UNIVERSITY OF SOUTHAMPTON

IMPROVEMENTS IN EDDY-CURRENT CALCULATION FOR POWER APPARATUS

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by Nigel Brooks

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ABSTRACT

FACULTY OF ENGINEERING AND APPLIED SCIENCE

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The solution of eddy-current problems is important in many areas of engineering design, these areas include motor and generator design, induction furnace design and the design of nuclear fusion reactors. The equation which describes a power frequency (ie displacement current may be ignored) eddy-current field is the diffusion equation.

The computer calculation of electromagnetic fields relies on approximate numerical methods such as the finite difference method, the finite element method and the boundary element method. An approximate method for calculating eddy-currents using variational principles is presented, this method may be used as the basis for a finite element solution technique. The development of the method is given showing the connection with analytical mechanics and the variational treatment of electrostatic and magnetostatic systems. The particular variational treatment which is used leads to a calculation scheme which gives dual bounds for global parameters (eg. inductance, resistance) in eddy-current problems. The advantages of a dual bound calculation scheme are explained and it is shown that a dual bound calculation scheme leads to a more efficient use of computing resources than an unbounded calculation scheme.

A practical example of the value of solving eddy-current problems is presented in Chapter 7. In this example an eddy-current probe is used to detect the temperature of an insulated conductor. A brief description of the international eddy-current workshops organised by Argonne National Laboratory U.S.A. is given in Chapter 8.

<u>List of Principal Symbols</u>

$\underline{A}$	magnetic vector potential
$\underline{B}$	magnetic flux density
$\underline{D}$	electric flux density
$\underline{E}$	electric field strength
$\underline{F}$	force
$\underline{H}$	magnetic field strength
I	electric current
$\underline{J}$	electric current density
J_1	Bessel function of first kind and order unity
Q	electric charge
P	power
R	resistance
Rh	resistivity
X	reactance
$\underline{a}$	acceleration
m	mass
t	time
ϵ_o	permittivity of free space
μ_o	permeability of free space
μ_r	relative permeability
ρ	volume charge density
σ	surface charge density, conductivity
ϕ	electric scalar potential
ω	angular frequency
Δ	eddy-current skin depth
Ψ	electric flux
<>	integration throughout a volume
[ ]	integration over a surface
$\underline{H}^*$	complex conjugate of $\underline{H}$

1. INTRODUCTION

If the quantity of magnetic flux linking a closed path is changing with respect to time, then an electromotive force will be generated around the closed path. This is in accordance with Faraday's law. A current will flow in the closed path if it is completely within a conducting medium. Currents of this type are called eddy-currents. In accordance with Lenz's law the eddy-currents oppose the change in magnetic flux linkage which causes them.

Eddy-currents may be beneficial or harmful, several situations are given below where eddy-currents are used for a specific purpose.

(a) When a conductor moves through a static magnetic field eddy-currents are induced in the conductor, these eddy-currents drive a magnetic reaction field. The magnetic reaction field interacts with the inducing field resulting in a force which opposes the motion of the conductor. This effect can be employed in an eddy-current brake.

(b) When a conductor is placed in an alternating magnetic field, eddy-currents are induced in the conductor, these eddy-currents cause a resistive power loss which can be used to heat the conductor. This effect can be employed in an induction furnace.

(c) When a conductor is placed in an alternating magnetic field, eddy-currents are induced in the conductor, these eddy-currents cause a magnetic reaction field which can be measured. Measurement of the reaction field can determine the geometry of the conductor (eg. crack detection) or the conductivity of the conductor.

The examples above show the beneficial use of eddy-currents. Two examples are given below where eddy-currents are harmful.

(a) When an alternating current is flowing through a conductor it produces an alternating magnetic field inside the conductor, this alternating magnetic field produces eddy-currents inside the conductor. This results in a skin effect where the current, due to its own magnetic field, tends to flow near the surface of the conductor. This is harmful because it increases the power loss in the conductor.

(b) When an alternating magnetic flux is passing through an iron core (as in a transformer), eddy-currents will flow in the core, causing losses. These losses may be reduced by using a laminated core.

The examples given above show the beneficial and harmful effects of eddy-currents. Eddy-currents exist in a wide variety of electrical equipment (e.g. induction motors, wave guides, transformer tanks). The accurate calculation of eddy-currents is important in order to determine; the localised heat loss, magnetic reaction field and forces resulting from the inducing and reaction field.

In order to calculate the distribution of eddy-currents in a conductor the diffusion equation (equation 1.14) must be applied to the problem. The accurate, analytical solution of this equation is only possible for problems with a simple geometry. For problems with a more complex geometry an approximate solution of the diffusion equation must be obtained by numerical methods. The accuracy of a numerical method depends on the efficiency of the method and the size of the computer used.

The aim of this thesis is to investigate improved methods for the calculation of eddy-currents in electrical power apparatus. In particular a method based on the study of energy [1,2], is considered. Computational work has been carried out to evaluate this approach to eddy-current problems. The formulation of the above method is considered as is the formulation of methods described by Ferrari [3], Penman and Fraser [4] and Rutherford Laboratory [5].

The application of eddy-current calculation to a particular problem is described in Chapter 7. The reaction field of eddy-currents is used to measure the conductivity of a metal object, hence its temperature. The experimental results obtained are compared with an analytical solution to the problem.

1.1 Time Varying Currents and Fields in Conductors

The analysis given below follows the analysis presented by Hammond [6]. How does a current distribute itself throughout the volume of a conductor? This question represents the starting point in any study of eddy-currents. To begin to answer this question the behaviour of electric charge in conductors is considered. The equation of continuity of charge and current is illustrated by Fig. 1.1. The equation is:

$$\sum_{i=1}^{n} I_i = -\frac{\partial Q}{\partial t} \qquad (1.1)$$

In terms of current and charge density

$$\oint_s \underline{J} \cdot ds = -\int_v \frac{\partial \rho}{\partial t} dv. \qquad (1.2)$$

at a point equation (1.2) may be written as below:

$$\text{div } \underline{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (1.3)$$

From Gauss's theorem we have

$$\oint \underline{D} \, .ds = Q \qquad (1.4)$$

which at a point may be written

$$\text{div } \underline{D} = \rho \qquad (1.5)$$

Using Ohm's law $J = \sigma E$ and the constitutive relation $D = \epsilon_0 E$ then equations (1.3) and (1.5) may be combined to give:

$$\frac{\sigma \rho}{\epsilon_0} + \frac{\partial \rho}{\partial t} = 0 \qquad (1.6)$$

this differential equation has the solution:

$$\rho = \rho_0 e^{-\left[\frac{\sigma}{\epsilon_0}\right] t} \qquad (1.7)$$

A typical conductivity for a metal is 10^7 siemens and ϵ_0 is the order of 10^{-11}. hence the exponential time constant is of the order 10^{-18} sec. Hence. from equations (1.3) and (1.7) we can see that because the charge diffuses very rapidly to the conductor surface

$$\text{div } \underline{J} = 0 \tag{1.8}$$

This argument that div J = 0 is preparatory work for the discussion of eddy-current behaviour, the argument from this point will aim directly at the mathematical formulation for describing the behaviour of eddy-currents.

Consider the full m.m.f. equations

$$\oint \underline{H} \cdot dl = I + \frac{d\Psi}{\partial t} \tag{1.9}$$

which for a point relationship becomes:

$$\text{Curl } \underline{H} = \underline{J} + \frac{\partial \underline{D}}{\partial t} \tag{1.10}$$

At an angular frequency ω the magnitude of the two terms J and $\partial D/\partial t$ have vastly different orders of magnitude (at power frequencies)

$$\frac{\underline{J}}{\frac{\partial \underline{D}}{\partial t}} = \frac{\sigma}{\omega \epsilon_0} \triangleq 10^{16} \text{ at power frequencies}$$

We can, therefore, adopt the simpler magnetostatic expression:

$$\oint \underline{H} \cdot dl = I \qquad (1.11)$$

which at a point becomes

$$\text{Curl}\underline{H} = \underline{J} \qquad (1.12)$$

this is consistent with equation (1.8) since the divergence of a curl is zero. When this equation is combined with Faraday's law

$$\text{Curl } \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Ohm's Law $J = \sigma E$ and the constitutive relation $\underline{B} = \mu_0 \mu_r \underline{H}$ it is possible to eliminate E, H and B to obtain:

$$\text{curl curl}\underline{J} = -\sigma \mu_0 \mu_r \frac{\partial \underline{J}}{\partial t} \qquad (1.13)$$

Using the vector identity curl curl$\underline{J}$ = grad div $\underline{J}$ - $\nabla^2 \underline{J}$ and noting that div$\underline{J}$ = 0.

$$\text{we obtain } \nabla^2 \underline{J} = \sigma \mu_0 \mu_r \frac{\partial \underline{J}}{\partial t} \qquad (1.14)$$

Similarly it can be shown that the form of the equation is the same if instead of J we write E, H or B. The equation is called the diffusion equation (This is because the diffusion of heat through matter is governed by the same form of equation).

1.2 Present Approaches to the Solution of Eddy-Current Problems

The requirement to predict eddy-current behaviour has existed for a long time. The various approaches which have been adopted are listed below.

Analytical methods

These methods form the foundation of the mathematical study of eddy-currents. Their use however, depends on the geometry of the problem being relatively simple and the existence of a suitable co-ordinate system. Stoll [7] discusses the analytical solution of several problems such as; eddy-currents in a long rectangular bar. and current distributions in rectangular and T-shaped conductors in slots. Tegopoulos and Kriezis [8] present an extensive collection of analytical results.

Finite difference methods

Finite difference schemes replace the original differential equation by a finite difference equation relating the value of the field component at a point to the value of neighbouring points in space. This results in a set of simultaneous equations which must be solved. The use of finite difference methods in eddy current problems is demonstrated by Stoll [7].

Finite element method

The starting point of this method is the minimisation of the variation integral, or functional, for the solution of a partial differential equation with given boundary conditions. The variation integral is the integral of a function, normally the local energy content. The first variation of the variation integral is zero about the correct solution to the problem. The evaluation and minimisation of the variation integral are carried out by splitting the air, conductor and iron regions into finite elements. This results in a set of simultaneous equations which must be solved. Trial or shape functions can be used as in the Galerkin method, the minimisation is then carried out with respect to the coefficients of these weighting functions. One of the main advantages of finite element methods is that the subdivision of the region can be graded to give small elements where it is anticipated that the field is varying rapidly, and larger elements elsewhere. A considerable amount of finite element literature exists and an introduction is provided by Silvester and Ferrari [9].

1.3 Links Between the Study of Mechanics and the Study of Electromagnetism

Force and Energy are defined by Newton's study of mass, length and time. Mechanical energy and heat were shown to be equivalent by Joule. The definition of the volt as "The potential difference between two points such that one joule of work is done if a coulomb of charge is moved from one point to the other" and the ampère as "That constant current which if maintained in two straight parallel conductors, of infinite length and negligible circular cross-section and placed one metre apart in a vacuum, would produce in each of them a force of 2×10^{-7} newtons per metre length" show how the study of electricity relies on the ideas of force and energy.

Hence, it can be seen that our inheritance in the physical sciences is one where force and energy are cornerstones of the entire building. It is not surprising, therefore, that a study of electromagnetism which centres on energy will lead to useful results.

In mechanics the idea that the energy of a system can determine its equilibrium or motion is embodied in analytical mechanics (Chapter 2). This approach involving the principles and mathematics of variations can be applied to the study of electromagnetism. This is the approach which is investigated in this thesis and one aspect of this method in particular is considered - the method of dual bounds.

1.4 The Advantages of a Dual Bound Method

What advantage is to be gained by using a dual bound method? This question must be answered by defining what is meant by a bounded solution and then explaining the use of a dual bound system.

1.4.1 Bounded solution

Any numerical method (eg. finite difference, finite element) provides an approximate solution for local parameters (eg. potential, field) and global parameters (eg. inductance, resistance). A typical method is the Rayleigh-Ritz method which is described below.

When considering the stationary value of the integral

$$I = \int_a^b f(x, y_1, y_2 \ldots y_m)dx \qquad (1.15)$$

If the values of y are expressed as below:

$$y = \sum_{r=1}^{n} a_r F_r(x) \qquad (1.16)$$

then the value of I is stationary if

$$\frac{\partial I}{\partial a_r} = 0 \qquad (r = 1, 2, \ldots n) \qquad (1.17)$$

The application of equation (1.17) results in a set of simultaneous linear equations which must be solved. The accuracy of the approximate solution depends mainly upon the number of indepedent variables and hence the number of simultaneous equations which must be solved. Ideally an infinite number of independent variables would be preferred, however, this would produce an infinite number of simultaneous linear equations which must be solved. The goal is to obtain a solution of known accuracy with the minimum number of independent variables.

A bounded solution is one in which the approximate solution is always geater than, or always less than the actual solution. If two bounded solutions, one which is known to be greater than the actual solution and one which is known to be less than the actual solution, can be found then it is possible to state with certainty that the actual solution lies between two error limits. The use of bounded solutions in analytical mechanics is shown in Chapter 2 and the use of bounded solutions in the study of electromagnetism is discussed in Chapters 3 to 6.

1.4.2 Dual bound methods

A method of solution which provides both an upper and a lower bound for a parameter by alternative formulation of the problem being considered is termed a dual bound method.

1.5 Dual Bounds and Simplified Solution Techniques

The idea of dual bound calculations are used in some of the simplest and most elegant solution techniques. An example is given here which is due to Maxwell [10].

Consider a resistive sheet of uniform thickness as shown in Fig. 1.2(a).

At the left hand side the electrode voltage is v_1 at the right hand side the electrode voltage is 0, the resistance of the sheet is R. If infinitely thin insulating strips are placed in the sheet as shown in Fig. 1.2(b) then the conducting path is divided into tubes. Unless the insulating strips are placed along original electric field lines the resistance of the sheet will increase giving upper bound R_+ (If the insulating strips are placed along original electric field lines the resistance will be unchanged). If infinitely thin infinitely conducting strips are placed in the sheet as shown in Fig. 1.2(c) then the conducting path is sliced. Unless the conducting strips are placed along the original equipotentials the resistance of the sheet will decrease giving a lower bound R_- (If the conducting strips are placed along the original equipotential lines the resistance will be unchanged).

Hence if a sheet of unknown resistance can be divided into tubes and slices of known resistance then R_- and R_- (dual bounds) can be found.

Similar schemes are given by Hammond et. al. [11] for electrostatic systems and Hammond and Zhan [12] for magnetostatic systems.

2. ANALYTICAL MECHANICS

Ever since the time of Sir Isaac Newton (1642-1727) the science of mechanics has developed along two main lines. Following the terminology of Lanczos [13] these two lines may be called vectorial mechanics and analytical mechanics.

Vectorial mechanics starts directly from Newton's laws of motion. It aims at recognising all the forces which are acting on any given particle. The motion of the particle is then uniquely determined by the known forces acting on it at any instant.

On the other hand analytical mechanics bases the entire study of equilibrium and motion on two fundamental scalar quantities the kinetic energy and the work function, the latter frequently replaceble by the potential energy.

A full account of analytical mechanics is given by Lanczos [13]. This chapter is devoted to those aspects of analytical mechanics which will be developed later for the study of eddy-current systems. By way of introduction to the subject the principle of virtual work is discussed and then the principle of least action is presented by means of an example.

2.1 Virtual Work

The first variational principle encountered in mechanics is the principle of virtual work. It controls the equilibrium of a mechanical system and is fundamental for the later development of analytical mechanics.

When considering a static mechanical system by vectorial mechanics it is usual to say that a particle is in equilibrium if the resulting force acting on the particle is zero.

When considering the same system using analytical mechanics the following procedure is used. Given the external forces F_1, F_2 .... F_n act at points P_1, P_2 ... P_n of the system. The virtual displacements of these points will be denoted by δR_1, δR_2 ... δR_n. These arbitrary, infinitesimal, virtual displacements must obey the given constraints. The principle of virtual work asserts that:- The given mechanical system will be in equilibrium if and only if the total virtual work of all the impressed forces vanishes. The total virtual work is $\delta\omega$ and the principle of virtual work is expressed by equation (2.1).

$$\delta\omega = F_1\delta R_1 + F_2\delta R_2 \ldots + F_n\delta R_n \qquad (2.1)$$

In order to apply this result to dynamics use is made of D'Alembert's principle. This replaces Newton's second law.

$$F = ma$$

by $$F - ma = 0$$

where -ma is called the force of inertia. This enables dynamics to be treated in the same manner as statics. Equation (2.1) can now be enlarged as shown below:

$$\delta\omega = \Sigma F_i \delta R_i - \Sigma m_i a_i \delta R_i$$

where F_i represents the given external forces and δR_i the corresponding virtual displacements.

hence $$\delta\omega = \Sigma[(F_i - m_i a_i) . \delta R_i] \qquad (2.2)$$

and at equilibrium $\delta\omega = 0$.

Further development of this argument [18] shows that D'Alembert's principle leads to Hamilton's principle which is discussed in the next section.

2.2 Principle of Least Action

Consider an uncharged particle moving in three dimensional space near the earth's surface. Using analytical mechanics it is possible to determine the motion of this particle by considering its kinetic energy and its potential energy.

Consider a particle at a point P_1 and time t_1 with known velocity at that time. Some time in the future the particle is at a point P_2. If a tentative path is assigned for the particle between P_1 and P_2 then conservation of energy will uniquely determine the motion along that path.

The time integral of the kinetic energy, extended over the entire motion from P_1 to P_2, can be calculated. This time integral multiplied by 2 is called action. Different tentative paths can be tried and each one will have a value of action associated with it. There must exist one definite path for which the action assumes a minimum value. The principle of least action asserts that this particular path is the actual path of motion.

This principle can clearly be used to obtain a bounded solution to the problem described above. If an approximate solution is obtained whose path is different from the actual path of motion, then the action associated with this path will be greater than the action associated with the actual path of the particle. Hence if any approximate path is chosen and its action calculated, this value of the action will provide an upper bound for the actual value of the action.

The principle of least action may be extended by use of a modified action integral to include systems in which the law of conservation of energy does not apply. This important extension is known as Hamilton's principle. The modified action used in Hamilton's principle is the time-integral of the difference between the kinetic and potential energies.

2.3 The Fundamental Process of the Calculus of Variations

The mathematical problem of minimising an integral was considered by Lagrange who developed a special branch of calculus called "calculus of variations" in order to solve the problem. The calculus of variations investigates the change in the value of an integral caused by infinitesimal variations, as is explained below:-

Consider an integral of the type:-

$$I = \int_a^b F(y, y', x)\,dx \qquad (\text{where } y' = \frac{dy}{dx})$$

A solution of the form $y = F(x)$ is required which will give a stationary value for I. In order to prove the existence of a stationary value the integral must be evaluated for a slightly modified function $y = \overline{F}(x)$, and it must be shown that the rate of change of the integral due to the change in the function becomes zero.

let $\overline{F}(x) = F(x) + \epsilon\,\phi(x)$

where $\phi(x)$ is some arbitrary new function (which must be continuous and differentiable)

ϵ is used to modify $F(x)$ by arbitrarily small amounts, for this purpose ϵ tends to zero.

