THEORY

OF

TOPOLOGICAL GROUPOIDS

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by

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ABSTRACT

FACULTY OF SCIENCE
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THEORY OF TOPOLOGICAL GROUPOIDS

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This thesis takes up the notion of topological and differentiable categories and groupoids and of their local triviality. Beginning with definitions of an (algebraic) category with object class X, and appropriate commutative diagrams. These are extended easily to the topological and differentiable cases in later chapters. For brevity here, suppose X is a Hausdorff space, path-connected (p.c.), locally path-connected (ℓ.p.c.), and locally simply connected (ℓ.s.c.). As important examples we consider PX (the set of all paths) as a topological category, and πX (the fundamental groupoid) as a topological groupoid, over X, also πX is a covering space of $X \times X$, and $\pi_1(\pi X, .)$ is computed. The relation between connected groupoids and fibre bundles is studied. We show that every connected locally trivial (ℓ.t.) groupoid over X has a bundle structure over $X \times X$, and for each $x \in X$, $St_G x$ is a principal bundle over X with group $G\{x\}$. Also, every connected ℓ.t. groupoid with discrete vertex groups over X is shown to be isomorphic to a quotient groupoid of πX; and if $\pi_1 X$ is abelian, then

$\pi_1(G, o_x) \approx \pi_1(St_G x, o_x) \oplus \pi_1(X, x)$.

The notion of topological covering morphism is introduced; if $p: X \longrightarrow Y$ is a covering map of Hausdorff spaces, then

$p_*: \pi X \longrightarrow \pi Y$ is a covering morphism of topological groupoids. In case G is a connected ℓ.t. Hausdorff groupoid, ∃ a 1 - 1 correspondence between the closed subgroups of its vertex group and its covering groupoids. If G is a connected ℓ.t. groupoid with discrete vertex groups over X, then the universal covering space $\tilde{G}$ of G is a groupoid over $\tilde{X}$, and in case $G = \pi X$, $\tilde{G}$ is the universal covering groupoid of G.

We consider the notion of $\mathcal{G}$d-transformation group (Γ, G), generalising the fundamental group of a transformation group. We show that this group is the set of all morphisms : $\tilde{G} \longrightarrow \tilde{G}$ lifting the elements of the group Γ. The set of all lifts of the identity in Γ is the group of cover transformations of $\tilde{G}$. Under certain conditions the orbit set $G/_{\Gamma}$ is a topological groupoid and the quotient morphism $q : G \longrightarrow G/_{\Gamma}$ is a covering morphism.

Finally, we study some examples of Lie categories and groupoids.

TO MY FAMILY.

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INTRODUCTION

In this chapter we first give a definition of categories, groupoids and functors in terms of maps and commutative diagrams. We take a path in a space X to be a continuous map $:\mathbb{R}^{+} \longrightarrow X$. This then allows us to define a reasonable topology on PX.

The plan of each chapter is given in the beginning of that chapter, except for Chapter I.

We follow the same terminology as in [2] throughout the thesis.

CHAPTER I

Definition 1.1.1

Let C and C_o be two classes. We call C a category over C_o if:-

(C_1) $\exists$ functions $i, \phi : C \longrightarrow C_o$,

Called the initial and final maps, respectively.

(C_2) If $D = \{(f, g) \in C \times C \mid \phi(f) = i(g)\}$,

then $\exists$ a function $\theta: D \longrightarrow C$, $(f, g) \rightsquigarrow f.g$,

called "the composition function" such that:-

$$\forall (f, g) \in D, \; i(f.g) = i(f) \;\; \& \;\; \phi(f.g) = \phi(g)$$

and which satisfies the:-

Associative Law: $\forall (f, g), (g, h) \in D, \; (f.g).h = f.(g.h)$

(C_3) $\exists$ a function $u: C_o \longrightarrow C$

called "the unit function" such that:-

$\forall x \in C_o$, $i(u(x)) = \phi(u(x)) = x$, and if $f, g \in C$ with $i(f) = \phi(g) = x$, then $u(x).f = f$ & $g.u(x) = g$

Notations and terminology:

For each $x \in C_o$, we call $u(x)$, the unit element of C at x, and $u(C_o) \subseteq C$ will be called the class of units of C, and denoted by O. We also write fg, gof, f + g for $\theta(f, g)$, as convenient. We denote $u(x)$ by o_x, when using additive notation, and 1_x in all other cases.

The set

$$C(x, y) = \{f \in C \mid i(f) = x \; \& \; \phi(f) = y\}$$

$x, y \in C_o$, will be called the set of morphisms from x to y.

If $f \in C(x, y)$, we may also write $f : x \longrightarrow y$, or $x \xrightarrow{f} y$. For each $f \in C$, we call $i(f)$, $\phi(f)$ the initial and final objects of f, respectively, and the class C_o will be called the class of objects of C, and the elements of C will be called the <u>morphisms</u> of the category C. In order to specify the functions in a category C, we sometimes write $C = (C, C_o, i, \phi, \theta, u)$. We also write C^{ob}, for the class of objects if it is not specified, and $St_C x, \mathcal{E}_x$ for $i^{-1}(x)$ and $\phi^{-1}(x)$, respectively.

<u>Remarks 1.1.2</u>

(1) For each $x \in C_o$, the unit element 1_x is unique.

(2) If $(x, y), (x', y') \in C_o \times C_o$ and $(x, y) \neq (x', y')$, then:-

$$C(x, y) \cap C(x', y') = \phi$$

(3) It is immediate from the definition that u is an injection i.e. C_o is bijective with O, the class of units in C. Hence one sometimes regards C_o as a sub-class of C, the class of morphisms.

<u>Definition 1.1.3</u>

A groupoid G over the class G_o is a category over G_o in which $\exists$ a function $\sigma : G \longrightarrow G$, $\sigma(g) = g^{-1}$ called the <u>inverse function</u>, satisfying the:-

<u>Inverse Law</u>: $\forall g \in G$, $i(g) = \phi(g^{-1})$, $\phi(g) = i(g^{-1})$;

thus $(g, g^{-1}) \in D$, $(g^{-1}, g) \in D$. Moreover

$$g \cdot g^{-1} = 1_{i(g)} \quad \text{and} \quad g^{-1} \cdot g = 1_{\phi(g)}$$

We call g^{-1}, the <u>inverse morphism</u> of g. It follows from the definition that $\sigma(g^{-1}) = g$. i.e. g is the inverse morphism of g^{-1}. And also, it follows that each $g \in G$, has a <u>unique</u> inverse.

<u>Remark 1.1.4:</u>

For each $x \in G_o$, $(i, \phi)^{-1}(x, x) = G(x, x)$, the set of all morphisms from x to x, forms a group called "<u>the vertex group</u>" at x, with 1_x

as its identity element. This group will be denoted by G {x} as in [2] .

<u>Definition 1.1.5:</u>

Let $C = C(, C_o, i, \phi, \theta, u)$, $C' = (C', C'_o, i', \phi', \theta', u')$ be two categories. By a functor $\Gamma: C \longrightarrow C'$,

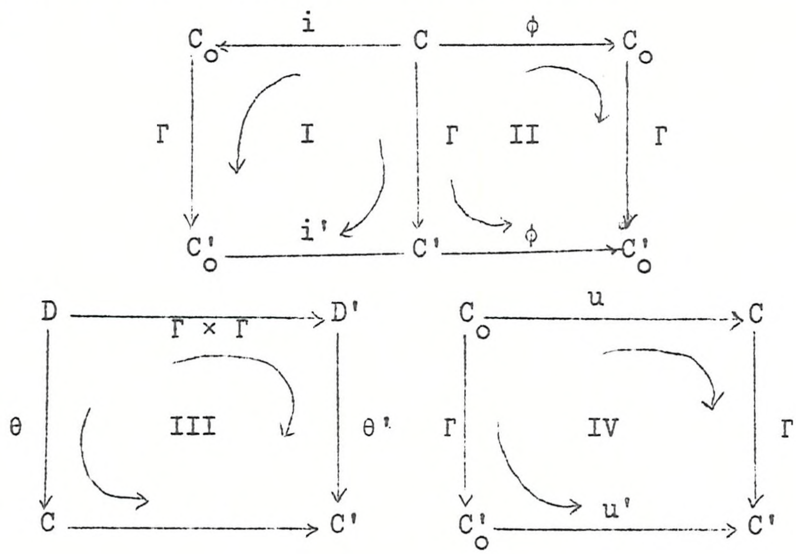
we mean a function $\Gamma: C \cup C_o \longrightarrow C' \cup C'_o$

satisfying:-

(F_1) $\Gamma(C) \subseteq C'$ & $\Gamma(C_o) \subseteq C'_o$

i.e.Γ preserves morphisms and objects.

(F_2) The following diagrams are commutative.



Diagrams I and II simply show that Γ commutes with initial and final maps, while diagram III indicates that:-

$\forall.(f, g) \varepsilon D, \Gamma(f.g.) = \Gamma(f).'\Gamma(g)$

Diagram IV is another form of the statement:-

$\forall\ x \varepsilon C_o, \Gamma(1_x) = 1_{\Gamma(x)}$

Lemma 1.1.6:

If C, C' are groupoids, then diagram IV is equivalent to the following commutative diagram.

$$\begin{array}{ccc} C & \xrightarrow{\Gamma} & C' \\ \sigma \downarrow & \overline{V} & \downarrow \sigma' \\ C & \xrightarrow{\Gamma} & C' \end{array}$$

Proof: Iv $\Longrightarrow$ V

$\forall\, g \in C,\ g.g^{-1} = 1_{i(g)} \Longrightarrow \Gamma(g.g^{-1}) = \Gamma(1_{i(g)})$

$\Longrightarrow \Gamma(g) \hat{.} \Gamma(g^{-1}) = 1_{\Gamma(ig)}$ (by III & IV)

Hence $\Gamma(g^{-1}) = (\Gamma(g))^{-1}$, and therefore we have:-

$$\Gamma(\sigma(g)) = \sigma'(\Gamma(g)) \text{ i.e. } \Gamma o\, \sigma(g) = \sigma' o \Gamma(g),\ g \in C.$$

Therefore V is commutative.

Conversely, V $\Longrightarrow$ IV

$$\forall\, x \in C_o,\ \Gamma(1_x) = \Gamma(f.f^{-1}) = \Gamma(f).'\Gamma(f^{-1}),\ f \in St_G x.$$

$$= \Gamma(f).'(\Gamma(f))^{-1} \quad \text{(by V)}$$

$$= 1_{i'(\Gamma(f))} = 1_{\Gamma(i(f))} = 1_{\Gamma(x)} \quad \text{(by I)}$$

q.e.d.

Definition 1.1.7:

Let $C = (C, C_o, i, \phi, \theta, u)$, $C' = (C', C'_o, i', \phi', \theta', u')$ be two categories. We call C' a subcategory of C if:-

(1) $C' \subseteq C$ and $C'_o \subseteq C_o$

(2) $i' = i \mid C'$, $\phi' = \phi \mid C'$, $\theta = \theta \mid D'$, $u' = u \mid C'_o$

If $C'_o = C_o$, we call C' a wide subcategory, and if $\forall\, (x, y) \in C'_o \times C'_o \subseteq C_o \times C_o$, $C'(x, y) = C(x, y)$, we call C' a full subcategory.

Definition 1.1.8:

Let G and G' be groupoids, then G' is called a subgroupoid of G if

(1) G' is a subcategory

(2) $\sigma' = \sigma \mid G'$

Definition 1.1.9:

Let G be a groupoid, and let A be a subgroupoid of G. Then A is called normal if:-

$\forall$ objects x, y of A, and $\forall$ $a \varepsilon G(x, y)$,

$$a^{-1} A\{x\} a = A\{y\}.$$

Definition 1.1.10:

A groupoid G over G_o is connected if:-

$\forall$ $x, y \varepsilon G_o$, $G(x, y) \neq \phi$

Definition 1.1.11:

A groupoid G over G_o is said to be totally disconnected if:

$\forall$ $x, y \varepsilon G_o$, $x \neq y \Longrightarrow G(x, y) = \phi$

Definition 1.1.12:

A groupoid G over G_o is said to be a tree if:-

$\forall$ $x, y \varepsilon G_o$, $G(x, y)$ has only one element.

Definition 1.1.13:

A groupoid G over G_o is called discrete if $u(G_o) = G$.

A first theorem on the structure of connected groupoids is that $G \approx G\{x\} * T$ (the free product), where T is any wide tree subgroupoid in G (see [2]).

SECTION 1.2

An important example to illustrate the definitions of section 1 arises from the set of paths on a topological space.

Definition 1.2.1:

Let X be a topological space. By a path of length r we mean a continuous function $f: \mathbb{R}^+ \longrightarrow X$, where $r \varepsilon \mathbb{R}^+$ is the smallest number such that $f \mid (\mathbb{R}^+ \setminus [o, r])$ is constant with the value f(r).

We call f(o), f(r), the initial and final points of f, respectively. To specify the length we may write f_r whenever needed. For each $x \in X$, we have a unique path of length zero, denoted by o_x, We also denote the set of all paths in X by PX.

Definition 1.2.2:

Let f, $g \in PX$ be paths of length r and s, respectively, such that $f(r) = g(o)$. Define the non-commutative addition of paths by:

$$(f + g)_t = \begin{cases} f(t) & o \leqslant t \leqslant r \\ g(t - r) & t \geqslant r \end{cases}$$

It is immediate from the definition that $f + g\colon \mathbb{R}^+ \longrightarrow X$ is a path of length r + s.

Define the functions i, ϕ, $PX \longrightarrow X$,

by:- $\forall f_r \in PX$, $i(f_r) = f_r(o)$ and $\phi(f_r) = f_r(r)$.

Also, let $D = \{(f_r, g_s) \in PX \times PX \mid f_r(r) = g_s(o)\}$

define the composition function $\phi : D \longrightarrow PX$

by:- $\forall (f_r, g_s) \in D$, $\phi(f_r, g_s) = f_r + g_s$.

Finally define the unit function

$$u\colon X \longrightarrow PX$$

by:- $\forall x \in X$, $u(x) = o_x$, the constant function at x.

From the definitions we see at once that θ is associative and u satisfies the required conditions for a unit map in a category so:-

Theorem 1.2.3: $(PX, X, i, \phi, u, \theta)$ is a category

The details of proof are straightforward and omitted.

Definition 1.2.4:

For each $f_r \in PX$, define the reverse map

$$-f_r\colon \mathbb{R}^+ \longrightarrow X$$

by:-

$$(-f_r)_s = \begin{cases} f_r(r + \ell - s) & o \leq s \leq r \\ f_r(o) & s \geq r \end{cases}$$

Where $\ell \in \mathbb{R}^+$ is the greatest number such that $f \mid [o, \ell]$ is constant. (Notice that this might happen, since the definition of f_r does not require that f_r should be non-constant in some neighbourhoods of $o \in \mathbb{R}^+$).

It is immediate that $-f_r$ is a path of length r.

<u>Note</u>: $\phi(-f_r) = i(f_r)$ and $i(-f_r) = \phi(f_r)$

This suggests PX with this inverse might be a groupoid. Unfortunately $f + (-f) \neq o$ in general. So we pass to homotopy classes.

$f_p \simeq g_q$ means $\exists$ a continuous function

$$F: \mathbb{R}^+ \times I \longrightarrow X \quad (I \text{ the unit interval})$$

such that $F(s, o) = f_p(s)$, $F(o, t) = f_p(o) = g_q(o)$

$F(s, 1) = g_q(s)$, $F(1, t) = g_q(q) = f_p(p)$.

<u>Lemma 1.2.5</u>:

(i) <u>if $f_p \simeq g_q$, then $-f_p \simeq -g_q$.</u>

(ii) <u>for each $f_r \in PX$, $f_r + (-f_r) \simeq o_{f_r(o)}$</u>

<u>$-f_r + f_r \simeq o_{f_r(r)}$</u>

<u>Proof</u>: (i) Let $F: \mathbb{R}^+ \times I \longrightarrow X$ be the homotopy: $f_p \simeq g_q$, then $\forall\, s \in \mathbb{R}^+$, $F(s, o) = f_p(s)$ and $F(s, 1) = g_q(s)$.

Define $H: \mathbb{R}^+ \times I \longrightarrow X$

by:

$$H(s, t) = \begin{cases} F\big((1 - t)(p + \ell) + t(q + \ell') - s, t\big) & o \leq s \leq (1-t)(p+\ell) + t(q+\ell') \\ F\ (o, t) & s \geq (1-t)(p+\ell) + t(q+\ell') \end{cases}$$

(ℓ and ℓ' the greatest numbers s.t. $f_p|[o,\ell]$ and $g_q|[o,\ell']$ are constant). Obviously H is continuous and we have:-

$$H(s, o) = \begin{cases} F(p+\ell-s, o) & o\leqslant s\leqslant p+\ell \\ F(o, o) & s\geqslant p+\ell \end{cases} = \begin{cases} f_p(p+\ell-s) & o\leqslant s\leqslant p+\ell \\ f_p(o) & s\geqslant p+\ell \end{cases}$$

$$= \begin{cases} f_p(p+\ell-s) & o\leqslant s\leqslant p \\ f_p(o) & s\geqslant p \end{cases} = -f_p(s)$$

(For $f_p(p+\ell-s) = f_p(o)$, $p\leqslant s\leqslant p+\ell$)

and $$H(s, 1) = \begin{cases} F(q+\ell'-s, 1) & o\leqslant s\leqslant q+\ell' \\ F(o, 1) & s\geqslant q+\ell' \end{cases} = \begin{cases} g_q(q+\ell'-s) & o\leqslant s\leqslant q+\ell' \\ g_q(o) & s\geqslant q+\ell' \end{cases}$$

$$= \begin{cases} g_q(q+\ell'-s) & o\leqslant s\leqslant q \\ g_q(o) & s\geqslant q \end{cases} = -g_q(s)$$

(For $g_q(q+\ell'-s) = g_q(o)$, $q\leqslant s< q+\ell'$).

Hence $-f_p \simeq -g_q$.

(ii) Define the homotopy $G, F: \mathbb{R}^+ \times I \longrightarrow X$

by:-

$$G(s, t) = \begin{cases} f(s), & o\leqslant s\leqslant tr \\ f(tr), & tr\leqslant s\leqslant tr+\ell \\ f(2tr+\ell-s), & tr+\ell\leqslant s\leqslant 2tr+\ell \\ f(o) & s\geqslant 2tr+\ell \end{cases}$$

$$F(s, t) = \begin{cases} f(r+t\ell-s), & o\leqslant s\leqslant tr \\ f(r+t\ell-tr), & tr\leqslant s\leqslant tr+\ell \\ f(s+(1-2t)r+(t-1)\ell), & s\geqslant tr+\ell \end{cases}$$

It is easily seen that G and F are continuous. We have:-

$F(s, o) = o_{f(r)}$ and $$F(s, 1) = \begin{cases} f(r+\ell-s) & o\leqslant s\leqslant r \\ f(\ell) & r\leqslant s\leqslant r+\ell \\ f(s-r) & s\geqslant r+\ell \end{cases}$$

$$= \begin{cases} f(r + \ell - s) & o \leq s \leq r \\ f(s - r) & s \geq r \end{cases} = (-f_r + f_r)_s$$

(For $f(s - r) = f(\ell)$, $r \leq s \leq r + \ell$)

similarly, $G(s, o) = o_{f(o)}$ and $G(s, 1) = (f_r + (-f_r))_s$

q.e.d.

Lemma 1.2.6:

Given $f_r \in PX$, then for any $r' \in \mathbb{R}^+$, $\exists$ a path f' of length r' such that $f' \simeq f_r$, and $f'(\mathbb{R}^+) = f_r(\mathbb{R}^+)$

Proof: Define $\sigma_{r'r}: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$

by: $\forall\, s \in \mathbb{R}^+$, $\sigma_{r'r}(s) = sr/_{r'}$

Then $\sigma_{r'r}$ is a continuous map, and hence

$$f_r \circ \sigma_{r'r}: \mathbb{R}^+ \longrightarrow X$$

is a path of length r'.

Claim: $f \simeq f_r \circ \sigma_{r'r}$

Define the homotopy $H: \mathbb{R}^+ \times I \longrightarrow X$

by:

$$H(s,t) = \begin{cases} f\left(\dfrac{sr}{tr + (1 - t)r'}\right) & o \leq s \leq tr + (1 - t)r' \\ f(r) & s \geq tr + (1 - t)r' \end{cases}$$

obviously H is continuous and is the required homotopy. q.e.d.

Lemma 1.2.7:

Let $(f_p, g_q) \in D$ and $(\lambda_{p'}, \mu_{q'}) \in D$. If $f_p \simeq \lambda_{p'}$, and $g_q \simeq \mu_{q'}$, then $f_p + g_q \simeq \lambda_{p'} + \mu_{q'}$

Proof: Let $F: f_p \simeq \lambda_{p'}$, and $G: g_q \simeq \mu_{q'}$, be the homotopies.

Define $H : \mathbb{R}^+ \times I \longrightarrow X$ by:-

$$H(s, t) = \begin{cases} F(s, t) & o \leq s \leq |F_t| \\ G(s - |F_t|, t) & s \geq |F_t| \end{cases}$$

where $|F_t|$ is the length of the path $F_t : \mathbb{R}^+ \longrightarrow X$ obtained from the homotopy F. Then, obviously $F(|F_t|, t) = G(o, t) = f_p(p)(=\lambda_{p'}(p'))$ Hence H is continuous and we have:-

$$H(s, o) = (f_p + g_q)_{(s)} \text{ and } H(s, 1) = (\lambda_{p'} + \mu_{q'})_{(s)}$$

q.e.d.

Denote the equivalence relation $\simeq$ by R. We now pass to cosets and prove:-

<u>Theorem 1.2.7:</u>

<u>Let $\pi X = PX_{/R}$, the set of all homotopy classes in PX. Then πX is a groupoid over X.</u>

<u>Proof:</u> For each $f_r \varepsilon PX$, let $\overline{f}_r$ denote the homotopy class of f_r.

(i) Define the initial and final maps

$$\overline{i}, \overline{\phi} : \pi X \longrightarrow X$$

by: $\forall \overline{f}_r \varepsilon \pi X$, $\overline{i}(\overline{f}_r) = f_r(o)$ and $\overline{\phi}(\overline{f}_r) = f_r(r)$.

obviously independent of coset representatives

(ii) Let $\overline{D} = \{(\overline{f}_r, \overline{g}_s) \varepsilon \pi X \times \pi X | f_r(r) = g_s(o)\}$. Define

$\overline{\theta}: \overline{D} \longrightarrow \pi X$ by: $\overline{\theta}(\overline{f}_r, \overline{g}_s) = \overline{f_r + g_s} \ (= \overline{f}_r + \overline{g}_s)$

It follows from 1.2.6 that $\overline{\theta}$ is well-defined.

Since addition in PX is associative and $q:PX \longrightarrow \pi X$, the quotient map, is onto and respects +, then:-

addition in πX is associative.

(iii) Define the unit function $\overline{u} : X \longrightarrow \pi X$

by: $\forall x \varepsilon X$, $\overline{u}(x) = \overline{o}_x$

Since o_x is a unit in PX, it follows the $\overline{o}_x$ serves as a unit in πX.

(iv) Define the inverse function $\overline{\sigma} : \pi X \longrightarrow \pi X$

by: $\forall \overline{f}_r \varepsilon \pi X$, $\overline{\sigma}(\overline{f}_r) = \overline{-f_r}$

By 1.2.5, $\overline{\sigma}$ is well-defined and we have:-

$$\overline{i}(\overline{f_r}) = f_r(o) = f_r(r) = \overline{\phi}(\overline{-f_r})$$
$$\overline{\phi}(\overline{f_r}) = f_r(r) = -f_r(o) = \overline{i}(\overline{-f_r})$$
$$\left.\begin{array}{l}\overline{f_r} + (\overline{-f_r}) = \overline{f_r + (-f_r)} = \overline{o_{f_r}(o)} \\ \overline{-f_r} + \overline{f_r} = \overline{-f_r + f_r} = \overline{o_{f_r}(r)}\end{array}\right\} \quad \text{(by 1.2.5)}$$

Therefore σ satisfies the required conditions.

q.e.d.

We now show that πX is isomorphic to the fundamental groupoid $\pi' X$ as defined in[2]. For this we need the following lemma:-

Lemma 1.2.8:

Let $f_q : \mathbb{R}^+ \longrightarrow X$ be a path, and let $[r, r'] \subseteq [o, q]$ be such that $f_q | [r, r']$ is constant. Then f_q is homotopic to the path

$$f' : \mathbb{R}^+ \longrightarrow X$$

defined by:- $$f'(s) = \begin{cases} f(s) & o \leq s \leq r \\ f(r' - r + s) & s \geq r \end{cases}$$

Proof: Obviously f' is a path of length $q - r' + r$. Define the homotopy $H : \mathbb{R}^+ \times I \longrightarrow X$

by:
$$H(s, t) = \begin{cases} f(s) & o \leq s < r \\ f(r) & r \leq s \leq r + t(r' - r) \\ f((1 - t)(r' - r) + s) & s \geq r + t(r' - r) \end{cases}$$

Since $t(r' - r) \leq r' - r$, $r + t(r' - r) \leq r'$. Hence it is easily seen that H agrees on the intersections, and therefore it is continuous.

We have:-
$$H(s, o) = \begin{cases} f(s) & o \leq s \leq r \\ f(r) & s = r \\ f(r' - r + s) & s \geq r \end{cases} = f'(s)$$

$$H(s, 1) = \begin{cases} f(s) & o \leq s \leq r \\ f(r) = f(s) & r \leq s \leq r' \\ f(s) & s \geq r' \end{cases} = f(s)$$

q.e.d.

We call f' the shrunk path of f.

Theorem 1.2.9: πX is isomorphic to $\pi' X$

Proof: Let f' be a path of length $|f'|$, as defined in[2], i.e. f' is a continuous map from the interval $[o, |f'|]$ into the space X.

Then we can extend f' to a path $f : \mathbb{R}^+ \longrightarrow X$ by:-

$$\forall\, t \in \mathbb{R}^+, \quad f(t) = \begin{cases} f'(t) & o \leq t \leq |f'| \\ f'(|f'|) & t \geq |f'| \end{cases}$$

obviously f' and f have the same initial and final points.

Define $\mu : \pi' X \longrightarrow \pi X$

by: (i) $\forall\, [f'] \in \pi' X$, $\mu([f']) = \bar{f}$, where f is the extension of f'.

(ii) $\forall\, x \in X$, $\mu(x) = x$

μ is well-defined: Let $f' \sim g'$, then $\exists\, r, r_1 \in \mathbb{R}^+$ such that $r +' f'$ is homotopic to $r_1 +' g'$, where $(+')$ is the addition of paths as defined in[2]i.e.

$$(f' + g')(s) = \begin{cases} f'(s) & o \leq s \leq |f'| \\ g'(s - |f'|, & |f'| \leq s \leq |f'| + |g'| \end{cases}$$

Let $F' : [o, |f'| + r] \times I \longrightarrow X$ be the homotopy: $r +' f' \simeq r_1 +' g'$.

We extend F' to $F : \mathbb{R}^+ \times I \longrightarrow X$

$$\text{by:} \quad F(s, t) = \begin{cases} F'(s, t) & o \leq s \leq (1 - t)|f'| + t\,|g'| \\ F'((1 - t)|f'| + t\,|g'|, t) & s \geq (1 - t)\,|f'| + t\,|g'| \end{cases}$$

obviously F is continuous and is a homotopy from f to g, the extensions of f' and g' respectively.

μ is a morphism of groupoids:

Since μ is identity on the set of objects and f, f' have the same initial and final points it is easily seen that diagrams I and II in (F_2) are satisfied. Obviously, the extension of $o'_x : [o] \longrightarrow X$, is the constant map $o_x : \mathbb{R}^+ \longrightarrow X$. Hence $\mu([o'_x]) = \bar{o}_x$, the unit element at x in πX. Therefore μ commutes with the unit maps, i.e. diagram IV of (F_2) is satisfied. Hence it remains to prove that μ commutes with the composition maps. Let D' be the set of all composable elements in $\pi'X \times \pi'X$, and let $\sigma' : D' \longrightarrow \pi'X$ be the composition map, then:-

$$\forall\ ([f'],[g']) \in D',\ \mu([f'] +'[g']) = ([f' +' g']) = \overline{h}$$

where h is the extension of f' +' g'. Let f and g be the extensions of f' and g', respectively. We show that $\overline{h} = \overline{f + g}$.

We have:-

$$(f' +' g')(s) = \begin{cases} f'(s) & 0 \leqslant s \leqslant |f'| \\ g'(s - |f'|) & |f'| \leqslant s \leqslant |f'| + |g'| \end{cases}$$

Let f and g be of length $p(\leqslant |f'|)$ and $q(\leqslant |g'|)$, respectively.

Then

$$(f + g)(s) = \begin{cases} f(s) & 0 \leqslant s \leqslant p \\ g(s - p) & s \geqslant p \end{cases}$$

If $p = |f'|$, then it is easily seen that $h = f + g$.

If $p \neq |f'|$, then $f + g$ is the shrunk path of h as defined in 1.2.8. (here $h|[p, |f'|]$ is constant). Hence $f + g \simeq h$.

Therefore $\mu([f'] +' [g']) = \overline{h} = \overline{f + g} = \overline{f} + \overline{g} = \mu([f']) + \mu([g'])$

Hence (F_2) is satisfied.

Now define $\nu : \pi X \longrightarrow \pi'X$

by: $\forall\ \overline{f}_q \in \pi X$, $\nu(\overline{f}) = [f_1]$, where $f_1 = f|[0, q]$.

It is easily verified that $(f + g)_1 = f_1 + g_1$ and $f \simeq g \Longrightarrow f_1 \sim g_1$. Therefore ν is a well-defined morphism.

It is also easy to see that $\mu \circ \nu = 1_{\pi X}$ and $\nu \circ \mu = 1_{\pi' X}$.

Hence μ is a bijection and so an isomorphism of groupoids.

q.e.d.

Next consider another, important, example.

Let T be a vector bundle over the space X with the fibre $\mathbb{R}^n$, and for each $z \in X$, let T_z denote the fibre over z, which is topologically isomorphic to $\mathbb{R}^n$

Let $$\mathcal{C}(T) = \bigcup_{x,y \in X} \operatorname{Hom}(T_x, T_y).$$

then we have:-

<u>Theorem 1.2.10</u>:

<u>$\mathcal{C}(T)$ is a category over X.</u>

<u>Proof</u>: (i) Define the initial and final maps

$$i, \phi : \mathcal{C}(T) \longrightarrow X$$

as follows:- given $f \in \mathcal{C}(T)$, then $\exists x, y \in X$ s.t. $f \in \operatorname{Hom}(T_x, T_y)$, then define $i(f) = x$ and $\phi(f) = y$

(ii) Let $D = \{(f, g) \in \mathcal{C}(T) \times \mathcal{C}(T) \mid \phi(f) = i(g)\}$,

define $$\theta : D \longrightarrow \mathcal{C}(T)$$

by: $$\theta(f, g) = gof \text{ (the composition of maps).}$$

Therefore θ satisfies the associative law.

(iii) define the Unit map $u : X \longrightarrow \mathcal{C}(T)$,

by:- $\forall x \in X$, $u(x) = id_{Tx}$

Obviously u satisfies the required condition.

q.e.d.

Another rather trivial but important example is:-

<u>Lemma 1.2.11</u>: For any group Γ and any class X, $X \times X \times \Gamma$ is a groupoid over X.

<u>Proof</u>: Define: (i) $i, \phi : X \times X \times \Gamma \longrightarrow X$ by:-

$$i(x, y, \gamma) = x \text{ and } \phi(x, y, \gamma) = y$$

(ii) Let $D = \{(x, y, \gamma), (y, z, \gamma')\}$

Define $\phi : D \longrightarrow X \times X \times \Gamma$ by:

$$\phi(\ (x, y, \gamma), (y, z, \gamma') = (x, z, \gamma\gamma')$$

(iii) $u : X \longrightarrow X \times X \times \Gamma$ by: $u(x) = (x, x, 1)$

(iv) $\sigma : X \times X \times \Gamma \longrightarrow X \times X \times \Gamma$ by: $\sigma(x, y, \gamma) = (y, x, \gamma^{-1})$

1.3: TOPOLOGY OF PX

Since $PX = X^{\mathbb{R}^+}$, it carries the compact-open topology. But this topology is not convenient for our purpose, so we will define a stronger topology for PX, which will turn it into a topological category, as we will see in chapter III.

Let $f_r \in PX$, and let $\eta : PX \longrightarrow \mathbb{R}^+$ be the length function, i.e. $\eta(f_r) = r$. Let $\{N_i\}$ be a basis for the system of neighbourhoods of f_r in C - O topology, and let $\{B_j\}$ be a basis for the nbds of $r \in \mathbb{R}^+$ in the usual topology. Then define the new topology for PX by taking as basis of neighbourhoods at f_r the family

$$\{N_\beta \cap \eta^{-1}(B_\alpha) \mid N_\beta \in \{N_i\}, B_\alpha \in \{B_j\}\}$$

We call this topology the LCO topology (L for length).

Remark

(i) Since $\mathbb{R}^+$ is a neighbourhood of r, and the length function η is onto, the LCO topology of PX contains the C - O topology.

(ii) The length map η is continuous in the LCO topology.

(iii) $i : PX \longrightarrow X$, $i(f) = f(o)$, has path lifting property (see [6] p. 83)

Note: Throughout the thesis, we will deal with the LCO topology of PX, and the topology of πX will be the quotient topology obtained from the LCO topology and denoted by Q-LCO topology.

CHAPTER II

THE FUNDAMENTAL GROUP OF πX

In this chapter, from any given wide normal subgroupoid A of a connected groupoid $(G, G^{ob}, i, \phi\ \theta\ , u, \sigma)$, we construct a groupoid, called E_A, over G^{ob} by defining an equivalence relation R_A in G; and take $E_A = {}^{G}/_{R_A}$. The process is analogous to the construction of covering spaces. We also construct another groupoid, called G_A, over E_A. Then, we will study the connection between G and these groupoids. It turns out that G_A is a covering groupoid of $G \times G$ with appropriate projection to be defined. If G arises as a fundamental groupoid of a path-connected, locally path connected and locally simply connected space X, then we show that E_A and hence πX is a covering space of $X \times X$. Finally we prove some theorems which enable us to compute $\pi_1\ (\pi\ X, 1)$, where X is as above. We close the chapter by introducing another topology for πX which is more convenient in practice and prove its equivalence to other existant topologies on πX.

2.1 CONSTRUCTION OF E_A

Let $(G, G^{ob}, i, \phi, \theta\ , \sigma, u)$ be a connected groupoid, and let A be any wide normal subgroupoid of G. Then it follows that:

$$\forall\ x \in G^{ob}\ ,\ A\ \{x\} \lhd G\ \{x\}$$

Conversely, given any normal subgroup A_z of the vertex group $G\ \{z\}$, $z \in G^{ob}$, then we construct a wide connected normal subgroupoid A of G as follows:-

Let T be any wide tree in G, then $G \approx G\{z\} * T$, with the isomorphism $\xi_T: a \rightsquigarrow \tau_{ia} + a' - \tau_{\phi a}$, where $a' \varepsilon G\{z\}$ and $\tau_x \varepsilon T(x, z)$. Take $A = \xi_T^{-1}(A_z * T)$, then:-

<u>Lemma 2.1.1</u>:

<u>A is a wide, normal, connected subgroupoid of G</u>

<u>Proof:</u> (i) Define the initial and final maps by:-

$$i_A = i|A \text{ and } \phi_A = \phi|A$$

(ii) Let $D_A = \{(a, b) \varepsilon A \times A | \phi(a) = i(b)\}$, then $D_A \subseteq D$, and we have:-

$\theta(D_A) \subseteq A$. For, let $(a, b) \varepsilon D_A$, then $a, b \varepsilon A$. Hence $\exists$ unique $a', b' \varepsilon A_z$ such that

$$a = \tau_{ia} + a' - \tau_{\phi a}$$
$$b = \tau_{ib} + b' - \tau_{\phi b}$$

But $(a, b) \varepsilon D_A \Longrightarrow \phi(a) = i(b) \Longrightarrow \tau_{\phi a} = \tau_{ib}$,

and $a', b' \varepsilon A_z \Longrightarrow a' + b' \varepsilon A_z$

Therefore $\theta(a, b) = a + b = (\tau_{ia} + a' - \tau_{\phi a}) + (\tau_{ib} + b' - \tau_{\phi b})$

$$= \tau_{ia} + (a' + b') - \tau_{\phi b}$$

Hence $a + b \varepsilon A$. So, we can define:-

$$\theta_A: D_A \longrightarrow A \text{ by } \theta_A = \theta|D_A.$$

(iii) Since A_z is a subgroup of $G\{z\}$, $o_z \varepsilon A_z$. Hence, it follows that $\forall y \varepsilon G^{ob}$, $o_y \varepsilon A$

Therefore, define $u_A: G^{ob} \longrightarrow A$

by:- $u_A(x) = u(x) = o_x$

So it meets the required conditions (For, u does).

(iv) Finally, define $\sigma_A: A \longrightarrow A$ by $\sigma_A = \sigma|A$.

This can be done, for:-

Let $d \varepsilon A$, then $\exists$ a unique $d' \varepsilon A_z$ such that $d = \tau_{id} + d' - \tau_{\phi d}$

Since $d' \in A_z \Longrightarrow -d' \in A_z$, we have:-

$\sigma(d) = -d = \tau_{\phi d} + (-d') - \tau_{id} = \tau_{i(-d)} + (-d') - \tau_{\phi(-d)}$

Hence $\sigma(d) \in A$. i.e. $\sigma(A) \subseteq A$

Therefore $(A, G^{ob}, i_A, \phi_A, \theta_A, u_A, \sigma_A)$ is a groupoid contained in G. It is immediate from the construction that A is a wide subgroupoid of G. It is easily seen that $T \subseteq A$, therefore A is connected.

It remains to show the normality of A in G. For this, we must show that:

$$\forall x, y \in G^{ob}, \forall g \in G(x,y), -g + A\{x\} + g \subseteq A\{y\}$$

We have:- $g \in G(x,y) \Longrightarrow \exists g' \in G\{z\}$ s.t. $g = \tau_x + g' - \tau_y$

$a \in A\{x\} \Longrightarrow \exists a' \in A_z$ s.t. $a = \tau_x + a' - \tau_x$

So, $-g + a + g = (\tau_y - g' - \tau_x) + (\tau_x + a' - \tau_x) + (\tau_x + g' - \tau_y)$

$= \tau_y + (-g' + a' + g') - \tau_y.$

But by normality of A_z in $G\{z\}$ we have $-g' + a' + g' \in A_z$.

Hence $-g + a + g \in A\{y\}$. q.e.d.

Thus normal subgroupoids exist. The independence of vertex groups of A from the tree T will be discussed later (see 2.1.4.)

<u>The relation R_A:</u>

Let A be a wide subgroupoid of G.

Define a relation R_A in G as follows:- let $a \in G$, then for any $b \in G$,

$$bR_A a \Longleftrightarrow ib = ia, \phi b = \phi a \text{ and } a - b \in A\{ia\} .$$

<u>R_A is an equivalence relation:-</u>

(i) R_A is reflexive: $\forall a \in G$, $a - a = o_{ia} \in A\{ia\} \Longrightarrow aR_A a$

(ii) R_A is symmetric: $\forall a, b \in G$,

$bR_A a \Longrightarrow a - b \in A\{ia\} \Longrightarrow b - a \in A\{ia\} \Longrightarrow aR_A b.$

(iii) R_A is transitive: $\forall$ a, b, c ε G s.t. bR_Aa and cR_Ab,

we have:-

$bR_Aa \Longrightarrow a - b \varepsilon A\{ia\}$, $ib = ia$ and $\phi b = \phi a$

$cR_Ab \Longrightarrow b - c \varepsilon A\{ib\} = A\{ia\}$, $ic = ib$ & $\phi c = \phi b$

By group property of $A\{ia\}$, $(a - c) = (a - b) + (b - c) \varepsilon A\{ia\}$.

Hence, cR_Aa.

Therefore R_A partitions G into disjoint equivalence classes.

We denote the equivalence class of any $a \varepsilon G$ by $[a]_A$.

Hence, $\forall a \varepsilon G$, $[a]_A = \{b \varepsilon G(ia, \phi a) | \; a - b \varepsilon A\{ia\}\}$

We also denote the quotient set $G/_{R_A}$ by E_A, and the quotient function: $G \longrightarrow E_A$ by p_A.

<u>Lemma 2.1.2:</u>

<u>If A is normal in G, then E_A is a groupoid over G^{ob} and $p_A: G \longrightarrow E_A$ can be extended to a functor of groupoids.</u>

<u>Proof:</u> (i) Define the functions $\bar{i}_A$, $\bar{\phi}_A$: $E_A \longrightarrow G^{ob}$

by:- $\forall [a]_A$, $\bar{i}_A[a]_A = i(a)$ & $\bar{\phi}_A[a]_A = \phi(a)$

It is immediate from the definition of R_A, that $\bar{i}_A$ and $\bar{\phi}_A$ are well-defined.

(ii) Let $\bar{D}_A = \{([a]_A, [b]_A) | \; \bar{\phi}_A[a]_A = \bar{i}_A[b]_A$ i.e. $\phi a = ib)\}$

Define $\bar{\theta}_A : \bar{D}_A \longrightarrow E_A$

by:- $\bar{\theta}_A([a]_A , [b]_A) = [a + b]_A$, where

$a + b = \theta(a , b)$.

<u>$\bar{\theta}_A$ is well-defined:</u> let $a', \varepsilon [a]_A$, $b' \varepsilon [b]_A$; we must show that $[a' + b']_A = [a + b]_A$. i.e. we must show that

$a + b - (a' + b') \varepsilon A\{ia\}$.

we have:-

$$a' \in [a]_A \Longrightarrow a - a' \in A\{ia\} \Longrightarrow \exists \alpha \in A\{ia\} \text{ s.t. } a = \alpha + a'$$

$$b' \in [a]_A \Longrightarrow b - b' \in A\{ib\} \Longrightarrow a' + (b - b') - a' \in A\{ia\},$$

by normality of A in G (this is the first place where normality is assumed).

Hence, $a + b - (a' + b') = \alpha + a' + b - b' - a' =$

$$\alpha + (a' + (b - b') - a') \in A\{ia\}.$$

We need to show the associative law for $\overline{\theta}_A$.

Let $([a]_A, [b]_A), ([b]_A, [c]_A) \in \overline{D}_A$. Then using the associativity of θ, we have:-

$$([a]_A + [b]_A) + [c]_A = [a + b]_A + [c]_A = [(a + b) + c]_A$$
$$= [a + (b + c)]_A = [a]_A + [b + c]_A = [a]_A + ([b]_A + [c]_A)$$

(iii) Define $\overline{u}_A : G^{ob} \longrightarrow E_A$

by:- $\forall x \in G^{ob}$, $\overline{u}_A(x) = [o_x]_A$, where $o_x = u(x)$.

$\overline{u}_A$ satisfies the required conditions for the unit map in E_A. For, let $[a]_A \in E_A$, then $\overline{i}_A [a]_A = i(a)$ and $\overline{\phi}_A [a]_A = \phi(a)$ and we have:-

$$[o_{ia}]_A + [a]_A = [o_{ia} + a]_A = [a]_A$$
$$[a]_A + [o_{\phi a}]_A = [a + o_{\phi a}] = [a]_A$$

(iv) Define $\overline{\sigma}_A : E_A \longrightarrow E_A$ by $\sigma_A [a]_A = [-a]_A$

<u>$\overline{\sigma}_A$ is well-defined:</u> Let $b \in [a]_A$, we must show that $[-b]_A = [-a]_A$.

To show this, we must prove $-a + b \in A\{\phi a\}$.

We have:-

$$b \in [a]_A \Longrightarrow b - a \in A\{ia\} \Longrightarrow -a + b = -a + (b - a) + a \in A\{\phi a\}.$$

(by normality of A).

