SUPERINSTANTONS

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ABSTRACT

FACULTY OF SCIENCE

Doctor of Philosophy

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We show how $O(g^2)$ divergent quantum corrections to the instanton effective action may be calculated in Yang-Mills theory. We verify that these are as required by a renormalisation group analysis of the semiclassical calculation. This requires a delicate treatment of the zero modes and of the jacobian corresponding to a change of variables between these zero modes and collective coordinates.

We generalise the instanton solution to a superfield solution of N=1 super Yang-Mills theory, and describe a general method of generating covariant expressions for the discrete zero modes. It is found that the linearly independent set of zero modes contains 4 more fermionic modes than were previously expected. These are anomalous supergauge modes. We show how to parameterize the continuous supergauge zero modes and the positive frequency modes. From this analysis we construct the full Green functions in the background of a super-instanton and projection operators onto the corresponding spaces. We generalise our previous $O(g^2)$ calculation to that of a superinstanton in super Yang-Mills. This allows a comparison with recent arguments that all the higher order quantum corrections in such a situation should vanish identically. We conclude that these arguments are invalid but investigate the possibility that quantum corrections do nevertheless vanish to all orders. The subtleties and complications of the cancellation mechanism make it difficult to imagine that this could be the case.

ACKNOWLEDGEMENTS

I would like to thank all the staff in the high energy physics group for their help in making my time here such a pleasant one. I am grateful to SERC for providing financial support, and indebted to Karen Booth for her moral support. Thanks go to Christopher Sachrajda and Douglas Ross for including me in some lively and stimulating research, and especially to Douglas for all his guidance and encouragement with numerous ventures in the last three years.

CONTENTS.

INTRODUCTION.

Instantons, topologically non-trivial solutions of the Euclidean equations of motion in Non-Abelian Gauge Theories, were discovered in the mid 1970's [1,2]. Although their significance may not yet be fully understood, it is clear that they constitute one of the most important non-perturbative effects in gauge theories, (for reviews of instanton physics see refs 3 - 5; The instanton itself is described in chapter 2.1). In this thesis we describe the results of a study of the generalisation of instantons to superfield solutions of Euclidean super Yang-Mills [6]. This we call the superinstanton [7]. The primary aim of the research was to calculate higher order quantum corrections to the superinstanton. With this end in mind we first attacked the problem of calculating the ultraviolet divergent quantum corrections to an instanton in Yang-Mills [8]. In the process of the calculation (to two loops) we discovered some new effects : we found that part of the divergent corrections came from new interactions which arose from a Jacobian (of a change of variables from zero modes to collective coordinates), and part of the corrections came from certain non-perturbative long distance effects. These divergences were cancelled by the renormalisation of the coupling constant that appears as a multiplicative factor in the semiclassical calculation [2] (see (2.3.2)). Together with the usual purely short distance divergences (which are cancelled by renormalisation of the instanton classical action) we had thus checked that the $O(g^2)$ explicit $\ln\mu$ dependence was as required by the renormalisation group invariance of the semiclassical result. This work is described in chapter 2. Our attention then turned to the superinstanton. Before carrying out the quantum corrections to such an object it is necessary to know the number and nature of the zero modes. And, for dealing with the non-zero modes in the semiclassical calculation, the generalisation of certain tricks [9] are needed. These convert the semiclassical problem into one of determining the determinant of background covariant-$\Box$. (See chapter 2.3 and 4.2). Chapter 3 describes the construction of the superinstanton, its zero modes and non-zero modes (through these tricks). That chapter starts with a discussion of Euclidean N=1 supersymmetry and its relevance to instantons. Chapter 4 describes the generalisation of the previous ideas on renormalisation around instantons (chapters 2.2 and 2.3) to the super-instanton and superfields. This generalisation is not as straightforward as one might imagine because of the complex non-linearity of super Yang-Mills and the fact that (unlike the component theory) even the ghosts have zero modes. Nevertheless we describe in detail in chapter 4 how one can compute to two loops the divergent quantum corrections to the superinstanton effective action.

We are then in a position (in chapter 5) to draw some conclusions from our work. In particular we are able to discuss the validity of a paper published by Novikov et al [10]. In this paper a general theorem about the vanishing of certain quantum corrections to instantons in super Yang-Mills led to a derivation of the β-function to all orders of perturbation theory. We have shown their proof of the theorem to be invalid [11] and we describe our reasoning in this chapter. Although the proof is wrong the theorem might still be correct. We discuss this possibility also in chapter 5. Our research has shown the theorem to be correct to two loops although we have unfortunately found no indication that this theorem should hold to all orders.

The work described in chapter 4 can be found in ref.[12] which also includes some component field calculations in the Wess Zumino gauge . The problem of including fermions in the quantum corrections to instantons in a general non-abelian gauge theory was included in ref.[8] but it is not discussed here.

The beginning of the thesis (chapters 1 and 2.1) is devoted to an introduction to the relevant background. Much of the thesis relies heavily on background field and superbackground field methods which is the subject of chapter 1.

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CHAPTER 1 THE BACKGROUND FIELD METHOD.

The prime concern of this thesis will be the calculation of a quantity called the background field effective action, when the background field is set equal to an instanton. This chapter will be devoted to discussing the background effective action and some tremendous simplifications in the method of calculation which can be gained by a judicious choice of gauge - the so-called "background gauge".
The background field method is important not only for the computational simplifications it allows (which are described at the end of this section) but also because it appears naturally in the calculation of quantities other than the background field effective action : for example the instanton contribution to vacuum-vacuum expectation values. (See chapters 2 and 3).

We will for the time being concern ourselves only with Yang-Mills. (It is trivial to include fermions and scalars).

The usual generating functional

$$Z_0[J] = \int \mathcal{D}A^f \; e^{S(A^f) + A^f \cdot J}$$

$$A^f \cdot J = \int d^4x \; A^f_\mu J_\mu \qquad (1.1.1)$$

can be turned into an effective action

$$\Gamma_0[\bar{Q}] = W_0[J] - J\cdot\bar{Q} \qquad (1.1.2)$$

where

$$W_0 = \ln Z_0 \qquad \text{and} \qquad \bar{Q} = \frac{\delta W_0}{\delta J} \qquad (1.1.3)$$

$\Gamma_0[\bar{Q}]$ is the generator of 1 particle irreducible diagrams.
The background field method does not use (1.1.1) but instead expands the 4-potential A^f_μ (the full field) in terms of a "background field" A_μ and a "quantum field" Q_μ by

$$A^f_\mu = A_\mu + Q_\mu \quad , \qquad (1.1.4)$$

integrates over Q_μ and couples the source to the quantum field:

$$Z_A[J] = \int \mathcal{D}Q \; e^{S(A+Q) + J\cdot Q} \qquad (1.1.5)$$

The effective action is constructed exactly as in (1.1.2) and (1.1.3) viz.

$$\Gamma_A[\tilde{Q}] = W_A[J] - J\cdot\tilde{Q}$$

$$W_A = \ln Z_A \qquad \tilde{Q} = \frac{\delta W_A}{\delta J} \tag{1.1.6}$$

and the background effective action is found by setting $\tilde{Q} = 0$,

$$\Gamma[A] \equiv \Gamma_A[0] \tag{1.1.7}$$

(1.1.7) has a loop expansion which is the sum of 1 particle irreducible graphs with no external $\tilde{Q}$ lines. That is they are 1 particle irreducible "vacuum" graphs from which we pull out interactions with the background field.

It is (1.1.7) which is referred to (when the background field is put equal to an instanton) as "the vacuum energy in the presence of an instanton" [1]; this is by analogy with the usual effective action (see (1.1.2)) for which

$$\Gamma_0[0] = \int d^4x \times \text{constant} \tag{1.1.8}$$

The constant can be thought of as the vacuum energy density and is calculated from 1 particle irreducible vacuum graphs.

When calculating (1.1.7) we will always subtract the contribution (1.1.8).

So far we have not discussed the gauge fixing that must be done in order to calculate Γ : The gauge invariance of the action S(A+Q) renders the quadratic action (in Q) non invertable and so we can not obtain the quantum field propagators.

A gauge transformation on the full field (see (1.1.4))

$$A^f_\mu \rightarrow e^{i\Omega} A^f_\mu e^{-i\Omega} + \frac{i}{g} e^{i\Omega} \partial_\mu e^{-i\Omega} \tag{1.1.9}$$

can be expressed as transformations on the component fields in a number of ways, of which two important ones are

(1)

$$A_\mu \rightarrow e^{i\Omega} A_\mu e^{-i\Omega}$$
$$Q_\mu \rightarrow e^{i\Omega} Q_\mu e^{-i\Omega} + i/g \, e^{i\Omega} \partial_\mu e^{-i\Omega} \qquad (1.1.10)$$

The background transforms homogeneously, but the quantum field undergoes a gauge transformation. It is the quantum field gauge invariance that makes the quadratic action non invertable.

(2)

$$A_\mu \rightarrow e^{i\Omega} A_\mu e^{-i\Omega} + i/g \, e^{i\Omega} \partial_\mu e^{-i\Omega}$$
$$Q_\mu \rightarrow e^{i\Omega} Q_\mu e^{-i\Omega} \qquad (1.1.11)$$

This transformation is the "background gauge transformation" so called because the background field undergoes a gauge transformation whereas the quantum field transforms homogeneously.

If we introduce a background dependent gauge fixing term of the form

$$S_{GF} = -\frac{1}{2\xi} \int (D_\mu Q_\mu)^2 \, d^4x$$
$$D_\mu = \partial_\mu - ig A_\mu \qquad (1.1.12)$$

then the quantum gauge invariance (1.1.12) will be broken so that the quadratic action can now be inverted (to form the propagator), but the background gauge invariance (1.1.11) is not broken.

Entirely in analogy with the usual method we introduce ghost terms that evaluate the determinant of the gauge fixing term under a gauge transformation

$$S_{gh} = \int d^4x \; \bar{\phi} \left(D_\mu - ig Q_\mu \right) D_\mu \phi \qquad (1.1.13)$$

and we can now proceed to calculate (1.1.7).

But note that since (1.1.12) remains unbroken (and noting that Q transforming homogeneously implies the same of $\tilde{Q}$) (1.1.7) is a <u>gauge invariant</u> functional of the background field.

Comparing (1.1.1) with (1.1.5) we see that

$$Z_A = Z_0 \, e^{-J\cdot A} \tag{1.1.14}$$

Hence $\quad W_A = W_0 - J\cdot A$

and differentiating w.r.t. J :

$$\tilde{Q} = \bar{Q} - J \tag{1.1.15}$$

Performing the Legendre transform on (1.1.14) (see (1.1.6)) and using (1.1.15) we obtain

$$\Gamma_A[\tilde{Q}] = \Gamma_0[\bar{Q}]\Big|_{\bar{Q} = \tilde{Q} + A} \tag{1.1.16}$$

Hence our gauge invariant effective action (1.1.7) is just the usual effective action but with $\bar{Q} = A$ and a peculiar gauge fixing term (containing A_μ). The choice of gauge fixing term can be proved not to affect the calculation of gauge invariant physical quantities (such as the β-function), and hence we can expect to extract the same physics from Γ as we did from Γ_0 .

The choice of a gauge invariant Γ greatly simplifies the divergent structure of the theory: this is because, although we could in principle find unrelated renormalisations

$$\begin{aligned} g_0 &\rightarrow Z_g \, g \\ A &\rightarrow Z_B^{\frac{1}{2}} \, A \end{aligned} \tag{1.1.17}$$

The preservation of gauge invariance requires

$$D_\mu = \partial_\mu - i g_0 A_{\mu 0} = \partial_\mu - i g A_\mu Z_g Z_B^{\frac{1}{2}} \tag{1.1.18}$$

to remain gauge invariant and so

$$Z_B = Z_g^{-2} \tag{1.1.19}$$

Furthermore in standard background field calculations we can ignore the renormalisation of the quantum field

$$Q \rightarrow Z_Q^{\frac{1}{2}} \, Q$$

(note we will find this is not the case in the calculation of "long distance" corrections in chapters 2.3 and 4.3)

This is because the quantum field lines only appear inside the graphs and the positive powers of $Z_a^{\frac{1}{2}}$ in the vertices will cancel the Z_a^{-1} associated with the propagators.

This leaves us with only one sort of counterterm diagram - that from renormalising the gauge fixing parameter

$$\xi \rightarrow Z_\xi \xi$$

(Note that, because the gauge fixing term is not renormalised
$Z_\xi = Z_a$)

Even this can be avoided by working in a general gauge (or the Landau gauge $\xi = 0$) [2].

(1.1.18) and (1.1.19) imply that the (U.V.) divergences in the background field action must take the form

$$Z_B F_{\mu\nu}^2 = Z_g^{-2} F_{\mu\nu}^2 \qquad (1.1.20)$$

(since they must be local, gauge invariant and, since the theory is renormalisable, be vertices found in the action).

(1.1.20) implies that the divergences can be found by expanding the vacuum diagrams up to quadratic in the background field and using

$$\int F_{\mu\nu}^2 \, d^4x = -2\int d^4x \, A_\mu (\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu) A_\nu + O(A^3) \qquad (1.1.21)$$

The β-function can be straightforwardly extracted from the divergences Z_g^{-2} .

In fact since (1.1.20) has the U.V. divergent structure of $1/g_0^2$ the renormalised quantity has explicit μ dependence

$$-\frac{1}{4}\int F_{\mu\nu}^2 \, d^4x \left(1 - \frac{2\beta_0}{(4\pi)^2} g^2(\mu) \ln\mu - \frac{2\beta_1}{(4\pi)^4} g^4(\mu) \ln\mu \ldots\ldots \right)$$

where (1.1.22)

$$\beta(g) \equiv \frac{\partial g(\mu)}{\partial \ln\mu} = -\frac{\beta_0}{(4\pi)^2} g^3(\mu) - \frac{\beta_1}{(4\pi)^4} g^5(\mu) + \cdots\cdots$$

Finally note that setting $\tilde{Q} = 0$ implies by (1.1.6)

$$Z_A = e^{\Gamma_A[0]}$$

(for a particular value of $J = -\frac{\delta\Gamma}{\delta\tilde{Q}}\Big|_{\tilde{Q}=0}$)

and similarly from (1.1.2) and (1.1.3) for $\bar{Q} = 0$

$$Z_0 = e^{\Gamma_0[0]}$$

Therefore

$$Z_A / Z_0 = e^{\Gamma_A[0] - \Gamma_0[0]} \tag{1.1.23}$$

and so subtraction of the (zero background) vacuum energy in our instanton calculations (see comments below (1.1.8)) can be regarded as normalising the calculation of Z_A by the zero instanton sector.

1.2 SUPERSPACE AND SUPER YANG-MILLS.

We begin with a discussion of the essential aspects of the superspace formulation of N=1 super Yang-Mills before going on to a brief discussion of the background superfield method in section 1.3, which will be the language used in chapters 3 and 4. We will not present here an introduction to supergraph techniques [3] nor the more advanced covariant supergraph techniques [4] : they are only used in the loop calculations of 4.3 and the methods themselves are not central to the argument. Let us start by fixing the notation: Our notation (with some minor changes made in chapter 3.1 when we change to euclidean supersymmetry) is that of [5] (see appendix A of that paper). We deal with an N=1 superspace consisting of x_μ , 2 Grassmann left handed spinorial coordinates θ^α and 2 Grassmann right handed spinorial coordinates $\bar\theta^{\dot\alpha}$.

$$\{\theta_\alpha , \theta_\beta\} = \{\theta_\alpha , \bar\theta_{\dot\beta}\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0$$

The α and $\dot\alpha$ serve to distinguish the fundamental and complex conjugate representations of the Lorentz covering group SL(2,$\mathbb{C}$).

(1.2.1)

Vectors (such as A_μ) lie in the $(\frac{1}{2},\frac{1}{2})$ representation and can thus be represented

$$A_{\alpha\dot\alpha} = A_\mu \, \sigma^\mu{}_{\alpha\dot\alpha}$$

(The $\dot\alpha$ is a different index not to be confused with α).
where our conventions are

$$\sigma_{\mu\,\alpha\dot\alpha} = \sigma_{\mu\,\dot\alpha\alpha} \equiv \sigma_\mu = (1, \underline{\sigma})$$
$$\sigma_\mu{}^{\alpha\dot\alpha} = \sigma_\mu{}^{\dot\alpha\alpha} \equiv \bar\sigma_\mu = (1, -\underline{\sigma})$$

We introduce Grassmann differentials

$$\partial_\alpha \theta^\beta = \delta_\alpha{}^\beta$$
$$\bar\partial_{\dot\alpha} \bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta} \qquad (1.2.2)$$

We can raise and lower the indices with a matrix proportional to the 2 by 2 alternating symbol (which is invariant under SL(2,$\mathbb{C}$) since M $\in$ SL(2,$\mathbb{C}$) $\Rightarrow$ detM = 1).

$$\varepsilon_{\alpha\beta} \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad (\text{i.e.} \quad \varepsilon_{01} = i\,) \tag{1.2.3}$$

$$\theta_\alpha = \theta^\beta \varepsilon_{\beta\alpha} \tag{1.2.4}$$

In this notation the natural contraction is always from top left to bottom right. We define

$$\varepsilon_{\dot\alpha\dot\beta} \equiv \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = -(\varepsilon_{\alpha\beta})^* \tag{1.2.5}$$

$$\bar\theta_{\dot\alpha} = \bar\theta^{\dot\beta} \varepsilon_{\dot\beta\dot\alpha} \tag{1.2.6}$$

so that for example

$$\bar\theta_{\dot\alpha} = -(\theta_\alpha)^* \tag{1.2.7}$$

An antisymmetric tensor $T_{\alpha\beta}$ is automatically proportional to $\varepsilon_{\alpha\beta}$.

$$T_{\alpha\beta} = -\tfrac{1}{2} T^\gamma{}_\gamma \, \varepsilon_{\alpha\beta} \tag{1.2.8}$$

The operators $\{ i\partial_\alpha , i\bar\partial_{\dot\alpha} , i\partial_{\alpha\dot\alpha} \}$ are hermitian (by which we mean, for the first two, that they obey the rules (1.2.1) and (1.2.7)). (1.2.9)

The supersymmetry algebra which we normalise as

$$\{ Q_\alpha , \bar Q_{\dot\alpha} \} = -2i\, \partial_{\alpha\dot\alpha} \tag{1.2.10}$$

has explicit representation

$$Q_\alpha = \partial_\alpha - i \bar\theta^{\dot\beta} \partial_{\alpha\dot\beta}$$

$$\bar Q_{\dot\alpha} = (Q_\alpha)^\dagger = \bar\partial_{\dot\alpha} - i \theta^\beta \partial_{\beta\dot\alpha} \tag{1.2.11}$$

The supersymmetry covariant derivatives anticommute with these generators

$$\{ D_\alpha , \overset{(-)}{Q} \} = \{ \bar D_{\dot\alpha} , \overset{(-)}{Q} \} = 0 \tag{1.2.12}$$

They are

$$D_\alpha = \partial_\alpha + i \bar\theta^{\dot\beta} \partial_{\alpha\dot\beta} \tag{1.2.13}$$

$$\bar D_{\dot\alpha} = \bar\partial_{\dot\alpha} + i \theta^\beta \partial_{\dot\alpha\beta} = (D_\alpha)^\dagger$$

with algebra $\{ D_\alpha , \bar D_{\dot\alpha} \} = 2i\, \partial_{\alpha\dot\alpha}$ (1.2.14)

It is useful to distinguish between Kronecker δ's in (1.2.2) and

$$\delta^{\beta}{}_{\alpha} = -\delta_{\alpha}{}^{\beta} \qquad (1.2.15)$$

so that identities such as

$$\varepsilon^{\beta\gamma}\varepsilon_{\gamma\alpha} = \delta^{\beta}{}_{\alpha} = -\delta_{\alpha}{}^{\beta} \qquad (1.2.16)$$

follow naturally.

Note the appearance of a factor 2 in the completeness relations for the σ matrices (above (1.2.2)):

$$\sigma^{\mu}{}_{\alpha\dot{\beta}}\,\sigma_{\mu}{}^{\gamma\dot{\delta}} = 2\,\delta_{\alpha}{}^{\gamma}\,\delta_{\dot{\beta}}{}^{\dot{\delta}}$$

$$\sigma_{\mu}{}^{\alpha\dot{\beta}}\,\sigma^{\nu}{}_{\dot{\beta}\alpha} = 2\,\delta_{\mu}{}^{\nu} \qquad (1.2.17)$$

so that

$$A^{\alpha\dot{\alpha}}B_{\alpha\dot{\alpha}} = 2A^{\mu}B_{\mu}$$

$$\partial_{\alpha\dot{\alpha}}x^{\beta\dot{\beta}} = 2\,\delta_{\alpha\dot{\alpha}}{}^{\beta\dot{\beta}} = 2\,\delta_{\alpha}{}^{\beta}\,\delta_{\dot{\alpha}}{}^{\dot{\beta}} \qquad (1.2.18)$$

Any superfield can be expanded as a Taylor series in θ and $\bar{\theta}$ using the fact that

$$\theta_{\alpha}\theta_{\beta}\theta_{\gamma} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\bar{\theta}_{\dot{\gamma}}$$

so that for example for a real superfield:

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & B(x) - \theta^{\alpha}\psi_{\alpha}(x) - \bar{\theta}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}(x) \\ & + \tfrac{1}{2}\theta^{2}H(x) + \tfrac{1}{2}\bar{\theta}^{2}\bar{H}(x) + \theta^{\alpha}\bar{\theta}^{\dot{\alpha}}A_{\alpha\dot{\alpha}}(x) \\ & - \bar{\theta}^{2}\theta^{\alpha}(\lambda_{\alpha} + \tfrac{i}{2}\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) - \theta^{2}\bar{\theta}^{\dot{\alpha}}(\bar{\lambda}_{\dot{\alpha}} + \tfrac{i}{2}\partial_{\alpha\dot{\alpha}}\psi^{\alpha}) \\ & + \tfrac{1}{2}\theta^{2}\bar{\theta}^{2}(D + \tfrac{1}{2}\Box B) \end{aligned} \qquad (1.2.19)$$

The component fields depend only on x and form a supermultiplet. The signs and the derivatives appear in (1.2.19) in this way to correspond with conventional definitions of the component fields in terms of supersymmetric derivatives [6] (a more convenient method of definition),

$$B(x) = V|$$
$$\psi_\alpha(x) = -D_\alpha V| \qquad \bar{\psi}_{\dot\alpha} = -\bar{D}_{\dot\alpha} V|$$
$$H(x) = -\tfrac{1}{2} D^2 V| \qquad \bar{H}(x) = -\tfrac{1}{2}\bar{D}^2 V|$$
$$A_{\alpha\dot\alpha}(x) = -\tfrac{1}{2}[D_\alpha, \bar{D}_{\dot\alpha}]V|$$
$$\lambda_\alpha(x) = \tfrac{1}{4}\bar{D}^2 D_\alpha V| \tag{1.2.20}$$
$$\bar{\lambda}_{\dot\alpha}(x) = \tfrac{1}{4} D^2 \bar{D}_{\dot\alpha} V|$$
$$D(x) = \tfrac{1}{8} D^\alpha \bar{D}^2 D_\alpha V| = \tfrac{1}{8}\bar{D}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha} V|$$

(since $D^\alpha \bar{D}^2 D_\alpha = \bar{D}^{\dot\alpha} D^2 \bar{D}_{\dot\alpha}$ is an identity).
The bar "|" indicates that one should set $\theta = \bar{\theta} = 0$ in the resulting superfield.
We can also define chiral superfields $\varphi(x,\theta,\bar\theta)$ such that

$$\bar{D}_{\dot\alpha}\varphi = 0 \tag{1.2.21}$$

Since $\{\bar{D}_{\dot\alpha}, \theta_\beta\} = 0$ (1.2.22)

and

$$[\bar{D}_{\dot\alpha}, x^+_{\beta\dot\beta}] = 0$$

where

$$x^+_{\beta\dot\beta} = x_{\beta\dot\beta} + 2i\theta_\beta\bar{\theta}_{\dot\beta} \tag{1.2.23}$$

is the chiral x space coordinate.
We can solve (1.2.21) $\quad \varphi \equiv \varphi(x_+, \theta)$ (1.2.24)
The component field content can be written down in a similar manner to (1.2.20) viz.

$$\Omega(x) = \varphi| \tag{1.2.25}$$
$$\varphi_\alpha(x) = D_\alpha\varphi|$$
$$F(x) = -\tfrac{1}{4} D^2\varphi|$$

If we substitute $\quad x \to x - 2i\theta\bar{\theta}$ (1.2.26) we go to the chiral representation where

$$\varphi \equiv \varphi(x,\theta) \qquad (1.2.27)$$

$$\bar{D}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} \qquad D_\alpha = \partial_\alpha + 2i\,\bar{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha}$$

We can do the hermitian conjugate manipulations on antichiral fields which satisfy

$$\bar{D}_{\dot\alpha}\bar{\varphi} = 0 \qquad \bar{\varphi} \equiv \bar{\varphi}(x,\theta,\bar{\theta}) \qquad (1.2.28)$$

The relevant coordinate is x_- , the antichiral coordinate.

$$x_- = x - 2i\,\theta\bar{\theta} \qquad (1.2.29)$$

and the substitution

$$x \rightarrow x + 2i\,\theta\bar{\theta} \qquad (1.2.30)$$

takes us to the antichiral representation

$$\bar{\varphi} \equiv \bar{\varphi}(x,\theta)$$

$$D_\alpha = \partial_\alpha \qquad \bar{D}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} + 2i\,\partial_{\alpha\dot\alpha}\theta^\alpha \qquad (1.2.31)$$

Grassmann integration is defined through the following rules

$$\int d^2\theta\; 1 = \int d^2\theta\; \theta_\alpha = 0$$

$$\int d^2\theta\; \theta^2 = 1 \qquad (1.2.32)$$

(and the hermitian conjugates for $\bar{\theta}$).

General superfield expressions can then be integrated over the full superspace

$$\int d^8z\; F(z) \equiv \int d^4x\, d^2\theta\, d^2\bar{\theta}\; F(x,\theta,\bar{\theta})$$

$$\equiv \int d^4x\, d^4\theta\; F(x,\theta,\bar{\theta}) \qquad (1.2.33)$$

Chiral superfield expressions are only integrated over θ (since they effectively do not depend on $\bar{\theta}$, see (1.2.27), an integral over $\bar{\theta}$ would cause the expression to vanish by rules (1.2.32)).

i.e. $G(z)$ such that $\bar{D}_{\dot\alpha} G = 0$

has

$$\int d^6z\; G(z) \equiv \int d^4x\, d^2\theta\; G(x,\theta,\bar{\theta}) \qquad (1.2.34)$$

Expressions (1.2.32) imply that

$$\int d^2\theta \;\equiv\; -\tfrac{1}{4}\,\partial^\alpha \partial_\alpha \qquad (1.2.35)$$

and in integrals such as (1.2.33) and (1.2.34) we may further replace the ∂^α's by covariant D^α's (1.2.13) since the differences involve x space integrals of total derivatives producing surface terms which can be dropped. i.e.

$$\int d^4x\, d^2\theta \;\equiv\; -\tfrac{1}{4}\int d^4x\; D^\alpha D_\alpha \qquad (1.2.36)$$

Similar remarks hold for $\bar\theta$, the antichiral superspace integral (analogous to (1.2.33)) sometimes being written

$$\int d^6\bar{z} \;\equiv\; \int d^4x\; d^2\bar\theta \qquad (1.2.37)$$

Note that with these definitions for superintegration the Grassmann derivatives (1.2.2) and supersymmetric derivatives (1.2.13) can be integrated by parts in the natural manner.

Now let us consider SU(n) super Yang-Mills. This is constructed from a real superfield $V(x,\theta,\bar\theta)$ (see (1.2.19) and (1.2.20)). It is known as the prepotential and takes values in the Lie algebra of SU(n) $(V \equiv V^i T_i)$. The action is

$$S_V = S + \bar{S}$$

where
$$S = \frac{tr}{128g^2}\int d^4x\, d^2\theta\; W^\gamma W_\gamma \qquad (1.2.38)$$

and
$$W^\gamma = \bar{D}^2\left(e^{-gV} D^\gamma e^{gV}\right) \qquad (1.2.39)$$

Note that since
$$\bar{D}_{\dot\alpha}\bar{D}_{\dot\beta}\bar{D}_{\dot\gamma} = 0 \qquad (1.2.40)$$

the <u>field strength</u> W_γ satisfies $\quad \bar{D}_{\dot\alpha} W^\gamma = 0 \qquad (1.2.41)$

which explains the chiral integral in (1.2.38).
This action is invariant under

$$e^{gV} \to e^{i\bar\Lambda}\, e^{gV}\, e^{-i\Lambda} \qquad (1.2.42)$$

where Λ is a chiral field: $\quad \bar{D}_{\dot\alpha}\Lambda = 0$

(Hence $\bar\Lambda$ is antichiral).

