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FACULTY OF MATHEMATICAL STUDIES

DEPARTMENT OF MATHEMATICS

Aspects of Hydrofracture and Heat Transfer In a Geothermal Energy Reservoir

BY

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Where is the wise man? Where is the scholar? Where is the philosopher of this age? Has not God made foolish the wisdom of the world? For since in the wisdom of God the world through its wisdom did not know him, God was pleased through the foolishness of what was preached to save those who believe.

1 Corinthians 1:20-21.

Dedicated to the memory of Grandad Arthur.

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ABSTRACT

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Doctor of Philosophy

ASPECTS OF HYDROFRACTURE AND HEAT TRANSFER IN A GEOTHERMAL ENERGY RESERVOIR

By Amanda Dawn Kelly

The problem of propagating a crack in a linearly elastic substance using a viscous fluid is considered. Conventionally, such problems assume that the elastic stress on the crack wall is supported entirely by the fluid pressure. Here, existing cracks are discussed for the case of the normal stress supported jointly by the fluid pressure and by the elastic deformation of local asperities in the crack. The resulting one-dimensional, second order, non-linear, partial, integro-differential equation is analysed. Analytical and numerical solutions of this equation are obtained using asymptotic analysis and similarity transformations for the cases of extreme values of the non-dimensional parameter, representing the balance between the two possible stress supporting mechanisms.

Additionally, the geothermal energy reservoir is considered on a macroscopic scale as a porous medium and the long term heat transfer effects are investigated. The permeability of the rock and the viscosity of the fluid are assumed to have a simple temperature dependence and a condition for the stability of an isotherm is determined.

CONTENTS

CHAPTER 1 - INTRODUCTION

In the last twenty years there has been an upsurge of interest, both from the general public and from industry, in alternative forms of energy. A new awareness of the greenhouse effect and air and sea pollution, for instance, has prompted investigations into renewable energy sources like solar, wind and wave power and geothermal energy.

Geothermal energy is the heat energy stored in the *hot rocks*, deep below the earth's surface. Twenty years ago in Los Alamos, New Mexico, engineers and geologists wondered whether they could fuel a power plant, by pumping cold water down a hole several kilometres deep, and then sucking the water back up, at a much higher temperature, through a second hole. Los Alamos National Laboratory sits on top of several old volcanoes and it is here that scientists drilled the first wells into the hot dry Precambrian granite to create a loop system into which they could pump the cold water and retrieve steam to drive turbines and then condense and re-use the water. Most of the heat in the Earth's outer crust comes from the disintegration of unstable forms of uranium, thorium and potassium in the rocks below the surface. The average temperature of the rocks worldwide increases with depth by about 30°C per kilometre although where the crust is thin or where volcanoes or earthquakes have disturbed the Earth, this geothermal gradient may be considerably steeper.

In 1977 The Camborne School of Mines (CSM) started work on its own version of the Los Alamos project at Rosemanowes Quarry, Penryn in Cornwall. Temperature gradients are not as steep in Cornwall as they are in New Mexico (30-40°C/Km compared to 55°C/Km at the Los Alamos site) and it took nearly three years to drill to a depth of 2 kilometres, where the rock temperature of around 80°C is above the minimum needed for a commercial system. A pair of wells were drilled forming huge Js which started vertically and then gently curved as they got deeper (see Figure 1.1).

The Cornish team used explosives to fracture the rock near the bore of the well, something which the American team think is unnecessary, and then pumped 26 million litres of water down the well at pressures of up to 14 megapascals.

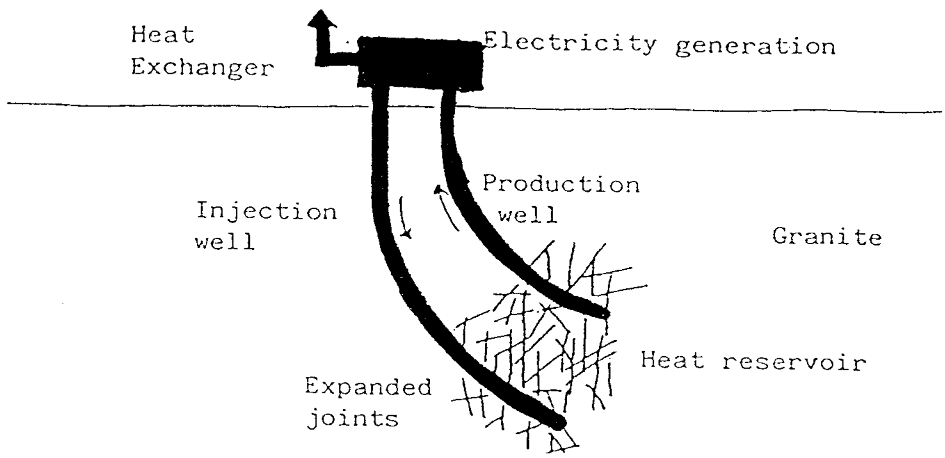

Figure 1.1 - The Wells

Experiments with hot dry rocks have shattered conventional notions about fracturing in granite. The rock is not homogeneous, nor will hydrofracturing necessarily create just a few large planar fractures exactly in line with the direction of maximum stress. Instead, injected water generates forces that shoot through a cobweb of natural joints, whose response may be determined by a pattern formed long ago. By using seismic sensors which were able to detect any small movement in the rock due to the shear stimulation of the rock joints, the group discovered that the hydrofracturing had created a vast cloud of vertical joints. Encompassing about 825 million cubic metres, it was the largest fracture field ever created, but it went the wrong way, projecting downward from the injection well rather than upward to the production well. This shearing mechanism is discussed in Pine and Batchelor (1984), with specific reference to the Camborne project. They conclude that the shearing is due to the locally high ratio of maximum to minimum principal effective stresses. Upward shear growth is predicted at greater depths where the ratio of horizontal stress gradients is lower. This disagreement with accepted practice, however, merely succeeded in pointing out how little is known about the motion of fluid filled cracks in a geothermal energy reservoir (GER).

The Cornwall project team came up with the idea of writing a computer code to model the shearing mechanism. The Fluid Rock Interaction Program (FRIP) developed by Cundall (1982, 1983), is a two-dimensional numerical code which models the behaviour of a network of fluid filled cracks. A one-dimensional, single crack version of FRIP, known as FBED (FRIP - Boundary Element Displacement Discontinuity Method), has since been written to try to improve the original model used in formulating FRIP. More recently FRIP has been extended to three dimensions and FBED has been extended to model a planar crack. The numerical models are discussed by Pine and Nicol (1988) with special reference to the Camborne project. The actual model used to develop the FBED code is discussed in Nicol (1988), and is the basis of the analytical model produced in this document.

The aim of this thesis is twofold. First, the one-dimensional crack problem is analysed, in order that a better understanding of the mechanisms involved in the geothermal energy project's hydrofracturing may be obtained. Second, the long term cooling effects within the GER are discussed in order to investigate the expected occurrence of instabilities in flow paths due to rock shrinkage.

Previous work is referred to as appropriate throughout the thesis, but a brief outline of some of the less directly relevant work is given here. A large amount of work has been carried out in the area of crack and fracture migration. Barenblatt (1962) gives a comprehensive study on the theory of equilibrium cracks and summarises much of the work done prior to 1962. Since then, Geertsma and Haafkens (1969) have presented a synopsis of several theories for the prediction of the behaviour of hydraulic fractures. The synopsis compares and contrasts previous models such that of Perkins and Kern (1961) (extended by Nordgren (1972)) and that of Geertsma and de Klerk (1969).

Spence and Turcotte have presented a considerable number of papers on a range of applications of the theory of cracks. The more relevant work is referred to later on in the thesis, but other applications include the behaviour of strike-slip faults (1974, 1975, 1979) and more recently bouyancy driven magma migration (1990), a topic also considered by Lister (1990).

The formation of rock joints is discussed in Kemeny and Cook (1985) and the properties and behaviour of the joints are further discussed by Cook (1988).

The current status of geothermal energy projects in the United States and in Japan are given by Franke (1988) and Tomita et al. (1988), respectively, while a series of progress reports on the Cornish project is available from the British Library Document Supply Centre at Boston Spa.

So, here in chapter one, the background history to the geothermal energy related problems that are discussed in this thesis, has been presented. In chapter two the required laws and equations are outlined and this chapter also serves to establish the notation used throughout. The content of chapter two is standard and well known. In chapter three a one-dimensional model for a typical fluid filled crack is presented and a single second order, non-linear, singular integro-differential equation for the crack height is obtained. The model outlined here is one that was developed by the CSM (Markland (1989)) although the analytical treatment of it, in the non-dimensionalisation and in the solutions obtained in chapter four, is new. In chapter five, numerical solutions to this equation are presented and a comparison with the analytical solutions is given in the last section. The numerical model used here is based on the CSM code FBED, and as such is not a new approach. The stability analysis carried out on the difference scheme in chapter five, however, has not been previously discussed.

In chapter six a different aspect of the behaviour of a GER is introduced and the long term heat transfer effects are considered. This involves developing a model for the heat transfer which includes a simple temperature dependent permeability for the rock. The model presented in this section is based on ideas that were discussed at the 1991

European Study Group with Industry. The analysis of the model, however, is original work and therefore has not been seen before. Solutions are obtained for the case of a planar isotherm within the rock and then this solution is perturbed in order to ascertain the conditions for stability of the isotherm. The motivation behind this work is to investigate the expected occurrence of cold spots in the reservoir. These cold spots are thought to be due to rock shrinkage which creates preferred paths for the fluid flow, the possible result of this being that large amounts of heat remain unrecovered.

Much of the work in this thesis was carried out with the aim of providing additional insight to the reservoir models used by the CSM. Analytical solutions are generally sought where possible, since the CSM already has comprehensive numerical models to describe the reservoir behaviour. The analytical results are sought in order to check on the accuracy of these numerical models. This underlying motivation for the work should be noted since it helps to explain the choice of methods used throughout the thesis.

CHAPTER 2 - BASIC EQUATIONS OF LINEAR ELASTICITY AND FLUID DYNAMICS

2.1 - Introduction

In this chapter the notation used throughout this thesis is established and the standard, basic equations of both linear elasticity and Newtonian fluid dynamics are introduced. Simplifications to these governing equations for the cases of plane strain elasticity and low Reynolds' number flow between lubricated plates are discussed since these are the particular cases which are relevant to the flow of a viscous fluid through cracks in an isotropic linearly elastic material.

2.2 - Linear Elasticity Equations for an Isotropic Material

The fundamental assumption concerning the behaviour of a perfectly elastic, isotropic solid, is that the six components of the stress tensor σ_{ij}, given by

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} ,$$

(six components because $\sigma_{ij} = \sigma_{ji}$) are linear functions of the six components of the strain tensor,

$$e_{ij} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix} .$$

The sign convention adopted here is that the stress component σ_{ij} is positive if it acts in the positive j direction on a plane whose outward normal points in the positive i direction. In other words positive normal stresses are tensile. Note that an isotropic solid is one in which its properties are independent of the direction considered. The stress-strain relations, known as the

Generalised Hooke's Law, for such a solid are given by

$$e_{xy} = \frac{1}{2G} \sigma_{xy} \qquad 2.2.01a$$

$$e_{xz} = \frac{1}{2G} \sigma_{xz} \qquad 2.2.01b$$

$$e_{yz} = \frac{1}{2G} \sigma_{yz} \qquad 2.2.01c$$

$$e_{xx} = \frac{1}{E} \left[\sigma_{xx} - \nu \, (\sigma_{yy} + \sigma_{zz}) \right] \qquad 2.2.01d$$

$$e_{yy} = \frac{1}{E} \left[\sigma_{yy} - \nu \, (\sigma_{xx} + \sigma_{zz}) \right] \qquad 2.2.01e$$

$$e_{zz} = \frac{1}{E} \left[\sigma_{zz} - \nu \, (\sigma_{xx} + \sigma_{yy}) \right] \qquad 2.2.01f$$

where E is Young's modulus, G is the shear modulus (or modulus of rigidity), and ν is Poisson's ratio. Young's modulus and the shear modulus are related by:

$$E = 2G \, (1 + \nu).$$

For the special case when the solid is in static equilibrium, with no body forces acting, the equations of motion are given by

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \qquad 2.2.02a$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0 \qquad 2.2.02b$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0. \qquad 2.2.02c$$

These equations are known as the equilibrium equations.

In most problems strains are very small quantities and so the product of two or more strains can be neglected in comparison to the strains themselves. Such strains are called infinitesimal strains. The relationship between strains and displacements (where the u_i are the displacements in the directions x_i) is given by the infinitesimal strain-displacement equations,

$$e_{xx} = \frac{\partial u_x}{\partial x} \qquad 2.2.03a$$

$$e_{yy} = \frac{\partial u_y}{\partial y} \qquad 2.2.03b$$

$$e_{zz} = \frac{\partial u_z}{\partial z} \qquad 2.2.03c$$

$$e_{xy} = e_{yx} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \qquad 2.2.03d$$

$$e_{xz} = e_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \qquad 2.2.03e$$

$$e_{yz} = e_{zy} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) . \qquad 2.2.03f$$

The complete formulation of problems in linear elasticity is now seen. The stresses must satisfy equations (2.2.02) although the boundary conditions may specify either stress or displacement. The strains are then found from the stresses via the stress-displacement relations (2.2.01). Finally, when the strains are known, their definitions (2.2.03) may be regarded as a set of equations from which the displacements can be found.

If one length scale in an elasticity problem is very large in comparison to the other two, it is often the case that the strains are very small across the large length scale, and that there is no displacement in such a direction. In such cases a plane strain approximation can be introduced. For example, picture a pipe with a uniform cross-section whose length scale is much greater than its dimensions of cross-section. Assume that the external force applied is parallel to the cross-section and is independent of position along the pipe. A cross-section at any point along the length of the pipe would be exactly the same as at any other (except perhaps at the ends of the pipe). So for an infinite pipe the problem can be reduced to two dimensions. If the cross-section of the pipe lies in the x,y plane, it is assumed that

$$e_{zz} = e_{xz} = e_{zx} = e_{yz} = e_{zy} = 0$$

and

$$u_z = 0$$

which in turn implies that

$$\sigma_{xz} = \sigma_{zx} = \sigma_{zy} = \sigma_{yz} = 0 \ .$$

These assumptions lead to the following reduced set of equations for the case of plane strain:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad \text{2.2.04a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad \text{2.2.04b}$$

$$\frac{\partial \sigma_{zz}}{\partial z} = 0 \qquad \text{2.2.04c}$$

$$\sigma_{xx} = 2G \left[e_{xx} + \frac{\nu}{1-2\nu} \left(e_{xx} + e_{yy} \right) \right] \qquad \text{2.2.04d}$$

$$\sigma_{yy} = 2G \left[e_{yy} + \frac{\nu}{1-2\nu} \left(e_{xx} + e_{yy} \right) \right] \qquad \text{2.2.04e}$$

$$\sigma_{zz} = \frac{2G\nu}{1-2\nu} \left(e_{xx} + e_{yy} \right) \qquad \text{2.2.04f}$$

$$\sigma_{xy} = 2G \ e_{xy} \qquad \text{2.2.04g}$$

$$e_{xx} = \frac{\partial u_x}{\partial x} \qquad \text{2.2.04h}$$

$$e_{yy} = \frac{\partial u_y}{\partial y} \qquad 2.2.04i$$

$$e_{xy} = e_{yx} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \qquad 2.2.04j$$

$$\frac{\partial u_x}{\partial z} = 0 \qquad 2.2.04k$$

$$\frac{\partial u_y}{\partial z} = 0 \qquad 2.2.04l$$

For further detail on the subject of linear elasticity see Jaeger (1969).

2.3 - Basic Solutions For An Elastic Half Plane

In this section a plane strain problem in an elastic half-plane, $y \geq 0$ is considered. The particular boundary conditions imposed are relevant to the problem of an infinite planar crack along $y = 0$. A known displacement along $y = 0$ is imposed and the resultant stresses and displacements in the half-plane $y > 0$ are sought, and standard relations between normal stress and normal displacement and between shear stress and shear displacement (along the crack) are derived. The Fourier Transform approach used in this section is the same as that used by Sneddon and Lowengrub (1969).

The governing equations (2.2.04) can be solved in two parts for the following sets of boundary conditions (Crouch (1976)),

(1)

$$\sigma_{xy} = 0 \qquad y = 0 \qquad 2.3.01a$$

$$u_y = \begin{cases} 0 & |x| > 1 \\ -\dfrac{D_y(x)}{2} & |x| < 1 \end{cases} \qquad y = 0 \qquad 2.3.01b$$

$$\sigma_{ij} \rightarrow 0 \qquad x,y \rightarrow \infty \quad \forall\ i,j \qquad 2.3.01c$$

(2)

$$\sigma_{yy} = 0 \qquad y = 0 \qquad 2.3.02a$$

$$u_x = \begin{cases} 0 & |x| > 1 \\ -\dfrac{D_x(x)}{2} & |x| < 1 \end{cases} \qquad y = 0 \qquad 2.3.02b$$

$$\sigma_{ij} \rightarrow 0 \qquad x,y \rightarrow \infty \quad \forall\ i,j \qquad 2.3.02c$$

where D_x and D_y, the given differences in displacement between the two sides of the crack, may be defined via

$$\begin{aligned} D_x &= u_x(x,0_-) - u_x(x,0_+) \\ D_y &= u_y(x,0_-) - u_y(x,0_+) \end{aligned} \qquad 2.3.03$$

where the crack surfaces are distinguished by assuming that one surface is on the positive side of $y = 0$, denoted by $y = 0_+$, and the other is on the negative side of $y = 0$, denoted by $y = 0_-$.

Eliminating the stresses and strains from the plane strain equations results in a pair of simultaneous, second order partial differential equations for the displacement components u_x and u_y

$$2(1-\nu)\frac{\partial^2 u_x}{\partial x^2} + (1-2\nu)\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y} = 0 \qquad 2.3.04$$

$$(1-2\nu)\frac{\partial^2 u_y}{\partial x^2} + 2(1-\nu)\frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_x}{\partial x \partial y} = 0 \qquad 2.3.05$$

In order to find solutions of (2.3.04) and (2.3.05), the equations are transformed to a pair of ordinary differential equations by taking Fourier transforms in x. This pair of linear equations can then be solved and the solutions inverted to obtain u_x and u_y. If the Fourier transform is defined by

$$F[\xi] = \mathcal{F}[f(t)] = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)\, e^{i\xi t}\, dt$$

and if $\hat{u}_x$ and $\hat{u}_y$ denote the Fourier transforms in the x variable of u_x and u_y respectively, so that

$$\hat{u}_x = \mathcal{F}[u_x(x,y)]$$

$$\hat{u}_y = \mathcal{F}[u_y(x,y)]$$

then the transforms of equations (2.3.04) and (2.3.05) are

$$(1-2\nu) \frac{\partial^2 \hat{u}_x}{\partial y^2} - 2(1-\nu)\, \xi^2\, \hat{u}_x - i\xi \frac{\partial \hat{u}_y}{\partial y} = 0 \qquad 2.3.06$$

$$-i\xi \frac{\partial \hat{u}_x}{\partial y} + 2(1-\nu) \frac{\partial^2 \hat{u}_y}{\partial y^2} - (1-2\nu)\, \xi^2\, \hat{u}_y = 0 \qquad 2.3.07$$

The boundary conditions are transformed to

(1)

$$\hat{\sigma}_{xy} = 0 \qquad y = 0 \qquad 2.3.08a$$

$$\hat{u}_y = \begin{cases} 0 & |x| > 1 \\ -\dfrac{\hat{D}_y(x)}{2} & |x| < 1 \end{cases} \qquad y = 0 \qquad 2.3.08b$$

$$\hat{\sigma}_{ij} \to 0 \qquad (x^2 + y^2)^{1/2} \to \infty \qquad \forall\ i,j \qquad 2.3.08c$$

(2)

$$\hat{\sigma}_{yy} = 0 \qquad y = 0 \qquad 2.3.09a$$

$$\hat{u}_x = \begin{cases} 0 & |x| > 1 \\ -\dfrac{\hat{D}_x(x)}{2} & |x| < 1 \end{cases} \qquad y = 0 \qquad 2.3.09b$$

$$\hat{\sigma}_{ij} \to 0 \qquad (x^2 + y^2)^{1/2} \to \infty \qquad \forall\ i,j \qquad 2.3.09c$$

The system of equations is solved in the usual way by seeking a solution of the form

$$\hat{u}_x = A_0(\xi)\ e^{\lambda y} \qquad \hat{u}_y = B_0(\xi)\ e^{\lambda y}.$$

Substituting these values into (2.3.06) and (2.3.07) gives

$$(1-2\nu)A_0\lambda^2 - 2(1-\nu)A_0\xi^2 - i\xi\lambda B_0 = 0$$

$$2(1-\nu)B_0\lambda^2 - (1-2\nu)B_0\xi^2 - i\xi A_0\lambda = 0$$

which can be written in matrix form as

$$\begin{pmatrix} (1-2\nu)\lambda^2-2(1-\nu)\xi^2 & -i\xi\lambda \\ -i\xi\lambda & 2(1-\nu)\lambda^2 - (1-2\nu)\xi^2 \end{pmatrix} \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For a non-trivial solution for A_0 and B_0 the determinant of this matrix must be zero

$$\begin{vmatrix} (1-2\nu)\lambda^2-2(1-\nu)\xi^2 & -i\xi\lambda \\ -i\xi\lambda & 2(1-\nu)\lambda^2 - (1-2\nu)\xi^2 \end{vmatrix} = 0 .$$

Evaluating the determinant gives us a quartic in λ, which is actually a quadratic in λ^2 and it is found that

$$\lambda = \pm\xi \quad \text{twice.} \qquad 2.3.10$$

$\hat{u}_x$ and $\hat{u}_y$, therefore, have solutions of the form

$$\hat{u}_x = A_1(\xi)e^{\xi y} + A_2(\xi)e^{-\xi y} + A_3(\xi)ye^{\xi y} + A_4(\xi)ye^{-\xi y}, \qquad 2.3.11$$

$$\hat{u}_y = B_1(\xi)e^{\xi y} + B_2(\xi)e^{-\xi y} + B_3(\xi)ye^{\xi y} + B_4(\xi)ye^{-\xi y}. \qquad 2.3.12$$

There are two possible ways of proceeding at this point. One is to find the corresponding eigenvectors of the matrix which would give us a relation between the coefficients $A_i(\xi)$ and $B_i(\xi)$. An alternative way to find this relation is to substitute these values for $\hat{u}_x$ and $\hat{u}_y$

back into equations (2.3.06) and (2.3.07). The second method is chosen because this is algebraically the less complicated of the two.

To ensure that $\hat{u}_x$ and $\hat{u}_y$ remain finite as $y \to \infty$, the exponentially growing terms in the expressions for $\hat{u}_x$ and $\hat{u}_y$ must be removed by choosing their coefficients to be zero. Hence if $\xi < 0$ then $A_2 = A_4 = B_2 = B_4 = 0$, if $\xi > 0$ then $A_1 = A_3 = B_1 = B_3 = 0$ and if $\xi = 0$ then, trivially, $\hat{u}_x$ and $\hat{u}_y$ are both identically zero.

Consider the modulus of ξ and introduce coefficients $C_1(\xi)$, $C_2(\xi)$, $D_1(\xi)$, $D_2(\xi)$ to obtain

$$\hat{u}_x = C_1(\xi)e^{-|\xi|y} + C_2(\xi)\ y\ e^{-|\xi|y} \qquad 2.3.13$$

$$\hat{u}_y = D_1(\xi)e^{-|\xi|y} + D_2(\xi)\ y\ e^{-|\xi|y} \qquad 2.3.14$$

In order to find the unknown coefficients, these results are substituted back into equations (2.3.06) and (2.3.07) and coefficients of y are equated to get

$$-C_1\xi^2-2(1-2\nu)C_2|\xi|-i\xi D_2 + i|\xi|\xi D_1=0 \qquad 2.3.15$$

$$i\xi|\xi|C_2 + D_2\xi^2=0 \qquad 2.3.16$$

$$-i\xi C_2+i\xi|\xi|C_1-4D_2|\xi|(1-\nu)+|\xi|^2D_1=0 \qquad 2.3.17$$

$$i\xi|\xi|C_2+D_2|\xi|^2=0 \qquad 2.3.18$$

Note equations (2.3.16) and (2.3.18) above are identical. By rearranging (2.3.16) and substituting $D_2 = \dfrac{-i|\xi|C_2}{\xi}$ into (2.3.15) and (2.3.18), the latter become

$$-C_1\xi^2 + (4\nu-3)|\xi|C_2 + i|\xi|\xi D_1 = 0 \qquad 2.3.19$$

$$(3-4\nu)\ \xi iC_2 + i\xi|\xi|C_1 + D_1|\xi|^2 = 0 \qquad 2.3.20$$

Notice that equation (2.3.20) is a factor of $\left(\dfrac{-i|\xi|}{\xi}\right)$ times equation (2.3.19), and so equations (2.3.19) and (2.3.20) are linearly dependent. Rearrange (2.3.19) to get

$$C_1 = \frac{-KC_2}{|\xi|} + i\frac{|\xi|}{\xi}D_1$$

where

$$K = 3-4\nu \qquad 2.3.21$$

and so $\hat{u}_x$ and $\hat{u}_y$ are given by

$$\hat{u}_x = \left(-\frac{KC_2}{|\xi|} + \frac{i|\xi|D_1}{\xi}\right) e^{-|\xi|y} + C_2\, ye^{-|\xi|y} \qquad 2.3.22$$

$$\hat{u}_y = D_1 e^{-|\xi|y} - \frac{i|\xi|}{\xi} C_2\, ye^{-|\xi|y} \qquad 2.3.23$$

In order to simplify the algebra that follows, set

$$D_1(\xi) = |\xi|^{-1}B(\xi)$$

$$C_2(\xi) = \frac{-|\xi|i}{K\xi}\left(A(\xi) - B(\xi)\right)$$

and solve for unknown functions $A(\xi)$ and $B(\xi)$. So for $y \geq 0$ the displacements are given by

$$\hat{u}_x = i\xi^{-1}\left[A(\xi) - K^{-1}\left(A(\xi) - B(\xi)\right)|\xi|y\right] e^{-|\xi|y} \qquad 2.3.24$$

$$\hat{u}_y = |\xi|^{-1}\left[B(\xi) - K^{-1}\left(A(\xi) - B(\xi)\right)|\xi|y\right] e^{-|\xi|y} \qquad 2.3.25$$

and along $y = 0$ these become

$$\hat{u}_x(\xi,0) = i\xi^{-1}A(\xi) \qquad 2.3.26$$

$$\hat{u}_y(\xi,0) = |\xi|^{-1}B(\xi) \qquad 2.3.27$$

Since the boundary conditions are also imposed on the stresses, in order to discover the functions $A(\xi)$ and $B(\xi)$, the stress–strain equations (2.2.04e,g) under Fourier transformation are needed. Let $\hat{\sigma}_{yy} = \mathcal{F}[\sigma_{yy}]$ and $\hat{\sigma}_{xy} = \mathcal{F}[\sigma_{xy}]$. Combining equations (2.2.04e & g) with equations (2.2.04h,i,j) gives a set of equations which relate the stress to the displacement. Taking the Fourier transformation in the x variable of these equations gives

$$\hat{\sigma}_{xy} = G\left\{\frac{\partial \hat{u}_x}{\partial y} - i\xi\hat{u}_y\right\}$$

$$= \frac{Gi|\xi|}{\xi K}\left\{-A(K+1) - B(K-1) + 2(A-b)|\xi|y\right\}e^{-|\xi|y} \qquad 2.3.28$$

$$\hat{\sigma}_{yy} = \frac{2G}{1-2\nu}\left\{\nu(-i\xi)\hat{u}_x + (1-\nu)\frac{\partial \hat{u}_y}{\partial y}\right\}$$

$$= \frac{G}{K}\left\{A(1-K) - B(1+K) + 2(A-B)|\xi|y\right\} e^{-|\xi|y} \qquad 2.3.29$$

First impose boundary condition (2.3.08a). Along $y = 0$ $\hat{\sigma}_{xy}$ is given by

$$\hat{\sigma}_{xy}(\xi,0) = -\frac{Gi|\xi|}{K\xi}\left[A(K+1) + B(K-1)\right] \qquad 2.3.30$$

so to ensure that $\hat{\sigma}_{xy} = 0$ along $y = 0$, the following relationship between A and B is chosen, $A(K+1) = - B(K-1)$. In order to simplify the algebra, choose

$$2GA(\xi) = -(1-2\nu)\psi(\xi) \qquad 2.3.31$$

$$2GB(\xi) = (2-2\nu)\psi(\xi) \qquad 2.3.32$$

and so, substituting (2.3.31) and (2.3.32) into (2.3.24), (2.3.25), (2.3.28) and (2.3.29) gives

$$2G\hat{u}_x = - i\xi^{-1}(1-2\nu-|\xi|y)\ \psi(\xi)\ e^{-|\xi|y}, \qquad 2.3.33$$

$$2G\hat{u}_y = |\xi|^{-1}(2-2\nu+|\xi|y)\ \psi(\xi)\ e^{-|\xi|y}, \qquad 2.3.34$$

$$\hat{\sigma}_{xy}(\xi,y) = -i\xi y\ \psi(\xi)\ e^{-|\xi|y}, \qquad 2.3.35$$

$$\hat{\sigma}_{yy}(\xi,y) = -(y|\xi| + 1)\ \psi(\xi)\ e^{-|\xi|y}. \qquad 2.3.36$$

Along y = 0 the stresses and displacements are

$$2G\hat{u}_x(\xi,0) = -(1-2\nu)\, i\xi^{-1}\psi(\xi) \qquad 2.3.37$$

$$2G\hat{u}_y(\xi,0) = 2(1-\nu)\,|\xi|^{-1}\psi(\xi) \qquad 2.3.38$$

$$\hat{\sigma}_{xy}(\xi,0) = 0 \qquad 2.3.39$$

$$\hat{\sigma}_{yy}(\xi,0) = -\psi(\xi). \qquad 2.3.40$$

$\psi(\xi)$ is found by imposing condition (2.3.08b) to be

$$\psi(\xi) = -\frac{G\hat{D}_y}{2(1-\nu)}|\xi| \qquad 2.3.41$$

and so substituting (2.3.41) into (2.3.37) and (2.3.40) gives the normal stress and the shear displacement along y = 0 as

$$\hat{u}_x(\xi,0) = \frac{1-2\nu}{4(1-\nu)}\, i\,\mathrm{sgn}\xi\,\hat{D}_y \qquad 2.3.42$$

$$\hat{\sigma}_{yy}(\xi,0) = \frac{G\hat{D}_y}{2(1-\nu)}|\xi|. \qquad 2.3.43$$

Note that the shear stress and the normal displacement along y = 0 are specified by the boundary conditions (2.3.08).

