UNIVERSITY OF SOUTHAMPTON

DISCRETE ABELIAN SYMMETRIES IN LATTICE GAUGE THEORY

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ABSTRACT

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DISCRETE ABELIAN SYMMETRIES IN LATTICE GAUGE THEORY

by Paul Purdon Martin

Wilson's proposed lattice approximation to Quantum Chromodynamics is reviewed, including a discussion of the approximate non-perturbative Monte Carlo method of calculation. The Transfer Matrix formulation of lattice models is discussed. This approach is used to confirm the exponential decay of the plaquette-plaquette correlation function at large distances in the Z(2) gauge model in three dimensions. Perturbative expressions are obtained for the inverse correlation length in both strong and weak coupling.

By raising the transfer matrix to a finite power the partition function for a finite lattice Z(2) gauge model is obtained as an exact polynomial in functions of the coupling constant. The zeros of this polynomial are found and some plaquette-plaquette expectation values are extracted to test the applicability of an exponential fit for the inverse correlation length at short distances.

Similar calculations for the three dimensional Ising model are discussed in an appendix.

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1. Introduction to Lattice Gauge Theory

We review aspects of Wilson's Lattice Gauge Theory [1 - 7]. In particular we discuss how physical observables might be extracted from a lattice model of Quantum Chromodynamics. We discuss some of the problems arising in practical lattice calculations with particular reference to some simpler problems.

1.1 Quantum Chromodynamics

The motivation behind Wilson's proposed lattice gauge theory has been the need for a non-perturbative method of calculation in Quantum Chromodynamics. Here we will restrict consideration to pure Yang-Mills fields in Euclidean space. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} \sum_{a = 1, 8} \sum_{\mu, \nu = 1, 4} F^a_{\mu\nu} F^a_{\mu\nu} \tag{1}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g \sum_{b, c} f^{abc} A^b_\mu A^c_\nu \, . \tag{2}$$

The f^{abc} are structure constants of SU(3)

$$\left[\lambda^a, \lambda^b \right] = 2i \sum_{c = 1}^{8} f^{abc} \lambda^c \tag{3}$$

where the λ's are Gell-Mann matrices [8]. The gauge fields $A^a_\mu(x)$ are vector fields carrying an index of the adjoint representation of SU(3). We construct an element of the Lie algebra

$$A_\mu(x) = \sum_{a = 1, 8} \frac{\lambda^a}{2} A^a_\mu(x) \tag{4}$$

The Lagrangian is invariant under the following local gauge transformation

$$A_\mu(x) \to {}^g A_\mu(x) = \frac{i}{g} \left[\partial_\mu g(x) \right] g^{-1}(x) + g(x) A_\mu(x) g^{-1}(x) \tag{5}$$

where g(x) is an element of SU(3).

With the gauge invariant addition of fermion fields carrying an index of the fundamental representation of SU(3) [9] this Lagrangian is hoped to give the (Euclidean) quantum field theory of strong interactions [10 - 12].

The Feynman path integral formulation [13, 14] of this theory becomes amenable to perturbative calculations upon addition to the Lagrangian of gauge fixing and ghost terms [15, 16]. In order to make predictions of physical quantities, however, the theory must be renormalised. This is because Euclidean integrals of the form

$$I = \int \frac{d^4K}{\left[K^2 + A^2 \right]^n} \tag{6}$$

appear in coefficients of the perturbation expansion [9]. These integrals are divergent for $n \leqslant 2$. The divergences which occur in the perturbation series may be systematically cancelled by renormalisations of the field and coupling constant appearing in $\mathcal{L}$ (eqn (1)) to leave finite coefficients [9 - 11]. Systematic treatment of the divergences is usually implemented by modifying the integrals, for example with $n = 2$

$$I = \lim_{\varepsilon \to 0} I_\varepsilon \quad \text{where} \quad I_\varepsilon = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty \frac{K^{D-1}\, dK}{\left[K^2 + A^2 \right]^2} \quad \text{and} \quad D = 4 - \varepsilon$$

or

$$I = \lim_{\Lambda \to \infty} I_\Lambda \quad \text{with} \quad I_{\Lambda_c} = 2\pi^2 \int_0^{\Lambda_c} \frac{K^3\, dK}{\left[K^2 + A^2 \right]^2} \tag{7}$$

These latter integrals are finite. Divergent parts of the limiting integrals manifest themselves as terms in $1/\varepsilon$ and $\ell n\ (\Lambda_c)$ respectively. These terms may be manipulated and cancelled, at least formally, in the perturbation series.

This is not to say that the perturbation series will converge, but we will see later that there are conditions under which the renormalised coupling becomes small. Then we might hope that the leading terms in

the series are significant*. However strong interactions have properties which are not manifest in the theory in this perturbative region [17]. With the above remarks about renormalisation in mind we proceed to consider Wilson's lattice approach [1].

1.2 Lattice Gauge Theory

Wilson proposed a lattice action which, for 'smooth' fields (see later) reduces to the action of QCD as the lattice spacing is set to zero. We will consider the pure gauge part (for proposals concerning the inclusion of fermionic fields see for instance [2, 3, 5, 18]). The gauge fields are thought of as being on the links of a hypercubical Euclidean lattice. Links are specified by a lattice site and an assigned 'forward' lattice direction out of the site.

$$S(g, a) = \frac{1}{2g^2} \sum_{\substack{\text{sites} \\ n}} \sum_{\substack{\mu = 1, 2, 3, 4 \\ \nu = 1, 2, 3, 4 \\ \neq \mu}} \mathrm{Tr}_{\substack{SU(3) \\ \text{matrices}}} \left[U_\mu(n)\, U_\nu(n + a(\mu))\, U_\mu^{-1}(n + a(\nu))\, U_\nu^{-1}(n) \right] \tag{8}$$

where $a(\mu)$ steps one lattice spacing in the μ direction and

$$U_\mu(n) = \exp \left[i\, a\, g \sum_{b=1}^{8} \frac{\lambda^b}{2} A_\mu^b(n) \right] \tag{9}$$

The Greek indices labelling lattice directions are a notational convention; they are not to be regarded as Lorentz-type indices; there is no Euclidean invariance and we will not attempt to make these indices match in equations.

We assume that we may Taylor expand the fields (see later), firstly

$$A_\nu(n + a(\mu)) = A_\nu(n) + a\frac{(A_\nu(n + a(\mu)) - A_\nu(n))}{a} \tag{10}$$

* Some phenomenological support is available on this basis. We will not discuss the point here (see [10 - 12]).

Anticipating the continuum limit we write

$$A_\nu(n + a(\mu)) = A_\nu(n) + a\, \partial_\mu A_\nu(n) + O(a^2) \tag{11}$$

and rewrite the product of U's in the action:

$$U_\mu(n)\, U_\nu(n + a(\mu))\, U_\mu^{-1}(n + a(\nu))\, U_\nu^{-1}(n) =$$

$$\exp\left[iag\, A_\mu(n) \right] \exp\left[iag\, (A_\nu(n) + a\, \partial_\mu A_\nu(n) + O(a^2)) \right]$$

$$\exp\left[-iag\, (A_\mu(n) + a\, \partial_\nu A_\mu(n) + O(a^2)) \right] \exp\left[-iag\, A_\nu(x) \right] \tag{12}$$

where we have used

$$A_\mu(n) = \sum_b \frac{\lambda^b}{2} A_\mu^b(n) \tag{13}$$

The Baker-Campbell-Hausdorff formula [19] gives

$$e^{ax}\, e^{ay} = e^{\sigma}$$

with

$$\sigma = ax + ay + \frac{a^2}{2}\left[x, y \right] + \frac{a^3}{12}\left(\left[x, \left[x, y \right] \right] - \left[y, \left[x, y \right] \right]\right)$$

$$- \frac{a^4}{24}\left[x, \left[y, \left[x, y \right] \right] \right] + O(a^5) \tag{14}$$

Repeated applications of eqn (14) will get (12) into the form of a single exponential

$$\exp\left\{ ia^2 g\, (\partial_\mu A_\nu(n) - \partial_\nu A_\mu(n) + ig\left[A_\mu(n), A_\nu(n) \right]) + O(a^3) \right\}$$

$$= \exp\left\{ ia^2 g \sum_d (\partial_\mu A_\nu^d(n) - \partial_\nu A_\mu^d(n) - g f^{dbc} A_\mu^b(n) A_\nu^c(n)) \frac{\lambda^d}{2} + O(a^3) \right\} \tag{15}$$

Altogether we have

$$S(g, a) = \frac{1}{2g^2} \sum_{n} \sum_{\mu, \nu} \mathrm{Tr} \exp \left(ia^2 g \sum_{a} F^a_{\mu\nu}(n) \frac{\lambda^a}{2} + O(a^3) \right) \qquad (16)$$

Equation (14) shows that the order a^3 terms in the exponential are traceless and we may write

$$S(g, a) = \frac{1}{2g^2} \sum_{n} \sum_{\mu, \nu} \left[3 - a^4 g^2 \left(\sum_{a} \sum_{b} \mathrm{Tr} \left(\frac{\lambda^a}{2} \frac{\lambda^b}{2} \right) F^a_{\mu\nu}(n) \, F^b_{\mu\nu}(n) \right) + O(a^5) \right] \qquad (17)$$

Before taking $a \to 0$ it is instructive to consider the infinite volume lattice Fourier transform, which should reduce to an ordinary Fourier transform as $a \to 0$. It will be sufficient to consider the one dimensional case (although see later). Since the sites of the lattice may be labelled by integers the momentum space is manifestly periodic

$$\phi(n) = c \int_{-\pi}^{\pi} \frac{dp}{(2\pi)} e^{ipn} \tilde{\phi}(p) \qquad (18)$$

since

$$\tilde{\phi}(p) = \frac{1}{c} \sum_{n = -\infty}^{\infty} e^{-ipn} \phi(n) \qquad (19)$$

with c an arbitrary constant. We introduce physical distances and momenta by specifying a lattice spacing a, whereupon

$$x_n = na$$

and

$$k = p/a \qquad (20)$$

so that

$$\phi(x) = ca \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} e^{ikx} \tilde{\phi}(k) \qquad (21)$$

Notice that introducing the lattice has had the effect of cutting off the momentum integral at $\Lambda_c = \pi/a$. Notice, however, that in higher dimensions this is not the same cut-off as shown in eqn (7). In four dimensions, for example, the lattice cut-off generalises to

$$\int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} \int_{-\pi/a}^{\pi/a} \frac{dk_1\, dk_2\, dk_3\, dk_4}{(2\pi)^4}$$

whereas in eqn (7) the cut-off is not introduced until after the angular integrations have been performed.

Returning to eqn (21) and putting $c = a^{-1}$, as $a \to 0$ we obtain

$$\phi(x) \to \int_{-\infty}^{\infty} \frac{dk}{(2\pi)} e^{ikx} \tilde{\phi}(k) \tag{22}$$

which is the usual definition. Also

$$\tilde{\phi}(k) = a \sum_{\frac{x}{a} = -\infty}^{\infty} e^{-ikx} \phi(x) \quad \to \quad \int_{-\infty}^{\infty} e^{-ikx} \phi(x)\, dx \qquad a \to 0 \tag{23}$$

provided

$$\lim_{a \to 0} a \sum_{x} \to \int_{-\infty}^{\infty} dx \tag{24}$$

Generalising (24) to four dimensions and dropping the constant term in (17) we obtain

$$S(g, 0) = \int d^4x \left(-\frac{1}{4} \sum_{\mu, \nu} \sum_{a} F^a_{\mu\nu} F^a_{\mu\nu} \right) \tag{25}$$

Finally, notice that the lattice action (8) is invariant under the following local gauge transformation

$$U_\mu(n) \to {}^{g}U_\mu(n) = g(n)\, U_\mu(n)\, g^{-1}(n + a(\mu)) \tag{26}$$

where g(n) is an element of SU(3) on the site n. Following the line of argument above we write

$$\exp(iag A_\mu(n)) \to g(n) \exp(iag A_\mu(n)) \left[g^{-1}(n) + a\partial_\mu g^{-1}(n) + O(a^2) \right] \tag{27}$$

then equating coefficients of order a we reproduce the continuum transformation of equation (5).

Unfortunately equation (11) is only valid for smoothly varying fields [2, 7]. Consider a fluctuation in A^a_μ of magnitude $\tilde{A}$ confined to a region of size a. For arbitrarily small a we can see the contribution to the action due to such a fluctuation [2]. $F^a_{\mu\nu}$ is locally dominated by its derivative part so $(F^a_{\mu\nu})^2 \sim (\tilde{A}/a)^2$. However this fluctuation only extends over a volume a^4. The contribution to the action is of order $\tilde{A}^2 a^2$ at most. In Euclidean quantum field theory fluctuations will only be damped when the action becomes large (and negative). Presumably then, we should take into account fluctuations $\tilde{A} \propto a^{-n}$ where $n \leqslant 1$. Under these conditions it does not seem justified to make the expansion in equation (11).

We see that the formal limit $a \to 0$ is insufficient to define a continuum quantum theory.

One way to proceed [17] would be to measure physical observables in the lattice system and find conditions under which they remain fixed as the lattice spacing is made arbitrarily small (for a given lattice size this means that observables with dimensions of length - correlation lengths - become very large compared with the spacing so that the granular effect is lost). Under these conditions Creutz [20] has tried to identify the lattice model with a regularised version of QCD. Before reviewing this approach we will try and see the attractive feature of Wilson's model which makes a connection with QCD desirable.

1.3 Strong coupling behaviour

We may write expectation values for lattice observables in Euclidean Space:

$$< \tilde{O} > = Z^{-1} \int DU \, \tilde{O} \exp \left[S(g, a) \right] \tag{28}$$

where

$$Z = \int DU \exp \left[S(g, a) \right] \tag{29}$$

is the partition function for the lattice model and DU means integration over all field configurations (that is $DU = \prod_{\text{links}} dU$ where dU is the

invariant group measure [16, 21]). When g^2 is large it is advantageous to expand the exponentiated action over characters (traces) of the irreducible representations of SU(3);

$$\exp\left[\frac{1}{2g^2}\sum_{n,\mu,\nu} \mathrm{Tr}\,(UUUU)\right]$$

$$= \prod_{n,\mu,\nu}\left(f_o\left(\frac{1}{2g^2}\right) + \sum_{r\neq 0} f_r\left(\frac{1}{2g^2}\right)\chi^r(UUUU)\right) \tag{30}$$

where

$$f_r\left(\frac{1}{2g^2}\right) = \int dU\, \chi^r(U^{-1})\exp\left(\left(\frac{1}{2g^2}\right)\mathrm{Tr}(U)\right),\ r \geq 0 \tag{31}$$

We put

$$\int d\,U = 1 \tag{32}$$

then

$$\int d\,U\ \chi^r(U^{-1})\,\chi^s(U) = \delta_{rs} \tag{33}$$

Thus f_o is independent of g^2 and the next dominant coefficient at large g^2 is $f_1(\frac{1}{2g^2}) = O(\frac{1}{g^2})$ where r = 1 gives the defining representation.

Consider for instance the observable

$$\tilde{O} = \mathrm{Tr}\left(\prod_{\substack{T\,x\,R\\ \text{planar}\\ \text{loop}}} U\right) \tag{34}$$

applying (33) to (30), the lowest non vanishing term in the expansion in $\frac{1}{g^2}$ of eqn (28) gives

$$\langle \tilde{O} \rangle \propto \left(\frac{1}{g^2}\right)^{TR} + \text{higher orders} \tag{35}$$

We have not attempted to write down a lattice version of the fermionic part of the action, therefore we shall not obtain expectation values for observables involving quarks. However Wilson [1] has interpreted the expectation value in eqn (35) as

$$< \tilde{O} > = \frac{Z[\,J\,]}{Z[\,0\,]} \tag{36}$$

where $Z[\,J\,]$ is the partition function modified by the introduction of an external current loop around the planar loop T x R (see also Kogut [4], Itzykson and Zuber [9]). Putting T >> R and taking slices through the loop in the long direction (we have a Euclidean lattice but we could think of this as the time direction) we see 'static' charges at a fixed separation R. We will see in the next chapter that

$$\frac{Z[\,J\,]}{Z[\,0\,]} \underset{T \to \infty}{\approx} \exp \{ (E(J) - E(0))T \} \tag{37}$$

where $E[\,J\,]$ is the lowest lying energy level for the system with static charges*. The energy difference is attributed to a potential between the charges and using (35) we obtain

$$V(R) \propto R \tag{38}$$

If this picture could be extended to include quarks then it would imply confinement (see [7] and references therein). This is perhaps the major reason for trying to establish a connection between the strong coupling lattice model and asymptotically free perturbative QCD.

1.4 The Renormalisation Group

The applications of the renormalisation group to perturbative QCD and to statistical mechanics have been extensively reviewed [10, 11, 14, 23, 24]. We will highlight some specific points.

The approach pioneered by Creutz and others [20, 25] has been to try and identify the known asymptotic behaviour of QCD in the intermediate/weak coupling region of the lattice model. The lattice is treated as a means of regularisation. It is instructive to start by comparing different methods of regularising divergences which appear in the continuum perturbation expansions.

* Strictly speaking we will consider an equivalent case for arbitrarily large finite lattices and arbitrarily large finite groups - see Camp and Fisher [22]. See also Refs [4] and [23] for the relationship between transfer matrix and Hamiltonian formalisms.

Consider, for example, the following integrals

$$I_\Lambda = \int_o^{\Lambda_c} \frac{k^3\, dk}{\left[\, k^2 + A^2 \,\right]^2} \;,\; I_\varepsilon = \int_o^{\infty} \frac{k^{D-1}\, dk}{\left[\, k^2 + A^2 \,\right]^2}$$

$$D = 4 - \varepsilon \tag{39}$$

$$I_\Lambda = \tfrac{1}{2} \left[\, \ell n \, (1 + (\frac{\Lambda_c^2}{A^2})) + \frac{1}{1 + (\frac{\Lambda_c^2}{A^2})} - 1 \,\right] \tag{40}$$

$$I_\varepsilon = \tfrac{1}{2} \left[\, \frac{1}{(A^2)^{\varepsilon/2}} \, \Gamma(2 - {}^{\varepsilon}/_{2}) \, \Gamma({}^{\varepsilon}/_{2}) \,\right] \tag{41}$$

for large Λ_c and small ε we may expand the logarithm and gamma functions to obtain

$$I_{\Lambda_c} = \tfrac{1}{2} \left[\, \ell n \, \frac{\Lambda_c^2}{A^2} + \text{constants} + O(\frac{1}{\Lambda_c^2}) \,\right] \tag{42}$$

$$I_\varepsilon = \tfrac{1}{2} \left[\, \frac{2}{\varepsilon} - \ell n \, A^2 + \text{constants} + O(\varepsilon) \,\right] \tag{43}$$

Notice that coefficients of $\ell n \, \Lambda_c^2$ and ${}^{2}/_{\varepsilon}$ are equal and that coefficients of $\ell n \, A^2$ are equal in the limits. If Λ_c, k and A are dimensionful quantities then I_{Λ_c} and $(\bar{\mu}^2)^{3/2} \, I_\varepsilon$ are dimensionless provided $\bar{\mu}$ has the same dimensions as Λ_c, k and A.

$\bar{\mu}$ is an arbitrary dimensionful parameter, we will also use it to separate off the divergence in I_{Λ_c}

$$(\bar{\mu}^2)^{\varepsilon/2} \, I_\varepsilon = \tfrac{1}{2} \left[\, \frac{2}{\varepsilon} - \ell n \, \frac{A^2}{\bar{\mu}^2} + \text{constants} + O(\varepsilon) \,\right] \tag{44}$$

$$I_{\Lambda_c} = \tfrac{1}{2} \left[\, \ell n \, \frac{\Lambda_c^2}{\bar{\mu}^2} - \ell n \, \frac{A^2}{\bar{\mu}^2} + \text{constants} + O(\frac{1}{\Lambda_c^2}) \,\right] \tag{45}$$

We may now consider the appearance of such integrals in the perturbative calculation of truncated vertex functions by Feynman graphs [10]. We

will write the unrenormalised truncated n-point vertex function obtained from the Feynman rules for QCD (see for instance Itzykson and Zuber [9]) as

$$\Gamma_u^{(n)}(P_i, \overset{\Lambda_c}{\varepsilon}, g) = S^{(n)}(P_i, g)\ (1 + f^{(n)}(P_i, \overset{\Lambda_c}{\varepsilon}, g)) \tag{46}$$

where $i = 1, \ldots, n$ and $\sum_i P_i = 0$. The P_i are the momenta on the external legs. Γ_u contains loop integrals which are only finite by virtue of finite ε or Λ_c (the integrals are well represented by those discussed above, see [9, 10, 26]), however we may construct

$$\Gamma_R^{(n)}(P_i, \mu, g_R) = Z_A^{n/2}\ \Gamma_u^{(n)} \tag{47}$$

where Γ_R is finite as $\Lambda_c \to \infty$ ($\varepsilon \to 0$) by rewriting the Lagrangian (1) in terms of renormalised fields and coupling constant. To see this consider the following procedure.

Neglecting gauge fixing and ghost terms (see Itzykson and Zuber [9] for details) we write (with repeated index summation)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4} Z_A\ (\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a) \\ &+ \frac{g_R}{2} Z_1\ (\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a)\ f^{abc}\ A_\mu{}^b A_\nu{}^c \\ &- \frac{g_R^2}{4} Z_4\ f^{abc}\ A_\mu{}^b A_\nu{}^c\ f^{abc}\ A_\mu{}^b A_\nu{}^c\end{aligned} \tag{48}$$

i.e. the unrenormalised quantities are rewritten (symbolically)

$$A_o = Z_A^{1/2}\ A,\quad g = Z_1\ Z_A^{-3/2}\ g_R \quad \text{etc} \tag{49}$$

We can see that we may rewrite the Feynman rules (and hence vertex functions) in terms of the renormalised quantities by introducing some multiplicative factors. The Z's are chosen so that these factors formally cancel the divergences in the vertex functions order by order [27]. Within this general requirement there is considerable freedom in the choice of renormalisation. Consider the two point function, which is well represented by

$$\Gamma_u^{(2)}\ (p,\ \varepsilon,\ g) \sim S^{(2)}\ (1 + Cg^2\ I_{\substack{\Lambda_c \\ \varepsilon}}\ (A^2 \propto p^2) + O(g^3)) \qquad (50)$$

where C is some (gauge dependent) quantity. Introducing

$$Z_A = 1 - \frac{Cg^2}{2}\ \ell n\ \frac{\Lambda_c^2}{\bar{\mu}^2} + O(g^3) \qquad (51)$$

removes the divergence at this order (we could equally well have chosen

$$\bar{Z}_A = 1 - Cg^2\ I_{\substack{\Lambda_c \\ \varepsilon}}\ (A^2 = \mu^2) + O(g^3) \qquad (52)$$

where μ is any arbitrary momentum value - see later).