The values of the modified function $\overline{F}(x)$ and the original functions $F(x)$ can be compared at a certain definite point x. The difference between $\overline{F}(x)$ and $F(x)$ is called the "variation" of the function $F(x)$ and is denoted by δy

$$\delta y = \overline{F}(x) - F(x) = \epsilon\phi(x)$$

The variation of a function is characterised by two fundamental features. It is an infinitesimal change, since the parameter ϵ descreases towards zero. It is also a virtual change, this means that it may be made in any arbitrary manner.

It is in the nature of the process of variation that only the dependent function y should be varied while the variation of x serves no useful purpose. Hence it is agreed that $\delta x = 0$.

The two limiting ordinates F(a) and F(b) are given and can not be varied, hence:

$$[\delta F(x)]_{x=a} = 0$$

$$[\delta F(x)]_{x=b} = 0$$

When the above conditions are met the variation may be described as a variation between limits.

If, in equation (2.1), y is replaced by $y - \delta y$ then $F(y,y',x)$ will be replaced by $F(y-\delta y, (y-\delta y)',x)$. The evaluation of F will contain the following terms.

$$F_1(y,y',x) + F_2(y,y',x)\delta y + F_3(y,y',x)(\delta y)^2$$

+ higher order terms.

In this case the first variation of F which is termed δF will be $F_2(y,y',x)\delta y$.

The second variation of F which is termed $\delta^2 F$ will be $F_3(y,y'x)\ (\delta y)^2$.

It can be shown that

$$\delta I = \int_a^b \delta F\,dx \text{ and } \delta^2 I = \int_a^b \delta^2 F dx.$$

These two terms δI and $\delta^2 I$ provide vital information when it is necessary to find a minimum or maximum value of I. In order for the value of I to be stationary the condition $\delta I = 0$ must be satisfied. In order for the stationary value of I to correspond to a minimum $\delta^2 I > 0$ must be satisfied. In order for the stationary value of I to correspond to a maximum $\delta^2 I < 0$ must be satisfied.

2.4 Euler-Lagrange Equation

When problems of motion are formulated in terms of analytical mechanics an integral of the type given below is often required to be minimised

$$I_1 = \int_{t_1}^{t_2} L(q,\dot{q},t)dt \qquad (2.3)$$

where $q = q(t)$, $\dot{q} = \frac{\partial q}{\partial t}$

L is the Lagrangian function and is generally some function of position and velocity in configuration space, and the time t (configuration space is described in ref. 14).

Using the calculus of variations Lanczos [15] shows that a stationary value of I_1 is obtained if the following differential equation is satisfied.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \qquad (2.4)$$

This equation is usually called the Euler-Lagrange differential equation.

Lanczos also shows that the problem of minimising the integral given below:

$$I_2 = \int_{t_1}^{t_2} L(q_1 \ldots\ldots q_n; \dot{q}_1, t)dt \qquad (2.5)$$

is equivalent to solving the system of simultaneous equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (i = 1, 2, \ldots n) \qquad (2.6)$$

2.5 Canonical Equations of Hamilton

The Lagrangian function L is given below

$$L = L\ (q_1 \ldots q_n\ ;\ \dot{q}_1 \ldots \dot{q}_n;\ t) \qquad (2.7)$$

This function may be transformed by the introduction of new variables p_i

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (2.8)$$

where p_i represents momentum in configuration space.

The Hamiltonian function H is defined below

$$H = \sum_{i=1}^{n} p_i\dot{q}_i - L \qquad (2.9)$$

By using equation (2.8) to express $\dot{q}_i$ in terms of p_i H may then be written:

$$H = H(q_1 \ldots q_n;\ p_1 \ldots p_n;\ t)$$

The Lagrangian function and the Hamiltonian function form the two systems of equations given below, which can be identified as Legendre's dual transformation [16].

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad (2.8)$$

$$H = \sum p_i \dot{q}_i - L \qquad (2.10)$$

$$H = H(q_1 \ldots q_n;\ p_1 \ldots p_n;\ t) \qquad (2.11)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (2.12)$$

$$L = \sum p_i \dot{q}_i - H \qquad (2.13)$$

$$L = L(q_i \ldots q_n;\ \dot{q}_1 \ldots \dot{q}_n;\ t) \qquad (2.14)$$

Using the results of Legendre's dual transformation the following equations can be obtained.

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i} \qquad (2.15)$$

and

$$\frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial t} \qquad (2.16)$$

equations (2.8), (2.12), (2.15) and (2.16) lead to the canonical equations of Hamilton:

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (2.17)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad (2.18)$$

These equations are entirely equivalent to the original Euler-Lagrange equations.

2.6 Duality

Hammond [17] introduces an original extension to the study of analytical mechanics, this extension allows problems to be formulated in a manner which gives upper and lower bounds to the correct solution.

Consider a new Lagrangian function.

$$L' = L' \, (p_1 \ldots p_n; \, \dot{p}_1 \ldots \dot{p}_n; \, t) \qquad (2.19)$$

Then similar treatment to that presented above leads to

$$L' = \sum_{i=1}^{n} (-q_i \dot{p}_i) - H. \qquad (2.20)$$

from which it can be seen that

$$L' = -\sum (q_i \dot{p}_i - p\dot{q}_i) + L \qquad (2.21)$$

this leads to

$$I_3 = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} L'\,dt + \Big[\Sigma p_i q_i\Big]_{t_1}^{t_2} \qquad (2.22)$$

L' is said to be the dual of L

It is noted that the difference between L and L' must be integrable, the minus sign in equation (2.20) facilitates this and the operators d/dt and -d/dt are said to be adjoint.

Hammond [17] goes on to show that at the correct solution $\delta I_3 = 0$ when L' is substituted for L, and that substitution of L' for L changes the sign of the second variation. Hence if an approximate solution for L leads to an upper bound for I_3, and approximate solution for L' will lead to a lower bound for I_3.

3. STATIC FIELDS

The aim of this thesis is to use the methods and results of analytical mechanics in order to find an improved method for calculating global parameters in eddy-current systems. Analytical mechanics involves the integration of a function over a time interval. In seeking to apply the method to eddy-current systems distributed in space it is necessary to integrate through a specified volume. In order to see how the methods and results of analytical mechanics have been applied to distributed field problems static fields will be considered first. This discussion of static fields is based on work by Hammond [1], Hammond and Penman [19], Hammond et. al.. [11] and Hammond and Zhan [12].

3.1 Electrostatics

In an electrostatic system the field quantities are the charge density ρ, the electric flux density D, the electric field strength E and the scalar potential ϕ. These quantities are related by the two equations given below:

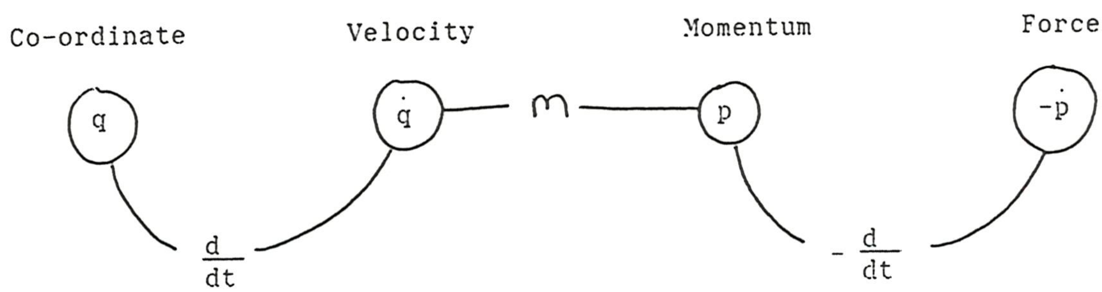
$$\underline{\nabla} \cdot \underline{D} = \rho \qquad (3.1)$$

and

$$-\underline{\nabla}\phi = \underline{E} \qquad (3.2)$$

equation (3.2) incorporates the statement that curl $\underline{E} = 0$, the constitutive field equation is $\underline{D} = \epsilon\underline{E}$.

In Chapter 2 it was shown that two systems, one a function of co-ordinates and velocity, the other a function of momentum and its time derivative, could be expected to have opposite signs for their second variations. This leads to dual bounds when used as the basis for a calculation scheme. The relationship between the two systems is shown below:



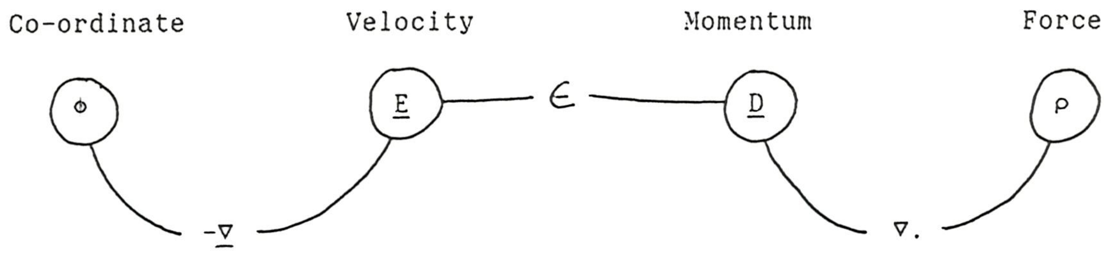
The two systems are L and L$'$ (section 2.6). An important feature of the two systems is their adjointness. The operators d/dt and -d/dt are said to be adjoint this is because

$$\frac{d}{dt} pq = p \frac{dq}{dt} + q \frac{dp}{dt}$$

This allows the difference between the two systems to be integrated (see section 2.6).

To fit electrostatics into this framework the field quantities ρ.D.E and ϕ must be described as generalised co-ordinate, generalised velocity, generalised momentum and generalised force. The operators between the co-ordinate and velocity and between the momentum and force must be adjoint.

The system adopted which is given below is applicable for non-relativistic three dimensional space.



The operators $-\underline{\nabla}$ and $\underline{\nabla}.$ are adjoint because

$$\underline{\nabla}\,(a\underline{A}) = a\,\underline{\nabla}\,.\,\underline{A} + \underline{A}\,.\,\underline{\nabla}a \qquad (3.3)$$

and this allows the difference between the two systems L and L$'$ to be integrated.

In analytical mechanics the quantities which characterise the system are the kinetic energy and the potential energy. In electrostatics the quantities which characterise the system are the field energy ($E.\underline{D}/2$ integrated throughout the system space) and the assembly work ($\rho\phi/2$ integrated throughout the system space).

3.1.1 System equilibrium

In analytical mechanics equilibrium is described by the principle of virtual work. In an electrostatic system a variational principle corresponding to the principle of virtual work can be written as

$$\langle(\rho - \mathrm{div}\underline{D}),\ \delta\phi\rangle = 0 \qquad (3.4)$$

Where the brackets <> indicate integration through the region of interest. Equation (3.4) states that an equilibrium condition is reached where $\rho = \mathrm{div}\underline{D}$ for a particular potential distribution, the condition that $\underline{E} = -\nabla o$ is implied. The equation can be integrated and expressed as the variation of a scalar energy functional $W(o)$. If the charge density ρ is given an assigned distribution $\bar{\rho}$ and the surface charge density $\bar{\sigma}$ is assigned, then the scalar energy functional $W_1(o)$ is given by equation (3.5).

$$\delta W_1(o) = \delta\left[\langle\bar{\rho}, o\rangle + [\bar{\sigma}, o] - \frac{1}{2}\langle\epsilon|\underline{\nabla}o|^2\rangle\right] = 0 \qquad (3.5)$$

The dual variational principle is obtained by varying the flux density $\underline{D}$. This is done by the introduction of curl sources this allows the condition $\rho = \mathrm{Div}\ \underline{D}$ to be maintained.

$$\langle(\underline{\nabla}o + \underline{E}),\ \delta\underline{D}\rangle = 0 \qquad (3.6)$$

Equation (3.6) can be integrated and expressed as the variation of a scalar energy functional $W(\underline{D})$. If the same specification of $\bar{\rho}$ and $\bar{\sigma}$ are used as in equation (3.5), then the scalar energy functional $W_1(\underline{D})$ is given by equation (3.7)

$$\delta\ W_1(\underline{D}) = \delta\left[\frac{1}{2}\left\langle\frac{|\underline{D}|^2}{\epsilon}\right\rangle\right] = 0 \qquad (3.7)$$

It can be seen that the second variation of $W_1(o)$ is negative and the second variation of $W_1(\underline{D})$ is positive. Hence, the dual system provides a second variation of opposite sign to the second variation of the original system.

An alternative system specification is to specify the potential $\circ$ where $\rho \neq 0$ and $\sigma \neq 0$. This alternative specification can be used with the equilibrium statement for the original system (equation 3.4) to obtain $W_2(\circ)$. The alternative specification can also be used with the equilibrium statement for the dual system (equation 3.6) to obtain $W_2(D)$.$\delta W_2(\circ)$ and $\delta W_2(D)$ are given below:-

$$\delta W_2(\circ) = \delta\left[\frac{1}{2}\langle\epsilon|\underline{\nabla}\circ|^2\rangle\right] = 0 \qquad (3.8)$$

$$\delta W_2(D) = \delta\left[<\rho,\bar{\circ}> + [\sigma\ \bar{\phi}] - \frac{1}{2}\left\langle\frac{|\underline{D}|^2}{\epsilon}\right\rangle\right] = 0 \qquad (3.9)$$

Hence it can be seen that the dual system provides a second variation of opposite sign to the second variation of the original system.

3.1.2 Physical features

In the preceeding sub-section we found two equations describing the equilibrium of an electrostatic system. The use of two different system specifications for each of the equilibrium equations has given rise to four variational principles.

Consider first the equilibrium equation (3.4). This equation allows the divergence of the field in the volume to be varied. However, if the system specification used for equation (3.5) is examined it is noticed that ϕ is free to vary at the surface, this causes a potential step at the surface corresponding to a double layer or curl source at the surface. If the alternative surface specification ($\phi = \bar{\phi}$ where $\rho \neq 0$, $\sigma \neq 0$) is used there is no curl source at the surface.

Now consider the equilibrium equation (3.6). This equation allows the curl of the field in the volume to be varied. If the system specification used for equation (3.5) is applied then only the curl of the field is varied. However, if the surface specification ($\phi = \bar{\phi}$ where $\rho \neq 0$, $\sigma \neq 0$) is used then both curl and divergence sources are allowed to vary.

It can be seen from the above discussion that when both the curl and divergence sources are allowed to vary this results in a variational principle with maximum energy at equilibrium. When only one type of source is allowed to vary this results in a variational principle with minimum energy at equilibrium. This is because the variation of both curl and divergence sources occurs when the first variation of the assembly work is non-zero.

3.1.2. Practical calculation schemes

Hammond et. al. [11] explains how the above analysis may be used to obtain dual bounds for the capacitance of a system. They show that any potential map produces an upper bound of capacitance and any flux map produces a lower bound. This dual bound behaviour is then exploited in an efficient calculation scheme for calculating the capacitance of a system.

3.2 Magnetostatics

The magnetostatic field can be treated in a similar manner to the electrostatic field. The field quantities are the current density J, the magnetic field strength H, and the magnetic vector potential A. These quantities are related by the equations given below.

$$\underline{\nabla} \times \underline{H} = \underline{J} \tag{3.10}$$

and $$\underline{\nabla} \cdot \underline{B} = 0 \tag{3.11}$$

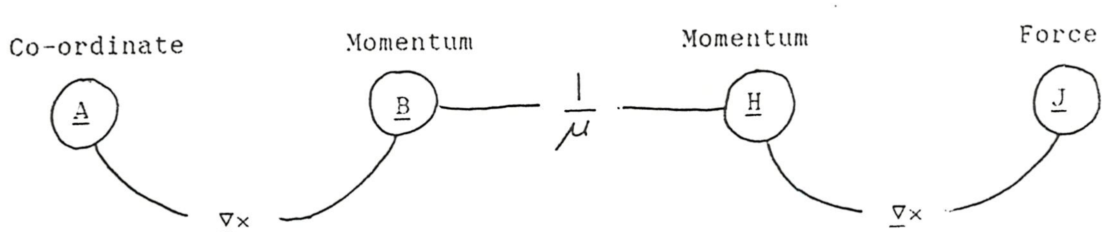
This equation allows the use of a vector potential where

$$\underline{B} = \underline{\nabla} \times \underline{A}$$

The constitutive field equation is:

$$\underline{B} = \mu \underline{H} \tag{3.12}$$

These equations fit into the framework described in section 3.1 in the following manner.



The operator $\underline{\nabla}\times$ is self-adjoint because:-

$$\nabla \cdot (\underline{F}\times\underline{G}) = \underline{G} \cdot (\underline{\nabla}\times\underline{F}) - \underline{F} \cdot (\underline{\nabla} \times \underline{G})$$

This allows the difference between the two systems L and L$'$ to be integrated.

In a magnetostatic system a variational principle corresponding to a principle of virtual work can be written as:

$$\langle (\underline{J} - \mathrm{curl}\underline{H}),\ \underline{\delta}A \rangle = 0 \qquad (3.13)$$

The dual variational principle is obtained by varying the magnetic field strength $\underline{H}$.

$$\langle (\underline{B} - \mathrm{curl}\underline{A}),\ \underline{\delta}H \rangle = 0. \qquad (3.14)$$

Examination of equations (3.13) and (3.14) leads to a bounded solution technique for calculating the inductance of a magnetostatic system [12].

4. EDDY-CURRENT CALCULATION BY OTHER AUTHORS

In Chapter 3 of this thesis the analogy between analytical mechanics and static electric and magnetic fields has been examined. This has led to the development of an efficient calculation scheme for global parameters in statics fields. The calculation scheme establishes upper and lower bounds for the global parameters. This enables error bounds to be stated for the correct solution.

It is expected that this analogy can be extended to systems containing eddy-currents, leading to an efficient calculation scheme for global parameters in eddy-current systems. The study of eddy-current systems by this method is discussed in Chapter 5.

The problem of obtaining approximate solutions for eddy-current systems has been discussed by many other authors. In this chapter the work of three of these authors is considered. This is done for the following reasons. Firstly to determine what has already been achieved. Secondly to investigate the differences between the methods used by these authors and the method based on the analogy between analytical mechanics and eddy-current systems.

Work by Ferrari [3] is discussed in section 4.1. Work by Penman and Fraser [4] is discussed in section 4.2. Work carried out at the Rutherford Appleton Laboratories [5] is discussed in section 4.3.

4.1 Complementary Variational Formulation for Eddy-Current Problems Using the Field Variables E and H Directly

This section investigates work done by Ferrari [3]. Ferrari introduces a variational analysis using H or E as the variable. Two functionals are introduced for an eddy-current system, one a function of H the other a function of E. The first variation of each of these functionals is said to be zero at the correct solution to the problem, provided that the correct boundary conditions have been applied. Ferrari describes the two functionals as complementary. However, he states that no attempt has been made to establish that the complementary approaches described would yield upper and lower bounds for any solution parameter.

Ferrari's work is of interest to the present author because the functionals he uses are similar (though different in one vital aspect) to the functionals yielded by the analogy of eddy-current systems and analytical mechanics. (Chapter 5).

At low frequencies (using the pre-Maxwell equations) the two complementary functionals introduced by Ferrari are:

(a) $$F_H = \langle(\nabla\times\underline{H}), (\nabla\times\underline{H})\rangle + j\,\mu\omega\sigma\,\langle\underline{H}.\underline{H}\rangle \qquad (4.1)$$

(b) $$F_E = \langle(\nabla\times\underline{E}), (\nabla\times\underline{E})\rangle \frac{1}{j\mu\omega\sigma} + \langle\underline{E}.\underline{E}\rangle \qquad (4.2)$$

Where H and E are phasor representations of periodic sinusoidal functions.