Inverting the Fourier transforms using the table in Appendix A leads to the solution

$$u_x(x,0) = -\frac{(1-2\nu)}{4\pi(1-\nu)}\int_{-1}^{1} D_y(s)\,\frac{ds}{s-x} \qquad 2.3.44$$

$$\sigma_{yy}(x,0) = \frac{-G}{2\pi(1-\nu)}\int_{-1}^{1}\frac{\partial D_y(s)}{\partial s}\,\frac{ds}{s-x}. \qquad 2.3.45$$

A similar method can be used to find the integral equation relating the shear stress to the relative shear displacement, for the case of zero normal stress along the x axis. This is done by imposing boundary conditions (2.3.09) on equations (2.3.24), (2.3.25), (2.3.28) and (2.3.29). The following solution for the normal displacement and the shear stress is obtained

$$u_y(x,0) = \frac{(1 - 2\nu)}{4\pi(1 - \nu)} \int_{-1}^{1} D_x(s) \frac{ds}{s - x} \qquad 2.3.46$$

$$\sigma_{xy}(x,0) = \frac{-G}{2\pi(1-\nu)} \int_{-1}^{1} \frac{\partial D_x(s)}{\partial s} \frac{ds}{s - x}. \qquad 2.3.47$$

Superimposing the two problems above leads to the solution for the stresses and displacements along a crack in an infinite elastic medium, for the case of zero far-field stresses (the crack is taken to extend from $x = -1$ to $x = 1$). The solution is

$$u_x(x,0) = -\frac{D_x}{2} - \frac{(1 - 2\nu)}{4\pi(1 - \nu)} \int_{-1}^{1} D_y(s) \frac{ds}{s - x} \qquad 2.3.48$$

$$u_y(x,0) = -\frac{D_y}{2} + \frac{(1 - 2\nu)}{4\pi(1 - \nu)} \int_{-1}^{1} D_x(s) \frac{ds}{s - x} \qquad 2.3.49$$

$$\sigma_{yy}(x,0) = \frac{-G}{2\pi(1-\nu)} \int_{-1}^{1} \frac{\partial D_y(s)}{\partial s} \frac{ds}{s - x} \qquad 2.3.50$$

$$\sigma_{xy}(x,0) = \frac{-G}{2\pi(1-\nu)} \int_{-1}^{1} \frac{\partial D_x(s)}{\partial s} \frac{ds}{s - x}. \qquad 2.3.51$$

where D_x and D_y are the relative displacements of the crack surfaces as described above (see definition 2.3.03).

2.4 - Incompressible Viscous Flow

In this section the governing equations for the flow of an incompressible fluid are outlined and simplifications are suggested for the case where lubrication theory is applicable.

The motion of a fluid is governed by several conservation laws and a set of constitutive equations. If $\rho(x,y,z,t)$ is the density of the fluid and $\underline{v}$ $(u(x,y,z,t), v(x,y,z,t), w(x,y,z,t))$ is the velocity vector of a fluid particle which has component u in the x direction, v in the y direction and w in the z direction, then by considering conservation of mass the continuity equation is obtained

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0 \qquad 2.4.01$$

The continuity equation for an incompressible fluid, is obtained from (2.4.01) by setting the material derivative of ρ to be zero (since the density of a fluid element does not change with the movement of the fluid element) and is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \qquad 2.4.02$$

Considering conservation of linear momentum of an incompressible Newtonian fluid leads to the familiar Navier-Stokes equations (no body forces acting)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u \qquad 2.4.03a$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\nabla^2 v \qquad 2.4.03b$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\nabla^2 w \qquad 2.4.03c$$

where ∇^2 is the standard Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

For further details see Batchelor (1970).

2.5 - Flow Between Two Moving Plates (Lubrication Theory)

In this section, the problem of slow viscous flow between two plates that are free to move in the direction perpendicular to the fluid layer is considered (see figure 2.1).

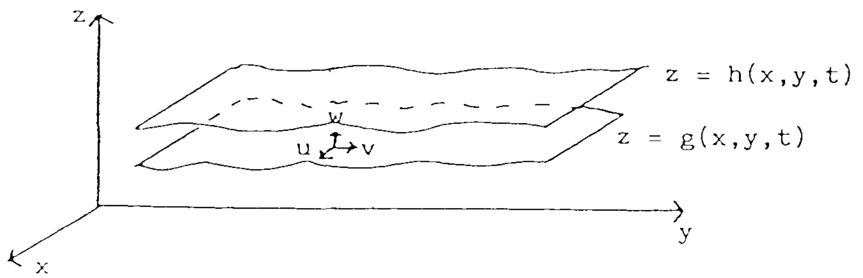

Fig. 2.1 - Flow Between Two Moving Plates

The aim of this chapter is to apply lubrication theory to the Navier-Stokes equations to obtain Reynolds' equation. Lubrication theory describes the phenomenon, often witnessed in practice, that a thin layer of fluid between two surfaces of much larger dimensions than the thin layer, serves to lubricate the subsequent motion. This is due to a large positive pressure set up in the fluid layer. This large pressure comes from the fact that because the fluid layer is so small, the rate of strain and stress due to viscosity in the fluid layer are very large, and this stress, under certain conditions (i.e. conditions which produce a low Reduced Reynolds' number) develops a large pressure. The reason for deriving this result is that a fluid filled crack in a solid such as rock, can be considered to hold to the approximations made in obtaining Reynolds' equation.

Consider a fluid, subject to a pressure $p(x,y,z,t)$, flowing between two plates $z = g(x,y,t)$ and $z = h(x,y,t)$, as pictured in figure 2.1. The governing equations are the Continuity equation (2.4.02) and Navier-Stokes equations (2.4.03) for an incompressible Newtonian fluid, where body forces are absent.

Consider the plate $z = h$. Define $f = h - z = 0$. If $\underline{t}_1$ and $\underline{t}_2$ are two vectors orthogonal to each other and to ∇f (i.e. tangential to f), and if $\nabla f = (f_x, f_y, -1)$, then set the vectors $\underline{t}_1$ and $\underline{t}_2$ to be

$$\underline{t}_1 = (0,\ 1,\ f_y) \qquad\qquad \underline{t}_2 = (f_y^{\,2} + 1,\ -f_x f_y,\ f_x)$$

Fluid cannot pass through the surface, hence

$$\frac{Df}{Dt} = 0 = \frac{\partial f}{\partial t} + \underline{v}.\nabla f$$

that is

$$0 = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - w \qquad z = h \qquad 2.5.01a$$

Since the fluid is viscous, a no slip condition on the moving plates is imposed. The plates are assumed to move only in the z direction, therefore the fluid velocity components in the directions of $\underline{t}_1$ and $\underline{t}_2$, on the plate surface $z = h$, are zero. This implies that

$$(0,\ 1,\ f_y)\ .\ (u,\ v,\ w)^T = 0$$

which is equivalent to

$$v + w\, f_y = 0 \qquad z = h \qquad 2.5.01b$$

and similarly

$$u\,(f_y^{\,2} + 1) - v\, f_x f_y + w\, f_x = 0 \qquad z = h. \qquad 2.5.01c$$

If X, Y and Z are typical length scales in the x, y and z directions respectively; and U, V and W are typical speeds in the x, y, and z directions, and T is a suitable time-scale, then in order to non-dimensionalise the system the following change of variables is introduced

$$x = X\bar{x} \qquad y = Y\bar{y} \qquad z = Z\bar{z} \qquad t = T\bar{t}$$
$$u = U\bar{u} \qquad v = V\bar{v} \qquad w = W\bar{w} \qquad f = Z\bar{f}$$

and finally to scale the pressure

$$p = \frac{\mu U X}{Z^2} \bar{p}.$$

Assume that $X = O(Y)$ and that $U = O(V)$, (i.e. $X = Y$ and $U = V$), and that $Z << X$. The non-dimensional model is then given by the following equations and conditions.

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{XW}{ZU}\frac{\partial \bar{w}}{\partial \bar{z}} = 0 \qquad 2.5.02a$$

Choosing $W = \frac{ZU}{X}$ leads to

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}} = 0 \qquad 2.5.03a$$

and the first Navier-Stokes equation becomes

$$\frac{\rho Z^2}{\mu T}\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{\rho Z^2 U}{\mu X}\left[\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}}\right] = -\frac{\partial \bar{p}}{\partial \bar{x}}$$
$$+ \frac{Z^2}{X^2}\left[\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right] + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}. \qquad 2.5.02b$$

Choosing $T = \frac{X}{U} = \frac{W}{Z}$, and making the usual low Reduced Reynolds' number assumption, this becomes

$$\frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \qquad 2.5.03b$$

and similarly the remaining Navier-Stokes equations become

$$\frac{\partial \bar{p}}{\partial \bar{y}} = \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} \qquad 2.5.03c$$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = 0 . \qquad 2.5.03d$$

Since variations in h(x, y, t) are of order Z, let

$$h(x, y, t) = Z\, H(\bar{x}, \bar{y}, \bar{t})$$

and so condition (2.5.01a) becomes

$$0 = \frac{\partial H}{\partial \bar{t}} + \bar{u}\frac{\partial H}{\partial \bar{x}} + \bar{v}\frac{\partial H}{\partial \bar{y}} - \bar{w} \qquad \text{on } \bar{z} = H. \qquad 2.5.04$$

Since Z << X the other boundary conditions (2.5.01 b & c) become

$$\bar{u} = 0 \qquad \text{on } \bar{z} = H \qquad 2.5.05a$$

$$\bar{v} = 0 \qquad \text{on } \bar{z} = H. \qquad 2.5.05b$$

Substituting these values into (2.5.04) gives

$$\bar{w} = \frac{\partial H}{\partial \bar{t}} \qquad \text{on } \bar{z} = H. \qquad 2.5.05c$$

Similarly for the lower plate, z = g(x, y, t), which becomes $\bar{z} = G(\bar{x}, \bar{y}, \bar{t})$, in terms of the non-dimensional variables, the boundary conditions are

$$\bar{u} = 0 \qquad \text{on } \bar{z} = G \qquad 2.5.06a$$

$$\bar{v} = 0 \qquad \text{on } \bar{z} = G \qquad 2.5.06b$$

$$\bar{w} = \frac{\partial G}{\partial \bar{t}} \qquad \text{on } \bar{z} = G. \qquad 2.5.06c$$

This system of equations (2.5.03) and boundary conditions (2.5.05 & 2.5.06) are easily solved to obtain the following solution for the velocity components

$$\bar{u} = \frac{1}{2}\frac{\partial \bar{p}}{\partial \bar{x}}\,[\bar{z}^2 - (H + G)\,\bar{z} + GH] \qquad 2.5.07a$$

$$\bar{v} = \frac{1}{2}\frac{\partial \bar{p}}{\partial \bar{y}}\,[\bar{z}^2 - (H + G)\,\bar{z} + GH]. \qquad 2.5.07b$$

These expressions for $\bar{u}$ and $\bar{v}$ are substituted into the continuity equation to obtain $\bar{w}$,

$$\bar{w} = -\frac{1}{2}\frac{\partial}{\partial \bar{x}}\left[\frac{\partial \bar{p}}{\partial \bar{x}}\left\{\frac{\bar{z}^3}{3} - (H + G)\frac{\bar{z}^2}{2} + GH\,\bar{z}\right\}\right]$$

$$-\frac{1}{2}\frac{\partial}{\partial \bar{y}}\left[\frac{\partial \bar{p}}{\partial \bar{y}}\left\{\frac{\bar{z}^3}{3} - (H + G)\frac{\bar{z}^2}{2} + GH\,\bar{z}\right\}\right]$$

$$+ C(\bar{x}, \bar{y}) \qquad 2.5.07c$$

where $C(\bar{x}, \bar{y})$ is an arbitrary constant.

Imposing boundary conditions (2.5.05c) and (2.5.06c) on(2.5.07c) and subtracting the resulting two expressions, gives an equation relating the pressure to the separation of the two plates

$$\frac{\partial (G-H)}{\partial \bar{t}} = \frac{1}{12}\frac{\partial}{\partial \bar{x}}\left[\frac{\partial \bar{p}}{\partial \bar{x}}(G - H)^3\right] + \frac{1}{12}\frac{\partial}{\partial \bar{y}}\left[\frac{\partial \bar{p}}{\partial \bar{y}}(G - H)^3\right]. \qquad 2.5.08$$

It should be noted that this equation can be obtained by considering the motion of the fluid between one moving plate and one fixed plate. This can be seen from the equation itself since it depends only on the distance between the two plates (G - H). Substituting the original variables back into this equation (2.5.08) and letting $h_* = H - G$ gives

$$\frac{\partial h_*}{\partial t} = \frac{1}{12\mu}\frac{\partial}{\partial x}\left[h_*^3\frac{\partial p}{\partial x}\right] + \frac{1}{12\mu}\frac{\partial}{\partial y}\left[h_*^3\frac{\partial p}{\partial y}\right] \qquad 2.5.09$$

which is the three dimensional Reynolds' equation for flow between moving plates.

CHAPTER 3 - A ONE-DIMENSIONAL MODEL OF A FLUID FILLED CRACK

3.1 - Introduction

In this section a one-dimensional model of a fluid filled crack is formulated. The model can be considered in three parts: (1) the sub-model of the rock, (2) the sub-model of the fluid and (3) the interaction of (1) and (2) known as the crack law. In (1) the rock is considered to be a two-dimensional linearly elastic material in a state of plane strain whereupon the stress/strain/displacement relations derived in chapter 2 (2.2.04) can be applied.

Part (2) of the model is concerned with the behaviour of a thin layer of an incompressible, Newtonian, viscous fluid. Since the crack width is much smaller than the crack length the lubrication approximation is taken to be valid and so Reynolds' equation (2.5.09), governs the fluid flow. It is noted that crack surfaces are not, in general, parallel to each other and since the surfaces touch at various points the flow through a crack is tortuous. It is because of this that the validity of the lubrication approximation has been discussed previously by Witherspoon et al. (1980). They summarise the arguments put forward by previous authors and they conclude that generally the cubic flow law is reasonable for flow through cracks.

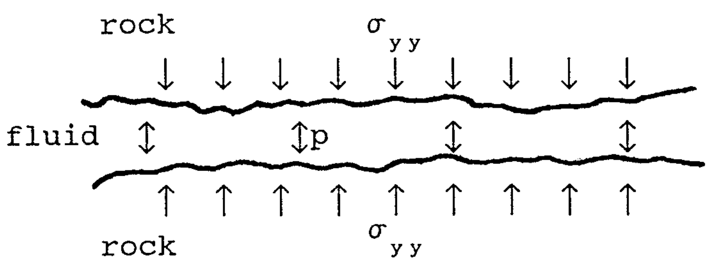

Fig. 3.1 - An Open Crack

Finally, part (3), to link these two sub-models an appropriate crack law is discussed, which allows the crack to be in one of two possible states; open or partially open.

An open crack is defined as the state when the fluid pressure, p, is equal to the elastic normal stress, σ_{yy}, so the crack faces are not exerting any force on each other (see figure 3.1). A partially open crack is then the case of the elastic normal stress being supported both by the fluid pressure and by deformations, δh, in local asperities (bumps) in the crack (see fig. 3.2). Note that a pre-existing crack is never completely closed since the roughness of the crack faces ensures a small separation even when very high stresses are imposed. The crack, however, is considered to be thin enough and flat enough that the elastic problem can be considered to have a flat boundary.

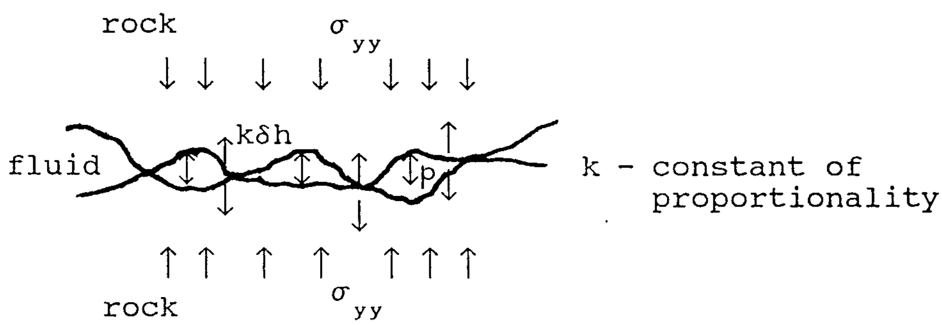

Fig. 3.2 - A Partially Open Crack

A partially open crack can itself be considered as having two separate states: (1) shearing, where the crack surfaces are allowed to slide over each other and (2) stuck, where the surfaces only move in the normal direction and do not slide over each other. The model presented here is the model used by the Camborne School of Mines to set up their one-dimensional numerical code called FBED which is discussed briefly in chapters one and five.

The final set of equations is non-dimensionalised and for the particular case of a partially open crack which does not shear, a single, non-linear, second order, partial integro-differential equation is obtained, containing just one parameter, which is physically small but not negligible. The following chapters then look at ways of solving this equation both analytically (first generally and then by considering limiting cases as the parameter tends to zero and infinity) and numerically.

3.2 - The Model

Consider a plane strain problem in the x, y plane. An infinite crack $y = 0$ splits the plane into two semi-infinite half-planes $y > 0$, $y < 0$. Since the problem is symmetric about $y = 0$ attention is restricted to the half-plane $y \geq 0$. The plane strain equations (2.2.04) for the case of zero stress at infinity and finite displacements, lead to expressions (2.3.50) and (2.3.51) for the elastic shear and normal stresses along the crack surface. Generalising for an infinite crack with stresses σ^o_{xy} and σ^o_{yy} at infinity, the following expressions are obtained

$$\sigma_{xy}(x,t) - \sigma^o_{xy} = \frac{-G}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial D_x(s)}{\partial s} \frac{ds}{s - x} \qquad 3.2.01a$$

$$\sigma_{yy}(x,t) - \sigma^o_{yy} = \frac{-G}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial D_y(s)}{\partial s} \frac{ds}{s - x} . \qquad 3.2.01b$$

where D_x and D_y are defined by (2.3.03) to be the relative displacements of the crack surfaces. G and ν are parameters determined by the properties rock.

These equations form the first part of this model of a one-dimensional crack. They therefore hold along the crack and it is assumed that the reader understands that $\sigma_{xy}(x,t) = \sigma_{xy}(x,0,t)$ and $\sigma_{yy}(x,t) = \sigma_{yy}(x,0,t)$. The equations for this problem all hold along $y = 0$, since the model is one-dimensional.

Now consider the crack surfaces to be lubricated by a thin layer of an incompressible, viscous, Newtonian fluid. The equation governing the flow of such a fluid is Reynolds' equation (2.5.09). The one-dimensional version of this equation forms part of the model and is

$$\frac{\partial h}{\partial t} = \frac{1}{12\mu} \frac{\partial}{\partial x} \left[h^3 \frac{\partial p}{\partial x} \right] . \qquad 3.2.01c$$

Note that because of the symmetry in this problem, the crack height, h, in this equation is minus the value of the normal displacement term, D_y, in equation (3.2.01b), that is

$$h = - D_y. \qquad 3.2.01d$$

The effective normal stress, σ_{yy}', is defined by

$$\sigma_{yy}' = \sigma_{yy} + p. \qquad 3.2.01e$$

Recall that positive normal stresses are tensile, so that for a typical crack, the elastic normal stress is negative since the rock faces are compressed, this in turn implies that the effective normal stress is negative as long as the fluid pressure is of a smaller magnitude than the normal stress. This can be used to test whether the crack is fully open. The crack opens at the point when the fluid pressure reaches the same size as the normal stress, so it is the point where the effective stress is zero.

To complete the model a crack law is required to relate the elasticity equations to the fluid flow equations. The crack law that is used in this model is discussed in Markland (1989). To help explain the crack law, the problem is considered in two parts: (1) Fully open crack (see figure 3.1) and (2) Partially open crack (see figure 3.2). The partially open problem is also considered in two parts: (A) that of a stuck crack when the displacement in the x direction, of one crack face relative to the other, is zero; and (B) that of a sliding or shearing crack, when the crack faces can move relative to each other along the direction of the crack. A simple friction law is used for deciding which state the partially open crack is in. The aperture of the crack, h, can be seen to have three contributory parts. First, there is a component, h^e, known as the elastic component of aperture, which is due to the stretching of the partially open crack when the shear force is insufficient to shear the crack. Second, there is a shear component, h^s, which arises as the shearing crack surfaces ride up over bumps in the crack. The final component, h^n, is the increase in aperture which occurs when the crack is fully open. The total aperture is therefore

$$h = h^e + h^s + h^n. \qquad 3.2.01f$$

Equations (3.2.01) hold for all modes of the crack, whether open or partially open, sliding or stuck. Each case is now considered separately to close the model.

OPEN CRACK

If the effective stress, σ_{yy}', is zero at any point in the crack then the crack is termed open at that point. This results from the fluid pressure being large enough to balance the normal stress

$$\sigma_{yy}' = \sigma_{yy} + p = 0. \qquad 3.2.02a$$

This is the crack law that is conventionally used in hydrofracture models (e.g. Spence and Turcotte (1985)).

When the crack is open, the only aperture gain will be due to the h^n component, h^e will be at its maximum value and h^s will remain constant for the period that the crack is open, so

$$h^e = h_{max} \qquad 3.2.02b$$

$$\frac{\partial h^s}{\partial t} = 0. \qquad 3.2.02c$$

PARTIALLY OPEN CRACK

As stated above this section will be split into two parts so that the case of a stuck crack is considered separately from that of a shearing crack. A crack is partially open if

$$\sigma_{yy}' < 0 \qquad 3.2.03a$$

and as a consequence, since there is by definition no open mode contribution to the aperture,

$$h^n = 0. \qquad 3.2.03b$$

If μ_f is the coefficient of friction then the frictional state of the crack is given by

$$|\sigma_{xy}| < \mu_f \, |\sigma_{yy}'| \qquad 3.2.04a$$

$\Rightarrow$ Crack is stuck

$$|\sigma_{xy}| = \mu_f \, |\sigma_{yy}'| \qquad 3.2.05a$$

$\Rightarrow$ Crack is sliding.

The equations that complete each section are considered in parts (A) Stuck and (B) Sliding.

<u>(A) Stuck</u>

Here a case is considered whereby the crack surfaces are not sliding over each other because the friction between them is too great, i.e.

$$|\sigma_{xy}| < \mu_f \, |\sigma_{yy}'|. \qquad 3.2.04a$$

The crack will effectively stretch as the fluid pressure increases and as the point at which slipping occurs approaches. The relative amount, D_x, that the crack stretches is governed by a spring law

$$\frac{\partial \sigma_{xy}}{\partial t} = k_s \frac{\partial D_x}{\partial t} \qquad 3.2.04b$$

where k_s is a type of spring constant for the crack.

The aperture gain due to shearing will not change if the crack is stuck so

$$\frac{\partial h^s}{\partial t} = 0. \qquad 3.2.04c$$

Assume that there is a minimum value for the thickness of the crack when the effective stress is large and negative. At this point, $h^s = h^n = 0$ and $h^e = h_{min}$. As the effective stress increases above some reference value, σ_{ref} it is assumed that h^e grows linearly in proportion to σ_{yy}' (Bandis et al. (1983), Barton (1986)), (see figure 3.3)

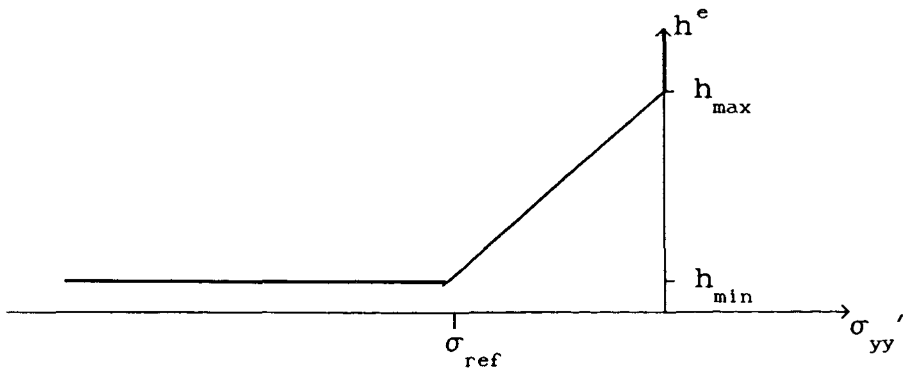

Fig. 3.3 - Elastic Aperture v Effective Stress

This means that h^e is given by

$$h^e = \begin{cases} h_{max} - \dfrac{(h_{max} - h_{min})}{\sigma_{ref}} \sigma_{yy}', & \sigma_{ref} \le \sigma_{yy}' < 0 \\ h_{min}, & \sigma_{yy}' \le \sigma_{ref} \end{cases} \qquad 3.2.04d$$

Although this is a piecewise linear relationship between the elastic aperture and the effective stress, there is no reason why an alternative functional relationship with a similar shape to that above, i.e. monotonically increasing in σ_{yy}' with a non-negative minimum value for h^e, could not have been chosen. It is shown later on in chapter four, however, that there is no power law dependency that gives a similarity solution to this problem. Using a piecewise linear function, though, enables similarity solutions to be found for certain cases of this problem, also discussed in chapter four.

(B) Sliding

When the fluid pressure is slowly increased, the effective stress decreases in magnitude and the surfaces begin to slide over each other until the point is reached where the effective stress is exactly balanced by the frictional force on the crack surface

$$|\sigma_{xy}| = \mu_f \, |\sigma_{yy}'| . \qquad 3.2.05a$$

It should be noted that it remains possible for the elastic aperture to increase so equation (3.2.04d) still holds.

The sliding aperture increases linearly with the distance that the crack surface is displaced, and also depends on the angle of the bump, known as the dilation angle, ψ. A maximum possible value for this sliding aperture (h^s_{max}) is assumed since this would seem reasonable in practice. So $\frac{\partial h^s}{\partial t}$ is defined by

$$\frac{\partial h^s}{\partial t} = \begin{cases} \tan\psi \left| \frac{\partial D_x}{\partial t} \right|, & 0 < h^s < h^s_{max} \\ 0, & h^s = h^s_{max} \end{cases} . \qquad 3.2.05b$$

It is the stretching and sliding that requires further study but the remainder of this thesis will concentrate mainly on the stretching, i.e. on the partially open crack problem where no shearing is involved, though the incorporation of shearing effects is briefly discussed at the end of this chapter.

Section 3.3 - Non-dimensionalisation of the Partially Open, No Shear Crack Problem

So far a set of equations governing the motion of a fluid filled crack has been written down and for the case of partially open crack with no shearing of the surfaces, these equations are (3.2.01, 3.2.03 and 3.2.04).

There are, therefore, ten equations in ten unknowns, although one of these unknowns (h^n) is zero, and D_y can be replaced by $-h$, so they can be reduced to eight equations in eight unknowns. The boundary conditions for these equations are discussed later on (chapter 4).

