UNIVERSITY OF SOUTHAMPTON

MAPS AND HYPERMAPS

- OPERATIONS AND SYMMETRY

by

Lynne Denise James

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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Doctor of Philosophy

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Just as a map on a surface is an imbedding of a topological realisation of a graph, a hypermap is an imbedding of a topological realisation of a hypergraph. The algebraic theory of (hyper)maps facilitates both a study of the symmetries of (hyper)maps and a study of the possible imbeddings of (hyper)graphs via an associated set of permutations. In chapter 1 we set out the established algebraic theory of maps on surfaces together with an extension to hypermaps and maps of higher dimension whose topological realisations include all cell decompositions of n-manifolds.

There is a group of six invertible topological operations on surface maps which includes the well-known duality that interchanges vertices and faces. These operations arise naturally in the algebraic theory, being induced by the outer automorphisms of a certain Coxeter group. In chapter 2 we study the analogous groups of operations on hypermaps and maps of higher dimension.

If the symmetry group of a map on a surface contains both a rotation centred on a face and a rotation centred on a vertex, each cyclically permuting successive incident edges, then the map is said to be regular. If, in addition, there is a symmetry which acts on an edge by interchanging the two incident vertices without interchanging the two incident faces then the map is said to be reflexible. In chapter 3 we consider a weaker version of these symmetry conditions, and in so doing we introduce a class of highly symmetric maps and hypermaps that remains invariant under the operations discussed in chapter 2. We find that every finitely generated group may be regarded as a group of symmetries of some highly symmetric hypermap.

Finally, in chapter 4 we give an application of the algebraic theory to an imbedding problem by classifying those imbeddings of complete graphs whose symmetry group acts transitively on edges.

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INTRODUCTION

Any imbedding of a connected graph in a connected possibly non-orientable surface, with or without boundary, decomposing that surface into topological open discs, or half discs, can be regarded as a transitive permutation representation of a certain Coxeter group. The concept of an algebraic theory of maps, that is, graph imbeddings of the above type, was presented by Tutte [27] who defined an associated set of permutations acting on the set of doubly directed edges. Firm foundations for this were set down by Bryant and Singerman [5] following a full treatment of the orientable case by Jones and Singerman [15]. We can consider hypergraph imbeddings in the same way. An algebraic theory of orientable hypermaps was first presented by Cori [6]. In [28], Vince considers a more general class of hypermap, to which he associates a transitive permutation representation of a specific Coxeter group. More generally, in [24] Ronan shows that any chamber system over a finite set, as defined by Tits [26], can be regarded as a cell complex. In chapter 1 we set out the established algebraic theory of maps on surfaces together with an extension to hypermaps and maps of higher dimension whose topological realisations include all cell decompositions of n-manifolds.

There is a group of six invertible topological operations on surface maps, isomorphic to the symmetric group on three elements, that was first described by Wilson [31] for reflexible maps and later by Lins [19] for all maps. These operations arise naturally in the algebraic theory, being induced by the outer automorphisms of the above Coxeter group, and were presented as such in [16] by Jones and Thornton. In chapter 2 we give generators for, and determine the isomorphism class of, the groups of outer automorphisms of an infinite family of Coxeter groups in order to study the analogous groups of operations on maps of higher dimension. This is done by viewing each Coxeter group as an amalgamated product, in several different ways, and uses induction on the dimension, the inductive step being provided by the determination of the centraliser of any element of finite order. Some light is thrown on the edge twists, or barrings, used by Edmonds [10] in his characterisation of graph imbeddings. Hypermaps are similarly treated.

If the group of symmetries of a map on a surface contains both a rotation centred on a face and a rotation centred on a vertex, each cyclically permuting successive incident edges, then the map is said to be regular. If, in addition, there is a symmetry which acts on an edge by interchanging the two incident vertices without interchanging the two incident faces then the map is said to be reflexible [7]. Every regular map on a non-orientable surface is reflexible. Regular maps have been extensively studied; on the sphere they are the platonic solids. In chapter 3 we consider a weaker version of these symmetry conditions, and in so doing we introduce a class of highly symmetric maps and hypermaps that remains invariant under the operations discussed in chapter 2. We find that every finitely generated group may be regarded as a group of symmetries of some highly symmetric hypermap.

Finally, in chapter 4 we give an application of the algebraic theory to an imbedding problem. In [1] Biggs showed that the complete graph on n vertices has a regular imbedding in an orientable surface without boundary if and only if n is a prime power. The examples he gave were Cayley maps based on the additive groups of finite fields. The symmetry group of any regular imbedding acts transitively on both vertices and edges. In [2] Biggs showed that any imbedding of a complete graph, in an orientable surface without boundary, whose symmetry group acts transitively on vertices can be described as a Cayley map. We classify those imbeddings of complete graphs, in a possibly non-orientable surface without boundary, whose symmetry group acts transitively on edges. This relies on the classifications of 2-homogeneous groups by Ito [12], Zassenhaus [32] and Kantor [17], and includes those surface maps that attain the upper bound on the number of symmetries of an imbedding of a simple graph with n vertices in a surface without boundary. We find that essentially each can be described as either a Cayley map based on the additive group of a finite field or as the image of such a map under one of the operations discussed in chapter 2.

CHAPTER 1

Maps and Hypermaps

1) First we briefly outline the algebraic theory of maps developed in [5] and [15]. A map $\mathcal{M}$ is a connected graph $\mathcal{G}$ imbedded (without crossings) in a connected surface $\mathcal{S}$ (possibly non-orientable or with boundary) such that each of the faces of $\mathcal{M}$ (the connected components of $\mathcal{S} \setminus \mathcal{G}$) is homeomorphic to an open disc or half disc. To each map $\mathcal{M}$ we associate a set Ω of blades: whenever an edge e meets a vertex v we draw on the surface a pair of blades, one on each side of e. We define three permutations of Ω as follows: r_2 transposes each such pair of blades, r_0 sends each blade to the blade at the other end of e and on the same side of e, and r_1 transposes pairs of blades with a vertex and face in common; Fig. 1 illustrates the effect of these permutations on a blade β. We refer to an orbit of Ω under the cyclic subgroup $\langle r_2 \rangle$ as a dart of the map. (These definitions

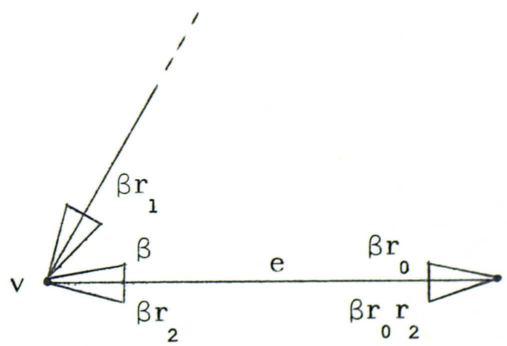


Figure 1

require some slight modifications, allowing r_0, r_1, r_2 or $r_0 r_2$ to have fixed points on Ω, when $\mathcal{M}$ is on a surface with boundary or when the underlying graph of $\mathcal{M}$ has free edges. Full details may be found in [5].) Clearly these permutations satisfy the relations

$$r_0^2 = r_1^2 = r_2^2 = (r_0 r_2)^2 = 1 ,$$

and by the connectedness of $\mathcal{M}$ they generate a transitive permutation representation of the group

$$\Gamma = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_0 r_2)^2 = 1 \rangle .$$

Conversely, given a transitive permutation representation of Γ on a set Ω, we can reconstruct the map $\mathcal{M}$: we define the vertices, edges and faces of $\mathcal{M}$ to be the orbits in Ω of the dihedral subgroups $\langle r_1, r_2 \rangle$, $\langle r_0, r_2 \rangle$ and $\langle r_0, r_1 \rangle$ of Γ, with incidence corresponding to non-empty intersection of orbits. We observe that a Petrie polygon of $\mathcal{M}$ is then an orbit of the subgroup $\langle r_0 r_2, r_1 \rangle$ as illustrated in Fig. 2. (A more sophisticated approach, described in [5] and [15], is to represent Γ as the

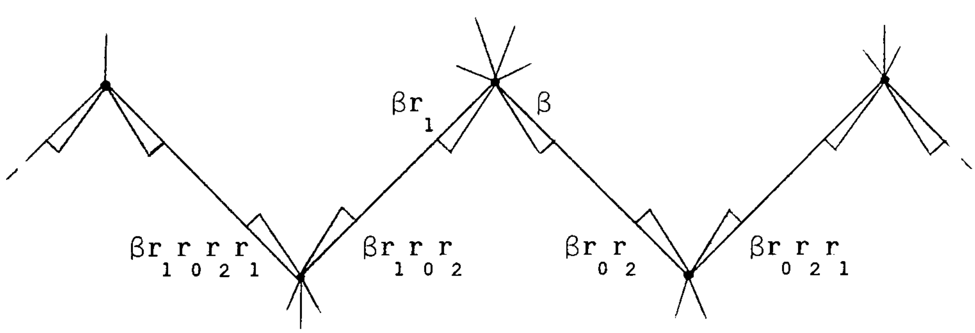


Figure 2

automorphism group of a certain tessellation on a Riemann surface, and to take $\mathcal{M}$ to be the quotient of this tessellation by a suitable subgroup M of Γ.) This gives a bijection between maps and transitive permutation representations of Γ (or more strictly between isomorphism classes in each category).

Each map $\mathcal{M}$ determines a permutation representation which is isomorphic to the action of Γ (by right multiplication) on the cosets Mg of a subgroup $M \leqslant \Gamma$; this subgroup M, the map subgroup associated with $\mathcal{M}$, is the stabiliser in Γ of an element of Ω, and is uniquely determined up to conjugacy. The automorphism group of the map can be realised as the action of N/M on the cosets of M by left multiplication, where N is the normaliser of M in Γ. A map is both orientable and without boundary if and only if $M \leqslant \Gamma^+$ [5], where Γ^+ is the even subgroup of Γ, generated by $r_0 r_2$ and $r_1 r_2$, which we denote by X and Y respectively, with presentation

$$\Gamma^+ = \langle X , Y \mid X^2 = 1 \rangle .$$

Two subgroups of Γ^+ determine equivalent maps, up to orientation-preserving isomorphism, if and only if they are conjugate in Γ^+. The orientation-preserving automorphism group of such a map can be realised as the action of N^+/M on the cosets of M by left multiplication, where N^+ is the normaliser of M in Γ^+.

2) There is an analogously developed algebraic theory of hypermaps in, for example, [6], [28] and [30]. A hypermap is a map on a surface S with underlying graph $\mathcal{G}$ satisfying the conditions:

(i) $\mathcal{G}$ is trivalent,

(ii) $\mathcal{G}$ has no looped edges,

(iii) no vertex of $\mathcal{G}$ lies on the boundary of S,

(iv) every free edge of $\mathcal{G}$ meets the boundary of S,

together with a colouring of the faces by $\{0, 1, 2\}$ such that every edge is bordered by two different coloured faces. Fig. 3 gives an example of a hypermap on a disc.

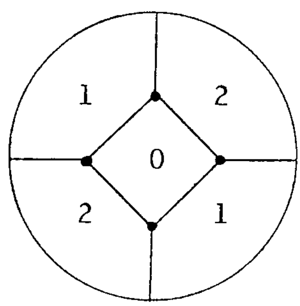


Figure 3

The faces of the map coloured $0, 1, 2$ are called the hypervertices, hyperedges and hyperfaces respectively, and we refer to an edge that is bordered by both a hypervertex and a hyperedge as a hyperdart.

This definition of a topological hypermap differs slightly from those of [6], [28] and [30]. Cori [6] takes the surface to be orientable and contracts each hyperdart to a point; Walsh [30] also takes an orientable surface, and considers the dual of the two coloured map formed by contracting each hyperface to a point; Vince [28] allows the surface to be non-orientable and contracts each hypervertex to a point (we refer to this as the underlying two-coloured map). Each represents an imbedding of a topological realisation of a hypergraph. We observe that by expanding vertices and edges we can regard any map on a surface as a hypermap.

To any hypermap we associate a transitive permutation representation of the group

$$G = \langle r_0 , r_1 , r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1 \rangle$$

by colouring each edge of the underlying map with the complement of the colours of its incident faces. This gives the Schreier coset graph for a subgroup in G, and thus a transitive permutation representation of G. In this case the permuted set is the set of vertices of the underlying map.

Conversely, given a transitive permutation representation of G on a set Ω, we can reconstruct a hypermap: we take the Schreier coset graph of the stabiliser in G of an element in Ω, replacing any looped edge by a free edge, then we attach a disc, or half disc, coloured c to each component formed by the deletion of all edges coloured c.

<u>Definition 1.1</u>. An <u>algebraic hypermap</u> is a transitive permutation representation ρ of the group

$$G = \langle r_0 , r_1 , r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1 \rangle$$

on a set Ω.

<u>Definition 1.2</u>. Two algebraic hypermaps ρ, ρ' are <u>isomorphic</u> if there exists a bijection $\phi : \Omega \longrightarrow \Omega'$ and a group isomorphism $\sigma : G\rho \longrightarrow G\rho'$ such that $r_i\rho\sigma = r_i\rho'$ for all $i \in \{0, 1, 2\}$ and the following diagram commutes

$$\begin{array}{ccccc} \Omega & \times & G\rho & \longrightarrow & \Omega \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\phi} \\ \Omega' & \times & G\rho' & \longrightarrow & \Omega' \end{array}$$

the horizontal arrows representing the group actions.

We shall identify the permutation r_i with its image $r_i\rho$ when no confusion is likely, and refer to the elements of Ω as hyperblades. The stabiliser in G of a hyperblade is referred to as a <u>hypermap subgroup</u>.

Many of the results of the algebraic theory of maps go over easily to the algebraic theory of hypermaps. For example, the automorphism group of a hypermap can be realised as the action of N/M on the cosets of M by left multiplication, with obvious definitions for N and M.

3) More generally, in [28] Vince defines a <u>combinatorial map</u> over a finite set I to be a connected graph $\mathcal{G}$, regular of degree $|I|$, whose edges are $|I|$-coloured such that no two incident edges are the same colour, and defines an isomorphism of two combinatorial maps to be a colour-preserving graph isomorphism. Clearly combin-

atorial maps are equivalent to Schreier coset graphs $\mathcal{G}(W,M)$ for groups W, generated by involutions, and subgroups $M \leqslant W$. We let $N_W(M)$ denote the normaliser in W of M.

<u>Proposition 1.3</u>, [28, 7.5]. If $\mathcal{G}$ is a combinatorial map with Schreier representation $\mathcal{G}(W,M)$, then $\mathrm{Aut}(\mathcal{G}) \simeq N_W(M)/M$.

<u>Proof</u>: For each $u \in N_W(M)$ the function $f_u : Mg \longmapsto Mu^{-1}g$ induces an automorphism of $\mathcal{G}$. Hence there is a homomorphism $\phi : N_W(M) \longrightarrow \mathrm{Aut}(\mathcal{G})$ given by $u \longmapsto f_u$. Since $\ker\phi = M$ we have only to show that ϕ is surjective. Let $f \in \mathrm{Aut}(\mathcal{G})$ and assume that $f : M \longmapsto Mu^{-1}$. This implies that $f : Mg \longmapsto Mu^{-1}g$ for all $g \in W$. Therefore $f = f_u$. Moreover, $u \in N_W(M)$ because for all $g \in M$ both $Mu^{-1}g$ and Mu^{-1} is the image of M under f_u ∎

4) Finally, motivated by the generalisation of topological maps on surfaces to maps of higher dimension, we consider a third family of combinatorial maps. Suppose that $\mathcal{M}$ is a cell decomposition of a connected n--manifold without boundary. Let $\Delta\mathcal{M}$ be its barycentric subdivision, and label each vertex of $\Delta\mathcal{M}$ with the dimension of the cell that it represents. We define a set of permutations on the n-simplices of $\Delta\mathcal{M}$ as follows. For each $i \in \{0, 1, 2, \ldots, n\}$ each (n-1)-simplex of $\Delta\mathcal{M}$ whose vertices are not labelled by i is contained in precisely two n-simplices of $\Delta\mathcal{M}$. We define r_i to

be the permutation that transposes each such pair of n-simplices. Since the boundary of each n-cell is itself a cell decomposition of a connected (n-1)-manifold, and each (n-1)-cell is contained in no more than two n-cells, there is an inductive proof that if $j > i+1$ then r_i and r_j commute.

<u>Definition 1.4</u>. An <u>n-dimensional algebraic map</u> is a transitive permutation representation ρ of the group

$$\Gamma_n = \langle r_0 , r_1 , \ldots , r_n \mid r_i^2 = (r_j r_k)^2 = 1 , \ k > j+1 \rangle$$

on a set Ω. Elements of Ω are referred to as <u>n-blades</u>.

<u>Definition 1.5</u>. Two n-dimensional algebraic maps ρ , ρ' are <u>isomorphic</u> if there is a bijection $\phi : \Omega \longrightarrow \Omega'$ and a group isomorphism $\sigma : \Gamma_n\rho \longrightarrow \Gamma_n\rho'$ such that $r_i\rho\sigma = r_i\rho'$ for all $i \in \{0, 1, \ldots , n\}$ and the following diagram commutes,

$$\begin{array}{ccccc} \Omega & \times & \Gamma_n\rho & \longrightarrow & \Omega \\ \phi\big\downarrow & & \big\downarrow\sigma & & \big\downarrow\phi \\ \Omega' & \times & \Gamma_n\rho' & \longrightarrow & \Omega' \end{array}$$

the horizontal arrows representing the group actions.

We shall identify the permutation r_i with its image under ρ when no confusion is likely, and refer to the stabiliser in Γ_n of an element in Ω as a <u>map subgroup</u>.

To each n-dimensional algebraic map $\mathcal{M}$ we associate an n-dimensional cell complex $\Delta\mathcal{M}$ as follows. Let

I denote the set $\{0, 1, \ldots, n\}$. For each element $\beta \in \Omega$, let $\Delta\beta$ be an n-simplex. Arbitrarily assign to each vertex of $\Delta\beta$ a distinct element of I. Call the set of elements assigned to a face s of $\Delta\beta$ the type of s. Let K be the disjoint union of the set $\{\Delta\beta \mid \beta \in \Omega\}$. In K identify two simplices $s \subseteq \Delta\beta$ and $s' \subseteq \Delta\beta'$ of the same type J if and only if β and β' are in the same orbit of the subgroup generated by $\{r_i \mid i \in I \setminus J\}$. If $\sim$ denotes this identification then take $\Delta\mathcal{M} = K/\sim$. Thus we have formed the cell complex associated by Ronan [24] to the partitions of the elements of Ω into their orbits under the subgroups $\langle r_i \rangle$ for $i \in I$. Intuitively $\Delta\mathcal{M}$ can be thought of as being built from n-simplices, one for each element of Ω, such that two n-simplices share a common codimension 1 face if the corresponding points are adjacent in the Schreier coset graph associated with the representation.

We note that $|\Delta\mathcal{M}|$ is not generally a manifold. For this it would at least be necessary for each orbit of the subgroup generated by $\{r_i \mid i \in I \setminus \{n\}\}$ to represent a map on an (n-1)-sphere. This can soon fail for $n > 2$. For representations of manifolds by edge-coloured graphs see [9] and [20].

The orbits in Ω of the subgroups $\langle r_j \mid j \in I \setminus \{i\} \rangle$ are called the <u>i-faces</u> of $\mathcal{M}$, with incidence corresponding to non-empty intersection of orbits. Furthermore, as observed by Vince [28, 4.2], there is a partial order on these faces defined as follows: let x be an i-face and

y a j-face of $\mathcal{M}$ then $x < y$ if and only if x is incident with y and $i \leq j$. We need only show that $<$ is transitive. Suppose that $y < z$. It is sufficient to show that $x \cap z \neq \emptyset$. Let g be a finite sequence $(g_1, g_2, \ldots, g_m)$ of elements in $\{r_\ell \mid \ell \in I \setminus \{j\}\}$ then g acts on Ω in the obvious way and we can choose g to satisfy the conditions:

(i) $xg \cap z \neq \emptyset$,

(ii) of all such sequences g is minimal with respect to its length m and,

(iii) of all such sequences g is minimal with respect to the length t of the initial subsequence of elements in $\{r_\ell \mid \ell \in I, \ell < j\}$.

We must show that $m = 0$. Assume that $m \neq 0$. If $t = 0$ then the first element in g can be removed, contradicting the minimality of m. Similarly, if r_ℓ is the last element in g then $\ell > j$. Thus $1 \leq t < m$ and we can contradict the minimality of t by interchanging the positions of g_t and $g_{(t+1)}$.

CHAPTER 2

Operations

1) There is a well-known duality for maps on surfaces that interchanges vertices and faces while retaining certain important features such as the automorphism group. Wilson [31] and Lins [19] gave topological descriptions for other similar invertible operations on surface maps which, together with the above duality, generate a group isomorphic to S_3. In [16] Jones and Thornton showed how these operations arise naturally in algebraic map theory, being induced by the outer automorphism group of Γ. In this chapter we study the analogous groups for hypermaps and maps of general dimension.

2) Suppose that $\mathcal{M}$ is a cell decomposition of a connected n-manifold without boundary. Let $\mathcal{M}^*$ denote the dual map. Cells of dimension i in $\mathcal{M}$ are transformed into cells of dimension $n-i$ in $\mathcal{M}^*$. If we consider the relationship between paths in $\mathcal{M}$ along the graph dual to $\Delta\mathcal{M}$ and paths in $\mathcal{M}^*$ along the graph dual to $\Delta\mathcal{M}^*$ then we see that the duality of maps on manifolds corresponds to a duality of sequences in $\{r_0, \ldots, r_n\}$ that acts by interchanging the symbols r_i and r_{n-i}. This duality preserves the set of sequences $\{ r_i^2, (r_j r_k)^2 : |k-j| > 1 \}$. It also preserves the juxtaposition of sequences. Thus we have a duality of elements in Γ_n that preserves group multiplication. In other words, we have a group automorphism of Γ_n. This group automorphism induces the duality of maps on manifolds by its action on map subgroups.

We define an operation on the topological realisation of n-dimensional algebraic maps to be any transformation induced by the action of a group automorphism of Γ_n on the map subgroups. This definition was first made by Jones and Thornton [16] for maps on surfaces. We define an operation on (topological) hypermaps analogously.

If a map $\mathcal{M}$ has map subgroup M then $\mathcal{M}^{\alpha}$, the image of $\mathcal{M}$ under the operator $\alpha \in \mathrm{Aut}(\Gamma_n)$, has map subgroup M^{α}. Thus the inner automorphism group $\mathrm{Inn}(\Gamma_n)$ acts trivially on maps, and so we have an induced action of the outer automorphism group

$$\mathrm{Out}(\Gamma_n) = \mathrm{Aut}(\Gamma_n)/\mathrm{Inn}(\Gamma_n) .$$

Similarly the action of Aut(G) induces an action of Out(G). We shall therefore determine the outer automorphisms of Γ_n and G. These appear to be original results.