Z_1 may be similarly calculated from the divergent parts of the three gluon vertex $\Gamma^{(3)}$ whereupon equation (49) may be written in the form

$$g = g_R\ (1 + \frac{\beta_o}{2}\ g_R^2\ \ell n\ \frac{\Lambda_c^2}{\bar{\mu}^2} + ..) \qquad (53)$$

where β_o is a constant.

The Λ_c dependence resides in g, while the $\bar{\mu}$ dependence is in g_R. Now putting $\Lambda_c = \bar{\mu}$ we find

$$g(\bar{\mu}) = g_R\ (\bar{\mu}) \qquad (54)$$

so that

$$g(\Lambda_c) = g(\bar{\mu}) + \frac{\beta_o}{2}\ g^3(\bar{\mu})\ \ell n\ \frac{\Lambda_c^2}{\bar{\mu}^2} + .. \qquad (55)$$

Also $\bar{\mu}$ is arbitrary so we may choose it very close to Λ_c. We write

$$g(\Lambda_c) \approx g(\bar{\mu}) + \delta\Lambda\ \frac{dg(\bar{\mu})}{d\bar{\mu}} \qquad (56)$$

with $\delta\Lambda = \Lambda_c - \bar{\mu}$. Since $\ell n\ (1 + \frac{\delta\Lambda}{\bar{\mu}}) \approx \frac{\delta\Lambda}{\bar{\mu}}$ we have

$$\bar{\mu}\ \frac{dg(\bar{\mu})}{d\bar{\mu}} = \beta_o\ g^3(\bar{\mu}) + \ldots \tag{57}$$

In fact β_o is negative [28] so g gets smaller as the cut-off gets bigger (for small g). We may obtain an approximate expression for the Λ_c dependence of $g(\Lambda_c)$:

$$\int_{g(\mu)}^{g(\Lambda_c)} \frac{dg}{g^3} \approx \beta_o \int_{\mu}^{\Lambda_c} \frac{d\bar{\mu}}{\bar{\mu}} \tag{58}$$

so

$$g^2(\Lambda_c) \sim \frac{1}{-2\beta_o\ \ell n\ (\frac{\Lambda_c}{\Lambda})} \tag{59}$$

where

$$\Lambda = \mu\ \exp\ (\frac{1}{2\beta_o\ g^2(\mu)}) \tag{60}$$

This result is modified by the inclusion of $O(g^5)$ terms in equation (53) but still $g(\Lambda_c) \to 0$ as $\Lambda_c \to \infty$. A few points are worth making. Firstly a different prescription for obtaining Z_A (such as equation (52)) changes equation (53) at $O(g^3)$. Equation (54) is then altered at $O(g^3)$ so that (at this order) equation (55) is unchanged.

Secondly, imagine that it were possible to measure a 'physical' observable (such as the string tension of section 1.3) as a function of the bare coupling in lattice gauge theory. If the claim that the lattice model is just QCD regulated by the lattice spacing is correct, then we now know what this coupling dependence must be. The dimensionless string tension measured on the lattice is $\sigma\ a^2$. We require that all the cut-off dependence comes from that implicit in g, and none from the physical quantity itself. This implies

$$\sigma\, a^2 = \frac{\sigma}{\left(\frac{\Lambda}{\pi}\right)^2} \exp\left(\frac{1}{\beta_o g^2 \left(\frac{\pi}{a}\right)}\right) \qquad (61)$$

(from equation (60)). Under these conditions the physical quantity remains fixed while a and $\sigma\, a^2$ go to zero in accordance with the β_o of QCD.

If $\sigma\, a^2$ varied in some other way from equation (61) at small g then the requirement that the physical quantity stayed fixed would imply a different β_o or a different form for equation (57) altogether. The lattice model would not be cut-off QCD.

From this point of view the interest in lattice gauge theory lies in trying to calculate the g dependence of observables. There is no more expectation of solving lattice gauge theory than solving QCD, however the lattice model is amenable to a powerful form of non-perturbative approximation. Before completing our review of Creutz's approach we will briefly outline this method of calculating with the lattice model.

1.5 Monte Carlo calculations [29]

We want to obtain an approximation for expectation values, which we will write symbolically as

$$\langle \tilde{O} \rangle = \frac{1}{Z} \int DU\, \tilde{O}(U) \exp\left[S(U) \right] \qquad (62)$$

Here we may think of { U } as a set of numbers describing a possible configuration of the system. If we make the set finite then we can write down a description of a configuration. There are still an infinite number of configurations, however, and it is only practical to approximate the full integral as a sum over a finite sequence of states. To obtain a reasonable approximation we will see later tha the density of states in the sequence should approach

$$\rho(U) = \frac{\exp\left[S(U) \right]}{Z} \qquad (63)$$

Since the set of numbers is finite we can generate successive configurations from preceding ones with a specific algorithm. That is the probability of obtaining configuration U' from U may be specified -

$P(U', U)$. Consider the following algorithm: A random generator is used to suggest a new configuration $U'_Q = Q(U)$, the U'_Q is then used or discarded by the following criteria

$$U' = \begin{cases} Q(U) & \text{if} \quad \rho(U') > \rho(U) \\ Q(U) & \quad\quad \rho(U') > x\rho(U) \\ U & \quad\quad \text{otherwise} \end{cases} \tag{64}$$

where x is a number generated randomly between 0 and 1. The probability that $Q(U)$ suggests U' is written $Q(U', U)$ whereupon

$$P(U', U) = \begin{cases} Q(U', U) \ldots \text{if (i) } \rho(U') \geqslant \rho(U),\ U' \neq U \\ Q(U', U)\,\dfrac{\rho(U')}{\rho(U)} \ldots \text{(ii) } \rho(U') < \rho(U) \\ \text{The rest} \quad \ldots \text{(iii) } U' = U \\ \text{(call it R)} \end{cases} \tag{65}$$

Now provided $Q(U', U) = Q(U, U')$ we have

$$\sum_{U'} P(U, U')\,\rho(U') = \sum_{U' \text{ in (i)}} Q(U, U')\,\rho(U') + \sum_{U' \text{ in (ii)}} Q(U, U')\,\rho(U) + R\rho(U)$$

$$= \sum_{U'} P(U', U)\,\rho(U) = \rho(U) \tag{66}$$

Writing the n-step probability as $P_n(U', U)$ we have

$$\sum_{U''} P(U', U'')\,P_m(U'', U) = P_{m+1}(U', U) \tag{67}$$

so that as $m \to \infty$ $P_m(U', U) \to \rho(U')$ or with

$$P_m(U', U) = \rho(U') + D_m(U', U)$$

$$D_m(U', U) \underset{m \to \infty}{\to} 0 \tag{68}$$

Under these conditions we will see that the expectation value of the sequence average of $\tilde{O}(U)$ over a sequence of configurations $U_o, \ldots, U_N$ approaches $< \tilde{O} >$ as $1/N \to 0$.

The expectation of the sequence average is

$$< \bar{\tilde{O}} > = \sum_{\substack{\text{all possible} \\ \text{sequences} \\ U_o, \ldots, U_N}} P(U_N, U_{N-1}) \ldots P(U_1, U_o) \frac{1}{N} \sum_{n=1}^{N} \tilde{O}(U_n)$$

$$= \frac{1}{N} \left(\sum_{U_1} \tilde{O}(U_1) P(U_1, U_o) + \sum_{U_1, U_2} \tilde{O}(U_2) P(U_1, U_o) P(U_2, U_1) + \ldots \right)$$

$$= \frac{1}{N} \left(\sum_{U} \tilde{O}(U) P(U, U_o) + \sum_{U} \tilde{O}(U) P_2(U, U_o) + \ldots \right)$$

$$= \frac{1}{N} \sum_{U} \sum_{n=1}^{N} \tilde{O}(U) P_n(U, U_o)$$

$$= \sum_{u} \tilde{O}(U) \left(\rho(U) + \frac{1}{N} \sum_{n=1}^{N} D_n(U, U_o) \right) \tag{69}$$

There is a positive constant λ such that

$$| D_n | \leq e^{-\lambda n} \tag{70}$$

so

$$\left| \sum_{n=1}^{\infty} D_n \right| < \frac{1}{\lambda} \tag{71}$$

and $< \bar{\tilde{O}} >$ approaches $< \tilde{O} >$ as $N \to \infty$.

A similar calculation shows that the standard deviation from $< \tilde{O} >$ also tends to zero as N tends to infinity [29].

In principle this method could demonstrate that the same model which confines quarks at large distances exhibits QCD asymptotic freedom at short distances. In practice the evidence it provides is weakened by technical difficulties. Many of these difficulties stem from the

requirement that configurations may be described by a finite set of numbers. This means that there must be a finite lattice, and yet we have seen that the region of interest (small g) should be a region of diverging (dimensionless) correlation lengths.

Fortunately this property of diverging correlation lengths at critical coupling values is not unique to the lattice SU(3) gauge model. We will be able to examine some simpler models with this behaviour.

Within the calculational scheme we have described above there is a freedom (in the choice of Q(U) for instance) in the precise implementation of equation (64). In fact there are other algorithms which reproduce equation (68). It is not known, except by trial and error, how these choices affect λ [7, 20]. Other things which have an effect on the rate of convergence of this approximation are the action S and (implicitly) the coupling g. Again the details are not known, but convergence is generally poor close to the critical coupling values we have discussed. As a rule of thumb convergence is best when a relatively small number of configurations have a relatively large $\rho(U)$. These considerations would not be important but for the fact that there are practical limitations on N.

Fortunately again we will be able to examine models with finite lattice size exactly. This corresponds to the $N \to \infty$ limit (see later).

Further severe practical problems due to finite lattice size arise when the observable $\tilde{O}$ itself is required to be non-local. In the case of the loop expectation value of section 1.3, the interpretation we made strictly required that the loop be very long. We will see in chapter 2 that similar requirements pertain to the extraction of correlation lengths from lattice observables. Nonetheless Creutz persisted with a calculation of this kind on a finite lattice and obtained a rough fit to equation (61) in the small g region [20]. The observable quantity used was the string tension. His results suggest that the continuum limit of Wilson's model has a non-vanishing string tension with the asymptotic behaviour required for QCD. His results also yield an approximate value for the Λ parameter.

The search for further evidence in this connection has taken two paths. Although the string tension has been the easiest quantity to extract from calculations with Wilson's model [30] the problems indicated above place severe restrictions on the accuracy of its measurement. It has become favourable to measure other quantities (now that the Λ parameter is found, comparisons with data are possible). Introducing quarks into the model introduces further parameters into the action [2, 3]. Once these are fixed from the data the remaining lowest lying masses of the model may be approximately calculated [31]. Agreement with data lends support to the interpretation which we have discussed, however the calculational problems we have described remain. The philosophy is to increase the weight of evidence available. Increasing the quality of the approximation is a task of considerable proportions [32].

As we have said, a continuum limit may be taken at any 'critical' point of the model where correlation lengths measured in terms of the lattice spacing diverge. Consider the two conjectures in figure 1 for a 'non-perturbative' version of equation (57), called the β-function [11]. The renormalisation scheme (equation (51) or (52), or holding observables fixed) will affect this picture only quantitatively. The β-function is no easier to extract from the theory than any other quantity, however the conjectures are instructive:

If the solid curve is qualitatively correct then, for asymptotically free QCD, at larger and larger distances the coupling tends to g_1 (which we have drawn outside the 'convergent' region of the strong coupling expansion). Then the lattice confinement argument is inapplicable. The strong coupling string tension could vanish as the spontaneous magnetisation vanishes above T_c in an Ising Ferromagnet (see later). The continuum theory associated with the lattice strong coupling region is not asymptotically free - we have drawn it everywhere outside the 'convergent' region of QCD perturbation theory.

If the dashed curve is correct then Creutz's approximate calculations are not misleading (although we have still to check if the lack of Euclidean invariance in our regularisation is a problem).

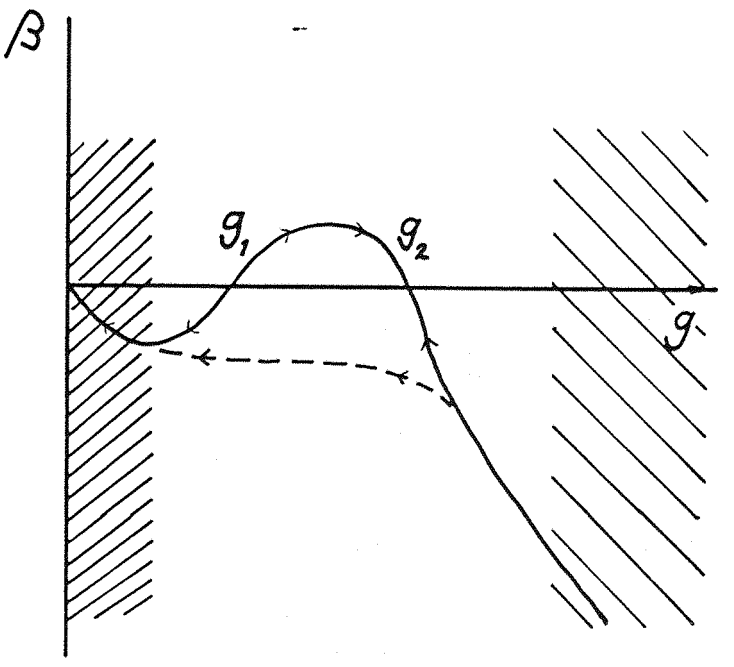

Figure 1

Conjectured forms for the β-function.

- The arrows show the effect on the bare coupling of reducing the lattice spacing a (or the effect on the renormalised coupling of increasing the momentum scale).

We will briefly mention an alternative approach to that of Creutz. Consider the partition function for the lattice model in equation (8). Schematically we have

$$Z = \int DU \exp \{ S(g, U) \} \tag{72}$$

We could try and rewrite this multiple integral as an integral over fewer variables (fewer sites). If we could rewrite the argument as a new action then this would describe a model with a bigger lattice spacing. For instance with

$$\exp \{ S_{new} (g_{new}, U_{new}) \}$$

$$= \int DU \, K(U, U_{new}) \exp \{ S(g, U) \} \tag{73}$$

and

$$\int DU_{new} \, K(U, U_{new}) = 1 \tag{74}$$

then

$$Z = \int DU_{new} \exp \{ S_{new} (g_{new}, U_{new}) \}. \tag{75}$$

The procedure could then be repeated to change the spacing again. In fact S_{new} and subsequent effective actions could contain any interaction provided the symmetries of the original action were preserved [3]. The nth action could be written

$$S_n (g_i^{(n)}, U)$$

- we have associated a coupling g_i with each interaction so that S_n and S_{n-1} differ only in coupling values. In principle we could extract figure 1 as a slice of the many dimensional plot of $(g_i^{(n)} - g_i^{(n-1)})$ against g_i [33].

A practical approximation to this process has been suggested by Wilson [34] (in the context of spin-models there are earlier suggestions - see [24]). A finite lattice and finite set of interactions are considered. The couplings are adjusted to hold

observables fixed on different lattices. Implementing this procedure requires a careful choice of retained interactions and observables to minimise finite size effects and statistical problems (from the Monte Carlo calculations). We will not discuss this approach further. See the paper of Wilson [34] and references therein.

1.6 Simpler Models

We have seen that (neglecting problems with Euclidean invariance and the inclusion of fermions [7]) the usefulness of lattice gauge theory depends crucially on the positions of the zeros of figure 1 (in statistical mechanics terms, the phase structure). Hopes for finding this structure depend in turn on the validity and correct interpretation of the approximations we have discussed. Even using these approximations the problem is hard and it is worthwhile to test them on some simpler models. A whole range of gauge and spin models have been examined in various dimensions to test the limitations of the Monte Carlo technique [7]. In the present work we will make exact calculations and concentrate on the effect of finite size.

Perhaps the simplest non-trivial lattice gauge model is that with Z(2) symmetry in three dimensions. The $a = 0$ limit discussed in section 1.2 has no equivalent here. The continuum limit associated with the second order phase transition in this model will not concern us.

We will start by briefly reviewing the Transfer Matrix formulation which, for lattice models, parallels the Hamiltonian formulation of quantum field theory [4]. In this framework, and specifically for the Z(2) case, we will show how to extract inverse correlation lengths or masses from the Euclidean lattice model by considering plaquette-plaquette expectation values.

Next we will obtain an exact expression for the partition function on a finite lattice. The zeros of the partition function in complex coupling constant space are found. The finite lattice image of a phase transition is identified with zeros close to the real axis. In fact on the infinite lattice a line of zeros is expected to cross the real axis. Since the partition function always appears in the denominator in expectation values this prevents using any series

expansion beyond such a line. In particular the strong coupling expansion for the 'string tension' could not be used to imply a non-vanishing value below the $g_{critical}$. The phase transition is precisely associated with vanishing string tension in the Z(2) model [4]. Unfortunately our lattice is too small to measure a string tension. Other observables vanish (or diverge, depending on dimensions) at the critical point and then reappear. For the mass gap in particular we will identify a finite lattice image of this behaviour.

In an appendix we discuss the Ising model (the action is

$$S_{Ising} = \beta \sum_{\text{links } ij} \phi_i \phi_j$$

with ϕ_i taking values from { ±1 } on the sites i of the lattice). This has a spontaneous magnetisation with behaviour analogous to the string tension at a phase transition. A finite lattice vestige of this behaviour is observed.

A number of avenues for further research are suggested by this work. We will mention them briefly in a discussion section.

We try to use conventional notations (in so far as they have been established). In particular we will use the coupling constant $\beta \propto \frac{1}{g^2}$ so that small β corresponds to 'strong coupling' (or to high temperature in the statistical mechanical equivalent: $\beta \propto \frac{1}{Temp}$). However we will define and use some new notations for labelling lattice variables with an eye towards uncluttered equations.

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$$\lambda^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^{(2)} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^{(4)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^{(5)} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^{(6)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^{(7)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^{(8)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

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2. Transfer Matrix Approach I. The infinite lattice

We confirm the exponential behaviour of plaquette correlation functions for a Z(2) gauge model at large distances and on an infinite lattice using the Transfer Matrix technique. We identify the inverse correlation length with the so called strong coupling glueball mass.

2.1 Introduction

The usefulness of the transfer matrix approach to infinite lattice problems rests on two key points. Firstly that the contribution to the Action of the lattice model due to a d-1 dimensional slice of the lattice may be written entirely in terms of the configuration of variables within that slice and the configuration of variables within an adjacent slice (as is the case for the spin and gauge models we have described). Secondly that the matrix formed by exponentiating these contributions, with rows and columns labelled by the relevant configurations, should have a unique largest positive eigenvalue (as is the case for all positive matrices).

Under these conditions, as we shall see, lattice problems effectively reduce to modified d-1 dimensional problems which are often much easier to solve. In particular thermodynamic quantities depend only on the largest eigenvalue of the matrix we have described, and all observables may be expressed in terms of the spectrum of this matrix. The crucial feature of such a problem, then, becomes the diagonalisability of this Transfer Matrix. In fact for the two dimensional Ising model the matrix turns out to be exactly diagonalisable (Schultz, Mattis and Lieb [1]), forming the basis of an exact solution to this problem. Both spin and gauge Z(2) models in arbitrary dimensions with intra layer interactions set to zero are exactly diagonalisable, whereupon the full models may be obtained perturbatively for small intra layer interactions. This approach has been discussed by Camp and Fisher [2] for the Z(2) spin model. Camp [3] goes on to demonstrate the exponential behaviour of the spin-spin correlation function, showing that the decay is governed by the model's mass gap (or inverse correlation length) at large distances.

Here we will use the transfer matrix approach to obtain the large distance behaviour of the plaquette correlation function in a 3d Z(2) gauge theory. We will first review the idea (as described in Camp and

Fisher, and Schultz, Mattis and Lieb) outlining a proof of the relevant property of positive matrices (the theorem of Perron). We will then proceed to give calculational details for the Z(2) gauge theory, obtaining the relevant mass to second non-trivial order and in agreement with the strong coupling results of Munster [4].

2.2 The Transfer Matrix

The configurations of a d-1 dimensional layer of a lattice of side L may be labelled by a number C. If the lattice variables only take on a finite range of values C is finite and in particular for a Z(M) gauge theory

$$C = 1, \ldots, M^{d\ L^{d-1}} \tag{1}$$

In this case contributions to the Action due to the addition of a new layer to an old one are completely determined by C_{old} and C_{new}. If $A(C_{old}, C_{new})$ gives these contributions we define the transfer matrix by

$$(K)\ c, c' \equiv \exp\left[-\beta\ A(c, c') \right] \tag{2}$$

Then for a system periodic in the layering direction (for example) we find the partition function for N layers in terms of K:

$$Z_N = \sum_c (K^N)_{c,\ c} \tag{3}$$

Furthermore if B(n) is an observable in the nth layer and

$$(\tilde{B})_{c,\ c'} = \delta_{c,\ c'}\ \tilde{B}(c) \tag{4}$$

where $\tilde{B}(c)$ is the value of B in configuration c then

$$< B(n)\ B(n + R) > = Z_N^{-1}\ \mathrm{Tr}\ (K^{N-R}\ \tilde{B}\ K^R\ \tilde{B}) \ . \tag{5}$$

Provided L and N are fairly small we may construct K and its higher powers explicitly, manipulating their entries algebraicaly as polynomials in exp [β]. This is the idea behind the small lattice calculations we will discuss in the next chapter. Here, however, we will make use of a theorem of Perron to obtain as ymptotic forms for such expectation values as N and R tend to infinity.

2.3 The theorem of Perron (non-degeneracy of largest eigenvalue of K)

First note that (from (2)) K has all positive entries for finite β. We shall say that K is a 'positive matrix'. Adopting the notation of Bellman [5] we write $K > 0$. For matrices with all non-negative entries we write $M \geqslant 0$, and we adopt an equivalent notation for vectors. It will be convenient in this proof to normalise vectors by

$$\sum_{i=1}^{c} x_i = 1 \tag{6}$$

Let $S(\lambda)$ be the set of non-negative numbers for which there exist vectors $x \gtrdot 0$ such that

$$Kx \geqslant \lambda x \tag{7}$$

Then

$$\sum_i \sum_j K_{ij}\, x_j \geqslant \lambda$$

and

$$\sum_i \sum_j K_{ij} \geqslant \sum_i \sum_j K_{ij}\, x_j \tag{8}$$

In general we would want to be able to extend consideration to infinite L and M, and the modifications to the present discussion strictly necessary for this are discussed in Camp and Fisher (and references therein). Here, however, we will use only finite (although arbitrarily large) K, and assume that there are no problems.

If λ_o is the largest $\lambda \in S(\lambda)$ then for some $x^{(o)}$

$$K\, x^{(o)} \geqslant \lambda_o\, x^{(o)} \quad . \tag{9}$$

We have

$$\sum_{j=1}^{c} K_{1j}\, x_j^{(o)} - \lambda_o\, x_1^{(o)} = d > 0$$

$$\sum_{j=1}^{c} K_{kj}\, x_j^{(o)} - \lambda_o\, x_k \geqslant 0 \qquad k = 2, \ldots, c \tag{10}$$

without loss of generality if λ_o is not in fact an eigenvalue of K. However for

$$y = x^{(o)} + \begin{bmatrix} d/2\lambda_o \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{11}$$

we have $Ky > \lambda_o y$ which contradicts the maximum property of λ_o. Thus $d = 0$ and in fact λ_o is an eigenvalue with $x^{(o)}$ a positive eigenvector (from (10) since K is positive).

