SOLUTION

OF

TWO-POINT BOUNDARY VALUE PROBLEMS

ASSOCIATED WITH

SUBMARINE PIPELINES

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ABSTRACT

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SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS ASSOCIATED WITH SUBMARINE PIPELINES

by

Ibrahim Konuk

This thesis is devoted to developing methods for qualitative and numerical treatment of some two-point boundary value problems arising in submarine pipelines and risers. A general problem is formulated in this thesis based on rod theories.

The boundary value problems treated in this thesis are all associated with the following ordinary differential system, which is defined along a space curve in R^3:

$$d\underline{u}/ds + æ(\underline{u}, s)\underline{u} + \underline{g}(\underline{u},s) = \underline{0} \qquad (1)$$

and defined on the interval $[L_1, L_2]$ and with various types of boundary conditions:

$$A_i\underline{u}(L_i) + B_i\dot{\underline{u}}(L_i) = \underline{c} \qquad (2)$$
(linear boundary condition)

$$\underline{f}_i[\underline{u}(L_i), \dot{\underline{u}}(L_i)] = \underline{0} \qquad (3)$$
(nonlinear boundary condition)

$$\underline{f}_i[L_1, L_2; \underline{u}(L_i), \dot{\underline{u}}(L_i)] = \underline{0} \qquad (4)$$
(free boundary condition)

where æ, A_i, B_i are matrices in $R^6 \times R^6$ and $\underline{u}$, $\underline{g}$, $\underline{f}_i$, $\underline{c}$ are vectors in R^6 and $i = 1$ or 2.

This thesis gives formulations of several practical pipeline problems and proves the existence and uniqueness of solutions. An asymptotic solution is obtained by using singular perturbation method.

This thesis also describes methods for obtaining discrete solutions for general forms of pipeline and riser problems.

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I owe a great deal to Mr John Stares, who has patiently put the manuscript into this form from my virtually illegible handwritten notes.

I would like to dedicate this work to my family, my wife Rengin and our little daughter Elif.

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1.1 INTRODUCTION

Advancement of offshore technology towards deeper waters presents challenges for more accurate and more general tools for pipeline and riser analysis. Many papers have appeared on this subject (see, for example, Offshore Technology Conference Proceedings). The recent progress in numerical solution techniques, such as finite elements, has led to an increase in this literature. However, although analysis of pipelines and risers involves mathematically interesting and elegant problems, the majority of these works are restricted to the application of various numerical techniques to obtaining discrete approximations to the solution. Computer implementations of various finite element techniques count for much of the published literature. So far, no systematic and rational treatment of the overall problem is available. It seems that the lack of such a unified and rigorous approach has led to confusion and to inconsistent formulations: in some cases, omission of some basic mathematical tools has caused serious errors. The aim of this thesis is to bridge this gap by providing a general treatment of the fundamental problems of the subject.

As it is developed in this thesis, the subject can be seen as a "jigsaw puzzle" picture, consisting of the following main elements:

1. Rational mechanics of rod theories
2. Differential geometry of space curves
3. Solution of two-point boundary value problems for ordinary differential equations and application of functional analysis
4. Numerical methods for stiff boundary value problems

A clear understanding of the problems requires approach from many directions, and it is usually essential to combine tools from all of the above fields in order to discover all the difficulties of the problems or see them in the right place of the picture. This thesis contains examples where interaction of methods from different fields provides a much better understanding than if the problem was treated with a method isolated in one field.

The subjects of rod theories and differential geometry of space curves provide the essential tools to lay a firm foundation for all of the problems arising in the analysis of slender bodies such as pipelines and risers. Fortunately, both of these subjects have been studied in great detail in the past. Chapter 2 of this thesis gives the basics of these subjects essential for the study of the problems falling within the scope of this thesis. The principles of rod theories are based on the ideas described in the fundamental paper by Antman [5]. Chapter 2 also presents a generalized, abstract rod theory model, independent of the description of the strain and constitutive relations. Therefore, future theoretical developments or experimental findings such as [37] can easily be incorproated into the general formulation developed in that Chapter.

It is quite interesting to note that the engineering literature on pipelines and risers has almost completely ignored rod theory literature. As shown in [27], some papers such as [1] contained serious omissions in equilibrium equations that have been known for at least a century. On the other hand, most of the engineering literature explicitly or implicitly rests on Kirchhoff's Hypothesis in terms of the description of strain in the pipelines. It also seems that omission of tools of differential geometry has led many authors to confusion in terms of twist. Many papers confuse the geometric torsion of a pipe axis with the torsion of the pipeline. Chapter 2 shows that these are not always related to each other.

Most pipeline and riser problems are eventually reduced to two-point boundary value problems on some one-dimensional Euclidean manifold in R^3, since a pipeline is characterized by a space curve, such as its axis, in the deformed state. Rod theory literature includes many interesting examples of studies of similar boundary value problems [3], [4]. However, with one exception (Plunkett [33]), the popular literature of the subject usually jumps very quickly to a discretized problem and its numerical solutions. Chapter 3 of this thesis, which transforms the problems into a form best suited for a qualitative study, exposes one of the basic features of the problems associated with slender bodies, namely that the occurrence of boundary layers in the solutions is one of the most essential points that must be well understood in order to develop effective solution methods. Chapter 3 gives alternatives to Plunkett's approach: an asymptotic approximation to the solution of the two-dimensional pipelaying problem is obtained by using a singular perturbation technique. Chapter 3 also gives proof of the existence of solution for the same problem, and finds *a priori* bounds which lead to estimates for the thickness of the boundary layers at both ends of the pipeline span. In order to complete the list of most useful methods that can be used to study similar two-point boundary value problems, Chapter 3 ends with a phase plane study of another two-dimensional pipeline problem.

Boundary layer behaviour of the solutions of pipeline and riser problems requires special numerical techniques. This type of problem corresponds to a special class of problems in numerical analysis, the stiff differential equations. The requirements from a stiff method are outlined in Chapter 4. Any numerical method used for such problems should be able to resolve boundary layers and the numerical solutions should behave like the asymptotic solution when the parameter ε governing the stiffness of the problem is reduced. In the past decade, the numerical analysis literature of stiff differential equations has increased dramatically. A general review of literature on stiff methods is given in Chapter 4. In spite of the great successes of numerical analysts on the stiff methods for boundary value problems, the pipeline literature does not take account of such developments. Apart from a few reports of convergence problems [12], no paper contained a systematic treatment of the associated numerical difficulties; apart from [28], all known works used standard finite difference or global approximation (finite element) methods.

Indirectly related to the stiffness of the problem is the strong nonlinearity arising in large deformation problems. Chapter 4 develops various methods for converting the nonlinear problem into a sequence of linear problems. The theoretical basis of these methods and their interrelationships are also given in the same Chapter. Physical interpretation of some of these methods interestingly provides a new meaning for the popular relaxation methods. Several variations of the methods presented in Chapter 4 cover all the situations that may arise in practice. Therefore, we can say that this thesis provides iterative methods which can solve approximately most of the problems that may arise from pipelines and risers. Chapter 4 of this thesis also describes the application of a numerical solution technique, based on the ideas developed earlier in that Chapter, to the three-dimensional pipelaying problem. The conclusions of this thesis are given in Chapter 5. This Chapter also describes several inconsistencies that appear quite frequently in the pipeline literature.

As is reflected in this introduction, in this thesis we have aimed to present detailed enough treatment of all the principal aspects of the boundary value problems that arise from pipelines and risers, so that it can serve as a source of analysis and solution techniques. However, it has not been intended to provide an exhaustive collection of solutions to all kinds of pipeline and riser problems. On the other hand, much attention is paid, throughout this thesis, to provide rigorous definitions for all the concepts used. We have also tried to give most general and rigorous arguments, usually by employing tools of functional analysis and linear algebra, rather than restricting ourselves to only formal and intuitive explanations. By adopting this approach, we have tried to lay the principal foundations of the subject in a more consistent way. We also hope that, in this way, pipeline and riser problems will attract more functional analysts and numerical analysts, thus opening up new fields for research.

1.2 NOTATION AND BACKGROUND MATERIAL

With few exceptions, which are described below, standard notation is used throughout this thesis. Since most of the basic material used in this thesis can be found in standard books, we did not attempt to make it self contained. However, the background material required is very briefly explained below. All notation is defined clearly as it appears in the text.

The equations of each Chapter are numbered separately, and any equation number refers to an equation within the same Chapter, unless stated otherwise. References are listed according to the alphabetical ordering of the last name of the author or editor, and a reference in the text is denoted by a bracketed number such as [1]. Theorems quoted from other sources are typed with bold letters. References and figures are given at the end of the text in respective order.

1.2.1 Linear algebra and tensor calculus

Matrix algebra of linear transformations, coordinate geometry and vector analysis constitute the most popular tools used in this thesis. The vectors are represented by underlined characters, such as $\underline{R}$.

Matrices are denoted by ordinary letters. However, in a matrix equation, $\underline{R}$ represents the column matrix that defines the vector $\underline{R}$ in the relevant coordinate system. The only exception to standard notation is that if:

$$\underline{n} = [n_1, n_2, n_3] \quad \text{and} \quad \underline{m} = [m_1, m_2, m_3] \tag{2}$$

then

$$[\underline{n}, \underline{m}] = [n_1, n_2, n_3, m_1, m_2, m_3] \tag{3}$$

The same notation is also used for matrices.

Standard Einstein notation for the summation of repeated indices, such as:

$$\underline{R} = R_i \underline{e}_i$$

is used throughout this thesis.

1.2.2 Functional analysis and differential equations

Notions from Banach spaces are regularly used in this thesis, as it is quite useful in differential equations. Various fixed point theorems are used, which are found in great detail in Smart [38]. Apart from some references on the numerical solutions, where weak solutions are almost a necessity, we have usually dealt with classical (strong) solutions of the boundary value problems studied. In this thesis, function spaces $C_m^k(R^n)$ constitute most of the Banach or normed spaces used, where k shows the order of differentiability of the functions considered, n means functions from the real line R into Euclidean space R^n, and m stands for the boundary conditions imposed. Where there is no boundary condition imposed, m is omitted.

1.2.3 Differential geometry

In this thesis, we quite commonly refer to a vector field as a local coordinate system, defined on a one-dimensional Euclidean manifold consitituted by a space curve in R^3. It can also be thought of as a coordinate curve of a curvilinear coordinate system in R^3.

CHAPTER 2 REVIEW OF ROD THEORIES AND GENERAL FORMULATION

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2.1 ROD THEORIES - A HISTORICAL ACCOUNT

The rod theories have a long history of development which goes back to James Bernoulli. An extensive historical account of rod theories, up to 1788, is given by Truesdell [39]. A recent and more technical review of rod theory literature is given by Antman [5].

The common scope of rod theories is to provide a rational one-dimensional scheme for approximating the system of classical field equations of elasticity by representing the rod as a space curve (which we will call the characteristic curve), and for expressing approximate solutions along this curve. The majority of rod theories can be grouped under three headings:

1. Projection methods

In these methods, the three-dimensional material position field $\underline{B}(\underline{R})$ is uniquely defined (or approximated) via a projection function:

$$\underline{b}(\underline{R};\ \underline{r}(s),\ s) \tag{1}$$

when the position of a space curve $\underline{r}(s)$, characterizing the rod (characteristic curve), is given. A well known example of this is Kirchhoff's Hypothesis, which requires plane cross-sections of the rod normal to the characteristic curve in the reference configuration to remain plane, undeformed and normal to the same characteristic curve in the arbitrary configuration. It is also possible to regard the approximation of (1) as an exact expression of permissible forms of $\underline{B}$.

2. Asymptotic methods

The construction of rod theories by the asymptotic expansion of the three-dimensional field equations of elasticity rests on a small parameter which usually reflects a measure of slenderness of the rod. By using this technique, the field equations are reduced to a sequence of one-dimensional equations which refer to a material reference curve (characteristic curve).

3. Director methods

In these methods, a rod is considered as a space curve attached to every point of which is a collection of direction vectors. These vectors are called directors, and they are susceptible to rotation and stretching independent of the deformation of the material reference curve (characteristic curve).

As far as the equilibrium equations are concerned, all three categories of theories lead basically to the same differential equations. The exact general equations that are needed to express the equilibrium of a bent and twisted rod were first given explicity by Clebsch [10]. Love [30] gives the derivation of these equations on a special positive orthogonal triad containing the tangent of a curve characterizing the rod and one of the principal axes of geometrical inertia. In rod theory literature, this reference system is known as the 'principal torsion-flexure' axes. Later, a careful modern generalization of these equations was given by Ericksen and Truesdell [15] for an arbitrary set of directors. Green [20] obtained these equations for the resultant forces and moments by integrating the three-dimensional equilibrium equations over a cross-section.

A simple derivation of these equations on a special orthonormal triad is given in [27]. However, the orthonormal system is kept more general than the principal torsion-flexure system in this latter paper. The details of this derivation are given in the next section.

The description of strain in a rod developed more slowly than the treatment of the stress. A clear analysis of strain in a rod, based on the concept of twist of the principal torsion-flexure axes, is given by Love [30]. There, twist is defined as the rate of rotation of the flexural axes along the tangent of the characteristic curve with respect to the arc length in the unstretched state. However, as remarked by Ericksen and Truesdell [15], this definition gives two different twists depending on which of the principal axes of inertia of the cross-section is selected as the base vector of the flexure reference system. In this respect, the asymptotic and director theories provide more consistent tools to describe the strain in a rod. The fundamental works on the description of strain in a rod are the papers by Hay [21] and Ericksen and Truesdell [15], on asymptotic methods and director theories respectively. Hay [21] employs a thickness parameter to obtain the asymptotic expansions of the strain field along the characterisitic curve and, by taking a five-parameter constitutive law, demonstrates the importance of some of the terms in the expansion obtained. Ericksen and Truesdell [15] give the kinematic description of strain, independent of constitutive relations, by using the displacement and deformations of the directors of the rod. They show that the lower order terms obtained by Hay [21] can be obtained by taking a special set of directors.

The work by Ericksen and Truesdell [15] was followed by a series of papers. However, if we ignore the rational mechanics side of these publications, most of them lead to the similar one-dimensional boundary-value problems, apart from the constitutive relations which relate the macroscopic equilibrium to the internal deformation field in the rod, in a rather general sense. The review of these works and derivation of one-dimensional constitutive relations is not included in the scope of this thesis. In this thesis, in section 2.2, in order to construct a well posed boundary value problem, we will postulate a general class of permissible constitutive laws by employing the embedding theory and differential geometry of space curves.

Although the general problem treated in this thesis can cover all slender rods with arbitrary cross-section, for the sake of simplicity and intuitive clearness of the arguments used, in the rest of this thesis, the term 'rod' will be replaced by the term 'pipe', which is sometimes used in a different context. However, the meaning of the word 'pipe' will be clear from the context. The pipe axis will represent the characteristic curve, unless explicitly stated otherwise. The term twist will be used to express both the geometric torsion of the characteristic curve and the twist of the local working coordinate system. The actual meaning will be clear from the context of the statement where it is used.

2.2 DERIVATION OF EQUILIBRIUM EQUATIONS

As shown in Figure 1, we adopt a right-handed, orthogonal coordinate system whose x-axis is tangential to the axis of the deformed pipe. The only requirement on the pipe axis is that there is a tangent defined at every point of it. By following sectional rod theories, we shall represent the stresses on any normal plane section of the deformed pipe by the resultant forces and moments acting on the pipe axis. The forces and moments will be denoted by the vectors $\underline{N}$ and $\underline{M}$ respectively.

If we take the stretched length of the axis from the starting point as the independent parameter, then any pipe element defined by $0 < S_1 \leq S \leq S_2$ will have the force $-\underline{N}(S_1)$ and the moment $-\underline{M}(S_1)$ on the left hand face. The vectors $\underline{N}(S_2)$ and $\underline{M}(S_2)$ will represent the corresponding forces and moments on the right hand face. Let us take an orthogonal positive triad of unit base vectors $\underline{f}_1$, $\underline{f}_2$ and $\underline{f}_3$ to define a fixed XYZ cartesian coordinate system and let the triad $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ specify the xyz local reference frame at the point S. We can write any vector $\underline{R}$ in the forms:

$$\underline{R} = r_i \underline{f}_i \qquad \text{or} \qquad \underline{R} = R_i \underline{e}_i$$

and in particular

$$\underline{N}(S) = N_i(S)\underline{e}_i, \qquad \underline{M}(S) = M_i(S)\underline{e}_i \tag{2}$$

where N_1 stands for the axial force while M_1 represents the torsional moment in the pipe. The derivative of the vector $\underline{R}$ can be written as:

$$d\underline{R}/dS = D\underline{R}/DS + \underline{X} \times \underline{R} \tag{3}$$

where D/DS denotes differentiation of components in a moving system, and $\underline{X}$ is the vector specifying the rotation of this reference system, that is, if we write $\underline{X} = X_i \underline{e}_i$, X_1 is the geometric torsion and X_2 and X_3 represent the curvatures of the projection of the pipe axis on the xz and xy planes respectively. The sign 'x' stands for the vectorial product. The equilibrium of forces acting on the pipe now gives:

$$D\underline{N}/DS + \underline{X} \times \underline{N} + \underline{B} = \underline{0} \tag{4}$$

where $\underline{B}$ denotes the external forces acting on the pipe. We can also derive the following moment equilibrium equation:

$$D\underline{M}/DS + \underline{X} \times \underline{M} + \underline{e}_1 \times \underline{N} + \underline{C} = \underline{0} \tag{5}$$

where $\underline{C}$ denotes the external moments acting on the pipe. Equations (4) and (5) can be written in a more compact form by using matrix notation:

$$D\underline{U}/DS + \text{Æ}\underline{U} + \underline{G} = \underline{0} \tag{6}$$

where the vectors $\underline{U}$ and $\underline{G}$ denote the generalized internal and external forces respectively, that is:

$$\underline{U} = [N_1, N_2, N_3, M_1, M_2, M_3] \tag{7}$$

$$\underline{G} = [B_1, B_2, B_3, C_1, C_2, C_3] \tag{8}$$

The matrix Æ can then be written in the explicit form as follows:

$$\text{Æ} = \begin{bmatrix} D & \emptyset \\ P & D \end{bmatrix} \tag{9}$$

where the skew symmetric submatrices D and P have the entries:

$$D = \begin{bmatrix} 0 & -X_3 & X_2 \\ X_3 & 0 & -X_1 \\ -X_2 & X_1 & 0 \end{bmatrix} \tag{10}$$

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{11}$$

and $\emptyset$ stands for a 3 x 3 matrix having all zero entries.

The equations (4) and (5) are obtained without any assumptions about the initial shape of the pipe. This means that the derived equations also apply in the case of initially curved pipes. This point will later be exploited to quasilinearize the equilibrium equations.

2.3 GEOMETRIC CONSIDERATIONS

The base vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ can be written as:

$$\underline{e}_i = a_{ij}\underline{f}_j \quad ; \qquad j = 1, 2, 3 \tag{12}$$

where the matrix $[a_{ij}]$ stands for the coordinate transformation matrix between the fixed coordinate system and the local coordinate system. It is clear that a_{ij} is the direction cosine between the local base vector $\underline{e}_i$ and the fixed vector $\underline{f}_j$.

Any vector:

$$\underline{R} = r_i\underline{f}_i = R_i\underline{e}_i \tag{13}$$

can then be written in both local and global coordinate systems as:

$$\underline{R} = a_{ij}r_j\underline{e}_i = a_{ij}R_i\underline{f}_j \tag{14}$$

The body force terms $\underline{B}$ and $\underline{C}$ are usually expressed in a fixed coordinate system and the components of these forces in the local coordinates must then be obtained by using the transformation:

$$B_i = a_{ij}b_j \text{ and } C_i = a_{ij}c_j \quad ; \qquad i = 1, 2, 3 \tag{15}$$

which introduces the direction cosines a_{ij} of the base vectors of the local coordinate system. We must remember that the local system, which we have taken as working coordinates in the derivation of the equilibrium equations, is not completely arbitrary. Along with the smoothness requirements, we have also assumed that the vector $\underline{e}_1$ will remain tangential to the pipe axis.

Therefore, for any point S on the pipe, the direction cosines a_{ij} must satisfy certain extra conditions as well as orthonormality. The well known Serret-Frenet equations, which express the derivatives of a_{ij} in terms of a_{ij} and X_i, can be used to obtain such conditions:

$$da_{1j}/dS = a_{2j}X_3 - a_{3j}X_2$$

$$da_{2j}/dS = a_{3j}X_1 - a_{1j}X_3 \quad ; \qquad j = 1, 2, 3 \tag{16}$$

$$da_{3j}/dS = a_{1j}X_2 - a_{2j}X_1$$

The derivation of these equations is given in Love [30] as a general theory of moving axes. They can be obtained directly by using equation (3) as follows:

$$d\underline{f}_j/dS = D\underline{f}_j/DS + \underline{X} \times \underline{f}_j = \underline{0} \tag{17}$$

2.4 GENERALIZATION OF ONE-DIMENSIONAL ROD THEORIES

The equilibrium equation (6) must be accompanied by some constitutive relations to be able to pose a well defined problem. But this requires, in general, a three-dimensional description of the strain of the rod, which in turn overrides the simplification obtained in the equilibrium equation by integrating the stresses over a cross-section. In order to obtain a similar simplification in the description of the strain in a pipe, further assumptions have to be utilized, such as Kirchoff's Hypothesis. A recent survey of the various strain theories is given by Antman [5]. For the purpose of this thesis, we will assume only that the local geometry of the pipe axis at any point can be uniquely described when the internal forces and moments are given. If we refer to a well known theorem of differential geometry:

A curve is uniquely defined, except as to position and orientation in space, when its curvature and its twist are given as functions of its arclength [18], we can see that this amounts to assuming the existence of relations of the following type:

$$Q_i(\underline{U}, \underline{X}, \underline{X}^0) = 0 \quad ; \quad i = 1, 2, 3 \tag{18}$$

where $\underline{X}$, $\underline{X}^0$ are the rotation vectors corresponding to the local reference system in the deformed and undeformed states. This hypothesis, however, does not influence the validity of the developed formulation: as noted by Basset [8], the equilibrium equation (6) corresponding to the actual position of the rod is exact, and no question of approximation arises unless we attempt to refer the equation to an undeformed configuration. Naturally, equation (18) amounts to some form of approximation of the strain field in the rod.

In relation to equation (18), we have to clarify one more important point which is essential for the formulation developed in this thesis. One of the three equations in (18) should be seen more as a kinematic condition specifying a relation between rotation vectors $\underline{X}$ and $\underline{X}^0$, rather than as a constitutive relation. In fact, as shown in [15], twist of the rod alone does not determine the geometric torsion of the rod axis, or vice versa, and there may exist a shift between them. Since we are working in a coordinate system defined in accordance with the deformed configuration of the rod, we can omit this extra relation at this stage. It can later be used to determine the convected position of any material coordinate system described originally in accordance with the undeformed rod configuration. In the same way, the extension of the pipe axis can be determined after the geometry of the deformed pipe axis and internal forces have been calculated.

In section 2.2, it was noted that the equilibrium equation (6) also applies to naturally curved rods, but this thesis is concerned only with the determination of the equilibrium configurations of pipelines that are normally straight in the undeformed state. Therefore, without loss of generality, we can assume that:

$$X_2^0 = 0, \quad X_3^0 = 0$$

and the two constitutive relations become:

$$Q_1(\underline{U}, \underline{X}) = 0, \quad Q_2(\underline{U}, \underline{X}) = 0 \tag{19}$$

Although the methods developed in the rest of this thesis can be applied with equations (18), for the sake of simplicity in description of the methods, the following constitutive relations will be used to replace equations (18):

$$Q(M_i) = X_i \quad ; \quad i = 2, 3 \tag{20}$$

where Q is a one-to-one function of M_i.

In order to obtain a consistent and well defined problem, in analogy with classical elasticity, the constitutive relation (20) is required to be invariant of the reference frame selected. Therefore, if we make a change of (cartesian) reference system such that:

$$a_{ij}M_i = a'_{ij}M'_i \quad ; \quad j = 1, 2, 3 \tag{21}$$

we must then have:

$$a_{ij}X_i = a'_{ij}X'_i \quad ; \quad j = 1, 2, 3 \tag{22}$$

2.5 CONSTRUCTION OF AN OPTIMIZED PROBLEM

In equation (6), the internal force $\underline{U}$ is always described in the local coordinate system. However, the external or body force $\underline{G}$ is mostly given in a global coordinate system. Therefore, the complexity of the mathematical or numerical problem to be solved will greatly depend both on the nature of the external or body forces and on the choice of local coordinate system. In general terms, the local coordinate system should be constructed in such a way that:

a) coordinate transformation matrix between local and global coordinates is bounded and has entries with continuous and bounded partial derivatives

b) the rotation vector $\underline{X}$ is continuous

c) the expression defining the vector $\underline{X}$ and the coordinate transformation matrix is as simple as possible, for computational purposes as well as for ease of handling

Consequently, the selection of the local coordinate system depends ultimately on the solution technique to be adopted as well as on the specific problem to be solved. Although there is no straightforward method for constructing the best local coordinate system for a given problem, a strong intuitive understanding of the problem should assist in reducing the effort required. The coordinate system described below is one of the simplest to construct: it satisfies the requirements set out above, and it is suitable for efficient implementation of several numerical techniques [28].