We first note the automorphisms θ_n and ϕ_n of

$$\Gamma_n = \langle r_0, r_1, \ldots, r_n \mid r_i^2 = (r_j r_k)^2 = 1 , \ k > j+1 \rangle$$

defined by $\theta_n : r_i \longmapsto r_{(n-i)}$ and $\phi_n : r_2 \longmapsto r_0 r_2$ where ϕ_n fixes r_i for all i other than $i = 2$. If we let H_n denote the subgroup $\langle \theta_n, \phi_n \rangle$ of $\mathrm{Aut}(\Gamma_n)$ then

$$H_0 = 1 ,$$
$$H_1 = \langle \theta_n \mid \theta_n^2 = 1 \rangle \simeq C_2 ,$$
$$H_2 = \langle \theta_n, \phi_n \mid \theta_n^2 = \phi_n^2 = (\theta_n \phi_n)^3 = 1 \rangle \simeq S_3 ,$$
$$H_n = \langle \theta_n, \phi_n \mid \theta_n^2 = \phi_n^2 = (\theta_n \phi_n)^4 = 1 \rangle \simeq D_4 , \quad n > 2 ,$$

and $H_n \cap \mathrm{Inn}(\Gamma_n) = 1$.

We also note that for each $i \in \{0, 1, \ldots, n\}$ Γ_n is an amalgamated product $P(n, r_i)$ of the subgroups $G_1(n, r_i)$ and $G_2(n, r_i)$ by the subgroup $A(n, r_i)$ where if R_n denotes $\{r_0, \ldots, r_n\}$ then

$$G_1(n, r_i) = \langle R_n \setminus \{r_i\} \mid r_j^2 = (r_k r_\ell)^2 = 1 , \ \ell > k+1 \rangle ,$$

$G_2(n,r_i) = \langle R_n \setminus \{r_{(i\pm 1)}\} \mid r_j^2 = (r_k r_\ell)^2 = 1, \ \ell > k+1 \rangle$,
$A\,(n,r_i) = \langle R_n \setminus \{r_i, r_{(i\pm 1)}\} \mid r_j^2 = (r_k r_\ell)^2 = 1, \ \ell > k+1 \rangle$.

If W is a group generated by a subset of involutions S then the couple (W,S) is called a Coxeter system if the following condition is satisfied

(C) For s, s' in S let $m(s,s')$ be the order of ss' ; let I be the set of couples (s,s') such that $m(s,s')$ is finite. The generating set S and the relations $(ss')^{m(s,s')} = 1$ for (s,s') in I forms a presentation of the group W .

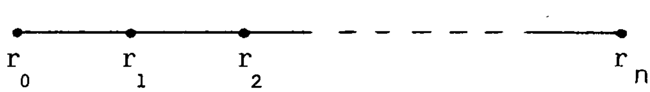
It is easy to see, by taking appropriate homomorphisms of Γ_n , that the couple (Γ_n, R_n) is a Coxeter system. We have.

<u>Theorem 2.1</u>, [4, IV.1.8]. If (W,S) is a Coxeter system then

(i) For all subsets X of S , the couple $(\langle X \rangle, X)$ is a Coxeter system

(ii) If $(X_\alpha)_{\alpha \in A}$ is a family of subsets of S then
$$\left\langle \bigcap_{\alpha \in A} X_\alpha \right\rangle = \bigcap_{\alpha \in A} \langle X_\alpha \rangle . \ \blacksquare$$

Finally, we draw attention to the representation of Γ_n by a Coxeter graph:



The nodes are to be interpreted as generating involutions for a group whose only other relations come from the commutativity of non-adjacent generators. (Note that we go against convention in that we do not interpret the cube of a product of adjacent generators as a relation.) Thus, for example, the graph below would be interpreted as $\Gamma_n \times \Gamma_m$ (or more strictly as being in the same isomorphism class).

In the following we use the notation $C_W(g)$ and $Z(W)$ for the centraliser of $g \in W$ and the centre of W respectively for any group W.

Lemma 2.2. There is a bijection between the sets of non-adjacent elements of R_n and the conjugacy classes of elements of finite order in Γ_n.

Proof: Let T be those $n \in \mathbb{N}$ for which any element of finite order in Γ_n is conjugate to a product of non-adjacent elements of R_n. Clearly $0 \in T$ and, by the torsion theorem for free products [21, IV.1.6], $1 \in T$. Suppose that $k \geq 2$ and that $n \in T$ for all $n < k$. By the torsion theorem for amalgamated products [21, IV.2.7], any element of finite order in Γ_k is conjugate to an element of either $G_1(k, r_k)$ or $G_2(k, r_k)$. But $G_1(k, r_k) = \Gamma_{(k-1)}$ and $G_2(k, r_k) = \Gamma_{(k-2)} \times \langle r_k \rangle$, and so by hypothesis $k \in T$. Suppose that the product of one set of non-adjacent elements is conjugate to the product of another. Then, by abelianisation, the two sets must be equal. ∎

Proposition 2.3. $C_{\Gamma_n}(r_j) = G_2(n, r_j)$ and if $\{s_1, .., s_m\}$ is a set of commuting elements of R_n then $C_{\Gamma_n}(s_1 \dots s_m) \subseteq \bigcap_{i=1}^{m} C_{\Gamma_n}(s_i)$.

Proof: Suppose that $t \in C_{\Gamma_n}(s_1 \dots s_m)$. Let $s = s_1 \dots s_m$ and let $i \in \{1, .., m\}$. Then $s \in G_2(n, s_i) \setminus A(n, s_i)$. Let $t = t_1 \dots t_k$ be in the reduced form of $P(n, s_i)$ and suppose that $k \geqslant 2$. If $t_1, t_k \in G_2(n, s_i)$ then $s^{t_1} \in G_2(n, s_i) \setminus A(n, s_i)$ and $t_k s \in A(n, s_i)$. Whence $t_{(k-1)} t_k s \in G_1(n, s_i) \setminus A(n, s_i)$ and so by the normal form theorem for amalgamated products [21, IV.2.6] $t^{-1} s t s \neq 1$. Thus $t \notin C_{\Gamma_n}(s_1 \dots s_m)$. If either t_1 or t_k are in $G_1(n, s_i)$ then we find a similar contradiction. Whence $k \leqslant 1$. Suppose that $t \in G_1(n, s_i) \setminus A(n, s_i)$ then again by the normal form theorem $t^{-1} s t s \neq 1$ and so $t \notin C_{\Gamma_n}(s_1 \dots s_m)$. Whence $t \in G_2(n, s_i)$ and $C_{\Gamma_n}(s_1 \dots s_m) \subseteq \bigcap_{i=1}^{m} G_2(n, s_i)$. If we put $m = 1$ and $s_1 = r_j$ then we have $C_{\Gamma_n}(r_j) \subseteq G_2(n, r_j)$ and thus $C_{\Gamma_n}(r_j) = G_2(n, r_j)$ ∎

The following proposition can be seen as a corollary to theorem 2.1, but we include a direct proof.

Proposition 2.4. If $\{s_1, .., s_m\}$ is a set of commuting elements of R_n then $\bigcap_{i=1}^{m} C_{\Gamma_n}(s_i) \subseteq \langle V_n\{s_1, .., s_m\}\rangle$ where $V_n\{s_1, .., s_m\}$ is the subset of elements of R_n that commute with every member of $\{s_1, .., s_m\}$.

Proof: Let $t \in \bigcap_{i=1}^{m} C_{\Gamma_n}(s_i)$ and assume that $t \neq 1$. Let $t = t_1 \dots t_k$ be a reduced decomposition of t with respect to R_n . Let $r_i \in \{s_1, .., s_m\}$ and suppose that $t_j = r_{(i+1)}$ for some $j : 1 \leqslant j \leqslant k$. Then $t \in C_{\Gamma_n}(r_i) = G_2(n, r_i) \subseteq G_1(n, r_{i+1})$.

Let $j(1) < j(2) < \ldots j(p)$ be those j for which $t_j = r_{i+1}$. Let $u_0 = t_1 \ldots t_{(j(1)-1)}$, let $u_q = t_{(j(q)+1)} \ldots t_{(j(q+1)-1)}$ for all $q : 0 < q < p$, and let $u_p = t_{(j(p)+1)} \ldots t_k$. So $t = u_0 r_{(i+1)} \cdot \ldots \cdot u_p$ where $u_q \in G_1(n, r_{i+1})$ for all q : $0 \leqslant q \leqslant p$. Suppose that $u_q \in A(n, r_{i+1})$ for some $q : 0 < q < p$. Then $t_{j(q)} u_q t_{j(q+1)} = r_{(i+1)} u_q r_{(i+1)} = u_q$ which contradicts the minimality of k. Thus $u_q \in G_1(n, r_{i+1}) \setminus A(n, r_{i+1})$ for all $q : 0 < q < p$. Let $|w|$ denote the length of w in $P(n, r_{i+1})$. If $p \geqslant 2$ then

$$|u_0 r_{(i+1)} u_1 \cdot \ldots \cdot u_{(p-1)} r_{(i+1)} u_p| \geqslant 2p-1 \geqslant 3 .$$

But $t \in G_1(n, r_{i+1})$ and so $|t| \leqslant 1$. Whence $p = 1$ and $u_0^{-1} t u_1^{-1} r_{(i+1)} = 1$. But $u_0^{-1} t u_1^{-1} \in G_1(n, r_{i+1})$ and $r_{i+1} \in G_2(n, r_{i+1}) \setminus A(n, r_{i+1})$ so $|u_0^{-1} t u_1^{-1} r_{(i+1)}| \geqslant 1$ which contradicts the normal form theorem for amalgamated products. Thus there is no j such that $t_j = r_{i+1}$. Similarly there is no j such that $t_j = r_{i-1}$. Whence $t_j \in V_n\{s_1, \ldots, s_m\}$ for all $j : 1 \leqslant j \leqslant k$. ∎

From propositions 2.3 and 2.4 we have

<u>Lemma 2.5</u>. The centraliser in Γ_n of a product of a set of commuting elements of R_n is presented by the Coxeter graph for Γ_n less those nodes adjacent to some element of that set. ∎

<u>Corollary 2.6</u>. $Z(\Gamma_n) = 1$ for all $n > 0$.

Proof: If $g \in Z(\Gamma_n)$ then

$$g \in C_{\Gamma_n}(r_0 r_2 r_4 \ldots) \cap C_{\Gamma_n}(r_1 r_3 r_5 \ldots)$$
$$= \langle r_0, r_2, r_4 \ldots \rangle \cap \langle r_1, r_3, r_5 \ldots \rangle$$
$$= 1, \text{ by abelianisation. } \blacksquare$$

For $g \in \Gamma_n$ let $\mu(g)$ be the set of those conjugacy classes C of elements of order two in Γ_n such that $ZC_{\Gamma_n}(h) \simeq C_2$, for all $h \in C$, and $g \in \bigcup_{h \in C} C_{\Gamma_n}(h)$. By lemma 2.2 $|\mu(g)|$ is finite.

Proposition 2.7. If $g \in \Gamma_n$ and $\alpha \in Aut(\Gamma_n)$ then $|\mu(g)| = |\mu(g^\alpha)|$.

Proof: $C \in \mu(g) \iff C^\alpha \in \mu(g^\alpha)$. $\blacksquare$

Proposition 2.8. $|\mu(r_i)| = |\{r_j \in R_n : j \notin \{2, (n-2), (i \pm 1)\}\}|$ for all $r_i \in R_n$.

Proof: By lemmas 2.2 and 2.5 $\{r_j \in R_n : j \notin \{2, (n-2)\}\}$ represents the conjugacy classes C of elements of order two such that $ZC_{\Gamma_n}(h) \simeq C_2$ for all $h \in C$. Suppose that r_j is adjacent to r_i. By abelianisation $r_i \notin C_{\Gamma_n}(r_j)^{\Gamma_n}$, and so the conjugacy class of r_j does not belong to $\mu(r_i)$. Conversely, if r_j is not adjacent to r_i and $j \notin \{2, (n-2)\}$ then its conjugacy class does belong to $\mu(r_i)$. $\blacksquare$

Corollary 2.9. If $|\mu(r_i)| = |\mu(r_n)|$ then $i \in \{0, 1, 3, (n-3), (n-1), n\}$. $\blacksquare$

Lemma 2.10. If $\alpha \in Aut(\Gamma_n)$ then there exists $\alpha_1 \in \langle \Gamma_n, H_n \rangle$ such that $r_n^{\alpha\alpha_1} = r_n$.

Proof: If $n = 2$ then let $W_n = \{r_0, r_2, r_0 r_2\}$ else let $W_n = \{r_j \in R_n \mid j \notin \{2, (n-2)\}\}$. By lemmas 2.2 and 2.5 W_n represents the conjugacy classes C of involutions that satisfy $ZC_{\Gamma_n}(h) \simeq ZC_{\Gamma_n}(r_n)$ for all $h \in C$. Thus $r_n^{\alpha g} \in W_n$ for some $g \in \Gamma_n$. If $n = 2$ then we are done, so we assume that $n \neq 2$. By proposition 2.7 $|\mu(r_n)| = |\mu(r_n^{\alpha g})|$ and so by corollary 2.9 $r_n^{\alpha g} = r_i$ for some i in $\{0, 1, 3, (n-3), (n-1), n\}$. Thus $r_n^{\alpha g \theta_n^{\delta}} \in \{r_{(n-3)}, r_{(n-1)}, r_n\}$ for some $\delta \in \{0, 1\}$.

Let P_n denote the set of sets of commuting elements of R_n and, for $g \in \Gamma_n$, let $U(g)$ be the group $C_{\Gamma_n}(g)$ modulo $ZC_{\Gamma_n}(g)$.

Suppose that $r_n^{\alpha g \theta_n^{\delta}} = r_{(n-1)}$. Then $U(r_n) \simeq U(r_{(n-1)})$.

If $n = 1$ then $r_n^{\alpha g \theta_n^{\delta+1}} = r_n$ and we are done.

If $n = 3$ then $\Gamma_1 \simeq 1$, which is false.

If $n > 3$ then $\Gamma_{(n-2)} \simeq \Gamma_{(n-3)}$, thus $|P_{(n-2)}| = |P_{(n-3)}|$, which is false.

Suppose that $r_n^{\alpha g \theta_n^{\delta}} = r_{(n-3)}$. Then $U(r_n) \simeq U(r_{(n-3)})$.

If $n = 3$ then $r_n^{\alpha g \theta_n^{\delta+1}} = r_n$ and we are done.

If $n \in \{4, 5\}$ then $\Gamma_{(n-2)} \simeq \Gamma_1$, thus $|P_{(n-2)}| = |P_1|$, which is false.

If $n > 5$ then $\Gamma_{(n-2)} \simeq \Gamma_{(n-5)} \times \Gamma_1$, thus $|P_{(n-2)}|$ equals $|P_{(n-5)}||P_1|$, that is, $3|P_{(n-5)}|$. But if $S \in P_{(n-5)}$ then $S, S \cup \{r_{(n-3)}\}, S \cup \{r_{(n-2)}\}, \{r_{(n-4)}\} \in P_{(n-2)}$. Whence $|P_{(n-2)}| \gneq 3|P_{(n-5)}|$ and so again a contradiction. ∎

<u>Proposition 2.11</u>. $\mathrm{Aut}(\Gamma_1) = \langle \Gamma_1, H_1 \rangle$

<u>Proof</u>: Let $\alpha \in \mathrm{Aut}(\Gamma_1)$. By lemma 2.10 there exists $\alpha_1 \in \langle \Gamma_1, H_1 \rangle$ such that $r_1^{\alpha\alpha_1} = r_1$. By the torsion theorem for free products there exists $g \in \Gamma_1$ such that $r_0^{\alpha\alpha_1 g}$ is in $\{r_0, r_1\}$. If $r_0^{\alpha\alpha_1 g} = r_1$ then $r_0^{\alpha\alpha_1} \sim r_1 = r_1^{\alpha\alpha_1}$, whence $r_0 \sim r_1$, which can be contradicted by abelianisation. Thus $r_0^{\alpha\alpha_1} = r_0^{g^{-1}}$ and so there exists $m \in \mathbb{Z}$ such that $r_0^{\alpha\alpha_1}$ is r_0 conjugated by $(r_0 r_1)^m$. Furthermore there exists $\delta \in \{0, 1\}$ and $k \in \mathbb{Z}$ such that $r_0^{(\alpha\alpha_1)^{-1}} = r_1^{\delta}(r_0 r_1)^k$. Thus $r_0 = \{r_1^{\delta}(r_0 r_1)^k\}^{\alpha\alpha_1} = r_1^{\delta}\{(r_1 r_0)^m r_0 (r_0 r_1)^m r_1\}^k$. By parity $\delta = 1$, and so $k(2m-1) = 1$. Thus $|2m-1| = 1$ and $m \in \{0, 1\}$. Whence $\alpha\alpha_1 r_1^m = 1$ and we are done. ■

<u>Proposition 2.12</u>. For any two groups G and H, if $\rho \in \mathrm{Aut}(G \times H)$ and $H^{\rho} = H$ then $\rho\pi_G \in \mathrm{Aut}(G)$ where π_G is the natural epimorphism $\pi_G : G \times H \longrightarrow G$.

<u>Proof</u>: $G = (G \times H)^{\pi_G} = (G \times H)^{\rho\pi_G} = G^{\rho\pi_G} \times H^{\rho\pi_G} = G^{\rho\pi_G}$.
Suppose that for some $g \in G$ we have $g^{\rho\pi_G} = 1$. Then $g^{\rho} \in H$ and so $g \in H^{\rho^{-1}} = H$. Whence $g = 1$. ■

<u>Theorem 2.13</u>. $\mathrm{Aut}(\Gamma_n)$ is a split extension of $\mathrm{Inn}(\Gamma_n)$ by H_n, and thus $\mathrm{Out}(\Gamma_n) \simeq H_n$.

<u>Proof</u>: We will show by induction on n that $\mathrm{Aut}(\Gamma_n)$ is generated by $\mathrm{Inn}(\Gamma_n)$ and H_n. This is clear for the case $n = 0$. We saw the case $n = 1$ in proposition 2.11, and the case $n = 2$ was proved by Jones and Thornton [16].
We now assume that $n > 2$ and that $\mathrm{Aut}(\Gamma_m) = \langle \Gamma_m, H_m \rangle$ for

all $m < n$.

Let $\alpha \in \mathrm{Aut}(\Gamma_n)$. By lemma 2.10 there exists $\alpha_1 \in \langle \Gamma_n, H_n \rangle$ such that $r_n^{\alpha\alpha_1} = r_n$. Thus $\alpha\alpha_1$ restricts to an automorphism of $C_{\Gamma_n}(r_n)$. By lemma 2.5 $C_{\Gamma_n}(r_n)$ is $\Gamma_m \times \langle r_n \rangle$, where $m = (n-2)$. So by proposition 2.12 $\alpha\alpha_1\pi \in \mathrm{Aut}(\Gamma_m)$, where π is the natural epimorphism $\pi ; \Gamma_m \times \langle r_n \rangle \longrightarrow \Gamma_m$. Thus by inductive hypothesis there exists $h \in H_m$ and $g \in \Gamma_m$ such that $r_i^{\alpha\alpha_1} = r_i^{hg} z_i$ for some $z_i \in \langle r_n \rangle$, $0 \leqslant i \leqslant m$. We note that

$$\begin{aligned} H_m &= \langle \theta_m \rangle \langle \theta_m \phi_m \rangle \\ &= \{1, \theta_m\}\{1, (\theta_m\phi_m), (\theta_m\phi_m)^2, (\theta_m\phi_m)^{-1}\} \\ &= \{1, \theta_m\}\{1, \theta_m\phi_m, \psi_m\phi_m, \theta_m\psi_m\}, \text{ where } \psi_m = \phi_m^{\theta_m}. \end{aligned}$$

Thus there exists $\beta, \gamma, \delta \in \{0, 1\}$ such that h is $\theta_m^{\beta}\psi_m^{\gamma}\phi_m^{\delta}$. We note that ϕ_m is the restriction of ϕ_n to Γ_m. Thus if $\alpha_2 = g^{-1}\phi_n^{\delta}$ then $r_i^{\alpha\alpha_1\alpha_2} = r_i^{\theta_m^{\beta}\psi_m^{\gamma}} z_i$ for $i : 0 \leqslant i \leqslant m$, and $r_n^{\alpha\alpha_1\alpha_2} = r_n$. Now if $i \notin \{2, m, (n-1), n\}$ then $|ZC_{\Gamma_n}(r_i)| \lneq |ZC_{\Gamma_n}(r_i^{\theta_m^{\beta}\psi_m^{\gamma}} r_n)|$. Whence $z_i = 1$ for all $i \notin \{2, m\}$. If $\beta = 1$ then $\alpha\alpha_1\alpha_2 : r_0 \longmapsto r_m$. But $|ZC_{\Gamma_n}(r_0)| \lneq |ZC_{\Gamma_n}(r_m)|$ and so $\beta = 0$. Thus we have

$$\begin{aligned} \alpha\alpha_1\alpha_2 : r_i &\mapsto r_i \quad \text{for } i : 0 \leqslant i \leqslant (n-3) \text{ and } i \notin \{2, (n-4)\} \\ r_2 &\mapsto r_2 z_2 \quad \text{for } n = 5 \text{ and } n > 6 \\ r_2 &\mapsto r_2^{\psi_m^{\gamma}} z_2 \quad \text{for } n = 6 \\ r_m &\mapsto r_m z_m \\ r_{n-4} &\mapsto r_{n-4}^{\psi_m^{\gamma}} \quad \text{for } n \neq 6 \\ r_n &\mapsto r_n \end{aligned}$$

which covers all cases except $i = (n-1)$.