(1.2.42) is the (super) gauge invariance of super Yang-Mills. For infinitessimal Λ, $\bar{\Lambda}$ (1.2.42) implies

$$\begin{aligned}\delta v &= -\tfrac{i}{2} L_v \{ (\bar{\Lambda}+\Lambda) + \coth(\tfrac{g}{2} L_v)(\Lambda-\bar{\Lambda})\} \\ &= \tfrac{i}{g}(\bar{\Lambda}-\Lambda) - \tfrac{i}{2} L_v(\Lambda+\bar{\Lambda}) \\ &\quad + \tfrac{ig}{3}\left(\tfrac{Lv}{2}\right)^2(\bar{\Lambda}-\Lambda) - \tfrac{ig^3}{45}\left(\tfrac{Lv}{2}\right)^4(\bar{\Lambda}-\Lambda) + \cdots\end{aligned} \qquad (1.2.43)$$

where $L_v X = [v, X]$ (1.2.44)

A careful study of (1.2.43) reveals that the fields $\psi, \bar{\psi}$, H, $\bar{H}$ and B defined in (1.2.20) (and (1.2.19)) can be gauged away algebraically; that is, there is a gauge - the so-called Wess-Zumino gauge in which the only non-zero component fields of v are A, λ, $\bar{\lambda}$ and D. In this gauge it is feasible to evaluate explicitlythe θ-integration of (1.2.38); we obtain the component form of super Yang-Mills theory

$$S_v = tr \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu}^2 + i\bar{\lambda}^{\dot{\alpha}} D_{\dot{\alpha}}{}^{\alpha} \lambda_\alpha + \tfrac{1}{2} D^2 \right) \qquad (1.2.45)$$

$F_{\mu\nu}$ is the Yang-Mills field strength with $A_{\alpha\dot{\alpha}}$ as 4-potential.

$$\begin{aligned} D_{\alpha\dot{\alpha}} &= D^\mu \sigma_{\mu\,\alpha\dot{\alpha}} \\ D_\mu &= \partial_\mu - ig/\sqrt{2}\, A_\mu \\ A_\mu &\equiv A_\mu^i T^i \end{aligned} \qquad (1.2.46)$$

The peculiar normalization in the covariant derivative (1.2.46) is due to a non standard normalization of the (fundamental representation) SU(n) generators

$$tr(T_i T_j) = \delta_{ij} \qquad (1.2.47)$$

(They are usually defined with a ½ here).
so that in terms of the standard structure constants f_{ijk}, we have

$$[T_i, T_j] = i\sqrt{2}\, f_{ijk} T_k \qquad (1.2.48)$$

Hence the covariant derivative of (1.2.46) is the usual one when expressed in terms of structure constants.
Note that $\bar{\lambda}$, λ and D are in the adjoint representation with e.g.

$$\lambda_\alpha \equiv \lambda_\alpha^i T^i \qquad (1.2.49)$$

and, as will be true always, expressions containing operators for example

$$D_{\dot{\alpha}}{}^{\alpha}\lambda_{\alpha} \qquad (1.2.50)$$

must be interpreted as the appropriate commutator/anticommutator

$$[D_{\dot{\alpha}}{}^{\alpha}, \lambda_{\alpha}] \qquad (1.2.51)$$

$D \equiv D^{i}T^{i}$ is an "auxiliary" needed to maintain the fermi-bose count of the theory when off-shell. (It should not be confused with supersymmetric derivatives or the gauge covariant derivative !)

Note that (1.2.45) is only supersymmetric up to a gauge transformation; but since the gauge symmetry must be broken if we are to quantize the theory, the Wess-Zumino gauge action (1.2.45) can not preserve supersymmetry at the quantum level.

Manifest supersymmetry is preserved by working on the full action of (1.2.38) and choosing a supersymmetric supergauge fixing condition. Such a supergauge fixing condition is provided by

$$D^2 V = 0$$

and

$$\bar{D}^2 V = 0 \qquad (1.2.52)$$

The first equation in (1.2.52) serves to project out the chiral gauge part of v (i.e. Λ) and the second equation the antichiral gauge part of v (i.e. $\bar{\Lambda}$).

We insert into the path integral

$$1 = N \int \mathcal{D}(a,\bar{a})\ \delta[D^2V - \bar{a}]\ \delta[\bar{D}^2 v - a]\ J\ e^{-\frac{tr}{16\xi}\int d^8z\, a\bar{a}} \qquad (1.2.53)$$

where a and $\bar{a}$ are chiral and antichiral fields (since D^2v is antichiral). N is the normalisation of the 't Hooft gaussian average which here can be ignored (but in the next chapter it will be important). The gaussian average gives us the gauge fixing term

$$S_{GF} = -\frac{tr}{16\xi}\int d^8z\ D^2V\,\bar{D}^2V \qquad (1.2.54)$$

The Jacobian J is, as usual, the determinant of the differential of the gauge fixing term (1.2.52) under a gauge transformation (1.2.43) and is evaluated by a ghost action, consisting in this case of anticommuting superfields.

$$S_{gh} = tr\int d^8z\,(\bar{c}'-c')\,\tfrac{g}{2}L_V\,[(\bar{c}+c) + \coth(\tfrac{g}{2}L_V)\,(c-\bar{c})]$$

$$= tr\int d^8z\,\{\,\bar{c}'c + c'\bar{c} + \text{interactions with } v\,\} \qquad (1.2.55)$$

where c and c′ are chiral superfields, and $\bar{c}$ and $\bar{c}'$ are antichiral superfields.

We can write the superfield strength W^γ in terms of (super)gauge covariant supersymmetric derivatives

$$W^\gamma = \bar{\nabla}^2(\nabla^\gamma) \equiv [\bar{\nabla}^{\dot{\gamma}},\{\bar{\nabla}_{\dot{\gamma}},\nabla^\gamma\}] \qquad (1.2.56)$$

In (1.2.39) these covariant derivatives are in the covariantly chiral representation

$$W^\gamma \equiv W^{c\gamma}$$

$$\bar{\nabla}^c_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} \qquad (1.2.57)$$

$$\nabla^c_\alpha = e^{-gV} D_\alpha\, e^{gV}$$

They are gauge covariant since under a gauge transformation (1.2.42)

$$\nabla^c_\alpha \rightarrow e^{i\Lambda}\,\nabla^c_\alpha\, e^{-i\Lambda} \qquad (1.2.58)$$

$$\bar{\nabla}^c_{\dot{\alpha}} \rightarrow e^{i\Lambda}\,\bar{\nabla}^c_{\dot{\alpha}}\, e^{-i\Lambda} \quad (=\bar{\nabla}^c_{\dot{\alpha}} \text{ since } \bar{D}_{\dot{\alpha}}\Lambda = 0\,)$$

One can also define a spatial supergauge covariant derivative by analogy with (1.2.14)

$$\{\nabla^c_\alpha,\bar{\nabla}^c_{\dot{\alpha}}\} = 2i\,\nabla^c_{\alpha\dot{\alpha}} \qquad (1.2.59)$$

In the Wess-Zumino gauge (see below (1.2.44))

$$\nabla^c_{\alpha\dot{\alpha}}\big|_{\theta=\bar{\theta}=0} = D_{\alpha\dot{\alpha}}$$

i.e. the Yang-Mills covariant derivative (1.2.46).

(1.2.57) is not the only representation: there are two others. Firstly we can define a covariantly antichiral representation

$$\bar{\nabla}^{A}_{\dot{\alpha}} = e^{gV} \bar{D}_{\dot{\alpha}} e^{-gV}$$

$$\nabla^{A}_{\alpha} = D_{\alpha} \qquad (1.2.60)$$

$$\{ \bar{\nabla}^{A}_{\dot{\alpha}} , \nabla^{A}_{\alpha} \} = 2i \nabla^{A}_{\alpha\dot{\alpha}}$$

Under a supergauge transformation

$$\overset{(-)}{\nabla}{}^{A} \rightarrow e^{i\bar{\Lambda}} \overset{(-)}{\nabla}{}^{A} e^{-i\bar{\Lambda}} \qquad (1.2.61)$$

Note that

$$\overset{(-)}{\nabla}{}^{A} = e^{gV} \overset{(-)}{\nabla}{}^{C} e^{-gV} \qquad (1.2.62)$$

so that from (1.2.56) $\quad W^{A\gamma} = e^{gV} W^{C\gamma} e^{-gV} \qquad (1.2.63)$

and so (using the cyclic properties of the trace) the action in (1.2.38) is independent of the representation. Note that in these representations the covariant derivatives are not hermitian in the sense of (1.2.9). In fact Hermitian conjugation maps between the two representations (e.g.

$$(\nabla^{C}{}_{\alpha})^{\dagger} = \bar{\nabla}^{A}_{\dot{\alpha}} \qquad).$$

We can however construct a hermitian representation - the vector representation. To achieve this we must split the prepotential v into two new prepotentials ω and $\bar{\omega}$. (ω is a general complex superfield).

$$e^{gV} = e^{g\omega} e^{g\bar{\omega}} \qquad (1.2.64)$$

The covariant derivatives are defined by

$$\nabla_{\alpha} = e^{-g\omega} D_{\alpha} e^{g\omega}$$

$$\bar{\nabla}_{\dot{\alpha}} = e^{g\bar{\omega}} \bar{D}_{\dot{\alpha}} e^{-g\bar{\omega}} \qquad (1.2.65)$$

$$\{ \nabla_{\alpha}, \bar{\nabla}_{\dot{\alpha}} \} = 2i \nabla_{\alpha\dot{\alpha}}$$

They are related to the other representations by a similarity transformation e.g.

$$\overset{(-)}{\nabla} = e^{g\bar{\omega}} \overset{(-)}{\nabla}{}^{C} e^{-g\bar{\omega}} \qquad (1.2.66)$$

so that the action (1.2.38) is the same in this representation also (compare comments about (1.2.62), (1.2.63)).

Making the splitting (1.2.64) has increased the gauge invariance of the theory (corresponding to the arbritrariness of definition of ω and $\bar{\omega}$ given by (1.2.64)).

The action is invariant under

$$e^{g\omega} \rightarrow e^{i\bar{\Lambda}} e^{g\omega} e^{-iK}$$
$$e^{g\bar{\omega}} \rightarrow e^{iK} e^{g\bar{\omega}} e^{-i\Lambda} \qquad (1.2.67)$$

where K is a real superfield $K \equiv K^i T^i$.

The covariant derivatives (1.2.65) transform homogeneously under (1.2.67) as

$$\overset{\leftrightarrow}{\nabla} \rightarrow e^{iK} \overset{\leftrightarrow}{\nabla} e^{-iK} \qquad (1.2.68)$$

The covariant derivatives ((1.2.57), (1.2.60) or (1.2.65)) can be used to define the component fields of v in a gauge covariant way (unlike those of (1.2.20)). With these definitions the component action turns out to be (1.2.45) in any gauge (not just the Wess-Zumino gauge). These covariant components are given in chapter 3.2 ; They are highly non-linear redefinitions of the component fields in (1.2.20).

Note that equation (1.2.41) is true for the chiral representation (1.2.56). It can also be written

$$\bar{\nabla}_{\dot{\alpha}} W^{\gamma} = 0 \qquad (1.2.69)$$

Since the different representations are connected by similarity transformations (1.2.69) is representation independent. It also follows directly, in an arbitrary representation, from the definition of W_{α} (1.2.56), the covariant derivatives and (1.2.40).

Fields that satisfy (1.2.69) are known as covariantly chiral. i.e. covariantly chiral (ζ) and covariantly antichiral fields ($\bar{\zeta}$) satisfy

$$\bar{\nabla}_{\dot{\alpha}} \zeta = 0$$

and

$$\nabla_{\alpha} \bar{\zeta} = 0 \qquad (1.2.70)$$

W_{α} also satisfies a Bianchi identity (proved from (1.2.56))

$$\nabla^{\alpha} W_{\alpha} = \bar{\nabla}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \qquad (1.2.71)$$

with definition $$\bar{W}_{\dot{\alpha}} = \nabla^2 (\bar{\nabla}_{\dot{\alpha}}) \tag{1.2.72}$$

Note that with this definition [5] $\bar{W}^{\dot{\alpha}} = - (W^{\alpha})^*$ (compare (1.2.1)).

The equations of motion are

$$\nabla^{\alpha} W_{\alpha} = 0 \tag{1.2.73}$$

Finally note that since we shall be dealing almost exclusively with covariantly chiral/antichiral representations ((1.2.57) and (1.2.60)) and covariantly chiral/antichiral fields (1.2.70) we shall henceforth use the term chiral/antichiral to mean covariantly chiral/antichiral unless otherwise stated.

1.3 THE BACKGROUND SUPERFIELD METHOD.

The background superfield method is set up in an analogous way to the Yang-Mills case described in the first chapter. Complications arise because the highly non-linear nature of the super Yang-Mills action ((1.2.38) and (1.2.39)) and the gauge transformation (1.2.42) require a non-linear splitting of the full field into background field and quantum field.

It is important in the background field method to have the quantum field transforming homogeneously under background gauge transformations (for example (1.1.11)) since it is (more or less) an essential step in proving the gauge invariance of the background action (see comments below (1.1.13)). It is convenient to keep the (covariantly) chiral representation for the quantum field rather than the vector representation since, as we have seen (1.2.66), the vector representation leads to a further gauge invariance which would have to be broken by further gauge fixing terms. It is essential (as it turns out [6]) to be in the vector representation for the background fieldif we wish to have the background field appearing only in the covariant derivatives (as we did in chapter 1.1). This final requirement proves to be very useful in improving methods of supergraph calculation of the background effective action ([4], see also chapter 4).

With the above requirements imposed there is a unique decomposition:

$$e^{gV_f} = e^{g\omega} e^{gv} e^{g\bar{\omega}} \tag{1.3.1}$$

where V_f is the full prepotential, ω and $\bar{\omega}$ contain only the background, and v is the quantum field.
The full field gauge invariance (1.2.42) is

$$e^{gV_f} \rightarrow e^{i\bar{\Lambda}'} e^{gV_f} e^{-i\Lambda'} \tag{1.3.2}$$

(where we have added the primes on the gauge fields which are ordinary chiral/antichiral (1.2.21/28) to distinguish them from covariantly chiral/antichiral gauge fields in (1.3.4)).

(1.3.2) can be expressed on the component fields in (1.3.1) in a number of ways, of which two important ones are

(1)

$$e^{g\omega}\ , e^{g\bar{\omega}} \qquad \text{unchanged}$$
$$e^{gV} \rightarrow e^{i\bar{\Lambda}}\, e^{gV}\, e^{-i\Lambda} \tag{1.3.3}$$

$$\bar{\Lambda} = e^{-g\omega}\, \bar{\Lambda}'\, e^{g\omega}$$
$$\Lambda = e^{g\bar{\omega}}\, \Lambda'\, e^{-g\bar{\omega}} \tag{1.3.4}$$

These gauge fields Λ, $\bar{\Lambda}$ are background covariantly chiral and background covariantly antichiral fields (see (1.2.64)).

(1) is an expression of the quantum field gauge invariance which must be broken (compare (1.1.10)).

(2)

$$e^{g\omega} \rightarrow e^{i\bar{\Lambda}'}\, e^{g\omega}\, e^{-iK}$$
$$e^{g\bar{\omega}} \rightarrow e^{iK}\, e^{g\bar{\omega}}\, e^{-i\Lambda'}$$
$$e^{gV} \rightarrow e^{iK}\, e^{gV}\, e^{-iK} \tag{1.3.5}$$

Here the background field gauges (see (1.2.66)) whereas the quantum field transforms homogeneously (as required above. Compare this with (1.1.11)).

By substituting (1.3.1) into (1.2.38) we can rewrite the action as

$$S = \frac{tr}{128 g^2} \int d^4x\, d^2\theta\, W_f^{\gamma}\, W_{f\,\gamma}$$
$$W_f^{\gamma} = \bar{\nabla}^2 \left(e^{-gV}\, \nabla^{\gamma}\, e^{gV} \right) \tag{1.3.6}$$

where $\overset{(-)}{\nabla}$ are covariant derivatives containing only the background field (i.e. as in (1.2.65)). To quantize the theory we, in close analogy with chapter 1.1, break the quantum gauge invariance (1.3.3) but keep the background gauge invariance (1.3.5) which we do by choosing the gauge

$$\nabla^2 v = 0$$
$$\bar{\nabla}^2 v = 0 \qquad (1.3.7)$$

This is background gauge invariant (from (1.3.5) and (1.2.67)) and projects out the (covariantly) chiral and antichiral gauge fields (1.3.4), (compare (1.2.53)).

The gauge fixing procedure follows in exact analogy with (1.2.53): we insert

$$1 = N \int \mathcal{D}(a,\bar{a})\ \delta[\nabla^2 v - \bar{a}]\ \delta[\bar{\nabla}^2 v - a]\ J\ e^{-\frac{tr}{16\xi}\int d^8z\, a\bar{a}} \qquad (1.3.8)$$

where a and $\bar{a}$ are background covariantly chiral/antichiral fields (since $\nabla^2 v$ is background covariantly antichiral).

The 't Hooft gaussian average gives the background gauge covariant fixing term

$$S_{GF} = \frac{-tr}{16\xi}\int d^8z\ \nabla^2 v\ \bar{\nabla}^2 v \qquad (1.3.9)$$

The Jacobian J is evaluated by ghost fields

$$S_{gh} = tr \int d^8z\ (\bar{c}' - c')\tfrac{g}{2}L_v[(\bar{c}+c) + \coth(\tfrac{g}{2}L_v)(c-\bar{c})]$$
$$= tr\int d^8z\ \{\bar{c}'c + c'\bar{c} + \text{interactions with } v\} \qquad (1.3.10)$$

which is exactly the same form as before (since the gauge transformation (1.3.3) is the same form as (1.2.42) and (1.2.43)). However in this case these ghosts (corresponding to the fields (1.3.4)) are background covariantly chiral/antichiral.

The normalisation factor N is now

$$N = \left\{\int \mathcal{D}(a,\bar{a})\ e^{-tr\int d^8z\, a\bar{a}}\right\}^{-1} \qquad (1.3.11)$$

(ignoring factors of $(1/16\xi)$ for the moment; we will not in chapter 4) and is not trivial because a and a , being background chiral/antichiral, contain interactions with the background field (see (1.3.4) and comments below). We can rewrite (1.3.11) by using <u>anticommuting</u> background chiral/antichiral fields η , $\bar{\eta}$ so

$$N = \int \mathcal{D}(\eta, \bar{\eta}) \; e^{-tr\int d^8z \; \eta\bar{\eta}} \tag{1.3.12}$$

These are a new pair of ghosts that only interact with the background field (and therefore occur only at one loop). They are known as Nielsen-Kallosh ghosts.

This completes the gauge fixing procedure. Let us finish this chapter by stating the form of S (1.3.6) expanded to order v^2 :

$$S = \frac{tr}{128g^2}\int d^4x \; d^2\theta \; W^\gamma W_\gamma \; +$$

$$+ \frac{tr}{32}\int d^8z \left\{ -\tfrac{2}{g} \, \nabla^\alpha W_\alpha v + v\left(\nabla^\alpha \bar{\nabla}^2 \nabla_\alpha - W^\alpha \nabla_\alpha\right) v \right\} \tag{1.3.13}$$

(where W_α contains only the background field).
Note that the quadratic operator is in fact hermitian: using a little algebra it can be shown that

$$\nabla^\alpha \bar{\nabla}^2 \nabla_\alpha \; - W^\alpha \nabla_\alpha \; - \tfrac{1}{2}\left(\nabla^\alpha W_\alpha\right)$$

$$= \; -4 \, \nabla^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \; + \tfrac{1}{2}\left(\nabla^2 \bar{\nabla}^2 + \bar{\nabla}^2 \nabla^2\right) - W^\alpha \nabla_\alpha \; + \bar{W}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}} \tag{1.3.14}$$

$$= \; \bar{\nabla}^{\dot{\alpha}} \nabla^2 \bar{\nabla}_{\dot{\alpha}} \; - \bar{W}^{\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}} \; - \tfrac{1}{2}\left(\bar{\nabla}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}\right)$$

(where we have included $\frac{1}{2}\nabla^\alpha W_\alpha$ in the above, with impunity, since

$$tr\int d^8z \;\; v\left(\nabla^\alpha W_\alpha\right) v$$

means

$$tr\int d^8z \;\; v\left[\nabla^\alpha W_\alpha \, , v\right]$$

which is $\propto \int d^8z \; f^{ijk} \; v^i \left(\nabla^\alpha W_\alpha\right)^j v^k \; = 0$).

The linear term in (1.3.13) verifies that (1.2.73) is the equations of motion for super Yang-Mills.

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CHAPTER 2 INSTANTONS.

2.1 EUCLIDEAN FIELD THEORY AND INSTANTONS.

We shall be interested, in this thesis, in a specific solution of the Euclidean Yang-Mills equations of motion known as a B.P.S.T. instanton [1]. This solution has important physical implications: it is related to quantum tunnelling between gauge equivalent (but homotopically gauge inequivalent) vacua, and it breaks chirality. This led, for example, to a solution of the U(1) problem by generation of an effective fermion interaction [2] and to the concept of θ-vacua.

We will not be concerned with these applications however but will instead be interested in using the instanton as an example of a non-trivial stationary point of the action about which we can expand and calculate perturbative quantum corrections.

The Euclidean action for SU(n) Yang-Mills theory is given by

$$S = -\frac{1}{4}\int d^4x \; F_{\mu\nu}^2 \tag{2.1.1}$$

where $x_\mu = (x_1, x_2, x_3, x_4)$

the metric is $g_{\mu\nu} = \delta_{\mu\nu}$

$F_{\mu\nu}$ is the field strength with

$$-ig\,F_{\mu\nu} = [D_\mu, D_\nu] \tag{2.1.2}$$

and D_μ is the covariant derivative

$$D_\mu = \partial_\mu - ig\,A_\mu \tag{2.1.3}$$

The fields ($F_{\mu\nu}$ and the potential A_μ) take values in the Lie algebra of SU(n). Note that a trace over the generators (divided by their norm) is always to be understood to have been taken in equations involving integrals (such as (2.1.1)).

Consider first the case of the group SU(2).

If we require finite action then it is clear that the field strength ($F_{\mu\nu}$) must tend to zero faster than $1/x^2$. This however does not imply that A_μ must decrease faster than $1/x$, since $F_{\mu\nu}$ will vanish if A_μ tends to a gauge transformation of zero. i.e.

$$A_\mu \to \frac{i}{g} U \partial_\mu U^\dagger \qquad (2.1.4)$$

$U(x \to \infty) \in SU(2)$ depends only on angles in Euclidean space.

Hence A_μ can have tangential components that fall off like $1/x$. If we gauge transform (2.1.4) by $S(x) \in SU(2)$ then

$$A_\mu \to \frac{i}{g} S^\dagger (U \partial_\mu U^\dagger) S + \frac{i}{g} S^\dagger \partial_\mu S$$

$$= \frac{i}{g} (U^\dagger S)^\dagger \partial_\mu (U^\dagger S)$$

Hence it would appear that if we choose S such that $S \to U$ as $x \to \infty$ we would be able to gauge transform away the $O(1/x)$ terms from A_μ. However this argument is only correct if the matrix $S(x)$ does not have any singularities at any value of x. Otherwise the problem of the behaviour of $A_\mu(x)$ is merely transferred from infinity to the position of the singularity in $S(x)$.

As a result, the problem of classifying the fields A_μ which give finite action becomes a problem of classifying the homotopy classes of the gauge transformations U. These transformations are mappings from the sphere at infinity in Euclidean space (S_3) onto the group manifold which, for SU(2), is also S_3 .

But the homotopy group of these maps is

$\Pi_3(S_3) = \mathbb{Z}$ (the integers)

Hence each class is labelled by an integer "N" which is known as the Pontryagin index or winding number.
The index "N" measures the number of times the map U "wraps around" the group manifold, and is negative if the map reverses the orientation of the space. It can be calculated from the volume covered by the map measured in units of the volume of the sphere

$$N = \frac{1}{48\pi^2} \int_{S_3} \varepsilon^{ijk} (u\partial_i u^{+})(u\partial_j u^{+})(u\partial_k u^{+})$$

We can rewrite this equation in terms of A_μ on a sphere at $x = R$ ($R \to \infty$), by using (2.1.4) and then turn it into a manifestly gauge invariant integral representation

$$N = \frac{g^2}{32\pi^2} \int d^4x \; F_{\mu\nu} \tilde{F}_{\mu\nu} \qquad (2.1.5)$$

where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\delta} F_{\sigma\delta} \qquad (\varepsilon_{1234} = 1) \qquad (2.1.6)$$

Although (2.1.8) is written as an integral over all space it depends only on the behaviour of A_μ at the sphere at ∞ because $F_{\mu\nu} \tilde{F}_{\mu\nu}$ can be rewritten as a total derivative.

We can rewrite the action

$$|S| = \frac{1}{4}\int d^4x\, F_{\mu\nu}^2 = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu}\tilde{F}_{\mu\nu} + \frac{1}{8}(F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 \right\}$$

$$= N\frac{8\pi^2}{g^2} + \frac{1}{8}\int d^4x\, (F_{\mu\nu} - \tilde{F}_{\mu\nu})^2 \qquad (2.1.7)$$

From this it is clear that fields A_μ that have Pontryagin index $N > 0$ and are (locally) a minimum of the action must have

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \qquad (2.1.8)$$

Field strengths satisfying (2.1.8) are known as self dual fields. Note then, that these solutions are stationary points of the action under local variations, that is to say, solutions of the field equations

$$D_\mu F_{\mu\nu} = 0 \qquad (2.1.9)$$

(note that here as elsewhere the action of the covariant derivative is to taken to be a commutator

$$D_\mu F_{\mu\nu} \equiv [D_\mu, F_{\mu\nu}]$$

In fact it is readily verified that (2.1.8) implies (2.1.9) and hence we have reduced the search for solutions from a second order differential equation (2.1.9) to a first order one (2.1.8).

When the Pontryagin index is negative we rewrite (2.1.7) so as to obtain

$$|S| = -N\frac{8\pi^2}{g^2} + \frac{1}{8}\int d^4x\, (F_{\mu\nu} + \tilde{F}_{\mu\nu})^2$$

which implies that local minima of the action (for $N < 0$) satisfy

$$F_{\mu\nu} = -\tilde{F}_{\mu\nu} \qquad (2.1.10)$$

i.e. are anti-self dual fields.

For the case N=1 the solution (together with a set of variable parameters) is unique [4] and is the B.P.S.T. instanton.

As we shall see, the solution in the case N = -1 is very similar; it is known as an anti-instanton. In fact it is clear already that the anti-instanton solution is obtained from the instanton solution merely by applying a parity transformation (e.g. $x_4 \to -x_4$) : this alters the sign of $\varepsilon_{\mu\nu\sigma\delta}$ (an axial tensor) and turns self dual tensors into anti self dual tensors.

In Euclidean 4-space the group of rotations is locally isomorphic to SU(2)xSU(2) (its covering group) i.e.

$$SO(4) \cong SU_L(2) \times SU_R(2) \tag{2.1.11}$$

("L" and "R" stand for left handed and right handed and merely serve to label the different groups). And (the 6 linearly independent) antisymmetric tensors, which transform under the adjoint of SO(4), can be expressed as a sum of self dual and anti self dual tensors :

$$\begin{aligned} F_{\mu\nu} &= F^+_{\mu\nu} + F^-_{\mu\nu} \\ F^+_{\mu\nu} &= \tfrac{1}{2}\left(F_{\mu\nu} + \tilde{F}_{\mu\nu} \right) \\ F^-_{\mu\nu} &= \tfrac{1}{2}\left(F_{\mu\nu} - \tilde{F}_{\mu\nu} \right) \end{aligned} \tag{2.1.12}$$

The "+" stands for self dual and the "-" for anti self dual.

These tensors ($F^+_{\mu\nu}$ and $F^-_{\mu\nu}$) have 3 linearly independent components and transform as adjoints under $SU_L(2)$ and $SU_R(2)$ respectively. The maps that interpolate between the tensors and adjoint vectors of the SU(2) groups are the 't Hooft symbols [2].

$$\begin{aligned} F^+_{\mu\nu} &= \eta^a{}_{\mu\nu}\, \xi^a_L \\ F^-_{\mu\nu} &= \bar{\eta}^a{}_{\mu\nu}\, \xi^a_R \end{aligned} \tag{2.1.13}$$

A representation of these is given by the following

$$\eta^a{}_{\mu\nu} = \varepsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4} \qquad (2.1.14)$$

$\bar{\eta}^a{}_{\mu\nu}$ is the parity converse ($x_4 \to -x_4$) ;

$$\bar{\eta}^a{}_{\mu\nu} = \varepsilon_{a\mu\nu} - \delta_{a\mu}\delta_{\nu 4} + \delta_{a\nu}\delta_{\mu 4} \qquad (2.1.15)$$

(note that $\varepsilon_{a\mu\nu}$ is the alternating symbol in 3 dimensions, $\varepsilon_{123} = 1$, $\varepsilon_{a\mu\nu} = 0$ if μ or $\nu = 4$).

These symbols satisfy certain properties which we list below.

$$\eta^a{}_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\sigma\delta}\,\eta^a{}_{\sigma\delta}$$

$$\eta^a_{\mu\nu} = -\eta^a_{\nu\mu}$$

$$\eta_{a\mu\nu}\eta_{b\mu\nu} = 4\delta_{ab}$$

$$\eta_{a\mu\nu}\eta_{a\mu\lambda} = 3\delta_{\nu\lambda}$$

$$\eta_{a\mu\nu}\eta_{a\mu\nu} = 12$$

$$\eta_{a\mu\nu}\eta_{a\gamma\lambda} = \delta_{\mu\gamma}\delta_{\nu\lambda} - \delta_{\mu\lambda}\delta_{\nu\gamma} + \varepsilon_{\mu\nu\gamma\lambda}$$

$$\varepsilon_{\mu\nu\lambda\sigma}\eta_{a\gamma\sigma} = \delta_{\gamma\mu}\eta_{a\nu\lambda} + \delta_{\gamma\nu}\eta_{a\lambda\mu} + \delta_{\gamma\lambda}\eta_{a\mu\nu}$$

$$\eta_{a\mu\nu}\eta_{b\mu\lambda} = \delta_{ab}\delta_{\nu\lambda} + \varepsilon_{abc}\eta_{c\nu\lambda}$$

$$\varepsilon_{abc}\eta_{b\mu\nu}\eta_{c\gamma\lambda} = \delta_{\mu\gamma}\eta_{a\nu\lambda} + \delta_{\nu\lambda}\eta_{a\mu\gamma} - \delta_{\mu\lambda}\eta_{a\nu\gamma} - \delta_{\nu\gamma}\eta_{a\mu\lambda}$$

$$\eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} = 0$$

$$\eta_{a\gamma\mu}\bar{\eta}_{b\gamma\lambda} = \eta_{a\gamma\lambda}\bar{\eta}_{b\gamma\mu} \qquad (2.1.18)$$

Needless to say these relations can be derived from the antisymmetry and self duality of $\eta^a_{\mu\nu}$, the symmetries of the ε symbols and the completeness relations which follow from the fact that the map is 1-1. (Note that the value of $\eta^a_{\mu\nu}\eta^a_{\mu\nu}$ is ofcourse representation dependent).