Let $\underline{f}_1$, $\underline{f}_2$, $\underline{f}_3$ denote the base unit vectors of an orthogonal reference frame XYZ. Let us define the third base vector of the local reference frame by:

$$\underline{e}_3 = (\underline{e}_1 \times \underline{f}_2)/|\underline{e}_1 \times \underline{f}_2| \tag{23}$$

and the second base vector by:

$$\underline{e}_2 = \underline{e}_3 \times \underline{e}_1 \tag{24}$$

where 'x' again stands for the operation of vector product.

Since the base vector $\underline{e}_1$ remains tangential to the pipe axis, the local coordinate system xyz is then well defined. It is very easy to visualize the geometry of this local reference system. Equation (23) states that $\underline{e}_3$ remains parallel to the XZ plane of the global reference frame, and the xy plane of the local system then remains parallel to the Y axis of the global reference frame. The description of this local reference frame can be more easily accomplished by using the following two angles, analogous to Eulerian angles. Let us denote the first angle by Θ (the angle between the x axis and the XZ plane of the global frame), and the second by Φ (the angle between the projection of the x axis on the XZ plane and the X axis). These angles and the relative positions of the local and global reference systems are illustrated in Figure 2.

The coordinate transformation matrix can now be written explicitly as:

$$\begin{pmatrix} \cos\Theta\cos\Phi & \sin\Theta & \cos\Theta\sin\Phi \\ -\sin\Theta\cos\Phi & \cos\Theta & -\sin\Theta\sin\Phi \\ -\sin\Phi & 0 & \cos\Phi \end{pmatrix} \tag{25}$$

The coordinate rotation vector:

$$\underline{X} = X_i\underline{e}_i \tag{26}$$

can then be calculated from the following relations, which are obtained from equation (16) as:

$$X_1 = \underline{e}_3 \cdot d\underline{e}_2/ds = -\dot{\Phi}\sin\Theta$$

$$X_2 = \underline{e}_1 \cdot d\underline{e}_3/ds = -\dot{\Phi}\cos\Theta \tag{27}$$

$$X_3 = \underline{e}_2 \cdot d\underline{e}_1/ds = \dot{\Theta}$$

where '.' stands for vectorial innner product, and '˙' denotes differentiation with respect to s.

It is interesting to note that we can also express $\underline{X}$ as

$$\underline{X} = \dot{\Theta}\underline{e}_3 - \dot{\Phi}\underline{f}_2 \tag{28}$$

which can be obtained directly from Figure 2.

A closer look at the overall problem shows that the introduction of this local reference frame permits a dramatic decrease in both the number of variables and the number of equations that must be handled. The equilibrium equation (6) and the two constitutive relations (20) comprise a well posed differential system with eight unknown variables. This gives an optimum formulation in the sense that eight variables constitute the minimum set of variables required to describe a pipe with a twist: six variables describe the state of stress, and two Eulerian angles describe the geometry of the pipe axis.

2.6 DEFINITION OF A DIMENSIONLESS PROBLEM

As shown in [27], further simplification can be obtained by introducing dimensionless parameters to represent the variables $\underline{U}$ and $\underline{X}(\Theta,\Phi)$. Let T and L denote some characteristic force and length parameters respectively. Then equation (6) is replaced by:

$$D\underline{u}/Ds + æ(\underline{x})\underline{u} + \underline{g}(\underline{x}) = \underline{0} \tag{29}$$

where:

$$\underline{u} = [\underline{N}/T, \underline{M}/LT] = [\underline{n}, \underline{m}] \tag{30}$$

$$\underline{g} = [\underline{B}/T, \underline{C}/LT] = [\underline{b}, \underline{c}] \tag{31}$$

$$æ = \begin{bmatrix} LD & 0 \\ P & LD \end{bmatrix} \tag{32}$$

$$s = S/L \tag{33}$$

$$\underline{x} = L\underline{X} \tag{34}$$

The constitutive relation (20), as analagous to the linear case, will be represented by:

$$q(m_i) = \varepsilon x_i \; ; \quad i = 2,\ 3 \tag{35}$$

where ε stands for the dimensionless pipe stiffness parameter [27], which is:

$$\varepsilon = EI/TL^2 \tag{36}$$

2.7 NUMERICAL STIFFNESS AND BOUNDARY LAYERS

We will now describe the source of the main difficulty associated with the numerical solution of thin rod problems. Let $\underline{u}_1$ and $\underline{u}_2$ be two solutions of the differential system (29). Then if $\underline{u}_2$ is sufficiently close to $\underline{u}_1$, by using the Jacobian matrix J of the system, we can write:

$$d(\underline{u}_1 - \underline{u}_2)/ds \simeq J(\underline{u}_1 - \underline{u}_2) \tag{37}$$

By formally integrating this equation, we can easily obtain [24]:

$$\underline{u}_1 - \underline{u}_2 \simeq \mathrm{Exp}\left(\int_0^s J[\underline{u}_1(x)]dx \right) \tag{38}$$

As can be seen from equations (9) and (32), the diagonal of the Jacobian matrix is dominated by skew symmetric terms in the order of $1/\varepsilon$. This means that, as $1/\varepsilon$ gets larger, there will be some eigenvalues with large real parts. However, if this is the case, equation (38) shows that the distance between the integral curves of equation (29) will change rapidly near the boundary points, where the independent variable has extreme values, as illustrated in Figure 3. In numerical analysis, a differential system with widely separated eigenvalues is called a stiff system. It is possible to use singular perturbation techniques to find asymptotic approximations to solutions of such problems [27]. The application of singular perturbation techniques will be given in Chapter 3.

Because of the large distance between some eigenvalues of the system (29), the Lipschitz coefficient of such a system will be very large. Iterative Newton-Raphson type methods will then require a very close starting solution and will normally fail to converge unless a good starting solution is known. Therefore, some special techniques which do not rely on a good initial estimate are required to handle the strong nonlinearity embedded in æ. On the other hand, any numerical technique to be used to solve such a problem should be able to resolve steep boundary layers.

2.8 DESCRIPTION OF COORDINATE AXES BY EULERIAN ANGLES

In the case of a vanishing external moment $\underline{C}$, equation (5) becomes analogous to the equation of a spinning top. In this analogy, $\underline{M}$ stands for the moment of the momentum of the top, s represents time, and $\underline{N}$ is the force acting on the pipe. The centre of gravity of the top is located by the local unit vector $\underline{e}_1$. This analogy is used in the study of the problem of elastica [30]. We will now use this analogy to describe a general material coordinate system which can also be taken as the torsion flexure axes of the rod.

As is done in rigid body dynamics, we will introduce the three Eulerian angles Θ, Φ, Ψ to describe the geometry of the pipe axis or, more specifically, the orientation of the torsion flexure axes of the pipe. We will take Θ as the angle between the tangent vector $\underline{e}_1$ to the pipe axis and the second base vector $\underline{f}_2$ of the global reference system. We then let Φ represent the angle which a plane parallel to these axes makes with the fixed plane XY, and let Ψ represent the angle between the principal plane xz of the flexure system and the plane defined by xY coordinate axes. These Eulerian angles are illustrated in Figure 4. We can easily obtain the direction cosines of the vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, or the coordinate transformation matrix $[a_{ij}]$, by using this Figure, as follows:

$$a_{11} = \sin\Theta\cos\Phi \tag{39}$$

$$a_{12} = \cos\Theta$$

$$a_{13} = \sin\Theta\sin\Phi$$

$$a_{21} = -\sin\Phi\cos\Psi - \cos\Theta\cos\Phi\sin\Psi$$

$$a_{22} = \sin\Theta\sin\Psi$$

$$a_{23} = \cos\Phi\cos\Psi - \cos\Theta\sin\Phi\sin\Psi$$

$$a_{31} = -\sin\Phi\sin\Psi + \cos\Theta\cos\Phi\cos\Psi$$

$$a_{32} = -\sin\Theta\cos\Psi$$

$$a_{33} = \cos\Phi\sin\Psi + \cos\Theta\sin\Phi\cos\Psi$$

The relations connecting $d\Theta/ds$, $d\Phi/ds$, $d\Psi/ds$ with the rotation vector $\underline{X}$ can be easily obtained by using Figure 4, or by using the equations:

$$x_1 = \underline{e}_3 \cdot d\underline{e}_2/ds \tag{40}$$

$$x_2 = \underline{e}_1 \cdot d\underline{e}_3/ds$$

$$x_3 = \underline{e}_2 \cdot d\underline{e}_1/ds$$

which are derived from equations (14). We can now substitute the components of the vectors $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$ as described in equations (39) into equations (40), and obtain:

$$x_1 = -\cos\Theta\dot{\Phi} - \dot{\Psi} \tag{41}$$

$$x_2 = -\cos\Psi\dot{\Theta} - \sin\Theta\sin\Psi\dot{\Phi}$$

$$x_3 = -\sin\Psi\dot{\Theta} + \sin\Theta\cos\Psi\dot{\Phi}$$

If we recall the fundamental theorem of differential geometry of space curves, which is stated in section 2.4, we can see that the three Eulerian angles Θ, Φ, Ψ cannot be, in general, independent. So we can introduce an additional constraint on the coordinate system described by the three Eulerian angles. For example, if we take:

$$\text{cotan}\Psi = -\sin\Theta\dot{\Phi}/\dot{\Theta} \tag{42}$$

the second component x_2 of the rotation vector $\underline{x}$ vanishes and this means that the base vector $\underline{e}_2$ is the principal normal of the pipe axis at any s. Therefore, equation (42) requires that the base vectors $\underline{e}_1$ and $\underline{e}_2$ remain in the osculating plane of the pipe axis for any s. If we take $\Psi = \pi/2$ and replace Θ by $\pi/2 - \Theta$, we then obtain the special coordinate system that is constructed in section 2.5. Comparison of equations (41) with (27) or equations (39) with (25) show how much simplification it is possible to obtain by choosing an appropriate coordinate system.

Up to now we have dealt with the coordinate systems that are defined in accordance with the deformed geometry of the pipe. This enabled us to drop one of the constitutive relations (15). However, if we wish to determine the orientation of a material coordinate system which is defined in accordance with the undeformed geometry of the pipe, then we can impose this constitutive equation as a constraint to define the convected orientation of a material coordinate system such as the flexure axes system. In the case of a cylindrically symmetric and initially straight pipe, the torsion-flexure system can be chosen to have no twist in the undeformed state, that is:

$$x_1^0 = 0$$

We can then express this extra constitutive relation in the dimensionless form:

$$q_t(m_1) = x_1 \tag{43}$$

where q_t stands for torsional constitutive law.

The converse of this is also true in the sense that if we wish to refer to a material coordinate system which is defined in the undeformed state of the pipe, then this coordinate system must be defined by three geometric parameters, unlike the coordinate system defined in section 2.5, which requires only two. This provides one more piece of evidence for the optimality of that coordinate system.

2.9 FORMULATION OF SOME PIPELINE PROBLEMS

2.9.1 Pipelaying in three dimensions

Final definition of any pipeline problem does not become complete unless the body and external force term

$$\underline{g}(\underline{x}) \tag{44}$$

is described. In the case of pipelaying, this term contains the weight, hydrostatic and drag forces due to sea water movements such as currents. In order to define these forces, we need to define the global coordinates XYZ precisely. Let the X coordinate be along the pipeline route on the seabed, and let Y point vertically upward. If $\underline{R}$ is a position vector in this coordinate system, we can describe the currents in the sea as a vector field:

$$\underline{V}(\underline{R}) \tag{45}$$

Then the dimensionless weight, hydrostatic and drag forces become respectively:

$$-W/T\underline{f}_2 \tag{46}$$

$$(\gamma A/T)\underline{f}_2 + d[(P/T)\underline{e}_1]/ds \tag{47}$$

$$\tfrac{1}{2}\rho C_D D|\underline{V} - (\underline{V}\cdot\underline{e}_1)\underline{e}_1|[\underline{V} - (\underline{V}\cdot\underline{e}_1)\underline{e}_1]/T \tag{48}$$

where:

W = the unit dry weight of the pipe
P = the total hydrostatic pressure on an area A equivalent to the pipe cross section
γ = the specific weight of sea water
ρ = the density of sea water
C_D = the drag coefficient
D = external diameter of the pipe

However, defining a transformed force by:

$$\underline{t} = \underline{n} + (P/T)\underline{e}_1 \tag{49}$$

and combining equation (46) with equation (47), we can obtain

$$[(\gamma A - W)/T]\underline{f}_2 \tag{50}$$

which is simply the submerged weight of the pipeline and will be denoted by w. Now the internal force term $\underline{u}$ in equation (29) will be:

$$\underline{u} = [\underline{t}, \underline{m}] \tag{51}$$

If we take L as the span length of the pipeline, then the interval of definition for equation (29) will become unity [0, 1]. Without loss of generality, we can take s = 0 as the touch-down point on the seabed, and s = 1 as the departure point on the laybarge stinger. The actual length, L^0, can later be determined from:

$$L^0/L = 1 - T[\int_0^1 n_1(s)ds]EA \tag{52}$$

In order to complete the definition of a well posed problem, we need to introduce eight boundary conditions. In a typical pipelaying problem, due to continuity requirements and by assuming that soil behaves as an elastic medium at s = 0, we would have:

$$b_1t_2 + a_1m_3 = 1 \tag{53}$$

$$b_2t_3 + a_2m_2 = 1 \tag{54}$$

where the coefficients a_1, a_2, b_1, b_2 depend on soil properties and pipe flexural rigidity.

In terms of the coordinate system defined in section 2.6, we can also assume:

$$\Phi(0) = 0 \tag{55}$$

$$\Theta(0) = \Theta_0 \tag{56}$$

where Θ_0 is the bottom slope. On the other hand, at s = 1, we have:

$$m_2(1) = 0 \tag{57}$$

$$m_3(1) = m_0 \tag{58}$$

due to matching requirements on the stinger. The definition of the problem becomes complete with the prescription of tension at the matching point s = 1

$$t_1(1) = t_0 \tag{59}$$

and the compatibility conditions [27]:

$$\tan\,(1) = d\Gamma[L \int_0^1 \sin\Theta(s)ds]/dX \tag{60}$$

which sets the geometric compatibility between the pipe span and the stinger, where:

$$Y = \Gamma(X)$$

describes the profile of the laybarge stinger. Ideally, the boundary conditions (58) through (60) should be replaced with the overbend analysis on the stinger. In that case, the problem is generalized to a multipoint boundary value problem. However, it is possible to treat the sagbend and overbend of the pipe span separately, and only bring them together at the analysis stage. This type of approach allows a modular and more general implementation of mixed numerical techniques. A similar technique can be used to treat the occurrence of plastic hinges in a pipe, as in the case of a dry buckle during laying.

One of the main difficulties associated with the pipelaying problem is the fact that one of the boundary points is not known until the solution is found. Therefore, the problem formulated above is a free-boundary problem and the boundary point is defined by the non-linear compatibility equation (60).

2.9.2 Problems with external forces which depend on the path of deformation

In many pipeline problems, the external forces acting on the pipeline depend on the trajectory of each point of the pipe axis. For example, a submarine pipeline which is being towed on the seabed encounters distributed friction forces which act along the tangent of the path of each point on the pipe axis. Another example is the off-bottom tow of submarine pipelines, where the pipeline is kept on the sea bottom by means of chains hanging down from the pipeline. In this latter case, the chain would be forced to a position tangential to the trajectory of the point that it is connected to. In most of these examples, the path of deformation depends on a parameter p, such as a lateral pull force at the head of the pipeline.

The coordinate rotation vector $\underline{x}$, which defines the geometry of the pipeline, now depends also on p, and constitutes a mapping

$$\underline{x}(s, p) \tag{61}$$

from R x R into R^3. In the same way, the trajectory of any point on the pipe axis is defined by a vector $\underline{R}(s, p)$, which can be calculated by:

$$R_i(s, p) = \int_0^s a_{1i} dt \tag{62}$$

Since the tangent to the trajectory of a point is defined by the partial derivatives of the components of $\underline{R}(s, p)$ with respect to p, the equilibrium equation (4) can now be written in dimensionless form as follows:

$$D\underline{n}/Ds + \underline{x} \times \underline{n} + P[\underline{R}]_p = \underline{0} \tag{63}$$

where P is usually a constant matrix and $[\underline{R}]_p$ stands for a tangent vector to the pipe axis at the point s.

In the special case of P being a zero matrix, equation (63) reduces to a homogeneous equation and the overall equilibrium equation (29) reduces to the homogeneous system:

$$D\underline{u}/Ds + æ\underline{u} = \underline{0} \quad (64)$$

This means that no external moments act on the pipe, and then equation (64) corresponds to a pipeline deformed by end forces and moments only.

We should note that equation (63) is not an ordinary differential equation, but a partial differential equation, and the associated problem of determining the path of deformation then becomes a Cauchy problem. The initial condition is the shape of pipe for the initial value of p.

CHAPTER 3 APPLICATION OF SPECIAL METHODS TO SOME SUBMARINE PIPELINE PROBLEMS

3.1 Some observations on a special class of pipeline problems

3.2 Pipelaying problems in two dimensions

3.3 Perturbation solution in two dimensions

3.4 Application of perturbation solutions

3.5 Qualitative behaviour of some pipeline problems

CHAPTER 3 APPLICATION OF SPECIAL METHODS TO SOME SUBMARINE PIPELINE PROBLEMS

3.1 SOME OBSERVATIONS ON A SPECIAL CLASS OF PIPELINE PROBLEMS

In this Chapter, several methods of nonlinear differential equations will be applied on an important class of pipeline problems. For the sake of clarity of the derivations and simplicity in illustration of the techniques employed, the following linear constitutive relation will be used throughout this Chapter:

$$m_i = \varepsilon x_i \quad ; \quad i = 2,\ 3 \tag{1}$$

We will also assume that no external moment $\underline{C}$ acts on the pipe. If we introduce the following notation:

$$\underline{x} = x_1 \underline{e}_1 + \underline{x}^0 \tag{2}$$

$$\underline{m} = m_1 \underline{e}_1 + \underline{m}^0 \tag{3}$$

$$\underline{n} = n_1 \underline{e}_1 + \underline{n}^0 \quad , \quad \underline{b} = b^1 \underline{e}_1 + \underline{b}^0 \tag{4}$$

we can write the equilibrium equations (4) and (5) of Chapter 2 on the different coordinate planes:

$$dn_1/ds\underline{e}_1 + \underline{x}^0 \times \underline{n}^0 + b^1 \underline{e}_1 = \underline{0} \tag{5}$$

$$D\underline{n}^0/DS + \underline{e}_1 \times (x_1 \underline{n}^0 - n_1 \underline{x}^0) + \underline{b}^0 = \underline{0} \tag{6}$$

and

$$dm_1/ds\underline{e}_1 = \underline{0} \tag{7}$$

$$\varepsilon D\underline{x}^0/Ds + \underline{e}_1 \times (\tau \underline{x}^0 + \underline{n}^0) = \underline{0} \tag{8}$$

where:

$$\tau = x_1 - m_1 \tag{9}$$

Equation (7) implies that if no external torsional moment is applied, the torsional moment in the pipe remains constant. Therefore, <u>if any torsional moment is applied to one end of a pipe which has linear constitutive relations, then a torsional moment with the same magnitude and opposite direction must act on the other end of the pipe, unless an external torsional moment is being applied at some point along the pipe.</u> This enables us to drop one equation from the differential system in equation (29) of Chapter 2.

If we use the coordinate system described in section 2.5, we can then rewrite equations (5), (6) and (8) in terms of the angles Θ and Φ:

$$\dot{n}_1 - \varepsilon\dot{\Phi}^2\dot{\Theta}\sin\Theta\cos\Theta + \varepsilon\ddot{\Phi}\dot{\Phi}\cos^2\Theta + \varepsilon\dot{\Theta}\ddot{\Theta} + b_1 = 0 \tag{10}$$

$$\dot{n}_2 + \dot{\Phi}\sin\Theta n_3 + n_1\dot{\Theta} + b_2 = 0 \tag{11}$$

$$\dot{n}_3 - \dot{\Phi}\sin\Theta n_2 + n_1\dot{\Phi}\cos\Theta + b_3 = 0 \tag{12}$$

$$n_3 = \varepsilon(\dot{\Phi}\dot{\Theta}\sin\Theta - \ddot{\Phi}\cos\Theta) - \tau\dot{\Theta} \tag{13}$$

$$n_2 = \tau\dot{\Phi}\cos\Theta - \varepsilon\ddot{\Theta} \tag{14}$$

where '$\cdot$' stands for the derivation with respect to the variable s.

Equations (10) through (14) constitute the state equations for the general pipeline problem with no external moments on the pipe (which is assumed to have linear constitutive relation (1)). A close look at equation (10) reveals that it can be written as:

$$d(2n_1 + \varepsilon^2\dot{\Phi}^2\cos^2\Theta + \varepsilon^2\dot{\Theta}^2)/ds + 2b_1 = 0 \tag{15}$$

From this, one can obtain the first integral as:

$$\varepsilon^2\dot{\phi}^2\cos^2\Theta + \varepsilon^2\dot{\Theta}^2 + 2[n_1 + \int_0^s b_1(x)dx] = z \tag{16}$$

where z is a constant. This equation can also be obtained by using the rigid body dynamics analogy as shown in Love [30]. However, in equation (16), the term corresponding to the contribution of the twist of the rod does not appear, because the torsional moment remains constant along the whole pipe. Note that this result has been obtained without any reference to the constitutive relation for the twist of the pipe. Thus it is possible to use any nonlinear constitutive law for the twist of the pipe. Equation (16) is analogous to the energy of a spinning top in rigid body dynamics.

3.1.1 First integral of state equation in two dimensions

If we assume that no external or body force acts in the direction of $\underline{e}_3$, and no external moment is applied on the pipe, equation (29) of Chapter 2 can be written as:

$$D\underline{u}/Ds + æ\underline{u} + \underline{g} = \underline{0} \tag{17}$$

where:

$$\underline{u} = [n_1,\ n_2,\ \varepsilon\dot{\Theta}] \tag{18}$$

$$\underline{g} = [(g_1\cos\Theta + g_2\sin\Theta + G_1),\ (-g_1\sin\Theta + g_2\cos\Theta + G_2),\ 0] \tag{19}$$

In this equation, G_i represents the components described on the local coordinate system of the dimensionless external force vector $\underline{g}$, and g_i denotes the components on the fixed coordinate system, that is, $\underline{g}$ has been decomposed into two parts as:

$$\underline{g} = G_i\underline{e}_i + g_i\underline{f}_i \tag{20}$$

The matrix æ now becomes:

$$æ = \begin{pmatrix} 0 & -\dot{\Theta} & 0 \\ \dot{\Theta} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{21}$$

We can now combine the first two scalar equations of the system (17) in two different ways: the first way is to multiply the first by $\cos\Theta$ and the second by $\sin\Theta$, and combine them; the second way is to multiply the first by $\sin\Theta$ and the second by $\cos\Theta$, and combine them. Eliminating the variable n_2 using the last equation of the system (17), we can obtain:

$$d(n_1\cos\Theta)/ds + \varepsilon d(\ddot{\Theta}\sin\Theta)/ds$$
$$= -(G_1\cos\Theta - G_2\sin\Theta + g_1) \tag{22}$$

$$d(n_1\sin\Theta)/ds - \varepsilon d(\ddot{\Theta}\cos\ \ )/ds$$
$$= -(G_1\sin\Theta + G_2\cos\Theta + g_2) \tag{23}$$

By integrating both sides from 0 to s, we can obtain:

$$[n_1\cos\Theta + \varepsilon\ddot{\Theta}\sin\Theta]_0^s = -\int_0^s(G_1\cos\Theta - G_2\sin\Theta + g_1)dx \tag{24}$$

$$[n_1\sin\Theta - \varepsilon\ddot{\Theta}\cos\Theta]_0^s = -\int_0^s(G_1\sin\Theta + G_2\cos\Theta + g_2)dx \tag{25}$$

Since

$$\int_0^s \cos\Theta(x)dx = X(s) \tag{26}$$

$$\int_0^s \sin\Theta(x)dx = Y(s) \tag{27}$$

if we assume that G_i, g_i are constant, we can write:

$$[n_1\cos\Theta + \varepsilon\ddot{\Theta}\sin\Theta]_0^s = -G_1X + G_2Y - g_1s \tag{28}$$

$$[n_1\sin\Theta - \varepsilon\ddot{\Theta}\cos\Theta]_0^s = -G_1Y - G_2X - g_2s \tag{29}$$

These equations are the first integrals of the state equations (10) through (14), and they simply express the equilibrium along the coordinate axis X, Y of the fixed reference system.

