E}^{\Gamma}(n+1, n+1)$ is generated by $(\gamma_1, 0), \ldots, (\gamma_s, 0), (0, 1)$ over $\mathcal{E}^{\Gamma}(n+1)$.

The proof of this lemma is straightforward and we omit it.

Proposition 2.7

The $\mathcal{E}^{\Gamma}(4)$-module $\mathcal{E}^{\Gamma}(4, 4)$ is generated by $(x_1, x_2, 0, 0)$, $(x_2, -x_1, 0, 0)$, $(0, 0, x_3, x_4)$ and $(0, 0, x_4, -x_3)$.

Proof

This proof is an extension of that given on page 5-2 of [8] for proving that $\mathcal{E}^{\Gamma}(2, 2)$ is generated over $\mathcal{E}^{\Gamma}(2)$ by (x, y) and $(-y, x)$.

We use complex coordinates $z = x_1 + ix_2$ and $w = x_3 + ix_4$. Given any polynomial in $\mathcal{E}^{\Gamma}(4, 4)$ we write it as

$$g(z, \bar{z}, w, \bar{w}) = \sum (a_{jk\ell m} + ib_{jk\ell m})\ z^j (\bar{z})^k w^\ell (\bar{w})^m$$

where $a_{jk\ell m}$ and $b_{jk\ell m}$ are real. The S^1-action is multiplication by $e^{i\theta}$, so the equivariance of g means that

$$\bar{e}^{i\theta} g(e^{i\theta} z, \bar{e}^{i\theta} \bar{z}, e^{i\theta} w, \bar{e}^{i\theta} \bar{w}) = g(z, \bar{z}, w, \bar{w}) .$$

Using the expression for g given above we have that

$$j + \ell = k + m + 1 ,$$

So we can write g as

$$g(z, \bar{z}, w, \bar{w}) = \sum (a_{jk\ell m} + ib_{jk\ell m}) (z \bar{z})^k (w \bar{w})^\ell z^{j-k} (\bar{w})^{m-\ell}$$

where $j - k = m - \ell + 1$ if $m - \ell \geqslant 0$, i.e. $k - j \leqslant -1$,

or

$$g(z, \bar{z}, w, \bar{w}) = \sum (a_{jk\ell m} + ib_{jk\ell m}) (z \bar{z})^j (w \bar{w})^m (\bar{z})^{k-j} w^{\ell-m}$$

where $\ell - m = k - j + 1$ if $k - j \geqslant 0$.

In the first case we have g equal to

$$\sum (a_{jk\ell m} + ib_{jk\ell m}) (z \bar{z})^k (w \bar{w})^\ell (z \bar{w})^{m-\ell} z$$

and in the second case

$$\sum (a_{jk\ell m} + ib_{jk\ell m}) (z \bar{z})^j (w \bar{w})^m (\bar{z} w)^{k-j} w .$$

Now $z \bar{z} = \sigma_1$, $w \bar{w} = \sigma_2$, $z \bar{w} = \sigma_4 - i\sigma_3$ and $\bar{z} w = \sigma_4 + i\sigma_3$, all of which belong to $\mathcal{E}^\Gamma(4)$ so $\mathcal{E}^\Gamma(4, 4)$ is the module generated by z, iz, w and iw. Translating back into x_1, x_2, x_3, x_4 terms and multiplying the second and fourth by -1 completes the proof. ■

We are now able to compute the linear space $\left\{DF(x)\ a(x); a \in \mathfrak{m}_1^\Gamma(5,5)\right\}$. From Proposition 1.9 we know that $T_4 F$ is made up of this linear space together with the set of maps formed by composing arbitrary maps that preserve the origin from R^3 to itself with F, since the S^1 action on R^3 is trivial.

Hence we can set about computing $T_{\mathcal{A}} F$ to decide if F is infinitesimally stable. To begin we will calculate the $\Gamma - \mathcal{A}$ codimension of F_3 as defined in Definition 1.7.

<u>The codimension of the 3-jet of F</u>

We will calculate the generators of $T_{\mathcal{A}} F_3$. Ignoring constants we have

$$F_3(x) = \left(\frac{1}{2}\left[x_3^2 + x_4^2 - x_1^2 - x_2^2\right], \; x_1x_4 - x_2x_3 - \frac{C}{2} x_5(x_1^2+x_2^2), \; Cx_5 \right)$$

and

$$DF_3(x) = \begin{pmatrix} -x_1 & -x_2 & x_3 & x_4 & 0 \\ x_4 - Cx_1x_5 & -x_3 - Cx_2x_5 & -x_2 & x_1 & -\frac{C}{2}(x_1^2+x_2^2) \\ 0 & 0 & 0 & 0 & C \end{pmatrix}$$

It is easier to perform the calculation if the σ_i's are used, rather than the x_i's. Using the generators of $\mathcal{E}^\Gamma(5, 5)$ over $\mathcal{E}^\Gamma(5)$ calculated in Proposition 2.7 with Lemma 2.6, and $DF_3(x)$ above, we find that the linear space $A = \left\{ DF(x)\, a(x) \;\; ; a \in \mathfrak{m}_1^\Gamma(5, 5) \right\}$ is generated as an $\mathcal{E}^\Gamma(5)$ - module by

$$\delta_1 = (-\sigma_1, \sigma_3 - C\sigma_1 \sigma_5, 0)$$
$$\delta_2 = (0, \sigma_4, 0)$$
$$\delta_3 = (\sigma_2, \sigma_3, 0)$$

and by $\mathfrak{m}^\Gamma(5)\, \delta_4$

where $\delta_4 = (0, -\frac{1}{2}\sigma_1, 1)$

To find the other generators of $T_{\mathcal{A}} F_3$ we must examine $B = \left\{ b(F_3(x)) ; \;\; b : (R^3, 0) \mapsto (R^3, 0) \right\}$ i.e.

$$\left\{ b(\sigma_2 - \sigma_1,\ \sigma_3 - \frac{C}{2}\sigma_1\sigma_5\ ,\ \sigma_5)\ ;\ b : (R^3, 0) \to (R^3, 0) \right\} .$$

Since $\mathcal{E}^{\Gamma}(5, 3)$ is generated, as an $\mathcal{E}^{\Gamma}(5)$ module, by triples of the form $(\sigma_i, 0, 0)$, $(0, \sigma_i, 0)$ and $(0, 0, \sigma_i)$ for $1, \ldots, 5$ we have to determine which of these triples belong to $T_{\mathcal{A}} F_3$. One of these triples can only belong to $T_{\mathcal{A}} F_3$ if either it is expressible as a function of the components of F_3 or is one of the δ's or a sum or multiple of them. In particular $(\sigma_4, 0, 0)$ cannot be an element of $T_{\mathcal{A}} F_3$ as there are no σ_4's in F_3 itself at all and no σ_4 in the first coordinate place of any δ_i. By a similar argument it can be shown that $(0, \sigma_1, 0)$ and $(0, \sigma_2, 0)$ are not elements of $T_{\mathcal{A}} F_3$ either (though $(0, \sigma_2 - \sigma_1, 0)$ is.) However we shall see which triples are elements of $T_{\mathcal{A}} F_3$.

(a) Clearly by choosing an appropriate map b we have $(\sigma_5, 0, 0)$, $(0, \sigma_5, 0)$ and $(0, 0, \sigma_5)$ in B.

(b) As A is a module over $\mathcal{E}^{\Gamma}(5)$ we have $\sigma_5\ \delta_4 = (0, -\frac{1}{2}\sigma_1\sigma_5, \sigma_5)$ in A, thus $(0, \sigma_1\sigma_5, 0)$ belongs to $A + B$, hence so does $(0, \sigma_3, 0)$ by choosing $b_1(y_1, y_2, y_3) = (0, y_2, 0)$.

(c) Now $\delta_1 = (-\sigma_1, \sigma_3 - C\sigma_1\sigma_5, 0)$ and $\delta_2 = (\sigma_2, \sigma_3, 0)$ both belong to A hence $(\sigma_1, 0, 0)$ and $(\sigma_2, 0, 0)$ belong to $A + B$.

(d) Taking multiples of δ_4 by $\sigma_1, \sigma_2, \sigma_3$ and σ_4 we have

$$\left.\begin{array}{l} (0, -\frac{1}{2}\sigma_1^2, \sigma_1) \\ (0, -\frac{1}{2}\sigma_1\sigma_2, \sigma_2) \\ (0, -\frac{1}{2}\sigma_1\sigma_3, \sigma_3) \\ (0, -\frac{1}{2}\sigma_1\sigma_4, \sigma_4) \end{array}\right\} \in A .$$

From δ_4 we know that $(0, \sigma_4, 0) \varepsilon A$ and as A is an $\mathcal{E}^{\Gamma}(5)$ - module $(0, \sigma_1\sigma_4, 0) \varepsilon A$ hence $(0, 0, \sigma_4) \varepsilon A$.

(e) We have $\sigma_5 \delta_1 = (- \sigma_1\sigma_5, \sigma_3\sigma_5 - C\sigma_1\sigma_5^2, 0) \varepsilon A$ and by letting $b_2(y_1, y_2, y_3) = (0, y_2y_3, 0)$ we obtain $(0, \sigma_3\sigma_5 - \frac{C}{2}\sigma_1\sigma_5^2, 0) \varepsilon B$ hence $(\sigma_1\sigma_5, \frac{C}{2}\sigma_1\sigma_5^2, 0) \varepsilon A + B$
But $\sigma_5^2\delta_4 = (0, -\frac{1}{2}\sigma_1\sigma_5^2, \sigma_5^2) \varepsilon A$ and $(0, 0, \sigma_5^2) \varepsilon B$, (by choosing $b_3(y_1, y_2, y_3) = (0, 0, y_3^2)$, so $(\sigma_1\sigma_5, 0, 0) \varepsilon A + B$.
Then letting $b_4(y_1, y_2, y_3) = (y_2, 0, 0)$ gives $(\sigma_3, 0, 0) \varepsilon A + B$.

(f) None of the three triples $(0, 0, \sigma_3)$, $(0, 0, \sigma_2)$ or $(0, 0, \sigma_1)$ can belong to $A + B$. For instance, if we consider $(0, 0, \sigma_3)$ the only possible way to try to show that it belongs is to use part (d) and write

$$(0 , 0, \sigma_3) = (0, -\frac{1}{2}\sigma_1\sigma_3, \sigma_3) + (0, \frac{1}{2}\sigma_1\sigma_3, 0)$$

and use Nakayama's Lemma (see page 102 of [5]). However to use Nakayama the triple $(0, \frac{1}{2}\sigma_1\sigma_3, 0)$ must belong to $\mathfrak{m}^{\Gamma}(5)$ A and as neither $(0, \sigma_1, 0)$ nor $(0, \sigma_3, 0)$ belong to A this is not the case.

As both $(0, \sigma_2 - \sigma_1, 0)$ and $(0, 0, \sigma_2 - \sigma_1)$ belong to $A + B$ the four triples $(\sigma_4, 0, 0)$, $(0, \sigma_1, 0)$, $(0, 0, \sigma_1)$, and $(0, 0, \sigma_3)$ form a basis for $\mathcal{E}^{\Gamma}(5, 3)/_{T_{\mathcal{A}}} F_3$, hence the Γ -$\mathcal{A}$ codimension is 4 .

We have now proved:

<u>Proposition 2.8</u>

The 3-jet of $E \times J$ has Γ-$\mathcal{A}$ codimension 4 and so in particular $E \times J$ is not Γ infinitesimally stable.

We will now decide on the Γ-determinacy of $E \times J$.

Right-Left Determinacy of $E \times J$.

Proposition 2.9

The map $E \times J$ is not $\Gamma - \mathcal{A}$ k determined for any k .

Proof

Using the notation from Chapter 1, in $\mathfrak{m}_k^\Gamma(5, 3)$ there is a term of the form $(\sigma_4 \sigma_5^{k-2}, 0, 0)$ and we will show that for any k this term cannot belong to $T_{\mathcal{A}} F_k$.

From Chapter 1

$$T_{\mathcal{A}} F_k = A + B$$

where

$$A = \left\{ DF_k(x) \; \zeta(x) \; ; \; \zeta \in \mathfrak{m}_1^\Gamma(5, 5)\right\}$$

and

$$B = \left\{\eta \; \left(F_k(x)\right) \; ; \; \eta \in \mathfrak{m}_1^\Gamma(3, 3)\right\} \; .$$

Generators of A as an $\mathcal{E}^\Gamma(5)$ module are

$$\Delta_1 = \left(- \sigma_1 + 0(3) \; , \; \sigma_3 + 0(3), \; 0\right)$$
$$\Delta_2 = \left(0(3), \; \sigma_4 + 0(3), \; 0\right)$$
$$\Delta_3 = \left(\sigma_2 + 0(3) \; , \; \sigma_3 + 0(3), \; 0\right)$$

and $\qquad \mathfrak{m}^\Gamma(5) \; \Delta_4$

where $\qquad \Delta_4 = \left(0, \; - \frac{1}{2} \sigma_1 + 0(3) \; , \; 1 + 0(3)\right).$

None of the terms of order 3 and above given in the first coordinate place of Δ_1, Δ_2 or Δ_3 contain any odd power of σ_4, (even powers are possible as $\sigma_4^2 = \sigma_1\sigma_2 + \sigma_3^2$)

Hence $\qquad (\sigma_4, \; 0, \; 0) \notin A$

and $\qquad (\sigma_5, \; 0, \; 0) \notin A$

thus $\qquad (\sigma_4 \; \sigma_5^p, \; 0, \; 0) \notin A$ for any p .

Now $B = \left\{ \eta \left[\frac{\sigma_2-\sigma_1}{2} + O(4),\ \sigma_3 + O(3),\ \sigma_5\right] ;\ \eta \in \mathfrak{m}_1^\Gamma(3, 3) \right\}$

where none of the higher order terms have a σ_4 in them.

Hence $(\sigma_4, 0, 0) \notin B$

thus $(\sigma_4\ \sigma_5^p, 0, 0) \notin B$ for any p

In fact if $(\tau, 0, 0) \in B$ then τ cannot involve σ_4 at all.

As there is no relationship expressing either σ_4 or σ_5^p, for any p, in terms of the other σ_i's then $\sigma_4\sigma_5^p$ cannot be written as the sum of two terms neither of which involves σ_4 multiplied by a power of σ_5, hence $(\sigma_4\ \sigma_5^p, 0, 0)$ cannot belong to $A + B$. This proves the proposition. ■

Although $E \times J$ is not Γ right-left finitely determined we shall show now that it is in fact Γ contact 2 determined.

Contact Determinacy of $E \times J$.

We will work with $F_3(x) = \left(\frac{\sigma_2-\sigma_1}{2},\ \sigma_3 - \frac{C}{2}\sigma_1\sigma_5,\ \sigma_5\right)$

Lemma 2.10.

Using the notation defined in Chapter 1, we have

$$\mathfrak{m}_3^\Gamma(5, 3) \subset \mathfrak{m}^\Gamma(5)\ T_K F_3 .$$

Proof

The linear space $\left\{ DF_3(x)\ a(x) ;\ a \in \mathfrak{m}_1^\Gamma(5, 5)\right\}$ is generated as a module over $\mathcal{E}^\Gamma(5)$ by

$$\delta_1 = (-\sigma_1,\ \sigma_3 - C\sigma_1\sigma_5,\ 0)$$
$$\delta_2 = (0,\ \sigma_4,\ 0)$$
$$\delta_3 = (\sigma_2,\ \sigma_3,\ 0)$$

and by

$$\mathfrak{m}^{\Gamma}(5)\delta_4$$

where

$$\delta_4 = (0, -\tfrac{1}{2}\sigma_1, 1) .$$

So $T_{\mathcal{K}}F_3$ is generated by these together with

$$L_{ij}\, F_3(x) \qquad (1 \leq i, j \leq 3)$$

where L_{ij} is a 3×3 matrix with 1 in the i-jth position and zeros elsewhere.

Now in this case, as the action of S^1 on R^3 is trivial,

$$\mathfrak{m}_3^{\Gamma}(5, 3) = \mathfrak{m}_3^{\Gamma}(5)\mathcal{E}^{\Gamma}(5, 3)$$

Denote the natural basis of $\mathcal{E}^{\Gamma}(5, 3)$ over $\mathcal{E}^{\Gamma}(5)$ by (e_1, e_2, e_3) .

(a) $\sigma_5\, e_i = L_{i3}\, F_3 \Rightarrow \sigma_5 e_i \;\varepsilon\, \mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3$.

(b) $\sigma_3\, e_i = L_{i2}\, F_3 + \frac{C}{2}\sigma_5\sigma_1 e_i$ hence by (a) $\sigma_3 e_i \;\varepsilon\, \mathfrak{m}^{\Gamma}(5) T_{\mathcal{K}}\, F_3$.

(c) It follows from (a) and (b) that $\sigma_1\, e_1 \;\varepsilon\, \mathfrak{m}^{\Gamma}(5) T_{\mathcal{K}}\, F_3$ because $\delta_1 = (-\sigma_1, \sigma_3 - C\sigma_1\sigma_5, 0) \;\varepsilon\, \mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3$.