The remainder of this section is devoted to a study of these functionals, in particular the functional F_H since the inherent symmetry of the two functionals F_H and F_E can be seen. The investigation of F_H will try to answer the following questions: What is the physical significance of the function? What is the first variation of the functional? What is the second variation of the functional?

4.1.1 Physical significance of the functional

Ferrari does not discuss the physical significance of the functional F_H. He is satisfied with calculating the first variation of F_H and considering if the functional is stationary at the correct solution to the problem. The present author considers that this approach discards some information which may be useful when considering the worth of a particular functional. The reason for this is explained below.

The functionals which have been developed for mechanics, electrostatics and magnetostatics have all been closely related to the energy of the system. In the frequency domain this would lead one to expect a functional which was closely related to the power of the system. Hence it is expected that in a sinusoidal eddy-current system the functional used would represent the real and reactive power used by the system. The functional F_H does not represent the real and reactive power in an eddy-current system. This is because when using a phasor representation of fields in a distributed system, or of currents and voltages in a circuit, the real and reactive power can only be calculated by use of complex conjugate quantities. This is explained by Yorke [20]. For this reason the present author cannot be entirely satisfied with the functional F_H.

4.1.2 The first variation of the functional

Despite the reservations given above, the functional F_H will provide a method for the approximate solution of eddy-current problems if its first variation is zero about the correct solution to the problem. The first variation of F_H is calculated below:

$$F_H = <(\nabla\times\underline{H}),\ (\nabla\times\underline{H})> + j\mu\omega\sigma <\underline{H}.\underline{H}> \qquad (4.1)$$

The phasor H may be written $\underline{H} = \underline{H}' + j\underline{H}''$, and the functional F_H may be expressed in terms of the components of $\underline{H}$.

$$F_H = <(\nabla\times\underline{H}'),(\nabla\times\underline{H}')> - <(\nabla\times\underline{H}'')(\nabla\times\underline{H}'')> + 2j<(\nabla\times\underline{H}')(\nabla\times\underline{H}'')>$$

$$+ j\mu\omega\sigma\ (<\underline{H}',\underline{H}'> - <\underline{H}'',\underline{H}''> + 2j<\underline{H}',\underline{H}''>) \qquad (4.3)$$

The first variation F_H is

$$\delta F_H = 2<(\nabla\times\underline{H}')\delta(\nabla\times\underline{H}')>-2<(\nabla\times\underline{H}'')\delta(\nabla\times\underline{H}'')>$$

$$+2j<\nabla\times\underline{H}')\delta(\nabla\times\underline{H}'')> +2j<(\nabla\times\underline{H}'')\delta(\nabla\times\underline{H}')>$$

$$+j\mu\omega\sigma(2<\underline{H}',\delta\underline{H}'>-2<\underline{H}'',\delta\underline{H}''>+2j<\underline{H}''\delta\underline{H}'>+2j<\underline{H}'\delta\underline{H}''>) \qquad (4.4)$$

Using the vector identity $\nabla\ .\ (\underline{F}\times\underline{G}) = \underline{G}\ .\ \underline{\nabla}\times\underline{F} - \underline{F}\ .\ \nabla\times\underline{G}$ and Gauss's theorem some of the terms from equation (4.4) may be rewritten as shown below:

$$<(\nabla\times\underline{H}').\delta(\nabla\times\underline{H}')> = <\nabla\times\nabla\times\underline{H}'.\delta\underline{H}'>-[(\nabla\times\underline{H}')\times\delta\underline{H}'.\hat{n}] \qquad (4.5)$$

$$<(\nabla\times\underline{H}'').\delta(\nabla\times\underline{H}'')>=<\nabla\times\nabla\times\underline{H}''.\delta\underline{H}''>-[(\nabla\times\underline{H}'')\times\delta\underline{H}''.\hat{n}] \qquad (4.6)$$

$$<(\nabla\times\underline{H}').\delta(\nabla\times\underline{H}'')>=<\nabla\times\nabla\times\underline{H}'.\delta\underline{H}''>-[(\nabla\times\underline{H}')\times\delta\underline{H}''.\hat{n}] \qquad (4.7)$$

$$<(\nabla\times\underline{H}'').\delta(\nabla\times\underline{H}')>=<\nabla\times\nabla\times\underline{H}''.\delta\underline{H}'>-[(\nabla\times\underline{H}'')\times\delta\underline{H}'.\hat{n}] \qquad (4.8)$$

If the surface values of H are held constant then surface terms in equation (4.5) to (4.8) are all zero. Hence, using equations (4.5) to (4.8) with specified surface values of H, equation (4.4) can be rewritten as shown below.

$$\delta F_H = 2<\nabla\times\nabla\times\underline{H}',\delta\underline{H}'>-2<\nabla\times\nabla\times\underline{H}'',\delta\underline{H}''>+2j<\nabla\times\nabla\times\underline{H}',\delta\underline{H}''>$$

$$+2j<\nabla\times\nabla\times\underline{H}'',\delta\underline{H}'>+j\mu\omega\sigma(2<\underline{H}',\delta\underline{H}'>-2<\underline{H}'',\delta\underline{H}''>$$

$$+2j<\underline{H}'',\delta\underline{H}'>+2j<\underline{H}',\delta\underline{H}''>) \qquad (4.9)$$

Equation (4.9) can be rearranged to give the coefficients of $\delta H'$ and $\delta H''$, it is these coefficients which determine whether the functional F_H is stationary about the correct solution to the eddy-current problem.

$$\delta F_H=2<\delta\underline{H}',(\nabla\times\nabla\times\underline{H}'-\mu\omega\sigma\underline{H}''+j(\nabla\times\nabla\times\underline{H}''+\mu\omega\sigma\underline{H}'))>$$

$$+2<\delta\underline{H}'',(-\nabla\times\nabla\times\underline{H}''-\mu\omega\sigma\underline{H}'+j(\nabla\times\nabla\times\underline{H}'-\mu\omega\sigma\underline{H}''))> \qquad (4.10)$$

The functional F_H has a stationary value at the correct solution of the eddy-current problem if the coefficients of $\delta H'$ and

$\delta H''$, in equation (4.10), are zero when the diffusion equation (the mathematical description of the behaviour of eddy-current systems) is satisfied. The phasor component solution of the diffusion equation is:

$$\nabla \times \nabla \times \underline{H}' - \mu\omega\sigma\underline{H}'' = 0 \qquad (4.11)$$

and

$$\nabla \times \nabla \times \underline{H}'' + \mu\omega\sigma\underline{H}' = 0 \qquad (4.12)$$

The application of equations (4.11) and (4.12) to the first variation of the functional F_H (equation 4.10) leads to the result that the coefficient of $\delta H'$ and the coefficient of $\delta H''$ are both zero. This result means that the functional F_H has a stationary value at the correct solution to the eddy-current problem. (F_H is complex and both the real and imaginary components of F_H have stationary values at the correct solution.) Hence the functional F_H will provide a method for the approximate solution of eddy-current problems. This fact is exploited by Ferrari [3].

4.1.3 The second variation of the functional

The existence of a stationary value for the functional F_H at the correct solution to the eddy-current problem is not sufficient to lead to a bounded solution technique for F_H. This requires that the second variation of F_H be either positive definite or negative definite at the stationary value. The second variation of the functional F_H is:

$$\delta^2 F_H = <\delta(\nabla\times\underline{H}')\delta(\nabla\times\underline{H}')> - <\delta(\nabla\times\underline{H}'').\delta(\nabla\times\underline{H}'')> - 2\mu\omega\sigma<\delta\underline{H}'.\delta\underline{H}''>$$

$$-j\{\mu\omega\sigma(<\delta\underline{H}'.\delta\underline{H}'> - <\delta\underline{H}''.\delta\underline{H}''>) + 2<\delta(\nabla\times\underline{H}'').\delta(\nabla\times\underline{H}')>\} \quad (4.13)$$

Examination of equation (4.13) reveals that the second variation of F_H is neither positive definite nor negative definite.

4.2 Complementary Variational Formulation for Finite Element Calculation of Eddy-Current Problems

This section investigates work done by Penman and Fraser [4]. Penman and Fraser introduce a variational analysis using $\underline{A}$ or $\underline{H}$ as the variable. The treatment considers two dimensional systems (Penman and Fraser do not say whether their analysis is suitable for three dimensional systems) with sinusoidal excitation. Two functionals are introduced for an eddy-current system, one a function of $\underline{A}$, the other a functional of $\underline{H}$. The first variation of each of these functionals is said to be zero at the correct solution to the problem, provided that the correct boundary conditions have been applied. Penman and Fraser describe their two functionals as complementary.

Starting with the eddy-current equation expressed in terms of the magnetic vector potential

$$\nabla \times \nabla \times \underline{A} + j\omega\sigma\mu\underline{A} = \underline{J} \quad (4.14)$$

where $\underline{A} = 0\underline{i} + 0\underline{j} + (A'+jA'')\underline{k}$

and $\underline{J} = 0\underline{i} + 0\underline{j} + J\ \underline{k}$

An analysis of equation (4.14) is presented by Penman and Frser which leads to two functionals F_A and F_H

$$F_A = \frac{1}{2\mu} <\nabla\times A', \nabla\times A'> - \frac{1}{2\mu} <\nabla\times A'', \nabla\times A''> - <\omega\sigma A'', A'> - <J, A'>$$

$$F_H = -\frac{\mu}{2} <H', H'> + \frac{\mu}{2} <H'', H''> - \frac{1}{\omega\sigma} <\nabla\times H'', \nabla\times H'> + \frac{1}{\omega\sigma} <J, \nabla\times H''>$$

Penman and Fraser claim that by comparison with results for static fields [21] that the functionals F_A and F_H provide bounds to the exact solution.

The present author has considered the functionals F_A and F_H in a similar manner to the discussion of the functionals presented by Ferrari (section 4.1). The following questions have been asked:- What is the physical significance of the functionals? Is the first variation of the functionals equal to zero at the correct solution to the problem? What is the second variation of the functionals? The answers to these questions are discussed below.

Penman and Fraser do not discuss the physical significance of the functionals F_A and F_H. The functionals do not represent complex power (sub-section 4.1.1) and do not appear to have physical significance. The first variation of the functionals F_A and F_H is zero about the correct solution to the eddy-current problem. This fact means that if the functionals F_A and F_H are used as the basis of a calculation scheme they will lead to an approximate solution of the eddy-current problem. The second variation of the functionals F_A and F_H are neither positive definite or negative definite.

In conclusion it is the present author's opinion that the functionals F_A and F_H may be used as the basis of a calculation scheme for obtaining an approximate solution to eddy-current problems, but this solution will not guarantee bounds for the global parameters of the eddy-current system.

4.3 Weighted Residual Finite-Element Calculation

This section describes the program theory used for the PE2D program [5]. The program has been developed at the Rutherford Appleton Laboratories. The approximate calculation methods described in sections 4.1 and 4.2 both use variational techniques. Another approach to the approximate solution of eddy-current problems is to use the method of weighted residuals. This method obtains an approximate solution for the diffusion equation without the need for a functional. The PE2D program uses the method of weighted residuals to solve two dimensional linear eddy-current problems. The solution is carried out using the magnetic vector potential.

4.3.1 Weighted residuals

The starting point for this method is the differential equation which governs the system under consideration. This equation can be expressed as below:

$$Lu = G$$

L is a differential operator

u is an unknown potential

G is a known quantity.

If ϕ is an approximate solution for u then a residual function e can be described

$$e = L\phi - G$$

The residual function (e) can be multiplied by a weighting function and then integrated over the region of the problem. When the integral of the weighted residual is equal to zero an approximate solution has been obtained. The weighting function may be chosen at will, however, a particular choice of weighting function called the Galerkin method is considered to lead to the most accurate solution [22]. The Galerkin method is used by PE2D.

The method of weighted residuals does not use a functional, so the question of boundedness by considering the functional does not arise. The method is discussed in order to illustrate the existence of an alternative approach to the approximate solution of eddy-current problems.

4.4 Comparison of Methods

Each of the three methods, for the approximate solution of sinusoidal steady-state eddy-current problems, discussed in this chapter leads to a similar type of calculation scheme. For a particular finite element sub-division with n degrees of freedom each of the methods discussed requires the inversion of an n by n matrix in order to solve the n simultaneous equations generated by the method. Hence the methods are approximately equally useful. The choice of $\underline{E}$, $\underline{H}$ or $\underline{A}$ as the most appropriate solution parameter depends upon the type of eddy-current problem under consideration and the boundary conditions which must be applied.

The three methods described all obtain an approximate solution to the eddy-current problem. Control of the first variation of a functional is sufficient for this purpose. However, none of the methods described control and second variation of the functional, it is the second variation of the functional which determines whether or not an approximate solution will bound the real solution. Also none of the methods described consider the physical meaning of the functionals which they use, and because of this fact these functionals are inappropriate for discussing bounded solutions for global parameters in eddy-current systems. A solution method which uses a functional with physical meaning, and controls the first and second variations of this function is given in Chapter 5. The advantages of a bounded solution technique are discussed in section 1.4.

5. DUAL BOUNDS FOR EDDY-CURRENT SYSTEMS

In Chapter 3 static electric and magnetic fields were analysed by methods analogous to those used in analytical mechanics. This analysis led directly to a bounded solution technique for global parameters in static systems. The analogy is now extended to include time-harmonic eddy-current systems. A variational principle is established which leads to functionals representing the real and reactive power of the system. Analysis of this variational principle leads to a bounded solution technique for the resistance and inductance of an eddy-current system.

5.1 Governing Equation

The governing equation for eddy-current systems is the diffusion equation

$$\nabla \times \nabla \times \underline{H} = -\mu\sigma \frac{\partial \underline{H}}{\partial t} \qquad (5.1)$$

This equation is not self adjoint due to the $-\partial/\partial t$ operator on the right hand side. This problem can be overcome by considering the physical aspects of the problem, this approach is suggested by Hammond [23]. The system has magnetic energy varying with time and it also has dissipation varying with time. A dissipative system requires sources of energy before its equilibrium can be discussed. The sources of energy must match the dissipation exactly. This is achieved by using an adjoint eddy-current system with negative conductivity. Such a system is described by equation (5.1a).

$$\nabla \times \nabla \times \underline{H}_a = -\mu(-\sigma) \frac{\partial \underline{H}_a}{\partial t} \tag{5.1a}$$

in the time domain this is equivalent to a negative time sequence with $H_a = H$ and $\partial H_a/\partial t = -\partial H/\partial t$. In phasor notation equations (5.1) and (5.1a) may be written

$$\nabla \times \nabla \times \underline{H} = -j\omega\mu\sigma\underline{H} \tag{5.2}$$

and $$\nabla \times \nabla \times \underline{H}_a = j\omega\mu\sigma\underline{H}_a \tag{5.3}$$

where $\underline{H} = (\underline{H}'+j\underline{H}'')e^{j\omega t}$ and $\underline{H}_a = (\underline{H}'-j\underline{H}'')e^{-j\omega t}$

5.2 Principle of Virtual Power

For a time harmonic eddy-current system equations (5.2) and (5.3) can be used to give a principle of virtual power, similar to the principle of virtual work

$$<\nabla \times \nabla \times \underline{H} + j\omega\sigma\underline{H},\ \delta\underline{H}_a> = 0 \tag{5.4}$$

$$=><(\nabla\times\nabla\times\underline{H}'-\omega\mu\sigma\underline{H}'')+j(\nabla\times\nabla\times\underline{H}''+\omega\mu\sigma\underline{H}'),\ \delta(\underline{H}'-j\underline{H}'')> = 0 \tag{5.5}$$

5.3 Boundary Conditions

Before investigating the principle of virtual power given in section 5.2 it is necessary to consider the type of systems to which it will be applied. Two specifications are possible, firstly the system may be described by a fixed current. This is equivalent to specifying the surface values of H (for example H' is fixed and $H''=0$). A system specified in this manner will have the real and reactive power given by equation (5.6)

$$P = \frac{1}{2\sigma} <\underline{J},\underline{J}^*> + \frac{j\mu\omega}{2} <\underline{H},\underline{H}^*> = - \frac{\overline{[\underline{E}\times\underline{H}^* . \hat{n}]}}{2} \qquad (5.6)$$

j^* represents the complex conjugate of J.

Since the total current is fixed an equivalent series circuit may be described with resistance R and reactance X the real and reactive power in this circuit is given by equation (5.7).

$$P = \frac{1}{2} I^2 (R + jX) \qquad (5.7)$$

A similar system of equations can be developed if the second specification is adopted, the specification of an applied voltage. This is equivalent to specifying the value of E at the surface ($\bar{E}$). A system specified in this manner would have real and reactive power given by equation (5.8)

$$P = \frac{\sigma}{2} <\underline{E},\underline{E}^*> + \frac{j\omega}{2\mu} <\underline{B},\underline{B}^*> = - \frac{\overline{[\underline{E}\times\underline{H}^* . \hat{n}]}}{2} \qquad (5.8)$$

Since the applied voltage is fixed an equivalent parallel circuit may be described with resistance R and reactance X. The real and reactive power in this circuit is given by equation (5.9)

$$P = \frac{1}{2} V^2 \left[\frac{1}{R} + j\frac{1}{X}\right] \qquad (5.9)$$

5.4 Development of Power Functionals

Using the principle of virtual power (equation 5.5) and the boundary condition $\underline{H}'$ fixed, $\underline{H}''$ equal 0, four power functionals can be developed which lead to upper and lower bounds for the equivalent circuit parameters R and X in section 5.3.

The principle of virtual power can be divided into two variational principles by specifying part of the system. If equation (5.10) is enforced throughout the problem then the variational statement is given by equation (5.11)

$$\nabla \times \nabla \times \underline{H}'' + \omega\mu\sigma\, \underline{H}' = 0 \qquad (5.10)$$

$$<(\nabla \times \nabla \times \underline{H}' - \omega\mu\sigma\, \underline{H}''),\ \delta(\underline{H}' - j\underline{H}'')> = 0 \qquad (5.11)$$

If equation (5.12) is enforced throughout the problem then the variational statement is given by equation (5.13).

$$\nabla \times \nabla \times \underline{H}' - \omega\mu\sigma\, \underline{H}'' = 0 \qquad (5.12)$$

$$j<(\nabla \times \nabla \times \underline{H}'' + \omega\mu\sigma\, \underline{H}'),\ \delta(\underline{H}' - j\underline{H}'')> = 0 \qquad (5.13)$$

5.4.1 Integration of the variational principles

The variational principles (5.11) and (5.13) have real and imaginary parts. these can be integrated separately.