3.2 PIPELAYING PROBLEM IN TWO DIMENSIONS

If we assume that there are no hydrodynamic forces acting on the pipe, then the only force acting on the pipe, according to the formulation developed in section 2.9.1, is the submerged weight of the pipe as expressed in equation (50) of Chapter 2, which can be written simply as:

$$\underline{g} = -w\underline{f}_2 \tag{30}$$

or, in terms of the notation used in equation (20):

$$g_2 = -w \tag{31}$$

Therefore, equations (28) and (29) can now be written as:

$$t_1\cos\Theta + \varepsilon\ddot{\Theta}\sin\Theta = t_1(0) \tag{32}$$

$$t_1\sin\Theta - \varepsilon\ddot{\Theta}\cos\Theta = ws + \varepsilon\ddot{\Theta}(0) \tag{33}$$

By choosing the value $N_1(0)$ as the scaling parameter T, we can get:

$$t_1(0) = 1 \tag{34}$$

Let us also denote the integration constant by:

$$q = \ddot{\Theta}(0) \tag{35}$$

We can then combine equations (32) and (33) as follows:

multiplying the first by $\cos\Theta$ and the second by $\sin\Theta$ gives:

$$\varepsilon\ddot{\Theta} + (ws + \varepsilon q)\cos\Theta - \sin\Theta = 0 \tag{36}$$

and multiplying the first by $\sin\Theta$ and the second by $\cos\Theta$ gives:

$$t_1 - (ws + \varepsilon q)\sin\Theta - \cos\Theta = 0 \tag{37}$$

A close look at equation (36) reveals that the tensile force $n_1(s)$ is now decoupled from the parameter $\Theta(s)$, which describes the pipe geometry.

It is also interesting to note that the integral

$$\int_0^s [\varepsilon\dot{\Theta}^2(x) + 2t_1(x)]dx \tag{38}$$

is the variational integral of equations (36) and (37). This means that any solution of the problem should minimize the functional integral (38). Moreover, this means that any solution of the problem should minimize an energy functional which is composed of the strain energy in the pipe excluding the stretching of the pipe axis and the work done by a force, equivalent to the axial force in the pipe, travelling along the pipe axis. The integral form (38) can also be generalized for the three-dimensional problem using equation (16).

We now have to recall the boundary condition expressed for this problem in section 2.9. Since we are dealing with a two-dimensional problem, we will only need the following conditions:

At $s = 0$, we must have:

$$-b_1\varepsilon\ddot{\Theta}(0) + a_1\varepsilon\dot{\Theta}(0) = 1 \tag{39}$$

$$\Theta(0) = \Theta_0 \tag{40}$$

and at $s = 1$, we must have:

$$m_3(1) = m_0 \tag{41}$$

$$t_1(1) = t_0 \tag{42}$$

Instead of imposing condition (39) on the differential equation (36), let us now introduce a family of equations by taking q as the free parameter in equation (36). We will assume that for every condition of type (39), we can find at least one q so that the final solution satisfies this condition. On the other hand, since we have already imposed condition (34) on the axial force t_1, we now have to determine w so that equation (37) is satisfied at $s = 1$, that is:

$$t_1(1) = (w + \varepsilon q)\sin\Theta(1) - \cos\Theta(1) = t_0 \tag{43}$$

We can also avoid this equation by making w a free parameter. Therefore, we can see equation (36) as a two parameter family differential system.

We are then left with conditions (40) and (41), which will be replaced by the following simplified conditions:

$$\Theta(0) = 0 \tag{44}$$

$$\dot{\Theta}(1) = -\rho \tag{45}$$

where ρ denotes the dimensionless stinger curvature at the departure point.

3.2.1 Existence and uniqueness of the solution

Let us now write the second order non-linear differential equation (36) in the following form:

$$\varepsilon d^2\Theta/ds^2 + f(s, \Theta(s)) = 0 \tag{46}$$

where:

$$f(s, \Theta(s)) = (ws + \varepsilon q)\cos\Theta(s) - \sin\Theta(s) \tag{47}$$

It is easy to see that

$$(1 - 2\Theta/\pi) \leq \cos\Theta \leq 1 \tag{48}$$

$$(2\Theta/\pi) \leq \sin\Theta \leq \Theta \tag{49}$$

hold on the interval $[0, \pi/2]$. This suggests that if we define a new function $F(s, \Theta)$ as follows

$$F(s, \Theta) = ws + \varepsilon q \quad \text{if } \Theta < 0$$

$$F(s, \Theta) = f(s, \Theta) \quad \text{if } 0 \leq \Theta \leq \pi/2$$

$$F(s, \Theta) = -1 \quad \text{if } \Theta > \pi/2$$

then by using the inequalities (48) and (49), we can show, for every s in [0, 1] and for every Θ, that:

$$F_1(s, \Theta) \leq F(s, \Theta) \leq F_2(s, \Theta) \tag{50}$$

where:

$$F_1(s, \Theta) = (1 - 2\Theta/\pi)\ (ws + \varepsilon q) - \Theta \tag{51}$$

$$F_2(s, \Theta) = (ws + \varepsilon q) - 2\Theta/\pi \tag{52}$$

This inequality (50) implies that for any pair Θ_1, Θ_2 we can write:

$$F(s,\Theta_2) - F(s,\Theta_1) \leq F_2(s,\Theta_2) - F_1(s,\Theta_1) \tag{53}$$

which leads to the following simplified condition:

$$|F(s, \Theta_2) - F(s,\Theta_1)| \leq K|\Theta_2 - \Theta_1| \tag{54}$$

where:

$$K = (w + \varepsilon q + 1) \quad (55)$$

Therefore, we have proven that the function $F(s, \Theta)$ is Lipschitz continuous with constant K. Instead of dealing with equation (46), it is preferable to deal with the equation:

$$\varepsilon d^2\Theta/ds^2 + F(s, \Theta) = 0 \quad (56)$$

Now, if we prove existence and uniqueness of a solution for equation (56) with appropriate boundary conditions, and if this solution remains in the interval $[0, \pi/2]$, then that solution is also the unique solution of the differential equation (46) that remains in this interval.

Let us now express the Sturm-Liouville problem defined by equation (56) in an integral operator form by using the following Green's functions:

$$G(t,s) = s(1-t)/\varepsilon \quad \text{if } 0 \leq s \leq t \leq 1 \quad (57)$$

$$G(t,s) = t(1-s)/\varepsilon \quad \text{if } 0 \leq t \leq s \leq 1$$

$$H(t,s) = s/\varepsilon \quad \text{if } 0 \leq s \leq t \leq 1 \quad (58)$$

$$H(t,s) = t/\varepsilon \quad \text{if } 0 \leq t \leq s \leq 1$$

The solution of equation (56) with boundary conditions

$$\Theta(0) = a, \quad \Theta(1) = A \quad (59)$$

corresponds to the integral equation:

$$\Theta(t) = \int_0^1 G(t, s)F(s, \Theta)ds + (A - a)t + a \quad (60)$$

which will be called the first boundary value problem.

The solution of equation (56) with the two following boundary conditions:

$$\Theta(0) = b, \quad d\Theta(1)/ds = B \tag{61}$$

corresponds to the following integral equation:

$$\Theta(t) = \int_0^1 H(t, s)F(s, \Theta)ds + Bt + b \tag{62}$$

which will be called the second boundary value problem.

We can easily obtain equation (56) from the integral equations (60) and (62) by simply differentiating these equations. We can write equations (60) and (61) in operator form:

$$\Theta = L[\Theta] \tag{63}$$

$$\Theta = T[\Theta] \tag{64}$$

respectively, where:

$$L[\Theta] = \int_0^1 G(t, s)F(s, \Theta)ds + (A-a)t + a \tag{65}$$

$$T[\Theta] = \int_0^1 H(t, s)F(s, \Theta)ds + Bt + b \tag{66}$$

Therefore, a solution $\Theta(s)$ of equation (56) will be a fixed point of either the operator T or the operator L, depending on the boundary conditions being used.

We can now use the Banach fixed point theorem to prove the existence and uniqueness of the solution, as described in [7]. Let us take the following norm on the space $C^0[0, 1]$ of continuous functions on the unit interval:

$$\|\Theta\|_v = \max[|\Theta(t)|/v(t)] \tag{67}$$

where max[.] denotes the maximum on the compact interval [0, 1], and v(t) is assumed to be a continuous, positive function on [0, 1]. It is easy to show that:

$$(1/v_s)\ \max|\Theta| \leq \|\Theta\|_v \leq (1/v_i)\ \max|\Theta| \tag{68}$$

for the supremum v_s and infimum v_i of the function v(t) on the interval [0, 1]. Therefore, the norm $\|.\|_v$ is equivalent to the maximum norm and the space $C^0[0, 1]$ becomes a Banach space with this norm.

To be able to use the Banach fixed theorem, we have to show that the operators L and T are contraction mappings on $C^0[0, 1]$. We can use equation (60) to write:

$$L[\Theta_2] - L[\Theta_1] = \int_0^1 G(t, s)[F(s, \Theta_2) - F(s, \Theta_1)]ds \tag{69}$$

or by using the inequality (54):

$$|L[\Theta_2] - L[\Theta_1]| \leq K \int_0^1 G(t, s)|\Theta_2 - \Theta_1|ds \tag{70}$$

We can then write:

$$|L[\Theta_2] - L[\Theta_1]|$$
$$\leq K\ \max[|\Theta_2 - \Theta_1|/v] \int_0^1 G(t, s)v(s)ds \tag{71}$$

Therefore we have:

$$\|L[\Theta_2] - L[\Theta_1]\|_v \leq M\,\|\Theta_2 - \Theta_1\|_v \tag{72}$$

where:

$$M = K\ \max\left[\int_0^1 G(t, s)v(s)ds/v(t)\right] \tag{73}$$

Now if we choose v(t) as a solution of

$$d^2v/dt^2 + kv(t) = 0 \tag{74}$$

then the function $v(t)$ will satisfy equation (60), and we can write:

$$v(t) = (1 - t)v(0) + tv(1) + k\int_0^1 G(t, s)v(s)ds \tag{75}$$

Since v is positive on $[0, 1]$, we have:

$$v(t) \geq k\int_0^1 G(t, s)v(s)ds \tag{76}$$

or

$$K/k \geq K \max[\int_0^1 G(t, s)v(s)ds/v(t)] \tag{77}$$

Therefore, we must have:

$$M \leq K/k < 1 \tag{78}$$

which means that k must satisfy:

$$K < k \tag{79}$$

However, in order to optimize this inequality, that is, to allow for largest K, we need to choose a function $v(s)$ such that the first two terms on the right hand side of equation (75) will be as small as possible. One such function satisfying this requirement, for small enough $\delta > 0$, can be written as:

$$\sin[\pi(t + \delta)/(1 + 2\delta)] \tag{80}$$

which means that

$$k = \pi^2/(1 + 2\delta)^2 \tag{81}$$

and then

$$K \leq \pi^2 \tag{82}$$

We can repeat the same procedure for equation (62), and obtain:

$$\| T[\Theta_2] - T[\Theta_1] \|_v \leq k \|\Theta_2 - \Theta_1\|_v \tag{83}$$

where

$$k = K \max[\textstyle\int_0^1 H(t, s)v(s)ds/v(t)] \tag{84}$$

In this case, instead of equation (75), we have to consider:

$$v(t) = v(0) + dv(1)/ds\ t + k\textstyle\int_0^1 H(t, s)v(s)ds \tag{85}$$

and then the function (80) is substituted by:

$$\sin[\pi(t + \delta)/2(1 + \delta)] \tag{86}$$

Therefore, we must have:

$$K \leq \pi^2/4 \tag{87}$$

We can now summarize the results by rewriting the inequalities (82) and (87) as follows:

$$(w + \varepsilon q + 1) \leq \pi^2 \tag{88}$$

$$(w + \varepsilon q + 1) \leq \pi^2/4 \tag{89}$$

respectively. More simply, we can state that for the first boundary value problem (59), we must have:

$$(w + \varepsilon q) \leq \pi^2 - 1 \tag{90}$$

and for the second boundary value problem (61):

$$(w + \varepsilon q) \leq \pi^2/4 - 1 \tag{91}$$

as the sufficient condition for the existence of a unique solution for equation (56).

3.2.2 A priori bounds for the solutions and thickness of boundary layers

In this sub-section, we will use some comparison theorems to obtain bounds for the solution of the problem stated in the beginning of section 3.2, that is, the second boundary value problem as defined in sub-section 3.2.1. Now let us refer to one of the basic comparison theorems given in [7] (see Chapter 5) which can be stated as:

If all initial value problems have a unique solution on the unit interval [0, 1], and solutions of the first and second boundary value problems, as defined in the previous sub-section, are unique, and if

$$d^2\Theta_1/ds^2 + F(s,\Theta_1) \geq 0 \tag{92}$$

or

$$d^2\Theta_2/ds^2 + F(s,\Theta_2) \leq 0 \tag{93}$$

then the solution of the second boundary value problem satisfies the following inequality:

$$\Theta_1(s) \leq \Theta(s) \leq \Theta_2(s) \tag{94}$$

on the open interval (0, 1). We can apply this theorem to the solutions of the equations:

$$d^2\Theta_1/ds^2 + F_1(s,\ \Theta_1(s)) = 0 \tag{95}$$

$$d^2\Theta_2/ds^2 + F_2(s,\ \Theta_2(s)) = 0 \tag{96}$$

to obtain bounds for the actual solutions $\Theta(s)$. We have shown in sub-section 3.2.1 that two boundary value problems have unique solutions, and it is also easy to show that the initial value problems have unique solutions on [0, 1] since F is a Lipschitz function on [0, 1]. (For an example, see Chapter 1 of [11].)

Let us first consider equation (95):

$$\varepsilon d^2\Theta_2/ds^2 + (ws + \varepsilon q) - 2\Theta_2/\pi = 0 \qquad (97)$$

or

$$\varepsilon d^2\Theta_2/ds^2 - (2/\pi)\Theta_2 + (ws + \varepsilon q) = 0 \qquad (98)$$

whose general solution is:

$$\Theta_2(s) = A_2\exp(ks) + B_2\exp(-ks) + \pi(ws + \varepsilon q)/2 \qquad (99)$$

where:

$$k = (2/\varepsilon\pi)^{1/2} \qquad (100)$$

which must satisfy:

$$\Theta_2(0) = A_2 + B_2 + \varepsilon q\pi/2 = 0 \qquad (101)$$

$$d\Theta_2(1)/ds = k(A_2 - B_2) + w\pi/2 = - \rho \qquad (102)$$

Therefore, we have:

$$A_2 = \pi\sqrt{\varepsilon}[\sqrt{2}\sqrt{\varepsilon} + (w + 2\rho/\pi)\sqrt{\pi}]/4\sqrt{2} \qquad (103)$$

$$B_2 = -\pi\sqrt{\varepsilon}[\sqrt{2}\sqrt{\varepsilon} + (w + 2\rho/\pi)\sqrt{\pi} + 4q\sqrt{\varepsilon}/\sqrt{2}]/4\sqrt{2} \qquad (104)$$

This bound function gives boundary layers with thickness $(\varepsilon\pi/2)^{1/2}$ at both ends of interval [0, 1].

The lower bound function Θ_1 can be obtained by solving equation (96), or:

$$\varepsilon d^2\Theta_1/ds^2 - (1 - 2\Theta/\pi)(ws + \varepsilon q) - \Theta_1 = 0 \qquad (105)$$

or

$$\varepsilon d^2\Theta_1/ds^2 - [2(ws + \varepsilon q)/\pi + 1]\Theta_1 + (ws + \varepsilon q) = 0 \qquad (106)$$

If we apply the transformation:

$$\zeta = (\varepsilon\pi/2w)^{-1/3}(2ws/\pi + 2\varepsilon q/\pi + 1) \tag{107}$$

to equation (61), we can obtain

$$d^2\Theta_1/d\zeta^2 - \zeta\Theta_1 = z(\zeta) \tag{108}$$

The general solution of this equation can be written as follows:

$$\Theta_1(\zeta) = A_1 Ai(\zeta) + B_1 Bi(\zeta) + \pi(ws + \varepsilon q)/2 + O(\varepsilon) \tag{109}$$

where Ai, Bi are Airy's functions, and $O(\varepsilon)$ represents terms of order ε. The exact meaning of this term will be given in section 3.3. Equation (109) shows that the lower bound for the solution has a boundary layer thickness of $(\varepsilon\pi/2)^{1/3}$ since the functions Ai and Bi are non-oscillatory smooth functions on the positive side of the real axis [2].

Therefore, we have found that the solutions of the second boundary value problem will have boundary layers of thickness δ which is bounded by:

$$(\varepsilon\pi/2)^{1/2} \leq \delta \leq (\varepsilon\pi/2)^{1/3} \tag{110}$$

The behaviour of the solution and the limiting functions Θ_1 and Θ_2 are illustrated in Figure 5, near the point $s = 0$.

We can use the upper bound $\Theta_2(s)$ for finding a sufficient condition which will guarantee that the solution to (56) will remain in the interval $[0, \pi/2]$ so that the findings of section 3.2 will apply to equation (46). A sufficient condition that will guarantee this is written as:

$$\Theta(s) \leq \Theta_2(s) \leq \pi/2 \tag{111}$$

on [0, 1]. This requires that

$$\pi(ws + \varepsilon q)/2 + \delta(s) \leq \pi/2 \tag{112}$$

where $\delta(s)$ is a function of order $O(\sqrt{\varepsilon})$. Therefore, we must have:

$$w < 1 - O(\sqrt{\varepsilon}) \tag{113}$$

3.3 PERTURBATION SOLUTION IN TWO DIMENSIONS

As shown in [27], a regular perturbation scheme cannot be used to obtain a solution to the second boundary value problem stated in the beginning of this section:

$$\varepsilon d^2\Theta/ds^2 + (ws + \varepsilon q)\cos\Theta - \sin\Theta = 0 \tag{114}$$

$$\Theta(0) = 0, \quad \dot{\Theta}(1) = -\rho \tag{115}$$

because the limiting equation, when $\varepsilon = 0$, does not in general have a solution satisfying the given boundary conditions. As it is convenient to use the Landau order symbols to show the order of approximation in perturbation problems, we will recall the following definition:

Given two functions $f(x, \varepsilon)$ and $g(x, \varepsilon)$, we write:

$$f = O(g); \quad \text{if} \quad \lim_{\varepsilon \to 0} |f(x, \varepsilon)/g(x, \varepsilon)| \leq B$$

where B is a non-negative real number.

We will usually attach the domain [0, 1] to each such relation when $f = O(g)$ is correct for every x in the domain [0, 1].

Let us now formally assume that the solution of our boundary value problem is analytic in ε. Then we can write:

$$\Theta(s, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Theta_i(s) \tag{116}$$

If we substitute this expansion into the differential equation (114), we obtain:

$$sw\cos\Theta_0 - \sin\Theta_0 + O(\varepsilon) = 0 \tag{117}$$

So we must have:

$$\Theta_0(s) = \arctan(ws) \tag{118}$$

This is the equation of a catenary. It may be seen as the possible configuration of the pipeline with zero stiffness. This function $\Theta_0(s)$ does not satisfy the boundary conditions in general. So, we have to use a singular perturbation technique to obtain an asymptotic solution which will satisfy the boundary conditions. The expansion given in equation (116) will be called the outer expansion. To be able to obtain a correct approximation near the boundary points $s = 0$, $s = 1$, we will employ the method of coordinate stretching and use the following transformations:

$$\eta = (1 - s)/\sqrt{\varepsilon} \qquad \text{if } s \text{ is close to } 1 \tag{119}$$

$$\zeta = s/\sqrt{\varepsilon} \qquad \text{if } s \text{ is close to } 0 \tag{120}$$

Let us start with the boundary point $s = 1$ corresponding to the departure point. By using the following notation:

$$\Psi(\eta, \sqrt{\varepsilon}) = \Theta(s, \varepsilon) - m \tag{121}$$

where:

$$m = \Theta_0(1) = \arctan(w) \tag{122}$$

is the slope at the departure point, we can obtain:

$$d^2\Psi/d\eta^2 + (1 - \sqrt{\varepsilon}\eta)w\cos(\Psi + m) + \varepsilon q\cos(\Psi + m) - \sin(\Psi + m) = 0 \tag{123}$$

It should be noted that the derivative is taken here with respect to the variable η. With the help of the relations:

$$w \cos(m) - \sin(m) = 0 \tag{124}$$

$$w \sin(m) + \cos(m) = (1 + w^2)^{\frac{1}{2}} \tag{125}$$

this equation can be written in the form:

$$d^2\Psi/d\eta^2 - \sqrt{\varepsilon}\,(\eta w - \sqrt{\varepsilon}\,q)\cos(\Psi + m)$$
$$- (1 + w^2)^{1/2}\sin\Psi = 0 \tag{126}$$

If the function $\Psi(\eta, \sqrt{\varepsilon})$ is assumed to be analytic in ε, then the leading term of the expansion:

$$\Psi(\eta, \sqrt{\varepsilon}) = \sum_{i=0}^{\infty} \varepsilon^{i/2}\Psi_i(\eta) \tag{127}$$

must satisfy

$$d^2\Psi_0/d\eta^2 - (1 + w^2)^{1/2}\sin\Psi_0 = 0 \tag{128}$$

together with the boundary condition at $\eta = 0$ and the matching of the zeroth order inner and outer terms. The boundary condition for $\Psi(\eta, \sqrt{\varepsilon})$ at $\eta = 0$ is:

$$d\Psi(0, \sqrt{\varepsilon})/d\eta = \sqrt{\varepsilon}\,\rho \tag{129}$$

so the zeroth order term must satisfy:

$$d\Psi_0(0)/d\eta = 0 \tag{130}$$

Matching of $\Psi_0(\eta)$ to arctan(ws) by using van Dyke's matching rule [42] gives:

$$\Psi_0 + m = \arctan(w) \tag{131}$$

so we have

$$\Psi_0 = 0 \tag{132}$$

The trivial solution, $\Psi_0 = 0$ for every η, satisfies these conditions. The first order terms of the inner expansion near s = 0 must then satisfy:

$$d^2\Psi_1/d\eta^2 - \eta w \cos(m) - (1 + w^2)^{1/2}\Psi_1 = 0 \tag{133}$$

or

$$d^2\Psi_1/d\eta^2 - (1 + w^2)^{1/2}\Psi_1 - \eta w/(1 + w^2)^{1/2} = 0 \tag{134}$$

It is interesting to note that this equation is analogous to the equation of a beam loaded with distributed forces:

$$w/(1 + w^2)^{1/2} \tag{135}$$

and under constant tension throughout:

$$(1 + w^2)^{1/2} \tag{136}$$

In fact, this is equal to the dimensionless tension at the upper end of the catenary and the distributed load may be seen as the component of the body force w in the perpendicular direction to the catenary end, because:

$$1/(1 + w^2)^{1/2} = \cos(m) \tag{137}$$

So we can attach a good physical significance to the terms in the expansions. The first order terms may be considered as a catenary approximation in the outer region and a solution of a beam under tension in the inner region. This can also be confirmed in the inner region near s = 0, because there is no significant difference between the regions near s = 0 and s = 1. Let us now continue to obtain a solution for the first order terms.

It is quite elementary to obtain the general solution of this equation:

$$\Psi_1(\eta) = A_1 \exp(-\alpha\eta) + A_2 \exp(\alpha\eta) - w\eta/\alpha^4 \qquad (138)$$

where:

$$\alpha = (1 + w^2)^{1/4} \qquad (139)$$

Now let us obtain inner expansion of zeroth order outer terms. For this, we first take a Taylor expansion near $s = 1$:

$$\Theta_0(\eta, \varepsilon) = \mathrm{Arctan}(w) - \sqrt{\varepsilon}\, w\eta/(1 + w^2) + O(\varepsilon) \qquad (140)$$

and we can then write:

$$\Theta_0(\eta) = m - \eta\sqrt{\varepsilon}w/\alpha^4 + O(\varepsilon) \qquad (141)$$

So by using Van Dykes matching rule, we must have:

$$A_1\exp(-\alpha(1 - s)/\sqrt{\varepsilon}) + A_2\exp(\alpha(1 - s)/\sqrt{\varepsilon}) \leq O(1) \qquad (142)$$

We also have the boundary condition at $\eta = 0$:

$$d\,\Psi_1(0)/d\eta = \rho \qquad (143)$$

or

$$-\alpha A_1 + \alpha A_2 - w/\alpha^4 = \rho \qquad (144)$$

The constants A_1, A_2 are obtained from conditions (142) and (144) as:

$$A_1 = -(\rho + w/\alpha^4)/\alpha \qquad (145)$$

$$A_2 = 0 \qquad (146)$$

We can finally write the solution as:

$$\Psi_1(\eta,\sqrt{\varepsilon}) = -\sqrt{\varepsilon}(\rho + w/\alpha^4)\,\exp(-\alpha\eta)/\alpha$$
$$- w\eta/\alpha^4 \tag{147}$$

or

$$\Psi(\eta,\sqrt{\varepsilon}) = -\sqrt{\varepsilon}(\rho + w/\alpha^4)\,\exp(-\alpha\eta)/\alpha$$
$$- (w/\alpha^4)\sqrt{\varepsilon}\eta + O(\varepsilon) \tag{148}$$

Before going to higher order terms, we will first repeat the same process for the inner expansion near $s = 0$ which will be taken as:

$$\Phi(\zeta,\ \sqrt{\varepsilon}) = \Theta(s,\ \varepsilon) \tag{149}$$

The differential equation will then become:

$$d^2\Phi/d\zeta^2 + \sqrt{\varepsilon}(w\zeta + \sqrt{\varepsilon}q)\cos\Phi - \sin\Phi = 0 \tag{150}$$

where the derivative is taken with respect to ζ. Since the outer solution $\Theta_0(s)$ satisfies the boundary condition for $s = 0$, we can omit the zeroth order term and assume an expansion of the form:

$$\Phi(\zeta,\ \sqrt{\varepsilon}) = \sum_{i=1}^{\infty} \varepsilon^{i/2}\Phi_i(\zeta) \tag{151}$$

The first order term will then satisfy:

$$d^2\Phi_1/d\zeta^2 + \zeta w - \Phi_1 = 0 \tag{152}$$

which has a general solution:

$$\Phi_1(\zeta) = B_1\exp(-\zeta) + B_2\exp(\zeta) + w\zeta \tag{153}$$

The outer term has the following inner expansion near this region:

$$\Theta_0(\zeta) = \sqrt{\varepsilon}w\zeta + O(\varepsilon) \tag{154}$$

so we must have:

$$B_1\exp(-1/\sqrt{\varepsilon}) + B_2\exp(1/\sqrt{\varepsilon}) = 0 \qquad (155)$$

The boundary condition at $\zeta = 0$ is:

$$B_1 + B_2 = 0 \qquad (156)$$

We therefore have $B_1 = 0$, $B_2 = 0$, and as a result we obtain:

$$\Phi(\zeta, \sqrt{\varepsilon}) = \sqrt{\varepsilon}w\zeta + O(\varepsilon) \qquad (157)$$

At this stage we have first order terms of two inner and one outer expansions. If we observe that we can use the idea of overlapping regions in the outer region, we can see that inner expansions behave similarly to the outer solution up to the same exponential order in ε and we can write a composite expansion as follows:

$$\Theta(s, \varepsilon) = \arctan(ws)$$
$$-\sqrt{\varepsilon}(\rho + w/\alpha^4) \exp\left[-\alpha(1-s)/\sqrt{\varepsilon}\right]/\alpha + O(\varepsilon) \qquad (158)$$

We shall denote the right hand side terms (excluding $O(\varepsilon)$) by z_1 and call it first order asymptotic approximation. It is easy to see that z_1 satisfies two requirements of a singular asymptotic expansion:

1. It satisfies the boundary conditions and the differential equation up to certain first order terms.

2. $z_1(s, \varepsilon)$ converges uniformly to the solution of the limiting equation (in our case, it ceases to be a differential equation) on any compact subdomain (any closed subinterval) of the solution domain $[0, 1]$.