(d) Likewise $\sigma_2 e_1 \;\varepsilon\, \mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3$ because $\delta_3 = (\sigma_2, \sigma_3, 0) \;\varepsilon\, \mathfrak{m}^{\Gamma}(5) T_{\mathcal{K}}\, F_3$

(e) Hence since $\mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}} F_3$ is an $\mathcal{E}^{\Gamma}(5)$ - module we see that

$$\left.\begin{array}{l} (\sigma_1^2, 0, 0) \\ (\sigma_1\sigma_2, 0, 0) \\ (\sigma_1\sigma_4, 0, 0) \\ (\sigma_2^2, 0, 0) \\ (\sigma_2\sigma_4, 0, 0) \end{array}\right\} \text{ all belong to } \mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3 .$$

(f) In a similar way

$$\left.\begin{array}{l} (0, \sigma_4^2, 0) \\ (0, \sigma_1\sigma_4, 0) \\ (0, \sigma_2\sigma_4, 0) \end{array}\right\} \varepsilon\, \mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3 \text{ because } \delta_2 = (0, \sigma_4, 0)\, \varepsilon\, \mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3$$

(g) Recall that $\sigma_1\sigma_2 = \sigma_3^2 + \sigma_4^2$

Hence as $(\sigma_3^2, 0, 0)$ and $(\sigma_1\sigma_2, 0, 0)$ belong to $\mathfrak{m}^{\Gamma}(5)\, T_{\mathcal{K}}\, F_3$ so must $(\sigma_4^2, 0, 0)$.

Similarly as $(0, \sigma_3^2, 0)$ and $(0, \sigma_4^2, 0)$ belong so does $(0, \sigma_1\sigma_2, 0)$.

(h) $(\sigma_2 - \sigma_1)e_2 = L_{21} F_3$

hence $(\sigma_2 - \sigma_1)e_2 \in T_{\mathcal{K}} F_3$

thus both $(\sigma_2^2 - \sigma_1\sigma_2)e_2$ and $(\sigma_2\sigma_1 - \sigma_1^2)e_2$ belong to $\mathfrak{m}^\Gamma(5)T_{\mathcal{K}} F_3$

we conclude that

$$\left.\begin{matrix}(0, \sigma_2^2, 0)\\(0, \sigma_1^2, 0)\end{matrix}\right\} \in \mathfrak{m}^\Gamma(5)T_{\mathcal{K}} F_3$$

(i) Let ρ stand for any element of $\left\{\sigma_1^2, \sigma_2^2, \sigma_1\sigma_2, \sigma_4^2, \sigma_1\sigma_4, \sigma_2\sigma_4\right\}$ $\subset \mathfrak{m}_3^\Gamma(5)$, so $\rho e_i \in \mathfrak{m}_3^\Gamma(5, 3)$ for $i = 1, 2, 3$.

Then

$$\rho\delta_4 = (0, -\tfrac{1}{2}\sigma_1\rho, \rho) \in \mathfrak{m}^\Gamma(5)\ T_{\mathcal{K}} F_3$$

Hence

$$(0, 0, \rho) \in \mathfrak{m}^\Gamma(5)\ T_{\mathcal{K}} F_3 + \mathfrak{m}^\Gamma(5)\left[\mathfrak{m}_3^\Gamma(5, 3)\right]$$

Thus

$$\mathfrak{m}_3^\Gamma(5, 3) \subset \mathfrak{m}^\Gamma(5)\ T_{\mathcal{K}} F_3 + \mathfrak{m}^\Gamma(5)\left[\mathfrak{m}_3^\Gamma(5, 3)\right]$$

Hence by Nakayama's Lemma we have

$$\mathfrak{m}_3^\Gamma(5, 3) \subset \mathfrak{m}^\Gamma(5)\ T_{\mathcal{K}} F_3 .$$ ■

<u>Corollary 2.11</u>

$E \times J$ is $\Gamma - \mathcal{K} - 2$ determined. ■

Thus we have proved :

<u>Theorem 2.12</u>

Locally near the origin $E \times J$ is Γ contact equivalent to the polynomial mapping :

$$(x_1, x_2, x_3, x_4, x_5) \mapsto \left(p_1(x_1,\ldots,x_5),\ p_2(x_1,\ldots,x_5),\ p_3(x_1,\ldots,x_5)\right)$$

where

$$p_1(x_1,\ldots,x_5) = \frac{1}{2}\left[Cx_5^2 - x_1^2 - x_2^2 + x_3^2 + x_4^2\right]$$

$$p_2(x_1,\ldots,x_5) = Cx_5 + x_1x_4 - x_2x_3$$

$$p_3(x_1,\ldots,x_5) = Cx_5$$ ■

We can now make a few remarks about $E \times J$. Firstly, from its 2-jet, we can see that E is a Morse function and thus particularly simple. Secondly, the 2-jet of $J_1 \times J_2$ is given by :

$$(x_1, x_2, x_3, x_4, x_5) \mapsto \left(Cx_5 + x_1x_4 - x_2x_3, Cx_5\right)$$

and so can be regarded as a family of nondegenerate quadratic forms parametrised by x_5 hence, by the Morse Lemma, the same is true for $J_1 \times J_2$ itself.

Looking at it in another way, taking J as given in Proposition 2.5, we see that because:

$$DJ(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 & C \end{pmatrix}$$

the kernel K of $DJ(0)$ is given by

$$K = \left\{(v_1, v_2, v_3, v_4, 0)\right\}$$

and the range R of $DJ(0)$ as

$$R = \left\{(x, y) \; ; \; x = y\right\}$$

If we define new coordinates in the codomain by

$$X = x - y \quad \text{and} \quad Y = x + y$$

we can express

$$J : K \times K^{\perp} \to R^{\perp} \times R$$

by

$$J : (x_1, x_2, x_3, x_4, x_5) \to (x_1x_4 - x_2x_3 + \text{h.o.t.}, 2Cx_5 + x_1x_4 - x_2x_3 + \text{h.o.t.})$$

If we denote $J\big|_K \to R^{\perp}$ by j then j is the essential "singular part" of J and we have :

$$j(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3 + \text{h.o.t.}$$

A quick calculation shows that the Hessian of j at 0 is non singular and hence by the Morse Lemma there exists a non-linear change of coordinates which will express j itself as a quadratic form.

This ties in with the result given on page 22 of Arms, Marsden and Moncrief [2] where they show that in fact there exists a <u>sympletic</u> change of coordinates which will perform the above operation.

Note from the proof of Lemma 2.10, that the map germ $E \times J$ itself has finite $\Gamma - \mathcal{K}$ codimension and a versal $\Gamma - \mathcal{K} - 2$ deformation can easily be written down by adding a multiple of $x_1 x_3 + x_2 x_4$ to E and a multiple of $x_1^2 + x_2^2$ to J_1. Whether such a deformation could have physical significance is not clear.

<u>The zero set of $E \times J$.</u>

As a consequence of Theorem 2.12 $E \times J$ has a zero set equivalent to that of the polynomial mapping (p_1, p_2, p_3) given there, namely

$$Z = \left\{(x_1, x_2, x_3, x_4, x_5);\ x_5 = 0,\ x_1x_4 - x_2x_3 = 0\ ,\ x_3^2 + x_4^2 - x_1^2 - x_2^2 = 0\right\}$$

As $x_5 = 0$, we can regard Z as a subset of R^4. Note that if x belongs to Z then all scalar multiples of x also belong to Z, and hence Z is a cone. To determine on what it is a cone we look at the intersection of Z with the unit sphere $S^3 \subset R^4$.

<u>Lemma 2.13</u>

The intersection of Z with the unit sphere S^3 is the set D given by:

$$D = \left\{ (x_1, x_2, x_3, x_4) \; ; \; x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2} \, , \; \frac{x_1}{x_2} = \frac{x_3}{x_4} \right\} .$$

The proof of this is clear. ■

The equations $x_1^2 + x_2^2 = \frac{1}{2}$ and $x_3^2 + x_4^2 = \frac{1}{2}$ define a torus in S^3 and the third equation $\frac{x_1}{x_2} = \frac{x_3}{x_4}$ determines two diagonals. In order to see this clearly take polar coordinates (r_1, θ_1) in the (x_1, x_2) plane and (r_2, θ_2) in the (x_3, x_4) plane. Then $\frac{x_1}{x_2} = \frac{x_3}{x_4}$ corresponds to $\theta_1 = \theta_2$ or $\theta_1 = \theta_2 + \pi$.

Thus the intersection of Z with the unit sphere S^3 is the pair of diagonals D shown in Figure 3.

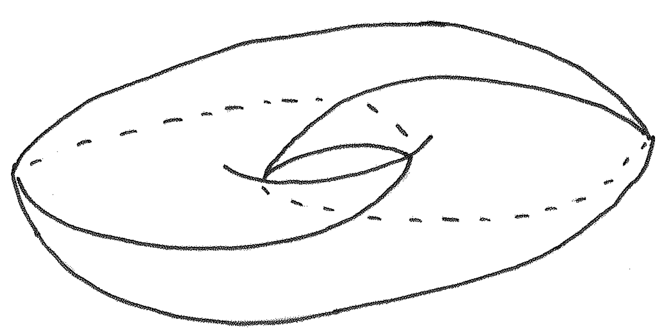


Figure 3.

Hence:

Corollary 2.14

The zero set Z of $E \times J$ is locally equivalent at $0 \in R^5$ to the cone on the pair of diagonals D. ■

By locally equivalent we mean that the conditions of Definition 1.4(b) hold on a neighbourhood of 0 rather than on the whole of R^n.

CHAPTER 3

STEADY PRECESSION OF THE TOP

In the tangent bundle to the configuration space we can change coordinates diffeomorphically from $(\dot\theta, \dot\phi, \dot\psi)$ to $(\dot\theta, \dot\phi, s)$ where $s = \dot\phi \cos\theta + \dot\psi$ as given above. As $J_2 = Cs$ is a multiple of one of the coordinates in the domain we will restrict attention to $E \times J_1$. In the configuration space we reduce the domain of $E \times J_1$ by factoring out by the symmetry group action and so regard $E \times J_1$ as being defined on the reduced phase space $\tilde{TM}$, hence as being a map from $(0, \pi) \times R^3$ to R^2.

For steady precession to take place both $\dot\theta$ and $\ddot\theta$ must be zero, so we look at $E \times J_1$ restricted to $\dot\theta = 0$. We will call this map Φ. It is defined by :

$$\Phi(\theta; \dot\phi, s) = (\tfrac{1}{2}\dot\phi^2 \sin^2\theta + \tfrac{1}{2} Cs^2 + \cos\theta, \dot\phi \sin^2\theta + Cs \cos\theta)$$

Of course as both $\ddot\theta$ and $\dot\theta$ are zero we have a constant value of θ. We will denote by Φ_{θ_0} the map Φ restricted to $\theta = \theta_0$, a constant. By considering the rate of growth of angular momentum about an axis perpendicular to the symmetry axis we find that

$$\ddot\theta = 0 \iff (Cs - \dot\phi \cos\theta)\dot\phi = 1 \qquad (3.1)$$

See, for instance, page 56 of [9].

This last equation defines a surface, <u>the steady precession surface K^2</u>, in the space $\tilde{TM}\big|_{\dot\theta=0}$

Considering Φ_{θ_0} as a map from R^2 to itself equation 3.1 gives a curve in $(\dot\phi, s)$ space, in fact a hyperbola as shown in Figure 4.

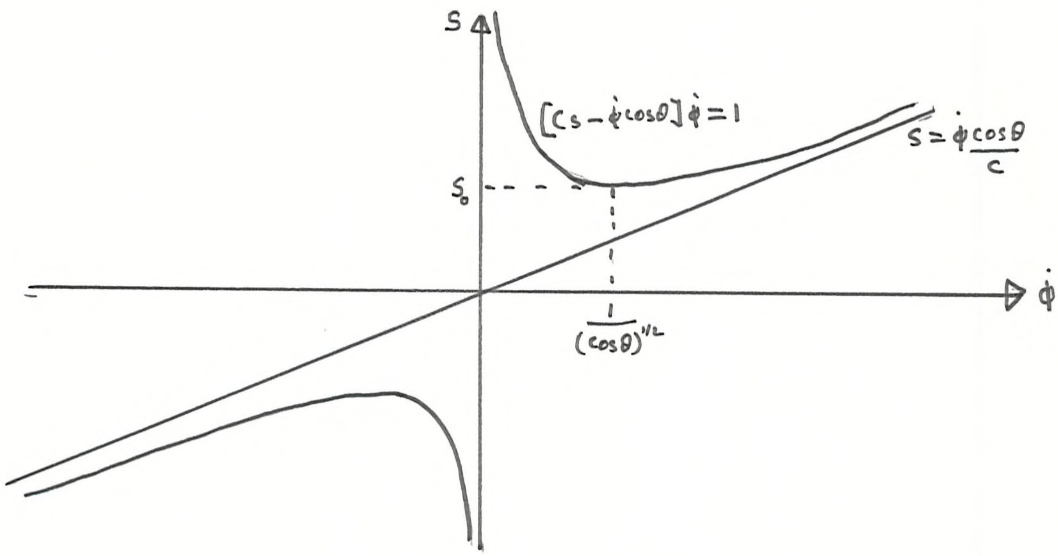


Figure 4

Clearly then, in order for steady precession to take place we must have $s^2 \geqslant \frac{4\cos\theta}{C^2}$. We will denote by s_o the minimum value of s that is $s_o = \frac{2}{C}(\cos\theta)^{\frac{1}{2}}$. For any given value of $s > s_o$ there are two different possible values of $\dot{\phi}$ given by :

$$\dot{\phi} = \frac{Cs}{2\cos\theta}\left\{1 \pm \left[1 - \left(\frac{s_o}{s}\right)^2\right]^{\frac{1}{2}}\right\}$$

called the slow and fast precession. In practice the fast precession is often unobtainable owing to the high energy requirement and of course when $\theta = \frac{\pi}{2}$ the fast precession speed becomes infinite.

The image under Φ_{θ_o} of this hyperbola is a curve in the (E, J_1) plane parametrised by $\dot{\phi}$ as follows :

$$E = \frac{1}{2}\dot{\phi}^2(\sin^2\theta + \frac{1}{C}\cos^2\theta) + (\frac{1}{C} + 1)\cos\theta + \frac{1}{2C}\cdot\frac{1}{\dot{\phi}^2}$$

$$J_1 = \frac{\cos\theta}{\dot{\phi}} + \dot{\phi}$$

A rough sketch of this curve is given in Figure 5 below.

We now look at the map Φ_{θ_o} in general. The first thing to notice is that not all points in the (E, J_1) plane are in the image of Φ_{θ_o} . More precisely :

Proposition 3.1

The set $\Phi_{\theta_o}^{-1}(E, J_1)$ is empty, consists of two distinct points or of one point precisely whenever E is respectively less than, greater than or equal to $\lambda J_1^2 + \cos\theta$,

where
$$\lambda = \frac{1}{2\left[\sin^2\theta + C\cos^2\theta\right]} .$$

Proof

Substituting $\dot{\phi} = \dfrac{J_1 - Cs\cos\theta}{\sin^2\theta}$ in the expression for E given at the beginning of this chapter gives a quadratic in s :-

$$C\, s^2\left[\sin^2\theta + C\cos^2\theta\right] - 2CJ_1 \cos\theta\, s + 2\left[\cos\theta - E\right]\sin^2\theta = 0$$

Thus there can be at most two distinct values of s , hence at most two points $(\dot{\phi}, s)$ in $\Phi_{\theta_o}^{-1}(E, J_1)$.

In order to get real roots for s we need

$$E - \cos\theta \geq \lambda J_1^2$$

and so for $\Phi_{\theta_o}^{-1}(E, J_1)$ to be empty, that is for there to be no real solutions for s , we have the first condition in the proposition. Repeated roots will occur when equality takes place, hence the third condition and the proposition is proved. ■

If we look at the domain of Φ_{θ_o} we find that

$$\Phi_{\theta_o}^{-1}(P) = \left\{(\dot{\phi},\ s);\ s = \dot{\phi}\cos\theta\right\}$$

where

$$P = \left\{(E,\ J_1);\ E = \lambda J_1^2 + \cos\theta\right\}$$

so Φ_{θ_o} looks like a fold along $s = \dot{\phi}\cos\theta$ but in order to show that it is a fold we need to look at the singularity sets of Φ_{θ_o} . This we do in the next section where we prove that we do have a line of fold points on $s = \dot{\phi}\cos\theta$.

So we can visualise Φ_{θ_o} by Figure 5.

