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SOME PROPERTIES OF TRANSPOSITION GRAPHS

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ABSTRACT

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SOME PROPERTIES OF TRANSPOSITION GRAPHS

by Nicholas John Prudden

For every finite graph G without isolated vertices, there is an associated set of transpositions $\Omega(G)$ which correspond in a natural way to the edges of G. $\Omega(G)$ generates a group H which is a symmetric group iff G is connected. The Cayley graph $\Gamma(H, \Omega)$ clearly depends only on G, and is called the transposition graph of G, $\Gamma(G)$.

The distance between any two vertices of a transposition graph $\Gamma(G)$ is established in the cases where G is a complete graph, a complete graph with an edge deleted, a path graph, or a star. The diameter of $\Gamma(G)$ is obtained as a corollary in these cases. General upper and lower bounds are found for the diameter of $\Gamma(G)$ which depend on the number of vertices and the diameter of G.

If G has no connected components isomorphic to C_4 or K_n then the automorphisms of $\Gamma(G)$ are completely determined by the automorphisms of G. In particular, if G is a connected graph on n vertices with no non-trivial automorphisms, then $\Gamma(G)$ is a graphical regular representation of S_n.

Every transposition graph with at least four vertices is hamiltonian.

If the complement of the line graph of a graph G is hamiltonian then the genus of $\Gamma(G)$ depends only on the number of vertices and edges of G. This result can be generalised if G has no circuits of length three.

Finally, it is proved that the complement of the line graph of a graph G is hamiltonian iff every vertex of G is incident to at most half the edges of G and every edge of G is non-incident to at least two other edges of G, provided G has at least thirty four edges.

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CONTENTS

INTRODUCTION

For every graph G without isolated vertices there is a corresponding transposition graph $\Gamma(G)$, which is a Cayley graph of the group generated by a set of transpositions corresponding to the edges of G. This correspondence is described in chapter 1, and a more explicit description of transposition graphs is given. In the following chapters, a number of properties of transposition graphs are studied. It is often the case that a problem about a transposition graph $\Gamma(G)$ can be reduced to a problem about G, although this is not necessarily possible for all graphs G.

Several straightforward properties of transposition graphs are dealt with in chapter 1, including colourings, edge colourings, connectedness and vertex transitivity. The problem of finding the distance between two arbitrary vertices of a transposition graph is also examined, but explicit formulae are only obtained in a few special cases. The problem of finding the diameter of a transposition graph appears to be no easier.

Chapter 2 is devoted to an examination of the subgraphs of a transposition graph. A complete classification is given of those subgraphs isomorphic to C_4, $K_{2,3}$ and $K_{3,3}$. The classification of subgraphs isomorphic to C_6 is far more complicated, and only that part of it needed in later chapters is proved here. Finally, the girth of all transposition graphs is established.

The automorphisms of a transposition graph are studied in chapter 3. One interesting result is that if G has no non-trivial automorphisms, then $\Gamma(G)$ is a graphical regular representation. This is a special case of the main result which states that for most graphs G, all the automorphisms of $\Gamma(G)$ are derived from the automorphisms of G. The only exceptional graphs are C_4 and K_n, and any graph containing one of these as a component. In each of these cases, $\Gamma(G)$ has additional automorphisms not accounted for by the automorphisms of G.

In chapter 4 it is proved that every non-trivial transposition graph has a hamiltonian circuit. The main result needed to prove this is that for every tree T on 3 or more vertices, $\Gamma(T)$ is hamiltonian. The proof divides into two main cases, depending on whether T is isomorphic to $K_{1,n-1}$ for some n. In this case $\Gamma(T)$ has no circuits of length 4, so the method used for other trees does not apply. The proof in the general case does not use any properties of trees except that for every end vertex of the tree there is another which is distance 3 or more from the first. The results in this chapter generalise a theorem of J. Dénes and E. Törők, (8).

The genus of a transposition graph is studied in chapter 5. The genus of $\Gamma(G)$ is established for those graphs G such that $\overline{L}(G)$, the complement of the line graph of G, is hamiltonian, or such that G has no circuits of length 3. The problem is much harder for graphs G which satisfy neither of these conditions, but has been solved in a few special cases. The question of which graphs have hamiltonian line graph complements is studied in the final section of the chapter, and a strong necessary and sufficient condition for this is established for all graphs with at least 34 edges.

CHAPTER 0: DEFINITIONS AND NOTATION

A graph G is an ordered pair (V, E) where V is a non-empty finite set of vertices and E is a set of pairs of (distinct) vertices of G, called edges. With this definition a graph is finite and has neither loops nor multiple edges. A multigraph is a graph which is allowed to have multiple edges, but no loops. A graph H = (V', E') is a subgraph of G = (V, E) if $V' \subset V$ and $E' \subset E$. H is a spanning subgraph of G if it is a subgraph of G and V' = V.

If $e = \{u, v\}$ is an edge of a graph G then u and v are the end vertices of e, and u and v are adjacent in G. This relation is often denoted by $u \sim_G v$, or simply $u \sim v$. A vertex u is incident to an edge e if u is an end vertex of e. Two edges e and e' are incident if they have a common end vertex. Otherwise, e and e' are non-incident or independent.

The degree or valency $d_G(v)$ of a vertex v of a graph G is the number of vertices of G adjacent to v. A vertex of degree 0 is an isolated vertex. Graphs in this thesis will normally have no isolated vertices. If G is a graph with vertices $v_1, v_2, \ldots, v_n$ then the degree sequence of G is the sequence $d_G(v_1), d_G(v_2), \ldots, d_G(v_n)$; it is usually ordered in such a way that $d_G(v_1) \leqslant d_G(v_2) \leqslant \ldots \leqslant d_G(v_n)$.

A walk of length k joining u and v in G is a sequence of vertices and edges of G of the form $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ where $v_0 = u$, $v_k = v$ and $e_i = \{v_{i-1}, v_i\}$ for $i = 1, \ldots, k$. A walk joining u and v is closed if u = v, and is a path if no two vertices of the walk (except possibly u and v) are equal; a closed path is called a circuit. Note that the edges $e_1, \ldots, e_k$ will frequently be omitted from the definition of a walk.

A graph G is connected if every pair of vertices of G are joined by some path; otherwise, G is disconnected. A connected component of G is a maximal connected subgraph of G. Each vertex and edge of G belongs to precisely one connected component of G.

If v is a vertex of a connected graph G, then $G - \{v\}$ will denote the subgraph of G with vertex set $V(G) - \{v\}$ and edge set $E(V(G)) - E(v)$, where $E(v)$ is the set of edges of G incident to v. A vertex v of a connected graph G is a cut vertex of G if $G - \{v\}$ is disconnected. A graph which has no cut vertices is called 2-connected. A block of a graph G is a maximal 2-connected subgraph of G.

If u and v are vertices of a connected graph G then the distance between u and v, $d_G(u, v)$, is the length of the shortest path in G joining u and v. The diameter of a graph G is the maximum distance between any two vertices.

A circuit is trivial if it is of the form u or u, v, u. A graph which contains no non-trivial circuits is called acyclic, or more normally, a forest. A connected forest is called a tree. The girth of a graph which is not a forest is the length of its shortest non-trivial circuit. A graph which has no circuits of odd length is called bipartite. Note that every forest is automatically bipartite.

A colouring of a graph G is a function which assigns a colour to each vertex of G, and which has the property that no two adjacent vertices are assigned the same colour. A graph G is k-colourable if there is a colouring of G which assigns k colours to the vertices of G. Note that a bipartite graph can also be defined as a graph which is 2-colourable. The chromatic number of G is the smallest value of k such that G is k-colourable.

An edge colouring, the k-edge colourability and the edge chromatic number of a graph are defined in the same way with edges replacing vertices and incidence replacing adjacency.

An isomorphism between two graphs G and G' is a bijection from V(G) to V(G') which preserves adjacencies. An automorphism of G is an isomorphism from G to itself. An automorphism may be regarded as an adjacency preserving permutation of V(G). A graph G is vertex transitive if for any two vertices u and v of G, there is an automorphism of G mapping u to v.

If G is any graph with a non-empty edge set E, then the line graph of G, L(G), is the graph with vertex set E with an edge $\{e_1, e_2\}$ iff e_1 is incident to e_2.

If G and G' are any graphs, then $G \times G'$ is the graph with vertex set $V(G) \times V(G')$ with an edge $\{(u, u'), (v, v')\}$ iff $u = v$ and $\{u', v'\} \in E(G')$ or $u' = v'$ and $\{u, v\} \in E(G)$. $G \times G'$ is called the cartesian product of G and G'.

The complete graph on n vertices, K_n, is the graph with vertex set $[n]$ and edge set $E(K_n) = \{\{i, j\}: i, j \in [n] \text{ and } i \neq j\}$. The complete bipartite graph $K_{m,n}$ is the graph defined by $V(K_{m,n}) = [m + n]$ and $E(K_{m,n}) = \{\{i, j\}: i \in [m] \text{ and } j \in [m+n] - [n]\}$. The path of length n-1, P_n, is defined by $V(P_n) = [n]$ and $E(P_n) = \{\{i, i+1\} : i = 1, \ldots, n-1\}$. The circuit of length n, C_n, is defined by $V(C_n) = V(P_n)$ and $E(C_n) = E(P_n) \cup \{\{1, n\}\}$.

In the above definitions and throughout most of this thesis, $[n]$ is used to denote the set of integers from 1 to n inclusive. Occasionally, however, $[x]$ is used to denote the integer part of x. It will normally be obvious which is meant. $\{x\}$ will mean the least integer not less than x. Curly brackets will occasionally be used as ordinary brackets and as set brackets.

Although the more general group-theoretical definitions will not be stated in this chapter, some definitions concerning permutations will, since they are frequently used in this thesis.

A permutation of a set X is a bijection from X to itself. The set of all permutations of X forms a group called the symmetric group on X, and is denoted by S(X). In this thesis, X will invariably be a finite set. In this case, if X has n elements (often called letters), then S(X) has n! elements. X will very often be the set $[n]$, and in this case, S(X) will be written as S_n.

Throughout this thesis, the image of a variable x under a function f will be denoted by xf rather than f(x). With this notation, the product of two functions f and g will be written as fg, where $x(fg) = (xf)g$. This notation will in particular be used for permutations.

If $x \in X$ and $\sigma \in S(X)$ then σ moves x if $x\sigma \neq x$; otherwise, σ fixes x. Two permutations ρ and σ of X are disjoint if ρ fixes every letter moved by σ, and vice versa.

A permutation σ of X is a cycle if for every x and y which are moved by σ, $y = x(\sigma^k)$ for some number k. Every cycle can be written in the form $(x_1\ x_2\ \dots\ x_r)$ where $x_i \in X$ and $x_{i+1} = x_i\sigma$; i = 1,..., r, subscripts mod r. A cycle of this form will be called a cycle of length r, or an r-cycle. 2-cycles are usually called transpositions. Note that the permutation which fixes every letter of X is trivially a cycle, but cannot be written in the above form. Instead it is written as (1), and it is called the identity permutation.

A well-known theorem states that every permutation can be expressed as a product of disjoint, non-trivial cycles in an essentially unique way. (It is unique up to the order of the disjoint cycles.) This representation will frequently be used throughout this thesis. Another well-known result states that every permutation can be expressed as a product of transpositions. This representation is far from unique.

CHAPTER 1: GRAPHS AND TRANSPOSITIONS

SECTION 1.1: INTRODUCTION

This chapter is concerned with a number of problems which arise from the study of a correspondence between graphs and sets of transpositions. This correspondence is well-known, and several papers have been published on the closely-related topic of the graphs connected with minimal products of transpositions. All this material is presented in section 1.2.

In section 1.3 a rather different connection between graphs and sets of transpositions is introduced, namely the transposition graph of a set of transpositions. Equivalently, the transposition graph can be derived from the graph corresponding to the set of transpositions. This is the more useful way of defining a transposition graph and is used continuously in this thesis. A number of simple properties of transposition graphs are established concerning regularity, connectedness, vertex transitivity and vertex and edge colourability. All the results in this section are either special cases of more general results or they are simple consequences of the definitions and the properties of transpositions and their products.

Section 1.4 is concerned with the problem of finding the distance between two vertices of a transposition graph. This problem may be thought of as generalising the results on minimal products of transpositions presented in section 1.2. In general, this problem appears to be very difficult, so most of the results in section 1.4 are concerned with special cases. Exact formulae are given for the distance between two arbitrary vertices in four special families of transposition graphs.

The problem of finding the diameter of a transposition graph is a special case of this problem, but seems to be no easier. Upper and lower bounds are given for the diameter of a transposition graph. Only one of these bounds is close to the true diameters in the four special cases which have been solved.

SECTION 2: TRANSPOSITIONS AND GRAPHS

There is a close connection between graphs and sets of transpositions. If $G=(V,E)$ is any graph without isolated vertices and without loss of generality $V=[n]$, then G is associated with a set of transpositions $\Omega(G):=\{(ij):\{i,j\}\in E\}$. Clearly, since G has no isolated vertices, each $i\in[n]$ is permuted by some $\omega\in\Omega(G)$.

Conversely, a set of transpositions Ω is associated with a graph $G(\Omega):=(V(\Omega),E(\Omega))$ where $V(\Omega)=\{i: i\omega\neq i \text{ for some } \omega\in\Omega\}$ and $E(\Omega)=\{\{i,j\}:(i\ j)\in\Omega\}$. Note that by the definition of $V(\Omega)$, $G(\Omega)$ has no isolated vertices. Thus there is a 1-1 connection between graphs without isolated vertices and sets of transpositions.

Examples.

If G_1 is the graph in fig. 1.2.1 then $\Omega(G_1)=\{(1\ 2),\ (3\ 4),\ (4\ 5)\}$.

If $\Omega_2=\{(1\ 2),\ (2\ 3),\ (3\ 4)\}$ then $G(\Omega_2)$ is the graph in fig. 1.2.2.

Fig. 1.2.1 G_1:

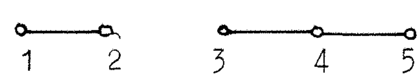


Fig. 1.2.2 $G(\Omega_2)$:

Theorem 1.2.1

If Ω is a (non-empty) set of transpositions and without loss of generality $V(\Omega)=[n]$, then Ω generates S_n iff $G(\Omega)$ is connected.

Proof

Suppose that Ω generates S_n; it is necessary to show that there is a path joining any two vertices of $G(\Omega)$.

Let i, j be any two vertices of $G(\Omega)$. Since i, $j \in V(\Omega)=[n]$, the transposition $(i\ j)\in S_n$. By hypothesis, Ω generates S_n so $\exists$ transpositions $\omega_1, \omega_2, \ldots, \omega_k \in \Omega$ such that $(i\ j)=W$, where $W \equiv \omega_1\omega_2\ldots\omega_k$. Note that the transpositions $\omega_1, \omega_2, \ldots \omega_k$ need not necessarily be distinct.

Let ω_1' be the first transposition in W which moves i; ω_1' must be of the form $\omega_1'=(i\ i_1)$ since $i\omega_1'\neq i$.

Let ω_2' be the first transposition in W after ω_1' moving i_1; ω_2' must be of the form $\omega_2'=(i_1\ i_2)$.

Similarly $\omega_3', \ldots, \omega_m'$ are defined, and $\omega_r'=(i_{r-1}\ i_r)$ for $r=3,\ldots,m$.

It is clear from the definition of $\omega_1', \omega_2', \ldots, \omega_m'$ that W moves i to i_1, then to i_2, and finally to i_m. Thus $iW=i_m$. However, $W=(i\ j)$ so $iW=j$. It follows that $i_m=j$.

Now $\omega_r'\in\Omega$ for $r=1,2,\ldots,m$, hence $e_r':=\{i_{r-1},i_r\}\in E(\Omega)$ for $r=1,2,\ldots,m$. $\{i,i_1\}, \{i_1,i_2\},\ldots,\{i_{m-1},i_m\}$ is a walk in $G(\Omega)$ from i to j since $j=i_m$. It follows immediately that there is a path in $G(\Omega)$ joining i to j, so $G(\Omega)$ is connected.

Conversely suppose that $G(\Omega)$ is connected and that $(i\ j)$ is any transposition in S_n. We show that $(i\ j)$ is generated by transpositions in Ω.

$G(\Omega)$ is connected and i and j are vertices of $G(\Omega)$ since $i, j\in[n]=V(\Omega)$. Thus there is a path $i,i_1,i_2,\ldots i_{m-1},j$ in $G(\Omega)$ joining i to j.

It is easy to check that

$$(i\ j)=(i\ i_1)(i_1\ i_2)\ldots(i_{m-2}\ i_{m-1})(i_{m-1}\ j)(i_{m-2}\ i_{m-1})\ldots(i_1\ i_2)(i\ i_1).$$

Also, since $\{i,i_1\}$, $\{i_{m-1},j\}$, and $\{i_{r-1},i_r\}$;$r=2,\ldots,m-1$ are edges of $G(\Omega)$, all the transpositions in the above product are elements of Ω. Thus Ω generates every transposition in S_n. A well-known result of elementary group theory states that every permutation in S_n may be written as a product of transpositions in S_n, so it follows that Ω generates S_n.

Theorem 1.2.1 has appeared several times in the literature, and appears to be due to Pólya (12). J. Dénes (7 , p 65) mentions in a somewhat confusing footnote a result in Pólya (12) which implies theorem 1.2.1 as an immediate corollary. J. Dénes and E. Török (8) give a direct proof of theorem 1.2.1 while mentioning Pólya's result in passing. Finally, C. Berge (3 , pp 141-142) also proves the result attributed by Dénes to Pólya, but does not himself attribute it to Pólya. This is strange as Berge is aware of Dénes' paper, which he cites as a reference. Unfortunately, some of the references to chapter 4 of Berge (3) are wrongly given after chapter 5. Pólya (12) does appear in this combined list of references, so it may be a reference to chapter 4. This, however, is unlikely since chapter 5 is concerned with enumeration, which is of course the main subject of Pólya (12).

<u>Corollary 1.2.2</u>

If $G(\Omega)$ has connected components $G_1, G_2, \ldots, G_k$ and $V_r = V(G_r)$; $r = 1, \ldots, k$, then Ω generates the group $S(V_1)S(V_2)\ldots S(V_k) \cong S(V_1) \times S(V_2) \times \ldots \times S(V_k)$.

<u>Proof</u>

Let $\Omega_r = \Omega(G_r)$; $r = 1, \ldots, k$. The sets $\Omega_1, \Omega_2, \ldots, \Omega_k$ form a partition of Ω. By theorem 1.2.1, Ω_r generates $S(V_r)$, so the group generated by Ω certainly contains $S(V_1)S(V_2)\ldots S(V_k)$.

If $r \neq s$, then G_r and G_s are distinct components of $G(\Omega)$ so $V_r \cap V_s = \phi$. This implies that if $\omega_r \in \Omega_r$ and $\omega_s \in \Omega_s$

then $\omega_r\omega_s = \omega_s\omega_r$.

Suppose that σ is any permutation generated by Ω ; then there exist transpositions $\omega_1, \omega_2, \ldots, \omega_m \in \Omega$ such that $\sigma = \omega_1\omega_2\ldots\omega_m$. Using the fact proved above that transpositions in distinct sets Ω_r and Ω_s commute, this may be rewritten in the form

$$\sigma = \omega_1^1\ldots\omega_{r_1}^1\omega_1^2\ldots\omega_{r_2}^2\ldots\omega_1^k\ldots\omega_{r_k}^k$$

where $\omega_s^j \in \Omega_j$ for $s = 1,\ldots, r_j$ and for $j = 1,\ldots, k$. In this product, $\omega_1^j, \omega_2^j, \ldots, \omega_{r_j}^j$ are in the same order with respect to one another as they were in the product $\omega_1\omega_2\ldots\omega_m$.

Since $\sigma_j = \omega_1^j\omega_2^j\ldots\omega_{r_j}^j$ is a product of transpositions in Ω_j, $\sigma_j \in S(V_j)$ for $j = 1,\ldots, k$. Hence σ is an element of $S(V_1)S(V_2)\ldots S(V_k)$. It follows that the group generated by Ω is $S(V_1)S(V_2)\ldots S(V_k)$.

To show that $S(V_1)\ldots S(V_k)$ is isomorphic to the group $S(V_1)\times S(V_2)\times\ldots\times S(V_k)$, consider the following mapping from $S(V_1)\times S(V_2)\times\ldots\times S(V_k)$ to $S(V_1)S(V_2)\ldots S(V_k)$:

$(\sigma_1, \sigma_2,\ldots, \sigma_k)\phi = \sigma_1\sigma_2\ldots\sigma_k$, where σ_j is an element of $S(V_j)$.

It is clear that if $r \neq s$, $\sigma_r \in S(V_r)$ and $\sigma_s \in S(V_s)$ then $\sigma_r\sigma_s = \sigma_s\sigma_r$. Also, $S(V_r) \cap S(V_s) = \{(1)\}$. Using these two facts it is easy to show that ϕ is an injective homomorphism. ϕ is obviously surjective, so it is an isomorphism and the result follows.

In the proof of theorem 1.2.1, a walk in $G(\Omega)$ was derived from a product of transpositions in Ω. This suggests that the connection between graphs and products of transpositions is worth investigating.

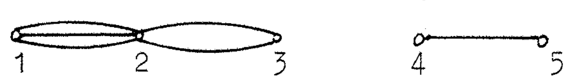
Definition 1.2.1 :A word is a product of transpositions. Two words W_1 and W_2 are identical ($W_1 \equiv W_2$) if they are identical as products of transpositions , and equal ($W_1 = W_2$) if they represent the same permutation. The length of a word W is the number of transpositions in W, and is written $l(W)$.

Definition 1.2.2: If W is a word, the multigraph of W, G(W), is the multigraph with $l(W)$ edges, one of which corresponds to each transposition in W in the same way as for the graph of a set of transpositions.

Example

If $W \equiv (1\ 2)(2\ 3)(1\ 2)(4\ 5)(1\ 2)(2\ 3)$, then G(W) is the multigraph in fig. 1.2.3.

Figure 1.2.3 G(W):



In general there is not a 1-1 correspondence between words and multigraphs; one multigraph may correspond to several words. For example, if $W' = (1\ 2)(4\ 5)(2\ 3)(1\ 2)(2\ 3)(1\ 2)$, then G(W') is again the multigraph in fig. 1.2.3. Thus $G(W)=G(W')$,but $W \not\equiv W'$. Note also that $W=(1\ 2)(4\ 5) \neq (2\ 3)(4\ 5)=W'$.

Definition 1.2.3: A multigraph G is related to a permutation σ, $G \sim \sigma$, if there is a word W such that $G=G(W)$ and $W=\sigma$. Let $\Sigma(G) = \{\sigma : G \sim \sigma\}$.

For example, if G is the multigraph in fig. 1.2.3, then $G \sim (1\ 2)(4\ 5)$ and $G \sim (2\ 3)(4\ 5)$, so $(1\ 2)(4\ 5), (2\ 3)(4\ 5) \in \Sigma(G)$.

This relation has been studied by M. Eden (9) in the special case when G is a graph without multiple edges. In particular, Eden found a number of constructions for graphs G such that

$G\sim(1)$ or "maps to the identity". The smallest such graph is the complete graph on 4 vertices, K_4, since $W\equiv(1\ 2)(3\ 4)(1\ 3)(2\ 4)(1\ 4)(2\ 3)=(1)$, and $G(W)=K_4$.

Using two constructions, Eden showed that each wheel $W_n:=(V_n,E_n)$ where $V=[n+1]$, and $E=\{\{n+1,i\},\{i,i+1\}$;$i=1,\ldots,n$, subscripts mod $n\}$ maps to the identity for $n\geq 3$. Finally, using further constructions, he showed that $\sum(K_n)=A_n$ if $n=0,1$ mod 4 and that $\sum(K_n)=S_n-A_n$ if $n=2,3$ mod 4.

This relation between graphs and words has an application to the genus of a family of graph embeddings which will be discussed in a later chapter.

The following result is implicit in the proof of theorem 1.2.1:

<u>Proposition 1.2.3</u>

If W is a word and $iW=j$, then there is a path in $G(W)$ joining i to j. □

The following simple result is useful in chapter 2:

<u>Proposition 1.2.4</u>

If W is a word and $W=(1)$, then $G(W)$ has no vertex of degree 1.

<u>Proof</u>

Suppose on the contrary that $W=(1)$ and that $G(W)$ has a vertex i of degree 1. Since i has degree 1, it is adjacent to exactly one other vertex j of $G(W)$. By the definition of $G(W)$, $(i\ j)$ appears exactly once in W, so $W=W_1(i\ j)W_2$, where $iW_1=iW_2=i$. Let $\sigma_k=W_k$;$k=1,2$. Since $W=(1)$, $\sigma_1(i\ j)\sigma_2=(1)$, so $(i\ j)=\sigma_1^{-1}\sigma_2^{-1}$. Thus $i\sigma_1^{-1}\sigma_2^{-1}\neq i$. However, $i\sigma_1=i$ and $i\sigma_2=i$, hence $i\sigma_1^{-1}=i$ and $i\sigma_2^{-1}=i$, so $i\sigma_1^{-1}\sigma_2^{-1}=i$, which is a contradiction. Thus $G(W)$ has no vertex of degree 1. □

A particularly important special case which has been studied by several authors is the relation between graphs and minimal words.

Definition 1.2.4 A word W is minimal if $W'=W \Rightarrow l(W') \geq l(W)$.

Notation Let $c(\sigma)$ denote the number of cycles (including 1-cycles) of a permutation $\sigma \in S_n$; let $c^*(\sigma)$ denote the number of non-trivial cycles in σ, and let $n^*(\sigma)$ denote the number of objects moved by σ.

The following result is well-known:

Proposition 1.2.5

If W is a word and $W=\sigma \in S_n$ then W is minimal iff $l(W)=n-c(\sigma)$.

Proof

A proof of this result is given in Chrystal (5). Another proof is given in Higgs & de Witte (11, p.378) which is based on the graph of W. □

Corollary 1.2.6

If W is a word and $W=\sigma$ then W is minimal iff $l(W)=n^*(\sigma)-c^*(\sigma)$.

Proof

Clearly, $\sigma \in S_n$ provided n is sufficiently large. For such an n, $n^*(\sigma)=|\{i \in [n] : i\sigma \neq i\}|$, so $n-n^*(\sigma)=|\{i \in [n] : i\sigma = i\}|$.

Also, $c(\sigma)-c^*(\sigma)=|\{\text{trivial cycles in } \sigma \in S_n\}|$

$$= |\{i : i\sigma = i\}|$$

$$= n-n^*(\sigma).$$

Hence $n-c(\sigma)= n^*(\sigma)-c^*(\sigma)$ and the result follows by proposition 1.2.5. □

Note that $n^*(\sigma)$ and $c^*(\sigma)$ do not depend on what symmetric group σ is a member of, hence the length of a minimal word representing σ does not either, although it appears to in proposition 1.2.5.

Theorem 1.2.7 (Higgs & de Witte)

A word W is minimal iff G(W) is acyclic, that is, G(W) is a forest.

Proof

A proof of this result may be found in Higgs & de Witte (11, p.378 theorem 3). □

Corollary 1.2.8 (Dénes)

If W is a word, then W represents an n-cycle and $l(W)=n-1$ iff G(W) is a tree on n vertices.

Proof

Proofs of this result may be found in Higgs & de Witte (11, p.379 corollary 3), Berge (3, p.143) and Dénes (7). Note, however, that Dénes' proof is incomplete. □

Corollary 1.2.9

If G is a tree on n vertices and $\sigma \in \Sigma(G)$ then σ is an n-cycle. □

Examples

$W \equiv (1\ 2)(2\ 3)(3\ 4)(2\ 3)=(1\ 4\ 2)$ is not a minimal word representing $(1\ 4\ 2)$ since

(i) $l(W)=4 > n^*((1\ 4\ 2))-c^*((1\ 4\ 2))=3-1=2$ (corollary 1.2.6)

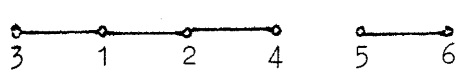
(ii) G(W) has a cycle of length 2, 2—3—2 (theorem 1.2.7).

$W' \equiv (1\ 2)(2\ 4)(5\ 6)(1\ 3)=(1\ 4\ 2\ 3)(5\ 6)$ is a minimal word representing $(1\ 4\ 2\ 3)(5\ 6)=\sigma$ since

(i) $l(W')=4=6-2=n^*(\sigma)-c^*(\sigma)$ (corollary 1.2.6)

(ii) G(W'), the (multi)graph in fig. 1.2.4, is acyclic (theorem 1.2.7)

Figure 1.2.4 G(W'):



If G is the graph in fig. 1.2.2 then $\Sigma(G)=\{(1\ 2)(2\ 3)(3\ 4),$ $(1\ 2)(3\ 4)(2\ 3),\ (2\ 3)(1\ 2)(3\ 4),\ (2\ 3)(3\ 4)(1\ 2),$

$(3\ 4)(1\ 2)(2\ 3),\ (3\ 4)(2\ 3)(1\ 2)\}$,

hence $\Sigma(G)=\{(1\ 4\ 3\ 2),\ (1\ 3\ 4\ 2),\ (1\ 2\ 4\ 3),\ (1\ 2\ 4\ 3),$
$(1\ 3\ 4\ 2),\ (1\ 2\ 3\ 4)\}$,

so $\Sigma(G)=\{(1\ 4\ 3\ 2),\ (1\ 3\ 4\ 2),\ (1\ 2\ 4\ 3),\ (1\ 2\ 3\ 4)\}$, and every $\sigma\in\Sigma(G)$ is a 4-cycle (corollary 1.2.9).

Theorem 1.2.10 (Eden & Schützenberger)

If T is a tree on n vertices with degree sequence $d_1, d_2, \ldots d_n$ then $|\Sigma(T)|=d_1!d_2!\ldots d_n!$.

Proof

Proofs of this result may be found in Eden & Schützenberger (10) and in Berge (3 ,p.147). □

Theorem 1.2.11 (Berge)

If T is a tree on n vertices, then $(i_1\ i_2\ \ldots i_n)\in\Sigma(T)$ iff the following diagram has no crossings:
$i_1, i_2, \ldots, i_n$ are drawn in a circle, and i_j is joined to i_k by a straight line iff i_j is adjacent to i_k in T.

Proof

A proof of this result may be found in Berge (3 ,p.145). Note that theorem 1.2.10 may be deduced as a corollary of this result. □

Examples

If T is the graph in fig.1.2.2, then $|\Sigma(T)|=1!2!2!1!=4$ by theorem 1.2.10. This agrees with the value obtained for $|\Sigma(T)|$ in the previous example. Further, $(1\ 3\ 2\ 4)\notin\Sigma(T)$ since the diagram in fig. 1.2.5 has a crossing, while $(1\ 4\ 3\ 2)\in\Sigma(T)$ since the diagram in fig. 1.2.6 has no crossing.

Figure 1.2.5

diagram of (1 3 2 4):

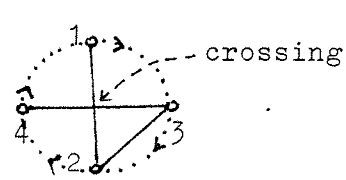
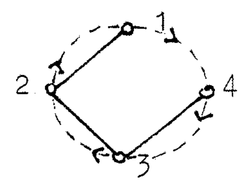


Figure 1.2.6

diagram of (1 4 3 2):



Theorems 1.2.10 and 1.2.11 can be easily generalised to hold for any forest; in fact theorem 1.2,10 is already true for any forest, while theorem 1.2.11 must be applied to each component of the forest. Thus it is known which graphs are graphs of minimal words, and which graphs are graphs of a minimal word representing a given permutation. Hence graphs of minimal words are very well understood.

By theorem 1.2.1, Ω generates S_n iff $G(\Omega)$ is connected and $V(\Omega)=[n]$. Thus it is reasonable to investigate the minimum length of a word W representing a given permutation σ, where transpositions in W are constrained to lie in some set Ω such that $G(\Omega)$ is connected. This seems to be a far harder problem and it has only been solved in a few special cases. It will be discussed in section 1.4.

SECTION 1.3: TRANSPOSITION GRAPHS

In this section, another type of graph associated with sets of transpositions is introduced, the transposition graph. All the results in section 1.2 can be interpreted as results about certain transposition graphs, although most of this interpretation will be left to later sections or omitted entirely as it is completely straightforward.

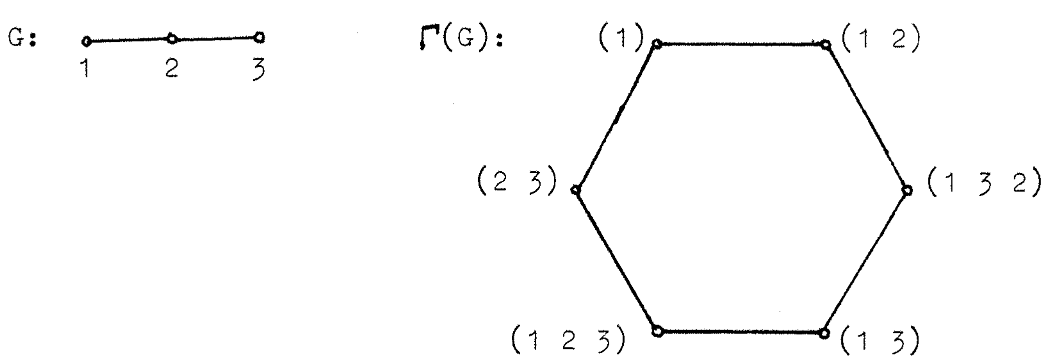
Definition1.3.1

If G is a graph without isolated vertices, the transposition graph of G, $\Gamma(G)$, is the graph $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)=\langle\Omega(G)\rangle$, the group generated by $\Omega(G)$, and $E(\Gamma)=\{\{\sigma_1, \sigma_2\}: \sigma_1, \sigma_2\in\langle\Omega(G)\rangle, \sigma_2=\sigma_1\omega$ and $\omega\in\Omega\}$. Since $\omega^2=(1)$, $\sigma_2=\sigma_1\omega$ iff $\sigma_1=\sigma_2\omega$. Thus the definition of an edge $\{\sigma_1, \sigma_2\}$ in $\Gamma(G)$ does not depend on the order of σ_1 and σ_2.

Example

If G is the smaller graph in fig. 1.3.1, then $\Omega(G)=\{(1\ 2), (2\ 3)\}$, so $\langle\Omega(G)\rangle= S_3$ by theorem 1.2.1 , and $\Gamma(G)$ is the larger graph in fig. 1.3.1. Note, for example, that in $\Gamma(G)$, $(1\ 3)\sim(1\ 3\ 2)$ since $(1\ 3)=(1\ 3\ 2)(1\ 2)$ and $(1\ 2)\in\Omega(G)$.

Figure 1.3.1



Other examples of transposition graphs may be found later in this section and in appendix 1 where all transposition graphs with 24 or fewer vertices are listed.

A transposition graph is clearly a special type of Cayley graph (as defined in White (15,p.22) , not as in Behzad & Chartrand (2 ,p.173)). However, this observation is not particularly helpful , and will not be used except in chapter 4 of this thesis. Thus a number of theorems stated here for transposition graphs will also hold for Cayley graphs, but these theorems are all of an elementary nature and are not worth stating more generally here since the general proofs often involve additional complications.

There is a natural labelling of the edges of a transposition graph $\Gamma(G)$; if $\{\sigma_1, \sigma_2\}$ is an edge of $\Gamma(G)$ then there is a transposition $\omega \in \Omega(G)$ such that $\sigma_2 = \sigma_1 \omega$. ω is clearly unique and is regarded as the label of $\{\sigma_1, \sigma_2\}$. Thus every edge of $\Gamma(G)$ is labelled with a unique element of $\Omega(G)$.

<u>Proposition 1.3.1</u>

Each vertex σ of a transposition graph $\Gamma(G)$ is adjacent to the vertices $\sigma\omega_1, \sigma\omega_2, \ldots, \sigma\omega_m$ where $\Omega(G) = \{\omega_1, \omega_2, \ldots, \omega_m\}$.

<u>Proof</u>

σ' is adjacent to σ in $\Gamma(G)$ iff $\{\sigma, \sigma'\}$ is an edge of $\Gamma(G)$, iff $\sigma' = \sigma\omega_i$ for some $\omega_i \in \Omega(G)$. Clearly, $\omega_i \neq \omega_j \Rightarrow \sigma\omega_i \neq \sigma\omega_j$, so the vertices $\sigma\omega_1, \sigma\omega_2, \ldots, \sigma\omega_m$ are distinct. □

<u>Corollary 1.3.2</u>

A transposition graph $\Gamma(G)$ is regular of degree m, where $m = |E(G)|$. □

It is convenient to give a preliminary result on the automorphisms of a transposition graph here. Further results will be given in chapter 4.

<u>Definition 1.3.2</u> An automorphism of a transposition graph $\Gamma(G)$ is <u>label-preserving</u> if it maps every edge to an edge with the

same label. Such an automorphism may also be called a <u>strong</u> automorphism. The group of strong automorphisms of a transposition graph $\Gamma(G)$ will be denoted by $A_s(\Gamma(G))$.

<u>Theorem 1.3.3</u>

$A_s(\Gamma(G)) \cong \langle\Omega(G)\rangle$, and is transitive on the vertices of $\Gamma(G)$.

<u>Proof</u>

Define a function $f: \langle\Omega(G)\rangle \longrightarrow A_s(\Gamma(G))$ by $\sigma f = \phi_{\sigma^{-1}}$ where $\phi_{\sigma^{-1}} \in S(\langle\Omega(G)\rangle)$ and is defined by $\sigma_1 \phi_{\sigma^{-1}} = \sigma^{-1}\sigma_1 \ \forall \sigma_1 \in \langle\Omega(G)\rangle$. It is first necessary to show that f is well-defined by showing that $\phi_{\sigma^{-1}} \in A_s(\Gamma(G))$.

$\phi_{\sigma^{-1}}$ is a permutation of the vertices of $\Gamma(G)$ and if $\{\sigma_1, \sigma_2\}$ is an edge of $\Gamma(G)$ then $\{\sigma_1, \sigma_2\}\phi_{\sigma^{-1}} = \{\sigma^{-1}\sigma_1, \sigma^{-1}\sigma_2\}$. Since $\{\sigma_1, \sigma_2\}$ is an edge of $\Gamma(G)$ there must be some $\omega \in \Omega(G)$ such that $\sigma_2 = \sigma_1\omega$, so $\omega = \sigma_1^{-1}\sigma_2 = \sigma_1^{-1}(\sigma\sigma^{-1})\sigma_2 = (\sigma^{-1}\sigma_1)^{-1}(\sigma^{-1}\sigma_2)$, so $\{\sigma_1, \sigma_2\}\phi_{\sigma^{-1}} = \{\sigma^{-1}\sigma_1, \sigma^{-1}\sigma_2\}$ is an edge of $\Gamma(G)$ labelled ω and f is well-defined.

The remainder of the proof consists of showing that f is a group isomorphism. This is done by showing that f preserves products so it is a homomorphism, and that f is injective and surjective.

$$\begin{aligned}(\sigma\sigma')f = \phi_{(\sigma\sigma')^{-1}} \ , \text{ and } \ \sigma_1 \phi_{(\sigma\sigma')^{-1}} &= (\sigma\sigma')^{-1}\sigma_1 = \sigma'^{-1}\sigma^{-1}\sigma_1 \\ &= (\sigma^{-1}\sigma_1)\phi_{\sigma'^{-1}} \\ &= (\sigma_1\phi_{\sigma^{-1}})\phi_{\sigma'^{-1}} \\ &= (\sigma_1)\phi_{\sigma^{-1}}\phi_{\sigma'^{-1}} \ \forall \sigma_1 \in \langle\Omega(G)\rangle.\end{aligned}$$

Thus $(\sigma\sigma')f = (\sigma f)(\sigma' f)$, and f is a group homomorphism.

Suppose that $\sigma f = \sigma' f$; then $\phi_{\sigma^{-1}} = \phi_{\sigma'^{-1}}$, so $\sigma_1\phi_{\sigma^{-1}} = \sigma_1\phi_{\sigma'^{-1}}$ $\forall \sigma_1 \in \langle\Omega(G)\rangle$. Thus taking $\sigma_1 = (1)$, we have $(1)\sigma^{-1} = (1)\sigma'^{-1}$, and hence $\sigma = \sigma'$, so f is injective.

Suppose $\phi \in A_s(\Gamma(G))$, and let $\sigma=(1)\phi$. Let σ_1 be any permutation in $\langle\Omega(G)\rangle$, and let ω be any transposition in $\Omega(G)$. Then $\{\sigma_1, \sigma_2\}$, where $\sigma_2=\sigma_1\omega$, is an edge of $\Gamma(G)$ labelled ω, and hence $\{\sigma_1, \sigma_2\}\phi = \{\sigma_1\phi, \sigma_2\phi\}$ is an edge of $\Gamma(G)$ labelled ω. Thus $(\sigma_1\omega)\phi = \sigma_2\phi = (\sigma_1\phi)\omega$ by the definition of an edge in $\Gamma(G)$.

By definition, Ω generates $\langle\Omega(G)\rangle$, so if σ' is any element of $\langle\Omega(G)\rangle$, then $\exists\ \omega_1, \omega_2, \ldots, \omega_k \in \Omega$ such that $\sigma' = \omega_1\omega_2\ldots\omega_k$. Now $(\sigma')\phi = (\omega_1\omega_2\ldots\omega_k)\phi$

$= (\omega_1\omega_2\ldots\omega_{k-1})\phi\,\omega_k$ by the above argument.

Repeating this argument k-1 times , we have

$$(\sigma')\phi = (1)\phi\,\omega_1\omega_2\ldots\omega_k = (1)\phi\,\sigma' = \sigma\sigma' = (\sigma')\phi_\sigma \quad \forall \sigma' \in \langle\Omega(G)\rangle.$$

Thus $\phi = (\sigma^{-1})f$, and f is surjective.

To show that $A_s(\Gamma(G))$ is transitive on the vertices of $\Gamma(G)$, it is necessary to show there is an automorphism in $A_s(\Gamma(G))$ mapping (1) to any given vertex of $\Gamma(G)$. Let σ be any vertex of $\Gamma(G)$; then $\sigma \in \langle\Omega(G)\rangle$, so $\phi_\sigma \in A_s(\Gamma(G))$, and $(1)\phi_\sigma = \sigma(1) = \sigma$. □

In its more general form, the above result is very well-known; the above proof is very similar to that of White (15,p.25).

<u>Corollary 1.3.4</u>

If $\Gamma(G)$ is a transposition graph and $\omega \in \Omega(G)$, then there is an automorphism of $\Gamma(G)$ mapping any edge of $\Gamma(G)$ labelled ω to any other edge of $\Gamma(G)$ labelled ω.

<u>Proof</u>

Let σ_1 be a vertex incident to the first edge, let σ_2 be a vertex incident to the second edge, and let $\sigma = \sigma_2\sigma_1^{-1}$. $\sigma_1\phi_\sigma = \sigma_2\sigma_1^{-1}\sigma_1 = \sigma_2$. There is only one edge incident to σ_1 or σ_2 labelled ω , and ϕ_σ is a label-preserving automorphism so ϕ_σ must have the required property. □

Now consider a walk $\sigma_1 \sim \sigma_2 \sim \dots \sim \sigma_k$ in a transposition graph $\Gamma(G)$. Since $\{\sigma_i, \sigma_{i-1}\}$ is an edge of $\Gamma(G)$ for i=1,..., k-1, $\exists\ \omega_1, \omega_2, \dots, \omega_{k-1} \in \Omega(G)$ such that $\sigma_{i+1} = \sigma_i \omega_i$ for $i = 1, \dots, k-1$. Thus $\sigma_2 = \sigma_1 \omega_1$, $\sigma_3 = \sigma_2 \omega_2 = \sigma_1 \omega_1 \omega_2$ and so on, until finally $\sigma_k = \sigma_1 \omega_1 \omega_2 \dots \omega_{k-1}$. Clearly, walks in $\Gamma(G)$ correspond to words in $\Omega(G)$.

Now suppose the above walk is closed, so $\sigma_k = \sigma_1$. Then $\sigma_1 = \sigma_k = \sigma_1 \omega_1 \omega_2 \dots \omega_{k-1}$, so $\omega_1 \omega_2 \dots \omega_{k-1} = (1)$. thus closed walks in $\Gamma(G)$ correspond to relations in $\Omega(G)$, that is, words in $\Omega(G)$ representing the identity (1).

Proposition 1.3.5

For any graph G, $\Gamma(G)$ is connected.

Proof

We will show there is a walk in $\Gamma(G)$ joining (1) to any other vertex σ of $\Gamma(G)$. Since σ is a vertex of $\Gamma(G)$, $\sigma \in \langle \Omega(G) \rangle$. By the definition of $\langle \Omega(G) \rangle$, $\exists\ \omega_1, \omega_2, \dots, \omega_k \in \Omega(G)$ such that $\sigma = \omega_1 \omega_2 \dots \omega_k = (1)\,\omega_1 \omega_2 \dots \omega_k$. Thus by the above observation, there is a walk in $\Gamma(G)$ from (1) to σ, and hence $\Gamma(G)$ is connected. □

Proposition 1.3.6

A transposition graph $\Gamma(G)$ is bipartite and the partition of $\langle \Omega(G) \rangle$ is $A \cup B$, where A is the set of even permutations in $\langle \Omega(G) \rangle$ and B is the set of odd permutations in $\langle \Omega(G) \rangle$.

Proof

If $A \cup B$ is not a suitable partition of $\langle \Omega(G) \rangle$ then there is an edge $\{\sigma_1, \sigma_2\}$ of $\Gamma(G)$ such that both σ_1 and σ_2 are even (or odd) permutations. However, $\sigma_1 \sim \sigma_2$ so $\exists\ \omega \in \Omega(G)$ such that $\sigma_2 = \sigma_1 \omega$. ω is a transposition, which is an odd permutation, so σ_1 is even iff σ_2 is odd, so σ_1 and σ_2 cannot both be even (or odd) . Hence $A \cup B$ is a suitable partition for $\Gamma(G)$ and $\Gamma(G)$ is bipartite. □

It follows that a transposition graph is 2-colourable and has no circuits of odd length. The edge-chromatic number of a transposition graph is equally easy to obtain.

Proposition 1.3.7

The edge-chromatic number of a transposition graph $\Gamma(G)$ is $m = |E(G)|$.

Proof

$\Gamma(G)$ is regular of degree m by corollary 1.3.2 so at least m colours are needed for the edges of $\Gamma(G)$. There is a natural colouring of the edges of $\Gamma(G)$ given by the natural labelling of the edges of $\Gamma(G)$ with elements of $\Omega(G)$, since no two edges incident to a vertex of $\Gamma(G)$ can have the same label. Since $|\Omega(G)| = |E(G)| = m$, it follows that the edge-chromatic number of $\Gamma(G)$ is m. □

Proposition 1.3.8

If $G_1 \cong G_2$, then $\Gamma(G_1) \cong \Gamma(G_2)$.

Proof

Let f be an isomorphism from G_1 to G_2 . f induces an isomorphism F from $\Gamma(G_1)$ to $\Gamma(G_2)$ defined by $\sigma_1 F = f^{-1} \sigma_1 f \;\; \forall \sigma_1 \in \langle \Omega(G_1) \rangle$. To prove F is an isomorphism, we first show that F maps vertices of $\Gamma(G_1)$ to vertices of $\Gamma(G_2)$. $(i_1\ j_1) \in \Omega(G_1)$ iff $\{i_1 , j_1\} \in E(G_1)$ iff $\{i_1 f , j_1 f\} \in E(G_2)$,since f is an isomorphism, iff $(i_1 f\ j_1 f) = f^{-1}(i_1\ j_1)f = (i_1\ j_1)F \in \Omega(G_2)$, Now suppose that σ_1 is a vertex of $\Gamma(G_1)$ so $\sigma_1 \in \langle \Omega(G_1) \rangle$. Then $\exists$ transpositions $\omega_1, \omega_2, \ldots, \omega_k \in \Omega(G_1)$ such that $\sigma_1 = \omega_1 \omega_2 \ldots \omega_k$.

Hence $\sigma_1 F = f^{-1}\sigma_1 f = f^{-1}\omega_1\omega_2\ldots\omega_k f$
$= f^{-1}\omega_1 f\, f^{-1}\omega_2 f \ldots f^{-1}\omega_k f$
$= (\omega_1 F)(\omega_2 F)\ldots(\omega_k F)$.

Since $\omega_i F \in \Omega(G_2)$ for all $\omega_i \in \Omega(G_1)$, it follows that $\sigma_1 F \in \langle\Omega(G_2)\rangle = V(\Gamma(G_2))$, so F maps vertices of $\Gamma(G_1)$ to vertices of $\Gamma(G_2)$.

F is clearly injective , hence F is surjective since $|\langle\Omega(G_1)\rangle| = |\langle\Omega(G_2)\rangle|$ by corollary 1.2.2 and the fact that $G_1 \cong G_2$, so F is a bijection.

If $\{\sigma_1, \sigma_1'\}$ is an edge of $\Gamma(G_1)$ labelled ω , then $\sigma_1' = \sigma_1\omega$. Hence $\sigma_1' F = (\sigma_1\omega)F = f^{-1}\sigma_1\omega f$
$= f^{-1}\sigma_1 f\, f^{-1}\omega f = \sigma_1 F\, \omega F$.

Since $\omega F \in \Omega(G_2)$, $\{\sigma_1 F, \sigma_1' F\}$ is an edge of $\Gamma(G_2)$ labelled ωF , and F is an isomorphism . □

Proposition 1.3.9

If G has connected components G_1 , G_2 ,..., G_k then $\Gamma(G) \cong \Gamma(G_1)\times\Gamma(G_2)\times\ldots\times\Gamma(G_k)$, where $\times$ denotes the product of two graphs.

Proof

Define a function $F : \Gamma(G_1)\times\Gamma(G_2)\times\ldots\times\Gamma(G_k)\to\Gamma(G)$ by $(\sigma_1, \sigma_2,\ldots, \sigma_k)F = \sigma_1\sigma_2\ldots\sigma_k$. This is well-defined since $\sigma_i \in \langle\Omega(G_i)\rangle$ for i = 1, 2,..., k, and since $\langle\Omega(G)\rangle = \langle\Omega(G_1)\rangle\langle\Omega(G_2)\rangle\ldots\langle\Omega(G_k)\rangle$ by corollary 1.2.2.

F is clearly a bijection by corollary 1.2.2 , so it remains to show that F maps edges to edges.

$(\sigma_1, \sigma_2,\ldots,\sigma_k) \sim (\sigma_1', \sigma_2',\ldots,\sigma_k')$ in $\Gamma(G_1)\times\ldots\times\Gamma(G_k)$ iff for some i such that $1\leqslant i\leqslant k$, $\sigma_i' = \sigma_i\omega_i$ for some $\omega_i \in$

$\Omega(G_i)$, and $\sigma'_j = \sigma_j \ \forall \ j \neq i$ by the definition of product.

Now $(\sigma'_1, \ldots, \sigma'_k)F = (\sigma_1, \ldots, \sigma_i\omega_i, \ldots, \sigma_k)F$

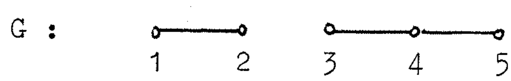
$$= \sigma_1\sigma_2 \cdots \sigma_i\omega_i \cdots \sigma_k$$
$$= \sigma_1\sigma_2 \cdots \sigma_k\omega_i \quad \text{since } \omega_i \in \langle\Omega(G_i)\rangle$$

and commutes with every $\sigma_j \in \langle\Omega(G_j)\rangle$ where $j \neq i$.

Thus $(\sigma'_1, \ldots, \sigma'_k)F = (\sigma_1, \ldots, \sigma_k)F\ \omega_i$, and $(\sigma'_1, \ldots, \sigma'_k)F \sim (\sigma_1, \ldots, \sigma_k)F$ in $\Gamma(G)$ so F is an isomorphism. □

We now give a detailed example to illustrate some of the above results. Let G be the graph in fig. 1.3.2 , so $\Omega(G) = \{(1\ 2), (3\ 4), (4\ 5)\}$, and $\langle\Omega(G)\rangle = S\{1, 2\}.S\{3,4,5\}$. By the above results, $\Gamma(G)$ is 3-regular, vertex transitive, connected, bipartite, 3-edge colourable and isomorphic to $\Gamma(G_1) \times \Gamma(G_2)$, where G_1 and G_2 are the two connected components of G. $\Gamma(G_1)$, $\Gamma(G_2)$ and $\Gamma(G)$ are shown in figs. 1.3.3 and 4.

<u>Figure 1.3.2</u>



The partition of $\Gamma(G)$ is shown by drawing vertices in A as • and vertices in B as ∘ . This partition gives a 2-colouring of $\Gamma(G)$. The edge colouring is given by the edge labels. It is obvious that $\Gamma(G)$ is connected, regular , vertex transitive and isomorphic to $\Gamma(G_1) \times \Gamma(G_2)$.

<u>Figure 1.3.3</u>

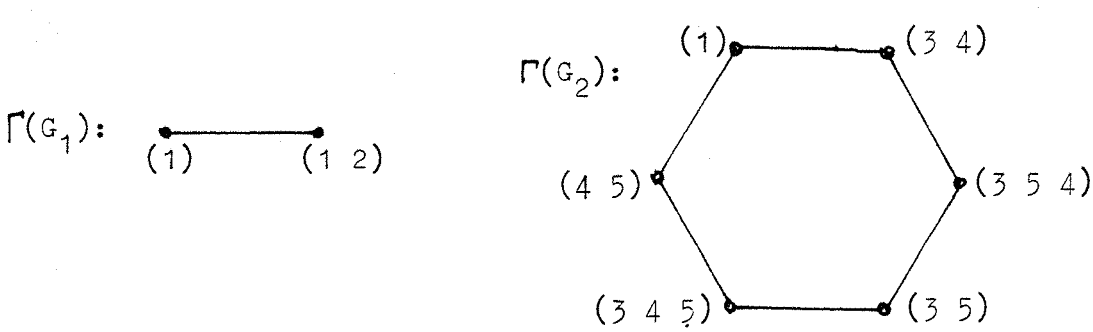
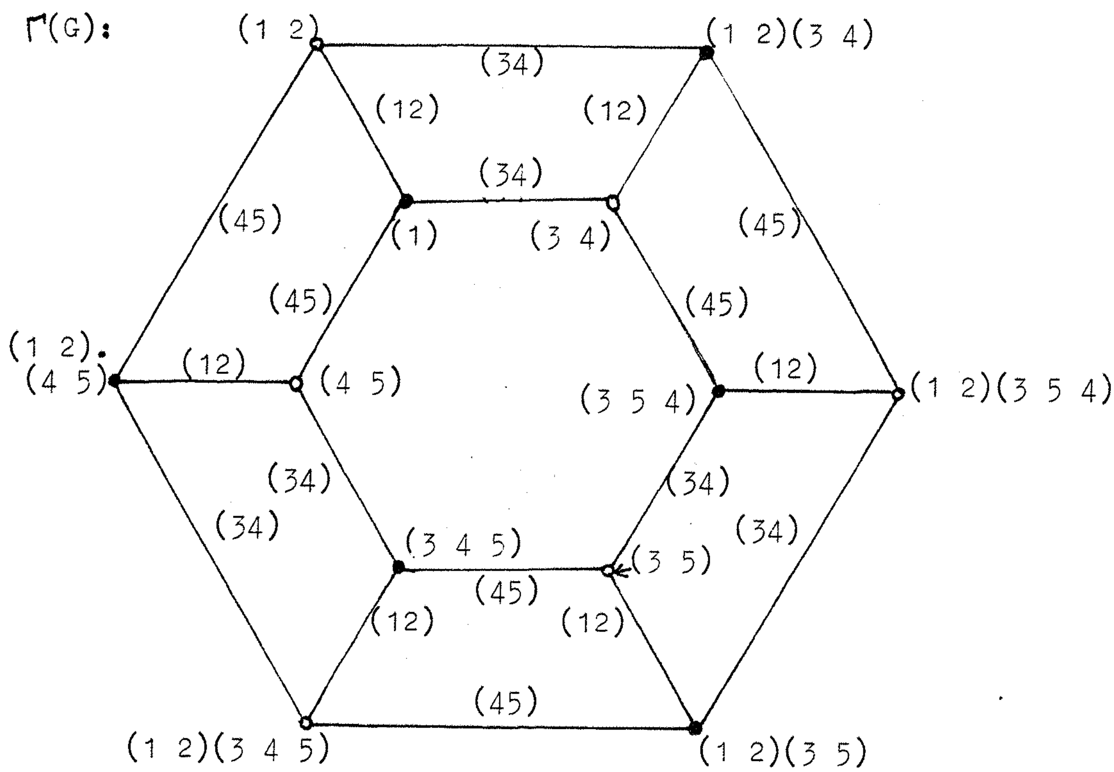


Figure 1.3.4



While transposition graphs do not seem to have been studied as a family before, there are several published papers in which particular transposition graphs are mentioned. The papers below are those which have the greatest relevence to this thesis. Dénes & Török (8 , section 2) prove that the graph obtained by replacing the (undirected) edges of $\Gamma(K_n)$ by pairs of directed edges has a directed hamiltonian path. This is of course equivalent to proving that $\Gamma(K_n)$ has a hamiltonian path. They then give a formula for the number of ways that a given element of S_n may be expressed as the product of a given number of transpositions in S_n. (This last restriction is clearly needed, though it is omitted in the paper.) This formula also gives the number of distinct paths of a given length joining two vertices of $\Gamma(K_n)$. Paths in transposition graphs are studied in the next section, although the emphasis is on their existence, not on the number of them.

G. Ringel (13) uses a special type of embedding of $\Gamma(P_n)$ in an orientable surface to establish an upper bound for the genus of the group S_n. A generalisation of this type of embedding is useful in establishing the genus of a very large class of transposition graphs. This type of embedding is discussed extensively in chapter 5.