Non-dimensionalising a problem helps determine the dominant terms in an equation and often leads to simplifying assumptions which enable approximate solutions to be found. In this case, if h_{max} is a typical crack width, p_o is a typical pressure, σ^o_{xy} and σ^o_{yy} are the stresses at infinity, and L is a typical length scale, then in order to non-dimensionalise the 1-D crack equations the following variables are introduced

$$h = h_{max}\,\bar{h},$$

$$p = p_o - (\sigma^o_{yy}+p_o)\,\bar{p}\ ,$$

$$\sigma_{yy} = \sigma^0_{yy} + \frac{Gh_{max}}{L}\,\bar{\sigma}_{yy}\ ,$$

$$\sigma_{xy} = \sigma^o_{xy}\,(1 - \bar{\sigma}_{xy})\ ,$$

$$\sigma_{yy}{}' = (\sigma^o_{yy}+p_o)\,(1 - \bar{\sigma}_{yy}{}')\ ,$$

$$t = T\,\bar{t}\ ,$$

$$x = L\,\bar{x}\ ,$$

$$D_x = \frac{\sigma^o_{xy}\,L}{G}\,\bar{D}_x\ .$$

Finally, in order to simplify the Reynolds' equation (3.2.01c) time is scaled by T where T is given by

$$T = \frac{-12\mu_f L^2}{h^2_{max}(\sigma^o_{yy} + p_o)} > 0 \quad \text{since } (\sigma^o_{yy}+p_o) < 0.$$

Consider the case when the crack is not clamped shut (i.e. $\sigma_{yy}{}' \geq \sigma_{ref}$). Eliminating h^n and D_y and setting

$h^s = 0$, the equations become

$$\bar{\sigma}_{xy} = \frac{-1}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial \bar{D}_x}{\partial \bar{s}} \frac{d\bar{s}}{\bar{s} - \bar{x}} \qquad 3.3.01a$$

$$\bar{\sigma}_{yy} = \frac{1}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial \bar{h}}{\partial \bar{s}} \frac{d\bar{s}}{\bar{s} - \bar{x}} \qquad 3.3.01b$$

$$\frac{\partial \bar{h}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left[\bar{h}^3 \frac{\partial \bar{p}}{\partial \bar{x}} \right] \qquad 3.3.01c$$

$$\bar{\sigma}_{yy}' = \bar{p} + A\,\bar{\sigma}_{yy} \qquad 3.3.01e$$

$$\frac{\partial \bar{\sigma}_{xy}}{\partial \bar{t}} = \gamma \frac{\partial \bar{D}_x}{\partial \bar{t}} \qquad 3.3.01f$$

$$\bar{h} = \begin{cases} 1+\frac{\alpha}{A}(1-\delta_h)(\bar{p}-1)+\alpha(1-\delta_h)\bar{\sigma}_{yy} & 1-A/\alpha \le \bar{\sigma}_{yy}' < 1 \\ \delta_h & \bar{\sigma}_{yy}' < 1-A/\alpha \end{cases} \qquad 3.3.01g$$

where

$$\alpha = -\frac{Gh_{max}}{\sigma_{ref} L} \qquad 3.3.02a$$

$$\delta_h = \frac{h_{min}}{h_{max}} \qquad 3.3.02b$$

$$\gamma = \frac{k_s L}{G} \qquad 3.3.02c$$

$$A = -\frac{Gh_{max}}{(\sigma^o_{yy} + p_o)L}. \qquad 3.3.02d$$

In standard S.I. units, typical magnitudes of the scale factors are (Pine and Batchelor (1984), CSM (1988)):

$G = O(10^{10})$

$\sigma_{ij} = O(10^7)$

$p_o = O(10^7)$

$h_{max} = O(10^{-4} - 10^{-3})$

$h_{min} = O(10^{-6})$

$$\sigma_{ref} = O(10^7)$$
$$k_s = O(10^8).$$

All other scale factors are taken to be O(1) and so the parameters have the following magnitudes:

$$\alpha = O(10^{-1} - 1),$$
$$\delta_h = O(10^{-3} - 10^{-2}),$$
$$\gamma = O(10^{-2}),$$
$$A = O(10^{-1} - 1).$$

Observe the equations containing the terms $\bar{\sigma}_{xy}$ and $\bar{u}_x$. Equation (3.3.01a) implies that $\bar{\sigma}_{xy}$ and $\bar{u}_x$ are of the same order of magnitude, whereas (3.3.01f) implies that $\bar{\sigma}_{xy}$ and $\bar{u}_x$ differ by a factor of γ, i.e. 10^{-2}. This is true if

$$\bar{\sigma}_{xy} = \bar{u}_x = 0. \qquad 3.3.03$$

Considering the limit as $\delta_h \to 0$, if $\bar{\sigma}_{yy}' < 1 - A/\alpha$ then $\bar{h} = 0$, otherwise the remaining equations in $\bar{h}$, $\bar{p}$ and $\bar{\sigma}_{yy}$ can be combined together to form one equation, valid for values of the effective stress in the region

$$1 - \frac{A}{\alpha} \le \bar{\sigma}_{yy}' < 1.$$

The equation is

$$\frac{\partial \bar{h}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left[\bar{h}^3 \frac{\partial}{\partial \bar{x}} \left\{ \frac{A}{\alpha} \left(\bar{h} - \alpha\, \bar{\sigma}_{yy} \right) \right\} \right] \qquad 3.3.04$$

where $\bar{\sigma}_{yy}$ is given by

$$\bar{\sigma}_{yy} = \frac{1}{2\pi(1-\nu)} \int_{-\infty}^{\infty} \frac{\partial \bar{h}}{\partial \bar{s}} \frac{d\bar{s}}{\bar{s} - \bar{x}}. \qquad 3.3.01b$$

Restricting attention to non-zero values of $\bar{h}$, the time variable is rescaled by a factor of A/α and so (3.3.04) becomes

$$\frac{\partial \bar{h}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left[\bar{h}^3 \frac{\partial}{\partial \bar{x}} \left\{ \bar{h} - \alpha \, \bar{\sigma}_{yy} \right\} \right]. \qquad 3.3.05$$

Defining a new parameter ε as

$$\varepsilon = \frac{Gh_{max}}{2\pi(1-\nu)\sigma^o_{ref}L} = \frac{\alpha}{2\pi(1-\nu)} = O(10^{-2} - 10^{-1}) \qquad 3.3.06$$

and substituting the normal stress term (3.3.01b) into (3.3.05), a singular integro-differential equation for the motion of a 1-D fluid filled, partially open, non-shearing crack is obtained

$$\frac{\partial \bar{h}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left[\bar{h}^3 \frac{\partial}{\partial \bar{x}} \left\{ \bar{h} - \varepsilon \int_{-\infty}^{\infty} \frac{\partial \bar{h}}{\partial \bar{s}} \frac{d\bar{s}}{\bar{s} - \bar{x}} \right\} \right]. \qquad 3.3.07$$

In the next chapter this equation is solved analytically using the techniques of asymptotic analysis and similarity solutions for the cases where ε is either very large or very small. Numerical solutions to (3.3.07) are obtained in chapter five and a comparison is made between the two sets of results.

Section 3.4 - The Shearing Problem

The above non-dimensional analysis is concerned strictly with a crack which is re-opened due to a fluid loading but which has zero elastic shear stress along the crack surfaces. This means that displacements only occur in the direction normal to the crack and not parallel to it. Here the shearing mechanism is briefly discussed and it is concluded that the inclusion of shearing results in a much more complicated mathematical problem.

Allowing the crack surfaces to slide over each other introduces three new variables to the problem. They are: (1) the elastic shear stress σ_{xy}, (2) the shear displacement u_x, and (3) an added component of normal displacement due to the shearing of the crack, h^s. These components are defined in the development of the model earlier in this chapter, by

(3.2.01, 3.2.04, 3.2.05a and 3.2.05b). For the case of zero elastic shear stress along the crack the problem can be written in terms of a singular integro-differential equation (3.3.07). When shearing is allowed to occur the elastic normal stress and displacement become dependent on the component of shear dilation, h^s, which in turn is dependent on the elastic shear stress. A complicated expression for u_x results which would almost certainly require a numerical integration technique to solve the problem. This problem is, therefore, considered no further.

CHAPTER 4 - ANALYTICAL SOLUTIONS OF THE PARTIALLY OPEN CRACK EQUATION

4.1 - Introduction

In the previous chapter a model for the behaviour of a partially open crack with zero shear stress along the crack surfaces is described. A non-linear, singular integro-differential equation for the non-dimensional crack width h, is obtained (3.3.07), containing one non-dimensional parameter, ε, whereby

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[h^3 \frac{\partial}{\partial x}\left\{ h - \varepsilon\, \mathcal{H}\left[\frac{\partial h}{\partial x}\right] \right\} \right]$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ 3.3.07

$$h = 0,$$

where $\mathcal{H}$ represents a Hilbert Transform defined by

$$\mathcal{H}[f(x)] = \int_{-\infty}^{\infty} f(s) \frac{ds}{s - x} .$$

In section 4.2 analytical solutions to equation (3.3.07) are sought. The problem is briefly outlined and then a general power law dependency of the crack width on the effective stress is discussed. It is found, however, that no analytical solutions can be obtained when such a law is introduced.

The cases of very large and very small values of ε in (3.3.07) are then considered. The parameter, ε, is a ratio of shear modulus/crack length to reference stress/maximum crack displacement and its value determines how changes in fluid pressure are compensated for. Large values of ε imply that changes in the fluid pressure are compensated for by a global redistribution of stresses along the crack, whereas small values of ε mean that a local deformation of asperities occurs. An asymptotic series is sought in each of these two cases and similarity solutions are found for the resulting first order equations.

Spence and Turcotte (1985) consider a problem that corresponds to the first order equation for large values of ε. Their analysis involves using similarity transformations and the resulting ordinary integro-differential equation is solved numerically using Chebychev polynomials. The problem of a pre-existing crack, with no stress singularity at the crack tip, is considered more carefully by Spence and Sharp (1985), and again numerical solutions are obtained. In this thesis, in order to obtain closed form analytical solutions (which can then be used as a test case for the CSM numerical code), the h^3 term is replaced by h. So, consider

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[h \frac{\partial}{\partial x}\left\{ h - \varepsilon \mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right\}\right]$$

or 4.1.01

$$h = 0$$

rather than (3.3.07).

The h^3 term in (3.3.07) arises from the assumption that a fluid filled crack behaves in the same way as lubricated parallel plates. Crack surfaces, however, are rough and not strictly parallel and, because of this, the validity of the cubic law has been a topic of discussion in the past (Witherspoon et al. (1980)). Additionally, for flow through partially open cracks the flow path is tortuous since the fluid moves around the touching asperities. It is not asserted that a linear law is more valid than the cubic law, but that, since the true nature of the flow law is unknown, a substitute law which allows us to obtain analytical solutions is justifiable. Smyth and Hill (1988) for example, made a similar assumption in order to obtain analytical solutions to high order non-linear diffusion equations. The cubic law equation is briefly discussed but is then not considered further in this thesis.

It is necessary, for the small ε case, to examine more closely the behaviour at the crack tip and this is done in section (4.5).

4.2 - Analysis of the Problem For a General value of ε

Equation (3.3.07) represents the width of a crack in a linearly elastic solid, which is being reopened by the injection of an incompressible viscous fluid. The crack faces are assumed only to move normal to each other and do not slide over each other. The crack pressure is assumed to be smaller in magnitude than the normal stress so that the crack never becomes fully open. A closed form analytical solution for equation (3.3.07) cannot be obtained since the h^3 term ensures that the equation is not integrable analytically. It is shown, however, that a similarity transformation for the equation exists, but that the solution does not satisfy any practical boundary conditions. An attempt to reconcile this difference is made by introducing a general power law relation between the crack width and the effective stress, but it is shown that all possible similarity transformations lead to physically unrealistic solutions, except for the open crack case where the normal effective stress is zero. Similarity solutions for this case have been obtained by Spence and Turcotte (1985) and Spence and Sharp (1985).

It is noted that $h = 0$ is the solution outside the crack; throughout this chapter, however, attention is restricted to the integro-differential equation in order to find non-zero solutions for h.

Look for a similarity solution of the form

$$h = t^{\alpha} f(\eta) \qquad \eta = x\, t^{-\beta} \qquad \bar{s} = s\, t^{-\beta}.$$

If $\eta_t = \dfrac{\partial \eta}{\partial t}$ and $\eta_x = \dfrac{\partial \eta}{\partial x}$ and if $f' = \dfrac{df}{d\eta}$ then

$$\eta_t = -\frac{\beta\eta}{t} \qquad \eta_x = t^{-\beta}.$$

Substituting these transformations into equation (3.3.07) gives

$$t^{\alpha-1}\,(\alpha f - \beta\eta f') = t^{-2\beta}\left(t^{3\alpha} f^3 \left[t^{\alpha} f - \varepsilon \int_{-\infty}^{\infty} t^{\alpha-\beta}\, f' \,\frac{d\bar{s}}{\bar{s} - \eta}\right]'\right)' \qquad 4.2.01$$

By choosing $\beta = 0$ and $\alpha = -1/3$, (4.2.03) can be reduced to an ordinary integro-differential equation in η,

$$-\frac{1}{3}\, f = \left(f^3 \left\{f - \varepsilon\, \mathcal{H}\left[f'\right]\right\}'\right)' . \qquad 4.2.02$$

The solution to equation (4.2.02) is of the form $h(x,t) = t^{-1/3} f(x)$ and since this solution seems to correspond to that of a distributed sink problem, it is of little physical interest.

Notice that one must choose $\beta = 0$ to ensure that the Hilbert transform has the same power dependency on t as the crack height term, h. Alternative power laws are investigated to see if there exists any law which gives a more useful similarity solution. The law used above is shown in figure 3.3.

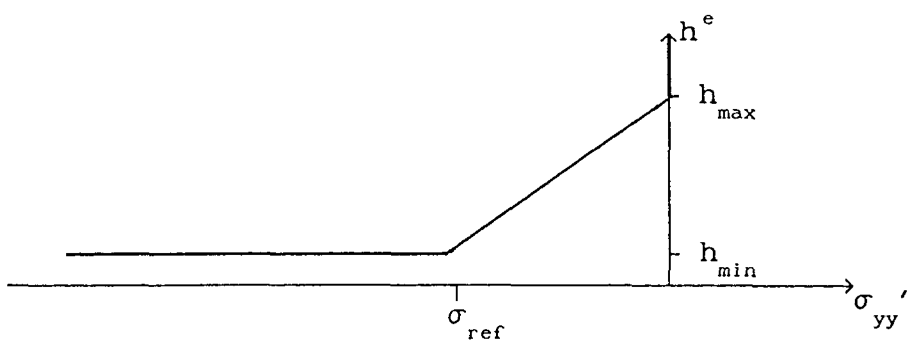

Fig. 3.3 - Elastic Aperture v Effective Stress (Piecewise Linear Law)

This is not ideal since it has a discontinuity in the first derivative at $\sigma_{yy}' = \sigma_{ref}$. A more realistic crack law would be one which is continuous and has continuous slope everywhere, but which has a similar shape to the above (monotonically increasing in σ_{yy}' with a non-negative minimum value $h = h_{min}$). What is actually required is a law of the kind pictured in figure 4.1, so if $F(\sigma_{yy}') = (\sigma_{yy}')^n$, h^e is defined as

$$h^e = \begin{cases} h_{max} - (h_{max} - h_{min}) \left[F(\sigma_{yy}')/F(\sigma_{ref}) \right], & \sigma_{ref} \leq \sigma_{yy}' < 0 \\ h_{min}, \quad \sigma_{yy}' \leq \sigma_{ref} & \end{cases} \qquad 4.2.03$$

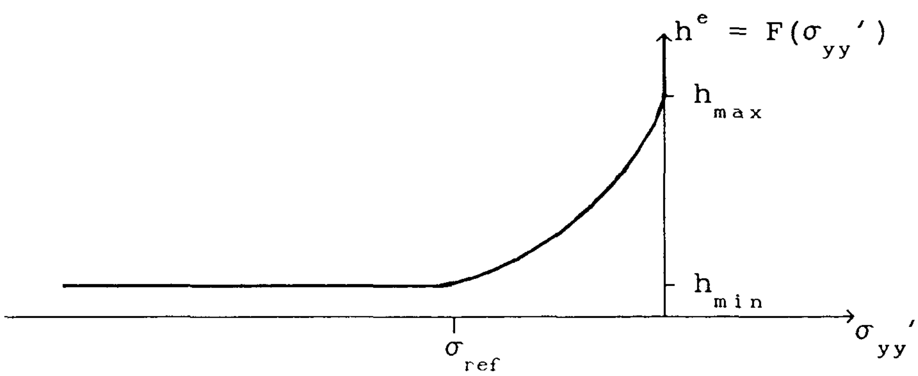

Fig. 4.1 - Elastic Aperture v Effective Stress (Non-linear Law)

Variations in F^{-1} are considered to be linearly dependent on σ_{ref}, since this ensures that the dimensions of the problem are again confined to just one parameter.

Restricting attention to the interval $\sigma_{ref} \leq \sigma_{yy}' \leq 0$, and non-dimensionalising and re-arranging in the same way as in section 3.3, the following equation is obtained

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x} \left[h^3 \frac{\partial}{\partial x} \left\{ F^{-1}[(1 - h)] - \varepsilon \mathcal{H}\left[\frac{\partial h}{\partial x}\right] \right\} \right]. \qquad 4.2.04$$

The problem here, however, is that because of the non-dimensionalisation, the function F depends on (1 - h) and not just on h, as previously expected. Therefore any power law, $(1 - h)^\gamma$, $\gamma \geq 0$, (other than $\gamma = 0$), automatically contains the linear terms in h which causeu the problem in the similarity solution above. With $\gamma = 0$ we have the first order equation for the large ε problem, which does indeed have a similarity solution which is found in section 4.3. It may be possible, however, to obtain a solution of (4.2.04) for the limiting case as $\varepsilon \to 0$. This has not yet been investigated but it is a possible line for future work to take.

The boundary and initial conditions have not yet been discussed. It is possible to impose a constant fluid flux condition as did Spence and Turcotte (1985), but in order to obtain closed form analytical solutions of this problem it is necessary to impose a point source, Barenblatt (1953) type, condition. This condition corresponds to assuming that the mass of fluid remains constant. Initially the height is given by a Dirac-delta function in x and by insisting that the height keeps compact support and showing this to be consistent with the solution obtained, this ensures that the mass of fluid in the crack remains constant. This means that the crack height is zero outside the fluid filled region and will take some positive value, inside the region. If the non-dimensional length of the crack is given by $2\ell(\bar{t})$, then the crack height has the form

$$h(x,t) \begin{cases} > 0 & |x| < \ell(t) \\ = 0 & |x| \geq \ell(t) \end{cases} . \qquad 4.2.05$$

The constant fluid mass condition can now be written as

$$\int_{-l(t)}^{l(t)} h(x,t)\ dx = 1 . \qquad 4.2.06$$

Finally, assume that the crack already exists and is being reopened by the fluid loading; this means that there is no stress singularity at the crack tip.

4.3 - $\varepsilon \gg 1$ Problem

Consider the simplified equation (4.1.01). Recall that large values of ε mean that the fluid pressure is supported preferentially by the elastic normal stress. This corresponds to the fully open crack case described in the model in chapter two, since deformations in asperities are negligible.

Let $\varepsilon = 1/\delta$, so that $\delta \ll 1$, rescale the time with δ and obtain

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[h \frac{\partial}{\partial x}\left\{ \delta h - \mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right\} \right] \qquad 4.3.01$$

Look for an asymptotic series solution of the form

$$h = h_0(x,t) + \delta\, h_1(x,t) + \delta^2\, h_2(x,t) + \ldots \qquad 4.3.02$$

and substitute this expression into equation (4.3.01) to obtain

$$\frac{\partial (h_0+\delta h_1+\ldots)}{\partial t} = \frac{\partial}{\partial x}\left[(h_0+\delta h_1+\ \ldots) \frac{\partial}{\partial x}\left\{ \delta (h_0+\delta h_1+\ \ldots) \right.\right.$$

$$\left.\left. - \int_{-\infty}^{\infty} \frac{\partial (h_0+\delta h_1+\ldots)}{\partial s} \frac{ds}{s - x} \right\} \right]. \qquad 4.3.03$$

As $\delta \to 0$ the first order equation and boundary conditions are

$$\frac{\partial h_0}{\partial t} = - \frac{\partial}{\partial x}\left[h_0 \frac{\partial}{\partial x}\left\{ \mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right\} \right] \qquad 4.3.04$$

$$h_0(x,t) \begin{cases} > 0 & |x| < \ell(t) \\ = 0 & |x| \geq \ell(t) \end{cases} \qquad 4.3.05$$

$$\int_{-1}^{1} h_0(x,t)\, dx = 1 \qquad 4.3.06$$

A similarity transformation to this problem exists and has the form

$$h_0(x,t) = t^{-1/4}\, H(\eta) \qquad\qquad \ell(t) = \lambda\, t^{1/4}$$

where $\eta = \frac{x\, t^{-1/4}}{\lambda}$ and $2\ell(t)$ denotes the length of the crack. This transformation differs from that of Spence and Turcotte (1985) because h is substituted for h^3 in the governing equation (3.3.07).

Using primes to denote the derivative with respect to the similarity variable, η, the ordinary integro-differential equation and its corresponding boundary conditions are

$$\frac{1}{4}\,(\eta H)' = \left[\frac{H}{\lambda^3}\left(\int_{-1}^{1}\frac{dH}{dr}\frac{dr}{r-\eta}\right)'\right]', \qquad 4.3.07$$

$$H\begin{cases} > 0 & |\eta| < 1 \\ = 0 & |\eta| \geq 1 \end{cases}, \qquad 4.3.08$$

$$\int_{-1}^{1} H\, d\eta = 1/\lambda\ . \qquad 4.3.09$$

Equation (4.3.07) can be integrated twice to obtain

$$\begin{aligned} \frac{\eta^2\lambda^3}{8} + C\lambda &= \int_{-1}^{1}\frac{dH}{dr}\frac{dr}{r-\eta} && |\eta| < 1 \\ H &= 0 && |\eta| > 1 \end{aligned} \qquad 4.3.10$$

where C is an arbitrary constant of integration. In order to find the form of H inside the crack, the finite range Hilbert transform is inverted (Tricomi (1951)) to obtain

$$\frac{dH}{dr} = -\frac{1}{\pi^2\sqrt{(1-\eta^2)}}\int_{-1}^{1}\frac{\sqrt{(1-r^2)}}{(r-\eta)}\left\{\frac{\eta^2\lambda^3}{8} + C\lambda\right\}dr + \frac{D}{\sqrt{(1-\eta^2)}}\ . \qquad 4.3.11$$

The inversion necessitated the switch from a cubic law to the linear law in equation (4.1.01), since a linear integral equation is then obtained, rather than a non-linear integral equation. The eigen-solution generated in this process is discarded since it is an odd function of x and since the crack width is assumed to be symmetric about the y-axis, so set $D = 0$. Evaluating the integral in (4.3.11), for $|\eta| < 1$, gives

$$\frac{dH}{d\eta} = \alpha \eta (1 - \eta^2)^{-1/2} + \beta \eta^3 (1 - \eta^2)^{-1/2} \qquad 4.3.12$$

where α and β are defined in terms of the unknown constants λ and C by

$$\alpha = \frac{\lambda}{\pi} \left(C - \frac{\lambda^2}{16} \right) \qquad 4.3.13$$

$$\beta = \frac{\lambda^3}{8\pi} \qquad 4.3.14$$

Equation (4.3.12) is then integrated to get

$$H = -(\alpha + \beta)(1 - \eta^2)^{1/2} + \frac{\beta}{3}(1 - \eta^2)^{3/2} + E \qquad |\eta| < 1$$

$$H = 0 \qquad |\eta| < 1 \qquad 4.3.15$$

where E is a constant of integration found to be zero since the crack height is zero at the ends of the crack, $|\eta| = 1$. To determine λ and C boundary condition (4.3.09) is imposed, and the following relation is obtained

$$\lambda^4 + 32C\lambda^2 + 64 = 0 \quad . \qquad 4.3.16$$

A second relation for the two constants can be found by examining the stress at the crack tip. As stated, the case under consideration is that of a pre-existing crack and so the stress at the crack tip is finite. The form of the similarity solution requires the normal elastic stress to have the form

$$\sigma_{yy} = t^{-1/2} \tilde{\sigma}_{yy} \, , \qquad 4.3.17$$

so the first order approximation to the stress/crack width relation (3.3.01b) can be rewritten

$$\tilde{\sigma}_{yy} = \frac{1}{\lambda}\int_{-1}^{1} \frac{dH}{ds}\,\frac{ds}{s - \eta} \quad . \qquad 4.3.18$$

Substituting the solution for $H(\eta)$, (4.3.15), into equation (4.3.18), the following expression for the normal stress is obtained

$$\tilde{\sigma}_{yy} = \begin{cases} \dfrac{1}{\lambda}\left\{(\alpha + \beta\eta^2)\pi + \dfrac{\beta\pi}{2}\right\} & |\eta| < 1 \\ \dfrac{1}{\lambda}\left\{(\alpha + \beta\eta^2)\pi + \dfrac{\beta\pi}{2} + \dfrac{2\eta(\alpha + \beta\eta^2)}{(\eta^2 - 1)} f(\eta)\right\} & |\eta| > 1 \end{cases} \qquad 4.3.19$$

where $f(\eta)$ is given by

$$f(\eta) = \arctan\left(\frac{1 - \eta}{(\eta^2 - 1)^{1/2}}\right) - \arctan\left(\frac{1 + \eta}{(\eta^2 - 1)^{1/2}}\right) \quad .$$

Thus, since it is known that the stress is finite as $\eta \to 1^+$, and since $f(\eta) \to -\pi/2$ as $\eta \to 1^+$, one must set $\alpha = -\beta$ in order to remove the singularity from equation (4.3.19). This condition is expected since it is also required that $dH/d\eta$ remains finite as $|\eta| \to 1^-$. Using this condition with equation (4.3.16) and the definitions of α and β (4.3.14, 4.3.15), the following values for the constants are obtained

$$\lambda = 2\sqrt{2} \qquad C = -\frac{1}{2} \qquad -\alpha = \beta = \frac{2\sqrt{2}}{\pi} \quad .$$

Hence, an approximation to the crack height for the problem for the small δ case is given by

$$H = \begin{cases} \dfrac{2\sqrt{2}}{3\pi}(1 - \eta^2)^{3/2} & |\eta| < 1 \\ 0 & |\eta| \ge 1 \end{cases} \quad . \qquad 4.3.20$$

In terms of the non-dimensional variables this gives

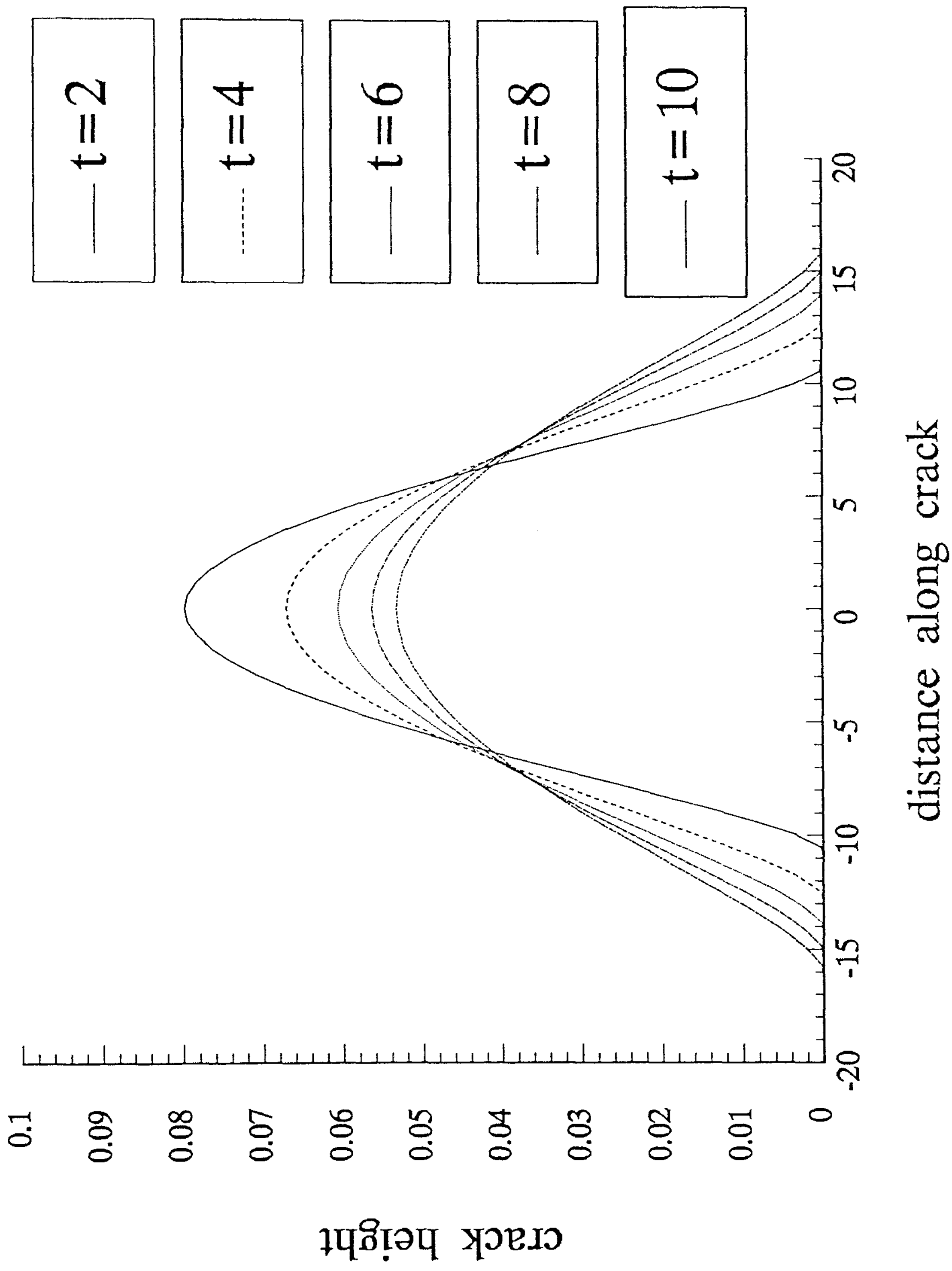

Figure 4.2 - Leading order solution as $\delta \rightarrow 0$

$$h_0 = \begin{cases} \dfrac{2\sqrt{2}}{3\pi}\, t^{-1/4}\left(1 - \dfrac{x^2}{8t^{1/2}}\right)^{3/2} & |x| < 2\sqrt{2}t^{1/4} \\ 0\ , & |x| \geq 2\sqrt{2}t^{1/4} \end{cases} \qquad 4.3.21$$

and h_0 is represented graphically in figure 4.2. This solution is consistent with the assumption that the Hilbert transform term in (4.1.01) is greater than the crack height term for sufficiently large values of ε. Therefore (4.3.21) is a uniformly valid leading order approximation for all values of the space variable.

The next term in the series expansion for the crack width, is found by equating O(δ) terms in (4.3.02) to obtain

$$\frac{\partial h_1}{\partial t} = \frac{\partial}{\partial x}\left[h_0 \frac{\partial h_0}{\partial x} - h_0 \frac{\partial}{\partial x}\left\{\mathcal{H}\left[\frac{\partial h_1}{\partial x}\right]\right\} - h_1 \frac{\partial}{\partial x}\left\{\mathcal{H}\left[\frac{\partial h_0}{\partial x}\right]\right\}\right] \qquad 4.3.22$$

with boundary conditions

$$h_1(x,t) \begin{cases} > 0 & |x| < \ell(t) \\ = 0 & |x| \geq \ell(t) \end{cases} \qquad 4.3.23$$

$$\int_{-l(t)}^{l(t)} h_1(x,t)\ dx = 0\ . \qquad 4.3.24$$

Substituting for h_0 (4.3.21) into (4.3.22), one proceeds in the same way as for the first order equation and looks for a similarity solution of the form,

$$h_1 = t^a\ H_1(2\sqrt{2}\eta), \qquad \ell(t) = 2\sqrt{2}\ t^{1/4}$$

where $\eta = \dfrac{xt^{-1/4}}{2\sqrt{2}}$.