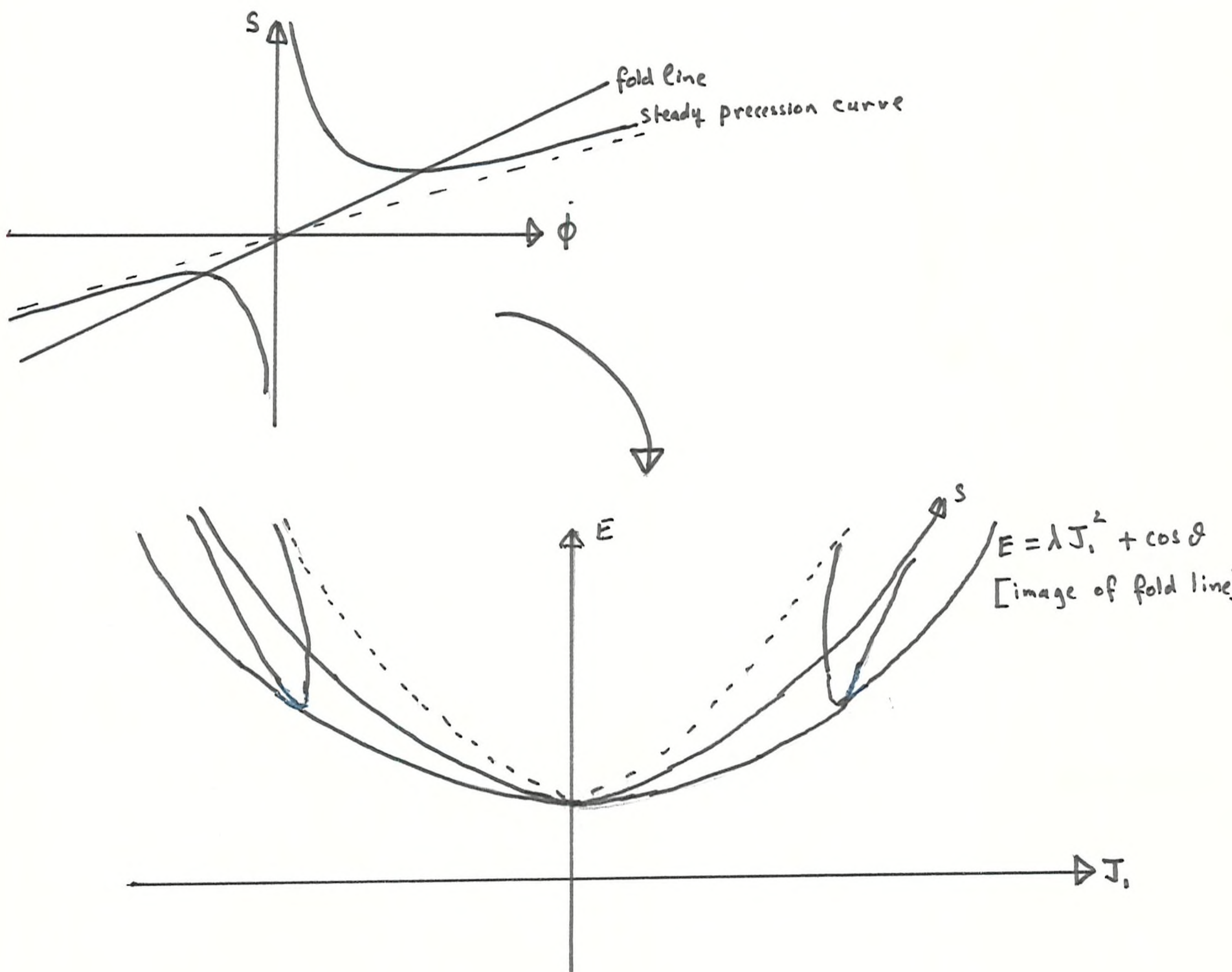


Figure 5

Singularities of Φ and Φ_{θ_o}

Proposition 3.2

For Φ_{θ_o} considered as a mapping from R^2 to R^2 the singularity set consists entirely of fold points.

Proof

$$\text{As}\quad \Phi_{\theta_o}(\dot\phi, s) = \left(\frac{1}{2}\dot\phi^2 \sin^2\theta + \frac{1}{2} Cs^2 + \cos\theta,\ \dot\phi \sin^2\theta + Cs\cos\theta\right)$$

we have that

$$D\Phi_{\theta_o}(\dot\phi, s) = \begin{pmatrix} \dot\phi \sin^2\theta & Cs \\ \sin^2\theta & C\cos\theta \end{pmatrix}$$

and as we are assuming that $\theta \neq 0$ we conclude that :

$$S_1(\Phi_{\theta_o}) = \left\{ (\dot\phi, s)\ ;\ s = \dot\phi \cos\theta \right\}$$

Referring to Theorem 1.7 we see that to prove that the singularity set consists entirely of fold points we need to show that Φ_{θ_o} is 1-generic and that the kernel of $D\Phi_{\theta_o}(\dot\phi, s)$ together with the tangent space to $S_1(\Phi_{\theta_o})$ span the tangent space of R^2 at $(\dot\phi, s)$.

Firstly 1 genericity.We will use Lemma 4.3 on page 52 of [6] which gives a necessary and sufficient condition for transversality of a mapping at a point to a submanifold. The condition is to choose a submersion into R^k (k is the codimension of the submanifold) such that the submanifold is the inverse image of 0 and then check that the composition of the mapping with the submersion is itself a submersion at the point in question.

So we define $P : J^1(R^2, R^2) \to R$, a submersion such that

$$P \circ j^1 \Phi_{\theta_o}(\dot\phi, s) = s - \dot\phi \cos\theta$$

Thus as

$$D(P \circ j^1 \Phi_{\theta_o})(\dot{\phi}, s) = (\cos\theta \quad 1)$$

is surjective at the origin we conclude that $j^1 \Phi_{\theta_o}$ is transverse to S_1

Secondly the condition on the tangent spaces and the kernel of the derivative. The kernel of $D\Phi_{\theta_o}$ at a point $(\dot{\phi}, s)$ in $S_1(\Phi_{\theta_o})$ is given by

$$\mathrm{Ker} D\Phi_{\theta_o}(\dot{\phi}, s) = \left\{(u, v);\ \frac{v}{u} = \frac{-\sin^2\theta}{C\cos\theta}\right\}$$

and $T_x S_1(\Phi_{\theta_o})$ is generated by the vector $(1, \cos\theta)$ and as this vector is never parallel to a vector in the kernel of the derivative, they span $T_x R^2$. $\blacksquare$

So we have shown that Φ_{θ_o} from the $(\dot{\phi}, s)$ plane to the (E, J_1) plane is indeed just a fold, but what happens if we look at the full map $\Phi : \widetilde{TM} \to R^2$?

Regarding Φ as given by :-

$$\Phi(\theta;\ \dot{\phi}, s) = \left(\frac{1}{2}\dot{\phi}^2 \sin^2\theta + \frac{1}{2} Cs^2 + \cos\theta\ ,\ \dot{\phi}\sin^2\theta + Cs\cos\theta\right)$$

where $\theta \neq 0$, then

$$D\Phi(\theta;\ \dot{\phi}, s) = \begin{pmatrix} (\dot{\phi}^2\cos\theta - 1)\sin\theta & \dot{\phi}\sin^2\theta & Cs \\ (2\dot{\phi}\cos\theta - Cs)\sin\theta & \sin^2\theta & C\cos\theta \end{pmatrix}$$

and

$$S_1(\Phi) = \left\{(\theta;\ \dot{\phi}, s);\ s = \dot{\phi}\cos\theta \text{ and } \dot{\phi}^2\cos\theta = \frac{1}{C-1}\right\}$$

Clearly $S_1(\Phi)$ is a subset of K^2.

Proposition 3.3

The 1-jet of Φ is transverse to S_1 .

Proof

Again we use Lemma 4.3 on page 52 of [6].

Let $P : J^1(S^1 \times R^2, R^2) \to R^2$ be a submersion chosen so that

$$P \circ j^1 \Phi : (\theta; \dot\phi, s) \mapsto \left(s - \dot\phi \cos\theta, \dot\phi^2 \cos\theta - \frac{1}{C-1}\right)$$

Then

$$D(P \circ j^1 \Phi)(\theta; \dot\phi, s) = \begin{pmatrix} \dot\phi \sin\theta & -\cos\theta & 1 \\ -\dot\phi^2 \sin\theta & 2\dot\phi \cos\theta & 0 \end{pmatrix}$$

which is surjective unless $\dot\phi = 0$ which cannot happen on $S_1(\Phi)$

So $j^1 \Phi$ is transverse to S_1 ■

Proposition 3.4

The set $S_{11}(\Phi)$ consists of just one point

Proof

Restricting Φ to $S_1(\Phi)$ and using θ as a parameter we get

$$\Phi\Big|_{S_1(\Phi)}(\theta) = \left(\frac{1 + 3[C-1]\cos^2\theta}{2[C-1]\cos\theta}, \frac{1 + [C-1]\cos^2\theta}{([C-1]\cos\theta)^{\frac{1}{2}}}\right)$$

So

$$D\Phi\Big|_{S_1(\Phi)}(\theta) = \left(\frac{\sin\theta}{2[C-1]\cos^2\theta} - \frac{3\sin\theta}{2}\quad \frac{\sin\theta}{2[(C-1)\cos\theta]^{\frac{1}{2}}\cos\theta} - \frac{3([C-1]\cos\theta)^{\frac{1}{2}}\sin\theta}{2}\right)$$

and this has rank 1 except when $\cos\theta = \left(\frac{1}{3[C-1]}\right)^{\frac{1}{2}}$ (remembering that $\theta \neq 0$).

So $S_{11}(\Phi)$ consists of the point Ω given by

$$\Omega = (\theta;\ \dot{\phi},\ s)$$

where

$$\theta = \arccos\left(\frac{1}{3[C-1]}\right)^{\frac{1}{2}}$$

$$\dot{\phi} = \left(\frac{3}{C-1}\right)^{\frac{1}{4}}$$

$$s = \left(3[C-1]^3\right)^{-\frac{1}{4}} \qquad \blacksquare$$

So we have one non fold point Ω , provided that $C > \frac{4}{3}$.

If we now look at $\Phi\left(S_1(\Phi)\right)$ we note that $\Phi\left(S_1(\Phi)\right)$ and $\Phi_{\theta_o}\left(S_1(\Phi_{\theta_o})\right)$ are tangent at the point on $\Phi\left(S_1(\Phi)\right)$ where the parameter θ takes the value θ_o , as $D\Phi(\theta;\ \dot{\phi},\ s)$ drops rank by 1 on $S_1(\Phi)$ and hence the whole tangent plane spanned by the tangent vectors to $S_1(\Phi)$ and $S_1(\Phi_{\theta_o})$ is mapped to a line.

Recalling the remarks after Theorem 1.8 we have the <u>standard Whitney cusp map from R^3 to R^2</u> given there with the point (0, 0, 0) called the <u>cusp point</u>. We will now show that Ω is a cusp point.

Since in Proposition 3.3 we have already shown that $j^1\Phi$ is transverse to S_1 , in order to show that Ω is a standard cusp point all that is needed, from Theorem 1.8, is to prove

Proposition 3.5

The 2-jet of Φ is transverse to S_{11} .

Proof

Using the construction of S_{11} given in Chapter 1 we can see that $j^2\,\Phi(x)$ will belong to S_{11} if and only if the symmetric mapping $\delta^2\Phi_x$ has corank 1, and transversality can then be proved by a method similar to that used in Proposition 3.3. So first of all we construct $\delta^2\Phi_x$.

Now

$$D\,\Phi(s,\,\theta,\,\dot{\phi}) = \begin{pmatrix} Cs & (\dot{\phi}^2\cos\theta - 1)\sin\theta & \dot{\phi}\sin^2\theta \\ C\cos\theta & (2\dot{\phi}\cos\theta - Cs)\sin\theta & \sin^2\theta \end{pmatrix}$$

We have changed the order of the coordinates from that used in Proposition 3.3 for computational convenience.

Also

$$S_1(\Phi) = \left\{(s,\,\theta,\,\dot{\phi});\ s = \dot{\phi}\cos\theta \ \text{ and } \ \dot{\phi}^2\cos\theta = \frac{1}{C-1}\right\}.$$

On $S_1(\Phi)$ we have

$$K_x = C\cos\theta\ u + \frac{(2-C)\sin\theta(\cos\theta)^{\frac{1}{2}}}{(C-1)^{\frac{1}{2}}}v + \sin^2\theta\ w = 0$$

We trivialise $T_x\,R^3$ by choosing the new coordinate system :-

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} C\cos\theta & \dfrac{(2-C)\ \sin\theta(\cos\theta)^{\frac{1}{2}}}{(C-1)^{\frac{1}{2}}} & \sin^2\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad (3.2)$$

and trivialise $T_x\,R^2$ by :

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} 1 & -\dot{\phi} \\ \dot{\phi} & 1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \qquad (3.3)$$

In terms of these coordinates $D\,\Phi_x$ has the form $\begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \end{pmatrix}$ where :

$$\alpha = \frac{s}{\cos\theta} - \dot\phi$$

$$\beta = (-\dot\phi^2 \cos\theta + Cs\dot\phi - 1)\sin\theta + \frac{(C-2)\sin\theta}{\left[(C-1)\cos\theta\right]^{\frac{1}{2}}}(s - \dot\phi\cos\theta)$$

$$\gamma = \left(\dot\phi - \frac{s}{\cos\theta}\right)\sin^2\theta$$

$$\delta = \frac{\dot\phi s}{\cos\theta} + 1$$

$$\varepsilon = \dot\phi\left(\left[\dot\phi^2\cos\theta - 1\right]\sin\theta\right) + (2\dot\phi\cos\theta - Cs)\sin\theta + \frac{(C-2)\sin\theta}{\left[(C-1)\cos\theta\right]^{\frac{1}{2}}}(\dot\phi s + \cos\theta)$$

$$\zeta = \left(\dot\phi - \frac{s}{\cos\theta}\right)\dot\phi\sin^2\theta$$

and for $x \varepsilon S_1(\Phi)$

$$D\,\Phi_x = \begin{pmatrix} 0 & 0 & 0 \\ 1+\dot\phi^2 & 0 & 0 \end{pmatrix}$$

and $K_x = \{(u', v', w')\ ;\ u' = 0\}$,

$L_x = \{(U', V')\ ;\ V' = 0\}$.

We can now calculate $D(D\,\Phi)_x$, together with $(d\,D\,\Phi)_x$ and $\delta^2\,\Phi_x$ but first we must change coordinates in the base space R^3 exactly as we have already done in the fibre R^3 .

If we regard $D\,\Phi_x = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta & \varepsilon & \zeta \end{pmatrix}$

then

$$\delta^2\,\Phi_x = \begin{pmatrix} \dfrac{\partial\beta}{\partial\theta} - \dfrac{(2-C)\sin\theta}{C\left((C-1)\cos\theta\right)^{\frac{1}{2}}}\dfrac{\partial\beta}{\partial s} & \dfrac{\partial\beta}{\partial\dot\phi} - \dfrac{\sin^2\theta}{C\cos\theta}\dfrac{\partial\beta}{\partial s} \\[2ex] \dfrac{\partial\gamma}{\partial\theta} - \dfrac{(2-C)\sin\theta}{C\left((C-1)\cos\theta\right)^{\frac{1}{2}}}\dfrac{\partial\gamma}{\partial s} & \dfrac{\partial\gamma}{\partial\dot\phi} - \dfrac{\sin^2\theta}{C\cos\theta}\dfrac{\partial\gamma}{\partial s} \end{pmatrix}$$

$$= \sin^2\theta \begin{pmatrix} \dfrac{3C-4}{C\cos\theta} & \dfrac{2(1-C)\sin\theta}{C(C-1)^{\frac{1}{2}}(\cos\theta)^{\frac{3}{2}}} \\ \dfrac{2(1-C)\sin\theta}{C(C-1)^{\frac{1}{2}}(\cos\theta)^{\frac{3}{2}}} & 1 + \dfrac{\sin^2\theta}{C\cos^2\theta} \end{pmatrix}$$

Thus

$$\det(\delta^2 \Phi_x) = \left[\frac{3C-4}{C\cos\theta} - \frac{\sin^2\theta}{C\cos^3\theta}\right]\sin^4\theta$$

As $\sin\theta \neq 0$, i.e. $\theta = 0$ or π are excluded, we have

$$\det(\delta^2 \Phi_x) = 0 \iff \tan^2\theta = 3C - 4$$

$$\iff \theta = \arctan(3C-4)^{\frac{1}{2}} = \arccos\left(\frac{1}{3(C-1)}\right)^{\frac{1}{2}} .$$

Hence $j^2 \Phi(x) \in S_{11} \iff x \in S_{11}(\Phi)$.