λ_o is in fact the eigenvalue with greatest absolute value. If there is an eigenvalue λ for which $|\lambda| \geqslant \lambda_o$ and

$$KZ = \lambda Z \tag{12}$$

then with $|Z|$ the vector whose components are the absolute values of the components of Z we have

$$K|Z| \geqslant |\lambda|\,|Z| \tag{13}$$

(the absolute value of the sum of two complex numbers is maximised if they are colinear) but (7) applies to $|Z|$ so that $|\lambda| = \lambda_o$ and

$$K|Z| = |\lambda|\,|Z| = |\lambda Z| = |KZ|. \tag{14}$$

Since K is positive all components of Z must be colinear in the complex plane and

$$KZ = \lambda Z \text{ is equivalent to } Kw = \lambda w \tag{15}$$

where $w > 0$ so that λ is real and positive and hence equal to λ_o.

Finally if there is a vector linearly independent of $x^{(o)}$ and not necessarily positive for which

$$KZ = \lambda_o Z \tag{16}$$

we may construct $x^{(o)} + \varepsilon Z$ where ε is the smallest scalar for which one or more components of $x^{(o)} + \varepsilon Z$ are zero, but we have

$$K\,(x^{(o)} + \varepsilon Z) = \lambda_o\,(x^{(o)} + \varepsilon Z) \tag{17}$$

and we have seen that for $(x^{(o)} + \varepsilon Z) \geqslant 0$ this implies $(x^{(o)} + \varepsilon Z) > 0$. Thus there can be no such Z and λ_o is non-degenerate. This proof appeals only to the positive quality of K, and is Perron's Theorem. The Jordan Form of K may thus be written:

$$K' = T^{-1} K\, T = \begin{bmatrix} \lambda_o & & & & & & & & & 0 \\ & \lambda_1 & 1 & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & 1 & & & & & & \\ & & & & \lambda_1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \lambda_m & 1 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & 1 & \\ & & & & & & & & \lambda_m & \\ & & & & & & & & & \ddots \\ & & & & & & & & & \lambda_b \; 1 \\ & & & & & & & & & \ddots \; \ddots \\ & & & & & & & & & 1 \\ 0 & & & & & & & & & \lambda_b \end{bmatrix} \tag{18}$$

Breaking this up into the (b + 1) blocks corresponding to degenerate eigenvalues we label the vectors

$$
\begin{array}{l}
0 \\
0 \qquad\qquad = \langle m, n |^{T} = | m, n \rangle \\
\cdot \\
\cdot \\
\cdot \\
0 \\
\hline
0 \\
\cdot \\
\cdot \\
1 \longleftarrow \text{nth of } d_m \text{ entries in mth block} \\
\cdot \\
\cdot \\
0 \\
\hline
0 \\
0 \\
\cdot \\
\cdot \\
\cdot \\
0
\end{array}
\tag{19}
$$

then with

$$
I_m = \left[\begin{array}{ccccccccccc}
0 & & & & & & & & & & \\
 & 0 & & & & & & & & & \\
 & & \cdot & & & & & & & & \\
 & & & \cdot & & & & & & & \\
 & & & & 0 & & & & & & \\
 & & & & & 1 & & & & & \\
 & & & & & & 1 & & & & \\
 & & & & & & & \cdot & & & \\
 & & & & & & & & \cdot & & \\
 & & & & & & & & & 1 & \\
 & & & & & & & & & & 0 \\
 & & & & & & & & & & & 0 \\
 & & & & & & & & & & & & \cdot \\
 & & & & & & & & & & & & & \cdot \\
 & & & & & & & & & & & & & & 0
\end{array}\right] \left.\vphantom{\begin{array}{c}a\\b\\c\\d\\e\\f\\g\\h\\i\\j\\k\\l\\m\\n\\o\end{array}}\right\} \text{mth block}
\tag{20}
$$

we have

$$
K' = \sum_{m=0}^{b} \left[\lambda_m I_m + \sum_{n=1}^{d_m - 1} | m, n \rangle \langle m, n + 1 | \right]
\tag{21}
$$

Now

$$\left[\sum_{n=1}^{d_m - 1} |m, n><m, n+1|\right]^{\rho} = \begin{cases} \sum_{n=1}^{d_m - \rho} |m, n><m, n+\rho| & \text{for } \rho < d_m \\ 0 \text{ otherwise} & \end{cases} \tag{22}$$

so that

$$(K')^{\rho} = \sum_{m=0}^{b} \sum_{\ell=0}^{\rho} \frac{\rho!}{(\rho-\ell)!\ell!} \sum_{n=1}^{d_m - \ell} |m, n><m, n+\ell| \lambda_m^{\rho-\ell} \tag{23}$$

using the binomial expansion. Tr K = Tr K' so that

$$Z_N = \sum_{m=0}^{b} \sum_{n=1}^{d_m} <m, n| (K')^N |m, n> = \lambda_o^N \left(1 + \sum_{m=1}^{b} d_m \left(\frac{\lambda_m}{\lambda_o}\right)^N\right) \tag{24}$$

for instance. Notice that as $N \to \infty$ $Z_N \to \lambda_o^N$. With $\tilde{B}' = T^{-1} \tilde{B} T$ and $<0| \equiv <0, 1|$ we obtain

$$\underset{N \to \infty}{<B>} = <0|\tilde{B}'|0> . \tag{25}$$

We define

$$<\delta B(n)\delta B(n+R)> \equiv <B(n)\,B(n+R)> - <B(n)><B(n+R)>$$

$$\underset{N >> R \to \infty}{=} \left(\frac{\lambda_i}{\lambda_o}\right)^R \sum_{\rho=0}^{d_i - 1} \frac{R!}{(R-\rho)!\rho!} \sum_{\ell=1}^{d_i - \rho} \frac{<0|\tilde{B}'|i, \ell><i, \ell+\rho|\tilde{B}'|0>}{\lambda_i^{\rho}} \tag{26}$$

where the λ_i is the largest eigenvalue for which

$$<0|\tilde{B}'|i, \ell> \neq 0 . \tag{27}$$

2.4 Perturbative calculation for the asymptotic form of the plaquette correlation function

The transfer matrix for the Z(2) gauge model in three dimensions is most easily written down in a gauge in which all the link variables in the layering direction are fixed equal to + 1 [6]. In lattice gauge theory one chooses a gauge by fixing any set of link variables which do not form a closed loop [7]. Strictly speaking our choice of gauge forms closed loops if we take periodic boundary conditions and we are thus no longer at liberty to do so. However it may be shown (Camp and Fisher) that in the infinite N limit equation (26) is the same for open or periodic boundaries.

For the three dimensional Z(2) gauge model in this gauge

$$A(c, c') = \beta_{\perp} \sum_{\substack{\text{plaquettes} \\ \text{in layer,} \\ ijk\ell}} u_i u_j u_k u_\ell + \beta \sum_{\substack{\text{links} \\ i}} u_i u_i' \qquad (28)$$

where u_i is a link variable taking values from { 1, -1 } and u_i' is the corresponding variable in the adjacent layer specified by c and c' respectively. Note that for convenience in this section we depart from the usual notation of labelling sites, and label links of the lattice.

We will obtain the plaquette correlation function by perturbation expansion about $\beta_{\perp} = 0$. The first step is to obtain $K \mid B_{\perp} = 0$ in diagonal form. We will order the configurations of the $2L^2$ link variables in a layer as follows. Arbitrarily assign each link with a number 1, ..., $2L^2$, write down a configuration by writing 1 for a link at +1 and 0 for a link at -1 for each of the links in order of their numbering. The binary number thus formed gives a Configuration C.

Notice that in this scheme if we write all 2^{2L^2} binary labelling numbers in ascending order,

000 .. 00, 000 .. 01, 000 .. 10, ..., 011 .. 11,
100 .. 00, 100 .. 01, 100 .. 10, ..., 111 .. 11

the first 2^{2L^2-1} correspond to all the configurations with link 1 set to -1 while the second 2^{2L^2-1} correspond to configurations with link 1 set to +1.

Subdividing each of these blocks, the first $2^{2L^2}/_4$ labels give configurations with link 2 set to -1 and the second $2^{2L^2}/_4$ give +1. And so it goes on.

We use this ordering for the rows and columns of K. It allows for K to be constructed in matrix notation by multiple direct products. We introduce a notation for direct products of matrices and vectors:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax & ay & bx & by \\ az & at & bz & bt \\ cx & cy & dx & dy \\ cz & ct & dz & dt \end{pmatrix}$$

where x, y, z and t are square matrices, and

$$\begin{pmatrix} e \\ f \end{pmatrix} \times \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} eg \\ eh \\ fg \\ fh \end{pmatrix} \tag{29}$$

where g and h are column vectors.

With this formalism K may be constructed as follows:

We define

$$\sigma(n) = \exp(\beta)\,\underline{1} + \exp(-\beta)\,\sigma^o \times \sigma^o \times \ldots \sigma^x \times \sigma^o \times \ldots \sigma^o \tag{30}$$

where $\underline{1}$ is the $2^{2L^2} \times 2^{2L^2}$ unit matrix and

$$\sigma^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{31}$$

n runs over the links in a layer and the σ^x is in the nth position in the multiple direct product. Then labelling the rows and columns of $\sigma(n)$ by the old and new configurations c and c' we may interpret $\sigma(n)$ as having non-zero entries $\exp(\beta)$ when all variables are unchanged and $\exp(-\beta)$ when all variables except n are unchanged. $\sigma(n)\,\sigma(m)$ then has entries giving the exponentiated action due to variables n and m where all other variables are unchanged and multiplication by successive σ's introduces the effect of all other variables. We write

$$K\,|_{B_\perp = 0} = \prod_{\substack{\text{links}\\ n}} \sigma(n) \quad . \tag{32}$$

With $\begin{pmatrix} a \\ b \end{pmatrix}_n$ as the vector formed by taking a multiple direct product of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$'s with $\begin{pmatrix} a \\ b \end{pmatrix}$ in the nth position we obtain

$$\begin{aligned} \sigma(n) \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n &= (e^{\beta} + e^{-\beta}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n \\ \sigma(n) \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n &= (e^{\beta} - e^{-\beta}) \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n \end{aligned} \tag{33}$$

Note that with $\psi_n = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}_n$ we have

$$\begin{aligned} \psi_n \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n \qquad \psi_n^T \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n = 0 \\ \psi_n \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n &= 0 \qquad \psi_n^T \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n \end{aligned} \tag{34}$$

so that

$$\begin{aligned} \psi_n \psi_n^T \begin{pmatrix} 1 \\ 1 \end{pmatrix}_n &= 0 \\ \psi_n \psi_n^T \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}_n \end{aligned} \tag{35}$$

and we may write $\sigma(n) = \cosh\beta \exp(-\ell n(\tanh\beta)\,\psi_n\psi_n^T)$ (36) whereupon $K\,|_{B_\perp = 0} = (\cosh\beta)^{2L^2} \exp\left[-\ell n(\tanh\beta) \sum_{\substack{\text{links}\\ n}} \psi_n \psi_n^T\right]$ (37). The maximum eigenvalue is $(\cosh\beta)^{2L^2}$, the next largest is $(\cosh\beta)^{2L^2 - 1}\sinh\beta$

and corresponds to eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}_n$ for any one of the $2L^2$ possible values of n, lower eigenvalues correspond to eigenvectors with $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ introduced at $m \neq n$ in the multiple direct product, and so on.

To introduce the contribution of plaquettes within the layer we note that these are independent of the configuration of the adjacent layer. We may introduce the contribution of a single plaquette

$$K \to K. \quad \exp \left[\beta_\perp \sigma^z(i)\, \sigma^z(j)\, \sigma^z(k)\, \sigma^z(\ell) \right] \tag{38}$$

where i, j, k, ℓ are the links round the plaquette and

$$\sigma^z(n) = \sigma^o \times \sigma^o \times \ldots \sigma^z \times \sigma^o \times .. \sigma^o \tag{39}$$

with $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in the nth position in the multiple direct product. Note that $\sigma^z(n)$ gives a factor of +1 when $U_n = +1$ and -1 when $U_n = -1$, thus a combination of four of these round a plaquette will give the matrix $\tilde{B}$ (of equation (4)) for the plaquette correlation function.

Introducing the effect of all plaquettes in the layer we obtain

$$K = \Big(\prod_{\substack{\text{links} \\ n}} \sigma(n) \Big) \Big(\prod_{\substack{\text{plaquettes} \\ \text{in} \\ \text{layer,} \\ ijk\ell}} \exp \beta_\perp \left[\sigma^z(i)\, \sigma^z(j)\, \sigma^z(k)\, \sigma^z(\ell) \right] \Big) \tag{40}$$

Since $\sigma^z(n) = \psi_n + \psi_n^T$ (41) the eigenvalue spectrum of K is no longer clear. We expand in powers of $\beta_\perp$

$$K = \Big(\prod_{\substack{\text{links} \\ n}} \sigma(n) \Big) + \Big(\prod_{\substack{\text{links} \\ n}} \sigma(n) \Big) \times$$

$$\left[\sum_{n=1}^{\alpha} \frac{1}{n!} \Big(\sum_{\substack{\text{plaquettes} \\ ijk\ell}} \beta_\perp \sigma^z_{(i)}\, \sigma^z_{(j)}\, \sigma^z_{(k)}\, \sigma^z_{(\ell)} \Big)^n \right] \tag{42}$$

From equation (26) we see that in order to calculate the plaquette correlation function we need to know the eigenvalues and eigenstates of the maximum eigenvalue state and the largest eigenvalue state which couples to this via the plaquette operator.

From the Perron theorem we know that the maximum state remains non-degenerate. We will therefore calculate the eigenvalue and eigenstate to lowest non-trivial order using Rayleigh-Schrödinger perturbation theory (see for instance Schiff [8], the only modification is that our operators are non-Hermitian so that the bra and ket states must be calculated separately*).

The zeroth order eigenstate is the unperturbed maximum eigenstate with eigenvalue $(\cosh \beta)^{2L^2}$ which will be written $| 0^{(o)} >$. The unperturbed eigenstates are orthonormal, although in general degenerate. Adopting the notation of the previous section we obtain:

$$\beta_\perp | 0^{(1)} > = \beta_\perp \sum_{\substack{\text{blocks} \\ m=1}}^{b} \left[\frac{(\cosh \beta)^{2L^2} - m(\sinh \beta)^m}{(\cosh \beta)^{2L^2} - (\cosh \beta)^{2L^2 - m}(\sinh \beta)^m} \right] x$$

$$\sum_{n=1}^{d_m} | m, n^{(o)} >< m, n^{(o)} | \sum_{\substack{\text{plaquettes} \\ ijk\ell}} \sigma^z_{(i)} \sigma^z_{(j)} \sigma^z_{(k)} \sigma^z_{(\ell)} | 0^{(o)} > \quad (43)$$

From equations (41) and (34) and the orthonormality condition we obtain

$$\beta_\perp | 0^{(1)} > = \beta_\perp \frac{(\tanh \beta)^4}{1 - (\tanh \beta)^4} \sum_{\substack{\text{plaquettes} \\ ijk\ell}} | 4, (ijk\ell)^{(o)} > \quad (44)$$

and $\beta_\perp < 0^{(1)} | = \beta_\perp \dfrac{1}{1 - \tanh^4 \beta} \sum_{\text{plaquettes}} < 4, ijk\ell^{(o)} |$ where

$\{ | 4, (ijk\ell)^{(o)} > \}$ are the subset of degenerate eigenstates with $m = 4$ for which the $\begin{pmatrix} 1 \\ & \\ -1 \end{pmatrix}$ links form a plaquette.

* We could have chosen a Hermitian Transfer Matrix by sharing out the intra layer interactions symmetrically. In any event we will obtain real eigenvalues.

There is no lowest order correction to the eigenvalue. At order $\beta_\perp^2$ we obtain

$$\beta_\perp^2 \lambda_o^{(2)} = \beta_\perp^2 (\cosh\ \beta)^{2L^2} \Big[\tfrac{1}{2} < 0^{(o)} \Big| \sum_{\text{plaquettes}} \sigma^z\sigma^z\sigma^z\sigma^z \sum_{\text{plaquettes}} \sigma^z\sigma^z\sigma^z\sigma^z \Big| 0^{(o)} >$$

$$+ \frac{(\tanh\ \beta)^4}{1 - (\tanh\ \beta)^4} < 0^{(o)} \Big| \sum_{\text{plaquettes}} \sigma^z\sigma^z\sigma^z\sigma^z \sum_{\substack{\text{plaquettes} \\ ijk\ell}} \Big| 4, (ijk\ell)^{(o)} > \Big]$$

$$= \beta_\perp^2\ (\cosh\ \beta)^{2L^2} \Big[\frac{L^2}{2} + \frac{(\tanh\ \beta)^4}{1 - (\tanh\ \beta)^4} L^2 \Big] \qquad (45)$$

again using equation (41) and (34) and noting that L^2 is the total number of plaquettes in a layer.

In fact the largest eigenvalue states coupled to the vacuum by

$$\tilde{B} = \sigma_i^z\ \sigma_j^z\ \sigma_k^z\ \sigma_\ell^z \qquad (46)$$

(where i, j, k, ℓ label the links of a plaquette) are just the subset of m = 4 states appearing in $| 0^{(1)} >$ (equation (44)).

This is at first sight a problem. The m = 4 states are initially degenerate and at whatever level the degeneracy turns out to be broken one might expect the subset to get mixed up with all the other m = 4 states. In fact we shall see that the states in the subset turn out to mix only among themselves, and that their degeneracy is broken in second order.

From degenerate perturbation theory (see for example Schiff 'Quantum Mechanics') we know that

$$< 4, xyzt^{(o)} \Big| \sum_{\substack{\text{plaquettes} \\ ijk\ell}} \sigma_i^z\ \sigma_j^z\ \sigma_k^z\ \sigma_\ell^z \Big| 4, abcd^{(o)} > = 0 \qquad (47)$$

if xyzt are the links of a plaquette and then the degeneracy is not broken in lowest order (note in particular that the plaquette operator does not couple $< 4, xyzt^{(o)} |$ to any other m = 4 state). At order $\beta_\perp^2$ we find

$$\cosh^{2L^2} - 4\beta \sinh^4\beta < 4, \text{xyzt}^{(o)} \Big| \Big[\frac{\beta_\perp^2}{2} \Big(\sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma \Big)^2 + \beta_\perp^2 \Big(\sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma \Big).$$

$$\sum_{m,n} \Big| m, n \Big>\Big< m, n \Big| \frac{\cosh^{2L^2}\beta \exp\left(-\ell n \tanh\beta \Sigma_x \psi_x \psi_x^T\right) \Big(\sum_{\text{plaquettes}} \sigma\sigma\sigma\sigma \Big)}{(\lambda_4 - \lambda_m)} \Big] \Big| 4, \text{abcd}^{(o)} \Big>$$

$$\neq 0 \qquad (48)$$

provided that the links a, b, c and d form a plaquette. The linear combinations of unperturbed eigenstates which are the eigenstates of the full transfer matrix will be those which diagonalise the matrix implied above.

If we choose periodic boundary conditions the matrix elements above will depend only on the relative position of plaquette ijkℓ and plaquette abcd. We may uniquely associate each plaquette in the two dimensional layer with a position on the lattice, the combinations required are thus

$$\Big| p \Big> = \frac{1}{\sqrt{L^2}} \sum_{\underline{r}} e^{i\underline{p}\underline{r}_{xyzt}} \Big| 4, \text{xyzt}^{(o)} \Big> \qquad (49)$$

where $\underline{p}$ is a position on the dual momentum lattice and $\underline{r}_{xyzt}$ is the position of plaquette xyzt on the original layer lattice.