Omitting the algebra and choosing a = 0, the following o.i.d.e. for H_1 is obtained

$$\eta H_1' = \frac{-1}{6\pi}\left\{(1-\eta^2)^{3/2}\left[\mathcal{H}\left[\frac{\partial H_1}{\partial x}\right]\right]'\right\}' - (\eta H_1)'$$

$$- \frac{4}{3\pi^2}(1-\eta^2)\ (5\eta^2 - 1) \qquad 4.3.25$$

Although the right hand side of this equation is integrable, the left hand side is not, so no further progress is made in generating a closed form analytical solution to the first order problem.

4.4 - $\varepsilon \ll 1$ Problem

Here an asymptotic series solution is sought for equation (4.1.01) as $\varepsilon \to 0$. Recall that ε is defined by (3.3.06) in such a way that small values of ε correspond to a small shear modulus/length ratio compared to the reference stress/maximum crack displacement ratio. This implies that changes in fluid pressure are compensated for by a local deformation of asperities rather than a global redistribution of stresses. Look for a solution of the form

$$h = h_0 + \varepsilon\, h_1 + \varepsilon^2\, h_2 + \ldots + \varepsilon^n\, h_n + \ldots\ .$$

In this section it is shown that the $O(\varepsilon)$ terms in the solution as $\varepsilon \to 0$, are singular at the crack tip. Since this invalidates the assumption that the $O(\varepsilon)$ terms are smaller than the $O(1)$ terms (local to the tip), the behaviour in the region close to the tip is investigated further in section 4.5.

Substituting this series expansion into equation (4.1.01) gives

$$\frac{\partial(h_0 + \varepsilon h_1 + \ldots)}{\partial t} = \frac{\partial}{\partial x}\left[(h_0 + \varepsilon h_1 + \ldots)\ \frac{\partial}{\partial x}\left\{(h_0 + \varepsilon h_1 + \ldots)\right.\right.$$

$$\left.\left. - \varepsilon\int_{-\infty}^{\infty} \frac{\partial(h_0 + \varepsilon h_1 + \ldots)}{\partial s}\ \frac{ds}{s - x}\right\}\right], \qquad 4.4.01$$

so as $\varepsilon \to 0$, the highest order equation is

$$\frac{\partial h_0}{\partial t} = \frac{\partial}{\partial x}\left[h_0 \frac{\partial h_0}{\partial x}\right] \qquad 4.4.02$$

and the boundary conditions are

$$h_0(x,t) \begin{cases} > 0 & |x| < \ell(t) \\ = 0 & |x| \geq \ell(t) \end{cases}, \qquad 4.4.03$$

$$\int_{-1}^{1} h_0(x,t)\ dx = 1, \qquad 4.4.04$$

where 2ℓ is the length of the crack as before.

The solution to this problem can be found in the usual way by looking for a similarity transformation of the form

$$h_0 = t^{\alpha}\ H_0(xt^{-\beta})$$

and is found to be

$$h_0(x,t) = \begin{cases} \frac{t^{-1/3}}{6}\left[\lambda^2 - x^2 t^{-2/3}\right] & |x| < \lambda t^{1/3} \\ 0 & |x| > \lambda t^{1/3} \end{cases} \qquad 4.4.05$$

where λ is a constant of integration which can be found using boundary condition (4.4.04) to be

$$\lambda = \left(\frac{9}{2}\right)^{1/3}. \qquad 4.4.06$$

Equation (4.4.05) is plotted in figure 4.3. This is a standard solution to the non-linear diffusion equation (4.4.02).

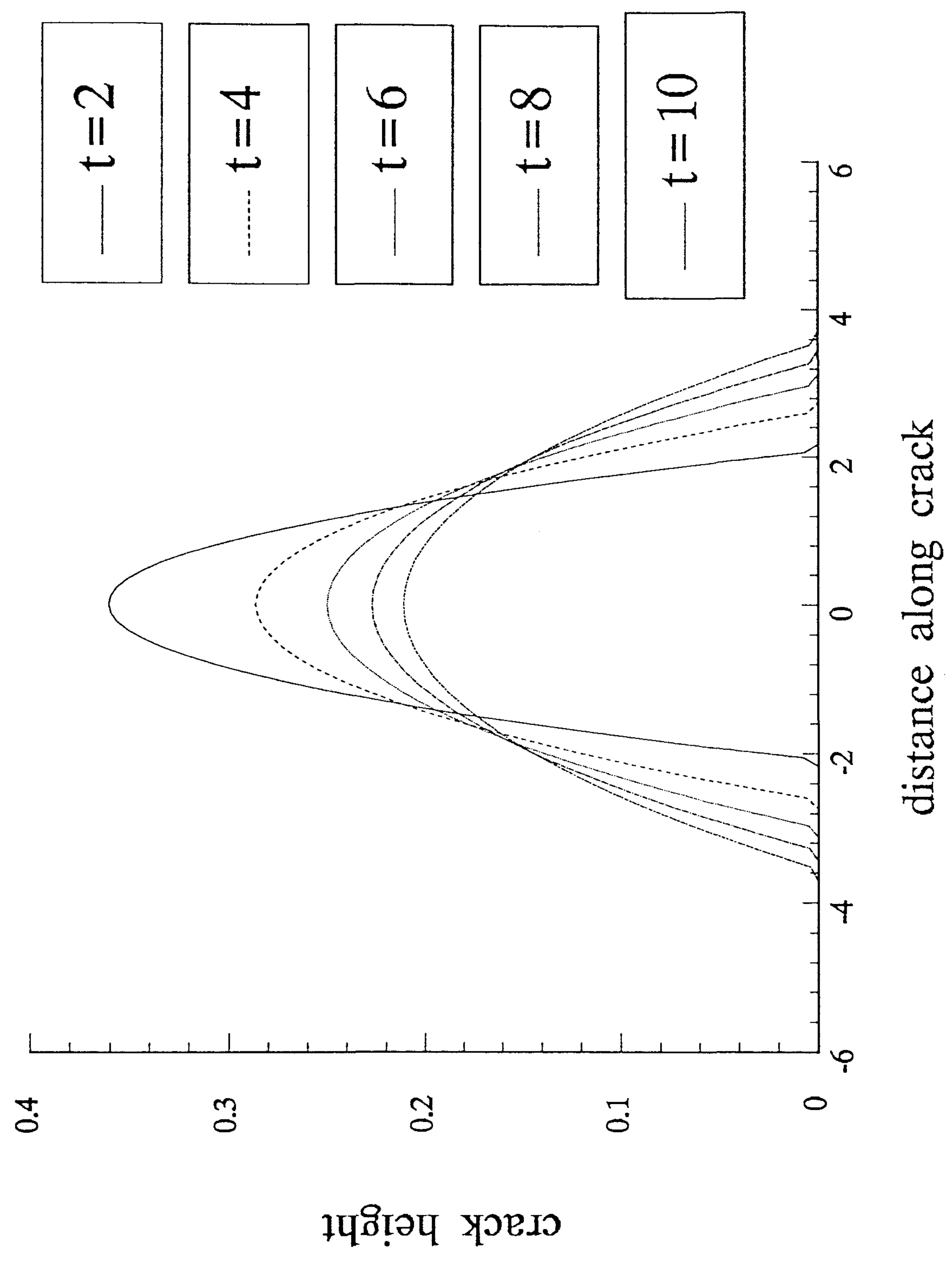

Figure 4.3 - Leading order solution as $\varepsilon \to 0$

The next step in obtaining a solution to the small ε problem is to consider the $O(\varepsilon)$ terms in the equation (4.4.01). Equating coefficients of ε gives

$$\frac{\partial h_1}{\partial t} = \frac{\partial}{\partial x}\left[h_0 \frac{\partial}{\partial x}\left\{h_1 - \mathcal{H}\left[\frac{\partial h_0}{\partial x}\right]\right\} + h_1 \frac{\partial h_0}{\partial x}\right]. \qquad 4.4.07$$

The boundary conditions for the first order problem are

$$h_1(x,t) \begin{cases} > 0 & |x| < \ell(t) \\ = 0 & |x| \geq \ell(t) \end{cases} \qquad 4.4.08$$

$$\int_{-l(t)}^{l(t)} h_1 \, dx = 0. \qquad 4.4.09$$

Rearranging (4.4.07) it is seen that

$$\frac{\partial h_1}{\partial t} = \frac{\partial^2 (h_0 h_1)}{\partial x^2} - f(h_0)$$

where $f(h_0)$ is given by the following expression

$$f(h_0) = + \frac{\partial}{\partial x}\left[h_0 \frac{\partial}{\partial x}\left\{\mathcal{H}\left[\frac{\partial h_0}{\partial x}\right]\right\}\right]. \qquad 4.4.10$$

The function f_0 can be obtained since h_0 is known (4.4.05) and integrable. After some algebra, $f(h_0)$ is found to be

$$f(h_0) = \begin{cases} \dfrac{x}{9t^2} \ln\left|\dfrac{\lambda t^{1/3} - x}{\lambda t^{1/3} + x}\right| + \dfrac{2\lambda}{9t^{5/3}} & |x| < \lambda t^{1/3} \\ 0 & |x| > \lambda t^{1/3}. \end{cases} \qquad 4.4.11$$

Outside the crack

$$\frac{\partial h_1}{\partial t} = 0 \qquad 4.4.12$$

and, since h_1 is initially zero outside the crack, this implies that

$$h_1 = 0 \qquad |x| > \lambda t^{1/3}. \qquad 4.4.13$$

Inside the crack, a similarity transformation for h_1 exists and is of the form

$$h_1 = t^{-2/3} H(\eta) \qquad \eta = \frac{xt^{-1/3}}{\lambda}$$

Again using primes to denote derivatives with respect to the variable η, the resulting ordinary differential equation is

$$(1 - \eta^2)\, H'' - 2\eta\, H' + 2H = F(\eta) \qquad 4.4.14$$

where $F(\eta)$ is given by

$$F(\eta) = \frac{2\lambda}{3} \left\{ \eta \ln\left|\frac{1 - \eta}{1 + \eta}\right| + 2 \right\}. \qquad 4.4.15$$

Equation (4.4.14) is a Legendre equation of order one. The complementary function for this equation is therefore

$$H = C_1 \left[\eta \ln\left|\frac{1 - \eta}{1 + \eta}\right| + 2 \right] + C_2\, \eta \qquad 4.4.16$$

where C_1 and C_2 are constants of integration, though since H is an even function of η one must choose $C_2 = 0$. The particular integral is given by

$$H_p = \int_0^{\eta} \frac{H_1(\xi) H_2(\eta) - H_1(\eta) H_2(\xi)}{(1 - \xi^2)\, \Delta_2(\xi)} F(\xi)\, d\xi \qquad 4.4.17$$

where Δ_2 is the wronskian, ie $\Delta_2 = H_1 H_2' - H_2 H_1'$, where

$$H_1 = \eta \qquad 4.4.18$$

and

$$H_2 = \eta \ln\left|\frac{1 - \eta}{1 + \eta}\right| + 2. \qquad 4.4.19$$

This can be expanded to give

$$H_p = \frac{-\lambda}{3}\left(\eta\ln\left|\frac{1-\eta}{1+\eta}\right| + 2\right)\int_0^{\eta}\xi\left(\xi\ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right)d\xi$$

$$+ \frac{\eta\lambda}{3}\int_0^{\eta}\left(\xi\ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right)^2 d\xi \;. \qquad 4.4.20$$

The general solution to equation (4.4.14) is the sum of the complementary function (4.4.16) and the particular integral (4.4.20)

$$H = \left\{C_1 - \frac{\lambda}{3}\int_0^{\eta}\xi\left(\xi\ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right)d\xi\right\}\left(\eta\ln\left|\frac{1-\eta}{1+\eta}\right| + 2\right)$$

$$+ \frac{\eta\lambda}{3}\int_0^{\eta}\left(\xi\ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right)^2 d\xi \;. \qquad 4.4.21$$

Simplifying this expression gives

$$H = C_1\left(\eta\ln\left|\frac{1-\eta}{1+\eta}\right| + 2\right) - \frac{4\lambda\eta}{9}\ln\left|\frac{1-\eta}{1+\eta}\right| + \frac{2\lambda}{9}\ln|1-\eta^2|$$

$$- \frac{\lambda\eta}{9}\ln\left|\frac{1-\eta}{1+\eta}\right|\ln|1-\eta^2| - \frac{4\lambda\eta}{9}\int_0^{\eta}\frac{\ln|1-\xi^2|}{1-\xi^2}d\xi \;. \qquad 4.4.22$$

The constant C_1 can be found by imposing the conservation of fluid mass boundary condition which, in terms of the similarity variable η, is

$$\int_{-1}^{1} H\, d\eta = 0. \qquad 4.4.23$$

A summary of the necessary integration is found in Appendix B. The constant C_1 is then given by

$$C_1 = -\frac{2\lambda}{9}\ln 2 - \frac{\lambda}{9} \qquad 4.4.24$$

The first order term for the crack height for the small ε problem is then given by

$$h_1 = \begin{cases} t^{-2/3} H(xt^{-1/3}/\lambda) & |x| < \lambda t^{1/3} \\ 0 & |x| > \lambda t^{1/3} \end{cases} \qquad 4.4.25$$

where λ is given by (4.4.06) and $H(xt^{-1/3}/\lambda)$ is given by (4.4.22) and (4.4.24). So an approximate solution to (4.4.01) is

$$h = \begin{cases} \dfrac{t^{-1/3}}{6}\left[\lambda^2 - x^2 t^{-2/3}\right] + \varepsilon t^{-2/3} H(xt^{-1/3}/\lambda) & |x| < \lambda t^{1/3} \\ 0 & |x| > \lambda t^{1/3} \end{cases} \qquad 4.4.26 .$$

For this asymptotic series solution to be a uniformly valid approximation of the crack width h, εh_1 is required to be smaller than h_0 as $\varepsilon \to 0$, for all values of x and t. Since h_1 becomes infinite at the crack tips $x = \pm\, \lambda t^{1/3}$,

$$h_1 \to \frac{\lambda}{3} \ln\left|\frac{x - \lambda t^{1/3}}{x + \lambda t^{1/3}}\right| \qquad |x| \to \lambda t^{1/3}$$

(see Appendix C for details), it is clear that the approximation does not hold in this region. This is expected since the limit $\varepsilon \to 0$ insists that changes in fluid pressure are compensated for by a local aperture change rather than a global redistribution in stresses. When the aperture is almost zero, i.e. at the crack tip, the otherwise small, elastic normal stress term becomes comparatively large and so the original assumption does not hold. Away from the crack tip, however, for $t > 1$ (as in the large ε case), the solution is reasonable.

The next step is then to investigate the behaviour at the ends of the crack, close to the tip.

Section 4.5 - Crack Tip Analysis

A uniformly valid approximate solution to equation (4.1.01) is now sought, for all values of x and t, for the case of $\varepsilon \ll 1$. To do this, consider more closely the behaviour of the crack near the crack tip. In section 4.4 an approximate solution (4.4.26) is obtained, valid everywhere except close to the crack tip. To obtain this solution a simplifying assumption that changes in the fluid pressure would be compensated for preferentially by a local deformation in the asperities rather than a global redistribution of stresses is made. In the notation used, this means that the term $\varepsilon\sigma_{yy}$ is small in comparison with the crack height h. Towards the ends of the crack the height decreases to zero and so it becomes impossible for changes in the fluid pressure to be balanced by changes in the crack height and the elastic normal stress term becomes significant. This implies that the initial assumption is invalid in the crack tip region and hence the solution is invalid in this region.

The increased importance of the stress redistribution term means that the behaviour at the crack tip affects the behaviour of the crack everywhere. It cannot be considered as a local effect and in this respect it is unlike usual boundary layer problems. In equation (4.1.01) it is the Hilbert transform term that ensures the global response. Bearing this in mind, local functions ϕ_L at the left-hand end and ϕ_R at the right-hand end of the crack are introduced, and a solution is sought of the form

$$h = h_0(x,t) + \varepsilon h_1(x,t) + \ldots + \varepsilon^n h_n^n + \ldots$$
$$+ \varepsilon\phi_L(\zeta_L,t,\varepsilon) + \varepsilon\phi_R(\zeta_R,t,\varepsilon) \qquad 4.5.01$$

where

$$\zeta_L = \frac{x + \lambda t^{1/3}}{\varepsilon} \qquad \zeta_R = \frac{x - \lambda t^{1/3}}{\varepsilon} .$$

Since ϕ_L and ϕ_R are local functions the following boundary conditions are imposed

$\phi_L \to 0$ $\qquad \zeta_L \to -a$ $\qquad$ 4.5.02a

$\phi_L \to 0$ $\qquad \zeta_L \to \infty$ $\qquad$ 4.5.02b

$\phi_R \to 0$ $\qquad \zeta_R \to a$ $\qquad$ 4.5.02c

$\phi_R \to 0$ $\qquad \zeta_R \to -\infty$ $\qquad$ 4.5.02d

where a is determined by the position of the front so that the new crack length is $\ell(t) = 2(\lambda t^{1/3} + \varepsilon a)$.

In the previous section solutions for the functions h_0 (4.4.05) and h_1 (4.4.25) are obtained. Notice that h_1 is singular at $|x| = \lambda t^{1/3}$. Since the crack height is not expected to be singular at any values of x and t (an exception being t = 0 because of the initial condition), the functions ϕ_L and ϕ_R are expected to remove these singularities. Also, notice that when the expression for h (4.5.01) is substituted into the equation (4.1.01), it is necessary to take the Hilbert transform of the functions $\partial h_1/\partial x$, $\partial\phi_L/\partial x$ and $\partial\phi_R/\partial x$. Since all three of these functions become singular as $|x| \to (\lambda t^{1/3})^-$, the Hilbert transforms do not exist there. To overcome this problem a function, f_1 is introduced and is of the form

$$f_1 = t^{-2/3}F(\eta),$$

where $F(\eta)$ is defined in terms of functions F_R and F_L by

$$\frac{dF}{d\eta} = \frac{dH}{d\eta} - \frac{dF_R}{d\eta} - \frac{dF_L}{d\eta} \qquad 4.5.03$$

where

$$\frac{dF_R}{d\eta} = \lim_{\eta\to 1^-}\frac{dH}{d\eta} \qquad \frac{dF_L}{d\eta} = \lim_{\eta\to -1^+}\frac{dH}{d\eta}\ .$$

Integrating (4.5.03) gives

$$F = H - F_R - F_L + C_f \qquad 4.5.04$$

where C_f is an arbitrary constant of integration. For simplicity choose C_f such that $\lim_{\eta\to\pm 1^-} F = 0$. By defining F via its derivative it is ensured that both F and $dF/d\eta$ are non-singular as $|\eta| \to 1^-$. It is not enough to define F by

$$F = H - \lim_{\eta\to 1^-} H - \lim_{\eta\to -1^+} H$$

since this does not ensure that $dF/d\eta$ is non-singular at the ends of the crack.

Note that the functions F_L, F_R, F and their derivatives are all taken to be zero for values of η such that $|\eta| > 1$. A detailed evaluation of the limiting values of H and $dH/d\eta$ at each end of the crack is given in Appendix B. For values of η where $|\eta| < 1$ the following expression exists

$$\frac{dF_L}{d\eta} = \lim_{\eta\to -1^-} \frac{dH}{d\eta} = \frac{-\lambda}{3(1+\eta)} + \frac{2\lambda}{3} \ln\left|\frac{1+\eta}{2}\right| + \frac{\lambda}{18} - \frac{\lambda\pi^2}{27} \qquad 4.5.05$$

and integrating with respect to η gives

$$F_L = -\frac{\lambda}{3} \ln|1+\eta| + \frac{2\lambda}{3}\left[(1+\eta)\ln|1+\eta| - (1+\eta)\right] + \left[-\frac{2\lambda}{3}\ln 2 + \frac{\lambda}{18} - \frac{\lambda\pi^2}{27}\right](1+\eta) + C_L \qquad 4.5.06$$

where C_L is an arbitrary constant of integration chosen to be

$$C_L = \frac{\lambda}{3}\ln 2 - \frac{2\lambda}{9} + \frac{\lambda\pi^2}{27} \qquad 4.5.07$$

such that

$$\lim_{\eta\to -1} F_L = \lim_{\eta\to -1} H$$

Similarly,

$$\frac{dF_R}{d\eta} = \lim_{\eta\to1^-} \frac{dH}{d\eta} = \frac{\lambda}{3(1-\eta)} - \frac{2\lambda}{3} \ln\left|\frac{1-\eta}{2}\right| - \frac{\lambda}{18} + \frac{\lambda\pi^2}{27} \qquad 4.5.08$$

and so

$$F_R = -\frac{\lambda}{3} \ln|1-\eta| + \frac{2\lambda}{3}\Big[(1-\eta)\ln|1-\eta| - (1-\eta)\Big] - \left[\frac{2\lambda}{3}\ln 2 - \frac{\lambda}{18} + \frac{\lambda\pi^2}{27}\right](1-\eta) + C_R \qquad 4.5.09$$

where C_R is an arbitrary constant of integration chosen to be

$$C_R = \frac{\lambda}{3}\ln 2 - \frac{2\lambda}{9} + \frac{\lambda\pi^2}{27} \qquad 4.5.10$$

such that

$$\lim_{\eta\to1^-} F_R = \lim_{\eta\to1^-} H.$$

The function F is then given by

$$F = H + \frac{\lambda}{3}\ln|1-\eta^2| - \frac{2\lambda}{3}\eta\ln\left|\frac{1-\eta}{1+\eta}\right| - \frac{2\lambda}{3}\ln 2 - \frac{4\lambda}{3} + C_f \qquad 4.5.11$$

and C_f is chosen to be

$$C_f = -\frac{13\lambda}{9} + \frac{\lambda\pi^2}{27} \qquad 4.5.12$$

so that $\lim_{\eta\to\pm1} F = 0$.

Recall that ϕ_L and ϕ_R and their first derivatives are expected to have singularities at the ends of the crack that are of the same order as the singularities in the function h_1 there. In order to remove these singularities the functions ψ_L and ψ_R are introduced and are of the forms

$$\psi_L = t^{-2/3}\Psi_L(\zeta_L) \qquad\qquad \psi_R = t^{-2/3}\Psi_R(\zeta_R)$$

where Ψ_L and Ψ_R are defined in terms of ϕ_L, ϕ_R, F_L and F_R by

$$\Psi_L = t^{2/3}\phi_L + F_L \qquad 4.5.13$$

$$\Psi_R = t^{2/3}\phi_R + F_R. \qquad 4.5.14$$

Rearranging equations (4.5.04), (4.5.13) and (4.5.14) and substituting h_1, ϕ_L and ϕ_R into the expression for h (4.5.01) gives

$$h = h_0 + \varepsilon t^{-2/3}[F - C_f + \Psi_L + \Psi_R] + O(\varepsilon^2) \qquad 4.5.15$$

where F, Ψ_L and Ψ_R are regular functions at the crack tip.

Consider equation (4.1.01). First, concentrate on the right-hand end of the crack as $x \to \lambda t^{1/3}$ and so change variables from (x,t) to (ζ_R,t) and consider the leading order terms as $\varepsilon \to 0$ and $\zeta_R \to 0^-$. Using primes to denote derivatives with respect to ζ_R, equation (4.1.01) becomes

$$\frac{\partial h}{\partial t} - \frac{\lambda}{3\varepsilon t^{2/3}} h' = \frac{1}{\varepsilon^2}\left(h\left(h - \varepsilon\mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right)'\right)'. \qquad 4.5.16$$

Notice that the Hilbert transform term has been left as a function of x. This is because it is necessary to determine the leading order terms in the Hilbert transform and so this term is dealt with separately, in more detail, later in this section. For now we concentrate on simplifying the rest of the equation.

In terms of the new variables (ζ_R,t), h_0 is given by

$$h_0 = \begin{cases} -\dfrac{\varepsilon\lambda\zeta_R}{3t^{2/3}} - \dfrac{\varepsilon^2\zeta_R^2}{6t} & \zeta_R < 0 \\ 0 & \zeta_R > 0 \end{cases}. \qquad 4.5.17$$

So the leading order terms in the crack height, as $\varepsilon \to 0$, for the region $\zeta_R < 0$, are

$$h = \varepsilon t^{-2/3}\left\{-\frac{\lambda\zeta_R}{3} + F - C_f + \Psi_L + \Psi_R\right\} = \varepsilon h \qquad 4.5.18$$

where

$$h = t^{-2/3}\left\{-\frac{\lambda\zeta_R}{3} + F - C_f + \Psi_L + \Psi_R\right\} . \qquad 4.5.19$$

Substituting this expression for h (4.5.18) into equation (4.5.16) gives

$$\varepsilon\frac{\partial h}{\partial t} - \frac{\lambda}{3t^{2/3}} h' = \left(h\left(h - \mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right)'\right)' \qquad 4.5.20$$

and so it is clear that the $\partial/\partial t$ term is negligible in comparison with the other terms in the equation. To highest order (4.5.20) is then

$$-\frac{\lambda}{3t^{2/3}} h' = \left(h\left(h - \mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right)'\right)' . \qquad 4.5.21$$

Integrating once with respect to ζ_R gives

$$-\frac{\lambda h}{3t^{2/3}} = \left(h - \mathcal{H}\left[\frac{\partial h}{\partial x}\right]\right)' h + g_1(t) \qquad 4.5.22$$

where $g_1(t)$ is an arbitrary function due to the integration. Since $h = 0$ for $|\eta| > 1$, $\forall t$, we must have $g_1(t) = 0$. It then follows that either

$$h = 0 \qquad 4.5.23$$

or

$$-\frac{\lambda\zeta_R}{3t^{2/3}} = h - \mathcal{H}\left[\frac{\partial h}{\partial x}\right] + g_2(t) \qquad 4.5.24$$

and again the arbitrary function of integration, $g_2(t)$ is chosen to be zero. Substituting the expression for h (4.5.19) into (4.5.24) gives

$$-\frac{\lambda\zeta_R}{3t^{2/3}} = t^{-2/3}\left\{-\frac{\lambda\zeta_R}{3} + F - C_f + \Psi_L + \Psi_R\right\} - \mathcal{H}\left[\frac{\partial h}{\partial x}\right]$$

4.5.25

which simplifies to

$$0 = t^{-2/3}\left\{ F - C_f + \Psi_L + \Psi_R \right\} - \mathcal{H}\left[\frac{\partial h}{\partial x}\right] . \qquad 4.5.26$$

As x tends to the right-hand end of the crack, $F \to 0$ and so

$$\Psi_L \to C_f . \qquad 4.5.27$$

Close to the right-hand tip, therefore, Ψ_R is given by

$$t^{-2/3}\,\Psi_R = \mathcal{H}\left[\frac{\partial h}{\partial x}\right] . \qquad 4.5.28$$

The Hilbert transform of the slope of the crack height is now investigated, in order to establish its leading order terms. The Hilbert transform term can be written as a sum of four integrals,

$$\mathcal{H}\left[\frac{\partial h}{\partial x}\right] = \int_{-\infty}^{\infty} \frac{\partial h}{\partial s}\frac{ds}{s - x} = \int_{-\infty}^{\infty} \frac{\partial h_0}{\partial s}\frac{ds}{s - x}$$
$$+ \varepsilon t^{-2/3}\left\{ \int_{-\infty}^{\infty} \frac{\partial F}{\partial s}\frac{ds}{s - x} + \int_{-\infty}^{\infty} \frac{\partial \Psi_L}{\partial s}\frac{ds}{s - x} + \int_{-\infty}^{\infty} \frac{\partial \Psi_R}{\partial s}\frac{ds}{s - x} \right\} \qquad 4.5.29$$

and each integral is considered in turn.

First, look at the Hilbert transform of $\partial h_0/dx$ and recall that $\eta = x/(\lambda t^{1/3})$, hence

$$\int_{-\infty}^{\infty} \frac{\partial h_0}{\partial s}\frac{ds}{s - x} = \frac{\lambda}{6t^{2/3}}\int_{-1}^{1} \frac{\partial (1 - r^2)}{\partial r}\frac{dr}{r - \eta}$$
$$= -\frac{\lambda}{3t^{2/3}}\left\{2 + \eta \ln\left|\frac{1 - \eta}{1 + \eta}\right|\right\}. \qquad 4.5.30$$

In terms of ζ_R, $\eta = 1 + \varepsilon\zeta_R/(\lambda t^{1/3})$ equation (4.5.30) can be written

$$\mathcal{H}\left[\frac{\partial h_0}{\partial x}\right] = -\frac{\lambda}{3t^{2/3}}\left\{2 + \left(1 + \frac{\varepsilon\zeta_R}{\lambda t^{1/3}}\right)\ln\left|\frac{\varepsilon\zeta_R}{2\lambda t^{1/3} + \varepsilon\zeta_R}\right|\right\}. \qquad 4.5.31$$

By expanding the argument of the natural log term as a Taylor series about $\zeta_R = 0$, the Hilbert transform is found to be

$$\mathcal{H}\left[\frac{\partial h_0}{\partial x}\right] = -\frac{\lambda}{3t^{2/3}}\left\{2 + \ln\left|\frac{\varepsilon\zeta_R}{2\lambda t^{1/3}}\right|\right\} + O(\varepsilon) \qquad 4.5.32$$

Now consider

$$\varepsilon t^{-2/3}\int_{-\infty}^{\infty}\frac{\partial F}{\partial s}\frac{ds}{s - x} = \frac{\varepsilon}{\lambda t}\int_{-1}^{1}\frac{\partial F}{\partial r}\frac{dr}{r - \eta} . \qquad 4.5.33$$

It is known that $\partial F/\partial\eta$ is finite at $\eta = 1$. (4.5.33) is, therefore, split into two parts so that the integral can be evaluated as $\eta \to 1^-$ (i.e. as $\zeta_R \to 0^-$), in the following way

$$\varepsilon t^{-2/3}\mathcal{H}\left[\frac{\partial F}{\partial x}\right] = \frac{\varepsilon}{\lambda t}\left[\int_{-1}^{1}\left\{\frac{\partial F(r)}{\partial r} - \frac{\partial F(1)}{\partial r}\right\}\frac{dr}{r - \eta} + \int_{-1}^{1}\frac{\partial F(1)}{\partial r}\frac{dr}{r - \eta}\right]$$

$$= \frac{\varepsilon}{\lambda t}\left[O(1) + \frac{\partial F(1)}{\partial\eta}\ln\left|\frac{\varepsilon\zeta_R}{2\lambda t^{1/3} + \varepsilon\zeta_R}\right|\right] = O(\varepsilon) \qquad 4.5.34$$

So as $\varepsilon \to 0$, the contribution from this term is negligible.