As $\text{codim}(S_{11}) = 3$, we project down onto R^3 by π and check whether $\pi \circ j^2 \Phi_x$ is a submersion when $\theta = \arctan(3C-4)^{\frac{1}{2}}$. Now

$$\pi \circ j^2 \Phi : (s, \theta, \dot\phi) \mapsto \left(s - \dot\phi\cos\theta,\ \dot\phi^2\cos\theta - \frac{1}{C-1},\ \frac{3C-4}{C\cos\theta} - \frac{\sin^2\theta}{C\cos^3\theta}\right)$$

so

$$D(\pi \circ j^2 \Phi) = \begin{pmatrix} 1 & \dot\phi\sin\theta & -\cos\theta \\ 0 & -\dot\phi^2\sin\theta & 2\dot\phi\cos\theta \\ 0 & \dfrac{\sin\theta}{C\cos^2\theta}\left[(3C-4) - \dfrac{2}{C} - \dfrac{3\tan^2\theta}{C}\right] & 0 \end{pmatrix}$$

On $S_{11}(\Phi)$ we see

$$D(\pi \circ j^2 \Phi) = \begin{pmatrix} 1 & \dfrac{(3C-4)^{\frac{1}{2}}}{3^{\frac{1}{4}}(C-1)^{\frac{3}{4}}} & -\left(\dfrac{1}{3(C-1)}\right)^{\frac{1}{2}} \\ 0 & -\dfrac{(3C-4)^{\frac{1}{2}}}{C-1} & \dfrac{2}{3^{\frac{1}{4}}(C-1)^{\frac{3}{4}}} \\ 0 & \dfrac{[(3C-4)(C-3)-2]\left([3C-4][3(C-1)]\right)^{\frac{1}{2}}}{C^2} & 0 \end{pmatrix}$$

which is surjective.

Hence $j^2\ \Phi$ is transverse to S_{11} . ■

So we have shown that :-

Theorem 3.6

The point Ω is a standard Whitney cusp point. ■

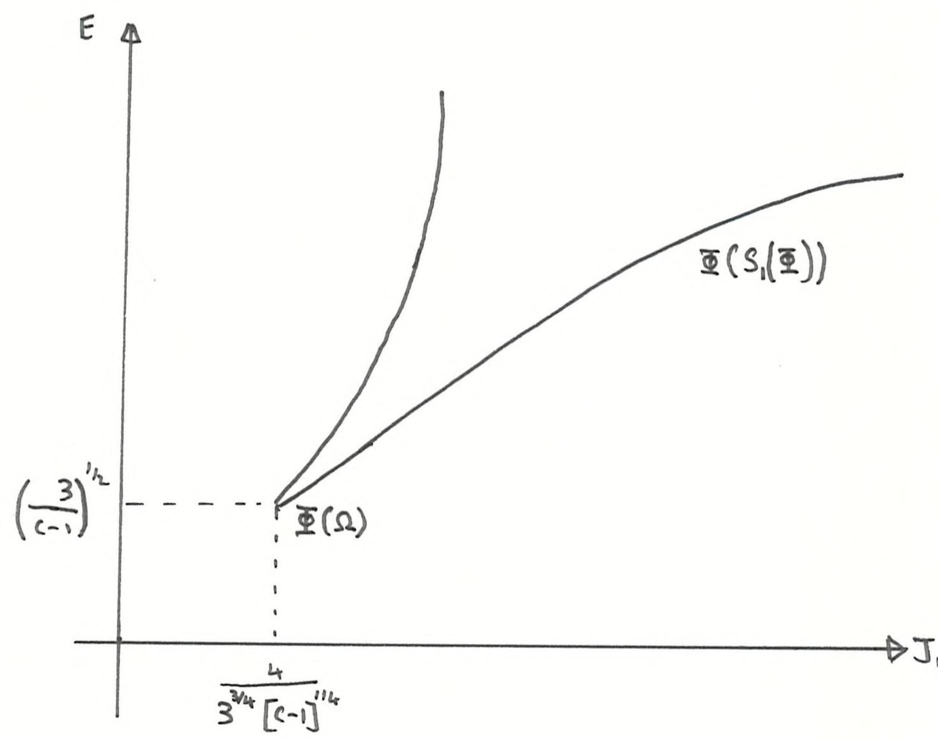


Figure 6

Structure of the Standard Whitney Cusp Map from R^3 to R^2

Since we have established that Φ in a neighbourhood of $S_{11}(\Phi)$, can be converted by a local diffeomorphism (local coordinate change) into the standard Whitney cusp map from R^3 to R^2, it is of interest to see exactly how that standard cusp map behaves.

The standard Whitney cusp map is given by :-

$$f : (x_1, x_2, x_3) \mapsto (x_1, x_1x_2 - x_2^3 - x_3^2)$$

and has singularity sets,

$$S_1(f) = \{(x_1, x_2, 0); x_1 = 3x_2^2\}$$

$$S_{11}(f) = \{(0, 0, 0)\}$$

The image of $S_1(f)$ under f is the cusp curve

$$\{(y_1, y_2); 4y_1^3 - 27y_2^2 = 0\}$$

and the image of $S_{11}(f)$ is the origin $\{(0, 0)\}$.

Restricted to the plane $x_3 = 0$ the map f behaves exactly like the standard $R^2 \to R^2$ cusp map so in this plane a second parabola viz

$$\left\{(x_1, x_2, 0) ; x_1 = \frac{3}{4} x_2^2\right\}$$

is also mapped onto the cusp curve.

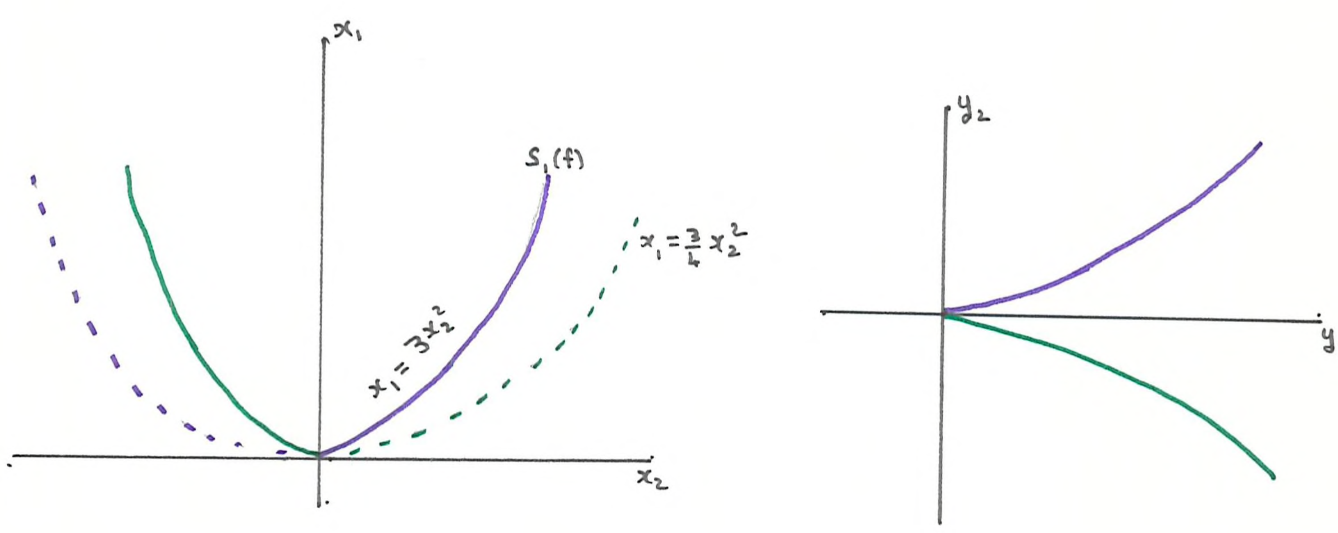


Figure 7.

If we look for $f^{-1}(0, 0)$, we get :

$$\{(x_1, x_2, x_3) ; x_1 = 0, x_2^3 = x_3^2\}$$

a cusp curve lying in the (x_2, x_3) plane. For the full inverse image of the cusp curve we have the equation :

$$4x_1^3 - 27(x_1x_2 - x_2^3 - x_3^2)^2 = 0 \qquad (3.4)$$

Taking a few x_2 = constant slices leads us to suspect that this is the equation of a swallowtail, see Figure 8.

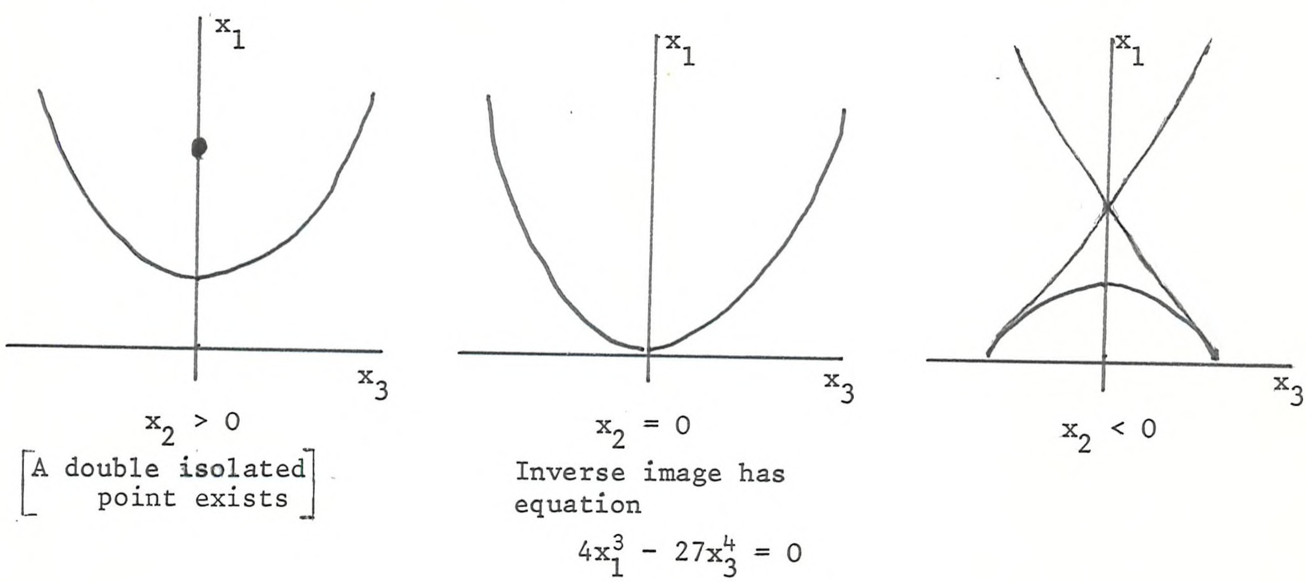


Figure 8.

So we prove:

Proposition 3.7

There is an origin preserving diffeomorphism from R^3 to itself which maps the surface given by Equation 3.4 to the canonical swallowtail surface.

Proof

The canonical swallowtail surface is defined as the set of points in (a, b, c) space for which the quartic equation

$$x^4 + ax^2 + bx + c = 0$$

has repeated roots. This is given algebraically by the vanishing of the discriminant, see for example page 120 of [15]

$$\Delta = S^3 - 27\, T^2$$

where $S = a^2/12 + c$

$$T = \frac{ac}{6} - \frac{b^2}{16} - \frac{a^3}{216}$$

We now define the diffeomorphism by:-

$$x_1 = 4^{-1/3}\left(\frac{a^2}{12} + c\right)$$

$$x_2 = 4^{1/3}\frac{a}{6}$$

$$x_3 = \frac{b}{4}$$

This map has non-vanishing Jacobian and as Δ corresponds to Equation 3.4 the proposition is proved. ■

We can now draw the swallowtail surface and see which parts of that surface map to which parts of the cusp curve . (See Figure 9)

If we look at the inverse image of points on the cusp curve we get the pictures shown in Figure 10.

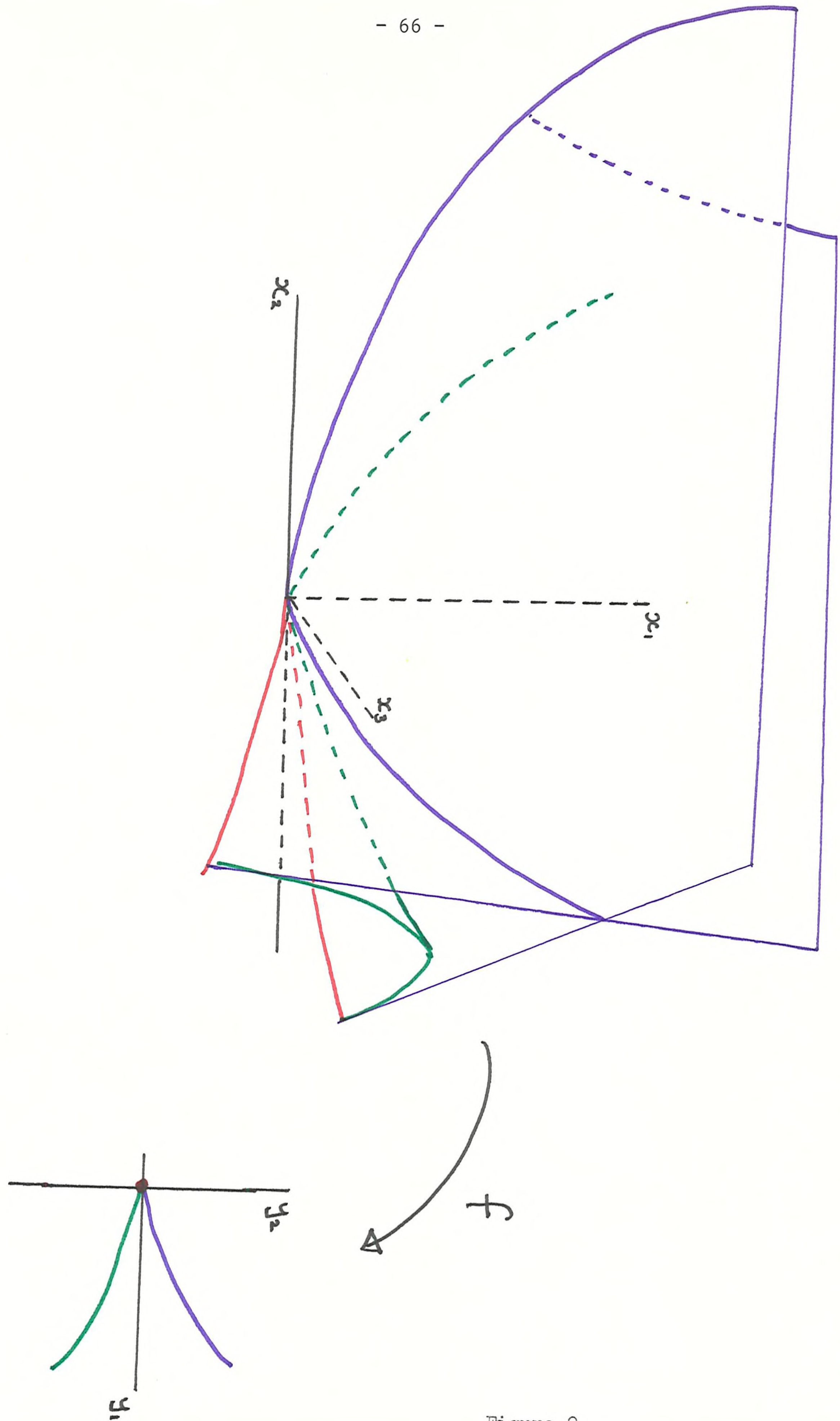


Figure 9.

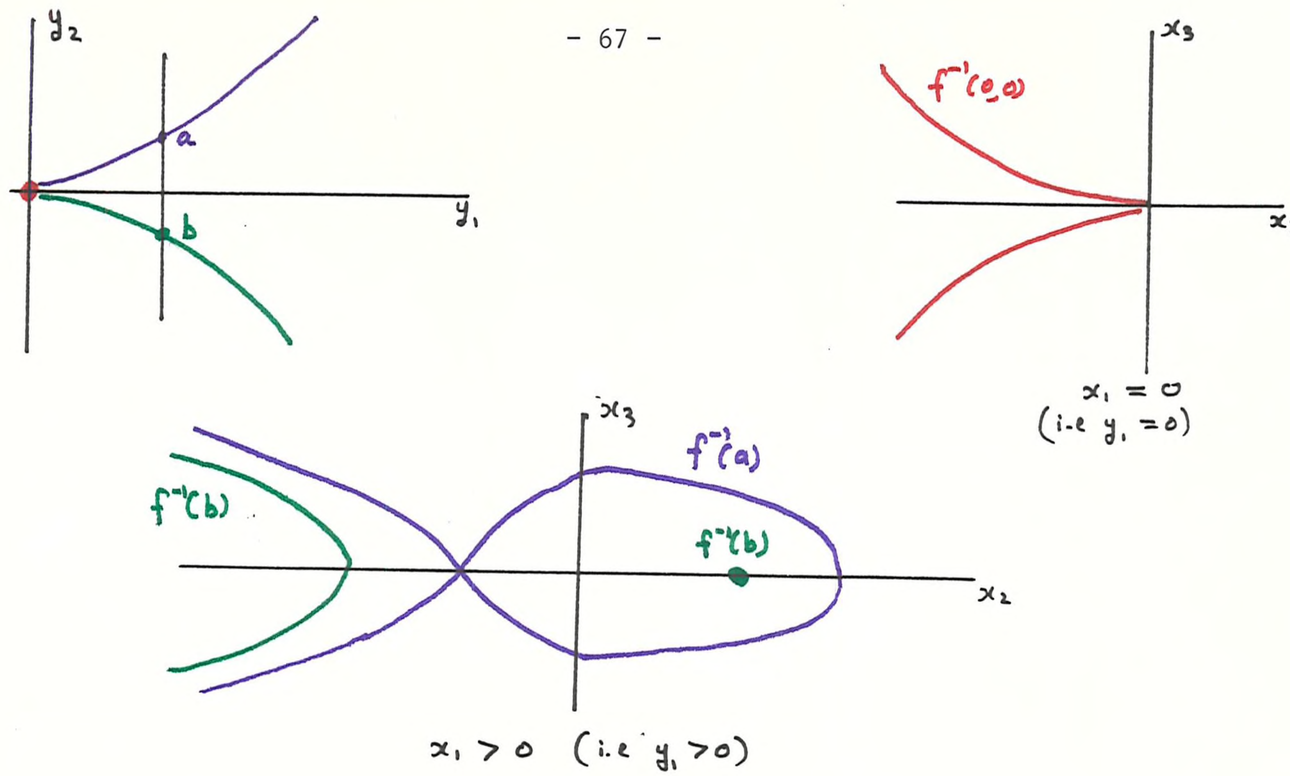


Figure 10.