We obtain (replacing $< 4, \text{xyzt}^{(o)}|$ and $| 4, \text{abcd}^{(o)} >$ with $< p |$ and $| p' >$ in (48))

$$\lambda_4 \frac{1}{L^2} \sum_{\underline{r}_{ijk\ell}} \sum_{\underline{r}_{abcd}} e^{-i\underline{p}\underline{r}_{ijk\ell}} e^{i\underline{p}'\underline{r}_{abcd}} \Big[\frac{\beta_\perp^2}{2} \{ L^2 \delta(\underline{r}_{ijk\ell}, \underline{r}_{abcd}) + \sum_{\substack{\text{non-zero lattice} \\ \text{vectors } \underline{v}}} 2\delta(\underline{r}_{ijk\ell}, \underline{r}_{abcd} + \underline{v}) \}$$

$$+ \beta_\perp^2 \{ \Big(\frac{\lambda_o}{(\lambda_4 - \lambda_o)} + \frac{4\lambda_6}{(\lambda_4 - \lambda_6)} + \frac{(N-5)\lambda_8}{(\lambda_4 - \lambda_8)} \Big) \delta(\underline{r}_{ijk\ell}, \underline{r}_{abcd})$$

$$+ \left(\frac{\lambda_o}{(\lambda_4 - \lambda_o)} + \frac{\lambda_6}{(\lambda_4 - \lambda_6)}\right) \sum_{\substack{\text{unit lattice vectors} \\ \underline{a}}} \delta(\underline{r}_{ijk\ell}, \underline{r}_{abcd} + \underline{a})$$

$$+ \left(\frac{\lambda_o}{(\lambda_4 - \lambda_o)} + \frac{\lambda_8}{(\lambda_4 - \lambda_8)}\right) \sum_{\substack{\text{non-unit lattice vectors} \\ \underline{w}}} \delta(\underline{r}_{ijk\ell}, \underline{r}_{abcd} + \underline{w}) \Big\} \Big]$$

$$= \lambda_4 \beta_\perp^2 \delta(\underline{p}, \underline{p}') \Big[\frac{L^2}{2} + \sum_{\underline{a}} e^{i\underline{p}\underline{a}} + \sum_{\underline{w}} e^{i\underline{p}\underline{w}}$$

$$+ \left(\frac{\lambda_o}{(\lambda_4 - \lambda_o)} + \frac{4\lambda_6}{(\lambda_4 - \lambda_6)} + \frac{(N-5)\lambda_8}{(\lambda_4 - \lambda_8)}\right)$$

$$+ \left(\frac{\lambda_o}{(\lambda_4 - \lambda_o)} + \frac{\lambda_6}{(\lambda_4 - \lambda_6)}\right) \sum_{\underline{a}} e^{i\underline{p}\underline{a}}$$

$$+ \left(\frac{\lambda_o}{(\lambda_4 - \lambda_o)} + \frac{\lambda_8}{(\lambda_4 - \lambda_8)}\right) \sum_{\underline{w}} e^{i\underline{p}\underline{w}} \Big] \qquad (50)$$

now

$$\frac{\cosh^{2L^2}\beta}{\cosh^{2L^2 - 4}\beta \sinh^4\beta - \cosh^{2L^2}\beta} + \frac{\cosh^{2L^2 - 8}\beta \sinh^8\beta}{\cosh^{2L^2 - 4}\beta \sinh^4\beta - \cosh^{2L^2 - 8}\beta \sinh^8\beta}$$

$$= \frac{(\tanh^{-4}\beta - 1) + (\tanh^4\beta - 1)}{(\tanh^4\beta - 1)(\tanh^{-4}\beta - 1)}$$

$$= -1 \qquad (51)$$

so we have

$$\beta_\perp^2 \lambda^{(2)}(p) = \lambda_4 \beta_\perp^2 \Big[\frac{L^2}{2} + \Big(\frac{1}{\tanh^4\beta - 1} + \frac{4}{\tanh^{-2}\beta - 1} + \frac{L^2 - 5}{\tanh^{-4}\beta - 1} \Big)$$

$$+ \Big(1 + \frac{1}{\tanh^4\beta - 1} + \frac{1}{\tanh^{-2}\beta - 1} \Big) \sum_{\underline{a}} e^{ip\underline{a}} \Big] \qquad (52)$$

with

$$\beta_\perp \mid p^{(1)} > = \beta_\perp \frac{1}{\sqrt{L^2}} \sum_{\underline{r}} e^{ip\underline{r}_{ijk\ell}} \Big[\frac{1}{\tanh^4\beta - 1} \mid 0^{(o)} >$$

$$+ \frac{1}{\tanh^{-2}\beta - 1} \sum_{\underline{a}} \mid 6, ijkbcd^{(o)} >$$

$$+ \frac{1}{\tanh^{-4}\beta - 1} \sum_{\underline{w}} \mid 8, ijk\ell\ abcd^{(o)} > \Big] \qquad (53)$$

(relative position of plaquette abcd)

and

$$\beta_\perp < p^{(1)} \mid = \beta_\perp \frac{1}{\sqrt{L^2}} \sum_{\underline{r}} e^{-ip\underline{r}_{ijk\ell}} \Big[\frac{1}{1 - \tanh^{-4}\beta} < 0^{(o)} \mid$$

$$+ \frac{1}{1 - \tanh^2\beta} \sum_{\underline{a}} < 4, ijkbcd^{(o)} \mid$$

$$+ \frac{1}{1 - \tanh^4\beta} \sum_{\underline{w}} < 8, ijk\ell\ abcd^{(o)} \mid \Big] \qquad (54)$$

From equation (52) we see that the degeneracy is broken. Equation (26) now gives (for plaquettes at r and r + R)

$$< \delta B(r)\ \delta B(r + R) > = \sum_p \Big(\frac{\lambda(p)}{\lambda_o} \Big)^R < 0 \mid \sigma\sigma\sigma\sigma_r \mid p >< p \mid \sigma\sigma\sigma\sigma_r \mid 0 >$$

$$= \sum_p \exp \Big[R(\ell n\ \lambda(p) - \ell n\ \lambda_o) \Big] \Big(\frac{1}{L^2} + O(\beta_\perp^2) \Big)$$

$$= \sum_p \exp \Big[R(\ell n\ (\tanh^4\beta) + \beta_\perp^2 \Big[\Big(\frac{1}{\tanh^4\beta - 1} + \frac{4}{\tanh^{-2}\beta - 1} - \frac{5}{\tanh^{-4}\beta - 1} \Big)$$

$$+ \left(1 + \frac{1}{\tanh^4\beta - 1} + \frac{1}{\tanh^{-2}\beta - 1}\right) . \Sigma_{\underline{a}} e^{i\underline{p}\underline{a}} \Big] + O(\beta^3)) \Big] \cdot \left(\frac{1}{L^2} + O(\beta^2)\right) \qquad (55)$$

Contributions at order β^2 come from

$$< 0^{(o)} \mid \sigma\sigma\sigma\sigma_r \mid p^{(o)} >< p^{(1)} \mid \sigma\sigma\sigma\sigma_r \mid 0^{(1)} > + < 0^{(1)} \mid \sigma\sigma\sigma\sigma_r \mid p^{(1)} >< p^{(o)} \mid \sigma\sigma\sigma\sigma_r \mid 0^{(o)} >$$

$$+ < 0^{(2)} \mid \sigma\sigma\sigma\sigma_r \mid p^{(o)} >< p^{(o)} \mid \sigma\sigma\sigma\sigma_r \mid 0^{(o)} > + < 0^{(o)} \mid \sigma\sigma\sigma\sigma_r \mid p^{(o)} >< p^{(o)} \mid$$

$$\sigma\sigma\sigma\sigma_r \mid 0^{(2)} > \qquad (56)$$

where

$$\beta^2 < 0^{(2)} \mid = \beta^2 \Big[\sum_{\text{plaquettes}} \sum_{\substack{\text{nearest} \\ \text{neighbours}}} \mid 6, xyzbcd^{(o)} >$$

$$\left\{ \frac{\tanh^4\beta}{(1 - \tanh^4\beta)(1 - \tanh^6\beta)} + \frac{\frac{1}{2}}{(1 - \tanh^6\beta)} \right\}$$

$$+ \sum_{\text{plaquettes}} \sum_{\substack{\text{non-} \\ \text{nearest} \\ \text{neighbours}}} \mid 8, xyzt\ abcd^{(o)} > \left\{ \frac{\tanh^4\beta}{(1 - \tanh^4\beta)(1 - \tanh^8\beta)} \right.$$

$$\left. + \frac{\frac{1}{2}}{(1 - \tanh^8\beta)} \right\} \Big] \qquad (57)$$

and

$$\beta^2 \mid 0^{(2)} > = \beta^2 \Big[\sum_{\text{plaquettes}} \sum_{\substack{\text{nearest} \\ \text{neighbours}}} < 6, xyzbcd^{(o)} \mid$$

$$\left\{ \frac{\tanh^{10}\beta}{(1 - \tanh^4\beta)(1 - \tanh^6\beta)} + \frac{\frac{1}{2}\tanh^6\beta}{(1 - \tanh^6\beta)} \right\}$$

$$+ \sum_{\text{plaquettes}} \sum_{\substack{\text{non-}\\ \text{nearest}\\ \text{neighbours}}} < 8, xyztabcd^{(o)} \mid \{ \frac{\tanh^{12}\beta}{(1-\tanh^{4}\beta)(1-\tanh^{8}\beta)}$$

$$+ \frac{\frac{1}{2}\tanh^{8}\beta}{(1-\tanh^{8}\beta)} \} \qquad (58)$$

combining these with equations (34), (43), (44), (53), and (54) we obtain

$$\frac{\beta_\perp^2}{L^2} \{ - (\frac{1}{1-\tanh^{4}\beta})^2 - (\frac{1}{1-\tanh^{-4}\beta})^2$$

$$+ \sum_{\substack{\text{nearest}\\ \text{neighbours}\\ \underline{a}}} e^{i\underline{p}\underline{a}} [-(\frac{1}{1-\tanh^{4}\beta})^2 - (\frac{1}{1-\tanh^{-4}\beta})$$

$$+ 2(\frac{1}{1-\tanh^{4}\beta})(\frac{1}{\tanh^{-2}\beta - 1}) + 2(\frac{1}{\tanh^{-4}\beta - 1})(\frac{1}{1-\tanh^{2}\beta})$$

$$+ \frac{2(\tanh^{4}\beta + \tanh^{10}\beta)}{(1-\tanh^{4}\beta)(1-\tanh^{6}\beta)} + \frac{(1+\tanh^{6}\beta)}{(1-\tanh^{6}\beta)}] \} \qquad (59)$$

Notice that potentially awkward terms involving sums over non- nearest neighbours have all cancelled. After some algebra we obtain

$$\frac{\beta_\perp^2}{L^2} \{ - \frac{(1+\tanh^{8}\beta)}{(1-\tanh^{4}\beta)^2} + \sum_{\substack{\text{nearest}\\ \text{neighbours}\\ \underline{a}}} e^{i\underline{p}.\underline{a}}$$

$$[\frac{2\tanh^{2}\beta(1 + 2\tanh^{2}\beta + 2\tanh^{4}\beta - 2\tanh^{6}\beta - 2\tanh^{8}\beta - \tanh^{10}\beta)}{(1-\tanh^{4}\beta)^2(1-\tanh^{6}\beta)}]$$

(60)

Since $\underline{p}$ is given by $(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L})$ where n_1, n_2 range from 1 to L, as L tends to infinity $p_i (i = 1, 2)$ tends to a continuous variable with range $(0, 2\pi)$. The sum over $\underline{p}$ is

$$\frac{1}{L^2} \sum_{\underline{p}} f(\underline{p}) = \frac{1}{L} \sum_{n_1 = 1}^{L} \frac{1}{L} \sum_{n_2 = 1}^{L} f(\frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}) \tag{61}$$

and

$$\lim_{L \to \infty} \frac{1}{L^2} \sum_{\underline{p}} f(\underline{p}) = \int_o^{2\pi} \frac{dp_1}{2\pi} \int_o^{2\pi} \frac{dp_2}{2\pi} f(p_1, p_2) \tag{62}$$

Therefore in the infinite volume limit

$$< \delta B(n) \delta B(n + R) > \approx \int_o^{2\pi} \int_o^{2\pi} \frac{dp_1 \, dp_2}{(2\pi)^2}$$

$$\exp \left[R \{ \ell n (\tanh^4 \beta) + \beta_\perp^2 \left[\frac{1}{\tanh^4 \beta - 1} + \frac{4}{\tanh^{-2} \beta - 1} - \frac{5}{\tanh^{-4} \beta - 1} \right] \right.$$

$$\left. + O(\beta_\perp^3) \} \right]$$

$$\exp \left[R \{ 2\beta_\perp^2 \left[1 + \frac{1}{\tanh^4 \beta - 1} + \frac{1}{\tanh^{-2} \beta - 1} \right] (\cos p_1 + \cos p_2) \} \right]$$

$$\cdot (1 + \beta_\perp^2 \left[\frac{-(1 + \tanh^8 \beta)}{(1 - \tanh^4 \beta)^2} + 2(\cos p_1 + \cos p_2) \left[\ldots \right] \right] + O(\beta_\perp^3)) \tag{63}$$

The leading term, then, is proportional to $I_o^2(x)$ where

$$I_\nu(x) = \int_o^{2\pi} \frac{dp_1}{2\pi} \cos(\nu p_1) \exp(x \cos p_1) \tag{64}$$

while the leading correction is proportional to $I_o I_1$. For large $R \sim x$ these have the same asymptotic behaviour (Camp [3]), in particular

$$I_o(x) \approx (2\pi x)^{-\frac{1}{2}} e^x \left[1 + \frac{\alpha}{x} + O(x^{-2}) \right] \tag{65}$$

so that for small $\beta_\perp$ we may neglect non-leading terms and obtain

$$< \delta\ B(n)\ \delta\ B(n + R) > \approx \frac{\exp\ [\ -\ R\ m\ ]}{R} \qquad (66)$$

which is the usual form provided m is the so called z(2) glueball mass. Reading m from (63) we obtain agreement with the strong coupling expansion of Munster [4] up to $O(\beta^2)$ (we cannot check beyond this order since there will be contributions at $O(\beta_{\perp}^3)$ to $O(\beta^3)$)

$$m = -\ 4\ \ell n\ (\tanh\ \beta) - \beta_{\perp}^2 + O(\beta^3) \qquad (67).$$

It should be straightforward to pursue this calculation both to higher orders and to include off-axis correlations. We could, for instance, obtain the asymptotic angular dependence of correlations (see Camp [3] and conclusions). For the purposes of this work, however, we are primarily interested in confirming equation (66) as the asymptotic form for plaquette correlations. In the next chapter we will observe the behaviour of plaquette correlations at much shorter distances.

Although we have achieved the expected result for this calculation, a result which is qualitatively very similar to that achieved by Camp and Fisher for the Ising model, it is interesting to note that the calculation has really followed a quite different quantitative path. Specifically, in the Ising model calculation the spin operator couples to the largest eigenvalue degenerate states, whose degeneracy is broken in lowest order. Thus the nearest neighbour interaction naturally leads to the degeneracy being broken by cos p terms in momentum space and the problem reduces to Bessel functions in the infinite limit. In contrast the largest eigenvalue degenerate states which couple in the gauge model have their degeneracy broken at second order, effectively introducing plaquette-plaquette couplings throughout the lattice. This gives rise to potentially divergent contributions as the infinite limit as approached. In this case, then, it is a notable sequence of exact cancellations of terms from different parts of the calculation which removes these problems!

In fact we have not shown that such cancellations continue to occur at higher orders. However the calculation is so far consistent with the

strong coupling expansion of Munster, which cannot break down at higher orders, and we assume that no terms occur at higher orders here to destroy the form of equation (66).

2.5 Large β behaviour

Finally in this section we briefly examine the behaviour of the mass gap at large β. Note from appendix I that the large β region of a Z(2) gauge model is transformed into the small β region of an Ising model in three dimensions. Note also that the quantity dual to a plaquette-plaquette expectation value is given by (Savit [9], appendix)

$$< \square_a \square_b > \Big|_{\beta^* = -\frac{1}{2} \ln \tanh \beta} = \frac{\tilde{Z}_I(\beta)}{Z_I(\beta)} \tag{68}$$

where $Z_I(\beta)$ is the Ising model p.f. and $\tilde{Z}_I(\beta)$ is a modified Ising model p.f. obtained by changing the sign of the coupling associated with links dual to plaquettes a and b. $\square$ signifies a plaquette.

In the transfer matrix formalism, for example with periodic boundaries, we write

$$\tilde{Z}_I(\beta) = \mathrm{Tr} \left[K^{N-R} \tilde{K} K^R \tilde{K} \right] \tag{69}$$

where $\tilde{K}$ gives the contribution of a layer modified as above. We may proceed as before (equation (26)) with $B = \tilde{K}$. We will not carry out a detailed calculation for the asymptotic form as $N >> R \to \infty$ but rather assume that the mass gap will as before be given by

$$m = \ln \lambda_o - \ln \lambda_n \tag{70}$$

where λ_n is the eigenvalue of the lowest eigenstate coupled to $< 0 \mid$ by $\tilde{K}$. The modified coupling in $\tilde{K}$ is on a link between layers and thus alters the $K \mid_{\beta_\perp = 0}$ part. We see from equation (33) that this involves only a sign change when the relevant layer site occurs in an eigenstate. The unperturbed eigenstates remain as eigenstates. As for K we expand $\tilde{K}$ in terms of β ($K_{\parallel} \equiv K \mid_{\beta_\perp = 0}$ etc)

$$\tilde{K} = \tilde{K}_{||} \quad K_{\perp} = \tilde{K}_{||} + \tilde{K}_{||} \quad (K_{\perp} - 1) \tag{71}$$

The first term has only diagonal matrix elements and from (41) and the Ising version of (42) all subsequent terms have diagonal elements and elements corresponding to changes from even to even eigenstates. A similar calculation to that indicated in equation (43) (for the Ising model in this case, see Camp [3]) shows that eigenstates with unperturbed eigenvalue λ_1 have only odd unperturbed states in them. λ_2 is the lowest eigenvalue required. Camp gives this as

$$\lambda_2(0) = \cosh^N\beta \tanh^2\beta \{ 1 + 8\beta_{\perp} + O(\beta_{\perp}^2) \} \tag{72}$$

and

$$\ell n \lambda_o - \ell n \lambda_2 = -2 \ell n \tanh \beta + 8\beta_{\perp} + O(\beta_{\perp}^2) \tag{73}$$

After duality transformation (with β now the dual gauge model coupling) we obtain for large $\beta_{\perp}$

$$m_{gauge} = 4\beta - 8e^{-2\beta_{\perp}} + O(e^{-4\beta_{\perp}}) \tag{74}$$

This note has provided a large β limit check on calculations to be discussed in the next chapter.

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3. Transfer Matrix Approach II. The small lattice

We examine the results of a small lattice calculation in which the transfer matrix is raised to a high power algebraically and expectation values appear as ratios of exact polynomials.

3.1 Introduction

In the previous chapter we discussed the behaviour of correlation functions on the lattice in the perturbative region of small β. The discussion concerned a discrete gauge group, although similar results from direct expansions apply to continuous groups (Münster [1]). Away from the perturbative regime these methods are unhelpful in either discrete or continuous models and one must turn to approximate non-perturbative techniques.

We saw in the first chapter (section 1.5) how a sequence of possible gauge configurations may be generated in such a way that their distribution approaches the Boltzmanian weighting of a path integral. In order to achieve this in practice it is necessary to parametrise the configurations. Each link gauge variable may be described by a finite number of parameters for any gauge group of physical interest, but we must also require that there be a finite number of variables. In other words, importance sampling Monte Carlo calculations imply finite lattices.

This means that results are at best only indicative of infinite lattice phenomena. One expects a very good approximation in regions where the correlation lengths of the model are small, and a poorer one as correlations grow to permeate the lattice.

In particular, phase transitions do not occur. Their presence on the infinite lattice has been successfully inferred from Monte Carlo results (Creutz [2]), but the finite lattice image appears dramatic only because of finite sample size. What does this mean? Monte Carlo calculations sample only a finite number of gauge configurations (out of an infinite number of possibilities in general), and although the ultimate Boltzmanian distribution of these configurations is of the essence of the importance sampling Monte Carlo algorithm, the rate at which this distribution is approached may be slow. The rate of

convergence of a Monte Carlo approximation is governed by the quantity λ (equation (70), section 1.5). This depends on the state of the model and the method of generating configurations.

We will evaluate directly the kind of path integral approximated by a sequence of Monte Carlo configurations on a finite lattice. We assume that finite lattice vestigial phase structures will look qualitatively similar in models with similar infinite lattice phase structures, regardless of the gauge group. Since we do not propose to examine the naive continuum limit (section 1.2) in this work, we may as well take advantage of the form of Wilson's Lattice Action [3] (gauge fields taking values from the group rather than the algebra) and consider a finite gauge group [4].

Although exact calculations on finite lattices are not a new idea ([5], [6]) they have yet to be applied to Lattice Gauge theories. We will see later that gauge models are more realistically modelled than conventional Z(2) spin models on a finite lattice. The idea is simply that, for fields taking values from a finite set, the relevant path integral becomes a finite sum and thus in principle trivial. The sum may be large, but practical limits on its size come from the solution of computing problems which need not concern us here.

The relevance to physics of a finite lattice model with a discrete gauge group is not immediate! However, exact solutions for non-trivial lattice models are not abundant, particularly in greater than two dimensions. We present exact results, applicable in all regions of coupling constant space, on (as far as we know) the largest lattice presently amenable to such treatment.

Firstly, this will allow us to examine the zeros of the finite lattice partition function, which should provide an indication of the global analytic structure of a lattice model [7]. It will also be possible to obtain subtracted plaquette-plaquette correlations exactly. Now in the case of a discrete global symmetry on an infinite lattice the symmetry is broken spontaneously at the critical temperature. This allows the use of a local order parameter. On a finite lattice no symmetry is spontaneously broken and the order parameter remains zero unless the symmetry is broken by hand. This is essential for the

measurement of subtracted spin-spin correlations (otherwise the subtraction is always zero and correlations do not decay exponentially) thus we are forced to use an artificial symmetry breaking (see Appendix II). Contrastingly a gauge symmetry is not spontaneously broken on any lattice (this is Elitzur's Theorem, see [8]) and a gauge invariant non-local order parameter is chosen from the start [9]. The need for artificial symmetry breaking never arises.

There has been considerable interest recently in the hadronic correlation functions in Monte Carlo lattice SU(3) gauge theory. The lattices have been of approximate size 6 x 6 x 6 x 12 [10]. The idea is to fit masses to the exponential decay of these correlation functions. At large distances

$$< A(o)\ A(R) > - <A(o)><A(R)> \sim \frac{\exp(-mR)}{R^{\frac{d-1}{2}}} \qquad (1)$$

where m is the lowest lying mass associated with A. Corrections to this form have higher powers of R in the denominator or larger masses in the exponential (see the previous chapter, for example). We see that fitting to this form is only absolutely safe as $R \to \infty$. With a periodic lattice of the size mentioned above we cannot have R greater than six lattice spacings!

We will use the exact expressions generated for this work to fit separated plaquette expectation values (for separations up to 4 lattice spacings) to the asymptotic form given in equation (1).

We have established in the previous section that this is the correct asymptotic form for the Z(2) gauge model in d = 3 dimensions. We will be able to compare with the small β expansion for the lowest lying mass [Münster [1]] and with leading terms in the small $e^{-2\beta}$ expansion (previous chapter). We will also see that the mass gap becomes small close to β_c.

Finally we will compare mass fits using different plaquette separations at fixed β and find that discrepancies are small, i.e. that separation dependent corrections to the exponential form remain small even close to the critical point in this small lattice model (in contrast with the Ising model, appendix I).

3.2 The Model

We discuss the requirements for a suitable model. It has been suggested [7], [11], [12] that vestiges of infinite lattice behaviour persist on extremely small (e.g. 2^3, 3^3) lattices. Specifically we find that although finite lattice models do not have phase transitions, measurable quantities such as specific heat will peak close to infinite lattice critical coupling values. However, while these small lattice sizes make partition functions (and their derivatives) relatively easy to obtain, they physically preclude the measurement of correlations between separated spins or plaquettes.

At the same time exact calculations on large lattices present computational difficulties. The compromise is to consider lattices of relatively large extent in one direction, and to measure correlations in this direction (cf [10], [13]). We find that a Z(2) gauge model on a 3 x 3 x 9 lattice is accessible. This will be sufficient for measurement of correlations. It implies a sum over roughly 10^{50} configurations after gauge fixing.

The partition function is

$$Z = \sum_{\text{configurations}} \exp \left[\beta \sum_{\substack{\text{plaquettes} \\ ijk\ell}} U_{ij} U_{jk} U_{k\ell} U_{\ell i} \right] \qquad (2)$$

where the link variable U_{ij} takes values from { 1, - 1 } on the link between sites i and j (cf. the direct labeling of links in the previous chapter). Z cannot be evaluated directly. The calculation is simple but too long winded. We proceed as follows: the dual Ising model partition function is evaluated using the finite transfer matrix approach suggested following equation (5) of the previous chapter; Z is then obtained by the duality transformation described in Appendix I. Checks on this method are outlined in appendix III, where Z is also listed. Periodic boundaries are used in the short directions while free ($\beta = 0$) boundaries are used in the long direction. We avoid periodic boundaries in the long direction so that separated plaquettes have a definite separation. We will discuss boundary effects in a subsequent section.

