UNIVERSITY OF SOUTHAMPTON

Young Tableaux and Modules of Groups and Lie Algebras

by

Trevor Alan Welsh

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This thesis was submitted for examination in September 1992.

This thesis is dedicated with love to my mum Noreen Welsh.
Thanks for everything Mummy.

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UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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YOUNG TABLEAUX AND MODULES OF GROUPS AND LIE ALGEBRAS

by Trevor Alan Welsh

In this thesis, Young tableaux are used to provide a very convenient explicit description of all the irreducible modules of the classical Lie groups and their Lie algebras, and a large class of irreducible modules of the general linear Lie supergroups and their Lie superalgebras. An original account of the Specht module techniques for the symmetric groups is also presented.

For each irreducible module, a basis is provided by a set of Young tableaux which index the weights of the module. The action of the group or algebra in question on these 'standard' tableaux is entirely natural. The result is, in general, a linear combination of non-standard tableaux. For each group, a standardisation algorithm is obtained which enables each non-standard tableaux to be expressed in terms of the basis of standard tableaux. For the symmetric groups and the general linear groups, this algorithm is provided by techniques developed by Garnir. This involves the Garnir relations which are closely related to the fundamental Young symmetrisers obtained by Young and based on the Young diagrams. Berele extended this construction by obtaining further relations between the tableaux based on Weyl's removal of trace tensors.

These ideas are extended to the mixed tensor representations of the general linear groups and to the orthogonal groups. In this latter case, new sets of standard tableaux are defined. For the spinor modules, it is necessary to develop a further class of relations. For the supergroups, a standardisation technique is obtained by coupling Garnir's methods with a graded symmetric group action.

In each construction, the standardisation algorithm involves simple coefficients, often integral. Consequently, the resulting matrix elements are especially simple. Each of the algorithms is exemplified, as well as the explicit construction of matrices representing elements of the various algebras.

Preface

At the time of submission, the work presented in this thesis has spawned the following papers:

R. C. King and T. A. Welsh, Construction of Orthogonal Group Modules Using Tableaux, *Linear and Multilinear Algebra*, to appear (1992);

R. C. King and T. A. Welsh, Construction of $GL(n)$-Modules Using Composite Tableaux, *Linear and Multilinear Algebra*, to appear (1992);

R. C. King and T. A. Welsh, Construction of Graded Covariant $GL(m/n)$-Modules Using Tableaux, *J. Algebraic Combinatorics* **1** (1992), 151–170.

All the module constructions described in this thesis, apart from the spinor modules of the orthogonal groups, have been implemented as computer programs. These are written in the language 'C' and are currently running in an MSDOS environment. These programs provide the following suite of facilities for each irreducible module:

(i) calculation of its dimension by means of a formula;

(ii) generation and display of the appropriate standard tableaux;

(iii) calculation of weight multiplicities;

(iv) standardisation of an arbitrary tableau — each step in the standardisation procedure is displayed, if desired;

(v) calculation of the explicit matrix representation of a specified element of the Lie algebra (or symmetric group element in the case of the Specht modules);

(vi) checking of the appropriate commutation relations;

(vii) generation of representation matrices via those of the simple root vectors (or simple transpositions in the case of the Specht modules).

In addition to the work presented in this thesis, research was undertaken to develop algorithms to determine weight multiplicities and tensor products of irreducible highest weight representations of affine Kac-Moody algebras. A concise summary of this work may be found in the following paper:

R. C. King and T. A. Welsh, Tensor Products of Affine Kac-Moody Algebras, in *Proceeding of the XVIII International Colloquium on Group Theoretical Methods in*

Physics, Eds. V. V. Dodonov and V. I. Mankov, *Lecture Notes in Physics* **382**, Springer-Verlag, Berlin (1991), 508–511.

A computer program has been written dealing with the constructions described in this paper for the two affine Kac-Moody algebras of rank one.

This thesis has been typeset using 'plain' TEX.

Trevor Welsh, September 1992.

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Chapter 1
Introduction

§1.1. Historical perspective

Over the past century, a great literature has amassed concerning the ubiquitous role of Young tableaux in the theory of group representations and characters. Many of these works occur in a purely combinatorial context, casting little light on the significance of the Young tableaux framework. In this thesis, the reader is guided on a journey through the classical groups and in each case, the utility of Young tableaux is pinpointed. More specifically, very convenient bases for the irreducible modules of the symmetric group, the classical Lie groups, their Lie algebras, the general linear supergroup, and its Lie superalgebra are constructed in terms of Young tableaux. The action of the group or algebra in question on these basis elements is determined and techniques are developed for rewriting the result in terms of the basis elements. In this way, the irreducible modules are constructed explicitly.

The origin of the usefulness of the Young diagrams may be traced to their enumeration of the classes of the symmetric group. Through the work of Frobenius at the beginning of the Twentieth Century, it was shown that partitions, and hence their diagrammatic representation as Young diagrams, serve to enumerate the irreducible representations of the symmetric groups. In the following two decades, Young brought the tableau to life through a diagrammatic construction [**Yo77,Ru68**] of the minimal idempotents of the symmetric group algebra. Although now bearing the name 'Frobenius algebra', the group algebra was at the time being used by Young, ignorant of Frobenius's work. The idempotents constructed by Young were termed 'Young symmetrisers' by Weyl. Young developed various constructions of irreducible S_l-modules based upon his tableaux. His notion of a standard tableau was a major step forward. Such tableaux are defined to have their entries ordered within their rows and columns. In this thesis they are referred to as S_l-standard tableaux, since many extensions of the notion will arise. Young showed that the S_l-standard tableaux serve to enumerate the dimension of the representation of S_l indexed by the underlying diagram. This fact was somewhat indirectly used in his module constructions. An altogether more direct construction was carried out much later, in the work of Specht [**Sp35**] and Garnir [**Ga50**]. Here, for each particular Young diagram, the S_l-standard tableaux are taken to form the basis for the irreducible S_l-module associated with that diagram. The symmetric group acts on these tableaux in a natural way, consistent with Young's original derivation of the symmetrisers. In general, the result of this action is a tableau which is not S_l-standard. Garnir devised an algorithm by which such non-standard tableaux may be written in terms of S_l-standard tableaux, thereby completing the specification of the explicit module. His techniques involve the so called Garnir relations which are intimately related to the Young symmetrisers. These relations prove extremely useful in the developments of this thesis. The construction of the Young symmetrisers, Specht modules and Garnir relations is detailed in Chapter 3.

Also around the turn of the century, Schur derived his double centraliser theorem [**Sc01**]. Using the work of Frobenius and Young, he exploited the dual centralising actions of S_l and $GL(m)$ on the l-fold tensor space $V^{\otimes l}$ of the defining $GL(m)$-module V, to decompose $V^{\otimes l}$ into irreducible covariant $GL(m)$-modules which are indexed by partitions of not more than m parts. Weyl [**We39**] utilised Schur's ideas and the Young symmetrisers to project the irreducible $GL(m)$-modules out of $V^{\otimes l}$. The appearance of a Young tableau in [**We39**] showed that, to some extent, they were being used in his methods. In the case of $GL(m)$, it was found that tableaux based on a particular diagram which obey a simple ordering condition (different to the S_l case) enumerate the dimension of the irreducible representation corresponding to that diagram. These tableaux are often referred to as semistandard, but in this thesis, the more descriptive '$GL(m)$-standard' is coined. Such tableaux seem to have first been defined by Littlewood [**Li50**] who implied their use as a basis for the irreducible $GL(m)$-modules although this was not described explicitly. Nevertheless, under the Weyl-Schur decomposition of $V^{\otimes l}$, the groups $GL(m)$ and $SL(m)$, and their Lie algebras $gl(m)$ and $sl(m)$, act naturally on these tableaux. However, it is doubtful as to whether Littlewood could deal with the resulting non-standard terms. As described in [**JK81**], the explicit use of the Young symmetrisers in the decomposition enable Garnir's techniques to be applied in order to effect a standardisation once more. The resultant $GL(m)$-modules, for which the sets of appropriate $GL(m)$-standard tableaux comprise bases, are known as Weyl modules.

A great convenience of the above constructions of irreducible modules by means of the Young symmetrisers is that standardisation necessarily results in terms with integral coefficients. Since the elements of S_l and $GL(m)$ act naturally on the respective standard tableaux, it follows that the matrix elements of the resulting explicit representations of elements of S_l or $gl(m)$ are all integral. A further elegant feature of this construction from the abstract weight viewpoint is that each of the $GL(m)$-standard tableaux which form the basis has a well defined weight. Thus these tableaux index the weights of each representation and thus, through their sum, yield its character. For $GL(m)$, these characters are the Schur functions which are thus endowed with a combinatorial definition [**Li50,Sta71**]. Many of their properties, for example the famous Littlewood-Richardson rule [**Li50,Ro61**], are best expressed, and indeed proved, using Young tableaux. Sections 1 and 2 of Chapter 4 describe the construction of the Weyl modules.

It seems somewhat surprising that work aimed at extending these simple constructions to other classical groups has only been undertaken recently. In fact, very little has been published on the construction of explicit modules at all. In 1950, Gelfand and Zeitlin published a paper [**GZ50a**] (see also [**BBi63**]) in which the basis states of the irreducible modules of $U(m)$ (and hence $GL(m)$) are indexed by Gelfand patterns. However, in order to have orthogonal basis elements, the module action is extremely complicated, involving irrational coefficients, in general. This work was extended to the orthogonal groups [**GZ50b**]. More recently, a Verma module approach [**LM86**] produced an explicit description of basis states for various simple Lie algebras of limited rank. However, it is not clear how the result of an algebra action on these basis states could be rewritten in terms of that basis in a systematic way.

By considering trace tensors formed by contraction of $V^{\otimes l}$ with an antisymmetric non-degenerate bilinear form, Weyl [**We39**] showed that the irreducible representations of $Sp(2r)$ may be labelled by a subset of those Young diagrams needed for the irreducible representations of $GL(2r)$; specifically they are those with not more than r rows. Some time later, King [**Ki76**] showed that the weights and characters of the irreducible representations of $Sp(2r)$ may be calculated by using certain tableaux, which are here called $Sp(2r)$-standard tableaux, based on the Young diagrams which index the particular representations. Berele amalgamated these two ideas to construct the irreducible $Sp(2r)$-modules [**Be86**]. His essential idea was to factor out the invariant trace submodules of $V^{\otimes l}$ described by Weyl and thus produce an extra relationship between the tableaux. He showed that this relationship, known as a Trace relation, together with the Garnir relations, enable any arbitrary tableau to be expressed in terms of the $Sp(2r)$-standard tableaux. This reduction involves only integers and thus the construction retains all the elegant features of the Weyl module. Berele's techniques, which are fundamental to the subsequent developments of this thesis, are elucidated in Section 4.3. This exposition differs considerably from that of Berele's concise account. The reason for this is twofold; firstly, it is desirable to expound on the elegance of the method; and secondly, the techniques provide a model upon which similar techniques for the other classical groups are developed.

In Section 4.4, the case of the mixed tensor (rational) representations of $GL(m)$ is considered. Weyl showed that these representations are indexed by certain generalised partitions which are permitted to have negative parts [**We39**]. Littlewood [**Li50**] noticed that this description is equivalent to that of an ordered pair of partitions satisfying a simple compatibility condition. These two partitions, in fact, specify Young symmetrisers acting independently on a covariant and a contravariant tensor space. This construction is conveniently depicted using composite Young diagrams and tableaux [**Ki70,Ki76**], upon both portions of which Garnir relations may be applied. In addition, Trace relations arise through Berele's process of factoring out the invariant subspace of, in this case, tensors resulting from the contraction of covariant and contravariant indices. By using the fact that the Kroneker product of a mixed tensor representation with a specific number of copies of the determinant representation is equivalent to a covariant tensor representation, King [**Ki76**] derived sets of composite tableaux which index the weights of these representations. As shown in Section 4.4, these once more provide bases for the irreducible mixed tensor $GL(m)$-modules. Section 4.5 is devoted to describing an association between the Garnir and Trace relations arising from the equivalence mentioned above.

In Chapter 5, an analogous construction for the irreducible $O(m)$- and $SO(m)$-modules is developed. For the $O(m)$ case, two difficulties need to be overcome. The first is that of the specification of an appropriate set of standard tableaux. It transpires that ever since Weyl specified the partitions that index the irreducible representations of $O(m)$ [**We39**], his index set has been ignored in favour of a subset appropriate to the $SO(m)$ case. This fact was recognised by Proctor [**Pr89**] who, using the ideas of King and El-Sharkaway [**KE83**], derived tableaux based on these partitions which index the weights of the representations. The second difficulty is that the appropriate Trace relations have to be applied over a pair of columns by virtue of the symmetry of the invariant form. This implies an interference between the Trace relations and the Garnir relations. This difficulty, coupled with the problem of the reduction to $SO(m)$, indicates that a different set of tableaux, which are closely related to Proctor's, should be used. These $O(m)$-standard tableaux are defined in Section 5.1. However, the standardisation procedure no longer involves only integers; factors of $1/2$ may appear. This construction and the reduction to $SO(m)$ are described in Sections 5.2 and 5.3 respectively. Section 5.4 is dedicated to the development of a Garnir-Trace relation duality similar to that which occurred for mixed tensor $GL(m)$-modules. An alternative construction of $O(m)$- and $SO(m)$-modules is outlined in Section 5.5, based on a further set of tableaux defined in **[Pr89]**, which do not index weights. Although only integers occur in the standardisation procedures, the reduction to $SO(m)$ involves complex numbers when $m = 2 \pmod 4$.

Since the groups $O(m)$ and $SO(m)$ are not simply but doubly connected, they necessarily possess two-valued representations. These provide genuine representations of the Lie algebras $so(m)$. Sets of tableaux which provide the weights and characters of these representations were first defined by King and El-Sharkaway **[KE83]**. The definitive investigation into the two-valued 'spinor' representations was carried by Brauer and Weyl **[BW35]**. In order to apply their techniques to the construction of the irreducible spinor $O(m)$-modules, it is necessary to generalise their use of Clifford algebras. This is carried out in Sections 6.1 and 6.2. By factoring out the invariant subspaces, relations between the tableaux, analogous to the Trace relations, are obtained. The standardisation procedure is developed in Sections 6.3, 6.4 and 6.5. Once more, this procedure involves relatively simple coefficients; they are simple rational numbers when m is even, and factors of $\sqrt{2}$ arise when m is odd. The reduction to $SO(m)$ is performed in Section 6.6.

Dondi and Jarvis **[DJ81]**, and later Berele and Regev **[BR83,BR87]**, discovered that a straightforward generalisation of Schur's action of the symmetric group on the l-fold tensor product $V^{\otimes l}$ of the defining module V of the general linear Lie supergroup $GL(m/n)$, enabled the double centraliser theorem to be applied in this case. Since this generalised (graded) symmetric group is isomorphic to the ordinary symmetric group, various properties of the irreducible representations of $GL(m/n)$ are similar to those of $GL(m)$. In particular, it was discovered that irreducible representations of $GL(m/n)$ are also indexed by partitions — in this case, those that lie in a 'hook' **[BR87]**. Correspondingly, the Young symmetrisers generalise. Thus Garnir's techniques may also be generalised to the case of the irreducible $GL(m/n)$-modules. The requisite $GL(m/n)$-standard tableaux were first defined in **[BR83]**. The irreducible $GL(m/n)$-modules are constructed in Chapter 7. The construction is extended to the Lie superalgebras $gl(m/n)$ and the basic classical Lie superalgebras $sl(m/n)$ in Section 7.4. Once more the standardisation algorithm involves only simple rational coefficients.

Each of the constructions described above, apart from that of the spinor modules of orthogonal groups, has been computer implemented. This has enabled the techniques of this thesis to be verified through checking that the representations generated satisfy the commutation relations of the appropriate Lie algebra. In addition, the programs have checked the examples presented.

The remainder of Chapter 1 is dedicated to the definition of the basic concepts in the theory of Lie groups, Lie algebras and representations.

In Chapter 2, the classical Lie groups and their Lie algebras are detailed. The classification of all their finite-dimensional irreducible representations is described, and formulae to calculate their dimensions presented. In addition, partitions, Young diagrams and Young tableaux are introduced, together with various associated notations.

§1.2. Lie groups

This thesis is concerned with various Lie groups, Lie algebras and their representations. These notions are introduced in this and the following sections. Since the general definition of a Lie group is not required in this thesis and embodies concepts beyond its scope, the more accessible definition of a linear Lie group is presented. The general definition may be found in [**Co84**], for example.

A faithful s-dimensional representation of a group $\mathcal{G}$ is an injective homomorphism $\Gamma : \mathcal{G} \to \mathcal{M}'_s$ of $\mathcal{G}$ into the set $\mathcal{M}'_s$ of invertible $s \times s$ matrices. Thus $\Gamma(g)\Gamma(h) = \Gamma(gh)$ for all $g, h \in \mathcal{G}$. If $\mathcal{G}$ possesses a faithful representation Γ then a metric $d_\Gamma : \mathcal{G} \times \mathcal{G} \to \mathbf{R}$ may be defined by:

$$d_\Gamma(g,h) = \left\{ \sum_{i=1}^{s} \sum_{j=1}^{s} |\Gamma(g)_{ij} - \Gamma(h)_{ij}|^2 \right\}^{\frac{1}{2}} .$$

This enables the group in question to be endowed with the topology of the m^2-dimensional complex Euclidean space $\mathbf{C}^{m^2}$.

Definition 1.2.1. *A linear Lie group $\mathcal{G}$ of dimension p is a group for which:*

(i) there exists a faithful finite-dimensional representation Γ;

(ii) there exists $\delta > 0$ such that the set $\mathcal{G}_\delta \subset \mathcal{G}$ for which $G \in \mathcal{G}_\delta$ if and only if $d_\Gamma(G, I) < \delta$, where $I \in \mathcal{G}$ is the identity element, is uniquely parameterised by p real parameters $x_1, x_2, \ldots, x_p$, with $I \in \mathcal{G}$ parameterised by $0, 0, \ldots, 0$;

(iii) there exists $\epsilon > 0$ such that if $\mathbf{R}^p_\epsilon$ denotes the set of all $(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p$ for which

$$\sum_{i=1}^{p} x_i^2 < \epsilon,$$

then each point $(x_1, x_2, \ldots, x_p) \in \mathbb{R}^p_\epsilon$ *corresponds to some group element* $G(x_1, x_2, \ldots, x_p) \in \mathcal{G}_\delta$*;*

(iv) each matrix element $\Gamma(G(x_1, x_2, \ldots, x_p))_{ij}$ *is an analytic function of the parameters* $x_1, x_2, \ldots, x_p$ *on the set* $\mathbb{R}^p_\epsilon$.

An immediate consequence of this definition is that if $\mathcal{G}$ is a linear Lie group then the matrices A_k, for $k = 1, 2, \ldots, p$, defined by:

$$(A_k)_{ij} = \left. \frac{\partial \Gamma(G(x_1, \ldots, x_p))_{ij}}{\partial x_k} \right|_{(0,\ldots,0)} \tag{1.2.2}$$

necessarily exist.

Definition 1.2.1 is especially convenient because the classical Lie groups, which are the main objects of study in this thesis, may be defined as matrix groups and thus a faithful representation is always readily available.

The following definitions and theorem are quoted from [**Co84**] where a full discussion is presented.

Definition 1.2.3. *A connected component of a linear Lie group is a maximal set of elements* $G \in \mathcal{G}$ *that can be obtained from each other by continuously varying the matrix elements* $\Gamma(G)_{ij}$ *of the faithful finite-dimensional representation* Γ *of* $\mathcal{G}$.

Definition 1.2.4. *A connected linear Lie group is a linear Lie group which possesses just one connected component.*

Definition 1.2.5. *A simply connected linear Lie group* $\mathcal{G}$ *is a connected linear Lie group for which every loop* $\rho : [0,1] \to \mathcal{M}'_s$ *for which* $\rho(0) = \rho(1)$*, with image in the domain* $\mathcal{M}_\Gamma$ *of the faithful s-dimensional representation* Γ*, is continuously contractible in* $\mathcal{M}_\Gamma$ *to a point.*

Theorem 1.2.6. *For every connected linear Lie group* $\mathcal{G}$ *there exists a simply connected group* $\tilde{\mathcal{G}}$ *for which* $\mathcal{G} = \tilde{\mathcal{G}}/K$ *for some discrete normal subgroup* K *of* $\tilde{\mathcal{G}}$.

The group $\tilde{\mathcal{G}}$ which appears in this theorem is known as the universal covering group of $\mathcal{G}$.

Definition 1.2.7. *Compact linear Lie group. A linear Lie group* $\mathcal{G}$ *of real dimension* p *having a finite number of connected components, is said to be compact if its real parameters* $x_1, x_2, \ldots, x_p$*, range over finite closed intervals.*

Since all the Lie groups encountered in this thesis are linear, the word 'linear' is omitted hereafter.

§1.3. Lie algebras

In this section Lie algebras are first introduced as infinitesimal generators of linear Lie groups, identified as matrix groups by means of some faithful finite-dimensional representation. They are then defined axiomatically.

Let $\mathcal{G}$ be a matrix Lie group of real dimension p. Consider a one-parameter subgroup $G(t) \in \mathcal{G}$ defined for $t \in \mathbf{R}$ in a small neighbourhood of 0 and for which $G(0) = I$, the identity of $\mathcal{G}$. Then, by virtue of Definition 1.2.1, the matrix:

$$A = \frac{d}{dt}G(t)\bigg|_{t=0} \tag{1.3.1}$$

exists. By considering all one-parameter families of $\mathcal{G}$, it is easily shown that the resulting matrices (1.3.1) form a p-dimensional vector space with basis $\{A_k : k = 1, 2, \ldots, p\}$. This vector space is known as the real Lie algebra $\mathcal{L}_{\mathcal{G}}$ of $\mathcal{G}$.

Let $A \in \mathcal{L}_{\mathcal{G}}$ be arbitrary. The matrix differential equation (1.3.1) may be solved with the constraints $G(0) = I$ and

$$G(t_1)G(t_2) = G(t_1 + t_2), \tag{1.3.2}$$

to yield $G(t) = \exp(At)$. Thus any element of $\mathcal{L}_{\mathcal{G}}$ may be exponentiated to obtain a one-parameter subgroup of $\mathcal{G}$.

It follows directly from the Campbell-Baker-Hausdorff formula (see [**Co84**]) that if $A, B \in \mathcal{L}_{\mathcal{G}}$ then the commutator $[A, B] \in \mathcal{L}_{\mathcal{G}}$, where:

$$[A, B] = AB - BA. \tag{1.3.3}$$

Abstractly, a Lie algebra is defined as follows.

Definition 1.3.4. *$\mathcal{L}$ is a Lie algebra over the field* F *if and only if $\mathcal{L}$ is a vector space over* F, *for which a product $[a, b]$ is defined such that:*

(i) $[a, b] \in \mathcal{L}$;

(ii) $[a, \beta b + \gamma c] = \beta[a, b] + \gamma[a, c]$;

(iii) $[a, b] = -[b, a]$;

(iv) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$,

for all $a, b, c \in \mathcal{L}$ and all $\beta, \gamma \in \mathsf{F}$. The first three of these requirements are known as the closure, linear, and anticommutation properties respectively; the fourth is known as the Jacobi identity. If $\mathsf{F} = \mathbf{C}$ then $\mathcal{L}$ is known as a complex Lie algebra and if $\mathsf{F} = \mathbf{R}$ then $\mathcal{L}$ is known as a real Lie algebra.

It is to be noted that any real Lie algebra becomes a complex Lie algebra on extending **R** to C. In general, each complex Lie algebra arises in this way from a number of distinct real Lie algebras. These real Lie algebras are known as real forms of the complex Lie algebra so obtained.

It is easily verified that any associative algebra which is closed under the product (1.3.3) satisfies the requirements of Definition 1.3.4. Thus, in particular, Definition 1.3.4 encompasses each Lie algebra $\mathcal{L}_{\mathcal{G}}$ resulting from a matrix Lie group $\mathcal{G}$.

The classification scheme for Lie algebras involves the following definitions.

Definition 1.3.5. *A Lie algebra $\mathcal{L}$ is said to be abelian if* $[a, b] = 0$ *for all* $a, b \in \mathcal{L}$.

Definition 1.3.6. *A Lie algebra $\mathcal{L}$ is said to be simple if it is not abelian and possesses no proper ideals.*

Definition 1.3.7. *A Lie algebra $\mathcal{L}$ is said to be semisimple if it possesses no proper abelian ideals.*

The classification of complex semisimple Lie algebras was completed in 1894 by Cartan [**Ca94**] who established the following theorem.