So now we turn our attention to the inverse images of points inside and outside the cusp curve. As seen above, the inverse image of the line $y_1 = k$ is the plane $x_1 = k$ and the inverse images of points on the line $y_1 = k$ are curves in the plane of the form illustrated in Figure 11.

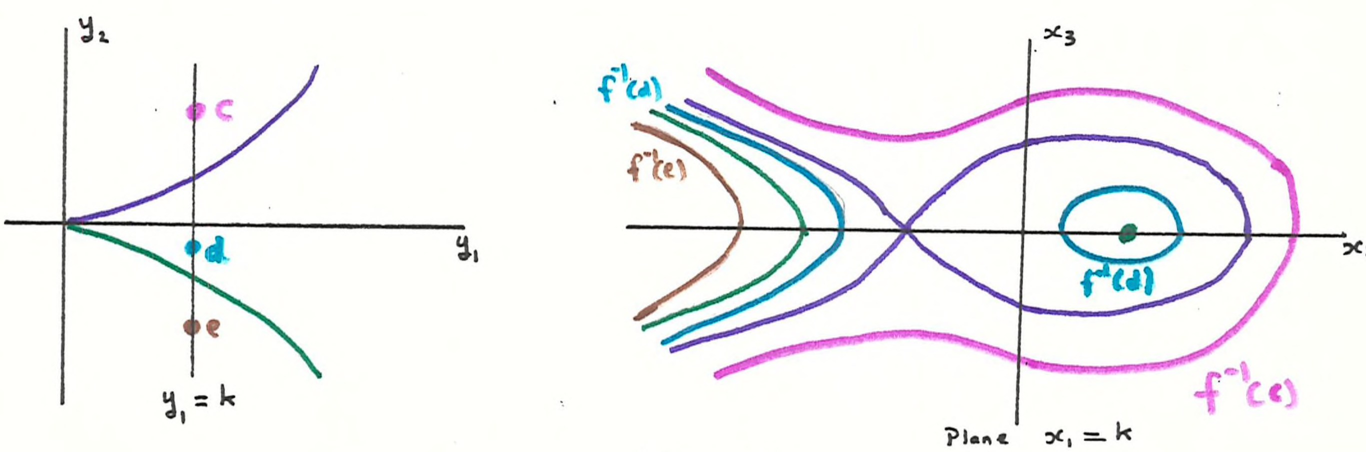


Figure 11.

So points outside the cusp have one curve as their inverse image while points inside the cusp have two curves as their inverse images, one a closed curve round the whisker, the other an open curve in the "tail portion" of the space enclosed by the swallowtail surface.

We have analysed $E \times J_1$ restricted to the subspace given by $\dot{\theta} = 0$ as it is on this subspace that steady precession takes place and therefore we would expect to see all the most important features of the map displayed there. We now examine $E \times J_1$ defined on the whole of the reduced phase space which we can do without having to do a great deal of extra work as all the essentials have been worked out.

We start by showing that $E \times J_1$ as a mapping from R^4 to R^2 is also a standard Whitney cusp map. Again returning to Theorem 1.8 put $p = 2$ and this time $n = 4$. The map given is the standard Whitney cusp map from R^4 to R^2. Then

Corollary 3.8

$E \times J_1 : \tilde{TM} \to R^2$ can be converted by a local coordinate change around Ω to the standard Whitney cusp map given by :-

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_1x_2 - x_2^3 - x_3^2 \pm x_4^2)$$

Proof

If we regard Φ as given as :

$$\Phi = (\Phi_1, \Phi_2)$$

then

$$E \times J_1 = \left(\frac{1}{2}\dot{\theta}^2 + \Phi_1, \Phi_2\right)$$

We have to prove that $j^1(E \times J)$ is transverse to S_1 and that $j^2(E \times J)$ is transverse to S_{11}. The proof of these results follows those of Propositions 3.3 and 3.5. For instance in the case of Proposition 3.5, letting F stand for $E \times J_1$, then

$$D\,F_x = \begin{pmatrix} D\,\Phi_x & \vdots & \dot\theta \\ & \vdots & 0 \end{pmatrix}$$

and $S_1(F) = \left\{(\theta, \dot\theta, \dot\phi, s);\ \dot\theta = 0 \text{ and the conditions for } S_1(\Phi) \text{ hold}\right\}$.

Hence K_x is the same as before and to trivialise $T_x\, R^4$ we use the matrix :

$$\begin{pmatrix} C\cos\theta & \dfrac{(2-C)\sin\theta(\cos\theta)^{\frac{1}{2}}}{(C-1)^{\frac{1}{2}}} & \sin^2\theta & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

After the change of coordinates in R^4 and R^2 we can write

$$D\,F_x = \begin{pmatrix} D\Phi_x & \vdots & \dot\theta \\ & \vdots & \dot\phi\dot\theta \end{pmatrix}$$

where $D\Phi_x$ is now in the form given after the coordinate changes in Proposition 3.6. The rest of the proof follows exactly as before. ■

From the proof of that corollary we can see that :

Corollary 3.9

$$S_1(E \times J_1) = S_1(\Phi)\ .$$ ■

If we now restrict Φ to the steady precession surface K^2 and ask about the singularities of $\Phi|_{K^2}$ we can say :

Proposition 3.10

$$S_1(\Phi|_{K^2}) = S_1(\Phi)$$

Proof

Restricting to K^2 we can express Φ by :

$$\Phi|_{K^2} : (\theta, \dot\phi) \mapsto \left(\frac{1}{2}\dot\phi^2 \sin^2\theta + \frac{1}{2C}\left(\dot\phi\cos\theta + \frac{1}{\dot\phi}\right)^2 + \cos\theta\, ,\ \dot\phi + \frac{\cos\theta}{\dot\phi} \right)$$

Then

$$\det\left[D\left(\Phi\Big|_{K^2}\right)\right] = (C-1)\cos\theta\left(\dot{\phi}^2 - \frac{1}{(C-1)\cos\theta}\right)(\dot{\phi}^4 - 2\cos\theta\,\dot{\phi}^2 + 1)\ .$$

which equals zero when $\dot{\phi}^2 = \dfrac{1}{(C-1)\cos\theta}$ which is the equation defining $S_1(\Phi)$. ■

So the singularity set for $E \times J_1$ defined on the phase space is the same as that for $E \times J_1$ restricted to the steady precession surface.

We would like to find the image of the steady precession surface K^2 under the change of coordinates that converts Φ to the standard Whitney cusp map so that we could see where it lies in the swallowtail but this would be extremely difficult to do. So we will at least investigate what happens to the tangent space to K^2 at the point Ω under the change of coordinates.

<u>Proposition 3.11</u>

Under the local diffeomorphism that gives the change of coordinates from $(s, \theta, \dot{\phi})$ to (x_1, x_2, x_3) converting Φ into the standard Whitney cusp map, the tangent space to K^2 becomes the plane Π given by :-

$$\Pi = \left\{(x_1, x_2, x_3);\ \lambda x_1 + \mu x_3 = 0\right\}$$

where

$$\lambda = \frac{(3\,[C-1])^{\frac{3}{4}}}{3^{\frac{1}{2}} + [C-1]^{\frac{1}{2}}}$$

and

$$\mu = (3)^{\frac{1}{2}}\,(3C-4) + 4[C-1]^{\frac{1}{2}}\ .$$

Proof

The steady precession surface is given by

$$K^2 = \left\{ (s,\ \theta,\ \dot{\phi})\ ;\ (Cs - \dot{\phi}\cos\theta)\dot{\phi} = 1 \right\}$$

Hence

$$T_\Omega K^2 = \left\{ \left[\frac{3^{\frac{1}{4}}C}{(C-1)^{\frac{1}{4}}},\ \frac{(3C-4)^{\frac{1}{2}}}{C-1},\ \frac{C-2}{3^{\frac{1}{4}}(C-1)^{\frac{3}{4}}} \right]^{\perp} \right\}$$

The derivative of the coordinate change diffeomorphism is given in two parts, the first by Equations 3.2 and 3.3 in the proof of Proposition 3.5, that is the change of coordinates in R^3 and R^2 used to simplify K_x and L_x respectively. Φ can then be expressed in the form

$$\Phi : (u',\ v',\ w') \to \left(u' + \text{ higher order terms},\ \ \phi_1(u',\ v',\ w') \right)$$

where $\phi_1(u', v', w')$ is of degree greater than or equal to 2.

The second part of the derivative of the coordinate change diffeomorphism is defined in Morin [14] as the change to a "system of quadratically adapted coordinates", by taking the quadratic form given by the terms of degree 2 in $\phi_1(0, v', w')$ and diagonalising it.

In this case the quadratic form is given by :

$$\begin{pmatrix} \dfrac{3C-4}{C\cos\theta} & \dfrac{2(1-C)\sin\theta}{C\left[(C-1)\cos\theta\right]^{\frac{1}{2}}\cos\theta} \\ \dfrac{2(1-C)\sin\theta}{C\left[(C-1)\cos\theta\right]^{\frac{1}{2}}\cos\theta} & 1 + \dfrac{\sin^2\theta}{C\cos^2\theta} \end{pmatrix}$$

and the diagonalising change of coordinates by

$$\begin{pmatrix} v'' \\ w'' \end{pmatrix} = \frac{1}{3^{\frac{1}{2}}(3C-4) + 4(C-1)^{\frac{1}{2}}} \begin{pmatrix} 2(C-1)^{\frac{1}{4}} & -3^{\frac{1}{4}}(3C-4)^{\frac{1}{2}} \\ 3^{\frac{1}{4}}(3C-4)^{\frac{1}{2}} & 2(C-1)^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} v' \\ w' \end{pmatrix}$$

Applying these changes of coordinates to $T_\Omega K^2$ gives the result. $\blacksquare$

The plane Π contains the x_2-axis that is the line which is tangent to $S_1(f)$ at 0 and corresponds to the line that is tangent to $S_1(\Phi)$ at Ω under the change of coordinates. This is as we would expect because $S_1(\Phi)$ is contained in K^2.

CHAPTER 4

THE AMENDED POTENTIAL

We will now carry out that part of the Smale programme outlined in Chapter 0 consisting of calculating and analysing the amended potential. In this Chapter we will use the cotangent bundle formulation as described in [1], in order to use the Hamiltonian.

The configuration space M is $SO(3)$, which we will examine away from the region where $\theta = 0$ or π, and the symmetry group action Ψ is given by

$$\Psi : (\gamma_1, \gamma_2; \theta, \phi, \psi) \mapsto (\theta, \phi + \gamma_1, \psi + \gamma_2)$$

while the induced action of the symmetry group T^2 on the cotangent bundle is :-

$$\Psi^{T^*} : (\gamma_1, \gamma_2 ; \theta, \phi, \psi, p_\theta, p_\phi, p_\psi) \mapsto (\theta, \phi + \gamma_1, \psi + \gamma_2, p_\theta, p_\phi, p_\psi)$$

The potential energy $V : M \to R$ and the kinetic energy $K : T^*M \to R$ defined by :

$$V(\theta, \phi, \psi) = \cos\theta$$

$$K(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi) = \frac{1}{2} p_\theta^2 + \frac{1}{2C} p_\psi^2 + \frac{1}{2} \frac{(p_\phi - p_\psi \cos\theta)^2}{\sin^2\theta}$$

are both clearly invariant under this induced action.

The momentum mapping $J : T^* M \to R^2$ calculated in Chapter 1 is also invariant under Ψ^{T^*}.

Following the calculations referred to in Chapter 1 and detailed in [1] Section 4.5 we look for the set Λ where $J_x : T^*_x M \to R^2$ is not surjective. A quick calculation shows that Λ consists of the subset defined by putting $\theta = 0$ or π .

Proposition 4.1

The amended potential V_μ is defined on the whole of $M - \Lambda$ and is given by :-

$$V_\mu(\theta, \phi, \psi) = \cos\theta + \frac{\mu_2^2}{2C} + \frac{(\mu_1 - \mu_2 \cos\theta)^2}{2 \sin^2\theta}$$

where $\mu = (\mu_1, \mu_2) \in R^2$.

Proof

For a point $x \in M$, $V_\mu(x)$ is defined by

$$V_\mu(x) = V(x) + K\big(\alpha_\mu(x)\big)$$

where $\alpha_\mu(x)$ is a 1-form in $J_x^{-1}(\mu)$ satisfying the condition that $K\big(\alpha_\mu(x)\big) = \inf_{\alpha \in J_x^{-1}(\mu)} K(\alpha)$.

Now letting $\mu = (\mu_1, \mu_2)$ we have that

$$J_x^{-1}(\mu) = \left\{(\theta, \phi, \psi, p_\theta, \mu_1, \mu_2) \; ; \; p_\theta \in R \right\}.$$

So for $\alpha \in J_x^{-1}(\mu)$ we can calculate

$$K(\alpha) = \frac{1}{2} p_\theta^2 + \frac{1}{2C} \mu_2^2 + \frac{1}{2} \frac{(\mu_1 - \mu_2 \cos\theta)^2}{\sin^2\theta}$$

and clearly the infimum of $K(\alpha)$ is obtained by letting p_θ be zero.

So $$\alpha_\mu(x) = (\theta, \phi, \psi, 0, \mu_1, \mu_2)$$

and hence

$$V_\mu(\theta, \phi, \psi) = \cos\theta + \frac{\mu_2^2}{2C} + \frac{(\mu_1 - \mu_2 \cos\theta)^2}{2\sin^2\theta} \qquad \blacksquare$$

We can now factor out by the symmetry group T^2 and let $z = \cos\theta$ to express the reduced amended potential, $\tilde{V}_\mu : (-1, 1) \to R$ as

$$\tilde{V}_\mu(z) = z + \frac{\mu_2^2}{2C} + \frac{(\mu_1 - \mu_2 z)^2}{2(1 - z^2)}$$

This has graph as illustrated in Figure 12.

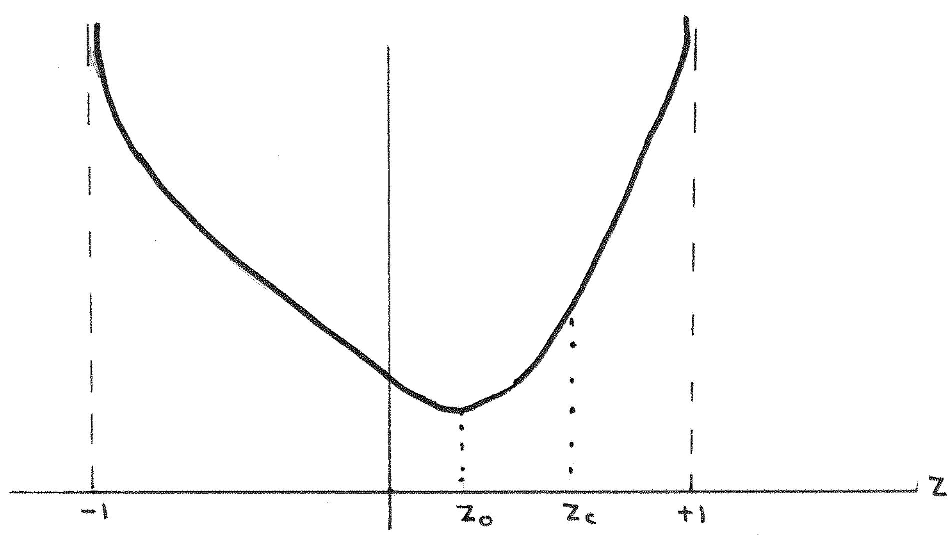


Figure 12.