$\overline{\sigma}_A$ <u>satisfies the inverse law</u>: For, $\overline{i}_A [a]_A = \phi_A [-a]_A$ and $\overline{\phi}_A [a]_A = \overline{i}_A [-a]_A$.

and we have:- $[a]_A + [-a]_A = [a - a]_A = [o_{ia}]_A = \bar{u}_A(ia)$

$$[-a]_A + [a]_A = [-a + a]_A = [0_{\phi a}]_A = \bar{u}_A(\phi a)$$

Hence it is the inverse function in E_A.

Define p_A to be the identity function on the set of objects, i.e. $\forall\ x \in G^{ob},\ p_A(x) = x$.

Then (F_1) is satisfied (by definition). So, we need only to show the commutativity of diagrams in (F_2). We have:-

(i) $\forall\ a \in G,\ p_A o \phi(a) = p_A(\phi a) = \phi a$

$\bar{\phi}_A o p_A(a) = \bar{\phi}_A(p_A(a)) = \bar{\phi}_A [a]_A = \phi a.$

Hence $p_A o \phi = \bar{\phi}_A o p_A$. Similarly $p_A o i = \bar{i}_A o p_A$.

(ii) $\forall\ (a,b) \in D,\ \bar{\theta}_A o\ (p_A \times p_A)(a, b) = \bar{\theta}_A([a]_A, [b]_A)$

$= [a + b]_A = [\theta(a, b)]_A$

$= p_A\ (\theta(a, b)) = p_A o \theta\ (a, b).$

(iii) $\forall\ a \in G,\ \bar{\sigma}_A o p_A(a) = \bar{\sigma}_A [a]_A = [-a]_A = [\sigma(a)]_A = p_A(\sigma(a))$

$= p_A o \sigma(a)$

Hence all the conditions for p_A to be a functor are satisfied.

q.e.d.

Remarks 2.1.3:

(i) the vertex groups of E_A are $G\{x\}_{/A\{x\}}$, $x \in G^{ob}$

(ii) If A is a tree in G, then $E_A = G$. It is also clear from the construction of A that if $A_z = \{o_z\}$, the trivial subgroup of G{z}, then A is a tree. Hence as in the case of groups, if we take A_z to be the trivial subgroup of G{z}, then $E_A = G$.

We showed in 2.1.1. how A might be obtained from a subgroup A_z of the vertex group G {z}. In that case E_A depends on the following choices:-

(i) The tree $T \subseteq G$

(ii) The object $z \in G^{ob}$

(iii) The normal subgroup A_z.

We now investigate the relation between different E_A's obtained by changing each of the above choices.

Lemma 2.1.4:

Let T' be another tree used in the construction of A in 2.1.1. Let $\xi_{T'}: G \simeq G\{z\} * T'$, and let $A' = \xi^{-1}_{T'}(A_z * T')$.

Then $\forall x \in G^{ob}$, $A\{x\} = A'\{x\}$.

Proof: For any $x \in G^{ob}$, let τ'_x denote the unique element of $T'(x, z)$.

Then, by construction, we have:-

$\forall \beta \in A\{x\}$, $\exists$ a unique $\alpha \in A_z$ s.t. $\beta = \tau_x + \alpha - \tau_x$.

Let $\lambda = -\tau_x + \tau'_x \in G\{z\}$, then:-

$$\tau_x = \tau'_x - \lambda.$$

Hence $\beta = \tau'_x - \lambda + \alpha + \lambda - \tau'_x$.

But $A_z \lhd G\{z\} \Longrightarrow -\lambda + \alpha + \lambda \in A_z$. Hence $\beta \in A'\{x\}$ (by construction). Therefore $A\{x\} \subseteq A'\{x\}$. Similarly $A'\{x\} \subseteq A\{x\}$

q.e.d.

Corollary 2.1.5: $E_A = E_{A'}$, in other words, the change of T has no effect on E_A.

Proof: $\forall [a]_A \in E_A$, we have:-

$$[a]_A = \{b \in G(ia, \phi a) \mid a - b \in A\{ia\}\}$$

$$= \{b \in G(ia, \phi a) \mid a - b \in A'\{ia\}\} \text{ (by 2.1.4)}$$

$$= [a]_{A'} \in E_{A'}.$$

Hence $E_A \subseteq E_{A'}$. Similarly $E_{A'} \subseteq E_A$.

q.e.d.

<u>Definition 2.1.6</u>: Let G be a groupoid, then subgroups $A \subseteq G\{x\}$, $B \subseteq G\{y\}$ are <u>said to be conjugate</u> in G, if $\exists \lambda \in G(x, y)$ such that $-\lambda + A + \lambda = B$.

<u>Notice</u> that if A and B are subgroups of the same vertex groups, then this definition is the usual definition for conjugate subgroups.

The other two cases are dealt with in the following lemma, simultaneously.

<u>Lemma 2.1.7</u>:

<u>Let $B_y \lhd G\{y\}$ be conjugate to A_z in G, then $E_A = E_B$.</u>

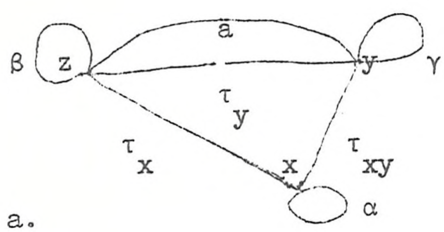
<u>Proof</u>: Let $\xi_T : G \approx G\{y\} * T$, and let $B = \xi_T^{-1}(B_y * T)$. Then we show that $\forall x \in G^{ob}$, $A\{x\} = B\{x\}$.

Let $\alpha \in B\{x\}$, then $\exists \gamma \in B_y$ s.t. $\alpha = \tau_{xy} + \gamma - \tau_{xy}$. (1)

Since B_y is conjugate to A_z,

$\exists a \in G(z, y)$ s.t.

$-a + A_z + a = B_y$.



Hence, $\gamma \in B_y \Longrightarrow \exists \beta \in A_z$ s.t.

$\gamma = -a + \beta + a$. (2)

Let $\beta^1 = a + \tau_y \in G\{z\}$, then it follows from (1) & (2) that:-

$\alpha = \tau_{xy} - a + \beta + a - \tau_{xy} = \tau_{xy} + \tau_y - \beta^1 + \beta + \beta^1 - \tau_y - \tau_{xy}$.

But $\tau_{xy} + \tau_y = \tau_x$, and by normality of A_z, $-\beta^1 + \beta + \beta^1 \in A_z$.

Hence $\alpha = \tau_x + (-\beta^1 + \beta + \beta^1) - \tau_x \in A\{x\}$.

Therefore $B\{x\} \subseteq A\{x\}$. Similarly $A\{x\} \subseteq B\{x\}$.

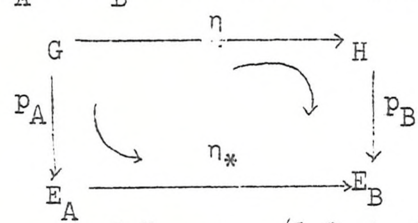
Hence, as with the proof of corollary 2.1.5, it follows that

$E_A = E_B$. q.e.d.

Let $\eta : G \longrightarrow H$ be a morphism of connected groupoids, and let $A \subseteq G$, $B \subseteq H$ be wide normal subgroupoids s.t. $\forall x \in G^{ob}$, $\eta(\Lambda\{x\}) \subseteq B\{\eta(x)\}$

We now investigate the effect on η of passage to the quotient.

η induces a morphism $\eta_*: E_\Lambda \longrightarrow E_B$, with the commutative diagram



as follows:-

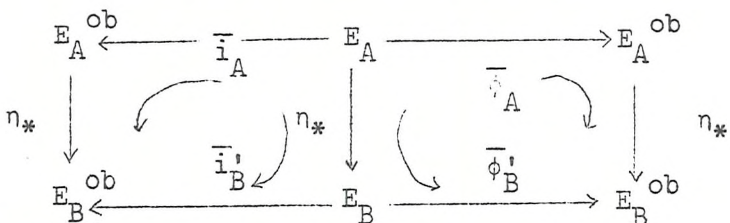
$$\eta_* | (E_\Lambda{}^{ob} = G^{ob}) = \eta, \text{ and } \forall [a]_\Lambda \in E_\Lambda, \ \eta([a]_\Lambda) = [\eta(a)]_B.$$

η_* is well-defined: Let $a' \in [a]_\Lambda$, then $a - a' \in \Lambda\{ia\}$. Hence, by hypothesis $\eta(a - a') = \eta(a) - \eta(a') \in B\{\eta(ia)\}$.

Therefore $[\eta(a')]_B = [\eta(a)]_B$.

Let $G = (G, G^{ob}, i, \phi, \theta, u, \sigma)$ and $H = (H, H^{ob}, i', \phi', \theta', u', \sigma')$.

Then the following diagrams are commutative:-



For, let $[a]_\Lambda \in E_\Lambda$, then:-

$$\eta_* \circ \bar{\phi}_\Lambda([a]_\Lambda) = \eta_*(\phi(a)) = \eta(\phi(a)) = \eta \circ \phi(a).$$

$$\bar{\phi}'_B \circ \eta_*([a]_\Lambda) = \bar{\phi}'_B([\eta(a)]_\Lambda) = \phi'(\eta(a)) = \phi' \circ \eta(a).$$

But η is a morphism of groupoids; so $\eta \circ \phi = \phi' \circ \eta$.

Hence, $\eta_* \circ \bar{\phi}_\Lambda = \bar{\phi}'_B \circ \eta_*$.

Similarly $\eta_* \circ \bar{i}_\Lambda = \bar{i}'_B \circ \eta_*$.

Next, let $([a]_\Lambda, [b]_\Lambda) \in \bar{D}_\Lambda$, then:-

$$\eta_*([a]_\Lambda + [b]_\Lambda) = \eta_*([a + b]_\Lambda) = [\eta(a + b)]_B = [\eta(a) + \eta(b)]_B$$
$$= [\eta(a)]_B + [\eta(b)]_B = \eta_*([a]_\Lambda) + \eta_*([b]_\Lambda).$$

For all $[a]_\Lambda$, $\eta_*(-[a]_\Lambda) = \eta_*([-a]_\Lambda) = [\eta(-a)]_B = [-\eta(a)]_B = -\eta_*[a]_\Lambda)$.

Hence (F_1) and (F_2) are satisfied.

q.e.d.

Remark

Let $\lambda : H \longrightarrow F$ be also a morphism of connected groupoids.

Let $M \subseteq F$ be a wide normal subgroupoid s.t.

$$\forall\, y \varepsilon H^{ob}, \quad \lambda\,(B\,\{y\}\,) \subseteq M\{\,\lambda\,(y)\}$$

Then $\forall\, x \,\varepsilon\, G^{ob}, \lambda \circ \eta\,(A\,\{x\}\,) = \lambda\,(\eta\,(A\{x\}\,)\,) \subseteq \lambda(B\{\eta(x)\}\,)$

$$\subseteq M\,\{\lambda(\eta(x)\,)\} = M\{\,\lambda \circ \eta\,(x)\}$$

Hence, we get induced morphisms:-

$$\lambda_*: E_B \longrightarrow E_M \quad \text{and} \quad (\lambda \circ \eta)_*: E_A \longrightarrow E_M.$$

It is easily seen that:-

$$(\lambda \circ \eta)_* = \lambda_* \circ \eta_* \quad \text{and} \quad (id_G)_* = id_{E_A}$$

i.e. the assignment $\lambda \longrightarrow \lambda_*$ is covariant. Hence we have at once:-

Lemma 2.1.8:

If $\eta: G \longrightarrow H$ is an isomorphism of connected groupoids, then $\eta_*: E_A \longrightarrow E_B$ is also an isomorphism.

2.2 THE GROUPOID G_A

Let A be a wide, connected normal subgroupoid of G, and let E_A be the set of equivalence classes as defined in section 2.1.

Define:-

$$G_A = \{(h,\, [a]_A,\, g) \mid (h,\, a)\,\varepsilon\, D,\ (a,\, g)\,\varepsilon\, D\} \subseteq G \times E_A \times G$$

Then we have:-

Theorem 2.2.1: G_A is a connected groupoid over E_A.

Proof: (i) Define $I_A, \Phi_A: G_A \longrightarrow E_A$ by:-

$$\forall\,(h,\,[a]_A,\,g)\,\varepsilon\, G_A,\quad I_A(h,\,[a]_A,\,g) = [a]_A \quad \ldots\ 2.2.1\ (a)$$

$$\Phi_{\Lambda}(h, [a]_{\Lambda}, g) = [h + a + g]_{\Lambda} \quad \ldots . \; 2.2.1 \text{ (b)}$$

It follows from the normality of Λ that the 'final' function ϕ_{Λ} is independent of representatives in $[a]_{\Lambda}$. For,

$$a^1 \varepsilon [a]_{\Lambda} \Longrightarrow a - a^1 \varepsilon \Lambda \{ia\} \Longrightarrow h + a - a^1 - h \varepsilon \Lambda \{ih\}$$

(by normality of Λ)

$$\Longrightarrow (h + a + g) - (h + a^1 + g) \varepsilon \Lambda \{ih\}$$

$$\Longrightarrow [h + a + g]_{\Lambda} = [h + a^1 + g]_{\Lambda}.$$

(ii) Let $\widetilde{D}_{\Lambda} = \left\{ \left((h, [a]_{\Lambda}, g), (h^1, [b]_{\Lambda}, g^1) \right) \middle| [h + a + g]_{\Lambda} = [b]_{\Lambda} \right\}$

Define $\Theta_{\Lambda}: \widetilde{D}_{\Lambda} \longrightarrow G_{\Lambda}$

by:- $\Theta\left((h, [a]_{\Lambda}, g), (h^1, [b]_{\Lambda}, g^1) \right) = (h, [a]_{\Lambda}, g) + (h^1, [b]_{\Lambda}, g^1)$

$$= (h^1 + h, [a]_{\Lambda}, g + g^1)$$

(<u>Note change of order in $h^1 + h$</u>)

Θ_{Λ} <u>satisfies the associative law:</u>

Let $(X, Y) = \left((h, [a]_{\Lambda}, g), (h^1, [b]_{\Lambda}, g^1) \right) \varepsilon \widetilde{D}_{\Lambda}$

and $(Y, Z) = \left((h^1, [b]_{\Lambda}, g^1), (h^2, [c]_{\Lambda}, g^2) \right) \varepsilon \widetilde{D}_{\Lambda}$

Then $[h + a + g]_{\Lambda} = [b]_{\Lambda}$ and $[h^1 + b + g^1]_{\Lambda} = [c]_{\Lambda}$ (by 2.2.1 (a) & (b))

Hence $[h^1 + (h + a + g) + g^1]_{\Lambda} = [c]_{\Lambda}$ i.e.

$$\phi_{\Lambda}(h^1 + h, [a]_{\Lambda}, g + g^1) = I_{\Lambda}(Z) = [c]_{\Lambda}.$$

Therefore:- $\left((h^1 + h, [a]_{\Lambda}, g + g^1), Z \right) \varepsilon \widetilde{D}_{\Lambda}$ (*)

Similarly, $\left(X, (h^2 + h^1, [b]_{\Lambda}, g^1 + g^2) \right) \varepsilon \widetilde{D}_{\Lambda}$

By associativity in G and E_{Λ}, $(X + Y) + Z = X + (Y + Z)$, since each sum exist (by *)

(iii) Define the unit map $U_{\Lambda}: E_{\Lambda} \longrightarrow G_{\Lambda}$ by:-

$$\forall [a]_{\Lambda} \varepsilon E_{\Lambda}, U_{\Lambda}([a]_{\Lambda}) = (0_{ia}, [a]_{\Lambda}, 0_{\phi a}).$$

It is easily seen that this satisfies the required conditions for the unit function.

(iv) Define the inverse function $\Sigma_A : G_A \longrightarrow G_A$

by:- $\forall\ (h, [a]_A, g) \varepsilon G_A$, $\Sigma_A(h, [a]_A, g) = (-h, [h + a + g]_A, -g)$

For, if $X = (h, [a]_A, g)$, it is easy to verify that:-

$\Phi_A (X) = I_A (\Sigma_A(X))$ & $I_A (X) = \Phi_A (\Sigma(X))$;

thus $(X, \Sigma_A(X))$, and $(\Sigma_A (X), X) \varepsilon \widetilde{D}_A$

and $X + \Sigma_A (X) = O_{[a]_A} = (o_{ia}, [a]_A, o_{\phi a})$

$\Sigma_A (X) + X = O_{[h + a + g]_A} = (o_{ih}, [h + a + g]_A, o_{\phi g})$.

So far, we have proved that $(G_A, E_A, I_A, \phi_A, U_A, \Sigma_A)$ is a groupoid. We next show that it is connected.

Let $[a]_A, [b]_A \varepsilon E_A$, then, by connectedness of $G, \exists g \varepsilon G(\phi(a), \phi(b))$

Let $h = b - g - a$, so $(h, [a]_A, g) \varepsilon G_A$, and by 2.2.1 (a) & (b),

$(h, [a]_A, g) \varepsilon G_A ([a]_A, [b]_A) \neq \Phi$

q.e.d.

Behaviour of G_A under morphisms

Let $\eta : G \longrightarrow H$ be a connected morphism of groupoids satisfying the conditions in (2.1.8). Then η induces a morphism

$$\bar{\eta}_* : G_A \longrightarrow H_B$$

as follows:-

(i) $\bar{\eta}_* | E_A = \eta_*$ (as defined earlier)

(ii) $\forall\ (h, [a]_A, g) \varepsilon G_A$, $\bar{\eta}_*(h, [a]_A, g) = (\eta(h), [\eta(a)]_B, \eta(g))$

Since $[\eta(a)]_B$ is independent of any representative in $[a]_A$ (see discussion preceding 2.1.8.), $\bar{\eta}_*$ is well-defined. Moreover:-

$\bar{\eta}_*$ is a morphism of groupoids: For, $\forall\ (h, [a]_A, g) \varepsilon G_A$, we have:-

$\bar{\eta}_* \circ I_A(h, [a]_A, g) = \bar{\eta}_*([a]_A) = \eta_*([a]_A) = [\eta(a)]_B$

$I_B \circ \bar{\eta}_*(h, [a]_A, g) = I_B(\eta(h), [\eta(a)]_B, \eta(g)) = [\eta(a)]_B$

Hence $\bar{\eta}_* \circ I_A = I_B \circ \bar{\eta}_*$. Similarly $\bar{\eta}_* \circ \Phi_A = \Phi_B \circ \bar{\eta}_*$.

Therefore $\bar{\eta}_*$ commutes with the initial and final functions.

Next, $\forall\ (o_{ia}, [a]_A, o_{\phi a})$, $\bar{\eta}_*(o_{ia}, [a]_A, o_{\phi a}) = \Big(\eta(o_{ia}), [\eta(a)]_B,$

$$\eta(o_{\phi a})\Big) = \left(o_{\eta(ia)}, [\eta(a)]_B, o_{\eta(\phi a)}\right) = \left(o_{i(\eta(a))}, [\eta(a)]_B, o_{\phi(\eta(a))}\right)$$ (for η is a morphism of groupoids)

Hence $\bar{\eta}_*$ commutes with unit functions. So, it remains to show

that $\bar{\eta}_*$ commutes with the composition functions.

Let $\big((h, [a]_A, g), (h', [b]_A, g')\big) \in \widetilde{D}_A$, then:-

$$\bar{\eta}_*\Big((h, [a]_A, g) + (h', [b]_A, g')\Big) = \bar{\eta}_*(h' + h, [a]_A, g + g')$$

$$= \Big(\eta(h' + h), [\eta(a)]_B, \eta(g + g')\Big)$$

$$= \Big(\eta(h') + \eta(h), [\eta(a)]_B, \eta(g) + \eta(g')\Big) \quad (\because \eta \text{ is a morphism})$$

$$= (\eta(h), [\eta(a)]_B, \eta(g)) + (\eta(h'), [\eta(b)]_B, \eta(g'))$$

$$= \bar{\eta}_*(h, [a]_A, g) + \bar{\eta}_*(h, [b]_A, g')$$

<u>Remark</u>:

Let $\lambda: H \longrightarrow F$ be a morphism of groupoids satisfying the same conditions as η. Then, in the same manner as in Remark preceding 2.1.8. the assignment $\eta \longrightarrow \bar{\eta}_*$ is covariant, so:-

<u>Theorem 2.2.2</u>:

<u>If $\eta: G \longrightarrow H$ is an isomorphism of connected groupoids. Then $\bar{\eta}_*: G_A \longrightarrow H_B$ is an isomorphism.</u>

q.e.d.

We now show the connection between G_A and G.

Define the functions $r, \ell: G_\Lambda \longrightarrow G$ by:-

$\ell | E_\Lambda = \bar{i}_\Lambda : E_\Lambda \longrightarrow G^{ob}$ & $\ell(h, [a]_\Lambda, g) = -h$.

$r | E_\Lambda = \bar{\Phi}_\Lambda : E_\Lambda \longrightarrow G^{ob}$ & $r(h, [a]_\Lambda, g) = g$

Reason for minus sign will be seen in 2.2.3. below. Recall change of order $h' + h$ in definition earlier.

<u>Lemma 2.2.3</u>:

$\ell, r: G_\Lambda \longrightarrow G$, <u>as defined above, are morphism of groupoids</u>.

<u>Proof</u>: Since the proofs in both cases are similar, we prove only one of them, say for ℓ. It follows from the definition that (F_1) is satisfied. $\forall (h, [a]_\Lambda, g) \in G_\Lambda$, we have:-

$$\ell \circ \bar{\phi}_\Lambda (h, [a]_\Lambda, g) = \ell\left(\phi_\Lambda(h, [a]_\Lambda, g)\right) = \ell\left([h + a + g]_\Lambda\right)$$
$$= \bar{i}_\Lambda([h + a + g]_\Lambda) = i(h + a + g) = i(h)$$

$$\phi \circ \ell(h, [a]_\Lambda, g) = \phi(-h) = i(h)$$

Hence $\ell \circ \bar{\phi}_\Lambda = \phi \circ \ell$. Similarly $\ell \circ \bar{i}_\Lambda = i \circ \ell$.

$$\forall [a]_\Lambda \in E_\Lambda = (G_\Lambda)^{ob}, \quad \ell(o_{ia}, [a]_\Lambda, o_{\phi a}) = -o_{ia} = o_{ia}.$$

$\forall \left((h, [a]_\Lambda, g), (h', [b]_\Lambda, g')\right) \in \tilde{D}_\Lambda$, we have:-

$$\ell(h, [a]_\Lambda, g) + (h', [b]_\Lambda, g') = \ell(h' + h, [a]_\Lambda, g + g')$$
$$= -(h' + h) = -h - h' = \ell(h, [a]_\Lambda, g) + \ell(h', [b]_\Lambda, g')$$

Hence F_2 is satisfied. Therefore ℓ is a covariant morphism.

q.e.d.

Obviously, $(\ell, r): G_\Lambda \longrightarrow G \times G$ is a morphism, but more can be said. One of the main and important connections between G and G_Λ is that (ℓ, r) is a <u>covering</u> morphism in sense of [2]. Thus:-

<u>Theorem 2.2.4</u>:

<u>$(\ell, r): G_\Lambda \longrightarrow G \times G$ is a covering morphism.</u>

Proof: We must show that for each $[a]_\Lambda \in E_\Lambda = (G_\Lambda)^{ob}$, the restriction of (ℓ, r) to $St_{G_\Lambda}[a]_\Lambda$ is a 1 - 1 function onto $St_{G \times G}(\ell, r)[a]_\Lambda = St_{G \times G}(ia, \phi a)$.

(i) $\forall (h, [a]_\Lambda, g), (h', [a]_\Lambda, g') \in St[a]_\Lambda$, we have:-

$$(\ell, r)(h, [a]_\Lambda, g) = (\ell, r)(h', [a]_\Lambda, g') \Longrightarrow (-h, g) = (-h', g')$$
$$\Longrightarrow h = h', g = g'$$

Hence $(h, [a]_\Lambda, g) = (h', [a]_\Lambda, g'$, which shows that (ℓ, r) is 1 - 1 on $St[a]_\Lambda$.

(ii) Given any $(h, g) \in St_{G\times G}(ia, \phi a)$, we have $i(h) = ia$ and $i(g) = \phi a$ (by definition of product of categories). Hence $(-h, a), (a, g) \in D$. So, $(-h, [a]_\Lambda, g) \in St[a]_\Lambda$, and we have $(\ell, r)(-h, [a]_\Lambda, g) = (h, g)$.

Therefore the restriction is onto.

q.e.d.

Consider now an application, to the case when $G = \pi X$.

Let $G = \pi X$, where X is a path-connected, locally path-connected and locally simply connected space; then $E_\Lambda = G_\Lambda{}^{ob}$ has a topology, called "the lifted topology" denoted by L - topology, which turn it into a covering space of $(G \times G)^{ob} = X \times X$ (see[2]p. 309).

Therefore we have:-

Corollary 2.2.5:

Let $G = \pi X$, where X is a path-connected, locally path-connected and locally simply connected space. Then E_Λ (and hence πX as a special case) with the lifted topology is a covering space of $X \times X$, with the projection $(\bar{i}_\Lambda, \bar{\phi}_\Lambda): E_\Lambda \longrightarrow X \times X$.

(The equivalence of this topology to that obtained from LCO-topology, as a quotient space, will be proved in next section).

Notation: In case $G = \pi X$, we denote E_Λ by $\pi_\Lambda X$, to distinguish it from the general case. The space X will be assumed to be path-connected, locally path-connected and locally simply connected for the rest of the chapter.

Lemma 2.2.6:

For any $[a]_\Lambda \in E_\Lambda$,

$$(\ell, r)(G_\Lambda\{[a]_\Lambda\}) = \{(f, g) \in G\{ia\} \times G\{\phi a\} \mid -f + a + g - a \in \Lambda\{ia\}\}$$

Proof: Let $(h, [a]_\Lambda, g) \in G_\Lambda\{[a]_\Lambda\}$, then $[h + a + g]_\Lambda = \Phi_\Lambda(h, [a]_\Lambda, g)$

$$= [a]_\Lambda$$

Therefore:- $h + a + g - a \in \Lambda\{ia\}$

and $i(h), = \phi(h) = ia,\ i(g) = \phi(g) = \phi a.$

so, $-h \in G\{ia\}$ and $g \in G\{\phi a\}$. Hence:-

$$(\ell, r)(h, [a]_\Lambda, g) = (-h, g) \in G\{ia\} \times G\{\phi a\}$$

Conversely, let $(f, g) \in G\{ia\} \times G\{\phi a\}$ such that $-f + a + g - a \in \Lambda\{ia\}$

Then $[-f + a + g]_\Lambda = [a]_\Lambda$ (by definition of R_Λ).

Hence $(-f, [a]_\Lambda, g) \in G_\Lambda\{[a]_\Lambda\}$, and we have:-

$$(f, g) = (\ell, r)(-f, [a]_\Lambda, g) \in (\ell, r)(G_\Lambda\{[a]_\Lambda\}).$$

q.e.d.

Now suppose that $\Lambda\{x\}$ is in the centre of $G\{x\}$, then:-

Theorem 2.2.7:

For each $x \in G^{ob}$, $G_\Lambda\{[o_x]_\Lambda\} \approx G\{x\} \times \Lambda\{x\}$

Proof: Since (ℓ, r) is a covering morphism, $G_\Lambda\{[a]_\Lambda\} \approx (\ell, r)(G_\Lambda\{[o_x]_\Lambda\})$

By 2.2.6 $(\ell, r)(G_\Lambda\{[o_x]_\Lambda\}) = \{(f, g) \in G\{x\} \times G\{x\} \mid -f + g \in \Lambda\{x\}\}$

$$= \{(f, f + a) \mid f \in G\{x\},\ a \in \Lambda\{x\}\}$$

$$= H, \text{ say. } (\subseteq G\{x\} \times G\{x\})$$

Define $\psi : H \longrightarrow G\{x\} \times \Lambda\{x\}$

by:- $\forall (f, f + a) \in H, \ \psi(f, f + a) = (f, a)$

ψ is a homomorphism:

$$\psi((f, f + a) + (g, g + b)) = \psi(f + g, f + a + g + b)$$
$$= \psi(f + g, f + g + a + b) \text{ (by hypothesis)}$$
$$= (f + g, a + b) = (f, a) + (g, b)$$
$$= \psi(f, f + a) + \psi(g, g + b)$$

Obviously ψ is 1 - 1 and onto. Hence ψ is an isomorphism.

q.e.d.

Corollary 2.2.8:

If Λ is a tree subgroupoid, then $G_\Lambda\{[o_x]_\Lambda\} \approx G\{x\}$

Corollary 2.2.9:

If $\Lambda\{x\}$ is in the centre of $\pi_1(X, x)$,

$$\pi_1(\pi_\Lambda X, o_{x_\Lambda}) \approx \pi_1(X, x) \times \Lambda\{x\}$$

Proof: By 2.2.7, $\pi X_\Lambda\{[o_x]_\Lambda\} \approx \pi X\{x\} \times \Lambda\{x\} = \pi_1(X, x) \times \Lambda\{x\}$.

On the other hand $\pi(\pi_\Lambda X) \approx (\pi X)_\Lambda$ (see [2], 9.5.5.)

Hence $\pi_1(\pi_\Lambda X, [o_x]_\Lambda) \approx (\pi X)_\Lambda\{[o_x]_\Lambda\} \approx \pi_1(X, x) \times \Lambda\{x\}$.

q.e.d.

Corollary 2.2.10:

$$\forall x \in X, \ \pi_1(\pi X, o_x) \approx \pi_1(X, x)$$

Proof: If we take Λ to be a tree groupoid, then $\pi_\Lambda X = \pi X$. Hence by 2.2.9, $\pi_1(\pi X, o_x) \approx \pi_1(X, x) \times \{o_x\} \approx \pi_1(X, x)$

q.e.d.

Theorem 2.2.11:

Let A and B be two wide connected normal subgroupoids of the connected groupoid G s.t. $\forall\ x \in G^{ob}$, $A\{x\} \subseteq B\{x\}$.

Then G_A is a covering groupoid of G_B.

Proof: It follows from 2.2.7. that the characteristic group of $(\ell, r): G_A \longrightarrow G \times G$ at $[a]_A$ is contained in the characteristic group of $(\ell, r)_B: G_B \longrightarrow G \times G$ at $[a]_B$.

Hence, $\exists$ a unique covering morphism

$$P_{AB}: G_A, [a]_A \longrightarrow G_B, [a]_B$$

s.t. $(\ell, r)_A = (\ell, r)_B \circ P_{AB}$.

(see[2]p. 300)

q.e.d.

Corollary 2.2.12:

If $G = \pi X$, then for any $A_x \subseteq B_x$ normal in $\pi_1(X, x)$, $\pi_A X$ is a covering space of $\pi_B X$.

Corollary 2.2.13:

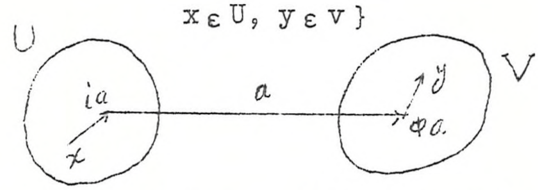
$\forall\ A_x \lhd \pi_1(X, x)$, πX is a covering space of $\pi_A X$.

3. EQUIVALENCE OF L-TOPOLOGY ON $\pi_A X$ TO THE QUOTIENT TOPOLOGY.

We saw in section 2 that $\pi_A X$ carries the L-topology which turn it into a covering space of $x \times X$. Since $\pi_A X$ is a quotient set of πX, it has a natural topology which is the quotient topology obtained from that of πX. Our aim is to show that these two topologies are equivalent. For this, we define another topology, suggested by the classical topology of covering spaces, which will be referred to as "C - N topology" (an abbreviation for canonical-neighbourhood topology), and show that both L-topology and the quotient topology on $\pi_\Lambda X$ are equivalent to this topology. It turns out that the C - N topology is more convenient in practice.

C - N topology: Let $[a]_\Lambda \epsilon \pi_\Lambda X$, and let U and V be two canonical neighbourhoods of ia and ϕa, respectively, obtained from the property of X, i.e. every loop in U and V is null-homotopic in X. Let PU and PV denote the categories of paths in U and V respectively, then define $< U, [a]_\Lambda, V> = \{[\gamma + a + \lambda]_\Lambda \mid \gamma \geq PU(x, ia), \lambda \geq PV(\phi a, y),$

$x \epsilon U, y \epsilon V\}$



In other words each element of $<U, [a]_\Lambda, V>$ is represented by an element of πX which is a composition of three roads, where the first road has a representative path in PU and the third one has a representative in PV, and the middle term is always a.

Let $\{U\}_{ia}$, $\{V\}_{\phi a}$ be bases for the canonical neighbourhoods of ia and ϕa, respectively, in X. Then, we define C - N topology for $\pi_\Lambda X$ by taking $\{<U, [a]_\Lambda, V> \mid U \epsilon \{U\}_{ia}, V \epsilon \{V\}_{\phi a}\}$ as a basis for the neighbourhoods of each $[a]_\Lambda$ in $\pi_\Lambda X$.

We now verify that this system of neighbourhoods of $[a]_\Lambda$ actually satisfies the conditions required for a basis of a topology on $\pi_\Lambda X$

(i) For any $<U, [a]_\Lambda, V>$, we have $[a]_\Lambda = [o_{ia} + a + o_{\phi a}]_\Lambda \epsilon <U, [a]_\Lambda, V>$

(ii) Given $<U, [a]_\Lambda, V>$, $<U', [a]_\Lambda, V'>$ with non-empty intersection, we must show that $\exists <U_1, [a]_\Lambda, V_1> \subseteq <U, [a]_\Lambda, V> \cap <U', [a]_\Lambda, V'>$.

Since $<U, [a]_\Lambda, V> \cap <U', [a]_\Lambda, V'> \neq \phi$

we have $U \cap U' \neq \phi$, $V \cap V' \neq \phi$

Hence, $\exists\, U_1 \varepsilon \{U\}_{ia}$ and $V_1 \varepsilon \{V\}_{\phi a}$ s.t. $U_1 \subseteq U \cap U'$ & $V_1 \subseteq V \cap V'$.

<u>Claim</u>: $<U_1, [a]_A, V_1> \;\subseteq\; <U, [a]_A, V> \cap < U', [a]_A, V'>$

Let $[b]_A \varepsilon <U_1, [a]_A, V_1>$, then $b = \gamma_1 + a + \lambda_1$, where $\gamma_1 \supseteq PU_1\ (ib, ia)$ and $\lambda_1 \supseteq PV_1(\phi a, \phi b)$. Moreover $ib \varepsilon\ U_1 \subseteq U \cap U'$ & $\phi b \varepsilon\ V_1 \subseteq V \cap V'$.

Then:-

$$U_1 \subseteq U \Longrightarrow PU_1 \subseteq PU \longrightarrow \gamma_1 \supseteq PU(ib, ia)$$

$$U_1 \subseteq U' \Longrightarrow PU_1 \subseteq PU' \longrightarrow \gamma_1 \supseteq PU'(ib, ia)$$

$$V_1 \subseteq V \longrightarrow PV_1 \subseteq PV \longrightarrow \gamma_1 \supseteq PV_1(\phi a, \phi b)$$

$$V_1 \subseteq V' \longrightarrow PV_1 \subseteq PV' \longrightarrow \gamma_1 \supseteq PV'(\phi a, \phi b)$$

Hence $[b]_A\ \varepsilon <U, [a]_A, V>$ & $[b]_A \varepsilon < U', [a]_A, V'>$. Therefore:-

$$[b]_A \varepsilon < U, [a]_A, V> \cap <U', [a]_A, V'>$$

(iii) Let $[b]_A \varepsilon < U, [a]_A, V>$ we must show that

$$\exists <U', [b]_A, V'> \text{s.t.} <U', [b]_A, V'> \subseteq <U, [a]_A, V>$$

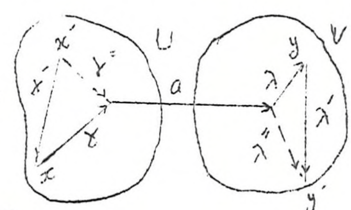
But this follows from the following lemma:-

<u>Lemma 2.3.1:</u>

<u>If $[b]_A\ \varepsilon <U, [a]_A, V>$, then $<U, [b]_A, V> \; = <U, [a]_A, V>$</u>

<u>Proof</u>: Let $ib = x$ and $\phi b = y$, then $\exists \gamma, \lambda \varepsilon\ \pi X$ s.t. $\gamma \supseteq PU(x, ia)$ and

$\lambda \supseteq PV(\phi a, y)$, and $b = \gamma + a + \lambda$



Let $[c]_A\ \varepsilon < U, [b]_A, V >$, then $c = \gamma' + a + \lambda'$ where $\gamma' \supseteq PU(x', x), \lambda' \supseteq PV(y, y')$

$x' = ic$ and $y' = \phi c$. Hence $[c]_A = [\gamma' + b + \lambda']_A =$

$[\gamma' + (\gamma + a + \lambda) + \lambda']_A = [\gamma'' + a + \lambda'']_A \varepsilon < U, [a]_A, V >$

where $\gamma'' = \gamma' + \gamma \supseteq PU(x', ia)$ and $\lambda'' = \lambda + \lambda' \supseteq PV'(\phi a, y')$

Therefore $<U, [b]_A, V > \subseteq <U, [a]_A, V>$

<u>Conversely</u>, let $[d]_A \varepsilon <U, [a]_A, V >$, then $d = \gamma_1 + a + \lambda_1$, where $\gamma_1 \supseteq PU(id, ia)$ and $\lambda_1 \supseteq PV\ (\phi a, \phi d)$. Taking $\gamma' = \gamma_1 - \gamma$ and $\lambda' = -\lambda + \lambda_1$, we get:-

$$[d]_A = [\gamma_1 + a + \lambda_1]_A = [\gamma' + \gamma + a + \lambda + \lambda']_A = [\gamma' + b + \lambda']_A$$
$$\varepsilon < U, [a]_A, V>$$

Hence $< U, [a]_A, V> \subseteq < U, [b]_A, V>$

Therefore $< U, [a]_A, V> = < U, [b]_A, V>$

q.e.d.

Remark: $St_{\pi_A} X$ is the usual covering space of X (with usual topology)

Theorem 2.3.2:

The C - N topology and the quotient topology on E_A are equivalent.

Proof: We first show that any basic open set $<U, [a]_A, V>$ in C - N topology is open in the quotient topology. For this we must show that $M = \{f \varepsilon pX \mid [\overline{f}]_A \varepsilon <U, [a]_A, V>\}$ is open in PX. Let $\lambda \varepsilon$ M be a path of length r, say, then $\lambda(o) \varepsilon$ U and $\lambda(r) \varepsilon$ V. Since $\lambda(I_r)$ is compact, it can be covered by a finite number of simply connected open sets $W_1 = U, W_2, \ldots\ldots, W_n = V$ of X. Hence, we can subdivide I_r into intervals

$$K_j = [t_{j-1}, t_j] , j = 1, \ldots\ldots, n$$

such that $t_o = o, t_n = r$ and $\lambda_r(K_j) \subseteq W_j$, $j = 1, \ldots\ldots, n$

Moreover we can choose K_j, j=1, ..., n in such a way that λ is not constant in some neighbourhood of t_j, $j = 1, \ldots\ldots, n - 1$.

For each $\lambda(t_j)$, we can choose a path-connected neighbourhood W'_j such that:

$$W'_o = U, W'_n = V \text{ and } W'_j \subseteq W_j \cap W_{j+1}, j \neq o, n.$$

Since $\lambda(t_j) \varepsilon W_j \cap W_{j+1}$, W'_j exists and even can be simply connected.

Let $o<\varepsilon<t_n - t_{n-1}$, and let $K'_n = \left[t_{n-1}, r+\varepsilon\right] \supseteq K_n$. Then clearly

$$N = \tau(K_1, W_1) \cap \ldots \cap \tau(K'_n, W_n) \cap \tau(\{t\}, W'_1) \cap \ldots \cap \tau(\{t_{n-1}\}, W'_{n-1}) \cap \eta^{-1}(r-\varepsilon, r+\varepsilon)$$

is an open neighbourhood of λ in PX (η being the length map).

<u>Claim</u>: $N \subseteq M$. Let $\mu_s \varepsilon N$, then it is easy to find t'_1, t'_2, $t'_{n-1} \varepsilon I_s$ such that $\mu(t'_i) \varepsilon W'_i$, $i = 1, \ldots, n-1$, and μ is not constant on some neighbourhood of t'_i, $i = 1, \ldots, n-1$. Clearly $\mu[t'_i, t'_{i+1}] \subseteq W_i$, $i = 1, \ldots, n-1$,

Define:-

$$\lambda_j : \mathbb{R}^+ \longrightarrow X, \ j = 1, \ldots, n \text{ by: } \lambda_j(t) = \begin{cases} \lambda(t+t_{j-1}), & o \leq t \leq t_j - t_{j-1} \\ \lambda(t_j), & t \geq t_j - t_{j-1} \end{cases}$$

$$\mu_j : \mathbb{R}^+ \longrightarrow X, \ j = 1, \ldots, n-1 \text{ by: } \mu_j(t) = \begin{cases} \mu(t+t'_{j-1}) & 0 \leq t \leq t'_j - t'_{j-1} \\ \mu(t'_j) & t \geq t'_j - t'_{j-1} \end{cases}$$

and $\mu_n : \mathbb{R}^+ \longrightarrow X$ by: $\forall\, t \,\varepsilon \mathbb{R}^+$, $\mu_n(t) = \mu(t + t'_{n-1})$

clearly $\lambda_j (i = 1, \ldots, n)$, $\mu_j (j = 1, \ldots, n-1)$ are of length $t_j - t_{j-1}$ and $t'_j - t'_{j-1}$, respectively, and $\eta(\mu_n) = s - t'_{n-1}$

We have:- $\lambda_j(o) = \lambda(t_{j-1}) = \lambda_{j-1}(t_{j-1} - t_{j-2})$ and $\lambda_j(\mathbb{R}^+) \subseteq W_j$, $j = 1, \ldots, n$.

$$\mu_j(o) = \mu(t'_{j-1}) = \mu_{j-1}(t'_{j-1} - t'_{j-2}) \text{ and } \mu_j(\mathbb{R}^+) \subseteq W_j, \ j = 1, \ldots, n.$$

Hence $\lambda_{j-1} + \lambda_j$ and $\mu_{j-1} + \mu_j$ are defined for $j = 1, \ldots, n$; and we have:-

$$\begin{aligned} \lambda &= \lambda_1 + \lambda_2 + \ldots + \lambda_n \\ \mu &= \mu_1 + \mu_2 + \ldots + \mu_n \end{aligned} \qquad (*)$$

Let $\gamma_j \varepsilon PW'_j(\lambda(t_j), \mu(t'_j))$, $j = o, 1, \ldots, n-1$ and

$$\gamma_n \varepsilon\ PW'_n(\lambda(r), \mu(s))$$

Then $-\gamma_{j-1} + \lambda_j + \gamma_j \in PW_j(\mu_j(o), \mu_j(t'_j - t'_{j-1}))$, $j = 1, \ldots, n-1$

and $-\gamma_{n-1} + \lambda_n + \gamma_n \in PW_n(\mu(t_{n-1}), \mu(s))$

Hence, by simply connectedness of W_j's, we have:-

$\overline{-\gamma_{j-1}} + \overline{\lambda_j} + \overline{\gamma_j} = \overline{\mu_j} \in \pi X(\mu(t'_{j-1}), \mu(t'_j))$, $j = 1, \ldots, n-1$

and $\overline{-\gamma_{n-1}} + \overline{\lambda_n} + \overline{\gamma_n} = \overline{\mu_n} \in \pi X(\lambda(r), \mu(s))$

$$\begin{aligned} \text{Therefore:-}\ \overline{\mu} &= (\overline{-\gamma_0} + \overline{\lambda_1} + \overline{\gamma_1}) + \ldots + (\overline{-\gamma_{n-1}} + \overline{\lambda_n} + \overline{\gamma_n}) \\ &= \overline{-\gamma_0} + \overline{\lambda_1} + \overline{\lambda_2} + \ldots + \overline{\lambda_n} + \overline{\gamma_n} \\ &= \overline{-\gamma_0} + \overline{\lambda} + \overline{\gamma_n}. \qquad \text{(by (*))} \end{aligned}$$

Since $\overline{\gamma_0} \supseteq PW'_0(\lambda(t_0), \mu(t_0)) = PU(\lambda(o), \mu(o))$

and $\overline{\gamma_n} \supseteq PW'_n(\lambda(r), \mu(s)) = PV(\lambda(r), \mu(s))$

we get $[\overline{\mu}]_A \in \langle U, [\overline{\lambda}]_a, V \rangle = \langle U, [a]_A, V \rangle$ (by 2.3.1.)