For $n > 4$ we have $|ZC_{\Gamma_n}(r_2)| \lneq |ZC_{\Gamma_n}(r_2 r_n)|$.
Thus $z_2 = 1$ for $n = 5$ and $n > 6$.
Also $|ZC_{\Gamma_6}(r_2)| \lneq |ZC_{\Gamma_6}(r_2^{\psi_4^\gamma} r_6)|$. Thus $z_2 = 1$ for $n = 6$.
Furthermore $|ZC_{\Gamma_n}(r_{(n-4)})| \lneq |ZC_{\Gamma_n}(r_{(n-4)} r_m)|$, so without loss of generality $\gamma = 0$. Thus if we put $z_m = r_n^{\varepsilon}$ and let $\rho = \alpha\alpha_1\alpha_2\psi_n^{\varepsilon}$ then ρ fixes r_i for all $i \neq (n-1)$, and so restricts to an automorphism of $C_{\Gamma_n}(r_{(n-3)} r_{(n-5)} r_{(n-7)} \cdots)$

If we let G be the subgroup $\langle r_n, r_{(n-1)} \rangle$ and H the subgroup $\langle r_{(n-3)}, r_{(n-5)}, r_{(n-7)}, \ldots \rangle$ then $C_{\Gamma_n}(r_{(n-3)} r_{(n-5)} r_{(n-7)} \cdots) = G \times H$ and $H^{\rho} = H$. So by proposition 2.12 $\rho\pi_G \in \mathrm{Aut}(G)$. Now $r_n^{\rho\pi_G} = r_n$ so we can deduce from the proof of proposition 2.11 that $r_{(n-1)}^{\rho\pi_G}$ is $r_{(n-1)}$ conjugated by r_n^{δ} for some $\delta \in \{0, 1\}$.
Whence $(r_{(n-1)})^{\rho} = (r_{(n-1)})^{r^{\delta}} h = (r_{(n-1)} h)^{r^{\delta}}$ for some $h \in H$.
Thus ρr_n^{δ} multiplies $r_{(n-1)}$ by h and fixes all other r_i.
If $n = 3$ then $h = r_0^{\eta}$ for some $\eta \in \{0, 1\}$ and so $\rho r_n^{\delta} \phi_n^{\eta} = 1$.
Whence $\alpha \in \langle \Gamma_n, H_n \rangle$.
If $n > 3$ then $ZC_{\Gamma_n}(r_{(n-1)} h) \simeq ZC_{\Gamma_n}(r_{(n-1)}) \simeq C_2$.
Thus $h = 1$ and $\rho r_n^{\delta} = 1$. Whence again $\alpha \in \langle \Gamma_n, H_n \rangle$.
It follows that $\mathrm{Aut}(\Gamma_n) = \langle \Gamma_n, H_n \rangle$ for all n. ■

We now consider the group

$$G = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = 1 \rangle .$$

<u>Theorem 2.14</u>. If S_3 is the group of automorphisms of G induced by the permutations of $\{r_0, r_1, r_2\}$ and ϕ is the automorphism that conjugates r_2 by r_0 then $\mathrm{Aut}(G)$ is generated by S_3 and ϕ, and $\mathrm{Out}(G) \simeq PGL(2,\mathbb{Z})$.

Proof: If $\alpha \in \mathrm{Aut}(G)$ then, by the torsion theorem for free products, for all $i \in \{0, 1, 2\}$ there exists j such that $r_i^\alpha \sim r_j$. Thus α restricts to an automorphism of G^+. If $x = r_0 r_1$ and $y = r_1 r_2$ then $\{x, y\}$ is a free basis for G^+, whence G^+ is a free group of rank 2 F_2.

Consider the natural composite map

$$\theta : \mathrm{Aut}(G) \to \mathrm{Aut}(G^+) = \mathrm{Aut}(F_2) \to \mathrm{Aut}(F_2/F_2') \simeq GL(2,\mathbb{Z}) \to PGL(2,\mathbb{Z}),$$

where F_2' denotes the commutator subgroup of F_2. We analogise the proof of lemma [21, I.4.5] which states that the kernel of the natural map from $\mathrm{Aut}(F_2)$ onto $GL(2,\mathbb{Z})$ is $\mathrm{Inn}(F_2)$.

For $\{i, j, k\} = \{0, 1, 2\}$ let σ_i be the automorphism that transposes r_j and r_k, and let ϕ_{ij} be the automorphism that conjugates r_i by r_j. If R_1 is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, R_2 the matrix $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$ and R_3 the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\overline{R}_1, \overline{R}_2, \overline{R}_3$ generate $PGL(2,\mathbb{Z})$ with defining relations

$$\overline{R}_1^2 = \overline{R}_2^2 = \overline{R}_3^2 = (\overline{R}_1\overline{R}_2)^3 = (\overline{R}_1\overline{R}_3)^2 = 1 \quad ,$$

where $\overline{R}_i$ denotes the image of R_i in $PGL(2,\mathbb{Z})$ [7, 7.2]. We have

$$x = r_0 r_1 \overset{\phi_{01}}{\longmapsto} r_1 r_0 \overset{\sigma_1}{\longmapsto} r_1 r_2 \overset{\phi_{01}}{\longmapsto} r_1 r_2 = y$$
$$y = r_1 r_2 \longmapsto r_1 r_2 \longmapsto r_1 r_0 \longmapsto r_0 r_1 = x$$

$$x = r_0 r_1 \overset{\sigma_2}{\longmapsto} r_1 r_0 = x^{-1} \qquad\qquad x = r_0 r_1 \overset{\phi_{01}}{\longmapsto} r_1 r_0 = x^{-1}$$
$$y = r_1 r_2 \longmapsto r_0 r_2 = xy \qquad\qquad y = r_1 r_2 \longmapsto r_1 r_2 = y \quad .$$

Whence if $a_1 = \phi_{01}\sigma_1\phi_{01}$, $a_2 = \sigma_2$ and $a_3 = \phi_{01}$ then $\overline{R}_i = a_i\theta$ for all $i \in \{1, 2, 3\}$, and so θ is onto.

We have $\sigma_i^2 = \phi_{ij}^2 = 1$, whence $a_1^2 = a_2^2 = a_3^2 = 1$.

Furthermore

$$\begin{aligned}(a_1a_2)^3 = (\phi_{01}\sigma_1\phi_{01}\sigma_2)^3 &= (\phi_{01}\phi_{21}\sigma_1\sigma_2)^3 = (r_1\sigma_1\sigma_2)^3 \\ &= r_1\sigma_1\sigma_2r_1\sigma_1\sigma_2r_1\sigma_1\sigma_2 = r_1r_2(\sigma_1\sigma_2)^2r_1\sigma_1\sigma_2 \\ &= r_1r_2r_0(\sigma_1\sigma_2)^3 = r_1r_2r_0 \in \mathrm{Inn}(G)\end{aligned}$$

and $(a_1a_3)^2 = (\phi_{01}\sigma_1)^2 = \phi_{01}\sigma_1\phi_{01}\sigma_1 = \phi_{01}\phi_{21}\sigma_1^2 = r_1 \in \mathrm{Inn}(G)$.

Let $\mathcal{R}$ be the set of defining relations for $PGL(2,\mathbb{Z})$ and F_3 the free group with basis $\{x_1, x_2, x_3\}$ and homomorphisms π, ρ defined by $\pi : x_i \mapsto a_i$ and $\rho : x_i \mapsto \overline{R}_i$. We have

$$\mathrm{Ker}(\theta) \subseteq (\mathrm{Ker}(\rho))\pi = \overline{\mathcal{R}}(x_1,x_2,x_3)\pi \subseteq \overline{\mathcal{R}}(a_1,a_2,a_3) \subseteq \mathrm{Inn}(G) ,$$

where $\overline{\mathcal{R}}$ denotes the normal closure of $\mathcal{R}$. Moreover,

$$x = r_0r_1 \xmapsto{r_0} r_1r_0 = x^{-1}$$
$$y = r_1r_2 \longmapsto r_0r_1r_2r_0 = xy^{-1}x^{-1}$$

$$x = r_0r_1 \xmapsto{r_1} r_1r_0 = x^{-1} \qquad\qquad x = r_0r_1 \xmapsto{r_2} r_2r_0r_1r_2 = y^{-1}x^{-1}y$$
$$y = r_1r_2 \longmapsto r_2r_1 = y^{-1} \qquad\qquad y = r_1r_2 \longmapsto r_2r_1 = y^{-1}$$

Whence $\mathrm{Ker}(\theta) = \mathrm{Inn}(G)$, and so $\mathrm{Out}(G) \simeq PGL(2,\mathbb{Z})$. Moreover,

$$\mathrm{Aut}(G) = PGL(2,\mathbb{Z})\theta^{-1} = \langle \overline{R}_1, \overline{R}_2, \overline{R}_3\rangle\theta^{-1} = \langle a_1, a_2, a_3\rangle \mathrm{Inn}(G)$$

and so $\mathrm{Aut}(G)$ is generated by S_3 and ϕ . ∎

3) We now consider the topological interpretation of some of these outer automorphisms. Clearly θ_2 corresponds to the well-known duality for surface maps, interchanging vertices and faces. Furthermore, the automorphism ϕ_2, which interchanges vertices and Petrie polygons, corresponds to Wilsons <u>opposite</u> operator [31] which is described as follows: make a directed cut along each edge and then rejoin corresponding sides in opposing directions. We illustrate this in Fig. 4.

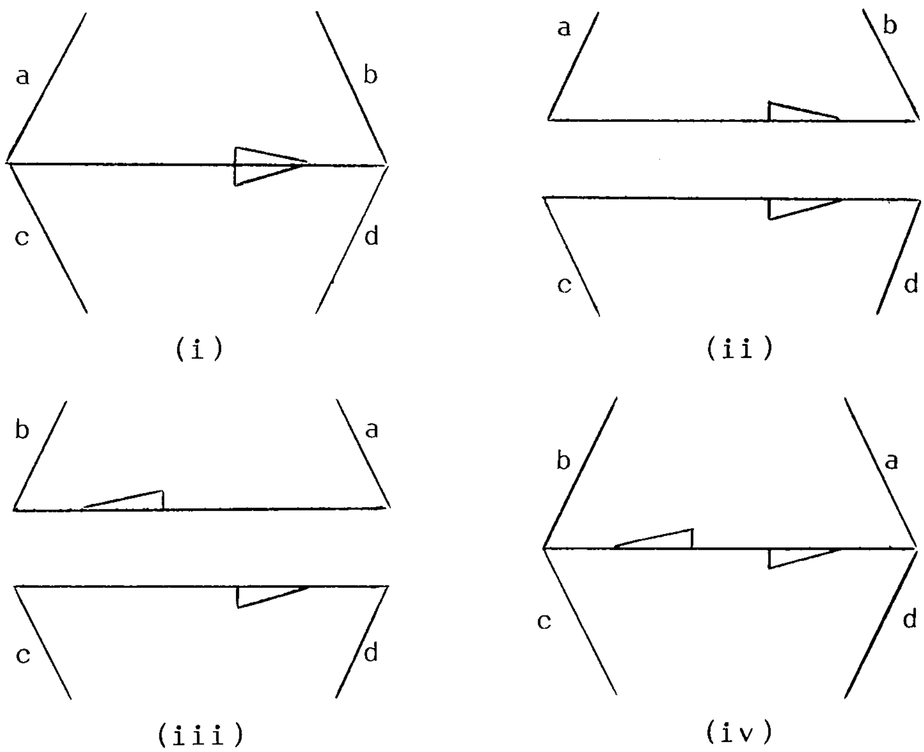


Figure 4

Clearly S_3 acts on hypermaps by permuting the hypervertices, edges and faces. We now verify that ϕ is the result of applying ϕ_2 to the underlying two-coloured map.

To any hyperblade β of a given hypermap we associate a blade β' of the underlying two-coloured map drawn on the incident edge coloured 0 at the incident (contracted) face coloured 0 and on the same side as the incident face coloured 2 (see Fig. 5) .

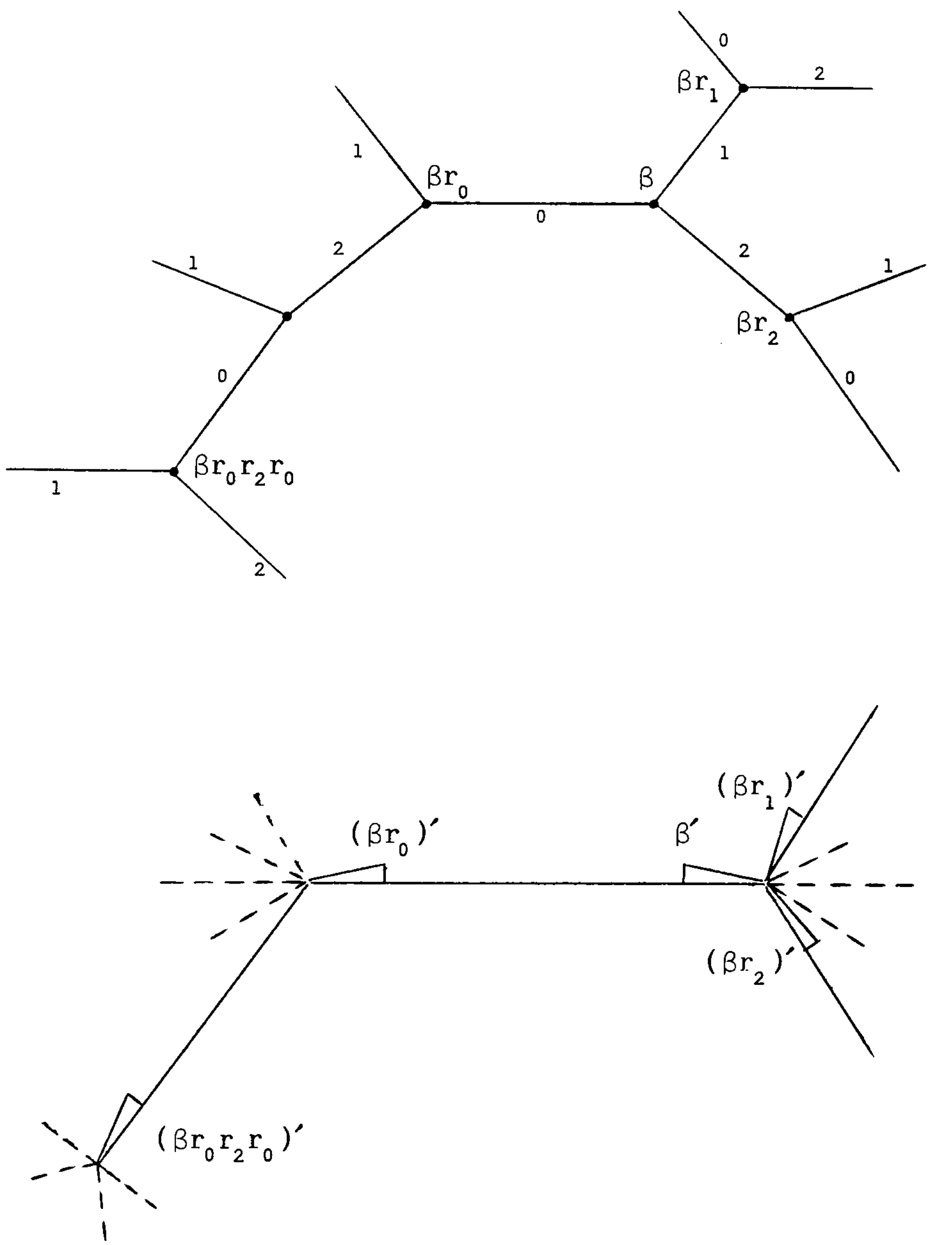


Figure 5

Then under ϕ_2 we have

$$(\beta r_0)' = \quad \beta' r_0 \longmapsto \beta' r_0 = (\beta r_0)'$$
$$(\beta r_1)' = \quad \beta' r_1 \longmapsto \beta' r_1 = (\beta r_1)'$$
$$(\beta r_2)' = \beta' r_2 r_1 r_2 \longmapsto \beta' r_0 r_2 r_1 r_0 r_2 = (\beta r_0 r_2 r_0)'$$

thus the action of ϕ_2 is equivalent to that of ϕ.

It may be of interest to note that the automorphism $\psi_2 = \phi_2^{\theta_2}$, which interchanges faces and Petrie polygons, leaving the underlying graph unchanged, may be topologically described, for a map with neither free edges nor boundary, as follows: remove a disc from each face of the map, then make a directed cut across each edge, rejoin in opposing directions and finally attach a disc to each boundary component. In Fig. 6(i)-6(iv) we show each stage of this operation on the usual imbedding of the complete graph on four vertices on a sphere, Fig. 6(iv) represents a cube which is to be antipodally identified.

(i)

(ii)

(iii)

(iv)

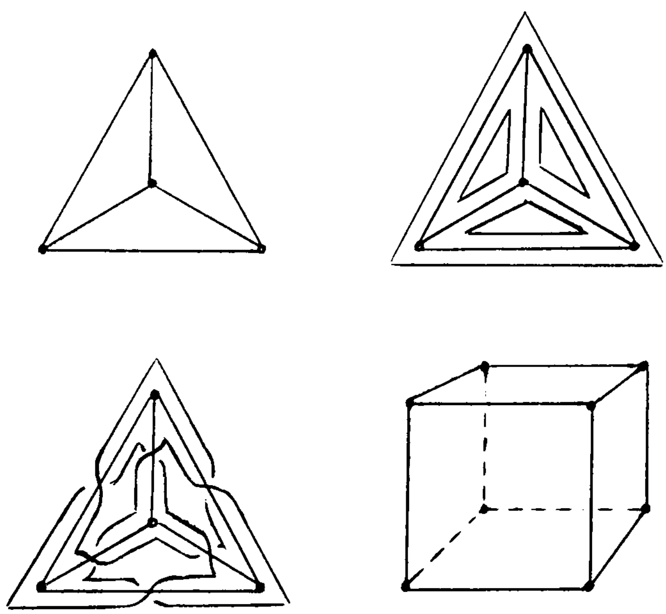
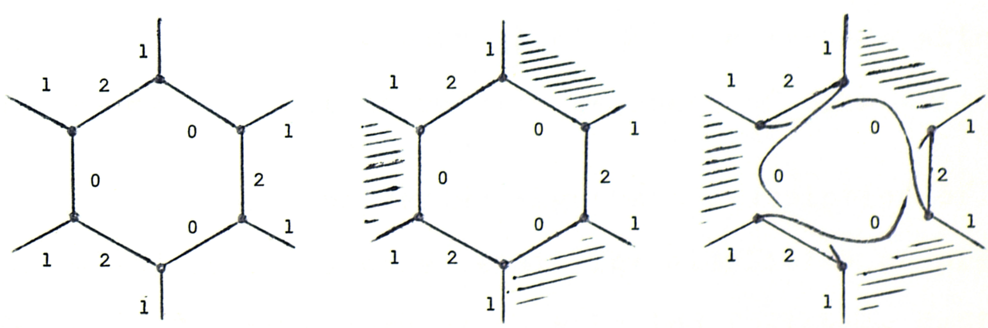


Figure 6

This operation restricts locally to the single edge twists used by Edmonds [10] in his characterisation of graph imbeddings. Every imbedding of a graph without free edges in a surface without boundary may be described in the following way: for each vertex of the graph give a cyclic ordering to the incidences of edges, thus orientably imbedding the graph [8], and then to each edge assign the value 1 or 0 according to whether or not it is to be twisted.

Analogously we consider the operation induced by the automorphism of G $\psi = \phi^{\theta}$ where θ transposes the symbols r_0 and r_2. Thus ψ conjugates r_0 by r_2. Topologically, for a hypermap without boundary, ψ has the following description: remove each hyperface of the hypermap, then make a directed cut along each hyperdart, rejoin in opposing directions and finally attach a disc to each boundary component. In Fig. 7 we show each stage of this operation on a simple example.



(i) (ii) (iii)

Figure 7

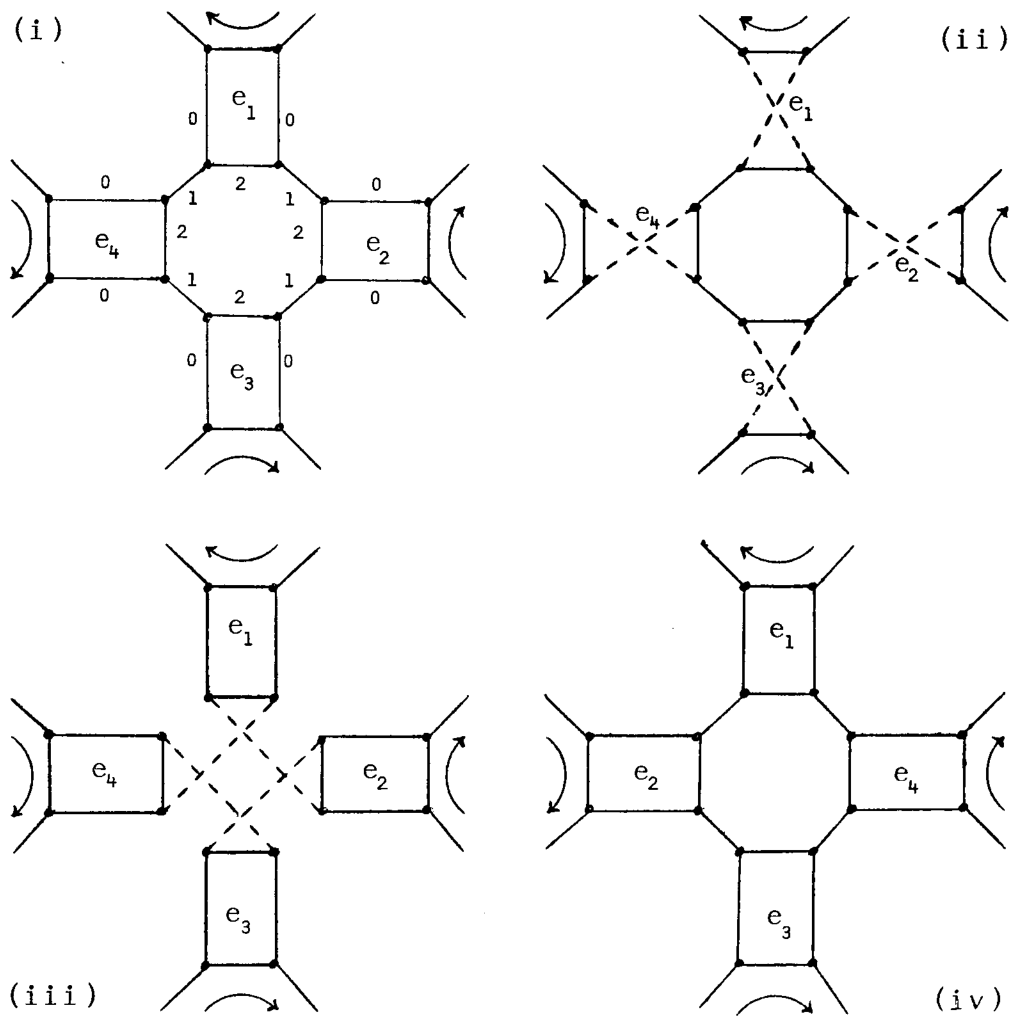
In fact a hyperdart twist corresponds to an edge twist of the underlying bipartite map considered by Walsh [30]. It is readily verified that every imbedding of a hypergraph in a surface without boundary may be described as follows: for each edge of the hypergraph give a cyclic ordering to the incidences of vertices and for each vertex give a cyclic ordering to the incidences of edges, thus orientably imbedding the hypergraph, finally to each vertex-edge incidence assign the value 1 or 0 according to whether or not it is to be twisted. (In essence we have defined the orbits of $\langle r_0, r_2 \rangle$ and $\langle r_1, r_2 \rangle$ in the Schreier coset graph, and then specified how they are to be identified.)