(In general we would have to factor out by the isotropy subgroup of the co-adjoint action of the symmetry group to calculate $\tilde{V}_\mu$ but in this case, where the group is abelian, this reduces to factoring out by the group itself).

The proof that it is impossible for $\tilde{V}_\mu$ to have more than one minimum and hence that the graph looks like that drawn above will be given below.

We can now calculate the condition for steady precession to take place. As this motion is a relative equilibrium we know from Theorem 1.2 that it will occur at the critical point or points of $\tilde{V}_\mu$.

<u>Proposition 4.2</u>

The motion of steady precession will take place if and only if the condition

$$(p_\psi - p_\phi \cos\theta)\ (p_\phi - p_\psi \cos\theta) = \sin^4\theta$$

holds.

<u>Proof</u>

We find the critical points of $\tilde{V}_\mu$ by setting $\frac{d\tilde{V}_\mu}{dz} = 0$.

Now $$\frac{d\tilde{V}_\mu}{dz} = 1 + \frac{(\mu_1 - \mu_2\, z)(\mu_1 z - \mu_2)}{(1 - z^2)^2}$$

which equals zero precisely when

$$z^4 - (\mu_1\mu_2 + 2)z^2 + (\mu_1^2 + \mu_2^2)z - \mu_1\mu_2 + 1 = 0$$

Replacing μ_1 by p_ϕ and μ_2 by p_ψ gives the condition. $\blacksquare$

On the graph we have denoted the critical point by z_o .

The condition of proposition 4.2 defines a sub-manifold in the cotangent bundle which is the analogue of the steady precession surface K^2 in the reduced tangent bundle as defined in Chapter 3. We will call this manifold the dual steady precession manifold and denote it by K^* . K^* has codimension one and is thus not a surface, but we can regard K^* as the cartesian product of a surface K_2^* with $R \times S^1 \times S^1$ where this last space has coordinate chart (p_θ, ϕ, ψ) . The surface K_2^* will be called the <u>dual steady precession surface.</u>

<u>The General Motion of the Top.</u>

In general the top is moving with a fixed value of the energy E which is made up of two components namely $E = \tilde{V}_\mu + \frac{1}{2} p_\theta^2$. When p_θ is zero E is equal to $\tilde{V}_\mu$ and assuming that $\tilde{V}_\mu$ is greater than its minimum value there are two possible values of z which correspond to this. The axis of the top must stay between the two values of θ, given by the two values of z, because if θ were to be outside of that range then $\tilde{V}_\mu$ would exceed E.

In addition to the critical point there is another interesting point on the graph of $\tilde{V}_\mu$. That is the point at which the rational expression $\dfrac{(\mu_1 - \mu_2 z)^2}{2(1 - z^2)}$ vanishes. We will call this point z_c, so $z_c = \mu_1/\mu_2$. We will now analyse what happens when E equals $\tilde{V}_\mu(z_c)$

<u>Lemma</u>

When $E = \tilde{V}_\mu(z_c)$ the value of θ given by $\cos\theta = z_c$ gives the highest point of the motion of the axis of the top, that is $z_c > z_o$.

<u>Proof</u>

We know that $\dfrac{d\tilde{V}_\mu}{dz}(z_c) = 1$ and $\dfrac{d\tilde{V}_\mu}{dz}(z_o) = 0$ so $z_c \neq z_o$.

Assume that $z_c < z_o$. Since $\dfrac{d\tilde{V}_\mu}{dz} \to -\infty$ as $z \to -1$ there must be a z in $(-1, z_o)$ with $\dfrac{d\tilde{V}_\mu}{dz} = 0$. However $\dfrac{d\tilde{V}_\mu}{dz} = 0$ at only one point in $(-1, 1)$ namely z_o. This is proved in corollary 4.11 below. (The proof is totally independent of this lemma!).

Hence $z_c > z_o$. ∎

<u>Proposition 4.3</u>

At $z = z_c$ we have the following :-

(1) $\dot{\theta} = \dot{\phi} = 0$

(2) $\ddot{\theta} \neq 0$, $\dddot{\theta} = 0$

(3) $\ddot{\phi} = 0$, $\dddot{\phi} \neq 0$.

Proof

(1) As $E = \tilde{V}_\mu(z_c)$, at $z = z_c$ we have $p_\theta = \dot{\theta} = 0$. Now we know from the construction of J carried out above in Chapter 1 that

$$p_\phi = p_\psi \cos\theta + \dot{\phi} \sin^2\theta$$

and we have let $(\mu_1, \mu_2) = (p_\phi, p_\psi)$. Furthermore at z_c we have $z_c = \mu_1/\mu_2$, that is $\cos\theta = p_\phi/p_\psi$. As $\theta \neq 0$ or π we know that $\sin\theta \neq 0$ hence $\dot{\phi} = 0$ at z_c.

(2) As $E = \frac{1}{2}\dot{\theta}^2 + \cos\theta + \dfrac{(\mu_1 - \mu_2 \cos\theta)^2}{2\sin^2\theta}$

on taking time derivatives, remembering that E, μ_1 and μ_2 are constant, we get

$$0 = \ddot{\theta} - \sin\theta + \frac{(\mu_1 - \mu_2 \cos\theta)\mu_2}{\sin\theta} - \frac{(\mu_1 - \mu_2 \cos\theta)^2 \cos\theta}{\sin^3\theta}.$$

(This result can also be obtained from the Euler equation : see [1] or [13].) z_c is given by $\mu_1 - \mu_2 \cos\theta = 0$ so this equation reduces to

$$\ddot{\theta} = \sin\theta \neq 0$$

However, if we differentiate the equation for $\ddot{\theta}$ again we get :-

$$0 = \dddot{\theta} - \cos\theta\, \dot{\theta} + \left[\mu_2^2 - \frac{3\mu_2\cos\theta(\mu_1-\mu_2\cos\theta)}{\sin^2\theta} + \frac{(\mu_1-\mu_2\cos\theta)^2(\sin^2\theta+3\cos^2\theta)}{\sin^4\theta}\right]\dot{\theta}$$

and as at z_c we have $\dot{\theta} = 0$ we must also have $\dddot{\theta} = 0$.

(3) We have $\mu_1 = \mu_2 \cos\theta + \dot{\phi}\sin^2\theta$ so taking the time derivatives we get :

$$0 = -\mu_2 \sin\theta\, \dot{\theta} + \ddot{\phi} \sin^2\theta + 2\dot{\phi}\dot{\theta} \sin\theta \cos\theta$$

that is

$$\ddot{\phi} \sin\theta + (2\dot{\phi} \cos\theta - \mu_2)\dot{\theta} = 0$$

(as $\sin\theta \neq 0$). (Again this result could be obtained from the Euler equations.) Hence as $\dot{\theta} = 0$ at z_c we must have $\ddot{\phi} = 0$.

Differentiating again we obtain

$$\dddot{\phi} \sin\theta + (2\,\dot{\phi}\cos\theta - \mu_2)\ddot{\theta} + \dot{\theta}(3\,\ddot{\phi}\cos\theta - 2\dot{\phi}\,\dot{\theta}\sin\theta) = 0$$

At z_c where $\dot{\theta} = \dot{\phi} = 0$ this equation becomes

$$\dddot{\phi}\sin\theta = \mu_2\,\ddot{\theta}$$

and as $\ddot{\theta} \neq 0$ we know that $\dddot{\phi} \neq 0$. ∎

If we regard the axis of the top as tracing out a curve parametrised by t on a sphere whilst the top is moving, then by choosing $t = 0$ to correspond to the axis being at the upper bounding circle $\theta = \theta_c$, we can say from the above :

Corollary 4.4

While the top is near the upper bounding circle, (θ, ϕ) are given by a curve

$$t \mapsto \left(\theta_c + P_o t^2 + O(4),\; Q_o t^3 + O(4)\right)$$ ∎

In fact we see that $2P_o = \ddot{\theta}(0) = \sin\theta_c = (1 - z_c^2)^{\frac{1}{2}}$ and $6Q_o = \dddot{\phi}(0) = \mu_2$.

Hence :

Corollary 4.5

There exist smooth invertible changes of coordinates $t \mapsto s$ and $(\theta, \phi) \mapsto (\chi, \rho)$ such that (χ, ρ) are given by the map

$$s \mapsto (s^2, s^3) .$$

Proof

A translation $\tau = \theta - \theta_c$ removes the constant term from the first coordinate, if we then prove that the map $g : s \mapsto (s^2, s^3)$ is 3-determined under right-left equivalence it follows immediately that the smooth changes of coordinates exist. Using the notation from Chapter 1 it suffices to prove that $\mathfrak{m}_3(1, 2) \subset \mathfrak{m}(1)\, T_{\mathcal{A}} g$ where $T_{\mathcal{A}} g$ is given in Proposition 1.9(a) (We can consider Γ to be the identity here as we have already factored out by the action of T^2). We will suppress the subscript $\mathcal{A}$ as we are only considering right-left equivalence from now on.

So we need to show that any map from R to R^2 of the form $\begin{pmatrix} as^3 + \ldots \\ bs^3 + \ldots \end{pmatrix}$ can be written as the product of a function that vanishes at 0, i.e. at least s, and a map of the form :

$$\left(G(s) \begin{pmatrix} 2s \\ 3s^2 \end{pmatrix} + H(s^2, s^3) \right)$$

where $G : R \to R$ and $H : R^2 \to R^2$.

This is clearly true, hence the corollary is proved. ■

By considering $\tilde{V}_\mu$ we have factored out by the precessional motion so this pattern will be repeated all round the bounding circle given by z_c . Thus the axis of the top will meet that circle in cusps though the cusps on the circle belong to infinitely many different trajectories of the top's axis.

At this juncture we will look at the previous result in a slightly different way. We can regard the motion of the top as given by a path in the phase space $T^* M$, that is a map $\zeta : R \to T^* M$

with $\zeta(0)$ giving the initial conditions with which the top is set in motion. As we are here only interested in the axis of the top we can compose ζ with the canonical projection $\pi: T^* M \to S^2$

$$\pi : (\theta, \phi, \psi ; p_\theta, p_\phi, p_\psi) \mapsto (\theta, \phi)$$

and then $\pi \circ \zeta : t \mapsto \big(\theta(t), \phi(t)\big)$ gives the curve on S^2 traced out by the axis of the top.

This map has singularities precisely when $\dot{\theta} = \dot{\phi} = 0$, that is $\dot{\theta} = 0$ and $\mu_1 - \mu_2 \cos\theta = 0$. These two equations define a codimension two sheet Σ in T^*M.

We know from the calculations carried out above leading to Corollary 4.5 that a choice of a point in Σ to be $\zeta(0)$ will result in the top executing the motion described above with cusps at the upper bounding circle given by $\theta = \theta(0)$.

However what will happen if we choose a point near to but not on Σ to be $\zeta(0)$? More precisely, let $(\varepsilon, 0, 0; 0, \mu_1, \mu_2)$ be a point in Σ and $(\varepsilon + \alpha, \delta, \psi; \omega, \mu_1 + \beta, \mu_2 + \gamma)$ be a typical point of $T^* M$, where $\alpha, \beta, \gamma, \delta$ and ω are small. (Note that Σ is invariant under the T^2-action, so we can always choose $\psi = \phi = 0$ for a point in Σ. The value of ψ does not enter into the calculations at all as we are only concerned with the axis of the top.)

Taking the typical point in $T^* M$ as $\zeta(0)$ we can calculate :-

<u>Proposition 4.6</u>

The first few terms of the Taylor series for $\pi \circ \zeta$ at $\zeta(0)$ are given by :-

$$\pi \circ \zeta: t \mapsto \big(\theta(t), \phi(t)\big)$$

where

$$\theta(t) = \varepsilon + \alpha + \omega t + Pt^2 + \text{ higher order terms}$$

$$\phi(t) = \delta + At + Bt^2 + Qt^3 + \text{ higher order terms}$$

where

$$A = \dot{\phi}(0) = \frac{\mu_1 + \beta - (\mu_2 + \gamma)\cos(\varepsilon + \alpha)}{\sin^2(\varepsilon + \alpha)}$$

$$B = \frac{1}{2}\ddot{\phi}(0) = \frac{[\mu_2 + \gamma - 2A\cos(\varepsilon + \alpha)]\omega}{2\sin(\varepsilon + \alpha)}$$

$$P = \frac{1}{2}\ddot{\theta}(0) = \frac{\sin(\varepsilon+\alpha)}{2} - \frac{[(\mu_1+\beta) - (\mu_2+\gamma)\cos(\varepsilon+\alpha)][(\mu_2+\gamma) - (\mu_1+\beta)\cos(\varepsilon+\alpha)]}{2\sin^3(\varepsilon + \alpha)}$$

$$Q = \frac{1}{6}\dddot{\phi}(0) = \frac{[\mu_2+\gamma-2A\cos(\varepsilon+\alpha)]2P - \omega[6B\cos(\varepsilon+\alpha)-2A\omega\sin(\varepsilon+\alpha)]}{6\sin(\varepsilon + \alpha)}$$

In R^2 we change coordinates by letting $\rho(t) = \theta(t) - \varepsilon$ and consider :

$$\widetilde{\pi\circ\zeta} : t \mapsto (\rho(t), \phi(t))$$

which is $\pi\circ\zeta$ expressed in the new coordinates.

We will define a map germ $\xi : (R^5 \times R, 0) \mapsto (R^2, 0)$ by

$$\xi(\alpha, \beta, \gamma, \delta, \omega; t) = \widetilde{\pi\circ\zeta}(t)$$

Now we define $\Xi : (R^5 \times R, 0) \mapsto (R^5 \times R^2, 0)$ by

$$\Xi : (\alpha, \beta, \gamma, \delta, \omega; t) \mapsto (\alpha, \beta, \gamma, \delta, \omega; \xi(\alpha, \beta, \gamma, \delta, \omega; t))$$

We will denote $\xi(0, 0, 0, 0, 0; t)$ by $\xi_o(t)$

Now we will show

<u>Theorem 4.7</u>

Ξ is a 5-parameter versal unfolding of ξ_o .

Proof

If we denote the values of P and Q (as calculated in Proposition 4.6) when $\alpha = \beta = \gamma = \omega = 0$ by P_o and Q_o then this agrees with P_o and Q_o as given in Corollary 4.4 so we can write

$$\xi_o : t \mapsto \left(P_o t^2 + 0(4),\ Q_o t^3 + 0(4)\right)$$

and

$$\xi(\alpha, \beta, \gamma, \delta, \omega; t) = (\rho, \phi)$$

where

$$\rho = \alpha + \omega t + P(\alpha, \beta, \gamma, \omega)t^2 + 0(3)$$

$$\phi = \delta + A(\alpha, \beta, \gamma, \omega)t + B(\alpha, \beta, \gamma, \omega)t^2 + Q(\alpha, \beta, \gamma, \omega)t^3 + 0(4)$$

We first compute $T\xi_o$. In fact let

$$g_o : t \mapsto (P_o t^2, Q_o t^3)$$

then we will show that $T\xi_o = Tg_o$.

Now

$$Tg_o = \left\{K \in \mathcal{E}(1, 2);\ K(t) = G(t)\begin{pmatrix} 2P_o t \\ 3Q_o t^2 \end{pmatrix} + H(P_o t^2, Q_o t^3)\right\}$$

where $G : R \to R$ and $H : R^2 \to R^2$ are arbitrary smooth germs.

Also

$$T\xi_o = \left\{K \in \mathcal{E}(1, 2);\ K(t) = G(t)\begin{pmatrix} 2P_o t + 0(3) \\ 3Q_o t^2 + 0(3) \end{pmatrix} + H\left(P_o t^2 + 0(4),\ Q_o t^3 + 0(4)\right)\right\}$$

$$= \left\{K \in \mathcal{E}(1, 2);\ K(t) = G(t)\begin{pmatrix} 2P_o t \\ 3Q_o t^2 \end{pmatrix} + 0(3) + H(P_o t^2, Q_o t^3) + 0(4)\right\}$$

So
$$T\xi_o \subset T_{g_o} + \mathfrak{m}_3(1, 2)$$

However we know from the proof of corollary 4.5 that $\mathfrak{m}_3(1, 2) \subset \mathfrak{m}(1)\, Tg_o \subset Tg_o$ so

$$T\xi_o \subset Tg_o$$

We could have said that

$$Tg_o \subset T\,\xi_o + \mathfrak{m}_3(1, 2)$$

then using the fact that $\mathfrak{m}_3(1, 2) \subset \mathfrak{m}(1)\, Tg_o$ we would have

$$Tg_o \subset T\xi_o + \mathfrak{m}(1)\, Tg_o$$

Then by Nakayama's Lemma (see e.g. page 102 of [5]) we have

$$Tg_o \subset T\xi_o$$

Hence $Tg_o = T\xi_o$, as claimed.