Hence $\mu \in M$, and $N \subseteq M$. Therefore $\langle U, [a]_A, V \rangle$ is open in the quotient topology.

<u>Conversely</u>, let $M \subseteq \pi_A X$ be open in the quotient topology; we show that it is open in C - N topology.

Let $[a]_A \in M$, then $N = \{f \in PX \mid [\overline{f}]_A \in M\}$

is open in PX and contains any representative path λ_r of a.

Then $\exists$ closed intervals $K_1, K_2, \ldots, K_m \subseteq I_r$, open sets $U_1, \ldots, U_m \subseteq X$,

and $\varepsilon > o$ such that $\lambda \in N' \cap \eta^{-1}(r - \varepsilon, r + \varepsilon) \subseteq N$, where $N' = \bigcap_{i=1}^{m} \tau(K_i, U_i)$.

Now, let $K = \{j \mid o \in K_j, o \leq j \leq m\}$

$L = \{i \mid r \in K_i, o \leq i \leq m\}$

Define $U = \begin{cases} \bigcap_{j \in K} U_j, & \text{if } K \neq \emptyset \\ X, & \text{otherwise} \end{cases}$, $V = \begin{cases} \bigcap_{i \in L} U_i, & \text{if } L \neq \emptyset \\ X, & \text{otherwise} \end{cases}$

It is immediate from the definition that $\lambda(o) \in U$, $\lambda(r) \in V$.

Therefore, $\exists$ simply connected open sets U' and V' such that $\lambda(o) \in U' \subseteq U$ and $\lambda(r) \in V' \subseteq V$.

<u>Claim:</u> $\langle U', [a]_A, V' \rangle \subseteq M$

By continuity of λ, $\exists \varepsilon_1, \varepsilon_2 \in R^+$ such that:-

$$\lambda([o, \varepsilon_1]) \subseteq U' \text{ and } \lambda([r - \varepsilon_2, r]) \subseteq V'$$

Moreover,
$$\begin{cases} t \in K_j,\ j \notin K & t > \varepsilon_1 \\ t \in K_i,\ i \notin L & t < r - \varepsilon_2 \end{cases} \quad (**)$$

Let $[b]_A \in \langle U', [a]_A, V' \rangle$, then $b = \bar{\gamma} + a + \bar{\nu}$, where $\bar{\gamma}$ and $\bar{\nu}$ have representatives in PU' and PV' respectively.

Hence, by lemma 1.2.6. $\exists f \in PU'$ and $g \in PV'$, of lengths $\bar{\varepsilon} = \text{minimum } \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ such that:-

$$b = \bar{f} + a + \bar{g} = \overline{f + \lambda + g} = \bar{\mu}, \text{ where } \mu = f + \lambda + g.$$

We now show that $\mu \in N' \cap \eta^{-1}(r - \varepsilon, r + \varepsilon) \subseteq N$

We have $\eta(f + \lambda + g) = \bar{\varepsilon} + r + \bar{\varepsilon} \leq r + \frac{2\varepsilon}{3} \left(\in (r - \varepsilon, r + \varepsilon)\right)$

and

$$\mu(t) = \begin{cases} f(t) & o \leq t \leq \bar{\varepsilon} \\ \lambda(t - \bar{\varepsilon}) & \bar{\varepsilon} \leq t \leq r + \bar{\varepsilon} \\ g(t - r - \bar{\varepsilon}) & t \geq r + \bar{\varepsilon} \end{cases}$$

It is now easily verified (using (**)) that

$$\forall t \in K_j,\ \mu(t) \in U_j,\ j = 1, \ldots, m.$$

Hence $\mu \in \bigcap_{i=1}^{m} \tau(K_i, U_i) \cap \eta^{-1}(r - \varepsilon, r + \varepsilon) \subseteq N$

Therefore $[\bar{\mu}]_A = [b]_A \in M$. This completes the proof of the claim.

Thus M is a neighbourhood of each of its points in the C - N topology. Hence it is open.

q.e.d.

Theorem 2.3.3:

The C - N topology is equivalent to the L - topology.

Proof: This follows from [2]9.5.5. and the following theorem.

Theorem 2.3.4:

$\pi_A X$ with C - N topology is a covering space of $X \times X$ with the projection: $(\ell, r): \pi_A X \longrightarrow X \times X$ as defined in 2.2.4.

Proof: Let $\{U_\alpha\}$ be a cover of X by simply connected open sets. Then $\{U_i \times U_j \mid U_i, U_j \varepsilon \{U_\alpha\}\}$ is an open cover of $X \times X$ and is a basis for the product topology on $X \times X$.

Let $< U, [a]_A, V>$ be a canonical neighbourhood of $[a]_A$ in $\pi_A X$, then:-

$(\ell, r) | < U, [a]_A, V>$ is a homeomorphism onto $U \times V$.

Let $[b]_A, [c]_A \varepsilon < U, [a]_A, V>$, then:-

$(\ell, r)([b]_A) = (\ell, r)([c]_A) \Longrightarrow (ib, \phi b) = (ic, \phi c) \Longrightarrow ib = ic$

$\& \ \phi b = \phi c$ (1)

and $[b]_A \varepsilon <U, [a]_A, V> \Longrightarrow b = \gamma_1 + a + \lambda_1$, where $\gamma_1 \supseteq PU(ib, ia)$,

$\lambda_1 \supseteq PV(\phi a, \phi b)$

$[C]_A \varepsilon <U, [a]_A, V> \Longrightarrow c = \gamma_2 + a + \lambda_2$, where $\gamma_2 \supseteq PU(ic, ia)$,

$\lambda_2 \supseteq PV(\phi a, \phi c)$.

It follows from (1) that:- $i\gamma_1 = i\gamma_2 \ \& \ \phi\lambda_1 = \phi\lambda_2$.

Hence, by property of U and V, $\gamma_1 = \gamma_2$ and $\lambda_1 = \lambda_2$

Therefore $[b]_A = [\gamma_1 + a + \lambda_1]_A = [\gamma_2 + a + \lambda_2]_A = [C]_A$, which shows that (ℓ, r) is 1 - 1 on $<U, [a]_A, V>$.

Next, let $(x, y) \varepsilon U \times V$, then $x \varepsilon U$ and $y \varepsilon V$. Let $\gamma_1 \supseteq PU(x, ia)$ and $\lambda_1 \supseteq PV(\phi a, y)$, then $[\gamma_1 + a + \lambda_1]_A \varepsilon <U, [a]_A, V>$, and we have:-

$(\ell, r)([\gamma_1 + a + \lambda]_A) = (i\gamma_1, \phi\lambda_1) = (x, y)$.

So (ℓ, r) maps $<U, [a]_A, V>$ onto $U \times V$.

But $\{<U, [a]_A, V> \mid U, V \varepsilon \{U_\alpha\}$ and $[a]_A \varepsilon \pi_A X\}$ is a basis for the C - N topology on $\pi_A X$ and $\{U \times V \mid U, V \varepsilon \{U_\alpha\}\}$ is a basis for the product topology on $X \times X$. Hence (ℓ, r) maps bijectively basis to basis, and so it is a homeomorphism.

We now show that for any $U \times V$, $U, V \varepsilon \{U_\alpha\}$

$(\ell, r)^{-1}(U \times V) = \bigcup < U, [a]_A, V>$ is a disjoint union.

Let $< U, [b]_A, V>$, $< U, [c]_A, V> \subseteq (\ell, r)^{-1} (U \times V)$ have non-empty intersection, then $\exists$ $[d]_A$ such that:-

$$[d]_A \varepsilon < U, [b]_A, V > \text{ and } [d]_A \varepsilon < U, [c]_A, V> .$$

But by 2.3.1. we have:-

$$<U, [b]_A, V > = <U, [d]_A, V > = <U, [c]_A, V>$$

Finally, we must show that (ℓ, r) maps $\pi_A X$ onto $X \times X$. Let $(x, y) \varepsilon$ $X \times X$, then since X is path-connected, $\exists$ a $\varepsilon \pi X(x, y)$, and we have:-

$$(\ell, r)([a]_A) = (ia, \phi a) = (x, y)$$

q.e.d.

CHAPTER III

TOPOLOGICAL STRUCTURE OF πX

Introduction: In this chapter, we first consider the question of topological categories and groupoids. These categories and groupoids were first introduced by C. Ehresmann [4] in 1958, in a rather special way without giving any examples. We introduce a new definition and give some nice examples, e.g. we show that if X is a Hausdorff space, then PX (the set of all paths) is a topological category and πX (the fundamental groupoid of X) is a topological groupoid over X. As another important example we prove that $\mathcal{C}(T)$, the set of all homorphisms between the fibres of a vector bundle T over a Hausdorff space X, is a topological category over X.

In section 2, we show that a special kind of these groupoids called "locally trivial" have bundle structures over the cartisian product of their object space with itself, and for each x, $st_G x$ ($= i^{-1}(x)$) is a principal bundle over X, under the projection ϕ and the structure group $G\{x\}$. Hence, every locally trivial groupoid G with discrete vertex groups over X is a covering space of $X \times X$, and $St_G x$ is a covering space of X. We show that if X is path-connected and locally path-connected and locally simply connected Hausdorff space, then πX is locally trivial, and since X has discrete fundamental group, once again, we obtain the covering space structure for πX over $X \times X$. We prove that every locally trivial groupoid G over a p.c., ℓ.p.c. and ℓ.s.c. space X, with discrete ~~fibres~~ vertex groups is isomorphic to $\pi_A X$ for some normal subgroupoid A of πX; and if X is p.c., ℓ.p.c. and ℓ.s.c. space with abelian fundamental group, then any covering space N of $X \times X$ corresponding to $A_0 \times \pi_1(X)$, A_0 a subgroup of $\pi_1(X)$, has a groupoid structure. This then tells us all connected locally trivial groupoids, over X, with discrete vertex groups. Moreover, in case X is p.c., ℓ.p.c. and ℓ.s.c. space, we show that πX has bundle structure over X.

In the last section, we prove some more facts about πX, e.g. we show that if $X \approx Y$, as spaces, then $\pi X \approx \pi Y$ as topological groupoids. Finally, we close the section and the chapter by introducing the notion of homotopy for topological groupoids and show that if $X \simeq Y$ as spaces, then $\pi X \simeq \pi Y$ as topological groupoids, if X and Y are p.c., ℓ.p.c. and ℓ.s.c. spaces.

1. TOPOLOGICAL CATEGORIES AND GROUPOIDS.

Definition 3.1.1:

A category $C = (C, C^{ob}, i, \phi, \theta, u)$ is called a topological category if:-

(1) C and C^{ob} are topological spaces with C^{ob} Hausdorff

(2) All the maps i, ϕ, θ, u are continuous.

(We take the relative topology for D, the set of composable pairs).

Definition 3.1.2:

A groupoid $G = (G, G^{ob}, i, \phi, \theta, u, \sigma)$ is called a topological groupoid if:-

(1) G is a topological category

(2) the inverse map σ is continuous

Before giving examples of topological categories and groupoids we make the following useful remarks about their properties.

Remarks: 3.1.3.

(1) Since C^{ob} is Hausdorff, then $\forall\, x \varepsilon C^{ob}$, $\{x\}$ is closed. Hence:-

In every topological category, for every object x, the sets

$$Stx = i^{-1}(x) \quad \text{and} \quad \mathscr{E}_x = \phi^{-1}(x)$$

are closed subsets of C.

(2) Since $\forall\, x, y \varepsilon C^{ob}$, $C(x, y) = Stx \cap \mathscr{E}y$, it follows that:-

In every topological category, the sets $C(x, y)$, $x, y \varepsilon C^{ob}$, are closed.

(3) Let $U, V \subseteq C^{ob}$ be open, then $C(U, V) = i^{-1}(U) \cap \phi^{-1}(V)$ is open

(4) Let G be a topological groupoid, then $\forall\, x \varepsilon G^{ob}$, the vertex group $G\{x\} = St_x \cap \mathscr{E}_x$ is closed in G.

(5) Let $f \in C(x, y)$, where C is a topological category, then f induces continuous functions:-

${}_f\Lambda : St_y \longrightarrow St_x$, defined by: ${}_f\Lambda(a) = f + a \; (= \theta(f, a))$

$\Lambda_f : \mathcal{E}_x \longrightarrow \mathcal{E}_y$, defined by: $\Lambda_f(b) = b + f \; (= \theta(b, f))$

Continuity of these functions follows from that of θ.

<u>In case of groupoids</u>, ${}_f\Lambda$, Λ_f are <u>homeomorphisms</u>.

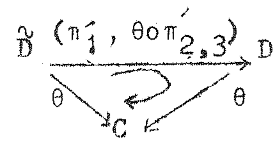
(6) Let $\tilde{D} = \{(f, g, h) \in C \times C \times C \mid (f, g) \in D \text{ and } (g, h) \in D\}$, then the function $\tilde{\theta} : \tilde{D} \longrightarrow G$

defined by: $\tilde{\theta}(f, g, h) = f + g + h$ is continuous

For, let $\pi_1 : C \times C \times C \longrightarrow C$ and $\pi_{2,3} : C \times C \times C \longrightarrow C \times C$ be the projections defined by: $\pi_1(a, b, c) = a$

$\pi_{2,3}(a, b, c) = (b, c)$

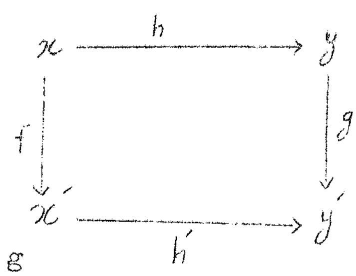
then $\pi_1' = \pi_1 \mid \tilde{D}$ and $\pi'_{2,3} \mid \tilde{D}$ are continuous. Now the continuity of θ follows from the commutative diagram:-



(7) Let G be a topological groupoid, connected in the groupoid sense. Then $\forall\, x, y, x', y' \in G^{ob}$, <u>$G(x, y)$ is homeomorphic to $G(x', y')$</u>

<u>Proof:</u> Since G is connected, $G(x, x') \neq \phi$ and $G(y, y') \neq \phi$

Let $f \in G(x, x')$ and $g \in G(y, y')$ be any elements. Define:-



$\eta : G(x, y) \longrightarrow G(x', y')$

$\mu : G(x', y') \longrightarrow G(x, y)$

by: $\forall\, h \in G(x, y)$, $\eta(h) = -f + h + g$

$\forall\, h' \in G(x', y')$, $\mu(h') = f + h' - g$

<u>η and μ are continuous:</u>

For, $\eta = {}_{-f}\Lambda \circ \Lambda_g$ and $\mu = {}_f\Lambda \circ \Lambda_{-g}$.

It is easily seen that:- $\mu \circ \eta = id_{G(x, y)}$ and $\eta \circ \mu = id_{G(x', y')}$

Hence η is a bijection and $\eta^{-1} = \mu$. Therefore it is a homeomorphism.

(8) <u>In every topological groupoid, connected in the abstract sense, the vertex groups are isomorphic topological groups.</u>

<u>Proof</u>: That the vertex groups are homeomorphic follows from (7). We only need to show that they are isomorphic as abstract groups. If in (7) we take $x = y$ and $x' = y'$ and $f = g$ then $\forall\, h \in G\{x\}$, $\eta(h) = -f + h + f$.

Let $h_1, h_2 \in G\{x\}$, then $\eta(h_1 + h_2) = -f + (h_1 + h_2) + f$

$$= -f + h_1 + f - f + h_2 + f = \eta(h_1) + \eta(h_2)$$

Hence η is an isomorphism.

(9) Let D denote the set of composable pairs in $C \times C$, C a topological category, then D <u>is a closed subset</u> of $C \times C$. For, let $\Delta = \{(x, x) \in C^{ob} \times C^{ob}\}$ be the diagonal, then Δ is closed, since C^{ob} is Hausdorff. Then $D = (\phi \times i)^{-1}(\Delta)$, and hence it is closed.

(10) <u>In every topological category C^{ob} is homeomorphic to O, the set of units in C.</u>

For, $u' = \phi|O = i|O$ is the inverse map of u, and we already know that $u: C^{ob} \longrightarrow O$ is bijective. Hence, it is a homeomorphism.

(11) O is closed in C if C is Hausdorff. For O is a retract of C, and any retract of a Hausdorff space is closed (see e.g. [6] p. 25)

Examples:

(1) Every topological semigroup C is a topological category over one object. For, in this case, i, ϕ and u are constant maps and hence continuous. $D = C \times C$ and θ is the composition of elements, and so by definition is continuous.

(2) Every category (groupoid) is a topological category (groupoid) with discrete topology on both, the set of objects and the set of morphisms.

(3) Every topological group is a topological groupoid over one object for the same reason as (1).

(4) Any union of topological (semigroups) groups is a topological (category) groupoid, with discrete topology for the set of objects, and the sum topology for the set of morphisms.

These are all rather trivial examples; we now proceed to show some non-trivial examples.

Theorem 3.1.4:

If X is a Hausdorff space, then PX is a topological category over X.

Proof: (i) For each $t \in \mathbb{R}^+$, let ν_t: $PX \longrightarrow X$ be the evaluation map, i.e. $\forall f \in pX$, $\nu_t(f) = f(t)$. Then ν_t is continuous (e.g. see [6] p. 74). Obviously $i = \nu_o$ and hence it is continuous.

Next, let U be an open neighbourhood of $\phi(f) = f(r) \in X$ (r being the length of f). Then since f is continuous, $\exists \varepsilon > o$ s.t. $f(r - \varepsilon, r + \varepsilon) \subseteq U$. Let $K = [r - \varepsilon/_2, r + \varepsilon/_2]$, then $\tau(k, U) \cap \eta^{-1}(r - \varepsilon/_2, r + \varepsilon/_2)$ is a neighbourhood of f in PX. For any $g \in \tau(K, U) \cap \eta^{-1}(r - \varepsilon/2, r + \varepsilon/_2)$ let s be the length of g, then $r - \varepsilon/_2 < s < r + \varepsilon/_2$ and so $g(s) \in U$.

Hence $\phi(\tau(K, U) \cap \eta^{-1}(r - \varepsilon/_2, r + \varepsilon/_2)) \subseteq U$.

Therefore ϕ is continuous.

(ii) The Unit map $u: X \longrightarrow PX$ is continuous

For, let $(\bigcap_{i=1}^{n} \tau(K_i, U_i)) \cap \eta^{-1}[o, \varepsilon)$ be any neighbourhood of $o_x = u(x)$ in PX, then $o_x(K_i) = x$, $i = 1, \ldots, n$. Hence $\bigcap_{i=1}^{n} U_i = U \neq \phi$. Therefore U is an open neighbourhood of x in X. Obviously

$$\forall\, y \in U, \quad o_y(K_i) = y \in U \subseteq U_i, \; i = 1, \ldots, n.$$

Hence $o_y \in (\bigcap_{i=1}^{n} \tau(K_i, U_i)) \cap \eta^{-1}([o, \varepsilon))$

i.e. $u(U) \subseteq (\bigcap_{i=1}^{n} \tau(K_i, U_i)) \cap \eta^{-1}[o, \varepsilon)$

Therefore u is continuous.

So, it remains only to show the continuity of $\theta: D \longrightarrow PX$.

Since the proof is lengthy, we prefer to do it in a separate lemma.

<u>Lemma 3.1.5</u>:

<u>The composition function $\theta: D \longrightarrow PX$ is continuous.</u>

<u>Proof</u>: Let $\tau(K, U)$ be any subbasic open set in PX, containing $\theta(f_r, g_s) = f_r + g_s$. Then for any $\varepsilon > o$, we show that $\exists\, \sigma_1, \sigma_2 > o$ and compact subsets $K_1, k_2 \subseteq \mathbb{R}^+$, and open sets $U_1, U_2 \subseteq X$ s.t.

$f_r \in \tau(K_1, U_1) \cap \eta^{-1}(r - \sigma_1, r + \sigma_1) = N_1$, say, and

$g_s \in \tau(K_2, U_2) \cap \eta^{-1}(r - \sigma_2, r + \sigma_2) = N_2$, say, and

$\theta((N_1 \times N_2) \cap D) \subseteq \tau(K, U) \cap \eta^{-1}(r + s - \varepsilon, r + s + \varepsilon)$

Since U is open and $f_r + g_s$ is continuous, $\exists\, \gamma > o$ s.t.

$$\forall\, t \in \mathbb{R}^+ \text{ s.t. } d(t, K) < \gamma \Longrightarrow (f_r + g_s)_t \in U.$$

Let $L = \{t \in \mathbb{R}^+ \mid d(t, K) < \gamma\}$, then $(f_r + g_s)(L) \subseteq U$. (We have $K \subseteq L$)

We consider 4 cases as follows:-

(1) <u>Suppose $K \cap [o, r] \neq \phi$ and $K \cap [r, r+s] \neq \phi$</u>

Then, let $\sigma = \sigma_1 = \sigma_2 = \text{minimum } \{\gamma, \varepsilon/3\}$, and let

$$K_1 = K \cap [o, r + \sigma] \text{ and } K_2 = K \cap [r - \sigma, r + s] \setminus (r - \sigma)$$

We have:-

$$(f_r + g_s)_t = \begin{cases} f(t) & o \leqslant t \leqslant r \\ g(t - r) & t \geqslant r \end{cases}$$

Hence $f_r + g_s \,\varepsilon\, \tau(K, U) \Longrightarrow \forall\, k_1 \varepsilon K_1,\ f_r(k_1) \varepsilon U$

and $\forall\, k_2 \,\varepsilon K_2,\ g_s(k_2) = g_s(\ell - r) \,\varepsilon\, U$, some $\ell \varepsilon L$.

Hence $f_r \,\varepsilon \tau(K_1, U) \cap \eta^{-1}(r - \sigma, r + \sigma)$ and $g_s \varepsilon \tau\ (K_2, U) \cap \eta^{-1}\ (s - \sigma, s + \sigma)$

Therefore $(f_r, g_s)\ \varepsilon \Big(\big(\tau(k_1, U) \cap \eta^{-1}(r - \sigma, r + \sigma)\big) \times \big(\tau(k_2, U) \cap$

$$\eta^{-1}(s - \sigma, s + \sigma)\big)\Big) \cap D = \underset{\sim}{N}, \text{ say.}$$

<u>Claim</u>: $\theta(\underset{\sim}{N}) \subseteq \tau(K, U) \cap \eta^{-1}(r + s - \varepsilon, r + s + \varepsilon)$

Let $(f'_p, g'_q)\ \varepsilon \underset{\sim}{N}$, then $\theta(f'_p, g'_q) = f'_p + g'_q$. We must show that:-

$$(f'_p + g'_q)(K) \subseteq U, \text{ and } |\ p + q - (r + s)| < \varepsilon$$

But

$$(f'_p, q'_q)\ \varepsilon\ \underset{\sim}{N} \Longrightarrow \begin{cases} f'_p\ \varepsilon \tau(K_1, U) \cap \eta^{-1}(r - \sigma, r + \sigma) \\ g'_q\ \varepsilon \tau(K_2, U) \cap \eta^{-1}(s - \sigma, s + \sigma) \end{cases} \qquad (*)$$

$$\Longrightarrow \begin{cases} f'_p(K_1) \subseteq U \text{ and } |\,p - r\,| < \sigma \\ g'_q(K_2) \subseteq U \text{ and } |\,q - s\,| < \sigma \end{cases}$$

$$\Longrightarrow \begin{cases} |p + q - (r + s)| \leqslant |\,p - r\,| + |\,q - s| < \sigma + \sigma = 2\sigma \leqslant 2\varepsilon/3 < \varepsilon \\ (f'_p + g'_q)(K) \subseteq U \end{cases}$$

For, $k \leqslant p \Longrightarrow k \,\varepsilon K_1 \Longrightarrow (f'_p + g'_q)_k = f'_p(k)\ \varepsilon\ U$ (by *)

$k \geqslant p \Longrightarrow k - p\, \varepsilon K_2 \Longrightarrow (f'_p + g'_q)_k = g'_q(k - p)\ \varepsilon\ U$ (by *)

Therefore $(f'_p + g'_q)(K) \subseteq U$. Hence $f'_p + g'_q\ \varepsilon \tau\ (K, U) \cap \eta^{-1}(r + s - \varepsilon,$
$r + s + \varepsilon)$

(2) <u>Let $K \cap [o, r] = \phi$, $K \cap [r, r + s] \neq \phi$.</u>

We can choose γ in such a way that $L \cap [o, r] = \phi$. Let $U_1 \subseteq X$

be any open set containing $f_r(r)$, then $\exists \gamma' > o$ such that $f_r(r - \gamma', r + \gamma') \subseteq U_1$. Let $\sigma = \text{minimum}\{\varepsilon/2, \gamma', \gamma, d(r, k)\}$, and let $K_1 = [r - \sigma, r + \sigma]$, $K_2 = L - r$

Then $f_r \varepsilon \tau(K_1, U_1)$ and $g_s \varepsilon \tau(K_2, U)$. For,

$$\forall k_2 \varepsilon K_2, \quad g_s(k_2) = g_s(\ell - r) \quad \text{for some } \ell \varepsilon L$$

$$= (f_r + g_s)(\ell) \varepsilon U.$$

clearly $(f_r, g_s) \varepsilon \underset{\sim}{N} = \Big(\big(\tau(K_1, U_1) \cap \eta^{-1}(r - \sigma, r + \sigma) \big) \times$

$$\big(\tau(K_2, U) \cap \eta^{-1}(s - \sigma, s + \sigma) \big) \Big) \cap D$$

It follows easily that $\theta(\underset{\sim}{N}) \subseteq \tau(K, U) \cap \eta^{-1}(r + s - \varepsilon, r + s + \varepsilon)$

For, let $(f'_p, g'_q) \varepsilon \underset{\sim}{N}$, then $\forall k \varepsilon K$, $(f'_p + g'_q)_k = g'_q(k - p)$. But $k - p \varepsilon K_2$, hence $g'_q(k - p) \varepsilon U$. (Notice that k is always $> p$.)

(3) <u>Let $K \cap [r, r + s] = \phi$, $K \cap [o, r] \neq \phi$</u>.

We take $K_1 = K$. Let U_2 be any open set containing $g_s(s)$, then $\exists \nu > o$ such that $g_s(s - \nu, s + \nu) \subseteq U_2$. Let $K_2 = [s - \nu/2, s + \nu/2]$, then $g_s \varepsilon \tau(K_2, U_2)$, and by definition we have:-

$$\forall k \varepsilon K, \ (f_r + g_s)_k = f_r(k) \varepsilon U. \quad (\because k \varepsilon K \text{ is less than } r).$$

Let $\sigma = \min(\varepsilon/2, d(r, K))$, then we have:-

$$(f_r, g_s) \varepsilon \underset{\sim}{N} = \Big(\big(\tau(K_1, U) \cap \eta^{-1}(r - \sigma, r + \sigma) \big)$$

$$\times \big(\tau(K_2, U_2) \cap \eta^{-1}(s - \sigma, s + \sigma) \big) \Big) \cap D$$

Given $(f'_p, g'_q) \varepsilon \underset{\sim}{N}$, then $p > k$, $\forall k \varepsilon K$. Hence:-

$(f'_p + g'_q)_k = f'_p(k) \varepsilon U$, by definition.

Therefore $f'_p + g'_q \ \varepsilon \tau(K, U)$, and hence

$$\theta(\underset{\sim}{N}) \subseteq \tau(K, U) \cap \eta^{-1}(r + s - \varepsilon, r + s + \varepsilon)$$

(4) <u>$K \cap [o, r] = K \cap [r, r + s] = \phi$</u>

This case is easily verified.

q.e.d.

Lemma 3.1.6:

The map σ: $PX \longrightarrow PX$, defined by $\sigma(f) = -f$, is continuous.

Proof: Let N be a basic neighbourhood of $-f$, and let $\eta(f) = r$. Then $\exists$ closed intervals $K_1, \ldots, K_n \subseteq \mathbb{R}^+$, open sets $U_1, \ldots U_n$ in X and $\varepsilon > 0$ such that $-f \varepsilon \bigcap_{i=1}^{n} \tau(K_i, U_i) \cap \eta^{-1}(r - \varepsilon, r + \varepsilon) = M$, say. Without loss of generality we may assume $K_i \subseteq [0, r]$, $i = 1, \ldots, n$. Let $K_i' = r + \ell - K_i$, where $[c, \ell]$ is the maximum interval such that $f| [0, \ell]$ is constant. Clearly K_i' is compact and we have:-

$$f \varepsilon \bigcap_{i=1}^{n} \tau(K_i', U_i) \cap \eta^{-1}(r - \varepsilon, r + \varepsilon) = N, \text{ say.}$$

Claim $\sigma(N) \subseteq M$.

Let $g_s \varepsilon N$, then $s \varepsilon (r - \varepsilon, r + \varepsilon)$ and so $-g_s \varepsilon\, \eta^{-1}(r - \varepsilon, r + \varepsilon)$.
We have $-g_s(K_i) = g(r + \ell - k_i) = g_s(K_i') \subseteq U_i$, $i = 1, \ldots, n$.
Hence $-g_s \varepsilon M$.

q.e.d.

Corollary 3.1.7:

The final map ϕ: $PX \longrightarrow X$ has path lifting property.

Proof: Given any path f in X, $\exists$ a path $\tilde{f}$ in PX such that $io\tilde{f} = f$ (For the initial map i has path lifting property, by remark (iii) in 1.3). Then $\sigma o\tilde{f}$ is a path in PX and we have:-

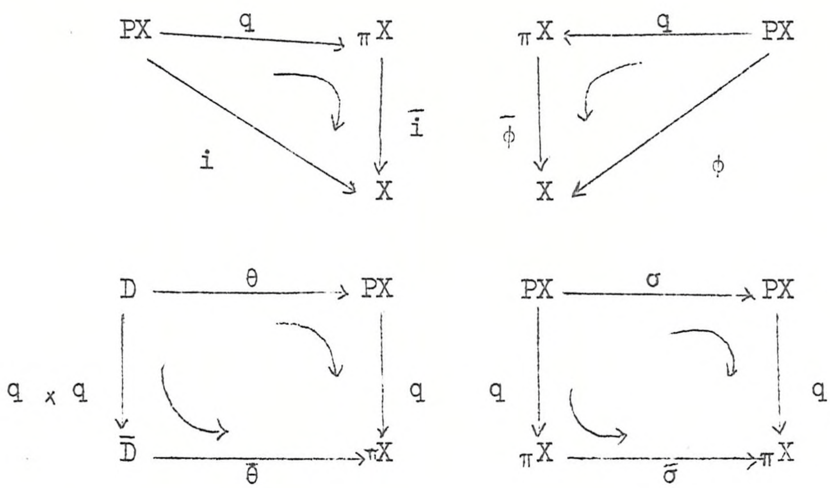
$$\phi o(\sigma o\tilde{f}) = (\phi o \sigma) o\tilde{f} = io\tilde{f} = f.$$

q.e.d.

Theorem 3.1.8:

If X is a Hausdorff space, then πX is a topological groupoid over X.

Proof: Let q: $PX \longrightarrow \pi X$ be the quotient map, then we have the following commutative diagrams:-



Hence continuity of the maps relating to πX follows from those relating to PX.

q.e.d.

<u>Definition 3.1.9</u>:

Let C, C' be topological categories. A function $\mu : C \longrightarrow C'$ is called a <u>functor of topological categories</u> if:

(1) μ is an abstract functor

(2) $\mu | C$ and $\mu | C^{ob}$ are continuous.

It is immediate from the above diagrams that:-

<u>Lemma 3.1.10:</u>

<u>The quotient map $q : PX \longrightarrow \pi X$ with $q | X$ = identity, is a functor of topological categories</u>.

Another example of topological groupoids is obtained from the space of all continuous functions $F(I_a, X) = \{f: I_a \longrightarrow X\}$, with the C - O topology, where $I_a = [o, a] \subseteq R^+$

Let $D = \{(f, g) \in F \times F \mid f(a) = g(o)\}$

Define $\theta : D \longrightarrow F(I_a, x)$ by $\theta(f, g) = f.g.$

Where f.g: $I_a \longrightarrow X$ is defined by:-

$$(f.g)_t = \begin{cases} f(2t) & o \leqslant t \leqslant a/_2 \\ g(2t - a) & a/_2 \leqslant t \leqslant a \end{cases}$$

(Notice that if a = 1, then elements of F are paths, and the above operation is the product of paths, in the old sense).

Let D carry the relative topology then:-

Lemma 3.1.10:

The map $\theta : D \longrightarrow F(I_a, X)$ is continuous.

Proof: Let $N = \tau(K, U)$ be any subbasic open set continuing f.g , we show that $\theta^{-1}(N)$ is open in D.

Let $\lambda_1, \lambda_2 : I_a \longrightarrow I_a$ be definted by:-

$\forall\, t \in I_a$, $\lambda_1(t) = t/_2$ and $\lambda_2(t) = t + \frac{a}{2}$ (*)

obviously λ_1, λ_2 are continuous. Let $K_i = \lambda_i^{-1}(K)$, i = 1, 2., then K_i, i = 1, 2 is compact. So $\tau(K_i, U)$ is a basic open set, hence $(\tau(K_1, U) \times \tau(K_2, U)) \cap D = \underset{\sim}{N}$ is open in D.

Claim: $\theta^{-1}(N) = \underset{\sim}{N}$:

Let $(f', g') \in \theta^{-1}(N)$, then $f'.g' \in \tau(K, U) = N$. Hence $f'.g'(K) \subsetneq U$.

But

$$(f'.g')_t = \begin{cases} f'(2t) & o \leqslant t \leqslant a/2 \\ g'(2t - a) & a/_2 \leqslant t \leqslant a \end{cases} \qquad (**)$$

Now, $\forall\, k_1 \in K_1$, let $k = \lambda_1(k_1) \in K$, then by (*) $k_1 = 2k$, $o \leqslant k \leqslant a/_2$

Hence $f'(k_1) = f'(2k)$, $o \leqslant k \leqslant a/_2$, and so by (**) $f'(k_1) = f'(2k)$

$$= f'.g'(k) \in U.$$

Therefore $f'(K_1) \subseteq U$, and $f' \in \tau(K_1, U)$.

Similarly, $\forall\, k_2 \in K_2$, $g'(k_2) \in U$, and hence $g'(K_2) \subseteq U$.

Hence $(f', g') \in \underset{\sim}{N}$ i.e. $\theta^{-1}(N) \subseteq \underset{\sim}{N}$.

Conversely, let $(f_1, g_1) \in \underset{\sim}{N}$, then:-

$$\left.\begin{array}{l} f_1 \in \tau(K_1, U) \Longrightarrow f_1(K_1) \subseteq U \\ g_1 \in \tau(K_2, U) \Longrightarrow g_1(K_2) \subseteq U \end{array}\right\} \qquad (***)$$

Let $k \in K$, then:-

$$\begin{cases} \text{if } o \leqslant k \leqslant a/_2, \ \exists\, k_1 \in K_1, \text{ s.t. } \lambda_1(k_1) = k, \text{ i.e. } k_1 = 2k \\ \text{if } a/_2 \leqslant k \leqslant a, \exists\, k_2 \in K_2 \quad \text{s.t. } \lambda_2(k_2) = k, \text{ i.e. } k_2 = 2k - a \end{cases}$$

Hence $\forall\, k \in K$, we have:-

$$(f_1 \cdot g_1)_k = \begin{cases} f_1(2k) \ , \ o \leqslant k \leqslant a/_2 \\ g_1(2k - a), \ \ a/_2 \leqslant k \leqslant a \end{cases} = \begin{cases} f_1(k_1) \in U \\ g_1(k_2) \in U \end{cases} \quad \text{by } (***)$$

Therefore $f_1 \cdot g_1 \in \tau\,(K, U) = N$. Hence $(f_1, g_1) \in \theta^{-1}(N)$.

Since any basic open set in F is a finite intersection of subbasic open sets and θ^{-1} preserves intersections, it follows that θ is continuous. q.e.d.

Lemma 3.1.11:

The map $\sigma: F \longrightarrow F$ defined by:-

$\forall\, f \in F, \sigma(f) = f^{-1}$, where $f^{-1}(t) = f(a - t)$, $t \in I_a$, is continuous

Proof: Let $N = \tau(K, U)$ contain f^{-1}, then $f^{-1}(K) \subseteq U$.

Define $\rho: I_a \longrightarrow I_a$ by $\rho(t) = a - t$.

then obviously ρ is continuous, and $K' = \rho^{-1}(K) = a - K$ is compact.

Claim: $\sigma^{-1}(N) = \tau(K', U)$

Let $g \in \sigma^{-1}(N)$, then $g^{-1} \in N$, therefore $g^{-1}(K) \subseteq U$. Hence:-

$$g(K') = g(a - K) = g^{-1}(K) \subseteq U.$$

So, $g \in \tau\,(K', U)$. Hence $\sigma^{-1}(N) \subseteq \tau(K', U)$.

Conversely, let $h \in \tau\,(K', U)$, then $h(K') \subseteq U$. i.e.

$$h(a - K) = h^{-1}(K) \subseteq U$$

Therefore $h^{-1} \in \tau\,(K, U) = N$, and so, $h \in \sigma^{-1}(N)$. Therefore $\tau(K', U) \subseteq \sigma^{-1}(N)$.

Then, continuity of σ follows from the fact that each open set contains a finite intersection of subbasic open sets and σ^{-1}

preserves the intersection. q.e.d.

The following is easily verified:-

Lemma 3.1.12:

(i) $f \simeq g \Longrightarrow f^{-1} \simeq g^{-1}$

(ii) $f.f^{-1} \simeq c_x$, $f^{-1}.f \simeq c_y$

(iii) $f.c_y \simeq f \simeq c_x.f$

where c_x, c_y are constant maps with the values $x = f(o)$ and $y = f(a)$, respectively.

(iv) Let (f, f_1), $(g, g_1) \varepsilon D$, then:-

$f \simeq g$ and $f_1 \simeq g_1 \Longrightarrow f.f_1 \simeq g.g_1$

(v) Let (f, g), $(g, h) \varepsilon D$, then $(f.g).h \simeq f.(g.h)$.

Theorem 3.1.13:

If X is Hausdorf, then $\tilde{F} = F(I_a, X)_{/R}$, the set of all homotopy classes in F with the quotient topology, is a topological groupoid over X.

Proof: For each $f \varepsilon F$, let cls(f) denote the homotopy class of f.

(i) Define the initial and final maps i_1, ϕ_1: $\tilde{F} \longrightarrow X$,

by:- $i_1(\text{cls}(f)) = f(o)$ and $\phi_1(\text{cls}(f)) = f(a)$.

(ii) Let $D = \{(\text{cls}(f), \text{cls}(g)) \mid f(a) = g(o)\}$

Define $\theta_1 : D_1 \longrightarrow \tilde{F}$ by:-

$$\theta_1(\text{cls}(f), \text{cls}(g)) = \text{cls}(f).\text{cls}(g) = \text{cls}(f.g).$$

By 3.1.12, θ_1 is well-defined and satisfies the associative law.

(iii) Define the unit map $u_1 : X \longrightarrow \tilde{F}$ by:-

$$u_1(x) = \text{cls}(c_x) = o_x.$$

It follows from 3.1.12. that u_1 satisfies the conditions required for a unit function.

(iv) Define the inverse function $\sigma_1 : \tilde{F} \longrightarrow \tilde{F}$

by:- $\sigma_1(c\ell s(f)) = c\ell s(f^{-1})$

Again by 3.1.12. it is well-defined and is the right function.

Therefore $\tilde{F}$ is an abstract groupoid over X.

We now show that the maps $i_1, \phi_1, \theta_1, u_1, \sigma_1$ are continuous.

Let $v_o, v_a : F(I_a, X) \longrightarrow X$ be the evaluation maps, and let

$q_1 : F \longrightarrow \tilde{F}$ be the quotient map. Then v_o, v_a, q_1, are continuous.

Let $u: X \longrightarrow F$ be defined by: $u(x) = c_x$, then u is continuous. For,

let $\tau(K, U)$ be any subbasic open set, containing c_x, then $x \in U$.

Clearly $u(U) \subseteq \tau(K, U)$. Now the continuity of $i_1, \phi_1, \theta_1, u_1, \sigma_1$

follows from the continuity of $v_o, v_a, \theta, u, \sigma$ from commutative

diagrams similar to those in 3.1.9.

q.e.d.

As another important example of topological categories, we now proceed to show that if X is Hausdorff, then the category $\mathcal{C}(T)$ (see 1.2.10) is a topological category over X.

$\mathcal{C}(T)$ has a natural topology defined as follows:-

Let $\mathcal{U} = \{U \subseteq X \mid U \text{ is open and } T|U \approx U \times \mathbb{R}^n\}$, and let for each

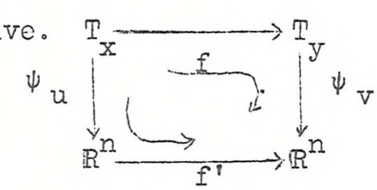
$U \in \mathcal{U}$, ψ_U be the homeomorphism: $T|U \approx U \times \mathbb{R}^n$. Then for each

$U, V \in \mathcal{U}$,

$$\mathcal{C}(T)(U, V) = \bigcup_{x \in U, y \in V} \mathcal{C}(T)(x, y) = \bigcup_{x \in U, y \in V} \text{Hom}(T_x, T_y),$$

and $\forall (x, y) \in U \times V$, $f \in \text{Hom}(T_x, T_y)$, $\exists$ a unique $f' \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$

such that the diagram is commutative.



Define $\eta_{u,v} : \mathcal{C}(T)(U, V) \longrightarrow U \times V \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$

by:- $\eta_{u,v}(f) = (i(f), \phi(f), f')$

and $\bar{\eta}_{u,v} : \mathcal{C}(T)(U, V) \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$

by:- $\bar{\eta}_{u,v}(f) = f'$

Clearly $\eta_{u,v}$ is a bijection. The following properties of $\bar{\eta}_{u,v}$ are immediate. Since we will refer to them later on we put them in the form of a lemma:-

Lemma 3.1.4:

Let $U, V, W \in \mathcal{U}$, then $\forall (x, y, z) \in U \times V \times W$, $f \in \mathrm{Hom}(T_x, T_y)$, $g \in \mathrm{Hom}(T_y, T_z)$ we have:-

$$\text{(i)}\ \bar{\eta}_{u,w}(g \circ f) = \bar{\eta}_{v,w}(g) \circ \bar{\eta}_{u,v}(f)$$

$$\text{(ii)}\ \bar{\eta}_{u,v}(f^{-1}) = (\bar{\eta}_{u,v}(f))^{-1}.$$

We topologize $\mathcal{C}(T)(U,V)$ by requiring $\eta_{u,v}$ to be a homeomorphism. Thus $\mathcal{C}(T)(U, V)$ has the structure of a product space, We now define a topology for $\mathcal{C}(T)$ as follows:-

Any $\mathcal{V} \subseteq \mathcal{C}(T)$ will be open if $\forall U, V \in \mathcal{U}$, $\mathcal{V} \cap \mathcal{C}(T)(U, V)$ is open in $\mathcal{C}(T)(U, V)$.

It follows immediately from the definition that $\forall U, V \in \mathcal{U}$, $\mathcal{C}(T)(U, V)$ is open in $\mathcal{C}(T)$.

Theorem 3.1.15:

$\mathcal{C}(T) = (\mathcal{C}(T), X, i, \phi, u, \theta)$ is a topological category.

Proof: We only need to verify the continuity of functions.