The third integral within the Hilbert transform term incorporates the effect of the function Ψ_L. Since $\partial h_0/\partial x$ is discontinuous at $x = \pm\lambda t^{1/3}$ and since it is this discontinuity which results in the singular behaviour of the Hilbert transform at $x = \pm\lambda t^{1/3}$, it is expected that the functions $\partial\Psi_L/\partial x$ and $\partial\Psi_R/\partial x$ are similarly discontinuous at $x = -\lambda t^{1/3}$ and at $x = \lambda t^{1/3}$, respectively. The Hilbert transforms of each of these functions should then introduce natural log singularities at the same points, which remove those introduced by $\mathcal{H}[\partial h_0/\partial x]$. Additionally, in order that no further singularities be introduced, it is expected that the gradients of Ψ_L and Ψ_R are continuous at the new positions of the crack tips, that is at $x = -\lambda t^{1/3} - \varepsilon a$ and at $x = \lambda t^{1/3} + \varepsilon a$, respectively.

At a distance greater than $O(\varepsilon)$ away from the left-hand end of the crack, the function Ψ_L looks like F_L (recall 4.5.02b). The region of integration is therefore split into two sub-regions, $(-\lambda t^{1/3} - \varepsilon a) < x < (-\lambda t^{1/3} + \varepsilon^b)$ and $(-\lambda t^{1/3} + \varepsilon^b) < x < (\lambda t^{1/3})$, where $0 < b < 1$, so that

$$\varepsilon t^{-2/3} \int_{-\infty}^{\infty} \frac{\partial \Psi_L}{\partial s} \frac{ds}{s - x} = I_1 + I_2 \qquad 4.5.35$$

where

$$I_1 = \varepsilon t^{-2/3} \int_{-\lambda t^{1/3} - \varepsilon a}^{-\lambda t^{1/3} + \varepsilon^b} \frac{\partial \Psi_L}{\partial s} \frac{ds}{s - x} \qquad 4.5.36$$

and

$$I_2 = \varepsilon t^{-2/3} \int_{-\lambda t^{1/3} + \varepsilon^b}^{\lambda t^{1/3}} \frac{\partial F_L}{\partial s} \frac{ds}{s - x} . \qquad 4.5.37$$

First, consider I_1. Since the integration is over a distance of $O(\varepsilon)$ around the point $x = -\lambda t^{1/3}$, to evaluate the integral close to $x = \lambda t^{1/3}$ a change of variables is introduced, from x to ζ_R and the dummy variable is rescaled as follows without invalidating the assumption that ε is small. Let

$$x = \varepsilon \zeta_R + \lambda t^{1/3} \qquad\qquad s = \varepsilon \tau - \lambda t^{1/3}$$

and substituting into (4.5.36) gives

$$I_1 = \varepsilon t^{-2/3} \int_{-a}^{\varepsilon^{b-1}} \frac{\partial \Psi_L}{\partial \tau} \frac{d\tau}{\varepsilon(\tau - \zeta_R) - 2\lambda t^{1/3}} . \qquad 4.5.38$$

Taking the limit $\varepsilon \to 0$ gives

$$I_1 = -\frac{\varepsilon t^{-1}}{2\lambda}\left[\Psi_L(\infty) - \Psi_L(-a)\right] \qquad 4.5.39$$

and imposing conditions (4.5.02a) and (4.5.02b) gives

$$I_1 = -\frac{\varepsilon t^{-1}}{2\lambda} \lim_{\eta \to -1+\varepsilon^b/\lambda t^{1/3}} [F_L]$$

$$= \frac{\varepsilon t^{-1}}{2\lambda}\left[\frac{\lambda b}{3}\ln\varepsilon - \frac{\lambda}{3}\ln|2\lambda t^{1/3}| + \frac{2\lambda}{9} - \frac{\lambda\pi^2}{27}\right]$$

$$= O(\varepsilon \ln\varepsilon) \qquad 4.5.40$$

Now consider I_2. $\partial F_L/\partial\eta$ is given by (4.5.05) so the integral I_2 can be written as

$$I_2 = I_{2a} + I_{2b} + O(\varepsilon) \qquad 4.5.41$$

where I_{2a} and I_{2b} are given by

$$I_{2a} = \frac{\varepsilon}{\lambda t}\int_{-1+\frac{\varepsilon^b}{\lambda t^{1/3}}}^{1} \frac{2\lambda}{3}\ln\left|\frac{1+r}{2}\right| \frac{dr}{r-\eta} \qquad 4.5.42$$

$$I_{2b} = \frac{\varepsilon}{\lambda t}\int_{-1+\frac{\varepsilon^b}{\lambda t^{1/3}}}^{1} \frac{-\lambda}{3(1+r)}\,\frac{dr}{r-\eta} \qquad 4.5.43$$

and I_{2a} and I_{2b} are both found to be $O(\varepsilon)$.

Now consider the final part of the Hilbert transform term in (4.5.29). Again it is known that at a distance greater than $O(\varepsilon)$ away from the right-hand end of the crack, the function Ψ_R looks like F_R (recall 4.5.02d). Once more, therefore, split the region of integration into two sub-regions so that

$$\varepsilon t^{-2/3}\int_{-\infty}^{\infty} \frac{\partial\Psi_R}{\partial s}\,\frac{ds}{s-x} = I_3 + I_4 \qquad 4.5.44$$

where

$$I_3 = \varepsilon t^{-2/3} \int_{-\lambda t^{1/3}}^{\lambda t^{1/3} - \varepsilon^b} \frac{\partial F_R}{\partial s} \frac{ds}{s - x} \qquad 4.5.45$$

and

$$I_4 = \varepsilon t^{-2/3} \int_{\lambda t^{1/3} - \varepsilon^b}^{\lambda t^{1/3} + \varepsilon a} \frac{\partial \Psi_R}{\partial s} \frac{ds}{s - x}. \qquad 4.5.46$$

First, consider I_3. Recall that $\partial F_R/\partial \eta$ is given by (4.5.08) and substituting this into (4.5.45) leads to considering I_3 in two parts. That is

$$I_3 = I_{3a} + I_{3b} + O(\varepsilon) \qquad 4.5.47$$

where I_{3a} and I_{3b} are given by

$$I_{3a} = \frac{\varepsilon}{\lambda t} \int_{-1}^{1 - \frac{\varepsilon^b}{\lambda t^{1/3}}} \frac{\lambda}{9(1 - r)} \frac{dr}{r - \eta} \qquad 4.5.48$$

$$I_{3b} = -\frac{\varepsilon}{\lambda t} \int_{-1}^{1 - \frac{\varepsilon^b}{\lambda t^{1/3}}} \frac{2\lambda}{3} \ln\left|\frac{1 - r}{2}\right| \frac{dr}{r - \eta} \qquad 4.5.49$$

and it is found that $I_{3a} = O(\varepsilon \ln\varepsilon)$ and $I_{3b} = O(\varepsilon (\ln\varepsilon)^2)$.

Lastly, consider I_4. The highest order terms of I_4 close to the right-hand end are required, so a change of variables from x to ζ_R is introduced and since s varies about $\lambda t^{1/3}$ with variations of order ε, s is scaled in the same way, so that

$$x = \varepsilon\zeta_R + \lambda t^{1/3} \qquad s = \varepsilon\tau + \lambda t^{1/3},$$

and so

$$I_4 = t^{-2/3} \int_{-\varepsilon^{b-1}}^{a} \frac{\partial \Psi_R}{\partial \tau} \frac{d\tau}{\tau - \zeta_R} = O(1). \qquad 4.5.50$$

Hence, the only terms in the Hilbert transform in (4.5.29) which make a significant contribution as $\varepsilon \to 0$, are the $\mathcal{H}[\partial h_0/\partial x]$ term (4.5.31) and the integral I_4 (4.5.50), so that

$$\mathcal{H}\left[\frac{\partial h}{\partial x}\right] = -\frac{\lambda}{3t^{2/3}}\left\{2 + \ln\left|\frac{\varepsilon\zeta_R}{2\lambda t^{1/3}}\right|\right\}$$

$$+ \frac{1}{t^{2/3}}\int_{-\varepsilon^{b-1}}^{a} \frac{\partial\Psi_R}{\partial\tau}\frac{\tau}{\tau - \zeta_R} + O(\varepsilon) \qquad 4.5.51$$

Substituting this into equation (4.5.28) gives

$$0 = \Psi_R + \frac{\lambda}{3}\left\{2 + \ln\left|\frac{\varepsilon\zeta_R}{2\lambda t^{1/3}}\right|\right\} - \int_{-\varepsilon^{b-1}}^{a} \frac{\partial\Psi_R}{\partial\tau}\frac{d\tau}{\tau - \zeta_R} \qquad 4.5.52$$

as the governing equation for the function Ψ_R in the region $-\varepsilon^{b-1} < \zeta_R < 0$, with the condition that

$$\Psi_R - F_R \to 0 \qquad \zeta_R \to -\varepsilon^{b-1}. \qquad 4.5.53$$

Note that the $\ln|\varepsilon\zeta_R|$ terms in (4.5.52) are retained. These are expected to be removed by the integral term.

Now examine the behaviour in the region $0 < \zeta_R < a$. Here the crack height is given by

$$h = \varepsilon t^{-2/3}\Psi_R + O(\varepsilon^2). \qquad 4.5.54$$

but the Hilbert transform term is the same as that given in (4.5.51) since ζ_R is still in the region close to zero. The governing equation in this region is then

$$-\frac{\lambda\zeta_R}{3} = \Psi_R + \frac{\lambda}{3}\left\{2 + \ln\left|\frac{\varepsilon\zeta_R}{2\lambda t^{1/3}}\right|\right\} - \int_{-\varepsilon^{b-1}}^{a} \frac{\partial\Psi_R}{\partial\tau}\frac{d\tau}{\tau - \zeta_R}$$

4.5.55

and in the region where $a < \zeta_R$, $\Psi_R = 0$.

A similar analysis can be carried out for the region around the left-hand tip of the crack as $x \to -\lambda t^{1/3}$, which leads to the governing equations for Ψ_L in terms of the local variable $\zeta_L = (x + \lambda t^{1/3})/\varepsilon$. It is noted, however, that because the crack height is an even function in x, it is necessary that $\Psi_L|_{x=-x_0} = \Psi_R|_{x=x_0}$. It follows, therefore, that

$$\Psi_L\left(\frac{-x_0 + \lambda t^{1/3}}{\varepsilon}\right) = \Psi_R\left(\frac{x_0 - \lambda t^{1/3}}{\varepsilon}\right) \qquad 4.5.56$$

that is

$$\Psi_L(\zeta_L) = \Psi_R(-\zeta_R). \qquad 4.5.57$$

So the equations for Ψ_L are found by substituting (4.5.57) into equations (4.5.52), (4.5.53) and (4.5.55). They are:

for $0 < \zeta_L < \varepsilon^{b-1}$

$$0 = \Psi_L + \frac{\lambda}{3}\left\{2 + \ln\left|\frac{\varepsilon\zeta_L}{2\lambda t^{1/3}}\right|\right\} - \int_{-a}^{\varepsilon^{b-1}} \frac{\partial\Psi_L}{\partial\tau}\frac{d\tau}{\tau - \zeta_L} \qquad 4.5.58$$

with boundary condition

$$\Psi_L - F_L \to 0 \qquad \zeta_L \to \varepsilon^{b-1}. \qquad 4.5.59$$

For $-a < \zeta_L < 0$

$$\frac{\lambda\zeta_L}{3} = \Psi_L + \frac{\lambda}{3}\left\{2 + \ln\left|\frac{\varepsilon\zeta_L}{2\lambda t^{1/3}}\right|\right\} - \int_{a}^{-\varepsilon^{b-1}} \frac{\partial\Psi_L}{\partial\tau}\frac{d\tau}{\tau - \zeta_L} \qquad 4.5.60$$

and in the region where $\zeta_L < -a$, $\Psi_L = 0$.

Closed form analytical solutions cannot be written down for either of the functions Ψ_L or Ψ_R. It is possible, however, that numerical solutions could be found and this is a likely route for future work to take although no numerical analysis is carried out here.

The solution to first order terms of equation (4.1.01) for the case when $\varepsilon \ll 1$ is then

$$h = \begin{cases} 0 & x < -\lambda t^{1/3} - \varepsilon a \\ \varepsilon t^{-2/3}\Psi_L(\zeta_L, t) & -\lambda t^{1/3} - \varepsilon a < x < -\lambda t^{1/3} \\ h_0(x,t) + \varepsilon t^{-2/3}[F - C_f + \Psi_L + \Psi_R] & -\lambda t^{1/3} < x < \lambda t^{1/3} \\ \varepsilon t^{-2/3}\Psi_R(\zeta_R, t) & \lambda t^{1/3} < x < \lambda t^{1/3} + \varepsilon a \\ 0 & \lambda t^{1/3} + \varepsilon a < x \end{cases}$$

4.5.61

where h_0, F and C_f are given by (4.4.05), (4.5.11) and (4.5.12) respectively. Ψ_L is defined by (4.5.58), (4.5.59) and (4.5.60) and Ψ_R is defined by (4.5.52), (4.5.53) and (4.5.56).

Section 4.6 - Summary

In section 4.3 a first order approximate solution for the one-dimensional crack equation (4.1.01), with no shearing is found, for the case where ε is very large. Since h_0, (4.3.21), is finite $\forall$ x,t and since $\sigma_{yy} \neq 0$ $\forall$ x,t (in the region of interest), the term $\varepsilon\sigma_{yy}$ is always greater than h for sufficiently large values of ε, except perhaps for very small time, so the approximation can be considered valid for all values of x and for time t > 1.

In section 4.4, the case where ε is very small is considered. The analysis is carried out in the same way as for the previous case, but leads to the result that the leading order solution is not valid close to the crack tip. The behaviour in this region is then investigated more closely in section 4.5 and the leading order approximate solution is improved upon by introducing local functions at the crack tip and taking into account that the effects produced by these functions have global significance.

CHAPTER 5 - NUMERICAL SOLUTION OF THE PARTIALLY OPEN CRACK EQUATION

5.1 - Introduction

In chapter three, a model for the behaviour of a one-dimensional, pre-existing crack in an infinite elastic material, for the case where the crack is re-opened by the injection of an incompressible viscous fluid is presented. In chapter four, one particular aspect of this model is investigated, namely the case where the fluid pressure is insufficient to completely separate the crack surfaces but where they do not shear over each other. Here, numerical solutions to this same problem are sought. The analytical and numerical results are then compared at the end of this chapter. It should be noted that large scale one, two and three dimensional codes for a single crack and for networks of cracks have been previously established by the CSMGEP. The codes FBED, FRIP-2D, and FRIP-3D incorporate the dilation due to shearing and the open crack state as described in chapter three. The numerical code used here (called SPOC - Stuck Partially Open Crack) aims only to solve equation (4.4.01), for a partially open crack with no shear. The code uses a similar numerical method to FBED, CSMGEP's one-dimensional code. The main reason for not using FBED directly is the incompatibility between computing facilities at Southampton University and the Cornwall project. Additionally, numerical results for the non-dimensional problem are required and until recently FBED contained only dimensional variables.

SPOC is written in PASCAL and uses a forward time, central space finite difference scheme for the flow problem. It also incorporates a displacement discontinuity boundary element technique to numerically evaluate the Hilbert transform in the elastic normal stress term. Both of these schemes, also employed in FBED, are described in section 5.2. A stability analysis is carried out in a different way to that employed in FBED and the results are discussed in section 5.3. Finally, in section 5.4, the numerical results are presented and compared with the analytical results.

Here, the one dimensional partially open crack equation is approximated by a set of difference equations. These are obtained in the usual way by considering a Taylor series expansion of the dependent variables and using this to obtain a local approximation for the derivatives.

To help explain the discretised version of the model that is used in the numerical code, equation (4.1.01) is rewritten in the following way

$$\frac{\partial h}{\partial t} = - \frac{\partial q}{\partial x} \qquad 5.2.01$$

$$q = - h \frac{\partial p}{\partial x} \qquad 5.2.02$$

$$p = h - \varepsilon \sigma \qquad 5.2.03$$

$$\sigma = \int_{-\infty}^{\infty} \frac{\partial h}{\partial s} \frac{ds}{s - x} \qquad 5.2.04$$

where q represents the fluid flux through the crack, p is a measure of the change in pressure and σ is a scaled component of the elastic normal stress.

The grid system used is shown in Figure 5.1. The domain is split into N discrete elements of width δx centred about the points x_i, $1 \le i \le N$. Let $x_i = i\delta x$, $t_n = n\delta t$ and h_i^n, q_i^n, σ_i^n and p_i^n represent the values of h, q, σ and p respectively, at the point x_i at time t_n. Most of the variables are placed at the nodal points with the exception of the flux variables, q_i^n, which are centred between the nodal points, at the end of each element.

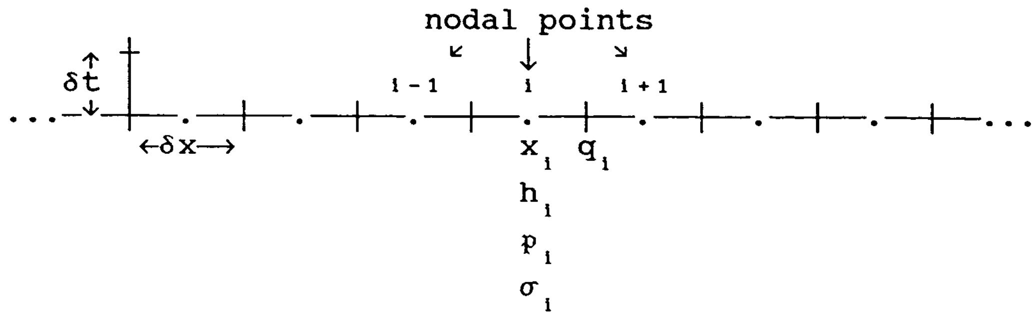

Figure 5.1 - Grid System in FBED and SPOC

Initially, the displacements are given a prescribed constant value for each element. The normal stress at each node is then calculated using a boundary element technique described below, so that

$$\sigma_i^n = \sum_{j=1}^{N} \frac{(\delta x/2)\ h_j^n}{(x_i - x_j)^2 - (\delta x/2)^2}. \qquad 5.2.05$$

The pressure change is then given by

$$p_i^n = h_i^n - \varepsilon\sigma_i^n, \qquad 5.2.06$$

and so the flux at the end of each element can be evaluated using a central difference approximation to $\partial p/\partial x$ and an average value of h, so that

$$q_i^n = - \left(\frac{h_{i+1}^n + h_i^n}{2}\right) \left(\frac{p_{i+1}^n - p_i^n}{\delta x}\right). \qquad 5.2.07$$

The crack height is then updated from the current flux values using a forward difference approximation to $\partial h/\partial t$, hence

$$h_i^{n+1} = h_i^n - \frac{\delta t}{\delta x}\left(q_i^n - q_{i-1}^n\right). \qquad 5.2.08$$

Once the crack height is updated the cycle is repeated to obtain the normal stress, pressure change and flux etc., for subsequent time steps.

For a fuller description of the CSM code FBED see the

Geothermal Energy Project report 3A-11 (Parker (1991)) and for an overall view of the FRIP codes see report 3A-5 (Markland (1989)).

A Displacement Discontinuity Boundary Element Method

The normal stress is calculated numerically using a displacement discontinuity method outlined in Crouch and Starfield (1983) from an original paper by Crouch (1976). The method is based on the analytical solution to a problem of a constant discontinuity in displacements over a finite line segment, representing a crack, along $y = 0$. Crack surfaces are displaced relatively by a constant amount.

The idea behind the numerical method is to approximate the continuous displacement of the crack surfaces by a summation of discrete constant displacements (see Figure 5.2). If the analytical solution for each single elemental displacement is known, then the numerical solution for the full crack is found by summing the contributions of all N elements.

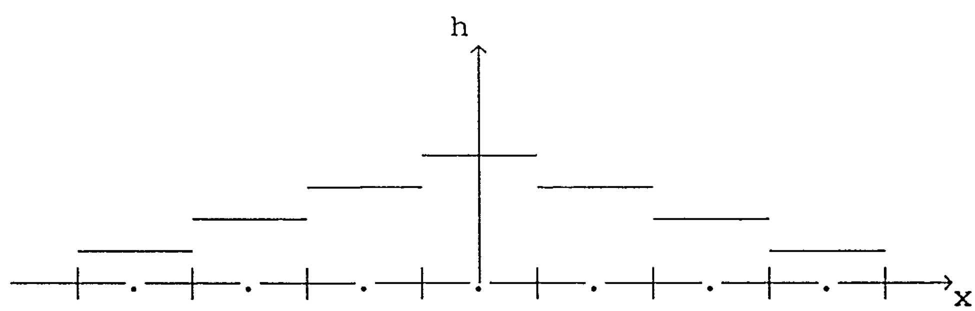

Figure 5.2 - Approximation of Crack Heights Using Displacement Discontinuities

The required analytical solution may be obtained from the results presented earlier in chapter two. To solve for a line displacement discontinuity, the problem of a point displacement discontinuity is considered and then a convolution integral is used to sum over a line of points. Although physically a point displacement is a somewhat strange concept, mathematically it can be dealt with in quite a straight forward manner. The problem considered is that of a constant displacement discontinuity, over a finite line segment in the x, y plane of an infinite elastic solid. This problem is specified by the condition that the displacements are continuous everywhere except over the line segment in question and so two separate point displacements are considered, first in the x direction and then in the y direction, and then the two solutions are superposed. Suppose the line segment is chosen to occupy a certain portion of the x axis, say $|x| \leq a$, $y = 0$; by thinking of this segment as a line crack, its two surfaces are distinguished by assuming that one surface is on the positive side of $y = 0$, denoted by $y = 0_+$, and the other is on the negative side of $y = 0$, denoted by $y = 0_-$ (as in chapter two). D_x and D_y, the differences in displacement between the two sides of the line segment (or point), may be defined via (2.3.03). It is assumed that the problem is symmetric in y, so that attention can be restricted to the region $y \geq 0$. If $\delta(x)$ is a Dirac delta function then the boundary conditions for the two problems are (c.f. 2.3.08 and 2.3.09):

<u>(1)</u>

$$\sigma_{xy} = 0 \qquad y = 0, \ \forall\, x \qquad 5.2.09a$$

$$u_y = -\frac{D_y\, \delta(x)}{2} \qquad y = 0, \ \forall\, x \qquad 5.2.09b$$

$$\sigma_{ij} \rightarrow 0 \qquad x, y \rightarrow \infty \qquad 5.2.09c$$

<u>(2)</u>

$$\sigma_{yy} = 0 \qquad y = 0, \ \forall\, x \qquad 5.2.10a$$

$$u_x = -\frac{D_x\, \delta(x)}{2} \qquad y = 0, \ \forall\, x \qquad 5.2.10b$$

$$\sigma_{ij} \rightarrow 0 \qquad x, y \rightarrow \infty. \qquad 5.2.10c$$

where here D_x and D_y are assumed constant. Note that the problem with boundary conditions

$$u_x = -\frac{D_x \delta(x)}{2} \qquad y = 0, \ \forall \ x \qquad 5.2.11a$$

$$u_y = -\frac{D_y \delta(x)}{2} \qquad y = 0, \ \forall \ x \qquad 5.2.11b$$

$$\sigma_{ij} \to 0 \qquad x, y \to \infty \qquad 5.2.11c$$

is a different problem since evidently, with conditions (5.2.11), the displacements are zero along y = 0 except at the origin. Whereas, with conditions (5.2.09) and (5.2.10), the displacements are merely continuous along y = 0 (except for the discontinuity at x = 0) and it is this condition which is physically appropriate. The solutions to the plane strain displacement equations (2.3.04 and 2.3.05), for these conditions (5.2.09 and 5.2.10) can be obtained in a similar way to that of chapter 2, the difference here being that D_x and D_y are point displacement discontinuities as opposed to the general functions of x considered previously. The solution for the problem with conditions (5.2.09) is then

$$\sigma_{xy} = -\frac{GD_y}{\pi(1-\nu)} \frac{xy(x^2 - 3y^2)}{(x^2 + y^2)^3} \qquad 5.2.12a$$

$$\sigma_{yy} = \frac{GD_y}{2\pi(1-\nu)} \frac{3y^4 - 6x^2y^2 - x^4}{(x^2 + y^2)^3} \qquad 5.2.12b$$

$$u_x = \frac{D_y}{4\pi(1-\nu)} \left[\frac{x^3(1-2\nu) + (3-2\nu)xy^2}{(x^2+y^2)^2}\right] \qquad 5.2.12c$$

$$u_y = \begin{cases} \dfrac{D_y}{4\pi(1-\nu)} \left[\dfrac{y^3(2\nu-3) + (2\nu-1)yx^2}{(x^2+y^2)^2}\right] & y \neq 0 \\ -\dfrac{D_y \delta(x)}{2} & y = 0 \end{cases} \qquad 5.2.12d$$

The problem with conditions (5.2.10) is solved in the same way. The solution is

$$\sigma_{xy} = - \frac{GD_x}{2\pi\ (1 - \nu)} \frac{y^4 - 6x^2y^2 + x^4}{(x^2 + y^2)^3} \qquad 5.2.13a$$

$$\sigma_{yy} = - \frac{GD_x}{\pi\ (1 - \nu)} \frac{xy\ (x^2 - 3y^2)}{(x^2 + y^2)^3} \qquad 5.2.13b$$

$$u_x = \begin{cases} \dfrac{D_x}{4\pi\ (1 - \nu)} \left[\dfrac{y^3 (2\nu - 1) + (2\nu - 3)\ yx^2}{(x^2 + y^2)^2} \right] & y \neq 0 \\ \dfrac{- D_x \delta(x)}{2} & y = 0 \end{cases} \qquad 5.2.13c$$

$$u_y = \frac{D_x}{4\pi\ (1 - \nu)} \left[\frac{x^3 (2\nu - 1) + (2\nu + 1)\ xy^2}{(x^2 + y^2)^2} \right] \qquad 5.1.13d$$

Recall that the solution required is the sum of the above two solutions. To obtain the solution for a point displacement discontinuity $D_i = (D_x, D_y)$, therefore, solutions (5.3.12) and (5.3.13) above are superimposed to obtain

$$\sigma_{xy} = - \frac{GD_x}{2\pi\ (1 - \nu)} \left[\frac{(y^4 - 6x^2y^2 + x^4) D_x}{(x^2 + y^2)^3} + \frac{2xy\ (x^2 - 3y^2) D_y}{(x^2 + y^2)^3} \right] \qquad 5.2.14a$$

$$\sigma_{yy} = - \frac{GD_x}{2\pi\ (1 - \nu)} \left[\frac{2xy\ (x^2 - 3y^2) D_x}{(x^2 + y^2)^3} + \frac{(3y^4 - 6x^2y^2 - x^4) D_y}{(x^2 + y^2)^3} \right] \qquad 5.2.14b$$

$$u_x = \begin{cases} \dfrac{D_x}{4\pi\ (1 - \nu)} \left[\dfrac{y^3(2\nu - 1) + (2\nu - 3)\ yx^2}{(x^2 + y^2)^2} \right] & y \neq 0 \\ -\dfrac{D_x \delta(x)}{2} & y = 0 \end{cases}$$

$$+ \frac{D_y}{4\pi(1 - \nu)} \left[\frac{x^3(1 - 2\nu) + (3 - 2\nu)\ xy^2}{(x^2 + y^2)^2} \right] \qquad 5.2.14c$$

$$u_y = \frac{D_x}{4\pi\ (1 - \nu)} \left[\frac{x^3(2\nu - 1) + (2\nu + 1)\ xy^2}{(x^2 + y^2)^2} \right]$$

$$+ \begin{cases} \dfrac{D_y}{4\pi\ (1 - \nu)} \left[\dfrac{y^3(2\nu - 3) + (2\nu - 1)\ yx^2}{(x^2 + y^2)^2} \right] & y \neq 0 \\ -\dfrac{D_y \delta(x)}{2} & y = 0 \end{cases} \qquad 5.2.14d$$

This solution for a point displacement discontinuity is used to obtain the solution for the constant displacement discontinuity problem over a line segment $y = 0$, $|x| \leq a$. Dividing the segment into small elements of length $d\eta$, the displacement discontinuity on an element which is centred at $x = \eta$, $y = 0$ is then:

$$D_i(\eta) = \bar{D}_i\ d\eta \qquad 5.2.15$$

where i is either x or y. The solution to the problem can be found by substituting $D_x(\eta)$ and $D_y(\eta)$ into (5.2.14), replacing x by $x-\eta$ and integrating the resulting expressions with respect to η between the limits -a and +a. The solution to this problem (Crouch and Starfield (1983)), can be written in terms of the function f(x,y), where

$$f(x,y) = \frac{-1}{4\pi(1 - \nu)} \left[y\left\{\arctan\left(\frac{y}{x - a} \right) - \arctan\left(\frac{y}{x + a} \right)\right\} - (x - a)\ \ln \sqrt{[(x - a)^2 + y^2]} + (x + a)\ \ln \sqrt{[(x + a)^2 + y^2]} \right]. \qquad 5.2.16$$

The first, second and third derivatives of f(x,y) are needed in order to be able to write down the solution. Introducing the following notation for partial derivatives

$$f_{,x} = \frac{\partial f}{\partial x} \qquad f_{,y} = \frac{\partial f}{\partial y},$$

the derivatives are then

$$f_{,x} = \frac{1}{4\pi(1 - \nu)} \left[\ln \sqrt{[(x - a)^2 + y^2]} - \ln \sqrt{[(x + a)^2 + y^2]} \right],$$

$$f_{,y} = \frac{-1}{4\pi(1 - \nu)} \left[\arctan\left(\frac{y}{x - a} \right) - \arctan\left(\frac{y}{x + a} \right) \right],$$

$$f_{,xy} = \frac{1}{4\pi(1 - \nu)} \left[\frac{y}{(x - a)^2 + y^2} - \frac{y}{(x + a)^2 + y^2} \right],$$

$$f_{,xx} = - f_{,yy} = \frac{1}{4\pi(1 - \nu)} \left[\frac{x - a}{(x - a)^2 + y^2} - \frac{x + a}{(x + a)^2 + y^2} \right],$$

$$f_{,xyy} = -f_{,xxx} = \frac{1}{4\pi(1 - \nu)} \left[\frac{(x - a)^2 - y^2}{\{(x - a)^2 + y^2\}^2} - \frac{(x + a)^2 - y^2}{\{(x + a)^2 + y^2\}^2} \right],$$

$$f_{,yyy} = -f_{,xxy} = \frac{2y}{4\pi(1 - \nu)} \left[\frac{x - a}{\{(x - a)^2 + y^2\}^2} - \frac{x + a}{\{(x + a)^2 + y^2\}^2} \right]$$

and omitting bars, the solutions are

$$\sigma_{xy} = 2G\, D_x \left[f_{,yy} + y f_{,yyy} \right] + 2G\, D_y \left[- y f_{,xyy} \right] \qquad 5.2.17a$$

$$\sigma_{yy} = 2G\, D_x\left[-\, yf_{,xyy}\right]$$

$$+\, 2G\, D_y\left[f_{,yy} - yf_{,yyy}\right] \qquad 5.2.17b$$

$$u_x = D_x\left[2(1-\nu)\, f_{,y} - yf_{,xx}\right]$$

$$+\, D_y\left[-\,(1-2\nu) f_{,x} - yf_{,xy}\right] \qquad 5.2.17c$$

$$u_y = D_x\left[(1-2\nu)\, f_{,x} - yf_{,xy}\right]$$

$$+\, D_y\left[2(1-\nu)\, f_{,y} - yf_{,yy}\right]. \qquad 5.2.17d$$

Note that D_x and D_y are *constant* displacement discontinuities. In practice, a better approximation to the displacements in each nodal element would be a function linear in x, which makes the above analytical problem considerably more difficult. In fact the above method cannot be solved analytically for a linear law, since the Fourier transform of the necessary boundary conditions cannot be expressed in terms of closed form analytical functions.