N. L. Biggs and A. T. White (4 , p 136) set as an extended exercise the study of two Cayley embeddings of $\Gamma(C_n)$ and $\Gamma(K_{1,n-1})$. The exercise is mainly concerned with proving that these embeddings are symmetrical. This term is defined in (4). Cayley embeddings are studied in section 5.3 of this thesis. An interesting generalisation of this exercise is as follows: which graphs G are such that $\Gamma(G)$ has a symmetrical Cayley embedding ? This seems to be equivalent to asking which graphs G have an automorphism which acts cyclically on E(G). A necessary condition for this is that G is edge transitive, but it is probably not sufficient. This problem is not examined elsewhere in this thesis since the emphasis in chapter 5 is on genus rather than symmetry.

SECTION 1.4 : DISTANCE IN TRANSPOSITION GRAPHS

The general problem considered in this section is to find the distance between any two vertices of a given transposition graph. The distance between σ_1 and σ_2 in $\Gamma(G)$ will be denoted by $D_{\Gamma(G)}(\sigma_1, \sigma_2)$, or simply by $D(\sigma_1, \sigma_2)$ if it is obvious from the context which transposition graph is referred to. In fact, it is not necessary to consider two arbitrary vertices; since transposition graphs are vertex transitive , one of the vertices may be chosen to be (1) . The following result implies that it is sufficient to consider transposition graphs of connected graphs.

Proposition 1.4.1

If G has connected components $G_1, G_2, \ldots, G_k$ and σ is a vertex of $\Gamma(G)$ then

$$D_{\Gamma(G)}((1), \sigma) = D_{\Gamma(G_1)}((1), \sigma_1) + D_{\Gamma(G_2)}((1), \sigma_2) + \ldots + D_{\Gamma(G_k)}((1), \sigma_k),$$

where $\sigma_i \in S(V(G_i))$ for $i = 1, 2, \ldots, k$, and $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k$.

Proof

This result is a simple consequence of the fact that $\Gamma(G)$ is isomorphic to $\Gamma(G_1) \times \Gamma(G_2) \times \ldots \times \Gamma(G_k)$. It is obvious that if H, H_1 and H_2 are graphs and $H = H_1 \times H_2$, then

$$d_H\{(u_1, u_2), (v_1, v_2)\} = d_{H_1}(u_1, v_1) + d_{H_2}(u_2, v_2),$$

from the definition of the product $\times$. The result now follows by using the isomorphism constructed in proposition 1.3.9. □

Note that it is not true that a shortest path from (1) to σ in a transposition graph $\Gamma(G)$ corresponds to a minimal word for σ . A simple counter-example is obtained by letting G be the graph in fig. 1.3.1 and taking $\sigma = (1\ 3)$. There are two paths of length 3 joining (1) to σ corresponding to the words (1 2)(2 3)(1 2) and (2 3)(1 2)(2 3). However, the minimal word for σ is simply (1 3) which does not correspond to any path in $\Gamma(G)$. The length of a minimal word does, however, give a lower bound for distance in a transposition graph.

Proposition 1.4.2

$D((1), \sigma) \geqslant n^*(\sigma) - c^*(\sigma)$ in any transposition graph $\Gamma(G)$ which has σ as a vertex.

Proof

Suppose $D_{\Gamma(G)}((1), \sigma) = k$; then there is a path in $\Gamma(G)$ of length k from (1) to σ, and hence a corresponding word $W = \sigma$ such that $l(W) = k$. By corollary 1.2.6, $l(W) \geqslant n^*(\sigma) - c^*(\sigma)$ so the result follows. □

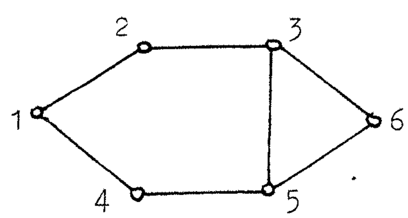
It is possible to give a necessary and sufficient condition for when $D_{\Gamma(G)}((1), \sigma) = n^*(\sigma) - c^*(\sigma)$. First it is necessary to introduce some notation. If $\rho = (i_1\ i_2 \dots i_k)$ and $i_1, i_2, \dots, i_k$ are vertices of G, the diagram of ρ in G is the subgraph of G induced by $\{i_1, i_2, \dots, i_k\}$ drawn with $i_1, i_2, \dots, i_k$ in cyclic order around a circle and joined by straight line segments. A subdiagram has the obvious meaning, and is plane if no two line segments of the subdiagram intersect inside the circle.

Example

If G is the graph in fig. 1.4.1, and $\sigma = (1\ 4)(2\ 5\ 3\ 6)$ then σ has cycles $\rho_1 = (1\ 4)$ and $\rho_2 = (2\ 5\ 3\ 6)$.

Figure 1.4.1

G:



The diagrams of ρ_1 and ρ_2 in G are shown in fig. 1.4.2 , and a connected, plane, spanning subdiagram of the diagram of ρ_2 in G is shown in fig. 1.4.3 .

Figure 1.4.2

Diagram of ρ_1 :

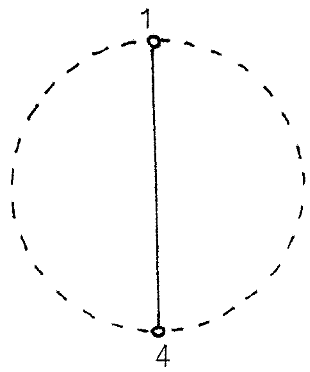


Diagram of ρ_2 :

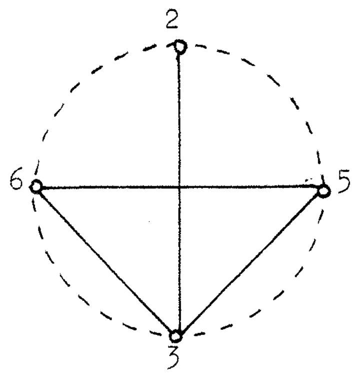
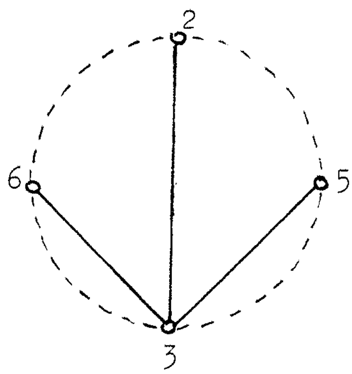


Figure 1.4.3

Subdiagram of ρ_2 :



Theorem 1.4.3

If σ is a vertex of $\Gamma(G)$, then $D((1), \sigma) = n^*(\sigma) - c^*(\sigma)$ iff the diagram of each cycle of σ has a connected, plane, spanning subdiagram.

Proof

Suppose first that $D((1), \sigma) = n^*(\sigma) - c^*(\sigma)$; then the path of length $n^* - c^*$ in $\Gamma(G)$ corresponds to a minimal word W for σ. Thus G(W) is acyclic by theorem 1.2.7 , and each component of G(W) corresponds to a cycle of σ, so without altering the order of transpositions in each component of $G(W)$, W may be rearranged to give another minimal word for σ, $W' = W_1 W_2 \ldots W_{c*}$, where W_i is a minimal word for ρ_i, the i th cycle of σ. $G(W_i)$ is a minimal word for a cycle, hence by corollary 1.2.8, $G(W_i)$ is a tree and by theorem 1.2.11, the diagram of ρ_i in $G(W_i)$ is plane . Now $G(W_i)$ is a tree on the vertices moved by ρ_i, and is a connected, spanning subgraph of the subgraph of G induced by the vertices moved by ρ_i. It follows that the diagram of ρ_i in G has a plane, connected, spanning subgraph .

The converse is similar, and uses the reverse implication in theorem 1.2.11 . □

For example, if G and σ are as in the previous example, figs.1.4.2 and 1.4.3 show that each cycle of σ has a connected plane spanning subdiagram in G, hence by theorem 1.4.3 ,

$$\begin{aligned} D_{\Gamma(G)}((1)\ ,\ (1\ 4)(2\ 5\ 3\ 6)) &= n^*(\ (1\ 4)(2\ 5\ 3\ 6)\ ) \\ &\quad - c^*(\ (1\ 4)(2\ 5\ 3\ 6)\ ) \\ &= 6 - 2 = 4\ . \end{aligned}$$

Theorem 1.4.3 implies that even to find whether or not a permutation is the minimum possible distance from (1) in some transposition graph is a complicated problem . Hence there is no hope of finding a general formula for $D_{\Gamma(G)}((1)\ ,\sigma)$ if both G and σ are arbitrary. It is possible to place restrictions on both G and σ , but it is more natural to place the restrictions on G. In a number of special cases it is possible to derive explicit formulae for $D((1)\ ,\sigma)$; the cases which will be considered here are $\Gamma(G)$ when G is K_n , $K_n - \{e\}$, $K_{1,n-1} = *_n$ and P_n .

First, however, we give a relatively simple condition for σ to be more than the minimum possible distance from (1) in $\Gamma(G)$.

Corollary 1.4.4

If σ is a vertex of $\Gamma(G)$, and has a cycle ρ such that the subgraph of G induced by the vertices moved by ρ is disconnected, then $D_{\Gamma(G)}((1), \sigma) > n^*(\sigma) - c^*(\sigma)$.

Proof

The hypothesis implies that the diagram of ρ in G is disconnected and cannot have a connected subdiagram. The result follows from theorem 1.4.3. □

Theorem 1.4.5

If σ is a vertex of $\Gamma(K_n)$ (i.e. $\sigma \in S_n$), then $D_{\Gamma(K_n)}((1), \sigma) = n^*(\sigma) - c^*(\sigma)$.

Proof

$\Omega(K_n)$ contains every transposition in S_n, so every word in S_n corresponds to a walk in $\Gamma(K_n)$. Also, if W is a minimal word representing σ, then σ moves every letter occuring in a transposition in W by a result of Higgs & de Witte (11, theorem 2). Since $\sigma \in S_n$, every transposition in W is in S_n, so W corresponds to a path in $\Gamma(K_n)$ joining (1) to σ, and $D_{\Gamma(K_n)}((1), \sigma) = n^*(\sigma) - c^*(\sigma)$. □

This result can also be proved as a corollary to theorem 1.4.3, since the diagram of any r-cycle in K_n must be isomorphic to K_r, and any subdiagram isomorphic to $K_{1,r-1}$ must be connected, plane and spanning.

Corollary 1.4.6

K_n ; $n \geq 2$ are the only connected graphs G such that $D_{\Gamma(G)}((1), \sigma) = n^*(\sigma) - c^*(\sigma)$ for all vertices σ of $\Gamma(G)$.

Proof

If G is not a complete graph, then G has two vertices, i and j say, which are not adjacent. Now let σ be any permutation which is a vertex of $\Gamma(G)$ and which has (i j) as a cycle. Since $i \not\sim j$ in G , the subgraph of G induced by i and j is disconnected, and the result follows from corollary 1.4.4. □

Theorem 1.4.7

If σ is a vertex of $\Gamma(K_n - \{1, 2\})$, then

$$D((1), \sigma) = \begin{cases} n^*(\sigma) - c^*(\sigma) & \text{if } (1\ 2) \text{ is not a cycle of } \sigma, \\ n^*(\sigma) - c^*(\sigma) + 2 & \text{if } (1\ 2) \text{ is a cycle of } \sigma. \end{cases}$$

Proof

For the duration of this proof, let $G = K_n - \{1, 2\}$. If ρ is a cycle of σ of length m , then the diagram of ρ in G is isomorphic to K_m unless 1 and 2 are both permuted by when the diagram is isomorphic to $K_m - \{1, 2\}$. In either case, a plane connected spanning subdiagram isomorphic to $K_{1, m-1}$ is obtained, provided $m > 2$, by choosing some i permuted by ρ which is distinct from 1 and 2 and joining it to every other vertex of the diagram . If m = 2, the diagram itself is plane, connected and spanning unless $\rho = (1\ 2)$. Thus if (1 2) is not a cycle of σ, then by theorem 1.4.3, $D((1), \sigma) = n^*(\sigma) - c^*(\sigma)$.

If (1 2) is a cycle of σ, let $\sigma' = (1\ 2)\sigma$. It is clear that $n^*(\sigma') = n^*(\sigma) - 2$ and $c^*(\sigma') = c^*(\sigma) - 1$ since σ' fixes 1 and 2 but otherwise is identical to σ. Thus there is a path from (1) to σ' in $\Gamma(G)$ of length $n^*(\sigma') - c^*(\sigma') = n^*(\sigma) - c^*(\sigma) - 2 + 1 = n^*(\sigma) - c^*(\sigma) - 1$. However, there is a path of length 3 from σ' to σ in $\Gamma(G)$ given by

$\sigma = \sigma'(1\ 3)(2\ 3)(1\ 3)$. Hence there is a path of length $n^*(\sigma) - c^*(\sigma) + 2$ in $\Gamma(G)$ from (1) to σ. It remains to show that this is the shortest possible path . Since the diagram of (1 2) in G is disconnected, it follows from theorem 1.4.3 and proposition 1.4.2 that $D((1), \sigma) > n^*(\sigma) - c^*(\sigma)$. Finally, if $D((1), \sigma) = n^*(\sigma) - c^*(\sigma) + 1$, there would be paths of both even and odd lengths from (1) to σ in $\Gamma(G)$ which would contradict the fact that $\Gamma(G)$ is bipartite. Thus the result follows. □

Theorem 1.4.8

If σ is a vertex of $\Gamma(K_{1,\ n-1})$, where 1 is the vertex of $K_{1,\ n-1}$ of degree n-1 and 2, 3,..., n are the vertices of degree 1 , then $D((1), \sigma) = n^*(\sigma) + c^*(\sigma) - 2$ if σ permutes 1
$n^*(\sigma) + c^*(\sigma)$ if σ fixes 1.

Proof

For the duration of this proof, let $G = K_{1,\ n-1}$.
Suppose that σ has disjoint, non-trivial cycles $\rho_1, \rho_2, \ldots, \rho_k$ of lengths $r_1, r_2, \ldots, r_k$ respectively which do not contain 1, and a possibly trivial cycle ρ_0 containing 1. If $i \geqslant 1$, then $\rho_i = (j_{i,1}\ \ j_{i,2}\ \cdots\ j_{i,r_i}) = (1\ \ j_{i,1})(1\ \ j_{i,2})\cdots$
$(1\ \ j_{i,r_i})(1\ \ j_{i,1})$,

so ρ_i may be written as a product of $r_i + 1$ transpositions in $\Omega(G)$. If $\rho_0 = (1\ \ j_1\ \ j_2 \ldots j_{r_0})$ is non-trivial, then $\rho_0 = (1\ \ j_1)(1\ \ j_2)\ldots(1\ \ j_{r_0})$, so ρ_0 may be written as a product of $r_0 - 1$ transpositions in $\Gamma(G)$. The same conclusion holds if ρ_0 is trivial. Thus σ may be written as a product of $r_0 - 1 + r_1 + 1 + \ldots + r_k + 1 = r_0 + r_1 + \ldots + r_k + k - 1$ transpositions in $\Omega(G)$. If ρ_0 is non-trivial, then $c^*(\sigma) = k + 1$ and $n^*(\sigma) = r_0 + r_1 + \ldots + r_k$ so there is a path in $\Gamma(G)$

from (1) to σ of length $n^*(\sigma) + c^*(\sigma) - 2$. If ρ_0 is trivial, then $n^*(\sigma) = r_1 + r_2 + \ldots + r_k = r_0 + r_1 + \ldots + r_k - 1$ and $c^*(\sigma) = k$, so there is a path in $\Gamma(G)$ of length $n^*(\sigma) + c^*(\sigma)$ joining (1) to σ.

It remains to show there are no shorter paths from (1) to σ in $\Gamma(G)$. Suppose that W is a word in $\Omega(G)$ representing σ, and that W has length m. Each letter moved by σ must occur in some transposition in W, so $m \geqslant n^*(\sigma) - 1$. We make the following claim: each cycle ρ_i of σ fixing 1 moves some letter j_i such that the transposition $(1 \ \ j_i)$ occurs at least twice in W . For suppose W contains each letter of the cycle $(j_1 \ \ j_2 \ \ldots \ j_r)$ once only; then $W = W_1(1 \ \ j_1')W_2(1 \ \ j_2')\ldots W_r(1 \ \ j_r')W_{r+1}$, where $(j_1', j_2', \ldots, j_r')$ is a permutation of $(j_1, j_2, \ldots, j_r)$ and $j_k'W_l = j_k'$ for $k = 1, \ldots, r$ and $l = 1, \ldots, r+1$. Let $j_0 = 1W_1^{-1}$, so $j_0 W_1 = 1$; then

$$j_0W = 1(1 \ \ j_1')W_2 \ldots W_{r+1}$$
$$= j_1' W_2 \ldots W_{r+1}$$
$$= j_1' \text{, since } j_1' \text{ is fixed by}$$

$W_2, \ldots, W_{r+1}$ and by $(1 \ \ j_2'), \ldots, (1 \ \ j_r')$. Thus j_0 is in the same cycle as j_1' , so j_0 must be one of j_1' , j_2' , ..., j_r' , and hence one of these must occur in two transpositions of W , which is a contradiction.

Suppose first that $1\sigma \neq 1$; then by the above argument, $m \geqslant n^*(\sigma) - 1 + (c^*(\sigma) - 1) = n^*(\sigma) + c^*(\sigma) - 2$. Now suppose that $1\sigma = 1$, so by the above argument, $m \geqslant n^*(\sigma) - 1 + c^*(\sigma)$. However, m must have the same parity as $n^*(\sigma) - c^*(\sigma)$, since σ may be represented as a product of $n^*(\sigma) - c^*(\sigma)$ transpositions . thus $m + n^*(\sigma) - c^*(\sigma)$ must be even. If $m = n(\sigma) + c(\sigma) - 1$, then we have $2n^*(\sigma) - 1$

is even, which is a contradiction. thus $m \geqslant n^*(\sigma) + c^*(\sigma)$ if σ fixes 1 , and the result follows. □

Definition 1.4.1

If $\sigma \in S_n$ and $1 \leqslant i < j \leqslant n$ then i and j <u>introduce an inversion</u> in σ if $i\sigma > j\sigma$. The <u>number of inversions</u> in σ is the sum of the inversions introduced by all pairs i, j such that $1 \leqslant i < j \leqslant n$. The number of inversions in σ is denoted by $I(\sigma)$.

Clearly, $I(\sigma) \leqslant \frac{1}{2} n (n - 1)$ since each unordered pair i, j can introduce at most one inversion in σ.

Example

If $\sigma = (1\ 2\ 3\ 5\ 4) \in S_5$, then $I(\sigma) = 4$ since $1\sigma > 4\sigma$, $2\sigma > 4\sigma$, $3\sigma > 4\sigma$,and $3\sigma > 5\sigma$.

Theorem 1.4.9

If P_n is the graph with vertex set $[n]$ and with edges $\{1, 2\}$, $\{2, 3\}$,..., $\{n-1, n\}$ and σ is a vertex of $\Gamma(P_n)$, then $D_{\Gamma(P_n)}((1), \sigma) = I(\sigma)$.

Proof

This result is essentially Theorem 1 of C. Berge (3 ,p128) in a disguised form. □

It should be possible to extend these results to other families of graphs. Particularly promising families include the complete bipartite graphs , of which $K_{1,n-1}$ is an example, complete graphs with a small number of edges deleted, and trees with a reasonably simple structure.

One possible simplification of the general distance problem is to study the diameter of a transposition graph. Unfortunately, very few exact values of this parameter are known, although a number of upper and lower bounds have been obtained.

Theorem 1.4.10

$\forall n \geq 2$, diam $\Gamma(K_n) = n - 1$, diam $\Gamma(K_{1,n-1}) = \left\{\frac{3}{2}n\right\} - 2$, diam $\Gamma(P_n) = \frac{1}{2}n(n - 1)$, and $\forall n \geq 3$, diam $\Gamma(K_n - e) = n$.

Proof

All these results are obtained by maximising the distance formulae given in theorems 1.4.5, 7, 8 and 9. The result is obvious for diam $\Gamma(K_n)$.

For diam $\Gamma(K_n - e)$, note that $\forall \sigma \in S_n$, $n^*(\sigma) \leq n$ and $c^*(\sigma) \geq 1$ so $n^*(\sigma) - c^*(\sigma) \leq n - 1$. Also, if $D((1),\sigma) = n^*(\sigma) - c^*(\sigma) + 2$, then $(1\ 2)$ is a cycle of σ, so $\sigma = (1\ 2)$ or $c^*(\sigma) \geq 2$,and in either case, $n^*(\sigma) - c^*(\sigma) + 2 \leq n$. If $\sigma = (1\ 2)(3\ 4\ \dots\ n)$, $D((1)\ ,\sigma) = n$, so the upper bound is attained.

For diam $\Gamma(P_n)$, it has already been noted that $I(\sigma) \leq \frac{1}{2}n(n - 1)$, so this is an upper bound for the diameter . $\sigma = (1\ n)(2\ n{-}1)(3\ n{-}2)\ \dots$ is the (unique) permutation such that $I(\sigma)$ attains this bound, since every pair $i < j$ introduces an inversion.

Diam $\Gamma(K_{1,n-1})$ must be considered in two special cases, n even and n odd.

Suppose first that n is even, and that $n = 2m$. If σ is a permutation fixing 1 , then $n^*(\sigma) \leq 2m - 1$ and $c^*(\sigma) \leq m - 1$ since each non-trivial cycle must permute at least two letters. Thus $D((1)\ ,\sigma) = n^*(\sigma) + c^*(\sigma) \leq 3m - 2$. If σ is a permutation which moves 1 , then $n^*(\sigma) \leq 2m$ and $c^*(\sigma) \leq m$. Thus $D((1)\ ,\sigma) = n^*(\sigma) + c^*(\sigma) - 2 \leq 3m - 2$. It follows that $3m - 2 = \left\{\frac{3}{2}n\right\} - 2$ is an upper bound for the diameter. $\sigma = (1\ 2)(3\ 4)(5\ 6)\dots(n{-}1\ n)$ is such that $D((1)\ ,\sigma) = 3m - 2$ so the bound is attained.

Now suppose that n is odd and that $n = 2m + 1$. If σ is a permutation fixing 1 , then $n^*(\sigma) \leq 2m$ and $c^*(\sigma) \leq m$, so $D((1), \sigma) = n^*(\sigma) + c^*(\sigma) \leq 3m$. If σ is a permutation moving 1 , then $n^*(\sigma) \leq 2m + 1$ and $c^*(\sigma) \leq m$, so $D((1), \sigma) = n^*(\sigma) + c^*(\sigma) - 2 \leq 3m - 1$. Thus an upper bound for the diameter is $3m = \left\{\frac{3}{2}n\right\} - 2$.
$\sigma = (2\ 3)(4\ 5)\ldots(n-1\ n)$ is a permutation such that $D((1), \sigma) = 3m$, so the upper bound is attained. □

Theorem 1.4.10 gives the diameters of all transposition graphs of connected graphs on 4 or fewer vertices with two exceptions which are dealt with in the next result.

Proposition 1.4.11

If G is either of the graphs in fig. 1.4.4 , then $\operatorname{diam} \Gamma(G) = 4$.

Proof

The simplest way to establish this result in either case is to use the diagram of $\Gamma(G)$ in appendix 1, label an arbitrary vertex 0 , label all adjacent vertices 1 , label all unlabelled vertices adjacent to a vertex labelled 1 with 2 ,and iterate this procedure until all vertices are labelled. The largest label in the graph is clearly the diameter; in both cases here it turns out to be 4. □

Figure 1.4.4

Three bounds are now given for the diameter of a transposition graph of any connected graph, and are compared with the exact values established in theorem 1.4.10 .

Theorem 1.4.12

Diam $\Gamma(G) \leqslant (n - 1)(2(\text{diam } G) - 1)$, where G is any connected graph on n vertices.

Proof

By proposition 1.2.5 , any permutation in S_n may be written as a product of at most n - 1 transpositions. Let (i j) be any transposition in S_n ; since G is connected, there is a path in G joining i to j whose length is $k \leqslant$ diam G . Suppose that the path is $i \sim i_1 \sim i_2 \sim \dots \sim i_{k-1} \sim j$. It is easy to check that $(i\ j) = (i\ i_1)(i_1\ i_2)\dots(i_{k-2}\ i_{k-1})(i_{k-1}\ j)(i_{k-2}\ i_{k-1})\dots$
$(i_1\ i_2)(i\ i_1)$,

so (i j) may be written as a product of 2k - 1 transpositions in $\Omega(G)$. It follows that any permutation in S_n may be written as a product of at most $(n - 1)(2(\text{diam } G) - 1)$ transpositions in $\Omega(G)$. The result follows from the correspondence between products of transpositions in $\Omega(G)$ and walks in $\Gamma(G)$. □

Every connected graph contains a vertex whose removal does not disconnect the graph. This follows from theorem 2.3 of Behzad & Chartrand (2 , p. 24). Hence if G is connected and has n vertices, there is a sequence $v_n, v_{n-1}, \dots, v_2$ of vertices of G such that all the graphs $G_n = G$, $G_{i-1} = G_i - \{v_i\}$; $i = n, \dots, 2$ are connected.

Theorem 1.4.13

If G is a connected graph on n vertices and $G_n, G_{n-1}, \dots, G_2$ are defined as above, then $\text{diam } \Gamma(G) \leqslant \sum_{i=2}^{n} \text{diam } G_i$.

Proof

Let σ be a vertex of $\Gamma(G)$, and let $v_n, v_{n-1}, \dots, v_2$ be defined as above. Since G_n is connected, there is a path of length $k \leqslant \text{diam } G_n$ joining v_n to $v_n\sigma$, $v_n \sim u_1 \sim u_2 \sim \dots \sim$

$u_{k-1} \sim v_n\sigma$. It is easy to check that

$\sigma_{n-1} = \sigma(v_n\sigma \;\; u_{k-1})(u_{k-1} \;\; u_{k-2})\ldots(u_2 \;\; u_1)(u_1 \;\; v_n)$

is a permutation fixing v_n. Thus σ_{n-1} is a vertex of $\Gamma(G_{n-1})$ and there is a path of length $k \leqslant \text{diam } G_n$ in $\Gamma(G)$ from σ to σ_{n-1} . This argument can be repeated until we reach a permutation σ_1 which fixes $v_n, v_{n-1}, \ldots, v_2$ and hence must be the identity . Clearly, σ_1 lies at a distance of at most $\sum_{i=2}^{n} \text{diam } G_i$ from σ in $\Gamma(G)$. □

Theorem 1.4.14

If G is a connected graph on n vertices and σ is any vertex of $\Gamma(G)$, then $\text{diam } \Gamma(G) \geqslant \frac{1}{2} \sum_{i=1}^{n} D_G(i, i\sigma)$.

Proof

For each vertex σ of $\Gamma(G)$, define $f_G(\sigma) = \sum_{i=1}^{n} D_G(i, i\sigma)$. Now let $\omega = (j\ k) \in \Omega(G)$ and let $\sigma' = \sigma(j\ k)$. If $i\sigma \neq j, k$ then $i\sigma' = i\sigma$; if $i\sigma = j$, then $i\sigma' = k$, and if $i\sigma = k$, then $i\sigma' = j$. Thus if $i \neq j, k$ then $D_G(i, i\sigma) = D_G(i, i\sigma')$ so $D_G(i, i\sigma) - D_G(i, i\sigma') = 0$. If $i\sigma = j$ then $D_G(i, i\sigma) - D_G(i, i\sigma') = D_G(i, j) - D_G(i, k)$

$= 0, 1, \text{ or } -1$ since $j \sim k$.

A similar result holds if $i\sigma = k$.

Hence $f_G(\sigma) - f_G(\sigma') = \sum_{i=1}^{n}\{D_G(i, i\sigma) - D_G(i, i\sigma')\}$

$= (D_G(i_0, j) - D_G(i_0, k)) + (D_G(i_1, k) - D_G(i_1, j))$

where $i_0\sigma = j$ and $i_1\sigma = k$,

$= 0, 1, -1, 2, \text{ or } -2$.

Now suppose that $D_{\Gamma(G)}((1), \sigma) = r$, so there are transpositions $\omega_1, \omega_2, \ldots, \omega_r$ such that $\sigma = \omega_1\omega_2\ldots\omega_r$. Let $\sigma_0 = (1)$ and let $\sigma_i = \sigma_{i-1}\omega_i$ for $i = 1, \ldots, r$.

By the previous argument, $f_G(\sigma_i) - f_G(\sigma_{i-1}) = 0, 1, -1, 2, -2$, and hence $|f_G(\sigma_i) - f_G(\sigma_{i-1})| \leq 2$ for $i = 1,\dots, r$.

Thus $|f_G(\sigma_r) - f_G(\sigma_0)| \leq |f_G(\sigma_r) - f_G(\sigma_{r-1})| + \dots +$

$$|f_G(\sigma_1) - f_G(\sigma_0)|$$

$$\leq 2r .$$

However, $\sigma_0 = (1)$, so $f_G(\sigma_0) = 0$ while $\sigma_r = \sigma$, so $|f_G(\sigma_r)| = \sum_{i=1}^{n} D_G(i, i\sigma)$. The result now follows from the fact that $\operatorname{diam}\Gamma(G) \geq r \geq \frac{1}{2}|f_G(\sigma_r)|$. □

To obtain the best lower bound for the diameter from this result it is necessary to choose different permutations according to the graph being considered.

The above bounds for the diameter are now compared with the exact values in four special cases.

<u>Case 1</u> : $G = K_n$.

By theorem 1.4.10, $\operatorname{diam}\Gamma(G) = n - 1$; the upper bound of theorem 1.4.12 is $(n - 1)(2\operatorname{diam} G - 1) = (n - 1)(2 - 1) = n - 1$ so the bound is exact. Taking $v_i = i$ for $i = n,\dots, 2$, then $G_i = K_i$, so $\operatorname{diam} G_i = 1$ for $i = n,\dots, 2$. Thus the upper bound of theorem 1.4,13 is $\sum_{i=2}^{n} \operatorname{diam} G_i = n - 1$. There are many permutations σ giving the best lower bound for the diameter; among them is $\sigma = (1\ 2\ \dots\ n)$ which gives a bound of $\frac{1}{2}n$ since $D_G(i, i\sigma) = 1$ for $i = 1,\dots, n$. Thus both upper bounds are exact while the lower bound is too small by a factor of about 2.

<u>Case 2</u> : $G = K_n - \{e\}$.

Diam $G = 2$, so the first upper bound is $(4 - 1)(n - 1) = 3(n - 1)$. If $e = \{n-1, n\}$, taking $v_i = i$ for $i = n,\dots, 2$ gives $G_n = K_n - \{e\}$ while $G_i = K_i$ for $i = n-1,\dots,2$.

Hence diam $G_n = 2$, and diam $G_i = 1$ for $i = n-1, \ldots, 2$, so the second upper bound for the diameter is
$2 + 1 + 1 + \ldots + 1 = n$.

Taking $\sigma = (1\ 2\ \ldots\ n-2)(n-1\ n)$ gives the best value for the lower bound of $\frac{1}{2}(1 + 1 + \ldots + 1 + 2 + 2) = \frac{1}{2}(n + 2)$.

Comparing these bounds with the actual diameter of n, the first upper bound is too large by a factor of about 3 , the second upper bound is exact and the lower bound is too small by a factor of about 2 .

Cases 3 and 4 will be dealt with more briefly as the results are similar to the first two cases .

Case 3 : $G = P_n$.

In this case, diam $\Gamma(G) = \frac{1}{2}n(n - 1)$. The first upper bound is $(n - 1)(2n - 3)$, the second upper bound is exact, and the lower bound is at best $\left[\frac{1}{4}(n^2)\right]$, so the first upper bound is too large by a factor of 4 and the lower bound is too small by a factor of 2.

Case 4 : $G = K_{1,\ n-1}$.

In this case, diam $\Gamma(G) = \left\{\frac{3}{2}n\right\} - 2$. the first upper bound is $3(n - 1)$, the second upper bound is $2n - 3$, and the lower bound is $n - 1$.

Thus in general none of the bounds is necessarily close to the actual diameter, although the second upper bound is normally much closer than the other two bounds. The lower bound is particularly weak as in three out of the four cases above it is no better than the trivial lower bound of $n - 1$ given by the length of a minimal word for $(1\ 2\ 3\ \ldots\ n)$. It does, however, give good results for graphs where every vertex has an antipodal vertex, a unique vertex at a distance from the first vertex equal to the diameter of the graph. An example of this is $G = C_4$, the graph with vertices 1, 2, 3, 4 and edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$, $\{1, 4\}$. Taking $\sigma = (1\ 3)(2\ 4)$ gives a lower bound for the diameter of 4, which by proposition 1.4.11 is the exact value . It is not known whether or not the lower bound always gives the exact diameter of $\Gamma(G)$ where G is an antipodal graph.

Note that the diameter of a transposition graph $\Gamma(G)$ does not depend only on the number of vertices and the diameter of G. For example, $G_1 = K_5 - \{e\}$ and $G_2 = C_5$ both have 5 vertices and diameter 2. However, $diam(\Gamma(G_1)) = 5$ by theorem 1.4.10, and if G_2 has $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 1$, then $D_{\Gamma(G_2)}((1), (1\ 3\ 5\ 2\ 4)) \geqslant 2 + 2 + 2 + 2 + 2 / 2 = 5$ by theorem 1.4.14. But $(1\ 3\ 5\ 2\ 4)$ is an even permutation, so $D((1), (1\ 3\ 5\ 2\ 4)) \geqslant 6$. It follows that $\Gamma(G_1)$ and $\Gamma(G_2)$ have different diameters.

CHAPTER 2 : TRANSPOSITION SUBGRAPHS

SECTION 2.1 : INTRODUCTION

The main purpose of this chapter is to introduce some theory concerning the subgraphs of transposition graphs. The most important idea is that of the type of a transposition subgraph. This is a graph associated with the edge labels of the transposition subgraph. Type is defined in section 2.2, as are the ideas of transposition subgraphs being identically labelled and equivalently labelled. These properties are both equivalence relations on the set of transposition subgraphs. A number of simple properties of these relations are proved, and are then used to classify transposition subgraphs isomorphic to C_4, $K_{2,3}$ and $K_{3,3}$. These classifications are very useful in the remainder of the thesis.

Section 2.3 presents without proof a similar classification for transposition subgraphs isomorphic to C_6. The reason for omitting the proof is that it is very long, and only part of the result is needed later in the thesis. This part of the result is proved. The section concludes with some simple results on the existence of circuits of certain lengths in transposition subgraphs. In particular it is shown that all but a small family of transposition graphs have girth 4. The remainder have girth 6.

The results presented in this chapter are confined largely to those needed in later chapters. However, a number of other problems concerning uniquely labellable transposition subgraphs and forbidden subgraphs of transposition graphs have also been studied. There is considerable scope for extending the results in this chapter.

SECTION 2 : CLASSIFICATION OF TRANSPOSITION SUBGRAPHS

Definition 2.2.1

Any subgraph Δ of a transposition graph $\Gamma(G)$ will be called a transposition subgraph; a transposition subgraph retains the vertex and edge labelling of the transposition graph containing it.

Definition 2.2.2

Given any transposition subgraph Δ there is an associated multigraph $G(\Delta)$, the type of Δ, defined as follows :
Let $\Omega(\Delta) = \{\omega: \omega$ is the label of some edge of $\Delta\}$. Now define $V(G(\Delta)) = V(\Omega(\Delta)) = \{i: i\omega \neq i$ for some $\omega \in \Omega(\Delta)\}$.
If $(i\ j) \in \Omega$ is the label of k edges of Δ then $G(\Delta)$ has k edges joining i to j. Note that by definition, i and j are vertices of $G(\Delta)$.

Example

If Δ is the graph in fig. 2.2.1 then Δ is a transposition subgraph as it is a subgraph of $\Gamma(G)$, where G is the graph in fig. 1.3.2. Clearly from fig 2.2.1, $\Omega(\Delta) = \{(3\ 4), (4\ 5)\}$, hence $V(G(\Delta)) = V(\Omega(\Delta)) = \{3, 4, 5\}$. Finally, $G(\Delta)$ has one edge joining 3 to 4 and two edges joining 4 to 5 since Δ has one edge labelled $(3\ 4)$ and two edges labelled $(4\ 5)$; $G(\Delta)$ is shown in fig. 2.2.2 .

Figure 2.2.1 Δ:

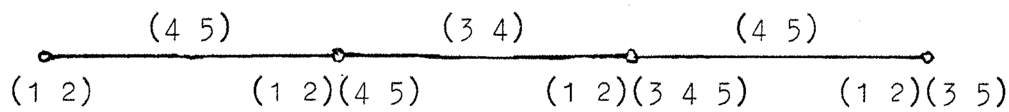
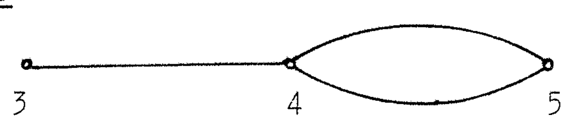


Figure 2.2.2

$G(\Delta)$:



If Δ is a walk in some transposition graph, then Δ is clearly a transposition subgraph. Since Δ is a walk, there is a corresponding product of transpositions W. It is clear that the type of Δ, $G(\Delta)$ is identical to the multigraph of W, G(W); hence type generalises the idea of the multigraph of a word. It follows that the results of Berge, Eden & Schützenberger and Higgs & de Witte in section 1.2 on the multigraphs of minimal words may be translated into results on the types of walks in transposition graphs. However, this must be done carefully for unless the transposition graph is $\Gamma(K_n)$, shortest paths do not necessarily correspond to minimal words.

Note: The word 'type' is used rather than the word 'multigraph' in the context of transposition subgraphs since to refer to the multigraph of a subgraph of a transposition graph (of a graph) would be rather confusing.

It is sometimes convenient to ignore the fact that $G(\Delta)$ has multiple edges and to consider instead the reduced type $\overline{G}(\Delta)$, the graph obtained by merging any multiple edges of $G(\Delta)$ into single edges.

Proposition 2.2.1

If $\Delta \subset \Gamma(G)$, then $\overline{G}(\Delta) \subset G$.

Proof

From its definition, $G(\Delta)$ is simply $G(\Omega(\Delta))$ with multiple edges, so $\overline{G}(\Delta)$ is identical to $G(\Omega(\Delta))$. Also, $\Omega(\Delta) \subset \Omega(G)$ since $\Omega(G)$ contains every edge label of $\Gamma(G)$, and hence of Δ since $\Delta \subset \Gamma(G)$. It follows that $G(\Omega(\Delta)) \subset G(\Omega(G)) = G$, and hence $\overline{G}(\Delta) \subset G$. □

This result has a near converse, which will be proved later in this section.

Definition 2.2.3

Two transposition subgraphs Δ and Δ' are identically labelled if there is an automorphism $f: \Delta \to \Delta'$ such that f maps an edge of Δ labelled ω to an edge of Δ' labelled ω.

Note that if Δ and Δ' are identically labelled, then $G(\Delta) = G(\Delta')$. Clearly the property of being identically labelled is an equivalence relation on the set of all transposition subgraphs. However, a more useful equivalence relation is defined below.

If $g: G \to G'$ is an isomorphism, then g maps edges of G to edges of G' and hence induces a bijection from $\Omega(G)$ to $\Omega(G')$ which will be denoted by h_g.

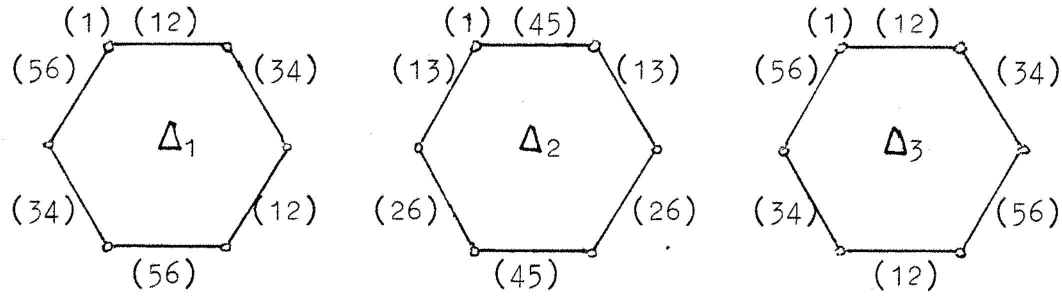
Definition 2.2.4

Two transposition subgraphs Δ and Δ' are equivalently labelled if there are isomorphisms $f: \Delta \to \Delta'$ and $g: \overline{G}(\Delta) \to \overline{G}(\Delta')$ such that $\forall\ \omega \in \Omega(\Delta)$, every edge ε of Δ labelled ω is mapped by f to an edge εf of Δ' labelled ωh_g, where h_g is the bijection from $\Omega(\Delta)$ to $\Omega(\Delta')$ induced by g.

Example

Let Δ_1, Δ_2, Δ_3 be the transposition subgraphs in fig. 2.2.3 ; then Δ_1 and Δ_2 are equivalently labelled, but neither of them is equivalently labelled to Δ_3.

Figure 2.2.3



For the sake of clarity, most of the vertex labels have been omitted from fig. 2.2.3 . They can easily be replaced by starting with the vertex (1) in each graph , and postmultiplying it by the adjacent edge labels to produce the adjacent vertices, and so on.

To see that Δ_1 is equivalently labelled to Δ_2 , consider the isomorphism f: $\Delta_1 \to \Delta_2$ defined by $(1) \to (13)$, $(12) \to (1)$, $(12)(34) \to (45)$, $(34) \to (13)(45)$, $(34)(56) \to (13)(26)(45)$, and $(56) \to (13)(26)$, and the isomorphism g: $\overline{G}(\Delta_1) \to \overline{G}(\Delta_2)$ defined by $1 \to 1$, $2 \to 3$, $3 \to 5$, $4 \to 4$, $5 \to 2$, $6 \to 6$. g induces the bijection h_g: $\Omega(\Delta_1) \to \Omega(\Delta_2)$ defined by $(12) \to (13)$, $(34) \to (45)$, $(56) \to (26)$. Now if ε is any edge of Δ_1 labelled (12) such as $\{(12)(34), (34)\} = \varepsilon$, then $\varepsilon f = \{(45), (13)(45)\}$ is an edge of Δ_2 labelled $(13) = (12)h_f$ as required. It is straightforward to check that this works for all other edge labels in $\Omega(\Delta_1)$ and for all other edges of Δ_1.

Suppose that Δ_1 is equivalently labelled to Δ_3 , so there exist isomorphisms f: $\Delta_1 \to \Delta_3$, g: $\overline{G}(\Delta_1) \to \overline{G}(\Delta_3')$ with the required properties. Let $(12)h_g = (ij)$, where $(ij) = (12)$, (34) or (56). The two edges of Δ_1 labelled (12) must both be mapped by f to edges of Δ_3 labelled (ij), and since the two edges labelled (12) are both incident to a common edge , they must be mapped by f to two edges of Δ_3 with this property. However, Δ_3 has no two edges with the same label which are both incident to some other edge of Δ_3, so Δ_1 is not equivalently labelled to Δ_3 .

The proof that Δ_2 is not equivalently labelled to Δ_3 is similar to this. In fact it follows from the fact that being equivalently labelled is an equivalence relation .

Proposition 2.2.2

Being equivalently labelled is an equivalence relation.

Proof

A transposition subgraph is equivalently labelled to itself since we may choose f and g to be the identity. If Δ is equivalently labelled to Δ', and f and g are isomorphisms with the required properties, then Δ' is equivalently labelled to Δ, since f^{-1} and g^{-1} are isomorphisms with the required properties. Finally, if Δ_1 is equivalently labelled to Δ_2 and Δ_2 is equivalently labelled to Δ_3 and $f_i: \Delta_i \to \Delta_{i+1}$ and $g_i: \overline{G}(\Delta_i) \to \overline{G}(\Delta_{i+1})$; $i = 1, 2$ are isomorphisms with the required properties, then $f_1 f_2$ is an isomorphism from Δ_1 to Δ_3 and $g_1 g_2$ is an isomorphism from $\overline{G}(\Delta_1)$ to $\overline{G}(\Delta_3)$ and $f_1 f_2$ and $g_1 g_2$ have the required property. Thus the relation is reflexive, symmetric and transitive and hence is an equivalence relation. □

We now establish a number of other results on equivalently labelled transposition subgraphs .

Proposition 2.2.3

If Δ is equivalently labelledto Δ', then $\Delta \cong \Delta'$ as graphs and $G(\Delta) \cong G(\Delta')$ as multigraphs.

Proof

It is clear from the definition that $\Delta \cong \Delta'$ and $\overline{G}(\Delta) \cong \overline{G}(\Delta')$ as graphs, so it is only necessary to show that the isomorphism $g: \overline{G}(\Delta) \to \overline{G}(\Delta')$ is an isomorphism from $G(\Delta)$ to $G(\Delta')$. This is the case iff g preserves the multiple edges of $G(\Delta)$. If some pair of vertices i, j of $G(\Delta)$ are joined by k edges, then $(i\ j)$ is the label of k edges of Δ . Each of these edges is mapped by f to an edge of Δ' labelled $(i\ j)h_g$, so there are k edges of Δ' labelled $(i\ j)h_g$. Now by the definition of h_g ,

$(i\ j)h_g$ is the transposition in $\Omega(\Delta')$ corresponding to $\{i, j\}g = \{ig, jg\}$, so $(i\ j)h_g = (ig\ jg)$. This is the label of k edges of Δ' iff ig is joined to jg in $G(\Delta')$ by k edges. Thus g does preserve edge multiplicities, so it is an isomorphism from $G(\Delta)$ to $G(\Delta')$. □

Note, however, the converse to this result does not hold. A counter-example is given by the transposition subgraphs Δ_1 and Δ_3 in fig.2.2.3 .

Proposition 2.2.4

If Δ and Δ' are identically labelled, then they are equivalently labelled.

Proof

This result is obvious from the two definitions. □

Definition 2.2.5

If Δ is a transposition subgraph and σ is a permutation then $\sigma\Delta$ is defined to be the transposition subgraph obtained by premultiplying every vertex of Δ by σ.

Proposition 2.2.5

For all Δ and σ, Δ is identically labelled to $\sigma\Delta$.

Proof

It suffices to show that $f: \Delta \to \sigma\Delta$ defined by $\rho \to \sigma\rho$ for all vertices ρ of Δ is a label-preserving isomorphism. If $\{\rho_1, \rho_2\}$ is an edge of Δ labelled ω, then $\rho_1^{-1}\rho_2 = \omega$. $\{\rho_1, \rho_2\}$ is mapped by f to $\{\sigma\rho_1, \sigma\rho_2\}$ and $(\sigma\rho_1)^{-1}(\sigma\rho_2) = \rho_1^{-1}\sigma^{-1}\sigma\rho_2 = \rho_1^{-1}\rho_2 = \omega$, so f is a label-preserving isomorphism. □

Proposition 2.2.6

If Δ is a connected transposition subgraph and G is a graph such that $\overline{G}(\Delta) \subset G$ then Δ is identically labelled to a subgraph

$\Delta' \subset \Gamma(G)$.

Proof

Suppose that ρ is a vertex of Δ, and let $\Delta' = \rho^{-1}\Delta$, so Δ' is identically labelled to Δ and has (1) as a vertex. Thus it suffices to prove that Δ' is a subgraph of $\Gamma(G)$. Since Δ' is connected, if σ is any vertex of Δ', there is a path in Δ' joining (1) to σ. Let this path be $(1) = \sigma_0 \sim \sigma_1 \sim \sigma_2 \sim \ldots \sim \sigma_k = \sigma$, where $\sigma_i = \sigma_{i-1}\omega_i$ for $i = 1, \ldots, k$.

Now $G(\Delta') = G(\Delta) \subset G$, so $\Omega(\Delta') \subset \Omega(G)$, and $\omega_1, \omega_2, \ldots, \omega_k \in \Omega(G)$. Also, (1) is a vertex of $\Gamma(G)$, so $\sigma_1 = (1)\omega_1$ is a vertex of $\Gamma(G)$. Similarly, $\sigma_2, \sigma_3, \ldots, \sigma_k$ are vertices of $\Gamma(G)$, so σ is a vertex of $\Gamma(G)$ and the result follows. □

The above result is the near-converse to proposition 2.2.1 referred to after the proof of 2.2.1 .

Proposition 2.2.7

If Δ and Δ' are equivalently labelled connected transposition subgraphs and $\Delta \subset \Lambda$, then there exists a transposition subgraph Λ' such that $\Delta' \subset \Lambda'$ and Λ is equivalently labelled to Λ'.

Proof

To prove this result, the following lemma is needed.

Lemma

If h is an isomorphism from a graph H to a graph H' and if G is a graph such that $H \subset G$, then $\exists$ a graph G' and an isomorphism $g: G \to G'$ such that $H' \subset G'$ and $g|_H = h$.

Proof of lemma

Let $V'(G - H)$ be any set such that $|V'(G - H)| = |V(G - H)|$ and $V'(G - H) \cap V(H') = \phi$, and let f be a bijection from $V(G - H)$ to $V'(G - H)$. (Note that G − H is the graph obtained by deleting the vertices of H from G.)

Now define a graph G' in the following way:
Let $V(G') = V'(G - H) \cup V(H')$, let $g: V(G) \to V(G')$ be the map defined by $vg = vh$ if $v \in V(H)$ and $vg = vf$ if $v \in V(G - H)$, and let $E(G') = \{\{ug, vg\} : \{u, v\} \in E(G)\}$.

Claim: g is an isomorphism from G to G'; for clearly g is a bijection from $V(G)$ to $V(G')$, and the definition of the edges of G' ensures that g maps edges to edges.

H' is a subgraph of G' since $V(H') \subset V(G')$ and $E(H)g = E(H)h$ by the definition of g, so $E(H)g = E(H')$ since by definition, h maps $E(H)$ to $E(H')$.

Finally, it is obvious from the definition of g that $g|_H = h$. $\square$

Proof of 2.2.7

Since Δ is equivalently labelled to Δ', there is an isomorphism $g: G(\Delta) \to G(\Delta')$, and since $\Delta \subset \Lambda$, $G(\Delta) \subset G(\Lambda)$ so by the lemma there is a graph G' and an isomorphism g' from $G(\Lambda)$ to G' such that $G(\Delta') \subset G'$ and $g'|_{G(\Delta)} = g$. There is also an isomorphism $f: \Delta \to \Delta'$ which maps edges of Δ labelled ω to edges of Δ' labelled ωh_g, where h_g is the bijection from $\Omega(\Delta)$ to $\Omega(\Delta')$ induced by g. Let σ be a vertex of Δ, and let $\sigma' = \sigma f$; σ' is a vertex of Δ'.

Now define f' by $\rho f' = \sigma'(g')^{-1}\sigma^{-1}\rho g'$ for all vertices ρ of Λ. f' is injective, for if $\rho_1 f' = \rho_2 f'$ then $\sigma'(g')^{-1}\sigma^{-1}\rho_1 g' = \sigma'(g')^{-1}\sigma^{-1}\rho_2 g'$ so $\rho_1 = \rho_2$. f' is used to define Λ' as follows:
Let $V(\Lambda') = \{\rho f' : \rho \text{ is a vertex of } \Lambda\}$, and let
$E(\Lambda') = \{\{\rho_1 f', \rho_2 f'\} : \{\rho_1, \rho_2\} \text{ is an edge of } \Lambda\}$. If $\{\rho_1, \rho_2\}$ is labelled $(i\ j)$, then $\{\rho_1 f', \rho_2 f'\}$ is labelled $(ig'\ \ jg')$. It is necessary to check that with this definition

Λ' is a correctly labelled transposition subgraph which satisfies the conditions of the proposition.

Certainly, $V(\Lambda')$ is a set of permutations, so it is only necessary to check that the edges of Λ' are well-defined. If $\{\rho_1, \rho_2\}$ is an edge of Λ labelled $(i\ j)$, then $\rho_1^{-1}\rho_2 = (i\ j)$. Also, $\rho_1 f'$ and $\rho_2 f'$ are vertices of Λ'. Now

$$\begin{aligned}(\rho_1 f')^{-1}(\rho_2 f') &= (\sigma' g'^{-1}\sigma^{-1}\rho_1 g')^{-1}(\sigma' g'^{-1}\sigma^{-1}\rho_2 g')\\ &= g'^{-1}\rho_1^{-1}\sigma g'\,\sigma'^{-1}\sigma' g'^{-1}\sigma^{-1}\rho_2 g'\\ &= g'^{-1}\rho_1^{-1}\rho_2 g'\\ &= g'^{-1}(i\ j)g'\\ &= (ig'\ \ jg')\end{aligned}$$

so $\{\rho_1 f', \rho_2 f'\}$ is well-defined as an edge labelled $(ig'\ jg')$. Clearly, f' is an isomorphism from Λ to Λ' mapping edges labelled $(i\ j)$ to edges labelled $(ig'\ jg') = (i\ j)h_{g'}$, thus Λ is equivalently labelled to Λ'.

Finally we must show that $\Delta' \subset \Lambda'$. Let ρ' be any vertex of Δ'; since Δ' is connected, there is a path $\sigma' = \sigma_0' \sim \sigma_1' \sim \sigma_2' \sim \cdots \sim \sigma_k' = \rho'$ in Δ' from σ' to ρ'. Hence $\exists$ transpositions $\omega_1', \omega_2', \ldots, \omega_k' \in \Omega(\Delta')$ such that $\sigma_i' = \sigma_{i-1}'\omega_i'$ for $i = 1, \ldots, k$.

Now f maps σ to σ' by definition, and it maps edges of Δ labelled ω to edges of Δ' labelled $\omega' = \omega h_g$. Let σ_i be vertices of Δ such that $\sigma_i f = \sigma_i'$ for $i = 1, \ldots, k$, and let $\{\sigma_{i-1}, \sigma_i\}$ be labelled ω_i so $\omega_i h_g = g^{-1}\omega_i g = \omega_i'$.

$$\begin{aligned}\sigma_k f' &= \sigma' g'^{-1}\sigma^{-1}\sigma_k g'\\ &= \sigma' g'^{-1}\sigma^{-1}(\sigma\omega_1\omega_2\ldots\omega_k)g'\\ &= \sigma' g'^{-1}(\omega_1\omega_2\ldots\omega_k)g'\\ &= \sigma'(g'^{-1}\omega_1 g')(g'^{-1}\omega_2 g')\ldots(g'^{-1}\omega_k g').\end{aligned}$$

However, ω_i corresponds to an edge of Δ, so if $\omega_i = (x\ y)$, then x and y are vertices of $G(\Delta)$. Using the lemma, g' was chosen so that $g'|_{G(\Delta)} = g$, so

$g'^{-1}\omega_i g' = \omega_i h_{g'} = \omega_i h_g$ for $i = 1, \ldots, k$.

Hence $\sigma'_k f' = \sigma'(\omega_1 h_g)(\omega_2 h_g)\ldots(\omega_k h_g)$

$= \sigma' \omega'_1 \omega'_2 \cdots \omega'_k$

$= \rho'$, so ρ' is a vertex of Λ'. It is easy to show that every edge of Δ' is an edge of Λ'. This completes the proof of proposition 2.2.7. □

Example

The graphs Δ, Δ' and Λ in fig 2.2.4 satisfy the hypotheses of proposition 2.2.7 and Λ' in fig 2.2.5 is equivalently labelled to Λ and $\Delta' \subset \Lambda'$. Λ' is constructed as in the proof of prop. 2.2.7 .

Figure 2.2.4

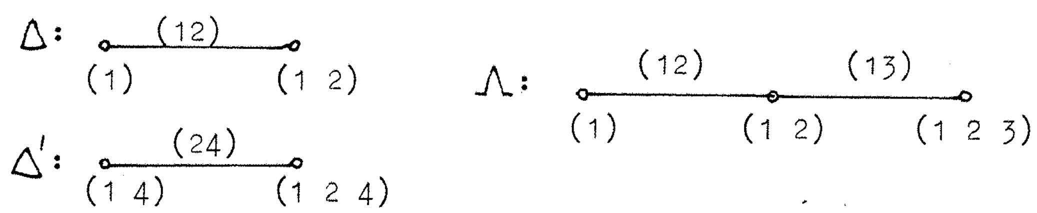
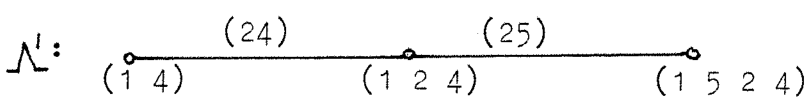


Figure 2.2.5



Definition 2.2.6

A transposition subgraph Δ has an induced labelling if it is an induced subgraph of some transposition graph. Otherwise it has a non-induced labelling.

Example

The transposition subgraph Δ_1 in figure 2.2.3 has a non-induced labelling. For suppose it is an induced subgraph of $\Gamma(G)$ for some graph G. Then $\Gamma(G)$ has edges labelled (1 2), (3 4) and (5 6) so $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$ are edges of G. However, (1) and (3 4) are vertices of $\Gamma(G)$ and $(3\ 4) \in \Omega(G)$ so $\{(1), (3\ 4)\}$ is an edge of $\Gamma(G)$ since $(1)^{-1}(3\ 4) = (3\ 4) \in \Omega(G)$. Also, (1) and (3 4) are vertices of Δ_1 but $\{(1), (3\ 4)\}$ is not an edge of Δ_1 so Δ_1 is not an induced subgraph of $\Gamma(G)$.

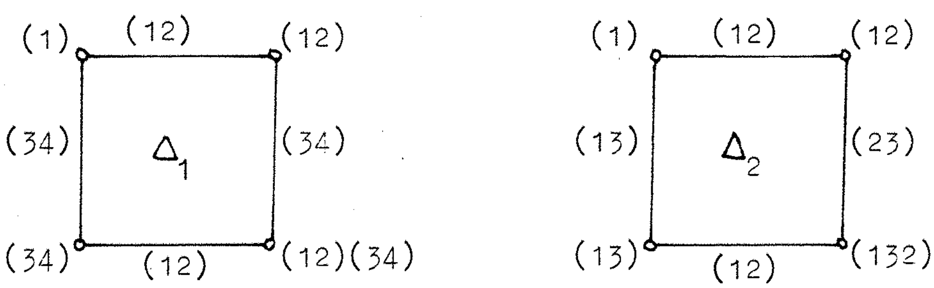
The graph Δ_3 in fig. 2.2.3 has an induced labelling since it is an induced subgraph of $\Gamma(G)$ where G is the graph with vertices 1, 2,..., 6 and edges $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

It is possible to classify transposition subgraphs according to which equivalence class of equivalently labelled graphs they belong to. This is particularly useful for small transposition subgraphs when it turns out that the number of equivalence classes is fairly small. In particular, the classification of transposition subgraphs isomorphic to C_4 is used repeatedly in this thesis, while the classification of transposition subgraphs isomorphic to $K_{3,3}$ and C_6 is vital to the study of the automorphisms of transposition graphs.