(i) The function $i: \mathcal{C}(T) \longrightarrow X$ is continuous. For, let $U \subseteq X$ be any open set, we must show $i^{-1}(U)$ is open. Let $g \in i^{-1}(U)$, then $\exists\ U' \in \mathcal{U}$ s.t. $i(g) \in U'$ and $U' \subseteq U$. Let $V \in \mathcal{U}$ be any element of $\mathcal{U}$ containing $\phi(g)$, then $g \in \mathcal{C}(T)(U', V) \subseteq i^{-1}(U)$.

Hence $i^{-1}(U)$ is a neighbourhood of each of its elements, and hence open. Similarly the final function $\phi: \mathcal{C}(T) \longrightarrow X$ is continuous.

(ii) The map $\theta: D \longrightarrow \mathcal{C}(T)$ is continuous.

For, let $(f, g) \in D$ and let $\tilde{W}$ be a neighbourhood of $\theta(f, g) = gof$, then $\exists\, U, V \in \mathcal{U}$ and open $N \subseteq \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ and $\exists\, w \subseteq \tilde{W}$ containing gof s.t. $U \times V \times N \approx W$.

(For, let $U', V' \in \mathcal{U}$ be s.t. $i(f) \in U$, $\phi(g) \in V$, then $gof \in \mathcal{C}(T)(U', V')$ and we can take $w = \tilde{W} \cap \mathcal{C}(T)(U', V')$. Since $\mathcal{C}(T)(U', V')$ is homeomorphic to $U' \times V' \times \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, $\exists$ open sets $U \subseteq U'$ and $V \subseteq V'$ and open $N \subseteq \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ s.t. $w \approx U \times V \times N$. Obviously $U, V \in \mathcal{U}$)

Let $U_1 \in \mathcal{U}$ and $i(g) = \phi(f) \in U_1$, then N is a neighbourhood of

$\bar{\eta}_{u,v}(gof) = \bar{\eta}_{u,v}(g) \circ \bar{\eta}_{u,u_1}(f)$ (by 3.1.14)

But $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is a topological semi-group; hence $\exists\, N_1, N_2$ neighbourhoods of $\bar{\eta}_{u_1,v}(g)$ and $\bar{\eta}_{u,u_1}(f)$, respectively, s.t. $N_1 o N_2 \subseteq N$ (*)

Then $W_f \approx U \times U_1 \times N_1$ and $W_g \approx U_1 \times V \times N_2$ are neighbourhoods of f and g, respectively. Therefore $W' = (W_f \times W_g) \cap D$ is a neighbourhood of (f, g) in D, and we have:-

$$\theta(W') \subseteq W \subseteq \tilde{W} \quad \text{(it follows from 3.1.14 and (*))}$$

Hence θ is continuous. It remains to prove the continuity of the unit map $u: X \longrightarrow \mathcal{C}(T)$

Let W be any open neighbourhood of $u(x) = id_{T_x}$, and let $U \in \mathcal{U}$ contain x. Then $W' = W \cap \mathcal{C}(T)(U, U)$ is open and contains id_{T_x}. Let $i(W') = U'$, $\phi(W') = V'$, then $\exists$ an open neighbourhood N of identity in $\text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ such that $W' \approx U' \times V' \times N$. Then $U' \cap V'$ is a neighbourhood of x, and it is easily seen that

$\forall\, y \in U' \cap V' \subseteq U$, $u(y) = id_{T_y} \in W' \subseteq W$.

Hence u is continuous.

q.e.d.

Let $\mathcal{G}(T)$ be the set of all invertable elements in $\mathcal{C}(T)$. Then $\mathcal{G}(T)$ with the restrictions of the maps relating to $\mathcal{C}(T)$ is a sub-category of $\mathcal{C}(T)$. Moreover:-

Corollary 3.1.16:

$\mathcal{G}(T)$, the set of all isomorphisms between the fibres of a vector bundle T, over the Hausdorff space X, is a topological groupoid over X.

Proof: It follows from 3.1.15. that $\mathcal{G}(T)$ is a topological category (a sub-category) of $\mathcal{C}(T)$ over X. So we need only to show that the inverse map $\sigma : \mathcal{G}(T) \longrightarrow \mathcal{G}(T)$, defined by $\sigma(f) = f^{-1}$ is continuous. Let W' be an open neighbourhood of f^{-1}, and let U, V $\in \mathcal{U}$ such that $i(f^{-1}) = \phi(f) \in U$ and $\phi(f^{-1}) = i(f) \in V$. Then $W' \cap \mathcal{G}(T)(U, V)$ is open in $\mathcal{G}(T)(U, V)$. Hence $\exists$ open sets $U' \subseteq U$, $V' \subseteq V$, $N \subseteq \text{Iso}\ (\mathbb{R}^n, \mathbb{R}^n)$ such that

$$W = W' \cap \mathcal{G}(U, V) \approx U' \times V' \times N.$$

Since Iso $(\mathbb{R}^n, \mathbb{R}^n)$ is a topological group N^{-1} is open and hence $V' \times U' \times N^{-1}$ is open in $V \times U \times \text{Iso}\ (\mathbb{R}^n, \mathbb{R}^n)$. Therefore $W_1 = \eta^{-1}_{v,u}(V' \times U' \times N^{-1})$ is an open neighbourhood of f. Clearly $\sigma(W_1) \subseteq W'$.

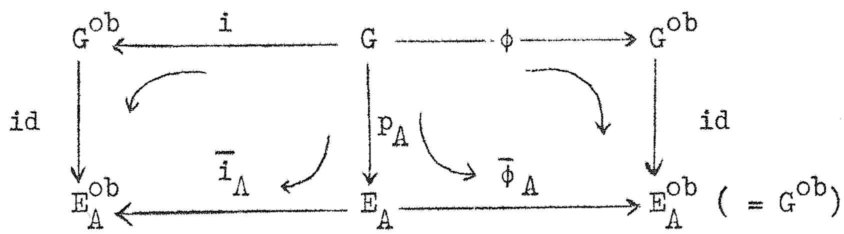
q.e.d.

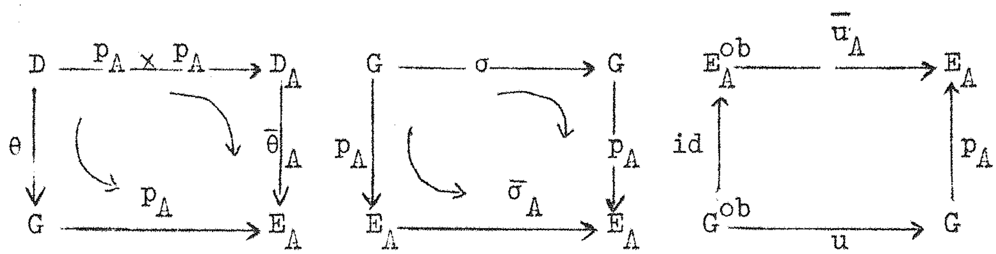
Definition 3.1.17:

By a connected topological groupoid we mean a topological groupoid which is connected as an abstract groupoid.

Theorem 3.1.18: If G is a connected topological groupoid, then for any wide normal connected subgroupoid $A \subseteq G$, E_A is a topological groupoid.

Proof: Let E_A carry the quotient topology, then the quotient morphism $p_A: G \longrightarrow E_A$ is continuous (Recall that $p_A \mid G^{ob}$ is identity) Now the continuity of the maps relating to E_A follows from the continuity of those relating to G, by using the following commutative diagrams





For example, to prove $\bar{i}_A$ is continuous, let $N \subseteq E_A^{ob}$ be open. Since E_A has the quotient topology, $\bar{i}_A^{-1}(N)$ is open in E_A if $p_A^{-1}(\ \bar{i}_A^{-1}(N)\ )$ is open in G. But, by commutativity of diagram, $p_A^{-1}(\bar{i}_A^{-1}(N) = i^{-1}(N)$. Hence it is open (For i is continuous) and therefore $\bar{i}_A$ is continuous.

q.e.d.

Theorem 3.1.19:

If G and H are topological categories, then the product category $G \times H$ is a topological category.

Proof: Let $G = (G,\ G^{ob},\ i_G, \phi_G, \theta_G, u_G)$

and $H = (H,\ H^{ob},\ i_H, \phi_H, \theta_H, u_H)$

Then, by definition $G \times H = (G \times H,\ G^{ob} \times H^{ob},\ i_G \times i_H, \phi_G \times \phi_H,$

$\theta_{G \times H}, u_G \times u_H)$. where $\theta_{G \times H}$ is defined by:-

$$\theta_{G \times H}((g_1\ h_1),(g_2,\ h_2)) = \left(\theta_G(g_1,\ g_2),\ \theta_H(h_1,\ h_2)\right)$$

We take the product topologies for $G \times H$ and $G^{ob} \times H^{ob}$

Then, continuity of $i_G \times i_H$, $\phi_G \times \phi_H$ and $u_G \times u_H$ is straightforward. Hence, we need only show the continuity of $\theta_{G \times H}$.

Let D_G, D_H and $D_{G \times H}$ denote the set of composable pairs in $G \times G$, $H \times H$ and $(G \times H) \times (G \times H)$, respectively.

Let $(g_1,\ h_1,\ g_2,\ h_2) \varepsilon\ D_{G\times H}$, and let $N \times M$ be a neighbourhood of $\theta_{G \times H}(g_1,\ h_1,\ g_2,\ h_2) = (\theta_G(g_1,\ g_2),\ \theta_H(h_1,\ h_2))$, then N is a neighbourhood of $\theta_G(g_1,\ g_2)$ in G and M is a neighbourhood of $\theta_H(h_1,\ h_2)$ in H. Since G and H are topological categories $\exists\ N_1,\ N_2$, neighbourhoods of $g_1,\ g_2$, respectively, in G s.t.

$$\theta_G((N_1 \times N_2) \cap D_G) \subseteq N$$

and $\exists\ M_1,\ M_2$, neighbourhoods of $h_1,\ h_2$, respectively, in H s.t.

$$\theta_H((M_1 \times M_2) \cap D_H) \subseteq M$$

But then $N_1 \times M_1$ and $N_2 \times M_2$ are neighbourhoods of (g_1, h_1) and $(g_2,\ h_2)$ respectively in $G \times H$. Hence:-

$$A = \left((N_1 \times M_1) \times (N_2 \times M_2)\right) \cap D_{G \times H}$$

is a neighbourhood of $(g_1,\ h_1,\ g_2,\ h_2)$ in $D_{G \times H}$, and we have:-

$$\forall (n_1,\ m_1,\ n_2,\ m_2)\ \varepsilon A,\ \theta_{G \times H}(n_1,\ m_1,\ n_2,\ m_2) =$$
$$\left(\theta_G(n_1,\ n_2),\ \theta_H(m_1,\ m_2)\right) \varepsilon\ N \times M$$

Hence $\theta_{G \times H}(A) \subseteq N \times M$, and so $\theta_{G \times H}$ is continuous. q.e.d.

<u>Corollary 3.1.20: If G and H are topological groupoids, then $G \times H$ is a topological groupoid</u>.

<u>Proof</u>: By 3.1.19, $G \times H$ is a topological category. So, we need only to verify the continuity of the inverse function:-

$$\sigma_{G \times H} : G \times H \longrightarrow G \times H$$

defined by:- $\sigma_{G \times H}(g, h) = (\sigma_G(g), \sigma_H(h)) = \sigma_G \times \sigma_H(g, h)$

But this follows from that of σ_G and σ_H. q.e.d.

2. LOCALLY TRIVIAL GROUPOIDS AND THE BUNDLE STRUCTURES OF πX.

In this section, we define the notion of "locally trivial" groupoids introduced by C. Ehresmann [4], and study its relations with the theory of fibre bundles.

Definition 3.2.1: A topological groupoid G over the space X is called locally trivial, if for each object $x_\alpha \in X$, $\exists$ an open neighbourhood U_α of x_α in X and a continuous map:

$$\lambda_\alpha : U_\alpha \longrightarrow G$$

such that $\forall\, x \in U_\alpha$, $i(\lambda_\alpha(x)) = x$ and $\phi(\lambda_\alpha(x)) = x_\alpha$, where i and ϕ are the initial and final maps, respectively, in G

Notice: We will refer to these maps as "continuous lifts", and U_α will be called "liftable".

Examples 3.2.2.:

(1) If X is a path-connected, locally path-connected and locally simply connected Hausdorff space, then πX is locally trivial.

Proof: By the local properties of X, for each $x_\alpha \in X$, there exists a simply connected neighbourhood U_α of x_α. Define $\lambda_\alpha : U_\alpha \longrightarrow \pi X$ by:- $\forall\, x \in U_\alpha$, $\lambda_\alpha(x) = \bar{\gamma} \in \pi X(x, x_\alpha)$, where $\gamma \in PU_\alpha(x, x_\alpha)$

Since all the paths in U_α with the same end points are homotopic in X, λ_α is well-defined.

λ_α is continuous: Let $< U_1, \bar{\gamma}, V_1 >$ be any basic open neighbourhood of $\bar{\gamma}$ in πX, then $U_\alpha \cap U_1$ is open and contains x. Hence $\exists$ a path-connected neighbourhood $\hat{U}$ of x contained in $U_\alpha \cap U_1$.

Then $\forall\, y \in \tilde{U}$, $\lambda_\alpha(y) \in \langle U_1, \bar{\gamma}, V_1 \rangle$. For, let $\nu \in P\tilde{U}(y, x)$, then:-

$\lambda_\alpha(y) = \overline{\nu + \gamma} = \bar{\nu} + \bar{\gamma} = \bar{\nu} + \bar{\gamma} + \bar{o}_{x_\alpha} \in \langle U_1, \bar{\gamma}, V_1 \rangle$.

Therefore λ_α is continuous.

q.e.d.

(2) <u>The groupoid $\mathcal{G}(T)$ (see 3.1.16) is locally trivial</u>

<u>Proof</u>: For each $x_\alpha \in X$, let $U_\alpha \in \mathcal{U}$ contain x_α .

Let $\eta_{U_\alpha, U_\alpha} : \mathcal{G}(T)(U_\alpha, U_\alpha) \approx U_\alpha \times U_\alpha \times \mathrm{Iso}(\mathbb{R}^n, \mathbb{R}^n)$, where

$$\eta'_{U_\alpha, U_\alpha} = \eta_{U_\alpha, U_\alpha} | \mathcal{G}(T)(U_\alpha, U_\alpha)$$

($\eta_{U_\alpha U_\alpha}$ as in the definition of the topology of $\mathcal{C}(T)$).

Define $\lambda_\alpha : U_\alpha \longrightarrow \mathcal{C}(T)$ by:-

$$\forall\, y \in U_\alpha,\ \lambda_\alpha(y) = \eta'^{-1}_{U_\alpha, U_\alpha}(y, x_\alpha, id_{\mathbb{R}^n})$$

Since $\eta'_{U_\alpha, U_\alpha}$ is a homeomorphism λ_α is continuous.

q.e.d.

<u>Lemma 3.2.3: If G is a locally trivial groupoid over X, so is E_Λ.</u>

<u>Proof</u>: Let $x_\alpha \in X = (E_\Lambda)^{ob} = G^{ob}$, then, by locally triviality of G, $\exists$ an open neighbourhood U_α of x_α and a continuous lift $\lambda_\alpha : U_\alpha \longrightarrow G$. Then $\bar{\lambda}_\alpha = p_\Lambda \circ \lambda_\alpha : U_\alpha \longrightarrow E_\Lambda$ is a continuous map, where $p_\Lambda : G \longrightarrow E_\Lambda$ is the quotient morphism. But $p_\Lambda \mid X$ is identity, so:-

$\forall\, x \in U_\alpha$, $\bar{i}_\Lambda(\bar{\lambda}_\alpha(x)) = \bar{i}_\Lambda(p_\Lambda \circ \lambda_\alpha(x)) = \bar{i}_\Lambda[\lambda_\alpha(x)]_\Lambda = i(\lambda_\alpha(x)) = x$

and $\bar{\phi}_\Lambda(\bar{\lambda}_\alpha(x)) = \bar{\phi}_\Lambda[\lambda_\alpha(x)]_\Lambda = \phi(\lambda_\alpha(x)) = x_\alpha$.

Hence $\bar{\lambda}_\alpha$ is the required local lifting.

q.e.d.

<u>Lemma 3.2.4</u>:

<u>If G and H are locally trivial topological groupoid over X and Y, respectively, then G × H is locally trivial over X × Y.</u>

Proof: Given $(x_\alpha, y_\alpha) \varepsilon X \times Y = (G \times H)^{ob}$, then $x_\alpha \varepsilon X$ and $y_\alpha \varepsilon Y$. Hence, $\exists\, U_\alpha, V_\alpha$ open neighbourhoods of x_α, y_α in X and Y, respectively, and the continuous lifts:

$$\lambda_\alpha : U_\alpha \longrightarrow G$$

$$\mu_\alpha : V_\alpha \longrightarrow H$$

s.t. $\forall (x, y) \varepsilon U_\alpha \times V_\alpha$, $\phi_G(\lambda_\alpha(x)) = x_\alpha$, $i_G(\lambda_\alpha(x)) = x$

$$\phi_H(\mu_\alpha(y)) = y_\alpha, \quad i_H(\mu_\alpha(y)) = y$$

Hence $\lambda_\alpha \times \mu_\alpha : U_\alpha \times V_\alpha \longrightarrow G \quad H$ is a continuous map, s.t.

$\forall (x, y) \varepsilon U_\alpha \times V_\alpha$, $i_{G \times H}(\lambda_\alpha \times \mu_\alpha(x, y)) = i_{G \times H}(\lambda_\alpha(x), \mu_\alpha(y))$

$$= (i_G(\lambda_\alpha(x)), i_H(\mu_\alpha(y))) = (x, y)$$

Similarly, $\phi_{G \times H}(\lambda_\alpha \times \mu_\alpha(x, y)) = (x_\alpha, y_\alpha)$, therefore $G \times H$ is locally trivial. q.e.d.

Given a topological group Γ, $\Gamma \times \Gamma$ acts continuously on Γ by:-

$$(\gamma_1, \gamma_2) \cdot g = \gamma_1 + g - \gamma_2$$

Clearly $\Gamma \times \Gamma$ acts effectively on Γ. Thus we may regard $\Gamma \times \Gamma$ as a subgroup of Aut Γ.

The next theorem shows the relations between fibre bundles and locally trivial groupoids (cf [12] theorem 2.5.)

Theorem 2.2.5:

Let $G = (G, X, i, \phi, \theta, \sigma, u)$ be a connected locally trivial topological groupoid, and let $x_o \varepsilon X$ be a fixed object. Then G is a coordinate bundle over $X \times X$, with the projection (i, ϕ), fibre $G\{x_o\}$ and group $G\{x_o\} \times G\{x_o\}$

Proof: Since G is locally trivial, we have an open cover $\{U_\alpha\}$ of X, and a family of continuous lifts $\{\lambda_\alpha : U_\alpha \longrightarrow G\}$. Let T be any wide tree in G, and let $\tau_\alpha \varepsilon T(x, x_o)$ be its unique element.

Then for each U_α we have a map: $\rho_\alpha : U_\alpha \longrightarrow G$

defined by:- $\forall\, x \in U_\alpha$, $\rho_\alpha(x) = \lambda_\alpha(x) + \tau_\alpha$

obviously $\rho_\alpha = \Lambda_{\tau_\alpha} \circ \lambda_\alpha$ and hence it is continuous. Moreover:-

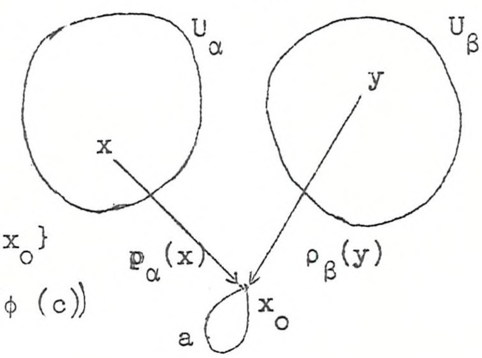
$$i(\rho_\alpha(x)) = x \text{ and } \phi(\rho_\alpha(x)) = x_o.$$

We will call ρ_α the <u>associated lift</u> of λ_α. (They are introduced only for simplification.)

The set $\mathcal{U} = \{U_\nu \times U_\beta \mid U_\nu , U_\beta \varepsilon \{U_\alpha\}\}$ is an open cover of $X \times X$, and we take it as the set of coordinate neighbourhoods.

For each $U_\alpha \times U_\beta \; \varepsilon \, \mathcal{U}$, define the coordinate function

$$\phi_{\alpha\beta} : U_\alpha \times U_\beta \times G\{x_o\} \longrightarrow (i,\phi)^{-1}(U_\alpha \times U_\beta))$$
$$= (G(U_\alpha, U_\beta))$$



by: $\phi_{\alpha\beta}(x, y, a) = \rho_\alpha(x) + a - \rho_\beta(y)$

<u>$\phi_{\alpha\beta}$ is bijective:</u>

Define $\psi_{\alpha\beta} : G(U_\alpha, U_\beta) \longrightarrow U_\alpha \times U_\beta \times G\{x_o\}$

by: $\psi_{\alpha\beta}(c) = (ic, \phi c, -\rho_\alpha(ic) + c + \rho_\beta(\phi(c))$

We have:-

$$\forall\, c \,\varepsilon\, G(U_\alpha, U_\beta),\; \phi_{\alpha\beta} \circ \psi_{\alpha\beta}(c) = \phi_{\alpha\beta}(ic, \phi c, -\rho_\alpha(ic) + c + \rho_\beta(\phi c))$$
$$= \rho_\alpha(ic) + (-\rho_\alpha(ic) + c + \rho_\beta(\phi c)) - \rho_\beta(\phi c) = C$$

Therefore $\phi_{\alpha\beta} \circ \psi_{\alpha\beta} = id_{G(U_\alpha, U_\beta)}$

Similarly $\psi_{\alpha\beta} \circ \phi_{\alpha\beta} = id_{U_\alpha \times U_\beta \times G\{x_o\}}$

Hence $\phi_{\alpha\beta}$ is a bijection with $\phi_{\alpha\beta}^{-1} = \psi_{\alpha\beta}$

We next show that $\phi_{\alpha\beta}$ and $\psi_{\alpha\beta}$ are continuous, and hence $\phi_{\alpha\beta}$ is a homeomorphism.

Let $t: G \times G \longrightarrow G \times G$ be the twisting map, i.e. $t(a, b) = (b, a)$; and let $j: G\{x_o\} \longrightarrow G$ be the inclusion map, then we have:-

$$\phi_{\alpha\beta} = \tilde{\theta}\circ(1_G \times t) \circ (\rho_\alpha \times (\sigma\circ\rho_\beta) \times j)$$

Where $\tilde{\theta}$ is as defined in 3.1.3(6). Since all maps in this composition are continuous, $\phi_{\alpha\beta}$ is continuous.

Let $\nu : G(U_\alpha, U_\beta) \longrightarrow G$ be defined by the following commutative diagram:-

$$\begin{array}{ccccc} G(U_\alpha, U_\beta) & \xrightarrow{(i,\phi\ ,\ j')} & U_\alpha \times U_\beta \times G & \xrightarrow{(\sigma\circ\rho_\alpha)\times\rho_\beta\times 1_G} & G\times G\times G \\ & & & & \downarrow 1_G \times t \\ & & & & D_{\tilde{\theta}} \\ & & & & \downarrow \tilde{\theta} \\ & \overset{\nu}{\dashrightarrow} & & & G \end{array}$$

where $j' : G(U_\alpha, U_\beta) \longrightarrow G$ denotes the inclusion map. (as in 3.1.3.(6))

Then, obviously ν is continuous, and we have:-

$\psi_{\alpha\beta} = (\overline{i}, \overline{\phi}\ ,\ \nu)$, where $\overline{i}$ and $\overline{\phi}$ are the restrictions of i and ϕ to $G(U, V) = (i, \phi)^{-1}(U_\alpha \times U_\beta)$. Hence $\psi_{\alpha\beta}$ is continuous.

We now show that these coordinate functions satisfy the required conditions for a coordinate bundle (see [10])

(i) $\forall\ (x', y') \in U_\alpha \times U_\beta$, define:-

$\phi_{\alpha\beta,\ x',\ y'} : G\{x_0\} \longrightarrow (i, \phi)^{-1}(x', y') = G(x', y')$

by: $\forall\ a \in G\{x_0\}$, $\phi_{\alpha\beta,x',y'}(a) = \phi_{\alpha\beta}(x', y', a)$.

Then $\forall\ (x, y) \in (U_\alpha \times U_\beta)\cap(U_{\alpha'} \times U_{\beta'}) = (U_\alpha \cap U_{\alpha'}) \times (U_\beta \cap U_{\beta'})$,

we have:- $\forall\ a \in G\{x_0\}$, $\phi^{-1}_{\alpha'\beta',\ x,y} \circ \phi_{\alpha\ \beta,x,y}(a) =$

$$\phi^{-1}_{\alpha'\beta',\ x,y}\ (\rho_\alpha(x) + a - \rho_\beta(y)\ )$$

$$= -\rho_{\alpha'}(x) + \rho_\alpha(x) + a - \rho_\beta(y) + \rho_{\beta'}(y) = \gamma_x + a - \gamma_y$$

$$= (\gamma_x, \gamma_y).\ a.$$

where $\gamma_x = -\rho_{\alpha'}(x) + \rho_\alpha(x) \in G\{x_0\}$ and $\gamma_y = -\rho_{\beta'}(y) + \rho_\beta(y) \in G\{x_0\}$

Hence $\phi^{-1}_{\alpha'\beta',x,y} \circ \phi_{\alpha\beta,\ x,y} = (\gamma_x, \gamma_y) \in G\{x_0\} \times G\{x_0\}$

(ii) The map $g\ : (U_\alpha \times U_\beta)\cap(U_{\alpha'} \times U_{\beta'}) \longrightarrow G\{x_0\} \times G\{x_0\}$

defined by:- $g_{\alpha\beta,\ \alpha'\beta'}(x, y) = \phi^{-1}_{\alpha'\beta',\ x,y} \circ \phi_{\alpha\beta,x,y}$

is continuous.

The continuous lifts $\rho_{\alpha}, \rho_{\alpha'}$ and $\rho_{\beta'}, \rho_{\beta}$ give rise to continuous maps $S_{\alpha\alpha'}: U_{\alpha} \cap U_{\alpha'} \longrightarrow G\{x_0\}$, $S_{\alpha\alpha'}(x) = -\rho_{\alpha'}(x) + \rho_{\alpha}(x)$

$S_{\beta\beta'}: U_{\beta'} \cap U_{\beta} \longrightarrow G\{x_0\}$, $S_{\beta\beta'}(y) = -\rho_{\beta'}(y) + \rho_{\beta}(y)$.

For, $S_{\alpha\alpha'} = \theta \circ (\sigma \circ \rho_{\alpha'}, \rho_{\alpha})$ and $S_{\beta\beta'} = \theta \circ (\sigma \circ \rho_{\beta'}, \rho_{\beta})$.

Clearly $g_{\alpha\beta,\alpha'\beta'} = S_{\alpha\alpha'} \times S_{\beta\beta'}$, and hence it is continuous.

q.e.d.

The coordinate bundle obtained as above, depends on the choices that were made, i.e. depends on the continuous lifts and the tree T. We now show that different choices give rise to equivalent coordinate bundles.

For each $U\alpha$, let $\lambda'_{\alpha}: U_{\alpha} \longrightarrow G$ be another continuous lift, and let T' be another tree in the construction of ρ_{α} Then, as before, we get a continuous map

$$\rho'_{\alpha} : U_{\alpha} \longrightarrow G .$$

Hence, for each coordinate neighbourhood $U_{\alpha} \times U_{\beta}$, the new coordinate function will be:-

$$\phi'_{\alpha\beta} : U_{\alpha} \times U_{\beta} \times G\{x_0\} \longrightarrow G(U_{\alpha}, U_{\beta})$$

defined by:- $\phi'_{\alpha\beta}(x, y, a) = \rho'_{\alpha}(x) + a - \rho'_{\beta}(y)$,

and for each pair $(U_{\alpha} \times U_{\beta}, U_{\alpha'} \times U_{\beta'})$, the new coordinate transformation will be:-

$$g'_{\alpha\beta,\alpha'\beta'}: (U_{\alpha} \times U_{\beta}) \cap (U_{\alpha'} \times U_{\beta'}) \longrightarrow G\{x_0\} \times G\{x_0\}$$

defined by:- $g_{\alpha\beta,\alpha'\beta'}(x, y) = \phi'^{-1}_{\alpha'\beta', x,y} \circ \phi'_{\alpha\beta\, x,y}$

$$= (-\rho'_{\alpha'}(x) + \rho'_{\alpha}(x) , \rho'_{\beta'}(y) + \rho'_{\beta}(y))$$

For each $U_{\alpha} \times U_{\beta}$, define

$$\mu_{\alpha\beta} : U_{\alpha} \times U_{\beta} \longrightarrow G\{x_0\} \times G\{x_0\}$$

by:- $\mu_{\alpha\beta}(x, y) = (-\rho_{\alpha}(x) + \rho'_{\alpha}(x), -\rho_{\beta}(y) + \rho'_{\beta}(y))$

obviously $\mu_{\alpha\beta}$ is continuous, and we have:-

$\forall(x, y)\varepsilon (U_\alpha \times U_\beta)\cap(U_{\alpha'}\times U_{\beta'})$,

$$g'_{\alpha\beta\alpha'\beta'}(x,y) = (-\rho'_{\alpha'}(x) + \rho'_\alpha(x), -\rho'_{\beta'}(y) + \rho'_\beta(y))$$

$$= (-\rho'_{\alpha'}(x) + \rho_{\alpha'}(x) - \rho_{\alpha'}(x) + \rho_\alpha(x) - \rho_\alpha(x) + \rho'_\alpha(x), -\rho'_{\beta'}(y) + \rho_{\beta'}(y) - \rho_{\beta'}(y) + \rho_\beta(y) - \rho_\beta(y) + \rho'_\beta(y))$$

$$= (-\rho'_{\alpha'}(x) + \rho_{\alpha'}(x), -\rho'_{\beta'}(y) + \rho_{\beta'}(y)) + (-\rho_{\alpha'}(x) + \rho_\alpha(x), -\rho_{\beta'}(y) + \rho_\beta(y)) + (-\rho_\alpha(x) + \rho'_\alpha(x), -\rho_\beta(y) + \rho'_\beta(y))$$

$$= -\mu_{\alpha'\beta'}(x,y) + g_{\alpha\beta\alpha'\beta'}(x,y) + \mu_{\alpha\beta}(x, y)$$

Hence, the two coordinate bundles are equivalent (see [10] p.12)

Corollary 3.2.6:

If the vertex groups of a connected locally trivial groupoid G over X are discrete, then G is a covering space of X × X.

Remarks 3.2.7.

(1) When X is p.c., ℓ.p.c. and ℓ.s.c., then $\pi_1(X,.)$ is a discrete topological group. Hence, once again, we get:-

If X is a p.c., ℓ.p.c. and ℓ.s.c. Hausdorff space, then πX, and hence $\pi_A X$, is a covering space of X × X. In case $\pi_1(X,.)$ is abelian, $\pi_A X$ corresponds to the subgroup $A\{\cdot\}\times \pi_1(X,\cdot)\subseteq\pi_1(X\times X)$.

(2) Since G is a bundle over X × X, with the projection (i,ϕ), it follows that in any connected locally trivial groupoid G, the initial and final maps are open.

(3) Given a εG, let U_{ia}, $U_{\phi a}$ be the liftable open neighbourhoods of ia and ϕa, respectively. Then for any open neighbourhoods $N_a\subseteq G(ia, \phi a)$ the set $M = \{\lambda_{ia}(x) + n - \lambda_{\phi a}(y) \mid x\varepsilon U_{ia}, y\varepsilon U_{\phi a}, n\varepsilon N_a\}$ is open in G.

For, let $\phi_{(ia)(\phi a)}: U_\alpha \times U_\beta \times G\{x_0\} \longrightarrow G(U_\alpha, U_\beta)$ be the coordinate function, then $M = \phi_{(ia)(\phi a)}(U_{ia} \times U_{\phi a} \times (-\tau_{ia} + N_a + \tau_{\phi a}))$.

(4) Let G be a connected locally trivial groupoid, and let τ be an element of the tree T with initial x_0. Then given any open neighbourhood N of τ, $\exists$ a basic open neighbourhood $< N_0, U_1 > = \{n + \rho(y) \mid n \in N_0, y \in U_1\}$ of τ contained in N.

Proof: Since $\phi(N)$ is open and contains $\phi(\tau)$, $\exists$ a liftable open neighbourhoods U_1 of $\phi(\tau)$ and the continuous lift $\rho : U_1 \longrightarrow G$. Let U_0 be any liftable neighbourhood of x_0, then $N_1 = N \cap G(U_0, U_1)$ is open in $G(U_0, U_1) \approx U_0 \times U_1 \times G\{x_0\}$ and contains τ. Hence $\exists$ neighbourhoods N_0 of o_{x_0} in $G\{x_0\}$ such that $N_1 = < N_0, U_1 >$.

q.e.d.

Theorem 3.2.8:

Let $(G, X, i, \phi, \theta, \sigma, u)$ be a connected locally trivial groupoid. Then for each $x_0 \in X$, Stx_0 is a principal fibre bundle over X with the projection ϕ and the group $G\{x_0\}$ acting on itself on the right.

Proof: Let $\{U_\alpha\}$ be a cover of X by liftable open sets, and for each α, let $\rho_\alpha : U_\alpha \longrightarrow G$ be an associated continuous lift, such that $\forall x \in U_\alpha$, $i(\rho_\alpha(x)) = x$ and $\phi(\rho_\alpha(x)) = x_0$.

For each α, define $\psi_\alpha : U_\alpha \times G\{x_0\} \longrightarrow \phi^{-1}(U_\alpha) \cap Stx_0$

by:- $\psi_\alpha(x, a) = a - \rho_\alpha(x)$.

By the same type of argument as in 3.2.5. it is easy to see that ψ_α is a homeomorphism.

Next, for each $x \in U_\alpha$, let $\psi_{\alpha, x} : G\{x_0\} \longrightarrow \phi^{-1}(U_\alpha) \cap Stx_0$ be defined by:- $\psi_{\alpha,x}(a) = \psi_\alpha(x, a)$

Let $U_\alpha \cap U_\beta \neq \phi$, then we have a homeomorphism

$$t = \psi^{-1}_{\beta,x} \circ \psi_{\alpha,x} : G\{x_o\} \longrightarrow G\{x_o\}$$

which is a right translation. For, we have:-

$\forall\, a \in G\{x_o\}$, $t(a) = \psi^{-1}_{\beta,x}(a - \rho_\alpha(x)) = a - \rho_\alpha(x) + \rho_\beta(x) = a + \gamma_x$.

where $\gamma_x = -\rho_\alpha(x) + \rho_\beta(x) \in G\{x_o\}$. Hence t is a right translation: $a \rightsquigarrow a + \gamma_x$ of $G\{x_o\}$ on itself. Therefore $t = \gamma_x \in G\{x_o\}$.

So, it remains to show that the map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G\{x_o\},\ x \rightsquigarrow \gamma_x$$

is continuous. $\forall\, x \in U_\alpha \cap U_\beta$ we have:-

$$\theta \circ (\sigma \circ \rho_\alpha, \rho_\beta)(x) = \theta \circ (\sigma \circ \rho_\alpha(x), \rho_\beta(x)) = \theta(-\rho_\alpha(x), \rho_\beta(x))$$
$$= -\rho_\alpha(x) + \rho_\beta(x) = \gamma_x = g_{\alpha\beta}(x).$$

Hence $g_{\alpha\beta} = \theta \circ (\sigma \circ \rho_\beta, \rho_\beta)$, and therefore its continuity follows from that of θ, σ, ρ_β and ρ_α. Thus we have verified the conditions of Steenrod ([10], p.7).

q.e.d.

The principal coordinate bundle obtained above depends on the choices of ρ's. We now show that if we choose different lifts $\rho'_\alpha : U_\alpha \longrightarrow G$, we get equivalent coordinate bundles.

For each U_α, define the continuous map

$$\mu_\alpha : U_\alpha \longrightarrow G \text{ by } \mu_\alpha(x) = -\rho_\alpha(x) + \rho'_\alpha(x)$$

For each pair U_α, U_β with $U_\alpha \cap U_\beta \neq \phi$, let

$$g'_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G\{x_o\}$$

be the new coordinate transformation. Then

$$\forall x \in U_\alpha \cap U_\beta,\ g'_{\alpha\beta}(x) = \gamma'_x = -\rho'_\alpha(x) + \rho'_\beta(x)$$

Now, $g'_{\alpha\beta}(x) = -\rho'_\alpha(x) + (\rho_\alpha(x) - \rho_\alpha(x) + \rho_\beta(x) - \rho_\beta(x)) + \rho'_\beta(x)$

$$= (-\rho_\alpha(x) + \rho_\alpha(x)) + (-\rho_\alpha(x) + \rho_\beta(x)) + (-\rho_\beta(x) + \rho'_\beta(x))$$

$= -\mu_\alpha(x) + g_{\alpha\beta}(x) + \mu_\beta(x)$.

Therefore the two coordinate bundles are equivalent .

Corollary 3.2.9:

For each wide connected normal subgroupoid A of the locally trivial groupoid G over X, $X_{A_{x_o}} = St_{E_A} x_o$ is a principal bundle over X with the projection $\bar{\phi}_A$.

In case A is not normal in G, E_A is not a groupoid. In this case $X_{A_{x_o}} = St\, x_o/_{R_A}$ is a bundle over X under the same projection and fibre but the group $G\{x_o\}$ (see[4]p. 147)

Corollary 3.2.10:

If G has discrete vertex groups, then $X_{A_{x_o}}$ is a covering space of X.

Theorem 3.2.11:

Every connected locally trivial groupoid G, over a p.c., ℓ.p. and ℓ.s.c. space X, with discrete vertex groups is isomorphic to $\pi_A X$ for some subgroupoid $A \subseteq \pi X$.

Proof: Let $x_o \in X$, then by 3.2.10. ϕ: $St_G x_o \longrightarrow X$ is a covering map. Hence ϕ_*: $\pi(St_G x_o) \longrightarrow \pi X$ is a covering morphism of groupoids. Let $G' = St_G x_o$ and q: $St_{\pi G'} {}^o x_o \longrightarrow St_{\pi X} x_o$ be the restriction of ϕ_*. Then as we see later (4.1.2.) q is a homeomorphism. Let $\bar{\phi}$: $\pi G' \longrightarrow G'$ be the final map, then $L = \bar{\phi} o q^{-1}: St_{\pi X} x_o \longrightarrow St_G x_o$ is continuous and open. Let T be a wide tree in πX, then $\pi X \approx T * G\{x_o\}$.

Define ξ: $\pi X \longrightarrow G$ by: $\xi|\ X = id$, and for any

$a = -\tau_y + a' + \tau_z \in \pi X(y,z)$, $\xi(a) = -L(\tau_y) + L(a') + L(\tau_z)$,

where τ_x denote the unique element of $T(x_o, x)$.

Clearly ξ is a morphism of abstract groupoids. Moreover ξ is onto.

For, given $g \varepsilon G(y_1, z_1)$, then $g = -L(\tau_{y_1}) + g' + L(\tau_{z_1})$, where $g' \varepsilon G\{x_o\}$.

(In fact $T_1 = \xi(T)$ is a tree in G). Let $\mu_{g'} \varepsilon PG'(o_x, g')$, then $\phi o \mu_{g'}$ is a loop at x_o in X. Hence $\phi_*(\overline{\mu}_{g'}) = \overline{\phi o \mu_{g'}} \varepsilon \pi_1(X, x_o)$.

Let $h = -\tau_{y_1} + \overline{\phi o \mu_{g'}} + \tau_{z_1}$, then
$$\xi(h) = -L(\tau_{y_1}) + L(\overline{\phi o \mu_{g'}}) + L(\tau_{z_1}) = -L(\tau_{y_1}) + g' + L(\tau_{z_1}) = g.$$

For, $L(\overline{\phi o \mu_{g'}}) = \overline{\phi} o q^{-1}(\phi_*(\overline{\mu_{g'}})) = \phi(\overline{\mu}_{g'}) = g'$

<u>ξ is continuous</u>: Let N be a neighbourhood of $\xi(a)$, then $\exists$ neighbourhoods N_1, N_2, N_3 of $L(\tau_y)$, $L(a')$, $L(\tau_z)$ in G such that

$$\tilde{\theta}(((-N_1) \times N_2 \times N_3) \cap \tilde{D}) \subseteq N \quad (\tilde{\theta}, \tilde{D} \text{ as in } 3.1.3(6)).$$

Let $N'_i = N_i \cap St_G x_o$, then N'_1, N'_2, N'_3 are neighbourhoods of $L(\tau_y)$, $L(a')$, $L(\tau_z)$, respectively, in $St_G x_o$. Hence by continuity of L, $\exists$ neighbourhoods, M_1, M_2, M_3 of τ_y, a', τ_z, respectively, such that $L(M_i) \subseteq N'_i$, i = 1, 2, 3. Using C - N topology of πX, $\exists$ basic neighbourhoods $<\tau_y, U>$, $<\tau_z, V>$ of τ_y and τ_z such that $<\tau_y, U> \subseteq M_1$, $<\tau_z, V> \subseteq M_2$. (Recall that $<b, w> = \{b + \overline{\gamma} \mid \gamma \varepsilon PW\}$). Then $<U, a, V>$ is a basic neighbourhood of a in πX. It is easy to see that $\xi(<U, a, V>) \subseteq N$.

Therefore ξ is a morphism of topological groupoids.

<u>Moreover ξ is open</u>. For, let $N = <U, a, V>$† be a basic neighbourhood of $a = -\tau_y + a' + \tau_z$, then $\exists$ neighbourhoods N_1, N_2 of τ_y and τ_z, respectively, in $St_{\pi X} x_o$ such that $N = -N_1 + a' + N_2$.

Hence $L(N_i)$, i = 1, 2 is open in $St_G x_o$, and therefore $L(N_1) = <L(\tau_y), U>$ and $L(N_2) = <L(\tau_z), V>$ (Recall that $<c, W_\alpha> = \{c + \lambda_\alpha(t) \mid t \varepsilon W_\alpha\}$). Since G has discrete vertex groups, $L(a')$ is open in $G\{x_o\}$ and hence:-

† We take U, V to be liftable in G.

$\xi(N) = -L(N_1) + L(a') + L(N_2) = \langle U, L(a'), V\rangle$. Thus ξ is open.

Next, let A be any subgroupoid of πX such that:-

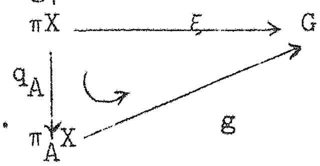
$$\forall\, x \in X,\ A\{x\} = \xi^{-1}(o_x)$$

Then $\pi_A X$ is a topological groupoid, and we show that $G \approx \pi_A X$.

Define $g : \pi_A X \longrightarrow G$ by $g([a]_A) = \xi(a)$, and $g|\ X = id$.

Then the diagram is commutative and

algebraically $g : \pi_A X \approx G$ (see [2], p. 279).



That g is a homeomorphism follows from the fact that $\pi_A X$ carries the quotient topology and ξ is open. q.e.d.

<u>Corollary 3.2.11a:</u> <u>Let G be a connected, locally trivial groupoid over the p.c., ℓ.p.c., ℓ.s.c. space X, then</u>

<u>$\pi_1(G, o_x) \approx \pi_1(St_G x, o_x) \times \pi_1(X, x)$.</u> (This follows from 3.2.11. & 2.2.9.)

<u>Remark 3.2.12:</u> Let X be a p.c., ℓ.p.c. and ℓ.s.c. space with abelian fundamental group, then every covering space N of $X \times X$ corresponding to the subgroup $A_o \times \pi_1(X)$ of $\pi_1(X) \times \pi_1(X)$ has a groupoid structure isomorphic to $\pi_A X$. Where A is the groupoid obtained from A_o as in chapter II. Because $\pi_A X$ is a covering space of $X \times X$ and by 2.2.9. has the fundamental group isomorphic to $A_o \times \pi_1(x)$. Therefore $\pi_A X$ is homeomorphic to N. This gives a locally trivial groupoid structure to N with discrete vertex groups. By the above theorem, N has only one such structure.