The relations for $\bar{\eta}$ follow from a parity transformation which simply changes the sign of all terms proportional to $\varepsilon_{\mu\nu\sigma\delta}$.

In view of the topological link between the gauge group SU(2) and angles in Euclidean space it should come as no surprise to find that the instanton is constructed through these 't Hooft symbols. Writing

$$\eta_{\mu\nu} = T^a \eta^a_{\mu\nu}$$

(T^a are generators of the gauge group SU(2))
the instanton solution is

$$A_\mu = 2/g \; \eta_{\mu\nu} x_\nu / 1+x^2$$

$$F_{\mu\nu} = -4/g \; \eta_{\mu\nu} / (1+x^2)^2$$

This is not the most general solution however.
That can be obtained by applying the generators of symmetries of the theory which are broken by this solution. Note that the SO(4) rotation group is, in a sense, not broken by this solution. This is because the solution does not transform under $SU_R(2)$ (see (2.1.13)) and the transformation under $SU_L(2)$ can be compensated for by applying the inverse transformation in the internal group SU(2). Also special conformal transformations can be undone by a gauge transformation [5]. This leaves general gauge transformations (Note that A_μ as written is in the Lorentz gauge $\partial_\mu A_\mu = 0$), translations (which move the center of the instanton $x \to x - a$) and dilatations (which change the "size" of the instanton to "ρ")

Therefore a general instanton is given by

$$A_\mu = e^{i\Omega(x)} A^I_\mu(x) e^{-i\Omega(x)} + i\sqrt{2}/g \; e^{i\Omega(x)} \partial_\mu e^{-i\Omega(x)} \qquad (2.1.17)$$

with

$$A^I_\mu = 2/g \; \eta_{\mu\nu} \frac{(x-a)^\nu}{(x-a)^2 + \rho^2} \qquad (2.1.18)$$

the general Lorentz gauge instanton and $\Omega(x)$ a general gauge function (taking values in the Lie algebra of SU(2)).

So far we have mentioned only the case where the gauge group is SU(2); What about SU(n) ?

In fact $\Pi_3(SU(n)) = \mathbb{Z}$ also and the instantons corresponding to the 1st Pontryagin class are simply those of (2.1.18) embedded in an SU(2) sub algebra, plus a general gauge transformation (2.1.17) ($\Omega(x)$ in the Lie algebra of SU(n)).

(There exists a more general statement due to Raoul Bott [6], which is that $\Pi_3 = \mathbb{Z}$ for any simple Lie group containing SU(2) as a subgroup.)

Finally note that (2.1.18) and (2.1.17) imply that the field strength is

$$F_{\mu\nu} = e^{i\Omega} F^I_{\mu\nu} e^{-i\Omega}$$

$$F^I_{\mu\nu} = -\frac{4}{g} \eta_{\mu\nu} \frac{\rho^2}{[(x-a)^2 + \rho^2]^2} \qquad (2.1.19)$$

and that the anti-instanton solution is obtained from these solutions by replacing $\eta_{\mu\nu}$ by $\bar{\eta}_{\mu\nu}$.

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2.2 ZERO MODES AND COLLECTIVE COORDINATES.

The next two sections will be devoted to a brief description of the treatment of zero modes and the calculation of quantum corrections in Yang-Mills (see ref.[1]). They are intended to set the stage for the more complicated application described in detail in chapters 3 and 4.

In chapter 1 (1.1) we discussed a particular gauge fixing term useful in evaluating Z_A (see (1.1.5) and (1.1.12)). There is an intuitively attractive idea due to Amati and Rouet [2], in which this gauge fixing term arises very naturally. The idea is to treat the infinite set of gauge degrees of freedom as a set of zero modes.

Zero modes are functions Q_ν for which

$$O_{\mu\nu} Q_\nu = 0 \tag{2.2.1}$$

Where $O_{\mu\nu}$ is the operator in the action S(A+Q) expanded up to quadratic terms in Q ; The quadratic part is

$$\delta S = \frac{1}{2} \int d^4x \; Q_\mu O_{\mu\nu} Q_\nu \tag{2.2.2}$$

$$O_{\mu\nu} = D^2 \delta_{\mu\nu} - D_\mu D_\nu - 2ig \, F_{\mu\nu} \tag{2.2.3}$$

Suppose we are dealing with a background field which is a solution of the equations of motion i.e.

$$\left. \frac{\delta S}{\delta A^f} \right|_{A^f = A} = 0 \qquad\qquad S \equiv S(A^f)$$

It follows that a change to another solution

$$A \to A + \delta A \tag{2.2.4}$$

implies

$$\left. \frac{\delta^2 S}{\delta A^f_x \, \delta A^f_y} \right|_{A^f = A} = 0$$

(suppressing integrals and indices).

But $\left.\frac{\delta^2 S}{\delta A^f_x \, \delta A^f_y}\right|_{A^f = A}$ is just the operator "$O_{\mu\nu}$" of (2.2.3).
Hence if Q is proportional to δA , Q is a zero mode.
An example of "δA" is provided by an infinitessimal change of gauge:

$$A_\mu \rightarrow e^{i\delta\Omega} A_\mu e^{-i\delta\Omega} + \tfrac{i}{g} e^{i\delta\Omega} \partial_\mu e^{-i\delta\Omega}$$

$$\Rightarrow \quad \delta A_\mu = i [\delta\Omega, A_\mu] + \tfrac{1}{g} \partial_\mu \delta\Omega$$

$$= \tfrac{1}{g} D_\mu \delta\Omega \qquad (2.2.5)$$

Since $\delta\Omega$ is arbitrary this leads to an infinite set of zero modes

$$Q_\mu(x) = \zeta^y_\mu(x) = D^x_\mu \, \delta(x-y) \qquad (2.2.6)$$

(There is also a colour index involved here - which we are suppressing).
If Q_μ is allowed to take values in the subspace defined by (2.2.6) (gauge zero mode space) the operator in the quadratic action (2.2.3) will not be able to be inverted to form the propagator; We will not be able to do perturbation theory.
We can remedy this problem by noting that the values Q_μ in (2.2.6) correspond to changes in the collective coordinates that parameterise the general gauge solution for A_μ . Hence if we could constrain Q_μ to remain "orthogonal" to the space defined by (2.2.6) and transfer the integration over this space to an integral over the A_μ collective coordinates we would have solved the problem. This we can do by applying the standard Fadeev-Popov trick and substituting

$$1 = \int \mathcal{D}(c.c) \prod_y \delta[(\zeta^y_\mu, Q_\mu)] \left| \frac{\partial}{\partial(c.c)} (\zeta^y_\mu, Q_\mu) \right| \qquad (2.2.7)$$

into the generating functional (1.1.5).
In this equation "c.c" stands for collective coordinates.

$$(\xi^y_\mu \, , Q_\mu) \quad \text{is} \quad \int d^4x \; \xi^y_\mu \, Q_\mu(x) = (D_\mu Q_\mu)(y) \tag{2.2.8}$$

The δ-functions in (2.2.7) constrain this to be zero, i.e. they force Q_μ to be orthogonal to $\{\xi^y_\mu(x)\}$ the zero mode space. From (2.2.8) we see that this is equivalent to the choice of background gauge (1.1.12) with $\xi \to 0$ (Landau gauge). We could have instead taken

$$\delta[\,(\xi^y_\mu, Q_\mu) - f(y)\,]$$

in which case a 't Hooft Gaussian average would lead to the general background gauge.

The last term in (2.2.7) is the Jacobian of the change of integration variable from Q_μ (in zero mode space) to collective coordinates. This is straightforward to evaluate if we use a system of collective coordinates such that the differential of A_μ with respect to one of them gives a term proportional to one of the zero modes

$$\frac{\partial A(x)}{\partial(c \cdot c^y)} \sim \xi^y(x) \tag{2.2.9}$$

Writing the matrix in (2.2.7) as

$$\left(\frac{\partial \xi^y_\mu}{\partial(c \cdot c)}, Q_\mu\right) + \left(\xi^y_\mu, \frac{\partial Q_\mu}{\partial(c \cdot c)}\right) \tag{2.2.10}$$

we use (2.2.9) directly on the first term (ξ^y_μ is a function of A) and in the second term we note that the full field A^f

($A^f = A + Q$ .... see (1.1.4))

does not depend on the background field parameters and hence

$$\frac{\partial Q}{\partial(c \cdot c)} = -\frac{\partial A}{\partial(c \cdot c)} \tag{2.2.11}$$

Finally we evaluate the determinant of (2.2.10) by introducing anticommuting integration variables and using the standard formula

$$\mathrm{Det}\, M = \int \mathcal{D}(\bar{\phi}, \phi)\; e^{\bar{\phi}^i M_i{}^j \phi_j} \tag{2.2.12}$$

$$\mathcal{D}(\bar{\phi}, \phi) \equiv \prod_j d\phi_j \, d\bar{\phi}_j$$

What we obtain from this process is, needless to say, the background gauge ghost action (1.1.13).

The procedure we have outlined above generalises in a very natural way when we apply it to instantons [3]. Differences arise simply because the operator (2.2.3) has more zero modes than just the gauge modes we have described.

This larger zero mode space is fixed out of the Q integration exactly as above. Before going on to outline the result let us pause for a moment to describe the instanton zero modes.

Five of the extra zero modes arise because δA (of (2.2.4)) can correspond to changes in the size and position of the instanton (2.1.18):

Applying the translation operator (contracted with small parameters δa^ν) we get

$$\delta A_\mu = \delta a^\nu \partial_\nu A_\mu \tag{2.2.13}$$

and if we add to this change a particular small gauge transformation (see (2.2.5))

$$\delta A_\mu = D_\mu (- \delta a^\nu A_\nu)$$

we "covariantize" (2.2.13) to

$$\delta A_\mu = \delta a^\nu F_{\mu\nu} \tag{2.2.14}$$

Similarly application of the dilatation operator

$$\delta A_\mu = \delta \rho \; (1 + x_\nu \partial_\nu) A_\mu$$

can be covariantized to

$$\delta A_\mu = \delta\rho \, F_{\mu\nu} x_\nu \tag{2.2.15}$$

These are not all the zero modes however (a point apparently missed in [3]).

Recall that the gauge zero modes when fixed out of the functional integral led to the gauge fixing term (1.1.12) so that the operator in the quadratic action becomes

$$O_{\mu\nu} - \frac{1}{\xi} D_\mu D_\nu \tag{2.2.16}$$

The non-gauge zero modes (2.2.14) and (2.2.15) are still zero modes of (2.2.16) but a general gauge mode (see (2.2.5))

$$Q_\nu = D_\nu \Omega \tag{2.2.17}$$

is of course no longer a zero mode: This was the raison d'être for the gauge fixing term.

However a mode such as (2.2.17) will not be gauge fixed (i.e. will remain a zero mode) if

$$D^2 \Omega = 0 \tag{2.2.18}$$

And there are 4n-5 "anomalous" [4] modes Ω_K for which

$$D^2 \Omega_K = 0 \tag{2.2.19}$$

but for which $Q_\nu = D_\nu \Omega_K$ (2.2.20)

is square integrable (so that these modes are included in the Q-functional measure).

Hence (2.2.20) gives us 4n-5 extra zero modes.

The Ω_K are not square integrable, nor can we integrate by parts (without producing surface terms) [1] and it is for this reason they were missed in the previous discussion.

They are indirectly connected to the 4n-5 global SU(n) rotations broken by the instanton (which fact can be seen by going to a normal gauge

$$\partial_\mu Q_\mu = 0$$

so that (2.2.19) is replaced by

$$\partial^2 \Omega_K = 0$$

and is satisfied by the aforementioned global rotations).

In the case of n=2 these modes have the explicit form [4] (for the special instanton, see above (1.1.17))

$$\Omega_K = \bar{\eta}^K{}_{\mu\nu}\, \eta^a_{\mu\sigma}\, T^a \frac{x_\nu x_\sigma}{1+x^2}$$

$$Q_\mu = \xi_\mu = D_\mu \Omega_K \propto \bar{\eta}^K{}_{\nu\sigma}\, x_\nu F_{\sigma\mu}$$

In the n=2 case these modes can be interpreted as arising from covariantized $SU_R(2)$ transformations (see chapter 3.2).

Finally let us mention that it proves convenient to expand the gauge modes in a different basis set from (2.2.6), using positive frequency eigenmodes of D^2

$$D^2 \Omega_\lambda = -\lambda^2 \Omega_\lambda \qquad \lambda > 0$$

$$Q_\mu = \xi^\lambda_\mu = D_\mu \Omega_\lambda \qquad (2.2.21)$$

(as in (2.2.6) we are suppressing colour indices which label degenerate modes).
With such a choice the 4n discrete modes (2.2.14), (2.2.15), (2.2.20) and these continuous modes (2.2.21) vary homogeneously under background gauge transformations (1.1.11).
The zero modes are mutually orthogonal and can be normalised :-

$$(\xi^k_\mu , \xi^l_\mu) = \delta^{kl} \qquad k,l = 1, \cdots, 4n$$

$$(\Omega_\lambda , \Omega_{\lambda'}) = \delta(\lambda - \lambda') \qquad (2.2.22)$$

$$(D_\mu \Omega_\lambda , \xi^k_\mu) = -(\Omega_\lambda , D_\mu \xi^k_\mu) = 0$$

Now let us return to our discussion about replacing the Q integration over the zero mode space by an integral over collective coordinates:

Once again this is achieved by using the Fadeev-Popov trick (2.2.7) where now the collective coordinates and δ-functions range over the continuous gauge modes (which are 't Hooft averaged) and the 4n discrete modes (which are not). The evaluation of the Jacobian follows in a similar manner to before.

In analogy to (2.2.9) we choose a convenient system of collective coordinates (a^λ, b^k) such that for small changes

$$\delta A_\mu = \frac{1}{g}\left\{ \int_\lambda \delta a^\lambda D_\mu \Omega_\lambda + \sum_{k=1}^{4n} \delta b^k \xi_\mu^k \right\} \qquad (2.2.23)$$

The collective coordinates do not contain g's nor do the zero modes (otherwise they would appear in the normalisation conditions (2.2.22)) but A_μ does ($A_\mu \sim 1/g$; see(2.1.18)), and this explains the factor of 1/g in (2.2.23).

(Note that this parameterization of the instanton is not at all the one given in (2.1.17) and (2.1.18)).

The Jacobian determinant is evaluated by introducing anticommuting integration variables as in (2.2.12), only this time there are also 4n discrete variables $\hat{c}^i$ (and $\bar{\hat{c}}^i$) for the 4n discrete modes.

As before the continuous parameter integration variables become the ghosts and the Jacobian yields the ghost action. However the off-diagonal terms in the Jacobian (in the discrete ⊗ continuous space sector) yield new interactions between the discrete zero mode "ghost"

$$\hat{c} = \sum_{i=1}^{4n} c^i \xi_i \qquad (2.2.24)$$

the ghosts ϕ, $\bar{\phi}$ and the vector field Q_μ . These are illustrated in fig.2.1 at the end of this section. (The stub lines represent these non-propagating discrete ghosts $\hat{c}$, $\bar{\hat{c}}$).

These extra interactions prove important in providing the correct $\ln\mu$ dependence for higher order quantum corrections to the instanton vacuum energy as we will see in the next section.

We have seen in this section how we eliminate the zero modes from the Q integration simultaneously with the usual background gauge fixing by treating the gauge degrees of freedom as zero modes, and this led to some new interactions (fig.2.1). It is worthwhile stressing that we must fix out all zero modes (continuous gauge and discrete modes) simultaneously. This is because the discrete zero modes change under a gauge transformation and gauge modes change under zero mode shifts of the discrete collective coordinates. Had we not fixed these two sets simultaneously we would not have obtained the new interactions in fig.2.1 and the resulting structure would have been inconsistent.

2.3 RENORMALISATION AROUND INSTANTONS.

In this section we will consider quantum corrections to the instanton action which make up the effective action described in chapter 1 (equation (1.1.7)).

The lowest order quantum corrections are provided by the semiclassical approximation. In this approximation we ignore all quantum interactions so that we are left with the action up to quadratic terms

$$\delta S_{full} = \delta S + \delta S_{gh} + S_{GF}$$

$$= -\frac{8\pi^2}{g^2} + \int d^4x \left\{ \frac{1}{2} Q_\mu \left(O_{\mu\nu} + \frac{1}{\xi} D_\mu D_\nu \right) Q_\nu + \bar{\varphi} D^2 \varphi - \frac{1}{g} \bar{c} \hat{c} \right\} \tag{2.3.1}$$

($\hat{c}$ is explained in (2.2.24))

(The ghost quadratic action is as in (1.1.13)).

These last two terms come from the diagonal part of the Jacobian. The $1/g$'s in the last term arise from the $1/g$'s in (2.2.23) (through the term like the second of (2.2.10) combined with (2.2.11)). They do not arise in the ghost term ($\bar{\varphi} D^2 \varphi$) because we have made a substitution

$$\bar{\varphi} \rightarrow \sqrt{g}\ \bar{\varphi}$$

$$\varphi \rightarrow \sqrt{g}\ \varphi$$

The Jacobian of this substitution leads to an infinite power of (g) which however is cancelled by the same power from ghosts in the zero instanton sector (see (1.1.23)). (In section 4.1 we will find that this is not the case for superfield super Yang-Mills where the ghosts themselves have zero modes).

Performing the integration over $Q, \varphi, \bar{\varphi}$ and $\hat{c}$ we obtain

$$Z = \int \frac{d^{4n}b}{g^{4n}} \mu^{4n} \det{}^{-\frac{1}{2}}\left(O_{\mu\nu} + \frac{1}{\xi} D_\mu D_\nu \right) \det(D^2)\, e^{-8\pi^2/g^2} \qquad (2.3.2)$$

The $1/g^{4n}$ comes from the $\hat{c}$ (and $\bar{\hat{c}}$) integration. The determinant of the Q quadratic operator is only over the non-zero mode space (due to the δ-function constraints inserted into the functional integral - see previous section).

Note that this expression should be divided by the determinants that come from the zero instanton sector (see (1.1.23)).

The integral over collective coordinates is accompanied by powers of the regularisation mass so as to make the integral dimensionless (in dimensional regularisation this mass is μ). Since the ζ_k have (mass) dimension 2 (from equation (2.2.22)) and A_μ has dimension 1, equation (2.2.23) implies that the b^k have dimension -1 and hence a power of μ^{4n} is required in this case.

This rule can be justified from dimensional arguments [1] though the easiest way to verify it is to use Pauli-Villars regularisation [4].

Luckily there are some tricks we can use [5,1] on the constrained vector determinant in (2.3.2) which turn it into $\det^{-2}(D^2)$ so that together with the ghost contribution (in (2.3.2)) and the inverse contribution from the zero instanton sector we obtain

$$\det(\partial^2) \Big/ \det(D^2) \qquad (2.3.3)$$

which can be expanded and evaluated perturbatively [1] (using the background field methods of (1.1.21)).

We will not pause to describe these tricks here but will treat their generalisation in chapters 3.3 and 4.2 in some detail.

The perturbative evaluation of (2.3.3) once renormalised provides together with the μ 's in (2.3.2) the correct explicit μ dependence required by the renormalisation group ($\propto \beta_0$, see (1.1.22)) to cancel the implicit μ dependence of the classical background action ($-8\pi^2/g^2$)

At this order the multiplicative factor of $(1/g^{4n})$ in (2.3.2) provides no implicit μ dependence. This μ dependence first appears at $O(g^2)$ and must be cancelled if the vacuum energy in the instanton background (2.3.2) is to be μ independent (as it must be for any physical quantity). We therefore move on to consider the $O(g^2)$ divergent corrections (which provide after renormalisation the explicit $\ln\mu$ dependence).

The order g^2 corrections are illustrated in figures 2.2 and 2.3. Those of fig.2.2 are "full" 2 loop vacuum diagrams in the instanton background field. They are drawn with thick lines to represent the fact that the propagators are the inverse of the quadratic operators in (2.3.1) which contain A_μ , the background field. The interactions also contain A_μ. We will discuss later how we may evaluate the divergences in these diagrams by expanding them as perturbative interactions with the background field.

The diagram in fig.2.3 is made from the new vertices in the Jacobian (fig.2.1 see discussion following (2.2.24)). Power counting shows that the divergence of this diagram appears when the propagators are taken to be zeroth order in the background field hence we may take the propagators to be the usual ones (i.e. those in the zero instanton sector).

Now let us consider the background field expansion of fig.2.2. If we expand these perturbatively (to 2nd order in the background field) and use the methods described in chapter 1 (see (1.1.21)) we obtain the "short distance" corrections to the background field action (some examples are shown in fig.2.4), in which the divergences arise from high momenta in both loops. These corrections are entirely insensitive to the particular form of the background field and yield the standard divergences of the background field method (1.2.20). Once renormalised they provide the $O(g^2)$ explicit $\ln\mu$ dependence ($\propto \beta_1$) required to cancel that of the instanton background action (see equation (1.2.22)). Perturbatively there are no other divergent corrections: they all arise from these short distance singularities [1].

There is however a non-perturbative contribution in which only 2 propagators carry divergent momentum. These propagators can be expanded perturbatively in the background field, whereas the remaining vector propagator must be left in its full non-perturbative form (examples are shown in fig.2.5). Note that the explicit form of this propagator is not required (although it has been evaluated [5,6]). The divergent loops (such as the thin ones of fig.2.5) sum together to produce the divergent quantum (vector) self energy in the presence of a background field. As should be expected this self energy is proportional to the transverse part of the operator in the quadratic action ($O_{\mu\nu}$, see (2.2.3)). The form of the contribution is entirely independent of the particular form of the background field (see comments above on short distance corrections) and is fixed by the appropriate Slavnov-Taylor identities in the background field [1,7]). When traced together with the long distance propagator (as implied in fig.2.5) we obtain a term proportional to the trace of the transverse projection operator (which is the projection operator onto the non-zero modes of $O_{\mu\nu}$). Subtracting a similar expression from the zero instanton sector we obtain a term proportional to minus the number of transverse zero modes of $O_{\mu\nu}$, which is $-4n$. This fact is arrived at by comparing the space of zero modes of $O_{\mu\nu}$ (which is the complement to the space of non-zero modes) in the 1-instanton and 0-instanton sectors : In the one instanton sector we have a full square integrable space of gauge modes and $4n$ further zero modes ($4n-5$ of these being non square integrable gauge modes) and in the zero instanton sector we have only the square integrable space of gauge modes. The reader will see this argument at work again in chapter 4.3 (or [1]).

Adding together the divergences from this "long distance" correction and the Jacobian (fig.2.3), and renormalising, we obtain the correct explicit $\ln\mu$ dependence to cancel the implicit μ dependence of the multiplicative factor of $1/g^{4n}$ in the semiclassical approximation (2.3.2).

We have seen in this chapter how the divergent quantum corrections to the instanton action arise from 3 distinct sources

1) The short distance singularities (fig.2.4)

2) The long distance singularities (fig.2.5)

3) The Jacobian (fig.2.3)

In chapter 4 we will see these again in much more detail, generalised to the case of a superinstanton. This is the subject of the next chapter.

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Fig. 2.1

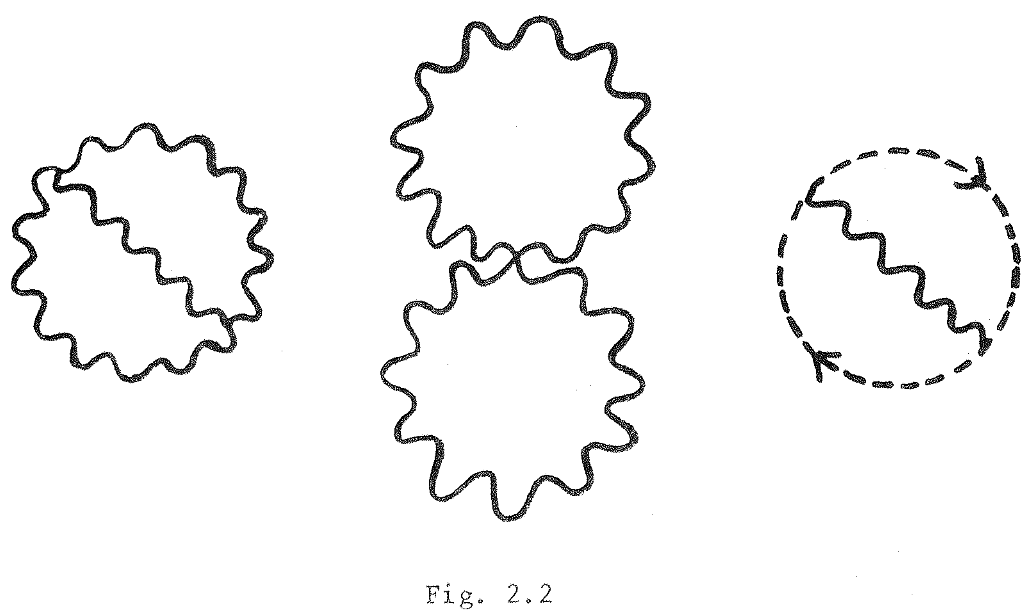

Fig. 2.2

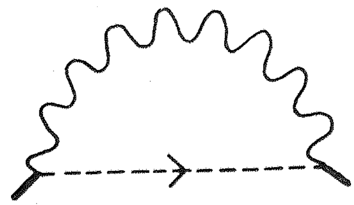

Fig. 2.3

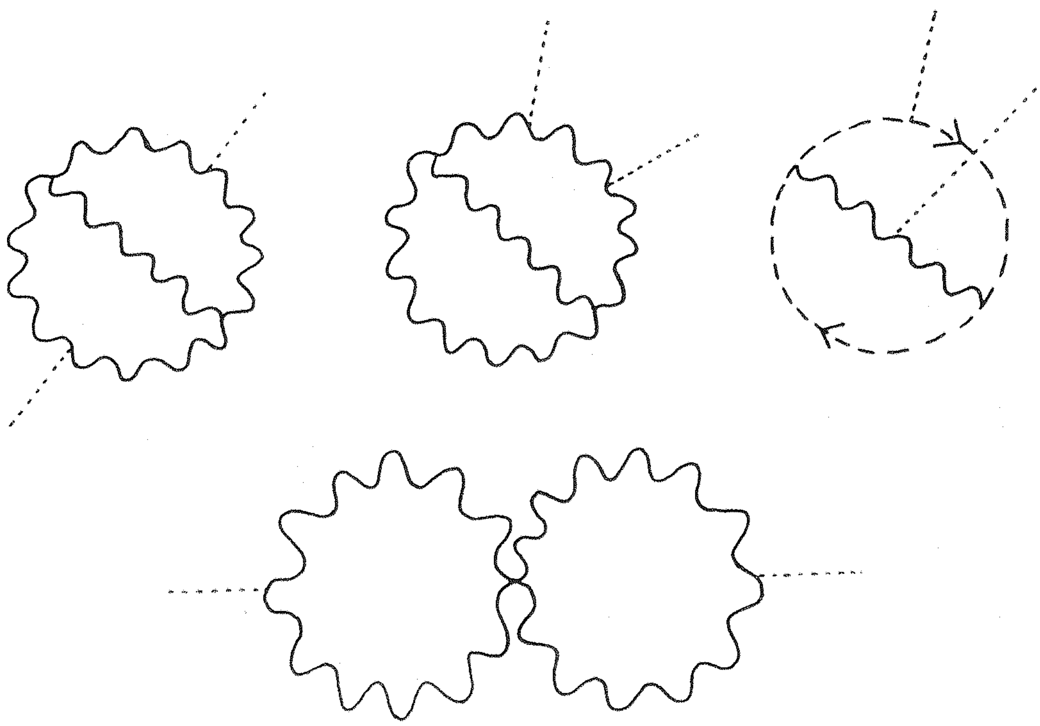

Fig. 2.4

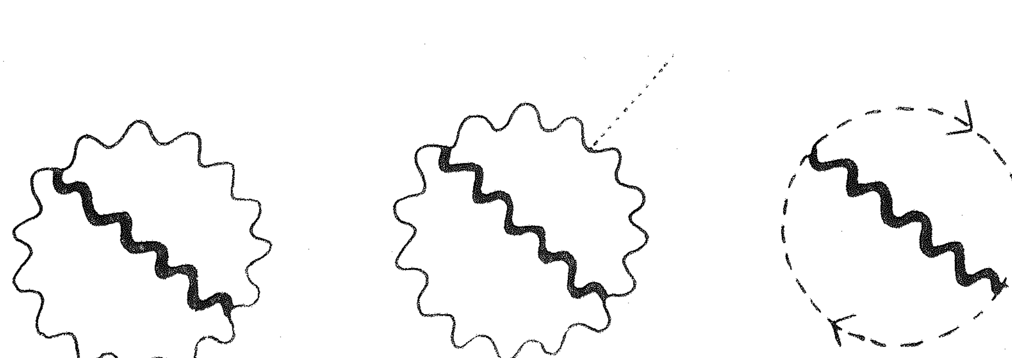

Fig. 2.5

CHAPTER 3 SUPERINSTANTONS.

In this chapter we construct the superfield version of the instanton and investigate the zero modes and the non-zero modes using the background superfield method. It is important in determining the classical contribution of the instanton to the functional integral that one knows the number and nature of the zero modes (see chapters 2.2 and 2.3). We therefore devote section 3.3 to a detailed discussion of these modes for the case where the gauge group is SU(2). A general method of deducing new discrete zero modes from ones already found is described . They are automatically generated in covariant form (compare (2.2.14) and discussion surrounding it). Many zero modes are generated by this method (one for each generator of the superconformal group) but orthogonality to the background gauge fixing term and a linear relation on the superinstanton show that the linearly independent set consists of 8 bosonic-parameter modes (which correspond to the translation, dilatation and SU(2) degrees of freedom in the instanton that we discussed in chapter 2.2) and 8 fermionic-parameter zero modes. 4 of the fermionic modes were expected: they correspond to supersymmetry (Q_α) and superconformal ($\bar{S}_{\dot{\beta}}$) degrees of freedom [10,14]. The remaining 4 are supergauge modes which, nevertheless, are not projected out by the gauge fixing condition. In this respect they are analogous to the 3 SU(2) bosonic gauge modes discovered by 't Hooft [2] (see below (2.2.19)).