Theorem 2.2.8

If Δ is a transposition subgraph isomorphic to C_4 then Δ is equivalently labelled to either Δ_1 or Δ_2, where Δ_1 and Δ_2 are the graphs in fig. 2.2.6.

Figure 2.2.6



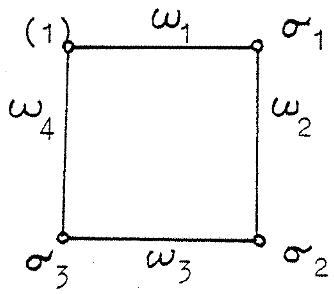
Proof

Note that Δ_1 and Δ_2 are not equivalently labelled since $G(\Delta_1)$ has 3 vertices and is not isomorphic to $G(\Delta_2)$ which has 4 .

If Δ has a vertex σ, then by premultiplying every vertex of Δ by σ^{-1} we obtain an identically labelled transposition subgraph $\sigma^{-1}\Delta$ which has (1) as a vertex. Thus we may assume without loss of generality that (1) is a vertex of Δ. Let the edge labels of Δ be ω_1, ω_2, ω_3, ω_4 in clockwise order starting from (1), so Δ is the graph in fig. 2.2.7 .

Figure 2.2.7

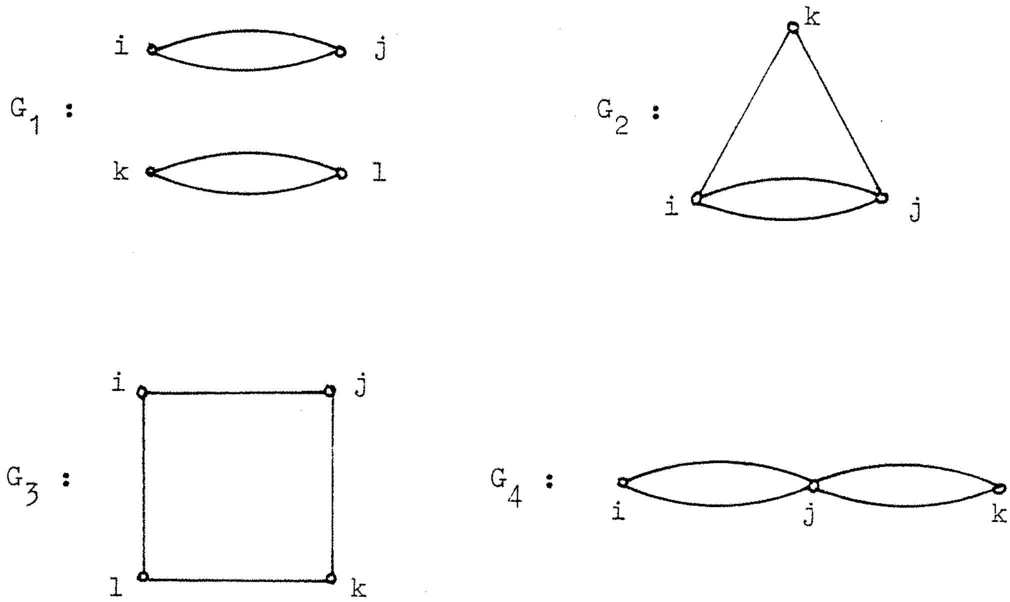
Δ :



Clearly, $\omega_i \neq \omega_{i+1}$, subscripts mod 4, or we would have either $(1) = \sigma_2$ or $\sigma_1 = \sigma_3$, in which case Δ would not be isomorphic to C_4. Hence there can be at most two edges in Δ with the same label. It follows that $G(\Delta)$ has at most two edges joining any pair of vertices.

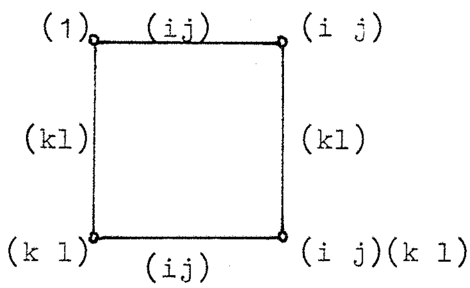
$W \equiv \omega_1\omega_2\omega_3\omega_4$ is an identity word since Δ is a circuit in some transposition graph, and hence $G(\Delta) = G(W)$ has no vertex of degree 1 by proposition 1.2.4 . Also, $G(\Delta)$ has exactly 4 edges, so $G(\Delta)$ must be isomorphic to one of the multigraphs $G_1, \ldots, G_4$ since they are the only multigraphs with 4 edges, no vertex of degree 1 and no more than 2 edges joining any pair of vertices. They are shown in fig. 2.2.8 .

Figure 2.2.8



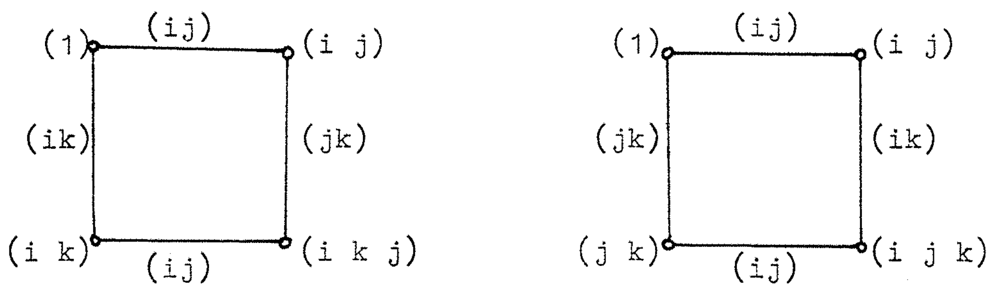
If $G(\Delta) = G_1$, then Δ must have 2 edges labelled (i j) and 2 edges labelled (k l), and these pairs must be non-incident. Hence Δ is the graph in fig 2.2.9 which is clearly equivalently labelled to Δ_1 .

Figure 2.2.9



If $G(\Delta) = G_2$, then Δ has 2 edges labelled (i j) which must be non-incident so Δ is one or other of the graphs in fig. 2.2.10. These two graphs are both equivalently labelled to Δ_2 ; for the first graph, take f: (1) $\rightarrow$ (1), (i j) $\rightarrow$ (1 2), (i k j) $\rightarrow$ (1 3 2) and (i k) $\rightarrow$ (1 3) , and take g: i $\rightarrow$ 1, j $\rightarrow$ 2, k $\rightarrow$ 3. For the second graph take f: (1) $\rightarrow$ (1), (i j) $\rightarrow$ (1 2), (i j k) $\rightarrow$ (1 3 2) and (j k) $\rightarrow$ (1 3) and take g: i $\rightarrow$ 2, j $\rightarrow$ 1 and k $\rightarrow$ 3 .

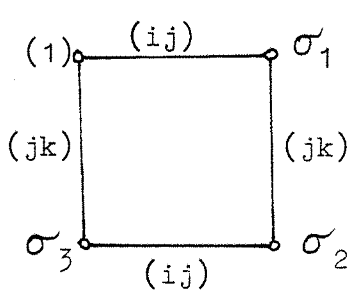
Figure 2.2.10



If $G(\Delta) = G_3$, then $G(\omega_1, \omega_2, \omega_3) = P_4$ whichever of the transpositions (i j), (j k), (k l), (i l) is ω_4 .
Hence by corollary 1.2.8, $W' \equiv \omega_1\omega_2\omega_3$ represents a 4-cycle , (x y z w), say. Now $W \equiv \omega_1\omega_2\omega_3\omega_4 = W'\omega_4$, and W represents the identity, so $(x\ y\ z\ w)\omega_4 = (1)$, and $\omega_4 = (x\ w\ z\ y)$, which is a contradiction since ω_4 is a transposition.

Finally, if $G(\Delta) = G_4$, by similar arguments to the first case, Δ must be the graph in fig. 2.2.11 . However, if this were true, W = (i j)(j k)(i j)(j k) = (i j k) = (1), which gives a contradiction.

Figure 2.2.11

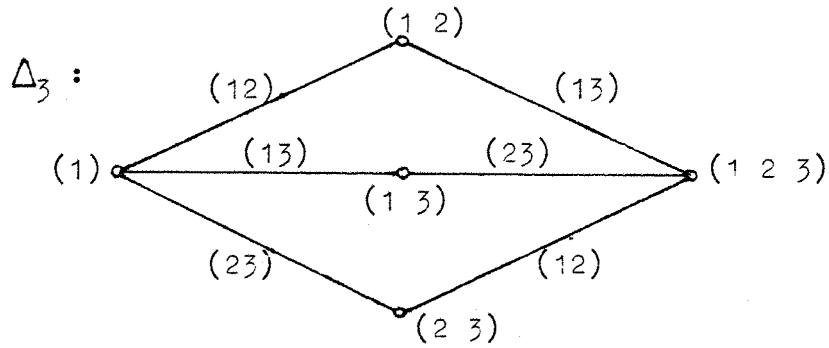


This completes the proof of theorem 2.2.8 . □

Theorem 2.2.9

If Δ is a transposition subgraph isomorphic to $K_{2,3}$, then Δ is equivalently labelled to the graph Δ_3 in fig. 2.2.11 .

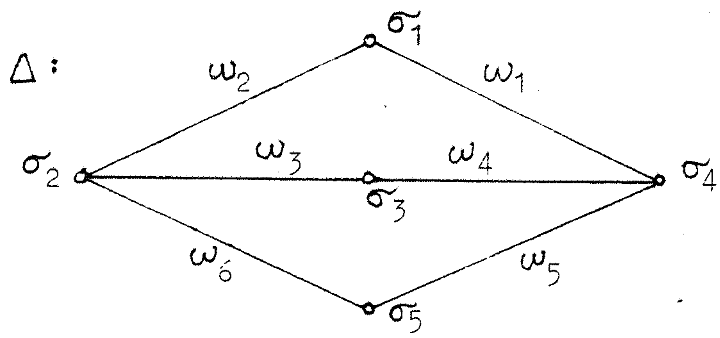
Figure 2.2.11



Proof

Let Δ be the graph in fig. 2.2.12.

Figure 2.2.12



Let Δ' be the subgraph of Δ induced by $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, and let Δ'' be the subgraph of Δ induced by $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5\}$. Δ' is isomorphic to C_4 so by theorem 2.2.8, Δ' is equivalently labelled to Δ_1 or Δ_2 , where Δ_1 and Δ_2 are the graphs in fig. 2.2.6 .

Suppose first that Δ' is equivalently labelled to Δ_1, so without loss of generality, $\omega_1 = \omega_3 =$ (i j) and $\omega_2 = \omega_4 =$ (k l). However, $\{\sigma_2, \sigma_3\}$ and $\{\sigma_3, \sigma_4\}$ are also edges of Δ'', so (i j) and (k l) are labels of edges of Δ'' and hence $\{i, j\}$ and $\{k, l\}$ are edges of $\overline{G}(\Delta'')$. Hence $\overline{G}(\Delta'') \ncong K_3$, so by theorem 2.2.8, Δ'' must be equivalently labelled to Δ_1 . Hence $\overline{G}(\Delta'')$

can only be the graph with vertices i, j, k, l and edges $\{i, j\}$ and $\{k, l\}$. Hence ω_6 must be either (i j) or (k l) . In either case this gives a contradiction, for if ω_6 = (i j), then $\omega_6 = \omega_3$ so $\sigma_3 = \sigma_5$, while if ω_6 = (k l) , then $\omega_6 = \omega_2$ so $\sigma_1 = \sigma_5$. Therefore Δ' must be equivalently labelled to Δ_2 , so without loss of generality, $\omega_1 = \omega_3$ = (i j), ω_2 = (i k) and ω_4 = (j k).

By a similar argument to the previous one, Δ'' must also be equivalently labelled to Δ_2 . The only possible choice for ω_6 is ω_6 = (j k), and the only choice for ω_5 is ω_5 = (i k). (Any other choice would imply either σ_1 or $\sigma_3 = \sigma_5$.)

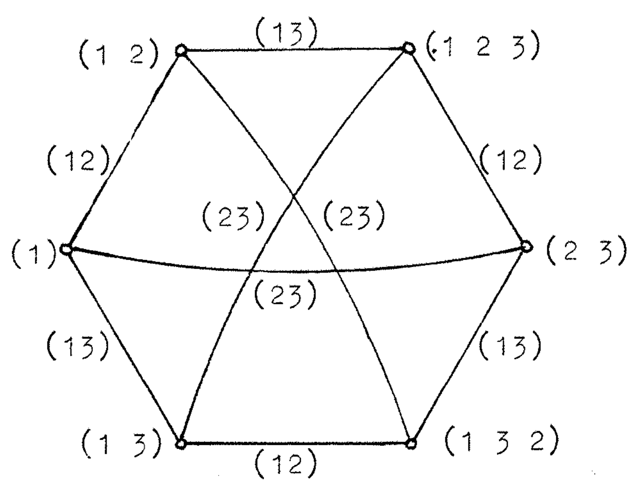
Now Δ is clearly equivalently labelled to Δ_3 ; one possible choice for f and g is f: $\sigma_1 \mapsto (1\ 2)$, $\sigma_2 \mapsto (1)$, $\sigma_3 \mapsto (1\ 3)$, $\sigma_4 \mapsto (1\ 2\ 3)$, $\sigma_5 \mapsto (2\ 3)$ and g: $i \mapsto 1$, $j \mapsto 3$, $k \mapsto 2$. □

Theorem 2.2.10

If Δ is a transposition subgraph isomorphic to $K_{3,3}$, then Δ is equivalently labelled to Δ_4, where Δ_4 is the graph in fig. 2.2.13 .

Figure 2.2.13

Δ_4 :



Proof

Suppose that Δ has vertices $\sigma_1, \ldots, \sigma_6$ and that $\sigma_i \sim \sigma_j$ iff iand j have different parities. Let $\Delta' = \Delta - \{\sigma_6\}$, so Δ' is isomorphic to $K_{2,3}$; then by theorem 2.2.9, Δ' is equivalently labelled to Δ_3, the graph in fig. 2.2.11 . Hence by proposition 2.2.7, there is a graph Δ^* such that Δ is equivalently labelled to Δ^* and $\Delta_3 \subset \Delta^*$. It is clear from fig 2.2.11 that $V(\Delta^*) = V(\Delta_3) \cup \{\sigma\}$, where σ is some permutation, and $E(\Delta^*) = E(\Delta_3) \cup \{\{(1\ 2), \sigma\}, \{(1\ 3), \sigma\}, \{(2\ 3), \sigma\}\}$. Let these three edges be labelled $\omega_1, \omega_2, \omega_3$ respectively.

Now the graph $\Delta'' = \Delta^* - \{(1\ 2\ 3)\} \cong K_{2,3}$, so by theorem 2.2.9 and definition 2.2.4 $\overline{G}(\Delta'') = \overline{G}(\Delta_3) = K_3$. Also, Δ'' contains edges labelled (1 2), (1 3), (2 3), $\omega_1, \omega_2, \omega_3$, and so $\{\omega_1, \omega_2, \omega_3\} = \{(1\ 2), (1\ 3), (2\ 3)\}$.

Finally, to avoid identifying σ with (1) or (1 2 3) in Δ^*, we must have $\omega_1 = (2\ 3)$, $\omega_2 = (1\ 2)$ and $\omega_3 = (1\ 3)$, so $\sigma = (1\ 3\ 2)$ and $\Delta^* = \Delta_4$. Thus Δ is equivalently labelled to Δ_4 . $\square$

SECTION 2.3 : CLASSIFICATION OF CIRCUITS OF LENGTH SIX

The classification of transposition subgraphs isomorphic to C_6 which will be described in this section is rather complicated, but a considerable part of the result is needed to study the automorphisms of a transposition graph. Since the proof of the full classification is very lengthy, splitting into fifteen separate cases, it will not be stated in full here; only that part of the classification needed to prove later results in this thesis will be proved here.

It is convenient to introduce an abbreviated notation for transposition subgraphs isomorphic to C_6 . If σ is a vertex of $\Delta \cong C_6$ and the edge labels of Δ are $\omega_1, \omega_2, \ldots, \omega_6$ in order from σ, Δ will be denoted by $\sigma ; \omega_1 \omega_2 \ldots \omega_6$. If $\sigma = (1)$, it will be suppressed in the notation. Also, if Δ has an induced labelling, it will be marked with an asterisk *. For example, if Δ_3 is the graph in fig. 2.2.3, (1 2)(3 4) is a vertex of Δ_3, so Δ_3 = (1 2)(3 4);(34)(12)(56)(34)(12)(56)*. More simply, since (1) is a vertex of Δ_3, Δ_3 = (12)(34)(56)(12)(34)(56)*.

Theorem 2.3.1

If Δ is a transposition subgraph isomorphic to C_6 , then Δ is equivalently labelled to one of the following graphs, which are grouped into nine classes. Every graph in class i has reduced type G_i , where G_i ; i = 1, 2,..., 9 are the graphs in fig. 2.3.1 .

Class 1 : (12)(34)(56)(12)(34)(56)*, (12)(34)(56)(12)(56)(34);

Class 2 : (12)(34)(45)(12)(45)(34);

Class 3 : (12)(23)(12)(23)(12)(23);

Class 4 : (12)(34)(12)(35)(34)(45) , (12)(34)(12)(35)(45)(35), (12)(34)(35)(12)(34)(45)*;

<u>Class 5</u> : (12)(34)(12)(23)(34)(24) , (12)(34)(12)(23)(24)(23);

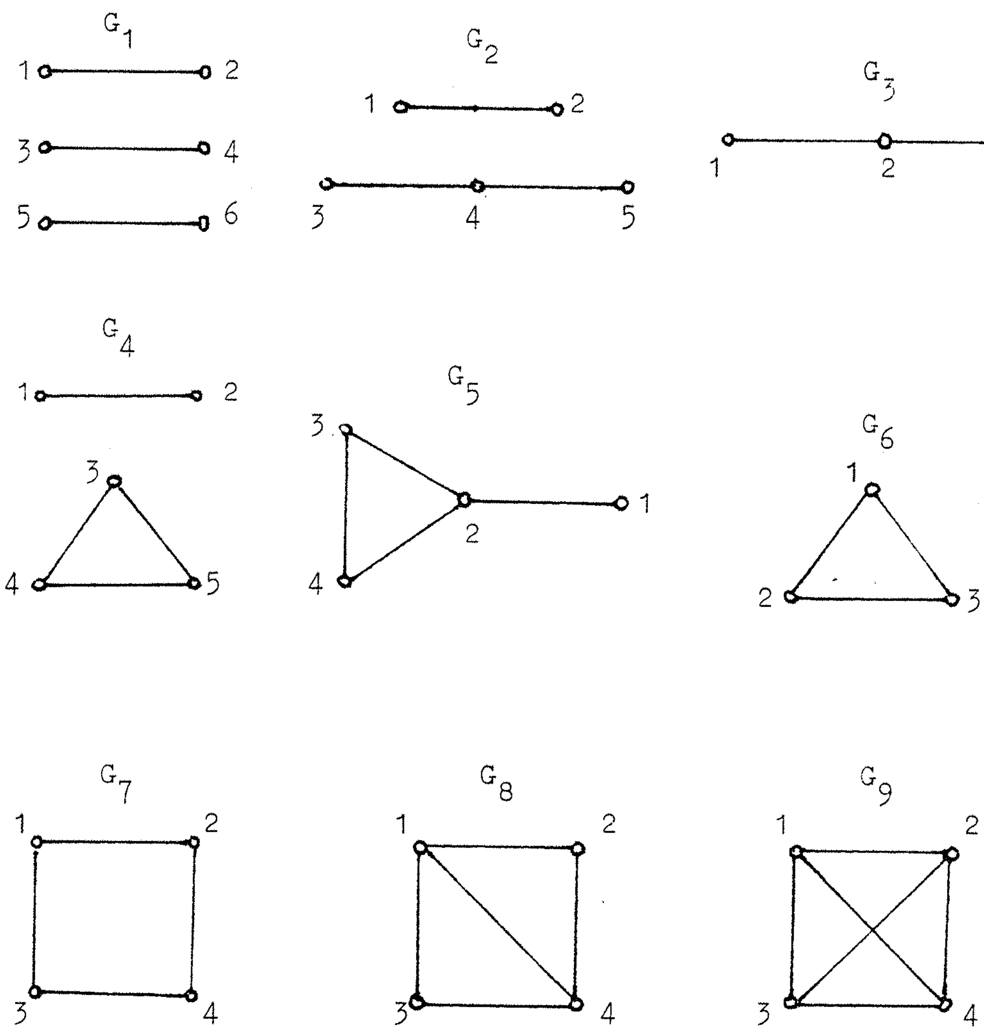
<u>Class 6</u> : (12)(23)(13)(12)(23)(13) ;

<u>Class 7</u> : (12)(23)(12)(34)(14)(34)*, (12)(23)(12)(14)(34)(14)*, (12)(23)(34)(12)(14)(34)*;

<u>Class 8</u> : (12)(13)(14)(23)(13)(34)*, (12)(13)(14)(12)(23)(34)*, (12)(13)(23)(34)(13)(14) , (12)(23)(12)(34)(13)(14) ;

<u>Class 9</u> : (12)(34)(13)(24)(14)(23)* .

<u>Figure 2.3.1</u>



<u>Proof</u>

Part of the proof of this result is given later in this section; the remainder is omitted. □

Theorem 2.3.2

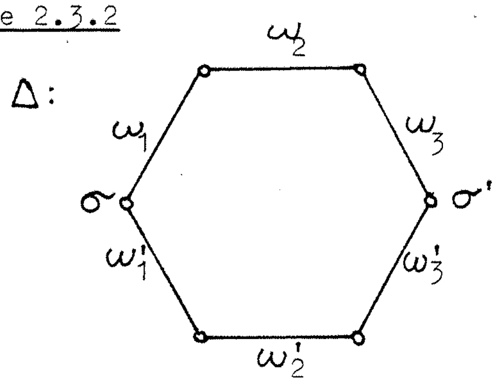
If Δ is a transposition subgraph isomorphic to C_6 which has an induced labelling, then its reduced type $\overline{G}(\Delta)$ is isomorphic to either G_1, G_3, G_4, G_7, G_8, or G_9, where $G_1, G_3, \ldots, G_9$ are some of the graphs in fig. 2.3.1 .

Proof

This result is clearly a corollary to theorem 2.3.1, but the proof given here is independent of theorem 2.3.1 .

Let σ be any vertex of Δ; since $\Delta \cong C_6$ there is a (unique) vertex σ' distance 3 from σ in Δ. In fact, Δ is the union of two edge-disjoint paths joining σ to σ' . Hence there exist transpositions $\omega_1, \omega_2, \omega_3, \omega_1', \omega_2', \omega_3' \in \Omega(\Delta)$ such that $\sigma' = \sigma\omega_1\omega_2\omega_3 = \sigma\omega_1'\omega_2'\omega_3'$. This situation is illustrated in fig. 2.3.2 .

Figure 2.3.2



Let $\rho = \omega_1\omega_2\omega_3 = \omega_1'\omega_2'\omega_3'$, and let $W \equiv \omega_1\omega_2\omega_3$ and $W' \equiv \omega_1'\omega_2'\omega_3'$ be words. It is clear that $G(\Delta)$ is the (multi)graph obtained by forming the union of $G(W)$ and $G(W')$, leaving any multiple edges distinct.

Since ρ is the product of three transpositions, it must be one of the following permutations, where a, b, ... are distinct: (a b c d) , (a b c)(d e), (a b)(c d)(e f), or (a b) . In each of the first three cases, W and W' are minimal words for ρ , hence G(W) and G(W') are forests whose connected components correspond to the disjoint cycles of ρ .

If $\rho = (a\ b\ c\ d)$ then G(W) and G(W') are trees on the vertices a, b, c, d, and hence $\overline{G}(\Delta)$ is a graph with four vertices. (In fact, it can be any connected graph on four vertices.)

If $\rho = (a\ b\ c)(d\ e)$, then both G(W) and G(W') must be one of the graphs in fig. 2.3.3 and hence $\overline{G}(\Delta)$ is one of the graphs in fig. 2.3.4 .

Figure 2.3.3

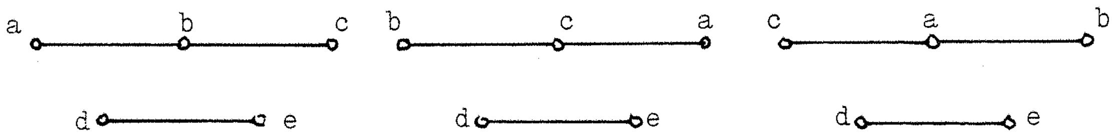
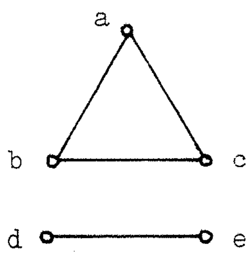
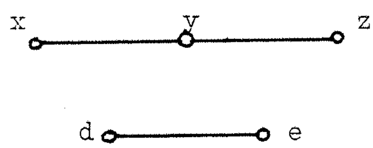


Figure 2.3.4

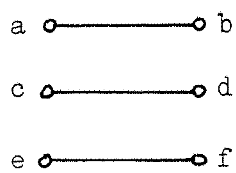




Where $\{x, y, z\} = \{a, b, c\}$.

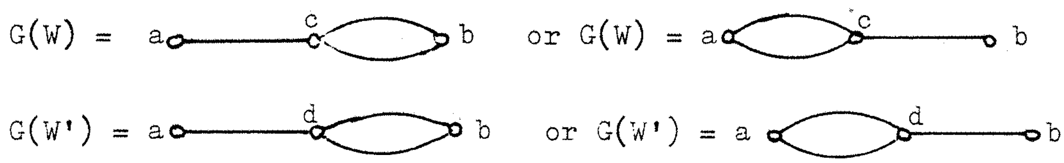
If $\rho = (a\ b)(c\ d)(e\ f)$, then both G(W) and G(W') must be the graph in fig. 2.3.5, and hence $\overline{G}(\Delta)$ is the graph in fig. 2.3.5 .

Figure 2.3.5



If $\rho = (a\ b)$ then W and W' are not minimal words, hence by theorem 1.2.8 the graphs G(W) and G(W') must contain circuits. Also, $\omega_1 \neq \omega_2$ and $\omega_1' \neq \omega_2'$, so $\overline{G}(W)$ and $\overline{G}(W')$ have at least two distinct edges. Finally, by hypothesis, Δ has an induced labelling, and hence $\{a,b\}$ cannot be an edge of either G(W) or G(W'). (If it were, (a b) would be the label of some edge of Δ, and σ would be joined to σ' by an edge labelled (a b) in any transposition graph containing Δ , which gives a contradiction.) By proposition 1.2.3 , a and b are in the same component of G(W) and G(W'), but by the above observation they are not adjacent in either graph. Since G(W) and G(W') have exactly 3 edges and must contain a circuit, they must be the graphs in fig.2.3.6 , where c and d may now be identical.

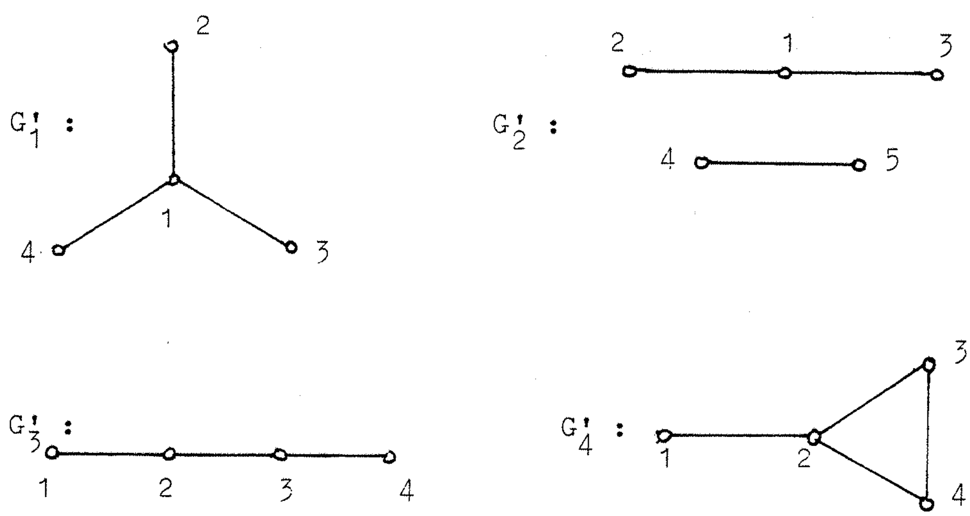
Figure 2.3.6



Hence $\overline{G}(\Delta) \cong C_4$ if $c \neq d$ and $\overline{G}(\Delta) \cong P_3$ if $c = d$.

Combining the above four cases, we have shown that $G(\Delta)$ is isomorphic to one of the six graphs in the hypothesis, or to one of the graphs in fig 2.3.7 .

Figure 2.3.7



To complete the proof of theorem 2.3.2 it is sufficient to show that there is no transposition subgraph Δ satisfying the hypotheses such that $\overline{G}(\Delta)$ is isomorphic to one of the graphs in fig. 2.3.7 . Let $\sigma, \sigma', \rho, \omega_1, \ldots, \omega_3', W, W'$ be defined as before. Note that if $\overline{G}(\Delta)$ is any of the graphs in fig. 2.3.7 then by the previous arguments, $\rho \neq (a\ b)$.

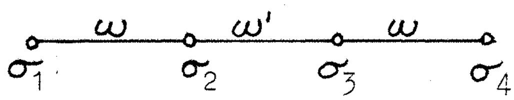
Lemma 2.3.3

If Δ has two edges with the same label ω, then these edges must be diametrically opposite in Δ.

Proof

Suppose that Δ has two such edges which are not diametrically opposite. Since they have the same label they cannot be incident, so the only remaining possibility is that there is a third edge incident to both of them. Let this edge be labelled ω', so contains the subgraph in fig. 2.3.8 . □

Figure 2.3.8



σ was chosen to be an arbitrary vertex of Δ', so we may choose $\sigma = \sigma_1$ so $\sigma' = \sigma_4$. Now $\sigma' = \sigma\rho$, and $\sigma' = \sigma\omega\omega'\omega = \sigma\,\omega^{-1}\omega'\omega = \sigma\omega^*$, where ω^* is a transposition. Hence $\rho = \omega^*$ is a transposition. This contradicts the observation made just before this lemma.

Lemma 2.3.4

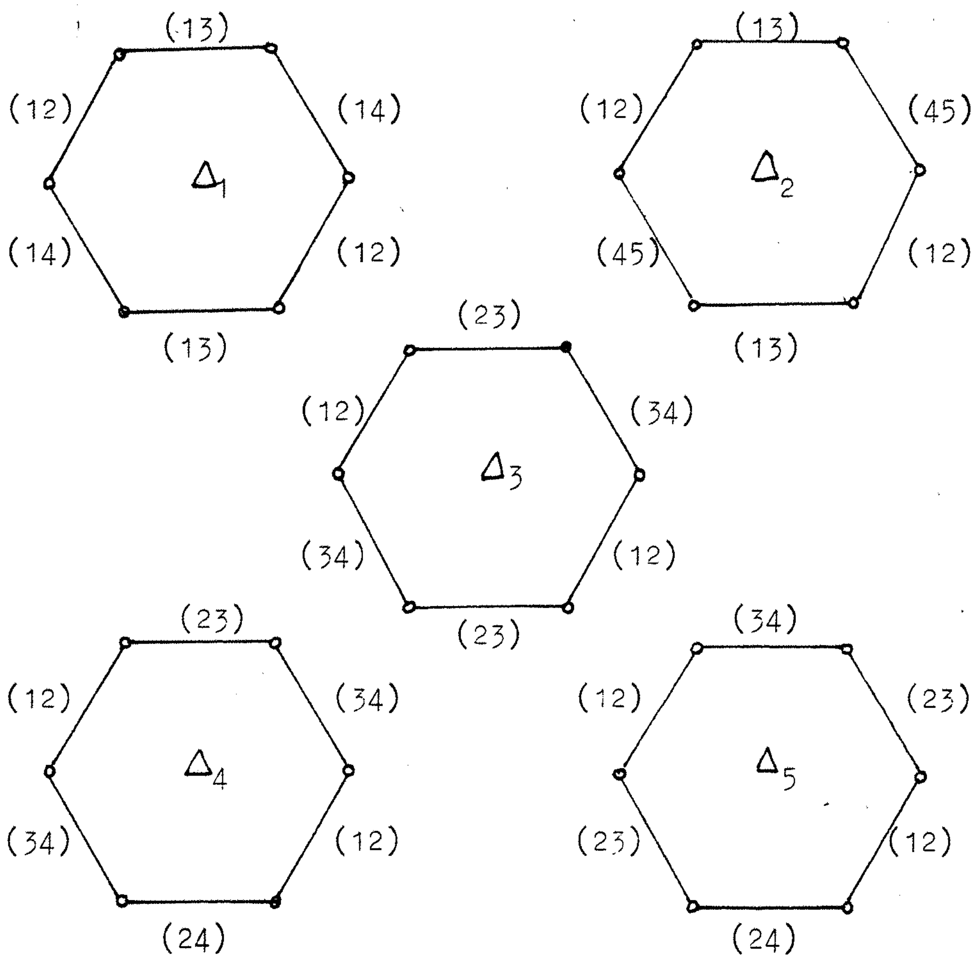
If $\{i\ j\}$ is an edge of $\overline{G}(\Delta)$ such that either i or j is a vertex of degree 1, then Δ has at least 2 edges labelled (i j).

Proof

This lemma follows immediately from proposition 1.2.4. The word W = (1) in the proposition is obtained by multiplying the labels of Δ in cyclic order. □

Using these two lemmas it is easy to see that Δ must be one of the graphs in fig. 2.3.9. Δ_i corresponds to $\overline{G}(\Delta) = G_i'$ for i = 1, 2, 3 , and Δ_4 and Δ_5 correspond to $\overline{G}(\Delta) = G_4'$.

Figure 2.3.9



However, each of these possibilities gives a contradiction. Since each of the graphs $\Delta_1, \ldots, \Delta_5$ is a transposition subgraph, the product of the edge labels in cyclic order must be an identity word. However, (12)(13)(14)(12)(13)(14) = (13)(24) , (12)(13)(45)(12)(13)(45) = (1 3 2), (12)(23)(34)(12)(23)(34) = (1 3)(2 4), (12)(23)(34)(12)(24)(34) = (1 2 3) and finally, (12)(34)(23)(12)(24)(23) = (1 2 4), giving a contradiction in each case. This completes the proof of theorem 2.3.2 . $\square$

Determining the existence of circuits of a given length in a transposition graph is a far easier problem than classifying them.

<u>Proposition 2.3.5</u>

$\Gamma(G)$ contains circuits of length 4 iff $G \not\cong K_{1,n}$ for all $n \geqslant 1$.

<u>Proof</u>

There exist transposition subgraphs $\Delta \cong C_4$ such that $\overline{G}(\Delta)$ is isomorphic to a graph in fig. 2.3.10 , by theorem 2.2.8. By proposition 2.2.6, if G is a graph containing either of these graphs as a subgraph, then $\Gamma(G)$ contains a subgraph isomorphic to C_4 . The result now follows from the observation that the only graphs without isolated vertices which do not contain two independent edges are K_3 and $K_{1,n}$; $n \geqslant 1$. □

<u>Figure 2.3.10</u>

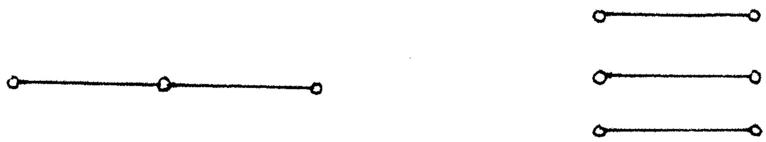
<u>Proposition 2.3.6</u>

Every transposition graph with 6 or more vertices contains a circuit of length 6.

<u>Proof</u>

Every graph contains a pair of incident edges unless all its edges are independent of each other. Hence every graph without isolated vertices except K_2 and G_1 , the graph in fig. 2.3.10, contain one or other of the graphs in fig.2.3.11 as a subgraph.

Figure 2.3.11



Now there exist transposition subgraphs $\Delta \cong C_6$ such that $\overline{G}(\Delta)$ is isomorphic to either of the graphs in fig. 2.3.11. In the condensed notation of theorem 2.3.1 they are (12)(23)(12)(23)(12)(23) and (12)(34)(56)(12)(34)(56) . The result now follows from proposition 2.2.6 and the observation that $\Gamma(K_2)$ and $\Gamma(G_1)$ have 2 and 4 vertices respectively. □

Corollary 2.3.7

A transposition graph $\Gamma(G)$ has girth 4 unless $G \cong K_{1,n}$, when $\Gamma(G)$ has girth 6 provided $n \geqslant 2$.

Proof

This result follows immediately from the two previous results and the fact that a transposition graph is bipartite. □

A similar result to proposition 2.3.6 can be proved for circuits of length 8. The proof uses transposition subgraphs isomorphic to C_8 with reduced types isomorphic to $K_{1,3}$, P_4 , G_1 and G_2 , where G_1 and G_2 are the graphs in fig. 2.3.1 . Thus it is possible to conjecture that the result holds for circuits of all even lengths $\geqslant 6$. An equivalent and more natural way to state this conjecture is as follows: A transposition graph with 2m vertices contains a circuit of length 2k for all k such that $3 \leqslant k \leqslant m$. (Note that all transposition graphs have an even number of vertices.) This alternative conjecture has been verified for all $m \leqslant 12$. The conjecture also implies that all transposition graphs are hamiltonian. This is proved in chapter 4 of this thesis.

CHAPTER 3: AUTOMORPHISMS OF TRANSPOSITION GRAPHS

SECTION 3.1: INTRODUCTION

In this chapter it is proved that any automorphism of a transposition graph can be expressed as the product of two or three special types of automorphism, a strong automorphism as defined in section 1.3, a weak automorphism fixing (1), and an irregular automorphism. Weak and irregular automorphisms are defined in section 3.2; weak automorphisms may be thought of as permuting the edge labels of the transposition graph, while irregular automorphisms destroy the edge labelling.

The weak automorphisms of a transposition graph are completely described in section 3.2. In fact the weak automorphisms of $\Gamma(G)$ are very closely related to the automorphisms of G. It is also shown that every automorphism of a transposition graph behaves 'locally' like a weak automorphism. This result is used to prove that $\Gamma(G)$ is a graphical regular representation iff G has no non-trivial automorphisms.

In section 3.3 it is proved that if G is a graph with no component isomorphic to a complete graph or to C_4 then $\Gamma(G)$ has no irregular automorphisms. In this case the automorphisms of $\Gamma(G)$ can be completely described in terms of automorphisms of G.

The irregular automorphisms of $\Gamma(K_n)$ and $\Gamma(C_4)$ are described in section 3.4, and the converse to the result of section 3.3 is proved.

Note that for most of the results in this chapter, graphs with a component isomorphic to K_2 are excluded since they complicate the statement of the results while adding little to the theory.

SECTION 3.2 : PRELIMINARY RESULTS

The automorphism group of a transposition graph $\Gamma(G)$ will be denoted by $A(\Gamma(G))$. Strong (or label-preserving) automorphisms of a transposition graph were defined in section 1.3 (definition 1.3.2), and the group of such automorphisms is denoted by $A_s(\Gamma(G))$. Clearly, $A_s(\Gamma(G)) \leqslant A(\Gamma(G))$. It is very useful in this chapter to distinguish an intermediate group of automorphisms, the weak automorphisms of a transposition graph.

Definition 3.2.1

An automorphism θ of a transposition graph $\Gamma(G)$ is weak or label-permuting if $\forall$ edges ε_1, ε_2 of $\Gamma(G)$, ε_1 and ε_2 have the same label iff $\varepsilon_1\theta$ and $\varepsilon_2\theta$ have the same label. The set of weak automorphisms of a transposition graph forms a group denoted by $A_w(\Gamma(G))$. Every strong automorphism is a weak automorphism, so $A_s(\Gamma(G)) \leqslant A_w(\Gamma(G)) \leqslant A(\Gamma(G))$.

Definition 3.2.2

If θ is an automorphism of $\Gamma(G)$, and σ is a vertex of $\Gamma(G)$, then θ fixes σ if $\sigma\theta = \sigma$ .. The set of all automorphisms of $\Gamma(G)$ fixing σ forms a group called the stabiliser of σ, denoted by $A(\Gamma(G),\sigma)$. The group $A_w(\Gamma(G),\sigma)$ is defined similarly.

Proposition 3.2.1

Every automorphism of $\Gamma(G)$ may be expressed as the product of a strong automorphism and an automorphism fixing (1); hence $A(\Gamma(G)) = A_s(\Gamma(G)).A(\Gamma(G), (1))$.

Proof

Let θ be an automorphism of $\Gamma(G)$, and let σ be such that $\sigma\theta = (1)$. Let ϕ_σ be the strong automorphism of $\Gamma(G)$ mapping ρ to $\sigma\rho$ for all vertices ρ of $\Gamma(G)$. Since the strong automorphisms of $\Gamma(G)$ form a group, ϕ_σ^{-1} is a strong automorphism.

Also, $(1)[\phi_\sigma\theta] = [\sigma(1)]\theta = \sigma\theta = (1)$, so $\phi_\sigma\theta$ is an automorphism fixing (1). Since $\theta = \phi_\sigma^{-1}(\phi_\sigma\theta)$, the result follows. □

In this section it will be shown that every element of $A(\Gamma(G), (1))$ is the product of an element of $A_w(\Gamma(G), (1))$ and an element of $A(\Gamma(G), (1), \Omega(G))$, the group of automorphisms of $\Gamma(G)$ fixing (1) and every vertex adjacent to (1). In section 3.3 it will be shown that this second group is the identity for almost all graphs G. Thus it is very useful to study the group $A_w(\Gamma(G), (1))$.

Lemma 3.2.2

If Δ is a subgraph of $\Gamma(G)$ and $\Delta \cong C_4$, then $\exists \Delta' \cong K_{3,3}$ such that $\Delta \subset \Delta' \subset \Gamma(G)$ iff $\bar{G}(\Delta) \cong K_3$.

Proof

By theorem 2.2.8 and theorem 2.2.10, if $\Delta \cong C_4$, $\Delta' \cong K_{3,3}$ and $\Delta \subset \Delta'$ then $\bar{G}(\Delta) \cong \bar{G}(\Delta') \cong K_3$.

Conversely, if $\bar{G}(\Delta) \cong K_3$, then (i j), (i k) and (j k) are labels of edges of Δ, where i, j, and k are the vertices of $\bar{G}(\Delta)$. Hence (i j), (i k), and (j k)$\in \Omega(G)$ so if σ is any vertex of Δ, then σ(i j), σ(i k), σ(j k), σ(i j k), σ(i k j) and σ are vertices of $\Gamma(G)$. Furthermore, they induce a subgraph Δ' of $\Gamma(G)$ isomorphic to $K_{3,3}$ containing Δ as a subgraph . □

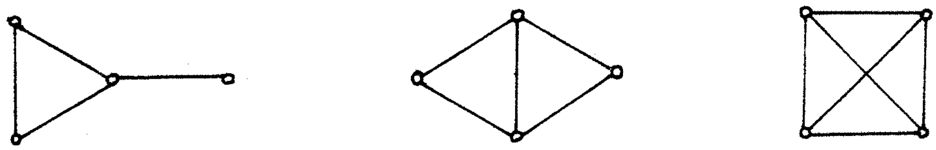
Lemma 3.2.3

If G is any connected graph with 3 or more vertices, then every edge automorphism of G is induced by a vertex automorphism unless G is one of the graphs in fig.3.2.1 .

Proof

See Behzad & Chartrand (2 , p.169) . □

Figure 3.2.1



Lemma 3.2.4

If G is as in lemma 3.2.3, then every edge automorphism of G which maps all subgraphs of G isomorphic to K_3 to subgraphs isomorphic to K_3 is induced by a vertex automorphism of G.

Proof

The result follows from lemma 3.2.3 unless G is one of the graphs in fig. 3.2.1 . If G is one of these graphs, then every edge automorphism of G not induced by a vertex automorphism is listed in Behzad & Chartrand (2 , p. 169); it is easy to check that none of them preserves triangles. □

Lemma 3.2.5

If g is an automorphism of G, then $\theta_g : \rho \mapsto g^{-1}\rho\, g$ for all vertices ρ of $\Gamma(G)$ is a weak automorphism of $\Gamma(G)$ fixing (1).

Proof

Let g be an automorphism of G and let $(i\ j) \in \Omega(G)$. From the definition of $\Omega(G)$ we have $\{i, j\} \in E(G)$, and since g is an automorphism, $\{i, j\} g = \{ig, jg\} \in E(G)$, so $(ig\ jg) \in \Omega(G)$. However, $(i\ j)\theta_g = g^{-1}(i\ j)g = (ig\ jg)$, so $(i\ j)\theta_g \in \Omega(G)$ for all transpositions $(i\ j) \in \Omega(G)$.

Now let ρ be any vertex of $\Gamma(G)$; hence there exist $\omega_1, \omega_2, \ldots, \omega_k \in \Omega(G)$ such that $\rho = \omega_1\omega_2\ldots\omega_k$. Therefore

$$\rho\,\theta_g = g^{-1}\rho\, g = g^{-1}(\omega_1\omega_2\ldots\omega_k)g$$
$$= (g^{-1}\omega_1 g)(g^{-1}\omega_2 g)\ldots(g^{-1}\omega_k g)$$
$$= \omega_1'\omega_2'\ldots\omega_k' \text{ , where } \omega_1', \omega_2', \ldots, \omega_k' \in \Omega(G).$$

Hence $\rho\,\theta_g$ is a vertex of $\Gamma(G)$, so θ_g maps vertices of $\Gamma(G)$ to vertices of $\Gamma(G)$.

Now let $\{\rho_1, \rho_2\}$ be an edge of $\Gamma(G)$ labelled ω, so
$\rho_1^{-1}\rho_2 = \omega \in \Omega(G)$. $\{\rho_1, \rho_2\}\theta_g = \{\rho_1\theta_g, \rho_2\theta_g\}$ and

$$
\begin{aligned}
(\rho_1\theta_g)^{-1}(\rho_2\theta_g) &= (g^{-1}\rho_1 g)^{-1}(g^{-1}\rho_2 g) \\
&= g^{-1}\rho_1^{-1} g g^{-1}\rho_2 g \\
&= g^{-1}\rho_1^{-1}\rho_2 g \\
&= g^{-1}\omega g \in \Omega(G).
\end{aligned}
$$

Therefore θ_g maps edges of $\Gamma(G)$ labelled ω to edges of $\Gamma(G)$ labelled $g^{-1}\omega g$, so θ_g is a weak automorphism of $\Gamma(G)$. Finally, $(1)\theta_g = g^{-1}(1)g = (1)$, so θ_g fixes (1). □

Theorem 3.2.6

For every automorphism $\theta \in A(\Gamma(G), (1))$, there is an automorphism $g \in A(G)$ such that $\phi = \theta_g^{-1}\theta$ is an automorphism of $\Gamma(G)$ fixing (1) and every vertex of $\Gamma(G)$ adjacent to (1), where θ_g is the automorphism of lemma 3.2.5.

Proof

The set of vertices of $\Gamma(G)$ adjacent to (1) is $\Omega(G)$. Since θ fixes (1), it must permute these vertices. Let $\hat{\theta}|_\Omega$ be the permutation of $\Omega(G)$ induced in this way, and let g^* be the corresponding permutation of $E(G)$.

We first show that g^* is an edge automorphism of G; that is, a permutation of $E(G)$ which preserves the incidence and independence of the edges of G. Suppose that e_1 and e_2 are non-incident edges of G which correspond to ω_1 and ω_2 in $\Omega(G)$. Then $(\omega_1\omega_2)^2 = (1)$. If Δ is the subgraph of $\Gamma(G)$ induced by the vertices (1), ω_1, ω_2, and $\omega_1\omega_2$ then $\Delta = C_4$ and $\bar{G}(\Delta)$ is isomorphic to G_1 in figure 2.3.10, so $\bar{G}(\Delta) \ncong K_3$. Hence by lemma 3.2.2, there is no graph $\Delta' \cong K_{3,3}$ such that $\Delta \subset \Delta' \subset \Gamma(G)$.

θ is an automorphism, so it must map Δ to a subgraph $\Delta\theta$ isomorphic to Δ and such that there is no subgraph Δ'' of $\Gamma(G)$ with $\Delta'' \cong K_{3,3}$ and $\Delta\theta \subset \Delta'' \subset \Gamma(G)$. Hence by lemma 3.2.2, $\overline{G}(\Delta\theta) \ncong K_3$, hence by theorem 2.2.8, $\overline{G}(\Delta\theta) \cong G_1$, the graph in fig. 2.3.10 . However, (1) is fixed by θ and ω_1 and ω_2 are mapped to $\omega_1\theta$ and $\omega_2\theta$ respectively, and these correspond to e_1g^* and e_2g^* by the definition of g^* . Since $\{(1), \omega_1\theta\}$ and $\{(1), \omega_2\theta\}$ are edges of $\Delta\theta$ labelled $\omega_1\theta$ and $\omega_2\theta$, e_1g^* and e_2g^* are edges of $\overline{G}(\Delta\theta)$, and since g^* is a permutation, they must be distinct edges. Since $\overline{G}(\Delta\theta) \cong G_1$, e_1g^* and e_2g^* must be non-incident. Hence g^* is an edge automorphism of G.

We now show that g^* preserves triangles in G. Let e_1 , e_2 and e_3 be edges of G forming a triangle (a subgraph isomorphic to K_3), and let $\omega_1, \omega_2, \omega_3$ be the corresponding elements of $\Omega(G)$. Then Δ , the subgraph of $\Gamma(G)$ induced by the vertices (1), ω_1, ω_2, ω_3, $\omega_1\omega_2$ is isomorphic to $K_{2,3}$. Δ is mapped by θ to a graph $\Delta\theta \cong K_{2,3}$, and by a similar argument to above, $\overline{G}(\Delta\theta)$ contains the edges e_1g^*, e_2g^*, e_3g^* which are distinct since g^* is a permutation. By theorem 2.2.9, $\overline{G}(\Delta\theta) \cong K_3$, so e_1g^*, e_2g^* and e_3g^* form a triangle in G, and g^* is triangle preserving. It follows from lemma 3.2.4 that g^* is induced by some automorphism g of G. That is, for all edges $\{i, j\}$ of G, $\{i, j\}g^* = \{ig, jg\}$.

Now suppose $(i\ j) \in \Omega(G)$, and $(i\ j)\theta = (k\ l) \in \Omega(G)$. Then $\{i, j\}g^* = \{ig, jg\} = \{k, l\}$, so $(i\ j)\theta = (k\ l) = (ig\ jg) = g^{-1}(i\ j)g = (i\ j)\theta_g$, where θ_g is the weak automorphism defined in lemma 3.2.5 . Hence if $\phi = \theta_g^{-1}\theta$, then $(1)\phi = (1)\theta_g^{-1}\theta = [g\ (1)\ g^{-1}]\theta = (1)\theta = (1)$ since θ fixes

(1) by hypothesis. Also, if ω is any element of $\Omega(G)$, then
$$\omega\phi = [\omega\,\theta_g^{-1}]\theta$$
$$= [\omega\,\theta_g^{-1}]\theta_g \quad \text{by the above argument ,}$$
$$= \omega\ .$$

Since the set of vertices adjacent to (1) in $\Gamma(G)$ is $\Omega(G)$, ϕ fixes every vertex adjacent to (1). This completes the proof of theorem 3.2.6 . □

Theorem 3.2.7

If G is a graph such that every connected component of G has at least three vertices, then $A_w(\Gamma(G), (1)) \cong A(G)$ and every element of $A_w(\Gamma(G), (1))$ is of the form $\theta_g : \rho \mapsto g^{-1}\rho\, g$ for all vertices ρ of $\Gamma(G)$, where $g \in A(G)$.

Proof

Consider the function $f: A(G) \to A_w(\Gamma(G), (1))$ defined by $g \mapsto \theta_g$ for all $g \in A(G)$; f is well defined since by lemma 3.2.5 , $\theta_g \in A_w(\Gamma(G), (1))$. We show that f is a group isomorphism, and begin by showing f is a homomorphism.

If $g_1, g_2 \in A(G)$, then $g_1 g_2 f = \theta_{g_1 g_2}$. If ρ is any vertex of $\Gamma(G)$, then
$$\rho\theta_{g_1 g_2} = (g_1\, g_2)^{-1}\rho\,(g_1\, g_2)$$
$$= g_2^{-1}\, g_1^{-1}\rho\, g_1\, g_2$$
$$= g_2^{-1}(\rho\theta_{g_1})g_2 = (\rho\theta_{g_1})\theta_{g_2}$$
$$= \rho\,\theta_{g_1}\theta_{g_2}\ .$$
Hence $\theta_{g_1 g_2} = \theta_{g_1}\theta_{g_2}$ so $(g_1\, g_2)f = (g_1 f)(g_2 f)$ and f is a homomorphism.

To show f is injective, suppose $g \in \ker(f)$, so f maps g to the identity in $A_w(\Gamma(G), (1))$. Then for all vertices of $\Gamma(G)$, $\rho\theta_g = g^{-1}\rho\, g = \rho$. Suppose that $g \neq (1)$, so g moves some vertex of G. Without loss of generality, suppose that $1g = 2$. The component of G containing 1 has at least three vertices by hypothesis, so it must contain at least one vertex distinct from both 1 and 2, which we may choose to be 3. (Note that 2 need not be in the same component of G as 1.) Since 1 and 3 are in the same component of G, $\Omega(G)$ generates (1 3) by corollary 1.2.2, so (1 3) is a vertex of $\Gamma(G)$. Since $g^{-1}\rho g = \rho$ for all vertices of $\Gamma(G)$ we have

$g^{-1}(1\ 3)\ g = (1g\ 3g) = (2\ 3g) = (1\ 3)$.

This is a contradiction since $2 \neq 1, 3$. Hence $g = (1)$, and f is injective.

Finally, if $\theta \in A_w(\Gamma(G), (1))$, then $\theta \in A(\Gamma(G), (1))$ so by theorem 3.2.6 and lemma 3.2.5 there is some $\theta_g \in A_w(\Gamma(G), (1))$ such that $\phi = \theta_g^{-1}\theta$ fixes (1) and every vertex of $\Gamma(G)$ adjacent to (1). Now since $A_w(\Gamma(G), (1))$ is a group and $\theta, \theta_g \in A_w(\Gamma(G), (1))$, $\phi \in A_w(\Gamma(G), (1))$. Since ϕ fixes (1) and every vertex adjacent to (1), ϕ fixes the edge label of every edge incident to (1). Hence the permutation of the edge labels of $\Gamma(G)$ induced by ϕ is the identity, so ϕ is a strong automorphism. Since ϕ fixes (1), ϕ is the identity, so $\theta = \theta_g$ and f is surjective. This completes the proof of theorem 3.2.7 . □

<u>Definition 3.2.3</u>

A non-trivial automorphism of a transposition graph which fixes (1) and every vertex adjacent to (1) is called an <u>irregular automorphism</u>. The set of all irregular automorphisms of $\Gamma(G)$, together with the identity, forms a group denoted by $A(\Gamma(G), (1), \Omega(G))$.

<u>Proposition 3.2.8</u>

Every automorphism of a transposition graph may be expressed as the product of a strong automorphism, a weak automorphism fixing (1), and (possibly) an irregular automorphism.

Proof

This result follows immediately from proposition 3.2.1, theorem 3.2.6 and the definition of an irregular automorphism. □

Irregular automorphisms of transposition graphs are studied in the next section. In general, a transposition graph has no irregular automorphisms, so its automorphisms are completely described by theorem 1.3.3 and theorem 3.2.7 .

Proposition 3.2.9

If σ is any vertex of a transposition graph $\Gamma(G)$ and ϕ is any automorphism of $\Gamma(G)$ fixing σ, then ϕ permutes the edges of $\Gamma(G)$ incident to σ, and hence ϕ permutes the labels of these edges. Thus ϕ induces a permutation of E(G); this permutation is induced by an automorphism of G.

Proof

This result follows from theorem 3.2.7 and the fact that there is a label-preserving automorphism of $\Gamma(G)$ from (1) to σ. □

Definition 3.2.4

A graphical regular representation of a group $\mathcal{G}$ is a graph G such that $A(G) \cong \mathcal{G}$ and $A(G, v) = \{1\}$ for all vertices v of G.

Graphical regular representations of the symmetric groups S_n have been studied by M.E. Watkins (14). Watkins' graphical regular representations are Cayley graphs generated by rather complicated sets of permutations, which are far from being minimal sets of generators . Using transposition graphs it is simple to construct numerous relatively simple graphical regular representations for S_n ; $n \geqslant 6$. For $n \geqslant 7$, some of these graphs are minimally generated.

Proposition 3.2.10

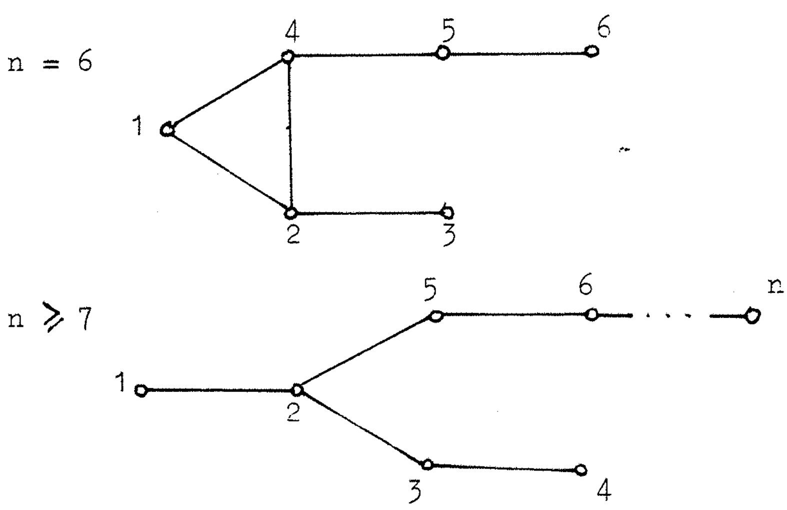
If G is a connected graph on n vertices such that $A(G) \cong \{1\}$, then $\Gamma(G)$ is a graphical regular representation for S_n .