The first one can be shown by direct computation and substituting z_1 into the differential equation and boundary conditions. The second follows from the fact that:

$$\exp(-\alpha\eta) \tag{159}$$

behaves like:

$$\exp(-\alpha(1 - s)/\sqrt{\varepsilon}) \tag{160}$$

as ε approaches zero and for any small $\delta > 0$ and any interval $[p, 1 - p]$, we can find small enough ε such that:

$$|\exp(-\alpha(1 - s)/\sqrt{\varepsilon})| < \delta$$

as s in $[p, 1 - p]$, for $0 < p < 1/2$.

In principle, this process can be continued indefinitely. In fact, the second order composite expansion can be obtained as:

$$\Theta(s, \varepsilon) = z_2(s, \varepsilon) + O(\varepsilon^{3/2}) \tag{161}$$

where:

$$\begin{aligned} z_2(s,\varepsilon) &= \arctan(ws) \\ &-(\rho + w/\alpha^4) \exp[-\alpha(1-s)/\sqrt{\varepsilon}]\{\sqrt{\varepsilon}/\alpha + \varepsilon[(1-s)^2/\varepsilon \\ &+(1-s)/\sqrt{\varepsilon}\alpha + 1/\alpha^2]/4\alpha^4\} - \varepsilon q \exp(-s/\sqrt{\varepsilon}) \\ &-2\varepsilon w^3 s/(1 + w^2 s^2)^{5/2} + \varepsilon q/(1 + w^2 s^2) \end{aligned} \tag{162}$$

The functions z_1 or z_2 both have boundary layers of thickness $\sqrt{\varepsilon}$, which naturally agrees with the findings of section 3.2.

3.4 APPLICATION OF PERTURBATION SOLUTIONS

The asymptotic solution obtained for the two-dimensional pipelaying problem in section 3.3 can provide accurate enough solutions for many practical problems. However, the solutions provided by equations (158) or (162) do not represent an explicit solution for a given problem. These solutions are actually parametrized by the parameters L and T, because the dimensionless parameters ε, w, ρ, can only be known when the scaling parameters L and T are determined. These parameters can be calculated by substituting either equation (158) or equation (162) into the boundary conditions which were not imposed on the singular perturbation problem:

$$(w + \varepsilon q)\sin z_i(1) - \cos z_i(1) = t_0 \qquad (163)$$

$$\tan(1) = d\Gamma[L \int_0^1 \sin z_i(s)ds]/dX \qquad (164)$$

for either i = 1 or i = 2. Equation (163) is equivalent to equation (43). Equation (164) represents the free-boundary condition (equation (60), Chapter 2). The parameter q, however, must be determined by using equation (39). Therefore, the problem is now reduced to the simultaneous solution of three transcendental equations for the parameters L, T, q. As can be seen from equation (162), the last parameter q does not appear in the lower order terms. This means that the term q and the associated boundary conditions (39) will not come into the picture in any solution containing terms with order lower than $O(\varepsilon)$. Therefore, the solution (158) can give an explicit solution of order $O(\sqrt{\varepsilon})$, together with equations (163) and (164).

However, the parameter t_0 in equation (163) actually corresponds to the tensile force T_0 at the departure point:

$$T_0 = Tt_0 \qquad (165)$$

and in order to determine T_0 we have to analyse the part of the pipeline from the point at which it touches the laybarge stinger up to the tensioner where a prescribed and constant tension is applied to the pipe. Assuming that the pipe is completely guided along this part, by the stinger and the ramp of the laybarge, the pipe geometry can be prescribed by a function $\Gamma(x)$.If we let $\rho(s)$ represent the curvature of the curve defined by $\Gamma(x)$ as a function of arc length of the pipe axis s from the starting point of the tensioners, then the equilibrium equation (17) would reduce to the following energy functional:

$$[\varepsilon\rho^2(s)/2 + t_i(s)]_{s*}^{0} = wh \qquad (166)$$

where s* is the length of pipe from the tensioners to the departure point.

This equation is in complete analogy with equation (38), and it can be seen as a conservation law combining the strain energy of the pipe with the potential energy change due to the effect of gravity.

An important consequence of equation (166) is that we can now replace the unknown parameter T_0 by the tension at tensioners T*, which is one of the few parameters prescribed in all normal pipelaying problems. It is now possible to find an explicit solution by using equations (43), (163), (164) and (166).

In order to devise an efficient solution method for determining the parameters L and T, we need to have good initial estimates and a proper domain of solutions for the above-mentioned equations. Naturally, one of the simplest approximations is obtained for the limiting case $\varepsilon = 0$, which corresponds to the catenary solution. Let us denote the corresponding values of these two parameters by L^0 and T^0. It is obvious that we have:

$$L^0 < L \quad \text{and } T^0 < T_0 \qquad (167)$$

This means that we can then take the solution domain as $[L^0, L] \times [T^0, T_0]$. A simple and reliable way of solving the parameters is to design a relaxation algorithm which decouples them from each other, that is, to fix one of the parameters (for example T^0) while trying to find a solution for the other.

If the following iteration process is inititated by the catenary parameters L^0 and T^0, then the results will obviously remain in the intervals $[L^0, L]$ and $[T^0, T_0]$.

Step I : Use L_k^n, T^n as scaling parameters to calculate ε, w, ρ and solve L_{k+1}^n from equation (164)

Step II : Check $|L_{k+1}^n - L_k^n|$: if it is greater than a specified accuracy, then set

$$k = k + 1$$

and go back to Step I

Step III : If the sequence has converged to L^n, then find T^{n+1} by using equations (166) and (164)

Step IV : Check if $|T^{n+1} - T^n|$ is less than a specified accuracy. If not, set

$$n = n + 1$$

and go back to step I

On the other hand, it is not difficult to see that the sequences $[L_j^i]$ and $[T^i]$ are strictly monotonic, ie:

$$L_{j+1}^i > L_j^i \qquad (168)$$

and

$$L^{i+1} > L^i \quad \text{and} \quad T^{i+1} > T^i \qquad (169)$$

for all values of i and j. Therefore, these sequences will converge to L and T_0 respectively. The inequality (168) can be proven by induction and using the fact that if

$$L_j^i > L_{j-1}^i \tag{170}$$

then

$$w_j^i > w_{j+1}^i \tag{171}$$

In the same way, the inequality (169) can be shown to hold, because if

$$T^i > T^{i-1} \tag{172}$$

then

$$L^i < L^{i+1} \tag{173}$$

3.5 QUALITATIVE BEHAVIOUR OF SOME PIPELINE PROBLEMS

In this section we will treat a special case of the problem described in section 2.9. In some cases, relatively short lengths of submarine pipelines are installed by fixing one end to a structure and deflecting the other end to a prescribed location. During this process, the pipeline is stabilized by chains which can be represented by some distributed forces $\underline{R}(s)$ acting on the pipeline. The corresponding Cauchy problem, described in section 2.9, can be parametrized by, for example, the distance between the pipeline end and the target point (see Figure 6). However, if we can assume that, at some stage during the process, chains induce forces of magnitude r perpendicular to the pipe axis, throughout the pipe length (or, if the axial component of $\underline{R}(s)$ is negligible), then the state equations (28) and (29) can be written as follows:

$$[n_1\cos\Theta + \varepsilon\ddot{\Theta}\sin\Theta]_0^s = + rY \tag{174}$$

$$[n_1\sin\Theta - \varepsilon\ddot{\Theta}\cos\Theta]_0^s = - rX \tag{175}$$

As was done in section 3.2, we can transform these equations back to local coordinates:

$$\varepsilon\ddot{\Theta} - \varepsilon q\sin\Theta - \cos\Theta = + r(X\cos\Theta + Y\sin\Theta) \tag{176}$$

$$n_1 - \varepsilon q\cos\Theta - \sin\Theta = - r(X\sin\Theta - Y\cos\Theta) \tag{177}$$

where we have taken:

$$\ddot{\Theta}(0) = 1/\varepsilon \tag{178}$$

$$\Theta(0) = 0 \tag{179}$$

The condition (178) amounts to taking the shear force N_2 at the point s = 0 as the scaling parameter T. Then by using equations (26) and (27), and integrating by parts, we can obtain:

$$X\dot{\Theta}\cos\Theta = - \sin\Theta\cos\Theta + d[X\sin\Theta]/ds \tag{180}$$

$$Y\dot{\Theta}\sin\Theta = + \sin\Theta\cos\Theta - d[Y\cos\Theta]/ds \tag{181}$$

With the help of equations (180) and (181), we can obtain the first integral of equation (176):

$$\varepsilon\dot{\Theta}^2/2 + \varepsilon q\cos\Theta + \sin\Theta = r[X\sin\Theta - Y\cos\Theta] + c \tag{182}$$

where c stands for the integration constant.

In the case of r = 0, equation (182) reduces to an interesting autonomous differential equation:

$$\varepsilon\dot{\Theta}^2/2 + \varepsilon q\cos\Theta + \sin\Theta = c \tag{183}$$

This equation corresponds to a pipeline being bent by end forces and/or moments only, which is illustrated in

Figure 6. The integration constant can be calculated explicitly in terms of the bending moment m_0 at the point $s = 0$:

$$c = m_0/2 + 1 \tag{184}$$

Since εq represents the dimensionless tensile force at the same end $s = 0$ of the pipeline, we can write:

$$\cot\beta = \varepsilon q$$

where β can be seen as the angle between the direction of the force applied at the other end $s = 1$, and the X axis of the global coordinates (see Figure 6).

We can rewrite equation (183) in the following form:

$$\dot{\Theta} = (\pm 2/\varepsilon)^{1/2}/[\pm h(c,\ \beta;\ \Theta)]^{1/2} \tag{185}$$

where:

$$h(c,\ \beta;\ \Theta) = c - \cot\beta\sin\Theta - \cos\Theta \tag{186}$$

The phase diagram of this equation is shown in Figure 7. The position of the $\dot{\Theta}$ axis naturally depends on the values of the parameters c, β, ε. Since any possible geometry must remain in the regions where $\dot{\Theta}$ takes only real values, in most cases the angle Θ has to be a bounded function. However, if m_0 is taken large enough, the solution would jump to outer continuous branches and the pipeline would assume a spiralling geometry.

The phase diagram shown in Figure 7 resembles the phase diagram of a physical pendulum which can be seen as an extension of the spinning top analogy. This analogy suggests the following (see Davis [13]):

$$s = (\varepsilon/2)^{1/2}\int_0^{\Theta}[h(c,\ \beta;\ t)]^{-1/2}dt \tag{187}$$

which gives us the inverse of the required solution

$$\Theta = DL(s) \tag{188}$$

If we take the scaling parameter L as the length of the pipeline, then s remains in the interval [0, 1], and the inverse of the function DL must satisfy

$$1 \geq DL^{-1}(\Theta) \tag{189}$$

or we must have

$$(2/\varepsilon)^{1/2} \geq \int_0^{\Theta} [h(c, \beta; t)]^{-1/2} dt \tag{190}$$

If we can find a majorant function H bounding the function h from above, then we can write:

$$(2/\varepsilon)^{1/2} \geq \int_0^{\Theta} [H(c, \beta; t)]^{-1/2} dt \tag{191}$$

If we restrict ourselves to the interval $[0, \pi/2]$ and $\beta \neq \pi/2$, we can easily obtain such a majorant function as:

$$H(c, \beta; t) = c - \text{cotan}\beta(2t/\pi) - (1 - 2t/\pi) \tag{192}$$

this gives:

$$(2/\varepsilon)^{1/2} \geq (2/3k)[(k\Theta + m_0^2/2)^{3/2} - (m_0^2/2)^{3/2}] \tag{193}$$

where

$$k = 2(1 - \text{cotan}\beta)/\pi \tag{194}$$

The inequality (193) can be used to predict the deflection angle $\Theta(1)$ or to approximate the solution DL. Bishop and Drucker [9] obtained the exact solution of this problem for the special case of $\beta = \pi/2$, in terms of Jacobi elliptic functions. It is possible to extend this solution by using numerical quadrature techniques for any value of β.

CHAPTER 4 LINEARIZATION AND GENERAL NUMERICAL SOLUTION TECHNIQUES

CHAPTER 4 LINEARIZATION AND GENERAL NUMERICAL SOLUTION TECHNIQUES

4.1 HEURISTIC DEVELOPMENT OF A CONTINUATION TECHNIQUE AND A FORMAL APPLICATION OF FICKEN'S THEOREM

The equilibrium equation (6) derived in Chapter 2 is also valid for naturally curved rods. Therefore, for a prescribed pipe geometry $\underline{x}(s)$, we can rewrite the dimensionless equilibrium equations (29) in operator form:

$$L[\underline{x}][\underline{u}] = -\underline{g}(\underline{x}) \qquad (1)$$

where $L[\underline{x}]$ is a linear operator on the space $C^1(R^6)$ of continuously differentiable functions on R^6. At this point, we assume that there exists a bounded, continuous inverse $L^{-1}[\underline{x}]$ of $L[\underline{x}]$. We can then write:

$$\underline{u} = L^{-1}[\underline{x}][-\underline{g}(\underline{x})] \qquad (2)$$

This operator equation actually determines the internal forces in the pipe if the final deformed geometry of the pipe is defined by the function $\underline{x}(s)$. Therefore, it is possible to see L and L^{-1} as operators mapping from $C^0(R^3) \times C^1(R^6)$ into $C^1(R^6)$.

The initial geometry, $\underline{x}^0$, of the pipe can be found simply by subtracting the deflection of the pipe

$$\underline{x}^0 = \underline{x} - \delta\underline{x}$$

where $\delta\underline{x}$ is solved using the constitutive relations (26), or more specifically by:

$$\underline{q}(m_2) = -\varepsilon\dot{\Phi}\cos\Theta \qquad (3)$$

$$\underline{q}(m_3) = \varepsilon\dot{\Theta} \qquad (4)$$

Let us denote this operation by:

$$\underline{x} = \underline{\Omega}(\underline{u}) \tag{5}$$

If the deflection vector $\underline{x}(s)$ is equal to $\underline{x}^0(s)$, the initial undeformed geometry of the pipe would be represented by a straight line. This means that $\underline{x}$ is then the desired solution $\underline{x}^*(s)$ of the equilibrium configuration of the pipe. Therefore, the problem is equivalent to determining the fixed point $\underline{x}^*$ of the following operator equation:

$$\underline{\Omega}(L^{-1}[\underline{x}][-\underline{g}(\underline{x})]) = \underline{x} \tag{6}$$

which we will denote by:

$$F(\underline{x};\ 1/\varepsilon) = \underline{x} \tag{7}$$

where $1/\varepsilon$ represents the dependence of this functional operator on the pipe stiffness parameter ε. Let us now define a one parameter t family of functionals F_t by:

$$F_t(\underline{x}) = F(\underline{x};\ t/\varepsilon) \tag{8}$$

from R x $C^0(R^3)$ into $C^0(R^3)$. Then the fixed points of this family:

$$F_t(\underline{x}_t(s)) = \underline{x}_t(s) \tag{9}$$

define a continuous curve in the function space $C^1(R^3)$.

It is obvious that:

$$\underline{x}_0(s) = 0 \tag{10}$$

$$\underline{x}_1(s) = \underline{x}^*(s) \tag{11}$$

If we can determine the curve drawn by x_t between $t = 0$ and $t = 1$, then the solution of the problem can be constructed by following the curve, starting from the trivial point $\underline{x}_0$. Thus we must construct a sequence:

$$0 = t_0 < t_1 < t_2 < \ldots\ldots < t_n < \ldots\ldots \leq 1$$

such that

$$\underline{x}_{t_i} \tag{12}$$

will be the starting solution for finding the next fixed point:

$$F_{t_{i+1}}(\underline{x}_{t_{i+1}}) = \underline{x}_{t_{i+1}} \tag{13}$$

The fixed point of this equation can now be found by using a Newton-Raphson type method if t_i is chosen close enough to t_{i+1}.

It is quite easy to attach a physical meaning to this abstract process. By replacing ε in the functional operator F by the ratio ε/t, we introduce a relaxation factor which corresponds to changing the dimensionless flexural rigidity ε of the pipe as a free parameter. This amounts to relaxing the pipe in a step-by-step manner from the undeformed position to its equilibrium position. Therefore, if $||\cdot||$ denotes some energy norm, we can see that the sequence:

$$\underline{x}_{t_i}(s) \tag{14}$$

has a monotonic ascend property, and we have:

$$0 = ||\underline{x}_{t_1}(s)|| \leq ||\underline{x}_{t_n}(s)|| \leq \ldots\ldots \leq ||\underline{x}^*(s)|| \tag{15}$$

The task is now to show that the sequence t_i reaches unity in a finite number of steps. It is important here to notice that proof of the existence of such a finite sequence is equivalent to proof of the existence of a solution. Therefore, we have developed a constructive existence proof and we can attach an intuitive meaning to the process of construction of the solution.

We will now give the outline of the proof of Ficken's theorem [17]. Let

$$DF_t(\underline{x})[\underline{y}] \tag{16}$$

be the Frechet derivative of the operator $F_t(\underline{x})$ at the point $\underline{x}$. Let us assume that

$$F_{t_0}(\underline{x}_0) = \underline{x}_0 \quad \text{and} \quad F_t(\underline{x}) = \underline{x} \tag{17}$$

We now start with the following identity:

$$DF_{t_0}(\underline{x}_0)[\underline{y}] = -F_t(\underline{x}_0 + \underline{y}) + \underline{x} + DF_{t_0}(\underline{x}_0)[\underline{y}] \tag{18}$$

where we write $\underline{x}$ as $\underline{x}_0 + \underline{y}$ for $\underline{y} = \underline{x} - \underline{x}_0$. Then we can write:

$$\underline{y} = DF_{t_0}^{-1}(\underline{x}_0)[-F_t(\underline{x}_0 + \underline{y}) + DF_{t_0}(\underline{x}_0)[\underline{y}] + \underline{x}] \tag{19}$$

or

$$\begin{aligned}\underline{y} = DF_{t_0}^{-1}(\underline{x}_0)[F_{t_0}(\underline{x}_0 + \underline{y}) - F_t(\underline{x}_0 + \underline{y}) + \underline{y} \\ + DF_{t_0}(\underline{x}_0)[\underline{y}] - F_{t_0}(\underline{x}_0 + \underline{y}) + F_{t_0}(\underline{x}_0)]\end{aligned} \tag{20}$$

Due to the definition of $DF_t(\underline{x}_0)$, we can conclude:

$$\underline{y} \simeq DF_{t_0}^{-1}(\underline{x}_0)[F_{t_0}(\underline{x}_0 + \underline{y}) - F_t(\underline{x}_0 + \underline{y}) + \underline{y}] \tag{21}$$

or

$$\underline{y} \simeq H(t_0, \underline{x}_0; t, \underline{y}) \tag{22}$$

where H can be made a contraction mapping if t is taken close enough to t_0. We can then obtain the required result by applying a fixed point theorem.

One of the important side effects of Ficken's work [17] is the proof of the existence of a uniform step size between t and t_0. However, calculation of the step size involves determination of several bounds and continuity constants which are either very difficult to obtain or too conservative to be of any use in practice. Therefore, an adaptive step size search algorithm is more practicable from the viewpoints of both implementation and numerical efficiency.

4.1.1 Continuation theorems and selection of the continuation parameter

Several continuation theorems have been proven with different requirements on the mapping F_t. A short survey of continuation theorems is given by Smart [38]. Most of the continuation theorems are based on the idea that if the mapping F_t is continuous on [0, 1] x B where B is a connected closed set in $C^0(R^3)$, and if F_t has no fixed point on the boundary ∂B, then the fixed points cannot escape from B through ∂B. The first major continuation theorem is referred to as the Leray-Schauder Theorem [38]. However, the application of this theorem seems to be very difficult, as it requires tools from the topological degree theory (for example, see [40]).

As in Ficken's work [17], several new theorems have now been introduced with stronger conditions which are easier to establish. Several theorems of this kind are given in Smart [38]. One of these theorems is quite similar to Ficken's theorem, but is easier to state:

If M is a closed convex subset of a normed space B, and if $U_t(x)$ is a continuous mapping of M x [0, 1] into a compact subset of B such that:

a) $U_t(\partial M)$ is a subset of M

b) U_t has no fixed point on ∂M x [0, 1]

then U_1 has no fixed point in M.

Although this theorem seems to be easier to apply than Ficken's theorem, the condition that F_t must map closed sets into compact sets is not easy to prove for our operator $F_t(\underline{x})$ on $C^0(R^3)$.

By taking the maximum norm on $C^0(R^3)$, we can make it a Banach space. On the other hand, we know that:

$$F_0(\underline{x}) = 0 \tag{23}$$

for any $\underline{x}$ in $C^0(R^3)$. Therefore, if the conditions of the theorem are satisfied, say for the closed ball B_r in $C^0(r^3)$, where:

$$r = \| F_1(\underline{o}) \| \tag{24}$$

then either we must be able to reach the fixed point inside B_r, or we hit the boundary ∂B_r of the closed ball for a t* less than unity. However, since F_t stands for $F(\underline{x}, t^*/\varepsilon)$, we have found fixed points of the operator F for all ε from $\varepsilon^* = \varepsilon/t^*$ to infinity. The only time this argument can be obstructed is when the function $\underline{x}$ ceases to be in $C^0(R^3)$ due to a singularity or if the function $\underline{g}(\underline{x})$ has a discontinuity.

The continuation technique described above seems to be quite reliable, as it always has a starting solution (see Konuk [28]):

$$\underline{x} = \underline{0} \tag{25}$$

However, if we can find a continuation parameter T and a corresponding f_T which has a fixed point $\underline{x}_0$ for $T = 0$, and if x is close to the fixed point $\underline{x}_1$ of $f_1(\underline{x})$, then the implementation of such a continuation technique would be relatively more efficient. Now let us define:

$$f_T(\underline{x}) = f(\underline{x};\ T\varepsilon) \tag{26}$$

where the operator f denotes the operator defined by equation (6). If we recall the results of sections 3.2 and 3.3, we can see the operator f_T as the singular perturbation of the operator f. The starting point $\underline{x}_0$ of the operator f_0 is simply the zeroth order solution from section 3.2, and $\underline{x}_0$ describes the geometry of the pipeline if it has no stiffness. That is, $\underline{x}_0$ is the catenary solution. Therefore, if we use the continuation operator f_T to find the fixed point $\underline{x}_1$ of the operator f which is the solution $\underline{x}^*$ of our problem, we can expect to move from the fixed point $\underline{x}_0$ towards the solution $\underline{x}^*$ more quickly than with the previous method, and the direction of travel would be the opposite of that technique. That is, we would move inward towards the origin. Unfortunately, the most important shortcoming of this latter technique is that f_0 does not always possess a fixed point as in the case of a pipeline problem with very low tensile force at the barge end.