The following features of the infinite lattice model should be noted:

The Ising model in three dimensions has a unique second order phase transition at $\beta_c \approx .22$ (Fisher and Burford [14]). Duality (appendix I) then implies a unique second order phase transition for the Z(2) gauge model at $\beta_c \approx .76$ (see also Balian, Drouffe and Itzykson [4]). Interest in Z(2) gauge models was initially stimulated by the qualitative change in the behaviour of Wilson loop expectation values from

$$< \prod_{\text{loop}} U_{ij} > \propto \exp \left[-f(\beta) \text{ (loop area)} \right] \tag{3}$$

at small β, to

$$< \prod_{\text{loop}} U_{ij} > \propto \exp \left[-g(\beta) \text{ (loop perimeter)} \right] \tag{4}$$

at large β (Wegner [9], Kogut [8]). Unfortunately exact calculations for the expectation values of different sizes of, say, square Wilson loops would require a lattice of large extent in at least two dimensions. The lattice described here could be used for N x 1 Wilson loops, but these are not sensitive to the crossover from area to perimeter dependence. Instead we will examine the plaquette correlation function. The correlation length associated with this function should become large close to the critical coupling, although as we have seen in the previous chapter we expect a finite mass gap in both the large and small coupling limits.

The Ising model partition function has a zero at β_c; in fact there must be an infinite number of zeros in the neighbourhood of β_c in the complex coupling constant plane (Abe [15], also see later). Since duality implies that the gauge model partition function is proportional to that for the Ising model (with transformed coupling; appendix I) we expect a similar distribution of zeros in this case.

It is found in the case of the 2 dimensional Ising model that (for some boundary conditions) the zeros of a finite lattice model all lie on the line of the infinite lattice distribution [7]. We proceed under the assumption of a similar correspondence in the next section.

3.3 Zeros of the partition function and behaviour of the specific heat

Before generating correlation functions we examine the partition function for the 3 x 3 x 9 Z(2) gauge model. Since the partition function is obtained as a polynomial (in this case a polynomial in tanh β, see appendices I and III) we may obtain the internal energy and specific heat directly:

$$\text{Average plaquette action} = \frac{1}{N_p} \frac{\partial \ \ell n \ Z(\beta)}{\partial \beta} \tag{5}$$

$$\text{Specific heat} = \frac{1}{N_p} \frac{\partial^2 \ \ell n \ \ (\beta)}{\partial \beta^2} \tag{6}$$

where N_p is the number of plaquettes.

The specific heat is shown as a function of β in figure 1. Similar results for the Ising model are given in appendix II. The peak in the specific heat close to β_c is more pronounced for the Ising model - we will be able to interpret this in terms of the distribution of zeros of the partition function.

First notice that we may write the partition function as

$$Z(\beta) \ \alpha \ \prod_{\substack{\text{zeros} \\ j}} (e^{-4\beta} - (a_j + ib_j)) \tag{7}$$

where $(a_j + ib_j)$ is the position of the jth zero of $Z(\beta)$ in the complex $e^{-4\beta}$ plane. Putting this into equation (6) we obtain

$$\text{Specific heat} = \frac{-16}{N_p} e^{-4\beta} \sum_j \frac{(a_j + ib_j)}{(e^{-4\beta} - (a_j + ib_j))^2} \tag{8}$$

Since the polynomial $Z(\beta)$ has real coefficents the zeros must occur in complex conjugate pairs. The contribution to the specific heat on the real axis due to such a pair (now jointly labelled j) is given by

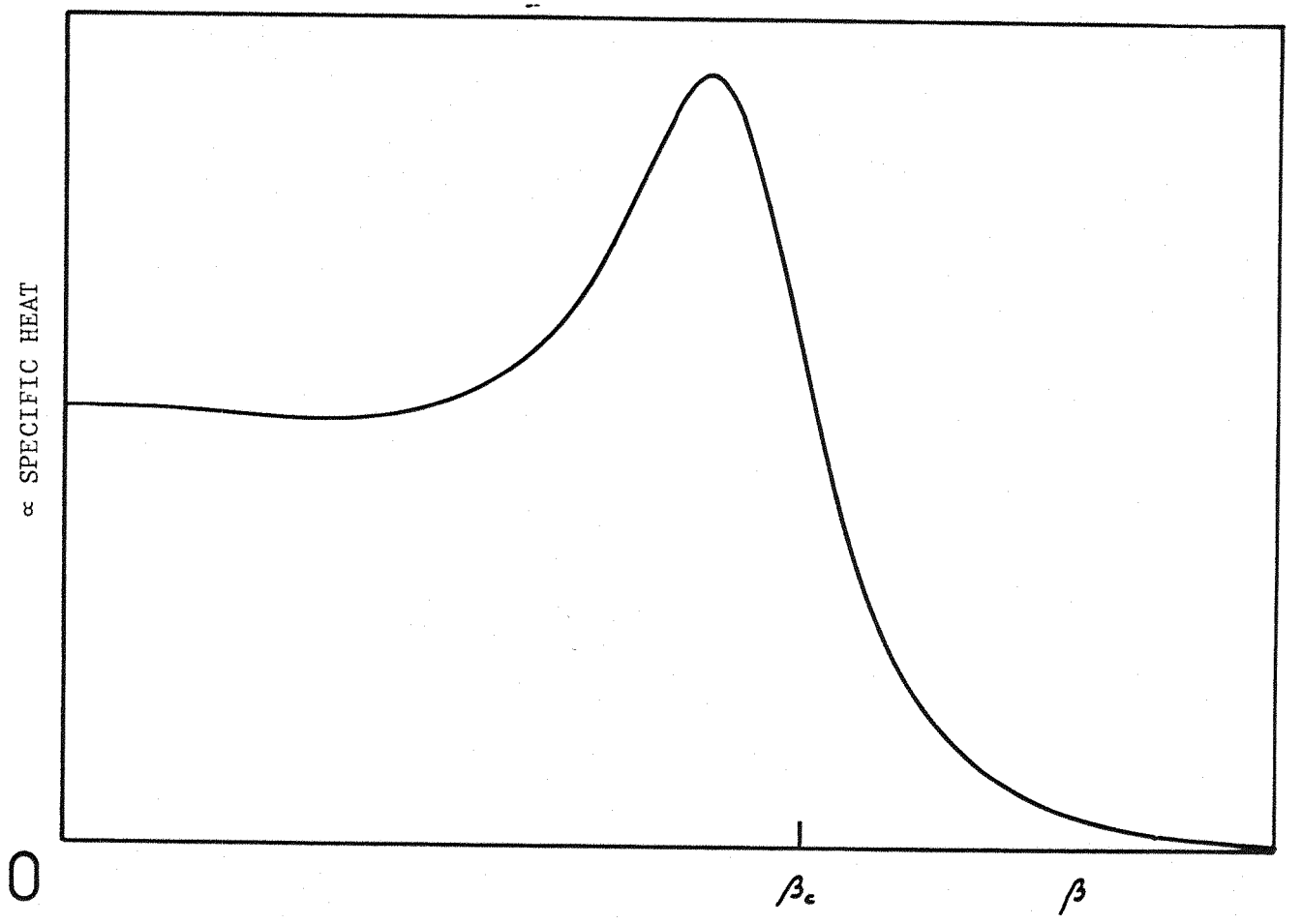


Figure 1

Specific heat vs. coupling constant for the 3 x 3 x 9 lattice Z(2) gauge model.

$$S_j(x) = \frac{x}{N_p} \frac{\left[-a_j \left((x - a_j)^2 + b_j^2\right) + 2b_j^2 x \right]}{\left[(x - a_j)^2 + b_j^2 \right]^2} \tag{9}$$

where $x = e^{-4\beta}$. $S_j(x)$ has maxima at

$$x = \pm \sqrt{a_j^2 + b_j^2} \ , \tag{10}$$

$0 < x < 1$ correspond to positive β and

$$S_j\left(\sqrt{a_j^2 + b_j^2}\right) = \frac{1}{N_p} \frac{1}{2\left(1 - \frac{a_j}{\sqrt{a_j^2 + b_j^2}}\right)} \tag{11}$$

The distribution of zeros in the complex x plane is shown in figure 2. This may be compared with the distribution for the Ising model given in appendix II. Figures 3 and 4 show the distribution of zeros in $\tanh \beta$ and $\tanh^2\beta$ for the gauge model. There are a few zeros which are not found (we use a modified Newton-Raphson technique [16]), and a few which lie well outside the unit circle in these plots.

Since the gauge model is obtained from a modified Ising model by the transformation

$$\beta \to -\tfrac{1}{2} \ln \tanh \beta \tag{12}$$

comparison with the conventional Ising model plot (appendix) shows the effect of the modified boundary conditions.

Equation (11) shows that the largest contribution to the specific heat from the nearest pair of zeros (j) pinching the positive x-axis is at

$$x = \sqrt{a_j^2 + b_j^2} \approx .07, \ \beta \approx .66$$

This compares with $e^{-4\beta_c} \approx .05$, $\beta_c \approx .76$, and corresponds closely to the peak in figure 1. In fact we may associate this peak with a 'coherent' distribution of zeros (i.e. a line with $a^2 + b^2 \approx$ constant). This parallels the anticipated infinite lattice behaviour [15]. Notice from appendix II that the corresponding zeros in the finite lattice Ising model give, in this sense, more coherent contributions at β_c, and produce a more pronounced peak.

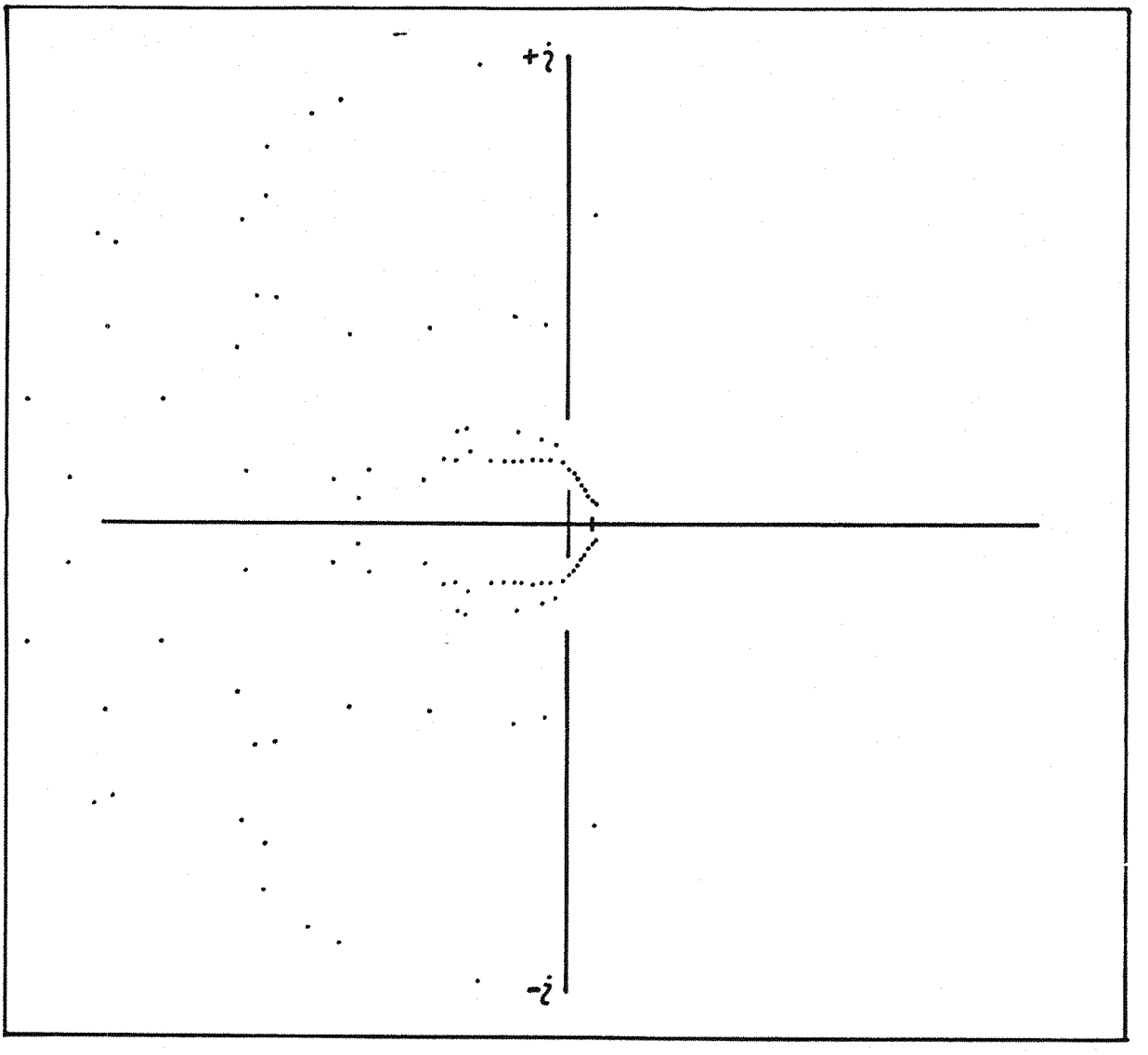


Figure 2

Some zeros of the partition function plotted in $e^{-4\beta}$ for the 3 x 3 x 9 lattice Z(2) gauge model.

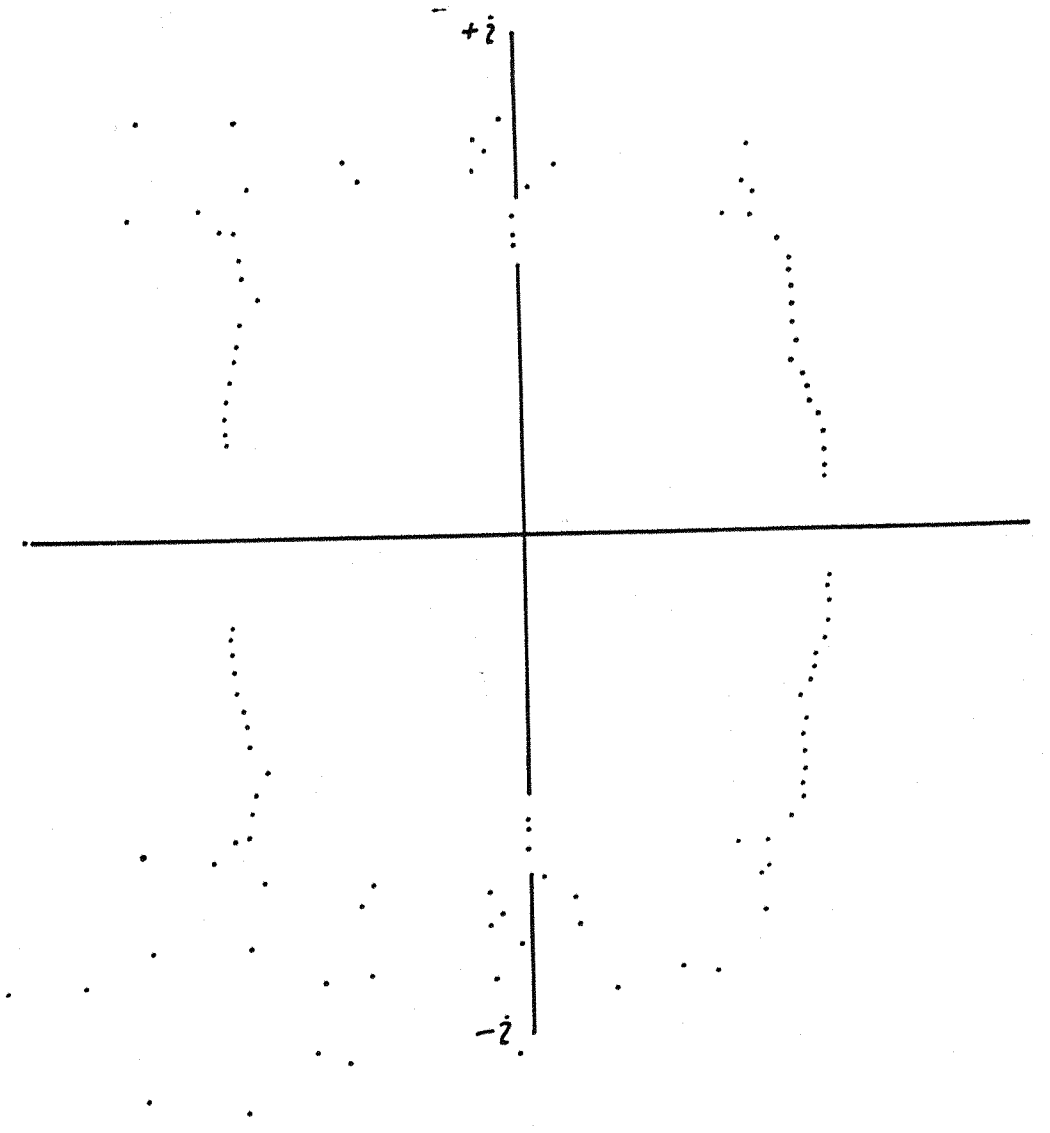


Figure 3

Some zeros of the partition function plotted in tanh β for the 3 x 3 x 9 lattice Z(2) gauge model.

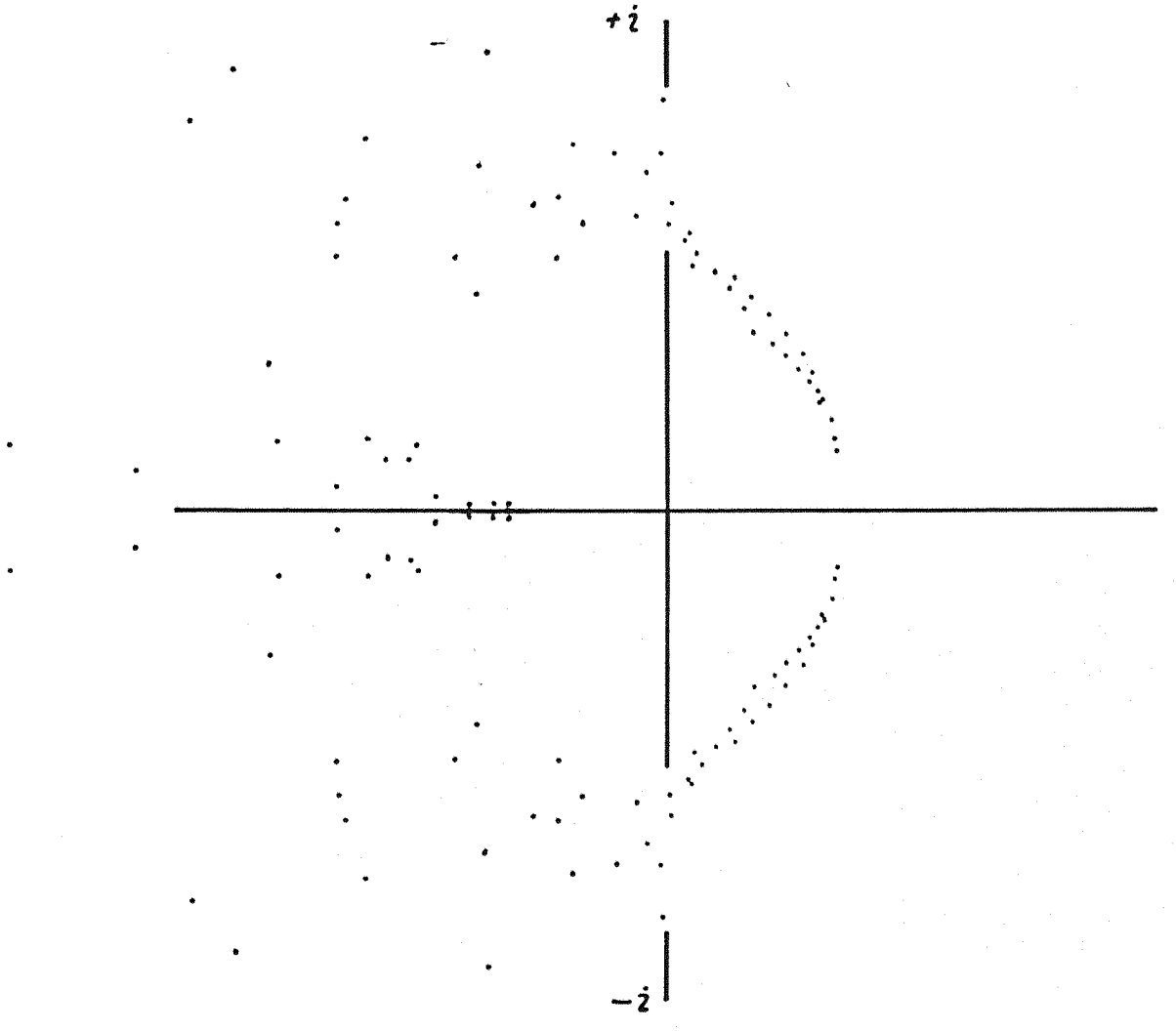


Figure 4

Some zeros of the partition function plotted in $\tanh^2 \beta$ for the 3 x 3 x 9 lattice Z(2) gauge model.

From these results it seems likely that the Z(2) gauge model has a line distribution of zeros passing through the real axis at β_c [7]. There is clearly some further structure away from this line although the present lattice is too small (and too anisotropic) to find it.

3.4 Short distance behaviour of the plaquette correlation function

We label the layers of the lattice as indicated in fig 5. The layers are periodically bounded. We arbitrarily introduce a labelling scheme for the plaquettes within a layer (figure 6). We will only deal with layer plaquettes in this work, so the layer number followed by the layer vector identifies a unique plaquette. We then have a shorthand for plaquette expectation values:

$$\langle \square_{(a,\,b,\,c)} \rangle = \frac{\sum_{\text{configs}} U_{ij}U_{jk}U_{k\ell}U_{\ell i} \exp\left[\beta \sum_{\text{plaquettes}} UUUU\right]}{\sum_{\text{configs}} \exp\left[\beta \sum_{\text{plaquettes}} UUUU\right]} \qquad (13)$$

where U_{ij}, U_{jk}, $U_{k\ell}$ and $U_{\ell i}$ are the link variables around plaquette (b, c) in the ath lattice layer. We obtain expressions for the following quantities:

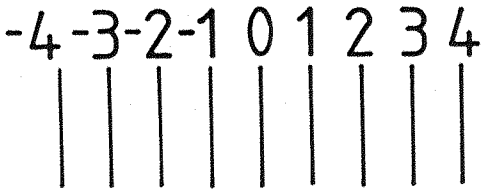
$$\begin{aligned}
&\langle \square_{(0,\,0,\,0)} \square_{(1,\,b,\,c)} \rangle, \langle \square_{(0,\,0,\,0)} \rangle, \\
&\langle \square_{(-1,\,0,\,0)} \square_{(1,\,b,\,c)} \rangle, \langle \square_{(1,\,0,\,0)} \rangle, \\
&\langle \square_{(-2,\,0,\,0)} \square_{(2,\,b,\,c)} \rangle, \langle \square_{(2,\,0,\,0)} \rangle
\end{aligned} \qquad (14)$$

for all b and c. Note that the lattice is symmetrical under (a, b, c) $\rightarrow$ (-a, b, c). We write the subtracted correlation function as

$$\begin{aligned}
D(|a_1 - a_2|, b, c) = &\langle \square_{(a_1,\,0,\,0)} \square_{(a_2,\,b,\,c)} \rangle \\
&- \langle \square_{(a_1,\,0,\,0)} \rangle \langle \square_{(a_2,\,b,\,c)} \rangle
\end{aligned} \qquad (15)$$

We may construct linear combinations within a layer:

Figure 5



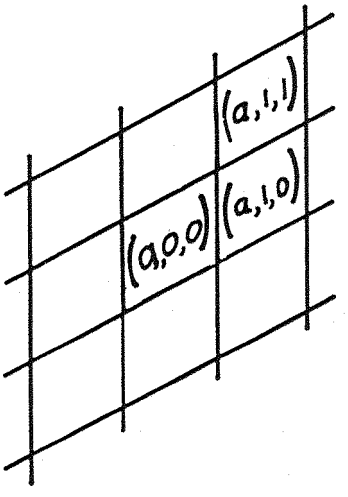


Figure 6

Figure 5: Labelling scheme for lattice layers.