Proof

By proposition 3.2.1, $A(\Gamma(G)) = A_s(\Gamma(G)).A(\Gamma(G), (1))$, and by theorems 1.3.3 and 1.2.1, $A_s(\Gamma(G)) = S_n$. Now suppose that $\phi \in A(\Gamma(G), (1))$; by theorem 3.2.7, $A_w(\Gamma(G), (1)) = \{1\}$, and hence by theorem 3.2.6, ϕ fixes every vertex of $\Gamma(G)$ adjacent to (1). Let σ be one of these vertices; by proposition 3.2.9, ϕ must fix every edge label incident to σ, so ϕ fixes every vertex adjacent to σ. It follows that ϕ fixes every vertex of $\Gamma(G)$ distance $\leqslant 2$ from (1). Repeating this argument as often as required, it is clear that ϕ is the identity, so $A(\Gamma(G), (1)) \cong \{1\}$ and $A(\Gamma(G)) \cong S_n$. □

Connected graphs G with n vertices such that $A(G) \cong \{1\}$ exist for all $n \geqslant 6$. A set of such graphs is shown in fig. 3.2.2; note that for $n \geqslant 7$ they correspond to minimal sets of generators for S_n by theorem 1.2.1 and the fact that trees are minimal connected graphs.

Figure 3.2.2



SECTION 3.3: RESTRICTIONS ON IRREGULAR AUTOMORPHISMS

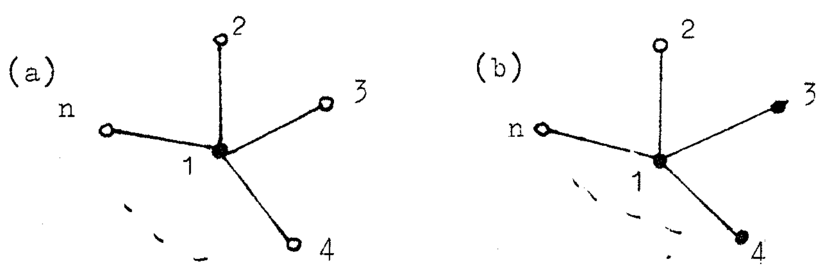
The main result in this section is that transposition graphs of almost all graphs have no irregular automorphisms. The exceptional graphs are also listed. Unfortunately, these results exclude graphs which have a connected component isomorphic to K_2 , since they are not covered by theorem 3.2.7. Before embarking on the proof of the main result, however, it is worth demonstrating that some transposition graphs do actually have irregular automorphisms. The simplest example is $\Gamma(K_3) \cong K_{3,3}$, the graph in fig. 2.2.13. By definition, an irregular automorphism is an automorphism which fixes (1) and every vertex adjacent to (1). ((1 2 3) (1 3 2)), the automorphism of $\Gamma(K_3)$ which transposes (1 2 3) and (1 3 2), is clearly irregular. The irregular automorphisms of $\Gamma(K_n)$; $n \geqslant 3$ are studied in section 3.4 .

It is convenient to introduce two special notations for diagrams of graphs. They will not be mixed in the same diagram.

Notation 1: if H is a subgraph of G, then a vertex of H will normally be denoted by o , but if H contains every edge of G adjacent to some vertex v , then v will be denoted by ● .

For example, if $H = K_{1,n-1}$ and $G = K_n$, then the vertices of fig.3.3.1(a) are correctly labelled, but those of fig. 3.3.1(b) are not.

Figure 3.3.1



<u>Notation 2</u>: if θ is an automorphism of $\Gamma(G)$, and if σ is a vertex of $\Gamma(G)$ fixed by θ , then σ may be denoted by ■ rather than o in a diagram showing the action of θ on $\Gamma(G)$.

It is also convenient to use the following notation: if e is an edge of a graph , then ω_e will be the transposition corresponding to e in the normal way.

<u>Definition 3.3.1</u>

An automorphism f of a graph G <u>fixes</u> a vertex v of G if vf = v, and f <u>fixes</u> an edge e of G if ef = e . Note that if f fixes e, then f does not necessarily fix the end vertices of e.

<u>Proposition 3.3.1</u>

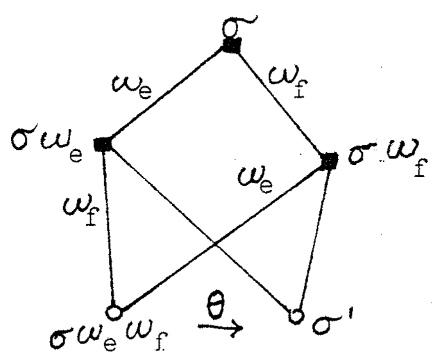
If e and f are independent edges of a graph G and θ is an automorphism of $\Gamma(G)$ fixing the vertices $\sigma, \sigma\omega_e$ and $\sigma\omega_f$ then θ also fixes the vertex $\sigma\omega_e\omega_f = \sigma\omega_f\omega_e$.

<u>Proof</u>

Suppose that $(\sigma\omega_e\omega_f)\theta = \sigma' \neq \sigma\omega_e\omega_f$. Now $\sigma\omega_e$ and $\sigma\omega_f$ are adjacent to $\sigma\omega_e\omega_f$ and θ is an automorphism, so $\sigma\omega_e$ and $\sigma\omega_f$ are adjacent to σ' . It follows that the graph Δ of fig. 3.3.2 is a subgraph of $\Gamma(G)$.

<u>Figure 3.3.2</u>

Δ:



Since $\Delta \cong K_{2,3}$, by theorem 2.2.9, $\overline{G}(\Delta) \cong K_3$. However, ω_e and ω_f are labels of edges of Δ, so e and f are edges of $\overline{G}(\Delta)$. This is a contradiction, since e and f are independent. □

Theorem 3.3.2

If e and f are edges of a graph G such that one of the graphs in fig. 3.3.3 is a subgraph of G, and if θ is an automorphism of $\Gamma(G)$ fixing the vertices ρ, $\rho\omega_e$ and $\rho\omega_f$ of $\Gamma(G)$, then θ also fixes the vertices $\rho\omega_e\omega_f$, $\rho\omega_f\omega_e$ and $\rho\omega_e\omega_f\omega_e = \rho\omega_f\omega_e\omega_f$.

Figure 3.3.3

Proof

Let Δ be the subgraph of $\Gamma(G)$ isomorphic to C_6 defined by $\Delta = \rho;\omega_e\omega_f\omega_e\omega_f\omega_e\omega_f$ in the notation introduced in section 2.3. By hypothesis, e and f cannot both be edges in a circuit of length 3 in G, and hence Δ is an induced subgraph of $\Gamma(G)$. (For if not, there would be a circuit of length 4 in $\Gamma(G)$ containing edges labelled ω_e and ω_f. Since e is incident to f the reduced type of this circuit of length 4 must be isomorphic to C_3. Since it is a subgraph of G and contains e and f, this is a contradiction.)

Let $\Delta' = \Delta\theta$, the image of Δ under θ . Since θ is an isomorphism, Δ' is an induced subgraph of $\Gamma(G)$ isomorphic to C_6. Hence by theorem 2.3.2, $\overline{G}(\Delta')$ is isomorphic to one of the graphs G_1, G_3, G_4, G_7, G_8, G_9 in fig.2.3.1 . Since θ fixes the vertices ρ, $\rho\omega_e$ and $\rho\omega_f$ of $\Gamma(G)$, Δ' contains edges labelled ω_e and ω_f , and hence $\overline{G}(\Delta')$ contains edges e and f. Also, $\overline{G}(\Delta')$ is a subgraph of G. The only one of the graphs G_1, G_3,..., G_9 consistent with these facts and with the restrictions on e and f imposed by hypothesis is G_3 . It follows that $\overline{G}(\Delta') = K_{1,2}$ and has edges e and f. The only subgraph Δ' of $\Gamma(G)$ containing the vertices ρ , $\rho\omega_e$, and $\rho\omega_f$ isomorphic to C_6 and with reduced type $\overline{G}(\Delta')$ as above is Δ. That is, θ maps Δ to itself. Since θ fixes two adjacent vertices of Δ , it is easy to see that θ fixes every vertex of Δ . □

Definition 3.3.2

Given an edge e of G, an automorphism g of G is of type A (w.r.t. e) if g fixes e, every edge of G not incident to e, and every edge f of G such that one of the graphs in fig. 3.3.3 is a subgraph of G.

Definition 3.3.3

An edge e of G is A-stable if the only automorphism of type A w.r.t. e is the identity.

Proposition 3.3.3

If e is an A-stable edge of G and θ is an automorphism of $\Gamma(G)$ fixing a vertex ρ of $\Gamma(G)$, and fixing every vertex of $\Gamma(G)$ adjacent to ρ , then θ fixes every vertex of $\Gamma(G)$ adjacent to $\rho\omega_e$.

Proof

By proposition 3.2.9, since θ fixes $\rho\omega_e$, it induces an automorphism g of G whose action on the edges of G is identical to the action of θ on the labels of the edges of $\Gamma(G)$ incident to $\rho\omega_e$. θ fixes ρ and $\rho\omega_e$, hence θ fixes the edge labelled ω_e incident to $\rho\omega_e$. It follows that g fixes e in G. We now show that g is of type A w.r.t. e.

Let f be any edge of G not incident to e ; by hypothesis, θ fixes ρ, $\rho\omega_e$,and $\rho\omega_f$, and hence by proposition 3.3.1, θ fixes $\rho\omega_e\omega_f$, so θ fixes the edge labelled ω_f incident to $\rho\omega_e$. It follows that g fixes f in G.

If f is an edge of G such that one of the graphs in fig. 3.3.3 is a subgraph of G, then $\rho\omega_e\omega_f$ is fixed by theorem 3.3.2, so θ fixes the edge labelled ω_f incident to $\rho\omega_e$. Hence g fixes f in G. By definition, g is of type A w.r.t. e. Since e is A-stable by hypothesis, g must be the identity, so θ fixes every vertex of $\Gamma(G)$ adjacent to $\rho\omega_e$. □

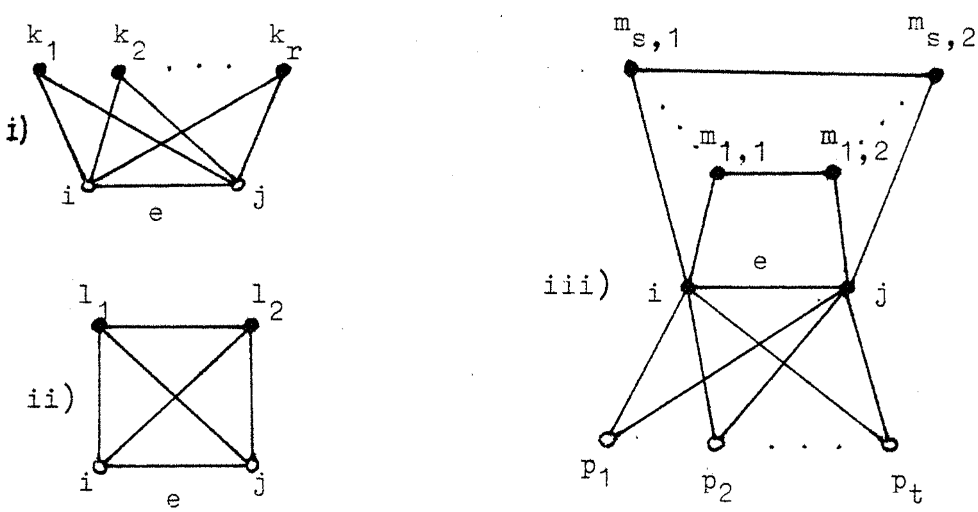
We now show that if G has an edge e which is not A-stable then G has a fairly special local structure.

Proposition 3.3.4

If G is a graph such that every component of G has at least three vertices, and if e is an edge of G which is not A-stable and g is a non-trivial automorphism of G of type A w.r.t. e then G contains one of the graphs in fig. 3.3.4 as a subgraph and g is a product of some of the following permutations:

i) any $\sigma \in S(k_1, k_2, \ldots, k_r)$, ii) $(l_1\ l_2)$,iii) $(i\ j)(m_{1,1}\ m_{1,2})\ldots(m_{s,1}\ m_{s,2})$.

Figure 3.3.4



Where $r \geq 2$, $s \geq 0$ and $t \geq 0$.

Proof

Let $e = \{i, j\}$ and let g be a non-trivial automorphism of G which is of type A w.r.t. e. Let G' be the graph obtained by deleting i and j from G. G' may be disconnected, so let $G_1, G_2, \ldots, G_n$ be the connected components of G'.

Lemma 3.3.5

The graph G_q ; $1 \leqslant q \leqslant n$ is fixed pointwise by g unless G_q has one or two vertices.

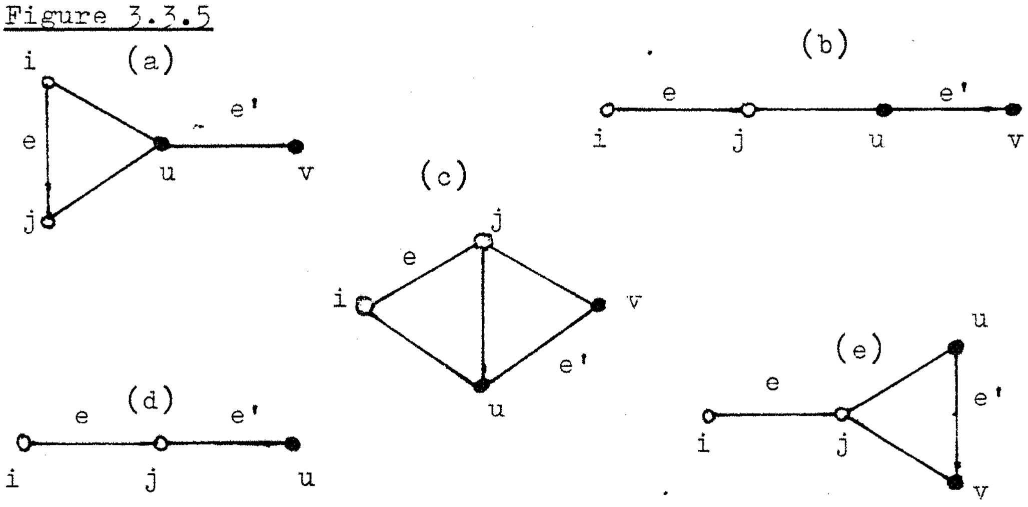
Proof of lemma

If G_q has 3 or more vertices, then it must clearly have 2 incident edges, since it is a connected graph. These two edges cannot be incident to e by the definition of G_q, so they are fixed by g. Since they have exactly one common vertex, it must also be fixed by g. Let this vertex be u. If v is any vertex adjacent to u in G_q then $e' = \{u, v\}$ is an edge of G_q. Both u and e' are fixed by g so v must be fixed by g. Since G_q is connected this argument can be extended to any vertex of G_q. □

Lemma 3.3.6

If G_q is any of the graphs in fig.3.3.5 then G_q is fixed pointwise by g.

Figure 3.3.5



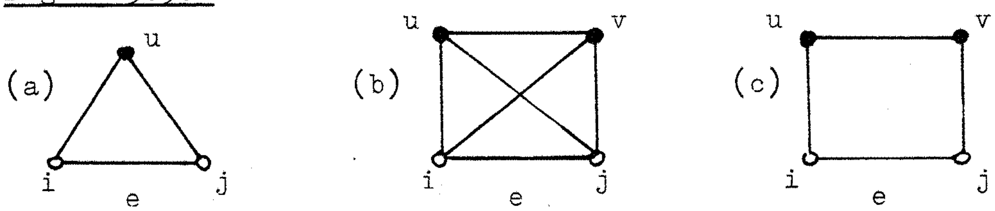
In case (d), $V(G_q) = \{u\}$. Otherwise, $V(G_q) = \{u, v\}$.

Proof of lemma

In cases (a), (b), (c), $e' = \{u, v\}$ is fixed by g, and u and v have different valencies, and cannot be in the same cycle of g. Hence they must be fixed by g. In case (d), g must fix u since it of type A w.r.t. e. In case (e), by a similar argument, g fixes $f = \{j, u\}$ and since j and u have different valencies, g must fix u. Similarly, g fixes v. □

By definition, g is non-trivial, so one of the graphs of fig. 3.3.6 must be a subgraph of G. (They are the only remaining possibilities, by the two lemmas.)

Figure 3.3.6



In case (a), G must have two of the subgraphs, or else u must be fixed by g. In case (c), (i j) must be a cycle of g for if i and j are fixed then u and v are also fixed. The result clearly follows from these observations. □

Definition 3.3.4

An edge of a graph G is B-stable if e is A-stable, or if e lies in a circuit of length 3 in G which also contains an A-stable edge.

Proposition 3.3.7

If $e = \{i, j\}$ is a B-stable edge of a graph G, where G has no component with 1 or 2 vertices, and if θ is an automorphism of $\Gamma(G)$ which fixes some vertex σ of $\Gamma(G)$ and every vertex of $\Gamma(G)$ adjacent to σ, then θ fixes every vertex adjacent to $\sigma\omega_e$.

Proof

If e is A-stsble then the result follows immediately from proposition 3.3.3, so suppose that e is not A-stable and lies in a circuit of length 3 with some A-stable edge c, say. Let the third edge in the circuit be d.

If θ fixes every vertex of $\Gamma(G)$ adjacent to $\sigma\omega_e$, then the result follows, so suppose that θ permutes the vertices of $\Gamma(G)$ in a non-trivial way. By proposition 3.2.9 there is a corresponding automorphism g of G whose action on the edges of G is identical to the action of θ on the labels of the edges of $\Gamma(G)$ incident to $\sigma\omega_e$; g is clearly a non-trivial automorphism. As in the proof of proposition 3.3.3, g is of type A w.r.t. e .

Let $e = \{i, j\}$ and let k be the other vertex in the special circuit of length 3, so without loss of generality, $c = \{i, k\}$ and $d = \{j, k\}$.

Lemma 3.3.8

The vertices i, j, k are fixed by g.

Proof of lemma

Since c is A-stable, θ fixes every vertex of $\Gamma(G)$ adjacent to $\sigma\omega_c$ by proposition 3.3.3. In particular, θ fixes the vertices $\sigma\omega_c\omega_d$ and $\sigma\omega_c\omega_e$. However, $\sigma\omega_c\omega_d = \sigma\omega_e\omega_c$ and $\sigma\omega_c\omega_e = \sigma\omega_e\omega_d$, so θ fixes the edges of $\Gamma(G)$ labelled ω_c and ω_d incident to the vertex $\sigma\omega_e$. Hence g fixes the edges c and d in G. It follows immediately that g fixes the vertices i, j and k of G. $\square$

Since g fixes i and j, the end vertices of e, and since g is of type A w.r.t. e, by proposition 3.3.4 the only vertices of G not necessarily fixed by g are $k_1, k_2, \ldots, k_r$ and l_1 and l_2, the vertices in subgraphs i) and ii) of fig. 3.3.4. (Either the k's or the l's must be vertices of G by proposition 3.3.4.)

Lemma 3.3.9

If k_1 is a vertex of G then $c' = \{i, k_1\}$ is A-stable.

Proof of lemma

Since $r \geqslant 2$ by proposition 3.3.4, i must have valency $\geqslant 3$. Also, k_1 has valency 2, $(i\ k_1)$ cannot be a cycle in any automorphism of G. Hence by proposition 3.3.4, if c' is not A-stable then one of the graphs in fig. 3.3.7 must be a subgraph of G.

Figure 3.3.7

In either case, k_1 has valency $\geqslant 3$, giving a contradiction. □

Hence by lemma 3.3.8, if k_1 is a vertex of G then k_1 is fixed by g. Since g is non-trivial, the only remaining possibility is that l_1 and l_2 are vertices of G.

Lemma 3.3.10

If l_1 and l_2 are vertices of G then the edge $c' = \{i, l_1\}$ is A-stable.

Proof of lemma

The proof of this lemma splits into two cases.

Case 1: both i and j are adjacent to vertices of G apart from each other and l_1 and l_2.

In this case, i has valency $\geqslant 4$ and l_1 has valency 3 so $(i\ l_1)$ cannot be a cycle in any automorphism of G. Hence by proposition 3.3.4, if c' is not A-stable then one of the graphs in fig. 3.3.7

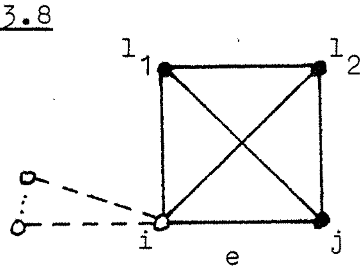
must be a subgraph of G, where k_1 is taken to be l_1.

Since l_1 is adjacent only to i, j and l_2, in either case, one of the black vertices must be j. This contradicts the fact that j has valency $\geqslant 4$.

Case 2: one of the vertices i and j has valency 3.

In this case, none of the edges in the subgraph of G induced by the vertices i, j, l_1, l_2 is A-stable. Since e = $\{i, j\}$ does not lie in any circuits of G outside this subgraph it follows that e is not B-stable, contrary to hypothesis. This situation is illustrated in fig. 3.3.8 in the case j has valency 3. □

Figure 3.3.8

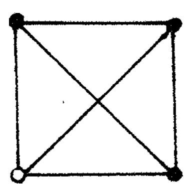


Again it follows by lemma 3.3.8 that g fixes l_1 . Similarly, g fixes l_2. Hence g fixes every vertex of G and so θ fixes every vertex of $\Gamma(G)$ adjacent to $\sigma\alpha_e$. □

Definition 3.3.5

An edge e of a graph G is C-stable if the component of G containing e is not isomorphic to K_4 , and if e lies in a subgraph of G isomorphic to the graph in fig. 3.3.9 .

Figure 3.3.9



Proposition 3.3.11

If G is a graph such that every component of G has at least three vertices, if e is a C-stable edge of G, and if θ is an automorphism of $\Gamma(G)$ such that θ fixes σ and every vertex of $\Gamma(G)$ adjacent to σ, then θ fixes every vertex of $\Gamma(G)$ adjacent to $\sigma\omega_e$.

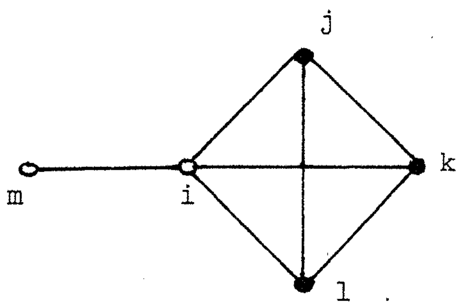
Proof

As in the proofs of the earlier results, θ induces an automorphism g of G by its action on the edges of $\Gamma(G)$ incident to $\sigma\omega_e$; as before, g is of type A w.r.t. e. Hence by proposition 3.3.4, one of the subgraphs of fig. 3.3.4 is a subgraph of G. (If g were a trivial automorphism of G, then the result would follow immediately.)

The proof of this result splits into two cases.

Case 1: e = $\{i,j\}$ in fig. 3.3.10 .

Figure 3.3.10



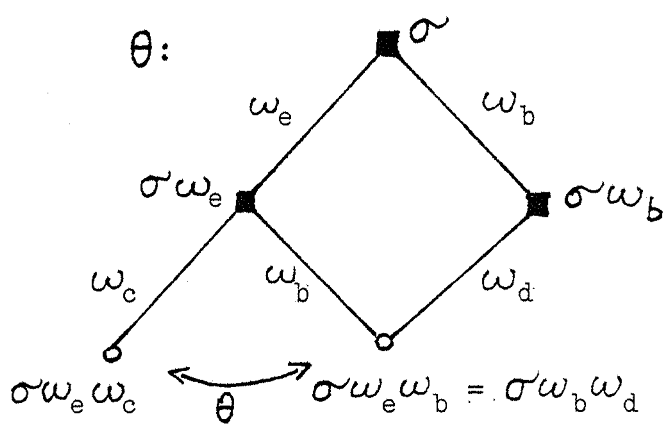
(Note that since the component of G containing e is not isomorphic to K_4 , then i must be adjacent to some vertex m other than j, k and l.)

Since i and j have different valencies, (ij) cannot be a cycle of g. Also, j is not adjacent to any vertices of valency 2, so by proposition 3.3.4, g = (k l), since it must fix every other vertex of G.

Let b = $\{i, k\}$, let c = $\{i, l\}$ and let d = $\{j, k\}$.

Clearly, g transposes edges b and c of G, and hence θ transposes the edges of $\Gamma(G)$ labelled ω_b and ω_c incident to $\sigma\omega_e$, by the definition of g. This is illustrated in fig. 3.3.12 .

Figure 3.3.12



Note that $\sigma\omega_e\omega_b = \sigma\omega_b\omega_d$ since b, d and e are the edges of a circuit of length 3 in G.

However, $\sigma\omega_e\omega_c$ is not adjacent to $\sigma\omega_b$ in $\Gamma(G)$, or there would be a circuit of length 4 in $\Gamma(G)$ with edges labelled ω_b, ω_c and ω_e (and one other label). This is a contradiction, by theorem 2.2.8 . In this case, θ does not preserve adjacencies in $\Gamma(G)$, which is also a contradiction.

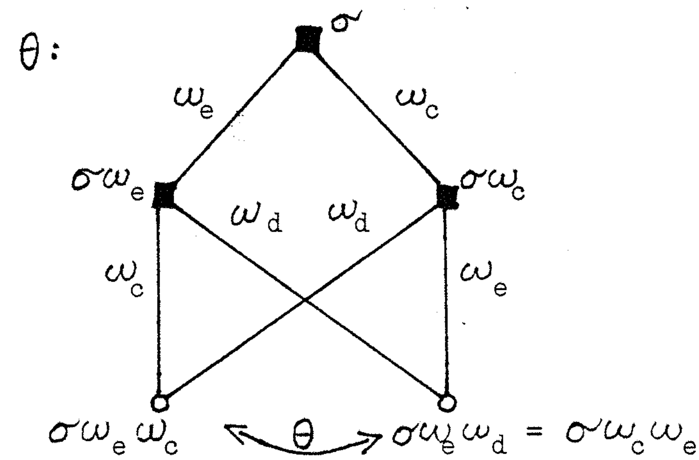
Case 2: $e = \{k, l\}$ in fig. 3.3.10 .

As in case 1, the only non-trivial possibility for g is $g = (k\ l)$.

Let $c = \{i, k\}$ and let $d = \{i, l\}$; g clearly transposes c and d. Hence θ acts on $\Gamma(G)$ as shown in fig. 3.3.13. Consider the action of θ on the edges of $\Gamma(G)$ incident to $\sigma\omega_c$. θ fixes the edge labelled ω_c, and transposes the edges labelled ω_d and ω_e . Hence the automorphism g' of G induced by the action of θ on the edges of $\Gamma(G)$ incident to $\sigma\omega_c$ fixes c and transposes d and e. This implies that (i k) is a cycle of g', which is a contradiction since i and k have different valencies in G.

Hence in either case, g is trivial and the result follows.

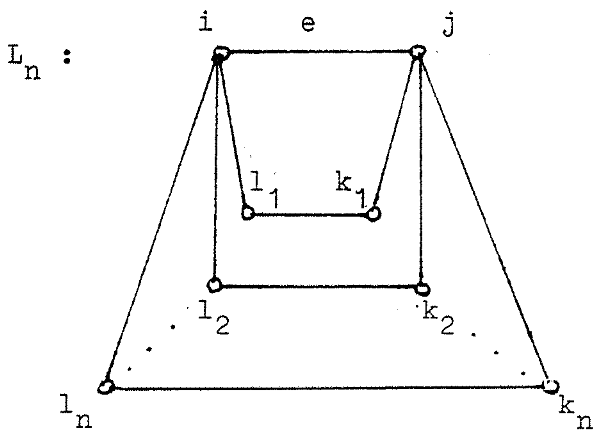
Figure 3.3.13



Proposition 3.3.12

If G is a graph such that every component of G has at least 3 vertices, and if e is an edge of G which is not B- or C-stable then the component of G containing e is isomorphic to K_n; $n \geqslant 3$, or L_n; $n \geqslant 1$, where L_n is the graph in fig. 3.3.14 .

Figure 3.3.14



Proof

Since $e = \{i, j\}$ is not B-stable, it follows that e is not A-stable, and hence by proposition 3.3.4, one of the graphs in fig. 3.3.4 is a subgraph of G. If i) or ii) is a subgraph of G then by lemma 3.3.9 or lemma 3.3.10 respectively, e lies in a circuit of length 3 with an A-stable edge and hence is B-stable, or else e is C-stable. In either case this is a contradiction.

The only remaining possibility is that iii) is a subgraph of G. Suppose first that both $s > 0$ and $t > 0$ in fig. 3.3.4 .

Lemma 3.3.13

In this case, $e' = \{i, p_1\}$ is A-stable.

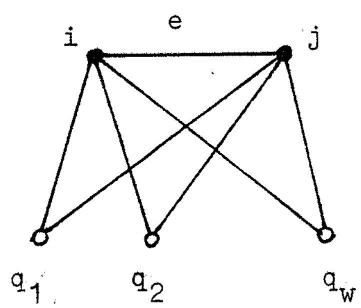
Proof of lemma

Suppose that g is an automorphism of G that is of type A w.r.t. e'. By definition, g fixes $d_1 = \{m_{1,1}, m_{1,2}\}$ and hence $m_{1,1}g = m_{1,1}$ or $m_{1,2}$. It follows that (i, p_1) cannot be a cycle of g since i is adjacent to $m_{1,1}$ but p_1 is not adjacent to $m_{1,1}g$. Hence if g is non-trivial, then by proposition 3.3.4, one of the graphs i) or ii) in fig. 3.3.4 is a subgraph of G. i) cannot be a subgraph of G since every vertex of G adjacent to i and to p_1 must also be adjacent to j, and hence has valency 3 or more. If ii) is a subgraph of G then the above argument implies that j must be one of the black vertices in the subgraph , and hence by an inspection of fig. 3.3.4 ii), every vertex adjacent to j is contained in a circuit of length 3 in G. This is a contradiction, since j is adjacent to $m_{1,2}$, which lies in no circuit of length 3 in G. It follows that g is trivial, and hence e' is A-stable. □

This result implies that e is B-stable, which is a contradiction. It follows that either $s = 0$ or $t = 0$.

If both s = 0 and t = 0 then the component of G containing e is isomorphic to K_2 . If t = 0 and s > 0 then the component of G containing e is isomorphic to L_s . In either case there is nothing remaining to prove. The only remaining case is s = 0 and t > 0. In this case G contains the graph in fig. 3.3.15 as a subgraph. (Note the relabelling of the vertices of G.)

Figure 3.3.15



If $e' = \{i, q_1\}$ is A-stable then e is B-stable. Hence by proposition 3.3.4, one of the graphs in fig. 3.3.4 is a subgraph of G. Note that if v is any vertex of G , then $i \sim v$ iff $j \sim v$, provided $v \neq i$ or j.

If i) or ii) is a subgraph of G then j is distinct from k_1 or l_1 respectively, without loss of generality. Hence k_1 has valency 3 if i) is a subgraph, since it is adjacent to i,j and q_1. This is a contradiction. If ii) is a subgraph of G then j is adjacent to i, q_1 and l_1 . The only possibility is that $j = l_2$, and hence i and j have valency 3. It is easy to check that in this case e is C-stable, which is a contradiction. The only remaining possibility is that iii) is a subgraph of G and that ($i\ q_1$) is a cycle of a non-trivial automorphism of type A w.r.t. e'. Note, however, that in fig. 3.3.4 iii), e must be changed to e' and j must be changed to q_1 , since we are now considering e', not e. The other labels of the graph do not clash with the new notation introduced in fig. 3.3.15.

First suppose that $s > 0$ in the relabelled version of fig. 3.3.4 iii). Since j lies in the circuit $i \sim j \sim q_1$ of G, $j \neq m_{1,2}$ since $m_{1,2}$ does not lie in a circuit of length 3 in G. Hence $i \sim m_{1,1}$ and $j \not\sim m_{1,1}$ which contradicts the observation that $i \sim v$ iff $j \sim v$. It follows that $s = 0$. Hence if v is any vertex of G distinct from i and q_1 then $i \sim v$ iff $q_1 \sim v$. A similar argument holds for $q_2, \ldots, q_w$ and hence the component of G containing e is a complete graph. This completes the proof of proposition 3.3.12. □

Proposition 3.3.14

If e is an edge of a graph G which has no component with less than 3 vertices, and if the component of G containing e is isomorphic to L_n ; $n \geqslant 2$, and if θ is an automorphism of $\Gamma(G)$ fixing σ and every vertex of $\Gamma(G)$ adjacent to σ, then fixes every vertex of $\Gamma(G)$ adjacent to $\sigma\omega_e$.

Proof Let g be the automorphism of G corresponding to the action of θ on the labels of the edges of $\Gamma(G)$ incident to $\sigma\omega_e$. As before, by proposition 3.3.1 and theorem 3.3.2, g is of type A w.r.t. e. If e is not the edge $\{i, j\}$ in fig. 3.3.14 then e is clearly A-stable and the result follows by proposition 3.3.3, so suppose that $e = \{i, j\}$. The only non-trivial automorphism of G which is of type A w.r.t. e is $(i\ j)(k_1\ l_1)(k_2\ l_2)\ldots(k_n\ l_n)$, so we must have $g = (i\ j)(k_1\ l_1)(k_2\ l_2)\ldots(k_n\ l_n)$. Hence if $b_r = \{j, k_r\}$, $c_r = \{k_r, l_r\}$ and $d_r = \{l_r, i\}$; $r = 1, 2, \ldots, n$, then g fixes e and c_r and transposes b_r and d_r. Hence θ fixes the edges of $\Gamma(G)$ labelled $(i\ j)$ and $(k_r\ l_r)$ incident to $\sigma\omega_e = \sigma(i\ j)$, and transposes the edges labelled $(j\ k_r)$ and $(i\ l_r)$.

Lemma 3.3.15

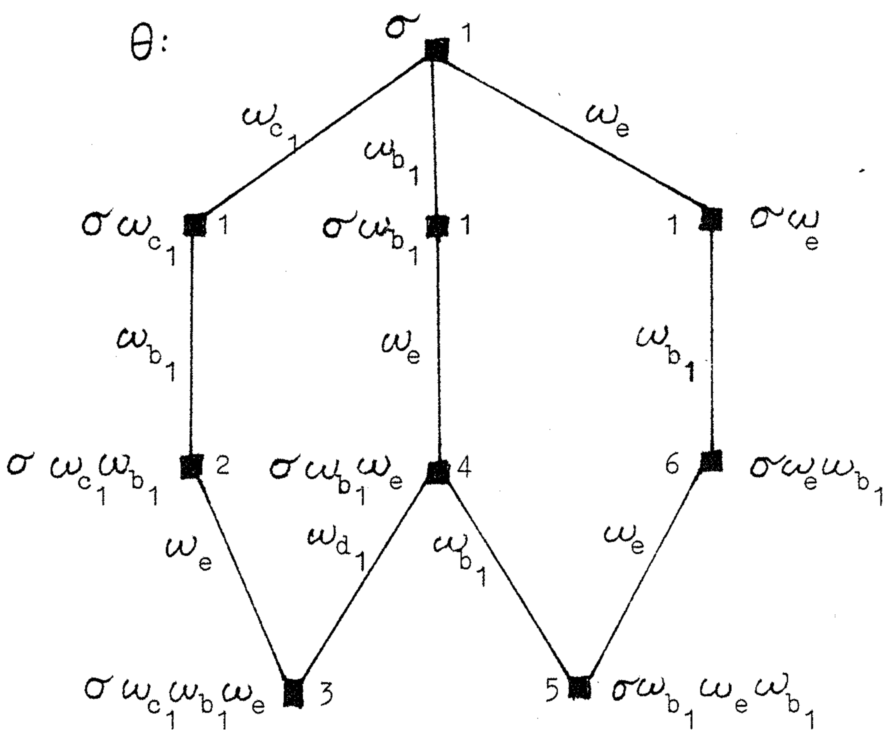
If a and a' are edges of a graph G such that every automorphism of G fixing a also fixes a', and if θ is an automorphism of $\Gamma(G)$ fixing ρ and $\rho\omega_a$, then θ also fixes $\rho\omega_a\omega_{a'}$.

Proof of lemma

The automorphism of G induced by the action of θ on the labels of edges of $\Gamma(G)$ incident to $\rho\omega_a$ must fix a in G, and hence it must fix a' by hypothesis. The result follows immediately. □

We now show that θ fixes all the vertices of $\Gamma(G)$ in fig. 3.3.16. Each vertex in the figure is numbered 1,..., 6; this number gives the reason that the vertex is fixed by θ . Since each numbered reason assumes that the previous reasons are true, they should be read in ascending order.

Figure 3.3.16



Note that $\omega_{c_1}\omega_{b_1}\omega_e\omega_{d_1}\omega_e\omega_{b_1} = (k_1\ l_1)(j\ k_1)(i\ j)$
$\cdot(i\ l_1)(i\ j)(j\ k_1) = (1)$,

and that $(\omega_e\omega_{b_1})^3 = ((i\ j)(j\ k_1))^3 = (1)$.

1: Vertices numbered 1 are fixed by hypothesis.

2: $\sigma\omega_{c_1}\omega_{b_1}$ is fixed by proposition 3.3.3 since c_1 is an A-stable edge of G.

3: This vertex is fixed by applying lemma 3.3.15 taking b_1, e and $\sigma\omega_{c_1}$ as a, a' and ρ respectively.

4: This vertex is fixed by applying lemma 3.3.15 to b_1, e and σ.

5: This vertex is fixed by lemma 3.3.15 applied to d_1, b_1 and $\sigma\omega_{c_1}\omega_{b_1}\omega_e$.

6: This vertex is fixed by lemma 3.3.15 applied to b_1, e and $\sigma\omega_{b_1}\omega_e$.

Hence θ fixes the edge labelled ω_{b_1} incident to $\sigma\omega_e$, which contradicts the earlier observation that θ must transpose this edge with the edge labelled ω_{d_1}. This completes the proof of proposition 3.3.14. $\square$

<u>Theorem 3.3.16</u>

If G is a graph without any connected components isomorphic to C_4 or to K_n ; $n \geqslant 1$ then $\Gamma(G)$ has no irregular automorphisms.

<u>Proof</u>

By proposition 3.3.12, every edge of G is B- or C-stable, or lies in a component of G isomorphic to L_n; $n \geqslant 2$. Let θ be any automorphism of $\Gamma(G)$ fixing (1) and every vertex of $\Gamma(G)$ adjacent to (1), and let σ be any vertex of $\Gamma(G)$ distance 2 from (1), so $\sigma = (1)\omega_e\omega_d$ where d and e are edges of G. Then θ fixes σ by proposition 3.3.7, proposition 3.3.11 or proposition 3.3.14 since e is either B-stable, C-stable or lies in a component of G isomorphic to L_n ; $n \geqslant 2$. Hence θ fixes every vertex of $\Gamma(G)$ distance 2 from (1). The same argument can be used to show that θ fixes vertices any distance from (1) and hence, since $\Gamma(G)$ is connected, θ fixes every vertex of $\Gamma(G)$. $\square$

Corollary 3.3.17

If G is a graph such that no component of G is isomorphic to C_4 or to K_n ; $n \geqslant 1$, then the stabiliser of $\Gamma(G)$, $A(\Gamma(G), (1)) \equiv A_w(\Gamma(G), (1))$, the group of weak automorphisms of $\Gamma(G)$ fixing (1).

Proof

This result follows from theorem 3.3.16 and theorem 3.2.6. □

In the next section it will be shown that the converse of this result also holds, so if G has a component isomorphic to C_4 or to K_n then $\Gamma(G)$ has irregular automorphisms.

Corollary 3.3.17 shows that for almost all graphs G, the stabiliser of $\Gamma(G)$ is isomorphic to A(G), and that all the elements of the stabiliser are very closely connected with automorphisms of G.

SECTION 3.4: IRREGULAR AUTOMORPHISMS OF C_4 AND K_n.

In the previous section it was shown that if G has no component isomorphic to C_4 or K_n then $\Gamma(G)$ has no irregular automorphisms. In the present section it will be shown that $\Gamma(C_4)$ and $\Gamma(K_n)$; $n \geqslant 3$ all have irregular automorphisms, and all the irregular automorphisms of these graphs will be described.

Theorem 3.4.1

For all $n \geqslant 3$, the bijection $\theta : S_n \to S_n$ defined by $\sigma\theta = \sigma^{-1}$ is an irregular automorphism of $\Gamma(K_n)$.

Proof

Clearly, θ permutes the vertices of $\Gamma(K_n)$, so to show that θ is an automorphism it suffices to show that θ preserves adjacency in $\Gamma(K_n)$. Let σ_1 and σ_2 be any two adjacent vertices of $\Gamma(K_n)$, so $\sigma_2 = \sigma_1\omega$ for some $\omega \in \Omega(K_n)$. Note that since K_n has an edge joining every possible pair of vertices, $\Omega(K_n)$ contains every transposition in S_n.

$$\sigma_2\theta = \sigma_2^{-1} = (\sigma_1\omega)^{-1} = \omega^{-1}\sigma_1^{-1} = \omega\sigma_1^{-1} \text{ since } \omega^2 = (1),$$
$$= \sigma_1^{-1}(\sigma_1\omega\sigma_1^{-1}) = \sigma_1^{-1}\omega' \text{ where } \omega' \text{ is a permutation}$$

of S_n conjugate to ω. Now conjugate permutations have the same cycle structure, so ω' is also a transposition, and hence by the earlier observation, $\omega' \in \Omega(K_n)$. It follows that σ_1^{-1} is adjacent to σ_2^{-1}, so θ is an automorphism of $\Gamma(K_n)$. To see that θ is irregular, note that $(1)^2 = (1)$ and $\omega^2 = (1)$ for all $\omega \in \Omega(K_n)$ so θ fixes (1) and every vertex of $\Gamma(K_n)$ adjacent to (1). Also, since $n \geqslant 3$, (1 2 3) is a vertex of $\Gamma(K_n)$, and $(1\ 2\ 3)\theta = (1\ 3\ 2) \neq (1\ 2\ 3)$ so θ is a non-trivial automorphism. □

In fact $\Gamma(K_n)$ has no other irregular automorphisms. This will be proved in the following result. It can also easily be shown that θ is not an automorphism of any other transposition graph.

Theorem 3.4.2

For all $n \geqslant 3$, θ is the only irregular automorphism of $\Gamma(K_n)$.

Proof

Suppose that ϕ is another irregular automorphism of $\Gamma(K_n)$. Define d to be the largest integer such that for all vertices ρ of $\Gamma(K_n)$ such that $D_{\Gamma(K_n)}((1), \rho) \leqslant d$, $\rho\theta = \rho\phi$. Note that $d \geqslant 1$ since both θ and ϕ fix (1) and every vertex adjacent to (1) by the definition of an irregular automorphism. Since $\Gamma(K_n)$ has diameter n-1 by theorem 1.4.10, $d \leqslant n-2$, for if $d = n-1$ then θ and ϕ would be identical.

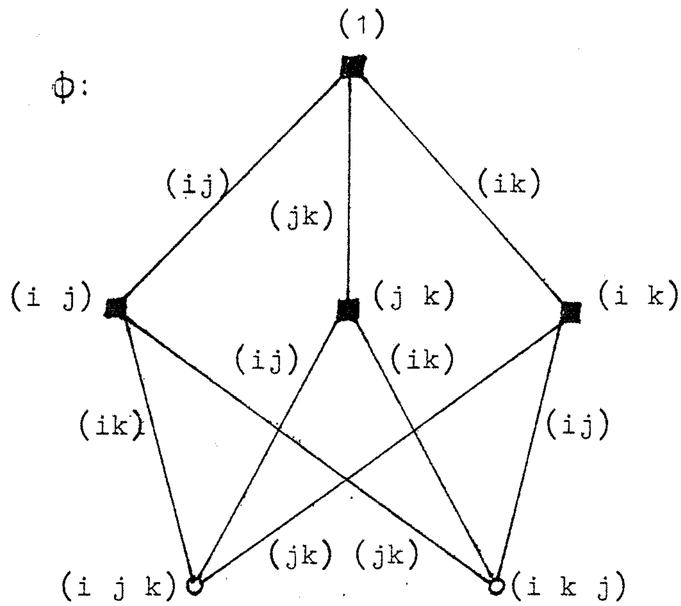
The proof now temporarily splits into two separate cases.

Case 1: d = 1.

In this case there is some vertex ρ of $\Gamma(K_n)$ such that $D_{\Gamma(K_n)}((1), \rho) = 2$ and $\rho\phi \neq \rho\theta = \rho^{-1}$. By theorem 1.4.5, $n^*(\rho) - c^*(\rho) = 2$, and since each cycle contributing to c^* must move at least 2 letters, the only solutions to this equation are $n^* = 3$, $c^* = 1$ and $n^* = 4$, $c^* = 2$. Hence ρ is a cycle of length 3 or a product of two disjoint cycles of length 2. (It is probably more easy to prove this directly using the fact that ρ is the product of two transpositions.)

If ρ is an involution then $\rho = (i\ j)(k\ l)$ for some i,...,l. Since $\rho\phi \neq \rho\theta$, $\rho\phi \neq ((i\ j)(k\ l))^{-1} = (i\ j)(k\ l)$. However, by proposition 3.3.1, taking $\sigma = (1)$, $\omega_e = (i\ j)$ and $\omega_f = (k\ l)$, ϕ fixes ρ, giving a contradiction. Hence $\rho = (i\ j\ k)$ for some i, j, k. Now consider the action of ϕ on the subgraph of $\Gamma(K_n)$ in fig. 3.4.1.

Figure 3.4.1



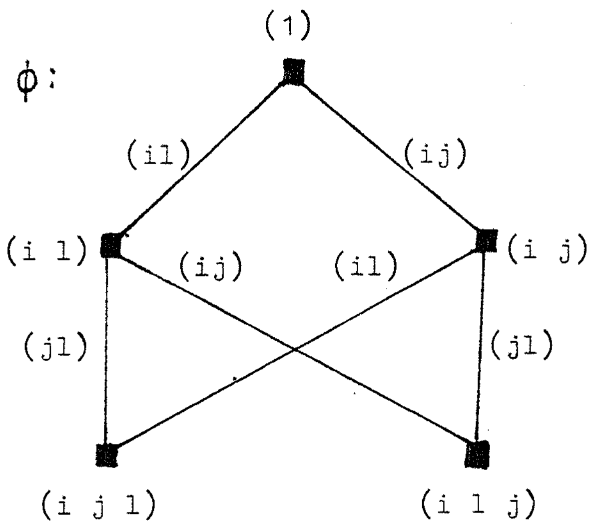
Let Δ be the subgraph of $\Gamma(K_n)$ in fig. 3.4.1. Note that $\Delta \cong K_{3,3}$ and that $\overline{G}(\Delta)$ is the complete graph with vertices i, j, and k. Also, $\Delta\phi$ contains the vertices (1), (i j), (j k) and (i k) since they are all fixed by ϕ. Since ϕ is an automorphism of $\Gamma(K_n)$, $\Delta\phi$ must be isomorphic to Δ. It follows from theorem 2.2.10 that $\overline{G}(\Delta\phi) \cong K_3$, and since contains edges labelled (ij), (jk) and (ik), $\overline{G}(\Delta\phi) \equiv \overline{G}(\Delta)$. Since $\Delta\phi$ and Δ contain common vertices it is easy to see that $\Delta\phi \equiv \Delta$. Hence $\rho\phi$ = (i j k) or (i k j) so $\rho\phi = \rho$ or $\rho\phi = \rho^{-1}$. If $\rho\phi = \rho^{-1}$ then $\rho\phi = \rho\theta$, which is a contradiction, so $\rho\phi = \rho$.

We now show that ϕ must fix every vertex of $\Gamma(K_n)$ distance 2 from (1) .

Consider the automorphism of K_n induced by the action of ϕ on the edges of $\Gamma(K_n)$ incident to (i j), g, say. From the above figure, g fixes $\{i, j\}$, $\{j, k\}$, $\{i, k\}$, and hence g fixes the vertices i, j, and k of K_n. If l is any other vertex of K_n, then we have already seen that ϕ fixes the vertex (i j)(k l) of $\Gamma(K_n)$ and hence ϕ fixes the edge of $\Gamma(K_n)$ labelled (kl) incident to (i j). It follows that g fixes the edge $\{k, l\}$ of K_n, and since g fixes k, g must also fix l. Hence g is the identity and ϕ fixes every vertex of $\Gamma(K_n)$ adjacent to (i j).

Let Δ' be the subgraph of $\Gamma(K_n)$ in fig. 3.4.2, where l is any vertex of K_n.

Figure 3.4.2



ϕ fixes (1), (i j), and (i l) by definition, and fixes (i j l) and (i l j) since they are adjacent to (i j). Hence ϕ fixes the edges of $\Gamma(K_n)$ labelled (il), (ij) and (jl) incident to (i l). This is a repeat of the earlier situation with l, i, j replacing i, j, k. Hence by the previous argument, ϕ fixes every vertex of $\Gamma(K_n)$ adjacent to (i l).

By a similar extension of this argument, if l and m are any two vertices of K_n then ϕ fixes every vertex of $\Gamma(K_n)$ adjacent to (l m). Hence ϕ fixes every vertex of $\Gamma(K_n)$ distance 2 from (1).

Case 2: $d \geq 2$.

In this case, $\rho\phi = \rho\theta$ for every vertex of $\Gamma(K_n)$ distance ≤ 2 from (1). Since $\rho(\theta)^2 = (\rho^{-1})\theta = (\rho^{-1})^{-1} = \rho$, θ^2 is the identity automorphism of $\Gamma(K_n)$. Hence if ρ is a vertex of $\Gamma(K_n)$ such that the distance from (1) to ρ is at most 2 then $\rho(\phi\theta) = (\rho\phi)\theta = (\rho\theta)\theta = \rho(\theta)^2 = \rho$. Hence if $\psi = \phi\theta$, then ψ is an automorphism of $\Gamma(K_n)$ fixing every vertex distance 2 or less from (1). Note that ψ cannot be the identity, for then we would have $\phi = \theta$ since θ is an involution.

Hence in either case we have an irregular automorphism of $\Gamma(K_n)$ which fixes every vertex distance d or less from (1), where $2 \leqslant d \leqslant n - 2$. From now on, this automorphism will be referred to as ψ . (Of course, in case 1, ψ is simply ϕ .)

Since d is chosen to be as large as possible, there is some vertex ρ of $\Gamma(K_n)$ such that $\rho\psi \neq \rho$ and $D_{\Gamma(K_n)}((1), \rho) = d+1$. Hence by theorem 1.4.5, $n^*(\rho) - c^*(\rho) = d+1$, and hence $n^*(\rho) - c^*(\rho) \geqslant 3$ since $d \geqslant 2$. Therefore ρ has at least one cycle of length 4 or more, or at least two cycles of length 2 or more in its cycle structure. These two cases are considered separately.

<u>Case 1</u>: $\rho = \sigma(i\ j\ k\ l \ldots)$, where σ is a permutation fixing i, j, k, l.

Hence $\rho(i\ j) = \sigma(j\ k\ l\ldots)(i)$,

$\rho(k\ l) = \sigma(i\ j\ l\ldots)(k)$,

and $\rho(i\ j)(k\ l) = \sigma(j\ l\ldots)(i)(k)$.

Now $n^*(\rho(i\ j)) - c^*(\rho(i\ j)) = n^*(\rho) - c^*(\rho) - 1 = d$, since $\rho(i\ j)$ fixes i, but otherwise moves the same vertices as ρ , and it has the same number of non-trivial cycles. Hence $D_{\Gamma(K_n)}((1), \rho(i\ j)) = d$ and $\rho(i\ j)$ is fixed by ψ. Similarly, $D_{\Gamma(K_n)}((1), \rho(k\ l)) = d$ so $\rho(k\ l)$ is fixed by ψ , and $D_{\Gamma(K_n)}((1), \rho(i\ j)(k\ l)) = d - 1$, so $\rho(i\ j)(k\ l)$ is also fixed by ψ .

Case 2: $\rho = \sigma(i\ j\ldots)(k\ l\ldots)$, where σ is a permutation fixing i, j,k and l.

Hence $\rho(i\ j) = \sigma(j\ldots)(i)(k\ l\ldots)$,

$\rho(k\ l) = \sigma(i\ j\ldots)(l\ldots)(k)$,

and $\rho(i\ j)(k\ l) = \sigma(j\ldots)(l\ldots)(i)(k)$.

It is easy to check that the distances from (1) to $\rho(i\ j)$, $\rho(k\ l)$ and $\rho(i\ j)(k\ l)$ are the same as in case 1, so all these vertices are fixed by ψ .

Thus in both cases, $\rho(i\ j)(k\ l)$, $\rho(i\ j)$ and $\rho(k\ l)$ are fixed by ψ . Hence taking $\sigma = \rho(i\ j)(k\ l)$, $\omega_e = (k\ l)$ and $\omega_f = (i\ j)$, by proposition 3.3.1, ψ fixes ρ , which gives a contradiction. This completes the proof of theorem 3.4.2. $\square$

Note that the group of irregular automorphisms of $\Gamma(K_n)$, together with the identity, is isomorphic to C_2, the cyclic group of order 2.

We now consider the irregular automorphisms of $\Gamma(C_4)$, where C_4 is of course the circuit of length 4. $\Gamma(C_4)$ turns out to have three irregular automorphisms, which are described in the next result. It is easy to check that the group of irregular automorphisms of $\Gamma(C_4)$, together with the identity, is K_4, the Klein-4 group, since all three irregular automorphisms are involutions.

In the statement of the next result, if ϕ is an automorphism of $\Gamma(G)$ and ρ and ρ' are vertices of $\Gamma(G)$ such that $\rho\phi = \rho'$ and $\rho'\phi = \rho$ then we will write $\phi : \rho \longleftrightarrow \rho'$.

Theorem 3.4.3

If C_4 is the graph with vertices 1, 2, 3 and 4 and edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$ then the only automorphisms of $\Gamma(C_4)$ fixing (1) and every vertex adjacent to (1) are 1, ϕ_1, ϕ_2, and ϕ_3 where 1 is the identity automorphism and

ϕ_1: (1 2 4)$\leftrightarrow$(1 3 2), (1 4 3)$\leftrightarrow$(2 3 4), (1 3)$\leftrightarrow$(1 2 3 4), (2 4)$\leftrightarrow$(1 4 3 2) and fixes every other vertex of $\Gamma(C_4)$,

ϕ_2: (1 4 2)$\leftrightarrow$(1 3 4), (1 2 3)$\leftrightarrow$(2 4 3), (2 4)$\leftrightarrow$(1 2 3 4), (1 3)$\leftrightarrow$(1 4 3 2) and fixes every other vertex of $\Gamma(C_4)$,

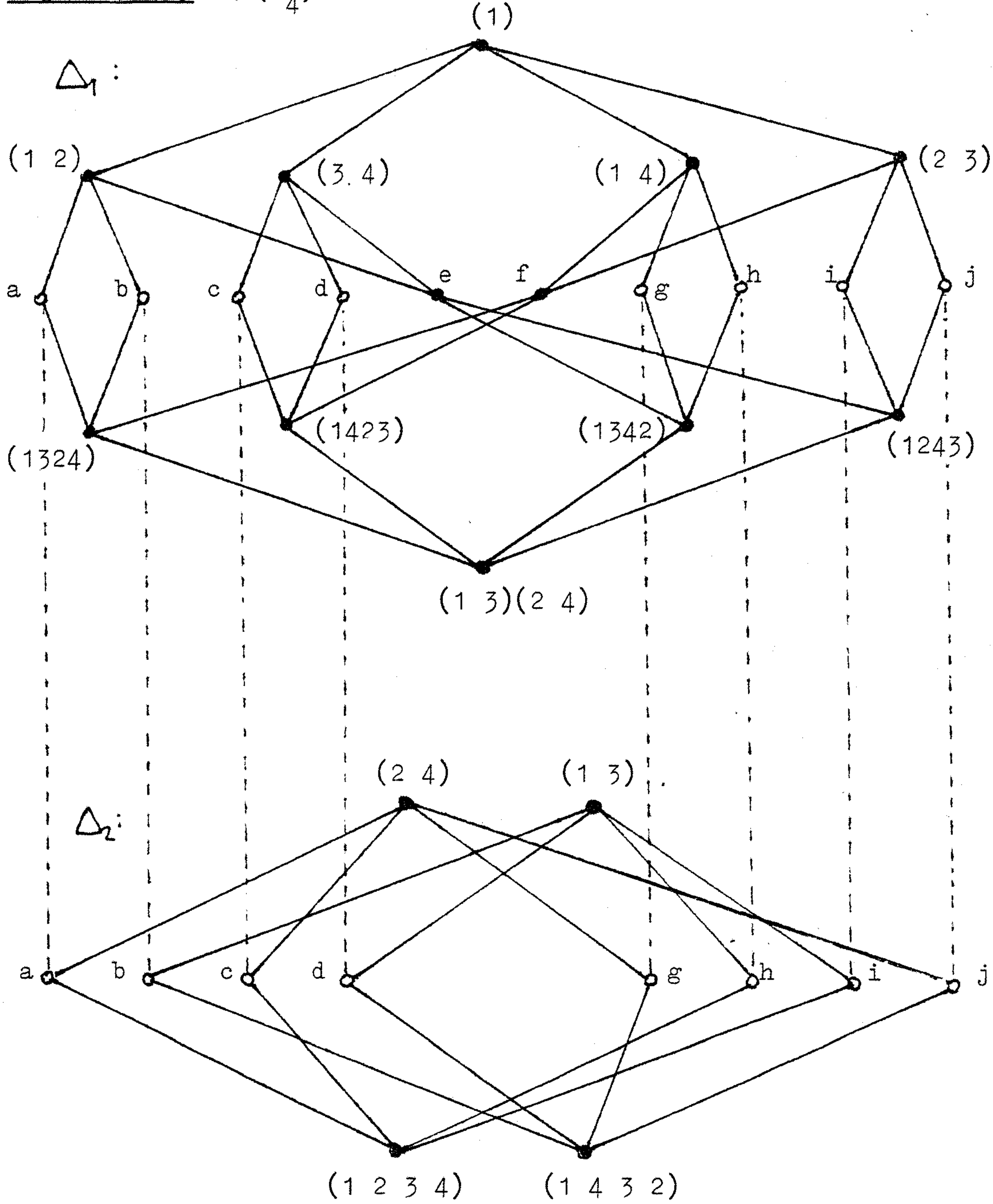
ϕ_3: (1 3)$\leftrightarrow$(2 4), (1 2 3 4)$\leftrightarrow$(1 4 3 2), (1 2 4)$\leftrightarrow$(1 3 2), (1 4 3)$\leftrightarrow$(2 3 4), (1 3 4)$\leftrightarrow$(1 4 2), (1 2 3)$\leftrightarrow$(2 4 3), and fixes every other vertex of $\Gamma(C_4)$.