Now $\mathcal{E}(1, 2)/_{Tg_o}$ is spanned by $(1, 0)$, $(0, 1)$ and $(0, t)$, thus both g_o and ξ_o have codimension 3 , so, providing that at least one of $\frac{\partial A}{\partial \alpha}$, $\frac{\partial A}{\partial \beta}$, $\frac{\partial A}{\partial \gamma}$ or $\frac{\partial A}{\partial \omega}$ is non-zero when $\alpha = \beta = \gamma = \omega = 0$ we will have, using the notation given in Theorem 1.12:

$$\mathcal{E}(1, 2) = T\xi_o + R\{\dot{F}_1, \dot{F}_2, \dot{F}_3, \dot{F}_4\}$$

From Proposition 4.6 we see that :

$$\left.\frac{\partial A}{\partial \beta}\right|_{\alpha=\beta=\gamma=\omega=0} = \frac{1}{\sin^2\theta_\sigma} \neq 0$$

Hence we conclude that Ξ is a versal unfolding of ξ_o . ∎

From corollary 4.5 we know that there exist smooth invertible changes of coordinates in R and R^2 such that, denoting ξ_o in these new coordinates by ξ_o' , we can write $\xi_o' : s \mapsto (s^2, s^3)$.

If we express the map Ξ as Ξ' in terms of these new coordinates we will have a mapping that will be a 5-parameter versal unfolding of ξ_o' , because it will be isomorphic (in the sense of Martinet [11]) to the original versal unfolding Ξ . Hence, by the remarks following the versal unfolding theorem of Martinet in [11] this 5-parameter versal unfolding of ξ_o' will be equivalent to a constant 2-parameter unfolding of a 3-parameter universal unfolding of ξ_o' , since ξ_o' has codimension 3. Thus we have proved :

Theorem 4.8

There exist local diffeomorphisms $H : (R^5 \times R, 0) \to (R^5 \times R, 0)$ and $X : (R^5 \times R^2, 0) \to (R^5 \times R^2, 0)$ given by

$$H(\alpha, \beta, \gamma, \delta, \omega; t) = (\alpha, \beta, \gamma, \delta, \omega; \eta(\alpha, \beta, \gamma, \delta, \omega; t)$$

$$X(\alpha, \beta, \gamma, \delta, \omega; \theta, \phi) = (\alpha, \beta, \gamma, \delta, \omega; \chi_1(\alpha, \beta, \gamma, \delta, \omega; \theta, \phi), \chi_2(\alpha, \beta, \gamma, \delta, \omega; \theta, \phi))$$

which are 5-parameter unfoldings of the identity of R , respectively R^2 , and a local diffeomorphism $h :(R^5, 0) \mapsto (R^5, 0)$ which we write as

$$h : (\alpha, \beta, \gamma, \delta,\omega) \mapsto (u_1, u_2, u_3, u_4, u_5)$$

such that we get a map $\lambda : R \mapsto R^2$ defined by :-

$$\lambda : s \mapsto (u_1 + s^2 , u_2 + u_3 s + s^3)$$

which we can regard as expressing $\pi \circ \zeta$ in "canonical form" in the sense that :-

$$(h \times id_2) \circ X \circ (id_5 \times [\pi \circ \zeta]) = (id_5 \times \lambda) \circ (h \times id_1) \circ H$$

where id_i means the identity map from R^i to itself. ■

From this we can immediately verify that the axis of the top will perform one of the two motions shown in Figure 13 near the upper bounding circle.

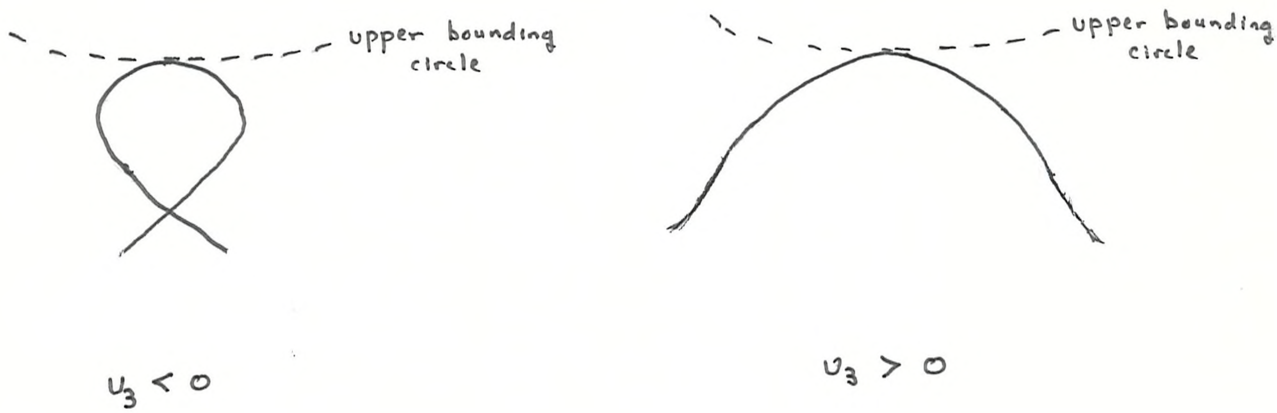


Figure 13.

Of course this description of the motion of the axis of the top is well known (see e.g. Figure 4.10 on page 153 in [13]). Our contribution here however, is to show that the motion can be put into canonical form by suitable smooth coordinate changes. Moreover, we will now use the above analysis to draw conclusions about the stability of the mathematical model, closely related to the phenomenon that a real top with all its attendant imperfections but with negligible friction actually appears to behave locally like the perfect top described above.

From Martinet [11] we know that any universal unfolding of a map germ is itself a stable germ, hence the expression given for λ in Theorem 4.8 above defines a stable germ. What we would like to be able to say is that this stability means that if we were to take a small perturbation of the system leading to a small perturbation of the unfolding Ξ then, with respect to some physically meaningful equivalence relation, the perturbed unfolding will be in the same equivalence class as the original unfolding. However, if F is an unfolding of a map germ at the origin and we perturb F slightly then F will not usually be equivalent to the new unfolding at the origin, so equivalence of unfoldings as defined in Chapter 1 is not a suitable relation to use.

Firstly we need a definition of F_1 at (u, x) in $R^q \times R^n$ being equivalent to F_2 at another point (v, y) in $R^q \times R^n$. Such a definition can easily be framed by shifting the origin. Secondly, following Wassermann [18], we can say :

Definition

Let $f_o : (R^n, 0) \to (R^p, 0)$ be a map germ and let $F : (R^q \times R^n, 0) \to (R^q \times R^p, 0)$ be an unfolding of f_o, as given in Definition 1.8. We say F is stable if for every open neighbourhood U of 0 in $R^q \times R^n$ and every representative F' of F defined on U, there is a neighbourhood V of F' in $C^\infty(U, R^q \times R^p)$ (with the weak C^∞-topology) such that for every $G' \in V$ there is a point $(u, x) \in U$ such that G' at (u, x) is equivalent to F' at 0.

We can now conjecture :

Proposition

The unfolding Ξ given in Theorem 4.7 is stable in the above sense.

We do not prove this claim. It ought to be possible to prove it by an analagous argument to that given by Wassermann for Theorem 4.11 on page 98 of [18], as we already known that Ξ is a versal unfolding. Assuming this proposition it follows that the motion sketched above will be locally stable under small perturbations of the equations of motion. Such perturbations could be brought about for instance by making the top slightly asymmetrical or alternatively spinning a magnetically sensitive top in a weak magnetic field. In such cases we would find that :

Theorem

Locally near the upper bounding curve of the motion of the axis of the top, the axis traces out either loops or waves as before.

We now turn our attention to the motion of steady precession.

Steady Precession

We are going to examine the geometry of the dual steady precession surface K_2^* and will begin by showing that the reduced amended potential $\tilde{V}_\mu$ has only one critical point in the range $-1 < z < +1$, i.e. the quartic equation

$$z^4 - (\mu_1\mu_2 + 2)z^2 + (\mu_1^2 + \mu_2^2)z + 1 - \mu_1\mu_2 = 0$$

has only one root in the indicated range.

To investigate the roots of $z^4 + az^2 + bz + c = 0$ we need only to look at the swallowtail catastrophe, as given on pages 176 - 178 of [16]. The bifurcation set, in (a, b, c) space, which gives the regions for different numbers of roots is parametrised by

$$(a, b, c) = (3q - 6r^2, - 6rq + 8r^3, 3qr^2 - 3r^4)$$

(There are a couple of small errors on page 177 of [16] itself).

In our case we are taking a plane slice through the bifurcation set, given by $a = c - 3$.

Proposition 4.9

The bifurcation set, in the plane $a = c - 3$, is given by

$b = \pm(2 - 2c)$ and $(b, c) = (2r^3 + 6r, -3r^2)$, $r^2 \neq 1$.

Proof

The parametrization of the bifurcation surface above, together with the equation $a = c - 3$, yields

$$3q = \frac{3r^4 - 6r^2 + 3}{r^2 - 1} = 3(r^2 - 1) \quad \text{providing} \quad r^2 \neq 1 .$$

Then

$$b = 8r^3 - 6rq = 2r^3 + 6r$$

and

$$c = 3qr^2 - 3r^4 = -3r^2$$

If $r^2 = 1$ then

$$c = 3qr^2 - 3r^4 = 3(q - 1)$$

and

$$b = 8r^3 - 6rq = 2r(4r^2 - 3q) = \pm 2(4 - 3q) .$$

So $b = \pm(2 - 2c)$, taking the plus sign if $r = 1$ and the minus when $r = -1$. ∎

The bifurcation set looks as in Figure 14.

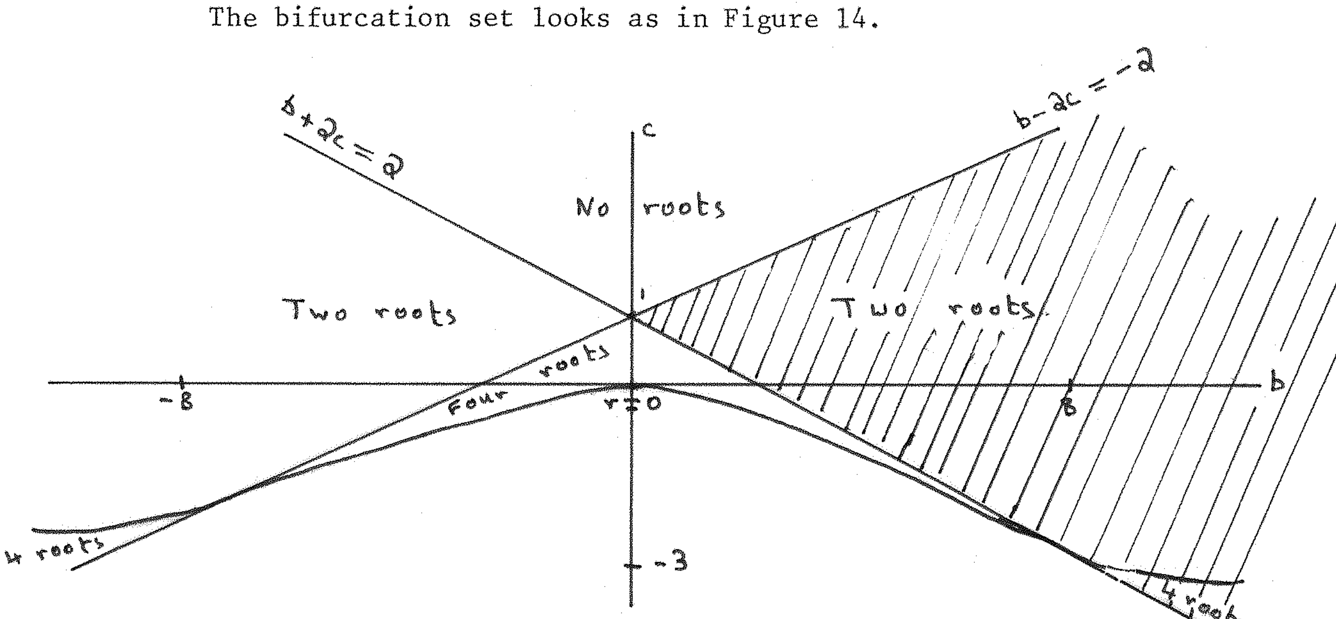


Figure 14.

The curve given by $(b, c) = (2r^3 + 6r, - 3r^2)$ intersects the lines $b = \pm(2 - 2c)$ at the points $(\pm 8, -3)$ with cubic tangency.

In order to decide about the number of roots of our equation, we need to pull back this picture to (μ_1, μ_2) space. As $b = \mu_1^2 + \mu_2^2$ and $c - 1 = - \mu_1\mu_2$ the line $b - 2c = - 2$ corresponds to $(\mu_1 + \mu_2)^2 = 0$ and the line $b + 2c = 2$ to $(\mu_1 - \mu_2)^2 = 0$. Therefore the only part of Figure 14 which has any significance in this case is that shown in Figure 15 which corresponds to the shaded region in Figure 14.

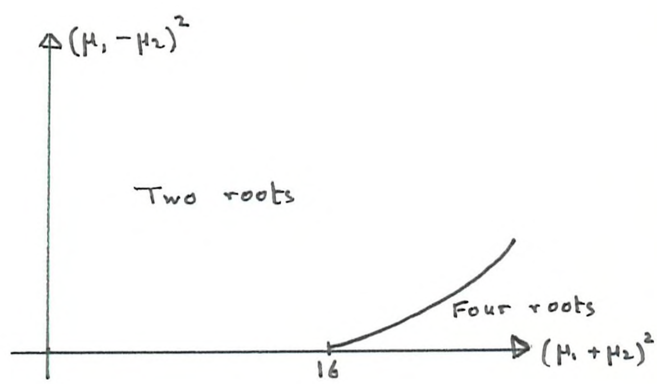


Figure 15.

In order to see where the roots lie we need to look at the catastrophe manifold over the bifurcation set. This will look very similar to the sketch given on page 178 of [16] with the addition of two extra "cusps" one on the upper sheet and one on the lower. Note that here we are not using "cusp" in the usual catastrophe theory sense, that of a $\frac{2}{3}$ power, but simply to mean "sharp point". Referring back to [16], page 177, we see that the parameter r, used above, can be identified with z, hence the additional cusp we are interested in is the one on the upper surface.

So the catastrophe manifold looks like Figure 16.

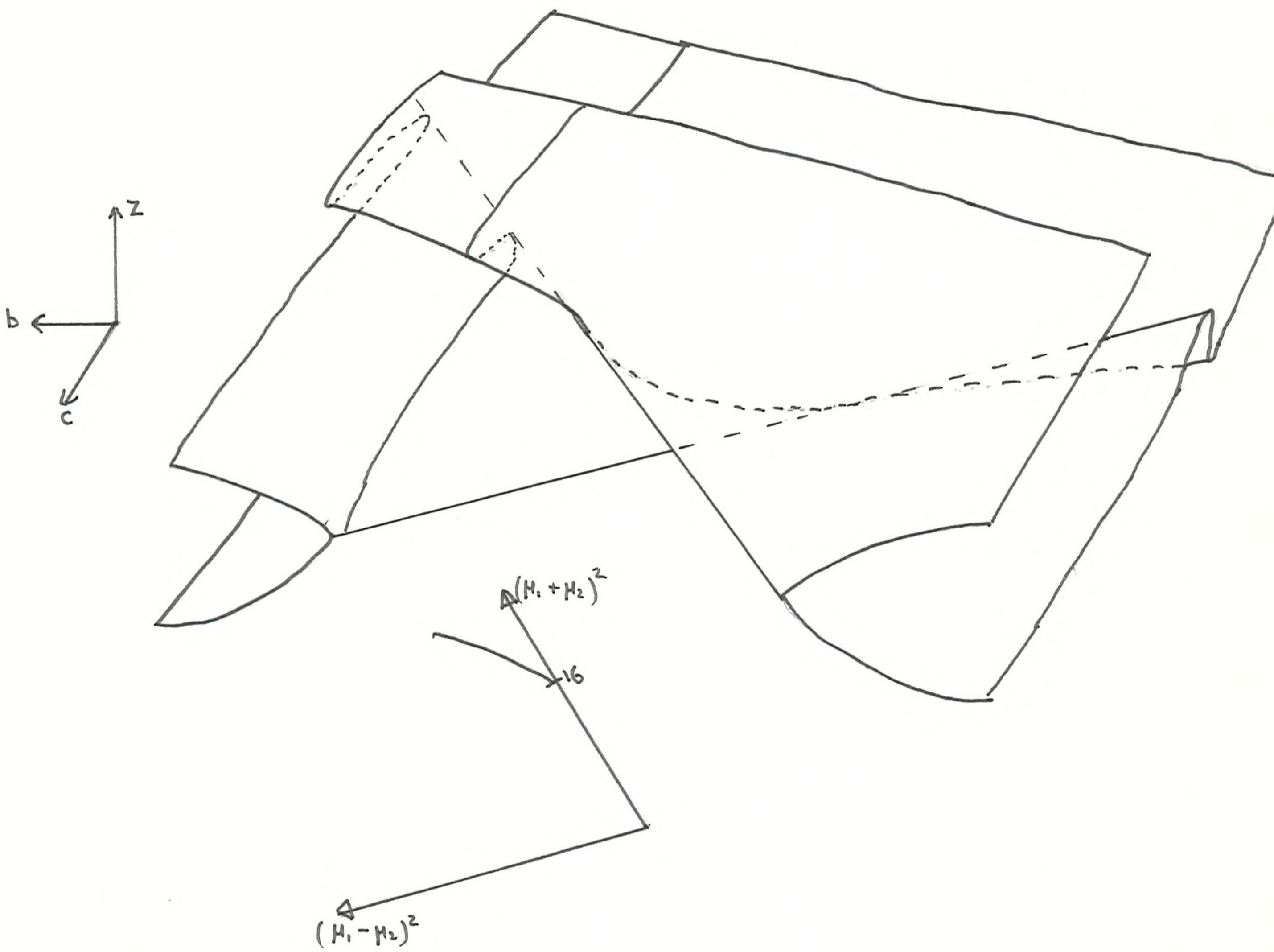


Figure 16.