The set of 8 fermionic and 8 bosonic zero modes have a natural orthonormality structure which displays another property of the 4 new zero modes - they serve to project out the supersymmetry and superconformal modes.

Attention is turned in chapter 3.4 to the continuous (i.e. these modes are labelled by a continuous parameter) supergauge zero modes. These modes are fixed by the background gauge fixing condition. In the case of an instanton background field the natural Laplace-type differential operator on (covariantly) anti-chiral fields is (background) $\Box(= \nabla_\mu\nabla_\mu$ where ∇_μ is the spatial covariant derivative defined in (1.2.59) or (1.2.65) and contains only the background field). This allows both the chiral and anti-chiral zero modes to be related to linear combinations of eigenstates of $\Box$. The vector superfield positive frequency modes are also related to the anti-chiral eigenstates of $\Box$ and this leads to the conclusion that chiral, anti-chiral and vector quantum fluctuations have the same spectrum of non-zero eigenvalues for the fields in the same representation of the gauge-group. This situation is analogous to the component case [3] where the same was proved for gluon, fermions and scalars. As in [3] we can use these expansions to construct projection operators and Greens functions.

The stage is then set to consider the non-perturbative quantum corrections to the instanton arising from the instanton measure, the Jacobian (of the change of variables between zero mode parameters and collective coordinates), and the functional integral over +ve frequency modes [9]. This will be discussed in chapter 4.

We start this chapter with a discussion of euclidean supersymmetry and the relevance of Osterwalder-Schrader (OS) conjugation [12,13] - a delicate matter since an instanton is not OS self conjugate.

3.1 EUCLIDEAN N=1 SUPERSYMMETRY AND INSTANTONS.

The BPST instanton [1] (see chapter 2.1) is a real solution of the euclidean space Yang Mills equations of motion. The construction of a supersymmetric version of this instanton must, therefore, involve the use of euclidean supersymmetry.

In Minkowski space, we have L.H. (left handed) and R.H. (right handed) superspace coordinates θ^{α}, $\bar{\theta}^{\dot{\alpha}}$ which transform in the fundamental and complex conjugate representations of the Lorentz covering group SL(2,C). They are therefore connected by complex conjugation -

$$(\theta^{\alpha})^* = \bar{\theta}^{\dot{\alpha}}$$

(see (1.2.1)).

On continuation to euclidean space the covering group changes to

$$SU_L(2) \times SU_R(2) \qquad [\ \simeq SO(4)\ ]$$

and the L and R superspace coordinates in the fundamental representations of the 2 (distinct) SU(2) groups: θ^{α}, $\theta^{\dot{\alpha}}$ are no longer connected by complex conjugation (hence the reason for dropping the bar on $\theta^{\dot{\alpha}}$)

$$\theta^{\dot{\alpha}} \neq (\theta^{\alpha})^*$$

In fact, complex conjugation produces coordinates transforming in the contragredient representations of the SU(2) groups:

$$(\theta^{\alpha})^* = \bar{\theta}_{\alpha} \qquad (3.1.1)$$

$$(\theta^{\dot{\alpha}})^* = \bar{\theta}_{\dot{\alpha}}$$

This implies that the minimal hermitian euclidean superspace would be

$$S = (x,\ \theta^{\alpha},\ \bar{\theta}_{\alpha},\ \theta^{\dot{\alpha}},\ \bar{\theta}_{\dot{\alpha}}) \qquad (3.1.2)$$

and corresponds to N=2 supersymmetry on continuation back to Minkowski space (as was first shown by Zumino [13]).

To obtain the same multiplet structure in euclidean space as N=1 supersymmetry we abandon the requirement of hermiticity and replace it with "hermiticity" under the unitary involution operator of Osterwalder and Schrader [12, 13]. This operation is Hermitian conjugation followed by time (x_4) reversal and is known as OS conjugation. Objects that are invariant under this operation are OS-self conjugate (or OS-real) and this concept replaces the concept of ordinary complex conjugation.

In particular, OS conjugation provides a map between $SU_L(2)$ and $SU_R(2)$ groups.
(-A general SO(4) generator $M_{\mu\nu}$ is expanded in terms of $SU_L(2)$ and $SU_R(2)$ generators by

$$M_{\mu\nu} = \eta^a{}_{\mu\nu}\, T_L{}^a + \bar{\eta}^a{}_{\mu\nu}\, T_R{}^a \tag{3.1.3}$$

where $\eta^a{}_{\mu\nu}$ and $\bar{\eta}^a{}_{\mu\nu}$ are the self dual and anti-self dual 't Hooft symbols (see below (2.1.12)). Time reversal reverses the orientation of the euclidean axes and turns $\eta^a{}_{\mu\nu}$ into $\bar{\eta}^a{}_{\mu\nu}$ (and vice versa)).
So under OS conjugation

$$\theta^{\alpha} \xleftrightarrow{OS} \bar{\theta}_{\dot{\alpha}} \qquad\qquad \bar{\theta}_{\alpha} \xleftrightarrow{OS} \theta^{\dot{\alpha}} \tag{3.1.4}$$

We now restrict our superspace and component field multiplet structure to N=1 by using OS self conjugacy. From (3.1.4) we find that we need only consider the "Grassmann-analytic"[13] superspace.

$$S^{+} = (x,\ \theta^{\alpha},\ \bar{\theta}_{\dot{\alpha}}) \tag{3.1.5}$$

in which the LH and $\overline{\text{RH}}$ Grassmann coordinates appear but not their complex conjugate partners.

Bosonic fields that are OS self conjugate become real on continuation to Minkowski space and, (2 component) fermionic fields and their OS-conjugates, become Weyl spinors and their complex conjugates on continuation to Minkowski space [12,13].

The close analogy between this OS conjugate superspace and Minkowski N=1 superspace leads to an exact correspondence with Minkowski Superspace notation [6], if the following notational changes are made :-

We call the fundamental representation of $SU_R(2)$ $\theta_{\dot{\alpha}}$ where $\dot{\alpha}$ is a

subscript .

Then S^+ appears as

$$S^+ = (x, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}) \qquad (2.6)$$

In addition, if we choose our euclidean σ matrices as

$$\sigma_\mu = (\underline{\sigma}, i)$$

$$\bar{\sigma}_\mu = (-\underline{\sigma}, i)$$

then each of these is OS self conjugate and the supersymmetry algebra takes the same form as Minkowski space [6] (see (1.2.10)).

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i\partial_{\alpha\dot{\alpha}}$$

Finally, to preserve the relation (see (1.2.18))

$$\partial_{\alpha\dot{\alpha}} x^{\beta\dot{\beta}} = 2 \delta_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}}$$

on continuation to euclidean space, we define our space coordinate

$$x_{\alpha\dot{\alpha}} = - x_\mu \sigma_{\mu\alpha\dot{\alpha}} \qquad (3.1.7)$$

This definition of σ matrices differs from the usual definition in euclidean space. But the fact that the σ matrices are OS self conjugate guarantees that real Minkowski vector fields (A_μ) are OS self-conjugate if the formulae are interpreted as in euclidean space.

Changes of sign from the Minkowski formula occur on moving between vector notation and two-component notation as a result of (c.f. (1.2.17))

$$\mathrm{Tr}\{\sigma_\mu \bar{\sigma}_\nu\} = -2\ \delta_{\mu\nu} \qquad (3.1.8)$$

e.g:

$$\partial^{\alpha\dot{\alpha}}\ \partial_{\alpha\dot{\alpha}} = -2\ \Box \qquad (3.1.9)$$

but this turns out to be required for consistency in euclidean space (see Section 3.2 and Section 3.4 for example, where $\Box$ having only negative eigenvalues is consistent with other results only if this sign change occurs.)
Apart from this change there is an exact correspondence with Minkowski formulae.

There is a problem with this formalism however since an instanton is not OS self conjugate.
Indeed, if the explicit formula (see equations above (2.1.17))

$$A^a_\mu = \frac{2}{g}\ \frac{\eta^a_{\mu\nu}\ x_\nu}{1 + x^2} \qquad (3.1.10)$$

is used and OS conjugation performed

$$A^a_\mu \longrightarrow (A_k(\underset{\sim}{x}, -x\ ),\ -A(\underset{\sim}{x}, -x\ )) \qquad (3.1.11)$$

it is readily seen that the new field corresponds to an anti-instanton (see comments below (2.1.14) and (2.1.19)).

$$A^a_\mu = \frac{2}{g}\ \frac{\bar{\eta}^a_{\mu\nu}\ x_\nu}{1 + x^2}$$

Recall that the concept of OS-conjugation was introduced to cut down the superspace and consequently the field multiplet from N=2 to that of N=1 supersymmetry, and that this implied that the superspace was Grassmann-analytic (see ref [13] and comments surrounding (3.1.5)). Let us generalise the concept to OS-analyticity: superfields that are real in Minkowski space (e.g. the prepotential v) are, in euclidean space, allowed to be OS-complex but Grassmann-analytic functions (of the superspace (3.1.6)). Our Lagrangians will no longer be OS-real but must nevertheless be OS-analytic functions of the superfields (e.g. must not contain the OS-conjugate $\bar{v}$). This guarantees the N=1 supermultiplet structure but also allows us to study instantons.

Gaussian integration (and functional integration) are treated in the usual spirit of analytic continuation:
For example such formulae as

$$\int \mathcal{D}v \, e^{\int d^8z \, v \, \Delta^{-1} \, v} = \mathrm{Sdet}^{\frac{1}{2}}\Delta$$

(up to numerical constants)
are taken to be true in general even when v is allowed to be OS-complex.
Note that, although the quantum field (v) will not be required to be OS-real, these considerations imply that further restrictions on the quantum field are unnecessary.

A 'bar' on fields (e.g. $\bar{W}_{\dot{\alpha}}$) and the word 'conjugation' will always refer to OS-conjugation. But for an instanton this will only be in a formal sense. This is because we wish to avoid introducing the OS conjugate prepotential $(v_B)^*$ which would destroy the N=1 multiplet structure (see previous comments). It will only correspond to OS-conjugation in the case of an OS-conjugate background field.
(For example, $\bar{W}_{\dot{\alpha}}$ will only be the OS-conjugate $((W_\alpha)^*)$ of W_α when the background field satisfies $v_B{}^* = v_B$, $A_\mu{}^{B*} = A_\mu{}^B$.) This means that, in the case of an instanton, the "conjugate" field needs further definition; however, we find that when we need to be precise about a certain "conjugate" field then further relations already exist determining the "conjugate" field unambiguously. (See, for example, the construction of W_γ and $\bar{W}_{\dot{\gamma}} \neq (W_\gamma)^*$ in Section 3.2, and the parameterisation of chiral and anti-chiral ($\bar{\Lambda} \neq (\Lambda)^*$) zero modes in Section

3.4). The fact that these further relations do exist supports the view that these problems do not indicate a fundamental inconsistency in the formulation of instantons with explicit N=1 supersymmetry.

Thus a suitable relaxation of the concept of OS-conjugacy allows us to consider instantons within a euclidean N=1 supermultiplet. We do not believe the above problems to be fundamental; the consistency of the resulting structure will become clearer on reading the rest of the thesis.

3.2 THE SUPERINSTANTON, TRANSLATION AND GAUGE MODES.

The super Yang-Mills action (1.2.38) is $S_V = S + \bar{S}$, (here $\bar{S}$ is the OS-conjugate to S) and can be expressed as

$$S = \frac{tr}{128 g^2} \int d^4x d^2\theta \, W^\gamma W_\gamma \qquad (3.2.1)$$

$$W^\gamma = [\bar{\nabla}^{\dot{\gamma}}, \{\bar{\nabla}_{\dot{\gamma}}, \nabla^\gamma\}] \qquad (3.2.2)$$

where, unless otherwise stated, we will use the vector representation for the covariant derivatives (1.2.64)

$$\nabla_\alpha = e^{-g\omega} D_\alpha \, e^{g\omega}$$

$$\bar{\nabla}_{\dot{\alpha}} = e^{g\bar{\omega}} \bar{D}_{\dot{\alpha}} \, e^{-g\bar{\omega}}$$

We can project out the component fields in W_γ, in covariant form, by using the covariant derivatives [15,16]:

Define

$$W_\alpha| = 4g \, \lambda_\alpha \sqrt{2} \qquad (3.2.3a)$$

$$\nabla^\alpha W_\alpha| = 8g \, D \sqrt{2} \qquad (3.2.3b)$$

$$\nabla_{(\alpha} W_{\beta)}| = 4ig \, f_{\alpha\beta} \sqrt{2} \qquad (3.2.3c)$$

(We use the notation $T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}$ and $T_{[\alpha\beta]} = T_{\alpha\beta} - T_{\beta\alpha}$)

and similarly for the conjugates $\bar{W}_{\dot{\alpha}}$, $\bar{\lambda}_{\dot{\alpha}}$ e.t.c.

The "|" means take $\theta = \bar{\theta} = 0$. (These definitions agree with those of (1.2.20) (up to numerical factors) only when the gauge group is Abelian, otherwise these represent highly non-linear redefinitions of the component fields).

$$f_{\alpha\beta} = F_\alpha{}^{\dot{\alpha}}{}_{,\beta\dot{\alpha}} = -2i \, (\sigma^2 \sigma_{\mu\nu})_{\alpha\beta} F^+_{\mu\nu} = i(\sigma^2 \sigma^a)_{\alpha\beta} \, \eta^a_{\mu\nu} F^+_{\mu\nu}$$

As in (2.1.2) and (2.1.12) $F_{\mu\nu}$ is the Yang Mills field strength, and $F^+_{\mu\nu}$ the self dual part.

We can also find the vector potential $A_{\alpha\dot{\alpha}} = \sigma_{\mu\alpha\dot{\alpha}} A_\mu$ from

$$\nabla_{\alpha\dot{\alpha}}| = \partial_{\alpha\dot{\alpha}} - i\sqrt{2} \, g \, A_{\alpha\dot{\alpha}} = D_\mu \, \sigma_{\mu\alpha\dot{\alpha}} \qquad (3.2.3d)$$

The factors of $\sqrt{2}$ in the equations (3.2.3) appear through a non-standard normalisation of the SU(n) generators (see discussion below (1.2.46)).

Now (see [6])-

$$\mathrm{tr}\int d^4x d^2\theta \; f \equiv -1/4 \; \mathrm{tr}\int d^4x \; \nabla^2 f \;| \qquad (3.2.4)$$

(which follows from (1.2.36) and expressing ∇_α in terms of a connection Γ_α by

$$\nabla_\alpha = D_\alpha - i\,\Gamma_\alpha$$

$$\Gamma_\alpha = \Gamma_\alpha^{\,i}\, T^i$$

so that

$$\mathrm{tr}\int d^4x \; [D_\alpha, G] = \mathrm{tr}\int d^4x \; [\nabla_\alpha, G]$$

for any superfield G, since $\mathrm{tr}\,[\Gamma_\alpha, G]$ vanishes).

Using (3.2.3) and (3.2.4) on (3.2.1) we obtain

$$S = \mathrm{tr}\int d^4x \; (-1/4 \; F^{+\,2}_{\mu\nu} + i/2 \; \lambda^\alpha D_\alpha^{\;\dot\alpha} \bar\lambda_{\dot\alpha} + 1/4 \; D^2 \;) \qquad (3.2.5)$$

$$\bar{S} = \mathrm{tr}\int d^4x \; (-1/4 \; F^{-\,2}_{\mu\nu} + i/2 \; \bar\lambda^{\dot\alpha} D_{\dot\alpha}^{\;\alpha} \lambda_\alpha + 1/4 \; D^2 \;) \qquad (3.2.6)$$

$f_{\dot\alpha\dot\beta} = F_{\dot\alpha}{}^{\alpha}{}_{,\dot\beta\alpha} = i(\sigma^2\,\sigma^a)_{\dot\alpha\dot\beta}\,\bar\eta^a{}_{\mu\nu}\,F^-{}_{\mu\nu}$ where $F^-{}_{\mu\nu}$ is the anti-selfdual part of $F_{\mu\nu}$(see (2.1.12)). (3.2.7)

Hence with definitions (3.2.3) we obtain the expected action S_V but note that these expressions are true in a general supergauge (and not just in the Wess-Zumino gauge as was the case with the previous definitions - see discussion above (1.2.45)) . Expressions (3.2.3) exhaust the covariant content of the theory (i.e. give all the fields that transform homogeneously under a supergauge transformation): Further applications of covariant derivatives to the superfield expressions yield only zero or $D_{\dot\alpha}^{\;\alpha}$ acting on one of the component fields already written down [15,16].

Now consider the case where the vector potential A_μ is the BPST instanton (see chapter 2.1) and all other component fields are zero. This is the "bosonic instanton" and the only non zero component is $f_{\alpha\beta}$; $f_{\dot{\alpha}\dot{\beta}} = 0$ since the instanton is self dual (see (3.2.7) and chapter 2.1).

Hence $$\bar{W}_{\dot{\alpha}} = 0 \qquad (3.2.8)$$

But this is a superfield equation and hence it must hold for the general instanton (which is the supersymmetric generalisation of the bosonic instanton).

The general instanton can be written down in component form by considering transformations of the superconformal group. (We will be considering only the SU(2) gauge theory. For SU(n), n>2, the fermionic instanton contains more parameters than the ones gained by this method. See chapter 4.1)

Since the superinstanton must satisfy its equations of motion (1.2.73):

$$\nabla^\alpha W_\alpha = 0 \qquad (3.2.9)$$

it follows immediately from (3.2.3b) that $D = 0$ (3.2.10)

is always the case.

For a general covariant superfield the supersymmetry transformations are given by

$$\delta\phi = (\delta\bar{\alpha}^{\dot{\gamma}}\bar{Q}_{\dot{\gamma}} + \delta\alpha^\gamma Q_\gamma) \, \phi$$

and taking $\theta = \bar{\theta} = 0$ components this is equivalent to

$$\delta\phi| = (\delta\bar{\alpha}^{\dot{\gamma}}\bar{D}_{\dot{\gamma}} + \delta\alpha^\gamma D_\gamma) \, \phi \, |$$

(see definitions (1.2.11) and (1.2.13)).

Covariantization by following this change with a supergauge transformation gives

$$\delta\phi| = (\delta\bar{\alpha}^{\dot{\gamma}}\bar{\nabla}_{\dot{\gamma}} + \delta\alpha^\gamma \nabla_\gamma) \, \phi \, | \qquad (3.2.11)$$

similarly a change under a special superconformal transformation

$$\delta\phi = (\delta\bar{\beta}^{\dot{\gamma}}\bar{S}_{\dot{\gamma}} + \delta\beta^\gamma S_\gamma) \, \phi$$

combined with a supergauge transformation leads to

$$\delta\phi| = -i\,(\delta\beta^{\gamma} x_{\gamma}{}^{\dot\gamma}\bar\nabla_{\dot\gamma} + \delta\bar\beta^{\dot\gamma} x_{\dot\gamma}{}^{\gamma}\nabla_{\gamma})\,\phi\,| \qquad (3.2.12)$$

Applying (3.2.11) and (3.2.12) to the superfield expressions in (3.2.3) allows us to calculate the so-called Wess-Zumino superconformal algebra on the component fields:

Supersymmetry:

$$\delta\lambda_{\alpha} = i/2\; f_{\alpha\beta}\,\delta\alpha^{\beta}$$
$$\delta A_{\alpha\dot\alpha} = -2\,\delta\bar\alpha_{\dot\alpha}\,\lambda_{\alpha} \qquad (3.2.13)$$

Superconformal:

$$\delta\lambda_{\alpha} = 1/2\;\delta\bar\beta^{\dot\gamma}\,x_{\dot\gamma}{}^{\beta}\,f_{\beta\alpha}$$
$$\delta A_{\alpha\dot\alpha} = 2i\;\delta\beta^{\gamma}\,x_{\gamma\dot\alpha}\,\lambda_{\alpha} \qquad (3.2.14)$$

The changes on D, $\lambda_{\dot\alpha}$, $f_{\dot\alpha\dot\beta}$ are not considered here since (3.2.10) and (3.2.8) imply that they are always zero. However (3.2.11) and (3.2.12) can be used to check this.

Remembering that Q, $\bar{Q}$, S and $\bar{S}$ generate the full superconformal algebra, equations (3.2.13) and (3.2.14) allow us to write down the components of the general superinstanton:-

$$A_{\mu} = e^{i\Omega}\left(\frac{2}{g}\;\frac{\eta^{a}{}_{\mu\nu}\,\tau^{a}(x-a)_{\nu}}{\rho^{2} + (x-a)^{2}}\right)e^{-i\Omega} + \frac{i\sqrt{2}}{g}\,e^{i\Omega}\,\partial_{\mu}\,e^{-i\Omega} \qquad (3.2.15)$$

(as in (2.1.17) and (2.1.18)),

$$\lambda_{\alpha} = i/2\;\alpha^{\gamma}\,f_{\gamma\alpha} + 1/2\;\bar\beta^{\dot\gamma}\,x_{\dot\gamma}{}^{\gamma}\,f_{\gamma\alpha}$$
$$\bar\lambda_{\dot\alpha} = D = 0 \qquad (3.2.16)$$

where $\Omega = \Omega^{a}(x)T^{a}$ is a general gauge transformation.

The expression in brackets in (3.2.15) is the general Lorentz gauge bosonic BPST instanton with size "ρ" and position "a_{μ}".

$f_{\alpha\beta}$ is calculated from A_{μ} .

Equations (3.2.16) can be shown to be the most general solutions for the instanton directly from the fact that the components must satisfy their equations of motion. (These can be derived from (3.2.5),(3.2.6) or from (3.2.9) and (3.2.3)).

These are

$$D = 0 \qquad \text{(a)}$$

$$D_\alpha{}^{\dot\alpha}\, \bar\lambda_{\dot\alpha} = 0 \qquad \text{(b)}$$

$$D_{\dot\alpha}{}^{\alpha}\, \lambda_\alpha = 0 \qquad \text{(c)}$$

(b) implies $D_\mu D_\mu \bar\lambda_{\dot\alpha} = 0$ (using $D^\alpha{}_{(\dot\alpha} D_{\dot\beta)\alpha} = -ig/2\, f_{\dot\alpha\dot\beta} = 0$) and the only square integrable solution to this equation is[2]

$$\bar\lambda_{\dot\alpha} = 0$$

(c) is known to have 4 solutions [2,10] and it is easily checked that the 4 given in (3.2.16) satisfy (c).

By temporarily going to the covariantly chiral representation (see (1.2.56)), expanding W_γ in terms of a power series in θ and functions of x_+ (the chiral coordinate (1.2.23)), and using definitions (3.2.3) we can solve for W_γ in terms of (3.2.15) and (3.2.16). After some algebra we obtain -

$$W_\beta = e^{g\bar\omega}\{(1 - L_{\theta^\alpha \tilde\Gamma_\alpha} + \theta^2/4\, L_{\tilde\gamma} - \theta^2/4\, L_{\tilde\Gamma^\alpha} L_{\tilde\Gamma_\alpha})\, \hat W_\beta\}\, e^{-g\bar\omega} \qquad (3.2.17a)$$

where "ω" (and "$\bar\omega$") are the prepotentials occuring in the vector representation of the covariant derivatives (1.2.64). L is the commutator:

$$L_V X = [V,X]$$

$\tilde\Gamma_\alpha$ and $\tilde\gamma$ are functions of x_+ constructed as follows:

$$\tilde\Gamma_\alpha(x) = e^{-gv}\, \partial_\alpha(e^{gv})\, |$$

$$\tilde\gamma(x) = \partial^\alpha\{e^{-gv}\, \partial_\alpha(e^{gv})\}\, |$$

and

$$\hat W_\alpha(x^+,\theta) = 2\sqrt{2}\,(2g\, \lambda_\alpha(x^+) + ig\, \theta^\beta\, f_{\beta\alpha}(x^+)) \qquad (3.2.17b)$$

But using (3.2.16) we obtain

$$\hat W_\alpha = 2\sqrt{2}\, ig\, (\theta^\beta + \alpha^\beta - i\bar\beta^{\dot\beta}\, x^+{}_{\dot\beta}{}^{\beta})\, f_{\beta\alpha}$$

Now since $f_{\alpha\beta}$ is a symmetric tensor, and the quantity in brackets above is an anticommuting Grassmann variable we have

$$(\theta^{\beta} + \alpha^{\beta} - i\bar{\beta}^{\dot{\beta}} x^{+}{}_{\dot{\beta}}{}^{\beta})\,\hat{W}_{\beta} = 0$$

Hence, using the linearity of the relation between W_α and $\hat{W}_\alpha$ in (3.2.17a), we find

$$(\theta^{\alpha} + \alpha^{\alpha} - i\bar{\beta}^{\dot{\beta}} x^{+}{}_{\dot{\beta}}{}^{\alpha})\,W_{\alpha} = 0 \qquad (3.2.18)$$

This is a linear relation on W_γ that holds for a superinstanton with fermionic "centre" at (α,β).
It is derived again in section 3.3 in a way that does not require the explicit solution (3.2.17).

Consider now, quantum fluctuations (v) around the instanton background field (ω and $\bar{\omega}$); The full field "v_f" is expressed in terms of the quantum and background fields by (1.3.1)-

$$e^{gv_f} = e^{g\omega}\, e^{gv}\, e^{g\bar{\omega}} \qquad (3.2.19)$$

In the usual background-vector quantum-chiral representation the action is (1.3.6) which can be written as-

$$S = -\frac{tr}{32g^2}\int d^8z\, e^{-gv}\,\nabla^{\alpha}\, e^{gv}\,\bar{\nabla}^2(\,e^{-gv}\,\nabla_{\alpha}\, e^{gv})$$

($d^8z \equiv d^4x d^2\theta d^2\bar{\theta} \equiv d^4x d^4\theta$ as in (1.2.33), and the conjugate of (3.2.4) has been used).
This can be expanded to 2nd order in the quantum field to give (compare (1.3.13))

$$S_v = -8\pi^2/g^2 + \delta S_v \qquad (3.2.20)$$

$$\delta S_v = \frac{tr}{16}\int d^8z\; v(\nabla^{\alpha}\bar{\nabla}^2\nabla_{\alpha} - W^{\alpha}\nabla_{\alpha})v$$

where we have inserted the instanton solution into (3.2.5) and used the fact that $\delta S = \delta\bar{S}$. The inverse propagator (the operator in the quadratic action) can be re-written in two other useful forms

$$1/8\;(\nabla^{\alpha}\bar{\nabla}^2\nabla_{\alpha} - W^{\alpha}\nabla_{\alpha}) \qquad (3.2.21a)$$

$$=1/8\;\bar{\nabla}^{\dot{\alpha}}\nabla^2\bar{\nabla}_{\dot{\alpha}} \qquad (3.2.21b)$$

$$= \Box + 1/16\,[\nabla^2\bar{\nabla}^2 + \bar{\nabla}^2\nabla^2] - 1/8\;W^{\alpha}\nabla_{\alpha} \qquad (3.2.21c)$$

(where we have used the equations of motion and $\bar{W}^{\dot{\alpha}} = 0$; compare (1.3.14))

$\Box = \nabla_\mu \nabla_\mu = -1/2\ \nabla^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}}$ is the background-Laplacian.

We gauge fix by adding the usual background gauge fixing term (1.3.9)

$$S_{GF} = -\frac{tr}{32\xi} \int d^8z\ v(\nabla^2 \bar{\nabla}^2 + \bar{\nabla}^2 \nabla^2) v \qquad (3.2.22)$$

Now suppose the quantum fluctuation v, in (3.2.19), corresponds to a shift in the parameters in the instanton, say $v = \zeta$. Then this shift can be re-written as changes in ω and $\bar{\omega}$:

$$e^{g\omega} \rightarrow e^{g\omega}\, e^{g\zeta/2}$$

$$e^{g\bar{\omega}} \rightarrow e^{g\zeta/2}\, e^{g\bar{\omega}}$$

If ζ is infinitessimal then $v = \zeta$ is a zero mode (i.e. (3.2.21) on v yields zero)

For ζ infinitessimal the above yields the following changes in the covariant derivatives:

$$\delta\nabla_\alpha = g/2\ \nabla_\alpha\ \zeta$$
$$\delta\bar{\nabla}_{\dot{\alpha}} = -\ g/2\ \bar{\nabla}_{\dot{\alpha}}\ \zeta \qquad (3.2.23)$$

Using these on (3.2.3) we can derive the following changes in the component fields

$$\delta A_{\alpha\dot{\alpha}} = -\frac{1}{2\sqrt{2}} [\nabla_\alpha, \bar{\nabla}_{\dot{\alpha}}]\ \zeta\ | \qquad (3.2.24a)$$

$$\delta\lambda_\alpha = \left(\frac{1}{4\sqrt{2}} \bar{\nabla}^2 \nabla_\alpha\ \zeta + g/2\ [\zeta, \lambda_\alpha]\right) | \qquad (3.2.24b)$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = \frac{1}{4\sqrt{2}} \nabla^2 \bar{\nabla}_{\dot{\alpha}}\ \zeta\ | \qquad (3.2.24c)$$

$$\delta D = \frac{1}{8\sqrt{2}} (\nabla^\alpha \bar{\nabla}^2 \nabla_\alpha - W^\alpha \nabla_\alpha)\ \zeta\ | \qquad (3.2.24d)$$

Unlike W_γ in (3.2.3), these definitions do not exhaust the (background) covariant content of ζ. This is because there are gauge fields appearing in v that do not appear in the action (3.2.5). We covariantize the usual definitions (1.2.20)

$$\delta B = \zeta| \qquad (3.2.24e)$$

$$\delta H = -1/2\ \nabla^2\ \zeta\ | \qquad (3.2.24f)$$

$$\delta \bar{H} = -1/2\ \bar{\nabla}^2\ \zeta\ | \qquad (3.2.24g)$$

$$\delta \psi_\alpha = -\ \nabla_\alpha\ \zeta\ | \qquad (3.2.24h)$$

$$\delta \bar{\psi}_{\dot{\alpha}} = -\ \bar{\nabla}_{\dot{\alpha}}\ \zeta\ | \qquad (3.2.24i)$$

Equations (3.2.24) can be used to find the changes in the components of the instanton field brought about by a zero mode "ζ".
(Note that comparison of (3.2.21a) and (3.2.24d) implies $\delta D = 0$ automatically - consistent with the general instanton (3.2.16)).
Gauge zero modes are described slightly differently. They arise from gauge transformations on the background field ("K" gauge transformations (1.2.66) are not considered here since they do not yield zero modes: the quantum field transforms homogeneously when K-gauging, see (1.3.5)).

$$e^{g\omega} \to e^{i\bar{\Lambda}_B}\ e^{g\omega}$$

$$e^{g\bar{\omega}} \to e^{g\bar{\omega}}\ e^{-i\Lambda_B}$$

where Λ_B and $\bar{\Lambda}_B$ are chiral and antichiral gauge transformations.
They can be re-expressed as a change in the quantum field (from (3.2.19))

$$e^{gv} \to e^{i\bar{\Lambda}}\ e^{gv}\ e^{-i\Lambda} \qquad (3.2.25)$$

where $\Lambda = e^{g\bar{\omega}}\ \Lambda_B\ e^{-g\bar{\omega}}$ and $\bar{\Lambda} = e^{-g\omega}\ \bar{\Lambda}_B\ e^{g\omega}$ (as in (1.3.4)) are covariantly chiral and covariantly antichiral gauge transformations. So if $v = \zeta$ corresponds to a small gauge transformation, (3.2.25) implies

$$\zeta = i/g\ (\bar{\Lambda}-\Lambda) \qquad (3.2.26)$$

The component content of Λ is exhausted by the following definitions (compare (1.2.25))

$$\surd 2\ \ \Omega = \Lambda\ | \qquad (3.2.27a)$$

$$\phi_\alpha = \nabla_\alpha\ \Lambda\ | \qquad (3.2.27b)$$

$$F = -1/4\ \nabla^2\ \Lambda\ | \qquad (3.2.27c)$$

and similarly for $\bar{\Lambda}$.