The real part of equation (5.11) is given by equation (5.14)

$$<(\nabla \times \nabla \times \underline{H}' - \omega\mu\sigma\underline{H}''), \delta\underline{H}'> = 0 \qquad (5.14)$$

$$\Rightarrow <\nabla \times \nabla \times \underline{H}', \delta\underline{H}'> - \omega\mu\sigma<\underline{H}'', \delta\underline{H}'> = 0$$

since $\quad \nabla \times \nabla \times \underline{H}'' = -\omega\mu\sigma\ \underline{H}'$ (equation 5.10)

$$\Rightarrow <\nabla \times \nabla \times \underline{H}', \delta\underline{H}'> + <\underline{H}'', \nabla \times \nabla \times \delta\underline{H}'> = 0$$

$$\Rightarrow <\nabla \times \underline{H}', \nabla \times \delta\underline{H}'> + <\nabla \times \underline{H}'', \nabla \times \delta\underline{H}'> = 0$$

$$\Rightarrow <\nabla \times \underline{H}', \delta(\nabla \times \underline{H}')> + <\nabla \times \underline{H}'', \delta(\nabla \times \underline{H}'')> = 0$$

$$\Rightarrow \delta(F_1(\underline{H})) = 0$$

where $\quad F_1(H) = \dfrac{<(\nabla \times \underline{H}), (\nabla \times \underline{H}^*)>}{2}$

Hence the correct solution $F_1(\underline{H}) = ½<\underline{J},\underline{J}^*> = \sigma I^2R/2$ (using eqns. 56 and 5.7). The first variation of $F_1(\underline{H})$ is zero at the correct solution and the second variation is positive definite. This means that $F_1(\underline{H})$ is a minimum a equilibrium and leads to an upper bound for R if $\underline{H}$ does not represent the correct solution to the eddy-current problem. The imaginary part of equation (5.11) is given by equation (5.15)

$$<(\nabla \times \nabla \times \underline{H}' - \omega\mu\sigma\underline{H}''), \delta\underline{H}''> = 0 \qquad (5.15)$$

$$\Rightarrow <\nabla \times \nabla \times \underline{H}', \delta\underline{H}''> - \omega\mu\sigma<\underline{H}'', \delta\underline{H}''> = 0$$

$$\Rightarrow \langle \nabla \times \underline{H}', \nabla \times \delta\underline{H}''\rangle - \omega\mu\sigma\langle \underline{H}'', \delta\underline{H}''\rangle = 0$$

$$\Rightarrow \langle \underline{H}', \nabla \times \nabla \times \delta\underline{H}''\rangle + [\underline{H}' \times (\nabla\times\delta\underline{H}'') \cdot \hat{\underline{n}}] - \omega\mu\sigma\langle \underline{H}'', \delta\underline{H}''\rangle = 0$$

Since $\nabla \times \nabla \times \underline{H}'' = -\omega\mu\sigma\underline{H}'$ (equation 5.10)

$$\Rightarrow [\underline{H}' \times (\nabla\times\delta\underline{H}'') \cdot \hat{\underline{n}}] - \omega\mu\sigma\langle \underline{H}', \delta\underline{H}'\rangle - \omega\mu\sigma\langle \underline{H}'', \delta\underline{H}''\rangle = 0$$

$$\Rightarrow \delta(F_2(\underline{H})) = 0$$

$$\Rightarrow F_2(\underline{H}) = -\text{Im}[(\nabla\times\underline{H})\times\underline{H}^* \cdot \hat{\underline{n}}] - \frac{\omega\mu\sigma}{2} \langle \underline{H}, \underline{H}^*\rangle$$

Hence at the correct solution (using eqns. 5.6 and 5.7)

$$F_2(H) = -\text{Im}\,[\sigma\underline{E}\times\underline{H}^* \cdot \hat{\underline{n}}] - \frac{\omega\mu\sigma}{2} \langle \underline{H}, \underline{H}^*\rangle = \frac{\sigma I^2 X}{2}$$

The first variation of $F_2(\underline{H})$ is zero at the correct solution, the second variation of $F_2(\underline{H})$ is negative definite. This means that $F_2(\underline{H})$ is a maximum at equilibrium and leads to a lower bound for X if $\underline{H}$ is not the correct solution to the eddy-current problem.

The second variational principle is given by equation (5.13). The real part of equation (5.13) is given by equation (5.16).

$$- \langle(\nabla \times \nabla \times \underline{H}'' + \omega\mu\sigma\underline{H}'), \delta\underline{H}''\rangle = 0 \qquad (5.16)$$

$$\Rightarrow - \langle \nabla \times \nabla \times \underline{H}'', \delta\underline{H}''\rangle - \omega\mu\sigma\langle \underline{H}', \delta\underline{H}''\rangle = 0$$

Since $\nabla \times \nabla \times \underline{H}' = \omega\mu\sigma\underline{H}''$ (equation 5.12)

$$\Rightarrow - \langle \nabla \times \nabla \times \underline{H}'', \delta\underline{H}'' \rangle - \langle \underline{H}', \nabla \times \nabla \times \delta\underline{H}' \rangle = 0$$

$$\Rightarrow - \langle \nabla\times\underline{H}'', \nabla\times\delta\underline{H}'' \rangle - \langle \nabla\times\underline{H}', \nabla\times\delta\underline{H}' \rangle - [(\nabla\times\delta\underline{H}')\times\underline{H}'.\hat{\underline{n}}] = 0$$

$$\Rightarrow - [\delta(\nabla\times\underline{H}')\times\underline{H}'.\hat{\underline{n}}] - \langle \nabla\times\underline{H}', \delta(\nabla\times\underline{H}') \rangle - \langle \nabla\times\underline{H}'', \delta(\nabla\times\underline{H}'') \rangle = 0$$

$$\Rightarrow \delta(F_3(\underline{H})) = 0$$

$$\Rightarrow F_3(\underline{H}) = - \mathrm{Re}[(\nabla\times\underline{H}) \times \underline{H}^*.\hat{\underline{n}}] - \frac{\langle \nabla\times\underline{H}, \nabla\times\underline{H}^* \rangle}{2}$$

Hence at the correct solution $F_3(H) = - \mathrm{Re}[\sigma\underline{E}\times\underline{H}^*.\hat{\underline{n}}] - 1/2 \langle \underline{J}, \underline{J}^* \rangle = \sigma I^2R/2$ (using eqns. 5.6 and 5.7). The first variation of $F_3(\underline{H})$ is zero at the correct solution and the second variation of $F_3(\underline{H})$ is negative definite. This means that $F_3(\underline{H})$ is a maximum at equilibrium and leads to a lower bound for R if $\underline{H}$ does not represent the correct solution to the eddy-current problem.

The imaginary part of equation (5.13) is given by equation (5.17)

$$\langle (\nabla \times \nabla \times \underline{H}'' + \omega\mu\sigma\underline{H}'), \delta\underline{H}' \rangle = 0 \qquad (5.17)$$

$$\Rightarrow \langle \nabla \times \nabla \times \underline{H}'', \delta\underline{H}' \rangle + \omega\mu\sigma\langle \underline{H}', \delta\underline{H}' \rangle = 0$$

$$\Rightarrow \langle \nabla \times \underline{H}'', \nabla \times \delta\underline{H}' \rangle + \omega\mu\sigma\langle \underline{H}', \delta\underline{H}' \rangle = 0$$

Since $\nabla \times \nabla \times \underline{H}' = \omega\mu\sigma\underline{H}''$ (equation 5.12)

$$\Rightarrow \omega\mu\sigma <\underline{H}'', \delta\underline{H}''> + \omega\mu\sigma<\underline{H}', \delta\underline{H}'> = 0$$

$$\Rightarrow \delta(F_4(\underline{H})) = 0$$

$$\Rightarrow F_4(\underline{H}) = \frac{\omega\mu\sigma}{2} <\underline{H}.\underline{H}^*>$$

Hence at the correct solution $F_4(\underline{H}) = \omega\mu\sigma/2 <\underline{H},\underline{H}^*> = \sigma I^2X/2$ (using eqns. 5.6 and 5.7). The first variation of $F_4(\underline{H})$ is zero at the correct solution and the second variation of $F_4(\underline{H})$ is positive definite. This means that $F_4(\underline{H})$ is a minimum at equilibrium and leads to an upper bound for R if $\underline{H}$ does not represent the correct solution to the eddy-current problem.

5.5 Discussion of the Functionals

The functionals $F_1(\underline{H})$, $F_2(\underline{H})$, $F_3(\underline{H})$ and $F_4(\underline{H})$ have all been developed for the surface specification $\underline{H}'$ known and $\underline{H}''=0$ for those parts of the surface where $[\underline{E}\times\underline{H}.\hat{\underline{n}}]\neq 0$. A similar set of functionals can be developed for the surface specification $\underline{E}$ known and $\underline{E}''=0$. The functionals $F_1(\underline{H})$ and $F_2(\underline{H})$ require the volume specification $\nabla \times \nabla \times \underline{H}'' = -\omega\mu\sigma\underline{H}'$ and give bounds R_+ and X_-. The functionals $F_3(\underline{H})$ and $F_4(\underline{H})$ require the volume specification $\nabla \times \nabla \times \underline{H}' = \omega\mu\sigma\underline{H}''$ and give bounds R_- and X_-. In chapter 6 an example is given in which the functionals $F_1(\underline{H})$, $F_2(\underline{H})$, $F_3(\underline{H})$ and $F_4(\underline{H})$ are used to obtain upper and lower bounds for the equivalent circuit parameters R and X.

5.6 Discussion of the Method

The method described by sections 5.1 to 5.5 is similar to a method described by Hammond and Penman [2]. A brief outline of part of that method is given below.

For time-harmonic eddy-current systems an adjoint system $\underline{H}^* = (\underline{H}' - j\underline{H}'')e^{-j\omega t}$ is specified. The variation is restricted by the condition $\nabla \times \underline{H} = \underline{J}$ or the condition $\nabla \times \underline{E} = -j\omega\underline{B}$. Using the condition $\nabla \times \underline{H} = \underline{J}$ and specified surface H the functional Y is developed which represents the real and reactive power in the system.

$$Y = \frac{1}{2\sigma} <\underline{J}.\underline{J}^*> + j\frac{\mu\omega}{2} <\underline{H}.\underline{H}^*> \qquad (5.18)$$

At the correct solution to the eddy-current problem the first variation of Y is shown to be given by equation (5.19).

$$\delta Y = \mu\omega<\underline{H}'', \delta\underline{H}'> - \mu\omega<\underline{H}', \delta\underline{H}''> - \frac{j}{\sigma} <\underline{J}'', \delta\underline{J}'> + \frac{j}{\sigma} <\underline{J}', \delta\underline{J}''> \qquad (5.19)$$

Hammond and Penman explain that $\delta Y=0$ because the variation is carried out on the power of the combined double system $<\underline{J},\underline{J}^*>$ and $<\underline{H},\underline{H}^*>$ and the phase angle at a particular place is not varied.

However, since $\underline{H}'$ and $\underline{H}''$ can be varied independently this reasoning does not appear to be strong enough. The real part of δY can be made equal to zero at the correct solution to the eddy-current problem by enforcing the condition $\nabla \times \nabla \times \underline{H}'' = -\omega\mu\sigma\underline{H}'$ and the surface condition $\underline{H}'' = 0$. The imaginary part of δY can be made equal to zero at the correct solution to the eddy-current problem by enforcing the condition $\nabla \times \nabla \times \underline{H}' = \omega\mu\sigma\underline{H}''$ and the surface condition $\underline{H}'' = 0$. These facts demonstrate the similarity between the method described by Hammond and Penman and the method described in sections 5.1 to 5.5.

6. PROBLEM SOLVING

The functionals developed in Chapter 5 are used to provide a bounded solution for global parameters in an eddy-current system. The system chosen is a large flat conductor. The geometry is shown in Fig. 6.

The conductor is considered to be infinite in the x and z directions. The analytical solution for the impedance per unit in the x and z directions is given by

$$R+jX = \frac{1}{2\sigma\Delta}\left[\frac{\sinh\frac{2b}{\Delta} + \sin\frac{2b}{\Delta}}{\cosh\frac{2b}{\Delta} - \cos\frac{2b}{\Delta}} + j\,\frac{\sinh\frac{2b}{\Delta} - \sin\frac{2b}{\Delta}}{\cosh\frac{2b}{\Delta} - \cos\frac{2b}{\Delta}}\right] \qquad (6.1)$$

where $\Delta = \sqrt{2/\mu\sigma\omega}$ is the eddy-current skin depth.

6.1 Bounded Solution

A total current of I per unit width is specified the current being in the z direction, by symmetry the magentic field at the top surface is -I/2 and the magnetic field at the bottom surface is I/2. At the centre of the conductor the magnetic field is zero. This symmetry means that odd functions should be used to describe the magnetic field within the conductor. For the functions $F_1(H)$ and $F_2(H)$ the specification of H'' is chosen as

$$H'' = \alpha_1 y^5 + \beta_1 y^3 + \gamma_1 y$$

and since

$$\nabla \times \nabla \times H'' = -\omega\mu\sigma H'$$

$$\Rightarrow \omega\mu\sigma H' = -20\alpha_1 y^3 - 6\beta_1 y$$

at the top surface

$$H' = -I/2 \quad H'' = 0$$

hence $\quad \alpha_1 b^5 + \beta_1 b^3 + \gamma_1 b = 0$

and $\quad 20\alpha_1 b^3 + 6\beta_1 b = \omega\mu\sigma I/2$

Using these definitions the system has one degree of freedom, the free variable is chosen to be α_1. For a value of $b = \Delta$ the values resistance and inductance are calculated from the functionals $F_1(H)$ and $F_2(H)$ and plotted in Figs. 6.2 and 6.3 for different values of α_1. The resistances have been normalized to the d.c. resistance $R_0 = 1/2\sigma b$ and the d.c. inductance $L_0 = \mu_0 b/6$. The behaviour of the functionals is clearly seen in these diagrams and it is clear that any value of α_1 will produce an upper bound for the resistance and a lower bound for the inductance.

For the functionals $F_3(H)$ and $F_4(H)$ the specification of H' is chosen as

$$H' = \alpha_2 y^5 + \beta_2 y^3 + \gamma_2 y$$

and since $\nabla \times \nabla \times H' = \omega\mu\sigma H''$

$$\Rightarrow \omega\mu\sigma H'' = 20\alpha_2 y^3 + 6\beta_2 y$$

at the top surface $H' = -I/2 \quad H'' = 0$

hence $\alpha_2 b^5 + \beta_2 b^3 + \gamma_2 b = -I/2$

and $20\alpha_2 b^3 + 6\beta_2 b = 0$

Using these definitions the system has one degree of freedom, the free variable is chosen to be α_2. For a value of $b = \Delta$ the values of resistance and inductance are calculated from the functionals $F_3(H)$ and $F_4(H)$ and plotted in Figs. 6.4 and 6.5 for different values of α_2. The resistance and inductance have been normalized to their respective d.c. values. Again the behaviour of the functionals is clearly seen in these diagrams and it is clear that any value of α_2 will produce a lower bound for the resistance and an upper bound for the inductance.

6.1.1 Discussion of results

The results for the large flat conductor demonstrate the bounds provided by the functions $F_1(H)$, $F_2(H)$, $F_3(H)$ and $F_4(H)$. The functionals can be used to provide bounded solutions either by guessing the form of solution for H or by using the Rayleigh-Ritz variational method to obtain an approximate solution for H. If the Rayleigh-Ritz method is used the following accuracy is obtained for the bounds for $b=\Delta$.

$R_+ = 1.000136R$ $\qquad$ $L_+ = 1.0000007L$

$R_- = 0.9999915R$ $\qquad$ $L_- = 0.9999904L$

where R and L represent the analytical solution given by equation (6.1).

7. PRACTICAL WORK

A practical project was carried out at the Central Electricity Generating Board, Central Engineering Reasearch Laboratories. The project was undertaken to demonstrate the interaction between the solution of eddy-current systems and the design of practical equipment.

7.1 Introduction to the Project

Many methods of temperature measurement exist. Those in common use today include; mercury-in-glass thermometer, platinum resistance thermometer, thermocouples and pyrometers. The particular interest of this work is the measurement of the temperature of an unseen and inaccessible conductor (such as a conductor which is covered with electrical insulation).

The techniques mentioned above are not suitable for this problem. The approach, suggested by Sutton [24] is to consider the disturbance of an alternating magnetic field due to the presence of the conductor. For a fixed geometry and frequency this is a function of the resistivity, and hence temperature, of the conductor. The geometry considered is that of a circular exciting coil with its axis perpendicular to a semi-infinite conductor (Fig. 7.2). This problem has an analytical solution which is given by Hammond [25].

The techniques used for measurement and data analysis are discussed, and results are presented which seem to indicate that further development would lead to a useful practical instrument.

Instruments using similar principles to determine distance already exist - however the problem of resistivity measurement is much more difficult as the changes in signal are smaller and the signal requires splitting into in-phase and quadrature components.

7.2 Analytical Solution

7.2.1 Description of the problem

A circular excitation coil carrying an alternating current generates an alternating magnetic field. If a circular pick-up coil is located on the same axis and in the same plane as the excitation coil, then the voltage output from the pick-up coil can be measured. The physical arrangement is shown in Fig. 7.1.

If a semi-infinite conductor is placed adjacent to the coils (Fig. 7.2) eddy-currents will be induced in the conductor. The magnetic field of the eddy-currents will alter the voltage output from the pick-up coil. In this chapter the change in magnetic vector potential A at the pick-up coil, due to the presence of the conductor, is calculated. The magnetic vector potential at the pick-up coil is proportional to the voltage output from the pick-up coil.

7.2.2 Coil in free space

The magnetic vector potential A1 at the pick-up coil, due to the excitation coil in free space, is given by equation (7.1). The coordinate system is the cylindrical (r, θ, z) system, the origin being chosen at the centre of the coils. Because of the symmetry of the problem the only component of $\underline{A}$ is A_θ and the suffix can be dropped.

$$A1 = \frac{\mu_o \mu_r I}{2} \int_0^\infty J_1(k) J_1 \left[\frac{kc}{a}\right] dk \qquad (7.1)$$

where I = current in the excitation coil

a = radius of the excitation coil

c = radius of the pick-up coil

k = variable

J_1 = Bessel function of the first kind and order unity

7.2.3 Coil adjacent to semi-infinite conductor

The magnetic vector potential A2 at the pick-up coil due to the excitation coil and the semi-infinite conductor (Fig. 7.2) is:

$$A2 = \frac{\mu_o \mu_r I}{2} \int_0^\infty J_1(k) J_1 \left[\frac{kc}{a}\right] \left[1 + \frac{(k\mu_r - (k^2 + jp^2)^{\frac{1}{2}})}{(k\mu_r + (k^2 + jp^2)^{\frac{1}{2}})} e^{-2kb/a}\right] dk \qquad (7.2)$$

where $p = \sqrt{2}/\Delta$

Δ = skin depth

b = distance of coils from the surface of the conductor

7.2.4 Change in magnetic field caused by the presence of the semi-infinite conductor

The difference A_D between A2 and A1 is caused by eddy-currents in the semi-infinite conductor.

$$A_D = \frac{\mu_o \mu_r I}{2} \int_0^\infty J_1(k) J_1\left[\frac{kc}{a}\right] \left[\frac{(k\mu_r - (k^2+jp^2)^{1/2}}{(k\mu_r + (k^2+jp^2)^{1/2}}\right] e^{-2kb/a}\, dk \qquad (7.3)$$

7.2.5 Determination of phase and quadrature components of A_D

The following analysis determines the components of A_D which are in-phase with and in phase-quadrature with the current in the excitation coil.