Proof

$\Gamma(C_4)$ may be conveniently divided into two edge-disjoint subgraphs Δ_1 and Δ_2 as shown in fig, 3.4.3. These two graphs have eight common vertices which are joined by dotted lines in fig. 3.4.3.

Let ϕ be any automorphism of $\Gamma(C_4)$ fixing the vertices (1), (1 2), (2 3), (3 4) and (1 4) . It is easy to check that (1 3)(2 4) is the only vertex of $\Gamma(C_4)$ distance 4 from (1), so (1 3)(2 4) must be fixed by ϕ . Also, ϕ fixes (1 2)(3 4) since it is the only vertex of $\Gamma(C_4)$ which is adjacent to both (1 2) and (3 4), apart from (1) which is already fixed by ϕ . Similarly, ϕ fixes (1 4)(2 3). ϕ fixes (1 3 2 4) since it is the only vertex of $\Gamma(C_4)$ adjacent to (1 4)(2 3) and distance 2 from (1 2), apart from (1 4) and (2 3) which are already fixed by ϕ . Similarly, ϕ fixes (1 4 2 3), (1 3 4 2) and (1 2 4 3). Thus ϕ fixes every vertex of $\Gamma(C_4)$ in Δ_1 except for those which are also vertices of Δ_2.

Figure 3.4.3 : $\Gamma(C_4)$.



Key: a = (1 2 4), b = (1 3 2), c = (2 3 4), d = (1 4 3),

g = (1 4 2), h = (1 3 4), i = (1 2 3), j = (2 4 3),

e = (1 2)(3 4), f = (1 4)(2 3).

Since (1 2 4) and (1 3 2) are the only vertices of $\Gamma(C_4)$ adjacent to both (1 2) and (1 3 2 4), and since (1 2) and (1 3 2 4) are fixed by ϕ, it follows that either ϕ fixes both (1 2 4) and (1 3 2) or ((1 2 4) (1 3 2)) is a cycle of ϕ. Similarly, either (2 3 4) and (1 4 3) are fixed by ϕ or ((2 3 4) (1 4 3)) is a cycle of ϕ; either (1 4 2) and (1 3 4) are both fixed by ϕ or ((1 4 2) (1 3 4)) is a cycle of ϕ; and finally, either (1 2 3) and (2 4 3) are both fixed by ϕ or ((1 2 3) (2 4 3)) is a cycle of ϕ.

Suppose that ((1 2 4) (1 3 2)) is a cycle of ϕ but that ((2 3 4) (1 4 3)) is not, so ϕ maps (1 2 4) to (1 3 2) and fixes (2 3 4). This gives a contradiction since

$D_{\Gamma(C_4)}((1\ 2\ 4),\ (2\ 3\ 4)) = 2 \neq 4 = D_{\Gamma(C_4)}((1\ 3\ 2),\ (2\ 3\ 4))$,

so ϕ does not preserve distance in $\Gamma(C_4)$. Similarly, if (2 3 4) (1 4 3)) is a cycle of ϕ then ((1 2 4) (1 3 2)) is a cycle of ϕ. Thus ((1 2 4) (1 3 2)) is a cycle of ϕ iff ((2 3 4) (1 4 3)) is. Similarly, ((1 4 2) (1 3 4)) is a cycle of ϕ iff ((1 2 3) (2 4 3)) is.

If none of the above cycles are cycles of ϕ then ϕ fixes (1 2 4), (1 3 2),..., (2 4 3). Hence ϕ also fixes (2 4) since it is the only vertex of $\Gamma(C_4)$ adjacent to both (1 2 4) and (2 4 3). Similarly, ϕ fixes (1 3), (1 2 3 4) and (1 4 3 2), and hence ϕ is the identity.

If ((1 2 4) (1 3 2)) and ((2 3 4) (1 4 3)) are cycles of ϕ but ((1 4 2) (1 3 4)) and ((1 2 3) (2 4 3)) are not, and if ϕ_1 is the automorphism defined in the statement of this theorem then $\phi\phi_1$ fixes (1 2 4), (1 3 2),..., (2 4 3) and hence by the above argument, $\phi\phi_1$ is the identity. Since ϕ_1 is an involution, $\phi = \phi_1$. (It is easy to check that ϕ_1 is an automorphism of $\Gamma(C_4)$, by studying its action on Δ_2 in fig. 3.4.3.)

By a similar argument, if ((1 4 2) (1 3 4)) and ((1 2 3) (2 4 3)) are cycles of ϕ but ((1 2 4) (1 3 2)) and ((2 3 4) (1 4 3)) are not, then $\phi = \phi_2$. Finally, if all four transpositions are cycles of ϕ then $\phi = \phi_3$.

This completes the proof of theorem 3.4.3. □

<u>Corollary 3.4.4</u>

If G is a graph with a component isomorphic to C_4 or to K_n; $n \geqslant 3$, then $\Gamma(G)$ has an irregular automorphism.

<u>Proof</u>

Let ϕ be an irregular automorphism of $\Gamma(H)$, where H is the component of G isomorphic to C_4 or K_n. By proposition 1.3.9 $\Gamma(G) \cong \Gamma(H) \times \Delta$, and every vertex of $\Gamma(G)$ can be written in the form $\sigma\tau$, where σ is a vertex of $\Gamma(H)$ and σ commutes with τ . Now define an automorphism ϕ' of $\Gamma(G)$ by $\rho\phi' = (\sigma\phi)\tau$, where $\rho = \sigma\tau$ and σ is a vertex of $\Gamma(H)$. It is easy to check that ϕ' is an irregular automorphism of $\Gamma(G)$. □

It is probably possible to extend the results in this chapter to all transposition graphs. However, in the remaining cases there is not such a natural connection between automorphisms of G and automorphisms of $\Gamma(G)$.

CHAPTER 4: HAMILTONIAN CIRCUITS IN TRANSPOSITION GRAPHS

SECTION 4.1: INTRODUCTION

The main aim of this chapter is to prove that all transposition graphs with four or more vertices are hamiltonian. This is not a particularly surprising result in view of the fact that only a few non-hamiltonian, vertex transitive graphs are known.

In section 4.2 some simple results are proved concerning the large scale structure of a transposition graph $\Gamma(G)$, where G is a connected graph. Of particular interest is the way in which $\Gamma(G')$ is contained in $\Gamma(G)$, where G' is a connected graph obtained by deleting a vertex of G. These results are useful in both section 4.3 and 4.4.

In section 4.3 it is proved that $\Gamma(K_{1,n-1})$ is hamiltonian for all $n \geqslant 3$. This case must be dealt with separately since $\Gamma(K_{1,n-1})$ contains no circuits of length 4, so the method of proof used in the general case does not work.

The main result is proved in section 4.4. It is in fact a simple corollary to the result that $\Gamma(T)$ is hamiltonian for any tree T with 3 or more vertices. The proof of this result takes up most of this section. The general method of proof is very simple but unfortunately does not work on trees with six or fewer vertices. These are dealt with by means of a laborious step by step argument which takes up much of the section.

SECTION 4.2: THE LARGE-SCALE STRUCTURE OF TRANSPOSITION GRAPHS

We begin by giving some results on the left cosets of S_r in S_n, where $r \leqslant n-1$.

Definition 4.2.1

If $r \leqslant n-1$ and $\sigma \in S_n$ then the left coset σS_r is defined by $\sigma S_r = \{\sigma\rho : \rho \in S_r\}$.

If σ_1 and σ_2 are elements of the same left coset of S_r in S_n then we will write $\sigma_1 \sim_r \sigma_2$; $\sim_r$ is clearly an equivalence relation.

Proposition 4.2.1

For all $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \sim_r \sigma_2$ iff $\forall$ s such that $r+1 \leqslant s \leqslant n$, $s\sigma_1^{-1} = s\sigma_2^{-1}$.

Proof

If $\sigma_1 \sim_r \sigma_2$ then there is some $\sigma \in S_n$ such that $\sigma_1, \sigma_2 \in \sigma S_r$. Hence there exist $\rho_1, \rho_2 \in S_r$ such that $\sigma_1 = \sigma\rho_1$ and $\sigma_2 = \sigma\rho_2$. Thus $\sigma = \sigma_1\rho_1^{-1} = \sigma_2\rho_2^{-1}$, and $\sigma_2 = \sigma_1\rho_1^{-1}\rho_2 = \sigma_1\rho$, where $\rho \in S_r$. If $r+1 \leqslant s \leqslant n$, then $s\sigma_2^{-1} = s(\sigma_1\rho)^{-1} = s(\rho^{-1}\sigma_1^{-1}) = (s\rho^{-1})\sigma_1^{-1}$

$= s\sigma_1^{-1}$ since $\rho \in S_r$ and fixes every s such that $s > r$.

Conversely, if $s\sigma_1^{-1} = s\sigma_2^{-1}$ for all s such that $r+1 \leqslant s \leqslant n$, let $\rho = \sigma_1^{-1}\sigma_2$, so $s\rho = s(\sigma_1^{-1}\sigma_2) = (s\sigma_1^{-1})\sigma_2$

$= (s\sigma_2^{-1})\sigma_2 = s$, and hence $\rho \in S_r$.

Also, σ_1 and σ_2 lie in the same left coset of S_r since $\rho \in S_r$ so the result follows. □

Notation: The left coset σS_r will be denoted by $\langle i_1, i_2, \ldots, i_{n-r}\rangle$ where $i_s = (n+1-s)\sigma^{-1}$ for $s = 1, 2, \ldots, n-r$. This symbol is well-defined by proposition 4.2.1.

Note that $i_1, \ldots, i_s$ are distinct and lie between 1 and n.

In this chapter we will mostly be interested in the cases $r = n-1$ and $r = n-2$, when the notation simplifies to $\langle i \rangle$ and $\langle i, j \rangle$ respectively.

Proposition 4.2.2

Every finite connected graph G has a vertex v which is not a cut-vertex of G.

Proof

This result is a simplified version of theorem 2.3 of Behzad and Chartrand, and is very easy to prove. □

Corollary 4.2.3

Every connected graph G on n vertices can be labelled in such a way that $G_r := G - \{v_{r+1}, v_{r+2}, \ldots, v_n\}$ is connected for all r such that $1 \leqslant r \leqslant n-1$.

Proof

Simply choose v_n to be any vertex of G which is not a cut vertex of G, choose v_{n-1} to be a vertex of G_{n-1} which is not a cut vertex of G_{n-1}, and so on. □

Theorem 4.2.4

Let G be a graph on the vertices 1, 2,..., n such that $G_r := G - \{r+1, r+2, \ldots, n\}$ is connected for all r. Then each left coset σS_r is a set of vertices of $\Gamma(G)$ and induces a subgraph Δ of $\Gamma(G)$ which is identically labelled to $\Gamma(G_r)$.

Proof

It is obvious that σS_r is a set of vertices of $\Gamma(G)$ since σS_r is a subset of S_n. Hence it induces a subgraph Δ of $\Gamma(G)$. Consider the map $\phi : \Gamma(G_r) \to \Delta$ defined by $\phi : \rho \longrightarrow \sigma\rho$. ϕ maps S_r to σS_r, and hence maps vertices of $\Gamma(G_r)$ to vertices of Δ. It remains to show that ϕ is an isomorphism and maps edges of $\Gamma(G_r)$ labelled ω to edges of Δ labelled ω .

Suppose that ρ_1 and ρ_2 are adjacent vertices of $\Gamma(G_r)$ and $\rho_2 = \rho_1\omega$ where $\omega \in \Omega(G_r)$. By definition, $\{\rho_1, \rho_2\}\phi = \{\sigma\rho_1, \sigma\rho_2\}$, and $\sigma\rho_2 = \sigma\rho_1\omega$, so $\sigma\rho_1$ and $\sigma\rho_2$ are adjacent vertices of $\Gamma(G)$ joined by an edge labelled ω. Since Δ is an induced subgraph of $\Gamma(G)$ and $\sigma\rho_1$ and $\sigma\rho_2$ are vertices of Δ, ϕ maps edges of $\Gamma(G_r)$ labelled ω to edges of Δ labelled ω, and the result follows. $\square$

The subgraph of $\Gamma(G)$ induced by the coset $\langle i_1, i_2, \ldots, i_{n-r}\rangle$ of S_r will be denoted by $\langle i_1, i_2, \ldots, i_{n-r}\rangle\Gamma(G_r)$, where G_r is the (connected) graph obtained by deleting the vertices r+1, r+2, ..., n from G.

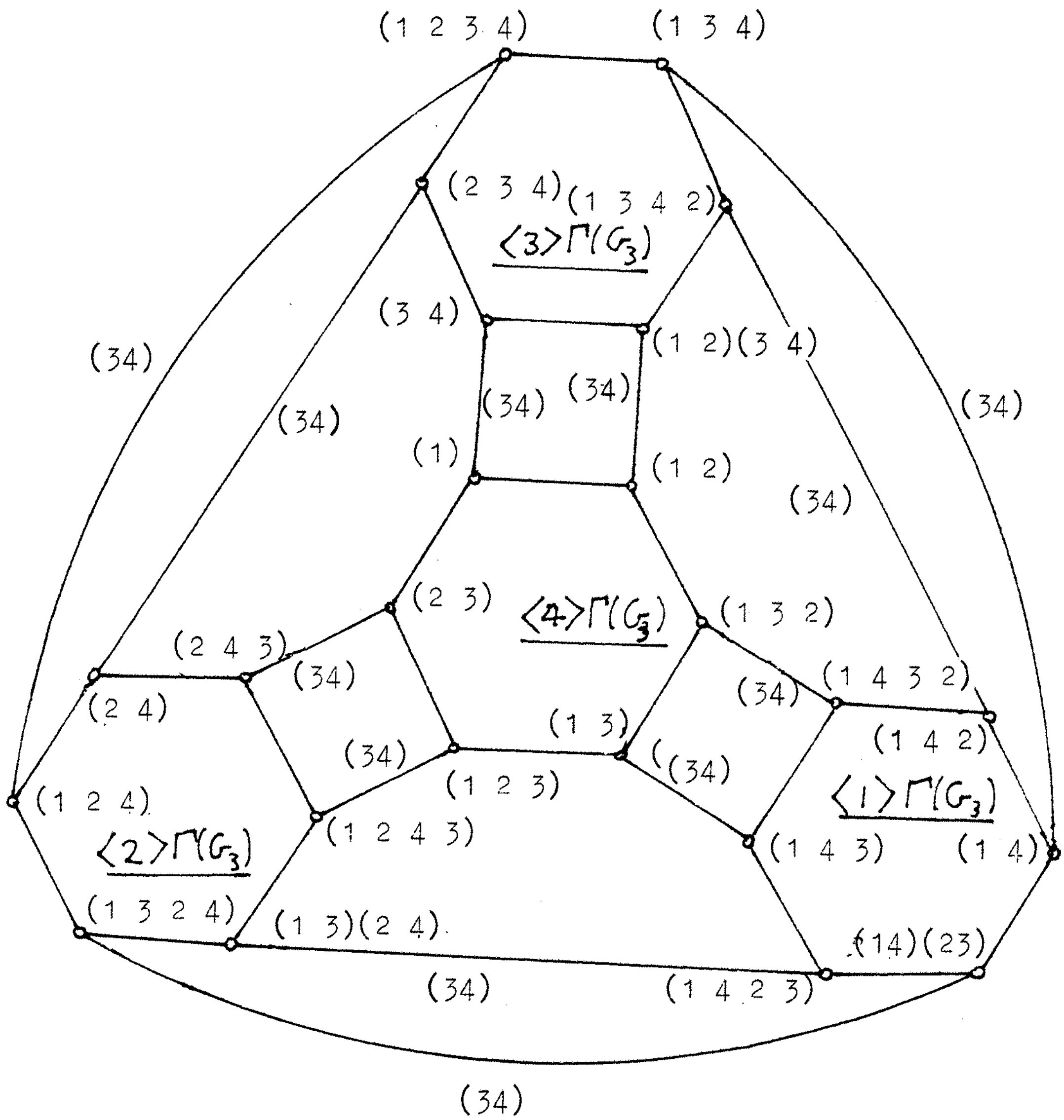
For example, if n = 4, r = 3, and G is the graph with vertices 1, 2, 3 and 4, and edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ then $\Gamma(G)$ is shown in fig. 4.2.1. There are four left cosets of S_3 in S_4, $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, $\langle 4\rangle$ in the above notation. Note that $\langle 4\rangle = (1)S_3$. These cosets and the subgraphs of $\Gamma(G)$ induced by them are also shown in fig. 4.2.1. It is easy to see that $\langle 4\rangle\Gamma(G_3) \equiv \Gamma(G_3)$, and that the other subgraphs are all identically labelled to $\Gamma(G_3)$.

Proposition 4.2.5

If G is a connected graph on n vertices and G is labelled in such a way that n is not a cut vertex of G, and if $\{i, n\}$ is an edge of G, and if $j \neq k$ are such that $1 \leqslant j,k \leqslant n$ then there are (n-2)! edges of $\Gamma(G)$ labelled (i n) joining $\langle j\rangle\Gamma(G_{n-1})$ to $\langle k\rangle\Gamma(G_{n-1})$. (That is, edges which have one end vertex in one coset, and the other end vertex in the other coset.)

An example of this result can be seen in fig. 4.2.1, where there are (4-2)! = 2 edges labelled (3 4) joining any two cosets.

Figure 4.2.1



Proof of proposition 4.2.5

We make the following claim: if $\sigma_2 = \sigma_1(i\ n)$ then $\sigma_1 \in \langle j \rangle$ and $\sigma_2 \in \langle k \rangle$ iff $i\sigma_1^{-1} = k$ and $n\sigma_1^{-1} = j$. For suppose that σ_1 and σ_2 are as above; since $\sigma_1 \in \langle j \rangle$, then $n\sigma_1^{-1} = j$ by definition. Also, by a similar argument, $k = n\sigma_2^{-1} = n(\sigma_1(i\ n))^{-1} = n\left\{(i\ n)^{-1}\sigma_1^{-1}\right\} = n\left\{(i\ n)\sigma_1^{-1}\right\}$ $= i\sigma_1^{-1}$ as claimed.

Hence there is one edge labelled $(i\ n)$ joining $\langle j \rangle$ to

$\langle k \rangle$ for each $\sigma \in S_n$ such that $i\sigma^{-1} = k$ and $n\sigma^{-1} = j$. There are clearly $(n-2)!$ permutations satisfying these constraints since $i \neq n$ and $j \neq k$. □

Proposition 4.2.6

If G is a connected graph on n vertices and n is not a cut vertex of G then every edge of $\Gamma(G)$ from $\langle j \rangle$ to $\langle k \rangle$, where $j \neq k$, is labelled $(i\ n)$ for some $i < n$

Proof

This result is obvious, for if $\sigma_2 = \sigma_1(i\ 1)$ where $i,1 < n$, then $n\sigma_2^{-1} = n(\sigma_1(i\ 1))^{-1} = n(i\ 1)\sigma_1^{-1} = n\sigma_1^{-1}$, and hence $\sigma_1 \sim_{n-1} \sigma_2$ by proposition 4.2.1. □

Hence the large-scale structure of $\Gamma(G)$ can be described as follows. There are n left cosets of S_{n-1} in S_n each of which induces a subgraph of $\Gamma(G)$ identically labelled to $\Gamma(G')$, where $G' = G - \{n\}$. Each pair of these subgraphs is joined by $(n-2)!$ edges of $\Gamma(G)$ labelled $(i\ n)$ for each i such that $\{i, n\}$ is an edge of G. In the special case where G is a tree, (and this case is very important in this chapter), n must be an end vertex of G, or else it would be a cut vertex. Hence n has valency 1 in G and there is a unique vertex i of G adjacent to n. Thus each pair of cosets of S_{n-1} in S_n are joined in $\Gamma(G)$ by $(n-2)!$ edges, all of which are labelled $(i\ n)$.

Proposition 4.2.7

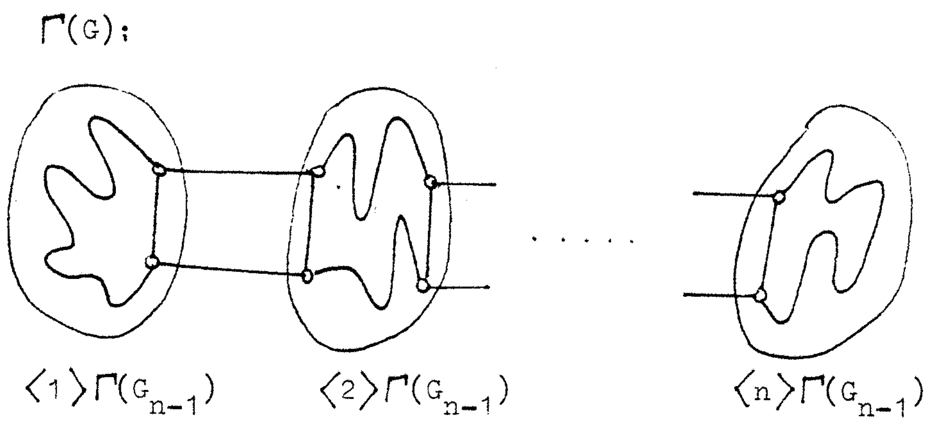
If G is a connected graph on n vertices such that $G_{n-1} = G - \{n\}$ and $G_{n-2} = G_{n-1} - \{n-1\}$ are connected, and if there exist vertices p and q of G such that $p \neq q$, $p \sim n-1$ and $q \sim n$, and if i_1, i_2, j_1, and j_2 are distinct integers such that $1 \leqslant i_1, i_2, j_1, j_2 \leqslant n$, then there is a circuit of length 4 $\Delta \subset \Gamma(G)$ such that $\overline{G}(\Delta)$ is the graph with edges

$\{p, n-1\}$ and $\{q, n\}$ and such that Δ has one vertex in each of the cosets $\langle i_1, j_1\rangle$, $\langle i_1, j_2\rangle$, $\langle i_2, j_1\rangle$ and $\langle i_2, j_2\rangle$.

Before proving this result, it is probably worth explaining its significance. No restriction at all is placed on G by the constraints that G_{n-1} and G_{n-2} are connected; by corollary 4.2.3 any connected graph can be labelled so as to make this true. The only connected graph for which p and q do not exist for any choice of n and n-1 is the graph $K_{1,n-1}$. Thus for all connected graphs except $K_{1,n-1}$ there is a choice of n, n-1, p and q satisfying all the hypotheses. The only constraint imposed by the choice of $i_1, \ldots, j_2$ is that $n \geqslant 4$.

The idea behind this result is that if a hamiltonian circuit exists in $\Gamma(G_{n-1})$, then this circuit and similar circuits in each coset give a set of circuits spanning the vertices of $\Gamma(G)$. The hope is to use n-1 of the squares constructed in this result to patch together the spanning set of circuits to give a hamiltonian circuit in $\Gamma(G)$. This idea is illustrated in fig. 4.2.2.

Figure 4.2.2



Proof of proposition 4.2.7

In the full permutation notation, let $\sigma_{1,1}$ be a permutation with the following form:

$$\sigma_{1,1} = \begin{pmatrix} \ldots i_1 \ldots j_1 \ldots i_2 \ldots j_2 \ldots \\ \ldots n \ldots n-1 \ldots q \ldots p \ldots \end{pmatrix} = \begin{pmatrix} i_1 & j_1 & i_2 & j_2 \\ n & n-1 & q & p \end{pmatrix}$$

in a more compressed notation. Such a permutation exists since i_1, j_1, i_2 and j_2 are all distinct, and since n, n-1, p and q are all distinct. Clearly, $\sigma_{1,1} \in S_n$ and since $n(\sigma_{1,1})^{-1} = i_1$ and $(n-1)(\sigma_{1,1})^{-1} = j_1$, $\sigma_{1,1} \in \langle i_1, j_1 \rangle$.

Let $\sigma_{1,2} = \sigma_{1,1}(p\ n-1)$,

so $$\sigma_{1,2} = \begin{pmatrix} i_1 & j_1 & i_2 & j_2 \\ n & n-1 & q & p \end{pmatrix}\begin{pmatrix} p & n-1 \\ n-1 & p \end{pmatrix}$$

$$= \begin{pmatrix} i_1 & j_1 & i_2 & j_2 \\ n & p & q & n-1 \end{pmatrix} \in \langle i_1, j_2 \rangle .$$

Similarly, $\sigma_{2,2} = \sigma_{1,2}\,(q\ n) \in \langle i_2, j_2 \rangle$ and

$$\sigma_{2,1} = \sigma_{2,2}\,(p\ n-1) \in \langle i_2, j_1 \rangle .$$

Finally, $$\sigma_{2,1}\,(q\ n) = \sigma_{2,2}\,(p\ n-1)(q\ n)$$
$$= \sigma_{1,2}\,(q\ n)(p\ n-1)(q\ n)$$
$$= \sigma_{1,1}\,(p\ n-1)(q\ n)(p\ n-1)(q\ n)$$
$$= \sigma_{1,1} .$$

Also, since $p \sim n-1$ and $q \sim n$ in G, $(p\ n-1)$, $(q\ n) \in \Omega(G)$, and hence the subgraph Δ of $\Gamma(G)$ induced by the vertices $\sigma_{1,1}$, $\sigma_{1,2}$, $\sigma_{2,2}$ and $\sigma_{2,1}$ is a circuit of length 4 and has one vertex in each of the required cosets. □

SECTION 4.3: HAMILTONIAN CIRCUITS IN $\Gamma(K_{1,n-1})$.

Throughout this section, $K_{1,n-1}$ will be the graph with vertices 1, 2,..., n and edges $\{1, 2\}$, $\{1, 3\}$,..., $\{1, n\}$.

Definition 4.3.1

For this section only, two vertices σ_1 and σ_2 of $\Gamma(K_{1,n-1})$ are related if there exist distinct numbers i, j, k such that $\sigma_2 = \sigma_1(1\ i\ j\ k)$. An equivalent definition is that σ_1 and σ_2 are distance 3 apart in $\Gamma(K_{1,n-1})$ but do not both lie in any circuit of length 6 in $\Gamma(K_{1,n-1})$.

Proposition 4.3.1

If σ_1 and σ_2 are both related to the identity (1) then there is an automorphism of $\Gamma(K_{1,n-1})$ fixing (1) and mapping σ_1 to σ_2.

Proof

By definition there exist a_1, b_1, c_1, a_2, b_2, c_2 such that $\sigma_1 = (1\ a_1\ b_1\ c_1)$ and $\sigma_2 = (1\ a_2\ b_2\ c_2)$. Since a_i, b_i, c_i are distinct for i = 1, 2, there is a permutation ρ of S_n mapping a_1 to a_2, b_1 to b_2 and c_1 to c_2. This permutation is clearly an automorphism of $K_{1,n-1}$. Hence by lemma 3.2.5,

$\theta_\rho : \sigma \to \rho^{-1}\sigma\rho$ is an automorphism of $\Gamma(K_{1,n-1})$ fixing (1).

Also,
$$\begin{aligned}(1\ a_1\ b_1\ c_1)\theta_\rho &= \rho^{-1}(1\ a_1\ b_1\ c_1)\rho \\ &= (1\ a_1\rho\ b_1\rho\ c_1\rho) \\ &= (1\ a_2\ b_2\ c_2) \text{ by the definition of } \rho.\end{aligned}$$

Hence θ_ρ is an automorphism of $\Gamma(K_{1,n-1})$ with the required properties. □

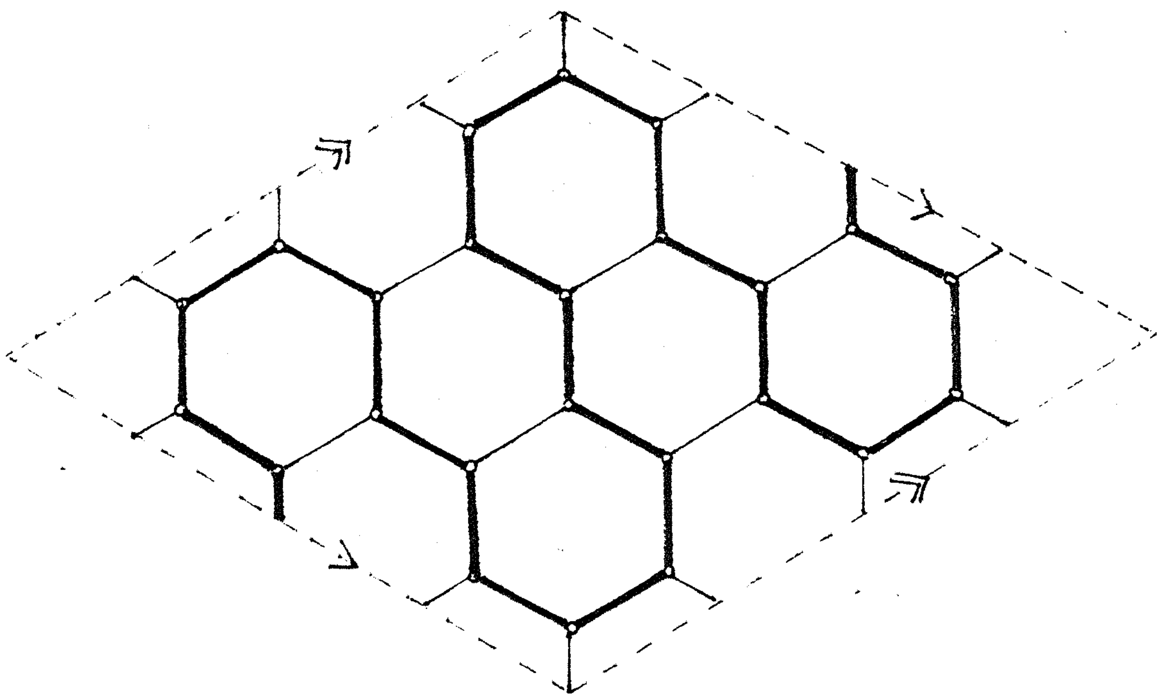
Proposition 4.3.2

$\Gamma(K_{1,3})$ is hamiltonian, and has a hamiltonian path joining any two related vertices.

Proof

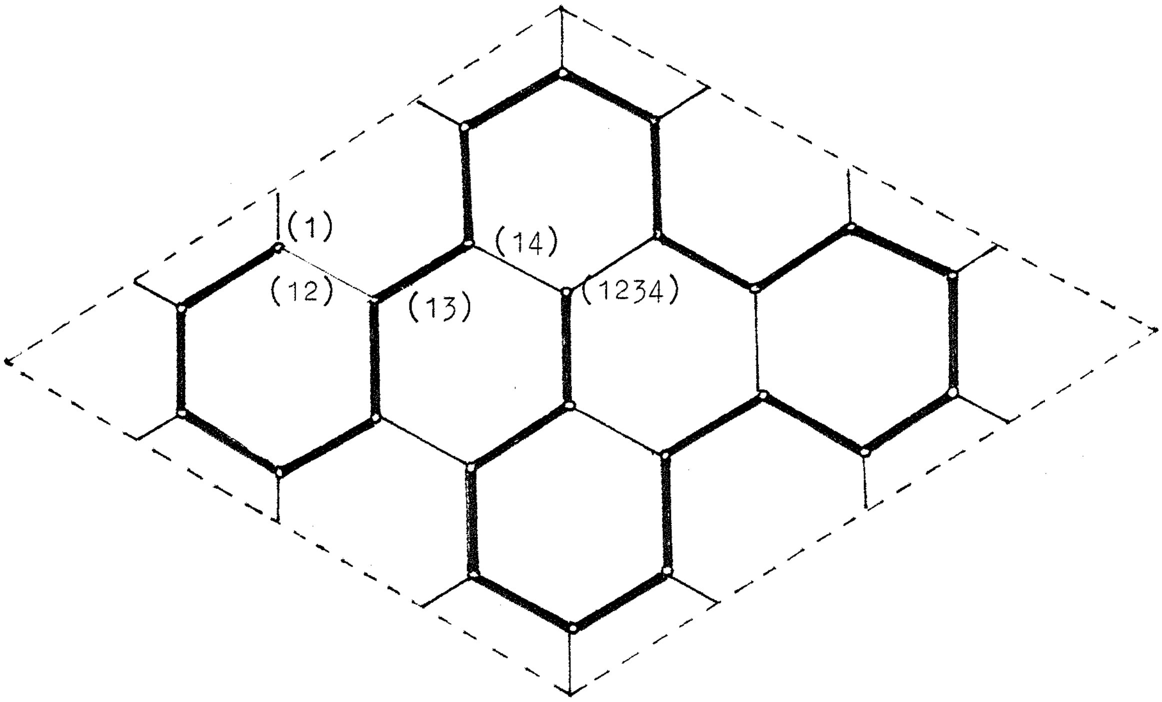
$\Gamma(K_{1,3})$ may be embedded in a torus as shown in fig. 4.3.1. The heavier lines in the figure form a hamiltonian circuit in the graph. Of course, the imbedding is irrelevant; it is simply a convenient way of drawing the graph, which is non-planar.

Figure 4.3.1



Since $\Gamma(K_{1,3})$ is vertex transitive, we may choose (1) as an end vertex of a hamiltonian path in $\Gamma(K_{1,3})$ without loss of generality. By proposition 4.3.1 we may choose the other end vertex of the path to be (1 2 3 4). A hamiltonian path joining (1) to (1 2 3 4) in $\Gamma(K_{1,3})$ is shown in fig. 4.3.2 . This completes the proof of proposition 4.3.2. □

Figure 4.3.2



Theorem 4.3.3

For all $n \geqslant 4$, if $\Gamma(K_{1,n-1})$ has a hamiltonian path joining any two related vertices, then $\Gamma(K_{1,n})$ is hamiltonian.

Proof

By theorem 4.2.4, the left cosets of S_n in S_{n+1} induce subgraphs $\langle i \rangle \Gamma(K_{1,n-1})$; $i = 1, 2, \ldots, n+1$ which are identically labelled to $\Gamma(K_{1,n-1})$.

Lemma 4.3.4

For all $i = 1, 2, \ldots, n-1$, there exist vertices σ_i and σ_i' of $\langle i \rangle \Gamma(K_{1,n-1})$ such that σ_i and σ_i' are related vertices of $\Gamma(K_{1,n})$ and such that $\sigma_{i+1} = \sigma_i'(1\ n+1)$ (mod n-1).

Proof of lemma

By definition, there exist a_i, b_i, c_i such that $\sigma_i' = \sigma_i(1\ a_i\ b_i\ c_i)$. Also, if a_i, b_i, or $c_i = n+1$, then it is easy to check that σ_i and σ_i' lie in different left cosets of S_n in S_{n+1}, giving a contradiction. Hence the vertices of $\Gamma(K_{1,n-1})$ which are mapped to σ_i and σ'_i by the label-preserving isomorphism of theorem 4.2.4 are related in $\Gamma(K_{1,n-1})$ as well as in $\Gamma(K_{1,n})$. Let $\rho_i = (1\ a_i\ b_i\ c_i)$; $i = 1,\ldots,n+1$.

For all n, choose $\sigma'_{n+1} = (1)$, so $\sigma_1 = (1\ n+1)$. After this choice, the proof divides into two cases.

<u>Case 1</u>: n = 4

Choose $\rho_1 = (1\ 3\ 4\ 2)$, $\rho_2 = (1\ 2\ 3\ 4)$, $\rho_3 = (1\ 2\ 4\ 3)$, $\rho_4 = (1\ 2\ 4\ 3)$ and $\rho_5 = (1\ 4\ 2\ 3)$.

With these choices, $\sigma_1, \sigma'_1, \ldots, \sigma'_5$ are as in the table below:

i	1	2	3	4	5
σ_i	(15)	(2534)	(12)(354)	(1453)	(1324)
σ'_i	(15342)	(12543)	(14)(35)	(13245)	(1)

It is easy to check that these permutations have the required properties.

<u>Case 2</u>: $n \geqslant 5$

Choose ρ_1, ρ_2, and ρ_3 as in case 1.

Choose $\rho_k = (1\ k-1\ k\ k+1)$ if $4 \leqslant k \leqslant n-1$.

Choose $\rho_n = (1\ n\ 2\ n-1)$, and finally choose $\rho_{n+1} = (1\ n\ n-1\ 2)$.

With these choices, $\sigma_1 = (1\ n+1)$, $\sigma'_1 = (1\ n+1\ 3\ 4\ 2)$,

$\sigma_2 = (2\ n+1\ 3\ 4)$, $\sigma'_2 = (1\ 2\ n+1\ 4\ 3)$,

$\sigma_3 = (1\ 2)(3\ n+1\ 4)$, and $\sigma'_3 = (1\ 4)(3\ n+1)$.

<u>Claim</u>: $\sigma'_k = (1\ k+1)(k\ n+1)$ for $k = 3,\ldots, n-1$

$\sigma_k = (1\ k\ n+1\ k-1)$ for $k = 4,\ldots, n-1$.

The claim is true for k = 3, so suppose it is true for all $k \leqslant i$, where $3 \leqslant i \leqslant n-1$.

By definition, $\sigma_{i+1} = \sigma'_i(1\ n+1) = (1\ i+1)(i\ n+1)(1\ n+1)$, so $\sigma_{i+1} = (1\ i{+}1\ n{+}1\ i)$, and

$\sigma'_{i+1} = \sigma_{i+1}\rho_{i+1} = (1\ i{+}1\ n{+}1\ i)(1\ i\ i{+}1\ i{+}2)$

$= (1\ i{+}2)(i{+}1\ n{+}1)$, and hence the claim is true for $k = i+1$. Hence the claim is true for all $k \leqslant n-1$. (Beyond this point, the definition of ρ_k changes so the result does not hold.)

Finally, $\sigma_n = \sigma'_{n-1}(1\ n{+}1) = (1\ n)(n{-}1\ n{+}1)(1\ n{+}1)$

$= (1\ n\ n{+}1\ n{-}1)$,

$\sigma'_n = (1\ n\ n{+}1\ n{-}1)(1\ n\ 2\ n{-}1) = (1\ 2\ n{-}1\ n\ n{+}1)$,

$\sigma_{n+1} = (1\ 2\ n{-}1\ n\ n{+}1)(1\ n{+}1) = (1\ 2\ n{-}1\ n)$, and finally,

$\sigma'_{n+1} = (1\ 2\ n{-}1\ n)(1\ n\ n{-}1\ 2) = (1)$.

It is now easy to check that $\sigma_k, \sigma'_k \in \langle k \rangle$ for $k = 1, 2, \ldots, n+1$. From the way they are defined in terms of the 4-cycles ρ_k, σ_k and σ'_k have the required properties. □

This completes the proof of the lemma. We now return to proving theorem 4.3.3.

Let τ_k and τ'_k respectively be the vertices of $\Gamma(K_{1,n-1})$ mapped to σ_k and σ'_k by the isomorphism of theorem 4.2.4. We have already seen in the proof of the lemma that τ_k and τ'_k are related vertices of $\Gamma(K_{1,n-1})$, and hence by hypothesis are joined by a hamiltonian path. This path is mapped by the isomorphism to a path joining σ_k to σ'_k which is a spanning subgraph of $\langle k \rangle \Gamma(K_{1,n-1})$. The union of all these paths, together with the edges joining σ'_{n+1} to σ_1, σ'_1 to $\sigma_2, \ldots,$ and joining σ'_n to σ_{n+1} is clearly a hamiltonian circuit in $\Gamma(K_{1,n})$. This is illustrated in fig. 4.3.3 . □

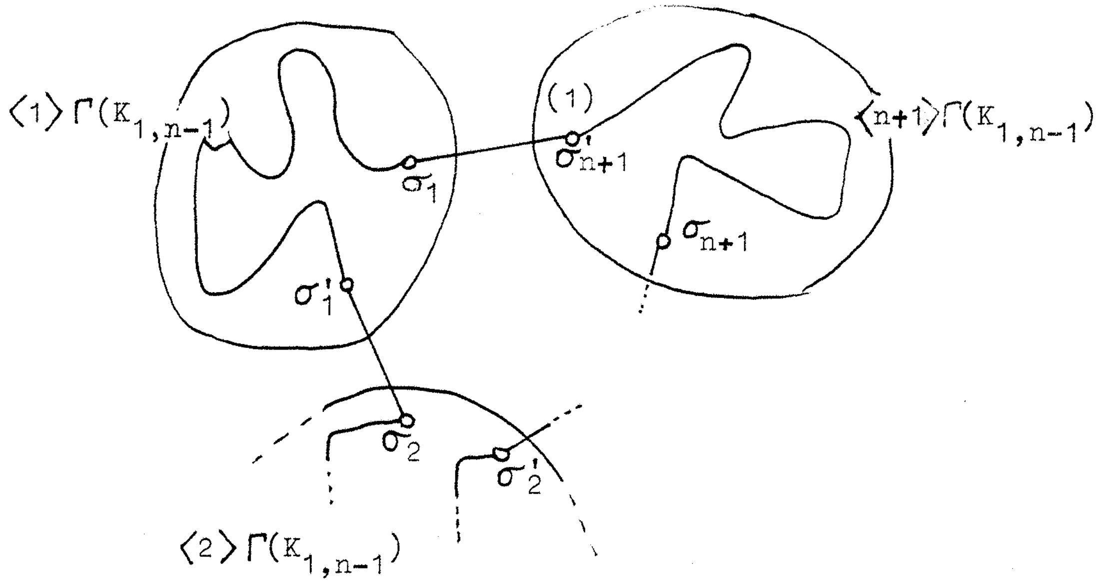
Corollary 4.3.5

$\Gamma(K_{1,4})$ is hamiltonian.

Proof

This follows immediately from theorems 4.3.3 and 4.3.2 . □

Figure 4.3.3



Corollary 4.3.6

If $\Gamma(K_{1,n-1})$ has a hamiltonian path joining any two related vertices, then $\Gamma(K_{1,n})$ has a hamiltonian path joining any two related vertices.

Proof

We may choose the two related vertices to be (1) and $\sigma = (1\ n{+}1\ 2\ 4)$ by vertex transitivity and by proposition 4.3.1.

By theorem 4.3.4 there is a hamiltonian circuit in $\Gamma(K_{1,n})$ containing the vertices $\sigma_1, \sigma'_1, \ldots, \sigma'_{n+1}$, where these are the vertices constructed in the proof of theorem 4.3.4. Let Δ be the subgraph of this circuit obtained by deleting all the vertices of the circuit in $\langle 1\rangle\Gamma(K_{1,n-1})$ except σ'_1. Hence Δ is a path joining $(1) = \sigma'_{n+1}$ to σ'_1.

Now $\sigma(1\ 2\ 3\ 4) = (1\ n{+}1\ 2\ 4)(1\ 2\ 3\ 4) = (1\ n{+}1\ 3\ 4\ 2)$
$= \sigma'_1$, so σ is related to σ'_1.

Hence as in the proof of theorem 4.3.4 there is a path joining σ to σ'_1 which spans $\langle 1\rangle\Gamma(K_{1,n-1})$. It is clear that the union of this path and Δ is a hamiltonian path in $\Gamma(K_{1,n})$ joining (1) to σ. □

Corollary 4.3.7

$\Gamma(K_{1,n-1})$ is hamiltonian for all $n \geqslant 4$.

Proof

This follows immediately from the preceding results. □

SECTION 4.4: HAMILTONIAN CIRCUITS IN $\Gamma(G)$

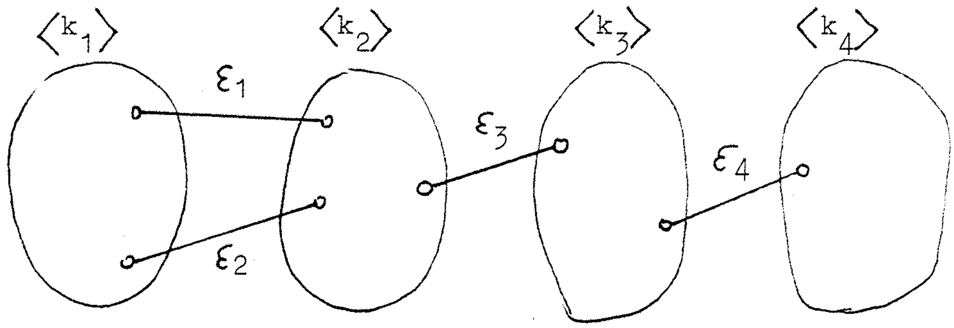
In fact, most of this section will be concerned with the existance of a hamiltonian circuit in $\Gamma(T)$, where T is an arbitrary tree. The more general result follows very easily from this special case.

It is not possible to use the same method of proof as in section 4.3, which depended on the rather special structure of $K_{1,n-1}$. The more general method of proof was discussed briefly before the proof of proposition 4.2.7 . If T_n is a tree on n vertices, and $T_{n-1} = T_n - \{n\}$ is a tree on n-1 vertices such that $\Gamma(T_{n-1})$ is hamiltonian, then we attempt to string together hamiltonian circuits in the cosets $\langle i \rangle \Gamma(T_{n-1})$ with circuits of length 4 to produce a hamiltonian circuit in $\Gamma(T_n)$: There is a fairly easy way of doing this, which works for all $n \geqslant 7$. Unfortunately, this method does not work at all for smaller values of n, for reasons which will be discussed later in this section, so laborious special arguments are needed for the first few values of n. These special arguments in fact make up the bulk of the proof that all transposition graphs are hamiltonian.

Definition 4.4.1

If T is a tree on n vertices and i is an end vertex of T adjacent to j, then two edges of $\Gamma(T)$ labelled (i j) are distant if they do not join the same two left cosets of $S([n] - \{i\})$. The two edges of $\Gamma(T)$ are properly distant if no two of their end vertices lie in the same left coset. Note that two properly distant edges are of course distant. These definitions are illustrated in fig. 4.4.1 .

Figure 4.4.1



If $T' = T - \{i\}$, and $\varepsilon_1, \ldots, \varepsilon_4$ are edges of $\Gamma(T)$ labelled $(i\ j)$ as in fig. 4.4.1, then ε_1 is distant from ε_3, and properly distant from ε_4, but is neither distant nor properly distant from ε_2. (It is assumed that $\langle k_1 \rangle, \ldots, \langle k_4 \rangle$ are distinct cosets of $S([n] - \{i\})$.)

Proposition 4.4.1

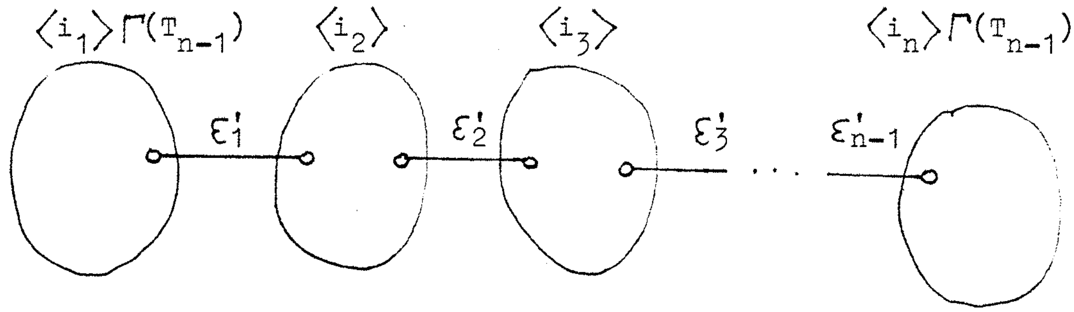
Let T_{n-1} be a tree on n-1 vertices such that u is not a vertex of T_{n-1}, v is an end vertex of T_{n-1} and such that $\Gamma(T_{n-1})$ is hamiltonian. If T_n is the tree obtained by adding the vertex u and the edge $\{u, v\}$ to T_{n-1}, then $\Gamma(T_n)$ has a hamiltonian circuit containing any two distant edges of $\Gamma(T_n)$ labelled $(u\ v)$.

Proof

Without loss of generality, suppose that $v = n-1$ and $u = n$. Since v is an end vertex of T_{n-1} there is a unique vertex w of T_{n-1} adjacent to v. Again without loss of generality, we may suppose that $w = n-2$.

Let ε_1 and ε_2 be any two distant edges of $\Gamma(T_n)$ labelled $(u\ v) = (n-1\ n)$. By proposition 4.2.5, there exist edges of $\Gamma(T_n)$ joining every pair of cosets $\langle i \rangle(T_{n-1})$ and $\langle j \rangle(T_{n-1})$. Hence there are edges $\varepsilon_3, \varepsilon_4, \ldots, \varepsilon_{n-1}$ of $\Gamma(T_n)$ labelled $(n-1\ n)$ such that the n cosets of S_{n-1} in S_n are joined in a chain by the edges $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ as shown in fig. 4.4.2.

Figure 4.4.2



In fig. 4.4.2, $i_1, i_2, \ldots, i_n$ are a permutation of the numbers $1, 2, \ldots, n$, and $\varepsilon'_1, \varepsilon'_2, \ldots, \varepsilon'_{n-1}$ are a permutation of the edges $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$. (It is not convenient to show $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$ in their true order since ε_1 and ε_2 may be either properly distant or distant but not properly distant. This would need two separate figures to show , and would be artificial since the circuit which will be constructed contains all the edges $\varepsilon'_1, \ldots, \varepsilon'_{n-1}$.)

There is a hamiltonian circuit in $\langle i_1\rangle\Gamma(T_{n-1})$ since it is isomorphic to $\Gamma(T_{n-1})$, and since $\Gamma(T_{n-1})$ is hamiltonian by hypothesis. Let $\varepsilon'_j = \{\rho_j, \sigma_j\}$ for $j = 1, \ldots, n-1$, where ρ_j is a vertex of $\langle i_j\rangle$ and σ_j is a vertex of $\langle i_{j+1}\rangle$. Hence ρ_1 is a vertex of $\langle i_1\rangle\Gamma(T_{n-1})$. In the hamiltonian circuit in $\langle i_1\rangle\Gamma(T_{n-1})$ there are two edges incident to ρ_1; let these edges be labelled ω_1 and ω_2, where ω_1 and ω_2 correspond to edges of T_{n-1}. They must be distinct since two distinct edges of a transposition graph which are incident cannot have the same label. Hence at least one of these transpositions must be distinct from $(n-2\ n-1)$. Since by hypothesis $n-1$ has valency 1 in T_{n-1}, it follows that ω_1, say, must fix $n-1$. Also, ω_1 fixes n, so $(\omega_1(n-1\ n))^2 = (1)$.

Hence there is a circuit of length 4 in $\Gamma(T_n)$ containing the vertices ρ_1, σ_1, $\rho_1\omega_1$, and $\sigma_1\omega_1$. By the contrapositive of proposition 4.2.6, $\rho_1\omega_1$ is a vertex of $\langle i_1\rangle\Gamma(T_{n-1})$ and $\sigma_1\omega_1$ is a vertex of $\langle i_2\rangle\Gamma(T_{n-1})$. For convenience, let the hamiltonian circuit in $\langle i_1\rangle\Gamma(T_{n-1})$ be Δ_1 and let the circuit of length 4 just constructed be $\Delta_{1,2}$.

We now show that there is a hamiltonian circuit in $\langle i_2\rangle\Gamma(T_{n-1})$ containing the edge $\{\sigma_1, \sigma_1\omega_1\}$. Suppose that a hamiltonian circuit Δ in $\Gamma(T_{n-1})$ contains no edge labelled ω_1. Clearly, $\bar{G}(\Delta) \subset T_{n-1} - \{e_1\}$, and hence by proposition 2.2.6, $\Delta \subset \Gamma(T_{n-1} - \{e_1\})$. However, $T_{n-1} - \{e_1\}$ is a disconnected graph with two components, and hence $\Gamma(T_{n-1} - \{e_1\})$ has $n_1!\ n_2!$ vertices, where $n_1 + n_2 = n-1$, and $1 \leqslant n_1, n_2 \leqslant n-2$. This is a contradiction, since Δ is a spanning subgraph of $\Gamma(T_{n-1})$, and has $(n-1)!$ vertices. Since $\langle i_2\rangle\Gamma(T_{n-1})$ is identically labelled to $\Gamma(T_{n-1})$, a similar result holds for a hamiltonian circuit in $\langle i_2\rangle\Gamma(T_{n-1})$. We may choose the edge labelled ω_1 to be incident to σ_1 by vertex transitivity. Hence as claimed there is a hamiltonian circuit Δ_2 in $\langle i_2\rangle\Gamma(T_{n-1})$ containing the edge $\{\sigma_1, \sigma_1\omega_1\}$.

These arguments may now be repeated to construct circuits $\Delta_{2,3}, \Delta_{3,4}, \ldots, \Delta_{n-1,n}$ of length 4, and hamiltonian circuits $\Delta_3, \Delta_4, \ldots, \Delta_n$ such that $\Delta_{j,j+1}$ has an edge in common with Δ_j and Δ_{j+1}. Deleting these common edges gives a hamiltonian circuit in $\Gamma(T_n)$ containing the edges $\varepsilon_1', \ldots, \varepsilon_{n-1}'$, and hence containing the edges ε_1 and ε_2. $\square$

<u>Corollary 4.4.2</u>

Let T_n be the tree obtained by adding an edge $\{n-1, n\}$ to $K_{1,n-2}$. Then $\Gamma(T_n)$ has a hamiltonian circuit containing any two distant edges of the graph labelled $(n-1\ n)$.

Proof

This follows immediately from corollary 4.3.7 and proposition 4.4.1. □

The next result in this section involves the lengthy technical proof mentioned at the start of this section.

Theorem 4.4.3

If T_6 is any tree on six vertices apart from $K_{1,5}$ then there is a hamiltonian circuit in $\Gamma(T_6)$ containing any two distant edges labelled (u v), where u is an end vertex of T_6 adjacent to v.

Proof

The proof of this result is given in a series of lemmas.

Lemma 4.4.4

If T_4 is any tree on 4 vertices and u is an end vertex of T_4 adjacent to v, then $\Gamma(T_4)$ has a hamiltonian circuit containing any two edges labelled (u v). (Note that the edges labelled (u v) are not required to be distant.)

Proof of lemma 4.4.4

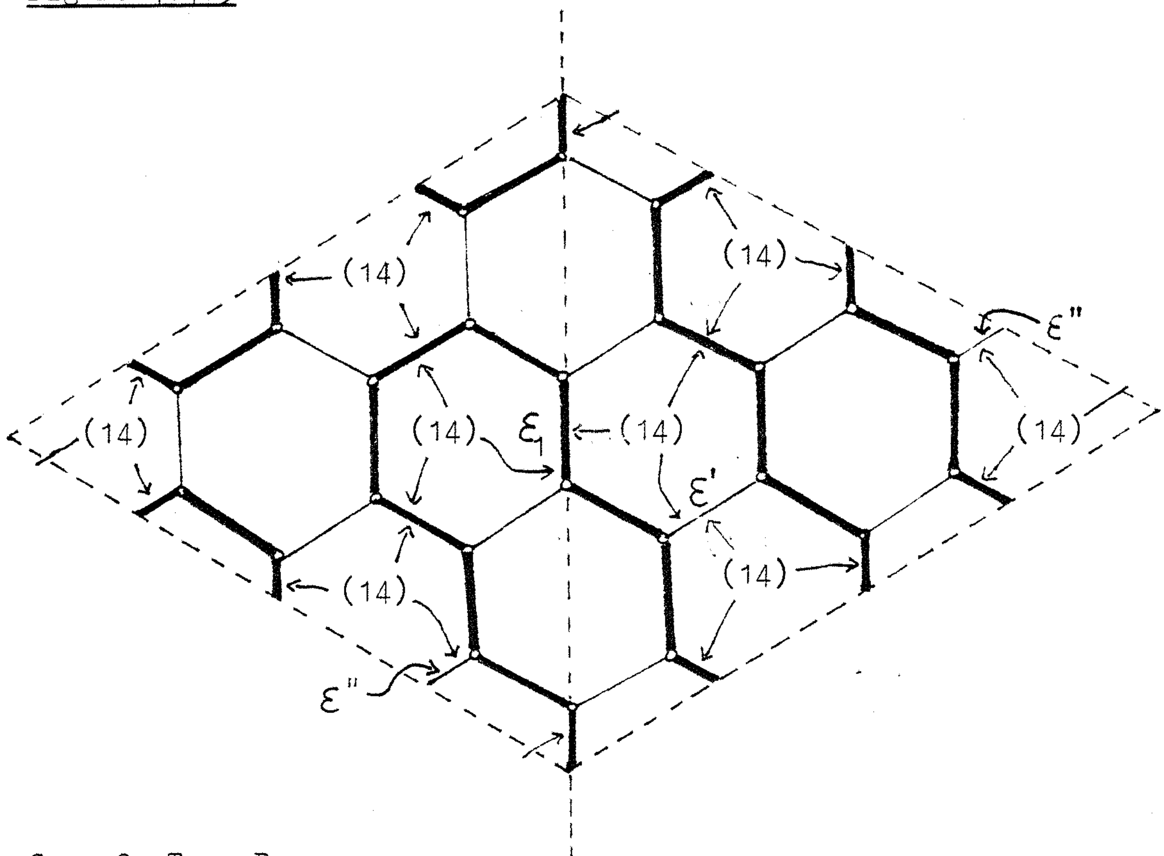
Without loss of generality, let the vertices of T_4 be 1, 2, 3, 4 and let u = 4 and v = 1. Let the two edges of $\Gamma(T_n)$ labelled (1 4) be ε_1 and ε_2. As in the proof of proposition 4.4.1, any hamiltonian circuit in $\Gamma(T_n)$ must contain at least one edge labelled (1 4), so by symmetry, we may assume that every hamiltonian circuit in $\Gamma(T_n)$ contains ε_1. It remains to show that one of these hamiltonian circuits also contains ε_2.