Theorem 1.3.8. *Every semisimple Lie algebra is a direct sum of simple Lie algebras.*

Cartan's classification scheme then involves four countable sequences of simple complex Lie algebras, denoted A_r for $r \geq 1$, B_r for $r \geq 2$, C_r for $r \geq 3$ and D_r for $r \geq 4$, and five exceptional Lie algebras, denoted G_2, F_4, E_6, E_7 and E_8. In this notation the integer appearing as the subscript is known as the rank of the algebra and gives the dimension of the maximal abelian subalgebra. The complex Lie algebras B_1, C_1, C_2, D_1, D_2 and D_3 are defined but are either isomorphic to those already given, or semisimple or, in the case of D_1, abelian. In fact,

$$C_1 \cong B_1 \cong A_1, \qquad C_2 \cong B_2, \qquad D_3 \cong A_3, \qquad D_2 \cong A_1 \oplus A_1. \tag{1.3.9}$$

In addition to the simple complex Lie algebras, Cartan determined the simple real Lie algebras. These are denoted using a notation similar to that used for the simple complex Lie algebras (see [**Co84**]).

§1.4. Representations and modules

In this section a representation is formally defined. Throughout, attention is confined to finite-dimensional representations. The equivalent notion of a module is also introduced. In addition, Schur's lemmas are stated and the adjoint representation defined.

Definition 1.4.1. *An s-dimensional representation of a group $\mathcal{G}$ is a map*

$$\Gamma : \mathcal{G} \to \mathcal{M}'_s(\mathsf{F}), \tag{1.4.1a}$$

onto $\mathcal{M}'_s(\mathsf{F})$, the set of $s \times s$ non-singular matrices over a field F, such that

$$\Gamma(gh) = \Gamma(g)\Gamma(h), \tag{1.4.1b}$$

for all $g, h \in \mathcal{G}$. If $\mathcal{G}$ is a Lie group, it is also required that Γ is a continuous map.

For groups actually defined in terms of matrices, for example the classical groups, there already exists a representation, called the defining representation, which maps every group element onto itself.

It is often convenient to discuss representations in terms of modules.

Definition 1.4.2. *An s-dimensional $\mathcal{G}$-module V is an s-dimensional vector space over F on which an action $\mathcal{G} : V \to V$ is defined such that:*

(i) $Iv = v$, where $I \in \mathcal{G}$ is the identity element;

(ii) $g(\mu v + \nu w) = \mu(gv) + \nu(gw)$;

(iii) $(gh)v = g(h(v))$,

for all $g, h \in \mathcal{G}$, all $\mu, \nu \in \mathsf{F}$, and all $v, w \in V$.

Given an s-dimensional representation Γ of $\mathcal{G}$, a $\mathcal{G}$-module is constructed by introducing a vector space V with basis $\{v_1, v_2, \ldots, v_s\}$ and defining:

$$gv_i = \sum_{j=1}^{s} \Gamma(g)_{ji} v_j, \tag{1.4.3}$$

for $i = 1, 2, \ldots, s$, and all $g \in \mathcal{G}$. By linearly extending this action to the whole of V, a $\mathcal{G}$-module is constructed, as is easily verified. Conversely, an s-dimensional $\mathcal{G}$-module V leads to a representation Γ by the introduction of a basis $\{v_1, v_2, \ldots, v_s\}$ for V, and for each $g \in \mathcal{G}$ defining $\Gamma(g)_{ji}$ for $1 \leq i, j \leq s$ by (1.4.3).

Definition 1.4.4. *Equivalent representations. Let Γ, Γ' be two s-dimensional representations of $\mathcal{G}$. If there exists a non-singular $s \times s$ matrix S such that $\Gamma'(g) =$*

$S^{-1}\Gamma(g)S$ for all $g \in \mathcal{G}$ then the representations Γ and Γ' are said to be equivalent and the notation $\Gamma \cong \Gamma'$ is used.

From the module viewpoint, equivalent representations correspond to nothing more than a change of basis. In fact, under the change of basis $v'_k = \sum_{l=1}^{s} S_{lk} v_l$ of V, the linear operator $\Gamma(g)$ acting on V with respect to the basis $\{v_1, v_2, \ldots, v_s\}$, gives rise to the linear operator $\Gamma'(g)$ acting on the basis $\{v'_1, v'_2, \ldots, v'_s\}$, where $\Gamma'(g) = S^{-1}\Gamma(g)S$. Therefore equivalent representations are essentially the same representation.

Definition 1.4.5. *Reducible representations. The s-dimensional representation Γ of $\mathcal{G}$ is said to be reducible if there exists a non-singular $s \times s$ matrix S and an integer a such that $0 < a < s$ and, for all $g \in \mathcal{G}$, $S^{-1}\Gamma(g)S$ is of the block matrix form:*

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{1.4.5}$$

where the submatrices A, B, C and 0 are $a \times a$, $(s-a) \times (s-a)$, $a \times (s-a)$ and $(s-a) \times a$ respectively and the matrix 0 consists entirely of zero elements. If no such S exists, then Γ is termed irreducible.

If V is the $\mathcal{G}$-module corresponding to Γ then this definition is equivalent to the statement that Γ is reducible if and only if V possesses a proper submodule W, in that the dimension of W is at least 1, $W \subset V$, $W \neq V$, and $gw \in W$ for all $w \in W$ and $g \in \mathcal{G}$. In such a case V is said to be a reducible $\mathcal{G}$-module. Conversely, if Γ is irreducible then V is also said to be irreducible.

Definition 1.4.6. *Decomposable representation. The s-dimensional representation Γ of $\mathcal{G}$ is said to be decomposable if there exists a non-singular $s \times s$ matrix S such that, for all $g \in \mathcal{G}$, $S^{-1}\Gamma(g)S$ is of the block matrix form:*

$$\begin{pmatrix} \Gamma^{(1)}(g) & 0 \\ 0 & \Gamma^{(2)}(g) \end{pmatrix}, \tag{1.4.6a}$$

where the submatrices $\Gamma^{(1)}(g)$ and $\Gamma^{(2)}(g)$ are $s_1 \times s_1$ and $s_2 \times s_2$ respectively for non-zero s_1, s_2 with $s_1 + s_2 = s$, and each 0 is the appropriately sized matrix consisting entirely of zero elements. If no such S exists, then Γ is termed indecomposable. If Γ is decomposable with $S^{-1}\Gamma(g)S$ given by (1.4.6a) then $\Gamma^{(1)}$ and $\Gamma^{(2)}$ define representations *of dimensions s_1 and s_2 respectively. This decomposition is known as a direct sum decomposition and is denoted:*

$$S^{-1}\Gamma(g)S = \Gamma^{(1)}(g) \oplus \Gamma^{(2)}(g). \tag{1.4.6b}$$

Similarly, the $\mathcal{G}$-module V is said to be decomposable if V can be written as the direct sum $V = W \oplus W'$ of two non-trivial $\mathcal{G}$-modules W and W'. If not, then V is indecomposable.

Definition 1.4.7. *Fully reducible representation. The representation Γ of $\mathcal{G}$ is said to be fully reducible if Γ can be expressed as the direct sum of a set of irreducible representations, in that there exist irreducible representations $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(p)}$ such that:*

$$\Gamma(g) \cong \Gamma^{(1)}(g) \oplus \Gamma^{(2)}(g) \oplus \cdots \oplus \Gamma^{(p)}(g), \tag{1.4.7}$$

for all $g \in \mathcal{G}$.

Once more this definition extends naturally to the module viewpoint.

Representations and modules of Lie algebras will now be introduced.

Definition 1.4.8. *An s-dimensional representation of a Lie algebra $\mathcal{L}$ is a map*

$$\Gamma : \mathcal{L} \to \mathcal{M}_s(\mathsf{F}), \tag{1.4.8a}$$

into $\mathcal{M}_s(\mathsf{F})$, the set of all $s \times s$ matrices over some field F, such that:

(i) $\Gamma(\alpha a + \beta b) = \alpha\Gamma(a) + \beta\Gamma(b)$;

(ii) $\Gamma([a, b]) = [\Gamma(a), \Gamma(b)]$,

for all $a, b \in \mathcal{L}$ and $\alpha, \beta \in \mathsf{F}$.

Once more, an equivalent notion of an $\mathcal{L}$-module exists which may be formally defined as follows.

Definition 1.4.9. *An s-dimensional $\mathcal{L}$-module V is an s-dimensional vector space on which an action $\mathcal{L} : V \to V$ is defined such that:*

(i) $a(\mu v + \nu w) = \mu(av) + \nu(aw)$;

(ii) $(\alpha a + \beta b)v = \alpha(av) + \beta(bv)$;

(iii) $[a, b]v = a(bv) - b(av)$,

for all $a, b \in \mathcal{L}$, all $v, w \in V$ and all $\alpha, \beta, \mu, \nu \in \mathsf{F}$.

The two notions of a representation of $\mathcal{L}$ and an $\mathcal{L}$-module may be shown to be equivalent in the same way as above for $\mathcal{G}$.

The concepts of reducibility and decomposability extend directly to the case of representations and modules of Lie algebras. A representation of a Lie group $\mathcal{G}$ gives rise to a representation of its Lie algebra $\mathcal{L}_{\mathcal{G}}$ and this representation is reducible, decomposable or fully reducible if that of $\mathcal{G}$ is reducible, decomposable or fully reducible respectively.

The following two lemmas which are known as Schur's lemmas apply to any group, or any algebra over an algebraically closed field. Proofs may be found in [**CR62,Bo63,Co84**].

Lemma 1.4.10. [**Sc01**] *Let $\mathcal{A}$ be either an algebra or a group, and let Γ, Γ' be irreducible representations of $\mathcal{A}$ of dimensions s and s' respectively. If there exists an $s \times s'$ matrix S such that:*

$$\Gamma(a)S = S\Gamma'(a), \tag{1.4.10}$$

for all $a \in \mathcal{A}$ then either $S = 0$, or $s = s'$ and S is non-singular.

Lemma 1.4.11. [**Sc01**] *Let $\mathcal{A}$ be either an algebra over an algebraically closed field or a group, and let Γ be an s-dimensional irreducible representation of $\mathcal{A}$. If there exists an $s \times s$ matrix S such that:*

$$\Gamma(a)S = S\Gamma(a), \tag{1.4.11}$$

for all $a \in \mathcal{A}$ then S is a multiple of the unit matrix I_s.

The following theorem of Weyl [**We39**] will be of great value.

Theorem 1.4.12. *Every representation of a compact Lie group $\mathcal{G}$ is fully reducible. Every representation of the corresponding complex Lie algebra $\mathcal{L}_{\mathcal{G}}$ is fully reducible.*

In the theory of group representations, the character of a representation contains much important information. It is introduced via the following definitions.

Definition 1.4.13. *The trace of a matrix. If M is an $s \times s$ matrix with elements M_{ij}, its trace, $\operatorname{tr} M$, is defined by:*

$$\operatorname{tr} M = \sum_{i=1}^{s} M_{ii}. \tag{1.4.13}$$

Definition 1.4.14. *The character of a representation. If Γ is a representation of $\mathcal{G}$, the function* $\mathrm{ch}(\Gamma) : \mathcal{G} \to \mathsf{F}$ *which assigns to each element of $\mathcal{G}$ the trace of its representation matrix:*

$$\mathrm{ch}(\Gamma)(g) = \mathrm{tr}\,\Gamma(g), \tag{1.4.14}$$

is known as the character of Γ.

The following two lemmas concerning the character have standard straightforward proofs.

Lemma 1.4.15. *The character of a representation of a group is a class function in that its value is constant within a class.*

Lemma 1.4.16. *Equivalent representations have the same character.*

The proof of the following lemma may be found in [**Co84**].

Lemma 1.4.17. *If two representations of a finite group or a compact Lie group have the same character, then the two representations are equivalent.*

Definition 1.4.18. *The adjoint representation of a Lie algebra. Let the Lie algebra $\mathcal{L}$ have a basis $\{a_1, a_2, \ldots, a_s\}$ and let the structure constants $c_{ij}^k \in \mathsf{F}$ be defined by:*

$$[a_i, a_j] = \sum_{k=1}^{s} c_{ij}^k a_k, \tag{1.4.18a}$$

for $i, j, k = 1, 2, \ldots, s$. For each a_i, define the matrix $\Gamma_{ad}(a_i)$ by

$$\Gamma_{ad}(a_i)_{kj} = c_{ij}^k, \tag{1.4.18b}$$

and extend this definition linearly to the whole of $\mathcal{L}$ by:

$$\Gamma_{ad}(\alpha a + \beta b)_{kj} = \alpha \Gamma_{ad}(a)_{kj} + \beta \Gamma_{ad}(b)_{kj}, \tag{1.4.18c}$$

for all $a, b \in \mathcal{L}$ and $\alpha, \beta \in \mathsf{F}$. By virtue of the constraints imposed on the structure constants by conditions (iii) and (iv) of Definition 1.3.4, *the matrices $\Gamma_{ad}(a)$, for $a \in \mathcal{L}$, form a representation of $\mathcal{L}$. Γ_{ad} is known as the adjoint representation of $\mathcal{L}$.*

The $\mathcal{L}$-module corresponding to the adjoint representation of $\mathcal{L}$ may be taken to be $\mathcal{L}$ itself, since if $b \in \mathcal{L}$ then, by (1.4.18*a*), (1.4.18*b*) and (1.4.18*c*):

$$[b, a_i] = \sum_{k=1}^{s} \Gamma_{ad}(b)_{ki} a_k. \tag{1.4.19}$$

The adjoint representation of $\mathcal{L}$ may be exponentiated to provide a representation of the corresponding connected Lie group. This representation is also known as the adjoint representation.

§1.5. Derived representations and modules

From a set of representations of a group or a Lie algebra, a number of other representations may be constructed. The most important are the contragredient of a representation, the direct sum of a pair (or more) of representations and the tensor product of a pair (or more) of representations. The following sequence of lemmas defines these representations and demonstrates that they are actually representations.

Lemma 1.5.1. *If Γ is an s-dimensional representation of $\mathcal{G}$ then the map $\hat{\Gamma} : \mathcal{G} \to \mathcal{M}'_s$ defined by:*

$$\hat{\Gamma}(g) = \widetilde{\Gamma(g)}^{-1}, \tag{1.5.1}$$

for $g \in \mathcal{G}$ where the tilde indicates transposition, is an s-dimensional representation of $\mathcal{G}$.

Proof.

$$\begin{aligned}\hat{\Gamma}(gh) &= \widetilde{\Gamma(gh)}^{-1} = \left(\widetilde{\Gamma(g)\Gamma(h)}\right)^{-1} \\ &= \left(\widetilde{\Gamma(h)}\widetilde{\Gamma(g)}\right)^{-1} = \widetilde{\Gamma(g)}^{-1}\widetilde{\Gamma(h)}^{-1} = \hat{\Gamma}(g)\hat{\Gamma}(h),\end{aligned}$$

where (1.4.1b) has been used.

The representation $\hat{\Gamma}$ defined by Lemma 1.5.1 is known as the contragredient of Γ. From the module viewpoint, the representation $\hat{\Gamma}$ arises from the action on a vector space V^* dual to V, this action preserving duality. Let V^* have basis $\{v^1, v^2, \ldots, v^s\}$ such that $v^i(v_k) = \delta^i_k$. Let the action of G on V^* be $gv^i = \Gamma'(g)_{ji} v^j$ for some matrix $\Gamma'(g)$. Then the preservation of duality requires that:

$$\begin{aligned}\delta^i_k &= (gv^i)(gv_k) \\ &= \sum_{j=1}^{s}\sum_{l=1}^{s} \Gamma'(g)_{ji} v^j (\Gamma(g)_{lk} v_l) \\ &= \sum_{j=1}^{s}\sum_{l=1}^{s} \Gamma'(g)_{ji} \Gamma(g)_{lk} \delta^j_l \\ &= \sum_{j=1}^{s} \Gamma'(g)_{ji} \Gamma(g)_{jk}.\end{aligned}$$

which implies that $\Gamma'(g) = \widetilde{\Gamma(g)}^{-1} = \hat{\Gamma}(g)$.

The contragredient representation $\hat{\Gamma}$ of $\mathcal{L}_{\mathcal{G}}$ is provided by differentiation of the representation $\hat{\Gamma}$ of $\mathcal{G}$ at the identity $I \in \mathcal{G}$. This yields:

$$\hat{\Gamma}(a) = -\widetilde{\Gamma(a)}, \tag{1.5.2}$$

for all $a \in \mathcal{L}$. It is easily verified that $\hat{\Gamma}$ satisfies Definition 1.4.8. Thus V^* is an $\mathcal{L}$-module on which each $a \in \mathcal{L}$ acts through the linear operator (1.5.2). The elements of V are known as covariant vectors and those of V^* as contravariant vectors.

The following lemma defines the direct sum of two representations. Its proof is straightforward.

Lemma 1.5.3. *If $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are representations of $\mathcal{G}$, having dimensions s_1 and s_2 respectively, then the map $\Gamma^{(1)} \oplus \Gamma^{(2)} : \mathcal{G} \to \mathcal{M}'_{s_1+s_2}$ defined by:*

$$(\Gamma^{(1)} \oplus \Gamma^{(2)})_{i,j} = \begin{cases} \Gamma^{(1)}(g)_{i,j} & \text{if } 1 \leq i,j \leq s_1; \\ \Gamma^{(2)}(g)_{i-s_1,j-s_1} & \text{if } s_1 < i,j \leq s_1 + s_2; \\ 0 & \text{otherwise,} \end{cases} \tag{1.5.3}$$

is an $(s_1 + s_2)$-dimensional representation of $\mathcal{G}$.

The representation $\Gamma^{(1)} \oplus \Gamma^{(2)}$ is known as the direct sum of $\Gamma^{(1)}$ and $\Gamma^{(2)}$. If $V^{(1)}$ and $V^{(2)}$ are $\mathcal{G}$-modules corresponding to the representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ respectively, then it is easily verified that $\Gamma^{(1)} \oplus \Gamma^{(2)}$ corresponds to $V^{(1)} \oplus V^{(2)}$ where:

$$g(v^{(1)} + v^{(2)}) = gv^{(1)} + gv^{(2)}. \tag{1.5.4}$$

The above notions of direct sum representations and direct sum $\mathcal{G}$-modules extend in a straightforward way to direct sums of more than two representations or $\mathcal{G}$-modules.

Direct sums of representations of Lie algebras and direct sums of $\mathcal{L}$-modules are defined in precisely the same way.

Lemma 1.5.5. *If $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are representations of $\mathcal{G}$, having dimensions s_1 and s_2 respectively, then the map $\Gamma^{(1)} \otimes \Gamma^{(2)} : \mathcal{G} \to \mathcal{M}'_{s_1 s_2}$ defined by:*

$$\left(\Gamma^{(1)} \otimes \Gamma^{(2)}\right)(g)_{ik,jl} = \Gamma^{(1)}(g)_{ij}\Gamma^{(2)}(g)_{kl}, \tag{1.5.5}$$

for $g \in \mathcal{G}$, is a $(s_1 s_2)$-dimensional representation of $\mathcal{G}$.

Proof.

$$\begin{aligned}\left(\Gamma^{(1)} \otimes \Gamma^{(2)}\right)(gh)_{ik,jl} &= \Gamma^{(1)}(gh)_{ij}\Gamma^{(2)}(gh)_{kl} \\ &= \sum_{m=1}^{s_1}\sum_{n=1}^{s_2} \Gamma^{(1)}(g)_{im}\Gamma^{(1)}(h)_{mj}\Gamma^{(2)}(g)_{kn}\Gamma^{(2)}(h)_{nl} \\ &= \sum_{m=1}^{s_1}\sum_{n=1}^{s_2} \left(\Gamma^{(1)} \otimes \Gamma^{(2)}\right)(g)_{ik,mn}\left(\Gamma^{(1)} \otimes \Gamma^{(2)}\right)(h)_{mn,jl},\end{aligned}$$

where, once more, (1.4.1b) has been used.

The representation $(\Gamma^{(1)} \otimes \Gamma^{(2)})$ is known as the tensor product of the representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ by virtue of this action arising from the tensor product of two $\mathcal{G}$-modules. Let $V^{(1)}$ and $V^{(2)}$ be the $\mathcal{G}$-modules corresponding to the representations $\Gamma^{(1)}$ and $\Gamma^{(2)}$ with bases $\{v_1^{(1)}, v_2^{(1)}, \ldots, v_{s_1}^{(1)}\}$ and $\{v_1^{(2)}, v_2^{(2)}, \ldots, v_{s_2}^{(2)}\}$ respectively. The tensor product $\mathcal{G}$-module $V^{(1)} \otimes V^{(2)}$ has a basis $\{v_j^{(1)} \otimes v_m^{(2)} : 1 \leq j \leq s_1, 1 \leq m \leq s_2\}$ for which:

$$g(v_j^{(1)} \otimes v_m^{(2)}) = \sum_{i=1}^{s_1} \sum_{k=1}^{s_2} \left(\Gamma^{(1)} \otimes \Gamma^{(2)}\right)(g)_{ik,jm}\, v_i^{(1)} \otimes v_k^{(1)}. \tag{1.5.6}$$

The notion of tensor product representations and modules may be further extended. If $V^{(i)}$ for $i = 1, 2, \ldots, l$ are each $\mathcal{G}$-modules, the direct product $V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(l)}$ also defines a $\mathcal{G}$-module for which the action of $\mathcal{G}$ on the derived basis is:

$$\begin{aligned} g(v_{i_1}^{(1)} \otimes v_{i_2}^{(2)} \otimes \cdots \otimes v_{i_l}^{(l)}) \\ &= g v_{i_1}^{(1)} \otimes g v_{i_2}^{(2)} \otimes \cdots \otimes g v_{i_l}^{(l)} \\ &= \sum_{j_1, \ldots, j_l} \Gamma^{(1)}(g)_{j_1 i_1} \Gamma^{(2)}(g)_{j_2 i_2} \cdots \Gamma^{(l)}(g)_{j_l i_l} v_{j_1}^{(1)} \otimes v_{j_2}^{(2)} \otimes \cdots \otimes v_{j_l}^{(l)}, \end{aligned} \tag{1.5.7}$$

so that

$$\left(\Gamma^{(1)} \otimes \Gamma^{(2)} \otimes \cdots \otimes \Gamma^{(l)}\right)(g)_{j_1 j_2 \cdots j_l, i_1 i_2 \cdots i_l} = \Gamma^{(1)}(g)_{j_1 i_1} \Gamma^{(2)}(g)_{j_2 i_2} \cdots \Gamma^{(l)}(g)_{j_l i_l}. \tag{1.5.8}$$

In the case of the Lie algebra $\mathcal{L}$, the tensor product of two representations is defined in the following lemma.

Lemma 1.5.9. *If $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are representations of $\mathcal{L}$, having dimensions s_1 and s_2 respectively, then the map $\Gamma^{(1)} \otimes \Gamma^{(2)} : \mathcal{L} \to \mathcal{M}_{s_1 s_2}$ defined by:*

$$\left(\Gamma^{(1)} \otimes \Gamma^{(2)}\right)(a)_{ik,jl} = \Gamma^{(1)}(a)_{ij}\delta_{kl} + \delta_{ij}\Gamma^{(2)}(a)_{kl}, \tag{1.5.9}$$

for $a \in \mathcal{L}$, is an $(s_1 s_2)$-dimensional representation of $\mathcal{L}$.