The components of equation (7.3) are all real apart from the term given below:

$$h = \left[\frac{(k\mu_r - (k^2+jp^2)^{1/2}}{(k\mu_r + (k^2+jp^2)^{1/2}}\right] \qquad (7.4)$$

It is necessary to express h in the form a+jb where a and b are real numbers. This is achieved as follows:

$$(k^2+jp^2)^{1/2} = [(k^4+p^4)^{1/2}\, e^{j\left[\tan^{-1}\frac{p^2}{k^2}\right]}]^{1/2} \text{ (by Euler's Identities)}$$

$$= (k^4+p^4)^{1/4}\, e^{\frac{j}{2}\left[\tan^{-1}\frac{p^2}{k^2}\right]} \qquad \text{(by De Moivre's Theorem)}$$

$$= y \cos x + jy \sin x \qquad (7.5)$$

where $y = (k^4+p^4)^{1/4}$ and $x = \frac{1}{2}\left[\tan^{-1}\left\{\frac{p^2}{k^2}\right\}\right]$

The substitution of equation (7.5) into equation (7.4) gives

$$h = \frac{k\mu_r - y \cos x + jy \sin x}{k\mu_r + y \cos x + jy \sin x} \tag{7.6}$$

Further analysis of equation (7.6) leads to

$$h = \left[\frac{(k\mu_r)^2 - y^2}{(k\mu_r)^2 + 2(k\mu_r)y \cos x + y^2}\right] + j\left[\frac{-2k\mu_r y \sin x}{(k\mu_r)^2 + 2(k\mu_r)y \cos x + y^2}\right] \tag{7.7}$$

This expression allows A_D to be written in the form

$A_D = A_D$ phase + jA_D quadrature.

Substituting equation (7.7) into equation (7.3) gives:

$$A_{Dphase} = \frac{\mu_o \mu_r I}{2} \int_0^\infty e^{-2kb/a} J_1(k) J_1\left[\frac{kc}{a}\right] \left[\frac{(k\mu_r)^2 - y^2}{(k\mu_r)^2 + 2k\mu_r y \cos x + y^2}\right] dk \tag{7.8}$$

$$A_{Dquad} = \frac{\mu_o \mu_r I}{2} \int_0^\infty e^{-2kb/a} J_1(k) J_1\left[\frac{kc}{a}\right] \left[\frac{-2k\mu_r y \sin x}{(k\mu_r)^2 + 2k\mu_r y \cos x + y^2}\right] dk \tag{7.9}$$

These values represent the changes in the amplitudes of the in-phase and quadrature components of magnetic vector potential at the pick-up coil, caused by the eddy currents in the conductor. The magnetic vector potential at the coil is proportional to the voltage output from the pick-up coil.

7.3 Experimental Work

The experimental work described in this section uses the same arrangement of coils and conductor as shown in Fig. 7.1 and Fig. 7.2. The experimental work is designed to accurately detect the change in the voltage output from the pick-up coil due to the presence of a semi-infinite conductor. This change in voltage can then be compared with the analytical solution for A_D. and this comparison enables the value of the resistivity of the conductor to be obtained.

7.3.1 Arrangement of the coils

The physical arrangement of the excitation and pick-up coils is designed to be equivalent to the theoretical coil arrangements in Fig. 7.1 and Fig. 7.2. However, since the measurement required is the difference in voltage output from the pick-up coil due to the presence of the conductor, the experiment has been designed to measure this directly. Two pairs of "identical" coils are used; the two excitation coils are connected in series and the two pick-up coils are connected in series opposite (Fig. 7.3). This arrangement gives zero output voltage in free space. Fine mechanical adjustment is used in order to obtain the above condition.

A conductor is then placed adjacent to one set of coils, the other set of coils are still effectively in free space. The voltage signal obtained is proportional to A_D. The remote coils are effectively in free space if $l/a > 5$ (where l is the distance between two sets of coils in Fig. 7.3) this is due to the term $e^{-2kb/a}$ in equation (7.8) and equation (7.9).

7.3.2 Experimental arrangement

The experimental arrangement is shown in Fig. 7.4.

The voltage output from the coil arrangement is fed into the precision lock-in amplifier. This allows the signal to be resolved into two components. The output from the precision lock-in amplifier is fed into a digital multimeter to enable more accurate readings to be taken. The precision lock-in amplifier works in the manner described below.

The input signal is multiplied by an internally generated square wave, this gives a measure of the signal which is in-phase with a preset reference (Fig. 7.5). By shifting the phase of the square wave by 90° a measure of the signal which is in phase quadrature with the reference is obtained (Fig. 7.6). The reference phase is the phase of the voltage induced in the pick-up coils when the coil arrangement is in free space.

The target conductor is an approximation to the semi-infinite conductor in Fig. 7.2. The approximation is good is the thickness of the conductor is greater than four skin depths and the distance from the axis of the coil arrangement is greater than eight times the radius of the excitation coil. (These figures are based on the analytical solution given by Hammond [25] and attested by varying the size of the target in the experiment).

Readings of the output voltage from the coil-arrangement are required when the target is at different temperatures. Hence, the thermal stability of the target conductor is important. A large target has greater thermal stability, this enables readings to be taken at constant temperature but means that more heat must be supplied to change the temperature of the target conductor.

7.3.3 Calibration of equipment

The coils within the coil arrangement are positioned to give a zero output when they are in free space. When a conducting target is moved towards one end of the coil arrangement an output voltage (V) from the pick-up coils is obtained. For a particular coil arrangement at a fixed operating frequency the measured in-phase and quadrature components of the output voltage are related to the in-phase and quadrature components of A_D (sub-section 7.2.5) by a scaling factor.

7.4 Data Analysis

The data obtained from the experiment is the phase (P) and quadrature (Q) signal from the pick-up coil. From experimental measurements it was found that $(|P| \div |Q|)$ is mainly sensitive to the distance of the coil arrangement from the target irrespective of the target resistivity. The phase of the signal $(\tan^{-1}(Q/P))$ is governed both by the resistivity of the target and by the distance of the coil arrangement from the target.

The measured voltages P and Q are compared with the theoretical values A_D phase and A_D quad given by equations (7.8) and (7.9). The following steps are therefore used in the computer program once the scale factor has been determined (sub-section 7.3.3).

(1) Generate a table of $A_{Dphase} + A_{Dquad}$.

(2) Compare (P-Q) with ($A_{Dphase} + A_{Dquad}$) in order to determine the distance of the coil arrangement from the target.

(3) Use information about P/Q to determine the resistivity (by comparing A_{Dphase}/A_{Dquad}).

A_{Dphase} and A_{Dquad} are calculated using the "Data Generation" program. The comparison with the P and Q values is carried out by the "Electromagnetic Thermometer" Program. The programs operate on a Tektronix 4042A computer.

7.4.1 Data generation program

This program calculates values of A_{Dphase} and A_{Dquad} by evaluating the integrals in equations (7.8) and (7.9). The Bessel functions in the integrands are evaluated using polynomial series given by Abromowitz and Stegun [26]. The integrands are evaluated at steps of k = 0.005 (over the range k = 0 to 10) and then integrated using Simpson's rule (at values of k greater than 10 the values of the integrands are negligible due to the terms $e^{-2kb/a}$ in equations (7.8) and (7.9)).

The values of A_{Dphase} and A_{Dquad} are calculated for the following range of variables:

Operating frequency (Fr)

Drive coil radius (A)

Pick-up coil radius (R)

Min. and Max. distance of coils from the target (B1, B2)

Min. and max. resistivity of target (Rh1, Rh2).

The program generates and stores on tape values of A_{Dphase} and A_{Dquad} for

B = B1 to B2 step 0.1mm

Rh = Rh1 to Rh2 step 0.25 (E-8 Ohm m)

It also computes and stores values of A_{Dphase} and A_{Dquad} for

B = B1

Rh = Rh1 to Rh2 step 0.03 (E-8 Ohm m)

Both sets of data are written onto tape file 'X' which is specified by the user.

7.4.2. Electromagnetic thermometer program

This program reads from tape the data generated by the data generation program and computes A_{Dphase} - A_{Dquad} and $A_{Dphaase}/A_{Dquad}$ and places these values in three tables (Fig. 7.7). The three tables are organised in this way to make use of the fact that

$$\frac{\left[\frac{A_{Dphase}}{A_{Dquad}}\right] \text{ evaluated at } Rh = \rho_1, \text{ distance } = D_s}{\left[\frac{A_{Dphase}}{A_{Dquad}}\right] \text{ evaluated at } Rh = \rho_2, \text{ distance } = D_s}$$

is virtually independent of the distance D_s. This enables less storage space to be used for a given accuracy.

Before the program can be used as a thermometer it requires calibration, which as mentioned in sub-section 7.3.3., is achieved by input of values of P and Q at known temperature (the determination of the scale factor is described in sub-section 7.4.3). Once the scale factor has been calculated from the calibration data, values of P and Q at unknown temperature may be input and the temperature of the target will be calculated by the program and displayed on the screen.

The program takes the following steps:-

1. Inputs P and Q (measured values)
2. Defines PPQ = (P + Q) * scale factor
3. Sets Rh = 1.75 R-8 Ohm m
4. Estimates the distance D_s at which PPQ = A_{Dphase} + A_{Dquad} (using linear interpolation between discrete points in Table 1, Fig. 7.7)
5. Calculates the value:

$$Lu = \frac{\left[\frac{A_{Dphase}}{A_{Dquad}}\right] \text{ evaluated at distance = B1, resistivity = Rh1}}{\left[\frac{A_{Dphase}}{A_{Dquad}}\right] \text{ evaluated at Rh distance} = D_s\text{, resistivity = Rh1}} \times \frac{P}{Q}$$

(using linear interpolation between discrete points (in Table 2, Fig. 7.7)

6. Estimates Rh by finding the value of Rh at which

$$Lu = \left[\frac{A_{Dphase}}{A_{Dquad}}\right]$$

evaluated at distance = B1 (using linear interpolation between discrete points in Table 3, Fig. 7.7)

7. With the newly calculated approximation for ρ repeat steps 4 to 7 until the iterations converge (this usually takes about 6 iterations).

7.4.3 Scale factors

The scale factor is obtained by using experimental measurements at known resistivity. The program guesses a scale factor. then compares the value of Rh obtained by performing steps 4-7 with the known value of Rh. A correction is made to the sale factor and this process is repeated until the correct scale factor is found.

7.5 Results

Fig. 7.8 shows a graph of results over the temperature range 33-47°C for a copper target. These results were obtained at a frequency of 10kHz, the coils were 12mm from the target. the coil dimensions were radius of pick-up 5.5mm, radius of excitation coil 5mm and the pick-up and exciting coil each had 20 turns.

Using the same experimental arrangement experiments were carried out over a wider temperature range which give the following results.

Temperature Rise (by thermocouple) = 92.3°C

Temperature Rise (by P and Q measurement) = 89.3°C.

7.6 Discussion of Future Work

Practical

The practical measurement of the Phase and Quadrature Components of the signal from the pick-up coil present the greatest challenge. Problems include stability of the Lock-in Amplifier and the temperature control of the exciting and pick-up coils.

The stability of the Lock-in Amplifier has two main considerations, these are:

(1) It is preferential to make the ratio P/Q as close to unity as possible. This requires a lower frequency (@

10 kHz P/Q = 17, @ 10 kHz P/Q = 5, @ 100 Hz P/Q = 1).

(2) It is preferential to increase the signal strength from the pick-up coil. This requires a higher frequency since the signal strength is proportional to frequency. Aternatively the signal strength could be increased by using more turns on the pick-up coil.

It is considered that the second consideration has more room for improvement and in future equipment a lower frequency may be prefered.

The effect of temperature on the coils and coil former requires further investigation into the use of materials whose physical properties are not sensitive to temperature. The thermal stability of the sensing coil arrangement could be improved by use of materials with low temperature expansion coefficients.

Programs

Analysis of the accuracy of the programs should be assessed more completely. This may lead to a reduction in the matrix size required for a particular range of distances from the target and of the target resistivity. The programs could be run on a microprocessor based instrument for greater practical convenience.

7.7 Discussion of Practical Work

Using the laboratory equipment described and a distance of 12mm between the coils and the target conductor, it has proved possible to measure temperature rises in a copper slab of up to 90°C with an error of less than 3°C. This is considered well within the permissible error for useful measurement techniques. The computer programs which have been written may be used as design tools to develop a microprocessor based instrument.

This practical project has demonstrated the interaction between the computation of approximate solutions to eddy-current problems and the design and use of practical equipment.

8. INTERNATIONAL ELECTROMAGNETIC WORKSHOP

A meeting was held at the 1985 COMPUMAG Conference to discuss the potential for initiating a series of international workshops on electromagnetics. The meeting was organised by Argonne National Laboratory, U.S.A. This meeting led to a series of international electromagnetic workshops with the goal of showing the effectiveness of numerical techniques and associated computer codes in solving electromagnetic field problems, and gaining confidence in their predictions. In order to achieve this goal six test problems were selected and participants were invited to solve these problems and submit results for comparison and discussion. To achieve some degree of standardisation the discretisation of the problem was specified as well as the problem itself. The problems chosen have either an analytical result, a set of experimental data from experiments carried out at Bath University or experimental data from the FELIX program (Fusion ELectromagnetic Induction eXperiments) which is being carried out at the Argonne National Laboratory.

The FELIX program is designed to study the effect of eddy-currents in the electrically conducting components of a fusion reactor. Changes in the magnetic field inside a fusion reactor in either normal operation or off-normal events, produce eddy currents with accompanying forces, torques, and voltages. A quantitive understanding of these electromagnetic effects is vital to the design of a fusion reactor. This new application for eddy-current analysis has provided the spur for the International Electromagnetic Workshops.

The FELIX facility includes [29]

(a) A solenoid magnet producing fields up to 1T.

(b) A dipole magnet surrounding the solenoid and producing fields up to 0.5T perpendicular to the solenoid field.

(c) A switching circuit capable of discharging the dipole field with a time constant as low as 10ms.

(d) A cylindrical experimental volume within the fields of the two magnets of dimensions 1.2m axially by 0.9m diameter.

(e) A non-conducting test-piece support frame centred on a support tube perpendicular to both magnetic fields and providing, through phosphor bronze leaf springs, restoring torques of up to 9kNm/rad.

(f) Instrumentation suitable for measuring currents, magnetic fields, angular displacements, temperatures and stresses.

(g) Computerised data acquisition with provision for recording values at 2048 times on as many as 30 data channels.

The six problems under consideration by the international electromagnetic workshops are described in reference [31]. A brief description of each of the problems is given below.

(a) The Bath Cube

This experiment was carried out at Bath University the results were published in 1981 [28]. Four identical aluminium cubes are symmetrically situated and enclosed within a laminated iron box under a laminated iron pole. A sinusoidal MMF of 100 + j0 A turns at a frequency of 50Hz is applied between the pole and the box. The problem is to calculate the magnitude and phase of the magnetic field at various locations. The geometry is shown in Fig. 8.1.

Experimental measurements were taken along the line z = 2mm x = 70mm using a Hall probe (active area 2 × 4.75mm) to measure the flux density and a search coil (3.2mm diameter) to measure the phase. The problem set by the International Electromagnetic Workshop is to compute the magnitude and phase of the magnetic flux density along the line z = 2mm x = 70mm.

(b) Bath Plate

The geometry of this problem is shown in Fig. 8.2 and 8.3. The problem consists of a conducting "ladder" with a current carrying coil above. The coil can have two positions (Fig. 8.3). The problem is to calculate the fields and eddy-currents flowing around the limbs of the ladder. For the purposes of comparison the following results are to be presented. The magnitude and phase of the z directed magnetic flux density along the line AB, (0.5mm above the top surface of the conducting ladder) with the driving coil at the two positions shown in Fig. 8.4 for the two frequencies 50Hz and 200Hz. Global quantities such as the total flux flowing through the two holes in the ladder, and the total current flowing through the central limb are also to be calculated.

(c) Infinitely Long Cylinder in a Sinusoidal Field

This problem (Fig. 8.4) consists of an infinitely long hollow alluminium cylinder placed in a uniform magnetic field. The magnetic field is perpendicular to the axis of the cylinder and varies sinusoidally with time. The problem is to calculate the induced eddy currents in the aluminium and the magnetic field inside and outside the cylinder. Global quantities such as power, losses stored energy, and forces are also to be calculated.

(d) The FELIX Long and Short Cylinder Experiments

These problems consist of hollow aluminium cylinders (Figs. 8.5 and 8.6) placed in uniform magnetic field. One cylinder is used for each experiment. The magnetic field is perpendicular to the axis of the cylinder and it decays exponentially with time. The problem is to calculate the induced eddy-currents in the aluminium and the magnetic field both inside and outside the cylinder at various axial positions and at various times. Global quantities such as power losses, stored energy, forces etc. should also be calculated.

(e) The FELIX Brick Experiment

A rectangular aluminium brick with a rectangular hole through it Fig. 8.7 is placed in a uniform magnetic field. The magnetic field is perpendicular to the faces with the hole, and decays exponentially with time. The problem is to calculate the total circulating current and the magnetic field at various positions. Global quantities such as power should also be calculated.

(f) Sphere in a Uniform Magnetic Field

This problem consists of a hollow sphere in a uniform sinusoidally varying magnetic field. The problem has an analytical solution [30] and it is to be used for the comparison of computer codes. The solution consists of the calculation of the magnetic fields and eddy-currents, as well as global quantities such as losses and stored energy.

Regional workshops have already been held and the first international workshop is in August 1987 at Graz in Austria. These workshops provide the first opportunity for a systematic comparison of the results obtained from different eddy-current computer codes and provide cooperation between workers in the field, leading to an interchange of ideas.

9. FUTURE WORK

The development of numerical methods for the approximate solution of eddy-current problems is essential in order to satisfy a large number of the existing engineering application requirements. This is because analytical solutions are only possible for problems with a high degree of symmetry and for materials with linear (or step function) magnetic properties [7, 8]. Numerical methods have been developed which will obtain an approximate solution to a much wider range of problems [27]. However, there are still many problems without a satisfactory solution. The greatest progress has been made for steady-state alternating current solutions for linear materials, but there is still much work to be done validating the results produced by computer codes which try to solve this type of problem. Areas which require even more work in the future are transient problems, non-linear problems, hysteresis effects and coupled thermal fields.

Validation of results is a severe problem when developing algorithms to solve problems for which no analytical solution exists. In order to overcome this problem a series of international workshops have been organised by the Argonne National Laboratory, U.S.A. These workshops consider several benchmark problems, these are problems for which experimental data already exists (eg. Bath Cube experiment [28]) or is being obtained as part of the FELIX program [29] (Fusion ELectromagnetic Induction eXperiments). The FELIX experiments are sponsored by the U.S. Department of Energy, they are designed to consider the effects of changes in magnetic field on the conducting components of a nuclear fusion reactor.

The interest in numerical methods for the solution of eddy-current problems is very great and it is clear that any improvement in the efficiency of computation schemes is very desirable. One possible way of improving the efficiency of computer calculation, and at the same time providing error limits for the solution of global parameters in eddy-current systems is the use of dual bounded solution techniques. A dual bounded solution technique for global parameters in steady-state alternating current eddy-current systems is given in Chapter 5. However, this solution technique is limited to conducting regions with particular types of boundary conditions. Further work is required to determine whether the method can be extended to include problems with both conducting and non-conducting regions. Further work is also required to extend the method to transient problems.

The advantages of a dual bound system are two-fold, firstly error bounds are established for the correct solution, this removes the need for the rather unsatisfactory system of increasing the number of degrees of freedom in a numerical solution and noting the change in the approximate solution yielded. Secondly a solution technique may be used which involves specifying a trial solution and then modifying this trial solution in such a way as to increase the lower bound of the solution or decrease the upper bound. A solution obtained in this manner does not require the solution of simultaneous equations which is one of the limiting factors in other approximate solution techniques.