Figure 6: Labelling scheme for plaquettes in layer a.

$$D_{(p_1, p_2)}(|a_1 - a_2|) = \sum_{b,c=-1,0,+1} \exp\left[\frac{2\pi i}{3}(p_1 b + p_2 c)\right]$$

$$< \square(a_1, b, c)\, \square(a_2, 0, 0) > - < \square(a_1, b, c) >< \square(a_2, 0, 0) >$$

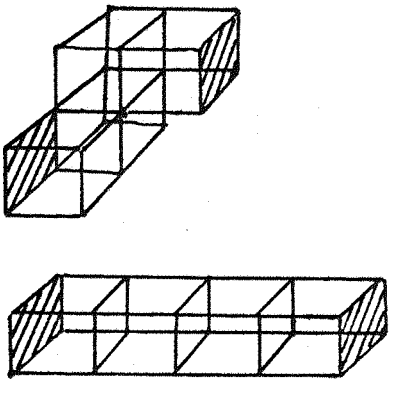
(16)

where p_1, p_2 take values from { 1, 0, -1 }.

The squared plaquette expectation value $< \square_{(a, b, c)} >^2$ is plotted against β in figure 7 for a = 0. Discrepancies with $a \neq 0$ are small. The worst case is a = 2 at β_c with a discrepancy of about 1 %. The D(n) are also small, however, and we avoid using plaquettes with $a \geqslant 3$ to try and minimise boundary effects.

The subtracted plaquette-plaquette expectation values are plotted against β in figure 8. In the small β region the lattice separation (2, 0, 0) gives a similar correlation to (1, 1, 0), and has a greater correlation than (1, 1, 1). In this sense the effective separation at (a, b, c) is given by a + b + c. Close to β_c, however, the true separation $\sqrt{a^2 + b^2 + c^2}$ pertains.

Separations (2, 1, 0) and (1, 1, 1) (and (4, 0, 0) with (2, 1, 1)) do not give identical small β behaviour on this small lattice because of the periodic boundaries. On an infinite lattice the lowest order strong coupling (small β) diagrams for these expectations are tubes connecting the relevant plaquettes (see appendix I, and for example figure 9).



$\} O(\tanh^{16} \beta)$

Fig 9

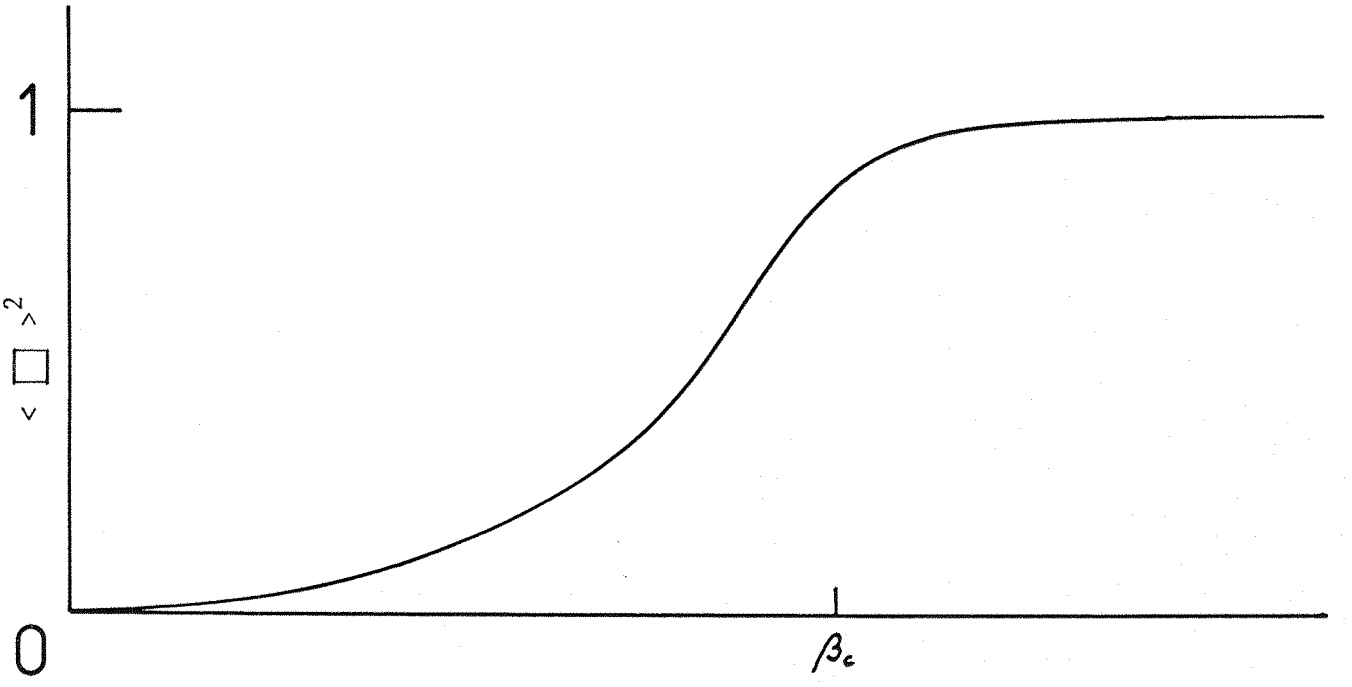


Figure 7

Plaquette expectation value squared vs. coupling for plaquette (0, 0, 0).

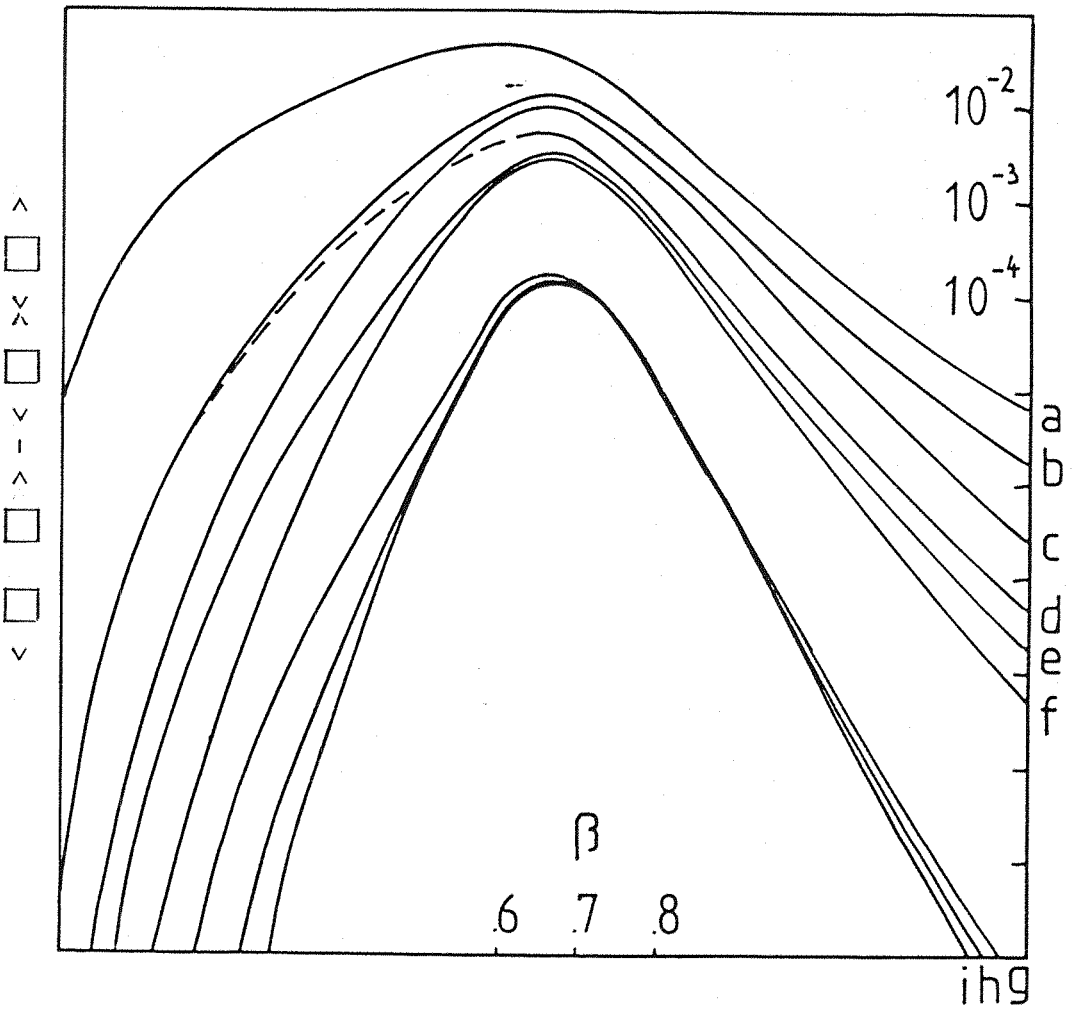
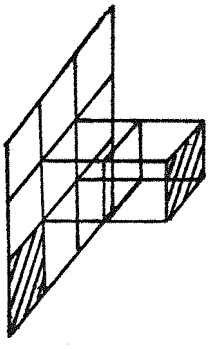


Figure 8

Subtracted plaquette-plaquette expectation values against coupling for the following plaquette-plaquette separations:

a) (1, 0, 0)	b) (1, 1, 0)	c) (1, 1, 1)
d) (2, 0, 0)	c) (2, 1, 0)	f) (2, 1, 1)
g) (4, 0, 0)	h) (4, 1, 0)	i) (4, 1, 1)

However with periodic boundaries it is possible to form diagrams at lower order (figure 10).



$O(\tanh^{15} \beta)$

Fig 10

Nonetheless we observe that true separation becomes more important as β_c is approached; specifically that D(4, 0, 0), D(4, 1, 0) and D(4, 1, 1) become more closely grouped than the other combinations (since $\sqrt{a^2 + b^2 + c^2} \simeq a$ when $a >> b, c$) and so on.

The simplest way to try and extract an approximate mass or inverse correlation length m_{approx} from these figures is to fit the (a, 0, 0) separations to the form obtained in the previous chapter. The results from fitting D(4, 0, 0)/D(2, 0, 0) and D(2, 0, 0)/D(1, 0, 0) are shown in figure 11.

To leading order (at small β) we have

$$D(a, 0, 0) = (\tanh \beta)^{4a} \tag{17}$$

and applying equation (66) of chapter 2 we find

$$m_{approx} = \frac{1}{a_1 - a_2} \left[\ell n \ \frac{a_2}{a_1} + \ell n \left(\frac{D(a_2, 0, 0)}{D(a_1, 0, 0)} \right) \right]$$

$$= \frac{1}{a_1 - a_2} \ell n \left(\frac{a_2}{a_1} \right) + m + O(\beta^2) \tag{18}$$

where m is the mass calculated in chapter 2 (equation(67)). The first term approaches zero as $a_1, a_2 \to \infty$, but for $a_1, a_2 \leqslant 4$ and small β it dominates the error in m_{approx}. This is not a very interesting short distance effect. We eliminate it by taking the linear combination

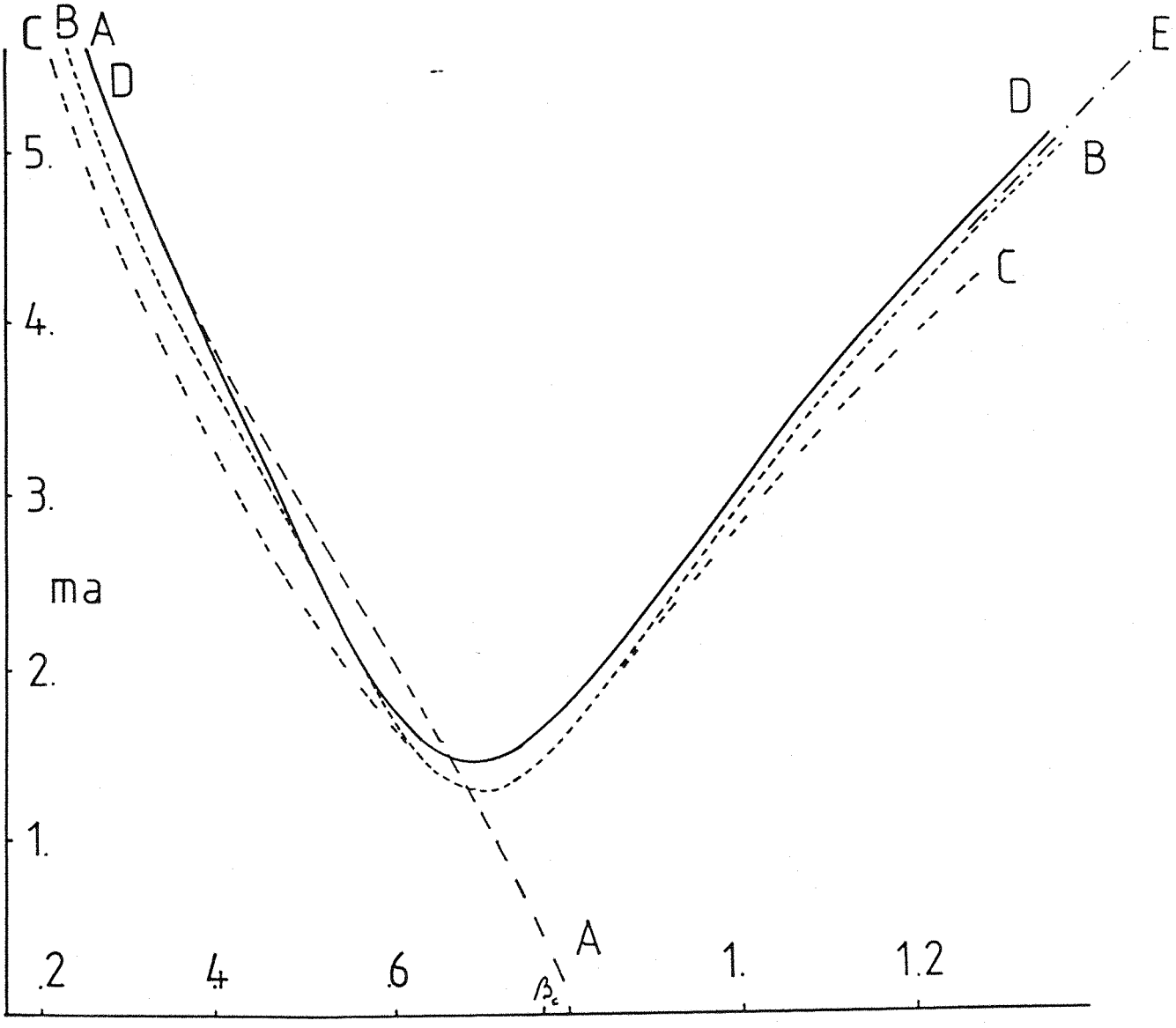


Figure 11

Approximations to the lowest lying mass (in units of the inverse lattice spacing) against coupling as follows:

A) Strong coupling expansion (Munster [1])

B) Exponential fit to D(4, 0, 0)/D(2, 0, 0)

C) Exponential fit to D(2, 0, 0)/D(1, 0, 0)

D) Exponential fit to $D_{(0,\ 0)}{}^{(4)}/D_{(0,\ 0)}{}^{(2)}$

E) Weak coupling expansion (Chapter 2).

$D_{(0,\ 0)}$ in equation 12. Then, in chapter 2, the implied modification of equation (55) will contain $\delta(p,\ 0)$ (see also equation (49)). This makes the sum over momentum states trivial and we obtain

$$D_{(0,\ 0)}\ (\ |\ a_1 - a_2\ |\ ) \approx \exp\ (-m\ |\ a_1 - a_2\ |\ ) \qquad (19)$$

The strong coupling (small β) expansion for m, including terms up to $\tanh^8\beta$ (Münster [1]), together with the large β expansion obtained in the previous chapter and $m_{4/2}$, the fit from $D_{(0,\ 0)}\ (4)/D_{(0,\ 0)}\ (2)$, are all shown in figure 11. It is interesting to try and see roughly where and how $m_{4/2}$ becomes unrealistic. First note that Münster's Expansion does not fall to zero particulary close to βc. For m greater than about two inverse lattice spacings (see fig 11) the expansion may be no better an approximation than $m_{4/2}$. Extending the series is an unnecessary labour however, since we can clearly see that $m_{4/2}$ has failed by the time the correlation length reaches about .7 lattice spacings. This is smaller than might have been expected, nonetheless it must be attributed directed to the finite lattice size and not to a breakdown in the fitting procedure. Figure 12 shows the fit $m_{4/2}$ compared with that obtained from $D_{(0,\ 0)}\ (2)/D_{(0,\ 0)}\ (1)$. If short distance effects of the kind anticipated in the previous chapter were already spoiling the exponential form at $m \simeq 1.5$ inverse lattice spacings one would expect an inconsistent prediction here, however the inconsistency is small.

This suggests that larger masses remain relatively large (in transfer matrix terms smaller eigenvalues remain relatively small) for this finite model as the ∞ lattice critical temperature is approached.

As we have noted before, the transfer matrix associated with a 3 x 3 layer in this model is a finite matrix. Thus it is possible in principle, to check the smallest masses by diagonalising the transfer matrix numerically, thus obtaining masses for a 3 x 3 x infinite system at any β (see section 2.3). In this approach, however, one must work with the gauge model transfer matrix from the start, and we have seen that the first, fifth and smaller eigenvalues would be required. This makes the calculation awkward in practice [17]. The corresponding calculation for the Ising model is relatively straightforward. We discuss this briefly in appendix II.

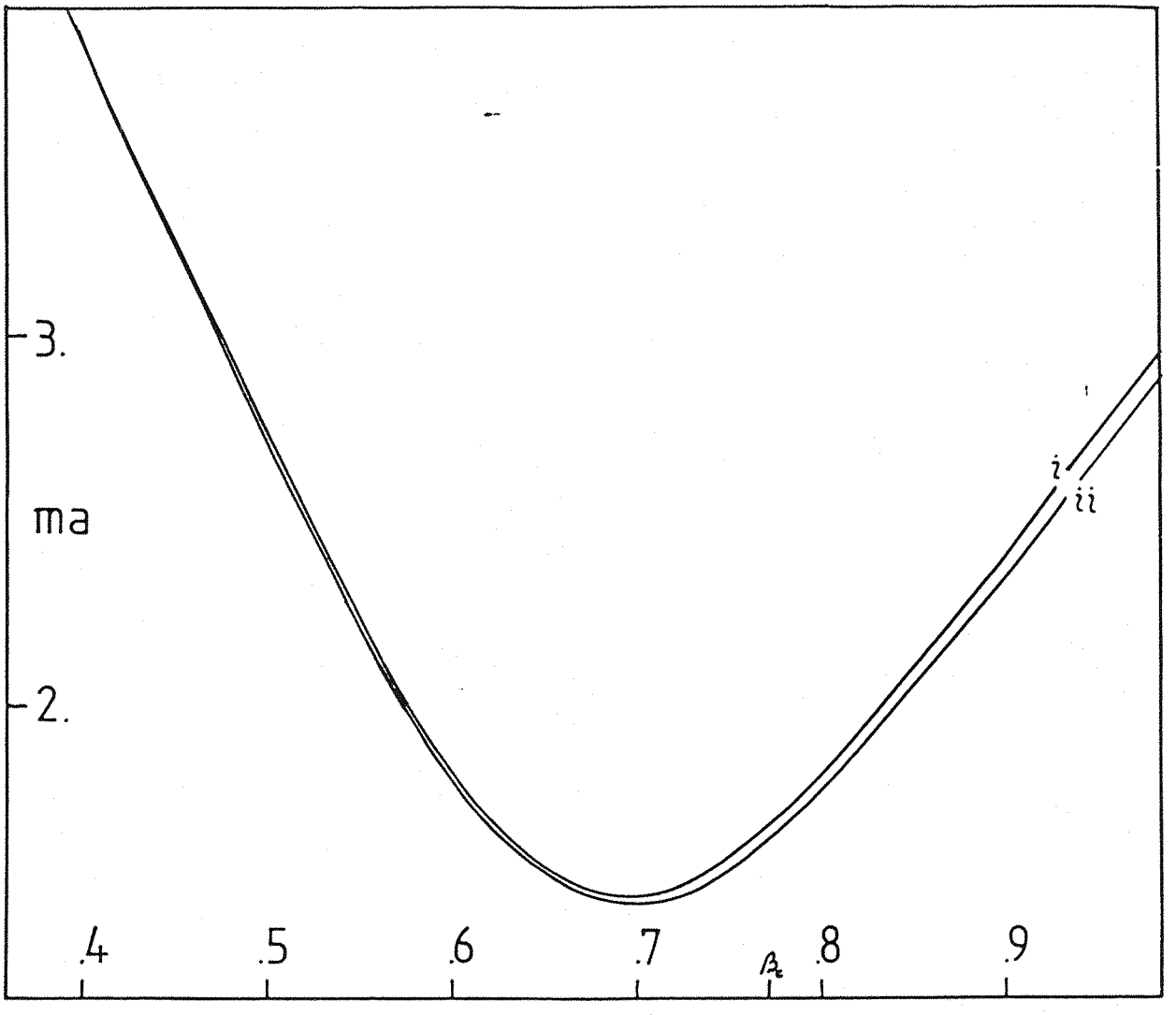


Figure 12

Approximations to the lowest lying mass (in units of the inverse lattice spacing) against coupling as follows:

i) exponential fit to $D_{(0,\ 0)}{}^{(4)}/D_{(0,\ 0)}{}^{(2)}$

ii) exponential fit to $D_{(0,\ 0)}{}^{(2)}/D_{(0,\ 0)}{}^{(1)}$

In this section we have confirmed that the path sums associated with a finite lattice $Z(2)$ gauge model form closed expressions approximating the thermodynamic limit for both strong and weak coupling. The finite lattice allows consistent measurement of a correlation length, and the approximation breaks down only as this length approaches the lattice size.

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* 165 out of 171 zeros have been found in the available computer time.