There are two non-isomorphic trees on 4 vertices, namely $K_{1,3}$ and P_4. These two cases must be considered separately.

Case 1: $T_4 = K_{1,3}$.

In this case, the hamiltonian circuit in fig. 4.4.3 contains ε_2 unless ε_2 is one of the edges ε' or ε'' . In this case the circuit obtained by reflecting the circuit in fig. 4.4.3 contains ε_1, ε' and ε'', and hence contains ε_1 and ε_2 .

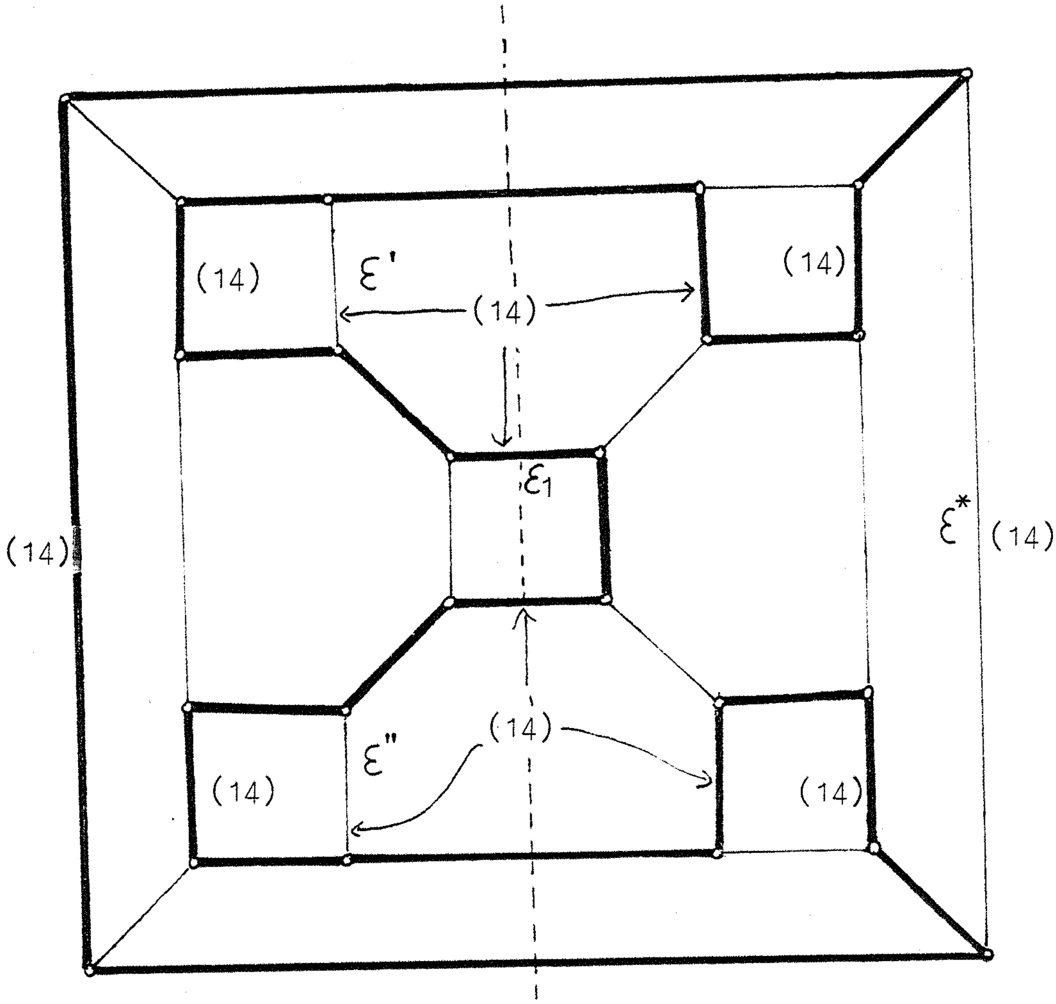
Figure 4.4.3



Case 2: $T_4 = P_4$

In this case, the hamiltonian circuit in fig. 4.4.4 contains ε_1 and ε_2 unless ε_2 is one of the edges ε', ε'' or ε^*. As in case 1, the hamiltonian circuit obtained by reflecting the hamiltonian circuit in fig. 4.4.4 in a vertical axis contains ε_1, ε', ε'' and ε^*, and hence contains ε_1 and ε_2 . This completes the proof of lemma 4.4.4.

Figure 4.4.4



Lemma 4.4.5

If T_4 is any tree on 4 vertices and u is an end vertex of T_4 adjacent to v, and if $\mathcal{E}_1$ and $\mathcal{E}_2$ are any two edges of $\Gamma(T_4)$ labelled (u v), then there are two circuits Δ_1 and Δ_2 in $\Gamma(T_4)$ such that 1): Δ_1 and Δ_2 together span $\Gamma(T_4)$; 2): Δ_1 and Δ_2 are disjoint ; and 3): $\mathcal{E}_1$ is an edge of Δ_1 and $\mathcal{E}_2$ is an edge of Δ_2.

Proof

Again, the proof separates into two cases, $T_4 = K_{1,3}$ and $T_4 = P_4$. As before, we assume that u = 4 and v = 1.

Case 1: $T_4 = K_{1,3}$

In this case, let Δ_1 and Δ_2 be the circuits in fig. 4.4.5;

Figure 4.4.5

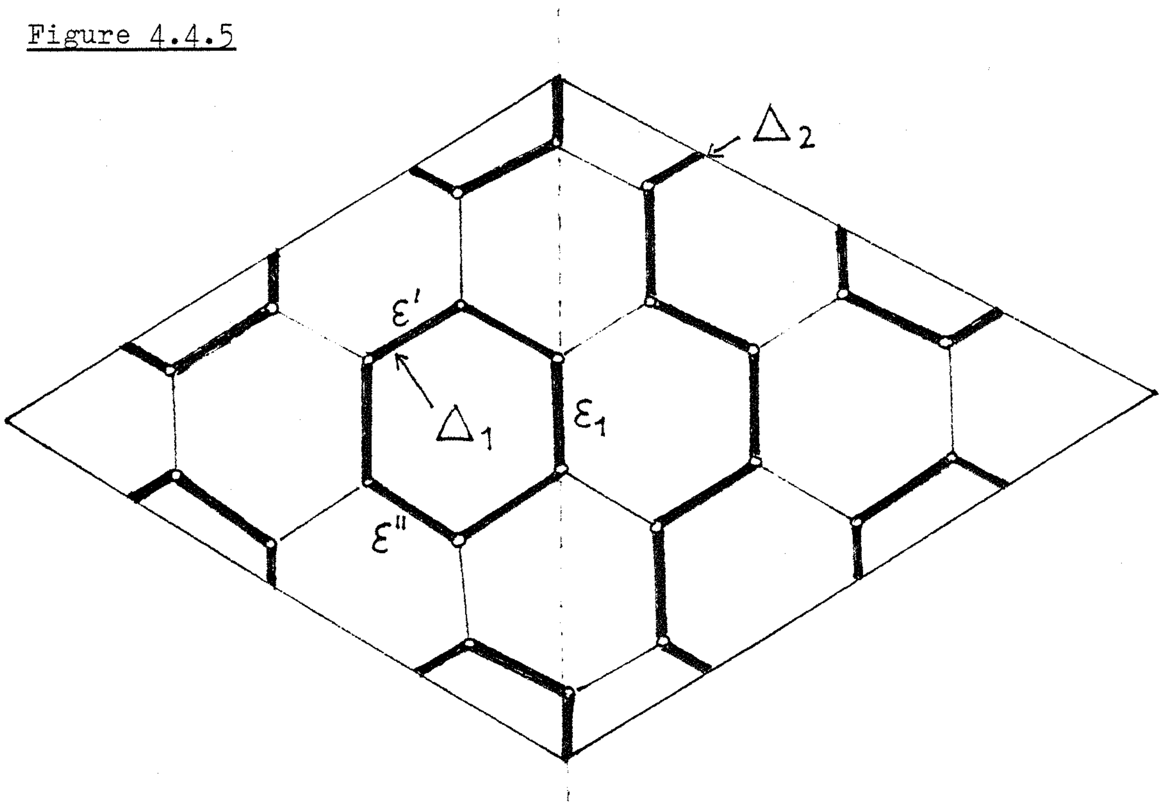


Figure 4.4.6

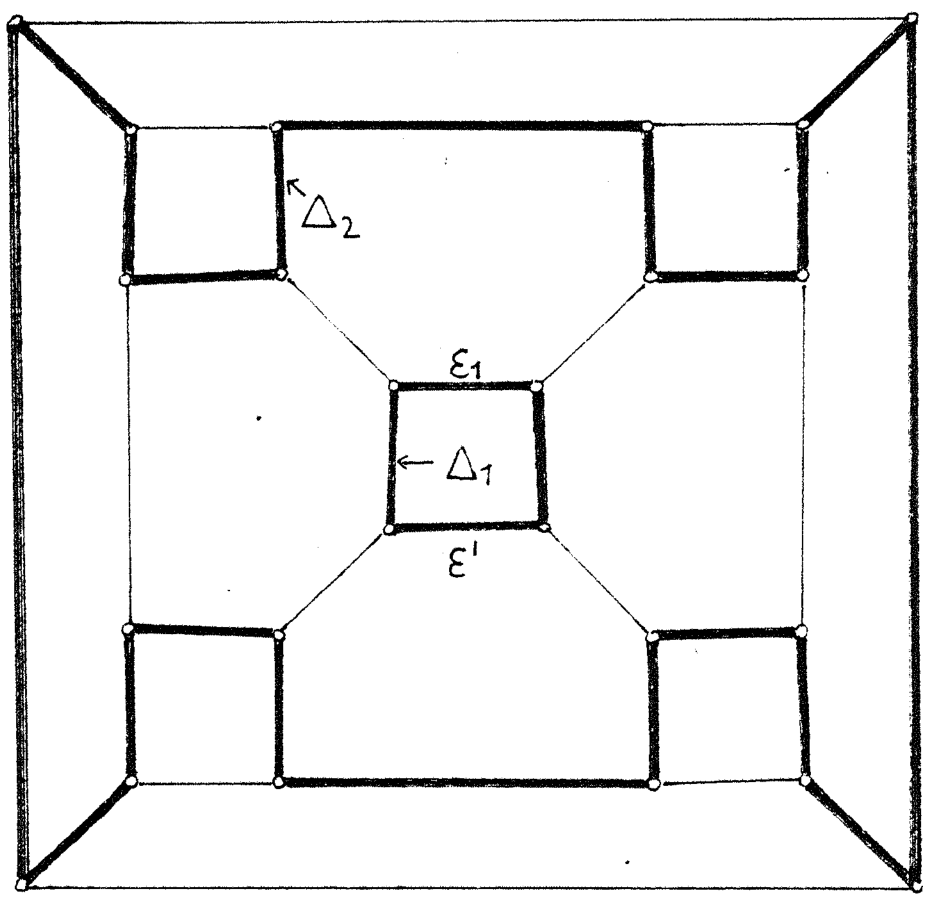
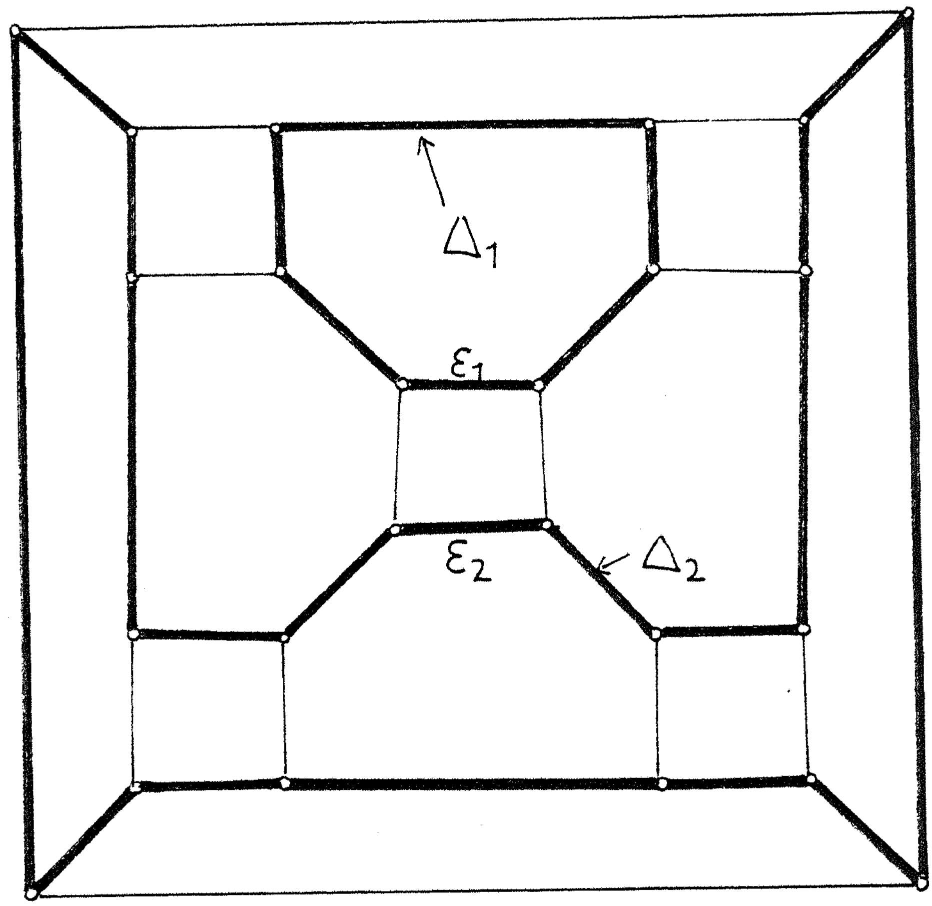


Figure 4.4.7



Note : The graph in fig. 4.4.5 is identically labelled to the graph in fig. 4.4.3, and the graphs in figs. 4.4.6 and 4.4.7 are identically labelled to the graph in fig. 4.4.4.

Then since Δ_1 and Δ_2 together contain all the edges of (T_4) labelled (1 4), Δ_1 and Δ_2 satisfy the hypotheses of the lemma unless $\mathcal{E}_2 = \mathcal{E}'$ or $\mathcal{E}''$. However, in this case, Δ_1' and Δ_2', the circuits obtained by reflecting Δ_1 and Δ_2 in a vertical axis have the required properties.

Case 2: $T_4 = P_4$.

In this case, Δ_1 and Δ_2, the circuits in fig. 4.4.6, have the required properties unless $\mathcal{E}_2 = \mathcal{E}'$. In this event, the circuits Δ_1 and Δ_2 in fig. 4.4.7 do have the required properties.

This completes the proof of lemma 4.4.5. $\square$

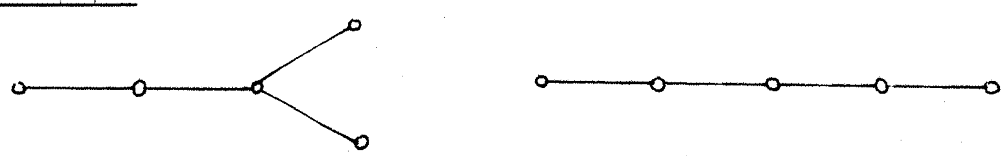
Lemma 4.4.6

If T_5 is any tree on 5 vertices except $K_{1,4}$, and if u is any end vertex of T_5 adjacent to v, and if $\mathcal{E}_1$ and $\mathcal{E}_2$ are any two edges of $\Gamma(T_5)$ labelled (u v), then there is a hamiltonian circuit in $\Gamma(T_5)$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$.

Proof

T_5 can be either of the graphs in fig. 4.4.8. In the first case, there are two possibilities for u up to isomorphism, and in the second case, one. In each of these cases we will assume without loss of generality that u = 5 and v = 2. Note that T_5 has another end vertex distance 3 or more from u. In each case, we will assume that this new end vertex is 4, and that the vertex adjacent to it is 1. The remaining vertex in T_5 will be 3.

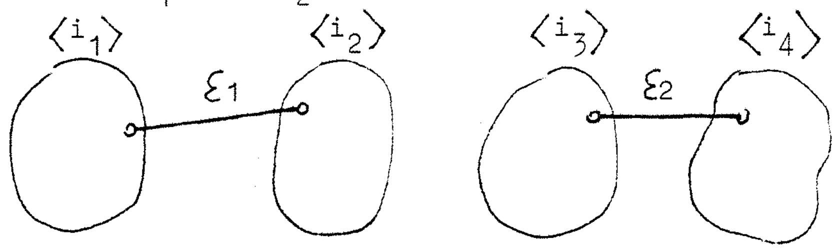
Figure 4.4.8



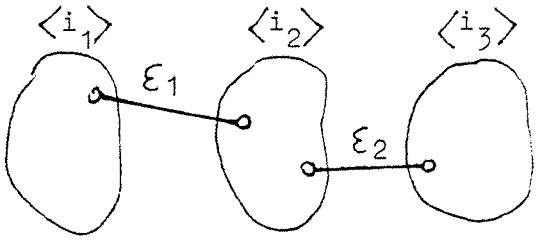
Let T_4 be the tree obtained by deleting 5 from T_5. Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are labelled (2 5), they must join cosets of S_4 in S_5, and hence they are either properly distant, distant but not properly distant, or not distant. In the case that they are not distant, $\mathcal{E}_1$ and $\mathcal{E}_2$ may both be edges in a circuit of length 4 in $\Gamma(T_5)$ with edges labelled (1 4) and (2 5). Otherwise, they must lie in two distinct circuits of this type. These four cases are illustrated in fig. 4.4.9.

Figure 4.4.9

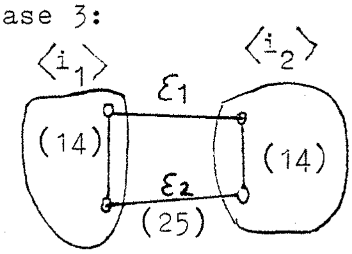
Case 1: $\mathcal{E}_1$ and $\mathcal{E}_2$ are properly distant.



Case 2: $\mathcal{E}_1$ and $\mathcal{E}_2$ are distant but not properly distant.

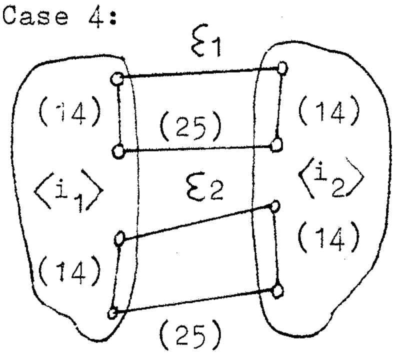


Case 3:



$\mathcal{E}_1$ and $\mathcal{E}_2$ are not distant and both lie in a circuit of length 4 with edges labelled (14) and (25).

Case 4:



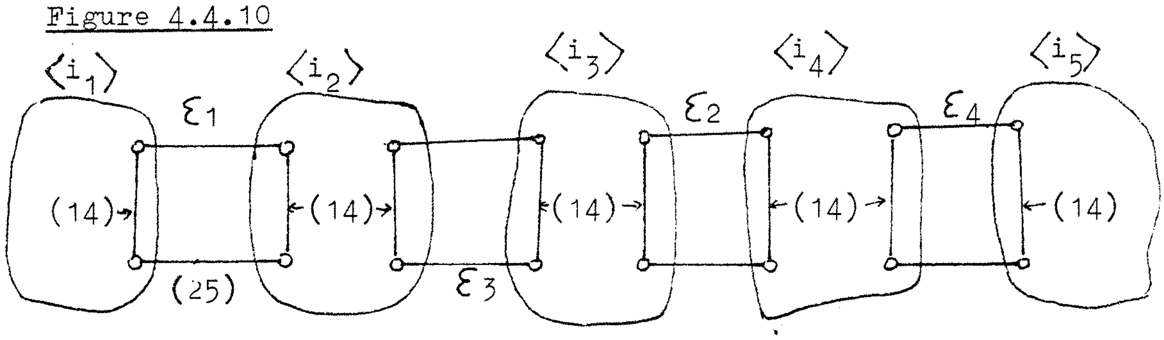
$\mathcal{E}_1$ and $\mathcal{E}_2$ are not distant and lie in different circuits of length 4 with edges labelled (14) and (25).

Note that all the vertices and edges in fig. 4.4.9 are distinct.

Case 1:

By proposition 4.2.5, there are edges $\mathcal{E}_3$ and $\mathcal{E}_4$ of $\Gamma(T_5)$ labelled (25) such that $\mathcal{E}_3$ joins $\langle i_2 \rangle$ to $\langle i_3 \rangle$ and $\mathcal{E}_4$ joins $\langle i_4 \rangle$ to $\langle i_5 \rangle$, where $\{i_1, i_2, \ldots, i_5\} = \{1, 2, \ldots, 5\}$. Each of the edges $\mathcal{E}_1, \ldots, \mathcal{E}_4$ lies in a distinct circuit of length 4, as shown in fig. 4.4.10.

Figure 4.4.10



For $j = 1,2,\ldots,5$, $\langle i_j \rangle \cong \Gamma(T_4)$. By lemma 4.4.4 there is a hamiltonian circuit in $\Gamma(T_4)$ containing any two edges labelled (14). Hence for j = 2, 3, 4, there is a hamiltonian circuit in $\langle i_j \rangle$ containing the two edges in fig. 4.4.10 labelled (14) . Similarly, there are hamiltonian circuits in $\langle i_1 \rangle$ and $\langle i_5 \rangle$ containing the (single) edge in fig. 4.4.10 labelled (14). A hamiltonian circuit in $\Gamma(T_5)$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$ is now obtained by taking the union of the hamiltonian circuits in $\langle i_j \rangle$; $j = 1,\ldots, 5$, and the four circuits of length 4 in fig. 4.4.10, and deleting the edges labelled (14) in fig. 4.4.10.

Case 2

In this case there exist edges $\mathcal{E}_3$ and $\mathcal{E}_4$ of $\Gamma(T_5)$ such that $\mathcal{E}_3$ joins $\langle i_3 \rangle$ to $\langle i_4 \rangle$, and $\mathcal{E}_4$ joins $\langle i_4 \rangle$ to $\langle i_5 \rangle$, where $\{i_1, i_2,\ldots, i_5\} = \{1, 2,\ldots, 5\}$. The remainder of the proof in this case is identical to the proof of case 1.

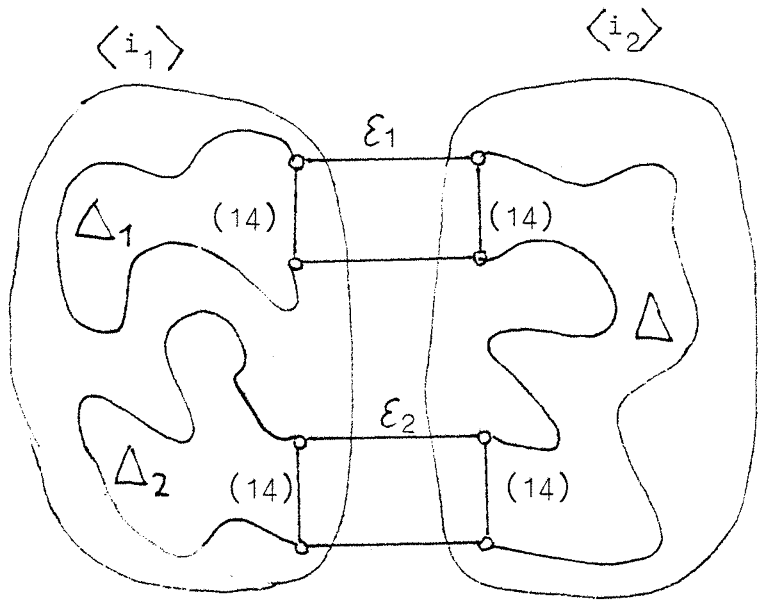
Case 3

This case is now clearly a special case of case 1, for we may choose an edge labelled (25) joining $\langle i_3 \rangle$ to $\langle i_4 \rangle$ and proceed as before.

Case 4

Since $\langle i_1 \rangle$ is identically labelled to $\Gamma(T_4)$, there are two circuits Δ_1 and Δ_2 of $\langle i_1 \rangle$ which are disjoint, span the vertices of $\langle i_1 \rangle$, and are such that one of the edges of $\langle i_1 \rangle$ labelled (14) in fig. 4.4.9 is an edge of Δ_1, while the other is an edge of Δ_2. As in the previous cases, there is a hamiltonian circuit in $\langle i_2 \rangle$ which contains the two edges of $\langle i_2 \rangle$ labelled (14) in fig. 4.4.9. Hence we have the situation in fig. 4.4.11 .

Figure 4.4.11



A careful examination of the hamiltonian circuits constructed in the proof of lemma 4.4.4 shows that Δ , the hamiltonian circuit in $\langle i_2 \rangle$, must contain at least 9 edges labelled (14). Hence there are another 7 edges labelled (14) in addition to the two in fig. 4.4.11. These edges give 14 additional vertices of $\langle i_2 \rangle$ incident to edges of Δ labelled (14). By proposition 4.2.5 , there are $(5-2)! = 6$ edges of $\Gamma(T_5)$ joining $\langle i_2 \rangle$ to $\langle i_1 \rangle$, and hence not all the 14 vertices above can be joined to $\langle i_1 \rangle$ by edges labelled (25). However, if σ is one of these 14 vertices, and $\sigma(2\ 5)$ is not a vetrex of $\langle i_1 \rangle$, then since $\sigma(2\ 5)$ cannot be a vertex of $\langle i_2 \rangle$, it must be a vertex of $\langle i_3 \rangle$, where $i_3 \neq i_1, i_2$. The proof in this case is now completed in much the same way as the proof in case 1.

This completes the proof of lemma 4.4.6. $\square$

It is now possible to prove theorem 4.4.3. There are two cases to consider. We assume without loss of generality that u = 6 and v = 3. Since $T_6 \ncong K_{1,5}$, there is an end vertex of T_6 which is distance 3 or more from 6. Let this vertex be 5 and let the vertex adjacent to it be 2.

<u>Case 1</u>: $T_6 - \{6\} \cong K_{1,4}$.

In this case the result follows from corollary 4.4.2.

<u>Case 2</u>: $T_6 - \{6\} \ncong K_{1,4}$

The proof in this case proceeds in an identical way to the proof of cases 1 and 2 of lemma 4.4.6.

This completes the proof of theorem 4.4.3. $\square$

<u>Theorem 4.4.7</u>

If T is any treeon n vertices such that $T \ncong K_{1,n-1}$, and if u is an end vertex of T adjacent to v, and if $\mathcal{E}_1$ and $\mathcal{E}_2$ are any two properly distant edges of $\Gamma(T)$ labelled (u v), then there is a hamiltonian circuit in $\Gamma(T)$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$.

<u>Proof</u>

<u>Case 1:</u> n = 7.

Without loss of generality, let u = 7 and let v = 4. Let T' be the tree obtained by deleting u = 7 from T. If $T' \cong K_{1,5}$ then the result follows from corollary 4.4.2, hence we assume that $T' \ncong K_{1,5}$.

Since $T \ncong K_{1,6}$, there is an end vertex 6, say, of T distance 3 or more from 7. Let the vertex of T adjacent to this vertex be 3. Since $\{3, 6\}$ and $\{4, 7\}$ are edges of T, every vertex of $\Gamma(T)$ is incident to edges labelled (3 6) and (4 7).

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be any two properly distant edges of $\Gamma(T)$ labelled (4 7). Each of these edges must lie in a circuit of length 4 in $\Gamma(T)$ with edges labelled (4 7) and (3 6), and these circuits must be distinct, or $\mathcal{E}_1$ and $\mathcal{E}_2$ would not be distant. Suppose that $\mathcal{E}_i = \{\rho_i, \sigma_i\}$ for i = 1, 2 .

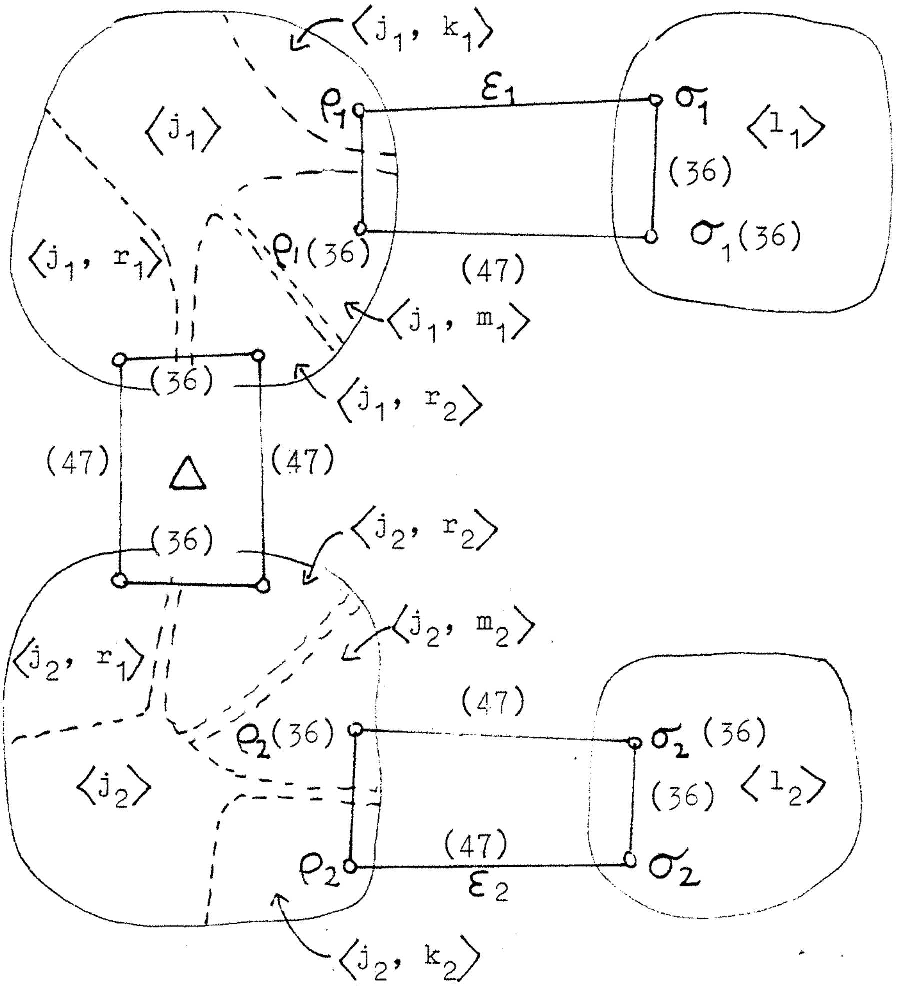
Now consider the left cosets of S_5 in S_7 , and suppose that $\rho_i \in \langle j_i, k_i \rangle$ and that $\sigma_i \in \langle l_i \rangle$, for i = 1,2 .

Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are properly distant, their end vertices must lie in four different left cosets of S_6 in S_7, so the numbers j_1, j_2, l_1 and l_2 are all distinct. $\rho_i(3\ 6)$ must lie in the same left coset of S_6 in S_7 as ρ_i, but it will lie in a different coset of S_5 in S_7 . Hence $\rho_i(3\ 6) \in \langle j_i, m_i \rangle$ where $k_i \neq m_i$ for i = 1, 2.
By the defininion of the symbol $\langle .,. \rangle$, each of the numbers j_i, k_i, l_i, m_i must lie between 1 and 7.

Let $r_1 \in [7] - \{j_1, j_2, k_1, k_2, m_1, m_2\}$. If $m_2 \neq j_1, j_2$ then let $r_2 = m_2$. Otherwise, choose r_2 in the same way as r_1. In either case, r_1 and r_2 are well-defined and distinct.

By proposition 4.2.7 there is a circuit Δ of length 4 in $\Gamma(T)$ with edges labelled (3 6) and (4 7) with one vertex in each of the cosets $\langle j_1, r_1 \rangle$, $\langle j_1, r_2 \rangle$, $\langle j_2, r_2 \rangle$, $\langle j_2, r_1 \rangle$. This situation is illustrated in fig. 4.4.12.

Figure 4.4.12



Note that $\langle j_i, r_2\rangle$ could be the same coset as $\langle j_i, m_i\rangle$ for $i = 1, 2$. All the other cosets in fig. 4.4.12 are definitely distinct. Also, Δ cannot have any vertices in common with the circuits of length 4 containing ε_1 and ε_2, for if it did it would be identical to one of them and would join the wrong cosets of S_6. Let the vertices of Δ be $\tau_{p,q}$, where $\tau_{p,q} \in \langle j_p, r_q\rangle$. $\langle j_1\rangle\Gamma(T')$ is identically labelled to $\Gamma(T')$, and it is clear that $\{\rho_1, \rho_1(36)\}$ and $\{\tau_{1,1}, \tau_{1,2}\}$ must correspond to distant edges in $\Gamma(T')$, since they join at least 3 cosets of S_5. Hence there is a hamiltonian circuit in $\langle j_1\rangle\Gamma(T')$ containing $\{\rho_1, \rho_1(36)\}$

and $\{\tau_{1,1}, \tau_{1,2}\}$. There is a similar hamiltonian circuit in $\langle j_2 \rangle \Gamma(T')$.

By repeating the above arguments, the other cosets of S_6 in S_7 may be connected in a chain to $\langle 1_1 \rangle$, the coset containing σ_1. (In fact, these constructions are much easier since no constraints are placed on the choice of Δ by the new coset being added.)

The hamiltonian circuit is completed in the same way as in the proof of lemma 4.4.6, case 1.

<u>Case 2:</u> $n \geqslant 8$

Now choose u = n and v = 4. As in case 1, if $T' = T - \{n\}$ is isomorphic to $K_{1,n-2}$ then the result follows from corollary 4.4.2, hence suppose that $T' \not\cong K_{1,n-2}$. An inductive proof is used, so suppose that the theorem holds for trees with n-1 vertices.

Since $T \not\cong K_{1,n-1}$, there is an end vertex n-1, say, of T distance 3 or more from n. Let n-1 be adjacent to 3 in T. Note that by the induction hypothesis, there is a hamiltonian circuit in $\Gamma(T')$ containing any two properly distant edges labelled (3 n-1).

Define ε_i, ρ_i, σ_i, j_i, k_i, l_i and m_i as in case 1, replacing 5, 6 and 7 by n-2, n-1 and n respectively, where necessary. Now r_1 and r_2 can both be chosen to be elements of $[n] - \{j_1, j_2, k_1, k_2, m_1, m_2\}$, and $r_1 \neq r_2$. This is because $[n]$ has at least 8 elements while the second set has at most 6.

Δ can now be chosen in the same way as in the previous case, and the edges $\{\rho_1, \rho_1(3\ n-1)\}$ and $\{\tau_{1,1}, \tau_{1,2}\}$ now correspond to properly distant edges of $\Gamma(T')$, since all the cosets in $\langle j_1 \rangle$ in a suitably modified version of fig. 4.4.12 are now distinct. The proof now continues in the same way as the proof of case 1.

This completes the proof of theorem 4.4.7. □

Corollary 4.4.8

If G is any connected graph on 3 or more vertices then $\Gamma(G)$ is hamiltonian.

Proof

By theorems 4.4.3 and 4.4.7 , if G is a tree on 4 or more vertices, then $\Gamma(G)$ is hamiltonian. If G is a tree on 3 vertices, then $G \cong K_{1,2}$ and $\Gamma(G) \cong C_6$, which is of course hamiltonian. If G is not a tree then G contains a spanning tree T. $\Gamma(T)$ is clearly a connected spanning subgraph of $\Gamma(G)$ which is hamiltonian by the above remarks. It follows immediately that $\Gamma(G)$ is hamiltonian. □

Corollary 4.4.9

If G is any graph with 3 or more vertices and without isolated vertices then $\Gamma(G)$ is hamiltonian.

Proof

Each component of G must have at least 2 vertices, and if the components of G are $G_1, G_2, \ldots, G_k$ then by proposition 1.3.9, $\Gamma(G) \cong \Gamma(G_1) \times \Gamma(G_1) \times \ldots \times \Gamma(G_k)$. Also, since G_i is connected, $\Gamma(G_i)$ has $n_i!$ vertices, where n_i is the number of vertices of G_i , for $i = 1, 2, \ldots, k$. Since $n_i \geqslant 2$, $n_i!$ is even.

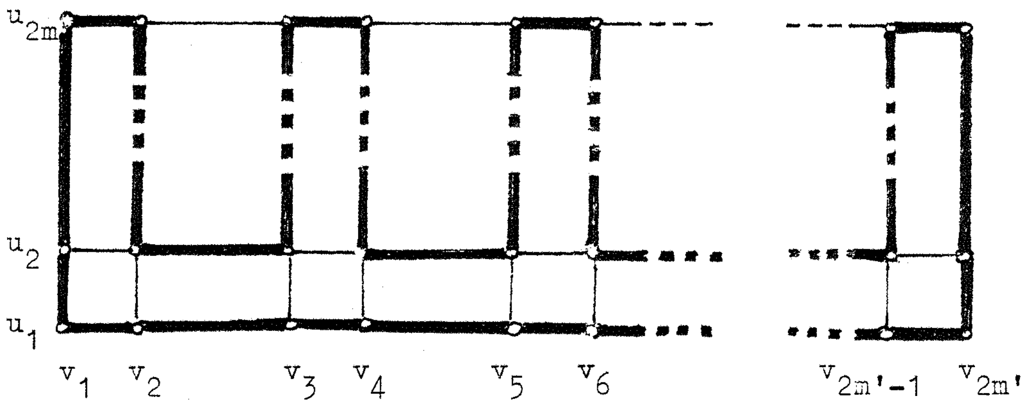
Lemma 4.4.10

If H and H' are path hamiltonian and have an even number of vertices, then $H \times H'$ is hamiltonian.

Proof of lemma

If H has 2m vertices and H' has 2m' vertices, then $H \times H'$ has 4mm' vertices and since H and H' contain hamiltonian paths, $H \times H'$ contains the graph in fig. 4.4.13 as a spanning subgraph.

Figure 4.4.13



As shown, the subgraph in fig, 4.4.13 has a hamiltonian circuit, and hence $H \times H'$ is hamiltonian. □

It immediately follows from the lemma that $\Gamma(G)$ is hamiltonian. This completes the proof of corollary 4.4.9. □

The results in this chapter generalise a theorem of J. Dénes, (8 , p. 262), which in effect states that $\Gamma(K_n)$ is path hamiltonian.

CHAPTER 5 : EMBEDDINGS OF TRANSPOSITION GRAPHS

SECTION 5.1 : INTRODUCTION

Attention in this chapter is concentrated on two special types of embedding of transposition graphs, the Cayley embedding and the alternating embedding. In both cases, the main problem considered here is that of finding the minimum genus of an embedding of the appropriate type for each transposition graph.

Section 5.2 is an informal introduction to the general theory of embeddings of graphs on (orientable) surfaces, and is intended only to introduce those results needed in the next two sections.

Section 5.3 is concerned with embeddings of transposition graphs, and in particular, with Cayley embeddings of transposition graphs. It is shown that the minimum genus of a Cayley embedding of a transposition graph $\Gamma(G)$ is connected with the minimum order of a product of all the transpositions in $\Omega(G)$. This problem is connected with a related problem of M. Eden, but is not studied in detail here.

Alternating embeddings are examined in detail in section 5.4, and it is proved that the minimum genus of an alternating embedding of $\Gamma(G)$ depends on how nearly $\overline{L}(G)$, the complement of the line graph of G, is hamiltonian. In particular, if $\overline{L}(G)$ is hamiltonian, then $\Gamma(G)$ has an alternating embedding whose genus is the minimum possible genus for any embedding of $\Gamma(G)$. This is also the case if G contains no circuits of length 3 .

Hamiltonian circuits in line graph complements are studied in section 5.5. The main result is that if G has sufficiently many ($\geqslant 34$) edges, then $\overline{L}(G)$ is hamiltonian iff G has no vertex incident to more than half the edges of G, and each edge of G is independent of at least two others. This second condition turns out to be relatively unimportant; only a rather small family of graphs with non-hamiltonian line graph complements satisfy the first condition but not the second. This result means that almost all graphs have hamiltonian line graph complements. It follows that the results in the previous section establish the genera of almost all transposition graphs. The only outstanding graphs $\Gamma(G)$ are those for which G contains circuits of length three and a vertex of very high degree, or else G is small. Finding the genus of such transposition graphs appears to be a difficult problem.

SECTION 5.2: EMBEDDINGS OF GRAPHS ON SURFACES

This section is intended only to introduce the basic terminology and results of the theory of embeddings needed in this chapter. It is not intended to be an introduction to the subject; for this, the reader is advised to consult the book 'Graphs, Groups and Surfaces' by A.T. White (15).

Following Biggs and White (4 , p.103), a surface will be defined as follows:

Definition 5.2.1

A surface is a compact topological space which is locally homeomorphic to the euclidean plane E^2 and which has a consistent global orientation.

The well-known Classification Theorem for surfaces implies that every surface (as defined here) is homeomorphic to a sphere with a number of handles attached.

Definition 5.2.2

The genus of a surface is the number of handles which must be attached to a sphere to make it homeomorphic to the surface. This is well-defined since the number of handles is a topological invariant; that is, is preserved by homeomorphisms.

Definition 5.2.3

An embedding of a graph $G = (V,E)$ in a surface S is a 1-dimensional subset M(G,S) of S consisting of a number of points corresponding to the vertices of G and a number of lines corresponding to the edges of G. Two points of M are joined by a line of M iff the corresponding vertices of G are joined by the corresponding edge. Also, two lines may only intersect at a point of M.

Intuitively, an embedding of G is simply a drawing of G on the surface.

Definition 5.2.4

A face of M(G,S) is a maximal connected subset of S - M(G,S). In all but one of the embeddings which will be considered in this chapter, every face will be simply-connected, or homeomorphic to an open disc.

Thus a face may be thought of as a region of the plane, and a surface is obtained by glueing together a number of faces along their edges. The glue lines form an embedding of some graph in the surface.

Note that if some face of an embedding is not simply connected, then we may remove that face from S and obtain a new surface by covering the holes with several simply-connected faces. This procedure gives an embedding of the same graph on a new surface which has a lower genus than the original one. Thus an embedding of a graph on a surface of minimum possible genus has all its faces simply-connected.

Since this chapter is almost entirely concerned with embeddings of this type, the restriction to simply-connected faces is not a serious one. A more formal proof of the result sketched out above is given by J.W.T.Youngs,(16).

With the restriction to simply-connected faces, the following result holds:

Theorem 5.2.1

If M(G,S) is an embedding of G on S, and if M has v points and e lines and f faces then $v - e + f = 2 - 2g$, where g is the genus of S.

Proof

This result is very well-known. One proof of it may be found in White (15, p.41). □

Theorem 5.2.2

If M(G,S) is an embedding of G in S, and M has e lines and f faces, and if f_i is the number of faces of M incident to i lines of M for $i \geqslant 3$, then $2e = 3f_3 + 4f_4 + 5f_5 + \dots$. Note that a face of M incident to 1 or 2 lines of M(G,S) would imply that G had a loop or a multiple edge, contradicting the fact that G is a graph.

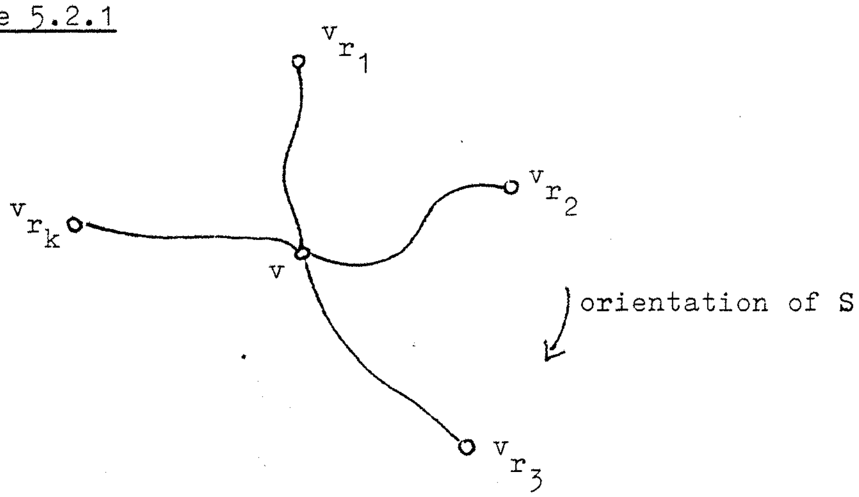
Proof

Let $\mathcal{F}_i$ denote the set of all faces of M(G,S) incident to i lines of M, so $|\mathcal{F}_i| = f_i$. If $F_i \in \mathcal{F}_i$, then F_i is incident to i lines of M. (Note that F_i may be incident to the same line twice; this must be counted twice.) Hence the faces in $\mathcal{F}_i$ are incident to a total of if_i lines of M.

However, since S is locally homeomorphic to the plane, each line of M is incident to two faces of M, again counting multiplicities. Hence equating the two different ways of counting the total number of incidences between lines and faces of M, $2e = 3f_3 + 4f_4 + 5f_5 + \dots$. □

In fact, embeddings of graphs on surfaces can be defined in a purely algebraic way, using the idea of a rotation of a graph. Consider a vertex v of a graph G, and suppose that G has an embedding M(G,S) in some surface. For the sake of convenience, each vertex of G will be considered to be identical to its corresponding point in M. Suppose that v is adjacent to $v_1, v_2, \dots, v_k$ in G, so there are lines of M joining v to each of these points in S. Since S is an orientable surface by definition, it has a consistent global orientation. Starting with one of the points adjacent to v and following the orientation around v, in turn, we reach all the other points of M joined to v, and finally return to the first point. This is illustrated in fig. 5.2.1.

Figure 5.2.1



Thus the orientation of S induces a cyclic permutation $(v_{r_1}\ v_{r_2}\ \ldots\ v_{r_k})$ of the vertices of G adjacent to v. Let this permutation be ρ_v. Thus the embedding of G on S gives rise to cyclic permutations ρ_v for every vertex v of G, where ρ_v is a cyclic permutation of the vertices of G adjacent to v.

Definition 5.2.5

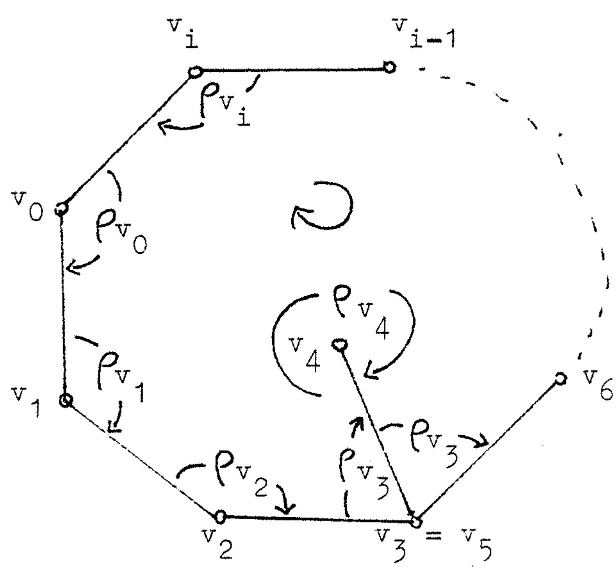
If G is a graph, then a rotation R of G is a family $R = \{\rho_v\}_{v \in V(G)}$, where ρ_v is a cyclic permutation of the vertices adjacent to v.

Clearly, by the above argument, each embedding M(G,S) gives rise to a unique rotation of G, for a given orientation of S. It is not so obvious that each rotation of G gives a distinct embedding of G, but this is the case.

Thus it suffices to show that an embedding can be constructed for each rotation of G.

Given a rotation of G, $R = \{\rho_v\}$, we first construct the faces of the embedding. Let $\{v_0, v_1\}$ be any edge of G, and let $v_0 \rho_{v_1} = v_2$. By definition, v_2 is a vertex of G adjacent to v_1, and is distinct from v_0 provided v_1 has valency 2 or more. Similarly, let $v_3 = v_1 \rho_{v_2}$, $v_4 = v_2 \rho_{v_3}$, and so on. Since G is finite, this process must eventually start repeating itself by reaching some vertex v_i of G such that $v_{i-1} \rho_{v_i} = v_0$ and $v_i \rho_{v_0} = v_1$. (Of course, it could be the case that the cycle began to repeat in the middle, but this would contradict the fact that each ρ_{v_i} is a permutation, since there would be two vertices v_i and v_j such that $v_i \rho_{v_{j+1}} = v_{j+2}$ and $v_j \rho_{v_{j+1}} = v_{j+2}$.) This process gives a face of the embedding, as shown in fig. 5.2.2.

<u>Figure 5.2.2.</u>



Note that the situation of a vertex of degree 1 is illustrated in the figure. The orientation of the face is given by the cycles at each vertex. The other face incident to $\{v_0, v_1\}$ is constructed similarly by going from v_1 to v_0. All the other faces of the embedding are constructed in a similar way to this. It remains to show that these faces fit together properly to give a surface. Each edge of G can only be incident to two faces, as was shown overleaf. It is possible for an edge to be incident to the same face twice; an example of this is the edge $\{v_3, v_4\}$ in fig. 5.2.2. Thus the faces may be glued together so that the vertices and edges meet properly. This procedure gives a manifold S. Each point of S in the interior of a face clearly has a neighbourhood which is locally homeomorphic to E^2, so the interiors of the faces arelocally flat. Each edge lies in two faces, so any point in the interior of the edge lies in a neighbourhood which is flat, as shown in fig. 5.2.3. Finally, S is flat at the vertices of G , as shown in fig. 5.2.4, so S is locally homeomorphic to E^2.

Figure 5.2.3

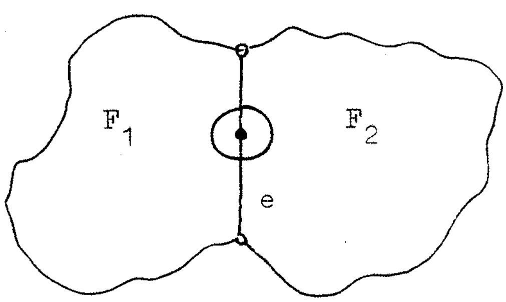
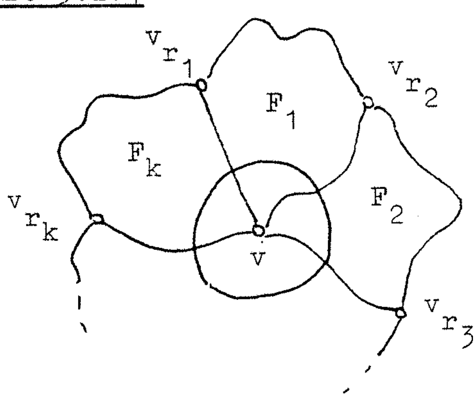


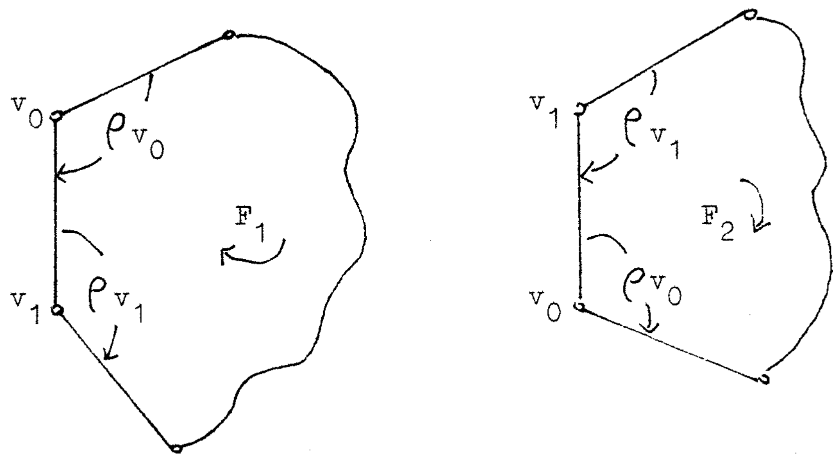
Figure 5.2.4



Where $\rho_v = (v_{r_1}\ v_{r_2}\ \dots\ v_{r_k})$.

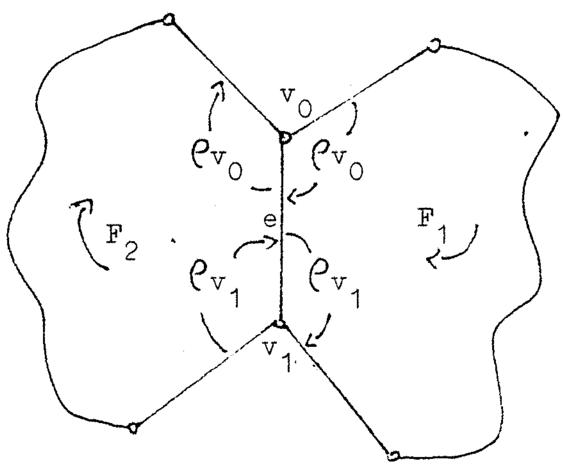
The orientation of S is given by the orientation of each face of S. If this was not globally consistent, there would be two adjacent faces of S with conficting orientations. Suppose that this is the case; since the two faces are adjacent, they must both be incident to some edge e of G. If $e = \{v_0, v_1\}$ then the two faces are shown in fig. 5.2.5.

Figure 5.2.5



These faces fit together as shown in fig. 5.2.6, and their orientations clearly agree. Thus S is orientable, and hence is a surface.

Figure 5.2.6



Thus embeddings of graphs on surfaces can be defined in terms of rotations of graphs. If an embedding of a graph G is defined in this way, and R is a rotation of G, then the embedding of G induced by R will be denoted by M'(G,R).

Definition 5.2.6

The genus of an embedding M'(G,R) is the genus of the surface induced by R on which G is embedded.

Definition 5.2.7

The genus of a graph G is the minimum genus of any embedding of G.

SECTION 5.3: CAYLEY EMBEDDINGS OF TRANSPOSITION GRAPHS

Note that the two types of embeddings of transposition graphs described in this and the next section can both be generalised to any Cayley graph. However, this involves additional work which is not necessary for the purposes of this chapter. Further details of these embeddings may be found in White (15, p.78) and Biggs and White (4 , sections 5.3, 5.6).

If σ is any vertex of a transposition graph $\Gamma(G)$, then the set of vertices of $\Gamma(G)$ adjacent to σ is $\sigma\omega_1, \sigma\omega_2, \ldots, \sigma\omega_m$ where $\{\omega_1, \omega_2, \ldots, \omega_m\} = \Omega(G)$. Thus any cyclic permutation of the vertices adjacent to σ will be of the form $\rho_\sigma = (\sigma\omega_{r_1}\ \sigma\omega_{r_2} \ldots \sigma\omega_{r_m})$, where $\{r_1, \ldots, r_m\} = \{1, \ldots, m\}$. Regarding ρ_σ as a permutation of $V(\Gamma(G))$, if $\theta_\sigma : \pi \to \sigma\pi$ for all vertices π of $\Gamma(G)$, then $\rho_\sigma = \theta_\sigma^{-1}(\omega_{r_1}\ \omega_{r_2} \ldots \omega_{r_m})\theta_\sigma$. It follows that if R is any rotation of $\Gamma(G)$, so $R = \{\rho_\sigma\}$ where ρ_σ is a cyclic permutation of the vertices adjacent to σ, then $R = \{\theta_\sigma^{-1}\rho_\sigma^*\theta_\sigma\}$, where ρ_σ^* is a cyclic permutation of $\Omega(G)$. Hence an embedding of $\Gamma(G)$ can be defined by a set of cyclic permutations of $\Omega(G)$. If an embedding of $\Gamma(G)$ is defined in this way, then it will be written as $M^*(\Gamma(G), R^*)$, where $R^* = \{\rho_\sigma^*\}$ and ρ_σ^* is a cyclic permutation of $\Omega(G)$ for all vertices σ of $\Gamma(G)$.

The simplest and most natural way to choose R^* is to let $R^* = \{\rho_\sigma^*\}$, where $\rho_\sigma^* = (\omega_1\omega_2 \ldots \omega_m)$, for all vertices σ of $\Gamma(G)$. (Any cyclic permutation of $\Omega(G)$ may be chosen instead of $(\omega_1\ \omega_2 \ldots \omega_m)$.)

Definition 5.3.1

If $M^*(\Gamma(G), R^*)$ is an embedding of this type then it will be called a Cayley embedding of $\Gamma(G)$. Note that if G has m edges then there are (m-1)! Cayley embeddings of $\Gamma(G)$.

Definition 5.3.2

The Cayley genus of a transposition graph, $\gamma_c(\Gamma(G))$, is the minimum genus of any Cayley embedding of $\Gamma(G)$.

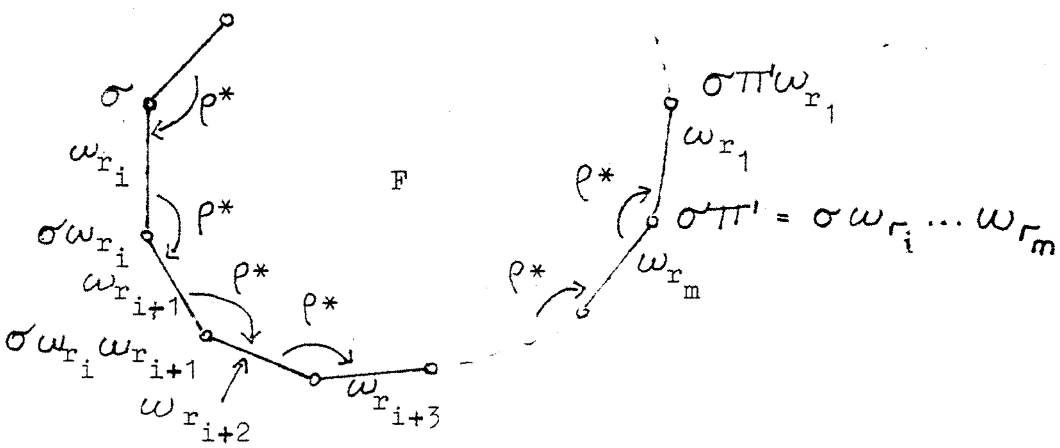
Theorem 5.3.1

If $M^*(\Gamma(G), R^*)$ is a Cayley embedding of $\Gamma(G)$, and if $R^* = \{\rho^*\}_\sigma$, where $\rho^* = (\omega_{r_1}\ \omega_{r_2}\ \dots\ \omega_{r_m})$, then every face of M^* is incident to mk edges of $\Gamma(G)$, where k is the order of $\pi = \omega_{r_1}\omega_{r_2}\dots\omega_{r_m}$ as a permutation.