10. CONCLUSION

The study of global parameters in eddy-current systems is of importance to the design of electrical power apparatus. These global parameters represent the equilibrium condition for a sinusoidal steady state system. The analysis of eddy-current systems and their behaviour at equilibrium can be treated in a manner analogous to the variational treatment of mechanics as used in analytical mechanics. A treatment of eddy-current systems by the above method has been shown to lead to a dual bounded solution for the global parameters (resistance inductance) in some eddy-current systems. The advantages of a dual bounded calculation scheme are discussed in section 1.4.

The method which has been developed is limited to conductors for which the magnetic field strength at each point on the surface is in phase with the total current in the conductor. The extension of the method to include regions of non-conducting material could be the subject of further study.

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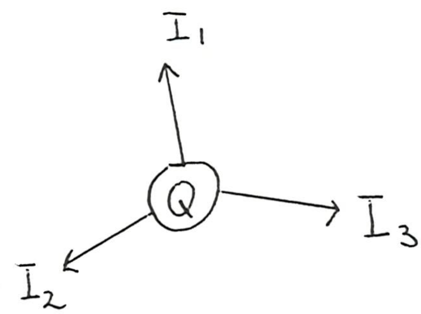


Fig. 1.1 Continuity of charge and current

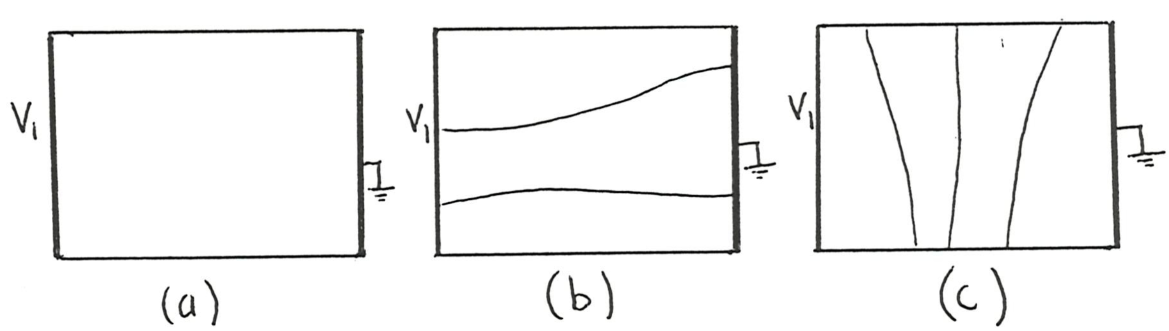


Fig. 1.2 Dual bound resistance calculation

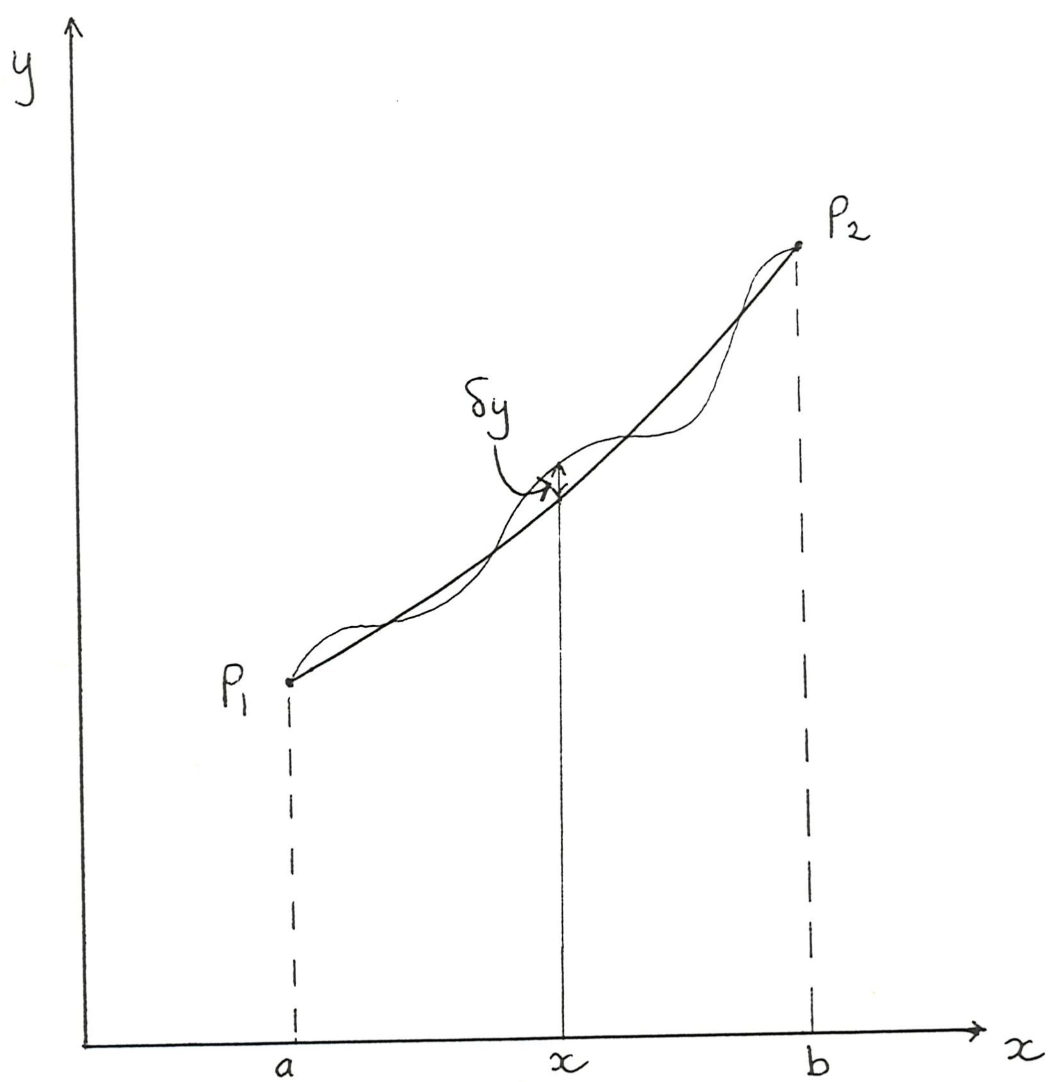


Fig. 2.1 Variation of a function

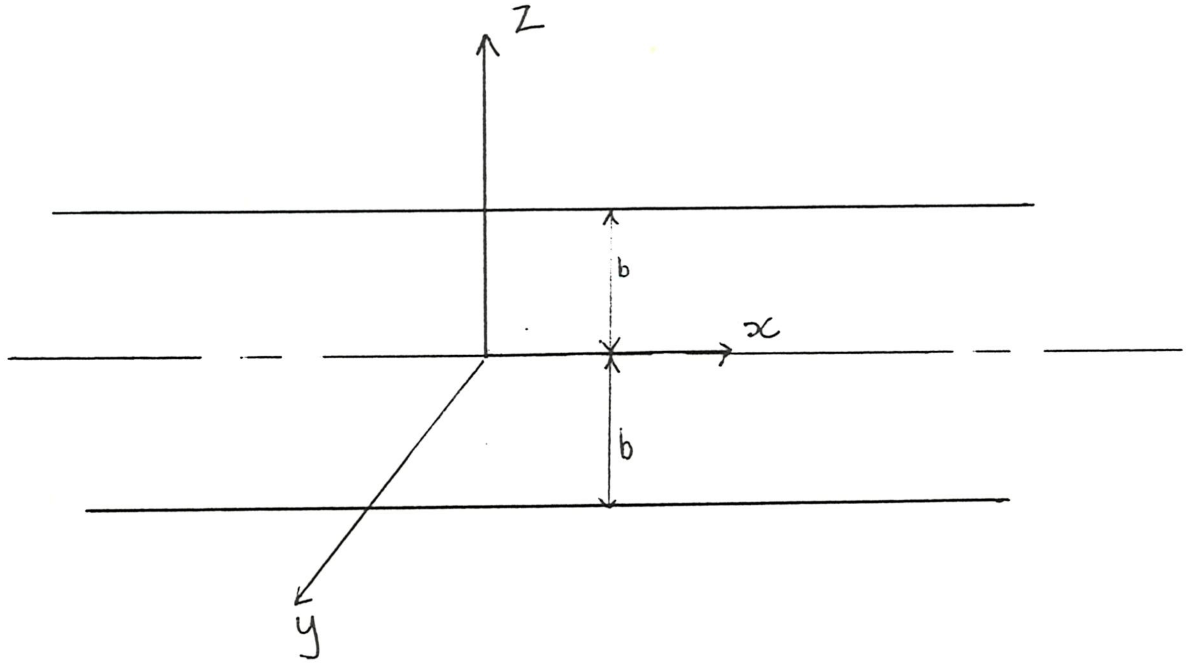


Fig. 6.1 Large flat conductor

Fig. 6.2 Graph of ${}^{R_+}/R_o$ against α_1 for $\frac{b}{\Delta} = 1$ (from functional $F_1(H)$)

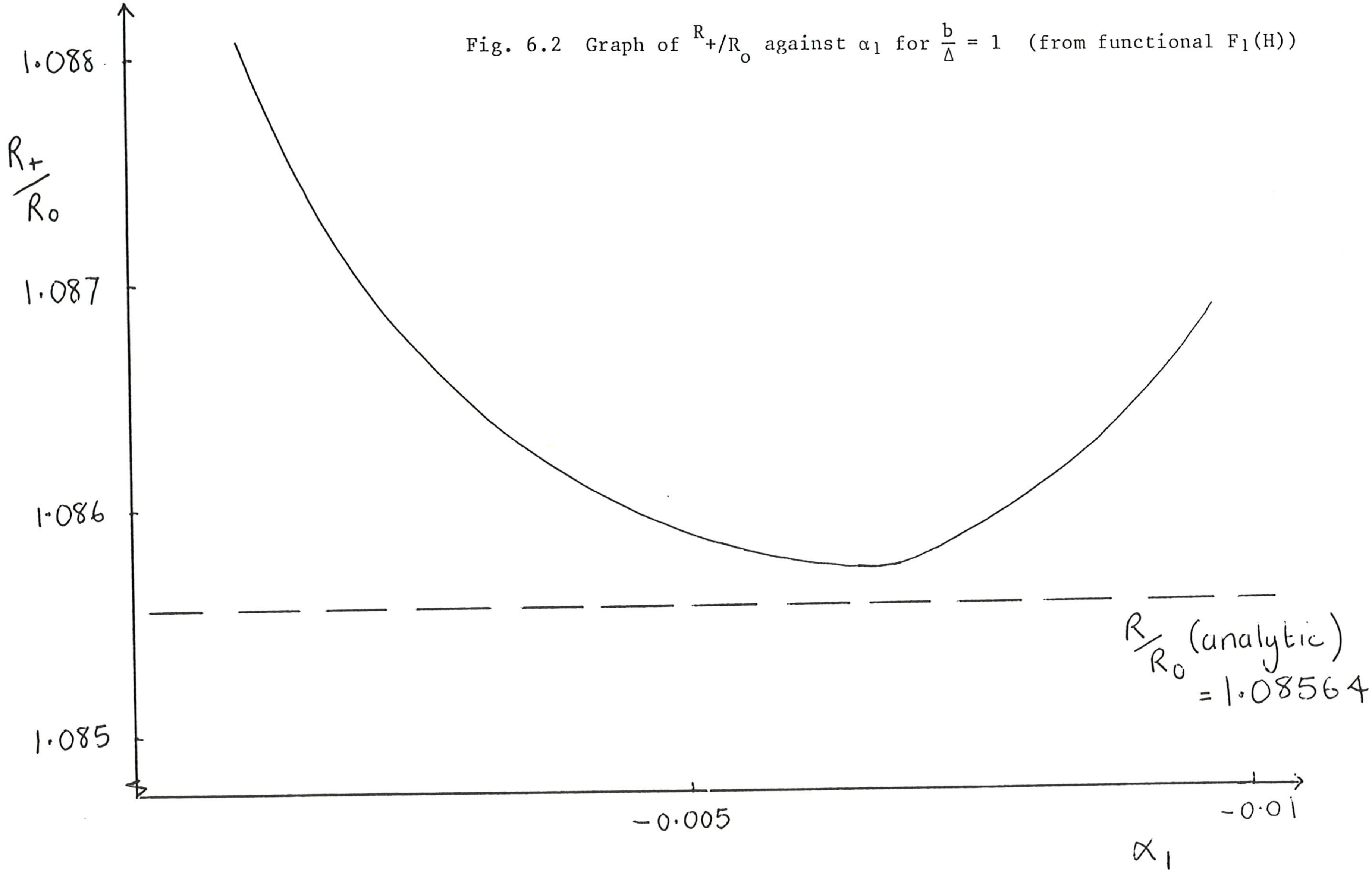


Fig. 6.3 Graph of $^{L}-/L_{o}$ against α_1 for $\frac{b}{\Delta} = 1$ (from functional $F_2(H)$)

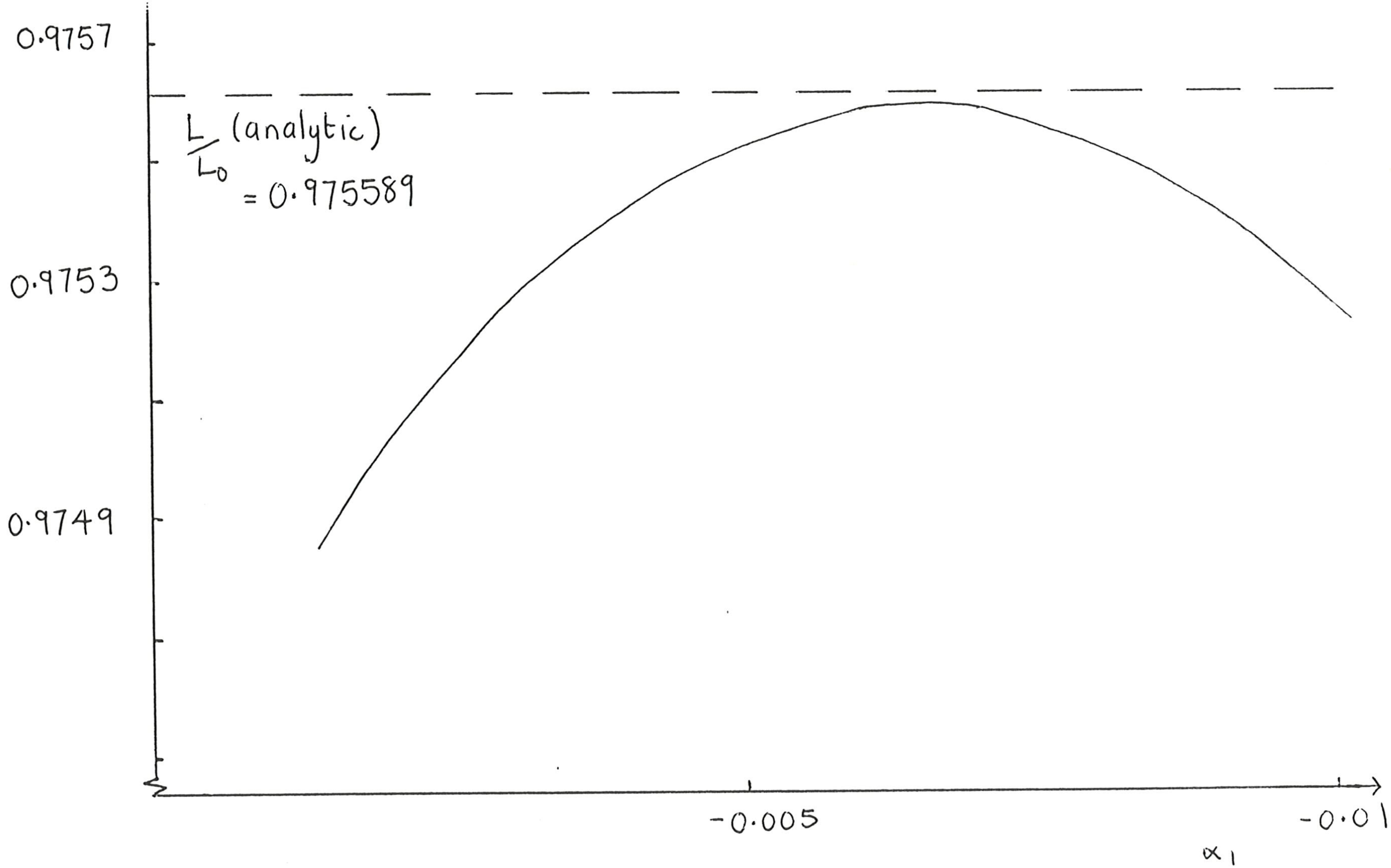


Fig. 6.4 Graph of R_-/R_o against α_2 for $\frac{b}{\Delta} = 1$ (from functional $F_3(H)$)