4.2 NEWTON'S METHOD AND QUASILINEARIZATION

In this section, we develop a method of solving the differential equations introduced in Chapters 2 and 3, based on abstract Newton's method. Newton's method and its certain modification are, at the present time, among the few methods which can be applied in practice to actually obtaining the solution of a nonlinear functional equation. In fact, most methods of linearization can be seen as a realization of the abstract Newton method applied to a certain mapping on some Banach space. Great credit for the development of this method goes to Kantorovich, and his most general results are translated into English by Feinstein [41]. In this section, we will develop one realization of the Newton method, and give an outline of the application of the results. Later, we will extend Newton's method to bridge the gap between the methods such as quasilinearization, continuation and imbedding.

Suppose that we have a operator P which maps some open sets B of a Banach space E into itself. Let us choose an arbitrary element z_0 of B. Assuming that P has a continuous Frechet derivative DP, then we can replace the identity:

$$P[z_0] = P[z_0] - P[z^*] \tag{27}$$

by the expression:

$$P[z_0] = DP(z_0)[z_0 - z^*] \tag{28}$$

where z^* is the desired root of the operator P. That is:

$$P[z^*] = 0 \tag{29}$$

Consequently, we can expect that the solution z of the equation:

$$DP(z_0)[z_0 - z] = P[z_0] \tag{30}$$

will be close to z^*. Since this equation is a linear operator equation, its solution can be found more easily than the original equation (29). By continuing this process, we can obtain the sequence $[z_n]$ satisfying:

$$z_{n+1} = z_n - DP(z_n)^{-1}[P[z_n]] \quad ; n = 0, 1, 2, \ldots.. \qquad (31)$$

Generally speaking, one expects that this sequence $[z_n]$ converges to the solution z^* of the operator P.

One of the most easy to apply versions of the Kantorovich theorems given in [41] can be stated as follows: **Let P be defined as the mapping of the open set B of a Banach space E into E, and let P have a continuous second Frechet derivative D^2P in the closed ball:**

$$B_0 = [z : \| z - z^0 \| \leq r]$$

where r satisfies the conditions shown below. Moreover, suppose that:

a. $\| DP(z^0)^{-1} \| \leq N$

b. $\| P[z^0] \| \leq k$

c. $\| D^2P(z) \| \leq K$

for any z in the ball B_0. Now if

$$h = KM^2k \leq 1/2 \qquad (32)$$

and

$$r \geq r_0 = [1 - (1 - 2h)^{1/2}]Mk/h \qquad (33)$$

then Newton's method defined by the process (31) is convergent to the solution z^* of the operator equation (29), satisfying:

$$\| z^* - z^0 \| \leq r_0 \qquad (34)$$

Furthermore, if, for $h < 1/2$,

$$r < r_1 = [1 + (1 - 2h)^{1/2}]Mk/h \tag{35}$$

or, for $h = 1/2$,

$$r \leq r_1 \tag{36}$$

the solution z^* will be unique in the ball B_0. The rate of convergence of the process is given by the inequality:

$$\| z^* - z^n \| \leq k[(2h)^2/2]^n/h \quad ; \quad n = 0, 1, 2, \ldots. \tag{37}$$

Let us now rewrite our operator equation (1) in the following form:

$$P[\underline{x}, \underline{u}] = L[\underline{x}][\underline{u}] + \underline{g}(\underline{x}) \tag{38}$$

It is easy to obtain the Frechet derivative of the operator P:

$$DP(\underline{x}, \underline{u})[\underline{v}] = D\underline{v}/Ds + J(\underline{x}, \underline{u})\underline{v} \tag{39}$$

where J denotes the Jacobian of the vector æ($\underline{x}$)$\underline{u}$, as defined in Chapter 2. Therefore, we have:

$$J(\underline{x}, \underline{u}) = æ(\underline{x}) + dæ(\underline{x}, \underline{u}) \tag{40}$$

where:

$$dæ = \begin{pmatrix} \emptyset & A(\underline{n}) \\ \emptyset & A(\underline{m}) \end{pmatrix} + [g_{ij}] \tag{41}$$

The matrix $A(\underline{r})$ for any vector $\underline{r}$ in R^3 can be written:

$$A(\underline{r}) = \begin{pmatrix} 0 & x_{22}r_3 & x_{33}r_2 \\ 0 & -x_{12}r_3 & x_{33}r_1 - x_{13}r_3 \\ 0 & x_{12}r_2 - x_{22}r_1 & x_{13}r_2 \end{pmatrix} \tag{42}$$

where x_{ij} represents the partial derivative of the component x_i of the rotation vector:

$$x_{ij} = (\partial x_i/\partial m_j) \tag{43}$$

The entries g_{ij} of the second matrix on the right hand side of equation (41) stand for the Jacobian of the external force vector $\underline{g}$:

$$g_{ij} = (\partial g_i/\partial x_k)(\partial x_k/\partial m_j) \tag{44}$$

We can now easily write equation (31) explicitly for our operator:

$$\begin{aligned} &D(\underline{u}^{n+1} - \underline{u}^n)/Ds + J(\underline{x}^n, \underline{u}^n)(\underline{u}^{n+1} - \underline{u}^n) \\ &+ D\underline{u}^n/Ds + æ(\underline{x}^n)\underline{u}^n + \underline{g}(\underline{x}^n) = \underline{0} \quad ; \quad n = 0, 1, 2.. \end{aligned} \tag{45}$$

where, as in equation (5), $\underline{x}^n$ of the nth iteration is determined by using:

$$\underline{x}^n = \underline{x}^{n-1} - \underline{\Omega}(\underline{u}^n) \tag{46}$$

Now, making use of equation (40), we can simplify (45):

$$\begin{aligned} &D\underline{u}^{n+1}/Ds + [æ(\underline{x}^n) + dæ(\underline{x}^n, \underline{u}^n)]\underline{u}^{n+1} \\ &+ [æ(\underline{x}^n) - dæ(\underline{x}^n, \underline{u}^n)]\underline{u}^n + \underline{g}(\underline{x}^n) = \underline{0} \end{aligned} \tag{47}$$

The starting solution $\underline{u}^0$ for the Newton process defined by equations (45) or (47) can ideally be the solution of the following operator equation:

$$P[\underline{0}, \underline{u}_0] = \underline{0} \tag{48}$$

which corresponds to the internal forces to maintain equilibrium if the deformed pipeline geometry is a straight line.

Before we give the outline of the application of the Kantorovich theorem, let us point out a very useful observation on equation (47). If we look at equation

(42) carefully, we see that, due to the definition of dimensionless constitutive relations in Chapter 2, the matrices $A(\underline{n})$ and $A(\underline{m})$ contain only terms of order ε. Therefore, if we assume that the terms g_{ij} defined in equation (44) are also of order ε, or that they vanish, then we can approximate (47) by the following ordinary successive approximation method:

$$D\underline{u}^{n+1}/Ds + æ(\underline{x}^n)\underline{u}^{n+1} + \underline{g}(\underline{x}^n) = \underline{0} \; ; \; n = 0, 1, 2.. \qquad (49)$$

As will be discussed in the following section, this equation provides a very simple and efficient method of solution for the problems with small ε.

In differential equations, the method developed (47), based on the Newton method, is known as quasilinearization, and proof of convergence of this method for boundary value problems is given by Roberts and Shipman [36]. In order to use the Kantorovich theorem, stated in the beginning of this section, they take the Banach space $C_0^1(R^6)$ of continuously differentiable functions satisfying the homogeneous boundary conditions, and they let the operator P map $C_0^1(R^6)$ into $C^0(R^6)$ of continuous functions. That result can easily be extended to general two-point boundary value problems by using the simple transformation:

$$\underline{z} = \underline{u} - [\underline{c}_1(b - s) - \underline{c}_2(s - a)] \qquad (50)$$

where the vectors $\underline{c}_1$ and $\underline{c}_2$ are selected so that the boundary conditions are satisfied at the ends of the interval of definition [a, b] for the problem. They equip the space $C_0(R^6)$ with the following norm:

$$||\underline{u}|| = \max_i[\max_s[u_i] + \lambda \max_s[|du_i/ds|]] \qquad (51)$$

where $\max_i[\ ]$ denotes the maximum of the six components of $\underline{u}$, and $\max_s$ stands for the maximum on the interval [a, b]. In the same way, the space $C^0(R^6)$ of external force functions $\underline{g}(\underline{x})$ is given the following norm:

$$\| \underline{g} \| = \max_i [\max_s [\,|g_i|\,]] \tag{52}$$

Then the operator norms can easily be obtained as:

$$\| J \| = \max_s [\max_i \sum_{j=1}^{n} J_{ij}] \quad ; \quad n = 6 \tag{53}$$

$$\| Q \| = \max_s [\sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} Q_{ij}^{k}] \quad ; \quad n = 6 \tag{54}$$

for the linear operator J (for example, the first Frechet derivative J of P) and the multilinear operator Q (for example, the second Frechet derivative D^2P of P), respectively. The parameter λ used in the definition (51) is determined so that the two conditions (35) or (36) can easily be satisfied.

4.2.1 Relation between continuation method and imbedding and extension of Newton's method

Let us now consider the following functional differential equation:

$$D_s D_t \underline{u}(s,\ t) + J(\underline{x}(s,\ t),\ \underline{u}(s,\ t)) D_t \underline{u} + P[\underline{x}^0,\ \underline{u}^0] = \underline{0} \tag{55}$$

where D_s, D_t represent the derivatives with respect to s and t respectively, and t denotes a second independent variable. Let us also assume that the solution of (55) satisfies the initial condition:

$$\underline{u}(s,\ 0) = \underline{u}^0(s), \quad \underline{x}(s,\ 0) = \underline{x}^0(s) \tag{56}$$

However, as shown in [14], if P satisfies some smoothness conditions, then (55) has a unique solution satisfying:

$$P[\underline{x}(s,\ t),\ \underline{u}(s,\ t)] = (1 - t) P[\underline{x}^0,\ \underline{u}^0] \tag{57}$$

for all t in [0, 1]. Therefore, $\underline{x}(s,\ 1)$ and $\underline{u}(s,\ 1)$

are solutions $\underline{x}^*(s)$ and $\underline{u}^*(s)$ of equation (38). The operator P defined by (57) on the functions from R x R into R^3 and R^6 respectively can now be seen as the continuation operator. Unfortunately, unlike the continuation parameters introduced in section 4.1, the parameter t cannot be easily obtained with a physcial meaning. However, the assertation that (57) is equivalent to equation (55) is quite significant, first because equation (55) is a partial differential equation that could be obtained by using imbedding techniques, and second because, as will be shown below, Newton's method can be seen as a special approximation to equation (55), therefore generating a connection to quasilinearization methods.

Let us assume that we can obtain a solution for the initial value problem (55) by using a numerical technique such as the method of lines described in [24]. In this way, we obtain an approximation:

$$\underline{u}^1(s) \simeq \underline{u}(s, 1), \quad \underline{x}^1(s) \simeq \underline{x}(s, 1) \tag{58}$$

If we denote the numerical integration operation by H, we can then write:

$$[\underline{x}^1(s), \underline{u}^1(s))] = H[\underline{x}^0(s), \underline{u}^0(s)] \tag{59}$$

If we use the numerical integration operator H iteratively, we can obtain a sequence $[\underline{x}^n(s), \underline{u}^n(s)]$ of successive approximations:

$$[\underline{x}^{n+1}(s), \underline{u}^{n+1}(s)] = H[\underline{x}^n(s), \underline{u}^n(s)] \;; n = 0,1,2,. \tag{60}$$

We can expect that this sequence will converge to the solutions $\underline{x}^*(s)$, $\underline{u}^*(s)$ of the basic problem defined by equation (38). In fact, if we use the most explicit difference scheme to find $D_t\underline{u}$, that is, Euler's difference scheme, with step length h = 1:

$$D_t\underline{u}(s, t) = \underline{u}^{n+1}(s) - \underline{u}^n(s) \tag{61}$$

then equation (55) would reduce to equation (45), and the scheme defined by equation (60) is therefore equivalent to Newton's method. As we have seen earlier in this section, Newton's scheme converges quadratically. This can mean that although Euler's difference scheme is a crude first order approximation, the integration process H leads to a good approximation $\underline{x}^1(s) \simeq \underline{x}(s, 1)$ and $\underline{u}^1(s) \simeq \underline{u}(s, 1)$. However, this depends on the closeness of the initial guess $\underline{x}^0$ and $\underline{u}^0$ to the solutions, and Newton's method may fail to converge if this starting solution is not close enough to the solution. This failure can be interpreted as the unstable behaviour of Euler's difference scheme when applied to the initial value problem (55). This observation leads Den Heijer [14] to use highly stable integration procedures for solving problems like (55) instead of very accurate ones. In the same way, by using the imbedding equation (55) and stable integration procedures, we can find schemes to replace our nonlinear problem with sequence of linear problems converging to our original problem.

It is also sometimes possible to combine several methods to obtain more desirable techniques. For example, we can use the continuation techniques described in section 4.1 in conjuction with Newton's method as explained in this section.

4.3 REVIEW OF NUMERICAL METHODS FOR STIFF PROBLEMS

Finding a solution to the operator equation (1) in the infinite dimensional function spaces is a very difficult task. However, if the problem can be approximated by an operator on some finite dimensional spaces, then some numerical technique can be used to obtain explicit solutions. This process of reducing the problem to a finite dimensional problem is called discretization. It usually involves construction of a finite dimensional space E to represent the solution and trial spaces (which are $C^1(R^6)$ in our case), and of a new representation L_E of the operator L in this finite dimensional space. The following diagram illustrates the discretization process:

$$\begin{tikzcd} (C^1(R^6), \; \|\cdot\| \;) \arrow[r, "L"] \arrow[d] & (C^1(R^6), \; \|\cdot\| \;) \arrow[d] \\ (E \quad , \; \|\cdot\| \;) \arrow[r, "L_E"] & (E \quad , \; \|\cdot\| \;) \end{tikzcd} \tag{62}$$

The method of definition of the approximation (that is, the definition of the norm $\|\cdot\|$) also forms a part of the proper definition of the discretized problem.

Since the solution of our problem is a vector valued function of a real variable, its discretized form has to be defined on some finite dimensional real Euclidean space R^k. Generally, a discrete problem can be defined in two ways.

a. Pointwise approximation

The real interval [0, 1] (on which our problem is assumed to be defined) is replaced by a finite sequence $[s_i : 0 \leq i \leq N]$, and the value of the solution $\underline{u}$ at each s_i is pointwise approximated. This category includes all finite difference methods.

b. Global approximation

The solution $\underline{u}$ is approximated by a linear combination of some functions $[\underline{v}_i : 0 \leq i \leq N]$ defined on the interval [0, 1], which are usually taken as the basis functions for finite dimensional solution or trial spaces. Finite element methods fall into this category.

However, this classification does not provide a rigid separation, and several methods can be seen as belonging to both of these two groups.

The requirements on the definition of approximation and the norm $\| \cdot \|$ are set in such a way that the solution of the discretized problem converges to the solution of the actual problem. These requirements for general numerical methods can be found in many numerical analysis books dealing with differential equations, such as [24]. The main aim of this Chapter is to show why the numerical solution of the problem studied in this thesis requires special attention, and to describe the desired features for numerical methods that suit best for our problem.

4.3.1 Failure of standard numerical methods

Let us now recall the differential equation (98) of Chapter 3, whose solution majorates the solution of the two-dimensional pipelaying problem. Let us take the first boundary value problem for the homogeneous equation which can easily be transformed into the following form:

$$\varepsilon d^2y/dx^2 - y = 0 \tag{63}$$

with the conditions:

$$y(0) = 1, \quad y(1) = 1 \tag{64}$$

The solution of this problem can be written as:

$$y(x) = [\exp((-2x + 1)/2\sqrt{\varepsilon}) + \exp((2x - 1)/2\sqrt{\varepsilon})]/[\exp(1/2\sqrt{\varepsilon}) + \exp(-1/2\sqrt{\varepsilon})] \tag{65}$$

which can be rewritten, for any $h > 0$, as:

$$y(x) = [d^{(-2x + 1)/2h} + d^{(2x - 1)/2h}]/[d^{1/2h} + d^{-1/2h}] \tag{66}$$

where:

$$d = \exp(h/\sqrt{\varepsilon}) \tag{67}$$

If we replace the differential equation (63) by the difference equation:

$$(y_{i+1} - 2y_i + y_{i-1})/h^2 - y_i = 0 \; ; \quad i = 1, 2, \ldots, N - 1 \tag{68}$$

with the conditions:

$$y_0 = 1, \quad y_N = 1 \tag{69}$$

we can obtain the following discrete solution:

$$y_i = [c^{-i+N/2} + c^{i-N/2}]/[c^{N/2} + c^{-N/2}] \tag{70}$$

where:

$$c = (1 + h^2/2\varepsilon) + [(1 + h^2/2\varepsilon)^2 - 1]^{1/2} \tag{71}$$

It is easy to see that the solutions (70) and (66) are of the same form.

Let us now investigate the behaviour of the discrete solution (70) for two conditions, for $h \ll \sqrt{\varepsilon}$ and for $h \gg \sqrt{\varepsilon}$. If $h \ll \sqrt{\varepsilon}$, the accuracy of approximation y_i is quite good as:

$$|d - c| = 0[(h/\sqrt{\varepsilon})^3] \tag{72}$$

On the other hand, when ε is reduced to zero while keeping h constant, that is, for $h \gg \sqrt{\varepsilon}$, we have:

$$c = h^2/\varepsilon + 2 - \varepsilon/h^2 + \ldots \tag{73}$$

and therefore we obtain:

$$\lim_{\varepsilon \to 0}[y_i] = 0 \quad ; \quad i = 1, 2, \ldots, N - 1 \tag{74}$$

which is the limiting solution of the problem (63). However, in this case, the accuracy of approximation gets poorer as ε gets smaller, since the rate of decay of the boundary layer of the discrete solution is not consistent with that of the actual solution:

$$d = \exp(-h/\varepsilon) << e \tag{75}$$

This accuracy cannot be improved unless h is taken to be of order ε in the boundary layer regions.

It should also be noted that, as illustrated by several examples by Hemker [22], condition (74) is not always satisfied, and in those cases the accuracy of the approximation degenerates throughout the interval [0, 1] as ε gets smaller. Therefore, application of a standard numerical technique to a stiff problem either leads to inaccurate representation of the solution for small values of stiffness parameter ε, and/or boundary layers in the solution are not properly resolved.

4.3.2 Norms of approximation and desirable features of stiff methods

Let us define a partition on the interval [0, 1]:

$$I = [s_i \ : \ 0 = s_0 < s_1 < s_2 < \dots < s_N = 1] \tag{76}$$

and let us write:

$$h = \min_i[(s_{i+1} - s_i) \ ; \quad i = 0, 1, 2, \dots, N - 1] \tag{77}$$

Then the following norms, which are quite commonly used for pointwise approximation, can be defined:

$$\| \underline{u} - \underline{v}_I \|_{I,k} = [\sum_{i=1}^{N} |\underline{u}(s_i) - \underline{v}_i|^k]^{1/k} \tag{78}$$

or in particular:

$$\| \underline{u} - \underline{v}_I \|_{I,0} = \max_i[|\underline{u}(s_i) - \underline{v}_i|] \tag{79}$$

where $\underline{v}_I$ represents the discrete solution defined on the partition I, and $\underline{v}_i$ gives its value at s_i. It is sometimes useful to replace this norm by the following:

$$\| \underline{u} - \underline{v}_I \| ^h_{I,k} = [\sum_{i=1}^{N} (h_i | \underline{u}(s_i) - v_i |)^k]^{1/k} \tag{80}$$

We should emphasize here that these norms depend crucially on the choice of the partition I which is a part of the definition of the numerical problem.

For measuring the accuracy of the global methods, the following integral norms are commonly used:

$$\| \underline{u} - \underline{v} \|_{0,2} = [\int_0^1 (\underline{u} - \underline{v}_E) \cdot (\underline{u} - \underline{v}_E) ds]^{1/2} \tag{81}$$

or:

$$\| \underline{u} - \underline{v} \|_{0,0} = \max_i [\max_s | u_i - v_i |] \tag{82}$$

where $\underline{v}_E$ is the approximating function taken from the finite dimensional space E.

If a good global approximation of the solution is required throughout the interval [0, 1], then the norm $\| \cdot \|_{0,0}$ is best suited. However, if the representation of boundary layer is not so important as long as the accuracy of the solution is not affected in the rest, then $\| \cdot \|_{0,2}$ can be used.

In the case of pointwise approximations, in order to decide on the accuracy of approximation, especially on certain parts of [0, 1], the partition I must be chosen accordingly, and the norm $\| \cdot \|_{I,0}$ would be best to achieve this objective. This means that the structure of a pointwise method would greatly depend on the selection of the partition I.

The definitions listed below give the basic requirements for stiff methods:

Definition: A numerical approximation $\underline{v}_I$ is called pointwise exact on a grid I if:

$$\| \underline{u} - \underline{v} \|_{I,1} = 0 \qquad (83)$$

Definition: A method is said to be uniformly ε-convergent of order p if there exist constants b, K independent of ε, such that:

$$\sup \| \underline{u}_\varepsilon - \underline{v}_{\varepsilon,I} \|_{I,0} \leq Kh^p \qquad (84)$$

if ε remains in the interval [0, b].

Definition: A method is called consistent with the reduced problem on the interval $[\delta, 1 - \delta]$ if:

$$\lim_{h \to 0} \underline{v}_{\varepsilon,I} = \underline{u}_0 \qquad (85)$$

for $0 < \delta < 1/2$.

These properties can be obtained in several ways. However, the actual choice of method depends on the problem being solved. Sometimes it is quite difficult to prove that a given method satisfies these requirements, or the converse.

The most potential techniques in both categories of the discretization methods are based on exponential fitting. In the case of finite differences, an exponentially fitted difference operator can be written as:

$$\underline{u}_{n+1} - \underline{u}_n = h[(1 - \lambda)\underline{f}_{n+1} + \lambda \underline{f}_n] \qquad (86)$$

for a differential equation of the form:

$$D\underline{u}/Ds = \underline{f}(\underline{u}, s) \qquad (87)$$

which corresponds to:

$$\underline{f}(\underline{u}, s) = - æ\underline{u} - \underline{g} \tag{88}$$

in our problem. The parameter λ is selected so that the numerical approximation $\underline{u}$ is made pointwise exact according to the definition given above. However, it should always be chosen less than 1/2 in order to satisfy some stability requirements (see [24]).

In the case of global methods, where the problem is defined by utilizing Sobolev norms:

$$\| \underline{u} - \underline{v}_E \| = [(\underline{u} - \underline{v}_E, \underline{u} - \underline{v}_E)_k]^{1/2} \tag{87}$$

where:

$$(\underline{w}, \underline{z})_k = \sum_{i=0}^{k} [\int_0^1 D^i\underline{w}.D^i\underline{z}ds] \tag{90}$$

the solution $\underline{u}$ satisfies:

$$(L[\underline{u}_E], \underline{v}_E)_k + (\underline{g}, \underline{v}_E)_k = \underline{0} \tag{91}$$

for all $\underline{v}_E$ in E, where L corresponds to our linear operator $L[\underline{x}]$, but is now defined on some finite dimensional subspace E of some Sobolev spaces containing $C^1(R^6)$. A detailed study of this method for stiff two point boundary value problems is given by Hemker [24]. However, in order to obtain uniform ε-convergence, Hemker [24] develops a class of exponentially fitted spaces with base functions behaving like exponential functions. He illustrates the poor performance of the classical global techniques compared to exponentially fitted techniques.

4.3.3 Selection of a code and available codes for numerical solution of two point boundary value problems

The selection of a discretization method is one of the most important steps in obtaining a numerical solution for a certain problem. Unfortunately, there is no straightforward guide for selection of the best method for a given problem. The question involves a number of practical considerations, such as:

- efficiency

- ease of implementation

- reliability

- flexibility

- modularity

as well as theoretical considerations such as order of accuracy of approximations and stability of the process. Usually the answer to this question is found by finding a compromise between available computer resources and effort to implement the method. In many cases, certain aspects or difficulties can only be solved by experimentation. In the case of two-point boundary value problems, the comparison of suitable methods mostly rests on the matter of efficiency since the normal advantages of flexibility and modularity of global methods in approximation of complicated regions for field problems is not valid in the case of one-dimensional problems. Therefore, there is no clear advantage in selecting finite element like methods.

In the past decade, numerical analysis literature on stiff methods has shown a dramatic explosion. Many algorithms have appeared in several conferences [6, 16, 19, 23, 25, 35, 43]. Two conferences [23, 43] especially were dedicated to methods for stiff problems. In [43], Willoughby gives an extensive historical review of stiff problems and methods. The most recent biography of the subject can be found in [19], which also gives the source of several computer codes for two point boundary value problems. Some of these codes were tested by Pereyra in [6]. One of the simple and well tested algorithms, initially developed by Pereyra, has been presented in a joint paper by Lentini and Pereyra [29]. This technique is actually based on an adaptive mesh generation algorithm. The boundary layers are resolved by refining the grid points as becomes necessary and the accuracy of approximation is controlled by using deferred corrections. The latest version of the program PASVAR based on this algorithm is presented by Pereyra in [19]. In [22], Hemker gives a program based on an exponentially fitted weighted residual method. However, adaptibility of a general purpose code such as PASVAR is far superior to programs such as the one given by Hemker [22], although efficiency of the exponentially fitted methods can be quite high for problems with steep boundary layers.