Proof. It is required to show that $\Gamma^{(1)} \otimes \Gamma^{(2)}$ satisfies (1.4.9). The first two conditions are seen to hold immediately, whereas for the third (using the convention of summing over repeated indices):

$$
\begin{aligned}
&\left(\Gamma^{(1)}\otimes\Gamma^{(2)}\right)([a,b])_{ik,jl}\\
&\quad=\Gamma^{(1)}([a,b])_{ij}\delta_{kl}+\delta_{ij}\Gamma^{(2)}([a,b])_{kl}\\
&\quad=\Gamma^{(1)}(a)_{im}\Gamma^{(1)}(b)_{mj}\delta_{kl}-\Gamma^{(1)}(b)_{im}\Gamma^{(1)}(a)_{mj}\delta_{kl}\\
&\qquad+\delta_{ij}\Gamma^{(2)}(a)_{kn}\Gamma^{(2)}(b)_{nl}-\delta_{ij}\Gamma^{(2)}(b)_{kn}\Gamma^{(2)}(a)_{nl}\qquad\text{(by (1.4.9)}(iii))\\
&\quad=\Gamma^{(1)}(a)_{im}\Gamma^{(1)}(b)_{mj}\delta_{kn}\delta_{nl}-\Gamma^{(1)}(b)_{im}\Gamma^{(1)}(a)_{mj}\delta_{kn}\delta_{nl}\\
&\qquad+\Gamma^{(1)}(a)_{im}\delta_{mj}\delta_{kn}\Gamma^{(2)}(b)_{nl}-\delta_{im}\Gamma^{(1)}(a)_{mj}\Gamma^{(2)}(b)_{kn}\delta_{nl}\\
&\qquad+\delta_{im}\Gamma^{(1)}(b)_{mj}\Gamma^{(2)}(a)_{kn}\delta_{nl}-\Gamma^{(1)}(b)_{im}\delta_{mj}\delta_{kn}\Gamma^{(2)}(a)_{nl}\\
&\qquad+\delta_{im}\delta_{mj}\Gamma^{(2)}(a)_{kn}\Gamma^{(2)}(b)_{nl}-\delta_{im}\delta_{mj}\Gamma^{(2)}(b)_{kn}\Gamma^{(2)}(a)_{nl}\\
&\quad=\left(\Gamma^{(1)}(a)\otimes I_{s_2}+I_{s_1}\otimes\Gamma^{(2)}(a)\right)_{ik,mn}\left(\Gamma^{(1)}(b)\otimes I_{s_2}+I_{s_1}\otimes\Gamma^{(2)}(b)\right)_{mn,jl}\\
&\qquad-\left(\Gamma^{(1)}(b)\otimes I_{s_2}+I_{s_1}\otimes\Gamma^{(2)}(b)\right)_{ik,mn}\left(\Gamma^{(1)}(a)\otimes I_{s_2}+I_{s_1}\otimes\Gamma^{(2)}(a)\right)_{mn,jl}\\
&\quad=\left[\left(\Gamma^{(1)}\otimes\Gamma^{(2)}\right)(a),\left(\Gamma^{(1)}\otimes\Gamma^{(2)}\right)(b)\right]_{ik,jl}.
\end{aligned}
$$

It follows from (1.5.9) that the action of $\mathcal{L}$ on the tensor product module $V^{(1)}\otimes V^{(2)}$ is governed by:

$$a\,(v_j^{(1)}\otimes v_l^{(2)})=(av_j^{(1)})\otimes v_l^{(2)}+v_j^{(1)}\otimes(av_l^{(2)}).\tag{1.5.10}$$

Alternatively, this result may be derived by considering the tensor product module (1.5.6) of the Lie group $\mathcal{G}$ corresponding to $\mathcal{L}$ and differentiating appropriate one-parameter subgroups. Generalisation to the tensor product $\mathcal{L}$-module $V^{(1)}\otimes V^{(2)}\otimes\cdots\otimes V^{(l)}$, gives:

$$a(v_{i_1}^{(1)}\otimes v_{i_2}^{(2)}\otimes\cdots\otimes v_{i_l}^{(l)})=\sum_{t=1}^{l}v_{i_1}^{(1)}\otimes\cdots\otimes v_{i_{t-1}}^{(t-1)}\otimes(av_{i_t}^{(t)})\otimes v_{i_{t+1}}^{(t+1)}\otimes\cdots\otimes v_{i_l}^{(l)},\tag{1.5.11}$$

where $a\in\mathcal{L}$ and where $\{v_1^{(k)},v_2^{(k)},\ldots,v_{s_k}^{(k)}\}$ is a basis for the s_k-dimensional $\mathcal{L}$-module $V^{(k)}$ for each $k=1,2,\ldots,l$.

In general, representations obtained as tensor products are reducible even if the original representations are not. When $V^{(1)}=V^{(2)}=\cdots V^{(l)}=V$, the tensor product representation is denoted $V^{\otimes l}$ and is referred to as the l-fold tensor power of V. As will be seen later, when V is the defining $\mathcal{G}$-module, the $\mathcal{G}$-module $V^{\otimes l}$ is of immense importance, playing a central role in the construction of the explicit irreducible representations of the classical groups and their Lie algebras.

§1.6. The structure of complex semisimple Lie algebras

In this section an overview is given of the structure of the complex semisimple Lie algebras as determined by Dynkin [**Dy50**]. Detailed expositions are given in [**Ja62,Hu72,Co84**] where, in addition, proofs omitted here may be located.

Since, by Theorem 1.3.8, each semisimple Lie algebra is a direct sum of simple Lie algebras, it is sufficient to consider the structure of the latter.

Fix a maximal abelian subalgebra $\mathcal{H}$ of the simple Lie algebra $\mathcal{L}$. Such a subalgebra is termed a Cartan subalgebra. The dimension r of $\mathcal{H}$, is referred to as the rank of $\mathcal{L}$. Since, in any representation Γ, the elements of $\mathcal{H}$ commute, it follows from a standard theorem in linear algebra that the matrices $\Gamma(h)$ for $h \in \mathcal{H}$, may be simultaneously diagonalised. In particular, for the adjoint representation Γ_{ad}, this implies that $\mathcal{L}$ may be written:

$$\mathcal{L} = \mathcal{H} \oplus \bigoplus_{\alpha \in \Delta} \mathcal{L}_\alpha, \tag{1.6.1}$$

where each $\alpha \in \mathcal{H}^*$, the dual of $\mathcal{H}$, $\mathcal{L}_\alpha \subset \mathcal{L}$ is defined by:

$$\mathcal{L}_\alpha = \{a \in \mathcal{L} : [h, a] = \alpha(h)a\}, \tag{1.6.2}$$

and the set $\Delta \subset \mathcal{H}^*$ is defined such that $\alpha \in \Delta$ if and only if $\alpha \neq 0$ and $\mathcal{L}_\alpha$ is non-trivial. These elements of Δ are known as the roots of $\mathcal{L}$. The elements of $\mathcal{L}_\alpha$ are root vectors corresponding to the root α. It may be shown that if $\alpha \in \Delta$ then $-\alpha \in \Delta$, $2\alpha \notin \Delta$ and that $\mathcal{L}_\alpha$ is one-dimensional. For each $\alpha \in \Delta$, fix an element $e_\alpha \in \mathcal{L}_\alpha$. This element spans $\mathcal{L}_\alpha$.

Definition 1.6.3. *The Killing form K of the Lie algebra $\mathcal{L}$ is defined by:*

$$K(a, b) = \operatorname{tr}(\Gamma_{ad}(a)\Gamma_{ad}(b)). \tag{1.6.3}$$

The Killing form is clearly bilinear and symmetric.

Lemma 1.6.4. *A Lie algebra $\mathcal{L}$ is semisimple if and only if its Killing form K is non-degenerate.*

Lemma 1.6.5. *If a Lie algebra $\mathcal{L}$ is semisimple then its Killing form K restricted to the Cartan subalgebra $\mathcal{H}$, is non-degenerate.*

Lemma 1.6.5 implies that for every $\alpha \in \mathcal{H}^*$, there exists a unique $h_\alpha \in \mathcal{H}$ such that:

$$K(h_\alpha, h) = \alpha(h), \tag{1.6.6}$$

for all $h \in \mathcal{H}$. Thereupon, since the Killing form is bilinear,

$$h_{\alpha+\beta} = h_\alpha + h_\beta, \tag{1.6.7}$$

for all $\alpha, \beta \in \mathcal{H}^*$. This enables a symmetric bilinear form on $\mathcal{H}^*$ to be defined by:

$$\langle \alpha, \beta \rangle = K(h_\alpha, h_\beta), \tag{1.6.8}$$

for all $\alpha, \beta \in \mathcal{H}^*$. From (1.6.6) it follows that $\langle \alpha, \beta \rangle = \alpha(h_\beta) = \beta(h_\alpha)$.

The following Lemma deals with products of root vectors.

Lemma 1.6.9. *For each root $\gamma \in \Delta$, let $e_\gamma \in \mathcal{L}_\gamma$. Then, for $\alpha, \beta \in \Delta$, the product $[e_\alpha, e_\beta]$ is given by one of the following three cases:*

$$[e_\alpha, e_\beta] = \begin{cases} kh_\alpha \quad (\text{for some } k \neq 0) & \text{if } \beta = -\alpha; \\ ke_{\alpha+\beta} \quad (\text{for some } k \neq 0) & \text{if } \alpha + \beta \in \Delta; \\ 0 & \text{if } \alpha + \beta \notin \Delta \cup \{0\}. \end{cases} \tag{1.6.9}$$

It may be shown that the set $\{h_\alpha : \alpha \in \Delta\}$ spans $\mathcal{H}$. Therefore a basis for $\mathcal{H}^*$ may be selected from Δ. Let the set of roots $\{\beta_1, \beta_2, \ldots, \beta_r\}$ be one such basis.

Definition 1.6.10. *Positive and negative roots. In terms of the given basis, each root $\alpha \in \Delta$ may be expanded:*

$$\alpha = \sum_{i=1}^{r} k_i \beta_i.$$

If the first non-zero coefficient of this expansion is positive, then α is said to be a positive root. Otherwise α is a negative root. Let the sets of positive and negative roots be denoted Δ_+ and Δ_- respectively. Then $\Delta = \Delta_+ \cup \Delta_-$.

Let $\mathcal{B}_+ = \bigoplus_{\alpha \in \Delta_+} \mathcal{L}_\alpha$ and $\mathcal{B}_- = \bigoplus_{\alpha \in \Delta_-} \mathcal{L}_\alpha$. The subalgebras $\mathcal{H} \cup \mathcal{B}_+$ and $\mathcal{H} \cup \mathcal{B}_-$ are known as Borel subalgebras of $\mathcal{L}$. In this thesis, $\mathcal{B}_+$ and $\mathcal{B}_-$ are referred to as nilpotent Borel subalgebras.

Definition 1.6.11. *Simple roots. If the positive root $\alpha \in \Delta_+$ cannot be expressed as a sum of two positive roots, then α is termed a simple root. The set of simple roots is denoted by Π_+.*

Lemma 1.6.12. *If the Lie algebra $\mathcal{L}$ has rank r then $\#\Pi_+ = r$ and Π_+ is a basis for $\mathcal{H}^*$. Moreover, if $\alpha \in \Delta_+$ then:*

$$\alpha = \sum_{\alpha_i \in \Pi_+} k_i \alpha_i, \tag{1.6.12}$$

where each k_i is a non-negative integer.

Lemmas 1.6.9 and 1.6.12 show that the whole Lie algebra $\mathcal{L}$ may be generated by the set of $2r$ root vectors $\{e_\alpha : e_\alpha \in \mathcal{L}_\alpha, \alpha \in \Pi_+ \cup \Pi_-\}$, where $\Pi_- = \{-\alpha : \alpha \in \Pi_+\}$.

Definition 1.6.13. *The Cartan matrix A of the semisimple Lie algebra $\mathcal{L}$ is the $r \times r$ matrix with elements*

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \tag{1.6.13}$$

for $\alpha_i, \alpha_j \in \Pi_+$ and $1 \le i, j \le r$.

It may be shown **[Hu72]** that the Cartan matrix determines a semisimple complex Lie algebra $\mathcal{L}$ uniquely, and that $\mathcal{L}$ may be constructed from its Cartan matrix. Cartan matrices for each of the simple Lie algebras A_r, B_r, C_r, D_r, G_2, F_4, E_6, E_7 and E_8 are listed in **[Hu72,Co84]**.

§1.7. Labelling the irreducible representations

In this section, a method is described for labelling the irreducible representations of semisimple complex Lie algebras and of simply connected compact Lie groups. Once more a detailed exposition is given in **[Co84]** where proofs and the original references may be found. In this section, the theorems and lemmas are stated mainly in terms of modules. The equivalence of the module and representation viewpoints implies that analogous results for the representations may be obtained merely by a substitution of words.

First consider the s-dimensional representation Γ of the rank r semisimple complex Lie algebra $\mathcal{L}$. As with the adjoint representation, the matrices $\Gamma(h)$ for $h \in \mathcal{H}$ mutually commute and can be simultaneously diagonalised. This implies that a basis $\{v_1, v_2, \ldots, v_s\}$ for the corresponding $\mathcal{L}$-module V may be chosen such that $v_1, v_2, \ldots, v_s$ are each eigenvectors of $\mathcal{H}$. For each $\mu \in \mathcal{H}^*$, define the subspace $V_\mu \subset V$ by:

$$V_\mu = \{v \in V : hv = \mu(h)v\}. \tag{1.7.1}$$

If V_μ is non-trivial then μ is known as a weight of the representation Γ or the $\mathcal{L}$-module V. V_μ is then known as a weight space. The dimension of V_μ is denoted m_μ and is known as the multiplicity of the weight μ in Γ or V. For example, in the adjoint representation Γ_{ad}, the non-zero weights may be identified with the roots. Consequently, their multiplicities are each unity. The zero weight of Γ_{ad} has a multiplicity r, since this is the dimension of the Cartan subalgebra of $\mathcal{L}$.

Lemma 1.7.2. *If μ is a weight of the $\mathcal{L}$-module V and α is a root of $\mathcal{L}$, then*

$$2\frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \tag{1.7.2}$$

is an integer.

Lemma 1.7.3. *If μ is a weight of the $\mathcal{L}$-module V and α is a root of $\mathcal{L}$ such that $e_\alpha v \neq 0$, then $\mu + \alpha$ is also a weight of V.*

Since the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_r$, form a basis of $\mathcal{H}^*$, each weight μ may be written:

$$\mu = \sum_{i=1}^{r} k_i \alpha_i. \tag{1.7.4}$$

It is then possible to compare two weights, μ and μ', by defining $\mu > \mu'$ if and only if the first non-vanishing component of $(\mu - \mu')$ is positive.

Definition 1.7.5. *Highest weight. If λ is a weight of the $\mathcal{L}$-module V such that $\lambda > \mu$ for every other weight μ of V, then λ is termed the highest weight of V.*

Lemma 1.7.6. *If $\mathcal{L}$ is a semisimple complex Lie algebra with simple roots $\Pi_+ = \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and λ is the highest weight of the irreducible $\mathcal{L}$-module V then:*

(i) λ has a multiplicity of one;

(ii) every weight μ of V may be written:

$$\mu = \lambda - \sum_{j=1}^{r} q_j \alpha_j,$$

where each q_j is a non-negative integer.

Theorem 1.7.7. *If $\mathcal{L}$ is a semisimple complex Lie algebra and V is an irreducible V-module with highest weight λ then, for each $i = 1, 2, \ldots, r$,*

$$a_i = 2\frac{\langle \lambda, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle} \tag{1.7.7}$$

is a non-negative integer. Moreover, each sequence $(a_1, a_2, \ldots, a_r)$ of non-negative integers identifies an irreducible $\mathcal{L}$-module which is unique, up to equivalence.

The integers defined by (1.7.7) are known as the Dynkin labels of the $\mathcal{L}$-module V or the corresponding representation Γ.

Theorem 1.7.8. *Each representation of the real Lie algebra $\mathcal{L}$, yields on exponentiation, a representation of the universal covering group $\tilde{\mathcal{G}}_{\mathcal{L}}$ whose Lie algebra is $\mathcal{L}$.*

Chapter 2
The Classical Lie Groups, Lie Algebras and their Representations

§2.1. The classical Lie groups

Definition 2.1.1. *The classical groups are the following groups of square matrices in which the matrix elements are members of the field* F *and the group composition law is matrix multiplication*:

(i) the general linear group $GL(m,\mathsf{F}) = \{G : G \text{ is } m \times m,\ \det G \neq 0\}$;

(ii) the unitary group $U(m,\mathsf{F}) = \{G : G \in GL(m,\mathsf{F}),\ G^\dagger G = I_m\}$, *where* $G^\dagger = \tilde{G}^*$, *the tilde denoting matrix transposition, the asterisk denoting complex conjugation, and* I_m *is the* $m \times m$ *unit matrix.*

(iii) the special linear group $SL(m,\mathsf{F}) = \{G : G \in GL(m,\mathsf{F}),\ \det G = 1\}$;

(iv) the special unitary group $SU(m,\mathsf{F}) = U(m,\mathsf{F}) \cap SL(m,\mathsf{F})$;

(v) the symplectic group $Sp(2r,\mathsf{F}) = \{G : G \in GL(2r,\mathsf{F}),\ \tilde{G}J_{2r}^- G = J_{2r}^-\}$, *where*

$$J_{2r}^- = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{pmatrix}; \tag{2.1.1a}$$

(vi) the orthogonal group $O(m,\mathsf{F}) = \{G : G \in GL(m,\mathsf{F}),\ \tilde{G}J_m^+ G = J_m^+\}$, *where*

$$J_{2r}^+ = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}, \tag{2.1.1b}$$

if $m = 2r$ *and*

$$J^+_{2r+1} = \begin{pmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 & \\ & & & & & & 1 & 0 & \\ & & & & & & & & 1 \end{pmatrix}, \tag{2.1.1c}$$

if $m = 2r + 1$*; and*

(vii) the special orthogonal group $SO(m, \mathsf{F}) = SL(m, \mathsf{F}) \cap O(m, \mathsf{F})$.

In addition to these define

(viii) $O'(m, \mathsf{F}) = \{G : G \in GL(m, \mathsf{F}),\ \tilde{G}G = I_m\}$*; and*

(ix) $SO'(m, \mathsf{F}) = SL(m, \mathsf{F}) \cap O'(m, \mathsf{F})$.

Note that $U(m, \mathbf{R}) \cong O(m, \mathbf{R})$ and $SU(m, \mathbf{R}) \cong SO(m, \mathbf{R})$.

In this thesis, the field F will be taken to be either the complex number field $\mathbf{C}$ or the real number field $\mathbf{R}$. F will be dropped from the notation only when it is irrelevant to the topic being discussed.

Definitions 2.1.1(vi) and 2.1.1(vii) for $O(m)$ and $SO(m)$ differ from those often used for the orthogonal groups. These more usual definitions are those given here for the groups $O'(m)$ and $SO'(m)$ as in Definitions 2.1.1$(viii)$ and 2.1.1(ix). If F is algebraically closed (i.e. $\mathsf{F} = \mathbf{C}$), there exists an $m \times m$ matrix S such that $J^+_m = \tilde{S}S$, whereupon, if $G \in O(m, \mathbf{C})$, the transformation $G' = SJG\tilde{S}$ gives $\tilde{G}'G' = I_m$. This demonstrates that $G' \in O'(m, \mathbf{C})$ and furthermore, since $(G_1G_2)' = G'_1G'_2$ as is easily shown, the groups $O(m, \mathbf{C})$ and $O'(m, \mathbf{C})$ are isomorphic. However, for $\mathsf{F} = \mathbf{R}$, a genuine distinction exists between the two groups when $m > 1$.

The following lemma will be useful later:

Lemma 2.1.2. *Let* $\mathcal{G}(m)$ *be any of the classical groups of Definition* 2.1.1. *If* $G \in \mathcal{G}(m)$ *then* $\tilde{G} \in \mathcal{G}(m)$.

Proof. Since $\det \tilde{G} = \det G$, this lemma follows immediately for the general and special linear groups. If $G \in U(m)$ then $GG^\dagger = I_m$ and

$$\tilde{G}^\dagger \tilde{G} = (\widetilde{GG^\dagger}) = \tilde{I}_m = I_m,$$

so that the lemma holds for the unitary groups $U(m)$ and $SU(m)$. The cases of the symplectic and orthogonal groups follow by noting that if $J^2 = \pm I_m$, so that $J^{-1} = \pm J$, then

$$\begin{aligned} \tilde{G}JG = J \quad &\Longleftrightarrow \quad G^{-1}J^{-1}\tilde{G}^{-1} = J^{-1} \\ &\Longleftrightarrow \quad J^{-1} = GJ^{-1}\tilde{G} \\ &\Longleftrightarrow \quad J = GJ\tilde{G}. \end{aligned}$$

§2.2. The classical Lie algebras

Let W be a m-dimensional vector space with basis $\{w_1, w_2, \ldots, w_m\}$ and let $E_a{}^b$ be a linear operator acting on W such that

$$E_a{}^b w_c = \delta^b_c w_a. \tag{2.2.1}$$

In the given basis, $E_a{}^b$ may be realised as an $m \times m$ matrix with the entry 1 at the intersection of the ath row and bth column and zeros everywhere else. No confusion will arise from denoting this matrix by the same symbol so that $(E_a{}^b)_{ij} = \delta_{ai}\delta_{bj}$. For $a, b \in \mathbb{N}_m$ the matrices $E_a{}^b$ span the vector space of all $m \times m$ matrices. These matrices satisfy the commutation relations:

$$\left[E_a{}^b, E_c{}^d\right] = \delta^b_c E_a{}^d - \delta^d_a E_c{}^b. \tag{2.2.2}$$

These matrices will be used to construct each of the Lie algebras of the classical groups of Definition 2.1.1. This construction proceeds via the following three lemmas.

Lemma 2.2.3. *If G is any square matrix then:*

$$\det(\exp G) = \exp(\operatorname{tr} G). \tag{2.2.3}$$

Proof. This result follows immediately on considering the Jordan normal form of G.

Lemma 2.2.4. *If $\mathcal{G}$ is a subgroup of $GL(m, \mathsf{F})$ for which $G \in \mathcal{G}$ only if $\tilde{G}JG = J$, then the Lie algebra $\mathcal{L}_{\mathcal{G}(m)}$ consists entirely of matrices A for which:*

$$\tilde{A}J + JA = 0. \tag{2.2.4}$$

Proof. Let $G(t)$ be a one-parameter subgroup of $\mathcal{G}$ and let:

$$A = \frac{d}{dt}G(t)\bigg|_{t=0}.$$

By hypothesis:

$$\widetilde{G(t)}JG(t) = J,$$

so that:

$$\frac{d}{dt}\widetilde{G(t)}\bigg|_{t=0} JG(0) + \widetilde{G(0)}J\,\frac{d}{dt}G(t)\bigg|_{t=0} = 0,$$

and hence:

$$\tilde{A}JI_m + \tilde{I}_m JA = 0,$$

whereupon:

$$\tilde{A}J + JA = 0.$$

Lemma 2.2.5. *If $J \in GL(m)$ is such that $\tilde{J} = \pm J$ and B is any $m \times m$ matrix, then the matrix $A = B - J^{-1}\tilde{B}J$ satisfies $\tilde{A}J + JA = 0$. Conversely, if the $m \times m$ matrix A satisfies $\tilde{A}J + JA = 0$ then $A = B - J^{-1}\tilde{B}J$ for some matrix B.*

Proof. If $A = B - J^{-1}\tilde{B}J$ then by direct substitution:

$$\begin{aligned}\tilde{A}J + JA &= \tilde{B}J - \tilde{J}B\tilde{J}^{-1}J + JB - JJ^{-1}\tilde{B}J \\ &= \tilde{B}J - (\pm J)B(\pm J^{-1})J + JB - \tilde{B}J \\ &= \tilde{B}J - JB + JB - \tilde{B}J \\ &= 0.\end{aligned}$$

Conversely, let A be such that $\tilde{A}J + JA = 0$. This implies that $\pm\tilde{A}\tilde{J} + JA = 0$ and hence that:

$$JA = \mp(\widetilde{JA}). \tag{2.2.5a}$$

Let $JA = C_- + C_0 + C_+$ where C_- is a strictly lower triangular matrix, C_0 is a diagonal matrix and C_+ is a strictly upper triangular matrix. Identity (2.2.5a) then implies that:

$$\tilde{C}_- = \mp C_+ \quad \text{and} \quad C_0 = \mp C_0.$$

Let:

$$B = J^{-1}C_- + \frac{1}{2}J^{-1}C_0,$$

whereupon:

$$\begin{aligned}\tilde{B} &= \tilde{C}_-\tilde{J}^{-1} + \frac{1}{2}\tilde{C}_0\tilde{J}^{-1} \\ &= (\mp C_+)(\pm J)^{-1} + \frac{1}{2}(\mp C_0)(\pm J)^{-1} \\ &= -C_+J^{-1} - \frac{1}{2}C_0J^{-1},\end{aligned}$$

so that:

$$\begin{aligned} B - J^{-1}\tilde{B}J &= J^{-1}C_- + \frac{1}{2}J^{-1}C_0 + J^{-1}C_+ + \frac{1}{2}J^{-1}C_0 \\ &= J^{-1}(C_- + C_0 + C_+) \\ &= A, \end{aligned}$$

which proves the lemma.

In what follows, it is convenient to introduce various index sets $\mathcal{I}^{\mathcal{G}(m)}$ for each of the classical groups $\mathcal{G}(m)$. These will be based on the sets $\mathsf{N}_l = \{1, 2, \ldots, l\}$ and $\overline{\mathsf{N}}_l = \{\bar{a} : a \in \mathsf{N}_l\}$, where $\bar{\bar{a}} = a$ and $\bar{0} = 0$. In addition, define:

$$\operatorname{sgn}(a) = \begin{cases} -1 & \text{if } a \in \overline{\mathsf{N}}_l; \\ 1 & \text{if } a \in \mathsf{N}_l, \end{cases} \tag{2.2.6}$$

and define $\operatorname{sgn}(ab) = \operatorname{sgn}(a)\operatorname{sgn}(b)$.

Each of the classical groups $\mathcal{G}(m)$ of Section 2.1 will be considered in turn and their Lie algebras $\mathcal{L}_{\mathcal{G}(m)}$ constructed in the defining representation. For each $\mathcal{L}_{\mathcal{G}(m)}$, a basis for the Cartan subalgebra $\mathcal{H}^{\mathcal{L}_{\mathcal{G}(m)}}$ and a set of simple root vectors $\Pi_+^{\mathcal{L}_{\mathcal{G}(m)}}$ will be specified. In general, many distinct choices exist for a set of simple root vectors. However, once specified, the set of positive root vectors $\Delta_+^{\mathcal{L}_{\mathcal{G}(m)}}$ is uniquely determined as is the positive nilpotent Borel subalgebra $\mathcal{B}_+^{\mathcal{L}_{\mathcal{G}(m)}}$ which they span. The same is true of the negative root vectors $\Delta_-^{\mathcal{L}_{\mathcal{G}(m)}}$ which span the negative nilpotent Borel subalgebra $\mathcal{B}_-^{\mathcal{L}_{\mathcal{G}(m)}}$.