We can see that over the "two root" section of the control plane one root of the quartic will be less than -1 and the other between - 1 and + 1 , whilst over the "four root" region, one root is less than - 1 , the next between - 1 and + 1 and the other two greater or equal to + 1 .

So we have proved :-

Proposition 4.10

The quartic $z^4 - (\mu_1\mu_2 + 2)z^2 + (\mu_1^2 + \mu_2^2)z + 1 - \mu_1\mu_2 = 0$ has only one root between -1 and $+1$ for all values of μ_1 and μ_2 such that $(\mu_1 - \mu_2)^2 > 0$ and $(\mu_1 + \mu_2)^2 > 0$. ■

As we know from elementary consideration of $\tilde{V}_\mu(z)$ (see immediately prior to Proposition 4.2) that there must be at least one minimum for $-1 < z < +1$, we conclude:

Corollary 4.11

$\tilde{V}_\mu(z)$ has only one critical point for z between -1 and $+1$ and that critical point is a minimum. ■

Thus :-

Corollary 4.12

Given values of μ_1 and μ_2 there is only one possible angle to the vertical at which steady precession can take place. ■

From page 360 of [1] we know that nondegenerate maxima or minima of the reduced amended potential give stable relative equilibria and so we can conclude that :

Corollary 4.13

Steady precession is a stable relative equilibrium. ■

This contrasts with sleeping which is stable provided that the angular velocity of the top about its axis is greater than a certain critical value. When the angular velocity is less than that value sleeping can still take place but it is an unstable motion. See, for example the discussion on page 156 of [13].

The relevant portion of the catastrophe manifold which we will denote Q^2 can now be pulled back to lie over (μ_1, μ_2) space where it will be the steady precession surface K_2^* given in Proposition 4.2.

We need also to examine the behaviour of K_2^* near the region corresponding to $\theta = 0$ or π . The analysis above excluded these values so we will need to investigate the situation using the $(\theta_1, \theta_2, \theta_3)$ coordinate chart. However, we can look at the behaviour of the catastrophe manifold over the bifurcation set to see the kind of behaviour we would expect.

Over the line $(\mu_1 + \mu_2)^2 = 0$ the only possibility is a repeated root of $z = -1$ and this must correspond to the top hanging vertically downwards, the ultimate in stable motion! Now we turn our attention to the portion of the catastrophe manifold over the line $(\mu_1 - \mu_2)^2 = 0$

We distinguish three cases:

Firstly, if $(\mu_1 + \mu_2)^2 > 16$ then we have a repeated root of $z = +1$ and no other admissable root.

Secondly, if $(\mu_1 + \mu_2)^2 < 16$ we have a repeated root at $z = + 1$ and another root between $+1$ and -1 .

Thirdly at $(\mu_1 + \mu_2)^2 = 16$ we have a threefold repeated root at $z = +1$.

At this stage we turn our attention more fully to the sleeping top and tie together the steady precession surface, investigated above using the Euler coordinate chart, with the sleeping conditions, investigated using the $(\theta_1, \theta_2, \theta_3)$ - coordinate chart.

The Sleeping Top.

We now wish to examine the behaviour of the sleeping top in a similar way to that which we used for steady precession. However, as remarked earlier, we need to use the $(\theta_1, \theta_2, \theta_3)$ - chart on M rather than the Euler angle chart.

In this chart we have the potential energy $V : M \to R$. given by

$$V(\theta_1, \theta_2, \theta_3) = \cos\theta_1 \cos\theta_2$$

and the kinetic energy $K : T^* M \to R$ by :-

$$K(\theta_1, \theta_2, \theta_3, p_1, p_2, p_3) = \frac{1}{2} p_2^2 + \frac{1}{2C} p_3^2 + \frac{1}{2} \frac{(p_1 - p_3 \sin\theta_2)^2}{\cos^2\theta_2}$$

The momentum mapping , $J : T^* M \to R^2$ is given, see Proposition 2.3 by

$$J : (\theta_1, \theta_2, \theta_3, p_1, p_2, p_3) \mapsto \left(- p_1 \frac{\cos\theta_1 \sin\theta_2}{\cos\theta_2} + p_2 \sin\theta_1 + p_3 \frac{\cos\theta_1}{\cos\theta_2}, p_3 \right)$$

Proposition 4.14

The set Λ of points x in M such that $J\big|_{T_x^*M} : T_x^* M \to R^2$ is not surjective consists of all points

$$x = (\theta_1, \theta_2, \theta_3) \text{ with } \theta_1 = \theta_2 = 0 .$$

Proof

$$DJ\big|_{T_x^*M} = \begin{pmatrix} \dfrac{- \cos\theta_1 \sin\theta_2}{\cos\theta_2} & \sin\theta_1 & \dfrac{\cos\theta_1}{\cos\theta_2} \\ 0 & 0 & 1 \end{pmatrix}$$

which has rank 1 if and only if $\sin\theta_1 = 0$ and $\sin\theta_2 = 0$ i.e. $\theta_1 = \theta_2 = 0$. ■

Hence we cannot carry out the construction of an amended potential to include the case of a sleeping top, as Λ corresponds to the sleeping position, and must be examined separately. (See remarks following Lemma 4.5.4 on page 343 of [1]). However, we can determine what happens to the steady precession surface K_2^* as we approach the sleeping condition.

In order to see this behaviour clearly we look at the situation in the tangent bundle rather than the cotangent bundle and then pull back the picture to the cotangent bundle.

Proposition 4.15

The steady precession manifold K is a submanifold of TM given by $K = \{(\theta_1, \theta_2, \theta_3; \dot\theta_1, \dot\theta_2, \dot\theta_3)\}$ where

(a) $\dot\theta_1 \sin\theta_1 \cos\theta_2 + \dot\theta_2 \cos\theta_1 \sin\theta_2 = 0$

and (b) $$\frac{\dot\theta_2}{\sin\theta_1}\left[C[\dot\theta_3 + \dot\theta_1 \sin\theta_2] - \dot\theta_2 \frac{\cos\theta_1 \cos\theta_2}{\sin\theta_1}\right] = 1$$

Proof

Using the relationships worked out in Chapter 2 between the new coordinates for the configuration space and the Euler angle coordinates, (a) corresponds to $\dot\theta = 0$ and (b) to $(Cs - \dot\phi \cos\theta)\dot\phi = 1$ which are the defining equations for K given in Chapter 3. ■

The steady precession surface K^2 is obtained from the steady precession manifold K by factoring out by the group action. The question we must now answer is : What does K and hence K^2 look like close to $\theta_1 = \theta_2 = 0$?

Proposition 4.16

As θ_1 and θ_2 approach 0 the time derivatives $\dot{\theta}_1$ and $\dot{\theta}_2$ also approach 0 for points $(\theta_1, \theta_2, \theta_3; \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$ in K

Proof

The defining equations of K are

(a) $$\dot{\theta}_1 \sin\theta_1 \cos\theta_2 + \dot{\theta}_2 \cos\theta_1 \sin\theta_2 = 0$$

and (b) $$\frac{\dot{\theta}_2}{\sin\theta_1}\left(C[\dot{\theta}_3 + \dot{\theta}_1 \sin\theta_2] - \dot{\theta}_2 \frac{\cos\theta_1 \cos\theta_2}{\sin\theta_1}\right) = 1 ,$$

i.e. $$\dot{\theta}_2 \left(C\, \dot{\theta}_3 \sin\theta_1 + C\dot{\theta}_1 \sin\theta_1 \sin\theta_2 - \dot{\theta}_2 \cos\theta_1 \cos\theta_2 \right) = \sin^2\theta_1 .$$

Replacing $\dot{\theta}_1$ by $-\dfrac{\cos\theta_1 \sin\theta_2}{\sin\theta_1 \cos\theta_2}\dot{\theta}_2$ we get

$$(\dot{\theta}_2)^2\left[\frac{\cos\theta_1}{\cos\theta_2}(\cos^2\theta_2 + C \sin^2\theta_2) \right] - [C\, \dot{\theta}_3 \sin\theta_1]\, \dot{\theta}_2 + \sin^2\theta_1 = 0$$

So

$$\dot{\theta}_2 = \sin\theta_1 (\cos\theta_2)^{\frac{1}{2}} \left[\frac{C\, \dot{\theta}_3 \pm \left(C^2\dot{\theta}_3^2 \cos\theta_2 - 4 \cos\theta_1 [\cos^2\theta_2 + C \sin^2\theta_2]\right)^{\frac{1}{2}}}{2 \cos\theta_1 [\cos^2\theta_2 + C \sin^2\theta_2]} \right]$$

thus

$$\dot{\theta}_1 = -\sin\theta_2 (\cos\theta_2)^{-\frac{1}{2}} \left[\frac{C\, \dot{\theta}_3 \pm \left(C^2\dot{\theta}_3^2 \cos\theta_2 - 4 \cos\theta_1 [\cos^2\theta_2 + C \sin^2\theta_2]\right)^{\frac{1}{2}}}{2 [\cos^2\theta_2 + C \sin^2\theta_2]} \right]$$

from which the Proposition follows. ∎

So we have shown that as θ_1 and θ_2 approach zero the manifold K tends to the set $\left\{(\theta_1, \theta_2, \theta_3; \dot\theta_1, \dot\theta_2, \dot\theta_3) = (0, 0, \theta_3; 0, 0, \dot\theta_3\right.$ or over our "factored out" set of coordinates, just the $\dot\theta_3$ axis, subject to the condition that $C^2\,\dot\theta_3{}^2 \geq 4\,\frac{\cos\theta_1}{\cos\theta_2}\left[\cos^2\theta_2 + C\sin^2\theta_2\right]$.

Theorem 4.17

The frontier of the surface K^2 of steady precession points (regarded as a subset of the reduced tangent space) is the line segment

$$L' = \left\{(\theta_1, \theta_2; \dot\theta_1, \dot\theta_2, \dot\theta_3) = (0, 0; 0, 0, \dot\theta_3);\ \dot\theta_3^2 \geq \frac{4}{C^2}\right\}$$

This line L' is exactly the set of points giving the motion of a stable sleeping top see, for example, page 156 of [13]. ■

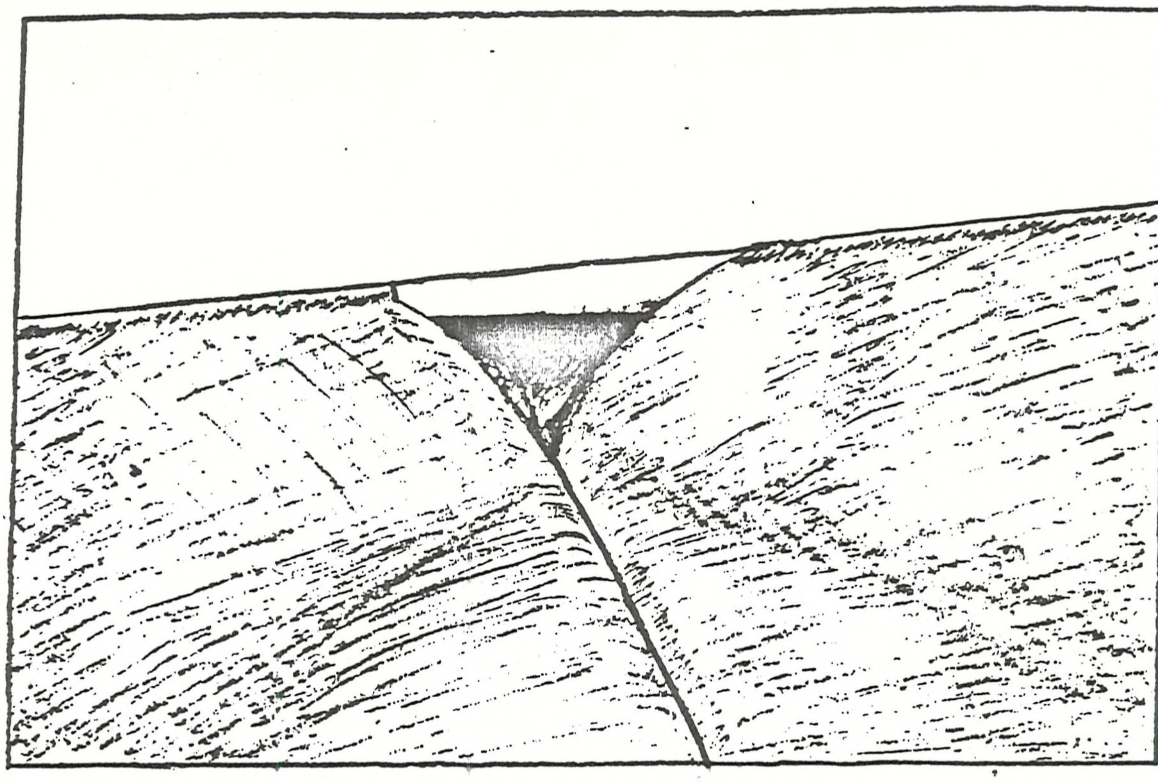
We now pull back Q^2 to lie over the (μ_1, μ_2) plane in T* M. As Q^2 lies over the quadrant given by $(\mu_1 + \mu_2)^2 \geq 0$ and $(\mu_1 - \mu_2)^2 \geq 0$ the surface K_2^* in (μ_1, μ_2, z) space lies over all the (μ_1, μ_2) plane but is symmetric around the planes

$$P_1 = \left\{(\mu_1, \mu_2, z);\ \mu_1 + \mu_2 = 0\right\} \text{ and } P_2 = \left\{(\mu_1, \mu_2, z);\ \mu_1 - \mu_2 = 0\right\}.$$

The surface K_2^* intersects the plane P_1 in the line $\{(\mu_1, \mu_2\ z)\ ;\ \mu_1 + \mu_2 = 0\ ,\ z = -1\}$ and intersects the plane P_2 in two segments of the line $\{(\mu_1, \mu_2, z);\ \mu_1 - \mu_2 = 0,\ z = +1\}$. The rest of K_2^* lies between these two levels of z . From the remarks at the end of the last section we see that the plane $z = +1$ intersects Q^2 and hence K_2^* only if $|\mu_1 + \mu_2| > 4$, the other part of the line $\{(\mu_1, \mu_2, z);\ \mu_1 - \mu_2 = 0,\ z = +1\}$ (giving the unstable sleeping top) being isolated.

Furthermore we know that Q^2 meets the lines corresponding to $z = -1$ and $z = +1$ quadratically so when pulled back to lie over the (μ_1, μ_2) plane K_2^* must meet the inverse images of those lines linearly at an acute angle. Putting all this information together we can make a sketch of K_2^* as in Figure 17.

From the sketch we can see that the case when $\mu_1 = \mu_2$ and $|\mu_1 + \mu_2| > 4$. corresponds to the stably sleeping top. However when $|\mu_1 + \mu_2| < 4$ we have the unstable sleeping top when a small perturbation will cause it to "wake-up". As we see, there is a possible steady precession that can take place with the given values of μ_1 and μ_2. The in-between point is the critical point between sleeping and wakefulness given as we see by $\mu_1 = \mu_2 = 2$.



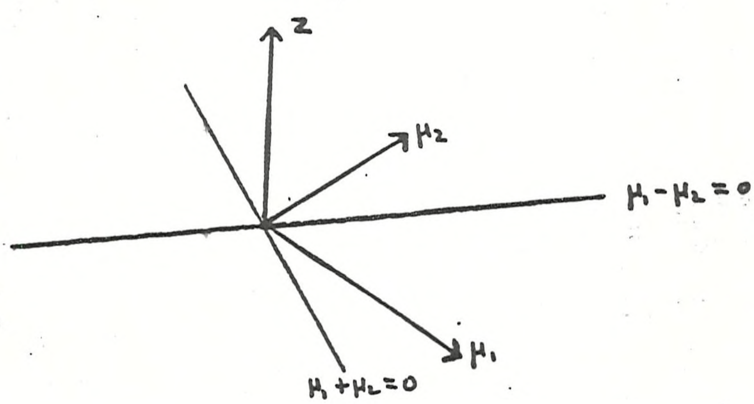


Figure 17

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