4.4 CONSTRUCTION OF A NUMERICAL ALGORITHM FOR THE THREE-DIMENSIONAL PIPELAYING PROBLEM

The pipelaying problem formulated in section 2.9 demonstrates most of the typical difficulties associated with pipeline and riser problems. On one hand, strong geometrical nonlinearity, due to the dependence of the external forces on the solution, makes it very difficult to set up the final state equations. On the other hand, the nonlinear free boundary condition (60) given in section 2.9 means that the scaling parameters L, T have to be added to the problem as additional unknowns. In general, there are two principal ways to obtain a numerical solution for a nonlinear problem: either the problem is first fully stated, and then a Newton-like method is used to solve the nonlinear discrete equations; or Newton's method is first used to linearize the problem, and then a discrete model of this linearized problem is constructed. At the end, one obtains a linear discrete problem on a finite dimensional Euclidean space R^n, whose solution is fairly straightforward to compute. Keeping in mind the difficulties of the pipelaying problem, as stated above, one can see that the latter method (linearizing the problem first and discretizing it in the second step) is much easier to implement than the other method. It is also quite easy to incorporate the method of continuation in this method.

The basic features of an algorithm designed in accordance with this method are shown in Figure 8. A more detailed description is given below.

Step I Initialize the starting value for the continuation parameter by taking $\lambda_0 = 1/N$ for a given N. Initialize a starting geometry by taking $\underline{x}_0(s) = 0$ on $[0, 1]$. Set the parameter $L_0 = NL^0$, where L^0 is the span length of the catenary solution (if it does not exist, take L^0 as the water depth). Take $T_0 = T^*$, which corresponds to the tension at the tensioner.

Step II Generate a grid distribution:

$$I_k = [0 = s_1 < s_2 < \ldots < s_k = 1]$$

by using the inverse of the asymptotic solution $z_1(s)$ obtained in Chapter 3.

Step III Calculate the external forces by using the geometry defined by $\underline{x}_i(s)$, transformation matrix $[a_{mn}]_i$, and the scaling parameters L_i and T_i.

Step IV Set up the system matrix of the discretized problem corresponding to the operator (1) by using Keller's mid-point scheme [22] (see Figure 9), and incorporate it in the overall system matrix if another part of the pipeline is to be analysed (such as overbend etc). Solve the resulting linear system for internal forces $\underline{u}_i$.

Step V Calculate the corresponding components of $\underline{x}_{i+1}$ by using $\underline{u}_i$ and constitutive relations defined in Chapter 2. Also calculate the angles Θ_i, Φ_i.

Step VIa Check if $\| \underline{x}_{i+1} - \underline{x}_i \|^{h}_{I,0}$ is less than a given accuracy: if not, jump to Step VII

VIb Check if $\lambda_i = 1$: if so, go to EXIT; if not increment $\lambda_{i+1} = \lambda_i + 1/N$

Step VII If it is required (can be switched on or off) use Newton's method (47) successively to obtain improved new estimate $\underline{x}_{i+1}(s)$ for the geometry of the pipe.

Step VIII Obtain a new transformation matrix $[a_{mn}]_{i+1}$ by integrating equation (16) of Chapter 2. Use a stiff initial value integrator for this purpose. In order to improve the stability of the scheme orthonormalize $[a_{mn}]_i$ at each grid point s_i.

Step IX Obtain a new L_{i+1} by using the new gemetry detailed by $\underline{x}_{i+1}$ and angles Θ_i, Φ_i, and compatibility condition (60) of Chapter 2. Analyse overbend either to obtain T_{i+1} or prepare discrete system matrix for the overbend to be incorporated into the overall system matrix.

Step X Increment $i = i + 1$ and if i is not greater than a given i_{max}, go to Step III.

It should be noted that all the entities used here actually contain now k components, that is, for example, $\underline{u}_i$ stands now for a matrix of $n \times k$ $(n = 5)$ containing k times the vector $\underline{u}$ defined at each s_i. The matrix $[a_{mn}]_i$ now denotes a matrix of $n \times n \times k$, that is, a collection of $n \times n$ matrices defined at each s_i. Scalar variables such as Θ_i or Φ_i are represented by k dimensional vectors. Therefore, the problem is now defined in a finite dimensional space $R^{5k} \times R^{5k}$, assuming no torsional moment is applied along the length of the pipeline.

4.4.1 Practical features of the pipelaying algorithm

The important features of the algorithm described above can be listed as follows:

1. It does not require a complicated mesh generation algorithm. Instead, the grid distribution is generated by using the information obtained from the asymptotic solution obtained in Chapter 3.

2. Step V enables the user to define his own constitutive relations - linear or nonlinear.

3. Step IX is actually equivalent to a multiplexing technique where a discretized version of overbend or other components of the pipeline can be built into the matrix of the overall system. This idea can be extended to include plastic hinges due to a local failure of the pipeline when the bending moment exceeds a certain limit.

4. It makes it possible to switch on or off both continuation and Newton's methods, thus enabling the user to optimize usage of his computer

resources and control convergence in the case of very stubborn problems.

5. This algorithm does not require any starting solution.

6. As can be seen from Figure 9, the matrix of the discretized problem is highly sparse, and can therefore be placed in a small core size computer. It is also possible to use a transfer matrix method to minimize the core requirements.

7. The algorithm has to converge to the right solution due to Steps V and VI.

A core wise version of this algorithm is implemented in order to demonstrate that the problem can be solved with very limited computer resources. Figure 10 gives the intermediate and final geometries of the pipeline span during laying for successive values of the continuation parameter λ. As can be seen, the algorithm converges fairly quickly to the solution. Figure 11 shows the comparison of the numerical solution obtained by this algorithm with the singular perturbation solution. As can be seen, the boundary layers are resolved more effectively than the asymptotic approximations. A detailed set of results and the required input data (prepared in a form compatible with an interpreter which was designed as a user interface for a mathematical software package for pipeline and riser problems) are also given in the Appendix. Those data correspond to the parameters of the 36 inch Ninian Field pipeline laid in the northern North Sea [34].

CHAPTER 5 CONCLUSIONS AND CRITICISM OF EXISTING LITERATURE

5.1 Conclusions

5.2 Inconsistencies in the implementation of numerical methods

CHAPTER 5 CONCLUSIONS AND CRITICISM OF EXISTING LITERATURE

5.1 CONCLUSIONS

As is seen from the developments in Chapter 2, great unification is obtained by taking the stress as a basis of formulation, rather than the strain. Although this may seem to lead to considerable simplification, nevertheless the task of description (or approximation) of the three-dimensional strain field in terms of the geometry of a space curve still constitutes a difficult step on the way to the final result. The reduction of the problem from R^3 into a one-dimensional Euclidean manifold is not so trivial if a significant improvement is to be obtained over Kirchoff's Hypothesis. The area of applicablity of the results obtained in this thesis will be significantly enlarged parallel to developments in this direction.

Apart from this point, the formulation developed in Chapter 2 represents the ultimate limit of one-dimensional rod theories. However, it is important to stress the fact that we have considerably deviated from classical elasticity in order to obtain the general results obtained in that Chapter. The essential component of such a formulation is basically differential geometry of space curves.

The results of Chapter 3 reveal some interesting relationships between intuitive concepts and mathematical notions. This not only enabled us to use the results of functional analysis to prove the existence and uniqueness of solutions, but also brought a clearer understanding of certain aspects of the problems connected with boundary layers in the solutions. It is important to note that the methods used in Chapter 3, as well as the results obtained there, can be applied to similar problems arising from pipelines or risers.

In Chapter 4, we have aimed to develop a firm mathematical footing for the numerical solution of general pipeline and riser problems. In this way, we have been able to clarify the interaction or relation between the discretization and linearizatioin steps and to attach a mathematical meaning to several intuitive methods which are presented in Chapter 4 in their most general form. One of the side products of Chapter 4 is that we have laid the principles of some constructive existence proofs. Although the proofs are not all completed in this thesis, we have raised the problem to a level where it can be picked up by a functional analyst without requiring any knowledge of the physical problem.

Chapter 4 also reveals the shortcomings of widely used classical numerical techniques when they are applied to some pipeline problems. We have illustrated in Chapter 4 that, by using the information about the qualitative behaviour of solutions obtained in Chapter 3, it is quite easy to develop stiff numerical solution techniques for pipeline problems. We have also demonstrated that the reliability and efficiency of any numerical solution technique can be greatly enhanced by choosing an optimal linearization method.

5.2 INCONSISTENCIES IN THE IMPLEMENTATION OF NUMERICAL METHODS

5.2.1 Inconsistent linearizations

It is quite important not to confuse the two different steps, linearization and discretization, as this can lead to inconsistent approximations. One of the common practices adopted by pipeline and riser literature is to combine linearization with discretization in one step. While doing this, some of the papers reduce the equilibrium equations into the following form

$$D\underline{u}/Ds + æ^0\underline{u} + \underline{g} = 0 \qquad (1)$$

where æ^0 is a constant matrix which corresponds to a prescribed geometry $\underline{x}^0(s)$. This procedure was sometimes supported by an intuitive argument that if the grid points are taken close enough (or if the finite elements are taken small enough) then the discrete equations can be replaced with some approximate equations that correspond to a linearized problem. This procedure is usually induced by the difficulty in describing the geometry of a space curve. Although for example Oran [31] admittedly states what his assumptions actually amount to, his results were quite commonly used to obtain discretized formulations of pipeline and riser problems [32], [34]. However, in spite of the fact that the problems that they treat involve large geometry changes, they do not seem to be aware of this discrepancy; it actually corresponds to approximating the problem by another rather than finding approximate solutions for the actual problem.

5.2.2 Consequence of usage of non-stiff numerical methods

In Chapter 4, we have illustrated the importance of selection of an appropriate stiff numerical method for a pipeline problem. We have also given criteria required for stiff methods in that Chapter. However, in pipeline literature, it is quite common to use uniformly distributed meshes with classical finite difference or finite element methods. These methods normally lead to incorrect representation of boundary layers. As can be seen from Figure 3, this inaccuracy in the boundary layers then forces the outer solutions to a different outer integral curve, thus resulting in a shift in the solution. Therefore, the solutions obtained by such methods do not converge to the limiting solution of the problem as ε gets smaller and so the condition of uniform-ε-convergence is not satisfied.

5.2.3 A simple test criterion for three-dimensional pipeline problems

One of the other points that reveals inconsistencies in the solution of three-dimensional pipeline and riser problems is the change of torsional moment along the pipe or riser span when linear constitutive relations are used. If we recall the proposition proven in section 3.1, the torsional moment should remain constant along the span length if there is no external torsional moment applied on the pipeline or riser. This point is usually related with the selection of the matrix $æ^0$ in equation (1) if the problem is replaced by an approximation, as described in section 5.2.1. Since this proposition does not necessarily hold for any matrix $æ^0$ selected, the results obtained may violate the statement of that proposition.

It seems that this criterion is not usually applied in the pipeline literature, and a number of programs, which were developed for the pipeline industry (such as the one referred to in [34]), seem not to satisfy this requirement.