Proof

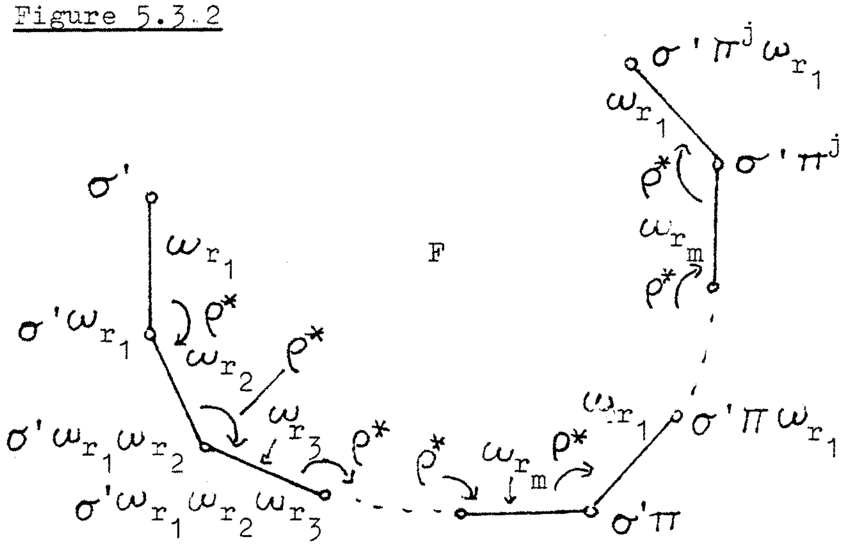
Consider a face F of M^* incident to the edge $\{\sigma, \sigma\omega_{r_i}\}$; F is shown in fig. 5.3.1.

Figure 5.3.1



Hence F is incident to the edge $\{\sigma', \sigma'\omega_{r_1}\}$, where $\sigma' = \sigma\pi'$. Starting again from σ', F is as in fig. 5.3.2. Clearly, the sequence of vertices and edges only starts repeating when $\pi^j = 1$, that is, when j = k by hypothesis. Thus F is incident to mk edges of $\Gamma(G)$. □

Figure 5.3.2



Corollary 5.3.2

If $M^*(\Gamma(G), R^*)$ is a Cayley embedding of $\Gamma(G)$, then M^* has genus $1 + \frac{1}{2}|V(\Gamma(G))|\left\{\frac{m}{2} - 1 - \frac{1}{k}\right\}$, where k is as defined in theorem 5.3.1 .

Proof

If g is the genus of M*, then by theorem 5.2.1, $g = 1 + \frac{1}{2}\left\{e - v - f\right\}$, where v, e and f are respectively the number of vertices and edges of $\Gamma(G)$, and f is the number of faces of M*. If G has m edges then $\Gamma(G)$ is m-valent, and hence $e = \left\{\frac{m}{2}\right\}v$. By theorem 5.2.2 and theorem 5.3.1, $2e = (mk)\cdot f_{mk} = mkf$ since every face of M* is incident to mk edges of $\Gamma(G)$. Hence $f = \frac{2e}{mk} = \frac{mv}{mk} = \frac{1}{k}\cdot v$. The result now follows by substuting $v = |V(\Gamma(G))|$ in the formula for the genus of M*. □

Corollary 5.3.3

$$\gamma_c(\Gamma(G)) \geqslant 1 + \frac{1}{2}|V(\Gamma(G))|\left\{\frac{m}{2} - 2\right\}.$$

Proof

This follows immediately from the fact that $k \geqslant 1$ in the formula proved in corollary 5.3.2. □

Note that in the equation in corollary 5.3.2 there is only one term which does not directly depend on G, namely k. It is fairly easy to show that there exist graphs for which k can take several different values. The simplest example is to take $G = C_4$, $\rho_1 = ((12)\ (34)\ (23)\ (14))$ and $\rho_2 = ((12)\ (23)\ (34)\ (14))$. Then $\pi_1 = (12)(34)(23)(14)$ so $\pi_1 = (13)(24)$ and $\pi_2 = (12)(23)(34)(14) = (243)$, and hence $k_1 = 2$ and $k_2 = 3$.

In section 1.2, definition 1.2.3, a graph G was defined to be related to a permutation σ if there exists a word W such that G = G(W) and $W = \sigma$ as a permutation. It is clear that if π is defined as in the proof of theorem 5.3.1 then G is related to π in this sense. It follows that the results of section 1.2 can be applied to find the genera of certain Cayley embeddings.

Proposition 5.3.4

A transposition graph $\Gamma(G)$ has a Cayley genus of $1 + \frac{1}{2}\,|V(\Gamma(G))|\left\{\frac{m}{2} - 2\right\}$ iff G maps to the identity (i.e. G is related to (1) in the sense of definition 1.2.3).

Proof

As in corollary 5.3.3, the result holds iff there is a cyclic permutation of $\Omega(G)$, $\rho = (\omega_1\ \omega_2\ \dots\ \omega_m)$ such that $\omega_1\omega_2\dots\omega_m$ has order 1, iff $\omega_1\omega_2\dots\omega_m = (1)$, iff G maps to the identity . □

Among the graphs mapping to the identity are K_n; $n \equiv 0, 1 \bmod 4$, and the wheel graphs W_n; $n \geqslant 3$ defined in section 1.2 . For most graphs which do not map to the identity, the Cayley genus of their transposition graph is hard to establish. However, if the graph is a tree, this is not the case.

Proposition 5.3.5

If T is a tree on n vertices then the Cayley genus of $\Gamma(T)$ is $1 + \frac{(n-1)!}{4}(n^2 - 3n - 2)$.

Proof

By corollary 1.2.9, if T is related to σ then σ is an n-cycle, which has order n. Thus whatever cyclic permutation ρ of $\Omega(G)$ is chosen, its order, $k = n$. The result follows after some algebraic manipulation of the expression in corollary 5.3.2. □

A similar result holds for the Cayley genus of $\Gamma(G)$ if G is a forest. However, the statement of the more general result is rather messy since it has to take into account the orders of the components of G, and involves their least common multiple. The proof is no more difficult, however.

SECTION 5.4: ALTERNATING EMBEDDINGS OF TRANSPOSITION GRAPHS

If G is any graph, then by proposition 1.3.6, $\Gamma(G)$ is a bipartite graph, and a bipartition for $\Gamma(G)$ is $V(\Gamma) = A \cup B$, where A is the set of even permutations in $V(\Gamma)$ and B is the set of odd permutations in $V(\Gamma)$.

Definition 5.4.1

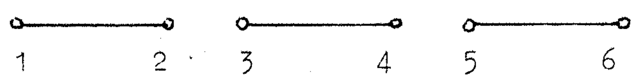
An embedding $M(\Gamma(G), R^*)$ of $\Gamma(G)$ which is defined in terms of a set of cyclic permutations of $\Omega(G)$ is alternating if $R^* = \{\rho^*_\sigma\}_{\sigma \in V(\Gamma(G))}$ satisfies the following condition: there exists a cyclic permutation ρ^* of $\Omega(G)$ such that $\rho^*_\sigma = \rho^*$ for all $\sigma \in A$ and $\rho^*_\sigma = \rho^{*-1}$ for all $\sigma \in B$, where A and B are defined as above.

Definition 5.4.2

The alternating genus $\gamma_a(\Gamma(G))$ of a transposition graph $\Gamma(G)$ is the smallest genus of any alternating embedding of the graph.

Example: Let G be the graph in fig. 5.4.1; then the embedding in fig. 5.4.2 is an alternating embedding of $\Gamma(G)$. The cyclic permutation of $\Omega(G)$ is $\rho^* = ((12)\ (34)\ (56))$.

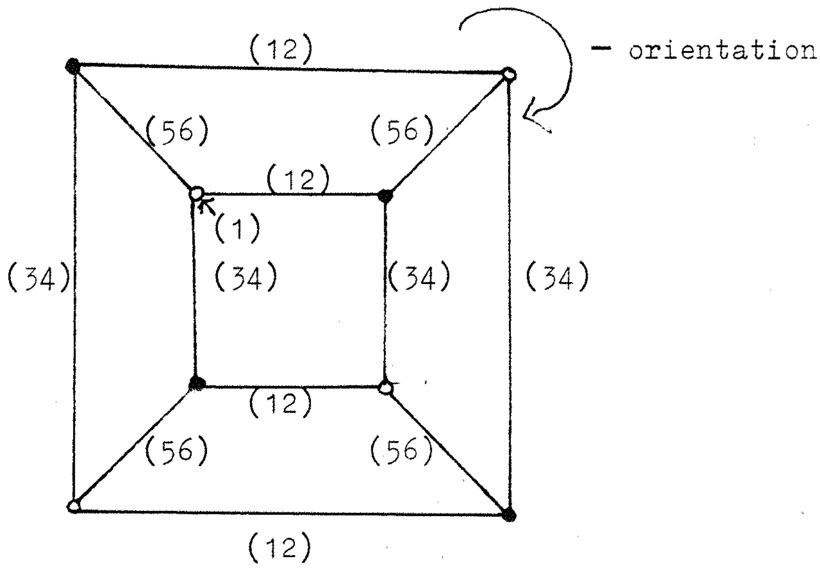
Figure 5.4.1

G:


Since there are $(m-1)!$ cyclic permutations of $\Gamma(G)$, where m is the number of edges of G, there are $(m-1)!$ alternating embeddings of $\Gamma(G)$. However, pairs of these embeddings are simply mirror images of one another, corresponding to interchanging ρ^* and ρ^{*-1} in definition 5.4.1.

Figure 5.4.2

$\Gamma(G)$:

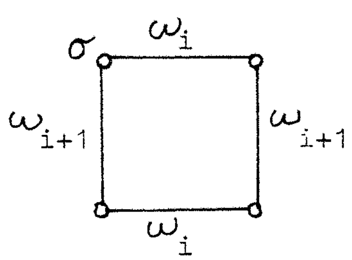


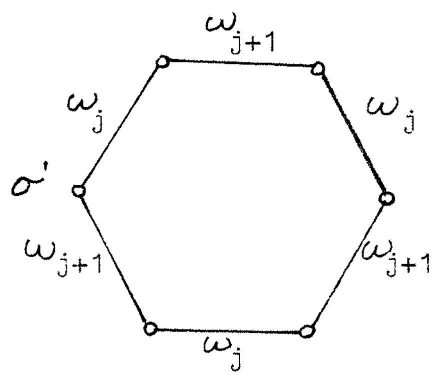
The even vertices of $\Gamma(G)$ are coloured white and the odd vertices are coloured black. The orientation of the surface is as shown in the figure.

Theorem 5.4.1

Let $M(\Gamma(G), R^*)$ be an alternating embedding of $\Gamma(G)$ and let $\rho^* = (\omega_1\ \omega_2\ \ldots\ \omega_m)$ be the cyclic permutation of $\Omega(G)$ in definition 5.4.1. Then each face of M is incident to either 4 or 6 edges of $\Gamma(G)$, and if Δ is a circuit of $\Gamma(G)$ bordering a face of M then Δ is one of the graphs in fig. 5.4.3.

Figure 5.4.3

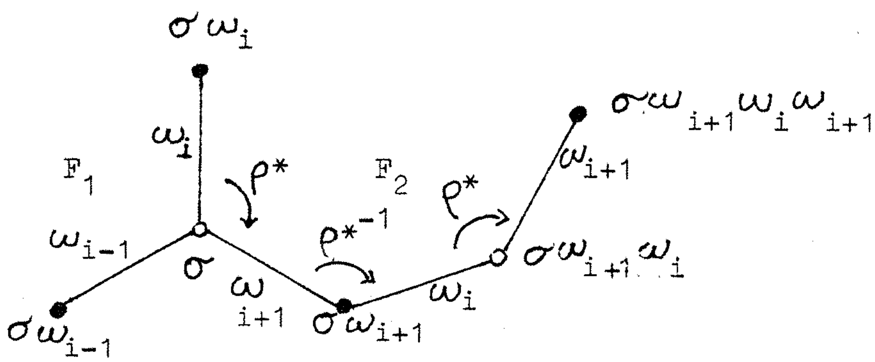




Proof

Let σ be any vertex of $\Gamma(G)$ and consider the two faces of M incident to the edge $\{\sigma, \sigma\omega_i\}$; let these faces be F_1 and F_2. Either σ or $\sigma\omega_i$ is an even permutation, so suppose without loss of generality that σ is even. Then F_1 and F_2 are as in fig. 5.4.4.

Figure 5.4.4



Concentrating on F_2, $\omega_i\rho^* = \omega_{i+1}$, and hence $\omega_{i+1}\rho^{*-1} = \omega_i$, so the edge labels of F_2 are as shown in fig. 5.4.4. If $(\omega_i\omega_{i+1})^2 = (1)$, then $\sigma\omega_{i+1}\omega_i\omega_{i+1} = \sigma\omega_i$, and hence F_2 is bounded by four edges, which form a subgraph of $\Gamma(G)$ isomorphic to the first graph in fig. 5.4.3.
If $(\omega_i\omega_{i+1})^2 \neq (1)$, then $(\omega_i\omega_{i+1})^3 = (1)$, and in a similar way, F_2 is bounded by six edges which form a subgraph of $\Gamma(G)$ isomorphic to the second graph in fig. 5.4.3. Clearly, a similar result holds for F_1. $\square$

Corollary 5.4.2

If $M(\Gamma(G), R^*)$ is an alternating embedding of $\Gamma(G)$ and if ρ^* is the cyclic permutation of $\Omega(G)$, and $\rho^* = (\omega_1\ \omega_2 \ldots \omega_m)$ then $\gamma(M) = 1 + \frac{|V(\Gamma(G))|}{24}\left\{4m - k - 12\right\}$, where k is the number of transpositions ω_i in $\Omega(G)$ such that $(\omega_i\omega_{i+1})^2 = 1$, (subscripts mod m).

Proof

Every vertex of $\Gamma(G)$ is incident to m faces of M, and by theorem 5.4.1 and by the definition of k, k of these faces are incident to 4 edges of $\Gamma(G)$ while the remaining m-k are incident to 6 edges of $\Gamma(G)$. If f_4 is the number of faces of M incident to 4 edges of $\Gamma(G)$ then clearly, $f_4 = k\frac{|V(\Gamma(G))|}{4}$, since each such face of M is incident to 4 vertices. Similarly, $f_6 = (m-k)\frac{|V(\Gamma(G))|}{6}$. If f is the total number of faces of M, then $f = f_4 + f_6$, by theorem 5.4.1. The result now follows by algebraic manipulation of Euler's formula (theorem 5.2.1).

Corollary 5.4.3

$$\gamma_a(\Gamma(G)) \geq 1 + \frac{|V(\Gamma(G))|}{8}(m - 4)$$

Proof

This follows from corollary 5.4.2 and the fact that $k \leq m$. □

If G is any graph with at least one edge, then $\overline{L}(G)$ will denote the complement of the line graph of G, or the line graph complement of G.

Theorem 5.4.4

Let $r \geq 0$ be the smallest number of edges which must be added to $\overline{L}(G)$ to make it hamiltonian; then

$$\gamma_a(\Gamma(G)) = 1 + \frac{|V(\Gamma(G))|}{24}(3m + r - 12)$$

Proof

We first show that $\Gamma(G)$ has an alternating embedding with this genus, then show that it has no alternating embedding with a smaller genus.

Note that the vertices of $\overline{L}(G)$ are the edges of G. Now suppose that by adding r edges to $\overline{L}(G)$ we obtain a hamiltonian circuit $e_1 \sim e_2 \sim \ldots \sim e_m \sim e_1$. Then r of these vertices are not adjacent in $\overline{L}(G)$, and the remaining $k = (m-r)$ are adjacent, by the definition of r.

For i = 1,..., m , let $\omega_i \in \Omega(G)$ be the transposition corresponding to e_i . If $e_i \sim e_{i+1}$ in $\overline{L}(G)$, then e_i is non-incident to e_{i+1} , and hence ω_i and ω_{i+1} are disjoint transpositions, so $(\omega_i \omega_{i+1})^2 = (1)$. Similarly, if $e_i \not\sim e_{i+1}$ then $(\omega_i \omega_{i+1})^3 = (1)$.

Now define a cyclic permutation ρ^* of $\Omega(G)$ by $\rho^* = (\omega_1\ \omega_2\ \omega_3\ \dots\ \omega_m)$, and let $M(\Gamma(G))$ be the alternating embedding of $\Gamma(G)$ defined by ρ^*. By corollary 5.4.2, $\gamma(M) = \frac{|V(\Gamma(G))|}{24}(4m - k - 12)$, where k is the number of transpositions $\omega_i \in \Omega(G)$ such that $(\omega_i \omega_{i+1})^2 = (1)$. However, by the above argument, this is the number of vertices e_i of $\overline{L}(G)$ such that $e_i \sim e_{i+1}$. There are (m-r) such vertices, hence $\gamma(M) = \frac{|V(\Gamma(G))|}{24}(3m + r - 12)$.

If there were an alternating embedding of $\Gamma(G)$ with a smaller genus than M, then it would follow by reversing the above argument that $\overline{L}(G)$ could be made hamiltonian by the addition of fewer than r edges, contradicting the definition of r. Hence the result holds. □

Note that a particular consequence of this result is that if $\overline{L}(G)$ is hamiltonian, then the alternating genus of $\Gamma(G)$ attains the bound of corollary 5.4.3.

Theorem 5.4.5

If G is a bipartite graph and M_1 and M_2 are embeddings of G such that $f_1^4 \geqslant f_2^4$ and $f_1^m = 0$ for all $m \geqslant 8$, where f_i^j is the number of faces of M_i incident to j edges, then $\gamma(M_1) \leqslant \gamma(M_2)$.

Proof

If f_i is the total number of faces of M_i, then

$f_1 = f_1^4 + f_1^6$ and

$f_2 = f_2^4 + f_2^6 + f_2^8 + f_2^{10} + \dots$

since G is bipartite and $f_1^m = 0$ if $m \geqslant 8$.

If G has v vertices and e edges then by theorem 5.2.2,

$2e = 4f_1^4 + 6f_1^6 = 4f_2^4 + 6f_2^6 + 8f_2^8 + 10f_2^{10} + \dots$, and hence

$f_1^6 - f_2^6 = \frac{-2}{3}(f_1^4 - f_2^4) + \frac{4}{3} f_2^8 + \frac{5}{3} f_2^{10} + \dots$.

By theorem 5.2.1,

$$\begin{aligned} \gamma(M_2) - \gamma(M_1) &= (1 + \tfrac{1}{2}(e - v - f_2)) - (1 + \tfrac{1}{2}(e - v - f_1)) \\ &= \tfrac{1}{2}(f_1 - f_2) \\ &= \tfrac{1}{2}(\ (f_1^4 - f_2^4) + (f_1^6 - f_2^6) - f_2^8 - f_2^{10} - \dots\ ) \\ &= \tfrac{1}{2}(\ \tfrac{1}{3}(f \quad - f \quad) + \tfrac{1}{3} f_2^8 + \tfrac{2}{3} f_2^{10} + \dots\ ) \\ &\geqslant 0 \text{ since } f_1^4 \geqslant f_2^4 . \quad \square \end{aligned}$$

Corollary 5.4.6

If G is a graph such that $\overline{L}(G)$ is hamiltonian, then

$$\gamma(\Gamma(G)) = 1 + \frac{|V(\Gamma(G))|}{8}(m - 4) \quad .$$

Proof

By theorem 5.4.4, $\Gamma(G)$ has an alternating embedding such that every face of the embedding is incident to 4 edges of $\Gamma(G)$, since $\overline{L}(G)$ is hamiltonian. By theorem 5.4.5, this is a minimum genus embedding for $\Gamma(G)$. The formula follows from that of theorem 5.4.4 with $r = 0$. □

Example: if G is the graph with 2n vertices and n disjoint edges $\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}$ then $\Gamma(G) \cong Q_n$, the n-cube, with 2^n vertices. Also, $\overline{L}(G) \cong K_n$, which is a hamiltonian graph. Hence by corollary 5.4.6, $\gamma(Q_n) = 1 + 2^{n-3}(n - 4)$.

Graphs G for which $\overline{L}(G)$ is hamiltonian are studied in the following section, and the following result is proved: If G is a graph with n vertices and $m \geqslant 34$ edges, then $\overline{L}(G)$ is hamiltonian iff G has no vertex v with degree $d(v) > \frac{m}{2}$, and every edge of G is non-incident to at least two others. In fact, this second condition is almost redundant.

Since large graphs with one vertex incident to more than half the edges are relatively rare, corollary 5.4.6 gives the genus of almost all transposition graphs. Some of the remaining graphs are covered by the next result.

Corollary 5.4.7

If G is a graph with no circuits of length 3, then

$$\gamma(\Gamma(G)) = 1 + \frac{|V(\Gamma(G))|}{24}(3m + r - 12) \quad .$$

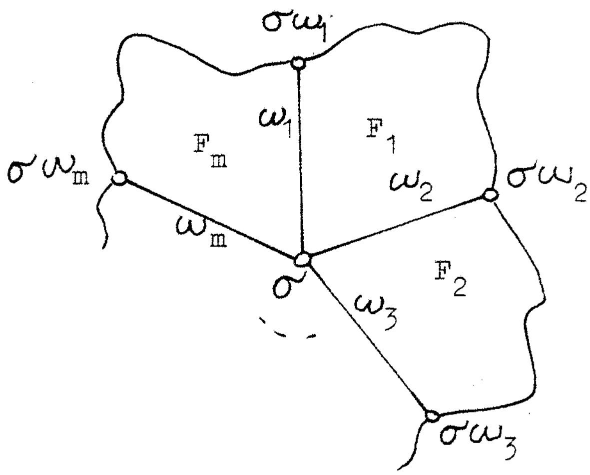
Proof

Let M' be any embedding of $\Gamma(G)$, and consider the faces of M' incident to some vertex σ of $\Gamma(G)$. Suppose that M' is defined in terms of some rotation R* of $\Gamma(G)$, and that

$\rho^*_\sigma = (\omega_1\ \omega_2\ \ldots\ \omega_m)$, where $\Omega(G) = \{\omega_1, \omega_2, \ldots, \omega_m\}$.

Then the faces of M' incident to σ are as shown in fig. 5.4.5.

Figure 5.4.5



Suppose that one of these faces F_i is incident to 4 edges of $\Gamma(G)$, and let Δ be the subgraph of $\Gamma(G)$ incident to F_i. Since $\Delta \cong C_4$, then $\overline{G}(\Delta)$ is isomorphic to one of the graphs in fig. 5.4.6, by theorem 2.2.8. However, by proposition 2.2.6, $\overline{G}(\Delta) \subset G$, so $\overline{G}(\Delta) \not\cong K_3$. Hence ω_i and ω_{i+1} are disjoint transpositions, so $(\omega_i\omega_{i+1})^2 = (1)$.

Figure 5.4.6

Suppose that k is the largest number of faces of size 4 of M' to which any vertex of $\Gamma(G)$ is incident, and suppose that σ is incident to k faces of M' of size 4. For each such face, $(\omega_i\omega_{i+1})^2 = (1)$, so there are at least k elements $\omega_i \in \Omega(G)$ such that $(\omega_i\ \omega_{i+1})^2 = (1)$. It follows that there are at least k vertices e_i of $\overline{L}(G)$ such that $e_i \sim e_{i+1}$. Hence by adding at most m-k edges $\{e_j, e_{j+1}\}$ to $\overline{L}(G)$, we obtain a hamiltonian circuit in $\overline{L}(G)$. Hence the alternating embedding M of $\Gamma(G)$ generated by this hamiltonian circuit has at least k circuits of length 4 incident to each vertex of $\Gamma(G)$. Thus this alternating embedding M has more faces of size 4 than M', and hence by the argument of theorem 5.4.5, M has a smaller genus than M'. It follows that at least one minimum genus embedding of $\Gamma(G)$ is alternating, and the result follows from theorem 5.4.4. □

Thus the genus of a transposition graph $\Gamma(G)$ has been established for all graphs G such that either G has no circuit of length 3 or $\overline{L}(G)$ is hamiltonian. Further, in both these cases, at least one minimum genus embedding is alternating. This is not necessarily the case for the remaining transposition graphs; several examples will be given of transposition graphs $\Gamma(G)$ for which $\gamma(\Gamma(G)) < \gamma_a(\Gamma(G))$. However, such embeddings are normally very difficult to construct, and it can be even harder to prove that such an embedding is minimum genus.

For the remainder of this section, we will establish the genus of all but one of the transposition graphs with at most 24 vertices. For the exceptional graph, there are two possible values for the genus.

It is easy to check that if $\Gamma(G)$ is a transposition graph with 24 or fewer vertices, then G is one of the graphs in fig. 5.4.7. This requires only corollary 1.2.2 and a list of small graphs. The line graph complements of these graphs are shown in fig. 5.4.8. The dotted lines in some of these graphs indicate the smallest set of edges which must be added to make the graph hamiltonian.

Figure 5.4.7

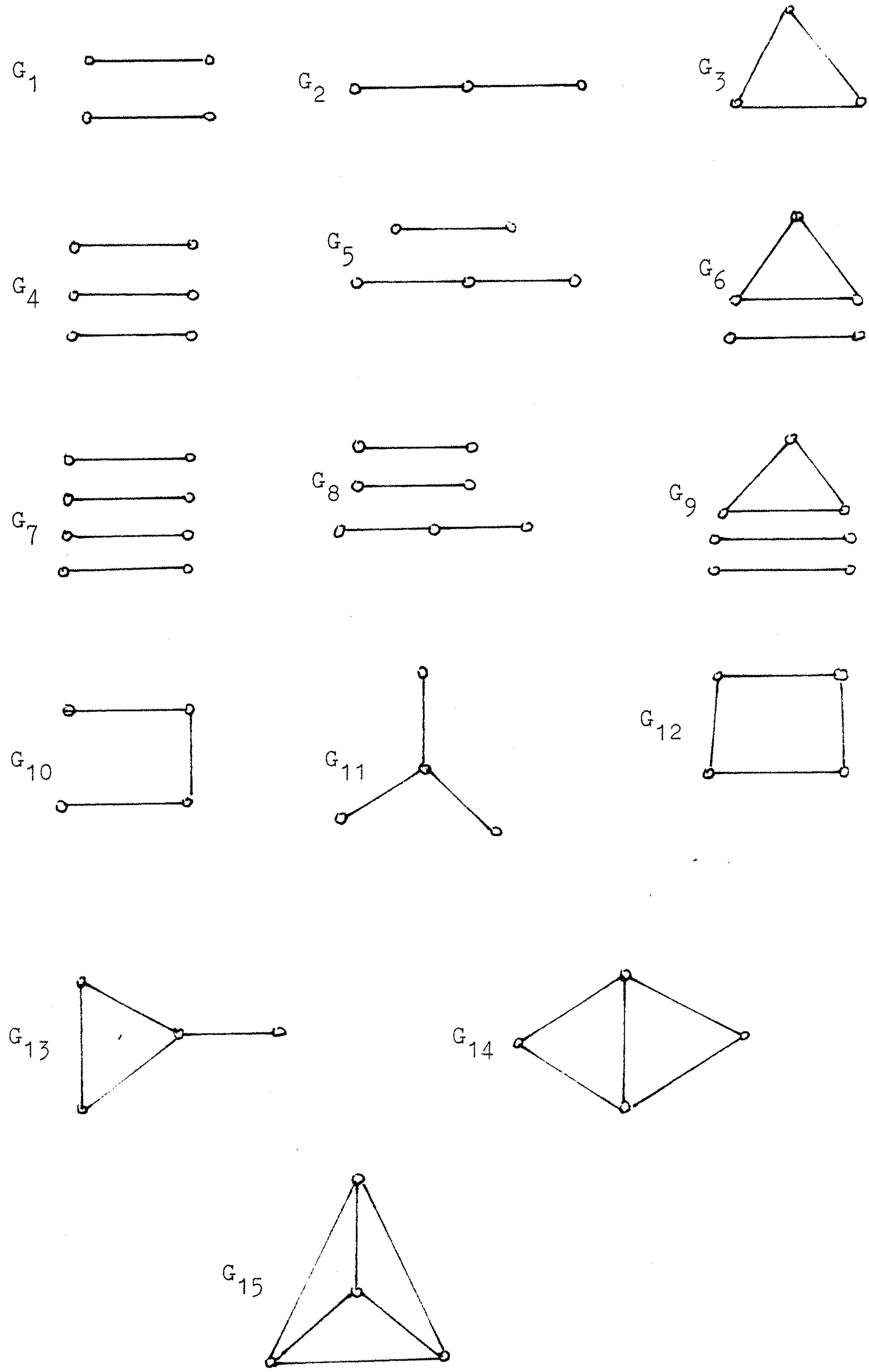
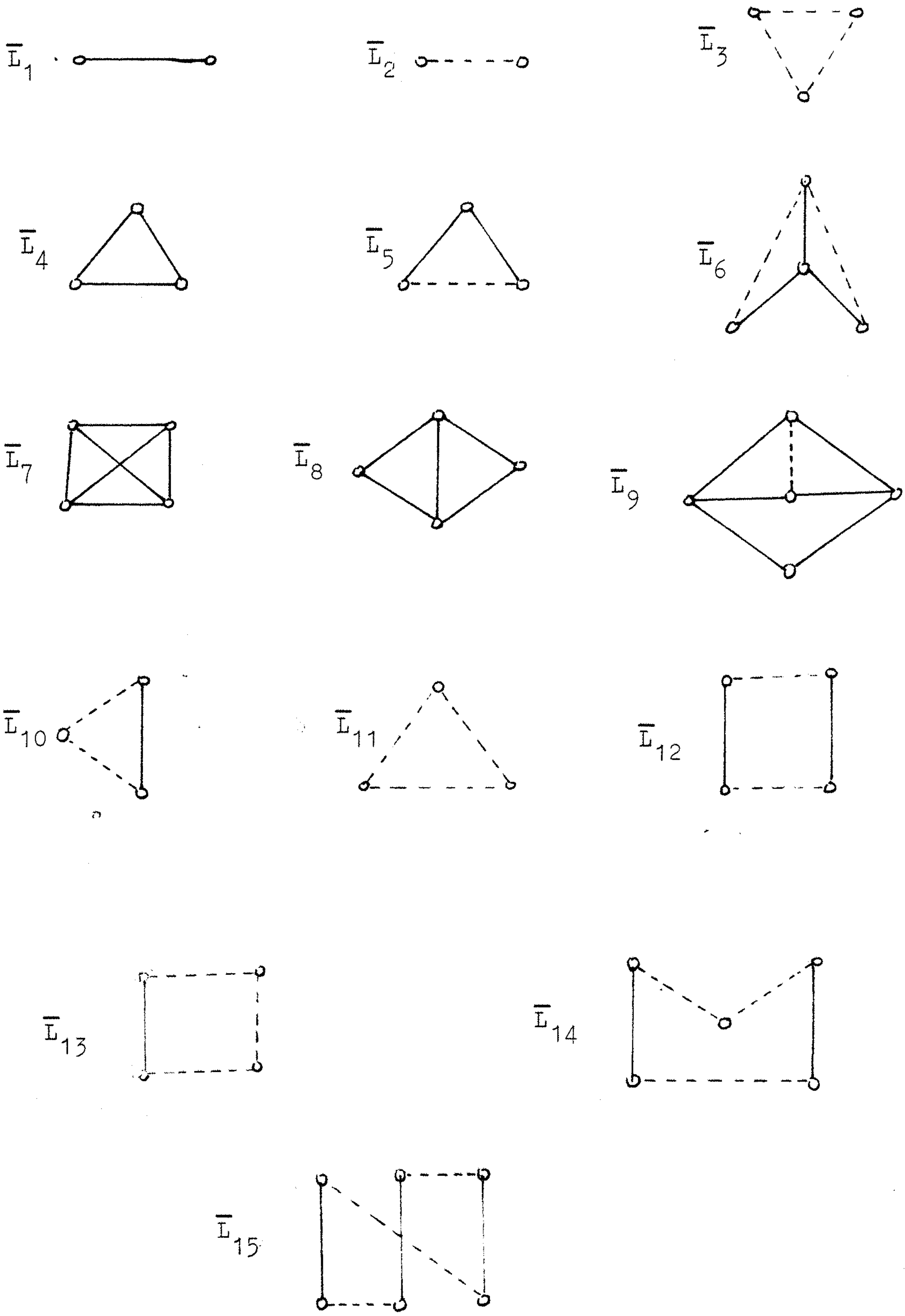


Figure 5.4.8


$\bar{L}_1$
$\bar{L}_2$
$\bar{L}_3$
$\bar{L}_4$
$\bar{L}_5$
$\bar{L}_6$
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$\bar{L}_{12}$
$\bar{L}_{13}$
$\bar{L}_{14}$
$\bar{L}_{15}$

Theorem 5.4.8

The genus of $\Gamma(G_i)$ is 0 if i = 1, 2, 4, 5 or 10, 1 if i = 7, 8 or 11, and 3 if i = 12.

Proof

For all these values of i, G_i has no circuits of length 3, so the genus of $\Gamma(G_i)$ is given by corollary 5.4.7. The value of r for each of these graphs is given by the number of dotted edges in fig. 5.4.8. □

Theorem 5.4.9

$\gamma(\Gamma(G_3)) = 1$, $\gamma(\Gamma(G_6)) = 2$ and $\gamma(\Gamma(G_{13})) = 4$.

Proof

Two general lemmas are useful in proving this result:

Lemma 1

The genus of a graph is equal to the sum of the genera of its components.

Proof of lemma 1

This is a corollary to the following theorem of Battle, Harary, Kodama and Youngs (1): The genus of a graph is equal to the sum of the genera of its blocks (maximal 2-connected subgraphs). □

Lemma 2

If $H \subseteq G$, then $\gamma(H) \leqslant \gamma(G)$.

Proof of lemma 2

This is obvious, since any embedding of G on a surface automatically gives an embedding of H on the same surface. □

It is easy to check that by theorem 5.4.4, $\gamma_a(\Gamma(G_3)) = 1$, $\gamma_a(\Gamma(G_6)) = 2$ and $\gamma_a(\Gamma(G_{13})) = 4$, giving upper bounds for the genera of these three graphs. However, $\Gamma(G_3) \cong K_{3,3}$, which is a well-known non-planar graph, so $\gamma(\Gamma(G_3)) \geqslant 1$. It follows that $\gamma(\Gamma(G_3)) = 1$.

By proposition 1.3.9, $\Gamma(G_6) \cong \Gamma(K_2) \times \Gamma(K_3)$ and $\Gamma(G_{13}) \cong \Gamma(K_2) \times \Gamma(K_2) \times \Gamma(K_3)$. Now $\Gamma(K_2) \cong K_2$ and $\Gamma(K_3) \cong K_{3,3}$ so $\Gamma(G_6) \cong K_2 \times K_{3,3}$ and $\Gamma(G_{13}) \cong K_2 \times K_2 \times K_{3,3} = C_4 \times K_{3,3}$. Hence $\Gamma(G_6)$ is spanned by two disjoint subgraphs isomorphic to $K_{3,3}$ and $\Gamma(G_{13})$ is spanned by four such subgraphs.

It follows from the two lemmas that $\gamma(\Gamma(G_6)) \geqslant 1 + 1 = 2$, and $\gamma(\Gamma(G_{13})) \geqslant 1 + 1 + 1 + 1 = 4$. The result now follows since $\gamma(\Gamma(G_6)) \leqslant \gamma_a(\Gamma(G_6)) = 2$, and $\gamma(\Gamma(G_{13})) \leqslant \gamma_a(\Gamma(G_{13})) = 4$. □

Theorem 5.4.10

$\gamma(\Gamma(G_9)) = 4$ and $\gamma(\Gamma(G_{15})) = 7$.

Proof

This result is proved by producing special embeddings for each of these graphs. The values of the genera of the two graphs stated above are both less than the alternating genera, namely 5 and 10. Hence the special embeddings are not alternating embeddings. The two embeddings are of minimum genus since all faces of the embeddings are of size 4. The embedding of $\Gamma(G_9)$ is shown in fig. 5.4.9, and that of $\Gamma(G_{15})$ in fig. 5.4.10. The genera of the two embeddings can be computed using Euler's formula (theorem 5.2.1). □

This leaves only G_{14} remaining. The following result will only be proved in outline since it is rather messy and does not completely solve the problem.

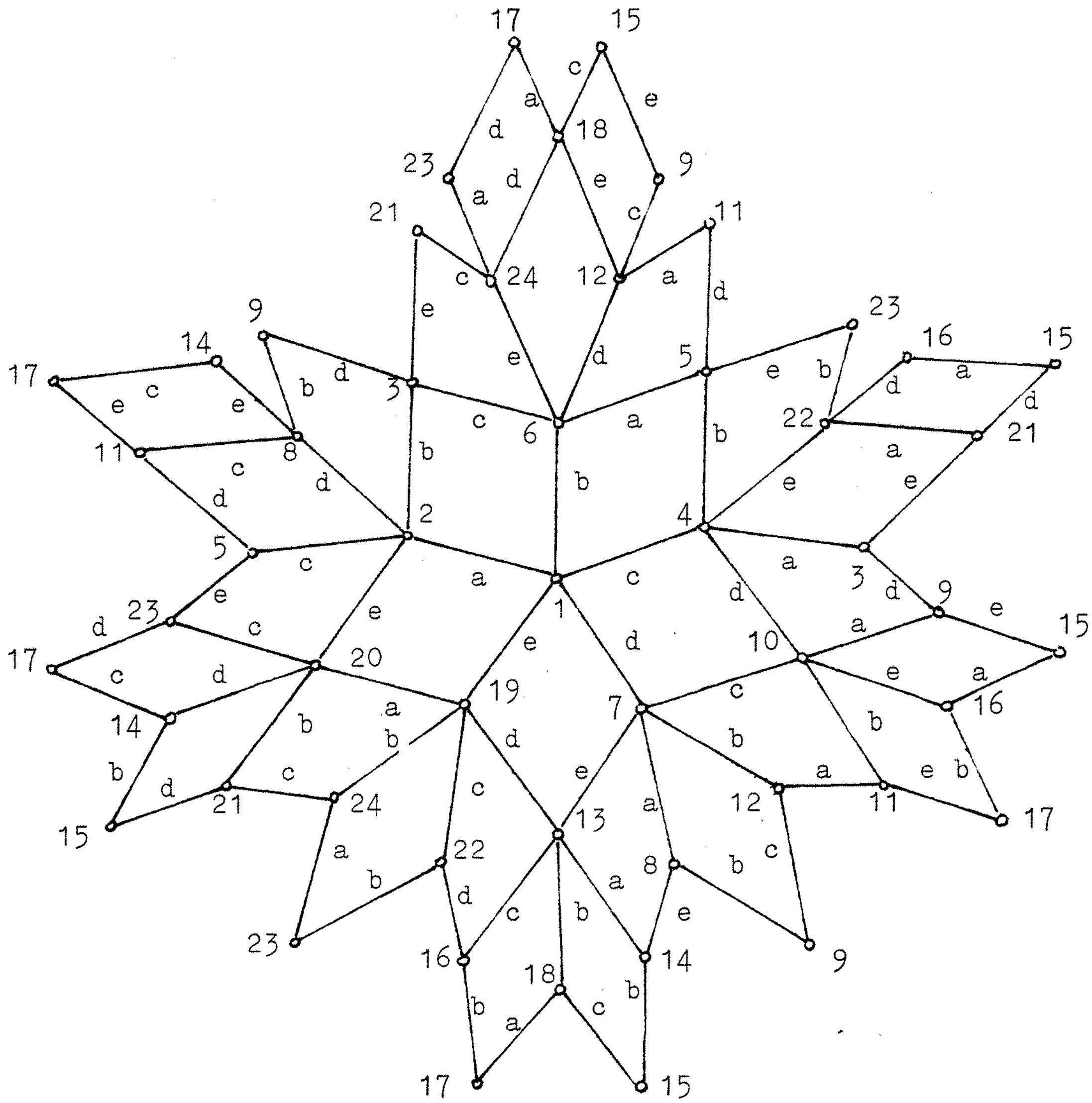
Theorem 5.4.11

$\gamma(\Gamma(G_{14})) = 5$ or 6.

Proof

If $\Gamma(G_{14})$ had an embedding with all faces of size 4, then this would be a minimum genus embedding by theorem 5.4.5, and

Figure 5.4.9

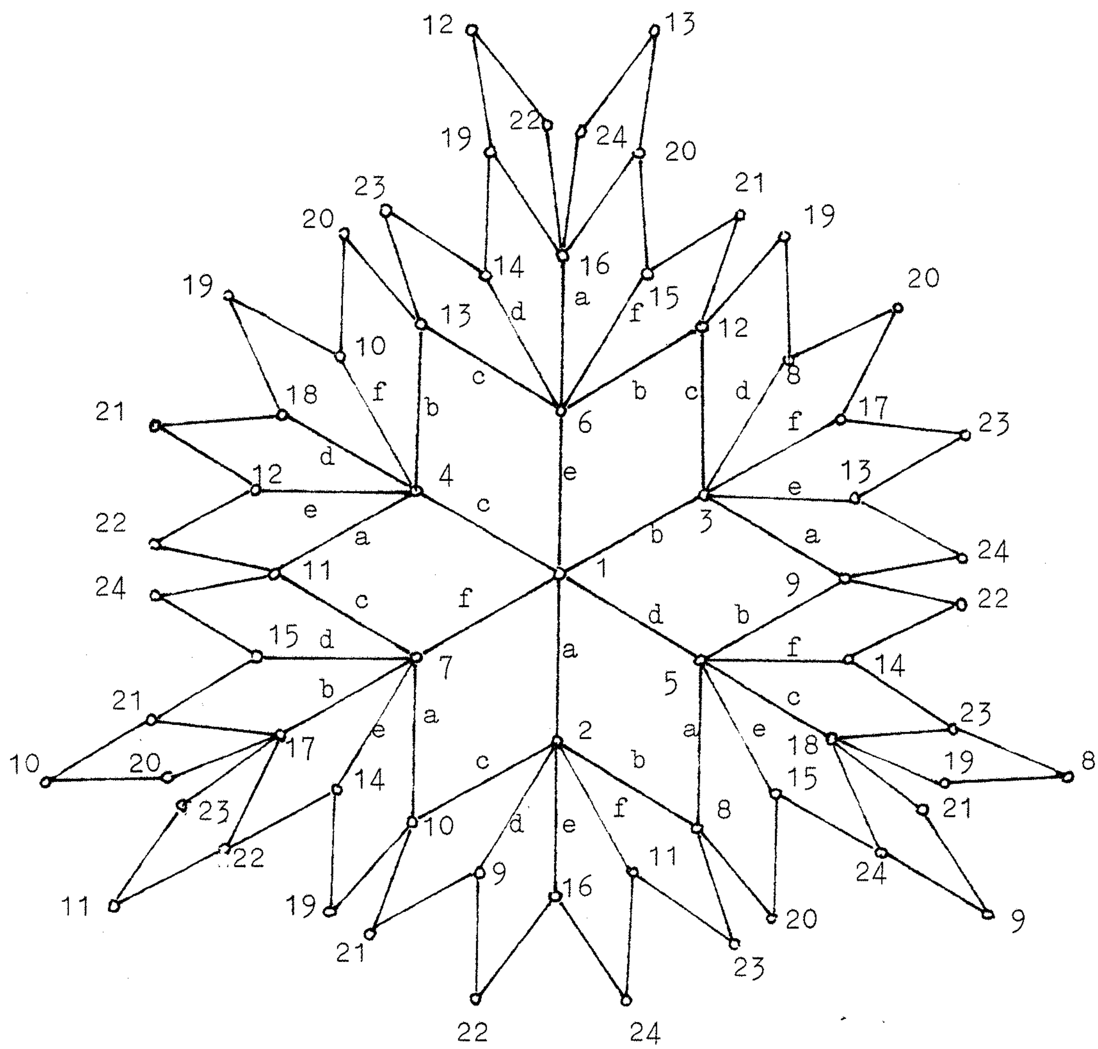


KEY

Edge labels: a = (12), b = (13), c = (23), d = (45), e = (67).

Vertex labels: 1 = (1), 2 = (12), 3 = (123), 4 = (23),
5 = (132), 6 = (13), 7 = (45), 8 = (12)(45), 9 = (123)(45),
10 = (23)(45), 11 = (132)(45), 12 = (13)(45), 13 = (45)(67),
14 = (12)(45)(67), 15 = (123)(45)(67), 16 = (23)(45)(67),
17 = (132)(45)(67), 18 = (13)(45)(67), 19 = (67), 20 = (12)(67),
21 = (123)(67), 22 = (23)(67), 23 = (132)(67), 24 = (13)(67).

Figure 5.4.10



KEY

Edge labels: a = (12), b = (13), c = (14), d = (23), e = (34), f = (24). Note that some of the edge labels have been omitted to improve clarity. They can be computed from the vertices.

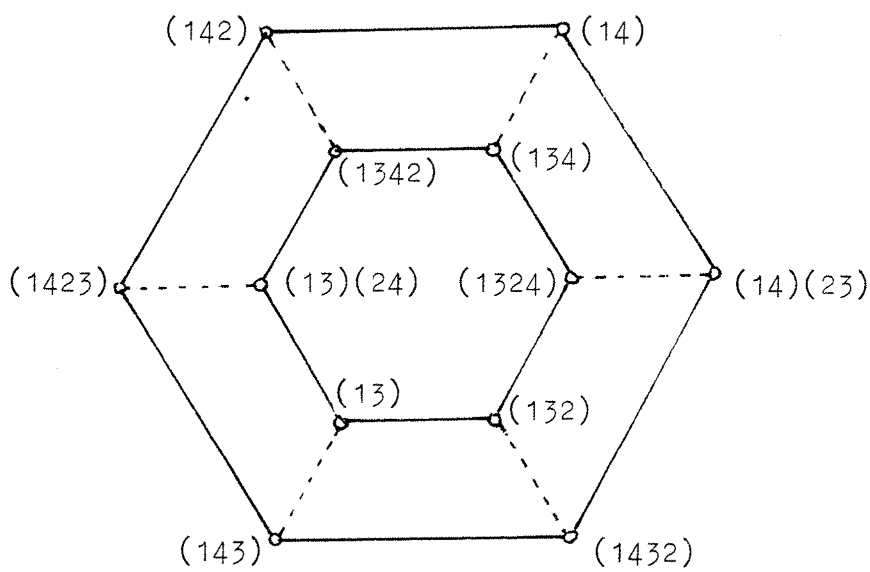
Vertex labels: 1 = (1), 2 = (12), 3 = (13), 4 = (14), 5 = (23), 6 = (34), 7 = (24), 8 = (123), 9 = (132), 10 = (124), 11 = (142), 12 = (134), 13 = (143), 14 = (234), 15 = (243), 16 = (12)(34), 17 = (13)(24), 18 = (14)(23), 19 = (1234), 20 = (1243), 21 = (1324), 22 = (1342), 23 = (1423), 24 = (1432).

this embedding would have genus 4. Hence $\gamma(\Gamma(G_{14})) \geq 4$. Further, if $\Gamma(G_{14})$ has no such embedding then $\gamma(\Gamma(G_{14})) \geq 5$. By considering a vertex of $\Gamma(G_{14})$ in such an embedding, and by examining all possible rotations of the edges incident to this vertex, it can be shown in each case that the embedding contains a Moebius strip and is hence non-orientable. In fact, the number of possible rotations is made small by symmetry and by the constraint that all five faces incident to the vertex have size 4. This 'establishes' the lower bound for the genus.

Since G_{14} is a subgraph of G_{15}, the embedding of $\Gamma(G_{15})$ in fig. 5.4.10 contains an embedding of $\Gamma(G_{14})$, which can be found by deleting all the edges of $\Gamma(G_{15})$ labelled (3 4). This procedure enlarges some of the faces, and in fact one of the faces is not simply-connected. This face can be removed and replaced by two simply-connected faces, giving an embedding of $\Gamma(G_{14})$ on a surface of genus 6. The face which is not simply-connected is shown in fig. 5.4.11. □

Figure 5.4.11

Note: the dotted edges are the deleted edges labelled (3 4).



SECTION 5.5: HAMILTONIAN CIRCUITS IN LINE GRAPH COMPLEMENTS

In section 5.4 it was shown that there is a close connection between the genus of a transposition graph $\Gamma(G)$ and the existence of a hamiltonian circuit in $\overline{L}(G)$, the line graph complement of G. In this section the existence of such circuits is investigated, and a simple necessary and sufficient condition for $\overline{L}(G)$ to be hamiltonian (provided G has $m \geqslant 34$ edges) is proved. The proof of this result depends on Chvátal's theorem on forcibly hamiltonian degree sequences (6).

Definition 5.5.1

A graph G is <u>normal</u> if it satisfies the following two conditions:

N1 : Each edge of G is non-incident to at least two others.

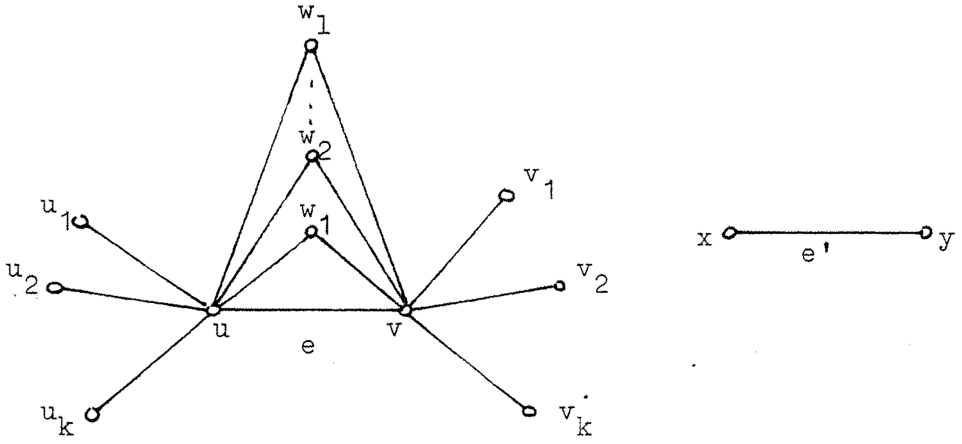
N2 : If G has m edges, then each vertex of G is incident to at most $\frac{m}{2}$ edges of G.

Proposition 5.5.1

Conditions N1 and N2 are equivalent to conditions N1' and N2, where N1' is as follows:

N1' : G is not isomorphic to the graph in fig. 5.5.1 for any values of k and l, and for any way of attaching e' to the rest of the graph in such a way that e is non-incident to e'.

Figure 5.5.1



Proof

We show that G satisfies N2 but not N1 iff G is isomorphic to the graph in fig. 5.5.1; that is, $(N2 \wedge \neg N1)$ iff $\neg N1'$. For then we have $(N1 \wedge N2)$ iff $((N1 \vee \neg N2) \wedge N2)$ iff $(\neg(N2 \wedge \neg N1) \wedge N2)$ iff $(\neg(\neg N1') \wedge N2)$ iff $(N1' \wedge N2)$, where $\neg$, $\wedge$ and $\vee$ denote the logical operations not, and and or respectively.

If G satisfies N2 but not N1, then it has some edge e, say, incident to all or all but one of the remaining edges of G. Let $e = \{u, v\}$, and if there is an edge of G not incident to e, let it be $e' = \{x, y\}$.

Suppose first that there is no such edge e', so every other edge of G is incident to either u or v. There are m-1 such edges, so if k are incident to u, then m-1-k are incident to v. Hence $d(u) + d(v) = (k + 1) + (m-1-k + 1) = m+1 > 2 \cdot \frac{m}{2}$. Hence either u or v has degree $> \frac{m}{2}$, contradicting the assertion that G satisfies N2. Hence there is an edge e' of G not incident to e. All the remaining m-2 edges of G are incident to u or v, so by a similar argument to the one above, both u and v have degree $\frac{m}{2}$. It is clear that G must be the graph in fig. 5.5.1.

Conversely, if G is isomorphic to the graph in fig. 5.5.1, then it is obvious that G satisfies N2 but not N1. □

The significance of this result is that only a very small family of graphs satisfies N1 but not N2. The significance of N1 and N2 themselves is that they are the necessary and sufficient conditions for a graph with more than 34 edges to have a hamiltonian line graph complement.

Theorem 5.5.2

If G is a graph with $m \geqslant 34$ edges, then $\overline{L}(G)$ is hamiltonian iff G is normal.

The proof of this result is rather complicated and takes up almost all of this section. The first stage of the proof is to prove the 'worst' case, where G has a vertex with degree $\frac{m}{2}$, the maximum possible degree for G to be normal. In fact, this is fairly easy to prove using Chvátal's theorem. This result is then generalised to graphs with a vertex of degree $\geqslant \left[\frac{m}{2}\right] - 4$. For graphs with maximum degree $\left[\frac{m}{2}\right] - 5$ or less, the result is proved by another method.

Proof

Lemma 5.5.3 (Chvátal)

If G is a graph on n vertices and G has degree sequence $d(v_1) \leqslant d(v_2) \leqslant \ldots \leqslant d(v_n)$, and for each $i < \frac{n}{2}$ either (i): $d(v_i) \geqslant i + 1$ or (ii): $d(v_{n-i}) \geqslant n-i$ holds, then G is hamiltonian. □

Lemma 5.5.4 (Chvátal)

If G is a bipartite graph on 2n vertices with a bipartition $V(G) = U \cup U'$ where $|U| = |U'| = n$, and if $U = \{u_1, u_2, \ldots, u_n\}$ where $d(u_1) \leqslant d(u_2) \leqslant \ldots \leqslant d(u_n)$ and $U' = \{u'_1, u'_2, \ldots, u'_n\}$ where $d(u'_1) \leqslant d(u'_2) \leqslant \ldots \leqslant d(u'_n)$, and for each $i < n$, either $d(u_i) \geqslant i + 1$ or $d(u'_{n-i}) \geqslant n - i + 1$, then G is hamiltonian. □

Proofs of these results may be found in Chvátal(6). In fact, the second result is a simple corollary of the first.

Lemma 5.5.5

If G is a graph with n vertices and has a vertex v such that $d(v) \geqslant \frac{n}{2}$, and $G - \{v\}$ is hamiltonian, then G is hamiltonian.

Proof of lemma 5.5.5

$G - \{v\}$ has $n - 1$ vertices, so v is adjacent to more than half of its vertices. Hence v is adjacent to two consecutive vertices in the hamiltonian circuit in $G - \{v\}$, u and u', say. Deleting the edge $\{u, u'\}$ and inserting the edges $\{u, v\}$ and $\{u', v\}$ gives a hamiltonian circuit in G. □

Lemma 5.5.6

If G is a hamiltonian graph, then there is no non-empty set of vertices $V' \subset V(G)$ such that $G - V'$ has more than $|V'|$ connected components.

Proof

G contains a spanning circuit C. Deleting one vertex from C clearly leaves a connected graph. Deleting a further vertex leaves either a path, or two disjoint paths. It is clear that deleting k vertices from C leaves at most k disjoint components. G will certainly have no more components than C after these deletions, so the result holds. □

Using these lemmas it is now possible to prove half of theorem 5.5.2 .

Theorem 5.5.2 (first half)

If G is a graph such that $\overline{L}(G)$ is hamiltonian then G is normal.

Proof

Each vertex of $\overline{L}(G)$ must have degree 2 or more for a circuit to pass through it. Hence each edge of G must be non-incident to at least two others. Hence G satisfies N1.

If some vertex of G is incident to $k > \frac{m}{2}$ edges, then these edges are all incident to one another and hence are all non-adjacent vertices of $\overline{L}(G)$. Let E' denote the set of edges of G not incident to this vertex. Then $|E'| = m - k < \frac{m}{2}$. Considering these edges as vertices of $\overline{L}(G)$, $\overline{L}(G) - E'$ consists of k mutually non-adjacent vertices, and hence has $k > |E'|$ components. This contradicts the hypothesis that $\overline{L}(G)$ is hamiltonian, by lemma 5.5.6. Hence G has no such vertex and G satisfies N2. Hence G is normal. □

Notice that this half of the proof is trivial, depending on only one straightforward lemma. It is very surprising that such weak conditions as N1 and N2 should turn out to be sufficient conditions for a graph to have a hamiltonian line graph complement, provided it has sufficiently many edges.

Lemma 5.5.7

If G is a graph with n vertices and m edges, and G has a vertex v with degree $d(v) = k = \frac{m}{2}$, and $k \geqslant 6$, then $\overline{L}(G)$ is hamiltonian.

Proof

Let E be the set of edges of G incident to v, and let E' be the set of edges not incident to v, so $|E| = |E'| = k$. Let $E = \{e_1, e_2, \ldots, e_k\}$ and let $E' = \{e_1', e_2', \ldots, e_k'\}$. Let H be the bipartite graph with vertex set E(G) and with an edge $\{e, e'\}$ iff $e \in E$, $e' \in E'$ and e is not incident to e' in G. H is clearly a subgraph of $\overline{L}(G)$, and if H is hamiltonian then $\overline{L}(G)$ is also. Thus we suppose that H is not hamiltonian.

If d(e) is the degree of e as a vertex of H, and if $d(e_1) \leqslant d(e_2) \leqslant \ldots \leqslant d(e_k)$ and $d(e_1') \leqslant d(e_2') \leqslant \ldots \leqslant d(e_k')$ then by lemma 5.5.4, since by assumption H is not hamiltonian, there is some $i < k$ such that $d(e_i) \leqslant i$ and $d(e_{k-i}') \leqslant k-i$.