We note that if we fix a vertex of an oriented surface map with neither free edges nor boundary and twist each incident edge then we reverse the cyclic ordering of incident edges at that vertex whilst preserving the orientability of the map (see Fig. 8). It follows that if $\mathcal{G}$ is a graph without free edges whose vertices are at most trivalent then any two imbeddings of $\mathcal{G}$ in a surface without boundary are equivalent under a sequence of edge twists.

4) We now consider an algebraic description of an edge twist, omitting some detail for simplicity of presentation. For this purpose we make the following definitions.

Let θ be the natural homomorphism from the free group F_3 generated by $\{r_0, r_1, r_2\}$, which we denote

Figure 8



by R , onto Γ and let $\mathcal{G}$ be any finite graph without free edges, with spanning tree T , imbedded as a map $\mathcal{M}$ in a surface without boundary. Let M denote the stabiliser in Γ of a blade β of $\mathcal{M}$ and let $\mathcal{G}(M)$ denote the Schreier coset graph for M in Γ with respect to R . We denote edge sets and vertex sets by the letters E and V respectively.

For $v \varepsilon V\mathcal{G}(M)$ let $e(v)$ denote the supporting edge of the associated blade in $\mathcal{M}$, and let $T(M)$ be the subgraph of $\mathcal{G}(M)$ with identical vertex set such that if v_1, v_2 are i-adjacent in $\mathcal{G}(M)$ then they are i-adjacent in $T(M)$ unless one of the following holds:

(i) $i = 2$ and $e(v_1) = e(v_2) \varepsilon T$

(ii) $i = 0$ and $e(v_1) = e(v_2) \not\varepsilon T$

(iii) $i = 1$ and $M \varepsilon \{v_1, v_2\}$

Clearly $T(M)$ is a spanning tree for $\mathcal{G}(M)$. Infact every vertex apart from M and Mr_1 has valency 2 in $T(M)$. We let $U(M)$ be the Schreier transversal for the pre-image of M in F_3 determined by $T(M)$ and let $X(M)$ denote the generating set for M determined in the usual way by $U(M)$, as presented in [21, II.4.1] for example. If $w \varepsilon F_3$ then w has a unique reduced decomposition $\underline{w}$ with respect to $R^{\pm 1}$. Let $\hat{\underline{w}}$ be the corresponding sequence in R (produced by changing negative powers to positive powers). We note that if $u \varepsilon U(M)$ then $\hat{u} = u$ and r_2 is never adjacent to r_0 in $\underline{u}$.

Given $e \varepsilon E\mathcal{G}$ we will define functions f_e from $U(M)$ onto $U(M_e)$ and g_e from $X(M)$ onto $X(M_e)$ such that f_e fixes the identity element, M_e is the stabiliser in Γ of a blade β_e in the imbedding $\mathcal{M}_e$ of $\mathcal{G}$ obtained from $\mathcal{M}$ by twisting e and the following diagram commutes, where darts$(\mathcal{G})$ is the set of directed edges of $\mathcal{G}$; darts$(\mathcal{G}(M))$ is the set of orbits of $\langle r_2 \rangle$ in $V\mathcal{G}(M)$; $\phi(\mathcal{M}, \beta)$ and $\phi(\mathcal{M}_e, \beta_e)$ are induced by the natural

$$\begin{array}{ccc} \mathrm{darts}(\mathcal{G}) & \xrightarrow{\text{id.}} & \mathrm{darts}(\mathcal{G}) \\ \Big\downarrow \phi(\mathcal{M},\beta) & & \Big\downarrow \phi(\mathcal{M}_e,\beta_e) \\ \mathrm{darts}(\mathcal{G}(M)) & \xrightarrow[\psi_e]{} & \mathrm{darts}(\mathcal{G}(M_e)) \end{array}$$

identifications of blades in $\mathcal{M}$ and $\mathcal{M}_e$ with $V\mathcal{G}(M)$ and $V\mathcal{G}(M_e)$ respectively; and, for $u \in U(M)$, ψ_e sends the dart $Mu\theta\langle r_2\rangle$ of $\mathcal{G}(M)$ to the dart $M_e u f_e \theta\langle r_2\rangle$ of $\mathcal{G}(M_e)$. Thus we assume that in specifying an edge of $\mathcal{G}$ we specify an orbit of $\langle r_0, r_2\rangle$ in $V\mathcal{G}(M)$.

For $e \in E\mathcal{G}$ we define a function f_e from $U(M)$ into F_3 as follows. For $u \in U(M)$ let $S_e(u)$ denote the set of initial sequences $\underline{u}_1$ of $\underline{u}$ such that $\underline{u}_1 r_0$ is also an initial sequence of $\underline{u}$ and $Mu_1\theta \in e$. Thus $S_e(u)$ has at most two elements and $e \in T$ if and only if $S_e(u)$ is non-empty for some $u \in U(M)$. There are three cases to consider.

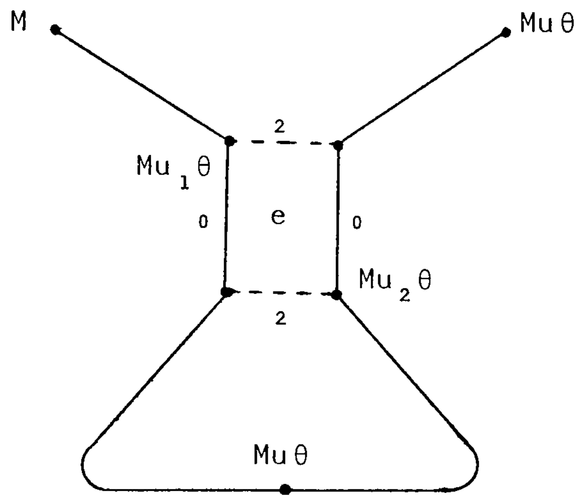
(i) If $S_e(u)$ is empty then define uf_e to be u.

(ii) If $S_e(u)$ consists of a single element $\underline{u}_1$ then there is a unique element $u_2 \in U(M)$ such that $Mu_1\theta r_0 r_2 = Mu_2\theta$. Define uf_e to be $(u_1 r_0 u_2^{-1} u)^\wedge$.

(iii) If $S_e(u)$ has two elements then let $\underline{u}_1$ be the shorter and $\underline{u}_2$ the longer. Define uf_e to be $(u_1 r_0 u_2^{-1} u_1 r_0 u_2^{-1} u)^\wedge$.

Fig. 9 gives an example of the latter two cases.

Figure 9



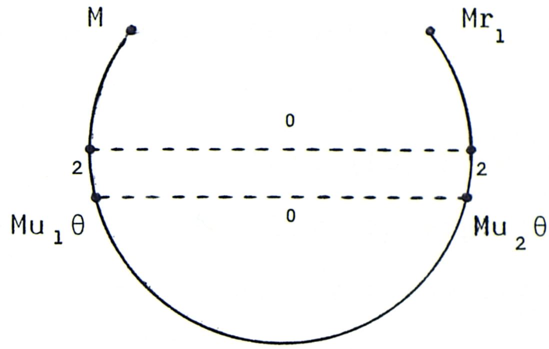
For $e \in E\mathcal{G}$ we define a function g_e from $X(M)$ into Γ as follows. First note that any reduced decomposition of w in Γ with respect to R is largely determined by the normal form of w in Γ considered as a free product of the groups

$$\langle r_1 \mid r_1^2 = 1 \rangle \text{ and } \langle r_0, r_2 \mid r_0^2 = r_2^2 = (r_0 r_2)^2 = 1 \rangle ,$$

and recall that if $u \in U(M)$ then r_2 cannot be adjacent to r_0 in $\underline{u}$. If $x \in X(M)$ then there exists $u_1, u_2 \in U(M)$ and $r_i \in R$ such that $(\underline{u}_1 r_i \underline{u}_2^{-1})^\wedge$ is a reduced decomposition of x with respect to R and $\underline{u}_1$ has maximal length. It follows from the above remarks that u_1, u_2 and r_i are unique. Fig. 10 gives an example. If $i = 0$ and $Mu_1\theta \in e$ then define xg_e to be $(u_1 f_e) r_0 (u f_e)^{-1}\theta$ where u is the unique element in $U(M)$ such that $Mu_1\theta r_0 r_2$ is $Mu\theta$, otherwise define xg_e to be $(u_1 f_e) r_i (u_2 f_e)^{-1}\theta$.

Finally, let M_e denote the subgroup generated

Figure 10



by $X(M)g_e$. Then M_e is the stabiliser in Γ of a blade β_e in the imbedding $\mathcal{M}_e$ of $\mathcal{G}$ obtained from $\mathcal{M}$ by twisting e, f_e is a function from $U(M)$ onto $U(M_e)$ and g_e is a function from $X(M)$ onto $X(M_e)$ such that f_e fixes the identity element, ψ_e is well-defined and the above diagram commutes.

In fact it is not hard to see now that M is a free group of rank $|E\mathcal{G}|+1$. An intuitive proof of this is as follows. Since $\mathcal{M}$ has neither boundary nor free edges M is torsion free by the torsion theorem for free products and so M is a free group by the Kurosh subgroup theorem for free products [21, III.3.6]. Suppose that a subset S of $X(M)$ is a free basis for M such that $S \cap \{x, x^{-1}\}$ is empty for some $x \in X(M)$. Consider the reduced decomposition $(\underline{u}_1 r_i \underline{u}_2^{-1})^\wedge$ for x with respect to R with $u_1, u_2 \in U(M)$, $r_i \in R$ and $\underline{u}_1$ maximal. Let w be any element of $\langle S \rangle$ and let $\underline{w}$ be any reduced decomposition of w with respect to R. By considering $\mathcal{G}(M)$ we see that there is no initial sequence $\underline{w}_1 r_i$ of

$\underline{w}$ such that $Mw_1 = Mu_1\theta$ and so $\underline{u}_1 r_i$ cannot be an initial sequence of $\underline{w}$, whence $x \notin \langle S \rangle$. Thus if $x \in X(M)$ then $S \cap \{x, x^{-1}\}$ is non-empty, in which case clearly S is of order $|E\mathcal{G}| + 1$.

Little and Ringeisen [18] have shown some interest in the use of topological edge twists to prove the double cover conjecture for bridgeless graphs, that is, that in every bridgeless graph one can find a family $\mathcal{C}$ of cycles such that each edge appears in exactly two cycles of $\mathcal{C}$. Clearly this is equivalent to imbedding the graph in a surface without boundary such that each edge borders precisely two faces.

If $e \in E\mathcal{G}$ then e is a bridge of $\mathcal{G}$ if and only if e is in every spanning tree T of $\mathcal{G}$. The two ways in which an edge may be monofacial are illustrated in Fig. 11.

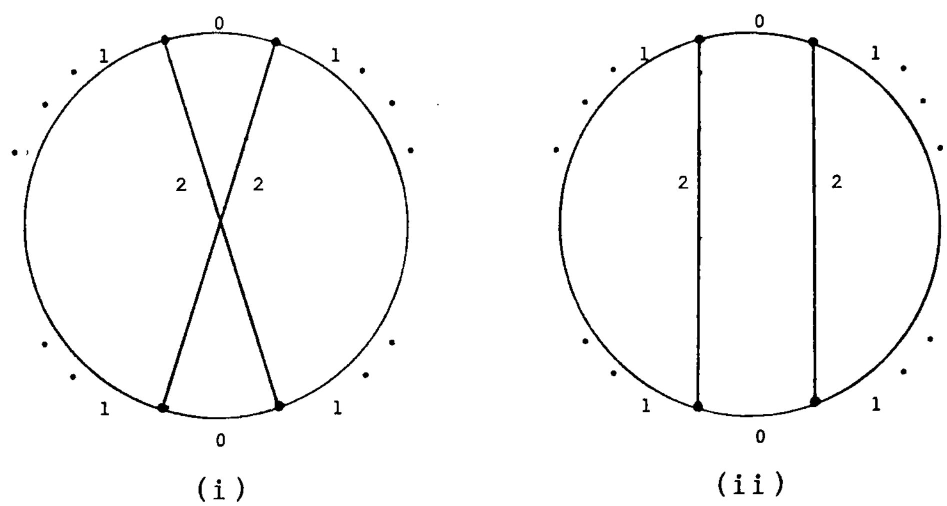


Figure 11

It is clear that in the first case there is an edge twist that strictly decreases the number of monofacial edges without changing their type. For $\mathcal{M}$ orientable Little and Ringeisen showed that in the second case there is a sequence of edge twists that strictly decreases the number of monofacial edges of an imbedding of a bridgeless graph.

Algebraically, $\mathcal{M}$ has m monofacial edges of the second type where m is one quarter of the number of cosets of M containing $(r_1 r_0)^n r_2 r_0$ for some $n \in \mathbb{N}$. Infact it is sufficient to prove the double cover conjecture for cubic graphs $\mathcal{G}$ and, as we have seen, any two imbeddings (without boundary) of $\mathcal{G}$ in this case are equivalent under a sequence of edge twists.

5) In [29] Vince defines an operator on combinatorial maps called the β-dual. Let W be a group generated by involutions $r_0, r_1, \ldots, r_n$ and let $\mathcal{G}$ be the Schreier coset graph for some subgroup of W. Call a word w in W an involution if every path of type w^2 is closed. Let $\beta = \{w_i \mid i \in \{0, 1, \ldots, n\}\}$ be an indexed set of involutions. Define a new $\{0, 1, \ldots, n\}$-labelled graph $\mathcal{G}_\beta$ as follows. The point set of $\mathcal{G}_\beta$ is that of $\mathcal{G}$, and two points are i-adjacent in $\mathcal{G}_\beta$ if and only if they are connected by a path of type w_i in $\mathcal{G}$. If $\mathcal{G}_\beta$ is connected then it is a combinatorial map which is called the β-dual of $\mathcal{G}$. It is clear that the operations induced by the outer automorphisms of Γ_n and G are equivalent to β-duals.

In [31] Wilson constructed an operator H_j. If j is relatively prime to the valency of each vertex of a surface map then H_j fixes the underlying graph and redefines the faces to be j th order holes, that is, cyclic sequences of edges, each two consecutive ones sharing a vertex, so that at each vertex the adjacent edges subtend j faces on one side, either the right or the left but consistently throughout. These operations were interpreted algebraically in [16] as being induced by the outer automorphisms of the group

$$\Gamma[2,m,\infty] = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_0r_2)^2 = (r_1r_2)^m = 1 \rangle$$

that fix r_0 and r_2 and send r_1 to $(r_1r_2)^{(j-1)}r_1$, where j is relatively prime to m.

The set of reflexible surface maps and hypermaps, that is, those that are in some sense most symmetric, is closed under the operations induced by the outer automorphisms of Γ and G. In the next chapter we consider the symmetry of surface maps and hypermaps and introduce a class of highly symmetric surface maps and hypermaps, containing the reflexible ones, that is also closed under all such operations.

CHAPTER 3

Symmetry

1) In this chapter we restrict our attention to surface maps and hypermaps of high symmetry. We start with some definitions.

A map $\mathcal{M}$ is <u>reflexible</u> [7], [16] if $\mathrm{Aut}(\mathcal{M})$ acts transitively on the set of blades. If $\mathcal{M}$ is orientable and without boundary and the subgroup of orientation-preserving automorphisms $\mathrm{Aut}^{+}(\mathcal{M})$ acts transitively on the set of darts then $\mathcal{M}$ is <u>regular</u> [7], [15]. A map $\mathcal{M}$ is <u>all-symmetric</u> if $\mathrm{Aut}(\mathcal{M})$ acts transitively on the vertices, edges and faces. If $\mathcal{M}$ is either reflexible or regular then $\mathcal{M}^{\alpha}$ is all-symmetric for every operation α (induced by an automorphism of Γ). We define such a map to be <u>highly-symmetric</u> (which we abbreviate to H.S.). If H is a subgroup of Γ then $\mathcal{M}$ is <u>H-symmetric</u> if $\mathrm{Aut}(\mathcal{M})$ acts transitively on H-orbits of blades under the action of Γ as a permutation group.

Reflexible, regular, all-symmetric, highly-symmetric and H-symmetric hypermaps are defined similarly.

<u>Proposition 3.1</u>. A map is H-symmetric if and only if $NH = \Gamma$ where N is the normaliser in Γ of the map subgroup.

<u>Proof</u>: If $\mathcal{M}$ is a map with map subgroup M then $\mathrm{Aut}(\mathcal{M})$ acts transitively on H-orbits if and only if for any $g \in \Gamma$ there exists $u \in N_{\Gamma}(M)$ such that $Mu^{-1}H = MgH$, that is, $g \in NH$. ∎

The analogous proposition for hypermaps, replacing Γ by G, is similarly proved.

It follows that a map or hypermap is reflexible if and only if the (hyper)map subgroup is a normal subgroup of Γ or G respectively, and a map or hypermap is regular if and only if the (hyper)map subgroup is a normal subgroup of Γ^+ or G^+ respectively.

We call the Schreier coset graph (with respect to $\{r_0, r_1, r_2\}$) of any subgroup of Γ a <u>Γ-graph</u>. If H is a subgroup of Γ then an <u>H-symmetric Γ-normaliser</u> is any subgroup N such that $NH = \Gamma$, and an <u>H-symmetric Γ-graph</u> is the Schreier coset graph of an H-symmetric Γ-normaliser. So a map is H-symmetric if and only if its map subgroup is normalised by an H-symmetric Γ-normaliser, that is, by a node stabiliser of an H-symmetric Γ-graph.

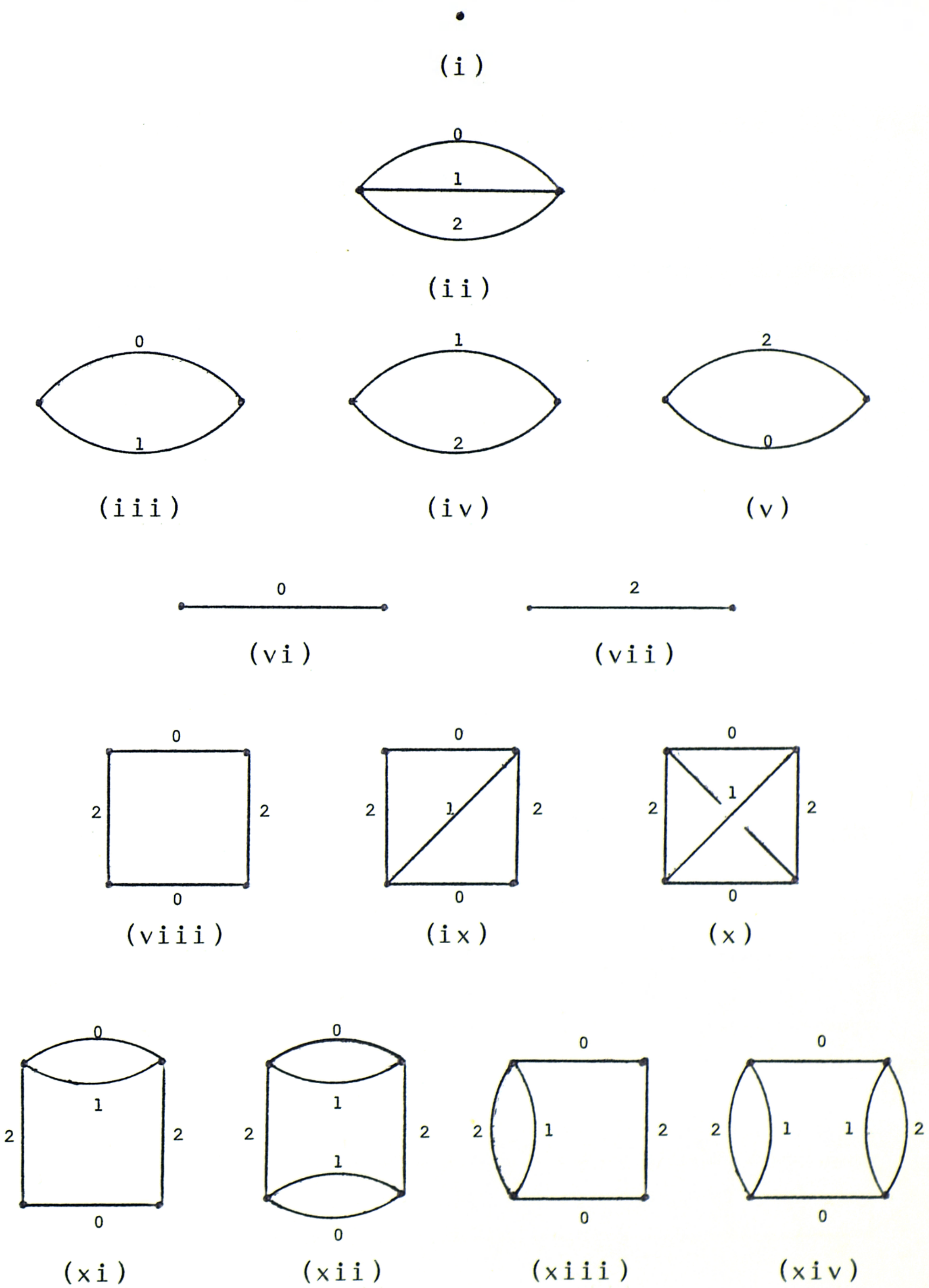
If $\mathcal{D}$ is the Γ-graph of a subgroup N of Γ and $\alpha \in \mathrm{Aut}(\Gamma)$ then we denote the Γ-graph of N^α by $\mathcal{D}^\alpha$ and identify the nodes of $\mathcal{D}$ with those of $\mathcal{D}^\alpha$ in the obvious way.

We make analogous definitions for hypermaps, replacing Γ by G.