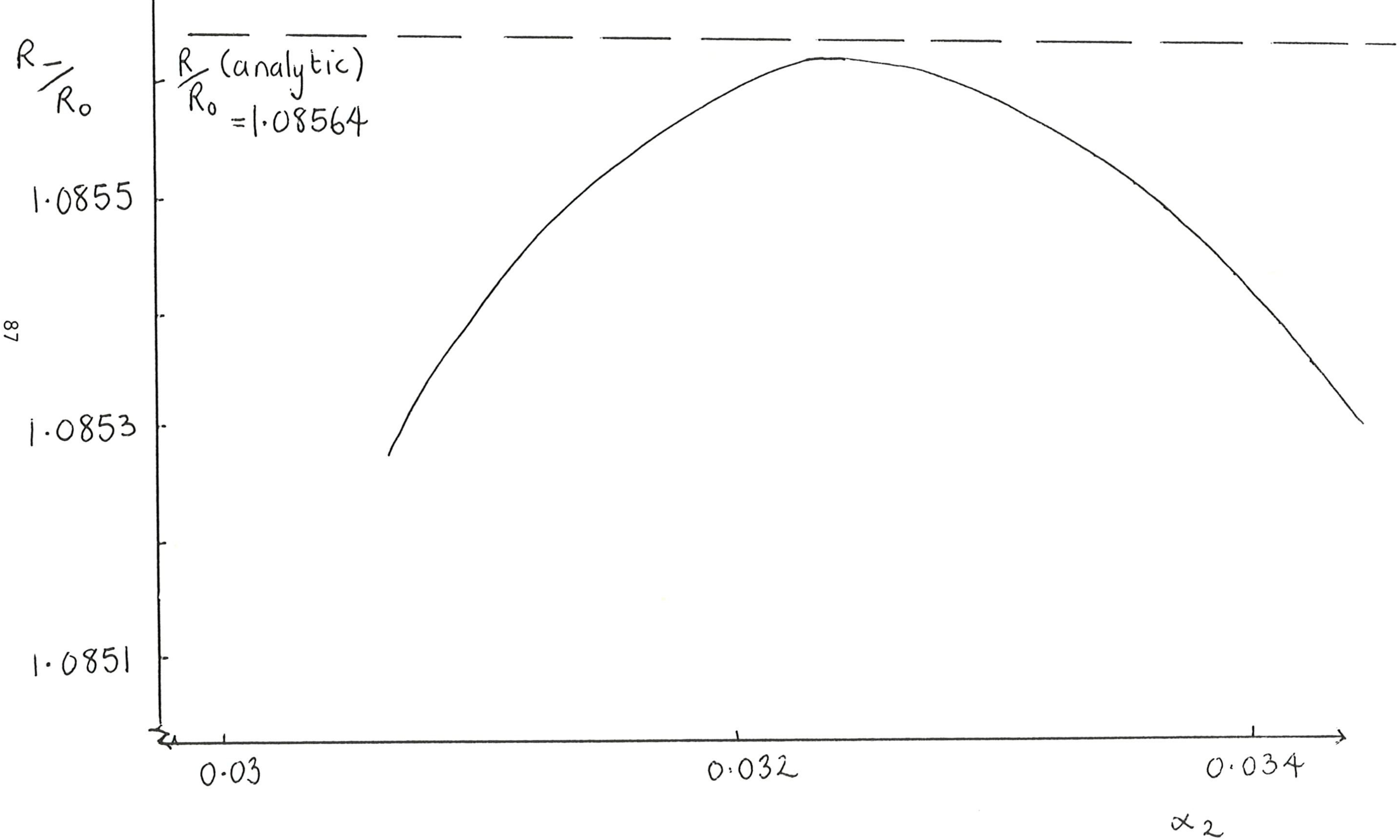
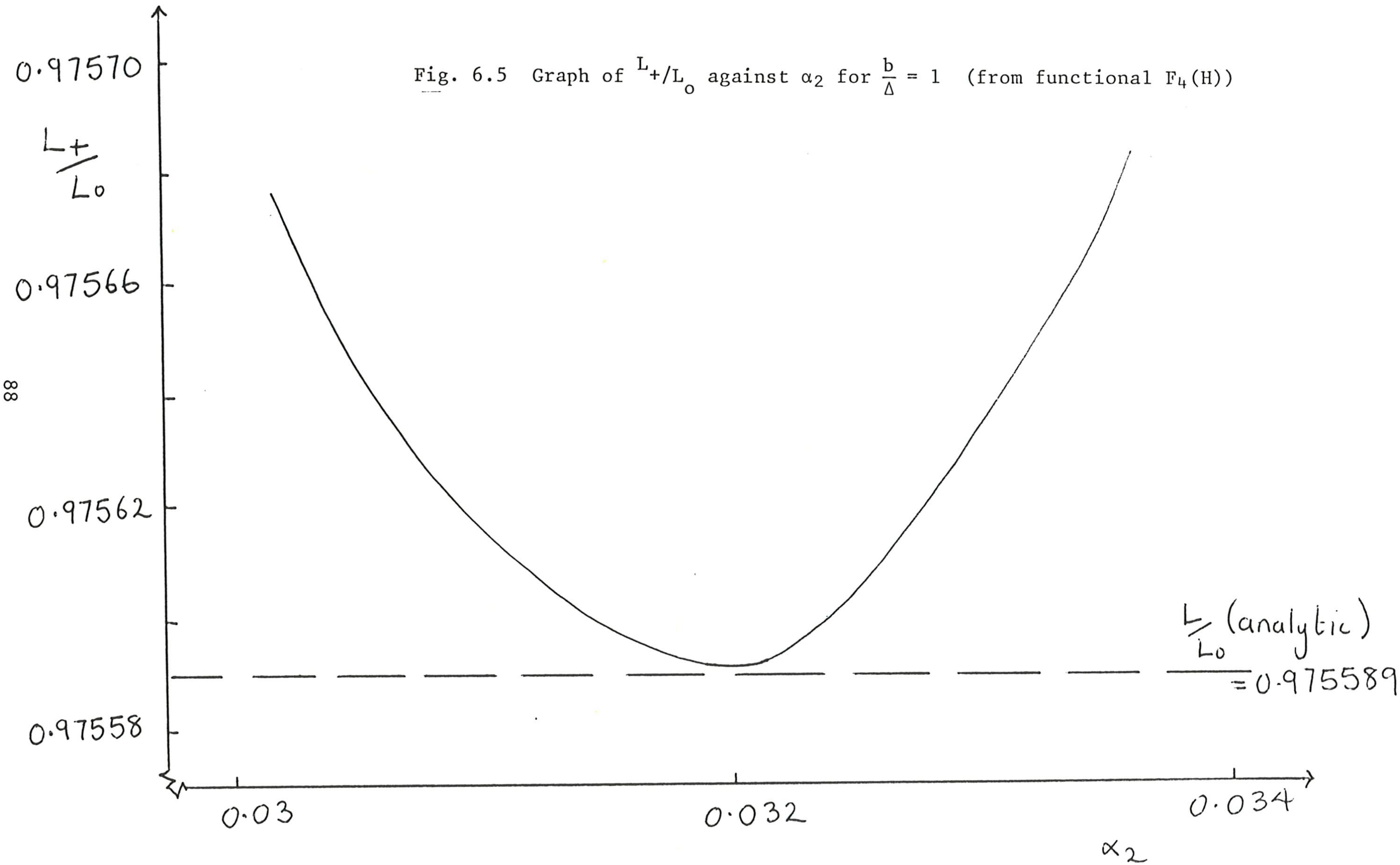


Fig. 6.5 Graph of L_+/L_o against α_2 for $\frac{b}{\Delta} = 1$ (from functional $F_4(H)$)



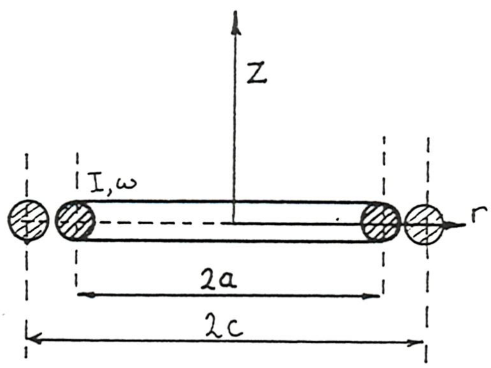


Fig. 7.1 Coils in free space

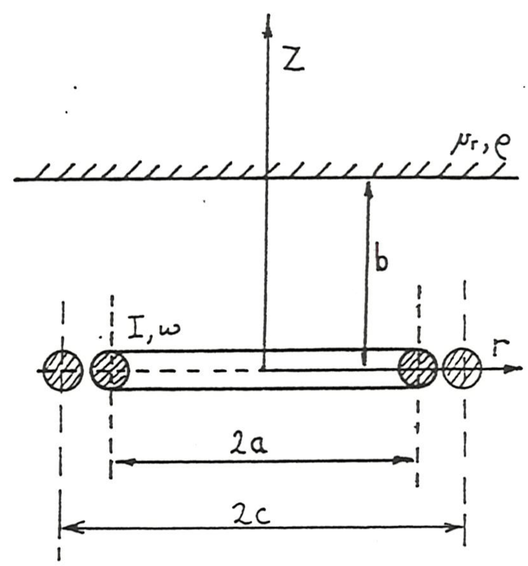


Fig. 7.2 Coils adjacent to semi-infinite conductor

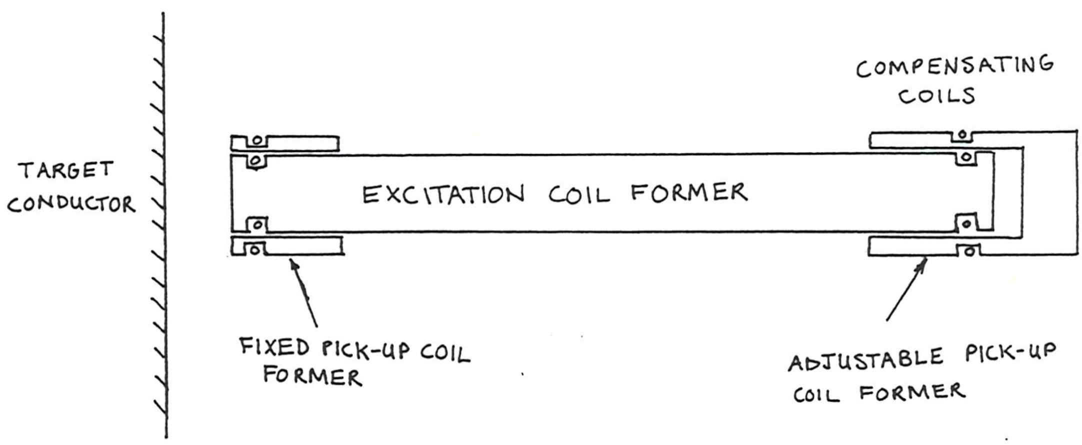


Fig. 7.3 Coil arrangement

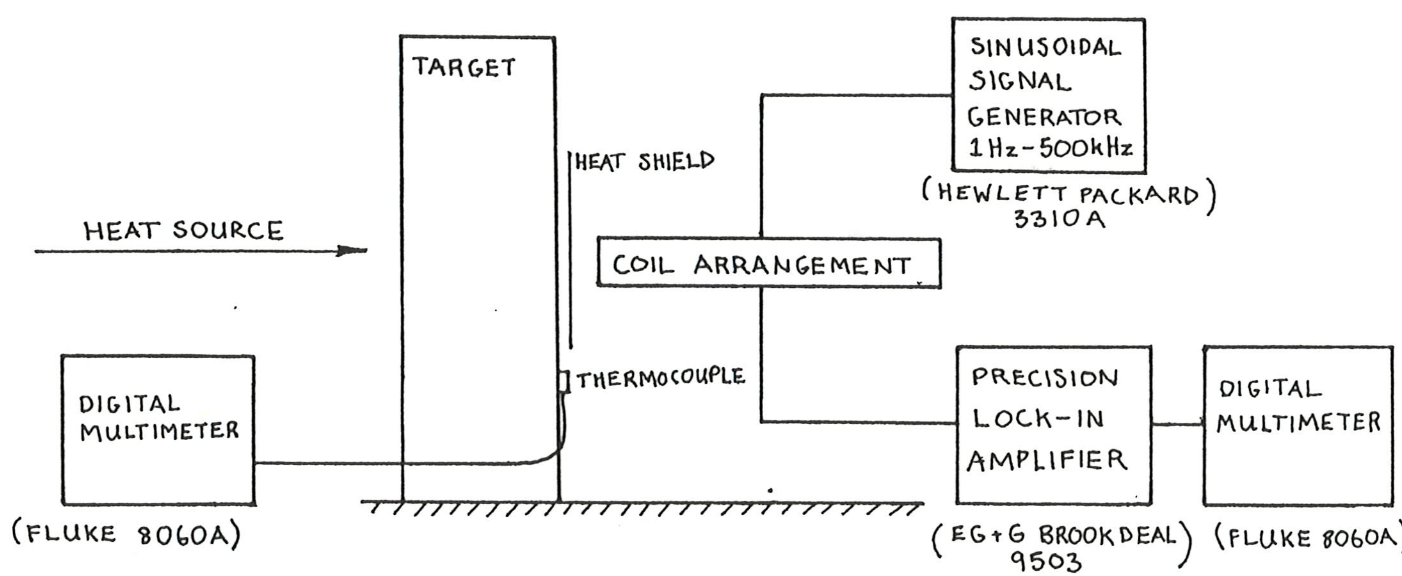


Fig. 7.4 Experimental arrangement

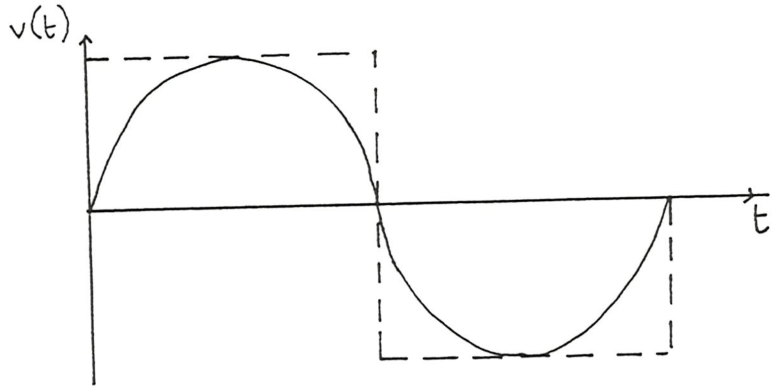


Fig. 7.5 Measurement of in-phase component

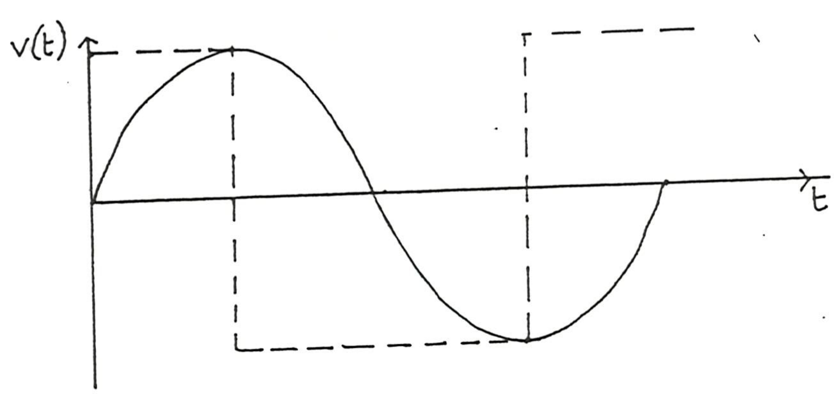


Fig. 7.6 Measurement of phase quadrature component

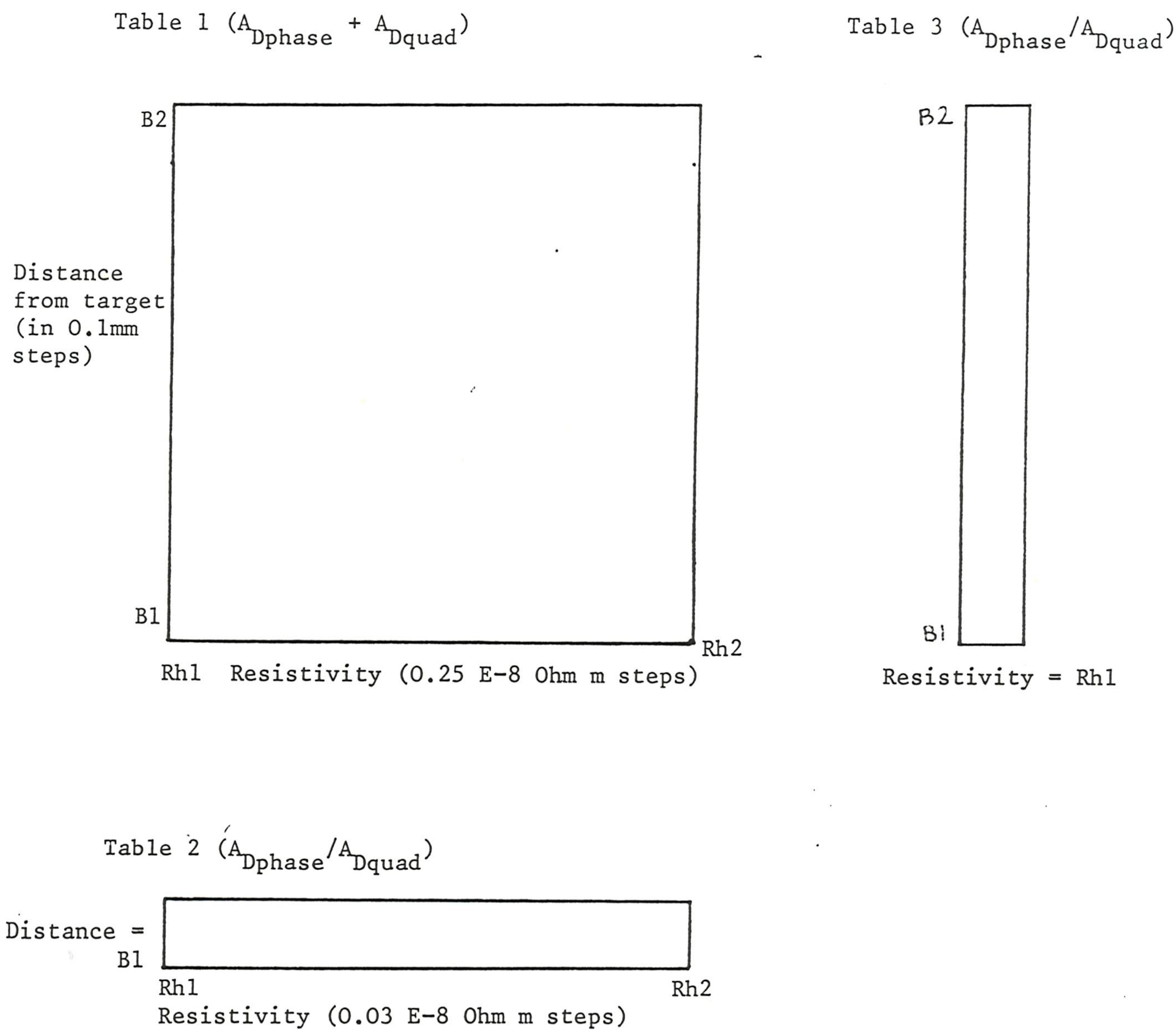
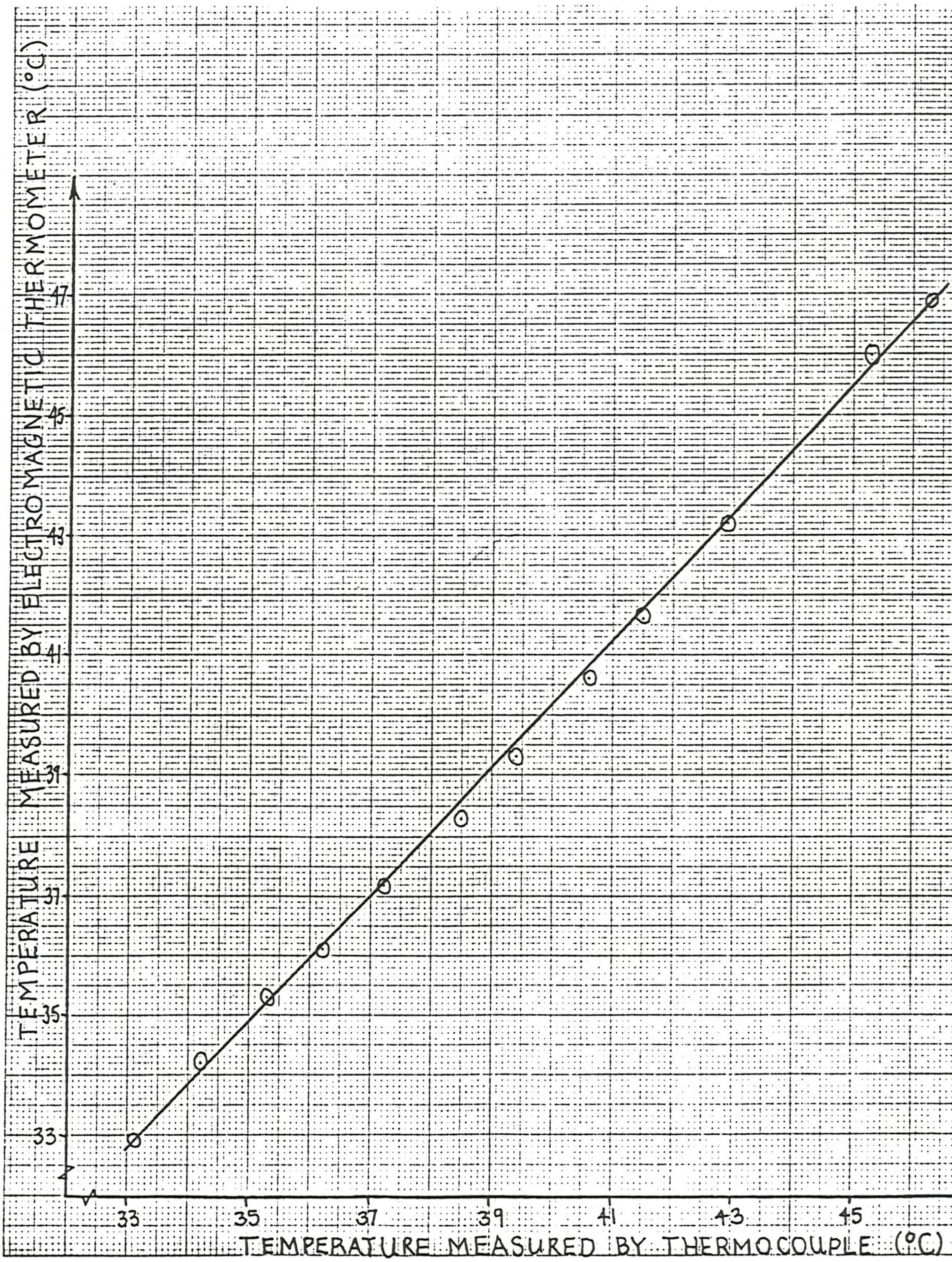