This leads to the expected changes in the instanton field (using (3.2.24))

$$\delta A_{\alpha\dot{\alpha}} = -1/g\ D_{\alpha\dot{\alpha}}\ (\Omega + \bar{\Omega})$$

$$\delta\lambda_{\alpha} = -i/2\ [\Omega + \bar{\Omega},\ \lambda_{\alpha}]$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = \delta D = 0 \qquad\qquad (3.2.28)$$

$$\delta B = i/g\ (\Omega - \bar{\Omega})\ \surd 2$$

$$\delta H = 2i/g\ F \qquad\qquad \delta\bar{H} = -2i/g\ \bar{F}$$

$$\delta\psi = -i/g\ \phi \qquad\qquad \delta\bar{\psi} = i/g\ \bar{\phi}$$

We now determine the superfield form of the translation zero mode. The change in the background field due to a shift of the instanton centre $a_{\mu} \rightarrow a_{\mu} - \delta a_{\mu}$ is

$$\delta e^{g\bar{\omega}} = \delta a^{\alpha\dot{\alpha}}\ \partial_{\alpha\dot{\alpha}}\ e^{g\bar{\omega}}$$

$$\delta e^{g\omega} = \delta a^{\alpha\dot{\alpha}}\ \partial_{\alpha\dot{\alpha}}\ e^{g\omega}$$

(up to irrelevent numerical factors, compare (2.2.13))
which can be expressed as a zero mode (using (3.2.19)) by

$$e^{g\omega}\, e^{g\zeta}\, e^{g\bar{\omega}} = \delta a^{\alpha\dot{\alpha}}\ (\ \partial_{\alpha\dot{\alpha}}\, e^{g\omega}\ e^{g\bar{\omega}} + e^{g\omega}\ \partial_{\alpha\dot{\alpha}}\, e^{g\bar{\omega}}\ )$$

$$\Rightarrow \quad \zeta = 1/g\ \delta a^{\alpha\dot{\alpha}}\ (e^{g\bar{\omega}}\ \partial_{\alpha\dot{\alpha}}\ e^{-g\bar{\omega}} - e^{-g\omega}\ \partial_{\alpha\dot{\alpha}}\ e^{g\omega})$$

Since this is true for a general perturbation δa_{μ} we have four linearly independent zero modes:

$$\zeta_{\alpha\dot{\alpha}} = 1/g\ (e^{g\bar{\omega}}\ \partial_{\alpha\dot{\alpha}}\ e^{-g\bar{\omega}} - e^{-g\omega}\ \partial_{\alpha\dot{\alpha}}\ e^{g\omega}) \qquad (3.2.29)$$

We now gauge covariantize the expression by adding a gauge zero mode of the form (3.2.26).

$$\Lambda_{\alpha\dot{\alpha}} = e^{g\bar{\omega}}\ \Gamma^{C}{}_{\alpha\dot{\alpha}}\ e^{-g\bar{\omega}} + 1/4\ \bar{\theta}_{\dot{\alpha}}\ W_{\alpha}$$

$$\bar{\Lambda}_{\alpha\dot{\alpha}} = e^{-g\omega}\ \Gamma^{A}{}_{\alpha\dot{\alpha}}\ e^{g\omega}$$

where $\nabla^{A,C}{}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} - i\Gamma^{A,C}{}_{\alpha\dot{\alpha}}$ and "C" and "A" refer to covariantly chiral (1.2.56) and antichiral (1.2.59) representations respectively.

Then (3.2.29) reads

$$\zeta_{\alpha\dot{\alpha}} = i/g\ \bar{\theta}_{\dot{\alpha}}\ W_{\alpha} \qquad (3.2.30)$$

(dropping an irrelevent numerical constant; Note that in general we obtain $\quad i/g(\bar{\theta}_{\dot{\alpha}}\ W_{\alpha} + \theta_{\alpha}\ \bar{W}_{\dot{\alpha}})$)

Using (3.2.24) we find the component transformations expected -

$$\delta A_{\beta\dot{\beta}} = \varepsilon_{\dot{\beta}\dot{\alpha}}\ f_{\beta\alpha}$$

(or in 4-component notation: $\delta A_{\mu} = -2\ F_{\mu\nu}\ \delta a^{\nu}$ as in (2.2.14))

$$\delta\lambda_{\beta} = 2\ D_{\beta\dot{\alpha}}\ \lambda_{\alpha}$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = \delta D = 0$$

but also one non-zero supergauge transformation -

$$\delta\bar{\psi}_{\dot{\beta}} = -4\surd 2\ i\varepsilon_{\dot{\beta}\dot{\alpha}}\ \lambda_{\alpha}$$

However $\zeta_{\alpha\dot{\alpha}}$ is orthogonal to the gauge fixing term (3.2.22), i.e.

$$(\nabla^2\bar{\nabla}^2 + \bar{\nabla}^2\nabla^2)\ \zeta_{\alpha\dot{\alpha}} = 0 \qquad (3.2.31)$$

This implies that the zero mode is orthogonal to all gauge zero modes (see section 3.4), and that it remains a zero mode after gauge fixing. (The fact that it is a zero mode can be checked straightforwardly by using (3.2.21b)).

In the next section we will determine covariant forms for the other zero modes directly from (3.2.30).

3.3 THE FULL SET OF DISCRETE ZERO MODES.

We start this section by describing a method for generating zero modes:

We can move to a new instanton solution by

(1) Changing the instanton parameters (e.g. $a_\mu \to a_\mu - \delta a_\mu$)

(2) Using the corresponding element of the underlying group (in this case P_μ) and shifting the arguments of the instanton (here $x_\mu \to x_\mu + \delta a_\mu$).

These two methods are equivalent for the instanton (i.e. they produce the same shifted instanton solution). But they differ when acting on the space of zero modes. The full shifted space obtained by each method must of course be the same but whereas the zero modes transform covariantly (i.e. into themselves) by method (1) (e.g. Dilatation mode (2.2.15) $\delta A_\mu = F_{\mu\nu}(a)x_\nu \to F_{\mu\nu}(a-\delta a)x_\nu$), they transform into each other by method (2) ($F_{\mu\nu}(a)x_\nu \to F_{\mu\nu}(a-\delta a)x_\nu + F_{\mu\nu}(a-\delta a)\delta a_\nu$). Linear combinations of other zero modes appear; In this example it is the translation zero mode $F_{\mu\nu}$ (2.2.14). The fact that the translation zero mode is obtained from the dilatation zero mode by performing a translation is of course no accident: It follows from the Lie algebra of the group

$$[P_\mu, \Delta] = -iP_\mu$$

These remarks hold for all the zero modes and all the generators of the underlying group. In this way one can generate new zero modes and identify their purpose from the corresponding Lie algebra relation. It is clear that starting from explicit expressions for one or more zero modes this method can be used to generate an invariant subspace of the space of zero modes and often to generate the full space.

Gauge zero modes are also generated by this technique; In particular if the commutator of some generator with the generator corresponding to the zero mode vanishes, it does not necessarily follow that no new zero modes are generated – gauge zero modes (with no obvious connection to the underlying group) are often generated. These gauge modes must be checked individually to see if they are fixed by the gauge fixing term (i.e. $[\nabla^2\bar{\nabla}^2 + \bar{\nabla}^2\nabla^2]\zeta \neq 0$). The fact that the gauge modes may be gauge fixed is a consequence of the fact that the gauge fixing term breaks superconformal invariance. If they are gauge fixed then they are just examples from the uncountably infinite set of gauge zero modes (see section 3.4) and of no interest to us here. If they are not gauge fixed by the (background) gauge fixing term then they are zero mode degrees of freedom for the instanton - even after gauge fixing. Therefore they must be treated separately (from those of section 3.4) - we call these modes super-gauge anomalous zero modes. From the structure of the superconformal algebra [15] it can be shown that starting with the instanton zero mode (3.2.30) and superspace expressions for the changes under the generators of the superconformal algebra ($Q,\bar{Q},S,\bar{S}$) that the full space of zero modes (apart from anomalous gauge modes) will be generated. These changes are ($x = \frac{1}{2}x_+ + \frac{1}{2}x_-$; $x_\pm = x \pm 2i\theta\bar{\theta}$ are the chiral/antichiral coordinates)

$\delta\alpha^\gamma Q_\gamma$:

$$\begin{aligned} \delta\theta &= \delta\alpha \\ \delta x_- &= -4i\delta\alpha\,\bar{\theta} \\ \delta x_+ &= \delta\bar{\theta} = 0 \end{aligned} \qquad (3.3.1a)$$

$\delta\bar{\alpha}^{\dot{\gamma}}\,\bar{Q}_{\dot{\gamma}}$:

$$\begin{aligned} \delta\bar{\theta} &= \delta\bar{\alpha} \\ \delta x_+ &= -4i\delta\bar{\alpha}\,\theta \\ \delta x_- &= \delta\theta = 0 \end{aligned} \qquad (3.3.1b)$$

$\delta\bar{\beta}^{\dot{\gamma}}\,\bar{S}_{\dot{\gamma}}$:

$$\begin{aligned} \delta\theta_\alpha &= i x^+{}_\alpha{}^{\dot{\alpha}}\,\delta\bar{\beta}_{\dot{\alpha}} \\ \delta\bar{\theta}_{\dot{\alpha}} &= -2\delta\bar{\beta}_{\dot{\alpha}}\,\bar{\theta}^2 \\ \delta x^-{}_{\alpha\dot{\alpha}} &= -4\delta\bar{\beta}^{\dot{\gamma}}\,x^-{}_{\dot{\gamma}\alpha}\,\bar{\theta}_{\dot{\alpha}} \\ \delta x_+ &= 0 \end{aligned} \qquad (3.3.1c)$$

$\delta\beta^\gamma\,S_\gamma$:

$$\begin{aligned} \delta\theta_\alpha &= -2\delta\beta_\alpha\,\theta^2 \\ \delta\bar{\theta}_{\dot{\alpha}} &= i x^-{}_{\dot{\alpha}}{}^{\alpha}\,\delta\beta_\alpha \\ \delta x^+{}_{\alpha\dot{\alpha}} &= -4\delta\beta^\gamma\,x^+{}_{\gamma\dot{\alpha}}\,\theta_\alpha \end{aligned} \qquad (3.3.1d)$$

For Q, $\bar{Q}$ and $\bar{S}$, W_γ changes to W'_γ the superinstanton field with appropriate changes in its parameters (use (3.2.13), (3.2.14) on (3.2.15), (3.2.16)). However under S_γ the instanton W_γ field changes non-trivially

$$W_\alpha \to W'_\alpha - \delta\beta^\gamma (8\, \theta_\gamma\, W'_\alpha - 4\theta_\alpha\, W'_\gamma)$$

(where W'_γ is constructed from appropriate changes in the instanton parameters). So that "S_γ" transformations must be accompanied by a compensating change on the W_γ field:

$$\delta W_\alpha = \delta\beta^\gamma (8\, \theta_\gamma\, W'_\alpha - 4\theta_\alpha\, W'_\gamma) \qquad (3.3.1e)$$

Using (3.3.1) on the translation zero mode (3.2.30) and recursively (on the new modes generated) we find all the zero modes and identify their nature from the Lie algebra of the group. They are

$i\bar\theta_{\dot\gamma}\, W_\gamma$	translation zero mode ($P_{\gamma\dot\gamma}$)	(3.3.2a)
$i\bar\theta_{\dot\gamma}\, W_\gamma\, x_+^{\gamma\dot\gamma}$	dilatation zero mode (Δ)	(3.3.2b)
$i\bar\theta^2\, W_\gamma$	supersymmetry zero mode (Q_γ)	(3.3.2c)
$\bar\theta^2\, x^+{}_{\dot\gamma}{}^\gamma\, W_\gamma$	superconformal zero mode ($\bar S_{\dot\gamma}$)	(3.3.2d)
$i\bar\theta_{(\dot\alpha}\, x^+{}_{\dot\gamma)}{}^\gamma\, W_\gamma$	$SU_R(2)$ zero mode ($J_{\dot\alpha\dot\gamma}$)	(3.3.2e)
$\bar\theta_{\dot\gamma}\, \theta^\gamma\, W_\gamma$	supersymmetry zero mode ($\bar Q_{\dot\gamma}$)	(3.3.3a)
$\bar\theta^{\dot\gamma}\, x_{\dot\gamma\gamma}\, \theta^\beta\, W_\beta$	superconformal zero mode (S_γ)	(3.3.3b)
$\bar\theta^2\, \theta^\gamma\, W_\gamma$	chiral (axial) charge zero mode (A)	(3.3.3c)
$i\bar\theta^{\dot\gamma}\, x_{\dot\gamma(\alpha}\, W_{\beta)}$	$SU_L(2)$ zero mode ($J_{\alpha\beta}$)	(3.3.3d)
$i\bar\theta^{\dot\alpha}\, x_{\dot\alpha\alpha}\, W^\gamma\, x^+{}_{\gamma\dot\gamma}$	special conformal zero mode ($K_{\alpha\dot\gamma}$)	(3.3.3e)

One can check, using (3.2.24), that the superinstanton component changes are as expected from these zero modes. In addition, by the same methods, we find the following supergauge anomalous modes -

$$i/g\; W_\alpha \qquad (3.3.4a)$$

$$i/g\; W^\alpha\, x^+{}_{\alpha\dot\alpha} \qquad (3.3.4b)$$

$$1/g\; \theta^\alpha\, W_\alpha \qquad (3.3.5)$$

They are supergauge because they are chiral.
(3.3.4a) and (3.3.4b) are associated with new fermionic parameters which parameterize the superinstantons discrete supergauge degrees of freedom.
e.g. in the expansion of the quantum field in terms of zero modes

$$v = \delta\tilde{\alpha}^{\beta}\ i/g\ W_{\beta} + \delta\tilde{\beta}^{\dot{\beta}}\ i/g\ W^{\beta}\ x^{+}_{\beta\dot{\beta}} + \ldots\ldots$$

and (3.3.5) is associated with a bosonic parameter, say $\delta\mu$. The components of the instanton change as in (3.2.28) with

$$\Omega = 4g\ \delta\tilde{\alpha}^{\beta}\ \lambda_{\beta}\ ,\quad \phi_{\alpha} = 2\sqrt{2}\ ig\ f_{\alpha\beta}\ \delta\tilde{\alpha}^{\beta}\ ,\ \text{all other fields}$$

vanishing (for (3.3.4a)) and,

$$\Omega = 4g\ \delta\tilde{\beta}^{\dot{\beta}}\ \lambda^{\beta}\ x_{\beta\dot{\beta}}\ ,\quad \phi_{\alpha} = 2\sqrt{2}\ ig\ f_{\alpha}^{\ \beta}\ x_{\beta\dot{\beta}}\ \delta\tilde{\beta}^{\dot{\beta}} \qquad (3.3.6)$$

all other fields vanishing

(for (3.3.4b)). In (3.3.5),

$$\delta\psi_{\alpha} = -4\sqrt{2}\ \lambda_{\alpha}\ \delta\mu \qquad (\phi_{\alpha} = -4\sqrt{2}\ ig\ \lambda_{\alpha}\ \delta\mu) \qquad (3.3.7)$$

is the only non-vanishing change in the instanton field.

Many other supergauge modes are generated than just (3.3.4) and (3.3.5); They are always chiral. Indeed, since transformations (3.3.1) preserve chirality, the effect of the transformations on these modes is often to produce further chiral supergauge modes but never a non-gauge zero mode. The supergauge zero modes (apart from (3.3.4) and (3.3.5)) are all gauge fixed and will not be considered further.

(3.3.2e) deserves further explanation. These are 3 bosonic zero modes carrying 3 real parameters, say Θ^{a} ($\Theta^{\dot{\beta}\dot{\gamma}} = \Theta^{a}\sigma^{a\dot{\beta}\dot{\gamma}}$). At the component level and in vector notation

$$\delta A_{\mu} \equiv x_{\nu}\bar{\eta}^{a}_{\nu\lambda}\ F_{\lambda\mu}$$

(where we have used $\bar{\sigma}_{\mu\nu} = -\frac{1}{2}\sigma^{a}\ \bar{\eta}^{a}_{\mu\nu}$)
and this can be written as a gauge transformation

$$\delta A_{\mu} = D_{\mu}\ \Psi(a)$$

$$\Psi(a) = \tau^{b}\ \eta^{b}_{\kappa\nu}\ \bar{\eta}^{a}_{\kappa\lambda}\ x^{\nu}\ x^{\lambda}\ (1+x^{2})$$

so these are the same as the 3 anomalous gauge zero modes discovered by 't Hooft [2].

Since (3.3.7) is the only non-zero component of (3.3.5) and this vanishes for a bosonic instanton (λ_γ=0) it must be that, for a bosonic instanton, (3.3.5) vanishes identically

i.e. $\theta^\gamma W_\gamma = 0.$

If we perform an α transformation ((3.3.1a),(3.2.13)) followed by a $\bar{\beta}$ transformation ((3.3.1c),(3.2.14)) we obtain the general instanton (as in (3.2.16)) and the above equation becomes

$$(\theta^\beta + \alpha^\beta - i\bar{\beta}^{\dot\beta} x^+{}_{\dot\beta}{}^\beta)W_\beta = 0$$

which is the linear relation (3.2.18).

(3.2.18) implies that only two of the three supergauge anomalous modes (3.3.4a),(3.3.4b),(3.3.5) are linearly independent. To obtain the linearly independent set which will parameterize the instanton in a non-singular fashion (at λ_γ=0) we must drop (3.3.5) and retain (3.3.4a) and (3.3.4b). (In fact, of course, we are allowed certain restricted linear combinations of the five modes (3.3.4) and (3.3.5), but note that it can be shown that such a combination must be fermionic (by considering the expansion of v in terms of these modes, at λ_γ = 0) and in any case we will see that the fermionic modes (3.3.4) are chosen automatically as the basis set when we go on to consider orthogonality to the supersymmetry and superconformal modes). Similar arguments hold for the α supersymmetry mode (3.3.3a) and the chiral charge mode (3.3.3c): By (3.2.18) they are not linearly independent of zero modes (3.3.2), and they vanish in the bosonic instanton case -so we drop them.

In order for the various quantities we will be considering to be well defined it is necessary to impose the restriction of square integrability on the components of the superfield. This excludes some of the modes in (3.3.3), which do not satisfy this requirement:

The β superconformal mode (3.3.3b) contains a component that is not square integrable:

$$\delta A_{\alpha\dot{\alpha}} = -4g\, x_{\dot{\alpha}\gamma}\, \lambda_{\alpha}$$

$$= -2g\, x_{\dot{\alpha}\gamma}\, x_{\dot{\gamma}}{}^{\beta}\, f_{\beta\alpha}\bar{\beta}^{\dot{\gamma}} + \ldots\ldots$$

$$\sim O(1/x^2) \quad \text{as } x \to \infty$$

It can be made square integrable by adding a linear combination of the special conformal modes (3.3.3e), namely

$$\bar{\beta}^{\dot{\gamma}}\, (i\bar{\theta}^{\dot{\beta}}\, x_{\dot{\beta}\beta}\, W^{\gamma}\, x^{+}{}_{\gamma\dot{\gamma}}\,)$$

The mode vanishes in the bosonic case and, using (3.2.18), it can be written as a linear combination of dilatation and $SU_L(2)$ zero modes. So we drop it.

The special conformal zero mode (3.3.3e) is not square integrable so this one is dropped.

And finally the $SU_L(2)$ zero mode (3.3.3d) can be re-expressed as a gauge zero mode (of form (3.2.28)) at the component level:

$$\delta A_{\mu} \equiv x_{\nu}\, \eta^{a}{}_{\nu\lambda}\, F_{\lambda\mu} = D_{\mu}\, \Omega(a)$$

$$\text{where } \Omega(a) = \frac{1}{2}\left[\frac{\tau^{a}}{1+x^2}\right]$$

and so, by using the definitions (3.2.27) and expanding Λ and Λ in terms of θ and θ we could construct explicit expressions for Λ and Λ. It follows that (3.3.3d) is a gauge mode at the superfield level (i.e. of form (3.2.26)) - but it is gauge fixed so we need not consider it further.

Note that the special conformal mode (3.3.3e), which we dropped because it is not square integrable, is also a gauge mode at the component level (see above (2.1.17)) and hence by the above argument a gauge mode at the superfield level. However it too is gauge fixed and this provides us with another reason for not considering it further.

This leaves us with the linearly independent set of zero modes (3.3.2) and (3.3.4). (Their linear independence is best seen by inspection at the component level). They are all orthogonal to the gauge fixing term.

The bosonic modes are the 4 translation, 1 dilatation, and 3 $SU_R(2)$ (or 't Hooft anomalous) zero modes ; they lead to changes in the instanton parameters a_μ, ρ, Θ^a.

The fermionic modes are the 2 supersymmetry, 2 superconformal and 4 supergauge anomalous modes which are associated with changes in the instanton parameters α_γ, $\bar{\beta}_{\dot{\gamma}}$, $\tilde{\alpha}_\gamma$, $\tilde{\beta}_{\dot{\gamma}}$ (see comments following (3.3.4)).

Hence there are a total of 8 bosonic and 8 fermionic zero modes.

In order to calculate with the superinstanton background field we will need a method of projecting out the coefficients multiplying the zero modes in the expansion of a general quantum field "v" (see discussion of general method in chapter 2.3). This is most conveniently done by defining an inner product on the superfields. The natural definition is

$$(U,V) = \mathrm{tr}\int d^8z\ U\ V \tag{3.3.8}$$

and this works for the bosonic modes e.g.

$$\zeta_{\alpha\dot{\alpha}} = i\bar{\theta}_{\dot{\alpha}}\, W_\alpha \qquad \text{the translation zero mode}$$

$$(\zeta_{\alpha\dot{\alpha}}\, ,\zeta_{\beta\dot{\beta}}\,) = \frac{1}{4}\, \epsilon_{\dot{\alpha}\dot{\beta}}\, \epsilon_{\alpha\beta}\, \mathrm{tr}\int d^4x\ d^2\theta\ W^\gamma\, W_\gamma$$

$$= -\,256\pi^2\, \epsilon_{\dot{\alpha}\dot{\beta}}\, \epsilon_{\alpha\beta}$$

or in 4-component notation (using (3.1.8))

$$(\zeta_\mu\, ,\zeta_\nu\,) = 512\pi^2\, \delta_{\mu\nu} \tag{3.3.9}$$

But in the case of the fermionic modes, the α and $\bar{\beta}$ modes both have zero norm $(\zeta,\zeta) = 0$ since θ^4 appears in the integrand of (3.3.8). And the $\tilde{\alpha}$ and $\tilde{\beta}$ modes have zero norm since the integrand is chiral and $\int d^2\bar{\theta} = 0$. However the norm between α and $\tilde{\alpha}$,and, β and $\tilde{\beta}$ zero modes is easily seen to be proportional to a non vanishing integral

$$(g^2\mathrm{tr}\int d^4x\ F^2_{\mu\nu} \text{ and } g^2\mathrm{tr}\int d^4x\ F^2_{\mu\nu}\ x^2 \text{ respectively}).$$

In fact it is straightforward to show that apart from this difference the 16 modes form an orthogonal set:

Let $\zeta_\kappa = (i\bar{\theta}_{\dot{\beta}}W_\beta;\ \frac{i}{\rho}\ \bar{\theta}_{\dot{\beta}}W_\beta x_+{}^{\dot{\beta}\beta};\ \frac{i}{\rho}\ \bar{\theta}_{(\dot{\beta}}x^+{}_{\dot{\gamma})}{}^\beta W_\beta;\ i\bar{\theta}^2 W_\beta\ \rho^{-\frac{1}{2}};\ iW_\beta\ \rho^{\frac{1}{2}};$

$\bar{\theta}^2 x^+{}_{\dot{\beta}}{}^\beta W_\beta\ \rho^{-3/2};\ W^\beta x^+{}_{\beta\dot{\beta}}\ \rho^{-\frac{1}{2}})$

where for convenience the zero modes are multiplied by powers of ρ so that the dimension $[\zeta_\kappa] = +1$. "κ" runs from 1 to 16 and is spinorial or bispinorial as appropriate.

And let ζ'_κ be as above except that the 3rd and 4th entries are swopped round and the same with the 5th and 6th. Then

$$(\zeta_\kappa\ ,\zeta'^{\kappa'}\ ) = \delta_\kappa{}^{\kappa'}$$

where the numerical constants have been ignored and " $\delta_\kappa{}^{\kappa'}$ " is interpreted appropriately.

If we set

$$b^\kappa = (\delta a^{\beta\dot{\beta}}\ ,\delta\rho\ ,\rho^{\frac{1}{2}}\ \delta\Theta^{\dot{\beta}\dot{\gamma}}\ ,\rho^{\frac{1}{2}}\ \delta\alpha^\beta\ ,\rho^{-\frac{1}{2}}\ \delta\tilde{\alpha}^\beta\ ,\rho^{3/2}\ \delta\bar{\beta}^{\dot{\beta}}\ ,\rho^{\frac{1}{2}}\ \delta\tilde{\beta}^{\dot{\beta}}\ )$$

then $[b^\kappa] = -1$

and a general quantum field "v" can be expressed as

$$v = \sum_\kappa b^\kappa\ \zeta_\kappa\ + v^+$$

$$b^\kappa = (v\ ,\zeta'^\kappa\ )$$

where v^+ contains no zero modes.

It does however contain +ve frequency modes and continuous gauge zero modes (which become +ve frequency modes once the gauge fixing term is added) .These are the subject of section 3.4.

3.4 GAUGE ZERO MODES, GREENS FUNCTIONS AND PROJECTION OPERATORS.

We start this section by considering the continuous parameter gauge modes, they are of the form (3.2.26)

$$v=i(\bar{\Lambda}-\Lambda) \qquad (3.4.1)$$

Although they are zero modes they are gauge fixed by the term (3.2.22)

$$S_{GF}=-\frac{tr}{32\xi}\int d^8z\, v(\nabla^2\bar{\nabla}^2+\bar{\nabla}^2\nabla^2)v$$

so we parameterize these modes by considering eigenmodes of

$$(\nabla^2\bar{\nabla}^2+\bar{\nabla}^2\nabla^2)v_{\lambda\kappa}=16\lambda^2\, v_{\lambda\kappa} \quad \lambda \text{ real and } >0 \qquad (3.4.2)$$

(The factor of 16 is just for convenience. The positivity of the eigenvalue ($16\lambda^2$) is required so that S_{GF} is negative definite as is the original action S_V). λ and a further discrete parameter κ (which can be spinorial) serve to label the eigenmodes.

Using (3.4.1) this is

$$\nabla^2\bar{\nabla}^2\,\bar{\Lambda}_{\lambda\kappa}=16\lambda^2\,\bar{\Lambda}_{\lambda\kappa} \qquad (3.4.3)$$

$$\bar{\nabla}^2\nabla^2\,\Lambda_{\lambda\kappa}=16\lambda^2\,\Lambda_{\lambda\kappa}$$

or

$$\Box_-\,\bar{\Lambda}_{\lambda\kappa}=-\lambda^2\,\bar{\Lambda}_{\lambda\kappa} \qquad (3.4.4)$$

$$\Box_+\,\Lambda_{\lambda\kappa}=-\lambda^2\,\Lambda_{\lambda\kappa}$$

(note that $\Lambda_{\lambda\kappa}$ and $\bar{\Lambda}_{\lambda\kappa}$ will be fermionic eigenstates when κ is spinorial). We have used $\{\bar{\nabla}^{\dot{\gamma}},[\bar{\nabla}_{\dot{\gamma}},\nabla^2]\}=-16\Box_+$ and similarly for $\Box_-$

where $\qquad \Box_+=\Box-1/8\, W^{\alpha}\,\nabla_{\alpha} \qquad (3.4.5)$

similarly $\qquad \Box_-=\Box+1/8\,\bar{W}^{\dot{\alpha}}\,\bar{\nabla}_{\dot{\alpha}}$

$\qquad\qquad =\Box \qquad$ for a superinstanton $\qquad (3.4.6)$

(3.4.4) would be sufficient for the definition of an orthonormal set of

gauge zero modes $\Lambda,\bar{\Lambda}$ if the background field was self conjugate since then $\Lambda=(\bar{\Lambda})^*$. This is not the case here and it is then not clear how to associate the $\Lambda_{\lambda\kappa}$ with $\bar{\Lambda}_{\lambda\kappa'}$. Fortunately we can give a more restrictive definition

$$\bar{\nabla}^2 \bar{\Lambda}_{\lambda\kappa} = 4\lambda \Lambda_{\lambda\kappa} \tag{3.4.7a}$$

$$\nabla^2 \Lambda_{\lambda\kappa} = 4\lambda \bar{\Lambda}_{\lambda\kappa} \tag{3.4.7b}$$

(3.4.4) follows from (3.4.7).