2) We now look at H-symmetric Γ-graphs and G-graphs for some choices of H. We note that an edge of a map is an orbit of the dihedral subgroup $H = \langle r_0, r_2 \mid r_0^2 = r_2^2 = (r_0 r_2)^2 = 1 \rangle$ of order 4. When H is defined in this way we read "edge-symmetric" for "H-symmetric". Thus a map $\mathcal{M}$ is edge-symmetric by definition if and only if $\mathrm{Aut}(\mathcal{M})$ acts transitively on edges. By the remarks of the above section, a map is edge-symmetric if and only if its map subgroup is normalised by a node stabiliser of an edge-symmetric Γ-graph. By definition, an edge-symmetric Γ-graph is the Schreier coset graph

of any subgroup N satisfying $NH = \Gamma$. It follows that if $\mathcal{G}$ is a graph whose edges are coloured by $\{0, 1, 2\}$ then $\mathcal{G}$ is an edge-symmetric Γ-graph if and only if there is a closed walk along four edges (not necessarily distinct, and possibly looped) coloured alternately 0 and 2 which spans the vertices of $\mathcal{G}$. We are free to choose the action of r_1 on the cosets. Fig. 12 thus gives all possible edge-symmetric Γ-graphs (with loops omitted).

Figure 12



We immediately observe the following.

<u>Theorem 3.2</u>.[13] If $\mathcal{M}$ is an edge-symmetric map on a surface and Aut($\mathcal{M}$) acts transitively on at least one of the set {vertices, faces, Petrie polygons} then Aut($\mathcal{M}$) acts transitively on at least two of that set. ■

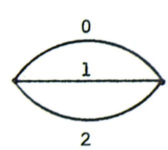
We also observe that the only all-symmetric Γ-graphs are graphs (i), (ii), (iii), (iv), (v), (ix), and (x) and that the only highly-symmetric Γ-graphs are graphs (i), (ii), (iii) and (iv) of Fig. 12 .

<u>Theorem 3.3</u>. There exists an all-symmetric G-graph on any number of nodes.

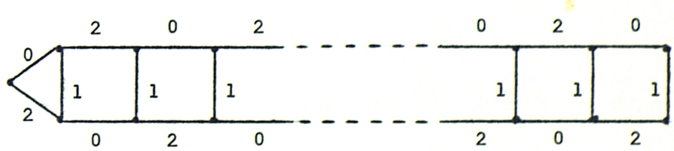
<u>Proof</u>: For any natural number n one of the following is an all-symmetric G-graph on n nodes. ■

n = 1

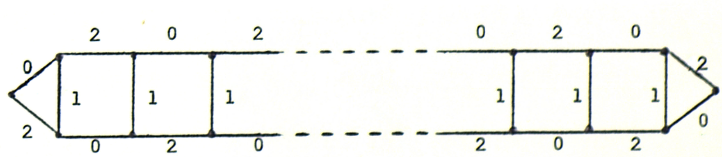
n = 2



n odd > 2



n even > 3



We now consider the existence of highly-symmetric G-graphs.

For $\{i, j, k\} = \{0, 1, 2\}$ if x and y are nodes of a G-graph that are connected by an $\{i, j\}$-path, that is, a path whose edges are coloured either i or j, then let $d_k(x,y)$ denote the length of a shortest $\{i, j\}$-path from x to y.

<u>Proposition 3.4</u>. If $\mathcal{D}$ is an H.S. G-graph on $n > 2$ nodes and $k \in \{0, 1, 2\}$ then $d_k(x,xr_k)$ is even for all nodes x.

<u>Proof</u>: If not then there exists an integer p such that $xr_k = x(r_i r_j)^p r_i (r_j r_i)^p$ where $\{i, j, k\} = \{0, 1, 2\}$.
If $y = x(r_i r_j)^p$ then $y(r_j r_i)^p r_k (r_i r_j)^p = yr_i$.
However, there exists $\alpha \in Aut(G)$ sending $(r_j r_i)^p r_k (r_i r_j)^p$ to r_k and fixing both r_i and r_j.
Whence in $\mathcal{D}^\alpha$ we have $yr_k = yr_i$, and so $\mathcal{D}^\alpha$ is not all-symmetric, which contradicts the choice of $\mathcal{D}$. ∎

<u>Proposition 3.5</u>. If $\mathcal{D}$ is an H.S. G-graph on $n > 2$ nodes and $i \in \{0, 1, 2\}$ then r_i has no fixed points in $\mathcal{D}$.

<u>Proof</u>: Suppose that there exists $i \in \{0, 1, 2\}$ such that $xr_i = x$ for some node x. If $\{i, j, k\} = \{0, 1, 2\}$ then $xr_k \neq x$ else $\mathcal{D}$ is not all-symmetric. By proposition 3.4 there exists an integer $p : 0 < 2p \leqslant (n-1)$ such that $xr_k = x(r_j r_i)^p$ and so $xr_i r_k r_i = x(r_j r_i)^p r_i$. We now choose $\alpha \in Aut(G)$ sending $r_i r_k r_i$ to r_k and fixing both r_i and r_j. Then in $\mathcal{D}^\alpha$ we have $xr_i = x$ and $xr_k = x(r_j r_i)^p r_i$, and so $d_k(x,xr_k) = (2p-1)$. Whence $\mathcal{D}^\alpha$ cannot be highly-

symmetric, which contradicts the choice of $\mathcal{D}$. ∎

Corollary 3.6. If $\{i, j, k\} = \{0, 1, 2\}$ then there is a spanning $\{i, j\}$-cycle in every H.S. G-graph on $n > 2$ nodes, thus n is even. ∎

For $\{i, j, k\} = \{0, 1, 2\}$ if x, y and z are nodes of an H.S. G-graph on $n > 2$ nodes then let $d_k^+(x,y)$ denote a directed distance from x to y along the $\{i, j\}$-cycle. Clearly $d_k^+(x,y) + d_k^+(y,z) \equiv d_k^+(x,z) \pmod n$ and $d_k^+(x,xr_k)$ is even.

Proposition 3.7. If $\mathcal{D}$ is an H.S. G-graph on $n > 2$ nodes and $k \in \{0, 1, 2\}$ then $d_k^+(x,xr_k) \equiv 2 \pmod 4$ for all nodes x.

Proof: Let $\{i, j, k\} = \{0, 1, 2\}$. Then $d_k(x,xr_k)$ is even and so there exists an integer p such that $xr_k = x(r_ir_j)^p$. Furthermore $d_i^+(x,xr_ir_j)$ is odd and so we have

$$d_i^+(x,x(r_ir_j)) + d_i^+(x(r_ir_j),x(r_ir_j)^2) + \ldots\ldots$$
$$\ldots + d_i^+(x(r_ir_j)^{(p-1)},x(r_ir_j)^p) \equiv d_i^+(x,xr_k)$$

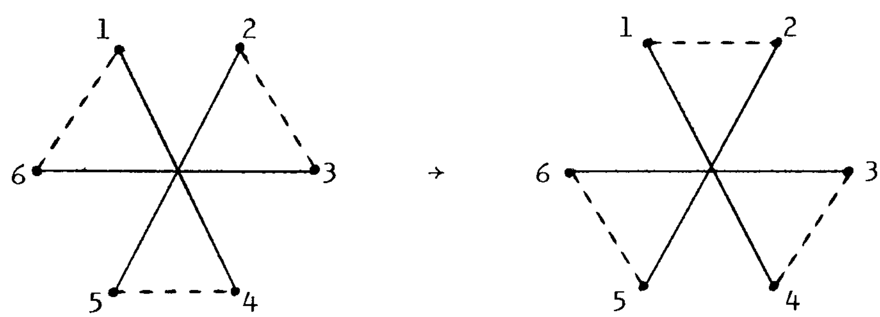
modulo n. Now the left hand side is the sum of p odd terms but $d_i^+(x,xr_k)$ is odd and n is even. Thus the left hand side must be odd. Whence p is odd. ∎

Corollary 3.8. If $\mathcal{D}$ is an H.S. G-graph on $n > 2$ nodes then $n \equiv 0 \pmod 4$.

Proof: $n = d_k^+(x,xr_k) + d_k^+(xr_k,xr_k^2)$. ∎

We need some more definitions. An alternating m-gon is a cycle on m nodes $\{1, 2, \ldots, m\}$ whose edges connect vertices of different parity. Let t_m be the permutation $(1\,2\ldots m)$ and for $\{x, y\} \subseteq \{1, 2, \ldots, m\}$ let $\{x, y\}t_m$ be be the pair $\{xt_m, yt_m\}$. A good alternating m-gon is one whose edges can be coloured alternately red and blue such that the union of the blue edges with the images of the red edges under t_m is again an alternating m-gon. An example of a good alternating hexagon is given in Fig. 13.

Figure 13



Proposition 3.9. If there exists an H.S. G-graph $\mathcal{D}$ on $n > 2$ nodes then there exists a good alternating $\frac{1}{2}n$-gon.

Proof: Let $\{i, j, k\} = \{0, 1, 2\}$ and let x be a distinguished node of $\mathcal{D}$. Colour a k-edge with vertex y blue if $d_k(x,y)$ is even, else colour it red. If we contract each i-edge $\{x(r_ir_j)^p, x(r_ir_j)^p r_i\}$ for $p \in \{1, 2, \ldots, \frac{1}{2}n\}$ to a point v_p, with index to be read modulo $\frac{1}{2}n$, then $\{v_1, v_2, \ldots, v_{\frac{1}{2}n}\}$, together with the k-edges, forms a $\frac{1}{2}n$-gon with edges coloured alternately blue and red. Similarly, if we contract each j-edge $\{x(r_ir_j)^p r_j, x(r_ir_j)^p\}$

for $p \in \{1, 2, \ldots, \frac{1}{2}n\}$ to a point w_p, with index to be read modulo $\frac{1}{2}n$, then $\{w_1, w_2, \ldots, w_{\frac{1}{2}n}\}$, together with the k-edges, forms a $\frac{1}{2}n$-gon with edges coloured alternately blue and red. Furthermore, for $p \in \{1, 2, \ldots, \frac{1}{2}n\}$ there exists $q = q(p)$ and $r = r(p)$ where both q and r are odd such that $x(r_i r_j)^p r_k = x(r_i r_j)^{(p+q)}$ and $x(r_i r_j)^p r_i r_k = x(r_i r_j)^{(p+r)} r_i$. Correspondingly, $\{v_p, v_{(p+q)}\}$ and $\{w_p, w_{(p+q)}\}$ are coloured blue whilst $\{v_p, v_{(p+r)}\}$ and $\{w_{(p+1)}, w_{(p+r+1)}\}$ are coloured red. Whence by identifying v_p with w_p it is clear that $\{v_1, v_2, \ldots, v_{\frac{1}{2}n}\}$, together with the k-edges, forms a good alternating $\frac{1}{2}n$-gon. ■

Proposition 3.10. There are no good alternating m-gons such that $m \equiv 0 \pmod 4$.

Proof: Suppose that we have a good alternating m-gon such that $m \equiv 0 \pmod 4$. Let r be the permutation of nodes that transposes the vertices of red edges and b that which transposes the vertices of blue edges. Then we may write $r = (a_1 a_2)(a_3 a_4)\ldots\ldots(a_{(m-1)} a_m)$ and $b = (a_2 a_3)(a_4 a_5)\ldots\ldots(a_m a_1)$. Let s be the permutation $(a_1 a_2 \ldots\ldots a_m)$ and t the permutation $(1\,2 \ldots\ldots m)$. For all $x \in \{1, 2, \ldots, m\}$ there exists an integer p such that $a_x t = a_x (rb)^p r$ since all three of t, r and b change parity of node. But r and b change parity of index, and so t changes parity of index. Furthermore, the images of the red edges under t_m are clearly the orbits of r^t.

Thus t is an m-cycle that changes parity of indices and is such that $b\,r^t$ is the product of two $\frac{1}{2}m$-cycles, fixing parity of indices. We note that $b = r^s$ and that $a_x r = a_x s^{-1}$ for x even, otherwise $a_x r = a_x s$.
Whence for x even we have

$$a_x r^s r^t = a_x s^{-1} r s t^{-1} r t = a_x s^{-1} s s t^{-1} s^{-1} t = a_x s t^{-1} s^{-1} t \quad .$$

Thus $st^{-1}s^{-1}t\,|_{\text{even indices}}$ is a $\frac{1}{2}m$-cycle.
But this is the product of st, t^{-2}, s^{-2} and st, each of which fixes parity of index.
Whence $st^{-1}s^{-1}t\,|_{\text{even indices}} \in A_{\frac{1}{2}m}$, the alternating group permuting $\frac{1}{2}m$ symbols, and so $\frac{1}{2}m$ must be odd, which contradicts the choice of m. ∎

Corollary 3.11. If $\mathcal{D}$ is an H.S. G-graph on $n > 2$ nodes then $n \equiv 4 \pmod 8$. ∎

For $n \equiv 0 \pmod 4$ consider a set of n points labelled a_r, b_r, c_r, d_r where r ranges from 1 to $\frac{1}{4}n$, with indices to be read modulo $\frac{1}{4}n$. For integers p and q we define a G-graph $\mathcal{D}_{n,p,q}$ on n nodes by

$$a_r r_0 = b_r \qquad b_r r_1 = c_r \qquad a_r r_2 = c_{(r+p)}$$

$$c_r r_0 = d_r \qquad d_{(r-1)} r_1 = a_r \qquad b_r r_2 = d_{(r+q)}$$

Clearly the nodes of $\mathcal{D}_{n,p,q}$ are covered by a $\{0,1\}$-path. The set of nodes of the longest $\{0,2\}$-path through a_r is

$$\{a_{(r+u(p-q))}, b_{(r+u(p-q))}, d_{(r+q+u(p-q))}, c_{(r+q+u(p-q))} \mid u \in \mathbb{Z}\}.$$

Whence the nodes of $\mathcal{D}_{n,p,q}$ are covered by a {0, 2}-path if $(\frac{1}{4}n,(p-q))=1$.

The set of nodes of the longest {1, 2}-path through a_r is

$$\{a_{(r+u(p+q+1))}, d_{(r-1+u(p+q+1))}, b_{(r-1-q+u(p+q+1))}, c_{(r-1-q+u(p+q+1))} \mid u \in \mathbb{Z}\}.$$

Whence the nodes of $\mathcal{D}_{n,p,q}$ are covered by a {1, 2}-path if $(\frac{1}{4}n,(p+q+1))=1$.

<u>Proposition 3.12</u>. If $\{i, j, k\}=\{0, 1, 2\}$ then $\mathcal{D}_{n,p,q}$ is invariant under the automorphism ϕ_{ij}.

<u>Proof</u>: Consider the relabelling

$$a_r' = c_{(1-r)}, \quad b_r' = d_{(-r)}, \quad c_r' = a_{(1-r)}, \quad d_r' = b_{(-r)}.$$

We have

$$a_r'(r_1 r_0 r_1) = c_{(1-r)}(r_1 r_0 r_1) = d_{(-r)} = b_r'$$
$$c_r'(r_1 r_0 r_1) = a_{(1-r)}(r_1 r_0 r_1) = b_{(-r)} = d_r'$$
$$b_r' r_1 = d_{(-r)} r_1 = a_{(1-r)} = c_r'$$
$$d_{(r-1)}' r_1 = b_{(1-r)} r_1 = c_{(1-r)} = a_r'$$
$$a_r' r_2 = c_{(1-r)} r_2 = a_{(1-r-p)} = c_{(r+p)}'$$
$$b_r' r_2 = d_{(-r)} r_2 = b_{(-r-q)} = d_{(r+q)}'$$

Whence $\mathcal{D}_{n,p,q}$ is invariant under ϕ_{01}.

Consider the relabelling

$$a_r' = d_{(q-p-r)}, \quad b_r' = c_{(-r)}, \quad c_r' = b_{(-r)}, \quad d_r' = a_{(q-p-r)}.$$

We have

$$a_r'(r_2 r_0 r_2) = d_{(q-p-r)}(r_2 r_0 r_2) = c_{(-r)} = b_r'$$
$$c_r'(r_2 r_0 r_2) = b_{(-r)}(r_2 r_0 r_2) = a_{(q-p-r)} = d_r'$$
$$b_r' r_1 = c_{(-r)} r_1 = b_{(-r)} = c_r'$$
$$d_{(r-1)}' r_1 = a_{(q-p-r+1)} r_1 = d_{(q-p-r)} = a_r'$$
$$a_r' r_2 = d_{(q-p-r)} r_2 = b_{(-p-r)} = c_{(r+p)}'$$
$$b_r' r_2 = c_{(-r)} r_2 = a_{(-p-r)} = d_{(r+q)}' \quad .$$

Whence $\mathcal{D}_{n,p,q}$ is invariant under ϕ_{02} .
Consider the relabelling

$$a_r' = d_{(-r)} , \qquad b_r' = c_{(-r)} , \qquad c_r' = b_{(-r)} , \qquad d_r' = a_{(-r)} \quad .$$

We have

$$a_r' r_0 = d_{(-r)} r_0 = c_{(-r)} = b_r'$$
$$c_r' r_0 = b_{(-r)} r_0 = a_{(-r)} = d_r'$$
$$b_r' r_1 = c_{(-r)} r_1 = b_{(-r)} = c_r'$$
$$d_{(r-1)}' r_1 = a_{(1-r)} r_1 = d_{(1-r)} = a_r'$$
$$a_r'(r_0 r_2 r_0) = d_{(-r)}(r_0 r_2 r_0) = b_{(-r-p)} = c_{(r+p)}'$$
$$b_r'(r_0 r_2 r_0) = c_{(-r)}(r_0 r_2 r_0) = a_{(-r-q)} = d_{(r+q)}' \quad .$$

Whence $\mathcal{D}_{n,p,q}$ is invariant under ϕ_{20} .
To complete the proof we note that $\phi_{ij}\phi_{kj} = r_j$ and that every G-graph is invariant under Inn(G) . ■

<u>Corollary 3.13</u>. If $(\frac{1}{4}n,(p-q)) = (\frac{1}{4}n,(p+q+1)) = 1$ then $\mathcal{D}_{n,p,q}$ is highly-symmetric.

<u>Proof</u>: By theorem 2.14 the automorphisms ϕ_{ij} generate a normal subgroup of Aut(G) with complement S_3 , thus $\mathcal{D}_{n,p,q}^{\mathrm{Aut}(G)} \subseteq \mathcal{D}_{n,p,q}^{S_3}$. Furthermore, since $\mathcal{D}_{n,p,q}$ is all-sym-

metric, each of $D^{S_3}_{n,p,q}$ is all-symmetric. ∎

Theorem 3.14. There exists an H.S. G-graph on n nodes if and only if $n \in \{1, 2\}$ or $n \equiv 4 \pmod 8$.

Proof: By corollary 3.11 we need only show existence. The coset graphs of G and G^+ in G give H.S. G-graphs on 1 and 2 nodes, and, by corollary 3.13, $D_{n,1,0}$ is highly-symmetric for $n \equiv 4 \pmod 8$. ∎

Theorem 3.15. The number of equivalence classes of H.S. G-normalisers of index $n \equiv 4 \pmod 8$ under Aut(G) tends to infinity with n .

Proof: We note that $D_{n,p,q}$ and $D_{n',p',q'}$ denote the same labelled graph if and only if $n = n'$ and $(p-p') \equiv (q-q') \equiv 0$ modulo $\frac{1}{4}n$. Whence we reduce the suffices p, q modulo $\frac{1}{4}n$. If $D_{n,p,q}$ and $D_{n',p',q'}$ denote the same unlabelled graph then there is a symmetry ρ of the $\{0, 1\}$-cycle such that $D^{\rho}_{n,p,q}$ has the same labelling as $D_{n',p',q'}$. If ρ is a rotational symmetry then $D^{\rho^2}_{n,p,q}$ has the same labelling as $D_{n,p,q}$. Whence the unlabelled graph denoted by $D_{n,p,q}$ has at most 4 notations. Furthermore, we have that $D^{Aut(G)}_{n,p,q} \subseteq D^{S_3}_{n,p,q}$, and so each unlabelled graph lies in an orbit of size at most 6. Whence there are at least $|T_n| / 24$ equivalence classes under Aut(G) of H.S. G-graphs on $n \equiv 4 \pmod 8$ nodes where T_n is the set of ordered pairs (p,q) such that $1 \leqslant p,q < \frac{1}{4}n$ and $(\frac{1}{4}n, (p-q)) = (\frac{1}{4}n, (p+q+1)) = 1$. Thus it only remains to show that $|T_n|$ tends to infinity with n .

For any real number x let $[x]$ denote the greatest integer less than or equal to x. Let $\ell(n) = [\log_2(\frac{1}{4}n)]$ and let Q_n be the set of pairs $\{2^m, (\frac{1}{4}n - 2^m)\}$ where m is any integer between 1 and $\ell(n)$. Then $|Q_n| = \ell(n)$. Furthermore, if $\{p, q\} \varepsilon Q_n$ then $1 \leqslant p,q \leqslant \frac{1}{4}n$ and $(\frac{1}{4}n,(p-q)) = (\frac{1}{4}n,(p+q+1)) = 1$. Whence $|T_n| \geqslant 2|Q_n| = 2\ell(n) > 2(\log_2(\frac{1}{4}n) - 1)$, and so $|T_n|$ tends to infinity with n as required. ■

Proposition 3.16. Every H.S. G-normaliser N of index $n > 2$ is a free group of rank $(\frac{1}{2}n + 1)$.

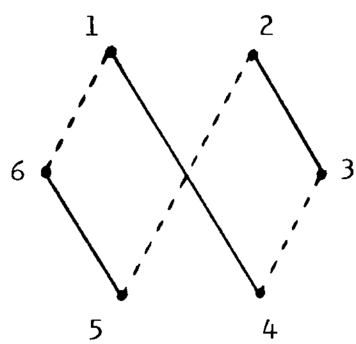
Proof: Let $U = \{1, r_0, r_0r_1, \ldots, (r_0r_1)^{(\frac{1}{2}n-1)}r_0\}$
and $B = \{ur_2\overline{ur_2}^{-1} \mid u \varepsilon U\} \cup \{(r_0r_1)^{\frac{1}{2}n}\}$

where, for $g \varepsilon G$, $\overline{g}$ is the unique element of U satisfying $Ng = N\overline{g}$. Then B clearly generates N (see for example [21, II.4.1]). Moreover, by considering the coset graph of N in G and applying the normal form theorem for free products [21, IV.1.2], it is clear that B' is a free basis for N of size $(\frac{1}{2}n + 1)$ where B' is any subset of B satisfying $|\{b, b^{-1}\} \cap B'| = 1$ for all $b \varepsilon B$. ■

Corollary 3.17. Every finitely generated group can be regarded as a group of automorphisms of some highly-symmetric hypermap. ■

We finish by giving one more property of an H.S. G-graph. We call an alternating m-gon <u>very good</u> if its edges can be coloured alternately red and blue such that the union of the blue edges with the images of the red edges under t_m^{ℓ} is again an alternating m-gon for all natural numbers ℓ. The example given in Fig. 13 is very good, whilst Fig. 14 shows a good alternating hexagon that is not very good.