(i) $gl(m, \mathsf{F})$. Let $\mathcal{I}^{GL(m)} = \mathsf{N}_m$. In view of Lemma 2.2.3, $G = \exp(A) \in GL(m, \mathsf{F})$ for any $m \times m$ matrix A with entries in F. Thus $gl(m, \mathsf{F})$ is the vector space of all matrices which is spanned by $\{E_a{}^b \in gl(m, \mathsf{F}) : a, b \in \mathcal{I}^{GL(m)}\}$. Since the $E_a{}^b$ are linearly independent, $gl(m, \mathsf{F})$ is an m^2-dimensional Lie algebra over F. However,

$$H = \sum_{a=1}^{m} E_a{}^a \tag{2.2.7}$$

generates a one-dimensional abelian ideal of $gl(m, \mathsf{F})$ which, therefore, is not semisimple. The following provides a convenient biography of $gl(m, \mathsf{F})$:

Basis: $\{E_a{}^b : a, b \in \mathcal{I}^{GL(m)}\}$;

Dimension over F : m^2;

Basis of $\mathcal{H}^{gl(m,\mathsf{F})}$: $\{E_a{}^a : a \in \mathcal{I}^{GL(m)}\}$;

Rank: m;

Simple root vectors, $\Pi_+^{sl(m,\mathsf{F})}$: $\{E_a{}^{a+1} : a \in \mathcal{I}^{GL(m-1)}\}$;

Positive root vectors, $\Delta_+^{gl(m,\mathsf{F})}$: $\{E_a{}^b : a, b \in \mathcal{I}^{GL(m)}, a < b\}$;

Negative root vectors, $\Delta_-^{gl(m,\mathsf{F})}$: $\{E_a{}^b : a, b \in \mathcal{I}^{GL(m)}, a > b\}$;

Dimension of $\mathcal{B}_+^{gl(m,\mathsf{F})}$: $m(m-1)/2$.

Here the nilpotent Borel subalgebra $\mathcal{B}_+^{gl(m,\mathsf{F})}$, which is spanned by $\Delta_+^{gl(m,\mathsf{F})}$, consists entirely of strictly upper triangular matrices. $\mathcal{B}_-^{gl(m,\mathsf{F})}$ consists entirely of strictly lower triangular matrices.

(ii) $sl(m, \mathsf{F})$. Let $\mathcal{I}^{SL(m)} = \mathsf{N}_m$. Since $G \in SL(m, \mathsf{F})$ if $\det G = 1$, Lemma 2.2.3 implies that $A \in sl(m, \mathsf{F})$ if $\operatorname{tr} A = 0$. For $a, b \in \mathcal{I}^{SL(m)}$ let:

$$A_a{}^b = \begin{cases} E_a{}^b & \text{if } a \neq b; \\ E_a{}^a - E_m{}^m & \text{if } a = b. \end{cases} \tag{2.2.8}$$

In terms of these matrices, $sl(m, \mathsf{F})$ has the following biography:

Basis: $\{A_a{}^b : a, b \in \mathcal{I}^{SL(m)}, (a, b) \neq (m, m)\}$;

Dimension over F : $m^2 - 1$;

Basis of $\mathcal{H}^{sl(m,\mathsf{F})}$: $\{A_a{}^a : a \in \mathcal{I}^{SL(m-1)}\}$;

Rank: $m - 1$;

Simple root vectors, $\Pi_+^{sl(m,\mathsf{F})}$: $\{A_a{}^{a+1} : a \in \mathcal{I}^{SL(m-1)}\}$;

Positive root vectors, $\Delta_+^{sl(m,\mathsf{F})}$: $\{A_a{}^b : a, b \in \mathcal{I}^{SL(m)}, a < b\}$;

Dimension of $\mathcal{B}_+^{sl(m,\mathsf{F})}$: $m(m-1)/2$.

Each positive root vector $A_a{}^b$, for $a < b$, may be generated according to:

$$A_a{}^b = [A_a{}^{a+1}, [A_{a+1}{}^{a+2}, [A_{a+2}{}^{a+3}, \cdots [A_{b-2}{}^{b-1}, A_{b-1}{}^b] \cdots]]]. \tag{2.2.9}$$

This is a direct consequence of (2.2.2), given that $A_i{}^{i+1} = E_i{}^{i+1}$. For $m > 0$, $sl(m, \mathbb{C})$ is isomorphic to Cartan's A_{m-1}, and $sl(m, \mathbb{R})$ is a particular (non-compact) real form.

(iii) $u(m)$. Let $G(t)$ be a one parameter subgroup of $U(m, \mathbb{C})$ and let:

$$A = \frac{d}{dt} G(t) \bigg|_{t=0}.$$

Since $G(t)G(t)^\dagger = I_m$,

$$\frac{d}{dt} G(t) \bigg|_{t=0} G(0)^\dagger + G(0) \frac{d}{dt} G(t)^\dagger \bigg|_{t=0} = 0,$$

$$A\tilde{I}_m + I_m A^\dagger = 0,$$

implying that:

$$A^\dagger = -A.$$

Therefore $u(m)$ consists entirely of antihermitian $m \times m$ matrices. Let $\mathcal{I}^{U(m)} = \mathbf{N}_m$ for $a, b \in \mathcal{I}^{U(m)}$ and let:

$$E_a^{(0)b} = E_a{}^b - E_b{}^a \tag{2.2.10a}$$

$$\text{and} \qquad E_a^{(1)b} = i(E_a{}^b + E_b{}^a). \tag{2.2.10b}$$

Thereupon $u(m)$ has:

Basis: $\{E_a^{(0)b} : a, b \in \mathcal{I}^{U(m)}, a < b\} \cup \{E_a^{(1)b} : a, b \in \mathcal{I}^{U(m)}, a \leq b\}$;

Dimension over $\mathbf{R}$: m^2.

With this basis, it is easily verified that $u(m)$ is a real Lie algebra. However, $u(m)$ is not semisimple since it possesses a one-dimensional abelian ideal spanned by $H^{(1)} = \sum_{a \in \mathcal{I}^{U(m)}} E_a^{(1)a}$.

(iv) $su(m)$. Lemma (2.2.3) implies that if $A \in su(m)$ then $\operatorname{tr} A = 0$. Let $\mathcal{I}^{SU(m)} = \mathbf{N}_m$. Using the notation above for $a, b \in \mathcal{I}^{SU(m)}$ let:

$$A_a^{(0)b} = E_a^{(0)b} \tag{2.2.11a}$$

$$\text{and} \qquad A_a^{(1)b} = \begin{cases} E_a^{(1)b} & \text{if } a \neq b; \\ E_a^{(1)a} - E_m^{(1)m} & \text{if } a = b. \end{cases} \tag{2.2.11b}$$

The simple real Lie algebra $su(m)$ then has:

Basis: $\{A_a^{(0)b}, A_a^{(1)b} : a, b \in \mathcal{I}^{SU(m)}, a < b\} \cup \{A_a^{(1)a} : a \in \mathcal{I}^{SU(m-1)}\}$;

Dimension over $\mathbf{R}$: $m^2 - 1$.

$su(m)$ is the compact simple real Lie algebra $A_{m-1}(\mathbf{R})$.

(v) $sp(2r, \mathsf{F})$. Define the index set $\mathcal{I}^{Sp(2r)} = \mathbf{N}_r \cup \overline{\mathbf{N}}_r$. From (2.1.1$a$),

$$(J_{2r}^-)_{ij} = \operatorname{sgn}(i)\, \delta_{i\bar{j}}, \tag{2.2.12}$$

with respect to the ordering $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r$, of the index set $\mathcal{I}^{Sp(2r)}$. In view of Lemma 2.2.5, since $\tilde{J}_{2r}^- = -J_{2r}^-$, let:

$$C_a{}^b = E_a{}^b - \operatorname{sgn}(ab)\, E_{\bar{b}}{}^{\bar{a}}, \tag{2.2.13}$$

for $a, b \in \mathcal{I}^{Sp(2r)}$. These operators satisfy the commutation relations:

$$\left[C_a{}^b, C_c{}^d\right] = \delta_c^b C_a{}^d - \delta_a^d C_c{}^b + \operatorname{sgn}(cd)\delta_a^{\bar{c}} C_{\bar{d}}{}^b - \operatorname{sgn}(cd)\delta_{\bar{d}}^b C_a{}^{\bar{c}}. \tag{2.2.14}$$

Note that $C_a{}^b = -\operatorname{sgn}(ab)\, C_{\bar{b}}{}^{\bar{a}}$, which leads to the following biography of $sp(2r, \mathsf{F})$:

Basis: $\{C_a{}^b : a, b \in \mathsf{N}_r\} \cup \{C_a{}^{\bar{b}} : a, b \in \mathsf{N}_r, a \leq b\} \cup \{C_{\bar{a}}{}^b : a, b \in \mathsf{N}_r, a \leq b\}$;
Dimension over F : $r(2r+1)$;
Basis of $\mathcal{H}^{sp(2r,\mathsf{F})}$: $\{C_a{}^a : a \in \mathsf{N}_r\}$;
Rank: r;
Simple root vectors, $\Pi_+^{sp(2r,\mathsf{F})}$: $\{C_a{}^{a+1} : a \in \mathsf{N}_{r-1}\} \cup \{C_r{}^{\bar{r}}\}$;
Positive root vectors, $\Delta_+^{sp(2r,\mathsf{F})}$: $\mathcal{C}_1 \cup \mathcal{C}_2$ where
$\mathcal{C}_1 = \{C_a{}^b : a, b \in \mathsf{N}_r, a < b\}$ and
$\mathcal{C}_2 = \{C_a{}^{\bar{b}} : a, b \in \mathsf{N}_r, a \leq b\}$;
Dimension of $\mathcal{B}_+^{sp(2r,\mathsf{F})}$: r^2.

Positive root vectors from the set $\mathcal{C}_1$ may be generated from $\Pi_+^{sp(2r,\mathsf{F})}$ as in (2.2.9), whereas for $\mathcal{C}_2$, (2.2.12) gives

$$C_a{}^{\bar{r}} = \frac{1}{2}\left[C_a{}^r, C_r{}^{\bar{r}}\right], \tag{2.2.15a}$$

and then, for $a \leq b$,

$$C_a{}^{\bar{b}} = [C_b{}^{b+1}, [C_{b+1}{}^{b+2}, [\cdots[C_{r-1}{}^r, C_a{}^{\bar{r}}]\cdots]]]. \tag{2.2.15b}$$

The nilpotent Borel subalgebra $\mathcal{B}_-^{sp(2r,\mathsf{F})}$ may be described and generated in a way analogous to that given here for $\mathcal{B}_+^{sp(2r,\mathsf{F})}$. The simple complex Lie algebra $sp(2r, \mathsf{C})$ is Cartan's C_r and the simple real Lie algebra $sp(2r, \mathsf{R})$ is $C_r(\mathsf{R})$.

(vi) $so(2r, \mathsf{F})$. Define the index set $\mathcal{I}^{O(2r)} = \mathsf{N}_r \cup \overline{\mathsf{N}}_r$. From (2.1.1b),

$$(J_{2r}^+)_{ij} = \delta_{i\bar{j}}, \tag{2.2.16}$$

with respect to the ordering $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r$, of the index set $\mathcal{I}^{O(2r)}$. In view of Lemma 2.2.5, since $\tilde{J}_{2r}^+ = J_{2r}^+$, let

$$D_a{}^b = E_a{}^b - E_{\bar{b}}{}^{\bar{a}}, \tag{2.2.17}$$

for $a, b \in \mathcal{I}^{O(2r)}$. These operators satisfy the commutation relations:

$$\left[D_a{}^b, D_c{}^d\right] = \delta_c^b D_a{}^d - \delta_a^d D_c{}^b + \delta_a^{\bar{c}} D_{\bar{d}}{}^b - \delta_{\bar{d}}^b D_a{}^{\bar{c}}. \tag{2.2.18}$$

Note that $D_a{}^b = -D_{\bar{b}}{}^{\bar{a}}$, which leads to the following biography of $so(2r, \mathsf{F})$:

Basis: $\{D_a{}^b : a, b \in \mathsf{N}_r\} \cup \{D_a{}^{\bar{b}} : a, b \in \mathsf{N}_r, a < b\} \cup \{D_{\bar{a}}{}^b : a, b \in \mathsf{N}_r, a < b\}$;
Dimension over F : $r(2r-1)$;

Basis of $\mathcal{H}^{so(2r,\mathsf{F})}$: $\{D_a{}^a : a \in \mathsf{N}_r\}$;

Rank: r;

Simple root vectors, $\Pi_+^{so(2r,\mathsf{F})}$: $\{D_a{}^{a+1} : a \in \mathsf{N}_{r-1}\} \cup \{D_r{}^{\overline{r-1}}\}$;

Positive root vectors, $\Delta_+^{so(2r,\mathsf{F})}$: $\mathcal{D}_1 \cup \mathcal{D}_2$ where

$\mathcal{D}_1 = \{D_a{}^b : a, b \in \mathsf{N}_r, a < b\}$ and

$\mathcal{D}_2 = \{D_a{}^{\bar{b}} : a, b \in \mathsf{N}_r, a < b\}$;

Dimension of $\mathcal{B}_+^{so(2r,\mathsf{F})}$: $r(r-1)$.

Positive root vectors from the set $\mathcal{D}_1$ may be generated from $\Pi_+^{so(2r,\mathsf{F})}$ as in (2.2.9), whereas for $\mathcal{D}_2$, (2.2.18) gives

$$D_a{}^{\bar{r}} = \left[D_r{}^{\overline{r-1}}, D_a{}^{r-1}\right], \tag{2.2.19a}$$

if $a < r-1$, or

$$D_a{}^{\overline{r-1}} = \left[D_a{}^r, D_r{}^{\overline{r-1}},\right] \tag{2.2.19b}$$

if $r \geq 2$; and then, for $a < b$,

$$D_a{}^{\bar{b}} = [D_b{}^{b+1}, [D_{b+1}{}^{b+2}, [\cdots [D_{r-2}{}^{r-1}, D_a{}^{\overline{r-1}}] \cdots]]]. \tag{2.2.19c}$$

The nilpotent Borel subalgebra $\mathcal{B}_-^{so(2r,\mathsf{F})}$ may be described and generated in a way similar to that given here for $\mathcal{B}_+^{so(2r,\mathsf{F})}$. The simple complex Lie algebra $so(2r,\mathbb{C})$ is Cartan's D_r and the simple real Lie algebra $so(2r,\mathbb{R})$ is a particular (non-compact) real form.

(vii) $so(2r+1,\mathsf{F})$. Although this case is very similar to the last, the existence of a few subtle differences justifies a reworking. Define the index set $\mathcal{I}^{O(2r+1)} = \mathsf{N}_r \cup \overline{\mathsf{N}}_r \cup \{0\}$. From (2.1.1$c$),

$$(J_{2r+1}^+)_{ij} = \delta_{i\bar{j}}, \tag{2.2.20}$$

with respect to the ordering $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r < 0$, of the index set $\mathcal{I}^{O(2r+1)}$. As with (2.2.17), since $\tilde{J}_{2r+1}^+ = J_{2r+1}^+$, let

$$B_a{}^b = E_a{}^b - E_{\bar{b}}{}^{\bar{a}}, \tag{2.2.21}$$

for $a, b \in \mathcal{I}^{SO(2r+1)}$. These operators satisfy the commutation relations (2.2.18), rewritten in the notation of this section:

$$\left[B_a{}^b, B_c{}^d\right] = \delta_c^b B_a{}^d - \delta_a^d B_c{}^b + \delta_a^{\bar{c}} B_{\bar{d}}{}^b - \delta_{\bar{d}}^b B_a{}^{\bar{c}}. \tag{2.2.22}$$

Note that $B_a{}^b = -B_{\bar{b}}{}^{\bar{a}}$, which leads to the following biography of $so(2r+1,\mathsf{F})$:

Basis: $\{B_a{}^b : a,b \in \mathsf{N}_r\} \cup \{B_a{}^{\bar{b}} : a,b \in \mathsf{N}_r, a < b\}$
$\cup \{B_{\bar{a}}{}^b : a,b \in \mathsf{N}_r, a < b\} \cup \{B_a{}^0 : a \in \mathsf{N}_r \cup \overline{\mathsf{N}}_r\}$;

Dimension over F : $r(2r+1)$;

Basis of $\mathcal{H}^{so(2r+1,\mathsf{F})}$: $\{B_a{}^a : a \in \mathsf{N}_r\}$;

Rank: r;

Simple root vectors, $\Pi_+^{so(2r+1,\mathsf{F})}$: $\{B_a{}^{a+1} : a \in \mathsf{N}_{r-1}\} \cup \{B_r{}^0\}$;

Positive root vectors, $\Delta_+^{so(2r+1,\mathsf{F})}$: $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ where

$\mathcal{B}_1 = \{B_a{}^b : a,b \in \mathsf{N}_r, a < b\}$, $\mathcal{B}_2 = \{B_a{}^0 : a \in \mathsf{N}_r\}$ and

$\mathcal{B}_3 = \{B_a{}^{\bar{b}} : a,b \in \mathsf{N}_r, a < b\}$;

Dimension of $\mathcal{B}_+^{so(2r+1,\mathsf{F})}$: r^2.

Positive root vectors from the set $\mathcal{B}_1$ may be generated from $\Pi_+^{so(2r+1,\mathsf{F})}$ as in (2.2.9), whereas for $\mathcal{B}_2$, (2.2.22) gives:

$$B_a{}^0 = [B_a{}^r, B_r{}^0], \tag{2.2.23a}$$

whereupon, for $\mathcal{B}_3$,

$$B_a{}^{\bar{r}} = [B_r{}^0, B_a{}^0], \tag{2.2.23b}$$

and

$$B_a{}^{\bar{b}} = [B_b{}^{b+1}, [B_{b+1}{}^{b+2}, [\cdots [B_{r-1}{}^r, B_a{}^{\bar{r}}] \cdots]]]. \tag{2.2.23c}$$

The nilpotent Borel subalgebra $\mathcal{B}_-^{so(2r+1,\mathsf{F})}$ may be described and generated in a way similar to that given here for $\mathcal{B}_+^{so(2r+1,\mathsf{F})}$. The simple complex Lie algebra $so(2r+1,\mathbb{C})$ is Cartan's B_r and the simple real Lie algebra $so(2r+1,\mathbb{R})$ is a particular (non-compact) real form.

(viii) $so'(m,\mathsf{F})$. Due to its lack of relevance to the results that follow in later chapters, this algebra will be given but cursory treatment here. However, by virtue of the fact that the corresponding Lie groups are isomorphic when $\mathsf{F} = \mathbb{C}$, $so'(m,\mathbb{C}) \cong so(m,\mathbb{C})$. Let $\mathcal{I}^{SO'(m)} = \mathsf{N}_m$. The techniques used in the previous cases may be employed here to show that if $B'^{b}_{a} = E_a{}^b - E_b{}^a$ then the set $\{B'^{b}_{a} : a,b \in \mathsf{N}_m, a < b\}$ is a basis for $so'(m,\mathsf{F})$. This Lie algebra is therefore $\frac{1}{2}m(m-1)$-dimensional, as expected from the results of cases (vi) and (vii). Since no diagonal matrices are present in this defining representation, there is no obvious choice for a basis of the Cartan subalgebra. Many are possible. It will be shown how this lack of a diagonal Cartan subalgebra leads to a certain inconvenience in the construction of orthogonal group modules. Nonetheless, the Lie algebra $so'(m, \mathbf{R})$ is compact, being Cartan's $D_r(\mathbf{R})$ if $m = 2r$ is even, or $B_r(\mathbf{R})$ if $m = 2r+1$ is odd. It therefore merits consideration.

§2.3. Partitions and Young diagrams

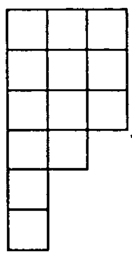
Partitions play a major role as a classification tool in the theory of representations [**We39,Li50**]. In this section, the notions of a partition and a Young diagram are introduced. In addition, all the associated notational developments that will be required when dealing with representations and modules are gathered here.

Definition 2.3.1. *Partition. The partition of the positive integer l into p positive integral parts $\lambda_1, \lambda_2, \ldots, \lambda_p$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_p = l$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0$ is denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. Partitions will always be denoted by lowercase Greek letters. It is convenient to define $\lambda_i = 0$ for $i > p$; two partitions being equal if and only if their non-zero parts are equal. On occasion, a partition with repeated parts will be denoted using exponents. For example, $(3^3, 2, 1^2)$ denotes the partition $(3,3,3,2,1,1)$.*

Let $P(l)$ denote the set of all partitions of l. For example, $P(2) = \{(2);(1,1)\}$, $P(3) = \{(3);(2,1);(1,1,1)\}$ and $P(4) = \{(4);(3,1);(2,2);(2,1,1);(1,1,1,1)\}$.

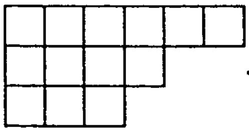
Definition 2.3.2. *Young diagram. Each partition $\lambda \in P(l)$ specifies a regular Young diagram, F^λ, consisting of l boxes arranged in p left-adjusted rows. The number of boxes in the ith row is λ_i for $i = 1, 2, \ldots, p$.*

This definition gives, for instance,

$F^{(3^3,2,1^2)} =$  . (2.3.3)

Definition 2.3.4. *Conjugate partition. Let $\lambda \in P(l)$, $q = \lambda_1$, and for $j = 1, 2, \ldots, q$, let $\tilde{\lambda}_j$ be the length of the jth column of F^λ. This defines $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_q)$, the partition conjugate to λ.*

As a consequence of this definition, the Young diagram $F^{\tilde{\lambda}}$ is obtained from F^λ by reflection in the main diagonal, that is interchanging rows and columns. Thus, for the example (2.3.3) where $\lambda = (3^3, 2, 1^2)$, $\tilde{\lambda} = (6, 4, 3)$ and

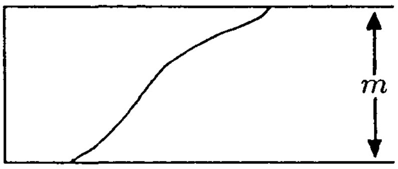
$F^{\tilde{\lambda}} =$  . (2.3.5)

Definition 2.3.6. *The partition sets $P(l;m)$ and $P(l;m/n)$. Define*

$$P(l;m) = \{\lambda \in P(l) : \tilde{\lambda}_1 \leq m\}; \qquad (2.3.6a)$$

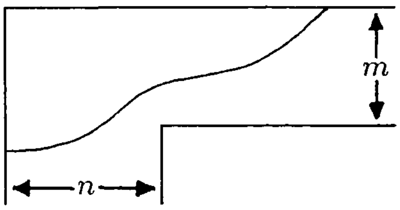
and
$$P(l;m/n) = \{\lambda \in P(l) : \tilde{\lambda}_{n+1} \leq m\}. \qquad (2.3.6b)$$

Note that if $\lambda \in P(l;m)$ then λ has, at most, m parts and F^λ fits within a horizontal strip of depth m:



. (2.3.7a)

Additionally, note that $P(l;m/0) = P(l;m)$ and if $\lambda \in P(l;m/n)$, then $\lambda_{m+1} \leq n$ and F^λ fits within a hook with arm depth m and leg width n:

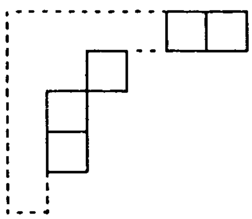


. (2.3.7b)

Let $\mu \in P(u)$ and $\nu \in P(v)$. If $\nu_i \leq \mu_i$ for $i = 1, 2, \ldots$ then ν is said to be contained in μ. This is denoted $\nu \leq \mu$ and defines a partial order on the set of all partitions.

Definition 2.3.8. *Skew Young diagram. If $\nu \leq \mu$ the skew Young diagram $F^{\mu/\nu}$ consisting of $u - v$ boxes, is defined as that diagram resulting from the removal from F^μ of all the v boxes corresponding to F^ν.*

If $\mu = (6, 3, 2^2, 1)$ and $\nu = (4, 2, 1^3)$ this definition gives:

$F^{\mu/\nu} =$  . (2.3.9)

Definition 2.3.10. *A generalised partition* $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_s)$ *consists of* s *parts* $\Lambda_1, \Lambda_2, \ldots, \Lambda_s$ *for which* $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_s$. *The parts are neither necessarily positive nor integral.*

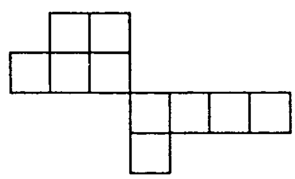
On considering generalised partitions with integral parts, Littlewood [**Li50**] devised the useful notion of a composite partition.