4. Discussion

We have checked the exponential fit method for extracting correlation lengths from finite lattice calculations. It is not sensible to use this method when the length approaches the lattice size (close to β_c). We have not yet been able to make a precise statement in this respect. Unfortunately a lattice with a 2 x 2 section is too small to extract masses in this way. A calculation with a 4 x 4 x n lattice is in progress, but will take considerable time. A line distribution of zeros of the partition function (through β_c) is suggested in agreement with the form expected from Ising model calculations [1]. It may be possible to extend consideration in this respect to Z(3) models.

The computing side of this work was aided greatly by the existence of a duality transformation (appendix I) between the Z(2) gauge and spin models. In four dimensions the Z(2) gauge model is self dual [2]. The duality transformation may be extended to Z(N) symmetries for $N \to \infty$ [3] although non-abelian groups have so far proved inaccesible. Recently Bhanot and Rebbi [4] and Petcher and Weingarten [5] have shown that the discrete subgroups of SU(2) reproduce SU(2) results over a wide range of coupling constant (although they have a phase transition at large β). Monte Carlo calculations are more efficiently implemented with discrete groups, but these results are ultimately prone to the same statistical problems with numerically small observables as for SU(2). It may be possible to obtain duality relations for discrete non-abelian groups. Calculations for appropriate quantities in small β could give accurate large β results by duality (c.f. section 2.5 and [6]).

In chapter 3 we saw how effective 'separations' on the lattice (as felt by correlation functions) changed towards true separations close to β_c. So far we have not considered the problem of absence of Euclidean invariance in the lattice scheme. We have used a cubic lattice because it makes duality transformations easy. Beyond this however, a program on a conventional computer (not a parallel processor) favours no particular lattice structure. A worrying feature of the cubic lattice is that it has preferred directions, however far away you look. A random lattice looks more isotropic at large distances. It would be worth trying to confirm universality (of critical behaviour among models

with the same dimensionality and interaction type) for lattice gauge theories. This should be relatively easy, even for SU(2). Since the usual hypercubical lattice is not expected to have physical relevance this would provide an important check on the lattice approach.

Finally it would be interesting to try and find β_c for the Z(2) gauge model using the finite lattice renormalisation transformations mentioned briefly in Chapter 1. Calculations of this kind have been performed for the Ising model [7], but it could be instructive to deal with the different problems posed by a gauge interaction while maintaining a simple and fairly well understood model.

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Appendix 1: The Ising model and Z(2) gauge model in three dimensions

A comprehensive survey of the relationship between spin and gauge models on flat three dimensional lattices is given by Savit [1] (see also the pioneering work of Wegner [3] and Balian, Drouffe and Itzykson [4]). We introduce modifications required for a periodic lattice [2].

The Z(2) gauge model partition function (eqn (2) of chapter 3 for example) may be written

$$Z(\beta) = \sum_{\substack{\text{configurations} \\ \text{of } U\text{'s}}} \prod_{\substack{\text{plaquettes} \\ ijk\ell}} \cosh\beta \sum_{K_p = \pm 1} (U_{ij}U_{jk}U_{k\ell}U_{\ell i} \tanh\beta)^{\frac{K_p + 1}{2}}$$

$$= \sum_{\substack{\text{configs} \\ U\text{'s}}} \sum_{\substack{\text{configs} \\ K\text{'s}}} \prod_{\substack{\text{plaquettes}}} (\cosh\beta \tanh^{\frac{K_p + 1}{2}}\beta) \prod_{\substack{\text{links} \\ ij}} U_{ij}^{\left(\sum_{p\,:\,ij} \frac{K_p}{2} \right)} \qquad (1)$$

- we have introduced $K_p = \pm 1$ on the plaquettes p and $\sum_{p\,:\,ij}$, a sum over the four plaquettes common to link ij. The advantage of this form is that terms in which any U_{ij} is raised to an odd power are cancelled in the sum over U_{ij} configurations. The remaining terms get a factor of 2 for each link and we may write

$$Z(\beta) \propto \sum_{\substack{\text{configs} \\ K_p\text{'s}}} \exp\left[\sum_{\text{plaquettes}} - K_p \tilde{\beta} \right] \left\{ \prod_{p\,:\,ij} K_p = +1 \right\} \qquad (2)$$

where $\tilde{\beta} = -\frac{1}{2} \ln(\tanh\beta)$. The expression in curly brackets means that we only consider configurations in which the product of four K_p's incident on every link gives + 1.

Our task is to show that this constrained sum may be written as an unconstrained sum over Ising spins with an Ising interaction. The situation we have at the moment is a sum over the small-β expansion terms for the gauge model partition function. We are summing over configurations of K_p's taking values on plaquettes under the constraint that the product of four incident on any link must be + 1. This means that the set of plaquettes with K_p taking value - 1 can be decomposed into closed surfaces in any allowed configuration. These configurations with closed surfaces are in one to one correspondence with strong coupling (small β) diagrams.

If the lattice is on a flat space then each of these closed surfaces may be considered as the boundary of a volume made of elementary cubes. We define a set of variables S_c on the cubes as follows. For an arbitrary first cube we put $S = + 1$ then the others are either put = + 1, if they are an even number of surface walls away, or otherwise put = - 1. The product of two cube variables with a common plaquette is now equal to K_p for that plaquette. In three dimensions there is a one to one correspondence between links and plaquettes and between sites and cubes (think of a lattice shifted so that sites lie at the centre of cubes of the original lattice).

We have seen that all the configurations in the constrained sum may be generated from configurations of S_c's taking values on cubes (or sites). The K_p's are obtained from the product of the two adjacent cubes. We have yet to show that all such cube configurations generate K_p configurations which satisfy the constraint. Consider a K_p constructed from adjacent S_c's

$$K_p = S_1 S_2 \tag{3}$$

We require

$$\prod_{\substack{\text{plaquettes} \\ \text{common to a link}}} K_p = + 1 \tag{4}$$

of course each S_c occurs twice in the product so the constraint is always satisfied. We write from (2) and (3):

$$Z(\beta) \propto \sum_{\substack{\text{configs} \\ S}} \exp\left[-\tilde{\beta} \sum_{\substack{\text{links} \\ ij}} S_i S_j \right] \qquad (5)$$

(we have chosen to put the S's on sites). This is the Ising model.

If the lattice is periodically bounded then the closed surfaces formed from plaquettes with $K_p = -1$ cannot always be considered as the boundaries of volumes. Consider a surface which is closed by virtue of the periodicity (a 'membrane' diagram in the strong coupling series). We cannot make a consistent assignment of cube variables. In other words there is no cube configuration which generates a 'membrane'.

Notice, however, that if we were to put an additional factor into the interaction so that

$$K_p = S_i \, \phi_{ij} \, S_j \qquad (6)$$

where $\phi_{ij}^{(12)} = -1$ for all links through a membrane
$= +1$ otherwise,

then this form generates all possible membrane configurations due to periodicity in a pair of directions. Further introducing $\phi_{ij}^{(13)}$ or $\phi_{ij}^{(23)}$ generates possible membranes due to periodicity in other pairs of directions. The complete partition function is thus obtained by summing over configurations of S_i's and over possible combinations of ϕ insertions.

Notice that a lattice periodic in two directions (as in chapter three) only requires one insertion.

It is possible to generalise this discussion to include Z(N) models. The non trivial ϕ's take values from the relevant representations of Z(N).

Expectation values of gauge invariant products of U's (closed loops, products of closed loops) are obtained as follows: Choose a surface of plaquettes bounded by the closed loop; change the sign of coupling on the corresponding links of the shifted 'dual' lattice Ising model; sum over configurations as above. This procedure has the effect of

generating configurations with $\prod_{p:ij} K_p = -1$ on the links of the closed loop so that a factor U_{ij} occurs in the expansion (1) to cancel that appearing from the expectation value.

References

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Appendix II: Small lattice results for the three dimensional Ising Model

We give the results of exact calculations for the specific heat, zeros of the partition function and spin-spin expectation values of the Ising model on a 3 x 3 x 8 lattice. These were preliminary calculations for the work discussed in Chapter 3.

Duality ensures that the locally and globally Z(2) symmetric lattice models have identical phase structure in three dimensions (reference [3] of appendix I). Nonetheless there are interesting differences between the two models. The Ising model has a spontaneous magnetisation or single spin expectation value as a local order parameter. Below the critical coupling value this quantity is zero as one might expect from the Z(2) symmetry of the Action:

$$A = \beta \sum_{\substack{\text{links} \\ ij}} \phi_i \, \phi_j \qquad (1)$$

(ϕ_i takes values from { $\pm$ 1 }). Above β_c this symmetry is spontaneously broken and the single spin expectation value becomes non-zero. This effect does not occur on a finite lattice so we break the symmetry by hand. In the model we discuss the symmetry is broken by fixing a layer of spins at each end of the lattice in the long direction to + 1. There are eight free layers. We may think of a periodic lattice with a single frozen layer or an infinite frozen lattice containing an unfrozen bubble. Either way one anticipates that at small β, where the correlation length is small, the effect of the frozen layers will not permeate into the lattice (and that the single spin expectation value remains very small), while at larger β the single spin expectation approaches unity as in the full model. We will see how these ideas are approximately realised.

The specific heat for the 3 x 3 x 8 model is given in figure 1. Intermediate results for a 3 x 3 x 4 model are also shown. There has been some use of finite size scaling arguments [1, 2, 3] in the literature to extract information about phase transitions from sequences of small lattices [4, 5, 6]. We have been dissuaded from attempting such calculations firstly because 2 x 2 x n lattices are too small and 4 x 4 x n lattices very awkward to calculate. Secondly

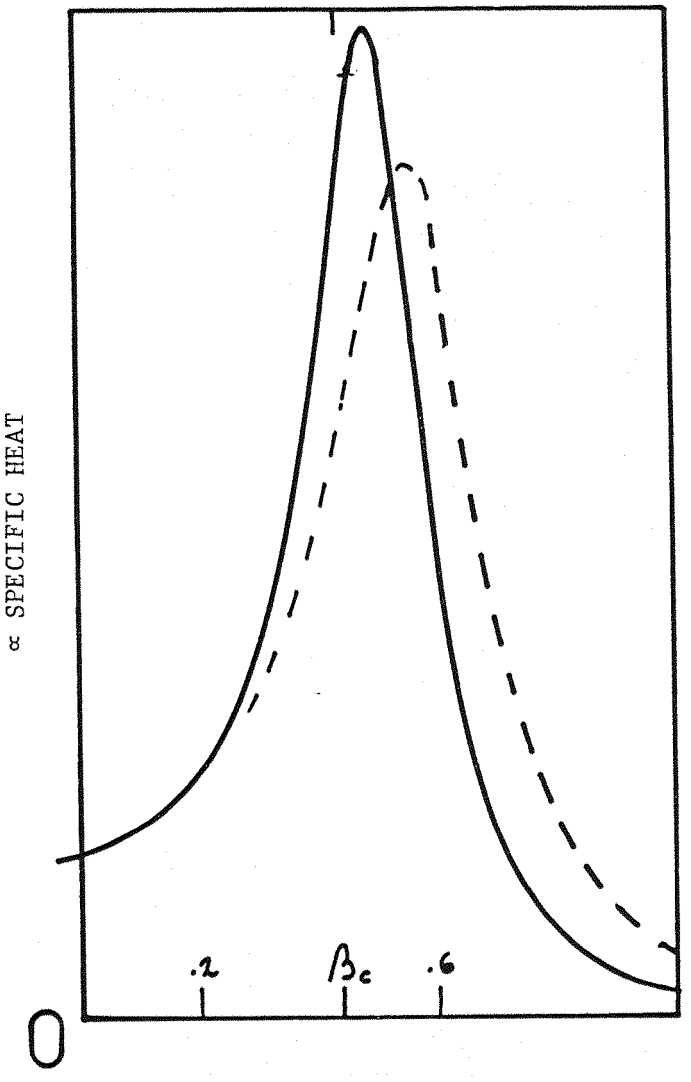

Figure 1

Specific heat vs. coupling constant for (solid line) 3 x 3 x 8 lattice and (dashed line) 3 x 3 x 4 lattice Ising models.

because a preliminary calculation using 2^3, 3^3 and 4^3 [7] lattices shows that the normal finite size scaling behaviour has not set in at these very small sizes. The specific heat curves are given in figure 2. Using the results of section 3.3 and the plots of zeros of the partition function for the 2^3 and 3^3 models in figure 3 it is easy to see why this is the case. In fact one may readily calculate the 2^3 partition function and obtain the zeros by hand:

$$Z(\beta) \alpha (e^{-4\beta} + 1)^4 (e^{-4\beta} + i)^2 (e^{-4\beta} - i)^2 (e^{-4\beta} - c)$$

$$(e^{-4\beta} - c^*)(e^{-4\beta} - \frac{1}{c})(e^{-4\beta} - \frac{1}{c^*})$$

where

$$c = (1 - \frac{1}{\sqrt{2}})(1 + i) \qquad (2)$$

Equation (11) of section 3.3 shows then that the peak in the specific heat will occur at approximately $\beta = .21$. Although the real component of the closest zero is closer and the imaginary component smaller for the 3^3 model, the magnitude is actually further away from β_c. In fact it is the size of the imaginary component of the nearest zero for the 2^3 model which pushes the magnitude close to β_c! In the light of these results the finite size scaling approach is not pursued.

The zeros of the partition function for the 3 x 3 x 8 model are given in figure 4. The situation is similar to that discussed in section 3.3 and reference [7].

In discussing spin-spin expectation values we adopt an equivalent notation to that of section 3.4, but label sites of the lattice rather than plaquettes. The unsubtracted spin-spin expectation values

$$< \phi_{(0,\,0,\,0)}\, \phi_{(1,\,0,\,0)} >, < \phi_{(1,\,0,\,0)} \phi_{(-1,\,0,\,0)} >< \phi_{(-1,\,0,\,0)} \phi_{(2,\,0,\,0)} >$$

and

$$< \phi_{(0,\,0,\,0)} >^2$$

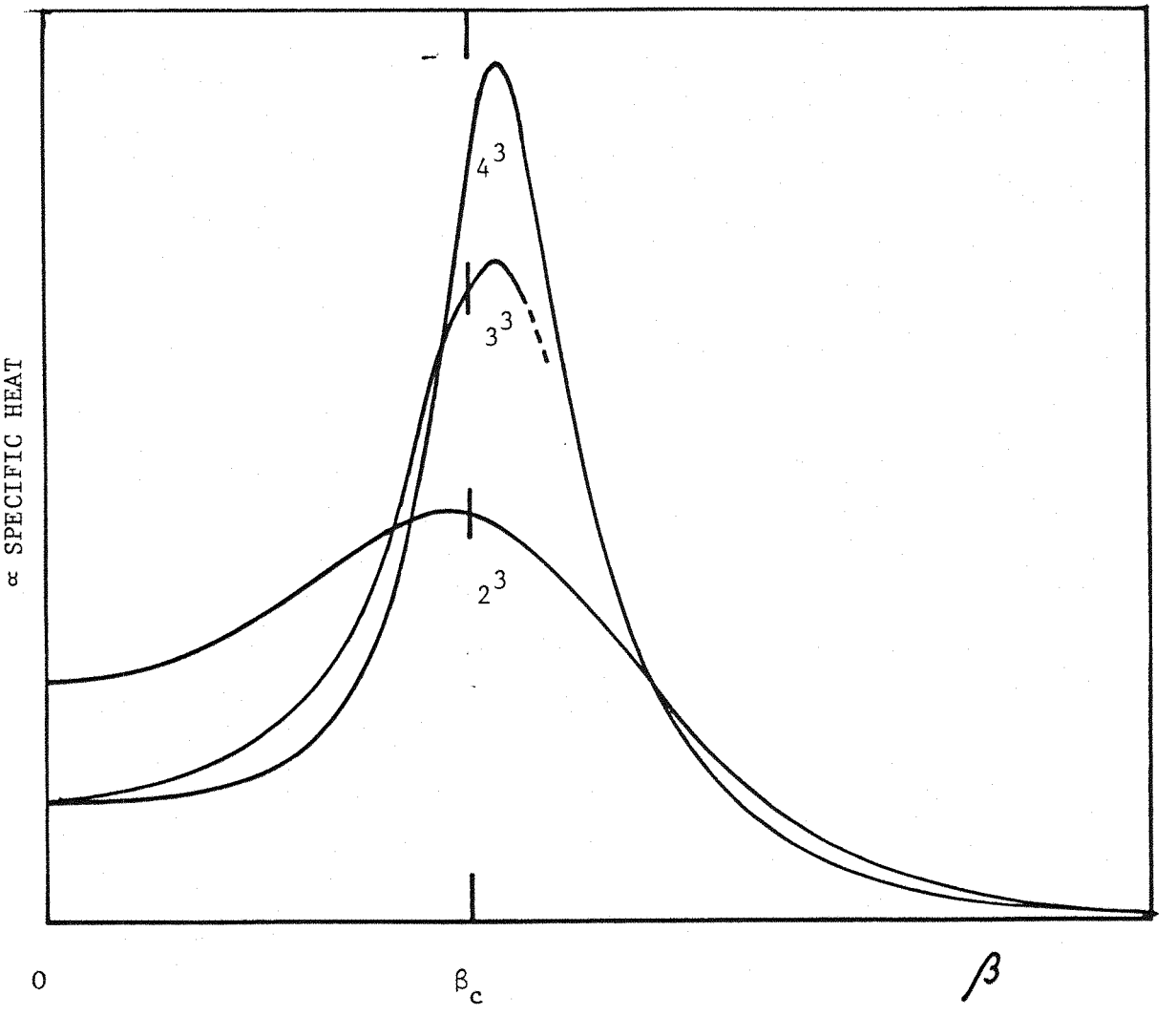

Figure 2

Specific heat vs. coupling constant for periodically bounded 2^3, 3^3 and 4^3 lattice Ising models.

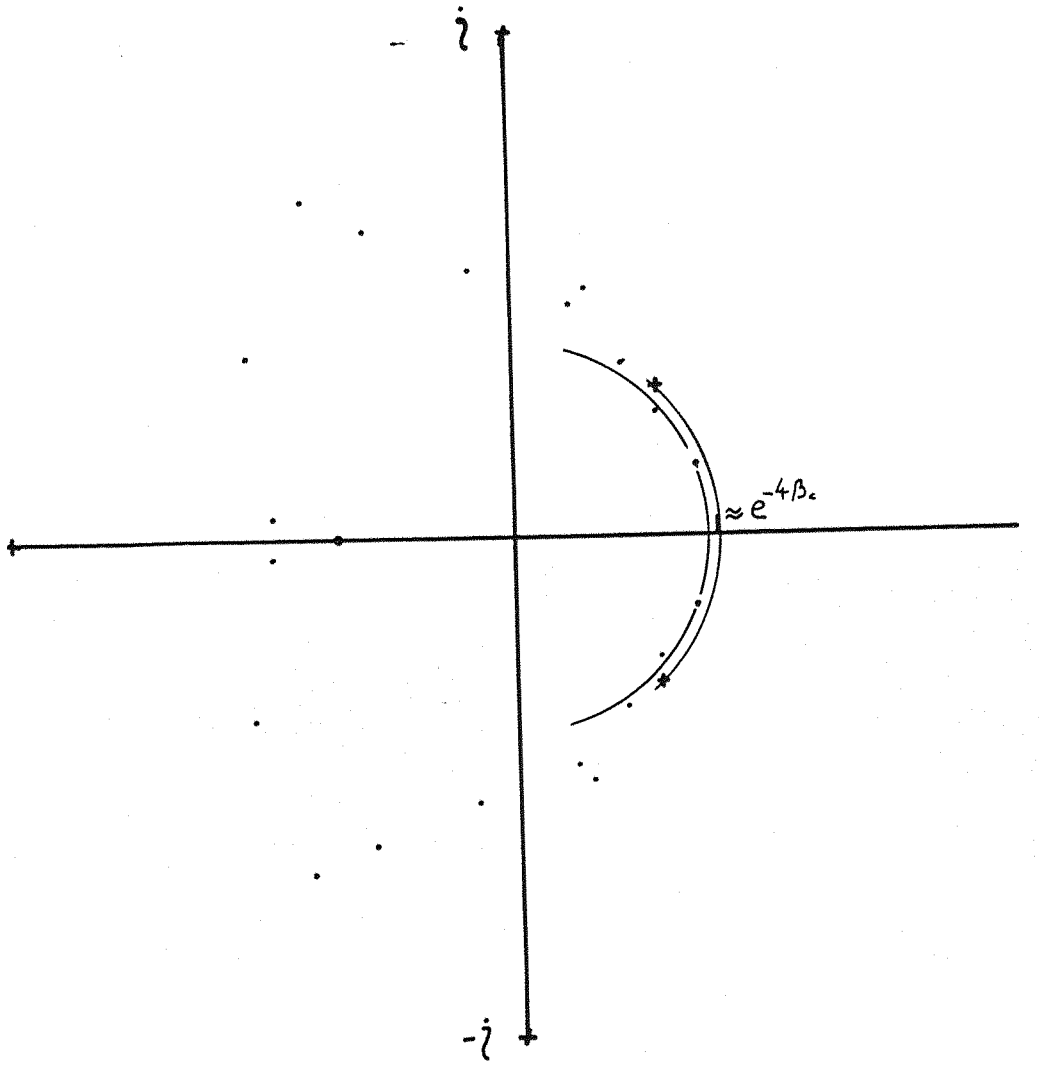

Figure 3

Zeros of the partition function in $e^{-4\beta}$ for periodically bounded 2^3 and 3^3 lattice Ising models (+'s and dots respectively). Lines of constant magnitude are drawn for the zeros nearest the real axis in each case (see section 3.3).