Figure 14



By the method of proof of proposition 3.10 it can be shown that very good alternating m-gons correspond to cyclic permutations t of $\{a_1, \ldots, a_m\}$ of length m that change parity of index and are such that the restriction of $st^{-\ell}s^{(-1)^{\ell}}t^{\ell}$ to symbols of even index is a $\frac{1}{2}m$-cycle for all natural numbers ℓ, where s is the permutation $(a_1 \ldots a_m)$.

If D is an H.S. G-graph on $n > 2$ nodes then let C be the alternating $\frac{1}{2}n$-gon obtained by contracting the i-edges as described in the proof of proposition 3.9.

<u>Proposition 3.18</u>. C is very good.

<u>Proof</u>: If x is the distinguished node used to construct C then

$$\{v_p, v_q\} \text{ is a blue edge of } C^{\sigma_k\phi_{ji}}$$
$$\Leftrightarrow \{x(r_ir_j)^p, x(r_ir_j)^q\} \text{ is a k-edge of } D^{\sigma_k\phi_{ji}}$$
$$\Leftrightarrow \{x(r_ir_j)^p, x(r_ir_j)^q\} \text{ is a k-edge of } D$$
$$\Leftrightarrow \{v_p, v_q\} \text{ is a blue edge of } C\text{, and}$$

$$\{v_p, v_q\} \text{ is a red edge of } C^{\sigma_k\phi_{ji}}$$
$$\Leftrightarrow \{x(r_ir_j)^p r_i, x(r_ir_j)^q r_i\} \text{ is a k-edge of } D^{\sigma_k\phi_{ji}}$$
$$\Leftrightarrow \{x(r_jr_i)^p r_i, x(r_jr_i)^q r_i\} \text{ is a k-edge of } D^{\sigma_k}$$
$$\Leftrightarrow \{x(r_ir_j)^p r_j, x(r_ir_j)^q r_j\} \text{ is a k-edge of } D$$
$$\Leftrightarrow \{v_{(p-1)}, v_{(q-1)}\} \text{ is a red edge of } C.$$

Whence $C^{(\sigma_k\phi_{ji})^\ell}$ is the union of the blue edges of C with the images of the red edges of C under t_m^ℓ. Furthermore, since the image of D under $(\sigma_k\phi_{ji})^\ell$ is highly-symmetric, the image of C under $(\sigma_k\phi_{ji})^\ell$ is an alternating $\frac{1}{2}n$-gon. ■

The final chapter demonstrates the possibility of classifying imbeddings with prescribed symmetry. We shall classify the edge-symmetric imbeddings of complete graphs in surfaces without boundary.

CHAPTER 4

Complete Maps

1) If $\mathcal{M}$ is an imbedding of a simple graph with n vertices in a possibly non-orientable surface, without boundary, then its automorphism group acts semi-regularly on the set of blades, so has order at most four times the number of edges, being at most $2n(n-1)$. This bound is attained if and only if $\mathcal{M}$ is a reflexible imbedding of a complete graph, thus we can regard such imbeddings as the most symmetric surface maps. Similarly, the regular imbeddings of complete graphs can be regarded as the most symmetric orientable surface maps.

In [1] Biggs showed that the complete graph on n vertices K_n has a regular imbedding if and only if n is a prime power. The examples he gave were Cayley maps based on the additive groups of finite fields, using the multiplicative action of a primitive element to generate the rotation.

Any reflexible or regular imbedding is also both vertex- and edge-symmetric. In [2] Biggs showed that any orientable vertex-symmetric imbedding of a complete graph in a surface without boundary can be described as a Cayley map. The aim of this chapter is to classify the edge-symmetric imbeddings of complete graphs in surfaces without boundary.

2) For motivation we start with an example of an orientable edge-symmetric non-regular imbedding of K_7. In fact we shall see later that this is unique. Consider the imbedding given in Fig. 16 of K_7 in an orientable surface of genus 3. There are three heptagonal faces, labedded 1, 2 and 3, and seven triangular faces. We have a rotation of order seven about the centre of a heptagonal face that cyclically permutes the vertices, and the stabiliser of each vertex is generated by a rotation of order 3. The imbedding is clearly edge-symmetric since the automorphism group is transitive on the edges incident with the centrally depicted vertex. In fact, if we regard the triangular faces as hyperedges and the heptagonal faces as hyperfaces then we have one of the two regular imbeddings of the Fano plane (see Fig. 15) regarded as a hypergraph [25]. The other is similarly related to the unique regular imbedding of K_7 on a torus [30].

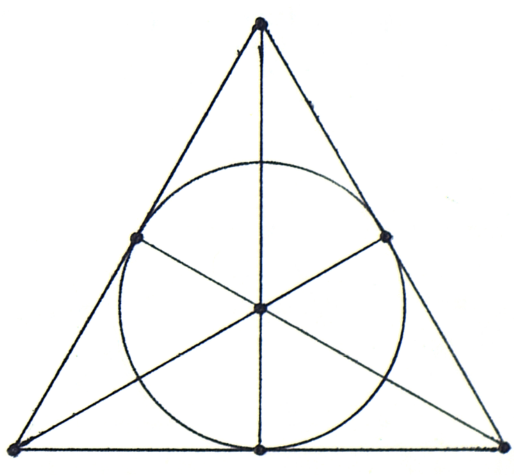
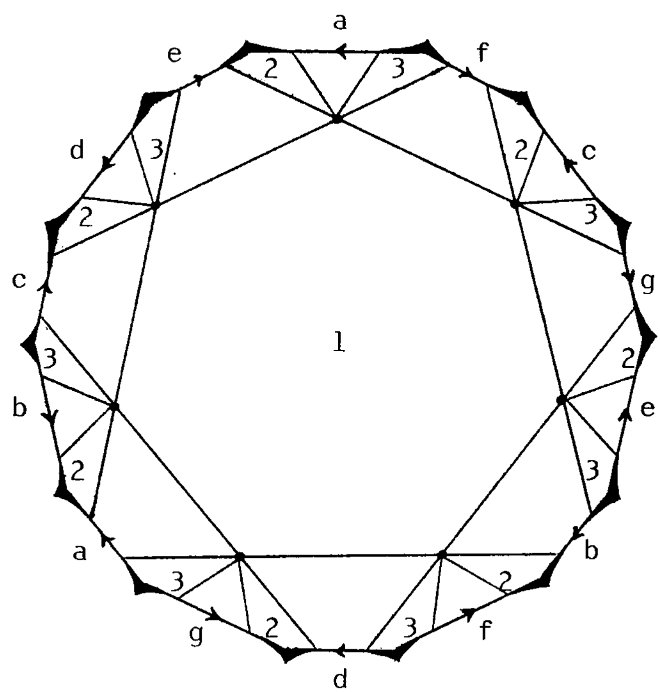


Figure 15

Figure 16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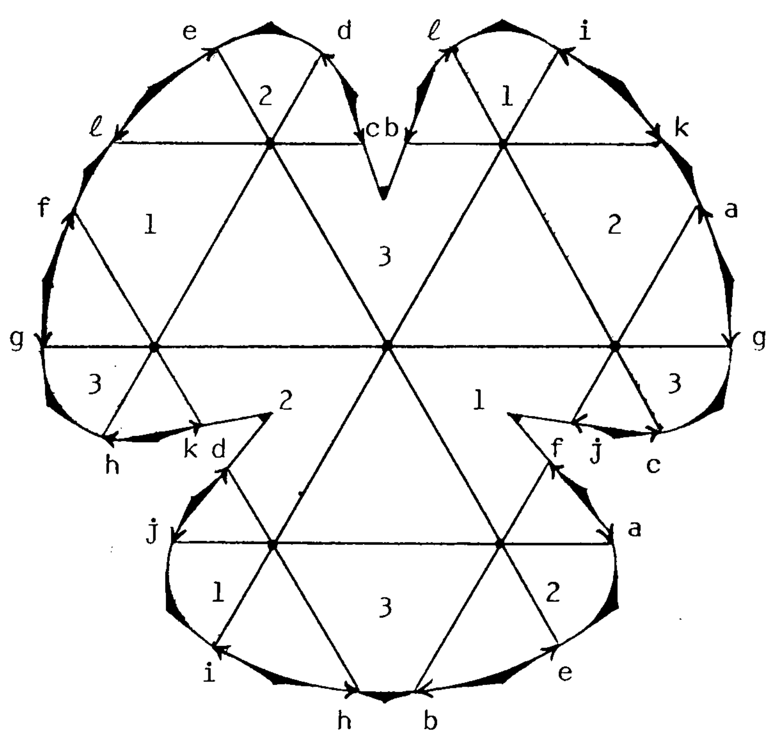
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3) By considering the possible Euler characteristics of the underlying surfaces it is easy to see that the only imbeddings of K_n for $n \leqslant 3$ in surfaces without boundary are the usual imbeddings in a sphere and the imbedding of K_3 in a projective plane obtained by antipodally identifying a spherical imbedding of a graph with six 2-valent vertices. Furthermore, each of these are reflexible imbeddings. We thus assume that $n > 3$.

Let $\mathcal{M}$ be an edge-symmetric imbedding of K_n with map subgroup M normalised in Γ by N. The coset graph of N in Γ must be one shown in Fig. 12, and so $[\Gamma : N]$ is either 1, 2 or 4. We may faithfully represent the map automorphism group $\mathcal{A}$ as a 2-homogeneous permutation group of order

$$[N : M] = [\Gamma : M] / [\Gamma : N] = 2n(n-1)/[\Gamma : N]$$

acting on the vertex set V. We note that if $\mathcal{A}$ has even order then it acts 2-transitively.

If $[\Gamma : N] = 1$ then N has graph (i) of Fig. 12 and $\mathcal{A}$ is a 2-transitive group of degree n and order $2n(n-1)$. In [12] Ito shows that such a group either has a regular normal subgroup or n is 6 and the group is isomorphic to A_5 or n is 28 and the group is isomorphic to a split extension of SL(2,8) by C_3. We will make use of Ito's methods later.

If $[\Gamma : N] = 2$ then $\mathcal{A}$ is a sharply 2-transitive group. By Zassenhaus's Theorem [32] (see also [11, 20.7.1]) we can identify the action of any such group with that of AGL(1,F) on some near field F, with $|F| = |V| = n$. In particular $\mathcal{A}$ is Frobenius and so a vertex stabiliser cannot be dihedral [23, 18.1]. If v is the vertex of the blade stabilised by M then

$$\mathcal{A}_v \simeq M\{\langle r_1, r_2\rangle \cap N\}/M$$

Thus the graph of N is either (ii) or (iv) of Fig. 12. In which case $\mathcal{A}_v$ is abelian and can be identified with F^*, that is, $F \setminus \{0\}$, and hence F is a field GF(n) where n is a prime power p^e.

If $[\Gamma : N] = 4$ then $\mathcal{A}$ is a sharply 2-homogeneous group. By a classification of Kantor [17] we can identify the action of any such group with the transitive action of a subgroup of index 2 in AGL(1,F) on a near field F where $|F| = |V| = n = p^e \equiv 3$ modulo 4. This is done in such a way that if G denotes the group AGL(1,F) then $\mathcal{A}_0$ is centralised by a element of $G_0 \setminus \mathcal{A}_0$. Since $\mathcal{A}$ acts transitively on vertices we have $\Gamma = N\langle r_1, r_2\rangle$ and so the graph of N is none of (viii), (xiii), (xiv) of Fig. 12. Furthermore, $\mathcal{A}_0$ has odd order and so cannot be dihedral. Whence, since

$$\mathcal{A}_0 \simeq M\{\langle r_1, r_2\rangle \cap N\} / M$$

the graph of N is neither (ix) nor (xi) of Fig. 12. Thus the graph of N is either (x) or (xii) of Fig. 12. In which case $\mathcal{A}_0$ is abelian and can be identified with a subgroup of index 2 in F^*. Moreover, $\mathcal{A}_0$ is centralised by an element of $G_0 \setminus \mathcal{A}_0$. By identifying G_0 with F^* we see that F^* is abelian. Hence F is a field GF(n).

Thus we have shown that the coset graph of N in Γ is one of (i), (ii), (iv), (x) and (xii) of Fig. 12. In cases (ii) and (iv) $\mathcal{A}$ is isomorphic to the group AGL(1,F) where F = GF(n) and $n = p^e$. In cases (x) and (xii) $\mathcal{A}$ is isomorphic to the unique subgroup of index 2 in AGL(1,F) where F = GF(n) and $n = p^e \equiv 3 \pmod 4$.

We recall the graph-preserving operation Ψ that is induced by the automorphism of Γ that multiplies r_0 by r_2. This preserves the subgroup $\langle r_0, r_2\rangle$, whose orbits are edges, and so Ψ preserves edge-symmetry. If $\mathcal{M}$ is an edge-symmetric imbedding of K_n with map subgroup M normalised in Γ by N where N has coset graph (iv) then $\mathcal{M}^\Psi$ is an edge-symmetric imbedding of K_n with map subgroup M^Ψ normalised in Γ by N^Ψ where N^Ψ has coset graph (ii), and vice versa. Similarly, if N has coset graph (x) then N^Ψ has coset graph (xii), and vice versa.

It follows that either $\mathcal{M}$ or $\mathcal{M}^{\psi}$ has map subgroup normaliser with coset graph (i), (ii) or (xii) of Fig. 12 where ψ is the graph-preserving operation induced by the outer automorphism of Γ that multiplies r_0 by r_2.

If N has graph (i) then $N = \Gamma$ and $\mathcal{M}$ is reflexible. If N has graph (ii) then $N = \Gamma^+$ which is presented by $\langle X, Y \mid X^2 = 1 \rangle$ where $X = r_0 r_2$ and $Y = r_1 r_2$ thus $\mathcal{M}$ is regular. If N has graph (xii) then $N = \Gamma^{++} = \langle P, Z \mid - \rangle$ where $P = Y^2$ and $Z = Y^{-1} X$, and so $\mathcal{M}$ is orientable. We shall call any orientable imbedding such that $N^+ = \Gamma^{++}$ a type 3 imbedding.

4) The following classification of the reflexible imbeddings of K_n in surfaces without boundary was first given in [13].

Theorem 4.1. Reflexible imbeddings of K_n in surfaces without boundary exist only for $n = 1, 2, 3, 4$ and 6, and they are: the usual imbeddings in a sphere for $n = 1, 2, 3$ and 4; the spherical imbedding of the graph with six 2-valent vertices, the cube and the icosahedron, with antipodal points identified in each case to give imbeddings in the projective plane for $n = 3, 4$ and 6 respectively; finally, the orientable imbedding of the 1-skeleton of the icosahedron formed by applying Wilson's operation H_2 to the icosahedron is antipodally identified to give a reflexible imbedding of K_6 in a non-orientable surface of genus 5.

Proof: It's easy to check that the above maps are reflexible imbeddings of complete graphs. We now verify that these are the only reflexible imbeddings of complete graphs in surfaces without boundary.

We have already seen that the theorem is true for $n \leqslant 3$. For $n > 3$ suppose that $\mathcal{M}$ is a reflexible imbedding of K_n. We make use of Ito's methods, the basic idea being to count the number of involutions in $\mathcal{A}$. Let v and w be vertices, let $H = \mathcal{A}_v$ and $K = \mathcal{A}_{(v,w)}$. Then $H \simeq D_{(n-1)}$, generated by reflections r and t, and $K \simeq C_2$, generated by t, with actions illustrated in Fig. 17. Furthermore, there exists a reflection ℓ that transposes v and w, and commutes with t. By its double transitivity we may decompose $\mathcal{A}$ into the disjoint union $\mathcal{A} = H + H\ell H$. If $h_1 \in H$ then

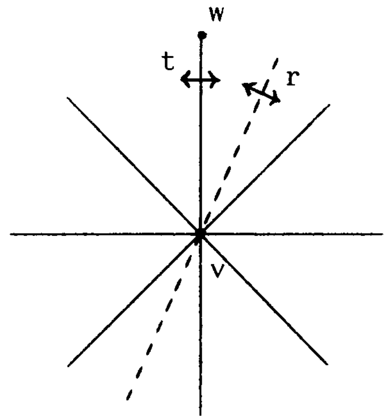


Figure 17

$H\ell h_1 = H\ell \Leftrightarrow \ell h_1 \ell \in H \Leftrightarrow \ell h_1 \ell(v) = v \Leftrightarrow h_1(w) = w \Leftrightarrow h_1 \in K$.

So the number of residue classes of the form $H\ell h_1$ is $|H|/|K| = (n-1)$. Suppose that $h_2 \in H$. Then

$(h_2 \ell h_1)^2 = 1 \Rightarrow h_2 \ell h_1 h_2 \ell h_1(v) = v \Leftrightarrow h_1 h_2 \ell h_1(v) = \ell h_2^{-1}(v) \Leftrightarrow$
$h_1 h_2(w) = w \Leftrightarrow h_1 h_2 \in K \Leftrightarrow h_2 \in \{h_1^{-1}, \; h_1^{-1}t\}$.

Thus the double coset $H\ell H$ is itself a disjoint union of $(n-1)$ right cosets of the form $H\ell h_1$, where $h_1 \in H$, each containing just two involutions. Thus there are $2(n-1)$ involutions in $H\ell H$.

We now take the cases n odd and n even separately. For n odd there are n involutions in H and so $n + 2(n-1)$ involutions in A . Each stabiliser contains $\frac{1}{2}(n+1)$ involutions with exactly one fixed point, that is, that lie in no other stabiliser. Thus we have at least $\frac{1}{2}n(n+1)$ involutions, and so $n + 2(n-1) \geqslant \frac{1}{2}n(n+1)$, giving $n \leqslant 4$, and thus a contradiction. For n even there are $(n-1)$ involutions in H , so $3(n-1)$ involutions in A. The number of involutions with exactly two fixed points equals the number of edges, that is, $\frac{1}{2}n(n-1)$, so $3(n-1) \geqslant \frac{1}{2}n(n-1)$, giving $n = 4$ or 6. We now examine these possibilities.

Let there be f faces, each an m-gon. For $n = 4$ the Euler characteristic of the underlying surface is $4 - 6 + f$ and so $f \leqslant 4$. We see that $A \leqslant S_4$ and that $r_1 r_0$ acts as a rotation of edges around a face. Thus $r_1 r_0$ has order m in S_4 and so $m \leqslant 4$. But fm is just twice the number of edges, that is, 12, and so $f \geqslant 3$. If $f = 4$ then $(r_1 r_2)^3, (r_1 r_0)^3 \in M$ and we are looking for $\overline{M}$ such that

$$\begin{aligned} \overline{M} &\trianglelefteq \Gamma[2,3,3] \\ &= \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_2 r_0)^2 = (r_1 r_2)^3 = (r_1 r_0)^3 = 1 \rangle \\ &\simeq S_4 \end{aligned}$$

But $|\Gamma/M| = 2n(n-1) = 24 = |S_4|$ and so $\overline{M} = 1$. If $f = 3$ then we are looking for $\overline{M}$ such that

$$\begin{aligned} \overline{M} &\trianglelefteq \Gamma[2,3,4] \\ &= \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_2 r_0)^2 = (r_1 r_2)^3 = (r_1 r_0)^4 = 1 \rangle \\ &\simeq S_4 \times C_2 \end{aligned}$$

But $|\Gamma/M| = 24$ and so $|\overline{M}| = 2$. Therefore $\overline{M} = Z(\Gamma[2,3,4])$ and so in either case $\overline{M}$, and thus M, is uniquely determined. Whence there are at most two reflexible imbeddings of K_4.

If $n = 6$ then A cannot have a regular normal subgroup since n is not a prime power [3, 1.7.6]. We now show that A must be simple. Suppose that L is a proper normal subgroup of A. Clearly L is transitive and thus L_v is a normal subgroup of $A_v \simeq D_5$ of order either 2 or 5. If $|L_v| = 2$ then $L_v \leqslant Z(D_5)$ against $Z(D_5) = 1$. If $|L_v| = 5$ then L is sharply 2-transitive and n would have to be a prime power by Zassenhaus' classification. Thus A is simple of order 60, and so M is the kernel of an epimorphism from Γ to A_5.

We need only look for images of r_0, r_1, r_2 in A_5 that form a generating set T of A_5 up to automorphism of A_5. Hence we treat conjugation by S_5 as an equivalence. We see that T contains two commuting involutions and we can assume that these are (12)(34) and (13)(24). The third element must move 5, and we can assume that it fixes 4 (else conjugate by (14)(23), (24)(13) or (34)(12)) so it is one of (12)(35), (13)(25) or (15)(23). Conjugation by (23) identifies the first two choices, so there are only two possibilities for T, namely
{(12)(34), (13)(24), (12)(35)} and
{(12)(34), (13)(24), (15)(23)}. In the first case commutativity and orders of elements imply that:
$r_1 \longmapsto (12)(35)$; $r_2 \longmapsto (13)(24)$; $r_0 \longmapsto (12)(34)$.
In the second case commutativity implies that
$r_1 \longmapsto (15)(23)$, and then conjugation by (23) fixes (15)(23) while transposing the other two, so either choice of images for r_2 and r_0 will give the same kernel. Thus there are at most two reflexible imbeddings of K_6. ∎

5) From now on we let G denote the group $AGL(1,F)$ where $F = GF(n)$ and $n = p^e$. Thus G is generated by the functions $s_a : f \mapsto af$, $t_b : f \mapsto f+b$ such that $a, b \in F$ with a non-zero. We let $\mathcal{P}$ be the set of primitive elements of F and let $\mathcal{Q} = F \setminus \{0\}$. Then $\mathcal{P}^2 = \{a^2 \mid a \in \mathcal{P}\}$ and $\mathcal{Q}^2 = \{b^2 \mid b \in \mathcal{Q}\}$. Finally, let H denote the unique subgroup of G of index 2 for $n \equiv 3 \pmod 4$ consisting of the functions $f \mapsto b^2 f + c$ for $b \in \mathcal{Q}$, $c \in F$.

The following classification and description of the regular imbeddings of K_n for $n > 3$ was first given in [14]. We use the same methods to classify and describe the type 3 imbeddings of K_n for $n > 3$.

If $M \cong N^+ = \Gamma^+$ then M is the kernel of an epimorphism from Γ^+ to G, sending (X,Y) to (x,y) where $o(x) = 2$ and $o(y) = (n-1)$. Moreover, we now show that such kernels must give rise to imbeddings of K_n. Since G acts transitively on vertices, the number of vertices is $[G : G_v] = [G : \langle y \rangle] = n$, and the number of edges is just half the order of G. So we need only check that these maps have neither loops nor multiple edges. If loops exist then all edges are loops, against the connectedness of the graph. If there is a multiple edge then all edges are multiple and so for some $i, j \not\equiv 0$ modulo $(n-1)$ we have $xy^i xy^j$ fixing a dart, and hence it is the identity in G. Hence $y^{(i+j)} = x^{-1} y^{-i} x y^i$ lies in the commutator subgroup, and thus the translation subgroup T, of G and therefore has order dividing $|T| = n$. However, y has order $(n-1)$ so $y^{(i+j)} = 1$.