Definition 2.3.11. *Composite partition. Let* $\mu \in P(u;p), \nu \in P(v;q)$ *with* $p + q \leq s$. *The composite partition* $(\bar{\nu};\mu)_s$, *denotes the* s *part generalised partition* $(\mu_1, \mu_2, \ldots, \mu_p, 0, \ldots, 0, -\nu_q, -\nu_{q-1}, \ldots, -\nu_1)$ *The subscript* s *may be dropped when the number of zero parts is irrelevant. If either* μ *or* ν *is the zero partition, the following notation will be adopted:* $(\bar{0};\mu) = \mu$ *and* $(\bar{\nu};0) = \bar{\nu}$.

Definition 2.3.12. *Canonical associate. With* $(\bar{\nu};\mu)_s$ *the composite partition corresponding to the* s *integral part generalised partition* Λ, *denote the ordinary partition* $(\Lambda_1 - \Lambda_s, \Lambda_2 - \Lambda_s, \ldots, \Lambda_{s-1} - \Lambda_s)$ *by both* $(\bar{\nu};\mu)^*_s$ *and* Λ^*. *This partition is known as the canonical associate of* $(\bar{\nu};\mu)_s$ *and* Λ.

Each composite partition and hence each generalised partition, may be used to specify a composite Young diagram [**Ki70,Ki89**].

Definition 2.3.13. *Composite Young diagram. For* $\nu \in P(v)$ *let* $F^{\bar{\nu}}$ *be the diagram obtained by reflecting the Young diagram* F^{ν} *successively in its topmost and leftmost edges. Thus* $F^{\bar{\nu}}$ *is a right-adjusted, bottom-adjusted array of boxes, the lengths of the rows of which decrease on passing up the diagram. The composite Young diagram* $F^{\bar{\nu};\mu}$ *is constructed by adjoining* $F^{\bar{\nu}}$ *and* F^{μ} *corner to corner as in the following example:*

$F^{\bar{3}\bar{2};41} =$  . (2.3.14)

It is convenient to define special symbols for certain generalised partitions whose parts are all half odd integers.

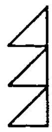
Definition 2.3.15. *Half partitions. Let* $\Delta_r = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) = (\frac{1}{2}^r)$ *be the generalised partition consisting of* r *parts each equal to* $\frac{1}{2}$. *More generally, for* $\lambda \in P(l;r)$, *define:*

$$(\Delta_r;\lambda) = (\lambda_1 + \tfrac{1}{2}, \lambda_2 + \tfrac{1}{2}, \ldots, \lambda_r + \tfrac{1}{2}). \qquad (2.3.15)$$

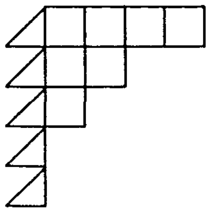
Note that $\Delta_r = (\Delta_r;0)$.

Definition 2.3.16. *Young half diagram. The Young diagram F^{Δ_r} is defined to be a column of r diagonal half boxes. The Young diagram $F^{\Delta_r;\lambda}$ is constructed by adjoining F^{Δ_r} to the left edge of F^{λ} with the topmost point of each at the same level.*

This definition implies that, for example:

$F^{\Delta_3} =$  (2.3.17a)

and

$F^{\Delta_5;421} =$  . (2.3.17b)

It will also be convenient to be able to refer to generalised partitions which are ordinary or half partitions with the last part having changed sign.

Definition 2.3.18. *If $\lambda \in P(l)$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, \lambda_p)$, define the p part generalised partition λ_- by:*

$$\lambda_- = (\lambda_1, \lambda_2, \ldots, \lambda_{p-1}, -\lambda_p) \tag{2.3.18a}$$

and if $\lambda \in P(l;r)$, define the r part generalised partition $(\Delta_r;\lambda)_-$ by:

$$(\Delta_r;\lambda)_- = (\lambda_1 + \tfrac{1}{2}, \lambda_2 + \tfrac{1}{2}, \ldots, \lambda_{r-1} + \tfrac{1}{2}, -\lambda_r - \tfrac{1}{2}); \tag{2.3.18b}$$

and denote $(\Delta_r, 0)_- = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2})$ by Δ_{r-}. In addition, let $\lambda_+ = \lambda$, $(\Delta_r;\lambda)_+ = (\Delta_r;\lambda)$ and $\Delta_{r+} = \Delta_r$.

§2.4. Partitions as representation labels

Through the work of Weyl [**We39**], Murnaghan [**Mu38**] and Littlewood [**Li50**], a means alternative to the Dynkin labelling of the irreducible finite dimensional representations of the classical groups arose. This scheme involves partitions which, together with the corresponding Young diagrams, have since proved very useful, particularly in the determination of characters, branching rules [**Ki75**] and tensor products [**BK83**] of these representations. In the notation of [**Li50**], the complete list of equivalence classes of finite dimensional irreducible representations of the classical Lie groups is given in Table 2.4.1.

Group	Representation	Restricitions
$GL(m)$	$\{\bar{\nu};\mu\}$	$\tilde{\mu}_1+\tilde{\nu}_1\leq m$
$SL(m)$	$\{\lambda\}$	$\tilde{\lambda}_1 < m$
$O(m)$	$[\lambda]$	$\tilde{\lambda}_1+\tilde{\lambda}_2\leq m$
$SO(2r+1)$	$[\lambda]$	$\tilde{\lambda}_1\leq r$
$SO(2r)$	$[\lambda]$	$\tilde{\lambda}_1 < r$
	$[\lambda]_\pm$	$\tilde{\lambda}_1 = r$
$Sp(2r)$	$\langle\lambda\rangle$	$\tilde{\lambda}_1\leq r$

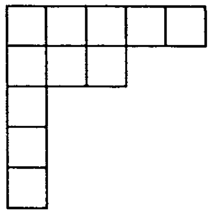
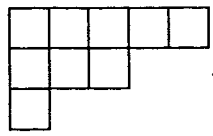
Table 2.4.1

In Table 2.4.1, the field F has been dropped from the notation since this list applies in each of the cases, F $=$ C or F $=$ R. Similarly, the irreducible representations of $U(m)$ and $SU(m)$ are closely related to, and have the same set of labels as, the irreducible representations of $GL(m)$ and $SL(m)$ respectively.

As will be demonstrated, each of the irreducible representations listed in Table 2.4.1, apart from $\{\bar{\nu};\mu\}$ of $GL(m)$ for $\nu\neq(0)$, occurs as an irreducible component in the tensor product of l copies of the defining representation for some l. In view of this, each of these is termed a covariant tensor representation. With the same exception, each element of the representation matrix is a polynomial function of the elements of the corresponding matrix of the defining representation. Thus the covariant representations may also be called polynomial. In contrast, each irreducible representation $\{\bar{\nu};\mu\}$ of $GL(m)$ for $\nu\neq(0)$ is referred to as a mixed tensor representation or as a rational representation. The representation $\{\bar{\mu};\nu\}$ of $GL(m)$ is contragredient to $\{\bar{\nu};\mu\}$.

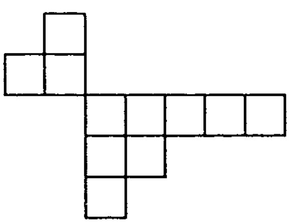
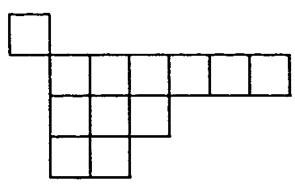
For the groups $GL(m)$ and $O(m)$, there exist irreducible covariant tensor representations labelled by the n part partition $\epsilon=(1,1,\ldots,1)$. These representations, $\{\epsilon\}$ and $[\epsilon]$, are each one dimensional, mapping each group element to its determinant: $\{\epsilon\}(A)=\det A$, for $A\in GL(m)$ and similarly $[\epsilon](B)=\det B$ for $B\in O(m)$. Denote the a-fold tensor product, $\{\epsilon\}\otimes\{\epsilon\}\otimes\cdots\otimes\{\epsilon\}$, by $\{\epsilon\}^a$, and define $[\epsilon]^b$ similarly. The representation contragredient to $\{\epsilon\}$ is denoted $\{\bar{\epsilon}\}$ and maps each group element to the inverse of its determinant: $\{\bar{\epsilon}\}(A)=(\det A)^{-1}$, for $A\in GL(m)$ and similarly $[\bar{\epsilon}](B)=(\det B)^{-1}$ for $B\in O(m)$. Since $\{\epsilon\}\otimes\{\bar{\epsilon}\}=\{0\}$ it is natural to denote $\{\bar{\epsilon}\}$ by $\{\epsilon\}^{-1}$ and to denote the a-fold tensor product $\{\bar{\epsilon}\}\otimes\{\bar{\epsilon}\}\otimes\cdots\otimes\{\bar{\epsilon}\}$, by

$\{\epsilon\}^{-a}$. $[\epsilon]^{-b}$ is defined similarly. As noted by Weyl [**We39**], the tensor product representations $\{\epsilon\}^a \otimes \{\lambda\}(A) = (\det A)^a (\{\lambda\}(A))$ and $[\epsilon]^b \otimes [\lambda](B) = (\det B)^b ([\lambda](B))$ are each irreducible and of the same dimension as $\{\lambda\}$ and $[\lambda]$ respectively. The representations $\{\epsilon\}^a \otimes \{\lambda\}$ and $[\epsilon]^b \otimes [\lambda]$ are termed associates of $\{\lambda\}$ and $[\lambda]$ respectively. In the case of $O(m)$, since $\det B = \pm 1$, only the cases $b = 0$ and $b = 1$ need be considered and each irreducible representation of $O(m)$ has a unique associate. In this case, it is conventional to write $[\lambda]^* = [\epsilon] \otimes [\lambda]$. In fact, $[\mu]^*$ and $[\nu]$ are equivalent if and only if $\tilde{\nu}_1 = m - \tilde{\mu}_1$ and $\tilde{\nu}_i = \tilde{\mu}_i$ for $i > 1$ [**Pr89**]. If, for example $\mu = (5,3,1^3)$, then for $O(8)$, $\nu = (5,3,1)$. The respective Young diagrams clarify the relationship:

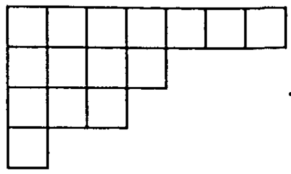
$F^{5,3,1^3} =$  and $F^{5,3,1} =$  . (2.4.2)

Note that if $m = 2\tilde{\lambda}_1$, then the representation $[\lambda]$ is self-associate.

For $GL(m)$ the situation is more complicated since the associated representations $\{\epsilon\}^a \otimes \{\bar{\nu};\mu\}$ are distinct for each $a \in \mathbf{Z}$. Weyl [**We39**] showed how these representations are related by employing generalised partitions (see Section 2.3). Let Λ be the generalised partition corresponding to $(\bar{\nu};\mu)_m$ and Γ that corresponding to $(\bar{\sigma};\rho)_m$. $\{\bar{\nu};\mu\}$ and $\{\bar{\sigma};\rho\}$ are associated irreducible representations of $GL(m)$ if and only if for some $a \in \mathbf{Z}$, $\Lambda_i = \Gamma_i + a$ for $i = 1,\ldots,m$. In this case $\{\bar{\nu};\mu\} = \{\epsilon\}^a \otimes \{\bar{\sigma};\rho\}$. Note that $\{\bar{\nu};\mu\} = \{\epsilon\}^{-\nu_1} \otimes \{\bar{\nu};\mu\}^*$. It is instructive to see how the composite Young diagrams of associate representations are related. Consider the composite Young diagram of (2.3.14) where $\mu = (4,1)$ and $\nu = (3,2)$. For $GL(5)$, the corresponding generalised partition is $(4,1,0,-2,-3)$ and that labelling the representation $\{\epsilon\}^a \otimes \{\bar{\nu};\mu\}$ is $(a+4, a+1, a, a-2, a-3)$. For $a = 1$ and $a = 2$ the corresponding composite partitions are $(\overline{2,1};5,2,1)$ and $(\bar{1};6,3,2)$ respectively with composite Young diagrams:

$F^{\overline{21};521} =$  and $F^{\bar{1};632} =$  (2.4.3a)

respectively. Setting $a = 3$ gives $(\bar{\nu}; \mu)^* = (7, 4, 3, 1)$ with composite Young diagram:

$F^{7,4,3,1} =$  . (2.4.3b)

Notice that, for each unit increase in a, the rightmost 'inverted' column is removed and replaced by an 'upright' column whose length is m minus the length of the former.

Under the restriction of the groups $GL(m)$ and $O(m)$ to their subgroups of unit determinant, namely $SL(m)$ and $SO(m)$ respectively, all representations that are associated to one another become equivalent. This is because, under such restrictions, the representations $\{\epsilon\}$, $\{\bar{\epsilon}\}$ and $[\epsilon]$ are each equivalent to the identity representation. These simple branchings are denoted $\{\epsilon\} \downarrow \{0\}$, $\{\bar{\epsilon}\} \downarrow \{0\}$ and $[\epsilon] \downarrow \{0\}$. All irreducible representations of $GL(m)$ and $O(m)$, apart from those of $O(m)$ that are self-associate, remain irreducible on restriction to the unimodular subgroups. The full list of such branchings is given in Table 2.4.4.

Group restriction	Rule	Range of validity
$GL(m) \downarrow SL(m)$	$\{\bar{\nu}; \mu\} \downarrow \{\bar{\nu}; \mu\}^*$	
$O(2r+1) \downarrow SO(2r+1)$	$[\lambda] \downarrow [\lambda]$	$\tilde{\lambda}_1 \leq r$
	$[\lambda] \downarrow [\lambda]^*$	$\tilde{\lambda}_1 > r$
$O(2r) \downarrow SO(2r)$	$[\lambda] \downarrow [\lambda]$	$\tilde{\lambda}_1 < r$
	$[\lambda] \downarrow [\lambda]^*$	$\tilde{\lambda}_1 > r$
	$[\lambda] \downarrow [\lambda]_+ \oplus [\lambda]_-$	$\tilde{\lambda}_1 = r$

Table 2.4.4

In addition to the true representations listed in Table 2.4.1, there exist irreducible two-valued 'spin' representations of the orthogonal groups which owe their existence to the double connectedness of $O(m)$ and $SO(m)$ for $m > 2$ (see Theorem 1.7.8). These two valued representations are genuine representations of the groups $Pin(m)$ and $Spin(m)$, as the simply connected universal covering groups of $O(m)$ and $SO(m)$ are respectively called. For the groups $O(2r)$ and $O(2r+1)$ these 'spin' representations may be denoted [**Mu38**] by $[\Delta_r; \lambda]$ where the partition λ is such that $\tilde{\lambda}_1 \leq r$. Δ_r is the basic spin representation first examined by Brauer and Weyl [**BW35**]. On restricting $O(2r+1)$ to the unimodular subgroup $SO(2r+1)$, all the representations $[\Delta_r;\lambda]$ remain irreducible and are labelled in the same way. However, for $O(2r)\downarrow SO(2r)$, the representation $[\Delta_r;\lambda]$ branches into a sum of two inequivalent irreducible representations of equal dimension. These representations are denoted $[\Delta_r;\lambda]_+$ and $[\Delta_r;\lambda]_-$.

Table 2.4.4 [**KA81**] gives the relationship between the Dynkin labels of the irreducible representations of the classical groups described in Section 2.1 and the partition labels described above.

Group	Algebra	Relationship between Dynkin label (a) and generalised partition label Λ	
$SL(r+1)$	A_r	$a_1=\Lambda_1-\Lambda_2$	$\Lambda_1= a_1+a_2+\cdots+a_{r-1}+a_r$
		$a_2=\Lambda_2-\Lambda_3$	$\Lambda_2= a_2+\cdots+a_{r-1}+a_r$
		$\vdots$	$\vdots$
		$a_{r-1}=\Lambda_{r-1}-\Lambda_r$	$\Lambda_{r-1}= a_{r-1}+a_r$
		$a_r=\Lambda_r$	$\Lambda_r= a_r$
$SO(2r+1)$	B_r	$a_1=\Lambda_1-\Lambda_2$	$\Lambda_1= a_1+a_2+\cdots+a_{r-1}+\frac{1}{2}a_r$
		$a_2=\Lambda_2-\Lambda_3$	$\Lambda_2= a_2+\cdots+a_{r-1}+\frac{1}{2}a_r$
		$\vdots$	$\vdots$
		$a_{r-1}=\Lambda_{r-1}-\Lambda_r$	$\Lambda_{r-1}= a_{r-1}+\frac{1}{2}a_r$
		$a_r=2\Lambda_r$	$\Lambda_r= \frac{1}{2}a_r$
$Sp(2r)$	C_r	$a_1=\Lambda_1-\Lambda_2$	$\Lambda_1= a_1+a_2+\cdots+a_{r-1}+a_r$
		$a_2=\Lambda_2-\Lambda_3$	$\Lambda_2= a_2+\cdots+a_{r-1}+a_r$
		$\vdots$	$\vdots$
		$a_{r-1}=\Lambda_{r-1}-\Lambda_r$	$\Lambda_{r-1}= a_{r-1}+a_r$
		$a_r=\Lambda_r$	$\Lambda_r= a_r$
$SO(2r)$	D_r	$a_1=\Lambda_1-\Lambda_2$	$\Lambda_1= a_1+a_2+\cdots+a_{r-2}+\frac{1}{2}a_{r-1}+\frac{1}{2}a_r$
		$a_2=\Lambda_2-\Lambda_3$	$\Lambda_2= a_2+\cdots+a_{r-2}+\frac{1}{2}a_{r-1}+\frac{1}{2}a_r$
		$\vdots$	$\vdots$
		$a_{r-2}=\Lambda_{r-2}-\Lambda_{r-1}$	$\Lambda_{r-2}= a_{r-2}+\frac{1}{2}a_{r-1}+\frac{1}{2}a_r$
		$a_{r-1}=\Lambda_{r-1}+\Lambda_r$	$\Lambda_{r-1}= \frac{1}{2}a_{r-1}+\frac{1}{2}a_r$
		$a_r=\Lambda_{r-1}-\Lambda_r$	$\Lambda_r= \frac{1}{2}a_{r-1}-\frac{1}{2}a_r$

Table 2.4.5

Since, by virtue of Theorem 1.7.7, the Dynkin labels $a_1,a_2,\ldots,a_r$, are non-negative integers, Table 2.4.5 shows that the irreducible representations of $SL(r+1)$ and $Sp(2r)$ are labelled by those Λ which are ordinary partitions with at most r parts. For $SO(2r+1)$, Λ is either an ordinary partitions with at most r parts or a r part half partition $(\Delta_r;\lambda)$. The case of $SO(2r)$ admits the same set of representation labels, as well as, in addition, r integral part generalised partitions λ_- and r half odd integral part partitions $(\Delta_r;\lambda)_-$, in which, in both cases, the first $r-1$ parts are non-negative and the last is negative. These last two cases correspond to the representations of $SO(2r)$ labelled by $[\lambda]_-$ in Table 2.4.1 and the spin representations of $SO(2r)$ labelled by $[\Delta_r;\lambda]_-$ respectively.

§2.5. Dimension formulae

One great benefit of the use of partitions for the labelling of irreducible representations of the classical groups is that they provide a very convenient means of obtaining the dimensions of these representations. For each partition λ, there exists an m-dependent formula based on the Young diagram F^λ, which yields the dimensions of the irreducible representations for each sequence of classical groups $\mathcal{G}(m)$. In each case, this formula is a polynomial in m divided by the product of hook lengths.

Definition 2.5.1. *Hook lengths. For the partition λ, define the hook length h_{ij} by*

$$h_{ij} = \lambda_i + \tilde{\lambda}_j - i - j + 1. \tag{2.5.1a}$$

For each (i,j) such that there exists a box at the intersection of the ith row and the jth column of the Young diagram F^λ, h_{ij} is given by the number of boxes in the hook consisting of that box together with all the boxes directly below it and all the boxes directly to its right. The product of hook lengths, $H(\lambda)$, is given by

$$H(\lambda) = \prod_{(i,j)\in F^\lambda} h_{ij}, \tag{2.5.1b}$$

the product being over all the boxes of F^λ.

As an example consider the partition $\lambda = (4,3,1)$. Writing in each box of F^λ the hook length associated with that box gives:

$$\begin{array}{|c|c|c|c|}\hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 & \\ \cline{1-3} 1 & & & \\ \cline{1-1} \end{array}\ . \tag{2.5.2}$$

Thereupon, $H(4,3,1) = 576$.

The following definition provides the numerators in the dimension formulae for irreducible representations of the classical Lie groups.

Definition 2.5.3. *Let*

$$N_m\{\lambda\} = \prod_{(i,j)\in F^\lambda} (m-i+j); \qquad (2.5.3a)$$

$$N_m\{\bar{\nu};\mu\} = \prod_{(i,j)\in F^\mu} (m-\tilde{\nu}_i-\tilde{\mu}_j+i+j-1) \prod_{(k,l)\in F^\nu} (m+\nu_k+\mu_l-k-l+1); \qquad (2.5.3b)$$

$$N_m[\lambda] = \prod_{\substack{(i,j)\in F^\lambda \\ i\geq j}} (m+\lambda_i+\lambda_j-i-j) \prod_{\substack{(i,j)\in F^\lambda \\ i<j}} (m-\tilde{\lambda}_i-\tilde{\lambda}_j+i+j-2); \qquad (2.5.3c)$$

and

$$N_m\langle\lambda\rangle = \prod_{\substack{(i,j)\in F^\lambda \\ i>j}} (m+\lambda_i+\lambda_j-i-j+2) \prod_{\substack{(i,j)\in F^\lambda \\ i\leq j}} (m-\tilde{\lambda}_i-\tilde{\lambda}_j+i+j). \qquad (2.5.3d)$$

In each of these cases, the polynomial is conveniently obtained by drawing the appropriate Young diagram and entering into each box the appropriate linear term in m. These terms are then multiplied together. As an example $N_m[4,3,1]$ is obtained via:

$m+6$	$m-4$	$m-3$	$m-1$
$m+4$	$m+2$	$m-1$	
$m+1$			

. (2.5.4a)

This gives:

$$N_m[4,3,1] = (m+6)(m+4)(m+2)(m+1)(m-1)^2(m-3)(m-4). \qquad (2.5.4b)$$

Further examples may be found in [**EK79**].

Theorem 2.5.5. *The dimensions* $D_m\{\lambda\}$, $D_m\{\bar{\nu};\mu\}$, $D_m[\lambda]$ *and* $D_{2r}\langle\lambda\rangle$ *of the irreducible representations* $\{\lambda\}$, $\{\bar{\nu};\mu\}$, $[\lambda]$, *and* $\langle\lambda\rangle$ *of the groups* $SL(m)$, $GL(m)$, $O(m)$ *and* $Sp(2r)$ *respectively, are given by:*

$$D_m\{\lambda\} = N_m\{\lambda\}/H(\lambda); \qquad (2.5.5a)$$

$$D_m\{\bar{\nu};\mu\} = N_m\{\bar{\nu};\mu\}/H(\nu)H(\mu); \qquad (2.5.5b)$$

$$D_m[\lambda] = N_m[\lambda]/H(\lambda); \qquad (2.5.5c)$$

and

$$D_{2r}\langle\lambda\rangle = N_{2r}\langle\lambda\rangle/H(\lambda). \qquad (2.5.5d)$$

The first of these formulae is the celebrated dimension formula of Robinson [**Ro58**], first suggested to him by Hall. It gives the dimensions of irreducible representations of $SU(m)$ as well as $SL(m)$. The other formulae were first obtained in this form by El-Samra and King [**EK79**]. It should be pointed out that since all but the self-associate irreducible representations of $O(m)$ are irreducible on restriction to

$SO(m)$, the dimension of the irreducible representation $[\lambda]$ of $SO(m)$ is also given by (2.5.5c) if $\tilde{\lambda}_1 \neq m/2$. Since, with $\tilde{\lambda}_1 = r$, the representations $[\lambda]_+$ and $[\lambda]_-$ of $SO(2r)$ are of the same dimension, it follows from (2.5.5c) that, in this case:

$$D_{2r}[\lambda]_+ = D_{2r}[\lambda]_- = N_m[\lambda]/2H(\lambda). \tag{2.5.6}$$

With $\lambda = (4,3,1)$, examples (2.5.2) and (2.5.4) give the dimensions of the representations $[4,3,1]$ of $O(m)$ to be:

$$D_m[4,3,1] = (m+6)(m+4)(m+2)(m+1)(m-1)^2(m-3)(m-4)/576. \tag{2.5.7}$$

This implies that for $m = 5,6,7,8$, the dimensions of the representations $[4,3,1]$ of $O(m)$ are 231, 1750, 7722 and 25725 respectively. From Table 2.4.4, the second of these representations is reducible on restriction to the unimodular subgroup. Thus, from (2.5.6), the dimensions of the irreducible representations $[4,3,1]_+$ and $[4,3,1]_-$ of $SO(6)$ are each 875.

For the irreducible 'spin' representations of the orthogonal groups, the dimensions are once again provided by [**EK79**].