The solutions for the stresses and displacements are included for completeness, though for the numerical code only σ_{yy} along y = 0 is required, and it is

$$\sigma_{yy} = 2G\, D_y\, f,_{yy} = \frac{-\, GD_y}{2\pi(1-\nu)}\left[\frac{2a}{x^2 - a^2}\right]. \qquad 5.2.18$$

In terms of the non-dimensional, discrete variables the sum of the contributions from each elemental displacement gives

$$\sigma_i^n = \sum_{j=1}^{N} \frac{\delta x\, h_j^n}{(x_i - x_j)^2 - (\delta x/2)^2}. \qquad 5.2.05$$

This expression completes the discretisation of the model.

Section 5.3 - Stability Analysis

To investigate the stability of SPOC, the linear equation

$$\frac{\partial h}{\partial t} = \frac{\partial^2}{\partial x^2}\left(h - \varepsilon\,\sigma\right) \qquad 5.3.01$$

is considered. Note that previously, stability of FBED was investigated by considering the equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left\{h^3\frac{\partial \sigma}{\partial x}\right\}$$

(Parker (1991)) which is only appropriate for values of $\varepsilon \gg 1$. The results presented here are applicable for any value of ε.

Written in terms of the discrete variables equation (5.3.01) becomes

$$\frac{h_i^{n+1} - h_i^n}{\delta t} = \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{\delta x^2} - \varepsilon\,\frac{\sigma_{i+1}^n - 2\sigma_i^n + \sigma_{i-1}^n}{\delta x^2}. \qquad 5.3.02$$

Recall that σ_i^n is given by

$$\sigma_i^n = \sum_{j=1}^{N} \frac{\delta x\, h_j^n}{(x_i - x_j)^2 - (\delta x/2)^2}, \qquad 5.2.05$$

substituting this into (5.3.02) and remembering that $x_i = i\delta x$ leads to

$$h_i^{n+1} = h_i^n + \frac{\delta t}{\delta x^2}\left(h_{i+1}^n - 2h_i^n + h_{i-1}^n\right) - \frac{\varepsilon\delta t}{\delta x^3}\sum_{j=1}^{N}\frac{96\, h_j^n}{(4(i-j)^2-1)(4(i-j)^2-9)}. \qquad 5.3.03$$

Introducing the vector $\underline{h}^n = (h_1^n, h_2^n, \ldots, h_N^n)^{-1}$, (5.3.03) can be written as

$$\underline{h}^{n+1} = A\,\underline{h}^{n} \qquad 5.3.04$$

where the N×N matrix A is given by

$$A = B - krC \qquad 5.3.05$$

with $k = 96\varepsilon/\delta x$, $r = \delta t/(\delta x^2)$ and where the components of the matrix B, $b_{i,j}$, are given by

$$b_{i,j} = \begin{cases} 1 - 2r & i = j \\ r & |i - j| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad 5.3.06$$

and the components of C, $c_{i,j}$, are given by

$$c_{i,j} = \frac{1}{(4(i-j)^2-1)(4(i-j)^2-9)} \qquad \forall\ i,j. \qquad 5.3.07$$

Note that i and j can take values 1, 2, ..., N.

The numerical scheme is stable if all of the eigenvalues of the matrix A are of magnitude less than or equal to 1. That is if

$$\rho(A) \le 1.$$

Since the spectral radius of a matrix is always less than the norm of the matrix (the maximum of the sums of the magnitudes of every element in each row or column), it is sufficient to have

$$\|A\| \le 1.$$

where $\|A\|$ represents the one norm of A.

Notice that that the one norm of A is obtain by summing elements in the middle row (or column since A is symmetric), or, if A has an even number of rows, by summing elements in either of the middle two rows (or columns). To simplify the notation it is assumed that A has an odd number of rows and so N = 2M - 1. The following calculation, however, can then be repeated for N = 2M to consider matrices with an even number of rows. Since B is a tri-diagonal matrix and since each row of C is symmetric about its central element, the norm of A can be written as

$$\|A\| = |1 - 2r - kr/9| + 2|r + kr/15| + 2kr \max(S_j) \qquad 5.3.08$$

where

$$S_j = \sum_{j=1}^{M-2} \frac{1}{(4(M-j)^2-1)(4(M-j)^2-9)} = \sum_{m=2}^{M-1} \frac{1}{(4m^2-1)(4m^2-9)}.$$

The argument of S_j can be split into partial fractions and in doing so it becomes clear that $1/(2m_0 - a)$ terms will cancel with $-1/(2m_1 + a)$ terms when $m_0 = m_1 + a$ (a is either 1 or 3). S_j then tends to its maximum value as $M \to \infty$, and so $\mathrm{Max}(S_j)$ is found to be

$$\lim_{M\to\infty} S_j = \frac{1}{90},$$

and therefore

$$\|A\| \le |1 - 2r - kr/9| + 2|r + kr/15| + \frac{2kr}{90}. \qquad 5.3.09$$

Gerschgorin's circle theorem implies that all the eigenvalues of the matrix A lie within or on the circle centre a, radius d, where a is the diagonal element of the norm row and d is the sum of the moduli of the off diagonal elements. For stability we then require

$$|a - d| \le 1$$

where a and d are given by

$$a = 1 - 2r - kr/9, \qquad d = 2r + 7kr/45.$$

For small values of ε then, the timestep has the limit $\delta t \le \delta x^2/2$, and for large ε the limit is found to be $\delta t \le 5\delta x^3/128$.

Alternatively, eigenvalues of the matrix A can be calculated numerically by employing standard Numerical Algorithms Group (NAG) routines. Use of these routines means that values for the parameters ε, dx and dt must be specified in advance which means that a general formula for the maximum possible timestep cannot be generated this way. Varying the parameters, however, to ensure that the maximum modulus eigenvalue is close to unity, is a simple task. In carrying out this investigation the maximum possible timestep, which guarantees stability of the SPOC code, is found for several values of grid size and epsilon. A table of results for a 20x20 matrix is presented in figure 5.3.

ε	dx	max dt
0.01	0.5	0.1118
0.01	1	0.4731
0.1	0.5	0.0557
0.1	1	0.3091
1	0.5	0.0093
1	1	0.0692
10	0.5	0.0010
10	1	0.0079
100	0.5	0.0001
100	1	0.0008

Figure 5.3 - Maximum Timestep for Stability

The values given in figure 5.3 are in good agreement with the limits established analytically using Gerschgorin's theorem and also with observed values of the maximum possible timestep. Any differences between observed values and the above calculated values are thought to be due to the fact that the stability analysis is carried out on the linear equation (5.3.01) and not on the non-linear system described by (5.2.01-5.2.04), which is approximated by the code. The code, however, is stable for all of the values presented in figure 5.3.

Section 5.4 - Comparison of Numerical and Analytical Solutions

In this section numerical solutions to (4.1.01) are presented and compared with the analytical solutions obtained in chapter four (see figures 4.2 and 4.3). Graphs of the crack height against distance along the crack are compared in each case.

The solutions given are calculated using twenty grid points with typical values of $\delta x = 0.5$ and δt given (approximately) by the corresponding values in figure 5.3. The code was tested using a range of values for N, δx and δt and the solutions appeared to be consistent.

Figures 5.4 and 5.5 are the numerical solutions for the cases where $\varepsilon \ll 1$ and $\varepsilon \gg 1$ respectively. These should be compared with the corresponding analytical solutions given in figures 4.3 and 4.2. Figures 5.6 and 5.7 pick out the curves at time = 10 from figures 4.3, 4.2, 5.4 and 5.5, and depict the analytical and numerical solutions for $\varepsilon \ll 1$ and $\varepsilon \gg 1$ respectively. Finally, in figure 5.8, one can see how the numerical solution varies with ε.

Notice that the analytical and numerical solutions are in good agreement for the $\varepsilon \ll 1$ case, except in the regions close to the crack tips. There are two reasons for the bad agreement in these regions. First, the analytical solution is only a valid approximation away from the crack tip, since the simplifying assumptions made in this case do not hold as the crack height tends to zero (see section 4.5). The second reason is that at the crack tip there is a discontinuity in the gradient of the height. This discontinuity is not modelled effectively by the numerical code since the actual position of the tip will not necessarily fall at a nodal point. The numerical method tries to smooth out the discontinuity, hence the discrepancy near the crack tip.

The graphs of the analytical and numerical solutions for the case where $\varepsilon \gg 1$, are in very good agreement for all positions along the crack.

Finally, figure 5.8 illustrates that the parameter ε can be thought of as the ratio of the stiffness of the rock to the stiffness of the crack. The stiffness is simply the resistance of the rock and the crack to deformation. So $\varepsilon \gg 1$ implies that the rock is much stiffer than the crack and so any change in crack height has a global effect and therefore the crack propagates quickly. Small values of ε, however, imply that the crack is much stiffer than the rock, so changes in height have an effect only locally and the crack, therefore, propagates much more slowly.

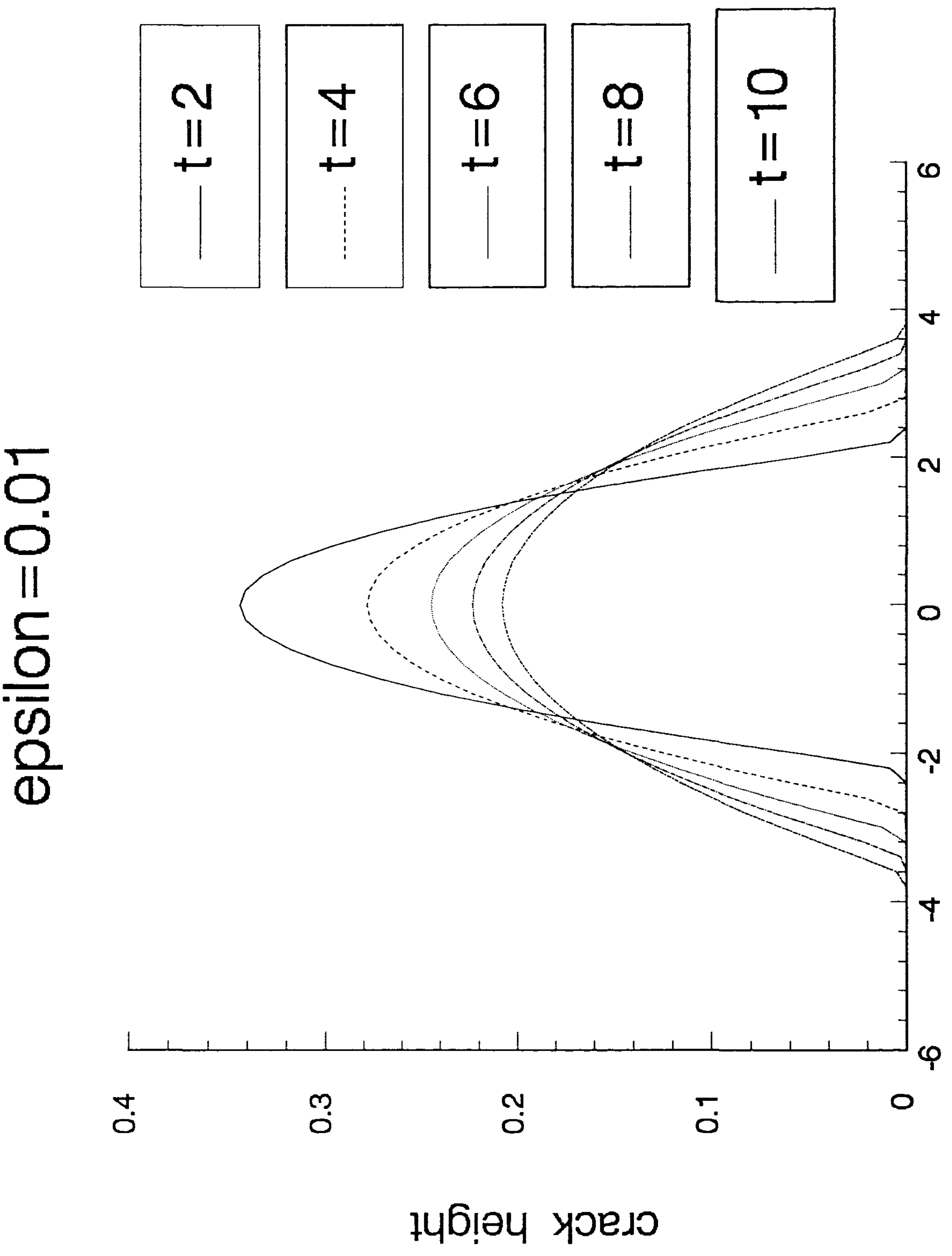

Figure 5.4 - Numerical Solutions for ε « 1 Case

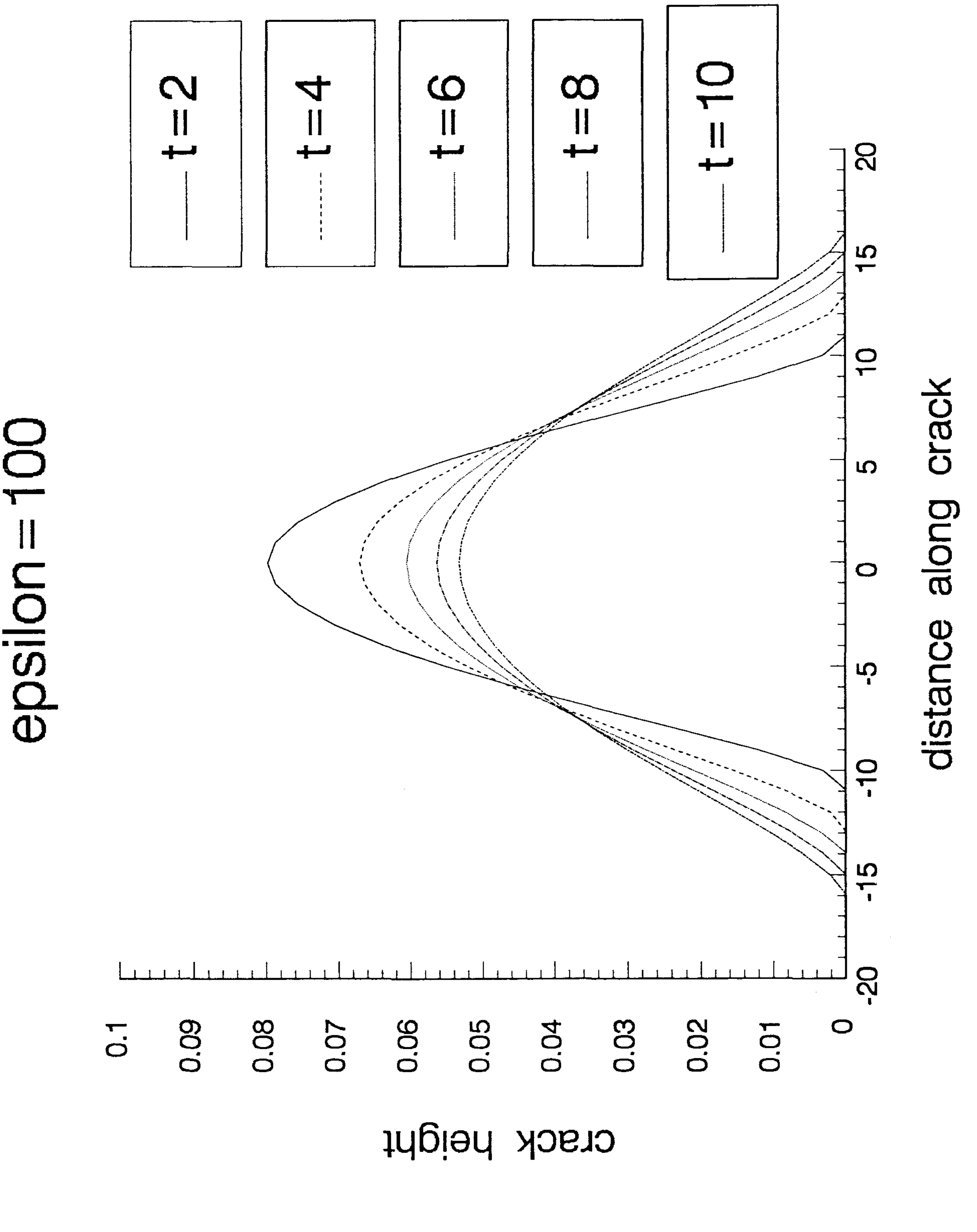

Figure 5.5 - Numerical Solutions for ε » 1 Case

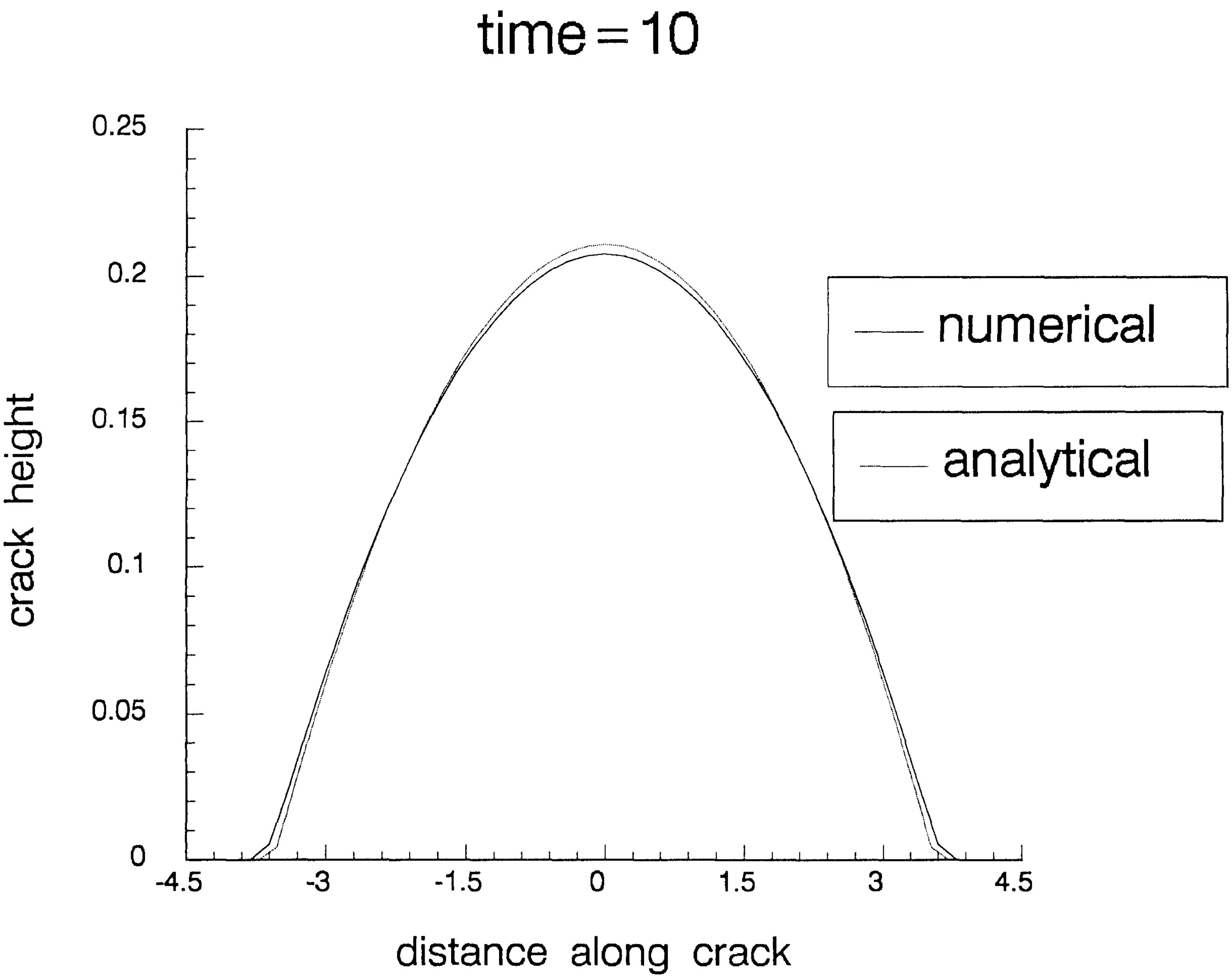

Figure 5.6 - Comparison Between Analytical and Numerical Solutions for $\varepsilon \ll 1$

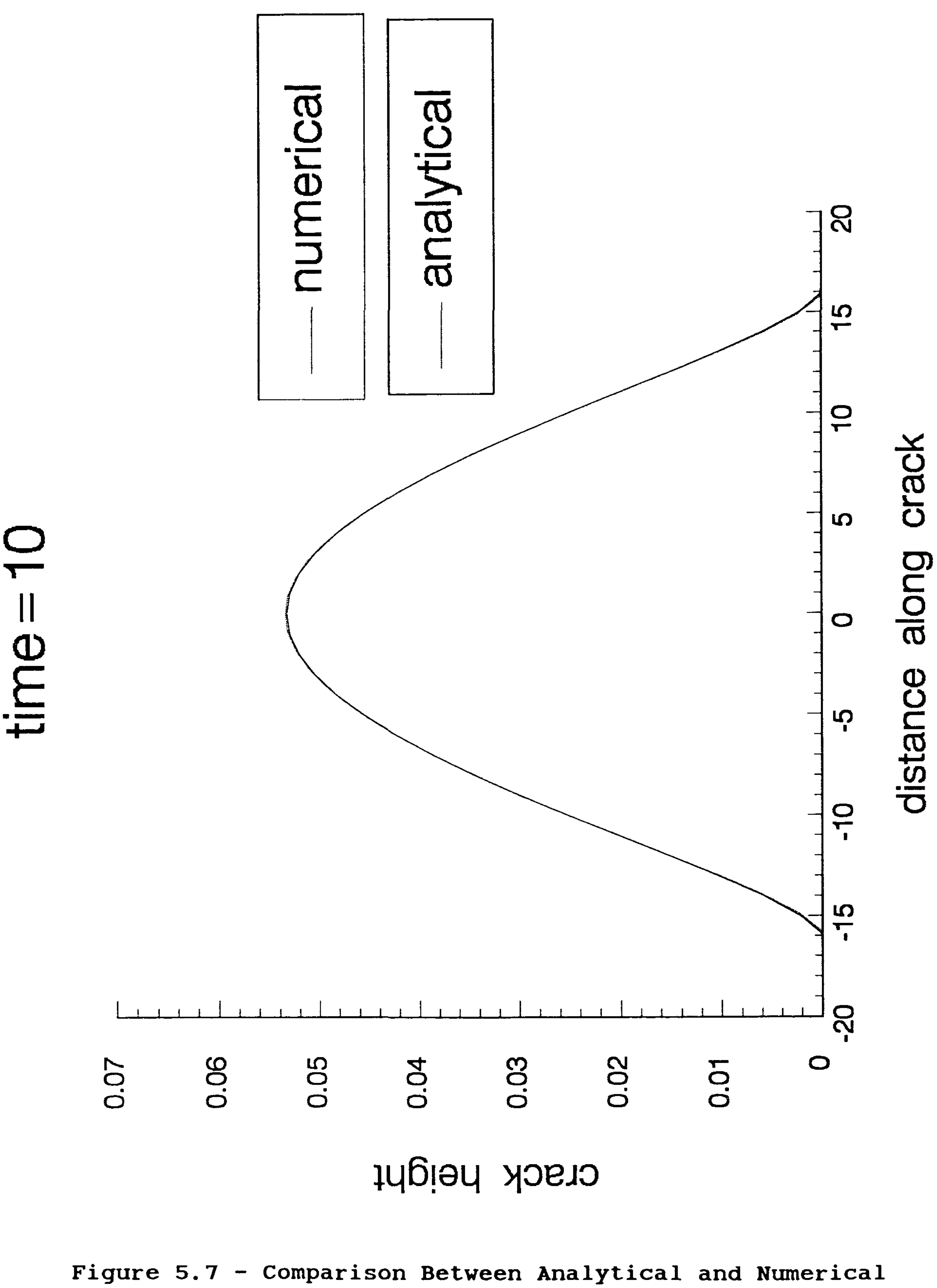

Figure 5.7 - Comparison Between Analytical and Numerical Solutions for $\varepsilon \gg 1$

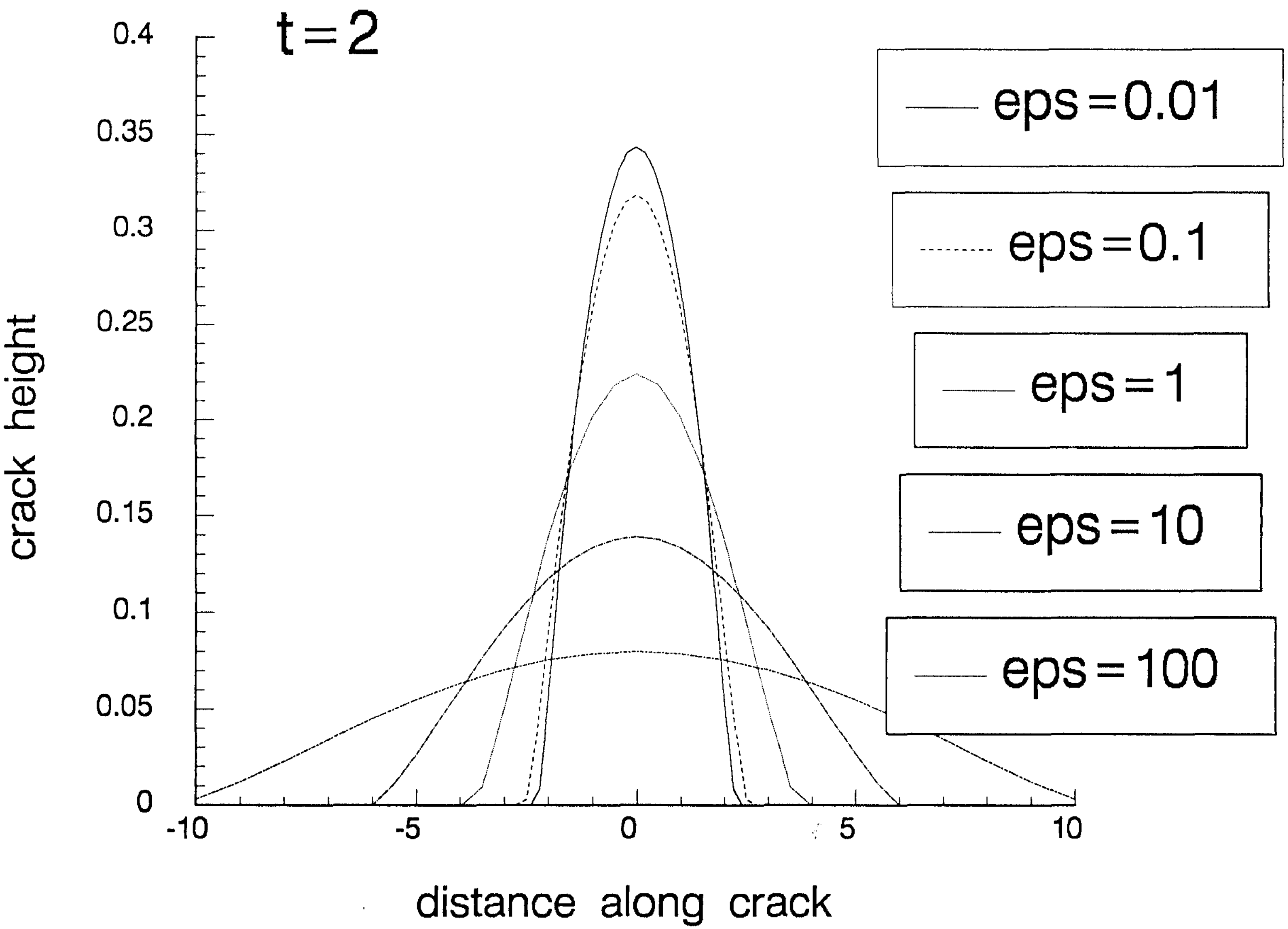

Figure 5.8 - Comparison in ε

CHAPTER 6 - LONG TERM HEAT TRANSFER IN A GEOTHERMAL ENERGY RESERVOIR

6.1 - Introduction

In this chapter the interaction between thermal and elastic effects in the extraction of geothermal energy from hot rock is considered. Recall that, as described in chapter one, this energy is extracted by pumping cold water through the rock extracting the heated water and passing this through a heat exchanger. The consideration of the coupling of heat transfer and elastic effects is motivated by the apparent occurrence of cold spots and of short circuits in the fluid flow through the rock. The hypothesis to explain such occurrences is that as water flows through the cracks and is heated, the rock cools and shrinks. This shrinkage widens the cracks allowing increased cooling of the rock and gives rise to preferred routes through the rock. As the time scale for heat conduction through the rock is large compared to the fluid residence time in the rock, once preferred fluid paths have been established, much of the geothermal energy becomes unrecoverable.