Thus y^i commutes with x and so lies in the centre of G; this is the trivial subgroup (since $n > 2$), so $i \equiv 0$ modulo $(n-1)$, against our hypothesis.

If $M \subseteq N^+ = \Gamma^{++}$ then M is the kernel of an epimorphism from Γ^{++} to H, sending (Z,P) to (z,ρ) where $o(\rho) = \frac{1}{2}(n-1)$. Moreover, we now show that such kernels must give rise to imbeddings of K_n. Since H acts transitively on vertices the number of vertices is $[H : H_v] = [H : \langle \rho \rangle] = n$, and the number of edges is exactly the order of H. So we need only check that these maps have neither loops nor multiple edges. As in the regular case, there can be no loops. If there is a multiple edge then all edges are multiple and so $XY^iXY^j \in M \subseteq \Gamma^{++}$ for some $i, j \not\equiv 0 \pmod{(n-1)}$ since the valency of the vertices must be $(n-1)$, there being $\frac{1}{2}n(n-1)$ edges and n vertices. If i is even then so is j, in which case $z^{-1}\rho^{\frac{1}{2}i}z\rho^{\frac{1}{2}j}$ is the identity in H. Hence $\rho^{\frac{1}{2}(i+j)} = z^{-1}\rho^{-\frac{1}{2}i}z\rho^{\frac{1}{2}i}$ lies in the commutator subgroup, and thus in the translation subgroup T, of H and therefore has order dividing $|T| = n$. However, ρ has order dividing $(n-1)$ and so $\rho^{\frac{1}{2}(i+j)} = 1$. Thus $\rho^{\frac{1}{2}i}$ commutes with z and so lies in the centre of H; this is the trivial subgroup, so $\frac{1}{2}i \equiv 0 \pmod{\frac{1}{2}(n-1)}$, against our hypothesis. If i is odd then so is j, in which case $z^{-1}\rho^{\frac{1}{2}(i-1)}z^{-1}\rho^{\frac{1}{2}(j-1)}$ is the identity in H. Now $o(\rho) = \frac{1}{2}(n-1)$ and so $\rho \notin T$. Thus ρ has a fixed point and so, by conjugation, we can assume that ρ fixes 0. Then $\rho = s_a^2$ and $z^{-1} = s_b^2 t_c$ for some $a, b, c \in \mathcal{Q}$, and so $z^{-1}\rho^{\frac{1}{2}(i-1)}z^{-1}\rho^{\frac{1}{2}(j-1)} = s_d^2 t_u$ where u is a non-zero multiple of $a^{(i-1)}b^{-2}+1$, against $-1 \notin Q^2$.

Two regular maps are +isomorphic, that is, there is an orientation-preserving isomorphism between them, if and only if they have the same map subgroup. If M is the kernel of an epimorphism from Γ^+ to G, sending (X,Y) to (x,y) where $o(x)=2$ and $o(y)=(n-1)$ and M' is a second such kernel, then $M=M'$ if and only if $[x,y]=[x',y']$, where square brackets denote the equivalence class under group automorphisms. It follows that the equivalence classes under orientation-preserving isomorphisms of regular imbeddings of K_n are in one-to-one correspondence with the equivalence classes under automorphisms of G of

$$\Sigma = \{ (x,y) \in G \times G \mid G = \langle x, y \rangle ,\ o(x)=2 \text{ and } o(y)=(n-1) \} .$$

Two type 3 imbeddings with associated map subgroups M and M' are +isomorphic if and only if M' is one of $\{M, M^Y\}$. We note that $Z^{Y^{-1}} = XY^{-1} = Z^{-1}P^{-1}$ and that $P^{Y^{-1}} = P$. If M is the kernel of an epimorphism from Γ^{++} to H, sending (Z,P) to (z,ρ) where $o(\rho)$ is $\frac{1}{2}(n-1)$, and M' is a second such kernel, then $M'=M$ if and only if $[z',\rho']=[z,\rho]$, and $M'=M^Y$ if and only if $[z',\rho']=[z^{-1}\rho^{-1},\rho]$. Furthermore, the normaliser of M in Γ^+ is strictly larger than Γ^{++} if and only if $M=M^Y$. For $(z,\rho),(z',\rho') \in H \times H$ we define $[z,\rho]$, $[z',\rho']$ to be <u>Y-paired</u> if and only if $[z',\rho']=[z^{-1}\rho^{-1},\rho]$. It is easy to see that this is a well-defined symmetric relation. We define a Y-pairing to be <u>proper</u> if it does not self-pair an equivalence class. It follows that the equivalence classes under orientation-preserving isomorphisms of type 3 imbeddings of K_n are in one-to-one correspondence with the proper Y-pairs of equivalence classes under automorphisms of H of

$$\Sigma' = \{ (z,\rho) \in H \times H \mid H = \langle z, \rho \rangle \text{ and } o(\rho) = \tfrac{1}{2}(n-1) \} .$$

We now classify the equivalence classes under automorphisms of G of Σ for $n > 1$. If $y \in G$ has order $(n-1) > 1$ then $y \notin T$ and so has a fixed point, and so is conjugate by an element of T to an element of G_0. Thus if $(x,y) \in \Sigma$ then $[x, y] = [s_c t_b, s_a]$ for some $a,b,c \in \mathcal{Q}$. Now t_b is conjugate by an element of G_0 to t_1. Thus $[x, y] = [s_c t_1, s_a]$. Moreover, $o(x) = 2$ and $o(y) = (n-1)$ if and only if $c = -1$ and $a \in \mathcal{P}$. Let $x_0 = s_{-1} t_1$, let $a \in \mathcal{P}$ and let $m = 0$ or $\frac{1}{2}(n-1)$ as n is even or odd, so that $a^m = -1$. Then $\langle x_0, s_a \rangle$ contains all non-trivial translations

$$t_{a^i} = s_a^{(m-i)} x_0 s_a^i \quad ;$$

thus it contains T and G_0 which together generate G, so $(x,y) \in \Sigma$ if and only if $[x, y] = [x_0, s_a]$ for some $a \in \mathcal{P}$.

Let Gal(F) denote the Galois group of F over its prime field $F_0 = GF(p)$; then each $\theta \in Gal(F)$ induces an automorphism $\overline{\theta}$ of G, sending each element $s_a t_b$ to $s_{a^\theta} t_{b^\theta}$. Clearly $\overline{\theta}$ fixes $x_0 = s_{-1} t_1$.

Lemma 4.2. Let $a, a' \in \mathcal{P}$. Then $[x_0, s_a] = [x_0, s_{a'}]$ if and only if a and a' are conjugate under Gal(F).

Proof: If $a' = a^\theta$ for some $\theta \in \mathrm{Gal}(F)$ then $\bar{\theta}$ sends (x_0, s_a) to $(x_0, s_{a'})$.

Conversely, suppose that $\alpha \in \mathrm{Aut}(G)$ fixes x_0 and sends s_a to $s_{a'}$. Since $a, a' \in \mathcal{P}$ we have $a' = a^i$ for some i coprime to $(n-1)$. The function $\theta : F \longrightarrow F$, $f \longmapsto f^i$, restricts to an automorphism of the multiplicative group F^*, taking a to a'; we shall show that θ is also an automorphism of the additive group of F, and hence an element of Gal(F), as required.

We must show that $(f_1 + f_2)^\theta = f_1^\theta + f_2^\theta$ for all $f_1, f_2 \in F$. Since $0^\theta = 0$, we can assume that $f_1, f_2 \neq 0$. If $f_1 + f_2 = 0$ then $(f_1 + f_2)^\theta = 0$ and

$$f_1^\theta + f_2^\theta = f_1^i + f_2^i = f_1^i + (-f_1)^i = f_1^i(1 + (-1)^i) = 0$$

(since if i is even then so is n); hence we can assume that $f_1 + f_2 \neq 0$.

For any integers j, k, ℓ consider the word

$$W(g,h) = h^{(m-j)}gh^{(j+m-k)}gh^{(k-\ell)}gh^{(\ell+m)} .$$

We see that $W(x_0, s_a)$ is a composite translation t_b where $b = a^j + a^k - a^\ell$. Since $s_{a'} = s_a^i$, if we replace h by h^i and use the fact that $im \equiv m \pmod{n-1}$, then we see that $W(x_0, s_{a'}) = t_{b'}$ where $b' = a^{ij} + a^{ik} - a^{i\ell}$.

Since $f_1, f_2, f_1 + f_2 \neq 0$, we have $f_1 = a^j$, $f_2 = a^k$ and $f_1 + f_2 = a^\ell$ for suitable integers j, k and ℓ;

in the above notation this gives $b = 0$, so that $W(x_0, s_a)$ is the identity in G and hence (applying α) we have $W(x_0, s_{a'}) = 1$. Thus $b' = 0$, so $a^{ij} + a^{ik} = a^{i\ell}$, that is, $f_1^\theta + f_2^\theta = (f_1 + f_2)^\theta$. ■

Let μ be the homomorphism from G to F^* sending $s_a t_b$ to a ($a \in F^*$, $b \in F$), then

<u>Corollary 4.3</u>. Two pairs $(x,y), (x',y') \in \Sigma$ are equivalent under $\mathrm{Aut}(G)$ if and only if $\mu(y)$ and $\mu(y')$ are conjugate under $\mathrm{Gal}(F)$. ■

<u>Remark</u>. In fact these arguments show that every automorphism of G is the composition of an inner automorphism and a "field automorphism" $\bar{\theta}$; in other words, $\mathrm{Aut}(G)$ can be identified with the group $A\Gamma L(1,F)$ of transformations $f \longmapsto af^\theta + b$ ($a \in F^*$, $b \in F$, $\theta \in \mathrm{Gal}(F)$) acting by conjugation on its normal subgroup G.

If $n = p^e$ (p prime) then $\mathrm{Gal}(F)$ is cyclic of order e, being generated by the Frobenius automorphism $f \longmapsto f^p$; thus $\mathrm{Gal}(F)$ has $\phi(n-1)/e$ orbits of size e on P, and so

<u>Theorem 4.4</u>.[14] For each prime power $n = p^e$ there are $\phi(n-1)/e$ orientation-preserving isomorphism classes of regular imbeddings of K_n. ■

We now classify the equivalence classes under automorphisms of H of Σ' for $n > 3$. If $\rho \in H$ has order $\frac{1}{2}(n-1)$ then $\rho \notin T$ and so has a fixed point, so is conjugate by an element of T to an element of H_0. Thus if $(z,\rho) \in \Sigma'$ then $[z,\rho] = [s_c^2 t_b, s_a^2]$ for some $a,b,c \in \mathcal{Q}$. Now G normalises H and t_b is conjugate by an element of G_0 to t_1. Thus $[z,\rho] = [s_c^2 t_1, s_a^2]$. Moreover, ρ has order $\frac{1}{2}(n-1)$ if and only if $a^2 \in \mathcal{P}^2$. Furthermore, if $a^2 \in \mathcal{P}^2$ then $\langle s_c^2 t_1, s_a^2 \rangle$ contains t_1 and so, since without loss of generality $a \in \mathcal{P}$, contains all non-trivial translations

$$t_{a^i} = (s_a^2)^{-\frac{1}{4}i(n+1)} t_1^{(-1)^i} (s_a^2)^{\frac{1}{4}i(n+1)} \quad ;$$

thus it contains T and H_0 which together generate H, so $(z,\rho) \in \Sigma'$ if and only if $[z,\rho] = [s_c^2 t_1, s_a^2]$ for some $a \in \mathcal{P}$, $c \in \mathcal{Q}$.

As in the regular case, each $\theta \in \mathrm{Gal}(F)$ induces an automorphism $\overline{\theta}$ of H. Clearly $\overline{\theta}$ fixes t_1.

<u>Lemma 4.5</u>. Let $a, a' \in \mathcal{P}$ and $c, c' \in \mathcal{Q}$. Then $[s_c^2 t_1, s_a^2] = [s_{c'}^2 t_1, s_{a'}^2]$ if and only if (c^2, a^2) and $((c')^2, (a')^2)$ are conjugate under $\mathrm{Gal}(F)$.

<u>Proof</u>: If $(a')^2 = (a^2)^\theta$ and $(c')^2 = (c^2)^\theta$ for some θ in $\mathrm{Gal}(F)$, then $\overline{\theta}$ sends $(s_c^2 t_1, s_a^2)$ to $(s_{c'}^2 t_1, s_{a'}^2)$.

Conversely, suppose that $\alpha \in \mathrm{Aut}(H)$ sends $(s_c^2 t_1, s_a^2)$ to $(s_{c'}^2 t_1, s_{a'}^2)$. Then α fixes both H_0 and T (since T is characteristic in H), and so α fixes t_1. Since $a, a' \in \mathcal{P}$ we have $a' = a^i$ for some i coprime to $(n-1)$. The function $\theta : F \longrightarrow F$, $f \longmapsto f^i$ restricts to an automorphism of the multiplicative group F^*, taking (c^2, a^2) to $((c')^2, (a')^2)$; we shall show that

θ is also an automorphism of the additive group of F, and hence an element of $\mathrm{Gal}(F)$, as required.

We must show that $(f_1 + f_2)^\theta = f_1^\theta + f_2^\theta$ for all $f_1, f_2 \in F$. As in the regular case, we can assume that $f_1, f_2, f_1 + f_2$ are all non-zero.

For any integers j, k, ℓ consider the word

$$W'(g,h) = h^{\lambda_1} g^{\lambda_2} h^{\lambda_3} g^{\lambda_4} h^{\lambda_5} g^{-\lambda_6} h^{\lambda_7} ,$$

where $\lambda_1 = -\frac{1}{4}j(n+1)$; $\lambda_2 = (-1)^j$; $\lambda_3 = \frac{1}{4}(j-k)(n+1)$; $\lambda_4 = (-1)^k$; $\lambda_5 = \frac{1}{4}(k-\ell)(n+1)$; $\lambda_6 = (-1)^\ell$; $\lambda_7 = \frac{1}{4}\ell(n+1)$.

We see that $W'(t_1, s_a^2)$ is a composite translation t_b where $b = a^j + a^k - a^\ell$. Since $(s_{a'})^2 = s_a^{2i}$, if we replace h by h^i and use the fact that i is odd then we see that $W'(t_1, s_{a'}^2) = t_{b'}$ where $b' = a^{ij} + a^{ik} - a^{i\ell}$.

Since $f_1, f_2, f_1 + f_2 \neq 0$ we have $f_1 = a^j$, $f_2 = a^k$, $f_1 + f_2 = a^\ell$ for suitable integers j, k and ℓ ; in the above notation this gives $b = 0$ so that $W'(t_1, s_a^2)$ is the identity in H, and hence (applying α) we have $W'(t_1, s_{a'}^2) = 1$. Thus $b' = 0$ so $a^{ij} + a^{ik} = a^{i\ell}$, that is, $f_1^\theta + f_2^\theta = (f_1 + f_2)^\theta$. ∎

<u>Corollary 4.6</u>. Two pairs $(z,\rho), (z',\rho') \in \Sigma'$ are equivalent under $\mathrm{Aut}(H)$ if and only if $(\mu(z),\mu(\rho))$ and $(\mu(z'),\mu(\rho'))$ are conjugate under $\mathrm{Gal}(F)$. ∎

<u>Remark</u>. These arguments in fact show that for $n > 3$ $\mathrm{Aut}(H)$ can also be identified with the group $A\Gamma L(1,F)$.

If $(z,\rho) \in \Sigma'$ then $[z',\rho'] = [z^{-1}\rho^{-1},\rho]$, where

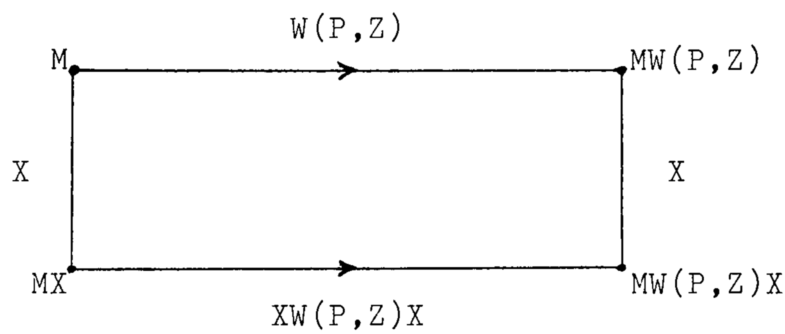
$(z',\rho') \in \Sigma'$, if and only if $(\mu(z'),\mu(\rho'))$ and $(\mu(z)^{-1}\mu(\rho)^{-1},\mu(\rho))$ are conjugate under Gal(F). Thus $[z,\rho]$ is self Y-paired if and only if $\mu(z)^2 = \mu(\rho)^{-1}$, which has precisely one solution for $\mu(z) \in \mathcal{Q}^2$, and so:

<u>Theorem 4.7</u>. For each prime power $n = p^e \equiv 3 \pmod 4$ greater than 3 there are $\frac{1}{4}(n-3)\phi(n-1)/e$ orientation-preserving isomorphism classes of type 3 imbeddings of K_n. ∎

By identifying Γ^+/M with G we have the coset graphs of M in Γ^+ corresponding to the regular imbeddings of K_n with vertices, edges and faces corresponding to orbits of $\langle Y \rangle$, $\langle X \rangle$ and $\langle Z \rangle$ respectively, and incidence given by non-empty intersection. This gives an orientable map, which we shall denote by $\mathcal{M}(x,y)$, and we have $\mathcal{M}(x,y) \simeq^+ \mathcal{M}(x',y')$, that is, there is an orientation-preserving isomorphism between them, if and only if $\mu(y)$ and $\mu(y')$ are conjugate under Gal(F). Each $^+$isomorphism class contains a map $\mathcal{M}(x,y)$ with $x = x_0$ and $y = s_a$ for some $a \in \mathcal{P}$; if we denote this map by $\mathcal{M}(a)$ then we have:

<u>Theorem 4.8</u>. The regular imbeddings of K_n $(n > 1)$ are $^+$isomorphic to the maps $\mathcal{M}(a)$ where a is a primitive element of GF(n). We have $\mathcal{M}(a) \simeq^+ \mathcal{M}(a')$ if and only if a and a' are conjugate under Gal(F). ∎

By identifying Γ^{++}/M with H and by using the automorphism of Γ^{++} induced by conjugation by X, we can construct the coset graphs of M in Γ^+ corresponding to the type 3 imbeddings of K_n, as illustrated below, with vertices, edges and faces corresponding to orbits of $\langle XZ^{-1} \rangle$, $\langle X \rangle$ and $\langle Z \rangle$ respectively, and incidence given by non-empty intersection.



This gives an orientable map, which we shall denote by $\mathcal{M}(\rho, z)$, and we have $\mathcal{M}(\rho, z) \simeq^+ \mathcal{M}(\rho', z')$ if and only if $(\mu(\rho'), \mu(z'))$ is conjugate under Gal(F) to either $(\mu(\rho), \mu(z))$ or $(\mu(\rho), \mu(z^{-1}\rho^{-1}))$. Each $^+$isomorphism class contains a map $\mathcal{M}(\rho, z)$ with $\rho = s_a^2$ and $z = s_c^2 t_1$ for some $a \in \mathcal{P}$, $c \in \mathcal{Q}$; if we denote this map by $\mathcal{M}(a, c)$ then we have:

<u>Theorem 4.9</u>. The type 3 imbeddings of K_n are $^+$isomorphic to the maps $\mathcal{M}(a, c)$ where a is a primitive element of GF(n) and $c \in GF(n) \setminus \{0\}$ such that $c^4 \neq a^{-2}$. We have $\mathcal{M}(a, c) \simeq^+ \mathcal{M}(a', c')$ if and only if $((a')^2, (c')^2)$ is conjugate under Gal(F) to either (a^2, c^2) or $(a^2, a^{-2}c^{-2})$. ■

We can give an alternative description of $\mathcal{M}(a)$ as one of the Cayley maps introduced by Biggs [2] (see also [3]). Let $g \in G$ be given by

$$g = s_b t_c : f \longmapsto bf + c$$

where $b, c \in F$ and $b \neq 0$. Then the coset $g \langle y \rangle$, consisting of the elements

$$gy^i : f \longmapsto a^i(bf + c) ,$$

contains a unique translation $t_{c/b}$ (with $a^i = b^{-1}$), so we label the vertex corresponding to $g \langle y \rangle$, via the identification of G with Γ^+/M, with the element $v = c/b \in F$; this gives a bijection between V and F. Now

$$gx : f \longmapsto -bf + (1-c)$$

and

$$gyx : f \longmapsto -abf + (1-ac)$$

are associated with vertices labelled $(1-c)/(-b) = v - b^{-1}$ and $(1-ac)/(-ab) = v - a^{-1}b^{-1}$ respectively; thus if we put $a^{-1} = u$ then the vertices adjacent to v are labelled with the elements of $F \setminus \{v\} = v + F^*$ in the cyclic order

$$v + 1,\ v + u,\ v + u^2,\ \ldots\ldots\ldots,\ v + u^{(n-2)}$$

corresponding to the orientation around v. We therefore have:

<u>Theorem 4.10</u>. The regular imbeddings of K_n $(n > 1)$ are $^+$isomorphic to the Cayley maps $\mathcal{M}(F, F^*, r)$ where $F = GF(n)$ as an additive group, the generating set for F is $F^* = F \setminus \{0\}$, and r is the cyclic permutation

$$s_u : f \longmapsto uf$$

of F^* for some primitive element u of F; two such maps are $^+$isomorphic if and only if the corresponding primitive elements are conjugate under $\mathrm{Gal}(F)$. ∎

We can also give an alternative description of $\mathcal{M}(a,c)$ as a Cayley map. Let $h \in H$ be given by

$$h = s_b^2 t_d : f \longmapsto b^2 f + d$$

where $b, d \in F$ and $b \neq 0$. Then the coset $h\langle\rho\rangle$ consisting of the elements

$$h\rho^i : f \longmapsto a^{2i}(b^2 f + d)$$

contains a unique translation t_{d/b^2} (with $a^{2i} = b^{-2}$) so we label the $\langle Y\rangle$-orbit of the coset $MU \in \Gamma^{++}/M$ corresponding to h with the element $v = d/b^2 \in F$. This gives a bijection between V and F. Now

$$MUX\langle Y\rangle = MUXY\langle Y\rangle = MUZ^{-1}\langle Y\rangle$$
and
$$MUY^2X\langle Y\rangle = MUY^2XY\langle Y\rangle = MUY^2Z^{-1}\langle Y\rangle .$$