In a superinstanton background (see (3.4.4) and (3.4.6))

$$\Box \bar{\Lambda}_{\lambda\kappa} = -\lambda^2 \bar{\Lambda}_{\lambda\kappa} \tag{3.4.8}$$

(3.4.8) and (3.4.7a) now serve to define the modes $\Lambda_{\lambda\kappa}$ and $\bar{\Lambda}_{\lambda\kappa}$ in terms of antichiral eigenstates of $\Box$.

Defining the chiral and antichiral norm by

$$(\ ,\ )^C = \mathrm{tr} \int d^4x d^2\theta$$

$$(\ ,\ )^A = \mathrm{tr} \int d^4x d^2\bar{\theta}$$

(3.4.8) allows us to show

$$(\bar{\Lambda}_{\lambda\kappa} , \bar{\Lambda}_{\lambda'\kappa'})^A = 0 \qquad \text{for } \lambda \neq \lambda'$$

So by orthogonalising the degenerate (κ) modes we can normalise such that

$$(\bar{\Lambda}_{\lambda\kappa} , \bar{\Lambda}_{\lambda'}{}^{\kappa'})^A = - \delta(\lambda - \lambda') \, \delta_\kappa{}^{\kappa'} \tag{3.4.9}$$

(The minus sign is included here since it can be shown, by for example using the component field definitions (3.2.27), that the (anti)chiral norm is negative definite in Euclidean space).
By using (3.4.7),(3.2.4) and super-integration by parts this implies

$$(\Lambda_{\lambda\kappa} , \bar{\Lambda}_{\lambda'}{}^{\kappa'}) = \lambda \, \delta(\lambda - \lambda') \, \delta_\kappa{}^{\kappa'} \tag{3.4.10}$$

and

$$(\Lambda_{\lambda\kappa} , \Lambda_{\lambda'}{}^{\kappa'})^C = - \delta(\lambda - \lambda') \, \delta_\kappa{}^{\kappa'} \tag{3.4.11}$$

This allows us to show that these modes are orthogonal to the discrete modes since (using (3.4.7))

$$(\zeta\ ,\Lambda_{\lambda\kappa}\ ) = \frac{1}{16\lambda^2}(\nabla^2\bar{\nabla}^2\ \zeta\ ,\Lambda_{\lambda\kappa}\ )$$

and $$(\zeta\ ,\bar{\Lambda}_{\lambda\kappa}) = \frac{1}{16\lambda^2}(\bar{\nabla}^2\nabla^2\ \zeta\ ,\bar{\Lambda}_{\lambda\kappa}\ )$$

and $\nabla^2\bar{\nabla}^2\ \zeta\ = \bar{\nabla}^2\nabla^2\ \zeta\ = 0$ for all discrete zero modes. These chiral and antichiral modes satisfy a completeness relation

$$\int_\lambda 16\lambda\ \Lambda_\lambda{}^\kappa(z)\ \bar{\Lambda}_{\lambda\kappa}(z') = \bar{\nabla}^2\ \nabla'^2\{\ \delta^8(z\ -z')\} \qquad (3.4.12)$$

(which can be checked using (3.4.10) and (3.4.7)).

The projection operator on chiral states is

$$P_C(z,z') = \int_\lambda \frac{1}{\lambda}\ \ \Lambda_\lambda{}^\kappa(z)\ \bar{\Lambda}_{\lambda\kappa}(z')$$

$$\Rightarrow \qquad P_C = -1/16\ \bar{\nabla}^2\ \Box^{-1}\ \nabla^2 \qquad (3.4.13)$$

(using (3.4.12) and (3.4.8))
and similarly, the projector on antichiral states is

$$P_A(z,z') = \int_\lambda \frac{1}{\lambda}\ \ \bar{\Lambda}_\lambda{}^\kappa(z)\ \Lambda_{\lambda k}(z')$$

$$\Rightarrow \qquad P_A = -1/16\ \nabla^2\ \Box^{-1}\ \bar{\nabla}^2 \qquad (3.4.14)$$

The properties of these projectors can be checked by using the identities

$$[\ \bar{\nabla}_{\dot{\gamma}},\ \Box_+\ ] = 0$$

$$[\ \nabla_\gamma,\ \Box\ ] = 0$$

$$\Box_+\ \nabla^2 = \Box\ \nabla^2 \qquad (3.4.15)$$

$$\nabla^2\ \Box_+ = \nabla^2\ \Box$$

which are true for also for any power of $\Box$ ($\Box_+$).

If we consider a Wess-Zumino action for some (background) chiral and antichiral fields ϕ and $\bar{\phi}$ (for example the ghost Lagrangian in SYM) then the propagator $\langle\phi(z)\bar{\phi}(z')\rangle$ can be defined through

$$\Box_+ \langle \phi(z)\bar{\phi}(z')\rangle = - P_C(z,z')$$

which implies

$$\langle \phi(z)\bar{\phi}(z')\rangle = \int_\lambda \frac{1}{\lambda^3} \Lambda_\lambda{}^\kappa(z)\, \bar{\Lambda}_{\lambda\kappa}(z')$$

(using (3.4.4) and (3.4.13))

$$= 1/16\ (\bar{\nabla}^2\, \Box^{-2}\, \nabla^2\ )_{zz'} \qquad (3.4.16)$$

(using (3.4.8) and (3.4.13))

and similarly $\langle \bar{\phi}(z)\phi(z')\rangle$ satisfies

$$\Box \langle \bar{\phi}(z)\phi(z')\rangle = - P_A(z,z')$$

so

$$\langle \bar{\phi}(z)\phi(z')\rangle = \int_\lambda \frac{1}{\lambda^3} \bar{\Lambda}_\lambda{}^\kappa(z)\, \Lambda_{\lambda\kappa}(z')$$

$$= 1/16\ (\nabla^2\, \Box^{-2}\, \bar{\nabla}^2\ )_{zz'} \qquad (3.4.17)$$

(Infact (3.4.16) and (3.4.17) follow easily by inspection of P_C and P_A , and (3.4.15)).

We turn our attention to the +ve frequency non-gauge modes. They must be eigen modes of (3.2.21). They turn out to be

$$v_{\lambda\kappa\dot{\alpha}} = \frac{1}{\sqrt{2}\,\lambda}\, \bar{\nabla}_{\dot{\alpha}}\, \bar{\Lambda}_{\lambda\kappa} \qquad (3.4.18)$$

since (using (3.2.21b))

$$\frac{1}{8}\, \bar{\nabla}^{\dot{\gamma}}\, \nabla^2\, \bar{\nabla}_{\dot{\gamma}}\, v_{\lambda\kappa\dot{\alpha}} = - \frac{1}{16\sqrt{2}\,\lambda}\ \bar{\nabla}_{\dot{\alpha}} \nabla^2 \bar{\nabla}^2\, \bar{\Lambda}_{\lambda\kappa}$$

$$= - \lambda^2\, v_{\lambda\kappa\dot{\alpha}} \qquad (3.4.19)$$

(using (3.4.3))

The "$\dot{\alpha}$" is an extra label and this means there are twice as many non-gauge vector modes as there are antichiral modes of $\Box$ – for each value of λ. Note that $v_{\lambda\kappa\dot{\alpha}}$ is fermionic when $\bar{\Lambda}_{\lambda\kappa}$ is bosonic and vice versa.

Equations (3.4.4), (3.4.2) and (3.4.19) show that the vector, chiral and antichiral fields have the same spectrum of non zero eigenvalues (apart from the multiplicative factor of "$1/\xi$" that appears for the vector gauge modes when not in the Feynman gauge).

Although these equations were formulated for the adjoint representation it is clear that the construction arguments hold for v, Λ, $\bar{\Lambda}$ in any representation of the gauge group.

The +ve frequency non-gauge modes are orthonormal

$$(v_{\lambda\kappa\dot{\alpha}}, v_{\lambda'}{}^{\kappa'\dot{\beta}}) = \delta(\lambda - \lambda')\, \delta_{\kappa}{}^{\kappa'}\, \delta_{\dot{\alpha}}{}^{\dot{\beta}} \qquad (3.4.20)$$

(follows from integration by parts and (3.4.7a), (3.4.10))

They are orthogonal to the gauge modes and the discrete zero modes because if ζ is one of these modes

$$\begin{aligned}(\zeta, v_{\lambda\kappa\dot{\alpha}}) &= -\frac{1}{8\lambda^2}(\zeta, \bar{\nabla}^{\dot{\beta}} \nabla^2 \bar{\nabla}_{\dot{\beta}} v_{\lambda\kappa\dot{\alpha}}) \\ &= -\frac{1}{8\lambda^2}(\bar{\nabla}^{\dot{\beta}} \nabla^2 \bar{\nabla}_{\dot{\beta}} \zeta, v_{\lambda\kappa\dot{\alpha}}) \\ &= 0\end{aligned}$$

(ζ is a zero mode of (3.2.21)).

We can construct the projection operator onto the +ve frequency non gauge modes

$$P(z,z') = \int_{\lambda} v_{\lambda}{}^{\kappa\dot{\alpha}}(z)\, v_{\lambda\kappa\dot{\alpha}}(z')$$

which implies

$$P = 1/8\; \bar{\nabla}^{\dot{\beta}} \nabla^2 \Box^{-1} \bar{\nabla}_{\dot{\beta}} \qquad (3.4.21)$$

(which can be derived from (3.4.18), (3.4.7b), (3.4.16) or (3.4.17) and properties (3.4.15)).
The term in the quadratic action (3.2.21) satisfies

$$(1/8\; \bar{\nabla}^{\dot{\alpha}} \nabla^2 \bar{\nabla}_{\dot{\alpha}})\, P = 1/8\; \bar{\nabla}^{\dot{\alpha}} \nabla^2 \bar{\nabla}_{\dot{\alpha}} \qquad (3.4.22)$$

which shows that (3.4.18) is the <u>complete set</u> of +ve frequency modes for (3.2.21).

We can define a Greens function in the space of these +ve frequency non-gauge modes by

$$1/8\ \bar{\nabla}^{\dot{\alpha}}\ \nabla^2\ \bar{\nabla}_{\dot{\alpha}}\ G(z,z') = -\ P(z,z') \tag{3.4.23}$$

and obtain $\quad G(z,z') = 1/8\ (\bar{\nabla}^{\dot{\alpha}}\nabla^2\ \Box^{-2}\ \bar{\nabla}_{\dot{\alpha}})_{zz'}$

And finally we can consider the propagator $\langle v(z)v(z')\rangle$ over +ve frequency modes. It satisfies

$$(1/8\bar{\nabla}^{\dot{\alpha}}\nabla^2\bar{\nabla}_{\dot{\alpha}} - \frac{1}{16\xi}[\nabla^2\bar{\nabla}^2+\bar{\nabla}^2\nabla^2])\langle v(z)v(z')\rangle = -\{P(z,z')+P_A(z,z')+P_C(z,z')\} \tag{3.4.24}$$

(The term on the left is the gauge fixed quadratic action, and on the right is the projector onto +ve frequency modes).

Hence from (3.4.16), (3.4.17), and (3.4.23)

$$\langle v(z)v(z')\rangle = (1/8\bar{\nabla}^{\dot{\alpha}}\nabla^2\Box^{-2}\bar{\nabla}_{\dot{\alpha}} - \xi/16\ [\bar{\nabla}^2\Box^{-2}\nabla^2 + \nabla^2\Box^{-2}\bar{\nabla}^2])_{zz'} \tag{3.4.25}$$

These results for the propagators and projectors agree with the relevant conclusions of [17].

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CHAPTER 4 RENORMALISATION AROUND SUPERINSTANTONS.

In this chapter we consider super Yang-Mills theory in its superfield form and evaluate (to two loops) the divergent quantum corrections to the classical action of the superinstanton. In component field theories [1,2](see sections 2.2 and 2.3) the first step is to factor out the zero modes from the measure in the generating functional. This is achieved by inserting δ-functions which constrain the quantum field to lie in the space of non-zero modes, replacing the integral over zero modes by an integral over collective coordinates [3]. When these δ-functions deal with gauge zero modes they generate the background gauge fixing term [1,4]. The Jacobian (of the change of variables from zero modes to collective coordinates) generates the ghost action together with new interactions involving the vector field, ghosts, and the discrete zero modes [1,2]. In superfield super Yang-Mills the scenario is the same as the above except that there is an additional complication: the ghost fields themselves have zero modes. To take care of these modes it is easiest to consider the above for a (general) background field for which the ghost measure includes the whole of (square integrable) function space. We can then factor out the ghost zero modes that occur when the background field is set equal to an instanton.

We divide this chapter into three sections. The first section is concerned with factoring out the zero modes (by replacing them with collective coordinates) and calculating the associated Jacobian. This is the analogue of section 2.2. The second section discusses the semiclassical approximation. The final section considers higher order quantum corrections. These last two sections are the appropriate generalisation the methods discussed in section 2.3. As in that section we investigate in particular the two loop contributions, splitting them into short and long distance parts. We check that the divergences are indeed those required to renormalise the semiclassical result.

Unlike the corresponding case in section 2.3 we find that there are no divergent corrections from the new interactions in the Jacobian (to any order) and that, once renormalization has taken place, the explicit $\ln\mu$ dependence of the full two loop graphs vanishes. Later (in chapter 5) we discuss some conclusions that can be drawn from this calculation, and in particular their relevance to gaining some understanding of a remarkable paper by Novikov et al [5]. (See also the Introduction).

4.1 COLLECTIVE COORDINATES AND THE JACOBIAN.

In this section we wish to perform the analogous steps to those described in section 2.2. However, as mentioned in the introduction to this chapter, we have to deal with a further problem in this superfield treatment: that of ghost zero modes. This is best dealt with if we begin the evaluation of the Jacobian by considering a general background field for which the ghost measure includes the whole of square integrable function space. We will see later that this is equivalent to the assumption that all square integrable gauge modes (and therefore all square integrable gauge transformations) are fixed by the background gauge fixing term. In chapter 3 a study was performed of the zero modes and the gauge modes for a superinstanton background field. We begin by generalizing the results of that chapter to the case of this background field. We work in Euclidean space and terms such as "real" and "conjugate" will refer to Osterwalder-Schrader conjugation [12] as explained in section 3.1.

We consider super Yang-Mills in the presence of a background field. (See section 1.2 and 1.3). The full prepotential v_f is expressed in terms of the quantum prepotential v and the background field (ω and $\bar{\omega}$) by (1.3.1)

$$e^{gv_f} = e^{g\omega}\, e^{gv}\, e^{g\bar{\omega}} \qquad (4.1)$$

The action is written in the quantum chiral background vector representation (see (1.3.6))

$$S_V = S + \bar{S}$$

$$S = \frac{tr}{128g^2}\int d^4x d^2\theta\ \bar{\nabla}^2(e^{-gv}\,\nabla^\alpha\, e^{gv})\ \bar{\nabla}^2(\ e^{-gv}\,\nabla_\alpha\, e^{gv}) \qquad (4.2)$$

where the covariant derivatives, which are in the vector representation (see (1.2.65)), contain only the background field (ω and $\bar{\omega}$).

$$\nabla_\alpha = e^{-g\omega}\, D_\alpha\, e^{g\omega}$$
$$\bar{\nabla}_{\dot{\alpha}} = e^{g\bar{\omega}}\, \bar{D}_{\dot{\alpha}}\, e^{-g\bar{\omega}} \qquad (4.3)$$

The background field strengths are defined by (compare (1.2.56))

$$W^{\gamma} = [\bar{\nabla}^{\dot{\gamma}}, \{\bar{\nabla}_{\dot{\gamma}}, \nabla^{\gamma}\}] \qquad (4.4)$$

and the conjugate relation for $\bar{W}_{\dot{\gamma}}$.

The quadratic action is (compare (3.2.20), note that (4.5) follows from (1.3.13) and comments below it)

$$\delta S_V = \frac{tr}{2}\int d^8z\ v\hat{O}v$$
$$\hat{O} = 1/8\ (\nabla^{\alpha}\bar{\nabla}^2\nabla_{\alpha} - W^{\alpha}\nabla_{\alpha}) \qquad (4.5)$$

When the background field satisfies its equations of motion $\hat{O}$ will have a number (m) of discrete zero modes "ζ_i",

$$\hat{O}\ \zeta_i = 0$$

which can be normalized:

$$(\zeta_i, \zeta^{i'}) = tr\int d^4x\ d^4\theta\ \zeta_i\ \zeta^{i'} = \delta_i^{\ i'} \qquad (4.6)$$

and chosen so that they transform homogeneously under background gauge transformations . (The index i runs over the m discrete modes and may be spinorial or bispinorial as appropriate). Specific examples of these modes were seen in sections 3.2 and 3.3. In addition there is a continuous parameter set of gauge zero modes which may be labelled through antichiral eigenstates of (as described in [10] and section 3.4)

$$\Box_- = \Box + 1/8\ \bar{W}^{\dot{\alpha}}\ \bar{\nabla}_{\dot{\alpha}} \qquad (4.7)$$

by

$$-16\ \nabla^2\bar{\nabla}^2\ \bar{\Lambda}_{\lambda\kappa} = \Box_-\ \bar{\Lambda}_{\lambda\kappa} = -\lambda^2\ \bar{\Lambda}_{\lambda\kappa}$$

The corresponding chiral eigenstates $\Lambda_{\lambda\kappa}$ being given by

$$\bar{\nabla}^2\ \bar{\Lambda}_{\lambda\kappa} = 4\lambda\ \Lambda_{\lambda\kappa} \qquad (4.8a)$$

which implies

$$\nabla^2\ \Lambda_{\lambda\kappa} = 4\lambda\ \bar{\Lambda}_{\lambda\kappa} \qquad (4.8b)$$

(κ is a discrete (spinorial or bi-spinorial) label which serves to distinguish the degenerate eigenmodes). We assume that the background field is such that the $\Lambda_{\lambda\kappa}$ and $\bar{\Lambda}_{\lambda\kappa}$ form a complete set of states that span the spaces of square integrable chiral and antichiral fields.

The modes in (4.8) are orthogonal to the discrete modes (see section 3.4 (or [10]) for the reasoning)

$$(\zeta_i, \Lambda_{\lambda\kappa}) = (\zeta_i, \bar{\Lambda}_{\lambda\kappa}) = 0$$

and can be chosen so that

$$(\Lambda_{\lambda\kappa}, \Lambda_{\lambda'}{}^{\kappa'})^C = -\delta(\lambda - \lambda')\,\delta_\kappa{}^{\kappa'}$$

$$(\bar{\Lambda}_{\lambda\kappa}, \bar{\Lambda}_{\lambda'}{}^{\kappa'})^A = -\delta(\lambda - \lambda')\,\delta_\kappa{}^{\kappa'}$$

$$(\Lambda_{\lambda\kappa}, \bar{\Lambda}_{\lambda'}{}^{\kappa'}) = \lambda\,\delta(\lambda - \lambda')\,\delta_\kappa{}^{\kappa'} \qquad (4.9)$$

$$(\ ,\ )^C = \mathrm{tr}\int d^4x d^2\theta = \mathrm{tr}\int d^6z$$

$$(\ ,\ )^A = \mathrm{tr}\int d^4x d^2\bar{\theta}$$

In addition they satisfy a completeness relation

$$\int_\lambda 16\lambda\; \Lambda_\lambda{}^\kappa(z)\,\bar{\Lambda}_{\lambda\kappa}(z') = \bar{\nabla}^2\,\nabla'^2\{\,\delta^8(z - z')\} \qquad (4.10)$$

(The integral over λ is written formally here to emphasize the fact that it may be interpreted, here and elsewhere, as a discrete sum when appropriate for parts of the positive eigenvalue spectrum [3,16]).

Note that the above equations follow straightforwardly in the same way as they did for instanton case in section 3.4 (equations (3.4.9) to (3.4.12)).

The background field solution contains parameters which may be put in one-one correspondence with these zero modes (call these parameters b^j, $a_{\lambda\kappa}$, $\bar{a}_{\lambda\kappa}$) if we choose them such that for small changes (see formulae below (3.2.22) and above (3.2.25))

$$e^{g\omega} \rightarrow e^{i\delta\bar{\Lambda}'}\, e^{g\omega}\, e^{\frac{1}{2}\delta\zeta}$$

$$e^{g\bar{\omega}} \rightarrow e^{\frac{1}{2}\delta\zeta}\, e^{g\bar{\omega}}\, e^{-i\delta\Lambda'} \qquad (4.11)$$

where

$$\delta\bar{\Lambda}' = \int_\lambda \delta\bar{a}_\lambda{}^\kappa\ \bar{\Lambda}'_{\lambda\kappa}$$

$$\delta\Lambda' = \int_\lambda \delta a_\lambda{}^\kappa\ \Lambda'_{\lambda\kappa} \qquad (4.12)$$

$$\delta\zeta = \sum_\kappa \delta b^\kappa\ \zeta_\kappa$$

The prime on Λ' and $\bar{\Lambda}'$ indicate that these are standard chiral and antichiral fields; they are related to the covariantly chiral and covariantly antichiral fields in the obvious manner (using the definitions (1.2.65), (1.2.70) and (1.2.21))

$$\Lambda' = e^{-g\bar{\omega}}\ \Lambda\ e^{g\bar{\omega}}$$

$$\bar{\Lambda}' = e^{g\omega}\ \bar{\Lambda}\ e^{-g\omega} \qquad (4.13)$$

In calculating the Jacobian it is necessary to evaluate the change in the quantum field with respect to a change in the background field parameters. This is found [2] by noting that the full field (4.1) is independent of the background-quantum splitting (see the discussion above (2.2.11) in section 2.2) and hence the change in the quantum field is such as to absorb the changes (4.11) i.e.

$$e^{gv} \to e^{-\frac{1}{2}\delta\zeta}\ e^{-i\delta\bar{\Lambda}}\ e^{gv}\ e^{i\delta\Lambda}\ e^{-\frac{1}{2}\delta\zeta} \qquad (4.14)$$

which implies (to first order in the background field parameters, compare (1.2.43) from which this equation can be derived)

$$\delta v = \tfrac{1}{2}iL_v\{(\delta\Lambda + \delta\bar{\Lambda}) + \coth(g/2\ L_v)(\delta\Lambda - \delta\bar{\Lambda})\} - \tfrac{1}{2}L_v\ \coth(g/2\ L_v)\delta\zeta$$

where $L_vX = [v,X]$. (4.15)

We insert into the generating functional

$$Z = \int \mathcal{D}(v)\ \ e^{S_v} \qquad (4.16)$$

a factor

$$1 = N^{-1} \int \mathcal{D}(a,\bar{a})\ \mathcal{D}(K,\bar{K})\ d^m b\ \exp\{\ -1/\xi \int_\lambda \lambda\ K_\lambda{}^\kappa\ \bar{K}_{\lambda\kappa}\ \}$$

$$\times \prod_{\lambda\kappa} \delta[(\Lambda_{\lambda\kappa},v) - \bar{K}_{\lambda\kappa}]\ \delta[(\bar{\Lambda}_{\lambda\kappa},v) - K_{\lambda\kappa}]\ \prod_j \delta[(v,\zeta_j)]\ \ \mathrm{Sdet}(J) \qquad (4.17)$$

We now explain the various terms in (4.17) (which is the analogue of (2.2.7) plus the appropriate 't Hooft gaussian average, see also [2] and (1.3.8)).

N is the normalisation factor for the Gaussian average

$$N = \int \mathcal{D}(K,\bar{K}) \quad \exp\{ -1/\xi \int_\lambda \lambda\, K_\lambda{}^\kappa\, \bar{K}_{\lambda\kappa} \}$$

$$= \prod_{\lambda\kappa} (\xi/\lambda)^{i_\kappa} \tag{4.18}$$

$$= \mathrm{Sdet}^{-1} (\Box_-^{\frac{1}{2}} / \xi)$$

where $i_\kappa = +1$ if $\Lambda_{\lambda\kappa}$ is a bosonic mode and $i_\kappa = -1$ if $\Lambda_{\lambda\kappa}$ is a fermionic mode. (This depends on the index κ : if κ is spinorial then $\Lambda_{\lambda\kappa}$ will be fermionic).

For any covariantly chiral field Λ :

$$\int \mathcal{D}(\Lambda) \exp\{ \int d^6z\ \tfrac{1}{2}\ \Lambda^2 \} = 1 \qquad , \tag{4.19}$$

(see equation (6.5.41) of [13]. (4.19) can be treated as a definition. It may be justified by defining $\mathcal{D}(\Lambda)$ as the integral over its component fields, or by an integral over the associated standard chiral field and its component fields [13].)

By expressing Λ in terms of the eigenmodes (4.8)

$$\Lambda = \int_\lambda a_\lambda{}^\kappa\ \Lambda_{\lambda\kappa}$$

we can use (4.9) and (4.19) to make the identification

$$\int \mathcal{D}\Lambda \equiv \int \prod_{\lambda\kappa} da_{\lambda\kappa} \quad \text{(which we will sometimes write as } \int \mathcal{D}(a) \text{).} \tag{4.20}$$

Hence for anticommuting covariantly chiral and antichiral fields, η and $\bar{\eta}$, we obtain (by expanding in terms of $\Lambda_{\lambda\kappa}$ and $\bar{\Lambda}_{\lambda\kappa}$ and using (4.9))

$$\int \mathcal{D}(\eta,\bar{\eta}) \exp\{ -\int d^8z\ \eta\bar{\eta} \} = \mathrm{Sdet}^{\frac{1}{2}} (\Box_-) \tag{4.21}$$

(in agreement with other methods [13]).

Hence the factor N^{-1} in (4.17) can be written

$$N^{-1} = \prod_{\kappa\lambda}(1/\xi) \int D(\eta,\bar{\eta}) \exp\{ -\int d^8z\, \eta\bar{\eta} \} \qquad (4.22)$$

These are the usual Nielsen-Kallosh ghosts.

Integration over $K_{\lambda\kappa}$ and $\bar{K}_{\lambda\kappa}$ in (4.17) gives (using (4.10)) the usual background gauge fixing term (as in (1.3.9))

$$\exp S_{GF} = \exp\{ -\frac{tr}{32\xi} \int d^8z\, v(\nabla^2\bar{\nabla}^2 + \bar{\nabla}^2\nabla^2)v \} \qquad (4.23)$$

The matrix J in (4.17) is the following :-

$$J^{[p],[q]} = \begin{bmatrix} \frac{\partial}{\partial a_{\lambda\kappa}}(\Lambda_{\lambda'\kappa'}, v) & \frac{\partial}{\partial a_{\lambda\kappa}}(\bar{\Lambda}_{\lambda'\kappa'}, v) & \frac{\partial}{\partial a_{\lambda\kappa}}(\zeta_{j'}, v) \\ \frac{\partial}{\partial \bar{a}_{\lambda\kappa}}(\Lambda_{\lambda'\kappa'}, v) & \frac{\partial}{\partial \bar{a}_{\lambda\kappa}}(\bar{\Lambda}_{\lambda'\kappa'}, v) & \frac{\partial}{\partial \bar{a}_{\lambda\kappa}}(\zeta_{j'}, v) \\ \frac{\partial}{\partial b_j}(\Lambda_{\lambda'\kappa'}, v) & \frac{\partial}{\partial b_j}(\bar{\Lambda}_{\lambda'\kappa'}, v) & \frac{\partial}{\partial b_j}(\zeta_{j'}, v) \end{bmatrix}$$

(4.24)

where [p] ([q]) run over $\lambda\ \kappa$ and j (and primed indices) as appropriate. Its super-determinant is evaluated by using opposite statistic (to $\{a_{\lambda\kappa}, \bar{a}_{\lambda\kappa}, b_j\}$ and $\{ \Lambda_{\lambda\kappa}, \bar{\Lambda}_{\lambda\kappa}, \zeta_j\}$ as appropriate) integration variables and using

$$\mathrm{Sdet}(J) = \int D(c',c) \exp\{ c'[q]\, J^{[p],[q]}\, c[p] \} \quad . \qquad (4.25)$$

(This is the equivalent of (2.2.12) for the superdeterminant).
We now evaluate the various terms in (4.24).

Using $\frac{\partial \Lambda_{\lambda'\kappa'}}{\partial a_{\lambda\kappa}} = \frac{\partial \bar{\Lambda}_{\lambda'\kappa'}}{\partial a_{\lambda\kappa}} = 0$

(which follows from the fact that $\Lambda_{\lambda'\kappa'}$ is defined through the covariant derivatives ∇_γ and $\bar{\nabla}_{\dot{\gamma}}$ which are invariant under the Λ' and $\bar{\Lambda}'$ transformations of equation (4.11) applied to (4.3), see also equation (1.2.68)).

we obtain

$$\begin{aligned} \frac{\partial}{\partial a_{\lambda\kappa}}(\Lambda_{\lambda'\kappa'}, v) &= (\Lambda_{\lambda'\kappa'}, \frac{\partial v}{\partial a_{\lambda\kappa}}) \\ \frac{\partial}{\partial \bar{a}_{\lambda\kappa}}(\Lambda_{\lambda'\kappa'}, v) &= (\Lambda_{\lambda'\kappa'}, \frac{\partial v}{\partial \bar{a}_{\lambda\kappa}}) \end{aligned} \qquad (4.26)$$

(Here, for simplicities sake, we just consider the case where the zero mode and the parameter differential are bosonic; the other cases follow straightforwardly).