<u>Example:</u> Let $f : S^1 \longrightarrow S^1$, $z \rightsquigarrow z^n$ be the n-fold covering map, then $f \times id : T = S^1 \times S^1 \longrightarrow S^1 \times S^1$ is also a covering map corresponding the subgroup $A \times \pi_1(S^1)(= n\mathbb{Z} \times \mathbb{Z})$. Hence $T \approx \pi_A S^1$.

Therefore:

<u>For any infinite subgroup of $\mathbb{Z}$, the Torus has only one locally trivial groupoid structure over S^1, with discrete vertex groups.</u>

We now show that if X is p.c., ℓ.p.c. and ℓ.s.c. Hausdorff space, then for each wide connected normal subgroupoid A of πX, $\pi_A X$ has a bundle structure over X with the projection $\bar{i}_A$(or $\bar{\phi}_A$), the group $\Gamma = \pi_A X\{x_o\}$ and the fibre $F = \bar{i}_A^{-1}(x_o)$, where $x_o \in X$ is a fixed element. With F and Γ as above we have:-

<u>Theorem 3.2.13:</u>

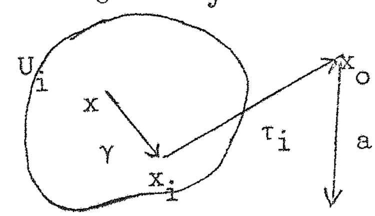
<u>$(E_A, X, F, \Gamma, \bar{i}_A)$ is a coordinate bundle.</u>

<u>Proof:</u> Let $\{U_\alpha\}$ be a cover of X by simply connected neighbourhoods (referred to as canonical neighbourhoods). For each $U_i \in \{U_\alpha\}$, let x_i be a fixed point in U_i, and let T be any wide tree in πX. For each $y \in X$, denote the unique element of $T(y, x_o)$ by τ_y.

Define $\qquad \Phi_i : U_i \times F \longrightarrow \bar{i}_A^{-1}(U_i)$

by:- $\qquad \Phi_i(x, [a]_A) = [\gamma + \tau_i + a]_A$,

where $\gamma \supsetneq PU_i(x, x_i)$



<u>Φ_i is well-defined:</u> Since γ is unique (by the property of U_i) and τ_i is fixed, we need only consider the case of taking different representatives of $[a]_A$.

Let $a' \in [a]_A$, we must show that $[\gamma + \tau_i + a]_A = [\gamma + \tau_i + a']_A$.

But:- $a' \in [a]_A \Longrightarrow a - a' \in A\{x_o\} \Longrightarrow$

$$(\gamma + \tau_i) + (a - a') - (\gamma + \tau_i) \in A\{x\} \text{ (by normality of A)}$$

$$\Longrightarrow \gamma + \tau_i + a - (\gamma + \tau_i + a') \in A\{x\}$$

$$\Longrightarrow [\gamma + \tau_i + a]_A = [\gamma + \tau_i + a']_A.$$

<u>Φ_i is 1 - 1:</u> Let $(x, [a]_A), (y, [b]_A) \in U_i \times F$, then:-

$\Phi_i(x, [a]_A) = \Phi_i(y, [b]_A) \Longrightarrow [\gamma + \tau_i + a]_A = [\gamma' + \tau_i + b]_A$.

$$\Longrightarrow \begin{cases} i\gamma = i\gamma' \Longrightarrow x = y \Longrightarrow \gamma = \gamma' \text{ (by the property of } U_i) \\ \phi a = \phi b \Longrightarrow a - b \text{ is defined} \end{cases}$$

Also, $[\gamma + \tau_i + a]_A = [\gamma' + \tau_i + b]_A$

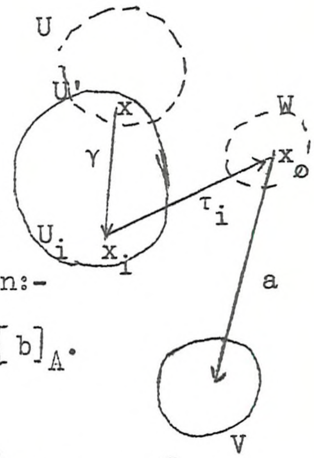
$$\Longrightarrow \quad \gamma + \tau_i + a - b - (\gamma' + \tau_i) \varepsilon A\{x\}$$

$$\Longrightarrow (\gamma + \tau_i) + (a - b) - (\gamma' + \tau_i) \varepsilon A\{x\}$$

$\Longrightarrow a - b \varepsilon A\{x_o\}$ (by normality of A)

$\Longrightarrow [a]_A = [b]_A$.

Hence $(x, [a]_A) = (y, [b]_A)$.



<u>Φ_i is onto:</u> Given $[b]_A \varepsilon \bar{i}_A^{-1}(U_i)$, then let $a = -\tau_i - \lambda + b$, where $\lambda \supseteq PU_i(ib, x_i)$. Then:-

$\Phi_i(ib, [a]_A) = [\lambda + \tau_i + (-\tau_i - \lambda + b)]_A = [b]_A$.

<u>Φ_i is continuous:</u>

Let N be a neighbourhood of $\Phi_i(x, [a]_A) = [\gamma + \tau_i + a]_A$, then N contains a basic open set

$$<U, [\gamma + \tau_i + a]_A, V>.$$

Let W be a canonical neighbourhood of $ia = x_o$, then $<W, [a]_A, V>$ is a basic open neighbourhood of $[a]_A$ in E_A. Hence $M = <W, [a]_A, V> \cap F$ is a neighbourhood of $[a]_A$ in F. Let $U' = U \cap U_i$, then $U' \times M$ is a neighbourhood of $(x, [a]_A)$, and we have:-

$$\Phi_i(U' \times M) \subseteq <U, [\gamma + \tau_i + a]_A, V> \subseteq N.$$

(This easily follows from the property of U_i).

<u>Φ_i is open:</u> Let $N \subseteq U_i \times F$ be open, then

$\exists$ open sets $U' \subseteq U_i$ and $V' \subseteq F$ s.t. $N = U' \times V'$.

We show that $\Phi_i(N)$ is open.

Let $[b]_A \varepsilon \Phi_i(N)$, then $\exists\, x \varepsilon U'$ and $[a]_A \varepsilon V'$ s.t.

$$\Phi_i(x, [a]_A) = [b]_A.$$

Let $\delta \varepsilon \pi X(x, x_i)$ contain $PU_i(x, x_i)$, then $[b]_A = [\delta + \tau_i + a]_A$.

Since V' is open, $\exists\, \tilde{V}$ open in E_A s.t. $V' = \tilde{V} \cap F$.

But then $\tilde{V}$ contains a basic neighbourhood $< W, [a]_A, V>$, where W is a canonical neighbourhood of x_o. Let U_1 be a canonical neighbourhood of x contained in U', then $< U_1, [b]_A, V>$ is a canonical neighbourhood of $[b]_A$ in E_A.

<u>Claim:</u> $< U_1, [b]_A, V> \subseteq \Phi_i(N)$

Let $[c]_A \varepsilon < U_1, [b]_A, V >$, and let $\nu \varepsilon \pi X(ic, x)$ contain $PU_1(ic,x)$, $\mu \varepsilon \pi X(\phi a, \phi c)$ contain $PV(\phi a, \phi c)$,

then $[c]_A = [\nu + b + \mu]_A = [\nu + \delta + \tau_i + a + \mu]_A = \Phi_i(ic, [a + \mu]_A)$.

But $ic \varepsilon N'$ and $[a + \mu]_A \varepsilon V'$, hence $[c]_A \varepsilon \Phi_i(N)$. Therefore $\Phi_i(N)$ is a neighbourhood of each of its elements, and hence open.

<u>Therefore Φ_i is a homeomorphism</u>

Next, we show that the collection $\{\Phi_i\}$, as constructed above, satisfy the required conditions for a system of coordinate functions.

(i) It is obvious that $\bar{i}_A \circ \Phi_i : U_i \times F \longrightarrow U_i$ is a projection.

(ii) For each $z \varepsilon U_i$, let $\Phi_{i,z} : F \longrightarrow \bar{i}_A^{-1}(z)$ be defined by:-

$$\Phi_{i,z}([a]_A) = \Phi_i(z, [a]_A).$$

We must show that for any $x \varepsilon U_i \cap U_j$, $U_i, U_j, \varepsilon \{U_\alpha\}$, the homeomorphism $\Phi_{j,x}^{-1} \circ \Phi_{i,x} : F \longrightarrow F$

is an element of Γ. We have:-

$$\forall [a]_A \varepsilon F, \Phi_{j,x}^{-1} \circ \Phi_{i,x}([a]_A) = \Phi_{j,x}^{-1}([\gamma_1 + \tau_i + a]_A)$$

$$= [- \tau_j - \gamma_2 + \gamma_1 + \tau_i + a]_A.$$

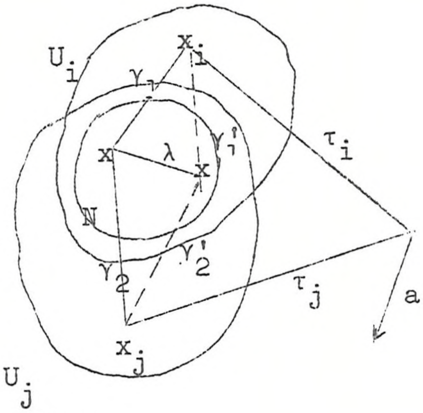
But $[-\tau_j - \gamma_2 + \gamma_1 + \tau_i]_A = \beta \varepsilon \Gamma$, and we have:-

$[-\tau_j - \gamma_2 + \gamma_1 + \tau_i + a]_A = \beta.[a]_A = \beta([a]_A)$. Hence

$\Phi_{j,x}^{-1} \circ \Phi_{i.x} = \beta \varepsilon \Gamma$

(iii) We must show that for each pair U_i, U_j, with non-empty intersection, the map: $g_{j,i} : U_i \cap U_i \longrightarrow \Gamma$

defined by $g_{ji}(x) = \Phi^{-1}_{j,x} \circ \Phi_{i,x}$ is continuous.



Let $[a]_A \in F$, and let

$\Phi_{i,x}([a]_A) = [\gamma_1 + \tau_i + a]_A$

$\Phi_{j,x}([a]_A) = [\gamma_2 + \tau_j + a]_A$,

then $\Phi^{-1}_{j,x} \circ \Phi_{i,x} = [-\tau_j - \gamma_2 + \gamma_1 + \tau_i]_A$.

Let $N \subseteq U_i \cap U_j$ be any path-connected neighbourhood of x. For any $x' \in N$, let $\lambda \supseteq PN(x', x)$ be the unique morphism, and let

$$\Phi_{j,x'}([a]_A) = [\gamma'_2 + \tau_j + a]_A, \quad \Phi_{i,x'}([a]_A) = [\gamma'_1 + \tau_i + a]_A$$

then $\Phi^{-1}_{j,x} \circ \Phi_{i,x} = [-\tau_j - \gamma'_2 + \gamma'_1 + \tau_i]_A$.

By simply connected property of U_i and U_j, we have:-

$$\gamma'_2 = \lambda + \gamma_2 \, , \quad \gamma'_1 = \lambda + \gamma_1$$

Hence $g_{j,i}(x') = [-\tau_j - \gamma'_2 + \gamma'_1 + \tau_i]_A = [-\tau_j - \gamma_2 - \lambda + \lambda + \gamma_1 + \tau_i]_A$

$$= [-\tau_j - \gamma_2 + \gamma_i + \tau_i]_A = g_{ji}(x)$$

Therefore $g_{j\,i}$ is constant on N and hence continuous over $U_i \cap U_j$.

This completes the proof of the theorem.

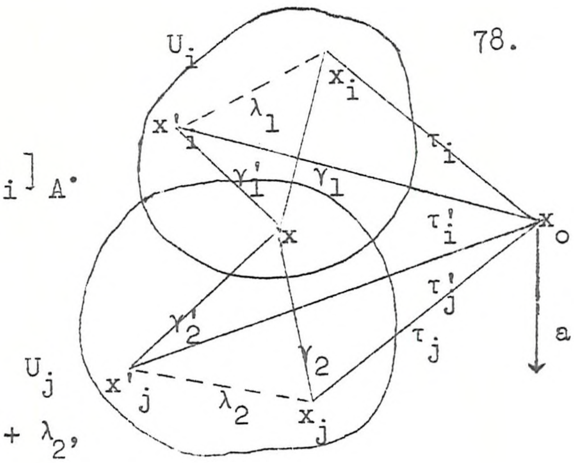
q.e.d.

The above coordinate bundle depends on the choices made, i.e. on the fixed point $x_i \in U_i$, and the tree T. We now show that if we choose a different tree T', say, and another fixed point $x'_i \in U_i$, then the resulting coordinate bundle is equivalent to the above.

Let, for any $U_i \in \{U_\alpha\}$

$$\Phi'_i : U_i \times F \longrightarrow \bar{i}_A^{-1}(U_i)$$

be defined by $\Phi'_i(x, [a]_A) = [\gamma' + \tau'_i + a]_A$.



Then, $\forall\ x \in U_i \cap U_j$,

$g'_{j\,i}(x) = [-\tau'_j - \gamma'_2 + \gamma'_1 + \tau'_i]_A$.

Let $\lambda_1 \supseteq PU_i(x_i, x'_i)$

$\lambda_2 \supseteq PU_j(x_j, x'_j)$

then we have:-

$$\gamma'_1 = \gamma_1 + \lambda_1 \quad \& \quad \gamma'_2 = \gamma_2 + \lambda_2,$$

and
$$\begin{aligned} g'_{ji}(x) &= [-\tau'_j - \lambda_2 - \gamma_2 + \gamma_1 + \lambda_1 + \tau'_i]_A \\ &= [(-\tau'_j - \lambda_2 + \tau_j) + (-\tau_j - \gamma_2 + \gamma_1 + \tau_i) + \\ &\quad (-\tau_i + \lambda_1 + \tau'_i)]_A \\ &= [-\tau'_j - \lambda_2 + \tau_j]_A + [-\tau_j - \gamma_2 + \gamma_1 + \tau_i]_A \\ &\quad + [-\tau_i + \lambda_1 + \tau'_i]_A \\ &= -\mu_j(x) + g_{ji}(x) + \mu_i(x) \end{aligned}$$

where $\mu_j : U_j \longrightarrow \Gamma$, defined by $\mu_j(x) = [-\tau_j + \lambda_2 + \tau'_j]_A$

$\mu_i : U_i \longrightarrow \Gamma$ defined by $\mu_i(x) = [-\tau_i + \lambda_1 + \tau'_i]_A$

Obviously μ_i & μ_j are continuous, and hence the above coordinate bundles are equivalent.

In a similar way it can be shown that $(E_A, X, F, \Gamma, \overline{\Phi}_A)$ is a coordinate bundle with the same set of coordinate neighbourhoods, but with $\{\Phi'_i\}$, as the set of coordinate functions where $\Phi'_i : U_i \times F \longrightarrow \overline{\phi}_A^{-1}(U_i)$ is defined by: $\Phi'_i(x, [a]_A) = -a - \tau_i - \gamma (= -\Phi_i(x, [a]_A))$.

These two coordinate bundles are equivalent as the following lemma shows:-

Lemma 3.2.14:

<u>The coordinate bundles $(E_A, X, F, \Gamma, \overline{i}_A)$ and $(E_A, X, F, \Gamma, \overline{\Phi}_A)$ are equivalent.</u>

Proof:- For each $x \in U_i \cap U_j$, we have:-

$$g_{ji}(x) = \phi^{-1}_{j,x} \circ \phi_{i,\bar{x}} = [-\tau_i - \gamma_1 + \gamma_2 + \tau_j]_A \quad \text{(as we saw in 3.2.11.)}$$

$$g'_{ji}(x) = \phi'^{-1}_{j,x} \circ \phi'_{i,x} = [-\tau_i - \gamma_1 + \gamma_2 + \tau_j]_A \quad \text{(this can be seen easily from the definition)}$$

Hence, $\forall\, x \in U_i \cap U_j$, $g_{ji}(x) = g'_{ji}(x)$. Therefore the two bundles are equivalent (see Steenrad p. 12). (here $\lambda_k : U_k \longrightarrow \Gamma$ is the constant map to the identity of Γ.)

Remark: The intersection of $\bar{i}_A$ and $\bar{\phi}_A$ fibres over each $x \in X$, is $\pi_A X\{x_0\}$ = the group of the bundle.

3. SOME MORE FACTS ABOUT πX:

Lemma 3.3.1:

Let $p: X \longrightarrow Y$ be a continuous map of p.c., ℓ.p.c., ℓ.s.c. Hausdorff spaces. Then $p_*: \pi X \longrightarrow \pi Y$ defined by $\bar{f} \rightsquigarrow \overline{pof}$ is a morphism of topological groupoids.

Proof: It is known (see[2]p. 187) that p_* is an abstract morphism, so we need only show the continuity of p_*.

Let $<U, \overline{pof}, V>$ a basic neighbourhood of $\overline{pof}$ in πy, then $p^{-1}(U)$ and $p^{-1}(V)$ are open neighbourhoods of $i(f)$ and $\phi(f)$, respectively. Hence $\exists$ simply connected neighbourhoods U_1, V_1 of $i(f)$ & $\phi(f)$, respectively, s.t. $p(U_1) \subseteq U$ and $p(V_1) \subseteq V$. Then $<U_1, \bar{f}, V_1>$ is a basic neighbourhood of $\bar{f}$, and it is easily seen that $p_*(<U_1, \bar{f}, V_1>) \subseteq <U, \overline{p_{of}}, V>$

q.e.d.

The following corollary is easily verified.

Corollary 3.3.2:

The function π: $\begin{cases} p \rightsquigarrow \pi p (= p_*) \\ X \rightsquigarrow \pi X \end{cases}$ is a functor from the category of p.c., ℓ.p., ℓ.s.c. Hausdorff spaces and continuous maps to the category of locally trivial topological groupoids and continuous morphisms.

In case of more general spaces Lemma 3.3.1. is true if we make the restriction that $p:X \longrightarrow Y$ be a light map, i.e. $\forall\, y \in Y$, $p^{-1}(y)$ be a discrete subspace of X.

Lemma 3.3.3:

Let $p:X \longrightarrow Y$ be a light map of Hausdorff spaces, then $p_{\#}: PX \longrightarrow PY$ is a functor of topological categories.

Proof:

It is easily seen that $p_{\#}$ is an abstract functor (see [2] p. 187) Since p is light, for each $f \in PX$, f and pof have the same lengths. Let $N \cap \eta^{-1}(r - \varepsilon, r + \varepsilon)$ be a basic neighbourhood of pof in PY, then $\exists$ closed intervals $K_1, \ldots\ldots, K_n \subseteq \mathbb{R}^+$, open sets $U_1, \ldots\ldots U_n \subseteq Y$ such that $N = \bigcap_{i=1}^{n} \tau(K_i, U_i)$.

Let $M = \bigcap_{i=1}^{n} \tau(K_i, p^{-1}(U_i))$, then clearly $M \cap \eta^{-1}(r - \varepsilon, r + \varepsilon)$ is a neighbourhood of f and it is easily seen that:-

$$p_{\#}(M \cap \eta^{-1}(r - \varepsilon, r + \varepsilon)) \subseteq N \cap \eta^{-1}(r - \varepsilon, r + \varepsilon)$$

Hence $p_{\#}$ is a continuous functor.

q.e.d.

Corollary 3.3.4:

If $p: X \longrightarrow Y$ is a light map of Hausdorff spaces, then $p_*: \pi X \longrightarrow \pi Y$ is a morphism of topological groupoids.

Proof:

Continuity of p_* is easily followed from that of $p_{\#}$ and the following commutative diagram, where q's denote the quotient morphisms.

$$\begin{array}{ccc} PX & \xrightarrow{p_{\#}} & PY \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ X & \xrightarrow{p_*} & Y \end{array}$$

q.e.d.

Theorem 3.3.5:

If $p: X \longrightarrow Y$ is a homeomorphism of Hausdorff spaces, then $p_* : \pi X \longrightarrow \pi Y$ is an isomorphism of topological groupoids.

Proof:

We know that p_* is an abstract morphism (see [2]). That p_* is a homeomorphism follows from 3.3.4.

Theorem 3.3.6:

Let $X = X_1 \times X_2$, where X_1, X_2 are p.c., ℓ.p.c. and ℓ.s.c. Hausdorff spaces, then $\pi X \approx \pi X_1 \times \pi X_2$

Proof:

By 3.1.19 $\pi X_1 \times \pi X_2$ is a topological groupoid over $X \times X$.

Let $p_i : X \longrightarrow X_i$, $i = 1, 2$, be the i^{th}-projection, then by 3.3.1 $p_{i*} : \pi X \longrightarrow \pi X_i$, $i = 1, 2$, is continuous.

Define $\zeta : \pi X \longrightarrow \pi X_1 \times \pi X_2$

by:- $\quad \zeta(f) = (p_{1*}(f) , p_{2*}(f))$.

It is known see[2] 6.4.4. that ζ is an isomorphism of abstract groupoids. So, we need only prove the continuity of ζ and its inverse.

ζ is continuous: Let $N_1 \times N_2$ be a neighbourhood of $(p_{1*}(f), p_{2*}(f))$, then N_i, $i = 1, 2$ is a neighbourhood of $p_{i*}(f)$, $i = 1, 2$ in $\pi_i X$. Hence $\exists$ canonical neighbourhoods $< U_i, p_{i*}(f), V_i >$ contained in N_i, $i = 1, 2$. Then $< U_1, p_{1*}(f), V_1 > \times < U_2, p_{2*}(f), V_2 > \subseteq N_1 \times N_2$. But then $B = < U_1 \times U_2, f, V_1 \times V_2 >$ is a neighbourhood of f in πX, and we have:-

$\forall\ \gamma + f + \lambda \ \varepsilon\ B(x, y),\ \zeta(\gamma + f + \lambda) = (p_{1*}(\gamma + f + \lambda),\ p_{2*}(\gamma + f + \lambda))$

$= (p_{1*}(\gamma) + p_{1*}(f) + p_{1*}(\lambda),\ p_{2*}(\gamma) + p_{2*}(f) + p_{2*}(\lambda)).$

But $\gamma \supseteq P(U_1 \times U_2)(x, if) \Longrightarrow p_{1*}(\gamma) \supseteq PU_1(p_1(x), i_1(p_{1*}(f)),$

$p_{2*}(\gamma) \supseteq PU_2(p_2(x), i_2(p_{2*}(f))$

$\lambda \supseteq P(V_1 \times V_2)(\phi f, y) \Longrightarrow p_{1*}(\lambda) \supseteq PV_1(\phi_1(p_{2*}(f)), p_1(y)),$

$p_{2*}(\lambda) \supseteq PV_2(\phi_2(p_{2*}(f)), p_2(y))$

Hence $\zeta(\gamma + f + \lambda)\ \varepsilon < U_1, p_{1*}(f), V_1 > \times < U_2, p_{2*}(f), V_2 > \subseteq N_1 \times N_2$

Therefore $\zeta(B) \subseteq N_1 \times N_2$.

<u>ζ is open:</u>

Let N be any open set in πX, we show that $\zeta(N)$ is open. Let $f \varepsilon N$, then $\exists$ a canonical neighbourhood $<U_1 \times U_2, f, V_1 \times V_2>$ of f contained in N. Obviously $<U_1, p_{1*}(f), V_1>$ and $< U_2, p_{2*}(f), V_2 >$ are canonical neighbourhoods of $p_{1*}(f)$ & $p_{2*}(f)$, respectively. Hence $< U_1, p_{1*}(f), V_1> \times <U_2, p_{2*}(f), V_2>$ is a neighbourhood of $(p_{1*}(f), p_{2*}(f))$

<u>Claim:</u> $< U_1, p_{1*}(f), V_1> \times <U_2, p_{2*}(f), V_2> \subseteq \zeta(N)$.

Let $\gamma_i + p_{i*}(f) + \lambda_i \varepsilon < U_i, p_{i*}(f), V_i>$, $i = 1, 2$; and let $\gamma'_i \varepsilon\ \gamma_i$ and $\lambda'_i \varepsilon\ \lambda_i$ $i = 1, 2$ be any representative paths. Then:-

$\gamma' = (\gamma'_1, \gamma'_2) \varepsilon\ P(U_1 \times U_2)$ & $\lambda' = (\lambda'_1, \lambda'_2) \varepsilon\ P(V_1 \times V_2)$.

Therefore $\gamma = \overline{\gamma'} \supseteq P(U_1 \times U_2)$ and $\lambda = \overline{\lambda'} \supseteq P(V_1 \times V_2)$.

Hence $\gamma + f + \lambda \varepsilon < U_1 \times U_2, f, V_1 \times V_2 > \subseteq N$.

Obviously, $\zeta(\gamma + f + \lambda) = (\gamma_1 + f_1 + \lambda_1, \gamma_2 + f_2 + \gamma_2)$

Therefore $(\gamma_1 + f_1 + \lambda_1, \gamma_2 + f_2 + \lambda_2) \varepsilon\ \zeta(N)$.

Hence $\zeta(N)$ is a neighbourhood of $\zeta(f)$. Since f was arbitrary, $\zeta(N)$ is open. Hence ζ is a homeomorphism.

q.e.d.

Theorem 3.3.7.

Let X and Y be p.c., ℓ.p.c. and ℓ.s.c. spaces, and let $f \simeq g: X \longrightarrow Y$, then $f_* \simeq g_*: \pi X \longrightarrow \pi Y$.

Proof: Let $H: X \times I \longrightarrow Y$ be the homotopy from f to g; i.e. $\forall\, x \in X$, $H(x, o) = f(x)$ and $H(x, 1) = g(x)$

Then by 3.3.1. $H_*: \pi(X \times I) \longrightarrow \pi Y$ is continuous. Let $\xi: \pi(X \times I) \longrightarrow \pi X \times \pi I$ be the homeomorphism constructed in 3.3.6., then the composition

$$\pi X \times I \xrightarrow{1 \times u} \pi X \times \pi I \xrightarrow{\xi^{-1}} \pi(X \times I) \xrightarrow{H_*} \pi Y$$

where $u: I \longrightarrow \pi I$ is the unit map, is continuous and is the required homotopy from f_* to g_*. For, $\forall \bar{\lambda} \in \pi X$ and $\forall t \in I$,

$$(\bar{\lambda}, t) \rightsquigarrow (\bar{\lambda}, o_t) \rightsquigarrow \overline{(\lambda, c_t)} \rightsquigarrow \overline{Ho(\lambda, c_t)}$$

where $c_t: \mathbb{R}^+ \longrightarrow I$ is the constant path at t.

It is easily seen that

$$Ho(\lambda, c_o) = fo\lambda \text{ and } Ho(\lambda, c_1) = go\lambda$$

Hence $\forall \bar{\lambda} \in \pi X$, $H_* o \xi^{-1} o (1 \times u)(\bar{\lambda}, o) = \overline{fo\lambda} = f_*(\bar{\lambda})$

$$H_* o \xi^{-1} o (1 \times u)(\bar{\lambda}, 1) = \overline{go\lambda} = g_*(\bar{\lambda})$$

q.e.d.

Definition 3.3.8: Two topological groupoids are said to be homotopic, if they are homotopic both as spaces and as abstract groupoids.

Theorem 3.3.9: Let X and Y be p.c., ℓ.p.c. and ℓ.s.c. Hausdorff spaces, and let $X \simeq Y$. Then $\pi X \simeq \pi Y$.

Proof: Brown ([2], 6.5.10) has shown that $\pi X \simeq \pi Y$ as abstract groupoids, and by 3.3.7. and the covariance property of *, $\pi X \simeq \pi Y$ as spaces.

q.e.d.

CHAPTER IV

COVERING GROUPOIDS

Introduction:

P.J. Higgins [5] has developed the notion of covering groupoids, and R. Brown[2] has shown that they model the covering map of spaces. We now consider the topological analogue, and show that if $p: X \longrightarrow Y$ be a covering map of Hausdorff spaces, then $p_*: \pi X \longrightarrow \pi Y$ is a covering morphism of topological groupoids. In section 2 we study the question of existence of these groupoids for a given connected locally trivial groupoid. In last section we prove that the universal covering space of any locally trivial groupoid G, with discrete vertex groups, over a path-connected, locally path-connected and locally simply connected space X has a locally trivial groupoid structure over the universal covering space of X. In case $G = \pi X$, we show that the universal covering space of G is a topological covering groupoid of G.

Section 1:

Definition 4.1.1:

Let $p: \tilde{G} \longrightarrow G$ be a morphism of topological groupoids; then p is called a covering morphism if:-

$$\forall \tilde{x} \in \tilde{G}^{ob}, \quad p\,|St_{\tilde{G}}\tilde{x} = p' : St_{\tilde{G}}\tilde{x} \longrightarrow St_G p(\tilde{x})$$

is a homeomorphism (i.e. p is fibre preserving).

Theorem 4.1.2.

If $p: X \longrightarrow Y$ is a covering map of Hausdorff spaces, then

$$p_*: \pi X \longrightarrow \pi Y$$

is a covering morphism of topological groupoids.

Proof:

It is shown in [2], p. 296, that $\forall\, x \in X$, $p_* \mid St_{\pi X}x : St_{\pi X}x \longrightarrow St_{\pi Y}p(x)$ is a bijection, and by 3.3.4, p_* and hence $p_*' = p_* \mid St\ x$ is continuous. We now show that $p_*' : St\ x \longrightarrow Stp(x)$ is open and hence a homeomorphism.

Let $q_X : PX \longrightarrow \pi X$ and $q_Y : PY \longrightarrow \pi Y$ be the quotient maps.

Let $\tilde{N}$ be a basic open subset of $St_{\pi X}x$, we must show that $p_*(\tilde{N})$ is open in $St_{\pi Y}p(X)$. For this we must prove that $q_Y^{-1}(p_*(\tilde{N}))$ is open in $St_{PY}p(x)$. Let $N = q_X^{-1}(\tilde{N})$, then N is open in $St_{PX}x$.

Let $g_r \in q_Y^{-1}(p_*(\tilde{N}))$, then by covering property of p, $\exists\, f_r \in St_{PX}x$ s.t. $g_r = p_{\#}(f_r)$, where $p_{\#} : PX \longrightarrow PY$ is the induced functor: $f \rightsquigarrow pof$ (see 3.3.3.). It is easily seen that $f_r \in N$. Hence $\exists$ open sets $U_1, \ldots\ldots, U_n \subseteq X$, closed intervals $K_1, \ldots\ldots, K_n \subseteq I_r$ and real $\varepsilon > o$ s.t.

$$f_r \in \tau(K_1, U_1) \cap \ldots\ldots \cap \tau(K_n, U_n) \cap \eta^{-1}(r-\varepsilon, r+\varepsilon) \subseteq N.$$

Since p is an open map (being a covering map), $p(U_i) \subseteq Y$, $i = 1, \ldots\ldots, n$ is open. Hence $\tau(K_1, p(U_1)) \cap \ldots\ldots \cap \tau(K_n, p(U_n)) \cap \eta^{-1}(r-\varepsilon, r+\varepsilon) = N$, say, is open in $St_{\pi Y}p(x)$, and we have $p_{\#}(f_r) = g_r \in N_1$. To show that N_1 is contained in $q_Y^{-1}(p_*(\tilde{N}))$, it suffices to show that $N_1 = p_{\#}(N)$.

It follows from the construction of N_1 that $p_{\#}(N) \subseteq N_1$.

Conversely, let $g_s \in N_1$, then $\exists$ a unique $f_s \in St_{PX}x$ s.t. $g_s = p_{\#}(f_s)$. But:-

$$g_s \in N_1 \Longrightarrow \forall\, t \in K_i,\ g_s(t_i) \in p(U_i),\ i = 1, \ldots\ldots, n$$
$$\Longrightarrow \forall\, t \in K_i,\ p \circ f_s(t_i)\,) = p(f_s(t_i)\,) \in p(U_i),\ i=1,\ldots,n$$
$$\Longrightarrow f_s(t_i) \in U_i,\ i = 1, \ldots\ldots, n$$

$\Longrightarrow f_s \varepsilon N.$

Hence $g_s \varepsilon p_{\#}(N)$; therefore $N_1 \subseteq p_{\#}(N)$. Thus every $g_r \varepsilon q_Y^{-1}(p_*(\tilde{N}))$ is an interior point, and so $q_Y^{-1}(p_*(\tilde{N}))$ is open.

q.e.d.

Lemma 4.1.3:

Let $p: \tilde{G}, \tilde{x} \longrightarrow G, x$ be a connected covering morphism of topological groupois, i.e. $\tilde{G}$ & G are connected). If G is locally trivial then $p|\tilde{G}^{ob}: \tilde{G}^{ob} \longrightarrow G^{ob}$ is an open map.

Proof:

Let $U \subseteq \tilde{G}^{ob}$ be open, and let $\tilde{\phi}$, ϕ be the final maps in $\tilde{G}$ and G, respectively. Then $K = \tilde{\phi}^{-1}(U) \cap St\,\tilde{x}$ is open in $St\,\tilde{x}$ Since p is a covering map, $p(K)$ is open in $St_G x$. Hence $\exists$ an open set $\mathcal{U} \subseteq G$ such that $p(K) = \mathcal{U} \cap St\,x$.

Since p is connected, $p(U) = \phi(p(K))$

Let $y \varepsilon p(U)$ be any point, and let V_1, V_2 be liftable open neighbourhoods of x, y, respectively. Then $G(V_1, V_2)$ is homeomorphic to $V_1 \times V_2 \times G\{x\}$, and $\mathcal{U} \cap G(V_1, V_2) = \mathcal{U}'$, say is open in $G(V_1, V_2)$. Hence $\exists$ open neighbourhoods V_1', V_2' of x and y, respectively, such that $\mathcal{U}' \approx V_1' \times V_2' \times N$, for some open set N in $G\{x\}$. Obviously,

$$\phi(\mathcal{U}') = \phi(\mathcal{U}' \cap St\,x) = V_2'$$

and $\mathcal{U}' \subseteq \mathcal{U} \Longrightarrow \mathcal{U}' \cap St\,x \subseteq \mathcal{U} \cap St\,x = p(K)$

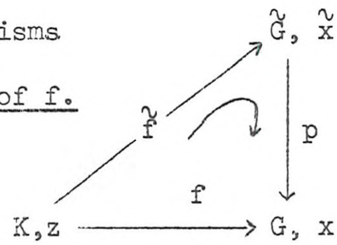
$$\Longrightarrow \phi(\mathcal{U}' \cap St\,x) \subseteq \phi(p(K)) = p(U)$$

Therefore $V_2' \subseteq p(U)$. Since y was arbitrary, $p(U)$ is open.

q.e.d.

Definition 4.1.4:

Let $p: \tilde{G}, \tilde{x} \longrightarrow G, x$ be a covering morphism of topological groupoids. If the diagram of pointed morphisms is commutative, then $\tilde{f}$ is called a lifting of f.



Theorem 4.1.5:

Let p: $\tilde{G}, \tilde{x} \longrightarrow G, x$ be a covering morphism, and let $f: K, z \longrightarrow G, x$ be any connected morphism of topological groupoids. Let G be locally trivial, then f lifts to a unique morphism $\tilde{f} : K, z \longrightarrow \tilde{G}, \tilde{x}$ of topological groupoids if $f(K\{z\}) \subseteq p(\tilde{G}\{\tilde{x}\})$.

Proof: The existence and uniqueness of $\tilde{f}$, as an abstract morphism is shown in ([2], 9.3.3.). So, it remains to show the continuity of $\tilde{f}$. Now $\tilde{f}$ is defined as follows:-

Let T be a wide tree in K, then any $a \in K$ can be written of the form $a = -\tau_1 + a' + \tau_2$, where $\tau_i \in T$ starting at z, and $a' \in K\{z\}$. Hence $f(a) = -f(\tau_1) + f(a') + f(\tau_2)$. Since p is a covering map, p_1: St $\tilde{x} \longrightarrow$ St x, the restriction of p, is a homeomorphism. Let $q = p_1^{-1}$, then define $\tilde{f}$ by:-

$$\forall\, y \in K^{ob},\ \tilde{f}(y) = \tilde{\phi}(q(f(\tau_y))),\ \tau_y \in T(z, y).$$

$$\forall\, a = -\tau_1 + a' + \tau_2,\ \tilde{f}(a) = -q(f(\tau_1)) + q(f(a')) + q(f(\tau_2)).$$

(Notice that $q(f(a')) \in G\{x\}$. For $f(a') \in p(G\{x\})$ by hypothesis). Clearly $\tilde{f}(z) = \tilde{x}$, and $\tilde{f}\,|K^{ob}$ is continuous. We must show that $\tilde{f}$ is continuous on the space of morphisms. Let N be an open neighbourhood of $\tilde{f}(a)$ in $\tilde{G}$, then $\exists$ open neighbourhoods N_1, N_2 of $q(f(\tau_1))$ and $q(f(\tau_2))$, respectively, in St $\tilde{x}$ and N_3 of $q(f(a'))$ in $\tilde{G}\{\tilde{x}\}$ such that $-N_1 + N_3 + N_2 \subseteq N$.

Then, since q is continuous, $\exists$ open neighbourhoods M_i of $f(\tau_i)$, $i = 1, 2$ in St x, and M_3 of $f(a')$ in $G\{x\}$ such that

$$q(M_i) \subseteq N_i,\ i = 1, 2, 3.$$

Let T' be the tree in G used in its bundle structure, then:-

$$f(\tau_i) = \alpha_i + t_i,\ i = 1, 2,\ t_i \in T',\ \alpha_i \in G\{x_o\}.$$

Hence $\exists$ neighbourhoods M_{t_i}, M_{α_i} of t_i, α_i, $i = 1, 2$ in St x and G {x}, respectively, such that

$$M_{\alpha_i} + M_{t_i} \subseteq M_i,\ i = 1, 2.$$

By 3.2.7.(4) M_{t_i}, i = 1, 2 contains a neighbourhood of the form $\langle B_i, U_i\rangle$ of t_i, i = 1, 2, where U_i is a liftable neighbourhood of $\phi(t_i)$, i = 1, 2. Then $M' = -B_1 - M_{\alpha_1} + M_3 + M_{\alpha_2} + B_2$ is a neighbourhood of $-\alpha_1 + a' + \alpha_2$ in $G\{x_o\}$.

But $a = -t_1 - \alpha_1 + a' + \alpha_2 + t_2$, hence $\langle U_1, M', U_2\rangle$ is a neighbourhood of a in G. Let $N' = f^{-1}(\langle U_1, M', U_2\rangle)$, then N' is a neighbourhood of a in K. It is easy to see that $\tilde{f}(N') \subseteq N$.

q.e.d.

Corollary 4.1.6:

Let p: $\tilde{G}, \tilde{x} \longrightarrow G, x$ and f: K, z $\longrightarrow$ G, x be connected covering morphisms of topological groupoids. Let G be locally trivial,

(i) If $f(K\{z\}) \subseteq p(\tilde{G}\{\tilde{x}\})$, then $\tilde{f}$: K, z $\longrightarrow \tilde{G}, \tilde{x}$, the lift of f is a covering morphism.

(ii) If $f(K\{z\}) = p(G\{x\})$, then f is an isomorphism of topological groupoids.

Definition 4.1.7:

Let G be a topological groupoid, then any covering groupoids $\tilde{G}$ of G which covers all other covering groupoids of G is called a universal covering groupoid of G.

Corollary 4.1.8:

Any tree covering groupoid of a connected locally trivial groupoid G is a universal covering groupoid of G.

This corollary easily follows from 4.1.5. q.e.d.

Remark:

By 4.1.6. all universal covering groupoids of a connected locally trivial groupoid G are topologically isomorphic.

Lemma 4.1.9:

Let $p : \tilde{G}, \tilde{x} \longrightarrow G, x$ be a covering morphism of topological groupoids, then the characteristic group of p(i.e. $p(\tilde{G}\{x\})$) is closed.

Proof:

Since $p \,|\, St\ \tilde{x}$ onto St x is a homeomorphism and $\tilde{G}\{\tilde{x}\}$ is closed, $p(\tilde{G}\{\tilde{x}\})$ is closed in St x and hence in $G\{x\}$.

q.e.d.

2. EXISTENCE OF COVERING GROUPOIDS

We now look at the question of existence of covering groupoids. Let G be a connected topological groupoid over the space X, and let R_A be the relation defined in chapter two. Let;

$X_{A_x} = St\ x/R_A$, $G_A^x = \{([a]_A, b) \varepsilon X_{A_x} \times G \,|\, (a, b) \varepsilon D\}$.

(Recall that if G is locally trivial with discrete vertex groups, then X_{A_x} is a covering space of X).

Then we have:-

Lemma 4.2.1:

G_A^x is a connected groupoid over X_{A_x}.

Proof:

Define $I_A, \Phi_A : G_A^x \longrightarrow X_{A_x}$

the initial and final maps by:-

$$I_A([a]_A, b) = [a]_A, \Phi_A([a]_A, b) = [a + b]_A.$$

We must show that ϕ_A is well-defined. Let $a' \varepsilon [a]_A$, then we must show that $[a' + b]_A = [a + b]_A$. For this, we must show that $a' + b - (a + b) = a' - a \varepsilon A\{x\}$.

But this follows from definition of R_A. (i.e. $a' \varepsilon [a]_A$

$\Longrightarrow a' - a \varepsilon A\{x\}$).

(ii) Let $\mathcal{D}_A = \{(([a]_A, b), ([c]_A, d)) \varepsilon G_A^x \times G_A^x \mid [a + b]_A = [c]_A\}$

Define the composition function $\Theta_A : \mathcal{D}_A \longrightarrow G_A^x$

by:- $\Theta_A (([a]_A, b), ([c]_A, d)) = ([a]_A, b) + ([c]_A, d) = ([a]_A, b+d)$

<u>Θ_A is associative:</u>

Let $(([a]_A, b), ([c]_A, d)) \varepsilon \mathcal{D}_A$, $(([c]_A, d), ([e]_A, f)) \varepsilon \mathcal{D}_A$, then:-

$[a + b]_A = [c]_A$, $[c + d]_A = [e]_A \Longrightarrow a + b - c \varepsilon A\{x\}$,

$c + d - e \varepsilon A\{x\}$.

Hence $a + b + d - e = a + b - c + c + d - e \varepsilon A\{x\}$. Therefore $[a + b + d]_A = [e]_A$.

Thus $([a]_A, b + d) + ([e]_A, f)$ is defined, and we have:-

$(([a]_A, b) + ([c]_A, d)) + ([e]_A, f) = ([a]_A, b + d) + ([e]_A, f)$

$= ([a]_A, (b + d) + f)$ (1)

$([a]_A, b) + (([c]_A, d) + ([e]_A, g)) = ([a]_A, b) + ([c]_A, d + f)$

$= ([a]_A, b + (d + f))$ (2)

Since G is a groupoid, $(b + d) + f = b + (d + f)$. Hence (1) = (2).

(iii) Define the unit map $u_A : X_{A_x} \longrightarrow G_A^x$

by:- $\forall [a]_A \varepsilon X_{A_x}$, $u_A([a]_A) = ([a]_A, 0_{\phi(a)})$.

Then, $\forall ([a]_A, b), ([c]_A, d)$ s.t. $[c + d]_A = [a]_A$, we have

$\phi a = ib$, $\phi d = \phi a$

$$([a]_A, o_{\phi a}) + ([a]_A, b) = ([a]_A, o_{\phi a} + b) = ([a]_A, o_{ib} + b)$$
$$= ([a]_A, b).$$

$$([c]_A, d) + ([a]_A, o_{\phi a}) = ([c]_A, d + o_{\phi a}) = ([c]_A, d + o_{\phi d})$$
$$= (([c]_A, d).$$

(iv) Define the inverse function $\Sigma_A : G_A^X \longrightarrow G_A^X$ by:-

$$\forall ([a]_A, b) \in G_A^X, \quad \Sigma_A([a]_A, b) = ([a + b]_A, -b).$$

Similar to that of Φ_A, it is easily seen that Σ_A is well-defined.