Theorem 2.5.8. *For $m = 2r$ or $m = 2r+1$, the dimension of the representation $[\Delta_r;\lambda]$ of $O(m)$ or $SO(m)$ is given by:*

$$D_m[\Delta_r;\lambda] = 2^r D_{m-1}\langle\lambda\rangle. \tag{2.5.8a}$$

Similarly, the dimensions of the irreducible representations $[\Delta_r;\lambda]_\pm$ of $SO(2r)$ are given by:

$$D_m[\Delta_r;\lambda]_\pm = 2^{r-1} D_{m-1}\langle\lambda\rangle. \tag{2.5.8b}$$

§2.6. Young tableaux

Definition 2.6.1. *A Young tableau, t^λ or T^λ, is a Young diagram F^λ in which the boxes each contain a single element from a specified set $\mathcal{I}$. F^λ will be referred to as the shape of t^λ or T^λ.*

In this thesis, a number of sets will be used to fill the role of the set $\mathcal{I}$ in this definition. Most often, these will be the sets $\mathcal{I}^{\mathcal{G}(m)}$ defined in Section 2.2 for the various classical groups $\mathcal{G}(m)$.

Definition 2.6.2. *The Young tableau t^λ is that tableau arising from the filling of the Young diagram F^λ with the integers $1,2,\ldots,$ passing first down the leftmost column and then the remaining columns taken consecutively, left to right.*

This definition gives, for example,

$$t^{(4,2^2,1)} = \begin{array}{|c|c|c|c|}\hline 1 & 5 & 8 & 9 \\ \hline 2 & 6 \\ \cline{1-2} 3 & 7 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \ . \tag{2.6.3}$$

It is often convenient to be able to refer to the entries of a particular tableau. There are two immediate ways of doing this and both have their uses.

Definition 2.6.4. *If T^λ is a Young tableau of shape F^λ, let $T^\lambda_{(i,j)}$ be the entry in the box at the intersection of the ith row and the jth column. For there to be such an entry, it is necessary that $j \leq \lambda_i$.*

Definition 2.6.5. *If T^λ is a Young tableau for which $\lambda \in P(l)$, let $T^\lambda_{(a)}$ be the entry at the position in which the integer a is located in t^λ. In order that $T^\lambda_{(a)}$ be defined it is necessary that $a \leq l$.*

Definition 2.6.4 provides the means to define, for each λ, a tableau which will play an important role in later chapters.

Definition 2.6.6. *Let $T^\lambda_>$ be such that $T^\lambda_{>(i,j)} = i$ for each $i = 1, 2, \ldots, \tilde{\lambda}_1$ and $j = 1, 2, \ldots, \lambda_i$.*

This definition gives, for example:

$$T^{4,3,1}_> = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array} \ . \tag{2.6.7}$$

A number of the proofs in later chapters will require an order to be defined on the set of all tableaux of one particular shape. The following will prove to be sufficient in most cases.

Definition 2.6.8. *Let t^b_x be the sum of the entries in the bth column of T^λ_x for $b = 1, 2, \ldots, q$ where $q = \lambda_1$. Define $\left|T^\lambda_x\right|$ to be the equivalence class of all tableaux which have their sequences of column sums identical to that of T^λ_x; that is $T^\lambda_y \in \left|T^\lambda_x\right|$ if $t^b_y = t^b_x$ for $b = 1, 2, \ldots, q$. A total order on the set of equivalence classes of tableaux is defined by $\left|T^\lambda_x\right| > \left|T^\lambda_y\right|$ if for some $k \leq q$, $t^k_x > t^k_y$ with $t^b_x = t^b_y$ for each $b = k+1, k+2, \ldots, q$. It is convenient to write $T^\lambda_x > T^\lambda_y$ when this strict inequality is true of the equivalence classes to which T^λ_x and T^λ_y belong and to say, in such a case, that T^λ_x is higher than T^λ_y.*

It will emerge that for each group a particular set of tableaux have a favoured status. These are the standard tableaux. Historically, the term 'standard' tableau has usually been reserved for those favoured tableaux associated with the symmetric group [**Yo77**], with various words such as 'semistandard' being used for other groups when necessary. This has led to inconsistencies. Here however, the word 'standard' will always be used and will be prefixed by the group under consideration.

Definition 2.6.9. *If $\lambda \in P(l)$, the tableau T^λ is S_l-standard if and only if:*

(i) the entries are distinct and taken from the set $\mathbb{N}_l$;

(ii) the entries increase from top to bottom down each column;

(iii) the entries increase from left to right across each row.

For example, if $l = 5$ and $\lambda = (3, 2)$ there are just five S_5-standard tableaux:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\,,\quad \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\,. \tag{2.6.10}$$

Let f^λ be the total number of S_l-standard tableaux. The following formula for f^λ was first proved by Young [**Yo77**] and first cast in this 'hook length' form by Frame, Robinson and Thrall [**FR54**].

Theorem 2.6.11. *If $\lambda \in P(l)$, then*

$$f^\lambda = \frac{l!}{H(\lambda)}. \tag{2.6.11}$$

For the example above, this gives

$$f^{(3,2)} = \frac{5!}{4.3.2} = 5, \tag{2.6.12}$$

verifying that there are just five S_5-standard tableaux of shape $F^{(3,2)}$. The following formula was also proved by Young [**Yo77**].

Theorem 2.6.13. *If $\lambda \in P(l)$, then*

$$\sum_{\lambda \in P(l)} (f^\lambda)^2 = l! \tag{2.6.13}$$

This theorem and the S_l-standard tableaux will be utilised in the next chapter.

The notation relevant to composite tableaux based on the composite Young diagrams of Definition (2.3.13) will now be defined.

Definition 2.6.14. *A composite Young tableau, $t^{\bar{\nu};\mu}$ or $T^{\bar{\nu};\mu}$, is a composite Young diagram $F^{\bar{\nu};\mu}$, in which the boxes of the F^μ portion each contain an entry from a set*

$\mathcal{I}$ and the boxes of the $F^{\bar{\nu}}$ portion each contain an entry from a, possibly different, set $\mathcal{J}$.

In this thesis, the set $\mathcal{I}$ of Definition 2.6.14 will always be a set of positive integers and the set $\mathcal{J}$ will always be a set of barred positive integers.

Definition 2.6.15. *The composite Young tableau $t^{\bar{\nu};\mu}$. With the notation of Definition 2.6.2, $t^{\bar{\nu}}$ is that diagram created by reflecting t^{ν} in its topmost and leftmost edges and barring its entries. $t^{\bar{\nu};\mu}$ is formed by bringing $t^{\bar{\nu}}$ and t^{μ} together to create a diagram of the same shape as $F^{\bar{\nu};\mu}$.*

This definition gives, for example:

$$t^{\overline{32};431} = \begin{array}{ccccccc} & \bar{4} & \bar{2} & & & & \\ \bar{5} & \bar{3} & \bar{1} & & & & \\ & & & 1 & 4 & 6 & 8 \\ & & & 2 & 5 & 7 & \\ & & & 3 & & & \end{array}. \qquad (2.6.16)$$

Definition 2.6.17. *Let $T^{\bar{\nu};\mu}$ be a composite tableau. Define $T^{\bar{\nu};\mu}_{(i,j)}$ to be the entry in the box at the intersection of the ith row and the jth column of the F^{μ} portion if $1 \leq i \leq \tilde{\mu}_1$ and $1 \leq j \leq \mu_i$. Define $T^{\bar{\nu};\mu}_{(\bar{i},\bar{j})}$ to be the entry in the box at the intersection of the ith row and the jth column of the $F^{\bar{\nu}}$ portion if $1 \leq i \leq \tilde{\nu}_1$ and $1 \leq j \leq \nu_i$, these rows and columns being counted from the bottommost row and the rightmost column respectively of $F^{\bar{\nu}}$.*

Definition 2.6.18. *If $T^{\bar{\nu};\mu}$ is a composite tableau for which $\mu \in P(u)$ and $\nu \in P(v)$, let $T^{\bar{\nu};\mu}_{(a)}$ be the entry at the position in which $a \in \mathbf{N}_u \cup \overline{\mathbf{N}}_v$ is located in $t^{\bar{\nu};\mu}$.*

An appropriate order on the set of composite tableaux is given by the following definition.

Definition 2.6.19. *Let μ and ν be partitions for which $\mu_1 = s$ and $\nu_1 = t$. Label the columns of $F^{\bar{\nu};\mu}$ left to right by the integers $-t, -t+1, \ldots, -1, 1, 2, \ldots, s$. Under the identification $\bar{i} = -i$, let t^b_x be the sum of the entries in the bth column of $T^{\bar{\nu};\mu}_x$ for $b \neq 0$. Let $t^0_x = 0$. Define $\left|T^{\bar{\nu};\mu}_x\right|$ to be the equivalence class of all composite tableaux which have their sequences of column sums identical to that of $T^{\bar{\nu};\mu}_x$; that is $T^{\bar{\nu};\mu}_y \in \left|T^{\bar{\nu};\mu}_x\right|$ if $t^b_y = t^b_x$ for $b = -t, -t+1, \ldots, s$. A total order on the set of equivalence classes of composite tableaux is defined by $\left|T^{\bar{\nu};\mu}_x\right| > \left|T^{\bar{\nu};\mu}_y\right|$ if there exists $k \in \{-t, -t+1, \ldots, s\}$ such that $t^k_x > t^k_y$ with $t^b_x = t^b_y$ for each $b = k+1, k+2, \ldots, s$. It is convenient to write $T^{\bar{\nu};\mu}_x > T^{\bar{\nu};\mu}_y$ when this strict inequality is true of the equivalence* *classes to which* $T_x^{\nu;\mu}$ *and* $T_y^{\nu;\mu}$ *belong and to say, in such a case, that* $T_x^{\nu;\mu}$ *is higher that* $T_y^{\nu;\mu}$.

When displaying tableaux and composite tableaux, it will often be convenient to omit the diagram F^λ or $F^{\nu;\mu}$ and display only the entries of T^λ or $T^{\nu;\mu}$ in their correct positions.

Tableaux based on the Young half diagrams $F^{\Delta_r;\lambda}$ of Definition 2.3.16 will now be introduced.

Definition 2.6.20. *Young half tableaux. A Young tableau* T^{Δ_r} *is a Young diagram* F^{Δ_r} *in which each half box contains entries from a set. A Young tableau* $T^{\Delta_r;\lambda}$ *may be constructed by adjoining a* T^{Δ_r} *to a* T^λ *by analogy with Definition* 2.3.16.

It will be convenient, when displaying a half tableau, to just write the entries in their correct positions and to distinguish the entries from the F^{Δ_r} portion by following them with full stops. Examples are provided by:

$$T^{\Delta_4} = \begin{matrix} \bar{1}\,\cdot \\ 2\,\cdot \\ \bar{3}\,\cdot \\ 4\,\cdot \end{matrix} \qquad \text{and} \qquad T^{\Delta_4;3,3,1} = \begin{matrix} 1\cdot 1 & 2 & 3 \\ \bar{2}\cdot 2 & 3 & \bar{4} \\ \bar{3}\cdot 4 & & \\ 4\,\cdot & & \end{matrix} \tag{2.6.21}$$

Definition 2.6.22. *Let* $T^{\Delta_r;\lambda}$ *be formed by adjoining* T^{Δ_r} *to* T^λ. *Define* $T^{\Delta_r;\lambda}_{(i,j)} = T^\lambda_{(i,j)}$ *for* $i, j \geq 1$, *and define* $T^{\Delta_r;\lambda}_{(i,0)}$ *to be the single entry in the ith row of* T^{Δ_r} *for* $1 \leq i \leq r$. *In addition, define* $T^{\Delta_r;\lambda}_{(a)} = T^\lambda_{(a)}$ *for* $1 \leq a \leq l$ *when* $\lambda \in P(l)$.

Definition 2.6.23. *For each u, let* $T_u^{\Delta_r;\lambda}$ *be formed by adjoining* $T_u^{\Delta_r}$ *to* T_u^λ. *Using the notation of Definition* 2.6.8, *define the equivalence class* $\left|T_x^{\Delta_r;\lambda}\right|$ *to be such that* $T_y^{\Delta_r;\lambda} \in \left|T_x^{\Delta_r;\lambda}\right|$ *if and only if* $T_y^\lambda \in \left|T_x^\lambda\right|$. *A total order on the set of equivalence classes of tableaux is defined by* $\left|T_x^{\Delta_r;\lambda}\right| > \left|T_y^{\Delta_r;\lambda}\right|$ *if and only if* $\left|T_x^\lambda\right| > \left|T_y^\lambda\right|$. *Then, as before, it will be convenient to write* $T_x^{\Delta_r;\lambda} > T_y^{\Delta_r;\lambda}$ *when this strict inequality is true of the equivalence classes to which* $T_x^{\Delta_r;\lambda}$ *and* $T_y^{\Delta_r;\lambda}$ *belong and to say, in such a case, that* $T_x^{\Delta_r;\lambda}$ *is higher than* $T_y^{\Delta_r;\lambda}$.

Standard composite and half tableaux will be defined when the need arises.

Chapter 3
The Symmetric Group and the Specht Module

§3.1. The symmetric group

Denote by N_l the set of integers $\{1, 2, \ldots, l\}$. Define S_l to be the group of permutations of N_l. Thus, if $\pi \in S_l$, $\pi : \mathsf{N}_l \to \mathsf{N}_l$ is a bijective map. S_l is called the symmetric group on l symbols. Its order is $l!$. Each specific element $\pi \in S_l$ may be denoted by the symbol

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & l-1 & l \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(l-1) & \pi(l) \end{pmatrix}, \tag{3.1.1}$$

in which each member of N_l is explicitly displayed above its image under π. The actual order of the columns in this symbol is, of course, irrelevant. Using this notation, the product of the elements $\pi, \sigma \in S_l$ is

$$\begin{aligned} \pi\sigma &= \begin{pmatrix} 1 & 2 & \cdots & l \\ \pi(1) & \pi(2) & \cdots & \pi(l) \end{pmatrix} \begin{pmatrix} 1 & 2 & \cdots & l \\ \sigma(1) & \sigma(2) & \cdots & \sigma(l) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & \cdots & l \\ \pi(\sigma(1)) & \pi(\sigma(2)) & \cdots & \pi(\sigma(l)) \end{pmatrix}. \end{aligned} \tag{3.1.2}$$

For $l > 2$, S_l is non-abelian.

Another useful way to denote a permutation of S_l is through cycles. A cycle consists of a subset of N_l written so that each member of the cycle is mapped to that member to its right. The final member of the cycle is mapped to the first. Each element $\pi \in S_l$ may be written as a product of disjoint cycles of elements of N_l. For example, the permutation denoted by

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 5 & 1 & 6 & 4 & 2 & 7 & 3 \end{pmatrix}, \tag{3.1.3}$$

may be written, $\pi = (183)(4625)(7)$ in cycle notation. Notice that the individual cycles, three in this case, may be permuted among themselves and that the elements of each cycle may be permuted cyclically without affecting the permutation so represented. In view of this, it is conventional to write a cycle such that the sequence of cycle lengths is non-increasing left to right and the first element of each cycle is the smallest in that cycle. In addition, cycles of unit length are omitted. Using this convention for the permutation of (3.1.3) gives $\pi = (2546)(183)$. The cycle structure of an element $\pi \in S_l$ denotes its set of cycle lengths and is thus unambiguously specified by a partition of l.

The note following (3.1.1) implies that the permutation given there may equivalently be written

$$\pi = \begin{pmatrix} \pi^{-1}(1) & \pi^{-1}(2) & \pi^{-1}(3) & \cdots & \pi^{-1}(l-1) & \pi^{-1}(l) \\ 1 & 2 & 3 & \cdots & l-1 & l \end{pmatrix}, \quad (3.1.4)$$

so that

$$\pi\sigma = \begin{pmatrix} \sigma^{-1}\pi^{-1}(1) & \sigma^{-1}\pi^{-1}(2) & \cdots & \sigma^{-1}\pi^{-1}(l) \\ 1 & 2 & \cdots & l \end{pmatrix} \quad (3.1.5a)$$

$$= \begin{pmatrix} \sigma^{-1}(1) & \sigma^{-1}(2) & \cdots & \sigma^{-1}(l) \\ \pi(1) & \pi(2) & \cdots & \pi(l) \end{pmatrix}. \quad (3.1.5b)$$

The conjugate of the permutation $\pi \in S_l$ by the permutation $\sigma \in S_l$ is then given by:

$$\begin{aligned} \sigma^{-1}\pi\sigma &= \begin{pmatrix} 1 & 2 & \cdots & l \\ \sigma^{-1}\pi\sigma(1) & \sigma^{-1}\pi\sigma(2) & \cdots & \sigma^{-1}\pi\sigma(l) \end{pmatrix} \\ &= \begin{pmatrix} \sigma^{-1}(1) & \sigma^{-1}(2) & \cdots & \sigma^{-1}(l) \\ \sigma^{-1}\pi(1) & \sigma^{-1}\pi(2) & \cdots & \sigma^{-1}\pi(l) \end{pmatrix} \end{aligned}. \quad (3.1.6)$$

Thus, if the class structure of the permutation $\pi \in S_l$ is given by the partition $\lambda \in P(l)$, the class structure of $\sigma^{-1}\pi\sigma$ is also given by λ since if

$$(a_{11}a_{12}\cdots a_{1\lambda_1})(a_{21}a_{22}\cdots a_{2\lambda_2})\cdots(a_{p1}\cdots a_{p\lambda_p}) \quad (3.1.7a)$$

is π in cycle notation, then

$$(\sigma^{-1}(a_{11})\sigma^{-1}(a_{12})\cdots\sigma^{-1}(a_{1\lambda_1}))(\sigma^{-1}(a_{21})\cdots\sigma^{-1}(a_{2\lambda_2}))\cdots(\sigma^{-1}(a_{p1})\cdots\sigma^{-1}(a_{p\lambda_p})) \quad (3.1.7b)$$

is the cycle notation for $\sigma^{-1}\pi\sigma$. Conversely, if $\pi, \pi' \in S_l$ have the same cycle structure, then there exists $\sigma \in S_l$ (not necessarily unique) such that $\pi' = \sigma^{-1}\pi\sigma$. Define the conjugacy class of $\pi \in S_l$ to be the set $\{\sigma^{-1}\pi\sigma : \sigma \in S_l\}$. The above argument establishes the following lemma.

Lemma 3.1.8. *There is a bijective map between $P(l)$ and the conjugacy classes of S_l.*

Definition 3.1.9. *Permutation length. For $\pi \in S_l$ let $L(\pi) = \{(a,b) : 1 \le a < b \le l,\ \pi^{-1}(a) > \pi^{-1}(b)\}$. $\hat{l}(\pi) = \#L(\pi)$ is known as the length of π.*

Lemma 3.1.10. *If $\pi \in S_l$ and $s = \hat{l}(\pi)$ then π may be written as a product of s simple transpositions of the form $(c, c+1)$ for $1 \le c < l$. Furthermore, if*

$L(\pi, b) = \{a : 1 \le a < b,\ \pi^{-1}(a) > \pi^{-1}(b)\}$ for $b = 2, \ldots, l$, then:

$$\pi = \prod_{b=2}^{l} (b-1, b)(b-2, b-1) \cdots (b - \#L(\pi, b), b + 1 - \#L(\pi, b)), \qquad (3.1.10a)$$

where the factors are combined left to right on increasing b. Alternatively, if $L(a, \pi) = \{b : a < b \le l,\ \pi^{-1}(a) > \pi^{-1}(b)\}$ for $a = 1, \ldots, l-1$, then:

$$\pi = \prod_{a=1}^{l-1} (a, a+1)(a+1, a+2) \cdots (a - 1 + \#L(a, \pi), a + \#L(a, \pi)), \qquad (3.1.10b)$$

where, this time, the factors are combined right to left on increasing a.

Proof. Consider the permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \cdots \end{pmatrix}.$$

By (3.1.5) postmultiplication of σ by the simple transposition $(c, c+1)$ serves to exchange the elements of the bottom row that lie beneath c and $c+1$ of the top row. For instance:

$$\begin{aligned} \sigma(2,3) &= \begin{pmatrix} 1 & 3 & 2 & 4 & \cdots \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots \\ \sigma(1) & \sigma(3) & \sigma(2) & \sigma(4) & \cdots \end{pmatrix}. \end{aligned}$$

By using this result, starting with the identity permutation, the permutation π may be constructed as a product of s simple transpositions as follows. Consider each of the integers $b = 2, 3, \ldots, l$, in turn. In the bottom row of the permutation constructed so far (the identity for $b = 2$), $\#L(\pi, b)$ is the number of integers to the left of b that occur to the right in the final permutation π. The sequence of simple transpositions $(b, b-1)$; $(b-1, b-2)$; $(b-2, b-3)$; $\cdots$; $(b+1-\#L(\pi, b), b-\#L(\pi, b))$ serves to move the integer b in the bottom row leftwards, so that there are now $\#L(\pi, b)$ integers less than b to the right of b. That these are the required set of integers $L(\pi, b)$ is ensured by dealing with $b = 2, 3, \ldots, l$ in ascending order.

The final part of the lemma is dealt with in a similar way but by considering instead, $a = l-1, l-2, \ldots, 1$ in this decreasing order.

To illustrate Lemma 3.1.10, consider the permutation π given by (3.1.3). This yields the following set of values for $\#L(\pi, i)$ and $\#L(i, \pi)$:

i	1	2	3	4	5	6	7	8
$\#L(\pi, i)$	–	0	0	2	4	3	1	7
$\#L(i, \pi)$	2	4	5	3	1	1	1	–

. (3.1.11)

Thus $\hat{l}(\pi) = 17$ and (3.1.10a) implies that:

$$\pi = (34)(23)(45)(34)(23)(12)(56)(45)(34)(67)(78)(67)(56)(45)(34)(23)(12), \tag{3.1.12a}$$

and (3.1.10b) implies that:

$$\pi = (78)(67)(56)(45)(56)(67)(34)(45)(56)(67)(78)(23)(34)(45)(56)(12)(23). \tag{3.1.12b}$$

Lemma 3.1.10 demonstrates that the symmetric group S_l is generated by the $l-1$ simple transpositions $(1,2)$; $(2,3)$; $\cdots$; $(l-1,l)$. However, by (3.1.7):

$$(1,2,3,\ldots,l)(c,c+1)(1,2,3,\ldots,l)^{-1} = (c+1,c+2), \tag{3.1.13}$$

for $c = 1,2,\ldots,l-1$, and therefore the two permutations $(1,2)$ and $(1,2,3,\ldots,l)$ also serve as generators for S_l.

Definition 3.1.14. *The permutation π is said to have even or odd parity depending on whether $\hat{l}(\pi)$ is even or odd. The signature of π, denoted $(-1)^\pi$, is given by:*

$$(-1)^\pi = (-1)^{\hat{l}(\pi)}. \tag{3.1.14}$$

Lemma 3.1.15. *If $\pi, \sigma \in S_l$ then:*

$$(-1)^{\pi\sigma} = (-1)^\pi(-1)^\sigma; \tag{3.1.15a}$$

$$(-1)^{\pi^{-1}} = (-1)^\pi; \tag{3.1.15b}$$

$$(-1)^{\sigma^{-1}\pi\sigma} = (-1)^\pi. \tag{3.1.15c}$$

Proof. Form the three disjoint sets:

$$\mathcal{C} = \{(a,b) : 1 \le a < b \le l, \pi^{-1}(a) > \pi^{-1}(b), \sigma^{-1}\pi^{-1}(a) > \sigma^{-1}\pi^{-1}(b)\},$$

$$\mathcal{D} = \{(a,b) : 1 \le a < b \le l, \pi^{-1}(a) > \pi^{-1}(b), \sigma^{-1}\pi^{-1}(a) < \sigma^{-1}\pi^{-1}(b)\},$$

and $$\mathcal{E} = \{(a,b) : 1 \le a < b \le l, \pi^{-1}(a) < \pi^{-1}(b), \sigma^{-1}\pi^{-1}(a) > \sigma^{-1}\pi^{-1}(b)\}.$$

Let the cardinalities of these sets be c, d and e respectively. It can immediately be seen that $\hat{l}(\pi) = c+d$ and $\hat{l}(\pi\sigma) = c+e$. Furthermore, $L(\sigma) = \{(\pi(a),\pi(b)) : (a,b) \in \mathcal{E}\} \cup \{(\pi(b),\pi(a)) : (a,b) \in \mathcal{D}\}$. Thus $\hat{l}(\sigma) = d+e$ and (3.1.15a) follows. Since $\pi^{-1}\pi = I$, the identity of the group, and $\hat{l}(I) = 0$, (3.1.15a) implies that $1 = (-1)^{\pi^{-1}}(-1)^\pi$ from which (3.1.15b) follows. From (3.1.15a), $(-1)^{\sigma^{-1}\pi\sigma} = (-1)^{\sigma^{-1}}(-1)^\pi(-1)^\sigma$, which equals $(-1)^\pi$ through (3.1.15b).