Since the discrete zero modes (ζ_j) transform homogeneously under background gauge transformations they are constructed from the covariant derivatives (∇_γ and $\bar{\nabla}_{\dot\gamma}$,see for example those in sections 3.2, 3.3 and (2.2.14), (2.2.15)). The same considerations then apply;

$$\frac{\partial}{\partial a_{\lambda\kappa}}(\zeta_j, v) = (\zeta_j, \frac{\partial v}{\partial a_{\lambda\kappa}}) . \qquad (4.27)$$

Also

$$\frac{\partial}{\partial b_j}(\zeta_j, v) = (\zeta_j, \frac{\partial v}{\partial b_j}) \qquad (4.28)$$

since the term $(\frac{\partial}{\partial b_j}\zeta_j, v)$ is linear in the quantum fields and therefore will not contribute. (In the background field method linear terms are dropped, see section 1.1). (4.26) to (4.28) are then completely evaluated by

$$\frac{\partial v}{\partial a_{\lambda\kappa}} = i/2\, L_v \{ \Lambda_{\lambda\kappa} + \coth(g/2\, L_v)\, \Lambda_{\lambda\kappa} \}$$

$$\frac{\partial v}{\partial \bar{a}_{\lambda\kappa}} = i/2\, L_v \{ \bar{\Lambda}_{\lambda\kappa} - \coth(g/2\, L_v)\, \bar{\Lambda}_{\lambda\kappa} \} \qquad (4.29)$$

$$\frac{\partial v}{\partial b_j} = -1/2\, L_v \coth(g/2\, L_v)\, \zeta_j$$

which follow readily from (4.12) and (4.15).

This leaves

$$\frac{\partial}{\partial b_j}(\Lambda_{\lambda'\kappa'}, v) = (\frac{\partial \Lambda_{\lambda'\kappa'}}{\partial b_j}, v) + (\Lambda_{\lambda'\kappa'}, \frac{\partial v}{\partial b_j}) \qquad (4.30)$$

since other terms in (4.24) not yet considered are OS-conjugates of (4.26) to (4.28).

To evaluate $\frac{\partial \Lambda_{\lambda\kappa}}{\partial b_j}$ we use (4.8) :

$$\frac{\partial \Lambda_{\lambda\kappa}}{\partial b_j} = \frac{1}{4\lambda} \{ \frac{\partial \bar{\nabla}^2}{\partial b_j} \bar{\Lambda}_{\lambda\kappa} + \bar{\nabla}^2 \frac{\partial \bar{\Lambda}_{\lambda\kappa}}{\partial b_j} \} . \qquad (4.31)$$

But using (4.11) we obtain

$$\delta\bar{\nabla}_{\dot{\alpha}} = -\tfrac{1}{2}\,\bar{\nabla}_{\dot{\alpha}}\,\delta\zeta$$

so that $\dfrac{\partial}{\partial b_j}\bar{\nabla}_{\dot{\alpha}} = -\tfrac{1}{2}\,\bar{\nabla}_{\dot{\alpha}}\,\zeta_j$.

Using this equation in (4.31) we obtain, after some manipulation,

$$\frac{\partial\Lambda_{\lambda\kappa}}{\partial b_j} = \tfrac{1}{2}[\zeta_j\,,\,\Lambda_{\lambda\kappa}] + \frac{1}{4\lambda}\,\bar{\nabla}^2\{\frac{\partial\bar{\Lambda}_{\lambda\kappa}}{\partial b_j} - \tfrac{1}{2}[\zeta_j\,,\,\bar{\Lambda}_{\lambda\kappa}]\} \quad . \qquad (4.32)$$

(Note that in the 1st term of this equation we have again used (4.8)). The last term in (4.32) can be ignored since it yields a bilinear interaction proportional to $\bar{\nabla}^2 v$, which by the Slavnov-Taylor identities [14] in the background field gauge, cannot contribute to physical quantities.

If we let $c[p] = (ic_\lambda{}^\kappa,\ -i\bar{c}_\lambda{}^\kappa,\ \hat{c}^j)$

$$c'[q] = (ic'_{\lambda'}{}^{\kappa'},\ -i\bar{c}'_{\lambda'}{}^{\kappa'},\ \hat{c}'^{j'})$$

and

$$c = \int_\lambda c_\lambda{}^\kappa\,\Lambda_{\lambda\kappa}$$

$$\bar{c} = \int_\lambda \bar{c}_\lambda{}^\kappa\,\bar{\Lambda}_{\lambda\kappa} \qquad (4.33)$$

$$\hat{c} = \sum_j \hat{c}^j\,\zeta_j$$

and similarly for the primed fields, then c, $\bar{c}$ and $\hat{c}$ are fermionic superfields. (c and $\bar{c}$ are conjugate pairs and $\hat{c}$ is real). Using (4.26) to (4.32), we can express Sdet(J) as

$$\mathrm{Sdet}(J) = \int D(\,c_\lambda{}^\kappa,\ \bar{c}_\lambda{}^\kappa,\ c'_\lambda{}^\kappa,\ \bar{c}'_\lambda{}^\kappa)\ \prod_{j=1}^{m}\ (dc_j\ dc'_j)\quad e^{(1/g\ S_{gh})} \qquad (4.34)$$

where Sgh is the properly normalized ghost action with extra interactions (compare equation (1.3.10)):

$$S_{gh} = \int d^8z\{\ -(c' + \bar{c}' + \hat{c}')\ g/2\ L_v(c - \bar{c} + \coth(g/2\ L_v)(c + \bar{c} + \hat{c})) + (c' - \bar{c}')\ g/2\ L_v\ \hat{c}\ \}$$

$$= -\int d^8z\{\ \bar{c}'c + c'\bar{c} + \hat{c}'\hat{c} + \text{interactions with } v\ \} \quad . \qquad (4.35)$$

We may use (4.20) to rewrite the integral over modes in (4.34) as the functional integral

$$\int D(c, \bar{c}, c', \bar{c}')\, d^m\hat{c}\, d^m\hat{c}' \qquad . \qquad (4.36)$$

We now consider the case where the background field is equal to a superinstanton and the gauge group is SU(n). In this case the number of vector field zero modes m=8n. These are 4n bosonic modes, 2n fermionic modes (of which 4 correspond to broken supersymmetry and superconformal symmetry) and 2n further fermionic modes that are anomalous supergauge modes. (The case n=2 is considered in [10]). These latter 2n modes are chiral square integrable modes, but they do not belong to the set (4.8) since

$$\nabla^2 \zeta_i = 0 \qquad (4.37)$$

(This zero mode counting follows from the components of the vector field: it is straightforward to establish that the vector quantum field component (Q_μ) has 4n zero modes - These are responsible for the 4n bosonic zero modes for v. Similarly L.H. fermions (λ) have 2n zero modes leading to 2n fermionic zero modes for v. But the fermion zero mode equation is a component of (4.37) also (ζ_i chiral). Therefore there are 2n fermionic supergauge anomalous modes.)

The fact that the modes in (4.37) are square integrable means that they must be integrated over in the chiral ghost integration in (4.36). The fact that they do not belong to the set of modes in (4.8) but instead satisfy (4.37) implies that the kinetic part of the action (4.35) vanishes when the chiral ghosts are equal to these modes, i.e. these modes are 2n zero modes for each chiral ghost.
This last statement is justified from the following:-
When the background field is set equal to an instanton $\bar{W}_{\dot{\alpha}}$ vanishes [9,10] (see (3.2.8)) so that $\Box_-$ (4.7) becomes background $\Box$ (see (3.4.6)). The latter has no zero modes, and is invertible in the space of square integrable functions [3] (and see (4.45)). Consequently the $\bar{\Lambda}$ modes of (4.8) form a complete set for the space of square integrable antichiral fields. But for any one of these modes (4.8) we have (using the vector norm of (4.6) and (3.3.8))

$$(\zeta_i \, , \, \bar{\Lambda}_{\lambda\kappa} \,) = \frac{16}{\lambda^2} \, (\zeta_i \, , \, \nabla^2 \bar{\nabla}^2 \, \bar{\Lambda}_{\lambda\kappa} \,)$$

$$= \frac{16}{\lambda^2} \, (\nabla^2 \, \zeta_i \, , \, \bar{\nabla}^2 \, \bar{\Lambda}_{\lambda\kappa} \,)$$

$$= 0 \qquad \text{by (4.37).}$$

Hence when the chiral ghosts (c, c', or η) are equal to these modes the kinetic action (of (4.35)) vanishes and so these modes are 2n zero modes for each chiral ghost.

Unlike the vector case (as in section 3.3), these modes when considered as belonging to the chiral field, must be normalized using the chiral norm of (4.8). i.e. they satisfy

$$(\Lambda_j \, , \, \Lambda^k \,)^c = - \, \delta_j^{\ k}$$

[For n=2 these are, up to numerical factors,

$$\Lambda_j = (iW_\beta \; ; \; W^\beta \, x^+_{\beta\beta} \, \rho^{-1})$$
]

(see the corresponding vector modes in (3.3.4) and below (3.3.9))

Expanding the ghost fields as

$$c = \sum_j e^j \, \Lambda_j + \text{continuous parameter modes (those of (4.33))}$$

$$c' = \sum_j e'^j \, \Lambda_j + \text{continuous parameter modes} \qquad (4.38)$$

$$\eta = \sum_j f^j \, \Lambda_j + \text{continuous parameter modes}$$

we insert into (4.34) the factor

$$1 = \int d^{2n}e \; d^{2n}e' \; d^{2n}f \; \prod_j \delta[(c,\Lambda_j)^c] \; \delta[(c',\Lambda_j)^c] \; \delta[(\eta,\Lambda_j)^c] \qquad (4.39)$$

(The Jacobian is readily seen to be "1")

Finally it is convenient to substitute

$$c \to c\sqrt{g} \;,\; c' \to c'\sqrt{g} \;,\; \hat{c} \to \hat{c}\sqrt{g}$$

(and similarly for barred fields) so as to cancel the factor of $1/g$ in (4.34). The Jacobian for this substitution leads to a continuous infinity of factors of $\sqrt{g}$ in the functional integration of (4.36) which cancel those from the zero instanton sector, together with 4n inverse powers of $\sqrt{g}$ which appear via the c and c' δ-function constraints of (4.38). No powers of $\sqrt{g}$ accompany the $\hat{c}$ and $\hat{c}'$ integrals in (4.36) because the 4n bose modes lead to a factor of $(\sqrt{g})^{4n}$ whereas the 4n fermi modes give $(\sqrt{g})^{-4n}$.

Piecing together (4.16),(4.17),(4.22),(4.23),(4.34),(4.36),(4.39) and this substitution we obtain

$$Z = \int \mathcal{D}'(v,c,\bar{c},c',\bar{c}',\eta,\bar{\eta})\; d^{8n}b\; d^{8n}\hat{c}\; d^{8n}\hat{c}'\; d^{2n}e\; d^{2n}e'\; d^{2n}f$$

$$g^{-2n} \left(\prod_{\kappa\lambda} 1/\epsilon\right) e^{\left(S_V + S_{GF} + S_{gh} + S_{NK}\right)}$$

$$\left(S_{NK} = - \int d^8z\; \eta\bar{\eta} \right) \qquad (4.40)$$

The prime on $\mathcal{D}$ indicates that integration is carried out only over the non-zero mode spaces.

This is the final result for the generating functional.

4.2 THE SEMICLASSICAL APPROXIMATION.

The semiclassical result is obtained by considering interactions up to quadratic in the action contained in (4.40) (see (2.3.1) and comments above this equation). This leads to the evaluation of the determinants of the appropriate Laplace-like operators (restricted to their non-zero mode spaces, see (2.3.2) and comments below it).

The action up to quadratic terms is (from (4.5), (4.23) and (4.35))

$$S = -\frac{8\pi^2}{g^2} + \mathrm{tr}\int d^8z\{\; v(\frac{1}{16}\bar{\nabla}^{\dot{\alpha}}\nabla^2\bar{\nabla}_{\dot{\alpha}} - \frac{1}{32\xi}(\nabla^2\bar{\nabla}^2 + \bar{\nabla}^2\nabla^2))v - \bar{c}'c - c'\bar{c}$$
$$- \hat{c}'\hat{c} - \eta\bar{\eta}\;\} \qquad (4.41)$$

The integration over $\hat{c}$ (and $\hat{c}'$) trivially gives 1. The integration over each pair of ghosts (c, c', η and their conjugates) gives a factor of (see equation(4.21))

$$\mathrm{Sdet}^{\frac{1}{2}}(\Box)$$

($\Box_{-} = \Box$ for a superinstanton, since $\bar{W}_{\dot{\alpha}} = 0$ see (4.7) and (3.2.8)).

The expansion of "v" in terms of non-zero eigen modes is

$$v = \int_{\lambda}\{\;(\bar{u}_{\lambda}{}^{\kappa}\,\bar{\Lambda}_{\lambda\kappa} + u_{\lambda}{}^{\kappa}\,\Lambda_{\lambda\kappa})1/\sqrt{\lambda} + u_{\lambda}{}^{\kappa\dot{\alpha}}\,v_{\lambda\kappa\dot{\alpha}}\}$$
$$v_{\lambda\kappa\dot{\alpha}} = \frac{1}{\sqrt{2}\,\lambda}\,\bar{\nabla}_{\dot{\alpha}}\,\bar{\Lambda}_{\lambda\kappa} \qquad (4.42)$$

where $\bar{u}_{\lambda}{}^{\kappa}$, $u_{\lambda}{}^{\kappa}$ and $u_{\lambda}{}^{\kappa\dot{\alpha}}$ are arbitrary coefficients. The terms containing Λ and $\bar{\Lambda}$ are the gauge modes that are gauge fixed in (4.41) and are hence non-zero modes of the gauge fixed action (which is the relevant action for the semiclassical calculation ([2] and (2.3.2)), see (4.23) and the discussion above). The last term contains the +ve frequency transverse modes. In chapter 3.4 we showed that these positive frequency modes span the full space of non-zero eigen modes of (4.7) (see also [10]).

The factor of $1/\surd\lambda$ in expression (4.42) means that (see norm in (4.9)) the coefficients are normalized so that[13]

$$\int D(v)\, \exp\{ \int d^8z\ v^2 \} = 1$$

implies $$\int D(v) \equiv \int D(\bar{u}_{\lambda\kappa},\ u_{\lambda\kappa},\ u_{\lambda\kappa\dot{\alpha}}) \quad . \qquad (4.43)$$

Putting (4.42) into (4.41) yields

$$\delta S_v = \frac{1}{2} \int_\lambda -\lambda^2\ (u_\lambda^{\ \kappa\dot{\alpha}}\ u_{\lambda\kappa\dot{\alpha}} + \frac{1}{\xi}\ a_\lambda^{\ \kappa}\ \bar{a}_{\lambda\kappa})$$

Performing the integration in (4.43) we obtain

$$\text{Sdet}\ \Box\quad \text{Sdet}^{-1}\ (\frac{1}{\xi}\ \Box)$$

$$= \prod_{\kappa\lambda}\ \xi \qquad (4.44)$$

(note that the $u_\lambda^{\ \kappa\dot{\alpha}}$ have opposite statistics to $\bar{\Lambda}_{\lambda\kappa}$ and that there twice as many for each eigenvalue λ).

These factors of ξ are cancelled, as expected, by those in (4.40) which arose from the normalisation (4.18) of the Gaussian average.

Hence the total positive frequency mode contribution comes just from the ghosts and is equal to

$$\text{Sdet}^{3/2}(\Box)\ .$$

However Sdet($\Box$) =1 . This was shown to be true by heat kernel methods [9]. We can verify this a number of ways :-

For example

$$\text{Sdet}(\Box/\Box_0) = \exp\{\text{tr}\int d^8z\ \ln(\Box/\Box_0)_{zz}\}$$

$$\Box_0 = \partial_\mu\partial_\mu$$

(where we have included the zero instanton sector contribution) leads to a sum of diagrams as in fig.4.1 when expanded in powers of the background field. But since no D_α's and $\bar{D}_{\dot{\alpha}}$'s appear inside the loops the rules of supergraphs [11] imply that these all vanish identically. However this argument is perturbative. The most convincing verification which is non-perturbative is by resorting to components: If we let

$$\omega = \bar{\omega} = \frac{1}{\sqrt{2}}\, \theta^{\alpha}\bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}$$

where $A_{\alpha\dot{\alpha}}$ is the instanton background field, we find, after some calculation, that

$$\Box\bar{\Lambda} = -\lambda^2\bar{\Lambda}$$

is equivalent to the three equations

$$D^2\,\bar{\Omega} = -\lambda^2\,\bar{\Omega}$$

$$D^2\,\bar{\phi}_{\dot{\alpha}} = -\lambda^2\,\bar{\phi}_{\dot{\alpha}} \qquad (4.45)$$

$$D^2\,\bar{F} = -\lambda^2\,\bar{F}$$

so that $\quad \mathrm{Sdet}\,\Box = \det D^2 \det^{-2}(D^2) \det D^2 = 1$

A similar analysis, though somewhat more complicated, can be carried out for a general superinstanton in the Wess Zumino K-gauge (i.e. the gauge in which ω and $\bar{\omega}$ are equal and are taken to be half the background prepotential (v_B) in the Wess Zumino gauge)

$$\omega = \bar{\omega} = \frac{1}{\sqrt{2}}\left(\theta^{\alpha}\bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}} - \bar{\theta}^2\theta^{\alpha}\lambda_{\alpha}\right) \qquad (4.46)$$

The conclusion is the same.

Through the zero modes we obtain powers of μ such as to cancel the dimensions of the integral over collective coordinates. (This rule can be justified on dimensional grounds or by regulating with (background) Pauli-Villars [3,2], see discussion above (2.3.3)).
The dimension of each b_j is -1 which follows from (4.11), (4.12) and the fact that ζ_j is normalized. It follows that the measure $d^{8n}b$ is dimensionless (since there are an equal number of fermionic and bosonic b_j). The dimension of e, e' and f is -1/2 and is given by (4.38) and the fact that the Λ_j are normalized by the chiral norm. Hence (note that e,e' and f are bosonic collective coordinates) we obtain

$$Z = \int d^{8n}b\; d^{2n}e\; d^{2n}e'd^{2n}f\; \mu^{3n}/g^{2n} \quad e^{-8\pi^2/g^2} \qquad (4.47)$$

as the final result from the semiclassical calculation.

Note that μ^{3n} is exactly the right power to cancel(to order g^0) the implicit μ dependence of the classical action (β_0 = 3n for SU(n)).

In the next chapter we will consider the divergent higher order corrections to this semiclassical result. These divergences are found to be cancelled, as expected, by renormalisation of (4.47).

4.3 HIGHER ORDER CORRECTIONS.

We consider here the corrections of order g^2 to the semiclassical action (4.47). They are illustrated in figures 4.2 and 4.3. The graphs in fig.4.2 are "vacuum" graphs in which the propagators are the full propagators in the background field. They are the analogues of figures 2.2 and 2.3 at the end of chapter 2. The graphs in fig.4.3 are graphs formed from the new interactions in the Jacobian. The stub-lines represent interactions containing $\hat{c}$, $\hat{c}'$. Power counting shows that the U.V. divergence from these graphs (which must be local) yields a contribution to the effective action

$$-Y\int \hat{c}\hat{c}'\, d^8z = -Y\,(\sum_j \hat{c}^j\hat{c}'_j)$$

where Y is a coefficient containing the divergences .
Integration over the $\hat{c}^j$ modes yields a multiplicative correction factor to the semiclassical result (4.47) (see (4.41) and (4.40)).

$$(1+Y)^{4n} / (1+Y)^{4n} = 1$$

Thus, because there are equal numbers of fermionic and bosonic collective coordinates (in the vector sector), there are no divergent corrections from the discrete part of the Jacobian. We note here that this conclusion holds true to all orders in the number of loops (since the divergences will always be of the above form).

We now consider the graphs of fig.4.2 in detail. As has previously been shown ([2] and see section 2.3) the divergences from these graphs can be split into "short distance" divergences in which momenta are large in both loops, (this can be calculated by a perturbative expansion in the background field) and "long distance" divergences in which one of the propagators is the full long distance propagator carrying no divergent momentum (the other propagators carry divergent momenta and can be expanded perturbatively).
The divergences in the short distance calculation are insensitive to the particular form of the background field and can be calculated by standard background field methods. The 2 loop calculation has been performed by Abbott et al [6] and has been further refined [7,8]. We

briefly review the methods developed in [7,8] and their application to this calculation. The essence of these developments is in the use of covariant-D algebra to generate terms that include W_γ ,$\bar{W}_{\dot\gamma}$ and $\nabla_{\gamma\dot\gamma}$ but nothing else [7]. Consequently the form of the divergences is highly restricted [8]: They must be local, form a background gauge invariant functional, be able to be written as an integral over the full superspace (d^8z), and contain the background field only as W_γ ,$\bar{W}_{\dot\gamma}$ or the connection $\Gamma_{\gamma\dot\gamma}$

$$(\quad \nabla_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} - i\Gamma_{\alpha\dot\alpha} \quad) \qquad .$$

The mass dimension of W_γ ,$\bar{W}_{\dot\gamma}$ is too high to be included and so any terms generated by the covariant-D algebra which contain these field strengths can be dropped. In fact, using dimensional reduction, the only divergent term that can appear, satisfying all the above requirements, is [8]

$$\mathrm{tr}\int d^4x\, d^4\theta\; \Gamma^{\alpha\dot\alpha}\, \Gamma_{\gamma\dot\gamma}\, (\delta_{\alpha\dot\alpha}{}^{\gamma\dot\gamma} - \hat\delta_{\alpha\dot\alpha}{}^{\gamma\dot\gamma}) \tag{4.48}$$

where $\quad \hat\delta_{\alpha\dot\alpha}{}^{\gamma\dot\gamma} = \hat\delta_\mu{}^\nu\, \sigma^\mu{}_{\alpha\dot\alpha}\, \sigma_\nu{}^{\gamma\dot\gamma}$

is the Kronecker δ in the $4\text{-}2\epsilon$ dimensional subspace. This term is background gauge invariant because the differential of the gauge field occuring in the gauge transformation of the connection

$$\partial_{\alpha\dot\alpha}\, K(x)$$

(x in the $4\text{-}2\epsilon$ dimensional subspace) satisfies

$$(\delta_{\alpha\dot\alpha}{}^{\gamma\dot\gamma} - \hat\delta_{\alpha\dot\alpha}{}^{\gamma\dot\gamma})\;\; \partial_{\gamma\dot\gamma}\, K(x) = 0 \qquad .$$

(4.48) can be written as

$$-\,\frac{\epsilon}{2}\;\mathrm{tr}(\int d^4x d^2\theta\; W^\gamma W_\gamma + \int d^4x d^2\bar\theta\; \bar W^{\dot\gamma} \bar W_{\dot\gamma}) \tag{4.49}$$

(we use the notation of "superspace" [13] and not sections 1.2 and 1.3 ([11]) here so as to make straightforward contact with [6,7,8]).

(4.49) differs from the corresponding equation in [8] only for instanton-like fields with non-vanishing boundary terms, however reality of (4.49) guarantees it to be of this form even in the latter case. (We have also checked that (4.49) is equal to (4.48) for the W.Z. K-gauge superinstanton (4.46)).

In the Feynman gauge the graphs c and d in fig.4.2 trivially vanish because there are insufficient ∇'s and $\bar{\nabla}$'s . Graph b vanishes by cancellation of the two ghost contributions. (There all possible couplings between the ghosts and they all have the same sign; however the two propagators have opposite signs). We are left with graph "a" which after some straightforward ∇-algebra reduces to a diagram equivalent to that of a scalar (ϕ^3) graph with covariant $\square$ propagators [8] (see fig.4.4).
Pulling out only terms of the form

$$\Gamma^{\alpha\dot{\alpha}}\Gamma_{\gamma\dot{\gamma}}\ \delta_{\alpha\dot{\alpha}}{}^{\gamma\dot{\gamma}}$$

we obtain just the tadpole diagram of fig.4.5 which is

$$-\frac{3}{2}\, g^2\, k_1 \int d^8z(\Gamma^{\alpha\dot{\alpha}}\Gamma_{\alpha\dot{\alpha}})_{ii} \int \frac{d^nq\ d^np}{(2\pi)^{2n}}\ \frac{1}{q^2(q+p)^2\ p^4} \tag{4.50}$$

$$= \frac{3g^2\ k_1}{4(4\pi)^4\varepsilon^2} \int d^8z(\Gamma^{\alpha\dot{\alpha}}\Gamma_{\alpha\dot{\alpha}})_{ii} \tag{4.51}$$

(where we have subtracted the U.V. subdivergence and taken care to separate out the I.R. divergence by e.g. first performing the q integration and then letting $p^2 \rightarrow p^2 + \lambda^2$ (λ^2 a small I.R. regulator mass) or by dimensional regularisation I.R. subtraction techniques [17,8]). Other terms in the background field expansion will covariantize this to the form (4.48) and hence we obtain (converting from the trace over adjoint representation generators in (4.51) to normalized generators)

$$\begin{aligned} &-\frac{3\ g^4\ k_1{}^2}{4(4\pi)^4\varepsilon}\left(\frac{tr}{2g^2}\int d^6z\ W^2\ \right) \\ &= \frac{3\ g^2\ n^2}{2(4\pi)^2\varepsilon} \end{aligned} \tag{4.52}$$

(we have used $\bar{W}_{\dot{\alpha}} = 0$ for an instanton. The term in brackets is

$\frac{tr}{128g^2}\int d^6z\, W^2$ in the notation of chapter 1 and is equal to $-\frac{8\pi^2}{g^2}$.

Note that $k_1 = 2n$ for SU(n) (see (1.2.48) and comments above). Although the short distance calculation was performed in the Feynman gauge the answer, which depends only on the β-function (see section 1.1), is gauge parameter independent. This fact allows us to perform the long distance calculation in a different, in fact general, gauge which we will do for reasons explained below (see discussion below equation (4.53)).

The divergence in (4.52) is cancelled by the counterterm ($\alpha\ \beta_1 = 6n^2$) coming from renormalising the classical action.
Inserting the factors of μ^ϵ (from $g_0 = g\mu^\epsilon$ to lowest order) (4.52) plus the counterterm leaves behind an explicit μ dependence equal to

$$\frac{6n^2g^2}{(4\pi)^2}\ \ln\mu \qquad (4.53)$$

(note that the counterterm contains a $\mu^{-2\epsilon}$ coming from the g_0 's in the classical action).

We now turn to the calculation of the long distance part of the graphs in fig.4.2 . We start by considering those diagrams containing a long distance ghost propagator. The divergent subdiagrams yield the ghost 1 loop self energy. This can be calculated by ignoring the background field and then covariantizing the final result. (This is because the divergent coefficient is proportional to the wavefunction renormalisation constant which is independent of the background field).The relevant diagrams are shown in fig.4.6. In the Feynman gauge both these diagrams vanish so it is useful (to provide a check on the ghost zero mode arguments in section 4.1) to work in a general gauge for which the vector propagator is

$$\langle vv\rangle = 1/p^2\ \{\ 1 + (\xi-1)\Pi_0\ \}$$

$$\Pi_0 = (D^2\bar{D}^2 + \bar{D}^2D^2)/\ 16p^2$$

(using Euclidean space conventions [10]).

In this case the 1st diagram of fig.4.6 still vanishes (by cancellation of the two ghost contributions) but the tadpole diagram gives a term proportional to

$$(\xi-1)\int\frac{d^np}{(2\pi)^n}\,\frac{1}{p^4}$$

which does contribute (providing we are careful to separate out the I.R. divergence , see comments below (4.51)).
so that we obtain

$$-\,\frac{g^2(\xi-1)k_1}{6\,(4\pi)^2\,\varepsilon}\,\{\mathrm{tr}\int d^8z\,(\bar{c}'c + c'\bar{c})\} \qquad (4.54)$$

(It is convenient to consider the graphs in fig.4.6 sandwiched between $\bar{c}'$ and c, and, c′ and $\bar{c}$ as in an effective action[11]).
As explained above (4.54) forms the 1 loop subdivergence of the full graphs in fig.4.2 when one of the ghost propagators is long distance. In order to obtain the final 2 loop result we must replace the ghost terms in (4.54) by the appropriate long distance propagators (together with the usual factors of ∇^2 and $\bar{\nabla}^2$ on the outgoing chiral and antichiral legs).
Using the explicit form of the non-perturbative propagators $\langle\bar{c}'c\rangle$ and $\langle c'\bar{c}\rangle$ (see [10,9] and (3.4.16), (3.4.17)) we find that the term in curly brackets in (4.54) becomes

$$-\,\mathrm{Str}(\,P_A + P_C) \qquad (4.55)$$

where P_A and P_C are the chiral and antichiral projectors of (3.4.13) and (3.4.14). (Note that (4.56) is quite general and simply arises because the propagators are minus the inverse of the Laplace-like term generated by the quadratic action on the appropriate (chiral or antichiral) spaces).

$$P_A = -1/16\ \nabla^2\ \Box^{-1}\ \bar{\nabla}^2$$

and $$P_C = -1/16\ \bar{\nabla}^2\ \Box^{-1}\ \nabla^2$$

So equation (4.37) implies that the combination $(P_A + P_C)$ has 2n fermi zero modes (when acting on the left or the right). Therefore

$$\mathrm{Str}(P_A + P_C) - \mathrm{Str}(P_A{}^0 + P_C{}^0) = +2n$$

where $P_A{}^0$ and $P_C{}^0$ are the corresponding projectors in the zero instanton sector (they do not have discrete zero modes) and this second term arises from subtracting the zero instanton contribution [2].

Hence the long distance contribution from the ghosts is (using (4.54) and (4.55))

$$\frac{2g^2(\xi-1)n^2}{3\,(4\pi)^2\,\varepsilon} \quad . \tag{4.57}$$

Now consider the diagrams containing a long distance vector propagator. The divergent subdiagrams yield the vector 1 loop self energy. Once again this may be calculated by ignoring the background field and covariantizing the final result. (Note that this time we need to use the appropriate background covariant Ward identity to fix the form of the covariantized result [2]). The relevant diagrams are those of fig.4.7. Note that the tadpole graph contributes outside the Feynman gauge (see comments above equation (4.54)).