It satisfies the required conditions. For,

$$\Phi_A([a]_A, b) = [a + b]_A = I_A([a + b]_A, -b)$$
$$\Phi_A([a + b]_A, - b) = [a + b - b]_A = [a]_A = I_A([a]_A, b).$$

and we have:-

$$([a]_A, b) + ([a + b]_A, -b) = ([a]_A, b - b) = ([a]_A, o_{ib})$$
$$= ([a]_A, o_{\phi a}).$$

$$([a + b]_A, -b) + ([a]_A, b) = ([a + b]_A, -b + b) = ([a + b]_A, o_{\phi b})$$
$$= ([a + b]_A, o_{\phi(a + b)})$$

<u>G_A^X is connected:</u> Let $[a]_A, [b]_A \in X_{A_X}$, then

$$([a]_A, - a + b) \in G^X{}_A ([a]_A, [b]_A).$$

q.e.d.

As we saw in 4.1.9, the characteristic group of any covering morphism is a closed group. We now show that in case G is a connected locally trivial groupoid, every closed subgroup of the vertex group $G\{x_o\}$, $x_o \in G^{ob}$ give rise to a covering groupoid of G. We first prove:-

<u>Theorem 4.2.2:</u>

<u>Let G be a connected locally trivial groupoid over the space X and let A be any wide connected closed subgroupoid of G, then G_A^X is a</u>

<u>topological groupoid over X_{A_x} .</u>

<u>Proof:</u>

By 4.2.1, G_A^x is an abstract groupoid over X_{A_x} . Since A is closed in G, A {x} is a closed subgroup of G{x} . Hence $G\{x\} /_{A\{x\}}$ is Hausdorff (see [1]p. 231). By 3.2.10, X_{A_x} is a bundle over X with fibre $G\{x\}/_{A\{x\}}$, and hence it is Hausdorff.

Since $G_A^x \subseteq X_{A_x} \times G$, we take its topology to be the induced topology from the product topology on $X_{A_x} \times G$. Now we verify the continuity of the functions

(i) Let $\pi_1 : X_{A_x} \times G \longrightarrow X_{A_x}$ be the first projection, then $I_A = \pi_1 | G_A^x$, and hence it is continuous.

(ii) Let q: St $x \longrightarrow X_{A_x}$ be the quotient map, then we have the following commutative diagram.

$$\begin{array}{ccc} (\mathrm{St}\ x \times G) \cap D & \xrightarrow{\ \theta\ } & \mathrm{St}\ x \\ {\scriptstyle q \times 1} \downarrow & & \downarrow {\scriptstyle q} \\ G_A^x & \xrightarrow[\ \phi_A\ ]{} & X_{A_x} \end{array}$$

which guarantees the continuity of ϕ_A. (θ is the composition map in G).

(iii) Let $\pi_1 : X_{A_x} \times G \times X_{A_x} \times G \longrightarrow X_{A_x}$ be the first projection

$\pi_{2,4} : X_{A_x} \times G \times X_{A_x} \times G \longrightarrow G \times G$ be the projection onto 2nd and 4th factors

Then
$$\pi_1' = \pi_1 | \mathcal{D}_A : \mathcal{D}_A \longrightarrow X_{A_x}$$
$$\pi'_{2,4} = \pi_{2,4} | \mathcal{D}_A : \mathcal{D}_A \longrightarrow D.$$

are continuous.

Hence $r = \theta \circ \pi'_{2,4} : \mathcal{D}_A \longrightarrow G$ is continuous. Therefore $\Theta_A = (\pi_1', r)$

is continuous.

(iv) Let $\bar{\phi}_\Lambda : E_\Lambda \longrightarrow X$ be the final map, as before, and let $u: X \longrightarrow G$ be the unit map in G, then $uo\bar{\phi}_\Lambda : X_{\Lambda_x} \longrightarrow G$ is continuous. Therefore $\mathcal{U}_\Lambda = (1, uo\bar{\phi}_\Lambda) : X_{\Lambda_x} \longrightarrow G^x_\Lambda$ is continuous, where $1: X_{\Lambda_x} \longrightarrow X_{\Lambda_x}$ is the identity map.

(v) The inverse function Σ_Λ is continuous. For, let $\pi_2 : X_{\Lambda_x} \times G \longrightarrow G$ be the 2nd projection, then $\pi_2' = \pi_2 | G^x_\Lambda : G^x_\Lambda \longrightarrow G$ is continuous. If σ denotes the inverse map in G, then $\nu = \sigma o \pi'_2$ is continuous. Now, $\forall ([a]_\Lambda, b) \in G^x_\Lambda$ we have:-

$$\Sigma_\Lambda([a]_\Lambda, b) = ([a+b]_\Lambda, -b) = (\Phi_\Lambda([a]_\Lambda, b), \nu([a]_\Lambda, b)).$$

Hence $\Sigma_\Lambda = (\Phi_\Lambda, \nu)$, and therefore it is continuous.

q.e.d.

Remark :

We need the condition Λ to be closed only for X_{Λ_x} to be Hausdorff. In case of locally trivial groupoids with discrete vertex groups we do not need to include this condition, for in that case X_{Λ_x} is a covering space of X and hence Hausdorff property of X_{Λ_x} follows from that of X.

As the proof shows, locally triviality condition on G has nothing to do with the continuity of the functions related to G^x_Λ. Now if G is a Hausdorff space, then $St_G x$ is Hausdorff. Therefore we have:-

Corollary 4.2.3:

Let G be a connected Hausdorff groupoid, then $G^x_o = \{(a, b) \in St\ x \times G \mid (a, b) \in D\}$ is a topological groupoid over St x. (G_o^x corresponds to $A = \{o_x\}$).

We now define $p : G_A^x \longrightarrow G$ by:-

$\forall$ $([a]_A, b) \varepsilon G_A^x$, $p([a]_A, b) = b$

$\forall$ $[a]_A \varepsilon X_{A_x}$, $p([a]_A) = \phi(a)$

(ϕ: $G \longrightarrow X$, is the final map)

We show that it is a covering morphism of topological groupoids.

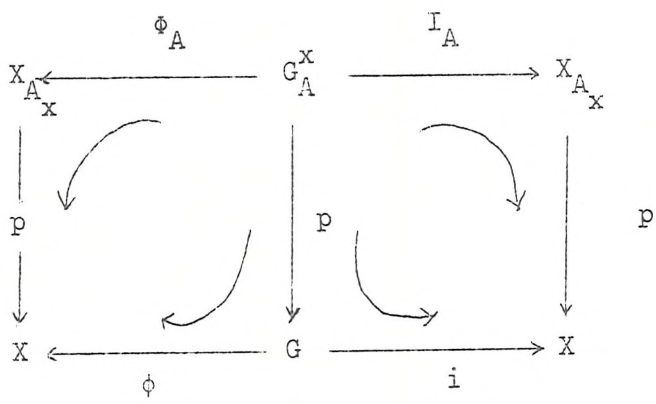
<u>Theorem 4.2.4:</u> Let A_x be a closed subgroup of $G\{x\}$, then:-

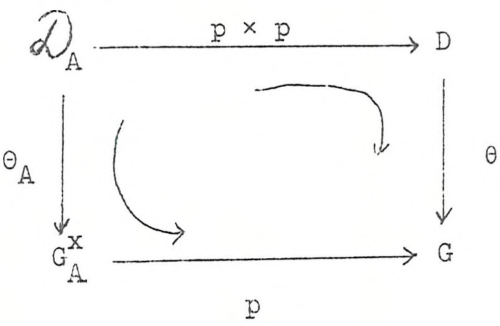
$p : G_A^x \longrightarrow G$, as defined above, is a connected covering morphism of topological groupoids. Moreover,

$\forall [o_x]_A \varepsilon X_{A_x}$, $p(G_A^x\{[o_x]_A\}) = A\{x\}$.

<u>Proof:</u>

<u>p is an abstract morphism:</u> it is easily seen that all the following diagrams are commutative.





$$\begin{array}{ccc} X_{A_x} & \xrightarrow{\;\;\mathcal{U}_A\;\;} & G_A^x \\ p\downarrow & & \downarrow p \\ X & \xrightarrow[\;\;u\;\;]{} & G \end{array}$$

p is continuous: For, $p = \pi_2 | G_\Lambda^x$, where $\pi_2 : X_{\Lambda_x} \times G \longrightarrow G$ is the 2nd projection, and $p | X_{\Lambda_x} = \overline{\phi}_\Lambda | X_{\Lambda_x}$, where $\overline{\phi}_\Lambda : E_\Lambda \longrightarrow X$ is the final map. Hence p is continuous <u>and open</u>.

Let $p' = p | St_{G_\Lambda^x}[a]_\Lambda : St\,[a]_\Lambda \longrightarrow St_G(\phi a)$, then:-

<u>p' is 1 - 1</u>: For, $p'([a]_\Lambda, b) = p'([a]_\Lambda, c) \Longrightarrow b = c$. Hence

$$([a]_\Lambda, b) = ([a]_\Lambda, c).$$

<u>p' is onto</u>: Given any $b \in St_G\phi a$, then $([a]_\Lambda, b) \in St\,[a]_\Lambda$, and we have $p'([a]_\Lambda, b) = b$

<u>p' is open</u>: Let $\mathcal{N}$ be a neighbourhood of $([a]_\Lambda, b) \in St[a]_\Lambda$, then $\exists\, N_{[a]} \subseteq X_{\Lambda_x}$ and $N_b \subseteq G$, neighbourhoods of $[a]_\Lambda$, b, respectively, such that

$$\mathcal{N} = ((N_{[a]} \times N_b) \cap G_\Lambda^x) \cap St\,[a]_\Lambda$$
$$= \{([a]_\Lambda, b) \mid b \in N_b \cap St_G\phi a\}.$$

Then $p'(\mathcal{N}) = N_b \cap St_G\phi a$ open in $St_G\phi a$.

Therefore p' is a homeomorphism, and hence p is a covering morphism of topological groupoids.

<u>$p(G_\Lambda^x\{[o_x]_\Lambda\}) = \Lambda\{x\}$</u>. Let $([o_x]_\Lambda, b) \in G_\Lambda^x\{[o_x]_\Lambda\}$, then

$$\phi([o_x]_\Lambda, b) = [o_x + b]_\Lambda = [b]_\Lambda = [o_x]_\Lambda = I_A([o_x]_\Lambda, b).$$

Hence $b \in \Lambda\{x\}$. i.e. $p([o_x]_\Lambda, b) \in \Lambda\{x\}$. Therefore

$p(G_\Lambda^x\{[o_x]_\Lambda\}) \subseteq \Lambda\{x\}$.

Conversely, given $b \varepsilon A\{x\}$, then $[b]_\Lambda = [o_x]_\Lambda$. Hence $([o_x]_\Lambda, b) \varepsilon G^x_\Lambda\{[o_x]_\Lambda\}$, and $b = p([o_x]_\Lambda, B) \varepsilon p(G^x_\Lambda\{[o_x]_\Lambda\})$.
Therefore $A\{x\} \subseteq p(G^x_\Lambda\{[o_x]_\Lambda\})$.

q.e.d.

Corollary 4.2.5:

For any closed subgroups $A_x \subseteq B_x$ of $G_\Lambda\{x\}$ in the connected locally trivial groupoid G, over X, G^x_Λ is a covering groupoid of G^x_B. (A and B are subgroupoids obtained as in Chapter Two.)

Proof: It is an immediate consequence of 4.1.5. and 4.2.5.

The following corollary is immediate from 4.2.3. and 4.2.5.

Corollary 4.2.7:

Let G be a connected Hausdorff groupoid, then $G_o{}^x$ is a covering groupoid of G with trivial characteristic group.

Corollary 4.2.8:

For any connected locally trivial Hausdorff groupoid G, G^x_o is the universal covering groupoid of G.

Corollary 4.2.9:

Let G, x be a pointed connected locally trivial Hausdorff groupoid, then $\exists$ a 1 - 1 correspondence between all connected covering groupoids of G and the closed subgroups of its vertex group $G\{x\}$.

Proof: It follows from 4.1.9., 4.2.5. and 4.1.6.

3. THE UNIVERSAL COVERING SPACE GROUPOID

Let $G = (G, X, i, \phi, \theta, u, \sigma)$ be a locally trivial groupoid, with discrete vertex groups, over the p.c, ℓ.p.c. and ℓ.s.c. space X. Then $\tilde{G} = St_{\pi G} o_x$, $x \varepsilon X$ is the universal covering space of G. We now show that $\tilde{G}$ is a topological groupoid over $\tilde{X} = St_{\pi X} x$.

We denote the homotopy class of each path f by $\overline{f}$.

Let $\overline{f}, \overline{g} \in \tilde{G}$ such that $\overline{iog} = \overline{\phi of}$, then $iog \simeq \phi of$.

Let F_t: $iog \simeq \phi of$ be the homotopy, then since i: $G \longrightarrow X$ is a fibre map, we can lift F_t to a homotopy $\mathcal{F}_t : \mathbb{R}^+ \longrightarrow G$, from g such that $io\,\mathcal{F}_t = F_t$. Let $g' = \mathcal{F}_1$, then we have $iog' = \phi of$.

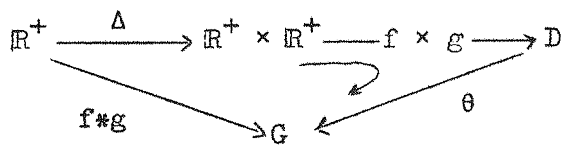
Hence, given $\overline{g}, \overline{f} \in \tilde{G}$ s.t. $\overline{iog} = \overline{\phi of}$, there always exist representatives $f_1 \in \overline{f}$, $g_1 \in \overline{g}$ s.t. $iog_1 = \phi\, of_1$. Hence $\forall\, t \in \mathbb{R}^+$, $i(g_1(t)) = \phi(f(t))$.

Let f, g be paths in G starting from o_x such that $iog = \phi\, of$.

Then we can define a function $f * g : \mathbb{R}^+ \longrightarrow G$ by:-

$$\forall\, t \in \mathbb{R}^+, \ (f * g)_{(t)} = f(t) + g(t) \ (= \theta(f(t), g(t))).$$

We have the following commutative diagram:-



where Δ is the diagonal map, and hence continuous. Therefore $f * g$ is continuous and hence it is a path in G. Moreover $(f * g)_{(o)} = f(o) + g(o) = o_x + o_x = o_x$. Hence $\overline{f * g} \in \tilde{G}$.

Lemma 4.3.1:

Let f, g, γ and ν be paths in G such that $f(o) = g(o) = o_x$ and $f + \gamma$, $g + \nu$ are defined and $\phi o(f + \gamma) = io(g + \nu)$ & $\phi of = iog$.

Then $\underline{(f + \gamma) * (g + \nu)} = \underline{(f * g) + (\gamma * \nu)}$.

Proof:

$\phi o(f + \gamma) = io(g + \nu) \Longrightarrow \phi of + \phi o\gamma = iog + io\nu$.

Since $\phi of = iog$, we get $\phi o \gamma = io\nu$. Hence $\gamma * \nu$ is defined. Now $\forall\, t \in \mathbb{R}^+$,

$$((f + \gamma) * (g + \nu))_t = (f + \gamma)_t + (g + \nu)_t$$

$$= \begin{cases} f(t) + g(t), & 0 \leq t \leq p \\ (t - p) + (t - p), & t \geq p \end{cases}, \text{ where } p \text{ is the common length of } f \text{ \& } g$$

$$= \begin{cases} (f * g)(t) & 0 \leq t \leq p \\ (\gamma * \nu)(t - p) & t \geq p \end{cases}$$

$$= \quad (f * g) + (\gamma * \nu))(t).$$

Therefore $(f + \gamma) * (g + \nu) = (f * g) + (\gamma * \nu)$.

q.e.d.

Theorem 4.3.2:

$\tilde{G}$ is a topological groupoid over $\tilde{X}$.

Proof:

(i) Define the initial and final maps $\tilde{i}, \tilde{\phi} : \tilde{G} \longrightarrow \tilde{X}$ by:

$\forall\, \bar{f} \in \tilde{G}$, $\tilde{i}(\bar{f}) = \overline{iof}$ & $\tilde{\phi}(\bar{f}) = \overline{\phi of}$.

Then $\tilde{i} = i_*| \tilde{G}$ and $\tilde{\phi} = \phi_* |\tilde{G}$, where $i_*, \phi_*: \pi G \longrightarrow \pi X$ are induced morphisms. Hence $\tilde{i}$ and $\tilde{\phi}$ are continuous.

(ii) Let $\tilde{D} = \{(\bar{f}, \bar{g}) \in \tilde{G} \times \tilde{G} \mid \overline{\phi of} = \overline{iog}\}$, define:

$\tilde{\theta} : \tilde{D} \longrightarrow \tilde{G}$ by $\tilde{\theta}(\bar{f}, \bar{g}) = \overline{f * g'}$, as obtained above, $(= \bar{f} \circledast \bar{g})$.

$\tilde{\theta}$ is well-defined:

Let $f_1 \in \bar{f}$, $g_1 \in \bar{g}$, then $\exists\, g_1' \in St_{PG} o_x$, s.t. $iog_1' = \phi of_1$.

Let $F: f \simeq f_1$ and $G: g' \simeq g_1'$, then:-

$\phi oF: \phi of \simeq \phi of_1$, and $ioG: iog' \simeq iog_1'$.

But we have, $\phi of = iog'$ and $\phi of_1 = iog_1'$, hence:-

$(\phi oF, ioG): (\phi of, \phi of) \simeq (\phi of_1, \phi of_1)$.

Since $(i, \phi): G \longrightarrow X \times X$ is a covering map ($\because$ G has discrete vertex groups), $\exists$ a homotopy $K: \mathcal{U} o \phi of \simeq K_1$ such that

$(i, \phi) oK = (\phi oF, ioG)$.

Hence $ioK = \phi oF$ and $\phi oK = ioG$. Therefore, for each $(s, t) \in \mathbb{R}^+ \times I$,

$i(K(s,t)) = \phi(F(s, t) \;\&\; \phi(K(s, t)) = i(G(s, t)).(*)$

We have:- $\quad ioK_1 = \phi of_1 = \phi oK_1$ and $K_1(o) = o_x$

on the other hand $uo\,\phi of_1$ is also a lift of ϕof_1 at o_x. For

$$\begin{aligned} i(uo\phi of_1) &= (iou)o(\phi of_1) = \phi of_1 \\ \phi(uo\phi of_1) &= (\phi ou)o(\phi of_1) = \phi of_1 \end{aligned} \quad \left(\because\ iou = \phi ou = id_X. \right)$$

Hence by unique lifting property of (i, ϕ), $K_1 = uo\phi of_1$.

Define $H: \mathbb{R}^+ \times I \longrightarrow G$ by:-

$$H(s, t) = F(s, t) + K(s, t) + G(s, t) \text{ (it is defined by *)}$$

obviously H is continuous and we have:-

$$H(s, o) = F(s, o) + K(s, o) + G(s, o) = f(s) + u(\phi of(s)) + g'(s) = f(s) + g'(s) = (f * g')(s)$$

$$H(s, 1) = F(s, 1) + K(s, 1) + G(s, 1) = f_1(s) + u(\phi of_1(s)) + g_1'(s) = f_1(s) + g_1'(s) = (f_1 * g_1')(s).$$

Hence $f * g' \simeq f * g_1'$, and so $\overline{f * g'} = \overline{f * g_1'}$.

<u>$\tilde{\theta}$ is associative:</u>

Let $\bar{f}, \bar{g}, \bar{h} \in G$ such that $\phi of = iog$ and $\phi og = ioh$, then:-

$(\bar{f} \circledast \bar{g}) \circledast \bar{h} = \overline{f * g} \circledast \bar{h} = \overline{(f * g) * h} = \overline{f * (g * h)} = \bar{f} \circledast (\bar{g} \circledast \bar{h})$.

For, $((f * g) * h)_t = f(t) + (f * h)_t = (f(t) + f(t) + h(t))$

By associativity of θ, $((f * g) * h)_t = (f * (g * h))_t$.

<u>$\tilde{\theta}$ is continuous:</u> Since $\tilde{G}$ is the subspace of πG, the sets of the form $<\bar{f}, w> = \{\bar{f} + \bar{\gamma} \mid \gamma \supseteq P\ W(f(p), y), y \in W\}$,

where W is simply connected neighbourhood of f(p)(p being the length of f), form a basis for the topology of $\tilde{G}$ (see C - N topology).

Now let $<\overline{f * g'}, W>$ be a basic open neighbourhood of $\theta(\bar{f}, \bar{g}) = \overline{f * g'}$, then W is an open neighbourhood of $(f * g')(p) = f(p) + g'(p)$. Since G is a topological groupoid, $\exists$ simply connected neighbourhoods U and V of f(p) and g'(p), respectively, such that

$$\theta(\ (U \times V) \cap D\ ) \subseteq W$$

Then $<\overline{f}, U>$ and $<\overline{g'}, V>$ are neighbourhoods of f and $\overline{g'} = \overline{g}$, respectively in $\tilde{G}$. Hence $N = (<\overline{f}, U> \times <\overline{g}, V>) \cap \tilde{D}$ is a neighbourhood of $(\overline{f}, \overline{g})$ in $\tilde{D}$.

Claim:

$$\tilde{\theta}(N) \subseteq <\overline{f * g'}, W>$$

Let $(\overline{f_1}, \overline{g_1}) \varepsilon N$, we may assume $\phi o f_1 = i o g_1$. Then

$f_1 = f + \gamma$ where $\gamma \varepsilon PU(f(p), f_1(p'))$

$g_1 = g' + \nu$ $\nu \varepsilon PV(g'(p), g_1(p'))$

where p' = common length of f_1 & g_1. Then

$$\tilde{\theta}(\overline{f_1}, \overline{g_1}) = \overline{f_1} \circledast \overline{g_1} = \overline{f_1 * g_1} = \overline{(f + \gamma) * (g' + \nu)}$$

$$= \overline{(f * g') + (\gamma * \nu)} \quad \text{by (4.3.1.)}$$

$$= (\overline{f * g'}) + (\overline{\gamma * \nu})$$

It is easily verified that $\overline{\gamma * \nu} \supseteq PW(f * g'(p), f_1 * g_1(p'))$.

Therefore $\tilde{\theta}(\overline{f_1}, \overline{g_1}) \varepsilon < \overline{f * g'}, W >$, and hence $\tilde{\theta}$ is continuous.

(iii) For each $f' \varepsilon St_{PX}x$, $Uof' \varepsilon St_{PG}ox$ (u is the unit map in G).

Hence, if $f \varepsilon PG$ such that $\phi o f = f' = io(uof')$, then

$(f * (uof'))_t = f(t) + u(f'(t)) = f_t + u(\phi(f(t))) = f(t)$

Similarly, if $g \varepsilon PG$ s.t. $iog = f'$, then $(uof') * g = g$. (*)

Define the unit map $\tilde{u} : \tilde{X} \longrightarrow \tilde{G}$ by $\tilde{u}(\overline{f}) = \overline{uof'}$

It follows from(*)that $\tilde{u}$ satisfies the required conditions.

$\tilde{u}$ is continuous:

$\tilde{u} = u_*|\tilde{X}$, where $u_* : \pi X \longrightarrow \pi G$ is the induced morphism.

Hence it is continuous.

(iv) Define the inverse map $\tilde{\sigma} : \tilde{G} \longrightarrow \tilde{G}$

by: $\tilde{\sigma}(\overline{f}) = \overline{\sigma o f}$

$\forall t \varepsilon R^+$, $(f*(\sigma o f))_t = f(t) + \sigma o f(t) = f(t) - f(t) = o_{i(f(t))} = u(i(f(t)))$)

Hence $(\bar{f} \circledast \overline{\sigma o f} = \overline{f * (\sigma o f)} = \overline{u o(i o f)} = \tilde{u}(\overline{i o f})$

Similarly, $\overline{\sigma o f} \circledast \bar{f} = \tilde{u}(\overline{\phi o f})$.

<u>$\tilde{\sigma}$ is continuous:</u>

For, $\tilde{\sigma} = \sigma_* \,|\tilde{G}$ onto $\tilde{G}$, where $\sigma_* : \pi G \longrightarrow \pi G$ is the induced morphism. Hence it is continuous. This completes the proof of the theorem.

q.e.d.

<u>Lemma 4.3.3:</u>

<u>$\tilde{G}$ is a locally trivial groupoid.</u>

<u>Proof:</u>

Let $\bar{f}\varepsilon\tilde{X}$, we must show that $\exists$ an open neighbourhood $N_{\bar{f}}$ in $\tilde{X}$ and a continuous lift $\Sigma_{\bar{f}}: N_{\bar{f}} \longrightarrow \tilde{G}$ such that

$\forall\ g\varepsilon N_{\bar{f}}$, $\tilde{i}(\Sigma_{\bar{f}}(\bar{g})) = \bar{g}$ and $\tilde{\phi}(\Sigma_{\bar{f}}(\bar{g})) = \bar{f}$

Let r be the length of f, and let $U_{f(r)}$ be a simply connected neighbourhood of f(r).

Then:-

$$<\bar{f}\ ,\ U_{f(r)}> \ = \{\bar{f} + \bar{\gamma} \mid \bar{\gamma} \supseteq PU_{f(r)}(f(r),\ y),\ y\varepsilon U_{f(r)}\}$$

is an open neighbourhood of $\bar{f}$.

Define:

$$\Sigma_{\bar{f}} : <f,\ U_{f(r)}> \longrightarrow \tilde{G}$$

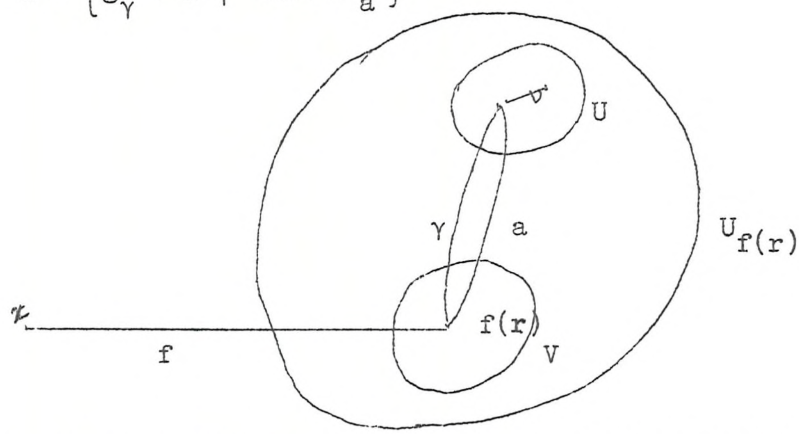
by:- $\Sigma_{\bar{f}}(\bar{f} + \bar{\gamma}) = \overline{g_{\bar{\gamma}}}$,

where $\overline{g_{\bar{\gamma}}}$ is the unique lift of $(\bar{f} + \bar{\gamma},\ \bar{f})$ at o_x, by the covering morphism $(i,\ \phi)_*: \pi G \longrightarrow \pi X \times \pi X$.

Let the end point of $\overline{g_{\bar{\gamma}}}$ be $a\varepsilon G$, then given any basic neighbourhood N of $\overline{g_{\bar{\gamma}}}$ in $\tilde{G}$, $\exists$ a simply connected open neighbourhood N'_a

of a in G such that

$$N = \left\{ \overline{g_\gamma} + \overline{h} \;\middle|\; h \varepsilon PN'_a \right\}$$



But by discreteness of vertex groups, each simply connected neighbourhood N'_a of a is of the form .

$\{\lambda'(x) + a - \lambda''(y) \mid x \varepsilon U, y \varepsilon V$, U, V some simply connected neighbourhoods of i(a) & $\phi(a)\}$

Where $\lambda' : U \longrightarrow G$.

and $\lambda'' : V \longrightarrow G$

are continuous lifts, then $<\overline{f + \gamma}, U>$ is a neighbourhood of $\overline{f + \gamma}$.

<u>Claim:</u> $\Sigma_{\overline{f}} (< \overline{f + \gamma}, U >) \subseteq N$.

Let $\overline{f + \gamma} + \overline{\nu} \;\varepsilon\; <\overline{f + \gamma}, U>$, then $\nu(\mathbb{R}^+) \subseteq U$.

Define $h: \mathbb{R}^+ \longrightarrow G$ by $h(t) = \lambda'(\nu(t)) + a$.

It is easy to see that h is continuous. By definition $h(\mathbb{R}^+) \subseteq N'_a$.

$(i, \phi)oh = (ioh, \phi oh) = (\nu, c_{f(r)})$ ($c_{f(r)}$ constant path at f(r).)

Hence $\overline{h}$ is the unique lift of $(\overline{\nu}, o_{f(r)})$. Therefore $\overline{g_\gamma} + \overline{h}$ is the unique lift of $(\overline{f + \gamma + \nu}, \overline{f})$, i.e. $\Sigma_{\overline{f}}(\overline{f + \gamma}, \overline{\nu}) = \overline{g_\gamma} + \overline{h} \varepsilon N$.

Hence $\Sigma_{\overline{f}}$ is continuous.

q.e.d.

Define $p : \tilde{G} \longrightarrow G$ by: $p(\bar{f})_r = f(r)$, $\bar{f} \in \tilde{G} \cup \tilde{X}$.

Then $p \,|\tilde{G}$, G and $p \,|\tilde{X}$, X are covering maps of spaces.

We now show that p is a morphism of topological groupoids.

(i) $\forall\, \bar{f} \in \tilde{G}$, $\begin{cases} iop(\bar{f}) = i(f(r)\ ) = (iof)(r) \\ po\tilde{i}(\bar{f}) = p(\overline{iof}) = (iof)(r) \end{cases} \Longrightarrow iop = po\tilde{i}$

Similarly $\phi op = po\tilde{\phi}$

(ii) $\forall\, (\bar{f}, \bar{g}) \in \tilde{D}$, $p(\bar{f} \circledast \bar{g}) = p(\overline{f * g}) = (f * g)(r) = f(r) + g(r)$

$= p(\bar{f}) + p(\bar{g}) = \theta(p(\bar{f}), p(\bar{g})\ )$

Hence $po\tilde{\theta} = \theta\ o(p \times p)$

(iii) $\forall\, \bar{\lambda} \in \tilde{X}$, $po\tilde{u}(\bar{\lambda}) = p(u(\lambda)\ ) = p(uo\lambda) = (uo\lambda)(s)$.

$uop(\lambda) = u(p(\lambda)\ ) = u(\lambda(s)\ ) = uo\lambda\ (s)$.

Hence $po\tilde{u} = uop$.

In general, the covering space groupoid $\tilde{G}$ need not be a covering groupoid. But if G is a fundamental groupoid of a p.c., ℓ.p.c. & ℓ.s.c. space, then p is a covering morphism of topological groupoids. In fact, it is a universal covering morphism. We first prove the following useful lemma.

Lemma 4.3.4:

Let $G = \pi X$, X a path-connected and ℓ.p.c. & ℓ.s.c. space, then $\forall\, f \in \tilde{G} = St_{\pi G} o_X$, $f(s) = -\overline{iof} + \overline{\phi of}$, s is the length of f.

Proof:

We have $(i.\phi);G \longrightarrow X \times X$ is a covering map, hence

$(i, \phi)_* : \pi G \longrightarrow \pi X \times \pi X$ is a covering morphism.

Therefore given any $(a, b) \in St_G x \times St_G x$, $\exists$ a unique $\bar{f} \in G$ such that $(i, \phi)_* (\bar{f}) = (a, b)$. i.e. $\overline{iof} = a$ & $\overline{\phi\, of} = b$.

Now, define $g: \mathbb{R}^+ \longrightarrow G$, by: $\forall\, t \in \mathbb{R}^+$, $g(t) = -\overline{(iof)}_t + \overline{(\phi of)}_t$ where, for each path λ, $\lambda_t(s) = \lambda(ts)$. Hence

$$\lambda_0 = \text{constant} \ \& \ \lambda_1 = \lambda.$$

Therefore $g(o) = -\overline{(iof)}_0 + \overline{(\phi of)}_0 = -\bar{o}_x + \bar{o}_x = \bar{o}_x = o_x$. Hence $\bar{g} \in \tilde{G}$, and we have $g(s) = -\overline{iof} + \overline{\phi of}$, s is the common length of f & g. Then by construction $\overline{iog} = \overline{iof}$ & $\overline{\phi og} = \overline{\phi of}$,

Hence $\bar{g}$ is a lift of $(\overline{iof}, \overline{\phi of})$. Therefore $\bar{f} = \bar{g}$, and,

$$f(s) = g(s) = -\overline{iof} + \overline{\phi of}.$$

q.e.d.

Theorem 4.3.5:

If $G = \pi X$, where X is a p.c., ℓ.p.c. and ℓ.s.c. Hausdorff space, then $p: \tilde{G} \longrightarrow G$ is a covering morphism.

Proof:

We need only show that $p' = p| St_{\tilde{G}}\bar{\gamma} : St_{\tilde{G}}\bar{\gamma} \longrightarrow St_G p(\bar{\gamma})$ is a bijection. For, then p being a covering map it is open and continuous hence p' is a homeomorphism.

p' is 1 - 1: Let $\bar{f}, \bar{g} \in St_{\tilde{G}}\bar{\gamma}$, then:-

$p'(\bar{f}) = p'(\bar{g}) \Longrightarrow \overline{iof} = \overline{iog} = \bar{\gamma}$, and $f(r) = f(s)$. But by 4.3.4.

$f(r) = -\overline{iof} + \overline{\phi of}$ & $g(s) = -\overline{iog} + \overline{\phi og}$.

Hence $-\overline{iof} + \overline{\phi of} = -\overline{iog} + \overline{\phi og} \Longrightarrow \overline{\phi of} = \overline{\phi og}$.

Then $\overline{f}$ and $\overline{g}$ are the lifts of the same morphism at o_x by $(i,\phi)_*$.

Hence $\overline{f} = \overline{g}$.

<u>p' is onto:</u>

Given $\overline{a} \varepsilon St_G p(\overline{\gamma})$, then $\overline{\gamma} + \overline{a}$ is defined and

$$(\overline{\gamma}, \overline{\gamma} + \overline{a}) \varepsilon St_G x \times St_G x.$$

Let $\overline{h}_q$ be the unique lift of $(\overline{\gamma}, \overline{\gamma} + \overline{a})$ at o_x, then

$$\overline{ioh} = \overline{\gamma} \ \& \ \overline{\phi oh} = \overline{\gamma} + \overline{a}$$

Therefore $p'(\overline{h}) = h(q) = -\overline{ioh} + \overline{\phi oh} = -\overline{\gamma} + \overline{\gamma} + \overline{a} = \overline{a}$.

Hence p' is onto.

q.e.d.

<u>Corollary 4.3.6:</u>

If $G = \pi X$, X p.c., ℓ.p.c. & ℓ.s.c. Hausdorff space then $\tilde{G}$ is isomorphic to G_o^x , the universal covering groupoid of G.

<u>Proof:</u>

We need only prove that p has trivial characteristic groups at any $\overline{\gamma} \varepsilon \tilde{X}$. Let $f \varepsilon \tilde{G}\{\tilde{\gamma}\}$, then $\overline{iof} = \overline{\phi of} = \overline{\gamma}$. Hence,

$$p'(\overline{f}) = f(r) = -\overline{iof} + \overline{\phi of} = -\overline{\gamma} + \overline{\gamma} = o_{p(\overline{\gamma})} \varepsilon G\{p(\overline{\gamma})\}.$$

Therefore the characteristic group of p at $\overline{\gamma}$ is trivial.

Hence $\tilde{G} \approx G_o^x$.

q.e.d.

CHAPTER V

$\mathcal{G}$d-TRANSFORMATION GROUPS

Introduction:

In this chapter we introduce the notion of $\mathcal{G}$d-transformation groups for groupoids, and show that if X is a Hausdorff space and (Γ, X) is a topological transformation group, then $(\Gamma, \pi X)$ is a topological $\mathcal{G}$d-transformation group. We generalise the notion of fundamental group of a transformation group introduced by F. Rhodes [8], to the fundamental groupoid of a $\mathcal{G}$d-transformation group and show that if (Γ, G) is a topological $\mathcal{G}$d-transformation group and G is connected, then the fundamental groupoid Γ_o of (Γ, G) is also a connected topological groupoid. In case $G = \pi X$, X path-connected Hausdorff, then for any $x \in \Gamma_o^{ob}$ $(= X)$ the vertex group $\Gamma_o\{x\}$ is $\sigma(X, x, \Gamma)$, the fundamental group of (Γ, X) as defined in[8]. Hence, since Γ_o is connected, then $\forall\, x,y \in X$, $\sigma(X, x, \Gamma)$, $\sigma(X, y, \Gamma)$ are isomorphic topological groups. We obtain exact **sequences** of abstract groupoids and morphisms, and groups and homomorphisms which in the special case $G = \pi X$, reduce to the sequences in[9]p. 906 . We show that if G is a connected locally trivial Hausdorff groupoid, then for each $x \in \Gamma_o^{ob}$, $\Gamma_o\{x\}$ is the set of all morphisms from G_o^x , the universal covering groupoid of G, onto itself which lift the elements of Γ; and show that the set of all lifts of the identity of Γ is the group of cover transformations of G_o^x and isomorphic to $G\{x\}$.

In section 3, we show that if Γ acts freely and property discontinuously on G, then the orbit set $G/_\Gamma$ is a topological groupoid over G^{ob}_Γ , and the quotient morphism: $G \longrightarrow G/_\Gamma$ is a covering morphism of topological groupoids.

1. DEFINITIONS AND EXAMPLES

Definition 5.1.1:

Let Γ be a group and G a groupoid over a set X. We say (Γ, G) is a $\mathcal{G}$d-transformation group, if Γ acts on G and X as sets such that $\forall \lambda \in \Gamma$, the following conditions are satisfied:

(i) $\forall\, a \in G,\quad i(\lambda.a) = \lambda.(ia)\ \&\ \phi(\lambda.a) = \lambda.(\phi a)$

(ii) $\forall (a, b) \in D,\quad \lambda.(a + b) = \lambda.a + \lambda.b$

(iii) $\forall\, a \in G,\quad \lambda.(-a) = -(\lambda.a).$

It is called effective if $\begin{cases} \lambda.g = g, \text{ all } g \in G \implies \lambda = e \\ \lambda.x = x, \text{ all } x \in X \implies \lambda = e \end{cases}$

Remark:

It is immediate from the definition that $\forall \lambda \in \Gamma$, the map

$$\lambda_* : G \longrightarrow G$$

defined by:- $\forall\, g \in G,\ \lambda_*(a) = \lambda.g$, $\forall\, x \in X,\ \lambda_*(x) = \lambda.x$

is a morphism of groupoids, in fact it is an isomorphism.

It is easily seen that $*$ is covariant; hence the map

$$\psi : \Gamma \longrightarrow \mathrm{Aut}(G),\ \lambda \rightsquigarrow \lambda_*,$$

is a homomorphism. ψ is an embedding if (Γ, G) is an effective $\mathcal{G}$d-transformation group. Hence, in this case, we may identify λ_* with λ or vice versa.

Definition 5.1.2:

A $\mathcal{G}$d-transformation group (Γ, G) is called a topological $\mathcal{G}$d-transformation group, if:-

(i) Γ is a topological group and G is a topological groupoid

(ii) $\forall \lambda \in \Gamma$, $\lambda_*: G \longrightarrow G$ is a morphism of topological groupoids.

Examples:

Let (Γ, X) be a topological transformation group, where X is a Hausdorff space, then $(\Gamma, \pi X)$ is a topological $\mathcal{G}$d-transformation group.

Proof:

By hypothesis, for each $\lambda \varepsilon \Gamma$, the map $\lambda_*: X \longrightarrow X$. defined by $\lambda_*(x) = \lambda . x$ is continuous. Hence for each $f \varepsilon PX(x, y)$, $\lambda_* o f \varepsilon PX(\lambda . x, \lambda . y)$. Moreover, $\forall \lambda, \mu \varepsilon \Gamma$, $f \varepsilon . PX$, we have:-

(i) $(\lambda\mu)_* = \lambda_* o \mu_*$ (ii) $e_* o f = f$,

where e is the identity of Γ.

Since $f' \simeq f \Longrightarrow \lambda_* o f' \simeq \lambda_* o f$, we have a well-defined action on πX, defined by:-

$$\forall \bar{f} \varepsilon \pi X, \lambda . \bar{f} = \overline{\lambda_* o f} \ (\varepsilon \ \pi X),$$

which satisfies:-

(i) $\left\{\begin{array}{l}\forall \lambda, \mu \varepsilon \Gamma, \\ \text{and } \bar{f} \varepsilon \pi X\end{array}\right.$ $(\lambda\mu).\bar{f} = \overline{(\lambda\mu)_* o f} = \overline{\lambda_* o \mu_* o f} = \lambda . \overline{\mu_* o f} = \lambda .(\mu . \bar{f}).$

(ii) $e . \bar{f} = \overline{e_* o f} = \bar{f}$

(iii) $\forall \bar{f} \varepsilon \pi X$, and $\forall \lambda \varepsilon \Gamma$, $i(\lambda . \bar{f}) = i(\overline{\lambda_* o f}) = (\lambda_* o f)(o) = \lambda_*(f(o)) =$

$$\lambda_*(i(\bar{f})) = \lambda .(i(\bar{f}))$$

Similarly, $\phi(\lambda . \bar{f}) = \lambda .(\phi(\bar{f}))$.

(iv) $\forall \bar{f}, \bar{g} \varepsilon \pi X$, such that $f(r) = g(o)$ (r being the length of f), we have:-

$$\lambda .(\bar{f} + \bar{g}) = \lambda .(\overline{f + g}) = \overline{\lambda_* o(f + g)} = \overline{\lambda_* o f + \lambda_* o g}$$

$$= \overline{\lambda_* o f} + \overline{\lambda_* o g} = \lambda . \bar{f} + \lambda . \bar{g}$$

(v) $\forall \lambda \varepsilon \Gamma$ and $\forall \bar{f}_r \varepsilon \pi X$, $\lambda .(-\bar{f}_r) = -(\lambda . \bar{f}_r)$. For, let $[o, \ell]$ be the maximum interval on which f is constant, then:-

$\forall\ t \in \mathbb{R}^+,\ \lambda_* o(-f)(t) = \lambda_*(-f(t)\ ) = \lambda_*(f\ (r + \ell - t) = \lambda_* of(r + \ell - t)$

$= -\lambda\ _* of(t).$

Hence $\lambda.(-\bar{f}) = \overline{\lambda.(-f)} = \overline{\lambda_* o(-f)} = -\overline{(\lambda_* of)} = -(\lambda.\bar{f}).$

Therefore $(\Gamma,\ \pi X)$ is a $\mathcal{G}$d-transformation group. We know πX is a topological groupoid and by hypothesis Γ is a topological group. So, it remains to show that $\lambda_* : \pi X \longrightarrow \pi X,\ \lambda \in \Gamma$, is continuous. Since $(\Gamma,\ X)$ is a topological transformation group $\lambda_*: X \longrightarrow X$ is homeomorphism; and by 3.3.5. $\lambda_*\ \pi X \longrightarrow \pi X$ is continuous.

q.e.d.

Following Rhodes [9], we give:-

Definition 5.1.3:

Let $(\Gamma,\ G)$ be a $\mathcal{G}$d-transformation group, with G connected, and let Λ be a wide connected normal subgroupoid of G.

We say Λ is invariant under Γ if:-

$\forall\ \lambda \in \Gamma$ and $\forall\ x \in G^{ob},\ \lambda.\ \Lambda\{x\} = \Lambda\{\lambda.\ x\}.$

Lemma 5.1.4:

Let $(\Gamma,\ G)$ be a $\mathcal{G}$d-transformation group, where G is connected, and let $\Lambda \subseteq G$ be any wide connected normal subgroupoid of G. If for some $x \in G^{ob},\ \lambda.\ \Lambda\{x\} = \Lambda\{\lambda.x\},\ \lambda \in \Gamma$, then Λ is invariant under Γ.