Furthermore, the cosets MUZ^{-1}, $MUYX$, MUY^2Z^{-1} correspond to the elements

$$hz^{-1} : f \longmapsto c^{-2}b^2 f + c^{-2}(d-1)$$
$$h\rho z : f \longmapsto c^2a^2b^2 f + (c^2a^2d + 1)$$
$$h\rho z^{-1} : f \longmapsto c^{-2}a^2b^2 f + c^{-2}(a^2d - 1)$$

and so have vertices labelled $(d-1)/b^2 = v - b^{-2}$, $(c^2a^2d+1)/c^2a^2b^2 = v + c^{-2}a^{-2}b^{-2}$ and $(a^2d-1)/a^2b^2 = v - a^{-2}b^{-2}$ respectively. Thus if we let $u = a^{-1}$ and let j be such that

$$-a^{-2}c^{-2} = u^{j} \qquad (*)$$

then since

$$c^{4} = a^{-2} \Rightarrow u^{j} = -c^{2} = -(\sqrt{a^{-2}} \cap \mathcal{Q}^{2}) = a^{-1} = u$$
$$\Rightarrow c^{4} = (-a^{-2}u^{-j})^{2} = a^{-4}u^{-2} = a^{-2}$$

we have that u^{j} ranges over all odd powers of u except u itself as c^{2} takes all values in $\mathcal{Q}^{2}$ except that value for which $c^{4} = a^{-2}$ and the vertices adjacent to v are labelled with the elements of $F \setminus \{v\} = v + F^{*}$ in the cyclic order

$$v-1,\ v-u^{j},\ v-u^{2},\ v-u^{(j+2)},\ v-u^{4},\ v-u^{(j+4)},\ \ldots\ldots\ldots,\ v-u^{(j+n-3)} .$$

Furthermore, if $((a')^{2\theta}, (c')^{2\theta}) = (a^{2}, a^{-2}c^{-2})$ where $a, a' \in \mathcal{P}$ and $c, c' \in \mathcal{Q}$ for some $\theta \in \mathrm{Gal}(F)$ and both

$$-a^{-2}c^{-2} = a^{-j} \qquad (1)$$

and

$$-(a')^{-2}(c')^{-2} = (a')^{-k} \qquad (2)$$

then $a = (a')^{\theta}$, and thus (applying θ to (2)) we have $-c^{-2} = a^{k}$ and so $j + k \equiv 2 \pmod{(n-1)}$ by (1). We therefore have:

<u>Theorem 4.11</u>. The type 3 imbeddings of K_n are $^{+}$isomorphic to the Cayley maps $\mathcal{M}(F, F^{*}, r)$ where $F = GF(n)$ as an additive group, the generating set for F is $F^{*} = F \setminus \{0\}$, and r is the permutation of F^{*} defined by

$$r|_{Q^2} = s_u^{(2-j)} : f \mapsto u^{(2-j)}f$$
$$r|_{Q\setminus Q^2} = s_u^j \ : f \mapsto u^j f$$

for some primitive element u of F and odd integer $j : 1 < j < (n-1)$. Furthermore, if we denote such a map by $\mathcal{M}(u,j)$ then $\mathcal{M}(u',k) \simeq^+ \mathcal{M}(u,j)$ if and only if u' and u are conjugate under $\mathrm{Gal}(F)$, and either $k \equiv j$ or $k \equiv (2-j) \pmod{(n-1)}$. ∎

The various regular imbeddings $\mathcal{M} = \mathcal{M}(a)$ and $\mathcal{M}' = \mathcal{M}(a')$ of K_n are related as follows by Wilson's operations. Since $a, a' \in \mathcal{P}$, we have $a' = a^i$ for some i coprime to $(n-1)$, so $\mathcal{M}'$ is obtained from $\mathcal{M}$ by using $y' = s_a^i$ instead of $y = s_a$ to describe the rotation of edges around each vertex; in other words, $\mathcal{M}'$ is the map $H_i(\mathcal{M})$. Similarly, if $\mathcal{M}(a,c)$ is a type 3 imbedding of K_n then H_i induces the representation

$$P \mapsto P^i \mapsto \rho^i = s_a^{2i}$$
$$Z = Y^{-1}X \mapsto P^{\frac{1}{2}(1-i)}Z \mapsto \rho^{\frac{1}{2}(1-i)}z = s_a^{(1-i)}s_c^2 t_1$$

corresponding to the map $\mathcal{M}(a^i, a^{\frac{1}{2}(1-i)}c)$; in other words, $H_i(\mathcal{M}(u,j)) = \mathcal{M}(u^i, (j-1)/i+1)$, by (*), giving the new rotation

$$r'|_{Q^2} : f \mapsto u^{(i-j+1)}f$$
$$r'|_{Q\setminus Q^2} : f \mapsto u^{(i+j-1)}f \quad .$$

We now consider which of the regular and type 3 imbeddings of K_n have map subgroup normalisers not contained in Γ^+. In the regular case such imbeddings would be reflexible, and so, as we have seen, this happens if and only if $n \leqslant 4$. Furthermore, since the only node stabilisers of graphs (i), (ii), or (xii) of Fig. 12 or their images under ψ to contain Γ^{++} are those of (i), (ii) and (xii) it follows that every map subgroup normaliser of a type 3 imbedding of K_n is contained in Γ^+. Thus for $n > 4$ no regular or type 3 imbedding of K_n is isomorphic to its own mirror image.

We now examine some of the topological features of these imbeddings for $n \geqslant 4$. If $\mathcal{M}$ is a regular imbedding of K_n then each face has the same number of sides, namely the order of $y^{-1}x$ in G; this number is $(n-1)$ unless $n \equiv 3 \pmod 4$, in which case it is $\frac{1}{2}(n-1)$, so the number of faces of $\mathcal{M}$ is n or $2n$, from which it follows that $\mathcal{M}$ has genus $\frac{1}{4}(n-1)(n-4)$ or $\frac{1}{4}(n^2 - 7n + 4)$ respectively (as in [3]). Since none of x, y, xy are in the commutator subgroup of G it follows that the length of each Petrie path in $\mathcal{M}$ is twice the order of the commutator $x^{-1}y^{-1}xy$ in G, that is, $2p$, whence $\mathcal{M}^\psi$ has Euler characteristic $n - \frac{1}{2}n(n-1) + \frac{1}{2}n(n-1)/p$. For n odd this is odd and so $\mathcal{M}^\psi$ is non-orientable. We note that r_1r_0 acts as a rotation of edges around a face and so for n even $(r_1r_0)^{(n-1)} \varepsilon M$. Then $(r_1r_0r_2)^{(n-1)} \varepsilon M^\psi$ and so $M^\psi \not\leqslant \Gamma^+$, whence $\mathcal{M}^\psi$ is non-orientable. So $\mathcal{M}^\psi$ is non-orientable of genus $2 - n + \frac{1}{2}n(n-1)(p-1)/p$.

If $\mathcal{M} = \mathcal{M}(u,j)$ is a type 3 imbedding of K_n then there are two orbits of faces, of sizes F_1 and F_2 say, containing $M\langle Z\rangle$, of valency α say, and $MX\langle Z\rangle$, of valency β say, respectively. Thus the two faces incident with any of the $\frac{1}{2}n(n-1)$ edges are in different orbits, and we have $F_1\alpha = F_2\beta = \frac{1}{2}n(n-1)$. Now α is the order of z in H , and β is the least positive integer such that $MXZ^{\beta} = MX$, that is, such that $MXZ^{\beta}X = M$. Equivalently, $M(PZ)^{-\beta} = M$, whence β is the order of ρz in H . By (*) we have $\mathcal{M}(u,j) \simeq^{+} \mathcal{M}(a,c)$ where $a = u^{-1}$ and $-a^{-2}c^{-2} = a^{-j}$, that is, $c^2 = -a^{(j-2)}$. Whence $z = s_c^2 t_1 = s_a^{(j-2+\frac{1}{2}(n-1))} t_1$. Thus if $2-j \equiv \frac{1}{2}(n-1) \pmod{(n-1)}$ then $\alpha = p$ else $\alpha = \frac{1}{2}(n-1)/(n-1,2-j)$. Furthermore, $\rho z = s_a^{(j+\frac{1}{2}(n-1))} t_1$ and so if $j \equiv \frac{1}{2}(n-1) \pmod{(n-1)}$ then $\beta = p$ else $\beta = \frac{1}{2}(n-1)/(n-1,j)$. From which it follows that $\mathcal{M}$ has F faces where if either $j \equiv \frac{1}{2}(n-1)$ or $2-j \equiv \frac{1}{2}(n-1) \pmod{(n-1)}$ then $F = n + \frac{1}{2}n(n-1)/p$ else $F = n\{(n-1,j) + (n-1,2-j)\}$. Thus $\mathcal{M}$ has genus g where if either $j \equiv \frac{1}{2}(n-1)$ or $2-j \equiv \frac{1}{2}(n-1) \pmod{(n-1)}$ then $g = \frac{1}{4}(n-1)\{n(p-1) - 4p\}/p$ else $g = \frac{1}{4}n\{(n-3) - 2(n-1,j) - 2(n-1,2-j)\} + 1$.

There is just one orbit of Petrie paths in $\mathcal{M}$. Suppose that $M(X^{-1}Y^{-1}XY)^{\gamma}X^{-1}Y^{-1} = M$ for some integer γ . Then $\rho z \in \langle z\rho z\rangle$ whence $z \in \langle z\rho z\rangle$, and so $H \subseteq \langle z\rho z\rangle$, against H being non abelian. Whence the length of each Petrie path is 2γ where γ is the order of $z\rho z$ in H , giving $\frac{1}{2}n(n-1)/\gamma$ Petrie paths in all. Now $\mu(z\rho z) = c^4a^2 = a^{2(j-1)}$ and so $\mu(z\rho z) \neq 1$. Whence $\gamma = (n-1)/(n-1,2(j-1))$. It follows that $\mathcal{M}^{\psi}$ has Euler characteristic χ where $\chi = n - \frac{1}{2}n(n-1) + \frac{1}{2}n(n-1,2(j-1))$, which is odd. Thus $\mathcal{M}^{\psi}$ is non-orientable of genus $\frac{1}{2}n\{(n-3) - (n-1,2(j-1))\} + 2$.

It is to be noted that these more obvious topological features do not therefore determine the isomorphism class of the imbedding. We have, however, proved the following:

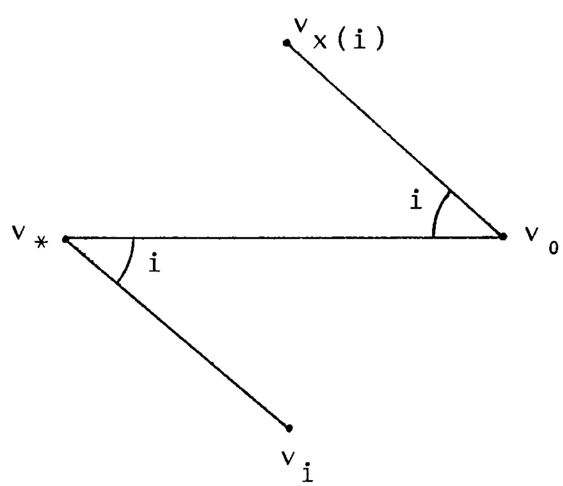
Theorem 4.12. The orientable non-regular edge-symmetric imbeddings of K_n in surfaces without boundary are precisely the type 3 imbeddings. ∎

6) We now give a simple algorithm to determine whether two given edge-symmetric imbeddings of K_n (drawn on polygons in $\mathbb{R}^2$ with sides identified, for instance) are isomorphic.

We have seen that an edge-symmetric imbedding of K_n is reflexible if and only if $n = 1, 2, 3, 4$ or 6. Furthermore, the isomorphism class of any reflexible imbedding of K_n is determined by its Euler characteristic. If we do not have a reflexible imbedding then we may use standard techniques to determine the orientability of the surface (see for example [22] in the case of a polygonal representation), and so, by applying the topological operation ψ if necessary, we need only determine whether two orientable non-reflexible edge-symmetric imbeddings of K_n are isomorphic. We may easily determine which such imbeddings are regular (by considering the symmetries induced by X and Y, for instance), and so, since the mirror image of a given orientable imbedding is readily obtainable, we need only give algorithms to determine whether two imbeddings of K_n that are either both regular or both of type 3 are $^{+}$isomorphic. We now do this.

Given any dart (v_*, v_0) of a regular imbedding of K_n the orientation of the map gives a cyclic ordering $(v_0\, v_1 \ldots\ldots v_{(n-2)})$ to the vertices adjacent to v_* and there is a unique map automorphism X_{v_*, v_0} sending (v_*, v_0) to (v_0, v_*) which is a half rotation of the map about the centre of the edge $\{v_*, v_0\}$ and which induces a permutation x_{v_*, v_0} of the index set $\{*, 0, 1, \ldots (n-2)\}$. By definition $x_{v_*, v_0} : (*, 0) \longmapsto (0, *)$ and its action on $\{1, \ldots, (n-2)\}$ is illustrated by Fig. 18, where x_{v_*, v_0} has been abbreviated to x.

Figure 18

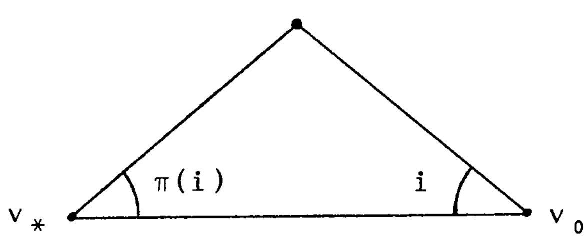


Conversely, x_{v_*, v_0} completely determines the map, for if y is the permutation $y = (0\,1 \ldots\ldots (n-2))^{-1}$ then x_{v_*, v_0} and y generate the map automorphism group A and the kernel of the epimorphism $\theta : \Gamma^+ \longrightarrow A$, $X \longmapsto x_{v_*, v_0}$, $Y \longmapsto y$ is a map subgroup.

If ϕ is a map $^{+}$isomorphism between two regular imbeddings $\mathcal{M}$ and $\mathcal{M}'$ with distinguished darts (v_*, v_0) and (v'_*, v'_0) then for some map $^+$automorphism α we have $\phi \circ \alpha : (v_*, v_0) \longmapsto (v'_*, v'_0)$. Thus a necessary and sufficient condition for two regular imbeddings $\mathcal{M}$ and $\mathcal{M}'$ with distinguished darts (v_*, v_0) and (v'_*, v'_0) to be $^+$isomorphic is that $x_{v'_*, v'_0} = x_{v_*, v_0}$.

If τ is the permutation of $\{1, \ldots, (n-2)\}$ that sends i to $(n-1-i)$ then we can clearly replace the role of x in the above condition by $\pi = \tau \circ x$. Furthermore, by definition $\pi_{v_*, v_0} : (*, 0) \longmapsto (0, *)$ and by Fig. 19, where π_{v_*, v_0} has been abbreviated to π, the action of π_{v_*, v_0} on $\{1, \ldots, (n-2)\}$ can easily be read off by inspection of any map.

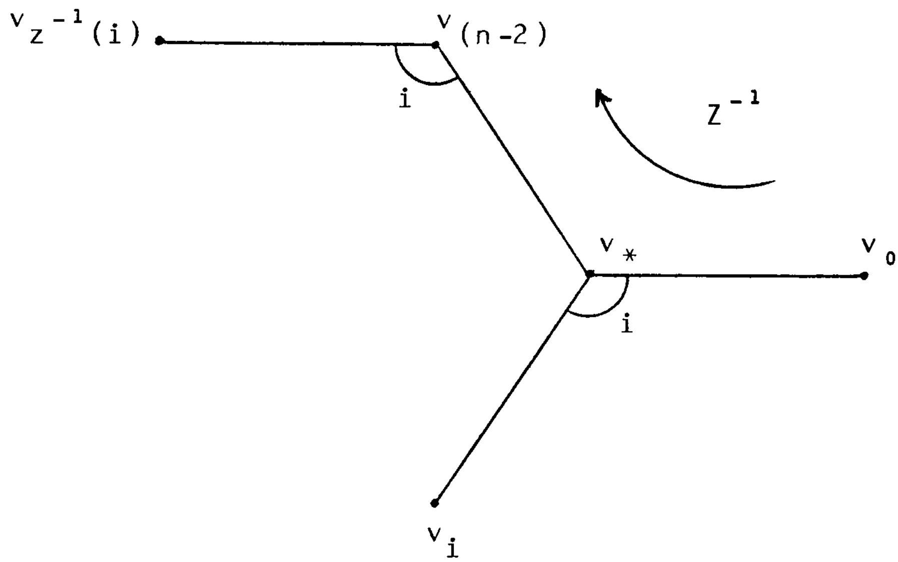
Figure 19



Given any dart (v_*, v_0) of a type 3 imbedding of K_n the orientation of the map gives a cyclic ordering $(v_0 v_1 \ldots\ldots v_{(n-2)})$ to the vertices adjacent to v_*, and there is a unique map automorphism $Z^{-1}_{v_*, v_0}$ that sends (v_*, v_0) to $(v_{(n-2)}, v_*)$ which rotates the map about the centre of the face $(\ldots\ldots v_0 v_* v_{(n-2)} \ldots\ldots)$ and which induces a permutation $z^{-1}_{v_*, v_0}$ of the index set $\{*, 0, 1, \ldots\ldots, (n-2)\}$. By definition $z^{-1}_{v_*, v_0}$ sends

$(*,0)$ to $((n-2),*)$ and its effect on $\{1, \ldots, (n-2)\}$ is illustrated by Fig. 20, where $z^{-1}_{v_*, v_0}$ is abbreviated to z^{-1}.

Figure 20

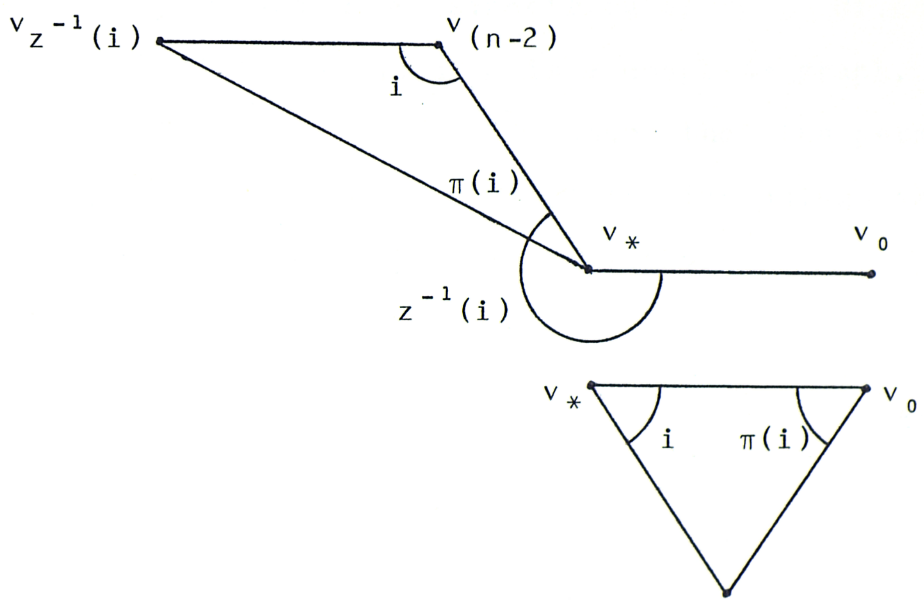


Conversely, $z^{-1}_{v_*, v_0}$ completely determines the map, for if ρ is the permutation y^2 then z_{v_*, v_0} and ρ generate the map automorphism group A and the kernel of the epimorphism $\theta : \Gamma^{++} \longrightarrow A$, $Z \longmapsto z_{v_*, v_0}$, $P \longmapsto \rho$ is a map subgroup.

If ϕ is a map $^+$isomorphism between two type 3 imbeddings $\mathcal{M}$ and $\mathcal{M}'$ with distinguished darts (v_*, v_0) and (v'_*, v'_0) then for some map $^+$automorphism α we have that $\phi \circ \alpha : (v_{\delta(*)}, v_{\delta(0)}) \longmapsto (v'_*, v'_0)$ where δ is some permutation of $\{*, 0\}$. Thus a necessary and sufficient condition for two type 3 imbeddings $\mathcal{M}$ and $\mathcal{M}'$ with distinguished darts (v_*, v_0) and (v'_*, v'_0) to be $^+$isomorphic is that $z^{-1}_{v'_*,v'_0} \varepsilon \{z^{-1}_{v_*,v_0}, z^{-1}_{v_0,v_*}\}$.

We can clearly replace the role of z^{-1} in the above condition by $\pi = \tau \circ y^{-1} \circ z^{-1}$. Furthermore, by definition $\pi_{v_*,v_0} : (*,0) \longmapsto (0,*)$ and by Fig. 21 , where $z^{-1}_{v_*,v_0}$ and π_{v_*,v_0} have been abbreviated to z^{-1} and π , the action of π_{v_*,v_0} on $\{1, ..., (n-2)\}$ for any distinguished dart (v_*,v_0) can easily be read off by inspection of any map.

Figure 21



It is not hard to see that this sort of algorithm has more general applications. For example, suppose that $\mathcal{M}$ and $\mathcal{M}'$ are regular imbeddings of any simple graph $\mathcal{G}$ with n vertices. Then for any distinguished directed edge of $\mathcal{G}$, from vertex w to vertex v say, Γ^+ acts as a group of symmetries of $\mathcal{M}$ and $\mathcal{M}'$: X rotates the maps about the centre of the edge {v, w} and Y rotates the maps about v, sending $\{v, w\}$ to the first edge met in a positive direction. Furthermore, there is a subset T of Γ^+ of order n such that the vertex set of $\mathcal{G}$ is the image set of v under the action of T as a set of symmetries of $\mathcal{M}$. If the same cannot be said of T applied as a set of symmetries of $\mathcal{M}'$ then the two imbeddings are non-isomorphic, else we label both copies of the vertex set by T in the obvious way. Then $\mathcal{M}$ and $\mathcal{M}'$ are isomorphic if and only if the vertex permutations induced by X and Y as symmetries of $\mathcal{M}$ are the same as those induced by X and Y as symmetries of $\mathcal{M}'$. Of course, in the special case when $\mathcal{G}$ is a complete graph, taking $T = \{1, X, XY, \ldots\ldots, XY^{(n-2)}\}$ means that the permutation induced by Y is independent of the imbedding, and thus the algorithm simplifies considerably.

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