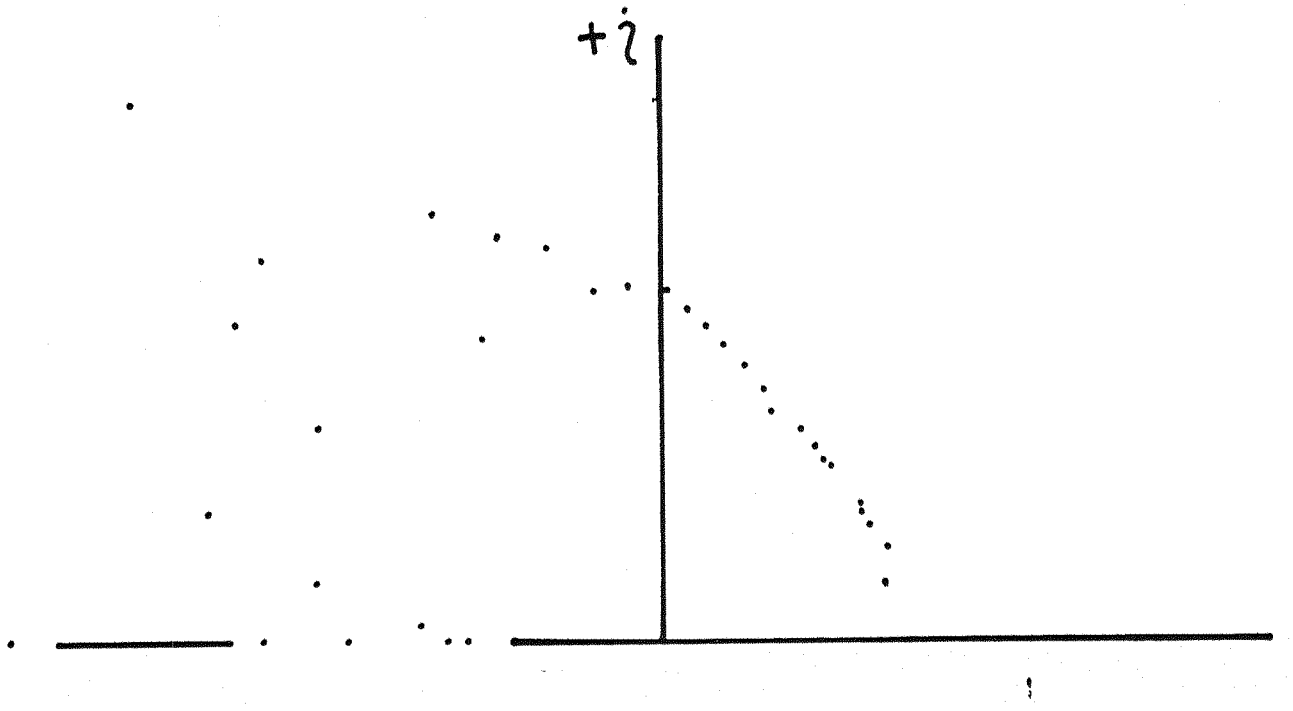

Figure 4

Zeros of the partition function in $e^{-4\beta}$ for the 3 x 3 x 8 lattice Ising model (zeros occur in complex conjugate pairs).

are shown as functions of β in figure 5. Notice that $< \phi_{(0,\,0,\,0)} >^2$ is extremely small at small β. In fact it has negligible effect up to $\beta \simeq .18$ (on the infinite lattice it would be zero up to $\beta_c \simeq .22$). Above β_c it rises very sharply to + 1, which is at least qualitatively the behaviour we require.

Approximate fits to the lowest lying mass (inverse correlation length) are obtained by comparing the subtracted values at separations 4/3, 3/2 and 2/1. We have not taken the linear combinations necessary for momentum eigenstates. We anticipate a poor approximation for intermediate and large β in any case because of the severe boundary conditions. The mass fits m_{43}, m_{32} and m_{21} are shown in figure 6. The small β expansion for the mass gap has been obtained to high order by Fisher and Burford [8]. This result is also shown in figure 6. The discrepancies at small β are exactly those described in section 3.4, so a momentum eigenstate fit would give good agreement with the series expansion result at small β. However at large β there are problems. We may write the subtracted expectation value for separated spins as

$$Z^2 \left(< \phi_{(0,\,0,\,0)}\, \phi_{(a,\,b,\,c)} > - < \phi_{(0,\,0,\,0)} >< \phi_{(a,\,b,\,c)} > \right)$$

$$= Z \left[\sum_{\substack{\text{configurations} \\ \text{with } \phi_{(0,\,0,\,0)} \cdot \phi_{(a,\,b,\,c)} = +1}} \exp\left[A\right] - \sum_{\substack{\text{configurations} \\ \text{with } \phi . \phi = -1}} \exp\left[A\right] \right]$$

$$- \left(\sum_{\{\, \phi_{(0,\,0,\,0)} = +1 \,\}} \exp\left[A\right] - \sum_{\{\, \phi = -1 \,\}} \exp\left[A\right] \right) .$$

$$\cdot \left(\sum_{\{\, \phi_{(a,\,b,\,c)} = +1 \,\}} \exp\left[A\right] - \sum_{\{\, \phi = -1 \,\}} \exp\left[A\right] \right) \qquad (3)$$

The dominant term at large β has all spins aligned, this cancels in (3). Next leading terms have one spin flipped, these also cancel. In general on an infinite lattice the cancellations proceed up to an order proportional to the separation of the spins. This makes a large β expansion for the subtracted quantity difficult to obtain. Consider the case of separation one lattice spacing. Here the leading non-vanishing term comes from two adjacent spins flipped. The configuration

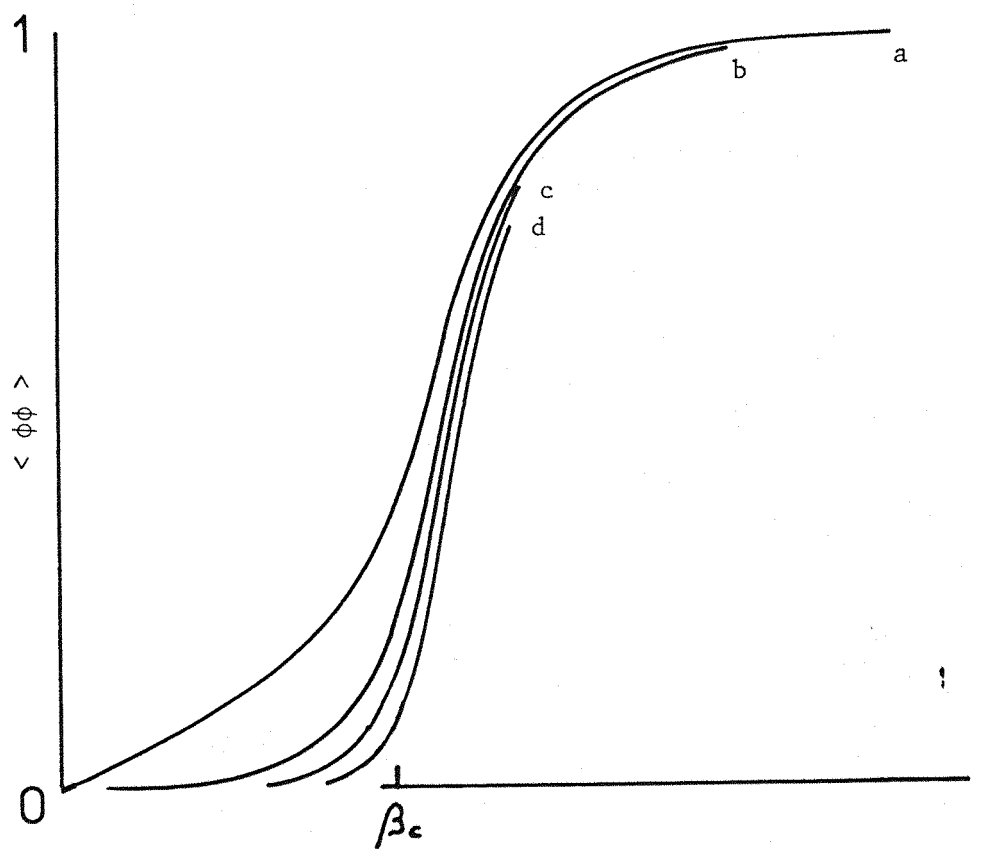

Figure 5

Unsubtracted spin-spin expectation values vs. coupling for the 3 x 3 x 8 lattice Ising model.

a) $< \phi_{(0,\ 0,\ 0)}\ \phi_{(1,\ 0,\ 0)} >$

b) $< \phi_{(-1,\ 0,\ 0)}\ \phi_{(1,\ 0,\ 0)} >$

c) $< \phi_{(-1,\ 0,\ 0)}\ \phi_{(2,\ 0,\ 0)} >$

d) $< \phi_{(0,\ 0,\ 0)} >^2$

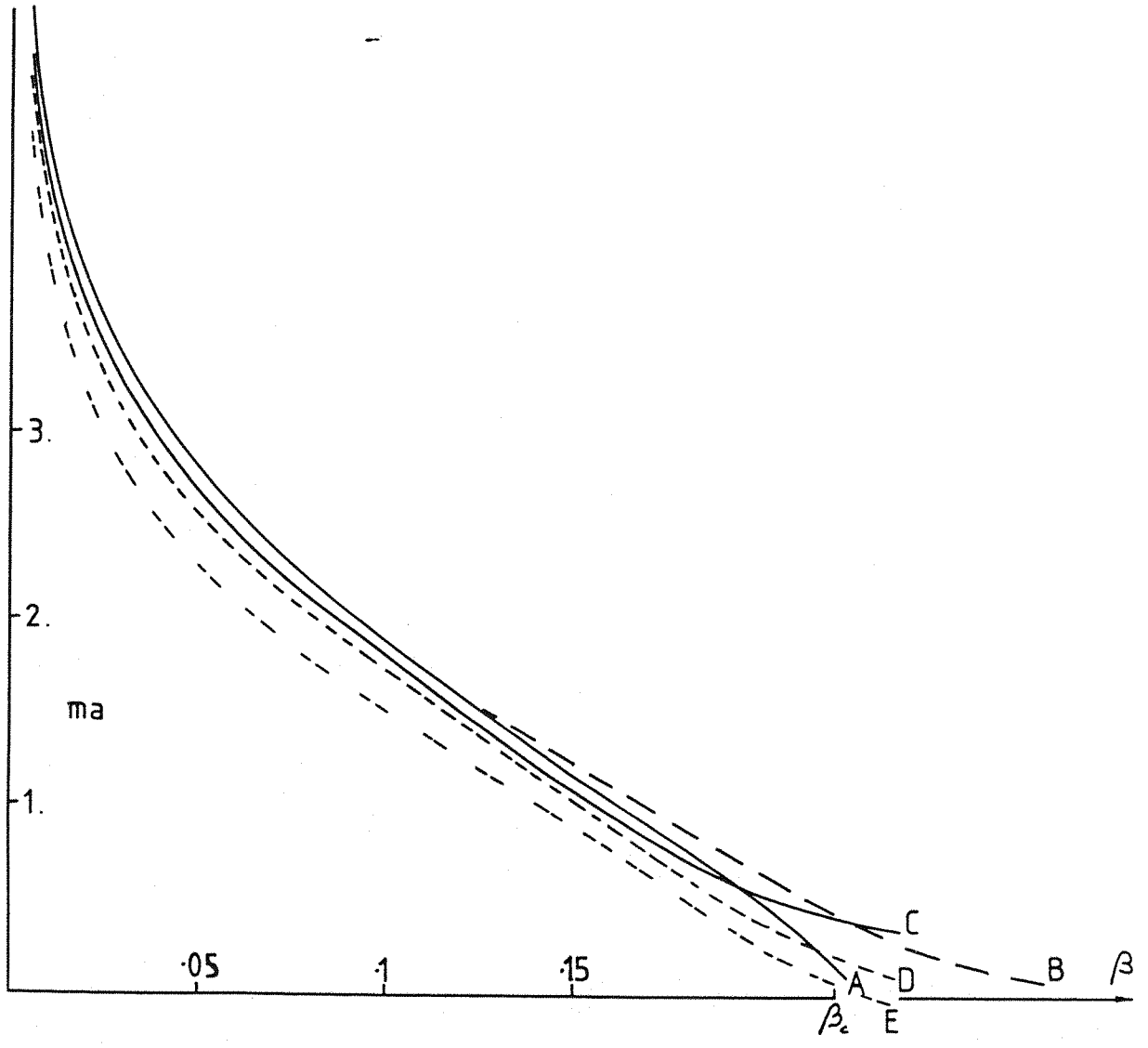

Figure 6

Approximations to the lowest lying mass (in units of the inverse lattice spacing) as follows:

A) Strong coupling expansion (Fisher and Burford [8])
B) Lowest lying mass for 3 x 3 x ∞ lattice
C) Exponential fit from D(4, 0, 0)/D(3, 0, 0)
D) Exponential fit from D(3, 0, 0)/D(2, 0, 0)
E) Exponential fit from D(2, 0, 0)/D(1, 0, 0)

- we have adopted the notation of section 3.4.

in which the two spins being measured are flipped contributes +1 to $< \phi\phi >$ and -1 to $< \phi >$. Since the flipped spins are adjacent, ten links make a negative contribution to the action. In the case of separation two lattice spacings the important configuration is that with three adjacent spins flipped in a row, including the two being measured. Here fourteen links make a negative contribution to the action, and so it goes on. Unfortunately at separation three the important configuration has eighteen negative links and this is precisely the number required to introduce an arbitrarily thick wall of flipped spins into the small lattice (by virtue of the periodicity). Configurations of this kind also contribute to the subtracted expectation value thus spoiling the large β behaviour at separation three and above.

With this in mind it is impossible to check the consistency of the pure exponential fit close to β_c by comparing results from different pairs of separations. However, this fit clearly breaks down for the separation one case. The mass gap estimate actually takes small negative values close to β_c! Presumably we should include the effect of further masses (see chapter 2).

We have obtained the transfer matrix for the 3 x 3 layer in this model. The largest eigenvalue and corresponding eigenvector were found by repeated application to a random vector at fixed β. The next largest eigenvalue was found by repeated application to a vector orthogonal to the first eigenvector. The gap obtained from this calculation gives the lowest lying mass for the 3 x 3 x ∞ lattice*. Results were obtained numerically for several values of β and are included in figure 6. The correlation length for this model is about $2\frac{1}{2}$ lattice spacings at β_c. If the large β behaviour discussed above has not set in at β_c then $\frac{1}{m_{43}}$ also gives an estimate of about $2\frac{1}{2}$ lattice spacings. This is roughly the effective size of the system (see [2, 3], in contrast with the gauge model case in chapter 3).

* without symmetry breaking.

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Appendix III. The partition functions

The Z(2) gauge model partition function is given in tables 1 and 2 in the following format:

Table 1 : N - 42 C_N

Table 2 : N - 41 C_N

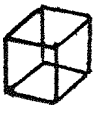
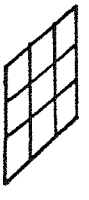
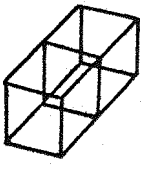
where

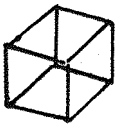
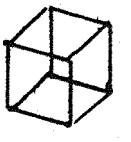
$$Z(\beta) = \sum_N C_N (\cosh \beta)^N (\sinh \beta)^{225-N} \qquad (1)$$

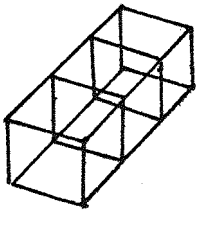
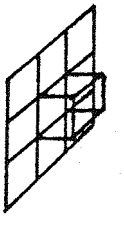
The Ising model partition function is then obtained from

$$Z_I(\beta) = \sum_{N \text{ odd}} C_N e^{(2N-225)\beta} \qquad (2)$$

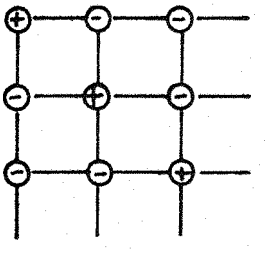
Now $\sum_{N \text{ even}} C_N = \sum_{N \text{ odd}} C_N = 2^{72}$. The leading diagrams in the strong coupling series (appendix 1) are:

$1 + (3\times3\times8)$  $+ 9$  $+ (3\times3\times7\times3 + 9\times2)$ 

$+ \dfrac{((6\times9\times(72-7) + 2\times9\times(72-6))}{2}$  

$+8\times6)$  $+ (7\times18 + 2\times9)$  $+$ etc.

and N_{min} is obtained from

(2×3) 

alternating + 1 $\leftrightarrows$ - 1 between each layer except where interlayer interactions are modified.

Table 1: N_{even}

N-42	C_N
12	6.000000000000000000000000000000
14	1305.000000000000000000000000000000
16	127935.000000000000000000000000000000
18	6294480.000000000000000000000000000000
20	178936560.000000000000000000000000000000
22	3296144790.000000000000000000000000000000
24	43830342258.000000000000000000000000000000
26	449839669126.000000000000000000000000000000
28	3729561635376.000000000000000000000000000000
30	25809635369638.000000000000000000000000000000
32	152806602513834.000000000000000000000000000000
34	788613026258034.000000000000000000000000000000
36	3599267018353176.000000000000000000000000000000
38	14689399268536762.000000000000000000000000000000
40	54064389147444999.000000000000000000000000000000
42	180602635931505621.000000000000000000000000000000
44	550214210941040979.000000000000000000000000000000
46	1534153492780610166.000000000000000000000000000000
48	3924843953565828075.000000000000000000000000000000
50	9228253330045442883.000000000000000000000000000000
52	19961872226508405663.000000000000000000000000000000
54	39745808295680145219.000000000000000000000000000000
56	72855773929650908646.000000000000000000000000000000
58	122946480389083087809.000000000000000000000000000000
60	191004471546768152144.000000000000000000000000000000
62	273223551471781489272.000000000000000000000000000000
64	360045960655956464673.000000000000000000000000000000
66	437502366787029346083.000000000000000000000000000000
68	490942710837563194845.000000000000000000000000000000
70	509782117035212661339.000000000000000000000000000000
72	491053633117520017659.000000000000000000000000000000
74	440081459479757594403.000000000000000000000000000000
76	368142678605547158913.000000000000000000000000000000
78	288481513009450650582.000000000000000000000000000000
80	212556361914823247649.000000000000000000000000000000
82	147840613979033860614.000000000000000000000000000000
84	97462337627090126584.000000000000000000000000000000
86	61148613890723145435.000000000000000000000000000000
88	36662043172657605010.000000000000000000000000000000
90	21088831329808529634.000000000000000000000000000000
92	11682412046185568535.000000000000000000000000000000
94	6254229898689498342.000000000000000000000000000000
96	3246005514708544638.000000000000000000000000000000
98	1637865875736981477.000000000000000000000000000000
100	805438012305423231.000000000000000000000000000000
102	386852981036077642.000000000000000000000000000000
104	181817533382490999.000000000000000000000000000000
106	83755364927526351.000000000000000000000000000000
108	37869997817512065.000000000000000000000000000000
110	16827424834909239.000000000000000000000000000000
112	7356053023241589.000000000000000000000000000000
114	3166465028057427.000000000000000000000000000000
116	1343213873721531.000000000000000000000000000000
118	561912740115957.000000000000000000000000000000
120	231924199734606.000000000000000000000000000000
122	94495805089977.000000000000000000000000000000
124	38018357846802.000000000000000000000000000000
126	15102008532000.000000000000000000000000000000
128	5928096572073.000000000000000000000000000000
130	2296108128537.000000000000000000000000000000
132	878363951991.000000000000000000000000000000
134	331960010211.000000000000000000000000000000
136	123255778320.000000000000000000000000000000
138	45388468785.000000000000000000000000000000
140	16373642877.000000000000000000000000000000
142	5787729027.000000000000000000000000000000
144	2049208632.000000000000000000000000000000
146	689284188.000000000000000000000000000000
148	232590726.000000000000000000000000000000
150	77747871.000000000000000000000000000000
152	23796432.000000000000000000000000000000
154	7679574.000000000000000000000000000000
156	2314095.000000000000000000000000000000
158	656532.000000000000000000000000000000
160	182934.000000000000000000000000000000
162	51621.000000000000000000000000000000
164	13437.000000000000000000000000000000
166	2214.000000000000000000000000000000
168	888.000000000000000000000000000000
170	144.000000000000000000000000000000
174	9.000000000000000000000000000000

Table 2: N_{odd}

N-41	C_N
16	11022.000000000000000000000000000000
18	967122.000000000000000000000000000000
20	35847522.000000000000000000000000000000
22	803417196.000000000000000000000000000000
24	12459583179.000000000000000000000000000000
26	144589216563.000000000000000000000000000000
28	1327273175916.000000000000000000000000000000
30	10018239598908.000000000000000000000000000000
32	63954934786818.000000000000000000000000000000
34	352798696037130.000000000000000000000000000000
36	1709543087153388.000000000000000000000000000000
38	7369288131395052.000000000000000000000000000000
40	28534828993879920.000000000000000000000000000000
42	99984895021006974.000000000000000000000000000000
44	318801975318792918.000000000000000000000000000000
46	928820470989290094.000000000000000000000000000000
48	2480079218869462932.000000000000000000000000000000
50	6081623733613464873.000000000000000000000000000000
52	13714153943172156393.000000000000000000000000000000
54	28460200048491737871.000000000000000000000000000000
56	54370909595013164766.000000000000000000000000000000
58	95627160731718259245.000000000000000000000000000000
60	154834800851024202645.000000000000000000000000000000
62	230809119841585027107.000000000000000000000000000000
64	316862925321589457331.000000000000000000000000000000
66	400900400043483977172.000000000000000000000000000000
68	468034270284993455493.000000000000000000000000000000
70	505075711259927490264.000000000000000000000000000000
72	504960624323900633952.000000000000000000000000000000
74	468989370429027447951.000000000000000000000000000000
76	405901868723822227773.000000000000000000000000000000
78	328484201241700196307.000000000000000000000000000000
80	249478919342471963511.000000000000000000000000000000
82	178506612971536404804.000000000000000000000000000000
84	120812960724554464407.000000000000000000000000000000
86	77658126688632679317.000000000000000000000000000000
88	47605529589757338714.000000000000000000000000000000
90	27943410113620492077.000000000000000000000000000000
92	15766649002107499140.000000000000000000000000000000
94	8582594165748597438.000000000000000000000000000000
96	4522346387357505345.000000000000000000000000000000
98	2313512352837345240.000000000000000000000000000000
100	1152093085841082420.000000000000000000000000000000
102	559775506039573599.000000000000000000000000000000
104	265904500429469988.000000000000000000000000000000
106	123704369853861270.000000000000000000000000000000
108	56448610811069949.000000000000000000000000000000
110	25298708862505995.000000000000000000000000000000
112	11148354240813582.000000000000000000000000000000
114	4835128418022933.000000000000000000000000000000
116	2065604617426779.000000000000000000000000000000
118	869875725514605.000000000000000000000000000000
120	361328267773071.000000000000000000000000000000
122	148129942020399.000000000000000000000000000000
124	59972655714345.000000000000000000000000000000
126	23987354978187.000000000000000000000000000000
128	9485213296329.000000000000000000000000000000
130	3710736664104.000000000000000000000000000000
132	1435922834049.000000000000000000000000000000
134	550738971306.000000000000000000000000000000
136	209335692708.000000000000000000000000000000
138	78754084695.000000000000000000000000000000
140	29544342363.000000000000000000000000000000
142	10936421133.000000000000000000000000000000
144	4033108449.000000000000000000000000000000
146	1491609852.000000000000000000000000000000
148	531575835.000000000000000000000000000000
150	198232443.000000000000000000000000000000
152	69860997.000000000000000000000000000000
154	24031740.000000000000000000000000000000
156	9567882.000000000000000000000000000000
158	2727216.000000000000000000000000000000
160	1081401.000000000000000000000000000000
162	421002.000000000000000000000000000000
164	71298.000000000000000000000000000000
166	53067.000000000000000000000000000000
168	13392.000000000000000000000000000000
170	846.000000000000000000000000000000
172	2397.000000000000000000000000000000
174	207.000000000000000000000000000000
178	72.000000000000000000000000000000
184	1.000000000000000000000000000000