We obtain (including the effective action combinatoric factor of ½)

$$-\frac{(7+2\xi)g^2\,k_1}{6(4\pi)^2\,\varepsilon}\,\{\,\frac{\mathrm{tr}}{16}\int d^8z\; v(\nabla^{\alpha}\bar{\nabla}^2\nabla_{\alpha} - W^{\alpha}\nabla_{\alpha})v\,\} \tag{4.58}$$

in agreement with previous results [15].

Converting the v's in (4.58) into a long distance propagator the term in curly brackets gives

$$-\tfrac{1}{2}\,\mathrm{Str}\,P$$

where P is the transverse positive mode projection operator which is for an instanton (see equation (3.4.21))

$$P = \frac{1}{8}\,\bar{\nabla}^{\dot{\alpha}}\nabla^2\,\Box^{-1}\,\bar{\nabla}_{\dot{\alpha}} \tag{4.59}$$

If we subtract the corresponding term in the zero instanton sector (4.59) will give half the number of bosonic transverse zero modes minus half the number of transverse fermionic modes. This is ½(4n-2n) = n . Note that the supergauge anomalous modes of (4.37) do not count because they are square integrable longitudinal (gauge) modes.

Alternatively note that the gauge fixed action has

$$\mathrm{Str}(P + P_A + P_C) = -4n + 4n = 0$$

(once the zero instanton sector term is subtracted)

so $$\mathrm{Str}\, P \equiv -\, \mathrm{Str}(P_A + P_C) \equiv -2n$$

(see (4.56))

substituting (4.59) in (4.58) we obtain

$$-\frac{(7+2\xi)g^2n^2}{3(4\pi)^2\, \varepsilon} \qquad (4.60)$$

as the long distance contribution from the vector field. Adding the contribution from the ghosts (4.57) we obtain the total long distance contribution

$$-\frac{3g^2n^2}{(4\pi)^2\varepsilon} \qquad (4.61)$$

This divergence is cancelled by the counterterm ($\propto -n\beta_0$; $\beta_0 = 3n$) coming from renormalizing the g^{-2n} term in the semiclassical result of (4.47). Once this counterterm is added to (4.61) we are left with an explicitly μ dependent term

$$-\frac{6g^2n^2}{(4\pi)^2} \ln\mu \qquad (4.62)$$

This completes the calculation of the $O(g^2)$ divergent (and explicitly $\ln\mu$ dependent) terms. In the next and final chapter we will be considering some conclusions that can be drawn from this calculation.

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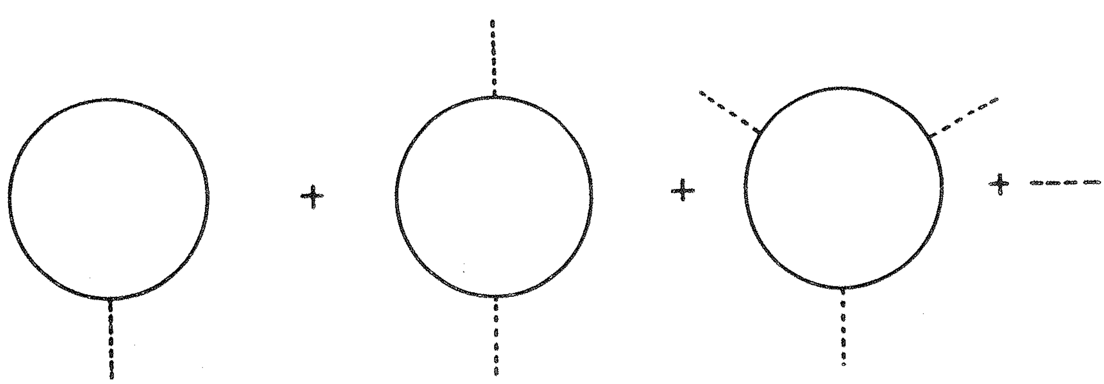

Fig. 4.1

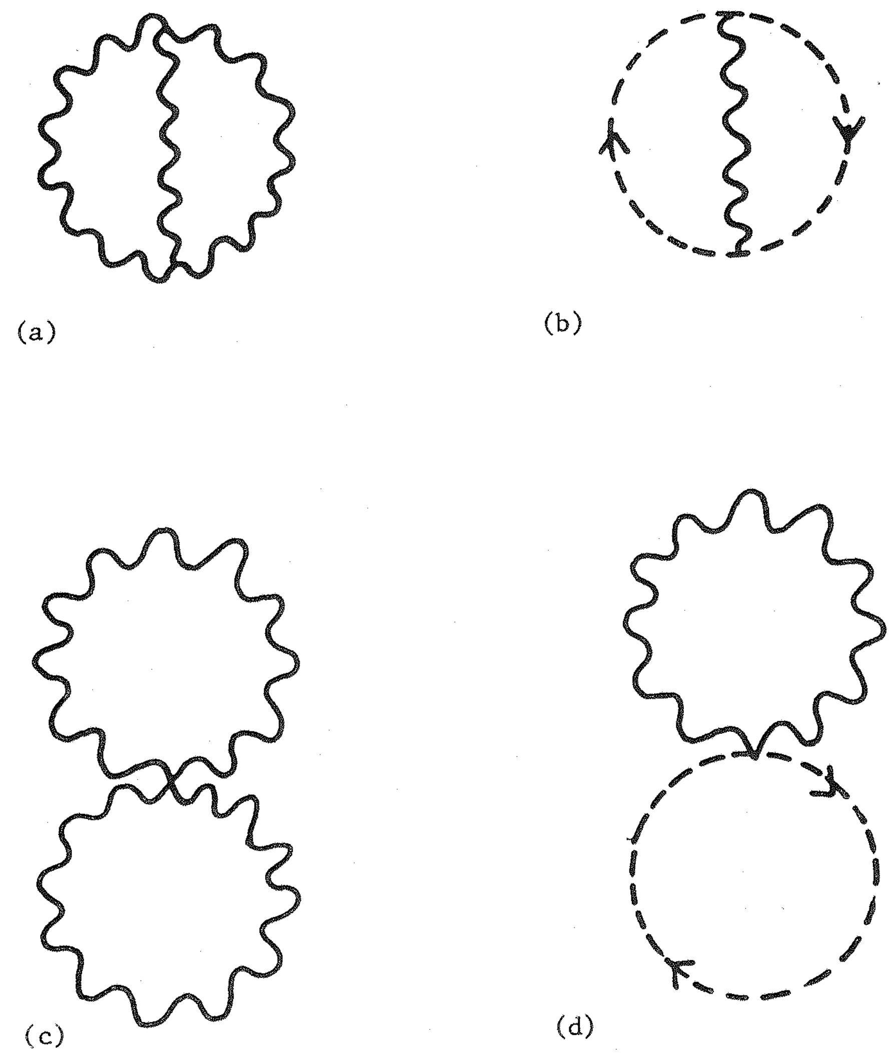

Fig. 4.2

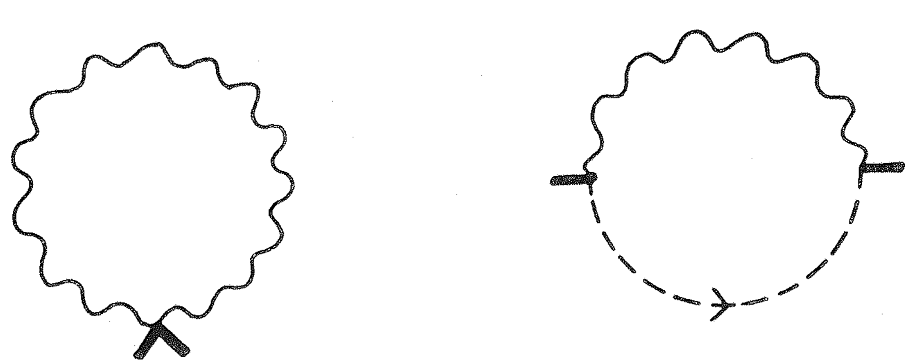

Fig. 4.3

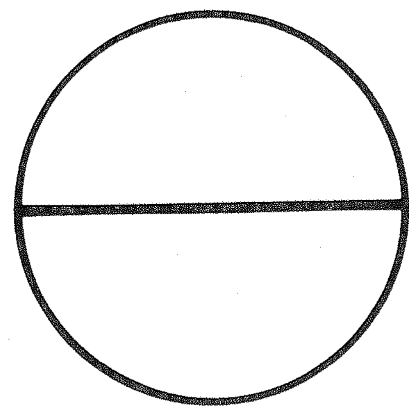

Fig. 4.4

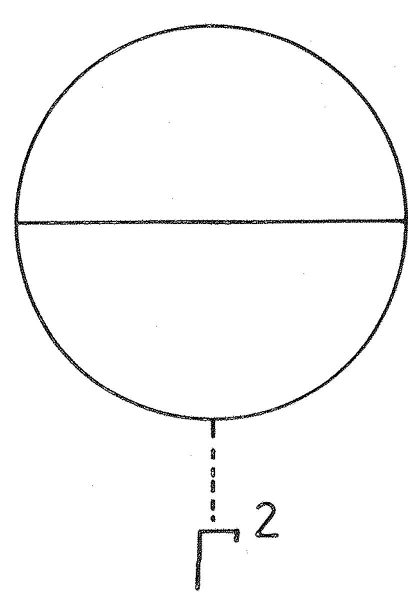

Fig. 4.5

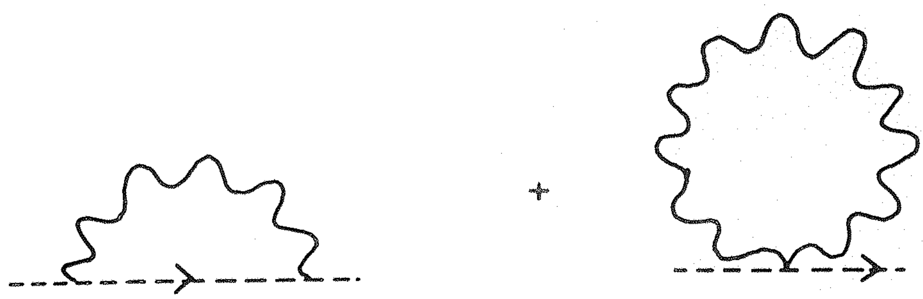

Fig. 4.6

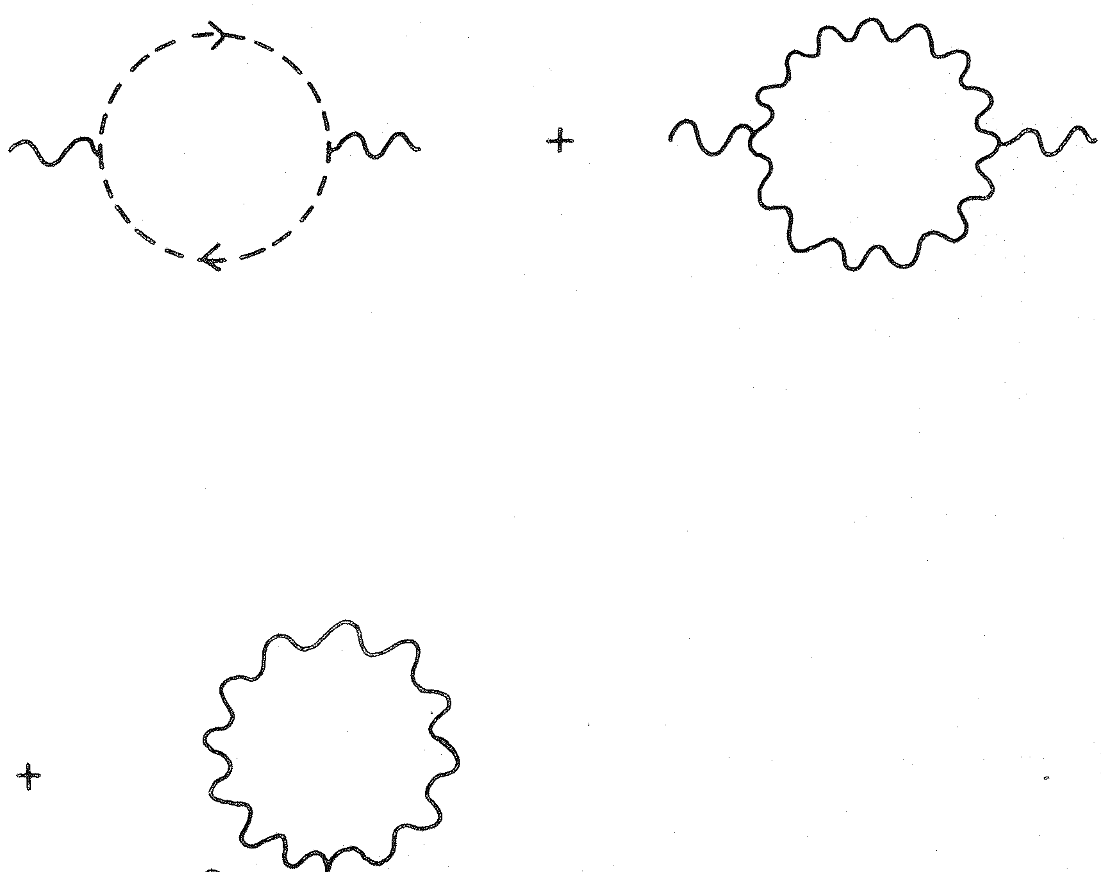

Fig. 4.7

CHAPTER 5 CONCLUSIONS.

We begin this final chapter with a summary and discussion of the research presented in the previous chapters. We then go on to compare our work with that of Novikov et al [1]. The derivation of the super Yang-Mills all orders β-function given in that paper relied on a theorem stating that all positive frequency contributions to the instanton action vanish. We show here why the proof presented in ref. [1] is invalid [2] and discuss the possibility that although the proof is invalid the theorem might still be shown to hold [2,3]. In the light of the subtleties discovered from our explicit calculations we argue that it is hard to see why this should be the case.

Chapter 2 outlined the effects that must be taken into account when calculating divergent quantum corrections to the instanton action in Yang-Mills [4]. A careful treatment of the zero modes was required, transferring the integration over zero mode space to an integration over instanton collective coordinates. Included in this treatment were the infinite set of gauge zero modes. It was important for consistency to include the gauge modes in this way since it led not only to generation of the (background) gauge fixing term and the ghost lagrangian but also to some extra interactions between the ghosts, the vector field and the discrete zero modes.

These latter interactions were one source of ultra violet divergences in the higher order quantum corrections. There were two others: the standard perturbative short distance corrections which were removed by renormalising the classical instanton action, and some non-perturbative U.V. divergences proportional to the number of transverse (non-gauge) zero modes. These latter "long distance" corrections together with those arising from the new ghost-vector interactions were cancelled by renormalising the factors of $1/g$ appearing in the semiclassical (1 loop) calculation.

Although chapter 2 dealt specifically with instanton background fields the analysis was in fact quite general and applied to any solution of the equations of motion which breaks certain symmetries of the action.

The 1 loop calculation however had to be treated differently. We verified that the μ dependence came from two sources:
(1) From the zero modes via powers of μ such as to cancel the dimension of the instanton measure.
(2) From the determinants of the inverse propagators, with the gluon determinant restricted to the non-zero mode space. This latter determinant could be calculated if certain tricks were used to turn it into the determinant of an invertable operator ($D_\mu D_\mu$ in fact). This method holds not just for the B.P.S.T. instanton but for any self dual field. (Note that any such field is automatically a solution of the equations of motion).

In chapters 3 and 4 we extended these ideas to super Yang-Mills in superfield form. The first step towards this generalisation was to find the generalisation of the instanton in super Yang-Mills i.e. a superfield solution to the euclidean super Yang-Mills equations of motion which contained the ordinary instanton but which had the possibility of non-vanishing values for the other component fields in the multiplet. We showed that the superinstanton was effectively described (up to a general supergauge transformation) by an ordinary bosonic instanton solution for the Yang-Mills field and a non-zero L.H. fermionic component $\lambda(\alpha,\bar{\beta})$ depending linearly on 4 fermionic parameters α_γ and $\bar{\beta}_{\dot{\gamma}}$ which correspond to chiral supersymmetry and antichiral superconformal transformations of the original bosonic instanton. The next step was to find all the zero modes which we did by using the superconformal algebra to generate covariant expressions for the zero modes. Anomalous supergauge modes were also generated. The methods used in that chapter (chapter 3) are applicable to any solution of the equations of motion of a theory. For a single superinstanton we found that the linearly independent set of zero modes that gave a non-singular parameterisation of the superinstanton contained 8 bosonic modes and 8 fermionic modes. (There were 4 more fermionic modes than had been expected from component analyses; these were supergauge anomalous modes). Discussed also in that chapter were the generalisation of the 1 loop tricks used in the Yang-Mills case, to the case of self dual superfields and quantum prepotential (v) fluctuations. Now let us pause to discuss the work of Novikov et al [1] already referred to at the beginning of this chapter, before going on to discuss and compare this with the work of chapter 5.

It has been argued by Novikov, Shifman, Vainshtein, and Zakharov [1], that in super Yang-Mills theories by using instanton calculus one can determine the β-function to all orders in perturbation theory. In particular they find that the vacuum energy in the presence of an instanton background field in super Yang-Mills theories is proportional to

$$\int d^4x_0 \frac{d\rho}{\rho} d^2\alpha\, d^2\bar{\beta} \frac{1}{g_0^{n_b-n_f}} (M\rho)^{n_b-n_f/2} e^{-8\pi^2/g_0^2} \tag{5.1}$$

where x_0 and ρ are the collective coordinates corresponding to the breaking of translational and scale invariance and α and β are grassmannian collective coordinates corresponding to the breaking of supersymmetry and superconformal invariance. M is the ultraviolet cutoff and n_b and n_f are the number of vector (Q_μ) and fermion (λ) zero modes respectively. The remarkable feature of eq. (5.1) is that there are no higher order corrections to the semiclassical result. Using the results that for an SU(n) gauge theory with N= 1,2, or 4 supersymmetry, n_b=4n and n_f=2Nn, the M independence of eq.(5.1) implies

$$\beta(g_0) \equiv \frac{\partial g_0}{\partial \ln M} = \frac{g_0^3}{16\pi^2} \frac{(N-4)n}{1-(2-N)2n(g_0^2/16\pi^2)} \tag{5.2}$$

Equation (5.2) agrees with results obtained from perturbation theory up to two loops. However for N=1 supersymmetry at the three loop level the result, which is renormalisation scheme dependent, differs from that obtained using the dimensional reduction scheme [5].

The argument of Novikov et al. [1] is based on the observation that whereas there is a pair of collective coordinates α_α corresponding to chiral supersymmetry transformations, there are no corresponding antichiral collective coordinates $\bar{\alpha}_{\dot{\alpha}}$. Indeed the antichiral transformation on the bosonic instanton $\theta\sigma^{\mu}\bar{\theta}A^{I}_{\mu}$ (where A^{I}_{μ} is the BPST instanton [6], see chapter 2) leads to a change in the superfield proportional to

$$\theta^2\bar{\theta}\bar{\sigma}^{\mu\nu}\bar{\alpha}F^{I}_{\mu\nu} \tag{5.3}$$

This is is zero since $F^{I}_{\mu\nu}$ is self dual and $\bar{\sigma}_{\mu\nu}$ is anti self dual. The next step in the argument is to note that the result of calculating higher order graphs such as those in Fig. 5.1, in which the propagators are those of the quantum superfield in the instanton background field, is of the form

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\; f(x,\theta,\bar{\theta}) \tag{5.4}$$

Supersymmetry invariance implies that any translation in $\bar{\theta}$ ($\bar{\theta} \to \bar{\theta} + \bar{\xi}$) can be compensated by a shift in the corresponding collective coordinates. The absence of the collective coordinates $\bar{\alpha}$ leads the authors of ref. [1] to conclude that the integrand in eq.(5.4) must be independent of $\bar{\theta}$ and hence that the integral is zero. They thus conclude that all higher order diagrams vanish. This, together with the result of ref. [7] that the contribution of positive modes at the semiclassical level is zero, gives us eq. (5.1).

We do not accept this argument. Indeed if it were correct the classical instanton action would also have to be zero for this is

$$\frac{1}{2g^2}\int d^4x\, d^2\theta\; W^{\alpha}W_{\alpha} \tag{5.5}$$

which can be written in the form

$$-\frac{1}{2g^2}\int d^4x\, d^2\theta\, d^2\bar{\theta}\; \left(e^{-gV}D^{\alpha}e^{gV}\right)\bar{D}^2\left(e^{-gV}D_{\alpha}e^{gV}\right) \tag{5.6}$$

(we are using the notation of ref.[8]).

Applying the argument outlined above would lead to zero for eq. (5.6) also, whereas we know that the classical action for the instanton is $8\pi^2/g^2$ (we have checked that the surface terms ignored in from (5.5) to (5.6) do indeed vanish). So how do we compensate for a translation in $\bar{\theta}$ by a shift in one of the collective coordinates, given that there is no $\bar{\alpha}$ collective coordinate? Recall that in addition to the collective coordinates corresponding to the discrete zero modes there is also an infinite number of collective coordinates corresponding to supergauge transformations. Indeed (5.3) is the change in the superinstanton after performing an antichiral supersymmetry transformation (see equation (3.3.1b))

$$\theta \to \theta \quad , \quad \bar{\theta} \to \bar{\theta} + \bar{\xi} \quad , \quad x \to x - 2i\bar{\xi}\theta \tag{5.7}$$

followed by a supergauge transformation selected so as to cancel the change in all the auxiliary fields. Since the combined supersymmetry plus supergauge transformation gives zero at $\lambda=0$ (note that (3.3.3a) vanishes from (3.2.18) with $\alpha=\bar{\beta}=0$) it follows that the supersymmetry transformation (5.7) is equivalent to a supergauge transformation. The "missing" collective coordinate is that corresponding to this supergauge transformation. The argument of ref. [1] would only be valid if the function f of eq.(5.4) were supergauge invariant, which is not the case here: the integrand in eq.(5.6) is not supergauge invariant. (Note that the integrand in eq. (5.5) (after taking the trace) is supergauge invariant and here their argument works: the integrand effectively does not depend on θ).

In spite of the failure of this argument for the vanishing of the higher order corrections to eq.(5.1), the β-function given in eq.(5.2) is correct for all the renormalisation scheme independent coefficients. This means that, at least up to two loop level the higher order corrections must indeed cancel.

Substituting $n_b = 4n$ and $n_f = 2n$ for the case we considered (N=1 supersymmetry) we see that our semiclassical result (4.47) agrees with (5.1) (disregarding irrelevent supergauge and superghost integrals and factors of ρ). However our zero mode counting was different : The 2n fermionic supergauge anomalous modes (see (4.37) and below) meant that the quantum prepotential had equal numbers of fermionic and bosonic modes so that the preexponential factor of (5.1)

$$\left(\frac{1}{g}\right)^{n_b - n_f}$$

is equal to unity. The Pauli-Villars regulator which is introduced as a term

$$-\int M^2 V^2 \, d^8 z$$

in the action yields (c.f. (5.1))

$$M^{n_b - n_f}$$

which is also unity. Our powers of g and M come purely from the ghost sector: there are 2n ghost zero modes for each chiral ghost. There are 3 chiral ghosts (c, c', and η - see (4.35) and (4.21)) and each yields a factor of $M^{\frac{1}{2}}$ (see discussion below (4.46)) so that we recover the factor of

$$M^{3n} = M^{n_b - n_f/2}$$

In addition the c-ghosts yield a factor of $(\sqrt{g})^{-4n} = g^{-2n} = g^{n_f - n_b}$ from the δ-function constraints which factor out the ghost zero modes (see below (4.39)). In this way we recover the factors of M and g as in (5.1).

The zero mode counting given in ref.[1] is the one appropriate for the Wess-Zumino gauge [3] and it would appear from ref.[1] that this is the gauge they were considering. This gauge however is not supersymmetric and the results of graphical calculations can not in general be cast in the form (5.4).

We have reached the conclusion that the argument given in ref.[1] for the vanishing of all higher order corrections to equation (5.1) is invalid. Does there exist some other proof that an all orders cancellation exists in some standard normalisation scheme? To try to gain some clue as to the answer to this question we turn to our calculation of section 4.3.

First note that the sum of the $\ln\mu$ dependent contributions at two loops ((4.53) and (4.62)) does indeed vanish. At two loop order this is however to be expected from the known β-function (as is readily verified by differentiating expression (4.47) by $\ln\mu$ and using $\beta_1 = 2n\beta_0$). At higher loop order the value of the explicit $\ln\mu$ dependent terms involves the third and higher order β-function coefficients. Therefore the vanishing or otherwise of these terms will depend on the renormalisation scheme. Ref.[1] however, requires that these higher order terms do indeed vanish. In our case (since the contributions from the discrete sector of the Jacobian vanish to all orders, see beginning of section 4.3) this implies that the short distance and long distance (explicitly) $\ln\mu$ dependent terms must cancel to all orders of perturbation theory, but we have gained no insight in the process of this calculation which would lead us to conclude that this holds true. In fact the theorem of ref.[1] implies that the full graphs (such as those in fig.5.1 and fig.4.2) must vanish identically - even before renormalisation; We see from equations (4.52) and (4.61) that this is not true for dimensional reduction even at two loops, (a fact which is readily verified by solving the β-function for the coupling constant renormalisation Z_g and applying this to the semiclassical result (4.40)). Thus not only is the proof in ref.[1] incorrect but it is impossible to construct a proof which will hold for graphs in dimensional reduction.

However the vanishing of the full graphs before renormalisation clearly depends on how one regulates: Note that, for example, in a gauge invariant higher derivative regularisation [9] we <u>will</u> find a vanishing 2 loop contribution (because the divergence is tied to the μ dependence through $\ln(\Lambda/\mu)$). This latter regularisation is more appropriate because it preserves identities relating to instantons (which only exist in 4 dimensions). Unfortunately it is not clear how one should use the background superfield method in this case (or for any other regularisation scheme that stays in 4 dimensions [10]) since the form of the divergence allowed by the present method (see equation (4.48)) is very specific to dimensional reduction.

If it were true that the full graphs in the background of an instanton vanished, to all orders, identically (using an appropriate regularisation scheme) then one might expect that such a cancellation could be deduced in a straightforward manner by formal manipulation on the full graphs using covariant ∇-algebra, instanton identities (such as $\bar{W}_{\dot{\gamma}} = 0$ and the equations of motion) and the full propagators of section 3.4. One can readily convince oneself however that such a program merely generates the scalar graph (of fig.4.4) plus terms containing one or more field strengths (W_Y), and consequently any cancellation between these terms is far from evident.

The higher order $\ln\mu$ dependent corrections (from three loop order upwards) are renormalisation scheme dependent and therefore can only cancel in some particular renormalisation scheme. Presumably we wish to preserve the gauge and supersymmetry Ward identities and instanton symmetries (such as self duality) but we do not expect this to be sufficient to determine the scheme, or equivalently the β-function, uniquely. Thus any general proof for the cancellation must depend on this particular renormalisation scheme; From our work we have no clues as to what this scheme might be.

In summary, despite the fact that we understand how to do higher order calculations in an instanton background field, we have been unable to construct a proof that an all orders cancellation of quantum corrections exists.

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   (This method requires a supplementary regularisation at 1 loop, for example a gauge invariant extension of Pauli-Villars is possible as explained in the above paper).

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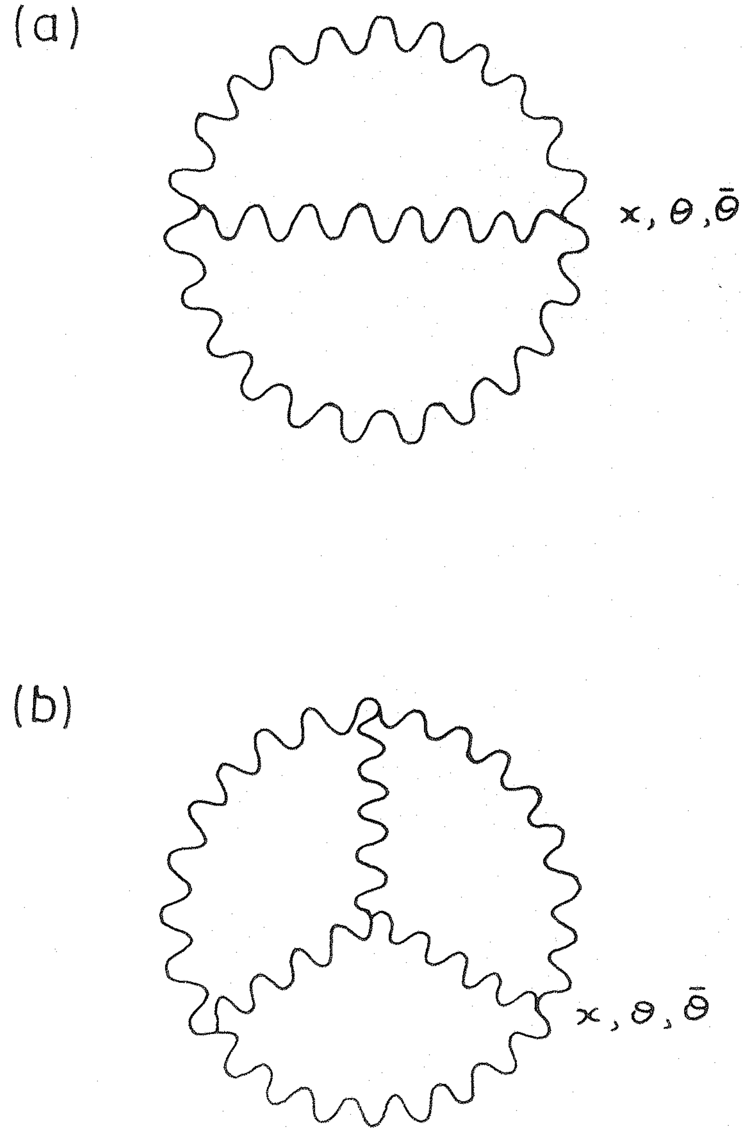

Fig. 5.1