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<u>FIGURES</u>

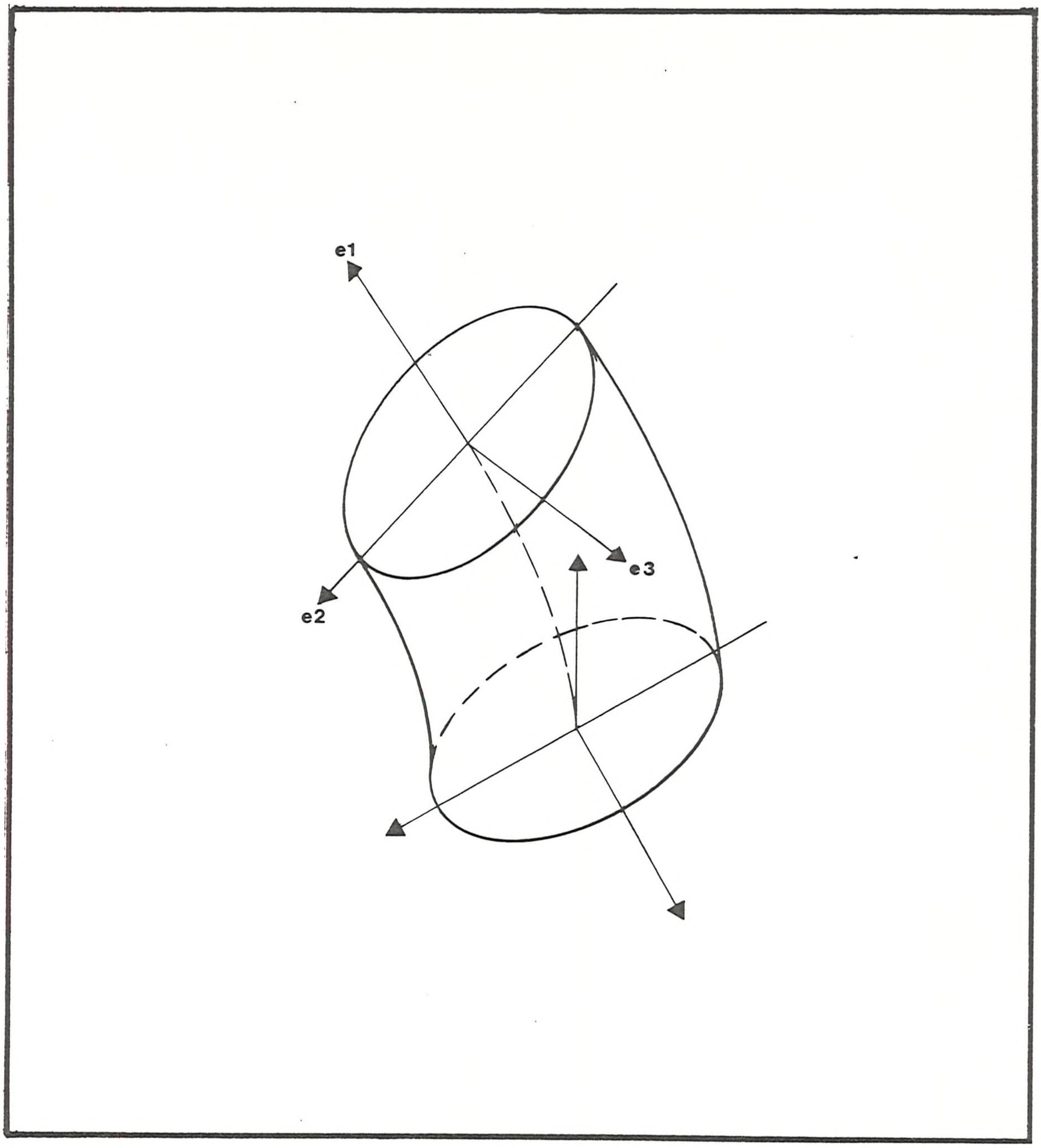

FIGURE 1 LOCAL COORDINATES

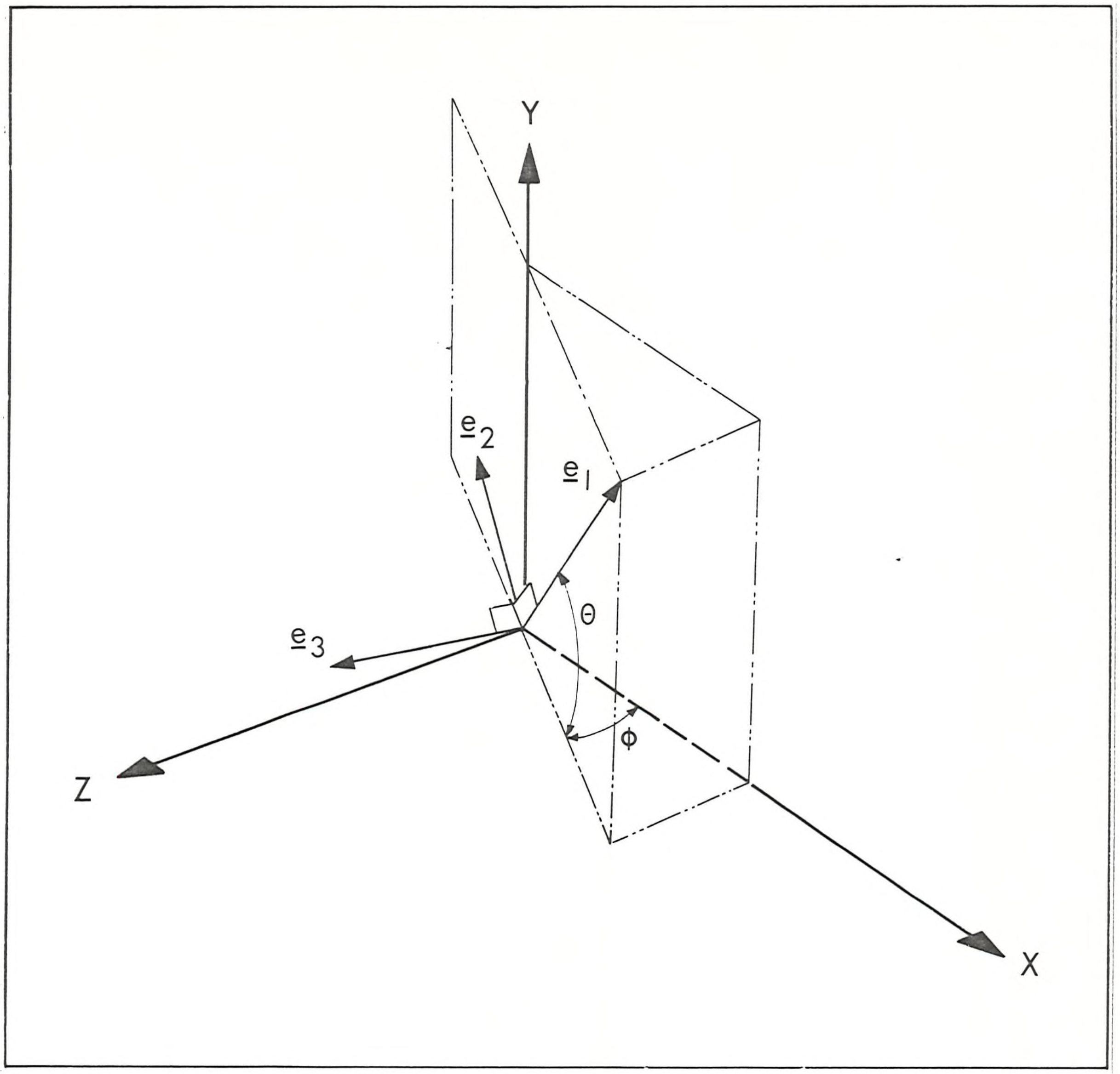

FIGURE 2 ILLUSTRATION OF LOCAL AND GLOBAL COORDINATES

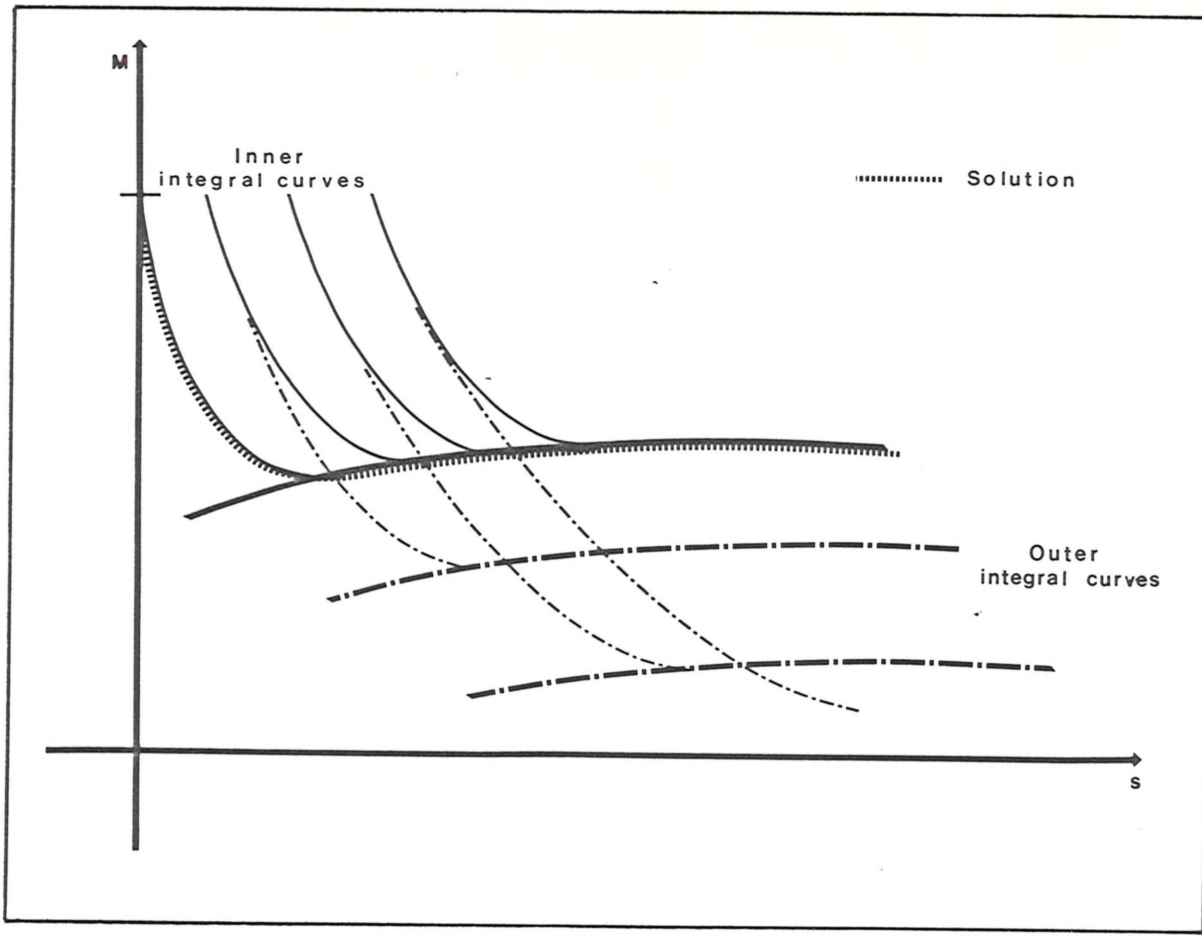

FIGURE 3 ILLUSTRATION OF BEHAVIOUR OF SOLUTIONS OF STIFF DIFFERENTIAL SYSTEMS

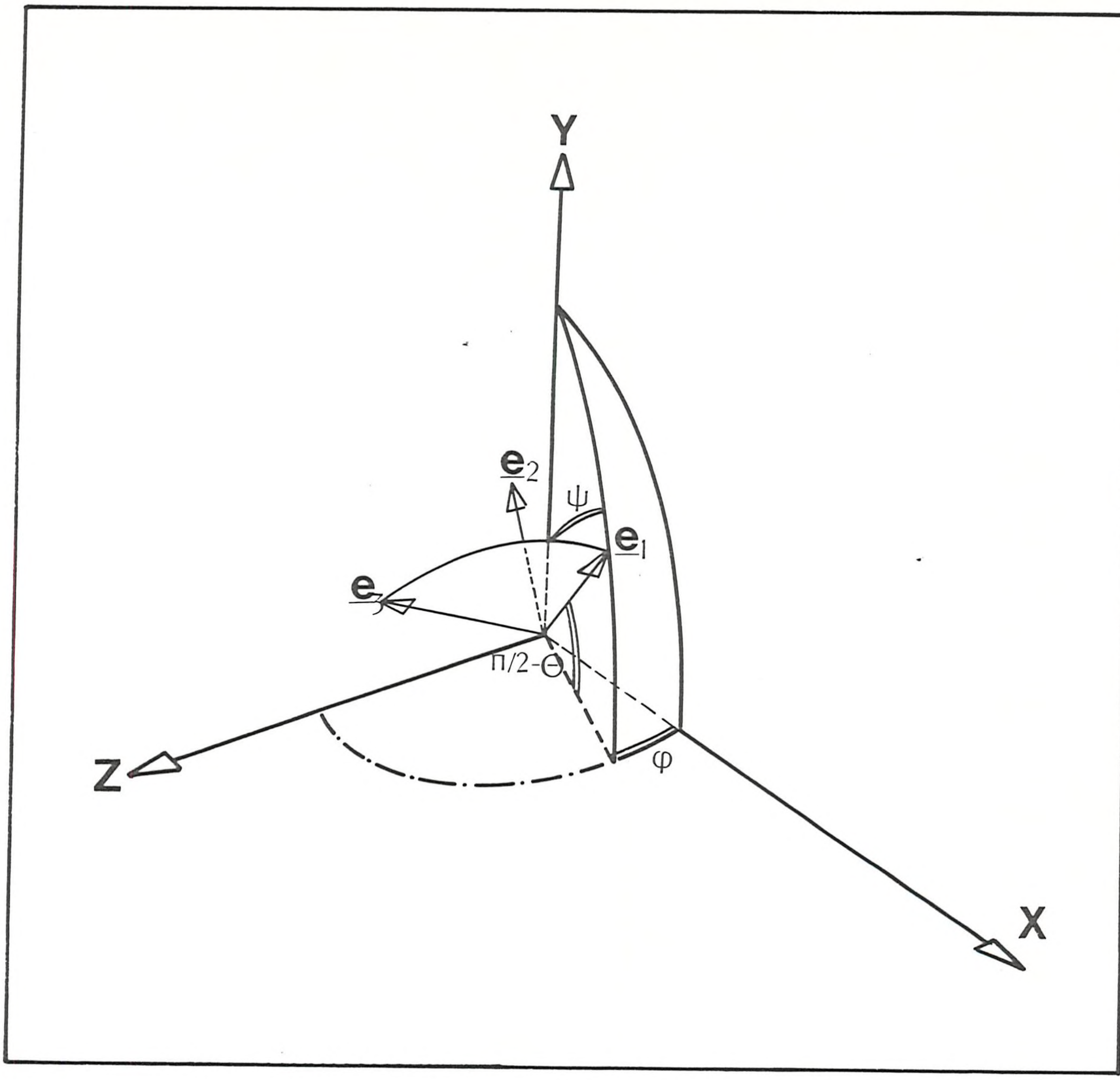

FIGURE 4 ILLUSTRATION OF EULERIAN COORDINATES

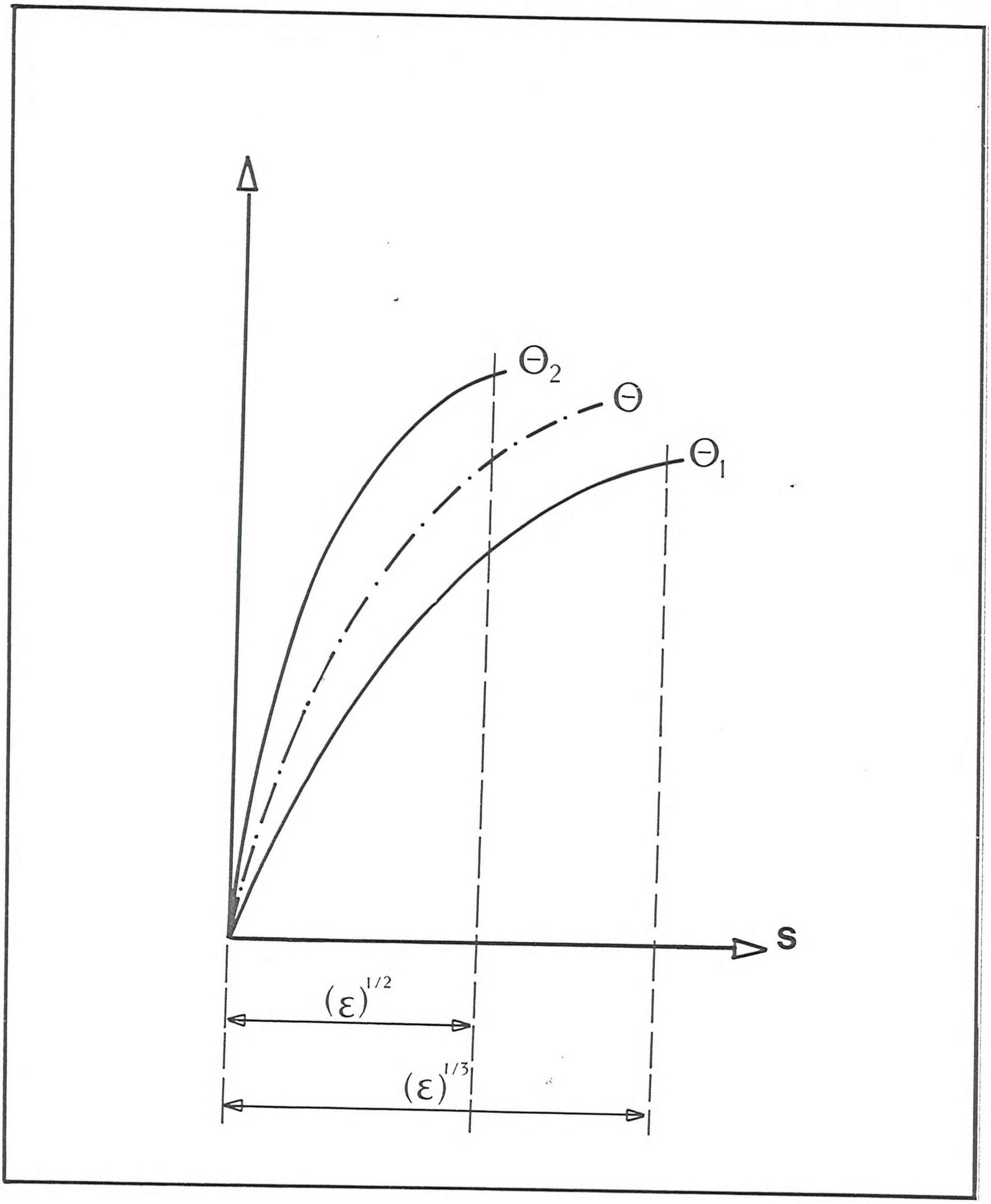

FIGURE 5 ILLUSTRATION OF BOUNDARY LAYER AT $s = 0$

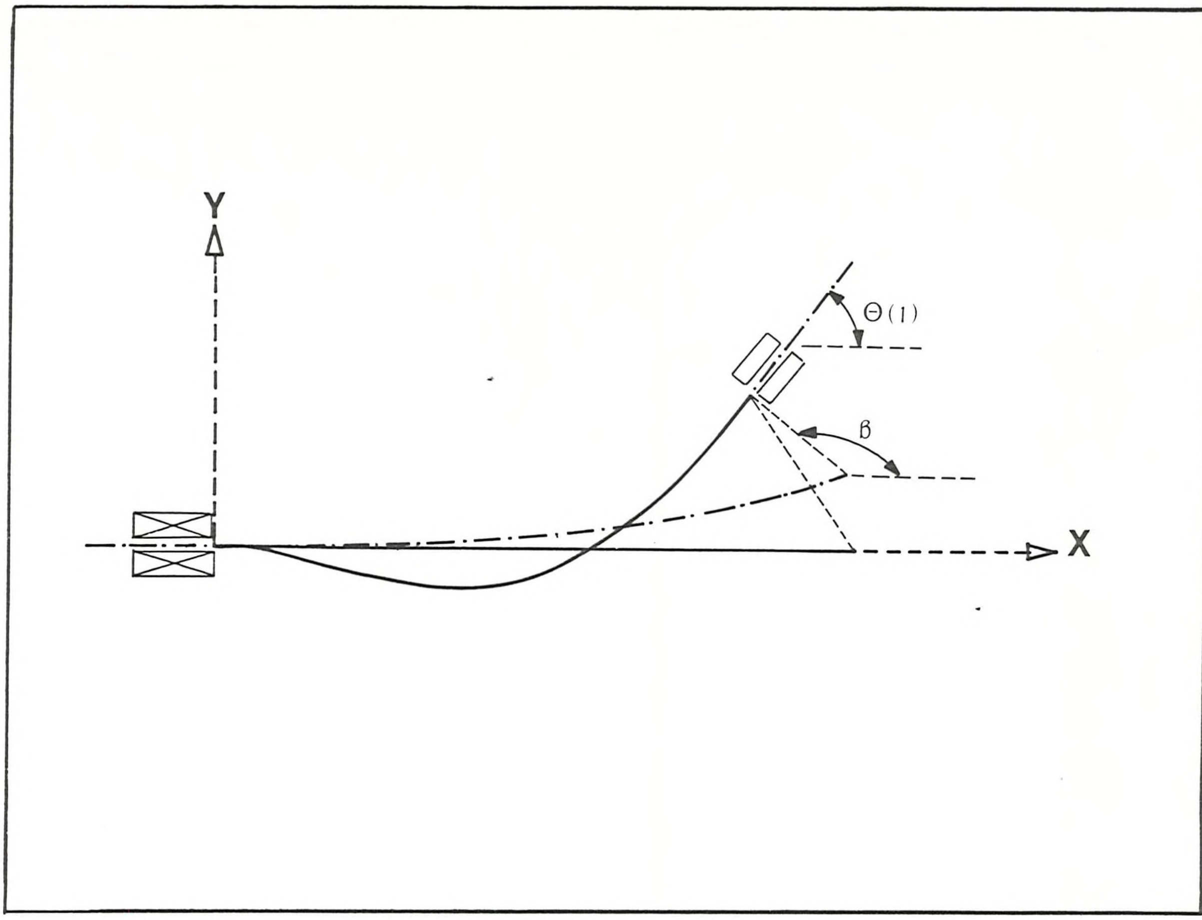

FIGURE 6 ILLUSTRATION OF DEFLECTION OF A PIPELINE BY END-FORCES ONLY

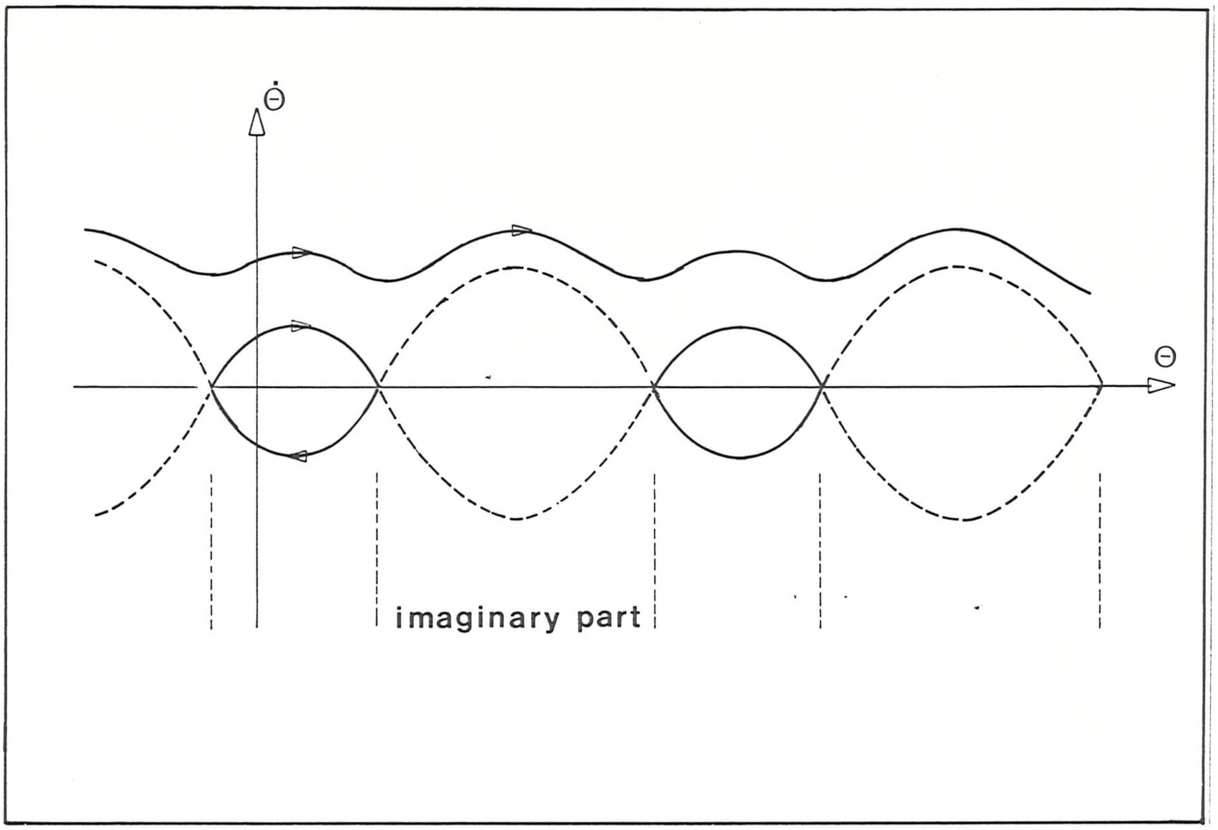

FIGURE 7 PHASE DIAGRAM OF THE EQUATION

$$\varepsilon\dot{\Theta}^2 = 2(c - \text{cotan}\beta \sin\Theta - \cos\Theta)$$

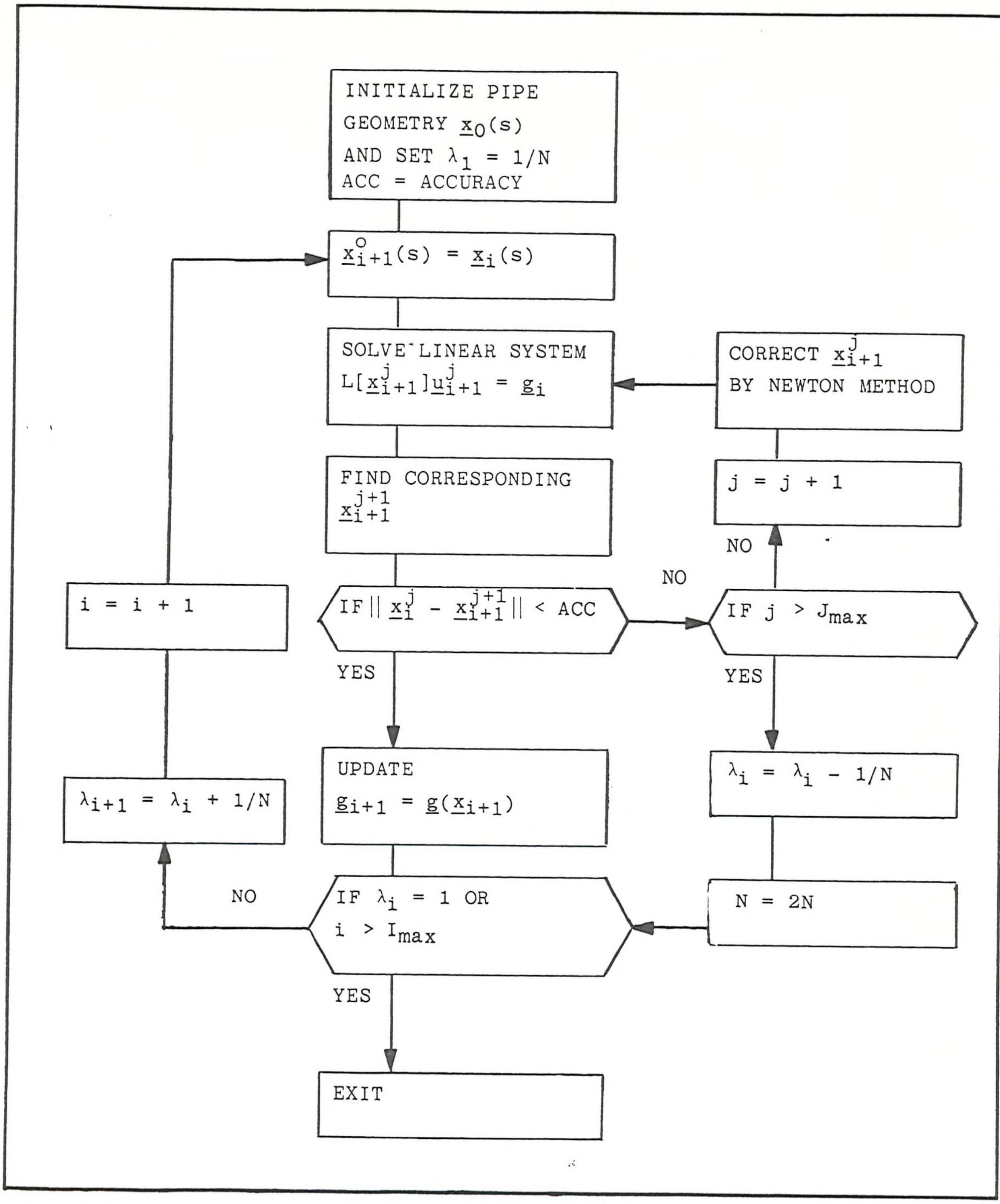

FIGURE 8 FLOWCHART OF THE TREX PROGRAM

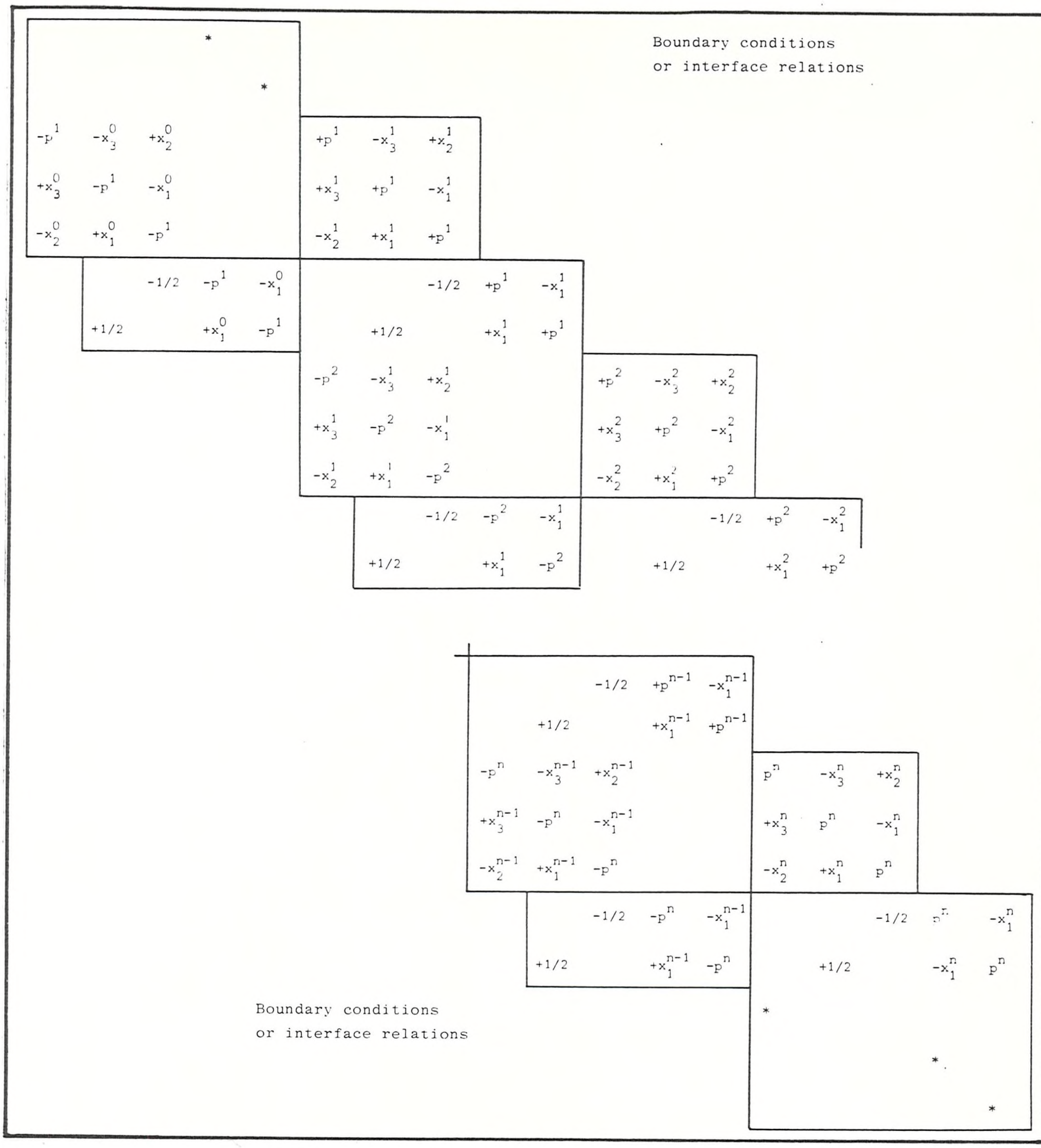

FIGURE 9 STRUCTURE OF THE DISCRETE SYSTEM MATRIX CORRESPONDING TO THE SAGBEND IN THE PIPELAYING PROBLEM

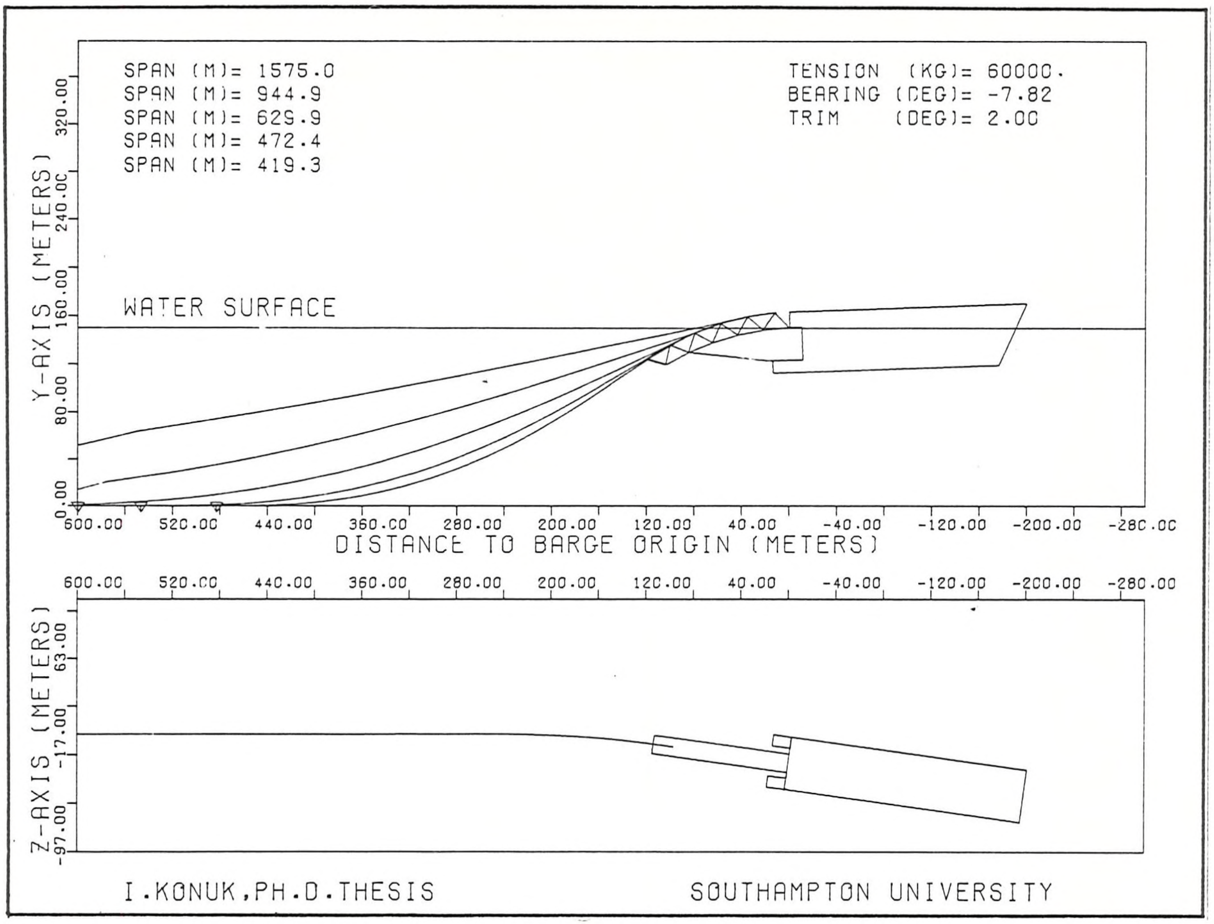

FIGURE 10 GRAPHICAL OUTPUT FROM THE PIPELAYING STRESS ANALYSIS PROGRAM TREX, WITH INTERMEDIATE AND FINAL RESULTS

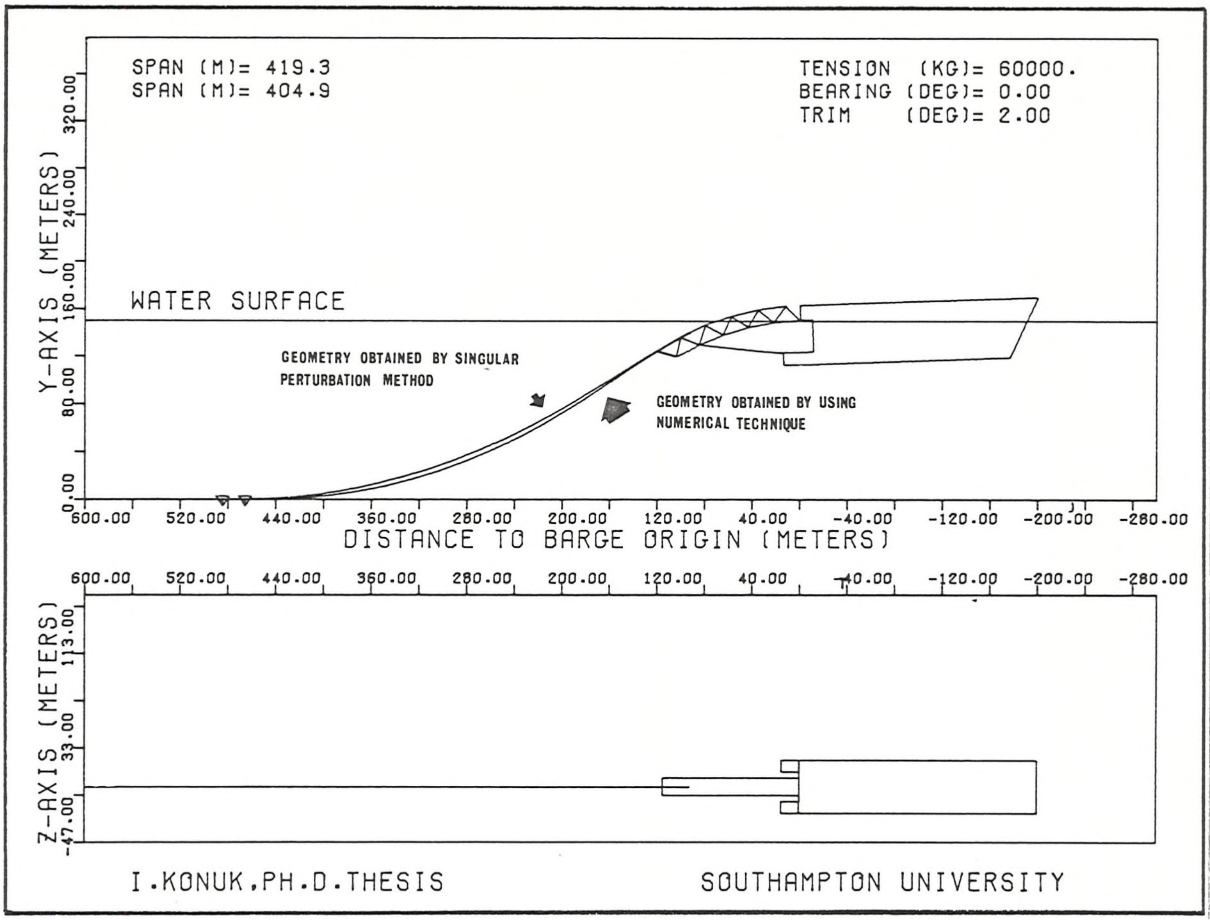

FIGURE 11 COMPARISON OF PIPELINE GEOMETRIES OBTAINED BY SINGULAR PERTURBATION AND BY A NUMERICAL METHOD

APPENDIX

SAMPLE INPUT AND OUTPUT FROM THE

PIPELAYING STRESS ANALYSIS PROGRAM

TREX

SAMPLE INPUT FOR THE PROGRAM TREX

```
FILENAME: TREXIN/SEPTEMBER 1977: SAMPLE INPUT FILE
         FOR PROGRAM TREX
STATUS ON THE FILE: ICL1907/METRIC, BRITISH/TREX1A
CONTENTS: 8 BLOCKS/IN RESPECTIVE ORDER
*PROJECT DESCRIPTION*--------------------------------
PROJECT DESCRIPTION     = PH.D THESIS
PROJECT NAME            = NINIAN FIELD MAIN LINE
AUTHORIZED USERS        = I.KONUK
EXPIRES ON              = 10.11.1978
*PIPE DATA*-----BLOCK NB=1--------------------------
1 7
0.9144      1-OUTSIDE DIAMETER      (M,FT)
984.997     2-DRY WEIGHT            (KG/M,LB/FT)
19.50E-3    3-WALL THICKNESS        (M,FT)
3.75E-3     4-COAT THICKNESS        (M,FT)
65.0E-3     5-CONCRETE THICKNESS    (M,FT)
1.2         6-DRAG COEFFICIENT
0.          7-SKIN FRICTION COEFFICIENT
*BARGE DATA*----BLOCK NB=2--------------------------
1 12
60.E3       1-TENSION               (KG,LB
228.6       2-STERN RADIUS          (M,FT)
0.0         3-X-POSITION OF BARGE   (M,FT)
0.0         4-Z-POSITION OF BARGE   (M,FT)
0.          5-PIPE/BARGE BEARING    DEGREES
2.          6-BARGE TRIM            DEGREES
20000.0     7-TORSION ON THE PIPE   (KG-M,LB-FT)
700.        8-ESTIMATED SAGBEND
0.          9-BARGE SIDEWAYS DIS.   (M,FT)
13.2       10-FREEBOARD             (M,FT)
550.       11-ESTIMATED UPPER LENGTH
155.0      12-STERN LENGTH
*CABLE DATA*----BLOCK NB=3----------------(NOT USED)
1 7
22.34       1-DRY WEIGHT OF CABLE   (KG/M,LB/FT)
19.44       2-SUBMERGED WEIGHT      (KG/M,LB/FT)
0.          3-WEIGHT OF A/R HEAD    (KG,LB)
0.          4-LENGTH OF A/R HEAD    (M,FT)
0.          5-WINCH FOOTAGE         (M,FT)
0.          6-BARGE MOVEMENT:A/R    (M,FT)
0.          7-HORIZONTAL DI.:A/R    (M,FT)
*MATERIAL DATA*-BLOCK NB=4--------------------------
1 6
2723.       1-CONCRET SP. WEIGHT    (KG/M3,LB/FT3)
7850.       2-STEEL SP.WEIGHT       (KG/M3,LB/FT3)
1442.       3-COAT SP. WEIGHT       (KG/M3,LB/FT3)
2.1E10      4-YOUNG S MODULUS       (KG/M2,LB/FT2)
8.9E5       5-SHEAR MODULUS         (KG/M2,LB/FT2)
4.574E7     6-YIELD STRESS          (KG/M2,LB/FT2)
```

```
*ENVIRONMENTAL DATA*K NB=5--------------------------
1 9
150.        1-WATER DEPTH AT OR.   (M,FT)
150.        2-WATER DEPTH AT BA    (M,FT)
1025.       3-WATER SPEC. WEIGHT   (KG/M3,LB/FT3)
2000.       4-DAMPING COEFF. -----
0.          5-SOIL SPRING CONSTANT  -----
0.          6-BOTTOM FRICTION COEF. -----
0.0         7-BOTTOM SLOPE   -----
0.0000      8-BOTTOM PROFILE CURVATURE(1/M,1/FT)
9.81        9-ACCELERATION OF GRAVITY
*SYSTEM PARAMETERS*CK NB=6--------------------------
1 15
1           1-PROGRAM USED         (1,2,3)
4           2-MODE OF PROGRAM      (1,2,ETC)
3           3-TYPE OF INPUT DATA   (1,2)
45          4-NO OF ITERATIONS     (1-50)
3           5-ACCURACY EXPONENT    (-1,6)
2           6-UNIT SYSTEM          (1-BR,2-MET)
2           7-INPUT FILE UNITS     (1,2,3-SI)
33          8-NO OF GRIDS          (3-21)
2           9-DEBUG MODE CHOOSEN
3          10-TEST 1, NORMAL 0, INTERACTIVE -1.
18         11-STEP LENGTH DURING MN CONVERGENCE.
75         12-SPEED OF DE-STIFFENING IN 3RD DIM.
8          13-NUMBER OF VALUES IN CURRENTS ARRAY
2          14-PLOT SWITCH:(1 OFF,2 ON)
7          15-ROLLER SWITCH
*CURRENT DATA*--BLOCK NB=7--------------------------
1 8
0.   0.   1.5
10.  0.   1.25
25.  0.   1.0
40.  0.   0.8
50.  0.   0.75
75.  0.   0.50
125. 0.   0.
135. 0.  -0.50
*EXECUTE*-------------------------------------------
--------------END OF INPUT FILE--------------------
```

SAMPLE OUTPUT FROM THE PROGRAM TREX

```
                PROGRAM :TREX1A                                              DATE: 29/09/77
                UNITS    :METRIC                                             PAGE:     1


#PIPE DATA#
 OUTSIDE DIAMETER                 =      0.914 METER
 DRY WEIGHT                       =    984.997 KG/M
 WALL THICKNESS                   =      0.019 METER
 COAT THICKNESS                   =      0.004 METER
 CONCRETE THICKNESS               =      0.065 METER
 DRAG COEFFICIENT                 =      1.200
 SKIN FRICTION COEFFICIENT        =      0.000


#BARGE DATA#
 TENSION                          =    60000.0  KG
 STERN RAMP RADIUS                =      228.6 METER
 X-POSITION OF BARGE              =        0.0 METER
 Z-POSITION OF BARGE              =        0.0 METER
 PIPE/BARGE BEARING DIFFER.       =        0.0 DEGREES
 BARGE TRIM                       =        2.0 DEGREES
 TORSION ON THE PIPE END          =    20000.0 KG-M
 ESTIMATED SAGBEND LENGTH         =      700.0 METER
 BARGE SIDEWAYS DISPLACEMENT      =        0.0 METER
 FREEBOARD                        =       13.2 METER


#CABLE DATA#
 DRY WEIGHT OF CABLE              =      22.34 KG/M
 SUBMERGED WEIGHT OF CABLE        =      19.44 KG/M
 WEIGHT OF A/R HEAD               =       0.00  KG
 LENGTH OF A/R HEAD               =       0.00 METER
 WINCH FOOTAGE                    =        0.0 METER
 BARGE MOVEMENT (A/R)             =        0.0 METER
 HORIZONTAL DI. (A/R)             =        0.0 METER
```

```
                PROGRAM :TREX1A                                              DATE: 29/09/77
                UNITS    :METRIC                                             PAGE:     2


#MATERIAL PROPERTIES#
 CONCRETE SPECIFIC WEIGHT         =    2723.00 KG/M3
 STEEL SPECIFIC WEIGHT            =    7850.00 KG/M3
 COAT SPECIFIC WEIGHT             =    1442.00 KG/M3
 YOUNG S MODULUS                  =   0.21E 11 KG/M2
 SHEAR MODULUS                    =   0.89E 06 KG/M2
 YIELD STRESS                     =   0.46E 08 KG/M2


#ENVIRONMENTAL DATA#
 WATER DEPTH AT ORIGIN            =      150.0 METER
 WATER DEPTH AT BARGE             =      150.0 METER
 WATER SPECIFIC WEIGHT            = 0.102E 04 KG/M3
 DAMPING COEFFICIENT              = 0.200E 04
 SOIL STIFFNESS COEFFICIENT       = 0.000E 00
 BOTTOM FRICTION COEFFICIENT      = 0.000E 00
 BOTTOM SLOPE                     = 0.000E 00
 BOTTOM PROFILE CUVATURE          = 0.000E 00  M-1
 ACCELARATION OF GRAVITY          = 0.981E 01 M/SEC2


#SYSTEM PARAMETERS#
 PROGRAM USED (1,2,3)             =   1
 MODE OF PROGRAM USED             =   4
 TYPE OF INPUT DATA (1,2)         =   3
 NUMBER OF ITERATIONS ALLOWED=  45
 ACCURACY EXPONENT REQUIRED       =   3
 UNIT SYSTEM :2-MET,1-BR,3-SI=   2
 INPUT FILE UNIT SYSTEM           =   2
 NUMBER OF GRIDS REQUIRED         =  33
```

PROGRAM :TREX1A
UNITS :METRIC

DATE: 29/09/77
PAGE: 3

CURRENT DATA

	WATER DEPTH METER	VEL-X M/SEC	VEL-Z M/SEC
1	0.00	0.00	1.50
2	10.00	0.00	1.25
3	25.00	0.00	1.00
4	40.00	0.00	0.80
5	50.00	0.00	0.75
6	75.00	0.00	0.50
7	125.00	0.00	0.00
8	135.00	0.00	-0.50

PROGRAM :TREX4A
UNITS :METRIC

DATE: 29/09/77
PAGE: 4

THE GEOMETRY OF THE PIPE:

ITERATION NUMBER= 28 STIFFNESS PARAMETER=0.1075E-01

DIRECTION COSINES:

GRID	11	12	13	21	22	23	31	32	33
1	1.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	1.00
2	1.00	0.00	0.00	-0.00	1.00	-0.00	-0.00	0.00	1.00
3	1.00	0.00	0.00	-0.00	1.00	-0.00	-0.00	0.00	1.00
4	1.00	0.00	0.00	-0.00	1.00	-0.00	-0.00	0.00	1.00
5	1.00	0.00	0.00	-0.00	1.00	-0.00	-0.00	0.00	1.00
6	1.00	0.01	0.00	-0.01	1.00	-0.00	-0.00	0.00	1.00
7	1.00	0.02	0.00	-0.02	1.00	-0.00	-0.00	0.00	1.00
8	1.00	0.03	0.00	-0.03	1.00	-0.00	-0.00	0.00	1.00
9	1.00	0.05	0.00	-0.05	1.00	-0.00	-0.00	0.00	1.00
10	1.00	0.07	0.00	-0.07	1.00	-0.00	-0.00	0.00	1.00
11	0.99	0.10	0.00	-0.10	0.99	-0.00	-0.00	0.00	1.00
12	0.99	0.13	0.00	-0.13	0.99	-0.00	-0.00	0.00	1.00
13	0.98	0.17	0.00	-0.17	0.98	-0.00	-0.00	0.00	1.00
14	0.98	0.22	0.00	-0.22	0.98	-0.00	-0.00	0.00	1.00
15	0.96	0.27	0.01	-0.27	0.96	-0.00	-0.01	0.00	1.00
16	0.95	0.32	0.00	-0.32	0.95	-0.00	-0.00	0.00	1.00
17	0.93	0.37	0.00	-0.37	0.93	-0.00	-0.00	0.00	1.00
18	0.91	0.42	-0.01	-0.42	0.91	0.00	0.01	0.00	1.00
19	0.89	0.46	-0.01	-0.46	0.89	0.01	0.02	0.00	1.00
20	0.87	0.49	-0.02	-0.49	0.87	0.01	0.03	0.00	1.00
21	0.85	0.52	-0.04	-0.52	0.86	0.02	0.04	0.00	1.00
22	0.84	0.54	-0.05	-0.54	0.84	0.03	0.06	0.00	1.00
23	0.83	0.55	-0.06	-0.55	0.84	0.04	0.07	0.00	1.00
24	0.83	0.55	-0.07	-0.55	0.83	0.05	0.09	0.00	1.00
25	0.83	0.55	-0.09	-0.55	0.83	0.06	0.10	0.00	0.99
26	0.83	0.55	-0.10	-0.54	0.84	0.06	0.11	0.00	0.99
27	0.84	0.54	-0.10	-0.53	0.84	0.07	0.12	0.00	0.99
28	0.85	0.52	-0.11	-0.52	0.85	0.07	0.13	0.00	0.99
29	0.86	0.50	-0.11	-0.50	0.86	0.07	0.13	0.00	0.99
30	0.86	0.49	-0.12	-0.48	0.87	0.06	0.13	0.00	0.99
31	0.87	0.48	-0.12	-0.47	0.88	0.06	0.13	0.00	0.99
32	0.88	0.47	-0.12	-0.46	0.88	0.06	0.13	0.00	0.99
33	0.88	0.46	-0.12	-0.46	0.89	0.06	0.13	0.00	0.99

COORDINATES :

GRID	*X* METER	*Y* METER	*Z* METER	*CURV1*	*CURV2*	*CURV3*
1	0.000	0.000	0.000	0.000	0.000	0.000
2	0.819	0.000	0.000	-0.000	-0.000	0.016
3	3.276	0.000	0.000	-0.000	-0.000	0.062
4	7.371	0.003	0.000	-0.000	-0.001	0.135
5	13.104	0.012	0.000	-0.000	-0.002	0.223
6	20.476	0.063	0.000	-0.000	-0.003	0.334
7	29.484	0.178	0.001	-0.000	-0.005	0.446
8	40.123	0.426	0.004	-0.000	-0.007	0.556
9	52.404	0.901	0.009	-0.000	-0.010	0.659
10	66.302	1.725	0.019	-0.001	-0.013	0.750
11	81.805	3.054	0.038	-0.002	-0.018	0.824
12	98.883	5.072	0.070	-0.003	-0.023	0.881
13	117.459	7.985	0.124	-0.005	-0.026	0.920
14	137.557	12.017	0.205	-0.005	-0.023	0.939
15	158.998	17.394	0.312	-0.001	-0.004	0.940
16	181.703	24.340	0.426	0.010	0.030	0.923
17	205.532	33.060	0.495	0.030	0.076	0.888
18	228.864	43.040	0.445	0.059	0.127	0.834
19	250.212	53.430	0.225	0.093	0.181	0.764
20	269.657	63.922	-0.192	0.132	0.234	0.678
21	287.332	74.250	-0.809	0.170	0.282	0.571
22	303.317	84.181	-1.612	0.203	0.319	0.440
23	317.723	93.522	-2.567	0.224	0.340	0.279
24	330.660	102.112	-3.628	0.229	0.344	0.086
25	342.194	109.833	-4.741	0.219	0.329	-0.140
26	352.383	116.601	-5.851	0.195	0.298	-0.395
27	361.275	122.371	-6.905	0.160	0.253	-0.670
28	368.863	127.135	-7.859	0.121	0.198	-0.953
29	375.137	130.918	-8.677	0.082	0.140	-1.226
30	380.069	133.767	-9.333	0.048	0.085	-1.469
31	383.625	135.743	-9.811	0.022	0.040	-1.660
32	385.773	136.900	-10.101	0.005	0.010	-1.782
33	386.492	137.281	-10.197	0.000	0.000	-1.824

DISTRIBUTION OF FORCES:

BODY FORCE DISTRIBUTION AND ELEMENT LENGTHS :

	X-LOCAL KG/M	Y-LOCAL KG/M	Z-LOCAL KG/M	ELEMENT METER	DISTANCE METER
1	0.000E 00	-0.942E 02	0.000E 00	0.8190E 00	0.0000E 00
2	-0.157E-02	-0.942E 02	-0.340E-11	0.2457E 01	0.8190E 00
3	-0.230E-01	-0.942E 02	-0.734E-08	0.4095E 01	0.3276E 01
4	-0.114E 00	-0.942E 02	-0.796E-06	0.5733E 01	0.7371E 01
5	-0.347E 00	-0.942E 02	-0.220E-04	0.7371E 01	0.1310E 02
6	-0.813E 00	-0.942E 02	-0.238E-03	0.9009E 01	0.2048E 02
7	-0.160E 01	-0.942E 02	-0.233E-02	0.1065E 02	0.2949E 02
8	-0.281E 01	-0.942E 02	-0.134E-01	0.1229E 02	0.4013E 02
9	-0.449E 01	-0.941E 02	-0.597E-01	0.1392E 02	0.5242E 02
10	-0.669E 01	-0.940E 02	-0.219E 00	0.1556E 02	0.6634E 02
11	-0.944E 01	-0.938E 02	-0.686E 00	0.1720E 02	0.8190E 02
12	-0.127E 02	-0.934E 02	-0.189E 01	0.1884E 02	0.9910E 02
13	-0.165E 02	-0.928E 02	-0.469E 01	0.2048E 02	0.1179E 03
14	-0.207E 02	-0.919E 02	-0.106E 02	0.2211E 02	0.1384E 03
15	-0.252E 02	-0.908E 02	-0.946E 01	0.2375E 02	0.1605E 03
16	-0.300E 02	-0.893E 02	-0.633E-01	0.2539E 02	0.1843E 03
17	-0.348E 02	-0.876E 02	0.434E 00	0.2539E 02	0.2097E 03
18	-0.393E 02	-0.856E 02	0.216E 01	0.2375E 02	0.2351E 03
19	-0.432E 02	-0.838E 02	0.535E 01	0.2211E 02	0.2568E 03
20	-0.463E 02	-0.821E 02	0.100E 02	0.2048E 02	0.2809E 03
21	-0.488E 02	-0.806E 02	0.160E 02	0.1884E 02	0.3014E 03
22	-0.506E 02	-0.795E 02	0.231E 02	0.1720E 02	0.3202E 03
23	-0.517E 02	-0.787E 02	0.310E 02	0.1556E 02	0.3374E 03
24	-0.523E 02	-0.784E 02	0.381E 02	0.1392E 02	0.3530E 03
25	-0.522E 02	-0.784E 02	0.421E 02	0.1229E 02	0.3669E 03
26	-0.516E 02	-0.788E 02	0.520E 02	0.1065E 02	0.3792E 03
27	-0.505E 02	-0.795E 02	0.614E 02	0.9009E 01	0.3899E 03
28	-0.491E 02	-0.804E 02	0.707E 02	0.7371E 01	0.3989E 03
29	-0.476E 02	-0.813E 02	0.795E 02	0.5733E 01	0.4062E 03
30	-0.460E 02	-0.822E 02	0.865E 02	0.4095E 01	0.4120E 03
31	-0.448E 02	-0.829E 02	0.916E 02	0.2457E 01	0.4161E 03
32	-0.439E 02	-0.833E 02	0.946E 02	0.8190E 00	0.4185E 03
33	-0.436E 02	-0.835E 02	0.956E 02	0.0000E 00	0.4193E 03

PROGRAM :TPEX4A
UNITS :METRIC

DATE: 29/09/77
PAGE: 7

THE INTERNAL FORCES IN THE PIPE:

	AXIAL KG	SHEARY KG	SHEARZ KG	MOMENTY KG-M	MOMENTZ KG-M	TOTAL STRESS KG/M2
1	-0.985E 05	-0.527E 04	-0.297E 02	0.000E 00	0.000E 00	-0.707E 07
2	-0.985E 05	-0.519E 04	-0.297E 02	-0.246E 02	0.428E 04	-0.709E 07
3	-0.985E 05	-0.497E 04	-0.297E 02	-0.102E 03	0.168E 05	-0.753E 07
4	-0.985E 05	-0.461E 04	-0.298E 02	-0.243E 03	0.364E 05	-0.835E 07
5	-0.985E 05	-0.416E 04	-0.300E 02	-0.464E 03	0.615E 05	-0.964E 07
6	-0.985E 05	-0.364E 04	-0.305E 02	-0.786E 03	0.903E 05	-0.114E 08
7	-0.984E 05	-0.309E 04	-0.315E 02	-0.123E 04	0.121E 06	-0.135E 08
8	-0.982E 05	-0.253E 04	-0.332E 02	-0.183E 04	0.151E 06	-0.158E 08
9	-0.977E 05	-0.200E 04	-0.359E 02	-0.261E 04	0.178E 06	-0.180E 08
10	-0.970E 05	-0.151E 04	-0.389E 02	-0.361E 04	0.203E 06	-0.199E 08
11	-0.957E 05	-0.108E 04	-0.394E 02	-0.480E 04	0.223E 06	-0.215E 08
12	-0.937E 05	-0.708E 03	-0.281E 02	-0.609E 04	0.238E 06	-0.227E 08
13	-0.909E 05	-0.392E 03	0.194E 02	-0.700E 04	0.249E 06	-0.235E 08
14	-0.869E 05	-0.127E 03	0.160E 03	-0.609E 04	0.254E 06	-0.239E 08
15	-0.816E 05	0.958E 02	0.374E 03	-0.119E 04	0.254E 06	-0.238E 08
16	-0.748E 05	0.291E 03	0.499E 03	0.815E 04	0.250E 06	-0.233E 08
17	-0.662E 05	0.477E 03	0.545E 03	0.204E 05	0.240E 06	-0.224E 08
18	-0.563E 05	0.673E 03	0.608E 03	0.343E 05	0.225E 06	-0.212E 08
19	-0.460E 05	0.903E 03	0.657E 03	0.488E 05	0.207E 06	-0.197E 08
20	-0.357E 05	0.120E 04	0.663E 03	0.631E 05	0.183E 06	-0.180E 08
21	-0.255E 05	0.161E 04	0.602E 03	0.759E 05	0.155E 06	-0.161E 08
22	-0.156E 05	0.216E 04	0.458E 03	0.859E 05	0.119E 06	-0.139E 08
23	-0.640E 04	0.288E 04	0.219E 03	0.918E 05	0.754E 05	-0.115E 08
24	0.209E 04	0.380E 04	-0.104E 03	0.927E 05	0.232E 05	-0.965E 07
25	0.969E 04	0.494E 04	-0.473E 03	0.887E 05	-0.378E 05	-0.950E 07
26	0.163E 05	0.626E 04	-0.897E 03	0.802E 05	-0.107E 06	0.125E 08
27	0.219E 05	0.773E 04	-0.138E 04	0.681E 05	-0.181E 06	0.175E 08
28	0.265E 05	0.926E 04	-0.190E 04	0.534E 05	-0.258E 06	0.234E 08
29	0.300E 05	0.108E 05	-0.240E 04	0.377E 05	-0.332E 06	0.294E 08
30	0.326E 05	0.121E 05	-0.285E 04	0.229E 05	-0.397E 06	0.348E 08
31	0.343E 05	0.132E 05	-0.321E 04	0.107E 05	-0.449E 06	0.392E 08
32	0.353E 05	0.138E 05	-0.343E 04	0.278E 04	-0.482E 06	0.420E 08
33	0.357E 05	0.141E 05	-0.351E 04	0.000E 00	-0.493E 06	0.430E 08

PROGRAM :TPEX4A
UNITS :METRIC

DATE: 29/09/77
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THE SUMMARY OF THE RESULTS:

PIPE LENGTH	=	419.34 METER
LENTGH ON STERN	=	102.22 METER
SLOPE AT DEPARTURE	=	27.62 DEGREES
DEPTH AT DEPARTURE	=	137.29 METER
BARGE OFF-SET (Z)	=	-23.28 METER
BARGE BEARING	=	-7.68 DEGREES
MAXIMUM STRESS	=	0.430E 08 KG/M2
BARGE ORIGIN (X)	=	483.52 METER

THE END

PROGRAM DEVELOPED BY I.KONUK

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SOUTHAMPTON UNIVERSITY
FEBRUARY 1977 ENGLAND.