This problem was brought to the European Study Group in April 1991 by the CSMGEP, where it was pointed out that the problem is similar to that which arises when attempting to drive water through an oil reservoir in order to flush out the oil. In this problem, it is known that the oil/water interface is unstable (Saffman & Taylor (1958)), and that viscous fingering leads to preferred paths for the injected water and leaves isolated oil pockets trapped in the reservoir. Chuoke et.al. (1959) present results of an experimental investigation into these instabilities along with photographs of the oil/water fingering. Here the Saffman-Taylor problem is outlined and a model for the geothermal energy problem is formulated and analysed. Both problems rely on linear stability analyses of planar interfaces; in the Saffman-Taylor problem the interface is between the water and the oil, and in the geothermal energy problem the interface is defined by an isotherm of the averaged temperature field in the rock (see section 6.3).

6.2 - The Muskat Problem

In 1958 P.G. Saffman and G.I. Taylor presented a paper in which they considered a problem which has since become known as the Muskat problem (see Muskat (1937)). The subject of the paper was the stability of an interface between two immiscible viscous fluids, in the event of one fluid being driven through a porous medium by the pressure of the second fluid. Saffman and Taylor established that the interface would be unstable if the driving fluid was the less viscous of the two and stable otherwise. In this section we discuss the assumptions made and the conditions imposed by Saffman and Taylor and go on to derive their results. For a synopsis on porous media flows see Wooding and Morel-Seytoux (1976).

Consider two fluids of viscosities μ^+ and μ^- which are being forced through a porous medium by an imposed pressure gradient, ∇p. Gravitational forces are neglected. Taking rectangular Cartesian co-ordinates (x,y,z), a situation is considered in which the interface is parallel to the y-z plane and moving with constant speed V in the x direction. Attention is restricted to the x-y plane. The superscript '+' is used to denote fluid 1, initially in the region $x > 0$, and the superscript '-' denotes fluid 2 initially in the region $x < 0$. At this point it is necessary to make some assumptions about the interface of the fluid. In practice there would not be a sharp interface separating one fluid from the other but instead they would mix together and form a very thin region of unknown viscosity. If the two fluids are oil and water, the thickness of this region is on a molecular length scale, i.e., much smaller than the macroscopic length scales of the problem; therefore in the following analysis it is reasonable to assume that the fluids are separated by a sharp interface, across which the pressure and the normal component of velocity are continuous. In an appendix to their paper Saffman and Taylor give the modifications required for the case in which one fluid does not completely expel the other but where a proportion of the fluid is left behind.

Governing Equations

If $\underline{u}(x,y,t)$ is the *superficial* fluid velocity (volume flow rate through a unit cross-sectional area of solid matter *plus* pore space) and ρ is the density of the fluid, then the continuity equation for flow through a medium of constant porosity ϵ, is

$$\epsilon\frac{\partial\rho}{\partial t} = -\nabla.(\rho\underline{u}). \qquad 6.2.01$$

For an incompressible fluid this becomes

$$\nabla.\underline{u} = 0 \qquad 6.2.02$$

and assuming flow through the medium to be governed by Darcy's law we have

$$\underline{u} = -\frac{k}{\mu}\nabla p = \nabla\phi \qquad 6.2.03$$

where k is the constant permeability of the medium, μ is the constant viscosity of the fluid and ϕ is a velocity potential.

Equations (6.2.02) and (6.2.03), for flow in a medium of uniform permeability, lead to Laplace's equation

$$\nabla^2\phi = 0. \qquad 6.2.04$$

Boundary Conditions

If the fluid velocity vector is

$$\underline{u}(x,y,t) = u_x(x,y,t)\underline{i} + u_y(x,y,t)\underline{j}$$

then the conditions of uniform parallel flow at infinity are imposed

$$\left.\begin{array}{l} u_x \to V \\ \\ u_y \to 0 \end{array}\right\} \qquad x \to \pm\infty \qquad 6.2.05$$

Now concentrate more closely on the behaviour at the interface. The two semi-infinite regions of constant viscosity have a common boundary at the interface between the two fluids. Since the position of this interface is unknown, and since (6.2.04) is elliptic, two conditions are needed along this boundary. Suppose that the interface between the two fluids is defined by

$$f(x,y,t) = 0. \qquad 6.2.06$$

At the interface, both fluid 1 and fluid 2 have the same normal velocity as the interface, so that

$$\underline{u}^{+}.\underline{n} = \underline{u}^{-}.\underline{n} = v_n \qquad 6.2.07$$

where $v_n(x,y,t)$ is the normal velocity of the interface and $\underline{n}$ is the unit normal to the interface defined by

$$\underline{n} = \frac{\nabla f}{|\nabla f|}. \qquad 6.2.08$$

At time t, the interface is represented by $f(x,y,t) = 0$ and so at a later time, $t + \delta t$, the interface is given by

$$f(x + v_n n_x \delta t, y + v_n n_y \delta t, t + \delta t) = 0 = f(x,y,t).$$

Straight away

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_n \underline{n}.\nabla f = 0, \qquad 6.2.09$$

and so the normal velocity of the interface is therefore

$$v_n = -\frac{1}{|\nabla f|}\frac{\partial f}{\partial t}. \qquad 6.2.10$$

Since $\underline{u} = \nabla\phi$, condition (6.2.07) shows that

$$-\frac{1}{|\nabla f|}\frac{\partial f}{\partial t} = \nabla\phi.\underline{n} \qquad 6.2.11$$

which can be rewritten as

$$\frac{\partial f}{\partial t} = -\nabla\phi.\nabla f. \qquad 6.2.12$$

As previously stated, the pressure is taken to be continuous across the interface. Since the pressure is simply

$$p= -\frac{\mu}{k}\phi,$$

then, in terms of velocity potentials $\phi^{+}(x,y,t)$ and $\phi^{-}(x,y,t)$, continuity of pressure across the interface becomes

$$\frac{\mu^{+}}{k^{+}}\phi^{+} = \frac{\mu^{-}}{k^{-}}\phi^{-} \qquad \text{on } f = 0. \qquad 6.2.13$$

Solution

First, a solution to the problem of a planar free boundary between the two fluids is sought. Then small perturbations to this solution are considered and the stability of the free boundary is analysed.

A travelling wave solution of the form

$$\phi(x,y,t) = \phi_0(x - Vt) \qquad 6.2.14$$

is sought, where the interface between the fluids is given by $x = Vt$ and so $f(x,t) = x - Vt$. The equations and boundary conditions for this problem are summarised diagrammatically in figure 6.1.

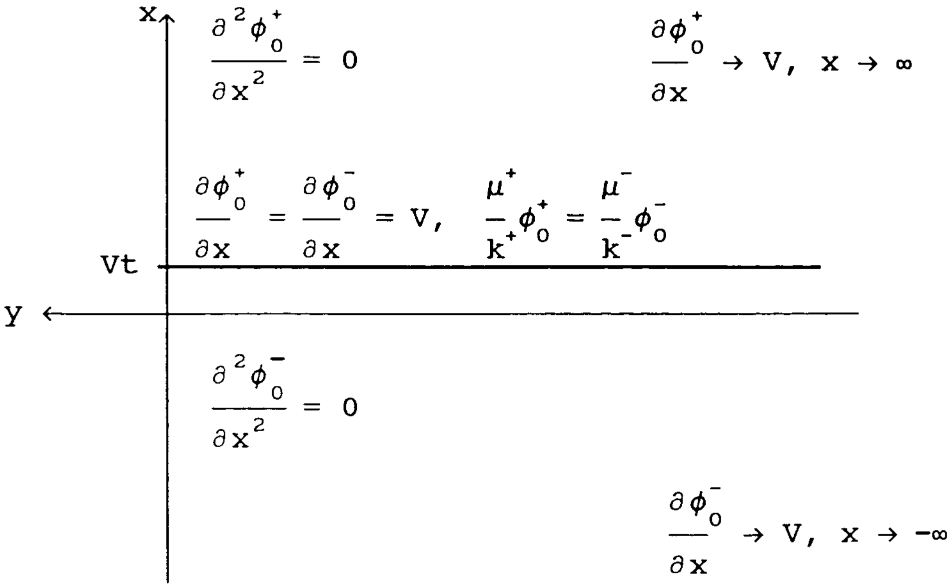

Figure 6.1 - O(1) Muskat Problem

This system is easily solved and the velocity potentials are found to be

$$\phi_0^+ = V(x - Vt) + c^+, \qquad \phi_0^- = V(x - Vt) + c^- \qquad 6.2.15$$

where $c^\pm$ are constants found by specifying the pressure at some point. For simplicity $c^\pm$ are taken to be zero.

Small, harmonic perturbations to the interface are imposed, of wavelength $2\pi/n$, so that

$$f = x - Vt - \varepsilon e^{\sigma t + iny} = 0, \qquad 6.2.16$$

where $\varepsilon \ll 1$, and a solution is sought of the form

$$\phi = \phi_0(x - Vt) + \varepsilon e^{\sigma t + iny}\phi_1(x - Vt) + O(\varepsilon^2). \qquad 6.2.17$$

Let $\zeta = x - Vt$ and let primes denote derivatives with respect to ζ, then substituting (6.2.17) into (6.2.04) gives

$$\phi_0^{+\prime\prime} + \varepsilon e^{\sigma t + iny}\,\phi_1^{\prime\prime} - \varepsilon n^2 e^{\sigma t + iny}\,\phi_1 + O(\varepsilon^2) = 0$$

$$6.2.18$$

and equating $O(\varepsilon)$ terms gives ϕ_1 defined by

$$\phi_1'' - n^2 \phi_1 = 0. \qquad 6.2.19$$

The continuity of normal velocity (6.2.12) along $f = 0$ gives, to $O(\varepsilon)$,

$$\left[\phi_0^{+\prime} + \varepsilon e^{\sigma t + iny} \phi_1^{+\prime}\right] \frac{\partial f}{\partial x} + \left[\varepsilon i n e^{\sigma t + iny} \phi_1^{+}\right] \frac{\partial f}{\partial y} = -\frac{\partial f}{\partial t}$$

6.2.20

and substituting in for $f = x - Vt - \varepsilon e^{\sigma t + iny}$ this becomes, to $O(\varepsilon)$,

$$\phi_0^{+\prime} + \varepsilon e^{\sigma t + iny} \phi_1^{+\prime} - (\varepsilon n e^{\sigma t + iny})^2 \phi_1^{+} = V + \varepsilon \sigma e^{\sigma t + iny}.$$

6.2.21

In order to be able to impose the interface conditions along the plane $x = Vt$, (6.2.21) is linearised about $x = Vt$, using Taylor's theorem to obtain

$$\phi_0^{+\prime} + \varepsilon e^{\sigma t + iny} \phi_0^{+\prime\prime} + \varepsilon e^{\sigma t + iny} \phi_1^{+\prime} + O(\varepsilon^2)$$

$$= V + \varepsilon \sigma e^{\sigma t + iny} \qquad \text{on } \zeta = 0 \qquad 6.2.22$$

and equating terms of $O(\varepsilon)$ gives the condition

$$\phi_1^{+\prime} = \sigma \qquad \text{on } \zeta = 0. \qquad 6.2.23$$

Substituting in (6.2.16) for ϕ in the second interface condition, (6.2.13), which ensures continuity of pressure, gives

$$\frac{\mu^+}{k^+}\left\{\phi_0^{+} + \varepsilon e^{\sigma t + iny} \phi_1^{+}\right\} = \frac{\mu^-}{k^-}\left\{\phi_0^{-} + \varepsilon e^{\sigma t + iny} \phi_1^{-}\right\} \qquad \text{on } f = 0.$$

6.2.24

Again linearise about $x = Vt$ using Taylor's theorem to obtain

$$\frac{\mu^+}{k^+}\Big\{\phi_0^+ + \varepsilon e^{\sigma t + iny}\,\phi_0^{+\prime} + \ldots + \varepsilon e^{\sigma t + iny}\Big[\phi_1^+ + \varepsilon e^{\sigma t + iny}\,\phi_1^{+\prime} + \ldots\Big]\Big\}$$

$$= \frac{\mu^-}{k^-}\Big\{\phi_0^- + \varepsilon e^{\sigma t + iny}\,\phi_0^{-\prime} + \ldots + \varepsilon e^{\sigma t + iny}\Big[\phi_1^- + \varepsilon e^{\sigma t + iny}\,\phi_1^{-\prime} + \ldots\Big]\Big\}$$

on $\zeta = 0$, 6.2.25

and equating $O(\varepsilon)$ terms gives

$$\frac{\mu^+}{k^+}\Big\{\phi_0^{+\prime} + \phi_1^+\Big\} = \frac{\mu^-}{k^-}\Big\{\phi_0^{-\prime} + \phi_1^-\Big\} \qquad \text{on } \zeta = 0. \qquad 6.2.26$$

Finally, since the perturbation ϕ_1 is required to be smaller than the leading order terms, we must have $\phi_1^+ = o(\zeta)$ as $\zeta \to \infty$ and $\phi_1^- = o(\zeta)$ as $\zeta \to -\infty$. The equations and boundary conditions for the first order ($O(\varepsilon)$) problem are summarised in figure 6.2.

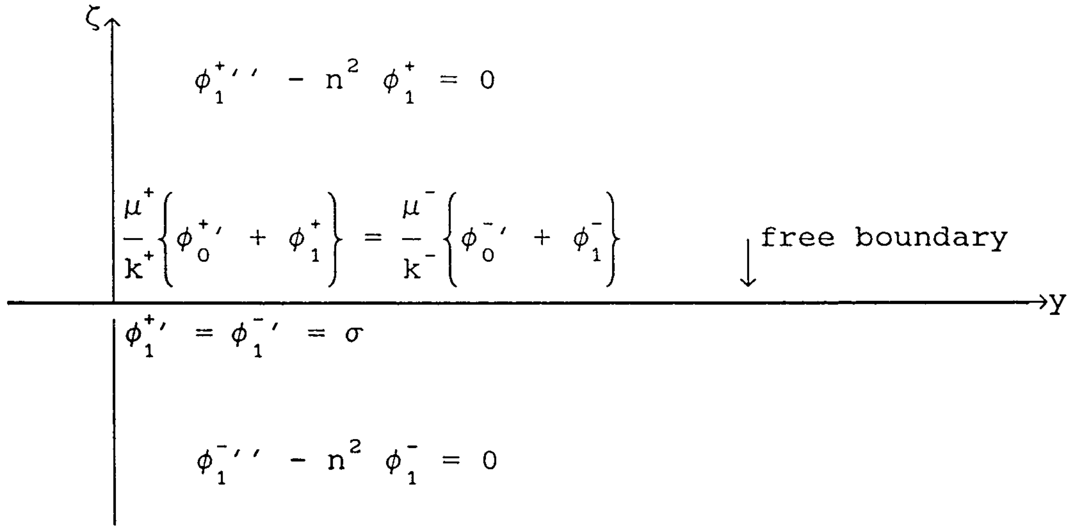

Figure 6.2 - O(ε) Muskat Problem

Since the governing equation for ϕ, (6.2.19), is well known, the solutions for both fluids can be written down straightaway

$$\phi_1^+ = -Vr\, e^{-n\zeta} \qquad 6.2.27$$

$$\phi_1^- = Vr\, e^{n\zeta}. \qquad 6.2.28$$

where r is given by the continuity of pressure condition and is

$$r = \frac{\left(\frac{\mu^+}{k^+} - \frac{\mu^-}{k^-}\right)}{\left(\frac{\mu^+}{k^+} + \frac{\mu^-}{k^-}\right)} .$$

So the solutions for the velocity potentials to the first order in the deviation are

$$\phi^+(x,y,t) = V(x - Vt) - \varepsilon\, Vr\, e^{\sigma t + iny - n(x-Vt)} \qquad 6.2.29$$

$$\phi^-(x,y,t) = V(x - Vt) + \varepsilon\, Vr\, e^{\sigma t + iny + n(x-Vt)}. \qquad 6.2.30$$

To determine the condition for the stability of the interface we impose continuity of normal velocity across the interface. This leads to an expression for σ,

$$\sigma = nVr. \qquad 6.2.31$$

To investigate stability, it is necessary to consider when σ is positive and when σ is negative. A positive value of σ implies that the $O(\varepsilon)$ terms grow exponentially and so the interface is unstable. Similarly $\sigma < 0$ ensures that $O(\varepsilon)$ terms decay exponentially and so the interface is stable.

For a medium of uniform permeability $k_1 = k_2$, and if $n > 0$ and if $V > 0$ then $\sigma > 0$ only if $\mu^+ > \mu^-$, that is only if the driving fluid is the least viscous of the two. If $\mu^+ < \mu^-$ then $\sigma < 0$, the interface is stable and fingering does not occur.

6.3 - Geothermal Energy Reservoir Problem

In this section the long term effects of heat transfer in a geothermal energy reservoir are considered. The fluid flow through individual cracks is considered in previous chapters where the heat transfer is assumed to have no effect on the flow path. In practice, as the heat is mined the cracks are expected to widen due to rock shrinkage and so the heat transfer process does affect the fluid flow. Two asymptotic cases of this coupled thermo-elastic problem can be treated by simple models. The first case is when the crack separation distance is much greater than the thermal diffusion length scale. This case (called model B by Jenkins and Aronofsky (1955)) has been considered before by the C.S.M. and is not presented here. The second case is when the separation distance is much smaller than the thermal length scale (model A in Jenkins and Aronofsky (1955)). This is the model which is considered in this section.

In practice the cracks are usually between one and ten metres apart and the thermal diffusion length is approximately twenty metres over the twenty-five year life time of the reservoir, which means that the physical problem lies somewhere between these two extremes.

In section two of this chapter we have seen how the ratio of permeability to fluid viscosity plays an important part in the stability of an interface between two fluids. Here we investigate the necessary conditions for the stability of flow in a geothermal energy reservoir, when the viscosity of the fluid and the permeability of the rock are assumed to have a simple temperature dependence.

Governing Equations

In order to model the heat transfer for closely spaced cracks we take a *macroscopic* view of the reservoir in a similar way to porous media models. In such a continuum model the volume elements to which the velocity, pressure and temperature refer are assumed to contain a large number of cracks and the dependent variables are averages over a large number of cracks, although in detail they may show large variations within the individual elements. The validity of this assumption for the case of the fluid flow through a geothermal energy reservoir is not certain, but the unknown nature of the crack distribution suggests that this is a case worth considering. If this assumption is valid, then for laminar flow through an isotropic medium, the motion of the fluid is governed by Darcy's law (6.2.03). Notice that the permeability, k, depends only on the structure of the medium and is independent of the fluid. The continuity equation for an incompressible fluid of constant viscosity flowing in a medium of uniform porosity is given by (6.2.02) and, as in the previous example, equations (6.2.02) and (6.2.03) combine together to give Laplace's equation for the velocity potential, ϕ, (6.2.04).

Denoting the average rock temperature by T_r and the average water temperature by T_w, the rate of change of the heat energy stored in the rock must equal the rate at which heat energy is lost to the water (since heat diffusion in the rock is neglected), thus

$$\rho_r c_r \frac{\partial T_r}{\partial t} = h(T_w - T_r) \qquad 6.3.01$$

where ρ_r and c_r are the density and the specific heat of the rock, respectively. Here h is a heat transfer coefficient, which takes into account the properties of the rock and fluid and the shape of the rock blocks.

Similarly, we consider a heat balance across a unit volume of water. Since we are considering a time scale of approximately twenty-five years, (the expected life span of a geothermal energy reservoir), we can consider the water temperature to be quasi-steady. So, in a unit time interval, the heat advected out of a volume element of water is equal to the amount of heat energy gained from the rock. Thus,

$$\rho_w c_w \underline{u}.\nabla T_w = h(T_r - T_w) \qquad 6.3.02$$

where ρ_w and c_w are the density and the specific heat of the water, respectively. We consider the problem to be independent of z and simply look at a the x-y plane with perturbations to a uniform flow in the x direction.

As seen in the crack models discussed in previous chapters, it is possible that small changes in the crack width could cause a wide-ranging redistribution of stresses. This means that a local rock shrinkage could affect the flow problem globally. These effects may be accounted for by considering the permeability to be a non-local function of rock temperature. Here, however, we consider the stress redistribution to occur over a length scale smaller than that of interest in the rock.

If the changes in permeability and viscosity can be assumed to occur rapidly at a known rock temperature, T^*, then the above model can be taken to hold in the two sub-regions where $T_r > T^*$ and $T_r < T^*$. It is understood that the temperature dependence is somewhat different to the step function behaviour suggested here, but since we wish to investigate the stability of the flow path rather than specify its exact nature, this assumption is thought to be acceptable.

Note that in the previous section the free boundary was defined to be the interface between two fluids of differing viscosities. Here the free boundary is an isotherm, or constant temperature front, across which both the fluid viscosity and the rock permeability are assumed to jump. Note also that physically the viscosity is dependent on the fluid temperature rather than the rock temperature. It is evident that this could be incorporated into the model by allowing the viscosity and the permeability to jump at different temperatures. The result would be two free boundaries and so the stability analysis would need to be carried out on both of these interfaces. Here, the jumps are considered to occur at the same temperature since this simplifies the algebra considerably.

Interface Boundary Conditions

As in the previous section the superscript + denotes values of the dependent variables on the side of the interface where $T_r > T^*$, and the superscript - those where $T_r > T^*$.

To determine the jump conditions across the interface a weak solution of the conservation laws could be considered (see Smoller (1983)). A physical argument, however, is thought to be sufficient here. In order to ensure conservation of mass across the interface it is necessary that

$$\left[\underline{u}.\underline{n}\right] = 0, \qquad 6.3.03$$

where square brackets, [], denote a jump in the specified variable. Conservation of momentum gives

$$[p] = 0 \qquad 6.3.04$$

and conservation of energy in both the fluid and the porous medium gives

$$[T_r] = 0 \qquad 6.3.05$$

$$[T_w] = 0. \qquad 6.3.06$$

Since the position of the interface is unknown a solution of the form

$$f(x,y,t) = x - f_0(y,t) = 0 \qquad 6.3.07$$

is sought for the position of the contour defined by $T_r = T^*$.

To complete the system, a condition on $\underline{u}$ as x tends to plus or minus infinity, a condition on T_w as x tends to minus infinity and also an initial condition on T_r, are required. The following conditions are imposed:

$$u_x = \frac{\partial \phi}{\partial x} \to V \qquad x \to \pm\infty, \qquad 6.3.08$$

$$u_y = \frac{\partial \phi}{\partial y} \to 0 \qquad x \to \pm\infty, \qquad 6.3.09$$

$$T_w \to 0 \qquad x \to -\infty, \qquad 6.3.10$$

$$T_r = \begin{cases} T^0 & x > 0 \\ 0 & x < 0 \end{cases} \qquad t = 0. \qquad 6.3.11$$

where T^0 is known ($T^0 < T^* < 0$).

Solution

Solutions for the average velocity potential, ϕ, and the average temperatures, T_r and T_w, of the form

$$\phi(x,y,t) = \phi_0(x,t),$$

$$T_r(x,y,t) = T_{r0}(x,t),$$

$$T_w(x,y,t) = T_{w0}(x,t),$$

are sought and then small perturbations to these solutions are considered and the stability of the free boundary is investigated.

The $\phi_0(x,t)$ problem is exactly the same as the corresponding problem in section two of this chapter and so the solution is

$$\phi_0^+ = V(x - f_0(t)), \qquad \phi_0^- = V(x - f_0(t)) \qquad 6.3.12$$

The position of the interface, however, is not now determined by the condition that the normal velocity of the front is the same as the normal fluid velocity at the front. Here the velocity of the front is governed by the change in temperature of the surrounding rock. The position of the undisturbed surface, $x = f_0(t)$, is determined implicitly by the condition $T_r = T^*$.

To solve the conservation of energy equations for the rock and water temperatures the two expressions are combined to get a single equation for T_{r0} and a new boundary condition for T_{r0} is also obtained,

$$\rho_w c_w V \left\{ \frac{\partial T_{r0}}{\partial x} + \frac{\rho_r c_r}{h} \frac{\partial^2 T_{r0}}{\partial x \partial t} \right\} = -\rho_r c_r \frac{\partial T_{r0}}{\partial t} \qquad 6.3.13$$

$$\frac{\rho_r c_r}{h} \frac{\partial T_{r0}}{\partial t} + T_{r0} \to 0 \qquad x \to -\infty \qquad 6.3.14$$

and recall the initial condition

$$T_{r0}(x,0) = \begin{cases} T^0 & x > 0 \\ 0 & x < 0 \end{cases} . \qquad 6.3.15$$

Physically this means that water at zero temperature is pushed from minus infinity through rock at zero temperature for $x < 0$, and at temperature T^0 for $x > 0$.

Introducing the non-dimensional variables

$$x = L\bar{x} \qquad t = \frac{\rho_r c_r}{h} \bar{t} \qquad T_{r0} = T^0 \bar{T}_{r0}$$

enables the equations and boundary and initial conditions to be rewritten in the following form

$$\left\{ \frac{\partial \bar{T}_{r0}}{\partial \bar{x}} + \frac{\partial^2 \bar{T}_{r0}}{\partial \bar{x} \partial \bar{t}} \right\} = -\alpha \frac{\partial \bar{T}_{r0}}{\partial \bar{t}} \qquad 6.3.16$$

$$\frac{\partial \bar{T}_{r0}}{\partial \bar{t}} + \bar{T}_{r0} \to 0 \qquad \bar{x} \to -\infty \qquad 6.3.17$$

$$\bar{T}_{r0}(\bar{x},0) = \begin{cases} 1 & \bar{x} > 0 \\ 0 & \bar{x} < 0 \end{cases} = H(\bar{x}) \qquad 6.3.18$$

where $H(\bar{x})$ is the Heaviside step function and α is the one non-dimensional parameter in the problem and is given by

$$\alpha = \frac{Lh}{\rho_w c_w V} . \qquad 6.3.19$$

Equation (6.3.16) is a form of the Telegraph equation and it can be solved using Laplace Transforms. Defining the Laplace transform in the time variable to be

$$\mathcal{L}[\bar{T}_{ro}(\bar{x},\bar{t})] = \hat{T}(\bar{x},p) = \int_0^\infty \bar{T}_{ro}(\bar{x},\bar{t})\, e^{-p\bar{t}}\, d\bar{t} \qquad 6.3.20$$

the transformed equation then becomes

$$\frac{\partial}{\partial \bar{x}}\left\{p\hat{T} - \bar{T}(\bar{x},0)\right\} + \frac{\partial \hat{T}}{\partial \bar{x}} = -\alpha\left\{p\hat{T} - \bar{T}(\bar{x},0)\right\}, \qquad 6.3.21$$

with boundary condition

$$\hat{T} \to 0 \qquad \bar{x} \to -\infty. \qquad 6.3.22$$

Substituting the initial data into (6.3.21) gives

$$\frac{\partial \hat{T}}{\partial \bar{x}} + \frac{\alpha p}{p+1}\hat{T} = \frac{1}{p+1}\left\{\frac{dH}{d\bar{x}} + \alpha H\right\}. \qquad 6.3.23$$

If a solution of the form

$$\hat{T} = \frac{H}{p+1} + \varphi(\bar{x},p) \qquad 6.3.24$$

is sought, an equation for $\varphi(\bar{x},p)$ is obtained,

$$\frac{\partial \varphi}{\partial \bar{x}} + \frac{\alpha p}{p+1}\varphi = \frac{\alpha H}{(p+1)^2}. \qquad 6.3.25$$

The solution for $\varphi(\bar{x},p)$, consistent with (6.3.22) is then found to be

$$\varphi = \begin{cases} 0 & \bar{x} < 0 \\ \dfrac{1}{p(p+1)}\left(1 - e^{-\alpha\bar{x}p/(p+1)}\right) & \bar{x} > 0 \end{cases} \qquad 6.3.26$$

The inverse Laplace transform is required, so let

$$\varphi(\bar{x},p) = \frac{1}{p(p+1)} - e^{-\alpha\bar{x}}\,\Xi(p) \qquad 6.3.27$$

where

$$\Xi(\bar{x},p) = \frac{e^{\alpha\bar{x}/(p+1)}}{p(p+1)} \qquad 6.3.28$$

then the inverse transform of $\frac{1}{p(p+1)}$ is trivially $1 - e^{-t}$ and, by convolution, the inverse transform of $\Xi(\bar{x},p)$ is

$$\mathscr{L}^{-1}[\Xi] = \int_0^{\bar{t}} e^{-\eta}\, I_0(2\surd(\alpha\bar{x}\eta))\, d\eta \qquad 6.3.29$$

where $I_0(\xi)$ is the usual zero order, modified Bessel function of the first kind. Substituting these inverse transforms into the inverse transform of equation (6.3.24) and simplifying gives

$$\bar{T}_{r0}(\bar{x},\bar{t}) = \begin{cases} 0 & \bar{x} < 0 \\ 1 - e^{-\alpha\bar{x}} \displaystyle\int_0^{\bar{t}} e^{-\eta}\, I_0(2\surd(\alpha\bar{x}\eta))\, d\eta & \bar{x} > 0 \end{cases} \qquad 6.3.30$$

This equation, (6.3.30), represents the leading order non-dimensional average of the temperature in the rock. The corresponding water temperature is obtained from the non-dimensional version of the heat conservation equation,

$$\bar{T}_{w0} = \frac{\partial\bar{T}_{r0}}{\partial\bar{t}} + \bar{T}_{r0} \qquad 6.3.31$$

(where $T_{w0} = T^0\bar{T}_{w0}$) and is

$$\bar{T}_{w0}(\bar{x},\bar{t}) = \begin{cases} 0, & \bar{x} < 0 \\ 1 - e^{-\alpha\bar{x}} \displaystyle\int_0^{\bar{t}} e^{-\eta}\, I_0(2\surd(\alpha\bar{x}\eta))\, d\eta & \\ \qquad - e^{-(\alpha\bar{x}+\bar{t})} I_0(2\surd(\alpha\bar{x}\bar{t})). & \bar{x} > 0 \end{cases} \qquad 6.3.32$$

In dimensional variables the temperatures are given by

$$\frac{T_{r0}(x,t)}{T^0} = \begin{cases} 0, & x < 0 \\ 1 - e^{-\alpha x/L}\int_0^{\beta t} e^{-\eta} I_0(2\surd(\alpha x\eta/L))\, d\eta, & x > 0 \end{cases} \qquad 6.3.33$$

$$\frac{T_{w0}(x,t)}{T^0} = \begin{cases} 0, & x < 0 \\ 1 - e^{-\alpha x/L}\int_0^{\beta t} e^{-\eta}\, I_0(2\surd(\alpha x\eta/L))\, d\eta \\ \quad - e^{-\beta t-\alpha x/L} I_0(2\surd(\alpha\beta xt/L)), & x > 0 \end{cases} \qquad 6.3.34$$

where

$$\beta = h/(\rho_r c_r). \qquad 6.3.35$$

Graphs of the non-dimensional expressions for the rock and water temperatures are given in figures 6.3 and 6.4 and it is seen that the functions behave as expected.