§3.2. The Frobenius algebra and the regular representation

The Frobenius algebra, or group ring, $\mathsf{F}\mathcal{G}$ of the group $\mathcal{G}$ is the formal vector space over F which has a basis comprising the elements of $\mathcal{G}$. Thus for $x \in \mathsf{F}\mathcal{G}$,

$$x = \sum_{\pi \in \mathcal{G}} x(\pi)\pi, \tag{3.2.1}$$

where $x(\pi) \in \mathsf{F}$ for each $\pi \in \mathcal{G}$. The product of any two elements is governed by the product in $\mathcal{G}$, this being extended linearly to the whole of $\mathsf{F}\mathcal{G}$. Thus if $x = \sum_{\pi \in \mathcal{G}} x(\pi)\pi$ and $y = \sum_{\sigma \in \mathcal{G}} y(\sigma)\sigma$ then:

$$xy = \sum_{\pi \in \mathcal{G}} \sum_{\sigma \in \mathcal{G}} x(\pi)y(\sigma)\pi\sigma = \sum_{\rho \in \mathcal{G}} \left(\sum_{\sigma \in \mathcal{G}} x(\rho\sigma^{-1})y(\sigma) \right) \rho. \tag{3.2.2}$$

Definition 3.2.3. *The regular representation. The Frobenius algebra may be regarded as an $\mathsf{F}\mathcal{G}$-module through its own natural left action. This defines a representation of $\mathsf{F}\mathcal{G}$ known as the regular representation.*

Since the elements of $\mathcal{G}$ serve as a basis for the $\mathsf{F}\mathcal{G}$-module corresponding to the regular representation, the dimension of each is equal to the order of the group. Through this natural $\mathsf{F}\mathcal{G}$-module, matrices forming the regular representation are readily obtained. The left action of $x = \sum_{\pi \in \mathcal{G}} x(\pi)\pi$ on each of the elements of $\mathcal{G}$ shows that x is represented by the matrix $\Gamma(x)$ whose elements are given by $\Gamma(x)_{\sigma\rho} = x(\sigma\rho^{-1})$ where $\sigma, \rho \in \mathcal{G}$ are used to index the rows and columns of Γ. It may be confirmed that this provides a representation of $\mathsf{F}\mathcal{G}$ by computing:

$$\begin{aligned} (\Gamma(x)\Gamma(y))_{\sigma\tau} &= \sum_{\rho \in \mathcal{G}} \Gamma(x)_{\sigma\rho}\Gamma(y)_{\rho\tau} = \sum_{\rho \in \mathcal{G}} x(\sigma\rho^{-1})y(\rho\tau^{-1}) \\ &= \sum_{\pi \in \mathcal{G}} x(\sigma\tau^{-1}\pi^{-1})y(\pi) = \Gamma(xy)_{\sigma\tau}. \end{aligned} \tag{3.2.4}$$

In general, the regular representation is reducible since the Frobenius algebra possesses proper left ideals. Those left ideals which do not themselves possess proper left subideals are termed minimal. Clearly minimal ideals give rise to irreducible $\mathsf{F}\mathcal{G}$-modules and hence irreducible representations. The importance of the Frobenius algebra and its minimal ideals in the generation of irreducible representations is a consequence of the following theorem.

Theorem 3.2.5. (see [**Bo63**]) *Every irreducible representation of the finite group $\mathcal{G}$ occurs as a direct summand in the regular representation.*

For each $x \in \mathsf{F}\mathcal{G}$, the set $U = \{yx : y \in \mathsf{F}\mathcal{G}\}$ clearly constitutes a left ideal. U is referred to as the left ideal generated by x. Similarly, $W = \{yxz : y, z \in \mathsf{F}\mathcal{G}\}$

constitutes a two-sided ideal which is referred to as the two-sided ideal generated by x. As will transpire, the ideals generated by idempotents are of particular importance.

Definition 3.2.6. *An idempotent is an element $e \in \mathsf{F}\mathcal{G}$ such that $e^2 = e$. Idempotents which cannot be written as the sum $e = e_1 + e_2$ of two non-zero idempotents e_1 and e_2 which satisfy $e_1 e_2 = e_2 e_1 = 0$ are termed primitive idempotents.*

The following sequence of classical theorems and lemmas relating to the structure of the Frobenius algebra $\mathsf{F}\mathcal{G}$ of the finite group $\mathcal{G}$, enable all the inequivalent irreducible representations of $\mathcal{G}$ to be obtained. These will be employed in the next section for the particular case of the symmetric group. Proofs may be found in [**Bo63**].

Theorem 3.2.7. *$\mathsf{F}\mathcal{G}$ may be decomposed into a set of minimal left ideals:*

$$\mathsf{F}\mathcal{G} = U_1 \oplus U_2 \oplus \cdots \oplus U_k, \tag{3.2.7}$$

which are unique up to order and equivalence. On writing $I = e_1 + e_2 + \cdots + e_k$, where each $e_i \in U_i$, idempotents are obtained which generate each such left ideal.

Lemma 3.2.8. *The left ideal generated by a primitive idempotent is minimal. Conversely, every minimal left ideal possesses (at least) one primitive idempotent which generates it.*

Definition 3.2.9. *Equivalent left ideals. The two left ideals U_1 and U_2 of $\mathsf{F}\mathcal{G}$ are said to be equivalent if and only if there exists a map $\zeta : U_1 \to U_2$ such that:*

$$\zeta(xu_1) = x\zeta(u_1) \tag{3.2.9}$$

for all $x \in \mathsf{F}\mathcal{G}$ and $u_1 \in U_1$.

Lemma 3.2.10. *If the left ideals U_1 and U_2 are equivalent, every equivalence map from U_1 to U_2 is provided by right multiplication.*

Lemma 3.2.11. *If the left ideals U_1 and U_2 are minimal with generating idempotents e_1 and e_2 respectively, then right multiplication with any $e_1 x e_2 \neq 0$, $x \in \mathsf{F}\mathcal{G}$, defines an equivalence map from U_1 to U_2. Such equivalence maps are only provided by elements of this form.*

Theorem 3.2.12. *If e is an idempotent, then e is primitive if and only if for each $x \in \mathsf{F}\mathcal{G}$, $exe = \alpha e$ for some $\alpha \in \mathsf{F}$.*

Consideration of the two-sided ideals via Lemma 3.2.10 puts some order into the multitude of possible left ideals.

Lemma 3.2.13. *If W is a two sided ideal and if a minimal left ideal U lies in W, every left ideal equivalent to U lies in W.*

Lemma 3.2.14. *$\mathsf{F}\mathcal{G}$ decomposes uniquely into a direct sum of minimal two-sided ideals:*

$$\mathsf{F}\mathcal{G} = W_1 \oplus W_2 \oplus \cdots \oplus W_r. \tag{3.2.14}$$

Elements from different two-sided ideals annihilate one another in that $w_i w_j = w_j w_i = 0$ for all $w_i \in W_i$, $w_j \in W_j$ and $i \neq j$. Each two-sided ideal W_i possesses a generating idempotent e_i which is unique and determined by the decomposition $I = e_1 + e_2 + \cdots + e_r$. Each e_i commutes with all elements of $\mathsf{F}\mathcal{G}$.

The following theorem is known as Wedderburn's theorem.

Theorem 3.2.15. *Each minimal two-sided ideal W_i is isomorphic to the complete ring of $f_i \times f_i$ matrices for some f_i.*

This theorem has the direct consequence that if $\mathsf{F}\mathcal{G}$ is the direct sum of r minimal two-sided ideals as in (3.2.14) then:

$$g = \sum_{i=1}^{r} f_i^2, \tag{3.2.16}$$

where g is the order of the group $\mathcal{G}$. It also provides the following theorem.

Theorem 3.2.17. *The minimal two-sided ideal W_i contains exactly f_i linearly independent minimal left ideals which are each of dimension f_i and equivalent to one another.*

Attention is now turned to the labelling of the irreducible representations.

Lemma 3.2.18. *The dimension of the centre Z of $\mathsf{F}\mathcal{G}$ is equal to the number of its minimal two sided ideals.*

This Theorem shows that Z has a basis $\{e_1, e_2, \ldots, e_r\}$ where $e_i \in W_i$ is given by Lemma 3.2.14.

Lemma 3.2.19. *The dimension of the centre, Z, of $\mathsf{F}\mathcal{G}$ is equal to the number of conjugacy classes of $\mathcal{G}$.*

Lemmas 3.2.13, 3.2.14, 3.2.18 and 3.2.19 may be combined to yield:

Theorem 3.2.20. *The number of inequivalent irreducible representations of $\mathcal{G}$ is equal to the number of its conjugacy classes.*

§3.3. Young symmetrisers

In this section, the results of the previous section are applied to the particular case of the symmetric group S_l. Theorem 3.2.20 in conjunction with Lemma 3.1.8 shows that the inequivalent irreducible representations of S_l are labelled by $P(l)$, the partitions of l. As will be seen, the representation associated with the partition $\lambda \in P(l)$ may be obtained through the Young tableaux of shape F^λ and their respective Young symmetrisers. Although initially a Young symmetriser is obtained from each tableau of shape F^λ, it will emerge that just one is required for the construction of each inequivalent irreducible representation.

In this chapter each tableau T^λ for $\lambda \in P(l)$ will be such that the entries are from the set N_l and distinct. For such tableaux a numeral permutation may be defined.

Definition 3.3.1. *Tableau numeral permutation. For $\lambda \in P(l)$ and T^λ any tableau with distinct entries from the set N_l, define the action of S_l on T^λ to be given by the action of S_l on each numeral of T^λ. That is:*

$$\pi T^\lambda = \begin{matrix} \pi\left(T^\lambda_{(1,1)}\right) & \pi\left(T^\lambda_{(1,2)}\right) & \pi\left(T^\lambda_{(1,3)}\right) & \cdots \\ \pi\left(T^\lambda_{(2,1)}\right) & \pi\left(T^\lambda_{(2,2)}\right) & \cdots & \\ \pi\left(T^\lambda_{(3,1)}\right) & \cdots & & \\ \vdots & \ddots & & \end{matrix} \quad . \tag{3.3.1a}$$

This action is extended linearly to both $\mathsf{F}S_l$ and to F-linear combinations of tableaux. Thus the action of $x \in \mathsf{F}S_l$ on $\hat{T}^\lambda = \sum_i s(i)T_i^\lambda$, where each $s(i) \in \mathsf{F}$, is given by:

$$x\hat{T}^\lambda = \sum_{\pi \in S_l} \sum_i x(\pi)s(i)(\pi T_i^\lambda), \tag{3.3.1b}$$

where $x = \sum_{\pi \in S_l} x(\pi)\pi$ and each $x(\pi) \in \mathsf{F}$.

For $\lambda \in P(l)$, let $\mathcal{R}_{T^\lambda}$ and $\mathcal{C}_{T^\lambda}$ be the subgroups of S_l which, when acting on the numerals of T^λ, stabilise the rows and columns respectively. Define $P_{T^\lambda} \in \mathsf{Z}S_l$ and $Q_{T^\lambda} \in \mathsf{Z}S_l$ according to:

$$P_{T^\lambda} = \sum_{\rho \in \mathcal{R}_{T^\lambda}} \rho \tag{3.3.2a}$$

$$\text{and} \qquad Q_{T^\lambda} = \sum_{\sigma \in \mathcal{C}_{T^\lambda}} (-1)^\sigma \sigma. \tag{3.3.2b}$$

Lemma 3.3.3. *If* $\rho \in \mathcal{R}_{T^\lambda}$ *and* $\sigma \in \mathcal{C}_{T^\lambda}$ *then*

$$\rho P_{T^\lambda} = P_{T^\lambda}\rho = P_{T^\lambda}; \tag{3.3.3a}$$

$$\sigma Q_{T^\lambda} = Q_{T^\lambda}\sigma = (-1)^\sigma Q_{T^\lambda}. \tag{3.3.3b}$$

Proof. As π runs through $\mathcal{R}_{T^\lambda}$ then $\rho\pi$ and $\pi\rho$ each run through $\mathcal{R}_{T^\lambda}$ since $\mathcal{R}_{T^\lambda}$ is a finite group. (3.3.3a) then follows from (3.3.2a), the definition of P_{T^λ}. Similarly, as π runs through $\mathcal{C}_{T^\lambda}$, so do $\pi\sigma$ and $\sigma\pi$. However, by (3.1.15a), the coefficient of each term in (3.3.2b) will have been multiplied by a factor of $(-1)^\sigma$. This proves (3.3.3b).

Definition 3.3.4. *The Young symmetriser* Y_{T^λ} *associated with the tableau* T^λ *is defined by*

$$\begin{aligned} Y_{T^\lambda} &= Q_{T^\lambda} P_{T^\lambda} \\ &= \sum_{\sigma\in\mathcal{C}_{T^\lambda}} \sum_{\rho\in\mathcal{R}_{T^\lambda}} (-1)^\sigma \sigma\rho. \end{aligned} \tag{3.3.4}$$

In this Chapter it is possible to proceed equally well with the order of the row and column permutations opposite to that defined here. This is done, for example, in [**Bo63**]. However Y_{T^λ} defined by (3.3.4) has definite advantages, as will become apparent in later chapters.

The action on T^λ of the corresponding Young symmetriser Y_{T^λ} produces a signed sum of tableaux known as a symmetrised (Young) tableau and denoted $\{T^\lambda\}$:

$$\{T^\lambda\} = Y_{T^\lambda} T^\lambda. \tag{3.3.5}$$

As an example, consider $\lambda = (2,2)$ and

$$T^\lambda = \begin{matrix} 12 \\ 34 \end{matrix}. \tag{3.3.6}$$

From (3.3.2a), $P_{T^\lambda} = (I + (12))(I + (34)) = (I + (12) + (34) + (12)(34))$ and $Q_{T^\lambda} = (I - (13))(I - (24)) = (I - (13) - (24) + (13)(24))$. Thereupon, with $Y_{T^\lambda} = Q_{T^\lambda} P_{T^\lambda}$, $\{T^\lambda\}$ is given as the following linear combination of tableaux through (3.3.5) and Definition 3.3.1:

$$\begin{aligned} &\begin{matrix}1&2\\3&4\end{matrix} + \begin{matrix}2&1\\3&4\end{matrix} + \begin{matrix}1&2\\4&3\end{matrix} + \begin{matrix}2&1\\4&3\end{matrix} \\ &- \begin{matrix}3&2\\1&4\end{matrix} - \begin{matrix}2&3\\1&4\end{matrix} - \begin{matrix}3&2\\4&1\end{matrix} - \begin{matrix}2&3\\4&1\end{matrix} \\ &- \begin{matrix}1&4\\3&2\end{matrix} - \begin{matrix}4&1\\3&2\end{matrix} - \begin{matrix}1&4\\2&3\end{matrix} - \begin{matrix}4&1\\2&3\end{matrix} \\ &+ \begin{matrix}3&4\\1&2\end{matrix} + \begin{matrix}4&3\\1&2\end{matrix} + \begin{matrix}3&4\\2&1\end{matrix} + \begin{matrix}4&3\\2&1\end{matrix}. \end{aligned} \tag{3.3.7}$$

This array has been obtained by first applying the four elements of $\mathcal{R}_{T^\lambda}$ to the numerals of T^λ, yielding the first four tableaux. Then, the tableaux in each column of this array is obtained through permuting the numerals of these tableaux according to the elements of $\mathcal{C}_{T^\lambda}$.

As may be seen from (3.3.7), the action on T^λ of a single summand $\sigma\rho$ of Y_{T^λ} is not to be thought of as a row permutation followed by a column permutation since the action of ρ disrupts the column structure. However, the action of $\sigma\rho$ may be regarded as a column permutation followed by a row permutation since, by (3.1.4), the integer m moves, under the action of $\sigma\rho$, to where the integer $\rho^{-1}\sigma^{-1}(m)$ originally resided. Thus the entry m is first moved within its column, to the original position of $\sigma^{-1}(m)$, and then within the row it then occupies under the action of ρ^{-1}. Note that as ρ and σ run through the groups $\mathcal{R}_{T^\lambda}$ and $\mathcal{C}_{T^\lambda}$ respectively then so do ρ^{-1} and σ^{-1}. This provides an alternative means of obtaining the symmetrised tableaux. Consider (3.3.7). Each of the tableaux in the first column of this array has been obtained by applying the four elements of $\mathcal{C}_{T^\lambda}$ to permute the elements within the columns of T^λ. Each of these tableaux then yields those to its right by permuting the elements among their rows. These notions of a *place* permutation will be developed below.

Let the permutation of S_l which maps the Young tableau T_i^λ into the Young tableau T_j^λ be denoted τ_{ji}:

$$T_j^\lambda = \tau_{ji} T_i^\lambda. \tag{3.3.8}$$

If $\rho \in \mathcal{R}_{T_i^\lambda}$, (3.1.6) shows that a corresponding row permutation of T_j^λ is given by $\tau_{ji}\rho\tau_{ij} \in \mathcal{R}_{T_j^\lambda}$. Similarly, to each $\sigma \in \mathcal{C}_{T_i^\lambda}$ corresponds $\tau_{ji}\sigma\tau_{ij} \in \mathcal{C}_{T_j^\lambda}$. Furthermore as ρ and σ run through the groups $\mathcal{R}_{T_i^\lambda}$ and $\mathcal{C}_{T_i^\lambda}$ respectively, then $\tau_{ji}\rho\tau_{ij}$ and $\tau_{ji}\sigma\tau_{ij}$ run through the groups $\mathcal{R}_{T_j^\lambda}$ and $\mathcal{C}_{T_j^\lambda}$ respectively. This implies that $P_{T_j^\lambda} = \tau_{ji}P_{T_i^\lambda}\tau_{ij}$ and also, since $(-1)^{\tau\sigma\tau^{-1}} = (-1)^\sigma$, that $Q_{T_j^\lambda} = \tau_{ji}Q_{T_i^\lambda}\tau_{ij}$. Thereupon:

$$Y_{T_j^\lambda} = \tau_{ji} Y_{T_i^\lambda} \tau_{ij}. \tag{3.3.9}$$

As already indicated, a single summand π of Y_{T^λ} acts on T^λ to move the entry a to the position originally occupied by $\pi^{-1}a$. If $T^\lambda = \tau t^\lambda$ then the entry $a = T^\lambda_{(\tau^{-1}a)}$ is moved to the position labelled by the entry $\tau^{-1}\pi^{-1}a$ of t^λ. Thus if $T^{*\lambda} = \pi T^\lambda$ then $T^{*\lambda}_{(\tau^{-1}\pi^{-1}a)} = T^\lambda_{(\tau^{-1}a)}$ and

$$T^{*\lambda}_{(b)} = T^\lambda_{(\tau^{-1}\pi\tau b)}, \tag{3.3.10}$$

for $b = 1, 2, \ldots, l$, as can be seen by putting $b = \tau^{-1}\pi^{-1}a$. Note that $\tau^{-1}\pi\tau$ is the summand of Y_{t^λ}, P_{t^λ} or Q_{t^λ} corresponding to the summand π of Y_{T^λ}, P_{T^λ} or Q_{T^λ} respectively. This motivates:

Definition 3.3.11. *Tableau place permutation. If $\lambda \in P(l)$ and $\pi \in S_l$ then the place permutation action π_* on T^λ results in $T^{*\lambda} = \pi_* T^\lambda$ where:*

$$T^{*\lambda}_{(a)} = T^\lambda_{(\pi^{-1}(a))}, \tag{3.3.11}$$

for $a = 1, 2, \ldots, l$.

This definition has been made using π^{-1} instead of π to ensure that $(\rho\sigma)_* = \rho_*\sigma_*$.

Let $\mathcal{C}^\lambda = \mathcal{C}_{t^\lambda}$ and $\mathcal{R}^\lambda = \mathcal{R}_{t^\lambda}$. For π a summand of Y_{T^λ}, $\pi = \sigma'\rho'$ for some $\sigma' \in \mathcal{C}_{T^\lambda}$ and some $\rho' \in \mathcal{R}_{T^\lambda}$. Corresponding to these are the elements $\tau^{-1}\sigma'\tau \in \mathcal{C}^\lambda$ and $\tau^{-1}\rho'\tau \in \mathcal{R}^\lambda$ respectively. Let $\sigma = \tau^{-1}\sigma'^{-1}\tau$ and $\rho = \tau^{-1}\rho'^{-1}\tau$ so that if $T^{*\lambda} = \pi T^\lambda$ then:

$$T^{*\lambda}_{(b)} = T^\lambda_{(\tau^{-1}\pi\tau b)} = T^\lambda_{(\sigma^{-1}\rho^{-1}b)} = (\sigma_* T^\lambda)_{(\rho^{-1}b)} = (\rho_*\sigma_* T^\lambda)_{(b)}, \tag{3.3.12}$$

which explains why the summands of Y_{T^λ} acting on T^λ may each be considered as a column permutation followed by a row permutation. As σ' runs through all the permutations of $\mathcal{C}_{T^\lambda}$, so does σ'^{-1} whereupon σ runs through all the permutations of $\mathcal{C}^\lambda$. Similarly, as ρ' runs through all the permutations of $\mathcal{R}_{T^\lambda}$, so does ρ'^{-1} and ρ runs through all the permutations of $\mathcal{R}^\lambda$. Lemma 3.1.13*c* implies that $(-1)^\sigma = (-1)^{\sigma'}$. This leads to the following alternative definition of a symmetrised tableau.

Definition 3.3.13. *Place symmetrisation. For $\lambda \in P(l)$, let:*

$$P^\lambda_* = \sum_{\rho \in \mathcal{R}^\lambda} \rho_*, \tag{3.3.13a}$$

$$Q^\lambda_* = \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \sigma_*, \tag{3.3.13b}$$

and define the Young symmetriser Y^λ_* by:

$$Y^\lambda_* = P^\lambda_* Q^\lambda_*. \tag{3.3.13c}$$

The symmetrised tableau $\{T^\lambda\}$ is then defined by:

$$\{T^\lambda\} = Y^\lambda_* T^\lambda = \sum_{\rho \in \mathcal{R}^\lambda} \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \rho_*\sigma_* T^\lambda. \tag{3.3.13d}$$

This definition is more useful than (3.3.5) since it is directly applicable to situations in which T^λ has repeated entries. Such situations do not arise until later chapters but nevertheless, many results in this are more clearly elucidated using this place permutation definition of a symmetrised tableau.

The remainder of this section is devoted to the exploitation of the Lemmas and Theorems of Section 3.2, in order to show how the Young symmetrisers may be used to obtain the irreducible representations of the symmetric groups. The proofs are modifications of those found in [**Bo63**].

Lemma 3.3.14. *If $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$ then two entries which occur in the same column of T^λ do not occur in the same row of $\sigma\rho T^\lambda$. Conversely, if every two entries which occur in the same column of T^λ do not occur in the same row of $T^{*\lambda} = \pi T^\lambda$ then $\pi = \rho\sigma$ for some $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$.*

Proof. In the first part $\sigma\rho T^\lambda$ occurs as a summand in $\{T^\lambda\}$ and consequently arises from T^λ through first a column, then a row permutation. The column permutation leaves the two entries in the same column and hence in different rows. The first part is proved since the row permutation does not then alter these rows. For the second part, the entries which occupy the first row of $T^{*\lambda}$ all lie in different columns of T^λ. Thus a column permutation exists which acts on T^λ to take each to the top of its column. Similarly a column permutation exists which acts on T^λ to take each of those entries from the second row of $T^{*\lambda}$ to the second row. In this way a column permutation can be found to put each entry in the correct row. A row permutation can then be found to produce $T^{*\lambda}$. This shows that $T^{*\lambda}$ is a summand of $\{T^\lambda\}$ and thus proves the lemma.

Lemma 3.3.15. *If $\mu, \nu \in P(l)$ and there exists s such that $\mu_i = \nu_i$ for $i = 1, 2, \ldots, s-1$, and $\mu_s > \nu_s$ then there are two entries which occur in the same row of T_1^μ and the same column of T_2^ν.*

Proof. Assume that every two entries which occur in the same row of T_1^μ occur in different columns of T_2^ν. Consider first the entries in the first row of T_1^μ which thus all occur in different columns of T_2^ν. Therefore $\mu_1 \leq \nu_1$. But by hypothesis $\mu_1 \geq \nu_1$. Therefore $\mu_1 = \nu_1$ and each of the (first ν_1) columns of T_2^ν contains an entry from the first row of T_1^μ. The entries in the second row of T_1^μ all occur in different columns of T_2^ν. Thus, since $\nu_2 \leq \nu_1$, each of these columns has at least two entries — the other from the first row of T_1^μ. Therefore $\mu_2 \leq \nu_2$ and hence $\mu_2 = \nu_2$. Proceeding in this way leads to the conclusion that $\mu = \nu$ contradicting the premise of the lemma. The initial assumption is therefore incorrect and the lemma is proved.