Fig. 7.7 Data tables

Fig. 7.8 Results of temperature measurement test



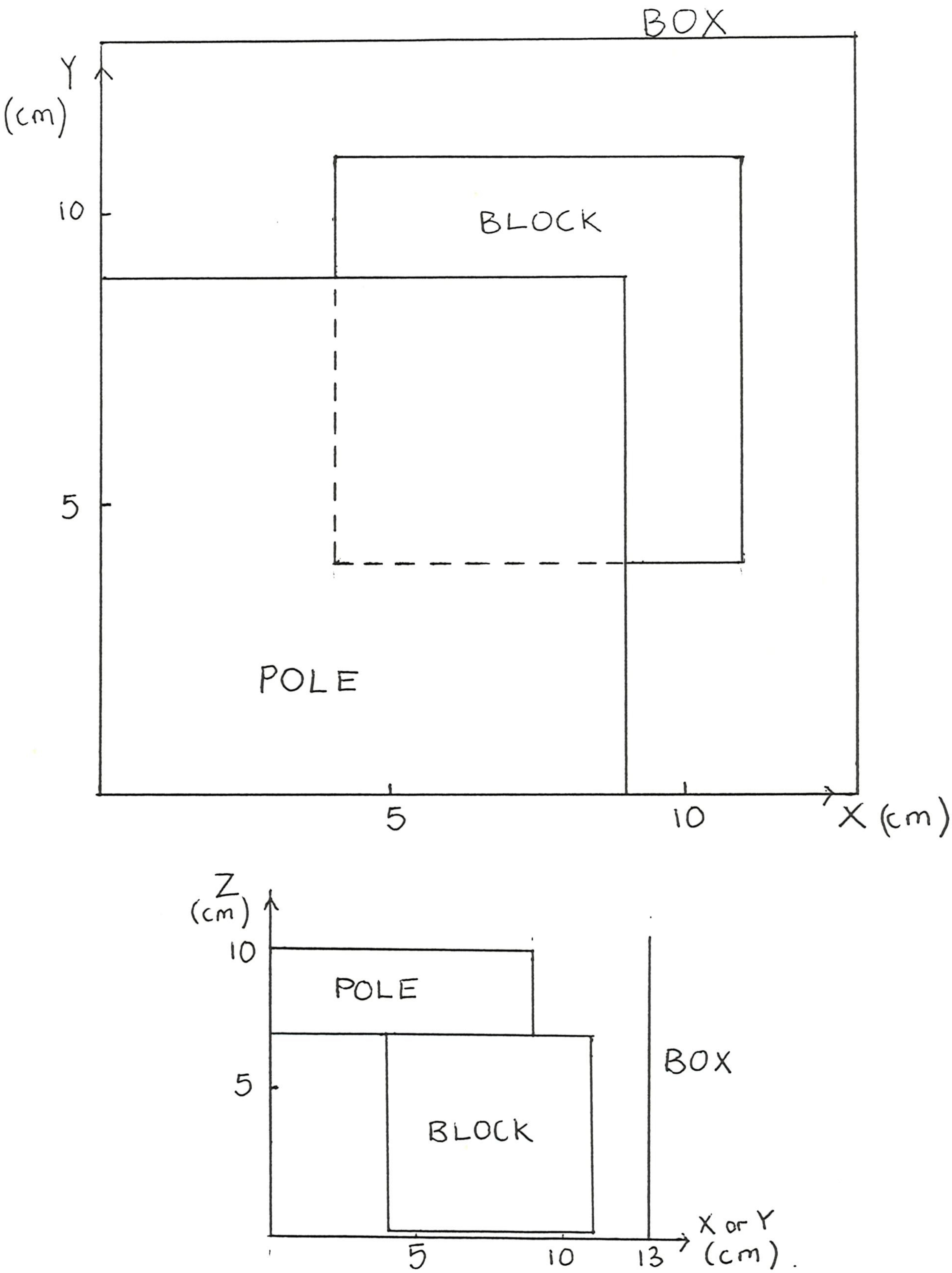


Fig. 8.1 Bath cube (plan view and elevation view)

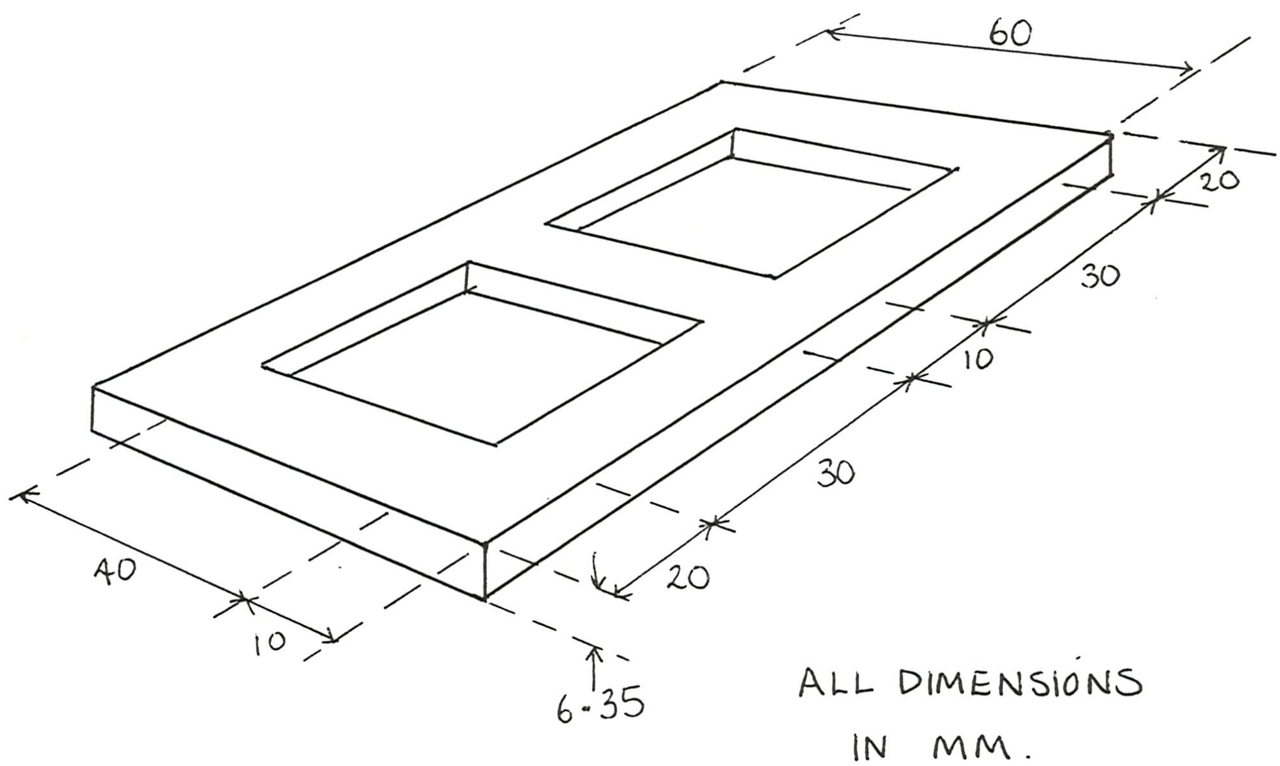


Fig. 8.2 Bath plate

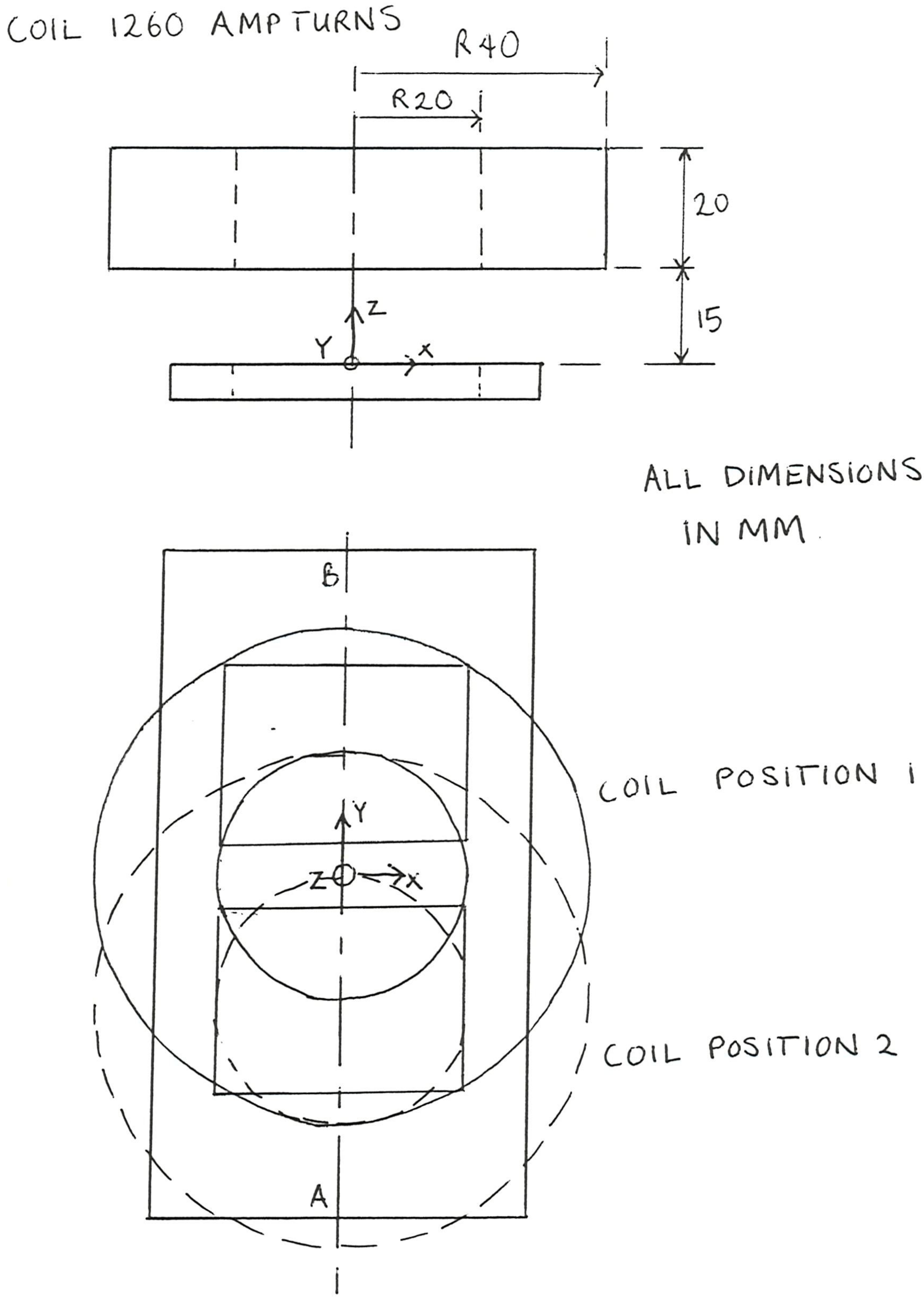


Fig. 8.3 Plan view and side view of bath plate

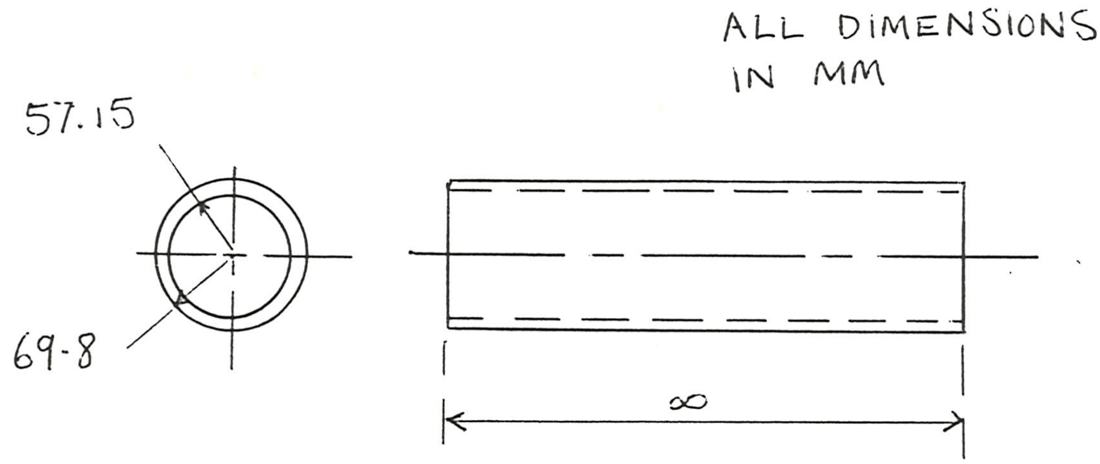


Fig. 8.4 Infinitely long conducting cylinder

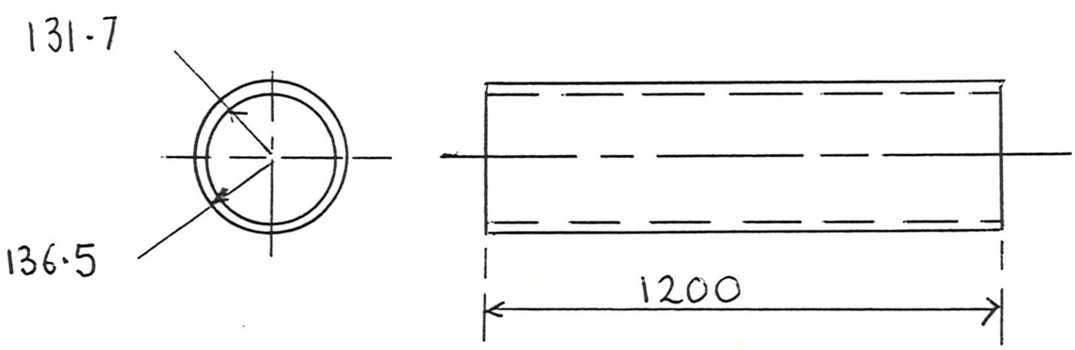


Fig. 8.5 FELIX long cylinder experiment

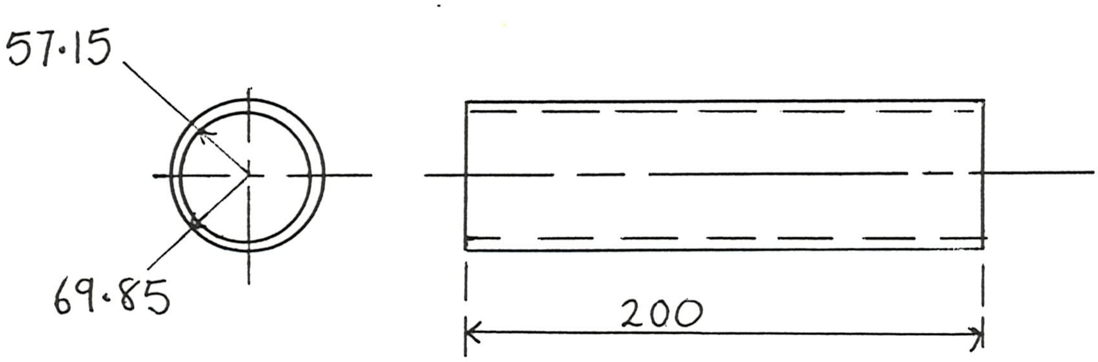


Fig. 8.6 FELIX short cylinder experiment

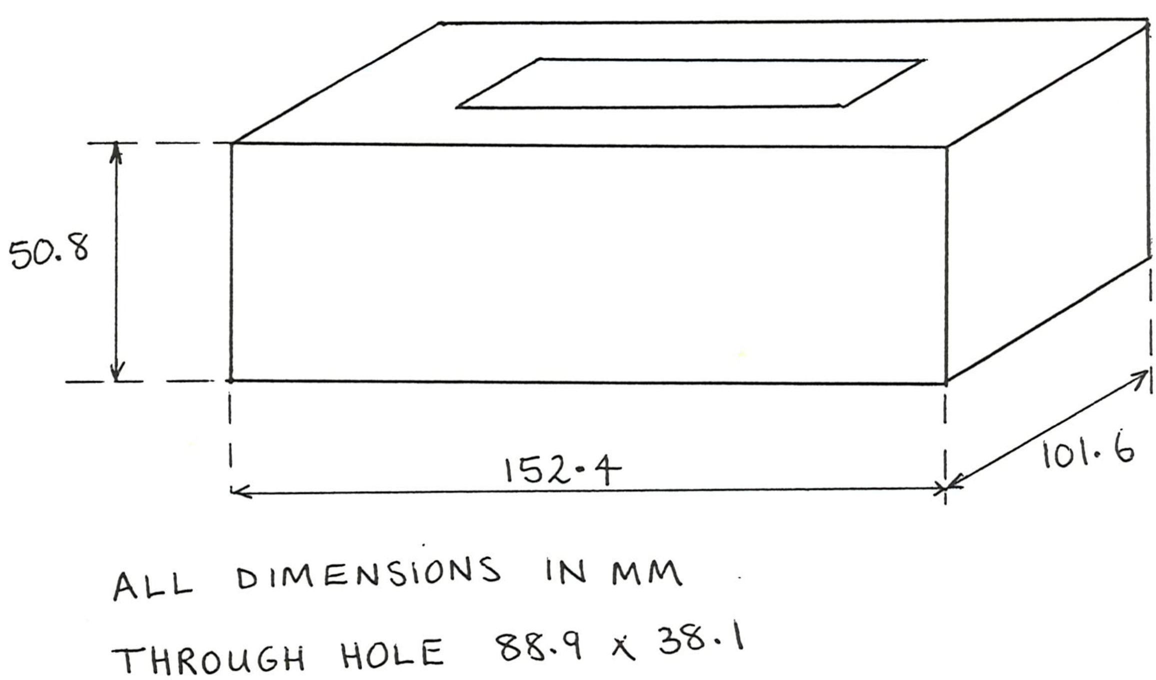


Fig. 8.7 FELIX Brick