The unperturbed interface position, $x = f_0$, defined by the rock temperature T^*, is then given implicitly by

$$\frac{T^*}{T^0} = 1 - e^{-\alpha f_0/L}\int_0^{\beta t} e^{-\eta}\, I_0(2\surd(\alpha f_0\eta/L))\, d\eta. \qquad 6.3.36$$

A small ($\varepsilon \ll 1$), harmonic perturbation to the solution is imposed and solutions of the form

$$T_r = T_{ro}(x,t) + \varepsilon T_{r1}(x,t)\, e^{iny} \qquad 6.3.37$$

$$T_w = T_{wo}(x,t) + \varepsilon T_{w1}(x,t)\, e^{iny} \qquad 6.3.38$$

$$\phi = \phi_o(x - f_0(t)) + \varepsilon\phi_1(x,t)\, e^{iny} \qquad 6.3.39$$

are sought, where the free boundary is now given by

$$f = x - f_0(t) - \varepsilon e^{iny} f_1(t). \qquad 6.3.40$$

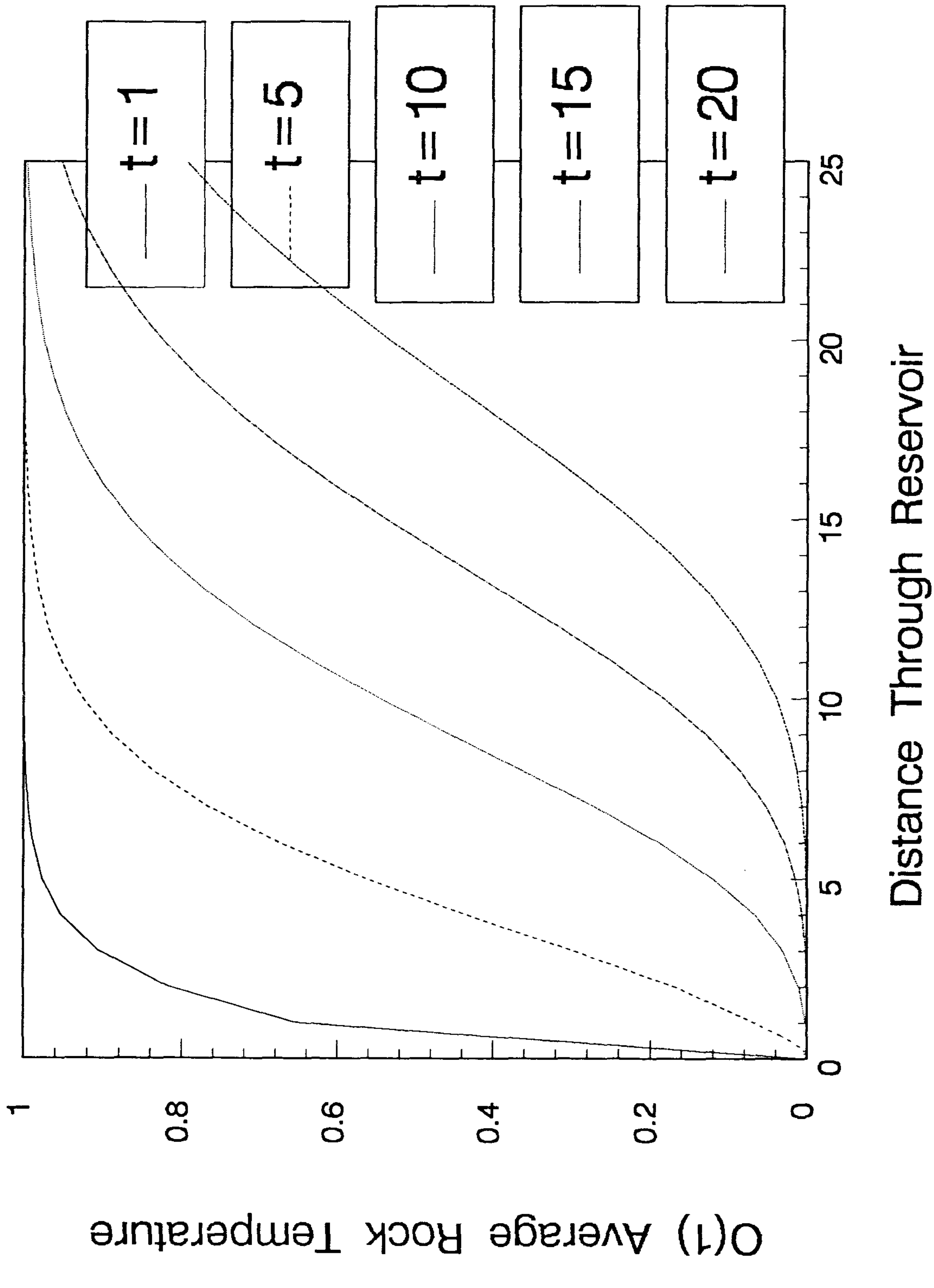

Figure 6.3 - Graph of the O(1) Rock Temperature

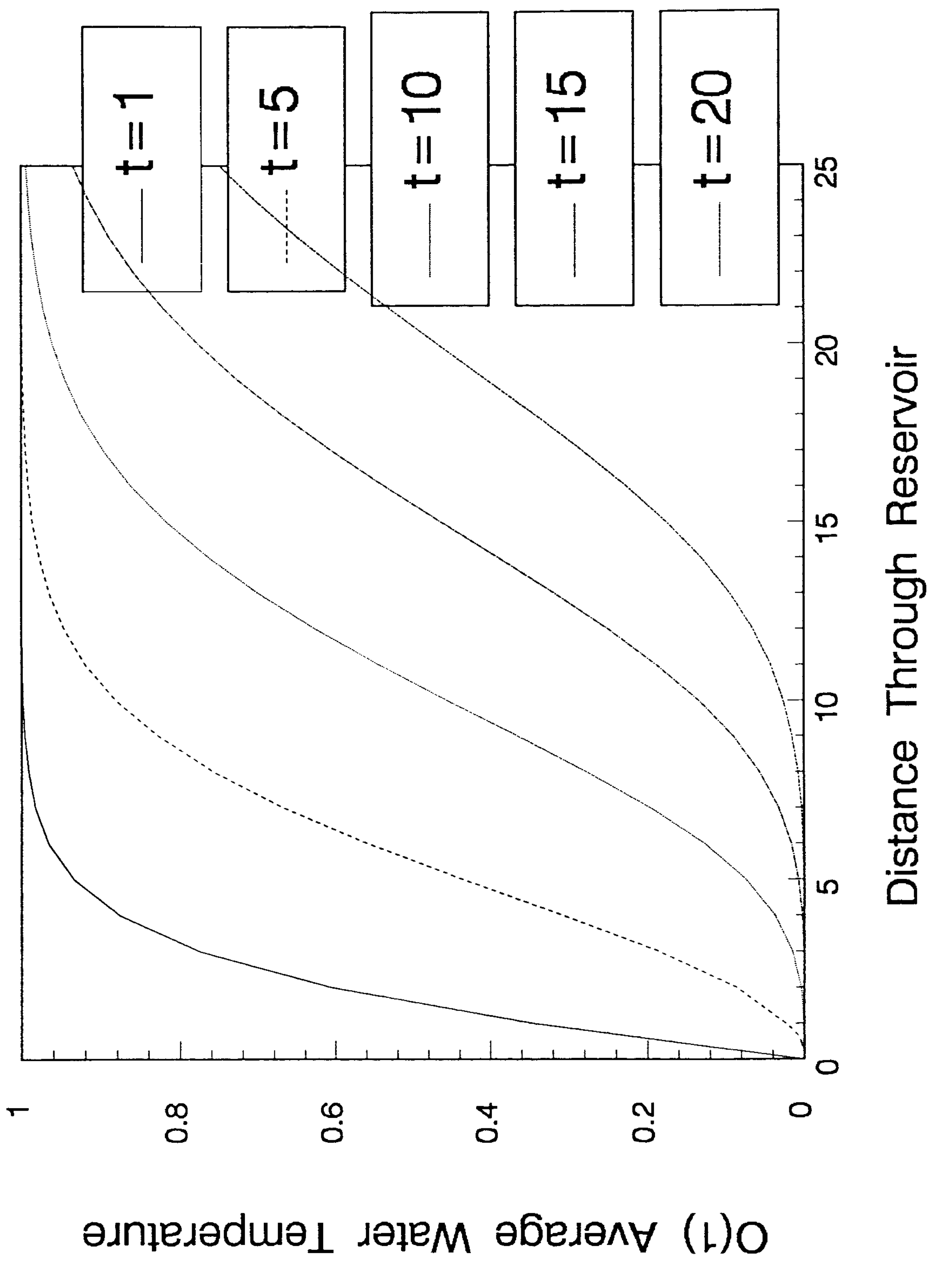

Figure 6.4 - Graph of the O(1) Water Temperature

The position of the free boundary is determined by the condition $T_r = T^*$,

$$T_{r0}(f_0 + \varepsilon e^{iny} f_1, t) + \varepsilon e^{iny} T_{r1}(f_0 + \varepsilon f_1, t) = T^*.$$

Expanding as a Taylor series about $x = f_0(t)$ and equating terms of $O(\varepsilon)$ gives

$$f_1 \frac{\partial T_{r0}}{\partial x} + T_{r1} = 0 \qquad x = f_0(t) \qquad 6.3.41$$

and so f_1 has the form

$$f_1 = - \left. \frac{T_{r1}}{\partial T_{r0}/\partial x} \right|_{x=f_0} . \qquad 6.3.42$$

Solutions for the velocity potentials are found in the same way as in section two to be

$$\phi_1^+ = -Vrf_1 \, e^{-n(x-f_0(t))}, \qquad 6.3.43$$

$$\phi_1^- = Vrf_1 \, e^{n(x-f_0(t))}. \qquad 6.3.44$$

The conservation of energy equations , can be combined together to give a second order, partial differential equation for the rock temperature

$$\rho_w c_w V \left\{ \frac{\partial T_{r1}}{\partial x} + \frac{\rho_r c_r}{h} \frac{\partial^2 T_{r1}}{\partial x \partial t} \right\} + \rho_r c_r \frac{\partial T_{r1}}{\partial t} = -\rho_w c_w \frac{\partial \phi_1}{\partial x} \frac{\partial T_{w0}}{\partial x}. \qquad 6.3.45$$

This equation is non-dimensionalised in the following way

$$x = L\bar{x}, \qquad t = \frac{\rho_r c_r}{h} \bar{t}, \qquad \frac{\partial \phi_1}{\partial x} = V \frac{\partial \bar{\phi}_1}{\partial \bar{x}},$$

$$T_{w0} = T^0 \bar{T}_{w0}, \qquad T_{r1} = T^0 \bar{T}_{r1}, \qquad f_1 = L\bar{f}_1,$$

$$f_0 = L\bar{f}_0,$$

to get

$$\frac{\partial \bar{T}_{r1}}{\partial \bar{x}} + \frac{\partial^2 \bar{T}_{r1}}{\partial \bar{x} \partial \bar{t}} + \alpha \frac{\partial \bar{T}_{r1}}{\partial \bar{t}} = g_1(\bar{x},\bar{t}) \qquad 6.3.46$$

where

$$\alpha = \frac{Lh}{\rho_w c_w V} \qquad 6.3.19$$

and

$$g_1(\bar{x},\bar{t}) = - \frac{\partial \bar{T}_{w0}}{\partial \bar{x}} \frac{\partial \bar{\phi}_1}{\partial \bar{x}}. \qquad 6.3.47$$

The non-dimensional boundary and initial conditions are

$$\bar{T}_{r1}(x,0) = 0 \qquad \forall x \qquad 6.3.48$$

$$\bar{T}_{r1}(0,t) = 0 \qquad \forall t \qquad 6.3.49$$

(this is true since for $-\infty < x < 0$ water at $\bar{T}_{w1} = 0$ is flowing through rock at $\bar{T}_{r1} = 0$).

Equation (6.3.46) with conditions (6.3.48) and (6.3.49) can be solved using Laplace Transforms in both the x and t variables. The Laplace transforms are defined by

$$\mathscr{L}[\bar{T}_{r1}(\bar{x},\bar{t})] = \hat{T}(\bar{x},p) = \int_0^\infty \bar{T}_{r1}(\bar{x},\bar{t})\; e^{-p\bar{t}}\, d\bar{t} \qquad 6.3.20$$

$$\mathscr{L}[\hat{T}(\bar{x},p)] = \tilde{T}(s,p) = \int_0^\infty \hat{T}(\bar{x},p)\; e^{-s\bar{x}}\, d\bar{x} \qquad 6.3.50$$

and so taking Laplace Transforms in x and t gives

$$(s + ps + \alpha p)\, \tilde{T} = \tilde{g}_1(s,p). \qquad 6.3.51$$

Using the convolution integral this transform can be inverted to obtain

$$\bar{T}_{r1}(\bar{x},\bar{t}) = g_1 * g_2 = \int_0^{\bar{x}}\int_0^{\bar{t}} g_1(\xi,\tau)\; g_2(\bar{x}-\xi,\bar{t}-\tau)\; d\tau d\xi. \qquad 6.3.52$$

where

$$g_1(\bar{x},\bar{t}) = nrf_1 e^{-nL|\bar{x}-\bar{f}_0(\bar{t})|} - \alpha\bar{x} \Bigg\{ e^{-\bar{t}} \Big[\alpha I_0(2\surd(\alpha\bar{x}\bar{t})) - \surd(\alpha\bar{t}\bar{x}^{-1})\; I_0'(2\surd(\alpha\bar{x}\bar{t}))\Big] + \int_0^{\bar{t}} e^{\eta}\Big[\alpha I_0(2\surd(\alpha\bar{x}\eta)) - \surd(\alpha\eta\bar{x}^{-1})\; I_0'(2\surd(\alpha\bar{x}\eta))\Big]\, d\eta\Bigg\}. \qquad 6.3.53$$

and

$$g_2(\bar{x},\bar{t}) = \mathcal{L}^{-1}\left[\frac{1}{s + ps + \alpha p}\right] = e^{-\alpha\bar{x}-\bar{t}}\; I_0(2\surd(\alpha\bar{x}\bar{t})). \qquad 6.3.54$$

(r is given in 6.2). Let $f_1 = L\bar{f}_1$, then (6.3.42) gives

$$\bar{f}_1 = \left.\frac{\bar{T}_{r1}}{\partial\bar{T}_{r0}/\partial\bar{x}}\right|_{\bar{x}=\bar{f}_0} \qquad 6.3.55$$

so to obtain $\bar{f}_1$, $\frac{\partial\bar{T}_{r0}}{\partial\bar{x}}$ evaluated at $\bar{x} = \bar{f}_0(\bar{t})$, is required,

$$\frac{\partial\bar{T}_{r0}(\bar{f}_0)}{\partial\bar{x}} = \alpha\left[1 - \frac{T^*}{T^0}\right] - e^{-\alpha\bar{f}_0}\int_0^{\bar{t}} e^{-\eta}\,\frac{\surd(\alpha\eta)}{\surd\bar{f}_0}\; I_0'(2\surd(\alpha\bar{f}_0\eta))\; d\eta. \qquad 6.3.56$$

Substituting (6.3.52) (evaluated at $x = \bar{f}_0$) and (6.3.56) into (6.3.55), gives an integral equation for $\bar{f}_1$. This could, presumably, be evaluated numerically, but unfortunately time did not permit this to be investigated for inclusion in the thesis.

Finally, note that the O(ε) water temperature term, T_{w1}, can be found by substituting the corresponding rock temperature term, T_{r1}, (6.3.52) into the following non-dimensional equation,

$$\bar{T}_{w1} = \bar{T}_{r1} + \frac{\partial \bar{T}_{r1}}{\partial \bar{t}}. \qquad 6.3.57$$

So a problem whereby a cold fluid is pushed through a hot porous medium is considered and the stability of an isotherm is investigated. The position of the isotherm is given by (6.3.36) and (6.3.40) and a triple integral definition for the function $\bar{f}_1$ is obtained (defined by (6.3.55), (6.3.52) and (6.3.56)), where $|\bar{f}_1|$ finite for all time means that the front is stable and $|\bar{f}_1| \to \infty$ as $t \to \infty$ means that the front is unstable.

CHAPTER 7 - CONCLUDING SUMMARY

The mathematical modelling of geothermal energy reservoirs is still very much in its infancy, albeit over twenty years old. Much remains unexplained and although some headway has been made in this thesis it is only a small part of the complicated mathematics which has still to be formulated.

A one-dimensional model of a fluid filled crack is presented in chapter three and analysed in chapters four and five, for the special case of a partially open crack with zero elastic shear stress along the crack walls. Asymptotic series solutions are obtained and the behaviour of the crack in the region close to the tip is investigated. Singular integral equations are obtained for local functions close to the tip. These functions are introduced so that a more accurate representation of the crack height near the tip can be obtained. Future work may possibly include seeking a numerical solution to the singular integral equations.

Numerical solutions of the partially open crack equation are presented in chapter five. A finite difference approximation to the flow problem is used along with a boundary element technique to evaluate the elastic normal stress along the crack. The numerical solutions compare favourably with the analytical solutions obtained in chapter four. A stability analysis for the numerical method is carried out using Gerschgorin's theorem and limits for the timestep are obtained and compared with limits obtained via the numerical evaluation of the maximum modulus eigenvalues of the governing matrix. The limits agree with observed values of stability.

In chapter six, the problem of the long term heat transfer effects on the flow paths is considered. The geothermal energy reservoir is modelled as a porous medium and the permeability of the rock and the viscosity of the fluid are given simple temperature dependencies in order that the stability of an isotherm in the rock can be established. The analysis of the long term heat transfer problem could be extended by the inclusion of some numerical integration. This would enable graphs of the shape of the free boundary, defined by an isotherm in the rock temperature, to be plotted. A clearer understanding of the stability of the flow paths would result. Additionally, the solutions for the rock and water temperatures could be analysed to determine the small and large time behaviour of the temperatures.

APPENDIX A - FOURIER TRANSFORMS

In this appendix a brief outline of the main properties of Fourier Transforms used in this thesis is given.

If f(t) is such that the integral

$$\int_{-\infty}^{\infty} f(t)e^{i\xi t}\,dt \qquad (1)$$

exists for all real values of ξ, (It is sufficient that f(t) is piecewise continuously differentiable and integrable on the whole real line), the *Fourier Transform* of f is defined by the equation:

$$F(\xi) \equiv \mathcal{F}[f(t)] = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t)\ e^{i\xi t}\,dt \qquad (2)$$

This says that the function f(t) is transformed by (2) to a function of ξ, $F(\xi)$. The Fourier Inversion theorem states

$$\mathcal{F}^{-1}[F(\xi)] = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} F(\xi)\ e^{-i\xi t}\,d\xi = f(t). \qquad (3)$$

If $F = \mathcal{F}[f(t);\xi]$ and $G = \mathcal{F}[g(t);\xi]$, the Convolution theorem is

$$\mathcal{F}^{-1}[F(\xi)G(\xi);t] = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(u)\ g(t-u)\,du. \qquad (4)$$

For $Re(\xi) > 0$, the following is a table of Fourier Transform pairs, f(t) and $F(\xi)$, used in chapter 2 (Sneddon (1972)):

f(t)	$F(\xi)$
1	$\sqrt{(2\pi)}\delta(\xi)$
t^{-1}	$\sqrt{(0.5\pi)}i\ \text{sgn}\xi$
t^{-2}	$-\sqrt{(0.5\pi)}\lvert\xi\rvert$

APPENDIX B

In section 4.4, in order to find the constant C_1, it is necessary to integrate the function H (4.4.22) with respect to η, between the limits (-1,1), i.e.

$$\int_{-1}^{1} H\, d\eta = \int_{-1}^{1} \left\{ C_1 \left(\eta \ln\left|\frac{1-\eta}{1+\eta}\right| + 2 \right) - \frac{4\lambda\eta}{9} \ln\left|\frac{1-\eta}{1+\eta}\right| \right.$$

$$+ \frac{2\lambda}{9} \ln|1-\eta^2| \qquad - \frac{\lambda\eta}{9} \ln\left|\frac{1-\eta}{1+\eta}\right| \ln|1-\eta^2|$$

$$\left. - \frac{4\lambda\eta}{9} \int_0^{\eta} \frac{\ln|1-\xi^2|}{1-\xi^2}\, d\xi \right\} d\eta\,. \qquad (1)$$

To evaluate this integral, consider the following simpler integrals and then substitute these back into (1) to give the final result:

$$\int_{-1}^{1} \ln|1-\eta^2|\, d\eta = \Big[- 2\eta + (1+\eta)\ln|1+\eta|$$

$$- (1-\eta)\ln|1-\eta| \Big]_{-1}^{1} = 4(\ln 2 - 1)$$

$$\int_{-1}^{1} \eta \ln\left|\frac{1-\eta}{1+\eta}\right| \ln|1-\eta^2|\, d\eta = \Big[3\eta - (1+\eta)\ln|1+\eta|$$

$$+ (1-\eta)\ln|1-\eta| + \frac{(\eta^2-1)}{2} \ln\left|\frac{1-\eta}{1+\eta}\right| \left(\ln|1-\eta^2| - 1 \right) \Big]_{-1}^{1}$$

$$= 2(3 - 2\ln 2)$$

To evaluate the double integral, (note that according to Fubini's theorem the order of integration cannot be reversed) infinite sum expressions for the natural logarithm terms are used and lead to the solution

$$\int_{-1}^{1} \eta \int_0^{\eta} \frac{\ln|1-\xi^2|}{1-\xi^2}\, d\xi\, d\eta = -2(1 - \ln 2)\,.$$

$$\int_{-1}^{1} \eta \ln\left|\frac{1 - \eta}{1 + \eta}\right| d\eta = \left[\frac{(\eta^2 - 1)}{2} \ln\left|\frac{1 - \eta}{1 + \eta}\right| - \eta\right]_{-1}^{1} = -2.$$

Substituting these expressions into (1) above leads us to the result

$$C_1 = -\frac{2\lambda}{9} \ln 2 - \frac{\lambda}{9}.$$

APPENDIX C

In section 4.5 the limiting values of the functions H and $\frac{dH}{d\eta}$ as $\eta \to 1^-$ and as $\eta \to -1^+$ are required. Recall that H is given by

$$H = \left\{C_1 - \frac{\lambda}{3}\int_0^\eta \xi \left(\xi \ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right) d\xi\right\} \left(\eta \ln\left|\frac{1-\eta}{1+\eta}\right| + 2\right)$$

$$+ \frac{\eta\lambda}{3}\int_0^\eta \left(\xi \ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right)^2 d\xi \ . \qquad 4.4.26$$

where C_1 is found to be

$$C_1 = -\frac{2\lambda}{9}\ln 2 - \frac{\lambda}{9} \ . \qquad 4.4.29$$

Rewriting this as

$$H = \left(C_1 - \frac{\lambda}{3} I_1\right) \left(\eta \ln\left|\frac{1-\eta}{1+\eta}\right| + 2\right) + \frac{\eta\lambda}{3} I_2$$

where I_1 and I_2 are

$$I_1 = \int_0^\eta \xi \left(\xi \ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right) d\xi$$

$$I_2 = \int_0^\eta \left(\xi \ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right)^2 d\xi \ ,$$

leads to $\frac{dH}{d\eta}$ given by

$$\frac{dH}{d\eta} = \left(C_1 - \frac{\lambda}{3} I_1\right) \left(\ln\left|\frac{1-\eta}{1+\eta}\right| - \frac{2\eta}{1-\eta^2}\right) + \frac{\lambda}{3} I_2 \ .$$

First, the limiting values of I_1 and I_2 as $\eta \to 1^-$ are evaluated. To do this let $\eta = 1 - \varepsilon$ and then let $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0} I_1 = \lim_{\varepsilon\to 0} \int_0^{1-\varepsilon} \xi \left(\xi \ln\left|\frac{1-\xi}{1+\xi}\right| + 2\right) d\xi$$

$$= \lim_{\varepsilon\to 0} \left[\frac{(\eta^3 - 1)}{3} \ln|1-\eta| - \frac{(\eta^3+1)}{3} \ln|1+\eta| + \frac{2\eta^2}{3}\right]_0^{1-\varepsilon} .$$

$$= \lim_{\varepsilon\to 0} \left(\frac{1}{3} \left[(1 - 2\varepsilon + \varepsilon^2)(1-\varepsilon) - 1\right] \ln\varepsilon \right.$$

$$\left. - \frac{1}{3}\left[2 - 3\varepsilon + 3\varepsilon^2 - \varepsilon^3\right] \ln|2-\varepsilon| + \frac{2}{3}(1 - 2\varepsilon + \varepsilon^2)\right) .$$

Since ε is small, expand $\ln|2 - \varepsilon|$ about the point 2 using Taylor's theorem, so that

$$\ln|2-\varepsilon| = \ln 2 - \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} - \ldots$$

and so as $\varepsilon \to 0$

$$I_1 \to -\frac{2}{3}\ln 2 + \frac{2}{3} + \varepsilon\left(-\ln\varepsilon + \ln 2 - 1\right) + O(\varepsilon^2) .$$

In a similar way using dilogarithms it can be shown that

$$\lim_{\eta\to 1} I_2 = \frac{4}{3} + \frac{\pi^2}{9} .$$

The following limits are also obtained

$$\lim_{\eta\to 1}\left(\eta\ln\left|\frac{1-\eta}{1+\eta}\right| + 2\right) = \lim_{\varepsilon\to 0}\left(\ln\varepsilon - \ln 2 + 2 \right.$$

$$\left. + \varepsilon\left[\ln 2 + \frac{1}{2} - \ln\varepsilon\right] + O(\varepsilon^2) \right),$$

$$\lim_{\eta\to 1} \ln\left|\frac{1-\eta}{1+\eta}\right| = \lim_{\varepsilon\to 0}\left(\ln\varepsilon - \ln 2 + \frac{\varepsilon}{2} + O(\varepsilon^2)\right),$$

and

$$\lim_{\eta\to 1}\left(\frac{-2\eta}{1-\eta^2}\right) = \lim_{\varepsilon\to 0}\left(-\frac{1}{\varepsilon}+\frac{1}{2}+\frac{\varepsilon}{4}+O(\varepsilon^2)\right).$$

Substituting these terms into the expressions for H and for $\frac{dH}{d\eta}$ lead us to the results

$$H \to -\frac{\lambda}{3}\ln|1-\eta| + \frac{\lambda}{3}\ln 2 - \frac{2\lambda}{9} + \frac{\lambda\pi^2}{27} \qquad \eta \to 1^-$$

$$\frac{dH}{d\eta} \to \frac{\lambda}{3(1-\eta)} - \frac{2\lambda}{3}\ln\left|\frac{1-\eta}{2}\right| - \frac{\lambda}{18} + \frac{2\lambda}{3}\ln 2 + \frac{\lambda\pi^2}{27} \qquad \eta \to 1^-$$

Since H is even and $\frac{dH}{d\eta}$ is odd in η, the following are consequential:

$$H \to -\frac{\lambda}{3}\ln|1+\eta| + \frac{\lambda}{3}\ln 2 - \frac{2\lambda}{9} + \frac{\lambda\pi^2}{27} \qquad \eta \to -1^+$$

$$\frac{dH}{d\eta} \to \frac{-\lambda}{3(1+\eta)} + \frac{2\lambda}{3}\ln\left|\frac{1+\eta}{2}\right| + \frac{\lambda}{18} - \frac{2\lambda}{3}\ln 2 - \frac{\lambda\pi^2}{27} \qquad \eta \to -1^+.$$

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