If $e' \in E'$ then e' is not incident to v so e' is incident to at most two edges in E. Hence $d(e') \geqslant k-2$ for all $e' \in E'$. Hence $k-i \geqslant d(e_{k-i}') \geqslant k-2$, so $i \leqslant 2$, and i = 1 or 2.

If i = 1, then $d(e_1) \leqslant 1$. Now e_1 is not adjacent in $\overline{L}(G)$ to any $e \in E$, since they are both incident to v in G. Hence every edge of H incident to e_1 is also an edge of $\overline{L}(G)$. It follows that e_1 has degree $\leqslant 1$ in $\overline{L}(G)$, and hence N1 does not hold for G, giving a contradiction. Thus $i = 2$. In this case, $d(e_1) = d(e_2) = 2$ and $d(e'_{k-2}) \geqslant k-2$.

Now partition E' into four sets:

$E'(e_1, e_2) = \{e' \in E' : e' \sim e_1 \text{ and } e' \sim e_2 \text{ in } H\}$,

$E'(e_1) = \{e' \in E' : e' \sim e_1 \text{ and } e' \not\sim e_2\}$,

$E'(e_2) = \{e' \in E' : e' \sim e_2 \text{ and } e' \not\sim e_1\}$,

$E'(.) = \{e' \in E' : e' \not\sim e_1 \text{ and } e' \not\sim e_2\}$.

Also, let $|E'(e_1, e_2)| = a$, $|E'(e_i)| = b_i$ for $i = 1, 2$ and $|E'(.)| = c$.

There is at most one element of $E'(.)$ since at most one edge of G in E' can be incident to both e_1 and e_2. Hence $c \leqslant 1$.

Also, $d(e_i) = a + b_i$, so $a + b_1 = a + b_2 = 2$.

$$\begin{aligned} \text{Finally, } k &= a + b_1 + b_2 + c \\ &\leqslant (a + b_1) + (a + b_2) + c \\ &\leqslant 2 + 2 + 1 = 5. \end{aligned}$$

It follows that if $k \geqslant 6$ then $\overline{L}(G)$ is hamiltonian. □

Before extending this result to graphs G with maximum degree less than $\frac{m}{2}$ it is necessary to deal with an exceptional family of graphs with maximum degree $\frac{m}{2} - 1$.

Lemma 5.5.8

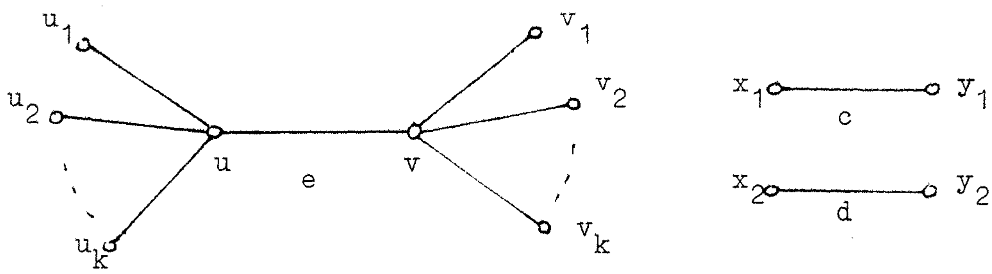
If G is isomorphic to the graph in fig. 5.5.2 then $\overline{L}(G)$ is hamiltonian, provided $k \geqslant 4$

Proof

Let $e = \{u, v\}$, $a_i = \{u, u_i\}$, $b_i = \{v, v_i\}$; $i = 1, \ldots, k$, $c = \{x_1, y_1\}$ and $d = \{x_2, y_2\}$. Let $A = \{a_i\}$ and let $B = \{b_i\}$

In $\overline{L}(G)$, each vertex a_i is adjacent to at least k-1 vertices of B, and if $a_{i_1} \sim b_j$ and $a_{i_2} \sim b_j$ then $i_1 = i_2$. Similar facts hold for each vertex b_j .

Figure 5.5.2



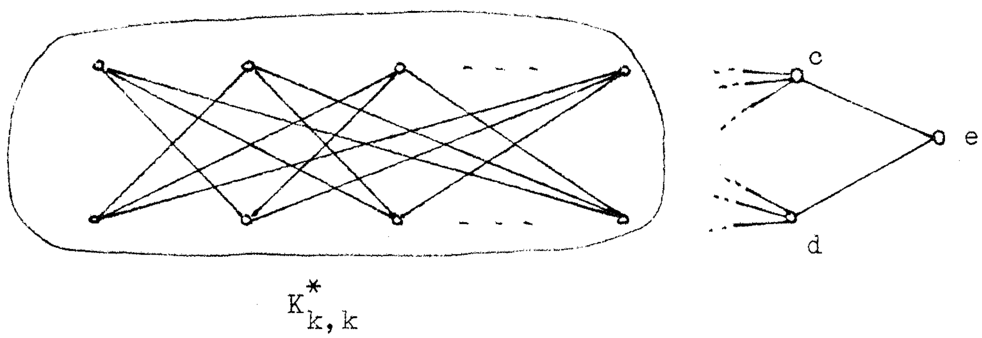
In fig. 5.5.2, all the vertices u_i are distinct from one another, as are the vertices v_i. Also, all the edges are distinct from one another, and x_1, y_1, x_2, y_2 are distinct from u and v.

By the argument on the previous page, the subgraph of $\overline{L}(G)$ induced by the vertices in A and B contains $K^*_{k,k}$, the graph consisting of $K_{k,k}$ with k mutually disjoint edges removed, as a spanning subgraph.

$K^*_{k,k}$ is a hamiltonian, edge-transitive graph and hence it contains a hamiltonian circuit containing any given edge, provided $k \geqslant 3$.

Also, in $\overline{L}(G)$, $e \sim c$ and $e \sim d$, and c and d may be adjacent to certain vertices in A and B. Hence $\overline{L}(G)$ contains a spanning subgraph isomorphic to the graph in fig. 5.5.3.

Figure 5.5.3



Now c is incident to at most 2 edges of A in G, and hence c is adjacent to at least k-2 vertices of A in $\overline{L}(G)$. But $k \geqslant 4$, so c is adjacent to at least 2 vertices of A. Similarly, d is adjacent to at least 2 vertices of B. Hence there is an edge $\{a_i, b_j\}$ in the subgraph isomorphic to $K^*_{k,k}$ such that $a_i \sim c$ and $b_j \sim d$ in $\overline{L}(G)$.

A hamiltonian circuit in $\overline{L}(G)$ is found by choosing a hamiltonian circuit in $K^*_{k,k}$ containing the edge $\{a_i, b_j\}$, deleting this edge and adding the edges $\{a_i, c\}$, $\{c, e\}$, $\{e, d\}$ and $\{d, b_j\}$. □

Lemma 5.5.9

If G is a normal graph with m edges and maximal degree $k = n - a$, where $n = \left[\frac{m}{2}\right]$, $a \geqslant 0$ if m is odd, $a \geqslant 1$ if m is even, and $n \geqslant 2a + 9$, and if u is a vertex of degree k, then there is some edge e' of G which is not incident to u, and is incident to at most n - 1 other edges of G.

Proof

Let v be a vertex of G such that $l = d(v) \geqslant d(v')$ for all vertices $v' \neq u$. Thus v has the second highest degree of all vertices of G.

Let E(u) be the set of edges of G incident to u, and let E(v) be similarly defined. Let $E(u,v) = E(u) \cup E(v)$, let $E'(u) = E(G) - E(u)$, and let $E'(u,v) = E(G) - E(u,v)$.

Now $|E(u,v)| = |E(u)| + |E(v)| - |(E(u) \cap E(v))|$, and there is at most one edge incident to both u and v. Hence $k + l - 1 \leqslant E(u,v) \leqslant k + l$.

Also, $m = 2n$ or $2n + 1$, and $|E'(u,v)| = m - |E(u,v)|$, so if $m = 2n + 1$, then substituting $k = n - a$ we obtain $n + a - l + 1 \leqslant |E'(u,v)| \leqslant n + a - l + 2$, and if $m = 2n$ then

$n + a - 1 \leqslant |E'(u,v)| \leqslant n + a - 1 + 1.$

If $m = 2n + 1$, then since $1 \leqslant k = n - a$, $E'(u,v) > 0$. If $m = 2n$, then $a \geqslant 1$, so $1 \leqslant k = n - a < n + a$, so in either case, $E'(u,v)$ is non- empty. Let $e' \in E'(u,v)$.

Since e' is not incident to u, e' is incident to at most two edges in $E(u)$. Also, e' is clearly incident to at most $|E'(u,v)|-1$ edges in $E'(u,v)$. Finally, e' is incident to at most two edges in $E(v)$. Hence whether m is odd or even, e' is incident to at most $4+|E'(u,v)|-1$ other edges of G, and taking the largest upper bound for $|E'(u,v)|$, e' is incident to at most $4 + (n + a - 1 + 2) - 1 = n + a - 1 + 5$ other edges of G. Hence the result holds unless $1 \leqslant a + 5$, which we now suppose to be the case.

For any end vertex v' of e', $d(v') \leqslant d(v) \leqslant a + 5$ since e' is not incident to u. Hence e' is incident to at most $2(a + 5 - 1) = 2a + 8 \leqslant n-1$ other edges of G, since by hypothesis $n \geqslant 2a + 9$. Hence the result holds. □

<u>Lemma 5.5.10</u>

If G is a normal graph with m edges and maximum degree $n - a$, where $a \geqslant 0$ and $n = \left[\frac{m}{2}\right]$ and $n \geqslant 2a + 9$, then $\overline{L}(G)$ is hamiltonian.

<u>Proof</u>

The proof is by induction on a.
If $a = 0$ and $m = 2n$, then $n \geqslant 9 \geqslant 6$ and the result follows by lemma 5.5.7 .

If $a = 0$ and $m = 2n+1$ then consider the edge e' whose existence was proved in lemma 5.5.9. If $G' = G - \{e'\}$ is a normal graph then it has maximum degree n and 2n edges, and $n \geqslant 9 \geqslant 6$, so $\overline{L}(G')$ is hamiltonian by lemma 5.5.7. Also, e' is incident to at most n-1 edges of G so it is adjacent to at least n vertices of $\overline{L}(G)$. Hence $\overline{L}(G)$ is hamiltonian by lemma 5.5.5 .

If G' is not normal, then either it has some vertex w, say, with degree $d(w) > n$ or some edge e, say, incident to all but one of the other edges of G. (If e were incident to all the other edges, then it would be incident to all but one of the edges of G, contradicting the fact that G is normal.)

However, if w has degree $\geqslant n+1$ in G', then it has degree n+1 or more in G, contradicting the hypothesis that G has maximum degree $\leqslant n$. Also, if G' has an edge e incident to all but one edges of G' then since G' satisfies condition N2, G' is the graph in fig. 5.5.1. Also, e' cannot be incident to e since G is normal, so G is isomorphic to the graph in fig. 5.5.2, which is hamiltonian by lemma 5.5.8.

Now suppose that the result is true for all $a \leqslant a_0$, and let $a = a_0 + 1$. Suppose first that $m = 2n$. By lemma 5.5.9 there is an edge $e' \in E'(u)$ incident to at most n-1 other edges of G. Now $G' = G - \{e'\}$ has $2n - 1 = 2(n - 1) + 1$ edges and maximum degree $n - a = (n - 1) - (a - 1) = (n - 1) - a_0$. Also, $n \geqslant 2a + 9$, hence $(n - 1) \geqslant 2a + 8 = 2(a - 1) + 10 \geqslant 2a_0 + 9$. Finally, G' is normal, since each edge of G is incident to at most $2(n - a - 1)$ other edges of G, and hence is non-incident to at least $(2n - 1) - (2n - 2a - 2) = 2a + 1 \geqslant 3$ edges of G. It follows that each edge of G' is non-incident to at least $3 - 1 = 2$ edges of G'. Hence by the induction hypothesis, $\overline{L}(G')$ is hamiltonian, and hence $\overline{L}(G)$ is hamiltonian by lemma 5.5.5.

Finally, if $m = 2n + 1$, then $G' = G - \{e'\}$ has 2n edges and maximum degree $n - a$. By a similar argument to the one above, G' is normal. Hence $\overline{L}(G')$ is hamiltonian by the arguments for $m = 2n$. It follows that $\overline{L}(G)$ is hamiltonian. This completes the proof of lemma 5.5.10 .

<u>Lemma 5.5.11</u>

If G is a normal graph with $m \geq 34$ edges and maximum degree $n - a$, where $n = \left\lceil \frac{m}{2} \right\rceil$ and $a \leq 4$, then $\overline{L}(G)$ is hamiltonian.

<u>Proof</u>

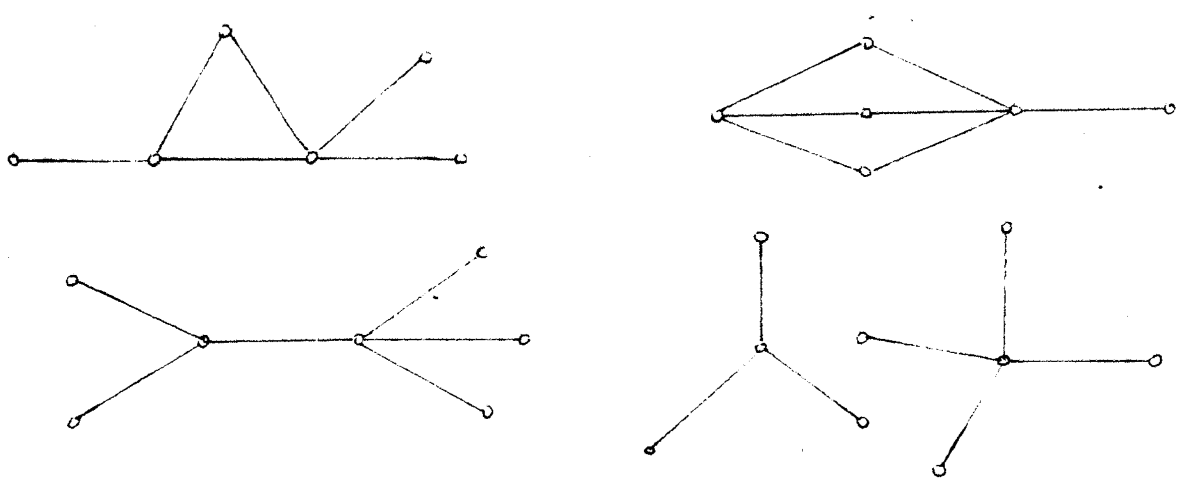
If $m \geq 34$ then $n \geq 17 = 8 + 9 \geq 2a + 9$. Hence the result follows immediately from lemma 5.5.10. □

<u>Definition 5.5.2</u>

If G_1 and G_2 are graphs, then the <u>sum</u> of G_1 and G_2, $G_1 + G_2$, is any graph which is obtained by identifying a number of pairs v_1, v_2 of vertices, where $v_i \in V(G_i)$ for $i = 1, 2$.

For example, if $G_1 = K_{1,3}$ and $G_2 = K_{1,4}$ then $G_1 + G_2$ could be any of the graphs in fig. 5.5.4, as well as other possible graphs.

<u>Figure 5.5.4</u> Some possibilities for $K_{1,3} + K_{1,4}$.



This definition is needed in the proof of theorem 5.5.2 in the case where G has no vertices with degree $\geq \left\lceil \frac{m}{2} \right\rceil - 4$. The following general lemma is also needed.

Lemma 5.5.12

If G is a graph with at least 6 vertices, with no isolated vertices, and with no three mutually non-incident edges, then $G = K_{1,m}$, $K_{1,m} + K_{1,n}$, $K_{1,m} + K_3$ or is the graph consisting of two disjoint copies of K_3.

Proof

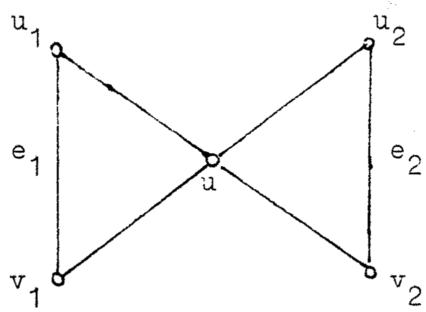
Note that all these graphs do have the required properties. The proof consists of showing that there are no other such graphs.

Suppose that G has c components. No component consists only of an isolated vertex, so if $c \geqslant 3$, then G contains three mutually disjoint edges. Hence $c \leqslant 2$. If $c = 2$, then if one of these components contains two disjoint edges, then taking any edge from the second component gives three mutually disjoint edges. Hence every edge in each component of G is incident to every other edge. Thus each component of G is either K_3 or $K_{1,m}$ for some value of m. Hence the result holds if G has 2 components.

The remaining possibility is that G has one component, and is connected. Let u be a vertex of maximum degree in G, and let $d(u) = d$. If $G' = G - \{u\} = K_3$ or $K_{1,m}$ then $G = K_{1,d} + K_3$ or $K_{1,d} + K_{1,m}$ and the result follows. Hence we assume that G' contains a pair of non-incident edges $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$. If $d(u) \geqslant 5$, then there is some vertex v of G adjacent to u which is distinct from u_1, v_1, u_2, v_2. But then $e = \{u, v\}$, e_1 and e_2 are mutually distinct. Hence $d(u) \leqslant 4$.

If $d(u) = 4$, then by the same argument, u must be adjacent to each of the vertices u_1, v_1, u_2, v_2. Hence the graph in fig. 5.5.5 is a subgraph of G. G is connected and has six vertices, so at least one of $u_1, \ldots, v_2$ is adjacent to some other vertex v of G. By symmetry, this vertex may be taken to be u_1.

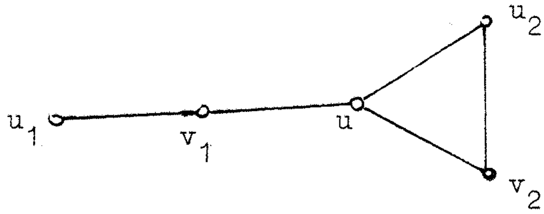
Figure 5.5.5



But in this case, $\{v, u_1\}$, $\{v_1, u\}$, and $\{u_2, v_2\}$ form a set of three mutually disjoint edges in G.

If $d(u) = 3$, then u is adjacent to three of the vertices $u_1, \ldots, v_2$. Let u_1 be the vertex not adjacent to u. Then G contains the graph in fig. 5.5.6 as a subgraph.

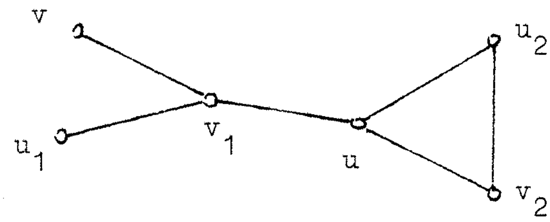
Figure 5.5.6



G is connected and has at least six vertices so there is another vertex of G adjacent to one of $u_1, \ldots, v_2$. There are essentially three possibilities: $v \sim u_1$, $v \sim u_2$ or $v \sim v_1$. If $v \sim u_1$, then the same three edges as in the case $d(u) = 4$ are mutually disjoint. If $v \sim u_2$ then $\{v, u_2\}$, $\{v_2, u\}$ and $\{v_1, u_1\}$ are mutually disjoint. Finally, if $v \sim v_1$ then G contains the graph in fig. 5.5.7. If G has no other vertices or edges then $G = K_{1,3} + K_3$. If G has another edge not incident to v_1 then it is easy to see that in every possible case G contains three mutually disjoint edges. If G has a number of other edges incident only to v_1 then $G = K_{1,m} + K_3$ for some m.

This completes the proof of lemma 5.5.12. $\square$

Figure 5.5.7



Lemma 5.5.13

If G is normal and has $m \geqslant 34$ edges and has maximum degree $n - a$ where $n = \left\lceil \frac{m}{2} \right\rceil$ and $a \geqslant 5$, then $\overline{L}(G)$ is hamiltonian.

Proof

The proof of this lemma is rather long, so to make it more readable it has been split into a number of shorter sublemmas.

Suppose that G is a graph satisfying the hypotheses of lemma 5.5.13, and that $\overline{L}(G)$ is not hamiltonian. Then by the contrapositive of lemma 5.5.3 there is some set of edges $E' < E(G)$ such that each edge $e' \in E'$ is independent of at most k other edges of G, where $k = |E'| < \frac{m}{2}$. Let H be the subgraph of G induced by the edges in E'.

The following sublemmas describe the structure of H.

Sublemma 1

H has no three mutually disjoint edges

Proof of sublemma 1

Suppose that e_1', e_2', and e_3' are mutually disjoint edges of G and are all elements of E'. Partition E(G) into the following subsets:

$E_1 = \{e_1', e_2', e_3'\}$,

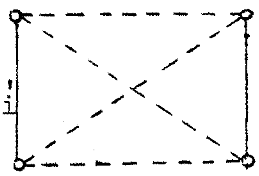
$E(e_i') = \{$ e: e is incident to e_i' but not to the other two edges in $E_1 \}$; i = 1, 2, 3 ,

$E(e_i', e_j') = \{$ e : e is incident to e_i and e_j but not to the third edge in $E_1 \}$

$E(.) = \{e : e$ is not incident to any edge in $E_1\}$.
Note that since the edges in E_1 are mutually disjoint, no edge of G can be incident to all three of them.

Let $|E(e_i')| = d_i$, let $|E(e_i', e_j')| = b_{i,j}$ and let $|E(.)| = c$. Figure 5.5.8 shows that there are at most 4 edges incident to both e_i' and e_j', so $b_{i,j} \leqslant 4$.

Figure 5.5.8



Now e_1' is non-incident to e_2', e_3', and to each edge in $E(e_2')$, $E(e_3')$, $E(e_2', e_3')$ and $E(.)$. Hence e_1' is adjacent to $2 + d_2 + d_3 + b_{2,3} + c$ vertices of $\overline{L}(G)$. Similarly,
$d(e_2') = 2 + d_1 + d_3 + b_{1,3} + c$ and
$d(e_3') = 2 + d_1 + d_2 + b_{1,2} + c$.

However, $d(e_i') \leqslant k < \frac{m}{2}$ and $m = 3 + d_1 + d_2 + d_3 + b_{1,2} + b_{1,3} + b_{2,3} + c$.

Hence $2m = 6 + 2d_1 + 2d_2 + 2d_3 + 2b_{1,2} + 2b_{1,3} + b_{2,3} + 2c$
$= d(e_1') + d(e_2') + d(e_3') + b_{1,2} + b_{1,3} + b_{2,3} - c$
$\leqslant \frac{m}{2} + \frac{m}{2} + \frac{m}{2} + 4 + 4 + 4 - 0$, maximising every term but the last, and minimising c.

Hence simplifying this expression, $m \leqslant 24 < 34$, which contradicts the hypothesis that $m \geqslant 34$. □

Hence by lemma 5.5.12, H has less than 6 vertices or else $H = K_{1,m} + K_{1,n}$, $K_{1,m} + K_3$ or $K_3 + K_3$, or $H = K_{1,m}$.

<u>Sublemma 2</u>

If $H \not\cong K_{1,r}$ or K_3 then $k = |E'| \geqslant \frac{m}{2} - 2$.

<u>Proof</u>

If H is not isomorphic to one of these graphs then H must contain two disjoint edges e_1' and e_2' . Partition E(G) as $E(G) = E(.) \cup E(e_1') \cup E(e_2') \cup E(e_1', e_2')$, where E(.) is the set of edges of G incident to neither e_1' nor e_2' , and so on. Let $d_i = |E(e_i')|$ for i = 1, 2 , let $c = |E(.)|$ and let $b = |E(e_1', e_2')|$. As before, $b \leqslant 4$.

Hence in $\overline{L}(G)$, $d(e_1') = 1 + d_2 + c$ and $d(e_2') = 1 + d_1 + c$. Also, since $e_i' \in E'$, $d(e_i') \leqslant k$.

Thus $m = c + d_1 + d_2 + b + 2$

$$\leqslant 2c + d_1 + d_2 + 4 + 2 = d(e') + d(e') + 4$$

$$\leqslant 2k + 4.$$

Hence the result follows.

<u>Sublemma 3</u>

If $H \cong K_{1,r}$ for some r, then $k \geqslant \frac{m}{2} - 1$.

<u>Proof</u>

Let u be the vertex of degree r in H, and partition E(G) as follows: $E(G) = E' \cup E_1 \cup E_2$, where E' is as always the set of edges generating H , E_1 is the set of edges of G incident to u but not in E', and E_2 is the set of edges not incident to u. Note that E' and E_2 are disjoint sets since every edge in E' is incident to u by hypothesis. Let $|E_1| = b$ and let $|E_2| = c$; $|E'| = k$ by definition.

Consider the number of incidences in G between edges in E' and edges in E_2, and let this number be s, say. Every edge in E' is by definition non-incident to at most k edges of E_2, so it is incident to at least c-k edges of E_2. Hence $s \geqslant k(c-k)$.

Since $m = b + c + k$, $s \geqslant k(m - 2k - b)$.

However, each edge $e_2 \in E_2$ is not incident to u and hence is incident to at most 2 edges which are incident to u. Since every edge in E' is incident to u, it follows that e_2 is incident to at most 2 edges of E'. Hence $s \leqslant 2c$, and after substituting for c, $s \leqslant 2(m - k - b)$. Combining this with the first inequality gives $k(m - 2k - b) \leqslant 2(m - k - b)$, and after some algebra, we have $(k - 2)(m - b) \leqslant 2k^2 - 2k$.

Note that if $k = 2$ then this inequality holds. This case must be considered separately. First suppose that $k \geqslant 3$.

Note that $b + k \leqslant \frac{m}{2} - 5$ since $b + k$ is the degree of u which is at most $\frac{m}{2} - 5$. Substituting in the previous inequality, we have

$$(k - 2)(m - (\tfrac{m}{2} - 5 - k)) \leqslant (k - 2)(m - b) \leqslant 2k^2 - 2k .$$

Afterfurther manipulation, this gives

$$m \leqslant \frac{2(k^2 - 5k + 10)}{(k - 2)} = 2k - 6 + \frac{8}{(k - 2)}$$

$$\leqslant 2k - 6 + 8 \text{ since } k \geqslant 3 \text{ so } \frac{1}{(k - 2)} \leqslant 1 .$$

Hence if $k \geqslant 3$, then $k \geqslant \frac{m}{2} - 1$.

If $k = 2$, then let $E' = \{e_1', e_2'\}$ and partition E(G) as before in this proof. Also, partition E_2 as follows:

$E_2 = E_2(.) \cup E_2(e_1') \cup E_2(e_2') \cup E_2(e_1', e_2')$, where $E_2(.)$ is the set of edges in E_2 incident to neither e_1' nor e_2', and so on. Let $|E_2(.)| = d_1$, $|E_2(e_1')| = d_2$, $|E_2(e_2')| = d_3$ and $|E_2(e_1', e_2')| = d_4$. Then $c = d_1 + d_2 + d_3 + d_4$.

Also, the degrees of e_1' and e_2' in $\overline{L}(G)$ are given by $d(e_1') = d_1 + d_3 \leqslant 2$, and $d(e_2') = d_1 + d_2$, since $k \leqslant 2$. Finally, $d_4 \leqslant 1$ since only one edge not incident to u may be incident to both e_1' and e_2'. Hence

$$\begin{aligned} c &= d_1 + d_2 + d_3 + d_4 \\ &\leqslant 2d_1 + d_2 + d_3 + d_4 \\ &\leqslant 2 + 2 + 1 = 5. \end{aligned}$$

However, G is normal so at least half of its edges are not incident to u. Hence $c \geqslant \frac{m}{2}$, so $m \leqslant 10$, which contradicts the hypothesis that $m \geqslant 34$. Hence $k \neq 2$.

This leaves only the case $k = 1$; but if this is the case then G has an edge which is non-incident to at most one other edge of G. This contradicts the fact that G is normal. This completes the proof of sublemma 3. □

Sublemma 4

$H \not\cong K_3$.

Proof

If $H = K_3$ then $E' = \{e_1', e_2', e_3'\}$. No edge of G not in E' can be incident to all three edges in E', since they form a subgraph isomorphic to K_3 by hypothesis. Also, no edge of G can be incident to only one of them, for the same reason. Hence E(G) may be partitioned as

$$E(G) = E' \cup E(e_1', e_2') \cup E(e_2', e_3') \cup E(e_3', e_1') \cup E(.)$$

where E(.) is the set of edges of G incident to none of the edges of E', $E(e_1', e_2')$ is the set of edges incident to e_1' and e_2' but not to e_3', and so on. Let $|E(e_i', e_j')| = b_{i,j}$ and let $|E(.)| = c$.

The degree of e_1' in $\overline{L}(G)$ is given by $d(e_1') = b_{2,3} + c \leqslant 3$. Similar formulae hold for e_2' and e_3' .

$$\begin{aligned} \text{Also, } m &= 3 + b_{1,2} + b_{2,3} + b_{3,1} + c \\ &\leqslant 3 + 3c + b_{1,2} + b_{2,3} + b_{3,1} \\ &\leqslant 3 + d(e_1') + d(e_2') + d(e_3') \leqslant 12. \end{aligned}$$

This contradicts the fact that $m \geqslant 34$ by hypothesis.

Sublemma 5

H has some vertex incident to at least $\frac{m}{4} - 1$ edges of H.

Proof of sublemma 5

By sublemma 1 and lemma 5.5.12, $H = K_{1,r}$, K_3, $K_{1,r} + K_{1,s}$, $K_{1,r} + K_3$, $K_3 + K_3$, or H has $\leqslant 5$ vertices. By sublemma 2, $H = K_{1,r}$ or K_3 or has $k \geqslant \frac{m}{2} - 2 \geqslant 15 > 6$ vertices. Thus $H = K_{1,r} + K_{1,s}$, $K_{1,r} + K_3$ or $K_{1,r}$.

By sublemmas 3 and 4, in each of these cases, H has at least $\frac{m}{2} - 2$ edges.

In each of the remaining cases, the central vertex of the (larger) star is incident to more than half the edges of H. The result follows. □

With these sublemmas it is possible to prove lemma 5.5.13.

Proof of lemma 5.5.13

By sublemma 5, H has some vertex u incident to at least $\frac{m}{4} - 1$ edges of G. Let E'(u) be the set of edges of E' incident to u, let E(u) be the remaining edges of G incident to u, and let E* be the set of all edges of G not incident to u. Note that E* will contain edges in E' and in E - E'.

Let $|E'(u)| = b$, $|E(u)| = c$ and $|E^*| = d$. The following equations all hold:

(1): $b \geqslant \frac{m}{4} - 1$ by sublemma 5 ;

(2): $b + c \leqslant n - a$, where $n = \left\lceil \frac{m}{2} \right\rceil$ and $a \geqslant 5$ by hypothesis ;

(3): $b + c + d = m$;

(4): $a \geqslant 5$;

(5): $m \geqslant 34$;

(6): $c \geqslant 0$.

Let e be the edge in E'(u) incident to the fewest edges in E*, and let s be the number of such edges. Since $e \in E'$, e is non-incident to at most k edges of G, so $d - s \leqslant k$, since e is non-incident only to edges in E*. However, $d = m - b - c$ so

$b + c + s \geqslant m - k$. Also, $k < \frac{m}{2}$ so $k \leqslant \left\{\frac{m}{2}\right\} - 1$ and

$b + c + s \geqslant \left[\frac{m}{2}\right] + 1$ ...(7).

Now consider the number of incidences between E'(u) and E*; this gives the inequality $2d \geqslant bs$ ...(8) , since each edge of E* is incident to at most 2 edges of E'(u), and each edge of E'(u) is incident to at least s edges of E*, by the definition of s.

(2) and (7) imply $s \geqslant a + 1$

$\geqslant 6$ by (4).

(3) and (8) imply that $2(m - b - c) \geqslant bs$, but by (7),

$b + c \geqslant \left[\frac{m}{2}\right] + 1 - s$, hence

$$2\left(m - \left(\left[\frac{m}{2}\right] + 1 - s\right)\right) \geqslant 2(m - b - c) \geqslant bs$$

$$\geqslant \left\{\frac{m}{4} - 1\right\} s \text{ by (1).}$$

Hence $2\left(\left\{\frac{m}{2}\right\} - 1 + s\right) \geqslant \frac{ms}{4} - s$.

If m is odd, then $2\left\{\frac{m}{2}\right\} = m + 1$, so $m + 2s - 1 \geqslant \frac{ms}{4} - s$

hence $m \leqslant \frac{4(3s - 1)}{s - 4}$ since $s \geqslant 6$ by an earlier inequality.

Hence $m \leqslant 12 + \frac{44}{s - 4} \leqslant 12 + \frac{44}{2} = 34$.

Since m is odd, we conclude that $m \leqslant 33$, which contradicts equation (5).

If m is even then $2\left\{\frac{m}{2}\right\} = m$, so $m + 2s - 2 \geqslant \frac{ms}{4} - s$.

After some manipulation this gives $m \leqslant 12 + \frac{40}{s-4} \leqslant 12 + \frac{40}{2} = 32$.

Again this contradicts the hypothesis that $m \geqslant 34$.

This completes the proof of lemma 5.5.13. □

Theorem 5.5.2 now follows from this result and from lemma 5.5.11. □

The proof of theorem 5.5.2 is very complicated, particularly the second half. However, the proof does not involve any particularly advanced ideas. The major weakness of the result is the value of m which must be assumed, namely $m \geqslant 34$. This seems to be the smallest value of m which can be obtained by the present method of proof, but it is probably far larger than is necessary. A number of examples are known of small graphs which are normal but do not have hamiltonian line graph complements. The largest of these has 10 edges, namely K_5. It is conjectured that this is the largest normal graph which does not have a hamiltonian line graph complement. The 'canonical' graphs are listed below. A number of other graphs can be obtained by modifying these graphs to produce further examples.

<u>Figure 5.5.9</u> Canonical normal graphs with non-hamiltonian LGC's.

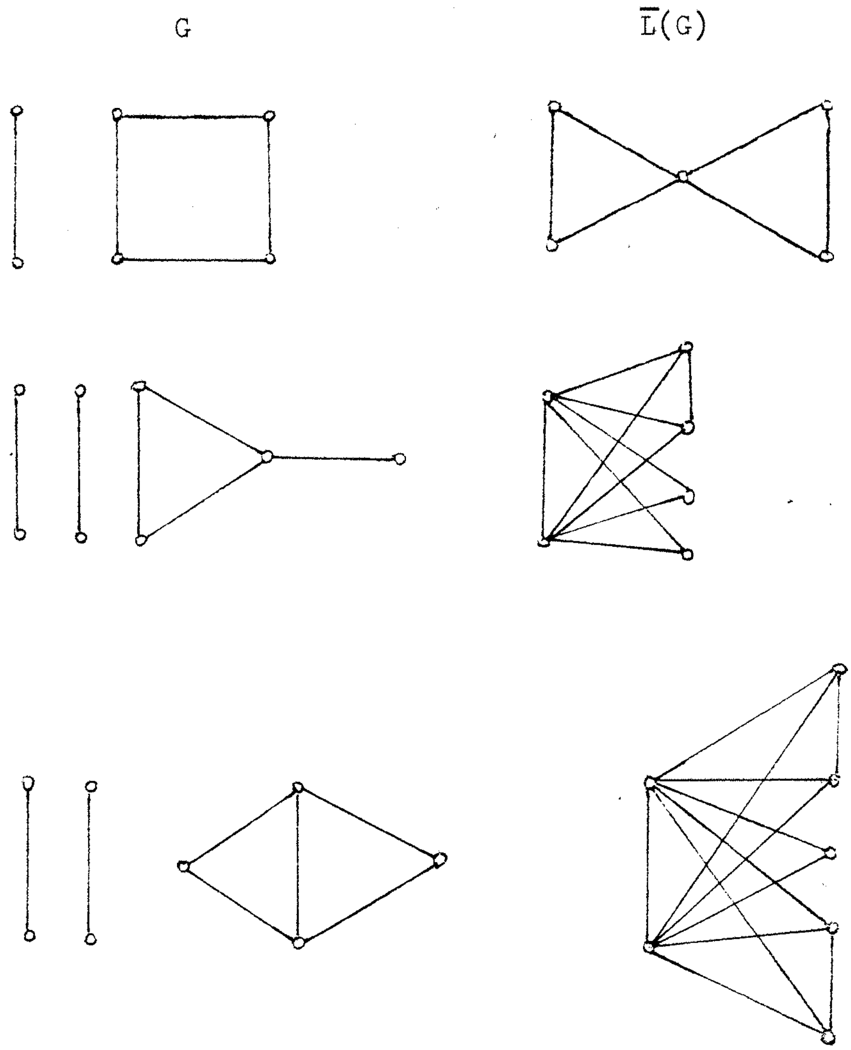
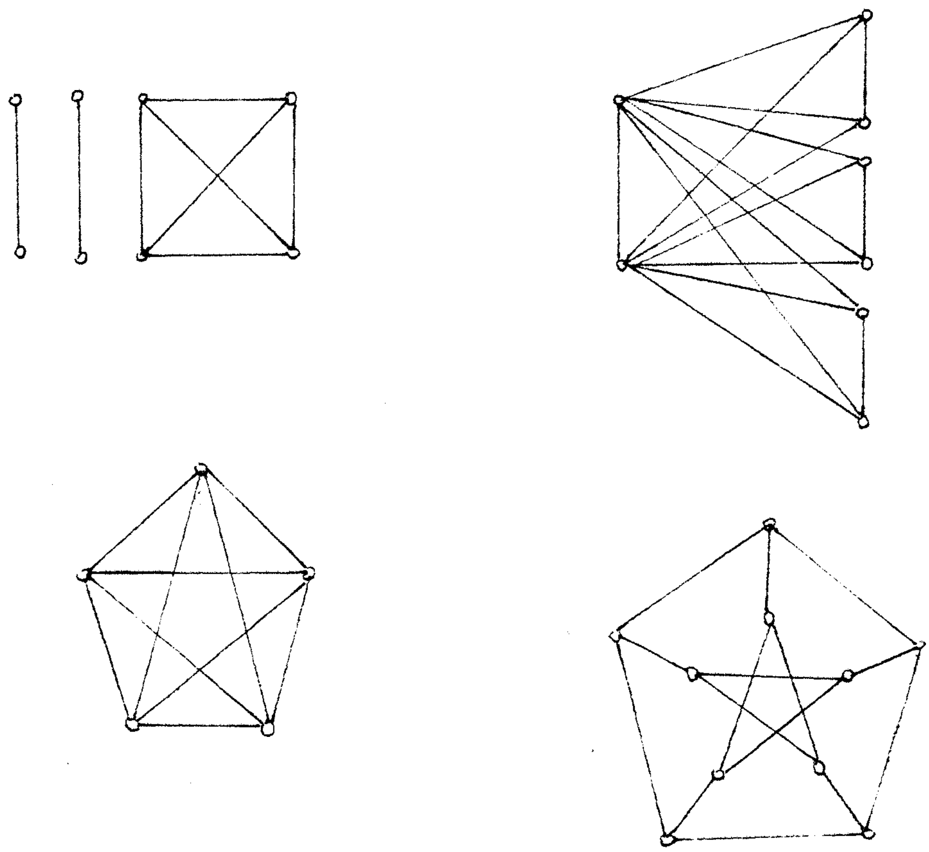


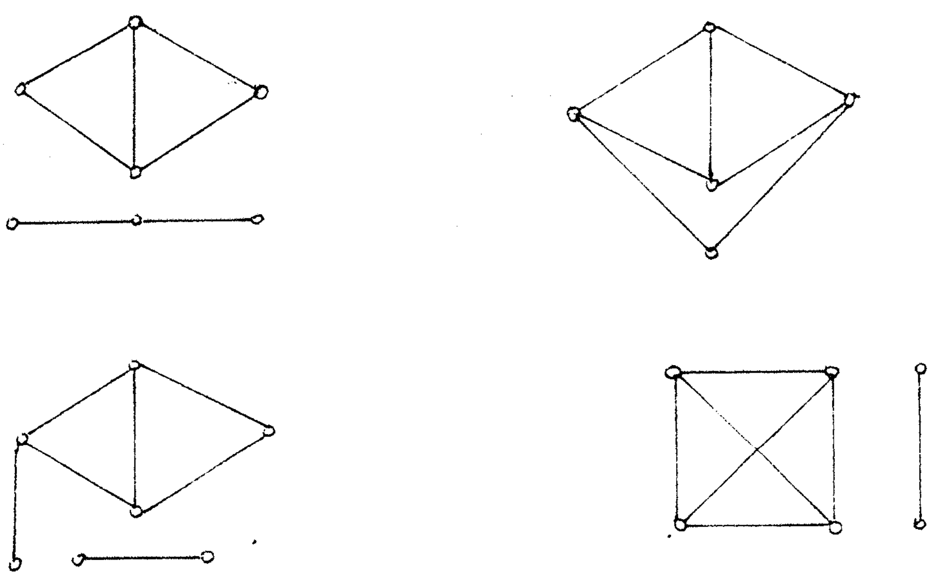
Figure 5.5.9 (continued)

G $\overline{L}(G)$



The third canonical graph, for example, can be modified to give the graphs in fig. 5,5.10, and two others.

Figure 5.5.10



The line graph complement of each of the graphs in fig. 5.5.10 is a spanning subgraph of the third line graph complement in fig. 5.5.9 and cannot be hamiltonian.

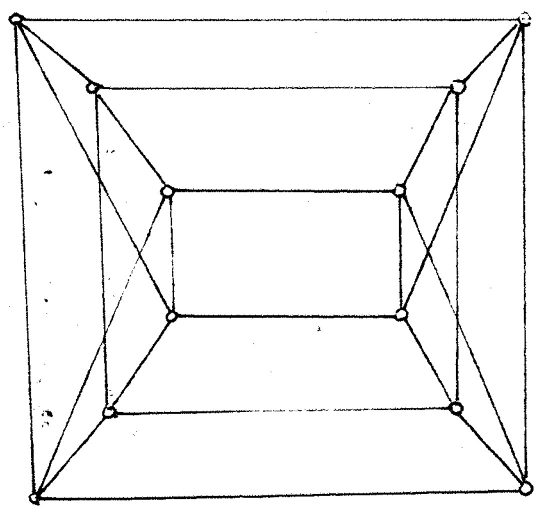
An exhaustive search has been made of all graphs with 8 or fewer edges but no further graphs were found. Numerous graphs with 9 or more edges have been examined, but so far no examples apart from K_5 have been found. It is tentatively conjectured that there are no other graphs apart from those mentioned above which are normal but do not have hamiltonian line graph complements. This conjecture is supported by the exhaustive search of small graphs, an unsuccessful search for likely counter-examples, the (fairly simple) propositions that all normal trees and all normal regular graphs except K_5 have hamiltonian line graph complements. Finally, it would be appropriate for K_5 to be the largest exceptional graph, since its line graph complement is Petersen's graph, an exceptional graph in other contexts.

However much it is improved, the present proof could not prove this conjecture. The proof of lemma 5.5.13 is useless for graphs with 21 or fewer edges since there exist normal graphs with hamiltonian line graph complements which have sets of edges E' with $k < \frac{m}{2}$ edges, each of which has valency k or less in $\overline{L}(G)$. The largest known example is K_7, where E' can be any set of 10 edges.

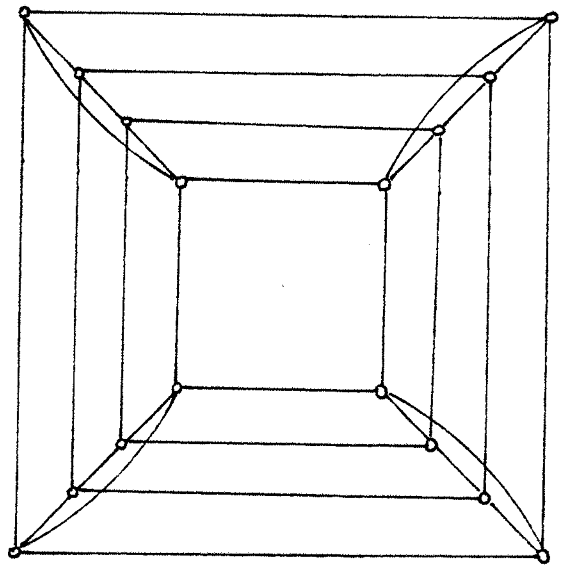
APPENDIX 1: A LIST OF TRANSPOSITION GRAPHS ON AT MOST 24 VERTICES

There are 15 graphs G_i ; i = 1, 2,..., 15 such that $\Gamma(G_i)$ has at most 24 vertices. These graphs are all shown in fig. 5.4.7. The notation of fig. 5.4.7 will also be used in this section. Transposition graphs which may be found elsewhere in this thesis will not be duplicated in this appendix. The graphs to which this remark applies are $\Gamma(G_1)$, which may be found in fig. 2.2.9, $\Gamma(G_2)$, (fig. 1.3.1), $\Gamma(G_3)$, (fig. 2.2.13), $\Gamma(G_4)$, (fig. 5.4.2), $\Gamma(G_5)$, (fig. 1.3.4), $\Gamma(G_{10})$, (fig. 4.2.1), $\Gamma(G_{11})$, (fig. 4.3.1) and $\Gamma(G_{12})$, (fig. 3.4.3) .

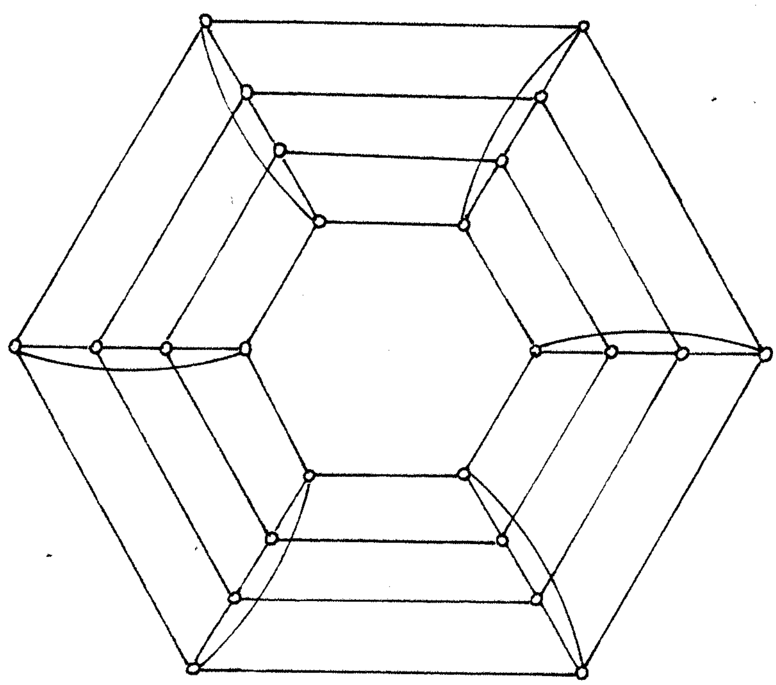
$\Gamma(G_6)$:



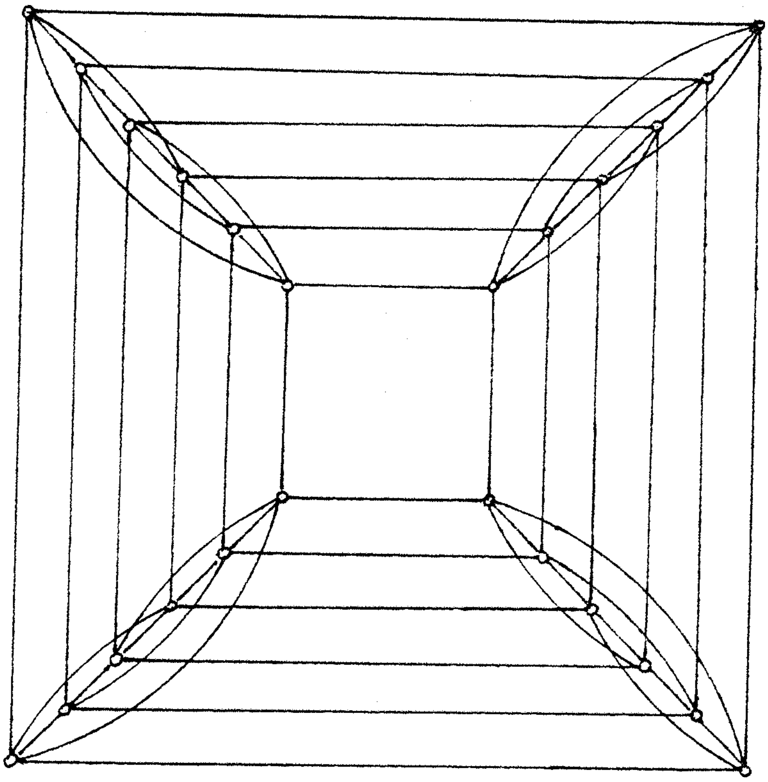
$\Gamma(G_7)$:



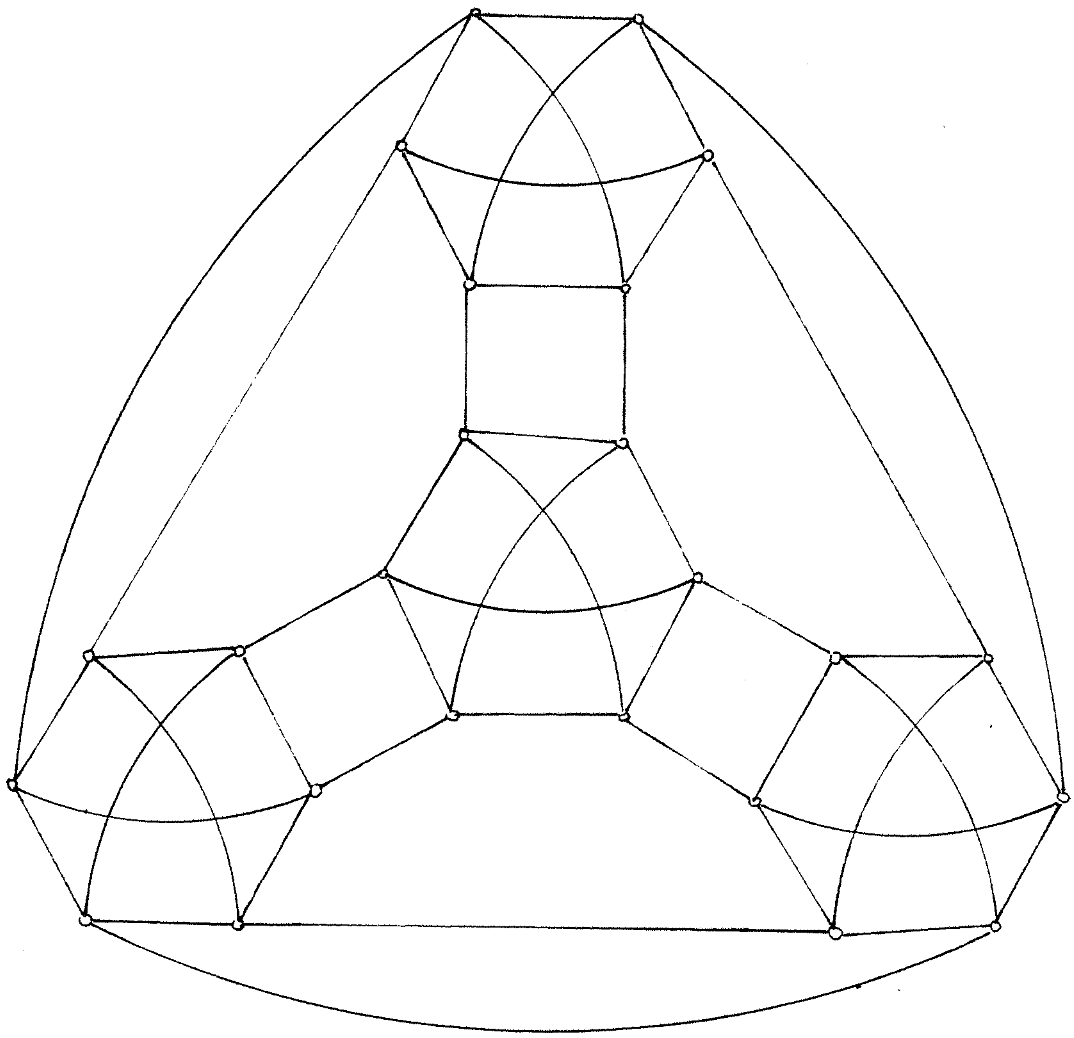
$\Gamma(G_8)$:



$\Gamma(G_9)$:

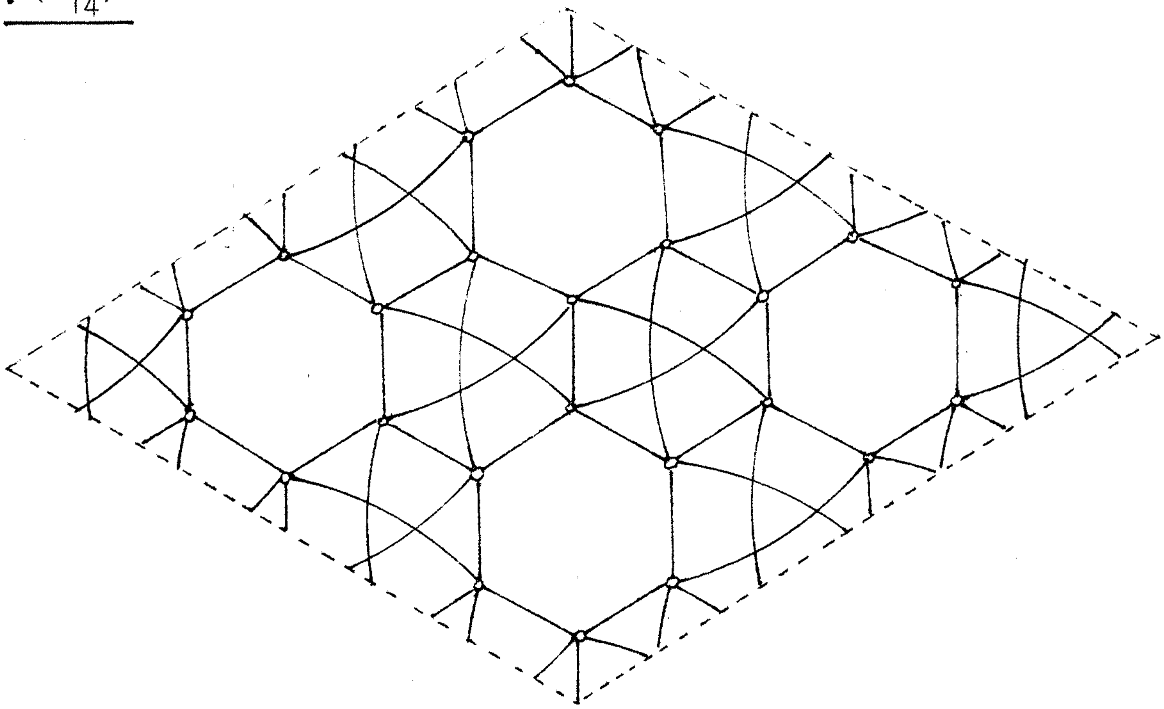


$\Gamma(G_{13})$:

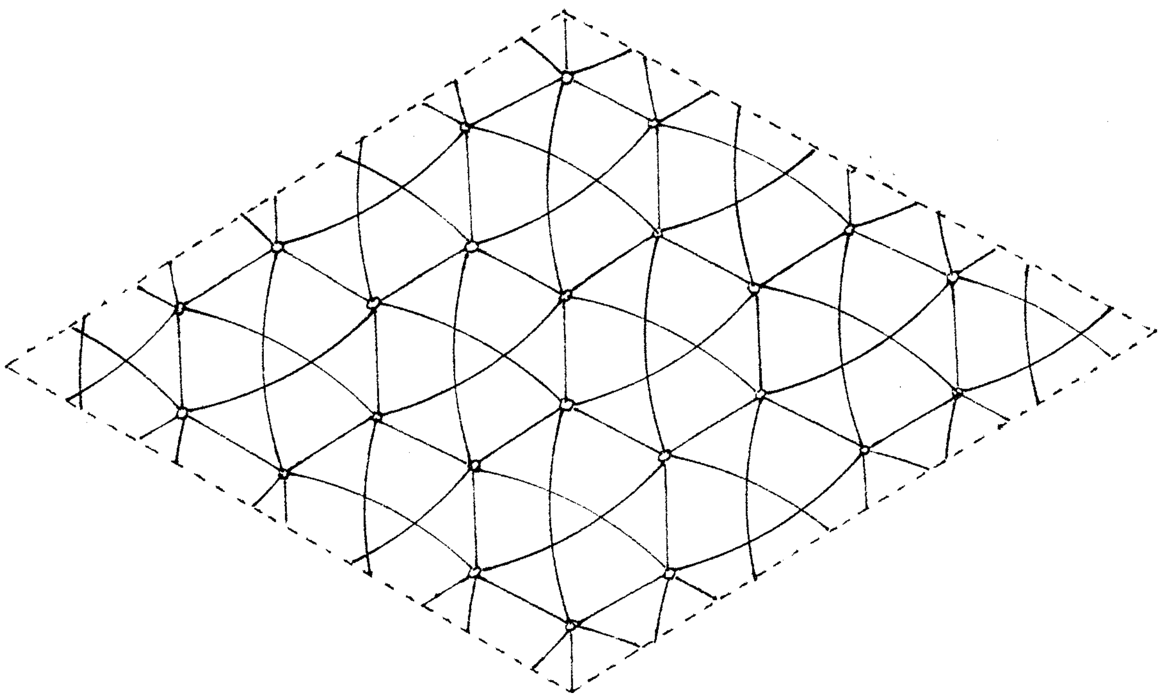


The drawings of $\Gamma(G_{14})$ and $\Gamma(G_{15})$ on the following page show the two graphs drawn on the torus, but with a number of lines crossing. This gives a more 'natural' picture of the graph than is possible drawing it in the plane. To make this convention more reasonable, those hexagons containing three intersecting lines may be regarded as cross-caps. Thus the drawings may be regarded as non-orientable embeddings of $\Gamma(G_{14})$ and $\Gamma(G_{15})$. This idea can be generalised.

$\Gamma(G_{14})$:



$\Gamma(G_{15})$:



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