Proof:

Let $y \in G^{ob}$, and let $\beta \in \Lambda\{y\}$, then given $a \in G(y,\ x),\ \exists\ \alpha \in \Lambda\{x\}$ such that $\beta = a + \alpha - a$

Hence $\lambda.\beta = \lambda.(a + \alpha - a) = \lambda.\ a + \lambda.\alpha + \lambda.(-a) = \lambda.\ a + \lambda.\ \alpha - \lambda.a.$

But $\lambda.a \in G(\lambda.y.\lambda.x)$ and by hypothesis $\lambda.\alpha \in \Lambda\{\ \lambda.\ x\}$, hence by normality of Λ, $\lambda.\ \beta \in \Lambda\{\ \lambda.\ y\}$.

Conversely, given $\beta' \in \Lambda\{\lambda.\ y\}$, then for any $b' \in G(\lambda.y,\ \lambda.x)$,

$\exists \alpha' \varepsilon \Lambda\{\lambda. x\}$ s.t. $\beta' = b' + \alpha' - b'$.

Let $b = \lambda^{-1}. b'$ and $\alpha = \lambda^{-1}.\alpha$, then $b \varepsilon G(y, x)$ and $\alpha \varepsilon \Lambda \{x\}$

and we have:- $\beta' = \lambda. b + \lambda.\alpha - \lambda. b = \lambda.(b + \alpha - b) \varepsilon \lambda.\Lambda\{y\}$.

Because $\alpha \varepsilon \Lambda \{x\} \Longrightarrow b + \alpha - b \varepsilon \Lambda \{y\}$, by normality of Λ.

Hence $\lambda. \Lambda \{y\} = \Lambda\{\lambda. y\}$.

Since y was an arbitrary object, we have proved that Λ is invariant under Γ. q.e.d.

Lemma 5.1.5:

Let $(\Gamma. G)$ be a topological $\mathcal{G}$d-transformation group, and let $\Lambda \subsetneq G$ be a wide connected invariant subgroupoid of G, then (Γ, E_Λ) is a topological $\mathcal{G}$d-transformation group.

Proof:

Define the action of Γ on E_Λ as follows:-

$$\forall [a]_\Lambda \varepsilon E_\Lambda, \ \forall \lambda \varepsilon \Gamma, \ \lambda. [a]_\Lambda = [\lambda. a]_\Lambda .$$

This action is well-defined: For, let $a' \varepsilon [a]_\Lambda$, then:-

$a - a' \varepsilon \Lambda \{ia\} \Longrightarrow \lambda.(a - a') = \lambda. a - \lambda. a' \varepsilon \Lambda\{\lambda.(ia)\} = \Lambda\{i(\lambda.a)\}$

Therefore $[\lambda. a]_\Lambda = [\lambda. a']_\Lambda$. We have:-

(i) $\forall \lambda, \mu \varepsilon \Gamma, \forall [a]_\Lambda, \ (\lambda\mu).[a]_\Lambda = [\lambda.(\mu.a)]_\Lambda = \lambda.(\mu.[a]_\Lambda)$.

(ii) Let $e \varepsilon \Gamma$ be the identity, then:-

$$e. [a]_\Lambda = [e. a]_\Lambda = [a]_\Lambda$$

(iii) $\forall [a]_\Lambda, \lambda. (\bar{i}_\Lambda([a]_\Lambda)) = \lambda.(i(a)) = i(\lambda. a) = \bar{i}_\Lambda(\lambda.[a]_\Lambda)$

Similarly, $\lambda.(\bar{\phi}_\Lambda([a]_\Lambda)) = \bar{\phi}_\Lambda(\lambda.[a]_\Lambda)$.

(iv) $\forall [a]_\Lambda, [b]_\Lambda \varepsilon E_\Lambda$ s.t. $(a, b) \varepsilon D$, we have:-

$$\lambda.([a]_\Lambda + [b]_\Lambda) = \lambda.[a + b]_\Lambda = [\lambda.(a + b)]_\Lambda = [\lambda. a + \lambda. b]_\Lambda$$
$$= [\lambda. a]_\Lambda + [\lambda. b]_\Lambda = \lambda. [a]_\Lambda + \lambda. [b]_\Lambda .$$

(v) $\forall [a]_\Lambda \varepsilon E_\Lambda, \ \lambda.(-[a]_\Lambda) = \lambda.[-a]_\Lambda = [\lambda.(-a)]_\Lambda = [-(\lambda. a)]_\Lambda$

$$= -[\lambda. a]_\Lambda$$

Hence (Γ, E_Λ) is a $\mathcal{G}$d-transformation group.

By 3.1.18. E_Λ is a topological groupoid.

So, we need only show that $\forall\ \lambda \epsilon \Gamma$,

$$\bar{\lambda} : E_\Lambda \longrightarrow E_\Lambda,$$

obtained from the action of λ on E_Λ is continuous.

But this follows from the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \lambda_*\ } & G \\ p_\Lambda \downarrow & & \downarrow p_\Lambda \\ E_\Lambda & \xrightarrow[\ \bar{\lambda}\ ]{} & E_\Lambda \end{array}$$

q.e.d.

2. FUNDAMENTAL GROUPOID OF A $\mathcal{G}$d-TRANSFORMATION GROUP

Let (Γ, G) be a connected $\mathcal{G}$d-transformation group, and let Λ be a wide connected normal subgroupoid of G invariant under Γ.

Then E_Λ is a groupoid and hence $\Gamma \times E_\Lambda = \Gamma_\Lambda$ is a product groupoid over G^{ob}. We now show that Γ_Λ has a different groupoid structure over G^{ob}, whose vertex groups are the fundamental group of the transformation group (Γ, X) in the case that G arises as a fundamental groupoid of the space X. (see [8]).

(i) Define $i^\Lambda, \phi^\Lambda : \Gamma_\Lambda \longrightarrow G^{ob}$ by:-

$$\forall\ (\lambda, [r]_\Lambda)\ \epsilon \Gamma_\Lambda, \quad i^\Lambda(\lambda, [r]_\Lambda) = \bar{i}_\Lambda\ [r]_\Lambda = i(r)$$

$$\phi^\Lambda(\lambda, [r]_\Lambda) = \lambda^{-1}.(\bar{\phi}_\Lambda\ [r]_\Lambda) = \lambda.^{-1}(\phi(r)\ ).$$

(ii) Let $D^\Lambda = \{((\lambda, [r]_\Lambda), (\mu, [s]_\Lambda)) \mid \phi^\Lambda(\lambda, [r]_\Lambda) = i^\Lambda(\mu, [s]_\Lambda)$ i.e. $\underline{\lambda^{-1}.(\phi(r) = i(s)}\ \}$

Define $\theta^\Lambda : D^\Lambda \longrightarrow \Gamma_\Lambda$ by:-

$$\theta^\Lambda(\ (\lambda, [r]_\Lambda), (\mu, [s]_\Lambda)\ ) = (\lambda\mu, [r + \lambda.\ s]_\Lambda)\ \ (= (\lambda, [r]_\Lambda) * (\mu, [s]_\Lambda))$$

<u>θ^A is well-defined:</u>

Let $r_1 \varepsilon [r]_A$, $s_1 \varepsilon [s]_A$, we must show $[r_1 + \lambda . s_1]_A = [r + \lambda . s]_A$

i.e. $$r_1 + \lambda . s_1 - \lambda . s - r \varepsilon A\{ i(r) \}$$

But: $s_1 \varepsilon [s]_A \Longrightarrow s_1 - s \varepsilon A\{ i(s) \} \Longrightarrow \lambda . s_1 - \lambda . s = \lambda . (s_1 - s)$

$\varepsilon A \{\lambda . (i(s))\}$ by invariance of A.

$$\Longrightarrow r_1 + \lambda . s_1 - \lambda . s - r_1 \varepsilon A\{i(r)\} \quad (I)$$

by normality of A,

and $r_1 \varepsilon [r]_A \Longrightarrow r_1 - r \varepsilon A\{ i(r) \}$ (II)

Then, it follows from (I) and (II) that $r_1 + \lambda . s_1 - \lambda . s - r \varepsilon A\{ i(r) \}$.

<u>θ^A is associative:</u>

Let $((\lambda, [r]_A), (\mu, [s]_A)) \varepsilon D^A$ and $((\mu, [s]_A), (\nu, [t]_A)) \varepsilon D^A$,

then:-

$$\begin{aligned}((\lambda, [r]_A) * (\mu, [s]_A)) * (\nu, [t]_A) &= (\lambda\mu, [r + \lambda . s]_A) * (\nu, [t]_A) \\ &= ((\lambda\mu)\nu, [r + \lambda . s + (\lambda\mu) . t]_A) \\ &= (\lambda\mu\nu, [r + \lambda . s + \lambda . (\mu . t)]_A)\end{aligned}$$

and

$$\begin{aligned}(\lambda, [r]_A) * ((\mu, [s]_A) * (\nu, [t]_A)) &= (\lambda, [r]_A) * (\mu\nu, [s + \mu . t]_A) \\ &= (\lambda(\mu\nu), [r + \lambda . (s + \mu . t)]_A) \\ &= (\lambda\mu\nu, [r + \lambda . s + \lambda . (\mu . t)]_A)\end{aligned}$$

(iii) Define $u^A : G^{ob} \longrightarrow \Gamma_A$ by:-

$\forall x \varepsilon G^{ob}$, $u^A(x) = (e, [o_x]_A)$.

Since $i^A(e, [o_x]_A) = i(o_x) = x$

$\phi^A(e, [o_x]_A) = e^{-1} . (\phi(o_x)) = e . (x) = x$

we have $i^A(e, [o_x]_A) = \phi^A(e, [o_x]_A)$.

Moreover:- Let $(\lambda, [r]_A)$, $(\mu, [s]_A) \varepsilon \Gamma_A$ be such that:

$$i^A(\lambda, [r]_A) = i(r) = x \quad \& \quad \phi^A(\mu, [s]_A) = \mu^{-1} . (\phi(s)) = x, \quad (*)$$

then $(e, [o_x]_A) * (\lambda, [r]_A) = (e\lambda, [o_x + e.r]_A) = (\lambda, [r]_A)$

$(\mu, [s]_A) * (e, [o_x]_A) = (\mu e, [s + \mu.o_x]_A) = (\mu, [s + o_{\mu.x}]_A)$

$= (\mu, [s]_A)$. For $\mu.x = \phi(s)$ (by *)

Hence u^A is the unit function in Γ_A.

(iv) Define the inverse function

$$\sigma^A : \Gamma_A \longrightarrow \Gamma_A$$

by:- $\sigma^A(\lambda, [r]_A) = (\lambda^{-1}, [\lambda^{-1}.(-r)]_A)$.

Then $i^A(\lambda^{-1}, [\lambda^{-1}.(-r)]_A) = i(\lambda^{-1}.(-r) = \lambda^{-1}.i(-r) = \lambda^{-1}.(\phi(r))$

$= \phi^A(\lambda, [r]_A)$,

and $\phi^A(\lambda^{-1}, [\lambda^{-1}.(-r)]_A = (\lambda^{-1})^{-1}.(\phi(\lambda^{-1}.(-r)) = \lambda.(\lambda^{-1}.\phi(-r))$

$= \lambda\lambda^{-1}.(\phi(-r))$

$= e.(i(r)) = i(r) = i^A(\lambda, [r]_A)$.

Hence $((\lambda, [r]_A), (\lambda^{-1}, [\lambda^{-1}.)-r)]_A)) \in D^A$

and $((\lambda^{-1}, [\lambda^{-1}.(-r)]_A), (\lambda, [r]_A)) \in D^A$

Moreover:-

$(\lambda, [r]_A) * (\lambda^{-1}, [\lambda^{-1}.(-r)]_A) = (\lambda\lambda^{-1}, [r + \lambda.(\lambda^{-1}.(-r)]_A)$

$= (e, [r + \lambda\lambda^{-1}.(-r)]_A) = (e, [r - r]_A)$

$= (e, [o_{i(r)}]_A)$.

$(\lambda^{-1}, [\lambda^{-1}.(-r)]_A) * (\lambda, [r]_A) = (\lambda^{-1}\lambda, [\lambda^{-1}.(-r) + \lambda^{-1}.r]_A)$

$= (e, [\lambda^{-1}.(-r + r)]_A) = (e, [\lambda^{-1}.o_{\phi(r)}]_A)$

$= (e, [o_{\lambda^{-1}.\phi(r)}]_A$

Hence $(\Gamma_A, G^{ob}, i^A, \phi^A, \theta^A, u^A, \sigma^A)$ is a groupoid.

<u>Γ_A is connected:</u> For any $x, y \in G^{ob}$, let $r \in G(x, \lambda.y), \lambda \in \Gamma$, then

$$(\lambda, [r]_A) \in \Gamma_A(x, y).$$

Since $\forall \lambda \in \Gamma$, $\lambda.o_x = o_{\lambda.x}$, $x \in G^{ob}$, it follows that any tree subgroupoid T of G is invariant under Γ. In this case $\Gamma_T = \Gamma \times G$, and we denote it by Γ_o.

<u>Definition 5.2.1:</u>

For any connected $\mathcal{G}$d-transformation group (Γ, G), Γ_o, with the above structure, is called <u>"the fundamental groupoid"</u> of (Γ, G), and <u>its vertex groups will be called "the fundamental groups" of (Γ, G).</u>

<u>Theorem 5.2.2:</u>

<u>If (Γ, G) is a topological $\mathcal{G}$d-transformation group and G is connected, then Γ_Λ is a topological groupoid.</u>

<u>Proof:</u>

(i) Let $\pi_2 : \Gamma \times E_\Lambda (=\Gamma_\Lambda) \longrightarrow E_\Lambda$ be the second projection, then $i^\Lambda = \bar{i}_\Lambda o \pi_2$. Hence it is continuous. (Recall that $E_\Lambda =$
$= (E_\Lambda, G^{ob}, \bar{i}_\Lambda, \bar{\phi}_\Lambda, \bar{\theta}_\Lambda, \bar{\sigma}_\Lambda, \bar{u}_\Lambda)$)

(ii) Let $\xi : \Gamma \times G^{ob} \longrightarrow G^{ob}$ be the continuous map defined by:-

$$\xi(\lambda, x) = \lambda . x \text{ and let } \sigma' : \Gamma \longrightarrow \Gamma \text{ be the inverse map.}$$

Let $\pi_1 : \Gamma_\Lambda \longrightarrow \Gamma$ be the first projection, then:-

$$\phi^\Lambda = \xi o(\sigma' o \pi_1, \bar{\phi}_\Lambda o \pi_2)$$

Hence it is continuous.

(iii) Let $\xi : \Gamma \times E_\Lambda \longrightarrow E_\Lambda$ be the continuous map defined by:-

$$\xi(\lambda, [r]_\Lambda) = \lambda . [r]_\Lambda = [\lambda . r]_\Lambda \quad \text{(by 5.1.5. this is defined).}$$

Let $\pi_i : (\Gamma \times E_\Lambda) \times (\Gamma \times E_\Lambda) \longrightarrow \Gamma$, $i = 1, 3$.

$\pi_j : (\Gamma \times E_\Lambda) \times (\Gamma \times E_\Lambda) \longrightarrow E_\Lambda$ $j = 2, 4$

be the obvious projections. Then it is easily seen that:-

$$\theta^\Lambda = (\theta' o(\pi_1, \pi_3), \bar{\theta}_\Lambda o(\pi_2, \xi\ o(\pi_1, \pi_4))),$$

where $\theta' : \Gamma \times \Gamma \longrightarrow \Gamma$ is the composition map.

Hence θ^Λ is continuous.

(iv) Let $c : G^{ob} \longrightarrow \Gamma$ be the constant map at e.

Then $u^{\Lambda} = (c, \bar{u}_{\Lambda})$.

Hence it is continuous.

(v) Let $\pi_1, \pi_2 : \Gamma_{\Lambda} \longrightarrow \Gamma$ and E_{Λ}, be the first and 2nd projections, respectively. Then it is easily seen that:-

$$\sigma^{\Lambda} = (\sigma' o \pi_1, \xi o(\sigma' o \pi_1, \bar{\sigma}_{\Lambda} o \pi_2))$$

where $\xi : \Gamma \times E_{\Lambda} \longrightarrow E_{\Lambda}$ is as above, and $\bar{\sigma}_{\Lambda}$ is the inverse map in E_{Λ}. Hence σ^{Λ} is continuous.

q.e.d.

Exact sequences

Let (Γ, G) be a $\mathcal{G}$d-transformation group and let $G^{ob} = X$. Then $X \times X \times \Gamma$ is a connected groupoid (see 1.2.11) over X.

We now investigate the relation of $X \times X \times \Gamma$ with Γ_{Λ} from the algebraic point of view.

Define $\eta : \Gamma_{\Lambda} \longrightarrow X \times X \times \Gamma$

by: $\eta(\lambda, [r]_{\Lambda}) = (i(r), \lambda^{-1}.\phi(r), \lambda)$ and $\eta | X$ is identity.

Lemma 5.2.3:

η is a morphism of groupoids.

Let $i_1, \phi_1, \theta_1, u_1, \sigma_1$ be the maps in $X \times X \times \Gamma$, then:-

(i) $\forall (\lambda, [r]_{\Lambda}) \varepsilon \Gamma_{\Lambda}$, $\eta o i^{\Lambda}(\lambda, [r]_{\Lambda}) = \eta(i(r)) = i(r)$

$$i_1 o \eta(\lambda, [r]_{\Lambda}) = (i(r), \lambda^{-1}.\phi(r), \lambda) = i(r)$$

Hence $\eta o i^{\Lambda} = i_1 o \eta$ Similarly $\eta o \phi^{\Lambda} = \phi_1 o \eta$

(ii) $\forall ((\lambda, [r]_{\Lambda}), (\mu, [s]_{\Lambda})) \varepsilon D^{\Lambda}$,

$$\eta((\lambda, [r]_{\Lambda}) * (\mu, [s]_{\Lambda})) = \eta(\lambda\mu, [r + \lambda.s]_{\Lambda})$$

$$= (i(r+\lambda.s), (\lambda\mu)^{-1}.\phi(r + \lambda.s), \lambda\mu)$$

$$= (i(r), \mu^{-1}\lambda^{-1}.\phi(\lambda.s), \lambda\mu)$$

$= (i(r), \mu^{-1}\lambda^{-1}.(\lambda.\phi(s)), \lambda\mu)$

(By definition of action)

$= (i(r), \mu^{-1}.\phi(s), \lambda\mu)$

$= (i(r), \lambda^{-1}.\phi(r), \lambda)(\lambda^{-1}.\phi(r), \mu^{-1}.\phi(s), \mu)$

$= (i(r), \lambda^{-1}.\phi(r), \lambda)(i(s), \mu^{-1}.\phi(s), \mu)$

$(\because \lambda.i(s) = \phi(r))$

$= \eta(\lambda, [r]_\Lambda)\,\eta(\mu, [s]_\Lambda)$

(iii) $\forall (e, [o_x]_\Lambda) \in \Gamma_\Lambda$, $\eta(e, [o_x]_\Lambda) = (i(o_x), e^{-1}.\phi(o_x), e)$

$= (x, x, e) = \tilde{o}_x \in X \times X \times \Gamma$

q.e.d.

Let $(\lambda, [r]_\Lambda) \in \operatorname{Ker} \eta$, then $\eta(\lambda, [r]_\Lambda) = (i(r), \lambda^{-1}.\phi(r), \lambda)$ is an identity in $X \times X \times \Gamma$. Hence $\lambda = e$, and $\lambda^{-1}.\phi(r) = \phi(r) = i(r)$. Hence $r \in G\{i(r)\}$. So:-

$\operatorname{Ker}\eta = \{(e.[r]_\Lambda) \mid r \in G\{i(r)\}\} \approx E^o_\Lambda$, where E^o_Λ denotes the wide and full totally disconnected subgroupoid of E_Λ. i.e. E^o_Λ is the union of all vertex groups of E_Λ.

Hence by ([2]8.3.2.) we have:-

$$\Gamma_{\Lambda/E^o_\Lambda} \approx \operatorname{im}(\eta)$$

<u>But η is onto</u>. For, given $(x, y, \lambda) \in X \times X \times \Gamma$, let $r \in G(x, \lambda.y)$, then $(\lambda, [r]_\Lambda) \in \Gamma_\Lambda$ and $\eta(\lambda, [r]_\Lambda) = (i(r), \lambda^{-1}.\phi(r), \lambda)$

$= (i(r), \lambda^{-1}(\lambda.y), \lambda) = (x, y, \lambda)$.

Therefore $\Gamma_{\Lambda/E^o_\Lambda} \approx X \times X \times \Gamma$.

Hence, we have the following exact sequence of groupoids:-

(5.2.4.) $o \longrightarrow E^o_\Lambda \longrightarrow \Gamma_\Lambda \xrightarrow[\eta]{} X \times X \times \Gamma \longrightarrow o$

Where o at the ends means the discrete groupoid of units in G.

Next, define $\Delta : \Gamma_o \longrightarrow \Gamma_\Lambda$ by:-

(i) $\Delta | \Gamma_o^{ob}$ = identity, (ii) $\Delta(\lambda, r) = (\lambda, [r]_\Lambda)$, $(\lambda, r) \varepsilon \Gamma_o$.

<u>Δ is a morphism of groupoids:</u>

(i) Let i^o, ϕ^o denote the initial and final maps in Γ^o, then:-

$$\forall (\lambda, r) \varepsilon \Gamma_o, \; \Delta o i^o(\lambda, r) = \Delta(i^o(\lambda, r)) = \Delta(i(r)) = i(r)$$

$$i^\Lambda o \Delta(\lambda, r) = i^\Lambda(\lambda, [r]_\Lambda) = i(r).$$

Hence $\Delta o i^o = i^\Lambda o \Delta$. Similarly $\Delta o \phi^o = \phi^\Lambda o \Delta$.

(ii) $\forall ((\lambda, r), (\mu, s)) \varepsilon D^o$, $\Delta((\lambda, r) * (\mu, s)) = \Delta(\lambda\mu, r + \lambda . s)$

$$= (\lambda\mu, [r + \lambda . s]_\Lambda) = (\lambda, [r]_\Lambda) * (\mu, [s]_\Lambda) = \Delta(\lambda, r) * \Delta(\mu, s).$$

(iii) $\forall (e, o_x) \varepsilon \Gamma_o$, $\Delta(e, o_x) = (e, [o_x]_\Lambda)$, identity of Γ_Λ at x.

Hence $\Delta : \Gamma_o \longrightarrow \Gamma_\Lambda$ is a morphism of groupoids.

Let $(\lambda, r) \varepsilon$ KerΔ, then $\Delta(\lambda, r) = (\lambda, [r]_\Lambda)$ is a unit element in Γ_Λ. Hence $\lambda = e$, $[r]_\Lambda = [o_x]_\Lambda$, where $x = i(r)$.

Therefore $r \varepsilon \Lambda\{x\}$, and we have:-

$$\text{Ker}\,\Delta = \{(e, r) | r \varepsilon \Lambda\{x\}, x \varepsilon G^{ob}\} \approx \Lambda^o,$$ the wide, full and totally disconnected subgroupoid of Λ.

<u>Δ is onto:</u> given $(\lambda, [r]_\Lambda) \varepsilon \Gamma_\Lambda$, $(\lambda, r) \varepsilon \Gamma_o$ and

$$\Delta(\lambda, r) = (\lambda, [r]_\Lambda)$$

Therefore $\Gamma_o/_{\Lambda^o} \approx \Gamma_\Lambda$, and we get the following short exact sequence of groupoids:-

(5.2.5) $$o \longrightarrow \Lambda^o \longrightarrow \Gamma_o \xrightarrow{\Delta} \Gamma_\Lambda \longrightarrow o$$

More generally, let $\Lambda \subseteq B$ be invariant subgroupoids of G, then we can define a morphism

$$\chi : \Gamma_\Lambda \longrightarrow \Gamma_B, \text{ by } \chi(\lambda, [r]_\Lambda) = (\lambda, [r]_B).$$

$$\chi | \Gamma_\Lambda^{ob} = \text{identity}$$

It is easily seen (following the same line as Δ) that χ is a morphism onto Γ_B, with

$$\mathrm{Ker}(\chi) = \{(e, [r]_A) \mid r \in B\{x\},\ x \in G^{ob}\} \approx B^o/A_o ,$$

where B^o, A^o are wide and full totally disconnected subgroupoids of B and A, respectively. Hence, we have the following short exact sequence:-

(5.2.6.) $$o \longrightarrow B^o/A_o \longrightarrow \Gamma_A \xrightarrow{\chi} \Gamma_B \longrightarrow o$$

Notice that B^o/A_o is a groupoid over G^{ob}. For A^o is a (wide totally disconnected) normal subgroupoid of B^o

Next, for each $x \in G^{ob}$, $\eta_x = \eta|\Gamma_A\{x\}$ is a homomorphism of the object group $\Gamma_A\{x\}$ onto the group Γ, with $\mathrm{Ker}\eta_x = \dfrac{G\{x\}}{A\{x\}}$;

and $\Delta_x = \Delta|\Gamma_o\{x\}$ is a homomorphism of the object group $\Gamma_o\{x\}$ onto $\Gamma_A\{x\}$ with $\mathrm{Ker}\Delta_x = A\{x\}$. Hence we get the following exact sequences of groups and homomorphisms

(5.2.7.)
$$o \longrightarrow \frac{G\{x\}}{A\{x\}} \longrightarrow \Gamma_A\{x\} \xrightarrow{\eta_x} \Gamma \longrightarrow o$$

$$o \longrightarrow A\{x\} \longrightarrow \Gamma_o\{x\} \longrightarrow \Gamma_A\{x\} \longrightarrow o$$

Application 5.2.8:

Let $G = \pi X$, the fundamental groupoid of a space X. Then $\Gamma_o\{x\} = \sigma(X, x, G)$, the fundamental group of the transformation group (Γ, X) as defined in [8]. By replacing $G\{x\}$, $\Gamma_o\{x\}$ and $\Gamma_A\{x\}$

by $\pi_1(X, x)$, $\sigma(X, x, \Gamma)$, $\sigma_{A_x}(X, x, \Gamma)$,

respectively, we get the exact sequences in [9] p. 906.

In case X is path-connected Hausdorff, πX and hence Γ_A are connected topological groupoids. Hence all object groups are isomorphic topological groups. Therefore $\forall\, x, y \in X$, $\sigma_{A_x}(X, x, \Gamma)$ and $\sigma_{A_x}(X, y, \Gamma)$ are isomorphic topological groups.

Given any $\mathcal{Y}$ d-transformation group (Γ, G), with G connected then, as we have seen, for any invariant subgroupoid A of G and any $x \varepsilon G^{ob}$, Γ_A and C_A^x are connected groupoids over G^{ob} and $St_{E_A} x$, respectively. Let $\Gamma_A^x = \Gamma_A \{x\}$ be the vertex group at x, and define an action of Γ_A^x on G_A^x as follows:-

$\forall(\lambda, [r]_A) \varepsilon \Gamma_A{}^x$, and $\forall([a], b) \varepsilon G_A^x$, $[a]_A \varepsilon St_{E_A} x = (G_A^x)^{ob}$,

$$(\lambda, [r]_A) \wedge ([a]_A, b) = ([r + \lambda . a]_A, \lambda . b)$$

$$(\lambda, [r]_A) \wedge [a]_A = [r + \lambda . a]_A .$$

<u>The action $\wedge$ is well-defined:</u> Let $s \varepsilon [r]_A$, $c \varepsilon [a]_A$, we must show $[r + \lambda . a]_A = [s + \lambda . c]_A$. i.e. $r + \lambda . a - \lambda . c - s \varepsilon A \{x\}$. But $c \varepsilon [a]_A \Longrightarrow a - c \varepsilon A \{x\} \Longrightarrow \lambda . a - \lambda . c = \lambda .(a - c) \varepsilon A \{\lambda . x\}$

(By invariance of A)

$\Longrightarrow r + \lambda . a - \lambda . c - r \varepsilon A \{x\}$ (By normality of A)

and $s \varepsilon [r]_A \Longrightarrow r - s \varepsilon A \{x\}$

Hence $r + \lambda . a - \lambda . c - r + r - s = r + \lambda . a - \lambda . c - s \varepsilon A \{x\}$.

<u>The operation does satisfy the required conditions:-</u>

(i) $(\lambda, [r]_A) \wedge ((\mu, [s]_A) \wedge ([a]_A, b)) = (\lambda, [r]_A) \wedge ([s + \mu . a]_A, \mu . b)$

$= ([r + \lambda .(s + \mu . a)]_A, \lambda .(\mu . b)) = ([r + \lambda . s + (\lambda\mu) . a]_A, (\lambda\mu) . b)$

$= (\lambda\mu, [r + \lambda . s]_A) \wedge ([a]_A, b) = ((\lambda, [r]_A) * (\mu, [s]_A)) \wedge ([a]_A, b)$

(ii) $(e, [o_x]_A) \wedge ([a]_A, b) = ([o_x + e . a]_A, e . b) = ([a]_A, b)$.

So, Γ_A^x acts on G_A^x as a set. (i), (ii) also shows that Γ_A^x acts on X_{A_x}.

(iii) $I((\lambda, [r]_A) \wedge ([a]_A, b)) = I([r + \lambda . a]_A, \lambda . b) = [r + \lambda . a]_A$

$= (\lambda, [r]_A) \wedge [a]_A$

$= (\lambda, [r]_A) \wedge (I([a]_A, b))$.

Similarly, $\phi((\lambda, [r]_A) \wedge ([a]_A, b)) = (\lambda, [r]_A) \wedge (\phi([a]_A, b))$.

(iv) $(\lambda, [r]_\Lambda) \wedge (([a]_\Lambda, b) + ([c]_\Lambda, d)) = (\lambda, [r]_\Lambda) \wedge ([a]_\Lambda, b+d)$

$$= ([r + \lambda . a]_\Lambda, \lambda . (b + d))$$

$$= ([r + \lambda . a]_\Lambda, \lambda . b + \lambda . d))$$

$$= ([r + \lambda . a]_\Lambda, \lambda . b) + ([r + \lambda . c]_\Lambda, \lambda . d)$$

$$= (\lambda, [r]_\Lambda) \wedge ([a]_\Lambda, b) + (\lambda, [r]_\Lambda) \wedge ([c]_\Lambda, d)$$

(v) $(\lambda, [r]_\Lambda) \wedge (-([a]_\Lambda, b)) = (\lambda, [r]_\Lambda) \wedge ([a + b]_\Lambda, - b)$

$$= ([r + \lambda . (a + b)]_\Lambda, \lambda . (-b))$$

$$= ([r + \lambda . a + \lambda . b]_\Lambda, -\lambda . b) = -([r + \lambda . a]_\Lambda, \lambda . b)$$

$$= -((\lambda, [r]_\Lambda) \wedge ([a]_\Lambda, b))$$

Therefore we have the following lemma and corollary.

Lemma 5.2.9:

If (Γ, G) is a $\mathcal{G}$d-transformation group, with G connected, and A is an invariant subgroupoid of G under Γ, then $\forall x \in G^{ob}$, $(\Gamma^x_\Lambda, G^x_\Lambda)$ is also a $\mathcal{G}$ d-transformation group.

Corollary 5.2.10:

If (Γ, G) is a $\mathcal{G}$d-transformation group and G is connected then (Γ^x_o, G^x_o), where G^x_o is the universal covering groupoid of G (in the abstract sense) and $\Gamma_o{}^x$ is the fundamental group of (Γ, G) at x, is a $\mathcal{G}$ d-transformation group.

In case of topological $\mathcal{G}$ d-transformation groups, since G^x_Λ need not be a topological groupoid (because of non-Hausdorff object space), we restrict ourselves to the case of locally trivial groupoids with Hausdorff vertex groups. So:-

Theorem 5.2.11:

Let G be a connected locally trivial groupoid with Hausdorff vertex groups, and let (Γ, G) be a topological $\mathcal{G}$ d-transformation group. Then for any invariant subgroupoid Λ of G under Γ, and for any $x \in G^{ob}$, $(\Gamma_{\Lambda}^{x}, G_{\Lambda}^{x})$ is a topological $\mathcal{G}$d-transformation group.

Proof:

By 5.2.10, $(\Gamma_{\Lambda}^{x}, G_{\Lambda}^{x})$ is a $\mathcal{G}$d-transformation group, and by 5.2.2. Γ_{Λ} is a topological groupoid and hence Γ_{Λ}^{x} is a topological group. By 4.2.2 G_{Λ}^{x} is a topological groupoid. So we need only prove the continuity of the action, i.e. the continuity of of the map $(\lambda, [r]_{\Lambda})_{*}: G_{\Lambda}^{x} \longrightarrow G_{\Lambda}^{x}$, $(\lambda, [r]_{\Lambda}) \in \Gamma_{\Lambda}^{x}$, defined by $(\lambda, [r]_{\Lambda})_{*}([a]_{\Lambda}, b) = (\lambda, [r]_{\Lambda}) \cdot [a]_{\Lambda}, b) = ([r + \lambda . a]_{\Lambda}, \lambda . b)$.

Let $p_{\Lambda} : G \longrightarrow E_{\Lambda}$ be the quotient morphism, $\theta : D \longrightarrow G$, the composition map in G and $\lambda_{*} : G \longrightarrow G$ that obtained from the action of λ on G. Then:-

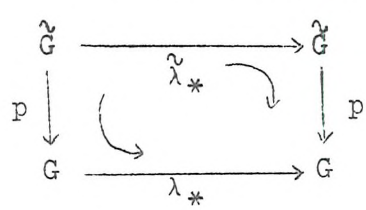
$$
\begin{aligned}
(\lambda, [r]_{\Lambda})_{*}([a]_{\Lambda}, b) &= ([r + \lambda . a]_{\Lambda}, \lambda . b) = (p_{\Lambda}(r + \lambda . a), \lambda . b) \\
&= (p_{\Lambda}(r + \lambda_{*}(a)), \lambda_{*}(b)) = (p_{\Lambda} o \theta (r, \lambda_{*}(a)), \\
&= (p_{\Lambda} o \theta (r, \lambda_{*}(a)), \lambda_{*}(b)).
\end{aligned}
$$

Therefore continuity of $(\lambda, [r]_{\Lambda})_{*}$ follows from those of θ, p_{Λ} and λ_{*}.

q.e.d.

Definition 5.2.12:

Let (Γ, G) and $(\tilde{\Gamma}, \tilde{G})$ be topological $\mathcal{G}$d-transformation groups, and let $p: \tilde{G} \longrightarrow G$ be a covering morphism of topological groupoids. Let $\lambda \in \Gamma$, then $\tilde{\lambda} \in \tilde{\Gamma}$ will be called a lift of λ, if the following diagram is commutative.



i.e. $\tilde{\lambda}_{*}$ is a lift of $\lambda_{*} o p$ in the usual sense.

<u>Theorem 5.2.13:</u>

<u>Let G be a connected locally trivial Hausdorff groupoid, and let (Γ, G) be an effective topological $\mathcal{G}$d-transformation group. Then Γ_Λ^x is the group of all morphisms from G_Λ^x onto itself which lift the elements of Γ.</u>

<u>Proof:</u>

For any $\lambda \varepsilon \Gamma$ and $r \varepsilon G(x, \lambda . x)$, $(\lambda, [r]_\Lambda) \varepsilon \Gamma_\Lambda^x$ and we have the following commutative diagram.

$$\begin{array}{ccc} G_\Lambda^x & \xrightarrow{(\lambda, [r]_\Lambda)_*} & G_\Lambda^x \\ \downarrow p & & \downarrow p \\ G & \xrightarrow{\lambda_*} & G \end{array}$$

Since p is a covering morphism, $(\lambda, [r]_\Lambda)$ is a lift of λ.

On the other hand, let $g : G_\Lambda^x \longrightarrow G_\Lambda^x$ be any lift of $\lambda_* o p$, then we show that $g = (\lambda, [s]_\Lambda)_*$, for some $s \varepsilon G$.

Let $([a]_\Lambda, b) \varepsilon G_\Lambda^x$, and let $g([a]_\Lambda, b) = ([d]_\Lambda, c) \varepsilon G_\Lambda^x$, then:-

$$\phi(a) = i(b) \text{ and } \phi(d) = i(c)$$

By definition $pog = \lambda_* op$, so we have $c = pog([a]_\Lambda, b)$

$$= \lambda_* op([a]_\Lambda, b) = \lambda_*(b).$$

Hence $\phi(d) = i(c) = i(\lambda_*(b)) = \lambda_*(ib)$ (For, λ_* is a morphism)

$$= \lambda_*(\phi(a)) = \lambda . \phi(a).$$

Therefore $d \varepsilon G(x, \lambda . \phi(a))$. Since $\lambda . a \varepsilon G(\lambda . x, \lambda . \phi(a))$, $\exists$ a unique $s \varepsilon G(x, \lambda . x)$ such that $s + \lambda . a = d$. Hence $[s + \lambda . a]_\Lambda = [d]_\Lambda$.

Therefore $g([a]_\Lambda, b) = ([d]_\Lambda, c) = ([s + \lambda . a]_\Lambda, \lambda . b)$

$$= (\lambda, [s]_\Lambda)_* ([a]_\Lambda, b).$$

Thus $g = (\lambda, [s]_\Lambda)_*$. So, for any fixed $\lambda \varepsilon \Gamma$, the set

$$\{(\lambda, [r]_\Lambda) \mid r \varepsilon G(x, \lambda . x)\}$$

is the set of all lifts of $\lambda \varepsilon \Gamma$. We now show that if $(\lambda, [r]_\Lambda)$, $(\mu, [s]_\Lambda)$ are lifts of λ and μ, respectively, then

$(\lambda, [r]_A)*(\mu, [s]_A) = (\lambda\mu, [r + \lambda.s]_A)$ is a lift of $\lambda\mu$. We must show that $p o(\lambda\mu, [r + \lambda.s]_A)_* = (\lambda\mu)_* o p$.

$\forall ([a]_A, b)$, $(p o(\lambda\mu, [r + \lambda.s]_A))([a]_A, b) =$

$$= p((\lambda\mu, [r + \lambda.s]_A)\hat{}([a]_A, b))$$

$$= p([r + \lambda.s + \lambda\mu.a]_A, (\lambda\mu).b) = (\lambda\mu).b$$

and $(\lambda\mu)_* o p([a]_A, b) = (\lambda\mu)_*(b) = (\lambda\mu).b$

Hence $p o(\lambda\mu, [r + \lambda.s]_A)_* = (\lambda\mu)_* o p$.

q.e.d.

Corollary 5.2.14:

$\Gamma_0^{\ x}$, the fundamental group of (Γ, G), is the group of all morphisms from $G_0^{\ x}$, the universal covering groupoid of G, onto itself which lift the elements of Γ.

As an application, consider the case $G = \pi X$, X.p.c., ℓ.p.c., ℓ.s.c., then $G_0^{\ x} \approx \tilde{G}$ and $Stx = \tilde{X}$, the universal covering space of X.

Corollary 5.2.15:

If $G = \pi X$, X p.c., ℓ.p.c. and ℓ.s.c. Hausdorff space, then $\sigma(X, x, \Gamma)$, the fundamental group of (Γ, X), is the group of all continuous maps from the universal covering space of X onto itself which lift the elements of Γ.

Definition 5.2.16:

Let $p : \tilde{G} \longrightarrow G$ be a covering morphism of topological groupoids. Then any morphism $h : \tilde{G} \longrightarrow \tilde{G}$ such that $p o h = h$, is called a cover transformation.

It is easily seen that the set of all cover transformations of $\tilde{G}$ form a group.

Let $\lambda = e$, the identity element of Γ, then $e_* : G \longrightarrow G$ is the identity morphism. Hence each lift of e is a cover transformation of G_Λ^x, As we saw in the proof of 5.2.13, the set of lifts of e is

$$\{(e, [r]_\Lambda) \mid r \in G(x, e.x)\} = \{(e, [r]_\Lambda) \mid r \in G\{x\}\} ,$$

which is clearly isomorphic to $E_\Lambda\{x\}$. Therefore:-

Corollary 5.2.17:

The group of cover transformations of G_Λ^x is isomorphic to the vertex group $E_\Lambda\{x\}$.

3. ORBIT GROUPOIDS

Definition 5.3.1:

A group Γ acts freely on a set X, if $g.x = x$ for some $x \in X$ and $g \in G$ implies $g = e$ (the identity).

Definition 5.3.2:

The group G is said to act properly discontinuously on a space X if the following conditions are satisfied:-

(i) If two points $x, x' \in X$ are not congruent modulo G, then x, x' have neighbourhoods U, U', respectively, such that:

$$\forall g \in G, \quad (g.U) \cap U' = \phi$$

(This condition implies that $X/_G$ is Hausdorff.)

(ii) For each $x \in X$, the isotropy group $G_x = \{g \in G \mid g.x = x\}$ is finite. (Note that if G acts freely this condition is satisfied).

(iii) Each $x \in X$ has a neighbourhood U, stable by G_x, such that:-

$$U \cap (g.U) = \Phi \quad \text{for every } g \in G \text{ and } g \notin G_x.$$

Lemma 5.3.3:

Let (Γ, G) be a $\mathcal{G}$d-transformation group with $X = G^{ob}$. If Γ acts freely on G, then $G/_{\Gamma} = \{\Gamma.g \mid g \in G\}$ is a groupoid over $X/_{\Gamma}$.

Proof:

Let $g \in G(x, y)$ be any morphism, then:-

$$\forall \gamma \in \Gamma \ , \ \gamma.g : \gamma.x \longrightarrow \gamma.y.$$

Hence
$$\bigsqcup_{\gamma \in \Gamma} \gamma.g : \bigsqcup_{\gamma \in \Gamma} \gamma.x \longrightarrow \bigsqcup_{\gamma \in \Gamma} \gamma.y$$
where " $\bigsqcup$ " denotes the disjoint sum. Since Γ acts freely, we have:-

$$\gamma \neq \lambda \Longrightarrow \gamma.g \neq \lambda.g, \ \gamma.x \neq \lambda.x, \ \gamma.y \neq \lambda.y$$

Hence the above sums are unions (see [2], p. 329)

Therefore we have:-

$$\bigcup_{\gamma \in \Gamma} \gamma.g : \bigcup_{\gamma \in \Gamma} \gamma.x \longrightarrow \bigcup_{\gamma \in \Gamma} \gamma.y.$$

Hence, $\Gamma.g : \Gamma.x \longrightarrow \Gamma.y$, and we can take $\Gamma.g$ to be a morphism from $\Gamma.x$ to $\Gamma.y$. So let $G = (G, X, i, \phi, \theta, u, \sigma)$, then:-

(i) Define the initial and final functions $i_1, \ \phi_1 : G/_{\Gamma} \longrightarrow X/_{\Gamma}$.

by:- $\forall \ \Gamma.g \in G/_{\Gamma}$, $i_1(\Gamma.g) = \Gamma.i(g)$ & $\phi_1(\Gamma.g) = \Gamma.\phi(g)$.

(ii) Let $D_1 = \{(\Gamma.f, \Gamma.g) \mid (f, g) \in D\}$

(D the set of composable pairs in $G \times G$)

Define $\theta_1 : D_1 \longrightarrow G/_{\Gamma}$

by:- $\theta_1(\Gamma.f, \Gamma.g) = \Gamma.f + \Gamma.g = \Gamma.(f + g)(= \Gamma.\theta(f, g))$

Since $\forall \gamma \in \Gamma$ & $\forall (f, g) \in D$, $\gamma.(f + g) = \gamma.f + \gamma.g$, the above definition makes sense. It satisfies the associative law:-

$$\begin{aligned}\forall (f, g), (g, h) \in D, \ (\Gamma.f + \Gamma.g) + \Gamma.h &= \Gamma.(f + g) + \Gamma.h \\ &= \Gamma.((f + g) + h) \\ &= \Gamma.(f + (g + h)) = \Gamma.f + \Gamma.(g + h) \\ &= \Gamma.f + (\Gamma.g + \Gamma.h)\end{aligned}$$

(iii) Define the unit function $u_1 : X/_{\Gamma} \longrightarrow G/_{\Gamma}$

by:- $u_1(\Gamma.x) = o_{\Gamma.x} = \Gamma.o_x$ ($= \Gamma.u(x)$)

Clearly $i_1(\Gamma.o_x) = \Gamma.x = \phi_1(\Gamma.o_x)$, and we have:-

$\forall f \in G(x, y), g \in G(z, x),\ \Gamma.g + \Gamma.o_x = \Gamma.(g + o_x) = \Gamma.g$

$\Gamma.o_x + \Gamma.f = \Gamma(o_x + f) = \Gamma.f.$

(iv) Define the inverse function $\sigma_1 : G/\Gamma \longrightarrow G/\Gamma$

by:- $\sigma_1(\Gamma.f) = \Gamma.(-f)(= \Gamma.\sigma(f))$

Clearly $i_1(\sigma_1(\Gamma.f)) = \phi_1(\Gamma.f) = \Gamma.\phi(f)$

$\phi_1(\sigma_1(\Gamma.f)) = i_1(\Gamma.f) = \Gamma.i(f)$

and $\Gamma.f + \Gamma.(-f) = \Gamma.(f - f) = \Gamma.(o_{i(f)}) = o_{\Gamma.i(f)}$

$\Gamma.(-f) + \Gamma.f = \Gamma(-f + f) = \Gamma.(o_{\phi(f)}) = o_{\Gamma.\phi(f)}.$

Hence G/Γ is a groupoid over X.

<u>Note:</u> G/Γ will be called "the orbit groupoid".

<u>Remark:</u>

If G is connected, then G/Γ is connected.

<u>Lemma 5.3.3:</u>

<u>Let (Γ, G) be a $\mathcal{G}$d-transformation group. If Γ acts freely on G, then $p : G \longrightarrow G/\Gamma$ defined by:-</u>

<u>$\forall g \in G,\ x \in X,\ p(g) = \Gamma.g$ and $p(x) = \Gamma.x$ is a covering morphism.</u>

<u>Proof:</u>

<u>p is a morphism of groupoids:</u> For,

(i) $\forall g \in G,\ po\phi(g) = p(\phi(g)) = \Gamma.\phi(g)$
$\phi_1 op(g) = \phi_1(\Gamma.g) = \Gamma.\phi(g)$ $\Big\} \Longrightarrow po\phi = \phi_1 op$

Similarly $poi = i_1 op$

(ii) $\forall (f, g) \in D,\ p(f + g) = \Gamma.(f + g) = \Gamma.f + \Gamma.g = p(f) + p(g)$

(iii) $\forall o_x \in G,\ p(o_x) = \Gamma.o_x = o_{\Gamma.x} = o_{p(x)}.$

Let $p' = p|\,St_G x$, $x \in X$, then we must show that

$p' : St_G x \longrightarrow St_{G/\Gamma}(\Gamma.x)$ is bijective. Let $f, g \in St_G x$, then:-

$$p'(f) = p'(g) \Longrightarrow \Gamma.f = \Gamma.g \Longrightarrow \Gamma.f - \Gamma.g = \Gamma.(f - g) = o_{\Gamma.x}$$

$$\Longrightarrow \forall \gamma \in \Gamma \;,\; \gamma.(f - g) = o_{\gamma.x}$$

$$\Longrightarrow f - g = o_x \text{ (by taking } \gamma = e)$$

$$\Longrightarrow f = g.$$

So p' is 1 - 1. Next, given $\bar{g} \in St_{G/\Gamma}\,\Gamma.x$, then $\bar{g} = \Gamma.g$ for some $g \in G$. Hence $p'(g) = \Gamma.g = \bar{g}$.

Therefore p' maps $St_G x$ onto $St_{G/\Gamma}(\Gamma.x)$.

Hence p is a covering morphism.

q.e.d.

Theorem 5.3.4:

Let (Γ, G) be a topological $\mathcal{G}$d-transformation group, and let Γ act freely and properly discontinuously on G, then $G/_\Gamma$ is a topological groupoid over $X/_\Gamma$

Proof:

Let $G/_\Gamma$ and $X/_\Gamma$ carry the quotient topology, then we must show the continuity of maps i_1, ϕ_1, θ_1, u_1, σ_1. Since p is continuous, and it is a quotient morphism, continuity of these maps follows from the continuity of $i, \phi, \theta, u, \sigma$. For example, to show the continuity of i_1, say, we know that $i_1 o p = p o i$ and p is a quotient map; hence continuity of i_1 follows from the familiar argument about the quotient topology.

q.e.d.

Corollary 5.3.5:

If (Γ, G) is a topological $\mathcal{G}$d-transformation group and Γ acts freely

<u>and properly discontinuously on G, then $p : G \longrightarrow G/_{\Gamma}$ is a covering morphism of topological groupoids.</u>

<u>Proof:</u>

By 5.3.3. p is a covering morphism in the abstract sense; by 5.3.4. $G/_{\Gamma}$ is a topological groupoid, and the quotient morphism p is continuous and open. Hence $\forall\ x \in X$, $p' : St_G x \longrightarrow St_{G/_{\Gamma}}\ \Gamma.x$ is a homeomorphism.

q.e.d.

CHAPTER VI

LIE CATEGORIES AND GROUPOIDS

Introduction:

These categories and groupoids were first intorduced by C. Ehresmann [4], and received more attention by Westman [12]. We give a new definition and produce some examples, e.g. For any vector bundle T over a differentiable manifold M, $\mathcal{C}(T)$, the set of all homomorphisms between the fibres is a lie category, with $\mathcal{G}(T)$, the set of isomorphisms as Lie groupoid. Most of the results obtained in our earlier chapters in the topological case go over to the differentiable case with minor modifications. We give some indications in section 2.

1. DEFINITIONS AND EXAMPLES

Before starting the definition of Lie categories we need:-

Lemma 6.1.1:

Let the n-dimensional C^r manifold C be a category over the C^r m-manifold M, and let i, ϕ, the initial and final maps, be C^r maps of rank m. Then D, the set of composable pairs, is a closed submanifold of C × C.

Proof:

Let $\Delta \subsetneq M \times M$ be the diagonal, then Δ (being the graph of $id: M \longrightarrow M$) is a closed submanifold of $M \times M$. The product map $\phi \times i : C \times C \longrightarrow M \times M$ is a C^r map of rank $2m = \dim (M \times M)$, since rank i = rank ϕ = m, by hypothesis. Hence for any $(a, b) \in D$, the differential

$$d(\phi \times i)(a, b): T_{(a, b)}(C \times C) \longrightarrow T_{(\phi a, ib)}(M \times M)$$

is surjective. Therefore $D = (\phi \times i)^{-1}(\Delta)$ is a closed submanifold of C × C (See [3] p. 21.) q.e.d.

Definition 6.1.2:

A set C is called a Lie category of class r, over a set M, if:-

(LC1) C is an algebraic category over M

(LC2) C and M are differentiable manifolds of class r with dim c > dim M

(LC3) i, ϕ & u, the initial, final and unit maps, are differentiable maps of class r and ranks m = dim M.

(LC4) The composition map $\theta : D \longrightarrow C$, is a C^r map (by 6.1 D is a submanifold of C × C).

Definition 6.1.3:

A set G is called a Lie groupoid of class r over a set M, if:-

(i) G is a Lie category of class r and an algebraic groupoid over M

(ii) The inverse map $\sigma : G \longrightarrow G$ is a C^r map.

Remarks 6.1.4:

(i) In every Lie category, for every object x, Stx and $\mathcal{E}_x$ are closed submanifolds. This is a consequence of the fact that i and ϕ are of maximum rank.

(ii) In every Lie category, $\forall$ objects x, y the sets C(x, y) are closed submanifolds.

(iii) O = u(M), the set of units in C is a closed submanifold of C diffeomorphic to M.

(Hence we may regard M as a submanifold of C).

For, $u o \phi$ (or $u o i$) $: C \longrightarrow C$ is a C^r map of rank m. Hence for each $c_x \varepsilon O$, $\exists$ a coordinate chart (U, λ) in C such that

$$O \cap U = \{a \varepsilon U \mid \lambda(a) = (\lambda^1(a), \ldots, \lambda^m(a), o, \ldots, o)\}$$

(see [11]p.41). Hence O is a closed submanifold of C (see [3] p.21).

(iv) Let $f \varepsilon C(x, y)$, then the map ${}_f\Lambda : St\, y \longrightarrow St\, x$

$$a \rightsquigarrow f + a$$

is a C^r map. For, let $C_f : St\, y \longrightarrow St\, x$ be the constant map: $a \longrightarrow f$ then ${}_f\Lambda = \theta o(c_f, id)$. Similarly

$\Lambda_f : \mathcal{E}_x \longrightarrow \mathcal{E}_y$, $a \rightsquigarrow a + f$ is a C^r map.

(v) Let G be a connected Lie groupoid, then $\forall$ x, y, x', y' ε G^{ob}, G(x, y) is diffeomorphic to G(x', y').

(vi) In every connected Lie groupoid G, the vertex groups are isomorphic Lie groups.

The proofs of (v), (vi) follow from (iv) similarly to 3.1.3(7, 8). Rather trivial examples of Lie groupoids are Lie groups with object set a point. For, i, ϕ, u are constant and hence are differentiable of class r, and obviously of constant rank. The other conditions follow from the definition of Lie groups.

(N.B. The groupoid structures on a torus are not trivial, because object spaces are not singleton.) As non-trivial and important examples, we will show that for any Manifold M, πM and in general $\pi_\Lambda M$ is a Lie groupoid, and for any differentiable vector bundle T over a manifold M, $\mathcal{C}(T)$ is a Lie **category** over M.

Theorem 6.1.5:

Let M be an n-dimensional manifold of class r, then for any connected wide normal subgroupoid Λ of πM, $\pi_\Lambda M = (\pi_\Lambda M, i_\Lambda, \phi_\Lambda, \theta_\Lambda, u_\Lambda, \sigma_\Lambda)$ is a Lie groupoid of class r.

Proof:

$M \times M$ is a C^r manifold of dimension 2n. By 3.2.2. πM and hence $\pi_\Lambda M$ is locally trivial. Therefore $\pi_\Lambda M$ is a covering space of $M \times M$, and hence it is a covering manifold of class r and of dimension 2n. Moreover the covering map $(\bar{i}_\Lambda, \bar{\phi}_\Lambda): \pi_\Lambda M \longrightarrow M \times M$ is of class r and hence $\bar{i}_\Lambda$ and $\bar{\phi}_\Lambda$ are of class r. The fact that $\bar{i}_\Lambda$ and $\bar{\phi}_\Lambda$ have ranks n is a consequence of local diffeomorphism and the projection properties of $(\bar{i}_\Lambda, \bar{\phi}_\Lambda)$ and $\pi_i : M_1 \times M_2 \longrightarrow M_i$, $i = 1, 2$, respectively.

Next, we show that the composition $\bar{\theta}_\Lambda: \bar{D}_\Lambda \longrightarrow E_\Lambda$ is class r. Let $([a]_\Lambda, [b]_\Lambda) \in \bar{D}_\Lambda$, and let $U = \langle U_1, [a]_\Lambda, V_1 \rangle \times \langle U_2, [b]_\Lambda, V_2 \rangle$

be a coordinate neighbourhood of $([a]_\Lambda, [b]_\Lambda)$. Then $<U_1, [a+b]_\Lambda, V_2>$ is a coordinate neighbourhood of $[a+b]_\Lambda$ in $\pi_\Lambda M$.

Let (λ_i, U_i), (γ_i, V_i) $i = 1, 2$ be coordinate charts in M, then:-

$$\psi_1 = (\lambda_1 \times \gamma_1) \circ (\bar{i}_\Lambda, \bar{\phi}_\Lambda) : <U_i, [a]_\Lambda, V_1> \longrightarrow \mathbb{R}^{2n}$$

$$\psi_2 = (\lambda_2 \times \gamma_2) \circ (\bar{i}_\Lambda, \bar{\phi}_\Lambda) : <U_2, [b]_\Lambda, V_2> \longrightarrow \mathbb{R}^{2n}$$

$$(\lambda_1 \times \gamma_2) \circ (\bar{i}_\Lambda, \bar{\phi}_\Lambda) : <U_1, [a+b]_\Lambda, V_2> \longrightarrow \mathbb{R}^{2n}$$

are charts in $\pi_\Lambda M$. Hence

$$\psi_1 \times \psi_2 : <U_1, [a]_\Lambda, V_1> \times <U_2, [b]_\Lambda, V_2> \longrightarrow \mathbb{R}^{4n}$$

is a chart in $\pi_\Lambda M \times \pi_\Lambda M$. We must show that the map

$$F = (\lambda_1 \times \gamma_2) \circ (\bar{i}_\Lambda, \bar{\phi}_\Lambda) \circ \bar{\theta}_\Lambda \circ (\psi_1 \times \psi_2)^{-1} :$$

$$\psi_1 \times \psi_2(U \cap D_\Lambda) \longrightarrow \mathbb{R}^{2n},$$ the representative function

of θ, is C^r. $\forall\ ([c]_\Lambda, [d]_\Lambda) \in \bar{D}_\Lambda$, we have:-

$$\psi_1 \times \psi_2([c]_\Lambda, [d]_\Lambda) = (\lambda_1(ic), \gamma_1(\phi c), \lambda_2(id), \gamma_2(\phi d))$$

$$= (\lambda_1^1\,(ic), \ldots\ldots, \lambda_1^n\,(ic), \lambda_1^1\,(\phi c), , \ldots\ldots, \lambda_2^1(id), \ldots\ldots, \gamma_2^1\,(\phi d), \ldots\ldots, \gamma_2^n\,(\phi d))$$

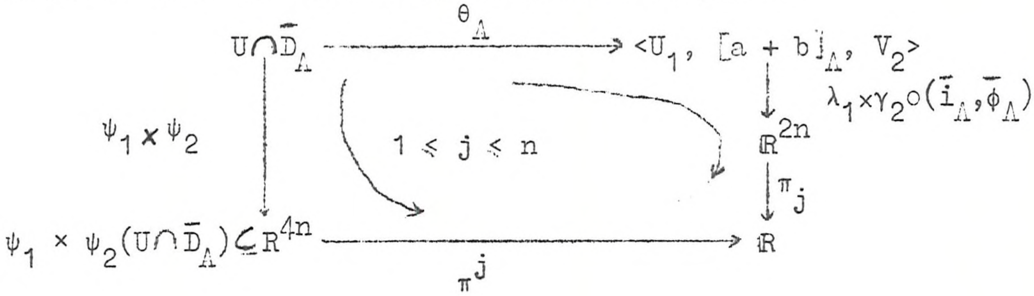
and

$$\lambda_1 \times \gamma_2 \circ (\bar{i}_\Lambda, \bar{\phi}_\Lambda) \circ \bar{\theta}_\Lambda([c]_\Lambda, [d]_\Lambda) = \lambda_1 \times \gamma_2 \circ (\bar{i}_\Lambda, \bar{\phi}_\Lambda)([c+d]_\Lambda)$$

$$= \lambda_1 \times \gamma_2(ic, \phi d)$$

$$= (\lambda_1^1(ic), \ldots\ldots, \lambda^n(ic), \gamma_2^1(\phi d), \ldots\ldots, {}_2^n(\phi d))$$

Hence we have the following commutative diagrams.



$$
\begin{array}{ccc}
U \cap \bar{D}_\Lambda & \xrightarrow{\quad \bar{\theta}_\Lambda \quad} & \langle U_1, [a + b]_\Lambda, V_2 \rangle \\
 & & \Big\downarrow {\scriptstyle \lambda_1 \times \gamma_2 o(\bar{i}_\Lambda, \bar{\phi}_\Lambda)} \\
{\scriptstyle \psi_1 \times \psi_2} \Big\downarrow & n < j \leqslant 2n & \mathbb{R}^{2n} \\
 & & \Big\downarrow {\scriptstyle \pi_j} \\
\psi_1 \times \psi_2(U \cap \bar{D}_\Lambda) \subseteq \mathbb{R}^{4n} & \xrightarrow[\quad \pi^{2n+j} \quad]{} & \mathbb{R}
\end{array}
$$

where π_j, π^j and π^{2n+j} are the obvious projections and hence C^r. Therefore F and hence $\bar{\theta}_\Lambda$ is of class r.

<u>Differentiability of $\bar{u}_\Lambda$</u>

We now show that the unit map $\bar{u}_\Lambda : M \longrightarrow \pi_\Lambda M$ is of class r. Let $x \varepsilon M$, and $U \subseteq M$ be a coordinate neighbourhood of x, then $\langle U, [o_x]_\Lambda, U \rangle$ is a coordinate neighbourhood of $[o_x]_\Lambda = \bar{u}_\Lambda(x)$ in $\pi_\Lambda M$. If $g : U \longrightarrow \mathbb{R}^n$ is a chart in M, then

$$g \times go(\bar{i}_\Lambda, \bar{\phi}_\Lambda) : \langle U \;\; [o_x]_\Lambda, U) \longrightarrow \mathbb{R}^{2n}$$

is a chart in $\pi_\Lambda M$. Then $P = g \times go(\bar{i}_\Lambda, \bar{\phi}_\Lambda) o\bar{u}_\Lambda o g^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^{2n}$ is a representative function for $\bar{u}_\Lambda$. It is easily seen that P is a projection and hence $\bar{u}_\Lambda$ is a C^r map with maximum rank = n.

<u>Differentiability of the inverse map</u>

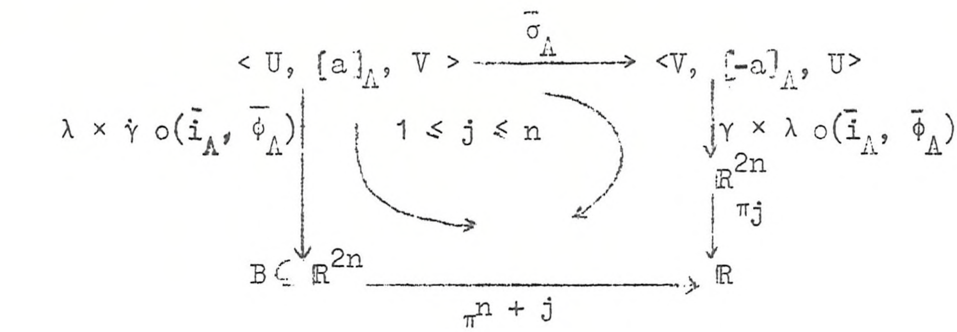
Finally we must show that the inverse map

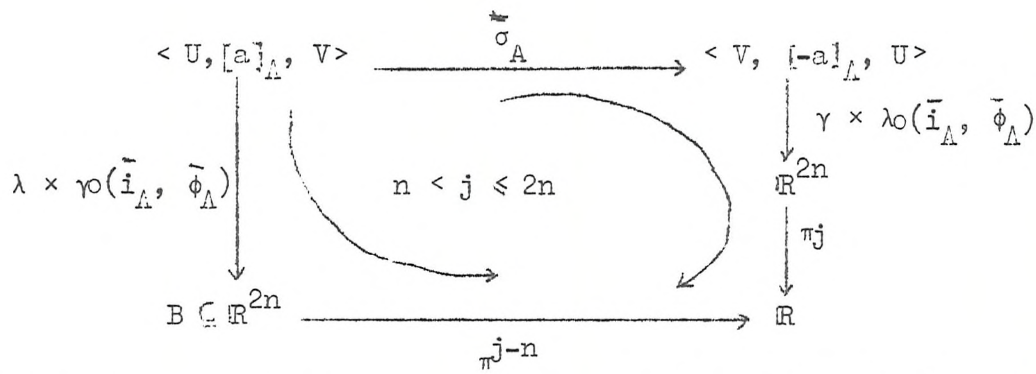
$$\bar{\sigma}_\Lambda : \pi_\Lambda M \longrightarrow \pi_\Lambda M, \; [a]_\Lambda \rightsquigarrow [-a]_\Lambda$$

is differentiable of class r. Let $\langle U, [a]_\Lambda, V \rangle$ be a coordinate neighbourhood of $[a]_\Lambda$, and let

$$\lambda \times \gamma o(\bar{i}_\Lambda, \bar{\phi}_\Lambda) : \langle U, [a]_\Lambda, V \rangle \longrightarrow B \subseteq \mathbb{R}^{2n}$$

be a chart where (λ, U), (γ, V) are coordinate charts in M. Then $\langle V, [-a]_\Lambda, U \rangle$ is a coordinate neighbourhood of $[-a]_\Lambda$ and $\gamma \times \lambda o(\bar{i}_\Lambda, \bar{\phi}_\Lambda) : \langle V, [-a]_\Lambda, U \rangle \longrightarrow \mathbb{R}^{2n}$ is also a chart in $\pi_\Lambda M$. It is easily verified that the following diagrams are commutative:





Therefore it follows that $\pi_j o(\gamma \times \lambda)o(\bar{i}_\Lambda, \bar{\phi}_\Lambda) o \bar{\sigma}_\Lambda o(\bar{i}_\Lambda, \bar{\phi}_\Lambda)^{-1} o(\lambda \times \gamma)^{-1}$ and hence $\bar{\sigma}_\Lambda$ is differentiable of class r. q.e.d.

Theorem 6.1.6:

Let T be a vector bundle over a differentiable n-manifold M of class r, then $\mathcal{G}(T)$, the category of homomorphisms between the fibres, is a Lie category over M.

Proof:

From Chapter III, we know that $\mathcal{G}(T)$ is a topological category over M. Topology of $\mathcal{G}(T)$ was defined by taking each

$\eta_{u,v} : \mathcal{G}(T)(U,V) \longrightarrow U \times V \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ to be a homeomorphism (see 3.1.14) where U, V are any coordinate neighbourhoods in M such that $T|U$ and $T|V$, are homeomorphic to $U \times \mathbb{R}^n$ and $V \times \mathbb{R}^n$, respectively. Since M and Hom $(\mathbb{R}^n, \mathbb{R}^n)$ are manifolds, $U \times V \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is a manifold of dim. $n(n + 2)$. We take each $\eta_{u,v}$ to be a diffeomorphism. Hence each

$\mathcal{G}(T)(U, V)$ will be a coordinate neighbourhood in $\mathcal{G}(T)$, and for any coordinate charts (λ, U), (ν, V), $(\lambda \times \nu \times \xi)o\eta_{u,v}$ is a chart in $\mathcal{G}(T)$, where ξ: Hom$(\mathbb{R}^n, \mathbb{R}^n) \longrightarrow \mathbb{R}^{n^2}$ is the unique chart defined by taking a standard basis in $\mathbb{R}^n$ and identifying each homomorphism with its matrix. We must verify that these charts are C^r related.

Let $\mathcal{G}(T)(U', V')$ be another coordinate neighbourhood such that $\mathcal{G}(T)(U,V) \cap \mathcal{G}(T)(U',V') \neq \phi$, then $(U' \times v') \cap (U \times v)$

$$= (U' \cap U) \times (V \cap V') \neq \phi$$

Let $\chi : \mathcal{G}(T)(U', v') \longrightarrow \mathbb{R}^{n(n+2)}$ be a chart, then $\exists$ charts $\lambda_1 : U_1 \longrightarrow \mathbb{R}^n$ and $\nu_1 : V_1 \longrightarrow \mathbb{R}^n$ such that $\chi = (\lambda_1 \times \nu_1 \times \xi) o \eta_{u_1, v_1}$

We must show that the change of coordinates

$S = (\lambda_1 \times \nu_1 \times \xi) o\eta_{u_1, v_1} o\, \eta^{-1}_{u,v}\, o(\lambda \times \nu \times \xi)^{-1} : A \longrightarrow B$,

where $A = (\lambda \times \nu \times \xi)\ o\eta_{u,v}(\mathcal{G}(T)(U \cap U_1, V \cap V_1))$

$B = (\lambda \times \nu \times \xi)o\eta_{u_1,v_1}(\mathcal{G}(Y)(U \cap U_1, V \cap V_1))$

is class r. Let $x \in A$, then $\exists b \in \mathcal{G}(T)(U \cap U_1, V \cap V_1)$ such that

$$x = (\lambda^1(ib), \ldots, \lambda^n(ib), \nu^1(\phi b), \ldots, \xi^1(b'), \ldots, \xi^{n^2}(b'))$$

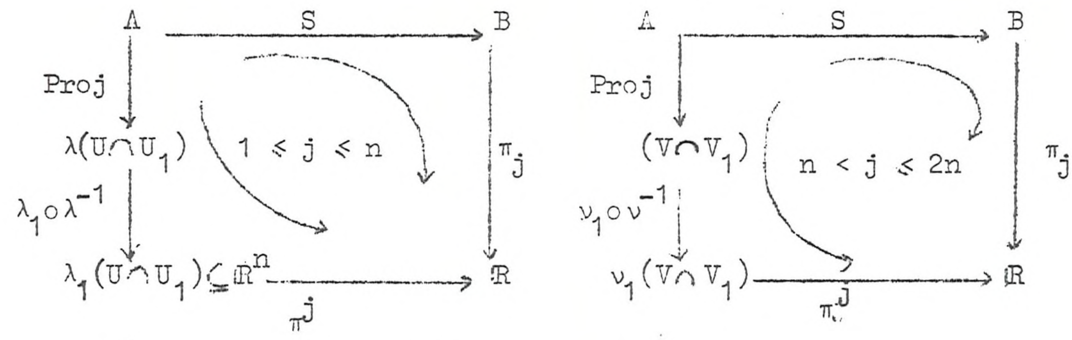
where $b' = \bar{\eta}_{u,v}(b) = \psi^{-1}_U o b o \psi_V$ (see 3.1.14).

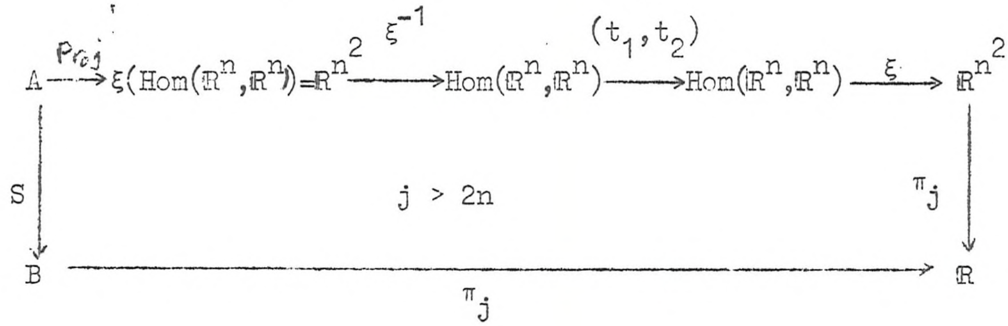
Then $S(x) = (\lambda_1^1(ib), \ldots, \nu_1^1(\phi b), \ldots, \xi^1(b''), \ldots, \xi^{n^2}(b''))$

where $b'' = \psi^{-1}_{U_1} o b o \psi_{V_1}$. Let $t_1 = \psi^{-1}_{U_1} o \psi_U$, $t_2 = \psi^{-1}_{V_1} o \psi_V$, then (t_1, t_2) can be regarded as an element of $GL(n^2, \mathbb{R}) \times GL(n^2, \mathbb{R})$, which acts differentiably of class r on Hom $(\mathbb{R}^n, \mathbb{R}^n)$, by:-

$$(t_1, t_2).C = t^{-1}_1 o c o t_2.$$

To show the differentiability of S, we must prove $\pi_j o s$, $j = 1, \ldots, n(n+2)$ is differentiable where π_j is the j^{th} projection. We have the following commutative diagrams:-





Now C^r differentiability of $So\pi_j$ follows from that of $\lambda_1 o \lambda^{-1}$, $\nu_1 o \nu^{-1}, \xi$ and the projections. Therefore the change of coordinates is differentiable of class r. Now we verify differentiability of the related maps:-

(i) the initial map $i : \mathcal{C}(T) \longrightarrow M$ is differentiable of class r. For, let $g \in \mathcal{C}(T)$, then $\exists\, U, V \in \mathcal{U}$ (as in 3.1.14) such that $g \in \mathcal{C}(T)(U, V)$. If (λ, U), (ν, V) are coordinate charts in M, then $((\lambda \times \nu \times \xi) o \eta_{u,v}, \mathcal{C}(T)(U,V))$ is a coordinate chart in $\mathcal{C}(T)$. We must show that

$$F = \lambda o i o \eta^{-1}_{u,v} o (\lambda \times \nu \times \xi)^{-1} : (\lambda \times \nu \times \xi) o \eta_{u,v}(\mathcal{C}(T)(U,V)) \longrightarrow \mathbb{R}^n,$$

the representative function of i, is differentiable.

We have:-

$$(\lambda \times \nu \times \xi) o \eta_{u,v}(g) = \lambda \times \nu \times \xi(ig, \phi g, g') = (\lambda(ig), \nu(\phi g), \xi(g'))$$
$$= (\lambda^1(ig), \ldots, \lambda^n(ig), \nu^1(\phi g), \ldots, \xi^1(g'), \ldots, \xi^{n^2}(g'))$$

and $\lambda o i(g) = \lambda(ig) = (\lambda^1(ig), \ldots, \lambda^n(ig))$.

Hence, F is the restriction of the projection:

$\mathbb{R}^{n(n+2)} \longrightarrow \mathbb{R}^n$ onto the first factors. Therefore F and hence i is differentiable of class r, and certainly of maximum rank.

(ii) Similarly, $\phi : \mathcal{C}(T) \longrightarrow M$ is differentiable of class r and of maximum rank.

Therefore by 6.1, $D \subseteq \mathcal{C}(T) \times \mathcal{C}(T)$ is a submanifold.

(iii) The composition map $\theta: D \longrightarrow \mathcal{C}(T)$ is differentiable of class r.

Let $(f, g)\varepsilon D$ and let $U, V, W \varepsilon \mathcal{U}$ such that

$$i(f)\varepsilon U, \ \phi(g)\varepsilon V, \ i(g) = \phi(f)\varepsilon W.$$

Then $\mathcal{C}(T)(U,V)$, $\mathcal{C}(T)(U, W)$, $\mathcal{C}(T)(W,V)$ are coordinate neighbourhoods containing gof, f and g, respectively. Hence,

$$\mathcal{V} = (\mathcal{C}(T)(U, V) \times \mathcal{C}(T)(W, V)) \cap D$$

is a coordinate neighbourhood of (f, g) in D such that

$$\theta(\mathcal{V}) \subseteq \mathcal{C}(T)(U, V).$$

Let ψ_1, ψ_2, ψ_3 be charts in M with the domains U, V and W respectively, then

$$\Sigma = (\psi_1 \times \psi_2 \times \xi) o \eta_{u,v}: \mathcal{C}(T)(U, V) \longrightarrow \mathbb{R}^{n(n+2)}$$

$$\chi = (\psi_1 \times \psi_3 \times \xi) o \eta_{u,v} \times (\psi_3 \times \psi_1 \times \xi) o \eta_{w,v}: \mathcal{V} \longrightarrow \mathbb{R}^{2n(n+2)}$$

are charts in $\mathcal{C}(T)$ and D, respectively, and we have:-

for any $(f_1, g_1)\varepsilon \mathcal{V}$,

$$\chi(f_1, g_1) = (\psi_1 \times \psi_3 \times \xi) o \eta_{u,w} \times (\psi_3 \times \psi_1 \times \xi) o \eta_{(w,v)}(f_1, g_1)$$

$$= (\psi_1 \times \psi_3 \times \xi(\eta_{u,w}(f_1)), \psi_3 \times \psi_1 \times \xi(\eta(g_1)_{w,v}))$$

$$= (\psi_1 \times \psi_3 \times \xi(if_1, (f_1), f'_1), \psi_3 \times \psi_2 \times \xi(ig_1, \phi g_1, g'_1))$$

$$= (\psi_1(if_1), \psi_3(\phi f_1), \xi(f'_1), \psi_3(ig\ ), \psi_1(\phi g_1), \xi(g'_1))$$

$= (\psi_1^1(if_1), \ldots, \psi_1^n(if_1), \psi_3^1(\phi f_1), \ldots, \xi^1(f'_1), \ldots,$

$\psi(ig_1), \ldots, \psi_1^1(\phi g_1), \ldots, \xi(g'_1), \ldots)$

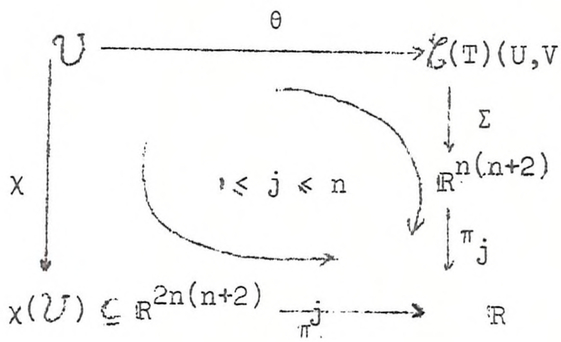
and $\Sigma(g_1 o f_1) =$

$(\psi_1 \times \psi_2 \times \xi)\eta_{u,v}(g\ of\ ) = \psi_1 \times \psi_2 \times \xi(if_1, \phi g_1, g'_1 o f'_1)$ (by property of $\overline{\eta}$, see 3.1.14).

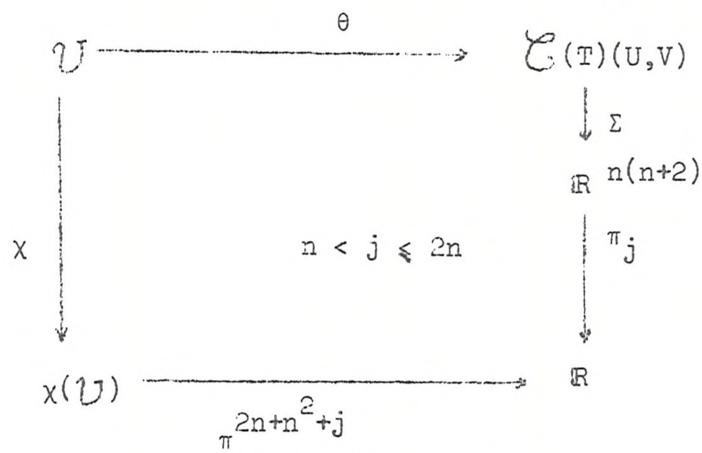
$= (\psi_1^1(if_1), \psi_2^1(\phi g_1), \xi(g'_1 o f'_1))$

$= (\psi_1^1(if\ ), \ldots, \psi_2(\phi g_1), \ldots, \xi^1(g_1'(g' o f'_1), \ldots)$

So, we get the following commutative diagrams:-



(where π_j, π^j are the jth projections).



In these cases $\pi_j o \Sigma o \theta o \chi^{-1}$ is a projection and hence differentiable of class r.

$$\begin{array}{ccc}
\mathcal{V} \xrightarrow{\chi} \mathbb{R}^{2n(n+2)} \xrightarrow{\mu} \mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \xrightarrow{\xi^{-1}\times\xi^{-1}} \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)\times\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n) \xrightarrow{\nu} \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n) & \longrightarrow & \mathbb{R}^{n^2} \\
\Big\downarrow \theta \qquad\qquad 1 \leqslant j \leqslant n^2 & & \Big\downarrow \pi^j \\
\mathcal{C}(T)(U,V) \xrightarrow{\Sigma} \mathbb{R}^{n(n+2)} & \xrightarrow{\pi_{2n+j}} & \mathbb{R}
\end{array}$$

where μ is the projection defined by:-

$\mu(\psi_1^1, \ldots, \psi_3^1, \ldots, \xi^1, \ldots, \psi_3^1, \ldots, \psi_2^1, \ldots, \xi^1, \ldots) =$
$(\xi^1, \ldots, \xi^{n^2}, \xi^1, \ldots, \xi^{n^2})$

and ν is the composition map (i.e. the product of matrices) in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$. Therefore μ and ν are differentiable of class r. Hence $\pi^j o \gamma o \nu o (\xi^{-1} \times \xi^{-1}) o \mu = \pi_{2n+j} o \Sigma o \theta o \chi^{-1}$ is differentiable of class r. Hence θ is differentiable of class r.

(iv) Finally, we must show that the unit map $u : M \longrightarrow \mathcal{C}(T)$

$$m \rightsquigarrow id_{T_m}$$

is differentiable of class r and is of maximum rank.

Let (ψ, U) be any coordinate chart in M such that $m \varepsilon U$, then $(\psi \times \psi \times \xi o \eta_{U,U}, \mathcal{C}(T)(U,U))$ is a coordinate chart in $\mathcal{C}(T)$ such that $u(U) \subseteq \mathcal{C}(T)(U,U)$. Now the differentiability of u and that it is of maximum rank follows from the same type of argument as the other maps.

<u>Corollary 6.7:</u>

<u>In every vector bundle T over a differentiable n-manifold M,</u>
<u>$\mathcal{G}(T) = \bigcup_{n,m \in M} \mathrm{Iso}(T_m, T_n)$ is a Lie groupoid over M.</u>

Proof:

We first show that $\mathcal{G}(T)$ with the relative topology is an open submanifold of $\mathcal{C}(T)$. For each U, V $\in \mathcal{U}$, $\mathcal{G}(T)(U,V)$
$= \eta^{-1}_{U,V}(U \times V \times \mathrm{Iso}(\mathbb{R}^n, \mathbb{R}^n))$.
But $\mathrm{Iso}(\mathbb{R}^n, \mathbb{R}^n)$ is open in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, therefore $\mathcal{G}(T)(U,V)$ is open in $\mathcal{C}(T)(U,V)$ and hence in $\mathcal{C}(T)$.
Thus $\mathcal{G}(T)$ is open in $\mathcal{C}(T)$. So, each $\mathcal{G}(T)(U,V)$, $U,V \in \mathcal{U}$ is a coordinate neighbourhood in $\mathcal{G}(T)$.
Therefore the differentiability of i, ϕ , u, θ , follows from that in $\mathcal{C}(T)$. Therefore we only need to verify the differentiability of

$$\sigma : \mathcal{G}(T) \longrightarrow \mathcal{G}(T)$$
$$f \rightsquigarrow f^{-1}$$

Let $\mathcal{G}(T)(U,V)$ be a coordinate neighbourhood of f, then $\mathcal{G}(T)(V,U)$ is a coordinate neighbourhood of f^{-1} such that $\sigma(\mathcal{G}(T)(U,V)) = \mathcal{G}(T)(V,U)$.
Let ψ and ν be charts in M with domains U and V, respectively, then $\psi \times \nu \times \xi \circ \eta'_{u,v}$ and $\nu \times \psi \times \bar{\xi} \circ \eta'_{v,u}$ are charts in $\mathcal{G}(T)$.
Now the differentiability of s follows by using the same method as in 6.1.5. and take into consideration the fact that $\mathrm{Iso}(\mathbb{R}^n, \mathbb{R}^n)$ is a Lie group.

q.e.d.

Definition 6.1.8.

A Lie groupoid G over M of class r is called locally trivial, if for any $m \in M$, $\exists$ a neighbourhood U and a C^r map $\lambda : U \longrightarrow G$ such that $\forall\, x \in U$, $\quad i(\lambda(x)) = x$ and $\phi(\lambda(x)) = m$

Lemma 6.1.9:

Let M be a differentiable manifold, then for each connected wide normal subgroupoid A of πM, $\pi_A M$ is locally trivial.

Proof:

As we have seen $\pi_A M$ is a locally trivial topological groupoid. Recall that the local lifts were defined as follows:-

Let $m \varepsilon M$, and let U be a simply connected neighbourhood of m, then $\forall\ x\ \varepsilon U$, $\lambda(x) = [a]_A$, where $a \supseteq PU(x, m)$.

Now let (ψ, U) be a coordinate chart in M, then $<U, [a]_A, U>$ is a coordinate neighbourhood in $\pi_A M$, such that $\lambda(U) \subseteq <U, [a]_A, U>$ and $\psi \times \psi\ o(\bar{i}_A, \bar{\phi}_A) : <U, [a]_A, U> \longrightarrow \mathbb{R}^{2n}$ $(n = \dim M)$ is a chart in $\pi_A M$. To show the differentiability of λ it suffices to show the differentiability of $\psi \times \psi\ o(\bar{i}_A, \bar{\phi}_A) o \lambda o \psi^{-1} : \psi(U) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{2n}$. But this easily follows using the standard arguments of earlier proofs.

q.e.d.

Lemma 6.1.10:

$\mathcal{G}(T)$ (as in 6.6) is a locally trivial Lie groupoid over M.

Proof:

Let $m \varepsilon M$, and let $U \varepsilon \mathcal{U}$ be a coordinate neighbourhood of m, then define $\lambda : U \longrightarrow \mathcal{G}(T)$ by:-

$\forall\ x \varepsilon M$, let $\lambda(x) = \eta'^{-1}_{u,u}(x, m, id_{\mathbb{R}^n})$,

where $\eta'_{u,u} = \eta_{u,u} | \mathcal{G}(T)(U,U)$.

Since $\eta'_{u,u}$ is a diffeomorphism, C^r differentiability of λ easily follows.

q.e.d.

SECTION 2

All the results about topological groupoids are true for Lie groupoids, if we change the hypotheses from 'continuous' to 'differentiable' in the obvious way. We confine ourselves to two examples.

6.2.1:

Every connected locally trivial Lie groupoid over M is a coordinate bundle over M × M.

A more complicated example occurs with transformation groups. Thus, definition of a Lie $\mathcal{G}$d-transformation group and covering morphism just as in the topological case, but replacing continuity by differentiability. Then we have the following theorem and corollary:-

Theorem 6.2.2:

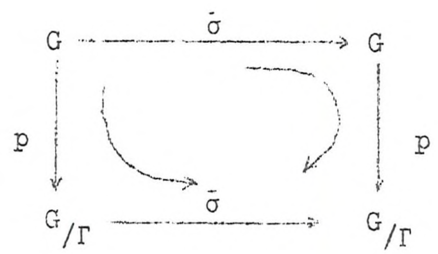
If (Γ, G) is a properly discontinuous Lie $\mathcal{G}$d-transformation group, and Γ acts freely on G, then the orbit manifold $G/_\Gamma$ is a Lie groupoid over $M/_\Gamma$. ($M = G^{ob}$).

Proof:

The fact that $G/_\Gamma$ and $X/_\Gamma$ are quotient manifolds follows from the theory of manifolds (see e.g. [7] p. 44). So, we need only prove the differentiability of the maps. But these follow from the commutativity of diagrams similar to those used in 5.3.4.

For example differentiability of the inverse map $\sigma : G/_\Gamma \longrightarrow G/_\Gamma$ is proved as follows:-

We have the following commutative diagram:-



Let $\Gamma.g = \bar{g}\varepsilon G/_\Gamma$, and let $(V, \bar{\psi})$ be a coordinate chart in $G/_\Gamma$, such that $\bar{g} \varepsilon V$. Then, by definition of the structure of $G/_{\Gamma}$, $\bar{\psi} = \psi o(p|U)^{-1}$, where U is a connected component of $p^{-1}(V)$, containing g, and (U, ψ) is a coordinate chart in G.

Let $(W, \bar{\gamma})$ be a coordinate chart in $G/_{\Gamma}$ such that $\sigma(V) \subseteq W$ and $\bar{\gamma} = \gamma o(p|U')^{-1}$, where (U', γ) is a coordinate chart in G, with $-g \varepsilon U'$. Since σ is differentiable $\gamma o \sigma o \psi^{-1}$ is differentiable.

To show the $\bar{\sigma}$ is differentiable, we must show that the representative function $\bar{\gamma} o \bar{\sigma} o \bar{\psi}^{-1}$ is differentiable.

But $\bar{\gamma} o \bar{\sigma} o \bar{\psi}^{-1} = \gamma o(p|U')^{-1} o \bar{\sigma} o(P|\ U) o\ \psi^{-1}$

$= \gamma o(p|U')^{-1} o(p|\ U') o \sigma o \psi^{-1}$ (by commutativity of diagram)

$= \gamma o \sigma o \psi^{-1}$

Hence $\bar{\gamma}\ o\ \bar{\sigma}\ o\ \bar{\sigma}^{-1}$ is differentiable. q.e.d.

<u>Corollary 6.9:</u>

<u>If (Γ, G) is a properly discontinuous Lie $\mathcal{G}$d-transformation group, and Γ acts freely on G, then $p : G \longrightarrow G/_{\Gamma}$ is a Lie covering morphism.</u>

It should be clear that we now have the foundations of a 'Lie' theory of groupoids, and further research is necessary to develop the subject along the lines of the theory of Lie groups, as indicated by Westman [12].

END.

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