Lemma 3.3.16. *Let $\mu, \nu \in P(l)$ (not necessarily distinct). If $a, b \in \mathbb{N}_l$ occur in the same row of T_1^μ and the same column of T_2^ν then:*

$$P_{T_1^\mu} Q_{T_2^\nu} = 0, \qquad (3.3.16a)$$

and consequently:

$$Y_{T_1^\mu} Y_{T_2^\nu} = 0. \qquad (3.3.16b)$$

Proof. The transposition $\pi = (a, b)$ is a member of both $\mathcal{R}_{T_1^\mu}$ and $\mathcal{C}_{T_2^\nu}$. By Lemma 3.3.3, $P_{T_1^\mu} = P_{T_1^\mu}\pi$ and $Q_{T_2^\nu} = -\pi Q_{T_2^\nu}$ since $(-1)^\pi = -1$. Combining these gives $P_{T_1^\mu} Q_{T_2^\nu} = -P_{T_1^\mu}\pi\pi Q_{T_2^\nu} = -P_{T_1^\mu} Q_{T_2^\nu}$ and thus (3.3.16a). (3.3.16b) then follows from (3.3.4).

Lemma 3.3.17. *If $\pi \neq \sigma'\rho'$ for all $\sigma' \in \mathcal{C}_{T^\lambda}$ and $\rho' \in \mathcal{R}_{T^\lambda}$, then there exist transpositions $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$ such that $\sigma\pi\rho = \pi$.*

Proof. From Lemma 3.3.14 there exist $a, b \in \mathsf{N}_l$ such that a and b occur in the same column of T^λ and in the same row of $T^{*\lambda} = \pi T^\lambda$. Let $\tau = (a, b)$ whereupon $\tau \in \mathcal{C}_{T^\lambda}$, $\tau \in \mathcal{R}_{T^{*\lambda}}$ and $\pi^{-1}\tau\pi \in \mathcal{R}_{T^\lambda}$. The selection of $\sigma = \tau$ and $\rho = \pi^{-1}\tau\pi$ proves the lemma since $\sigma\pi\rho = \tau\pi\pi^{-1}\tau\pi = \tau\tau\pi = \pi$.

Lemma 3.3.18. *If $x \in \mathsf{F}S_l$, $\lambda \in P(l)$ and*

$$\sigma x \rho = (-1)^\sigma x \tag{3.3.18}$$

for all $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$, then $x = \alpha Y_{T^\lambda}$ where $\alpha = x(I)$.

Proof. With $x = \sum_{\pi \in S_l} x(\pi)\pi$, (3.3.18) gives:

$$\sum_{\pi \in S_l} x(\pi)\sigma\pi\rho = (-1)^\sigma \sum_{\pi \in S_l} x(\pi)\pi. \tag{3.3.18a}$$

Since as π runs through S_l, so do $\sigma\pi$ and $\sigma\pi\rho$, then each permutation occurs only once on each side. Consider first the coefficients of each $\sigma\rho$. On the left side the coefficient is $\alpha = x(I)$ whereas on the right it is $(-1)^\sigma x(\sigma\rho)$, so that $x(\sigma\rho) = (-1)^\sigma\alpha$ for each $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$. Now consider the permutation τ for which $\tau \neq \sigma\rho$ for all $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$. By Lemma 3.3.17 there exist transpositions $\sigma \in \mathcal{C}_{T^\lambda}$ and $\rho \in \mathcal{R}_{T^\lambda}$ such that $\sigma\tau\rho = \tau$. On using these values in (3.3.18a), the coefficients of τ imply that $x(\tau) = -x(\tau)$ and thence that $x(\tau) = 0$. Thus

$$x = \sum_{\sigma \in \mathcal{C}_{T^\lambda}} \sum_{\rho \in \mathcal{R}_{T^\lambda}} x(\sigma\rho)\sigma\rho = \alpha \sum_{\sigma \in \mathcal{C}_{T^\lambda}} \sum_{\rho \in \mathcal{R}_{T^\lambda}} (-1)^\sigma \sigma\rho = \alpha Y_{T^\lambda}.$$

Theorem 3.3.19. *There exists a non-zero $\alpha \in \mathbf{Z}$ such that the normalised Young symmetriser $\frac{1}{\alpha}Y_{T^\lambda}$ is a primitive idempotent. Y_{T^λ} generates a minimal left ideal and thence an irreducible $\mathsf{F}S_l$-module.*

Proof. (see **[Yo77,Bo63]**) Since $Y_{T^\lambda}^2$ satisfies the premise of Lemma 3.3.18, it follows that:

$$Y_{T^\lambda}^2 = \alpha Y_{T^\lambda}.$$

Since $Y^2_{T^\lambda} \in \mathbb{Z}S_l$ and the coefficient of I in Y_{T^λ} is 1, then $\alpha \in \mathbb{Z}$. It is required to show that α is non-zero. If $x = \sum_{\pi \in \mathcal{G}} x(\pi)\pi$ and $y = \sum_{\tau \in \mathcal{G}} y(\tau)\tau$ then the coefficients of I in xy and yx are equal since, by (3.2.2), they are each equal to $\sum_{\pi \in \mathcal{G}} x(\pi)y(\pi^{-1})$. Thus the coefficients of I in $P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$ and $P_{T^\lambda}P_{T^\lambda}Q_{T^\lambda}$ are equal. By (3.3.2), $P_{T^\lambda}P_{T^\lambda} = p_\# P_{T^\lambda}$ where $p_\# \in \mathbb{N}$ is the order of $\mathcal{R}_{T^\lambda}$. The coefficient of I in $P_{T^\lambda}Q_{T^\lambda}$ is 1, therefore the coefficient of I in $P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$ is $p_\#$. If $\rho_1\sigma\rho_2$ is a summand of $P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$ then its inverse $\rho_2^{-1}\sigma^{-1}\rho_1^{-1}$ also occurs with the same sign. Thus the coefficient of I occurs as a sum of squares in $P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$ and so is at least $p^2_\#$ by virtue of $p_\#$ being the coefficient of I in $P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$. Since

$$P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda} = p_\# P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda},$$

it follows that the coefficient of I is positive in $P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$, showing that this term is not zero. Thus $Q_{T^\lambda}P_{T^\lambda}Q_{T^\lambda}P_{T^\lambda}$ is non-zero implying that $\alpha \neq 0$.

It remains to be shown that Y_{T^λ} is primitive and therefore generates a minimal left ideal by virtue of Lemma 3.2.8. For arbitrary $x' \in \mathbb{F}\mathcal{G}$, $Y_{T^\lambda}x'Y_{T^\lambda}$ satisfies the premise of Lemma 3.3.18 and is therefore a multiple of Y_{T^λ}. That Y_{T^λ} is primitive now follows from Theorem 3.2.12.

Theorem 3.3.20. *Young symmetrisers which arise from the Young tableaux T_1^μ and T_2^ν generate equivalent irreducible $\mathbb{F}S_l$-modules if and only if $\mu = \nu$.*

Proof. If $\mu = \nu$ then there exists τ such that $T_2^\mu = \tau T_1^\nu$ whereupon, from (3.3.9), $Y_{T_2^\nu} = \tau Y_{T_1^\mu}\tau^{-1}$. Theorem 3.3.19 shows that the left ideals generated by $Y_{T_2^\nu}$ and $Y_{T_1^\mu}$ are minimal, whereupon their equivalence follows from Lemma 3.2.11 since

$$Y_{T_1^\mu}\tau^{-1}Y_{T_2^\nu} = Y_{T_1^\mu}\tau^{-1}\tau Y_{T_1^\mu}\tau^{-1} = Y_{T_1^\mu}Y_{T_1^\mu}\tau^{-1} = \alpha Y_{T_1^\mu}\tau^{-1} \neq 0.$$

It follows that the corresponding irreducible $\mathbb{F}S_l$-modules are equivalent.

If $\mu \neq \nu$ then without loss of generality assume that $\mu_i = \nu_i$ for $i = 1, 2, \ldots, s-1$ and $\mu_s > \nu_s$ for some s. Lemmas 3.3.15 and 3.3.16 then show that $Y_{T_1^\mu}Y_{T_2^{*\nu}} = 0$ for each $T_2^{*\nu} = \tau T_2^\nu$. Thus $Y_{T_1^\mu}\tau Y_{T_2^\nu}\tau^{-1} = 0$ and hence $Y_{T_1^\mu}\tau Y_{T_2^\nu} = 0$. For arbitrary $x \in \mathbb{F}S_l$, $x = \sum_{\tau \in S_l} x(\tau)\tau$, this implies that:

$$Y_{T_1^\mu}xY_{T_2^\nu} = \sum_{\tau \in S_l} x(\tau)Y_{T_1^\mu}\tau Y_{T_2^\nu} = 0.$$

So for $\mu \neq \nu$, the $\mathbb{F}S_l$-modules generated by $Y_{T_1^\mu}$ and $Y_{T_2^\nu}$ are inequivalent by virtue of Lemma 3.2.11

Theorem 3.3.21. *Let T_1^λ and T_2^λ be distinct S_l-standard tableaux such that for some s, $T_{1(s)}^\lambda > T_{2(s)}^\lambda$ with $T_{1(a)}^\lambda = T_{2(a)}^\lambda$ for each $a = 1, 2, \ldots, s-1$. Then $Y_{T_1^\lambda}Y_{T_2^\lambda} = 0$.*

Proof. By Lemma 3.3.16, it is sufficient to show that there exists two entries which occur in the same row of T_1^λ and the same column of T_2^λ. Let $b = T_{1(s)}^\lambda$ and $c = T_{2(s)}^\lambda$, and let i and j specify the row and column of these entries so that $b = T_{1(i,j)}^\lambda$ and $c = T_{2(i,j)}^\lambda$. Note that $i = 1$ may be excluded since, for S_l-standard tableaux, the entry at the top of a column is uniquely determined as the least of those that are not in the columns to the left. Thus $i > 1$. Now $c = T_{1(k,l)}^\lambda$ for some k and l. The possibility that $l < j$ may be excluded since T_1^λ and T_2^λ coincide at all such positions. For the same reason the possibility that $l = j$ and $k \leq i$ may be excluded. Since $b > c$, T_1^λ being S_l-standard disallows the possibility that $l \geq j$ and $k \geq i$, since all such positions must contain an entry greater than b. The remaining possibility is that $l > j$ and $k < i$. Then $T_{1(k,j)}^\lambda = T_{2(k,j)}^\lambda$. Thus $T_{1(k,j)}^\lambda$ and c both occur in the kth row of T_1^λ and in the jth column of T_2^λ.

Theorem 3.3.22. *If $\lambda \in P(l)$ and $T_1^\lambda, T_2^\lambda, \ldots, T_f^\lambda$ are S_l-standard tableaux, then the left ideals generated by $Y_{T_1^\lambda}, Y_{T_2^\lambda}, \ldots, Y_{T_f^\lambda}$ are linearly independent.*

Proof. Without loss of generality it may be assumed that the S_l-standard tableaux are ordered such that if $i < j$ then there exists some s such that $T_{i(s)}^\lambda < T_{j(s)}^\lambda$ with $T_{i(a)}^\lambda = T_{j(a)}^\lambda$ for $a = 1, 2, \ldots, s-1$. Let:

$$x_1 Y_{T_1^\lambda} + x_2 Y_{T_2^\lambda} + \cdots + x_f Y_{T_f^\lambda} = 0,$$

for elements $x_i \in FS_l$. By Theorem 3.3.21 right multiplication with $Y_{T_1^\lambda}$ annihilates all terms but the first, implying that $0 = x_1 Y_{T_1^\lambda} Y_{T_1^\lambda} = \alpha x_1 Y_{T_1^\lambda}$ for some non-zero $\alpha \in \mathbb{Z}$. Therefore $x_1 Y_{T_1^\lambda} = 0$. Right multiplication with $Y_{T_2^\lambda}$ now shows that $x_2 Y_{T_2^\lambda} = 0$. Similarly each term is zero and the theorem is proved.

Theorem 3.3.23. *The dimension of the left ideal and hence the irreducible representation generated by Y_{T^λ} is equal to f^λ, the number of S_l-standard tableaux of shape F^λ.*

Proof. Let the dimension of the irreducible left ideal generated by Y_{T^λ} be f_*^λ. By Theorem 3.2.17, the minimal two-sided ideal W_λ generated by Y_{T^λ} is of dimension $(f_*^\lambda)^2$ and contains exactly f_*^λ independent left ideals. Theorem 3.3.22 then implies that $(f_*^\lambda)^2 \geq f_*^\lambda f^\lambda$ and hence $f_*^\lambda \geq f^\lambda$. Since S_l has order $l!$, (3.2.16) gives:

$$l! = \sum_{\lambda \in P(l)} (f_*^\lambda)^2.$$

However, Theorem 2.6.13 then implies that:

$$\sum_{\lambda \in P(l)} (f_*^\lambda)^2 = \sum_{\lambda \in P(l)} (f^\lambda)^2.$$

With $f_*^\lambda \geq f^\lambda$ for each $\lambda \in P(l)$, the only possibility is that $f^\lambda = f_*^\lambda$ for all $\lambda \in P(l)$. This proves the theorem.

§3.4. The Garnir relations and standardisation

The symmetrised tableaux of a given shape are not linearly independent. It is the purpose of this section to describe relations between the symmetrised tableaux and to describe how an arbitrary symmetrised tableau may be written in terms of the S_l-standard tableaux.

Definition 3.4.1. *Column strict tableaux. If the entries of the tableau T^λ are strictly increasing down each column then T^λ is termed column strict.*

Lemma 3.4.2. *If $\sigma \in C^\lambda$ then for any tableau T^λ:*

$$\{T^\lambda\} = (-1)^\sigma \{\sigma_* T^\lambda\}. \tag{3.4.2}$$

Proof. From Definition 3.3.13:

$$\{\sigma_* T^\lambda\} = P_*^\lambda Q_*^\lambda \sigma_* T^\lambda = P_*^\lambda (Q^\lambda \sigma)_* T^\lambda = (-1)^\sigma P_*^\lambda Q_*^\lambda T^\lambda = (-1)^\sigma \{T^\lambda\},$$

where (3.3.3b) has been used.

This Lemma provides what are known as the Column relations. It implies that $\{T^\lambda\}$ may be expressed as $\pm\{T'^\lambda\}$ for some column strict tableau T'^λ. Furthermore, when the generalisation to tableaux which may possess repeated entries is made, it implies that if T^λ has an entry repeated in any column, then $\{T^\lambda\}$ vanishes.

Lemma 3.4.3. *For $i < j$, let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the entries in the ith and jth columns, respectively, of t^λ such that $\#(\mathcal{X} \cup \mathcal{Y}) > \tilde{\lambda}_i$. Let $S(\mathcal{X})$, $S(\mathcal{Y})$ and $S(\mathcal{X} \cup \mathcal{Y})$ be the subgroups of S_l preserving $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Y}$, respectively. Then if $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ is a set of right coset representatives for $S(\mathcal{X}) \otimes S(\mathcal{Y})$ in $S(\mathcal{X} \cup \mathcal{Y})$,*

$$\sum_{\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \{\eta_* T^\lambda\} = 0. \tag{3.4.3}$$

Proof. If $G_*^\lambda = \sum_{\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \eta_*$ then the expression on the left side of (3.4.3) may be written $P_*^\lambda Q_*^\lambda G_*^\lambda T^\lambda$. Consider the tableau $T_1^\lambda = \sigma_* \eta_* T^\lambda$ which has a coefficient β in the expression $Q_*^\lambda G_*^\lambda T^\lambda$. It will not be assumed that T_1^λ arises in only one way. Since $\#(\mathcal{X} \cup \mathcal{Y}) \geq \tilde{\lambda}_i$, there exists at least one pair of entries from $\mathcal{X} \cup \mathcal{Y}$ lying in the same row of t^λ. Fix one such pair and let a and b be the entries of T_1^λ lying in those positions. Let τ_* be the place transposition which swaps a and b in T^λ to give $T_+^\lambda = \tau_* T^\lambda$. If $T_1^\lambda = \sigma_* \eta_* T^\lambda$ for some specific $\sigma \in C^\lambda$ and

$\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})$ then $T^\lambda_{1+} = \sigma_*.\eta_*.T^\lambda_+ = \sigma_*.\eta_*.\tau_*.T^\lambda$ differs from T^λ_1 only in that a and b have swapped places. Since $\eta\tau \in S(\mathcal{X}\cup\mathcal{Y})$ and $\mathcal{G}(\mathcal{X},\mathcal{Y})$ is a set of right coset representatives of $S(\mathcal{X})\otimes S(\mathcal{Y})$ in $S(\mathcal{X}\cup\mathcal{Y})$ then $\eta\tau$ can uniquely be expressed $\pi\eta'$ for $\pi \in S(\mathcal{X})\otimes S(\mathcal{Y})$ and $\eta' \in \mathcal{G}(\mathcal{X},\mathcal{Y})$. Thus $T^\lambda_{1+} = \sigma'_*\eta'_*T^\lambda$ where $\sigma' = \sigma\pi \in \mathcal{C}^\lambda$ with $(-1)^{\sigma'\eta'} = (-1)^{\sigma\pi\eta'} = (-1)^{\sigma\eta\tau} = -(-1)^{\sigma\eta}$. So for each occurrence of T^λ_1 in $Q^\lambda_*G^\lambda_*T^\lambda$, T^λ_{1+} occurs with an opposite sign. Thus if T^λ_1 has coefficient β in $Q^\lambda_*G^\lambda_*T^\lambda$ then T^λ_{1+} has coefficient $-\beta$. However, $P^\lambda_*T^\lambda_{1+} = P^\lambda_*T^\lambda_1$ since T^λ_{1+} differs from T^λ_1 by a simple place transposition from $\mathcal{R}^\lambda$. Thus the application of P^λ_* to $Q^\lambda_*G^\lambda_*T^\lambda$ produces a set of tableaux whose coefficients cancel. Therefore $P^\lambda_*Q^\lambda_*G^\lambda_*T^\lambda = 0$ and the lemma is proved.

Identities of the type appearing in Lemma 3.4.3 are known as Garnir relations after [**Ga50**] where they were first obtained. The elements $G^\lambda_{(\mathcal{X},\mathcal{Y})} \in \mathbf{Z}S_l$ defined by:

$$G^\lambda_{(\mathcal{X},\mathcal{Y})} = \sum_{\eta\in\mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \eta, \tag{3.4.4}$$

are known as Garnir elements. Lemma 3.4.3 shows that they satisfy:

$$Y^\lambda G^\lambda_{(\mathcal{X},\mathcal{Y})} = 0. \tag{3.4.5}$$

To illustrate a Garnir relation, let $\lambda = (4,3,1)$ whereupon:

$$t^\lambda = \begin{matrix} 1 & 4 & 6 & 8 \\ 2 & 5 & 7 & \\ 3 & & & \end{matrix} . \tag{3.4.6}$$

Let $i = 1$, $j = 2$, $\mathcal{X} = \{1,3\}$ and $\mathcal{Y} = \{4,5\}$. Right coset representatives for $S(\mathcal{X})\otimes S(\mathcal{Y})$ in $S(\mathcal{X}\cup\mathcal{Y})$ are provided by $\mathcal{G}(\mathcal{X},\mathcal{Y}) = \{I, (14), (354), (145), (35), (14)(35)\}$ which yields the Garnir element:

$$G^\lambda_{(\mathcal{X},\mathcal{Y})} = I - (14) + (354) + (145) - (35) + (14)(35). \tag{3.4.7}$$

Lemma 3.4.3 then implies, for example, the Garnir relation:

$$\begin{gathered} \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 7 & 3 & 8 & \\ 4 & & & \end{matrix}\right\} - \left\{\begin{matrix} 2 & 1 & 5 & 6 \\ 7 & 3 & 8 & \\ 4 & & & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 3 & 5 & 6 \\ 7 & 4 & 8 & \\ 2 & & & \end{matrix}\right\} \\ + \left\{\begin{matrix} 3 & 1 & 5 & 6 \\ 7 & 2 & 8 & \\ 4 & & & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 7 & 4 & 8 & \\ 3 & & & \end{matrix}\right\} + \left\{\begin{matrix} 2 & 1 & 5 & 6 \\ 7 & 4 & 8 & \\ 3 & & & \end{matrix}\right\} = 0. \end{gathered} \tag{3.4.8}$$

It will now be shown that the Column and Garnir relations can be used to express each symmetrised tableau in terms of the standard S_l-tableaux of Definition

2.6.9. The following algorithmic procedure for accomplishing this is described in [**Ga50**] and [**JK81**]. It is sufficient to consider column strict tableaux since Lemma 3.4.2 enables each symmetrised tableau to be expressed as such. If the column strict tableau T^λ is not standard then condition (iii) of Definition 2.6.9 implies that there exists a neighbouring pair of entries, $T^\lambda_{(a,b)}$ and $T^\lambda_{(a,b+1)}$, such that $T^\lambda_{(a,b)} > T^\lambda_{(a,b+1)}$. Let $\mathcal{X}$ be the set of positions below and including that of $T^\lambda_{(a,b)}$ in the bth column and let $\mathcal{Y}$ be the set of positions above and including that of $T^\lambda_{(a,b+1)}$ in the $(b+1)$th column. The relevant entries of T^λ are then as follows:

$$\begin{array}{ccc} \vdots & & T^\lambda_{(1,b+1)} \\ \vdots & & \wedge \\ \vdots & & \vdots \\ \vdots & & \wedge \\ \vdots & & T^\lambda_{(a-1,b+1)} \\ \vdots & & \wedge \\ T^\lambda_{(a,b)} & > & T^\lambda_{(a,b+1)} \\ \wedge & & \\ T^\lambda_{(a+1,b)} & & \\ \wedge & & \\ \vdots & & \\ \wedge & & \\ T^\lambda_{(\lambda_b,b)}. & & \end{array} \tag{3.4.9}$$

Since, with $\mathcal{X}$ and $\mathcal{Y}$ so defined, $\#(\mathcal{X} \cup \mathcal{Y}) = \tilde{\lambda}_b + 1$, Lemma 3.4.3 may be used to express $\{T^\lambda\}$ in terms of other tableaux. With $\eta \in \mathcal{G}(\mathcal{X}, \mathcal{Y})$ and $\eta \notin S(\mathcal{X}) \otimes S(\mathcal{Y})$, $T^\lambda_\eta = \eta T^\lambda$ has necessarily been formed from T^λ by swapping the columns of at least one pair of elements from the positions $\mathcal{X} \cup \mathcal{Y}$. Since the entries of T^λ at positions $\mathcal{X}$ are all larger than those at positions $\mathcal{Y}$, it follows that $T^\lambda_\eta > T^\lambda$ in terms of the tableaux ordering of Definition 2.6.8. Hence this algorithm enables $\{T^\lambda\}$ to be written in terms of higher tableaux. It may then be iterated until solely S_l-tableaux result. That this procedure terminates is guaranteed by the ordering on the set of all tableaux of shape F^λ and their finite number.

To illustrate this procedure, consider the non-standard tableau:

$$T^\lambda = \begin{array}{cccc} 1 & 2 & 5 & 6 \\ 7 & 3 & 8 & \\ 4 & & & \end{array} . \tag{3.4.10}$$

Lemma 3.4.2 implies that:

$$\left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 7 & 3 & 8 & \\ 4 & & & \end{matrix}\right\} = -\left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 4 & 3 & 8 & \\ 7 & & & \end{matrix}\right\}. \tag{3.4.11}$$

The algorithm described above then dictates the use of a Garnir relation resulting from the adoption of $\mathcal{X} = \{2,3\}$ and $\mathcal{Y} = \{4,5\}$. An appropriate choice of coset representatives then yields the Garnir element:

$$G^{\lambda}_{(\mathcal{X},\mathcal{Y})} = I + (254) - (2354) - (25) + (235) + (24)(35), \tag{3.4.12}$$

and the Garnir identity:

$$\begin{aligned} &\left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 4 & 3 & 8 & \\ 7 & & & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 4 & 8 & \\ 7 & & & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 7 & 8 & \\ 4 & & & \end{matrix}\right\} \\ &\quad - \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 8 & \\ 7 & & & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 8 & \\ 4 & & & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 4 & 5 & 6 \\ 2 & 7 & 8 & \\ 3 & & & \end{matrix}\right\} = 0. \end{aligned} \tag{3.4.13}$$

The coset representatives of (3.4.12) have been selected so that each of the tableau in (3.4.13) are column strict in the portions specified by the sets $\mathcal{X}$ and $\mathcal{Y}$. This minimalises further usage of Lemma 3.4.2. Combining (3.4.11) and (3.4.13) now gives

$$\begin{aligned} \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 7 & 3 & 8 & \\ 4 & & & \end{matrix}\right\} &= \left\{\begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 4 & 8 & \\ 7 & & & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 7 & 8 & \\ 4 & & & \end{matrix}\right\} \\ &\quad - \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 8 & \\ 7 & & & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 2 & 5 & 6 \\ 3 & 7 & 8 & \\ 4 & & & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 4 & 5 & 6 \\ 2 & 7 & 8 & \\ 3 & & & \end{matrix}\right\}. \end{aligned} \tag{3.4.14}$$

Thus $\{T^{\lambda}\}$ has been expressed in terms of higher tableaux. In this case each of the tableaux on the right is S_8-standard. However, in general, the standardisation procedure will need to be iterated a number of times before solely S_l-standard tableaux result.

§3.5. The Specht module

The following lemma provides the means of constructing the irreducible S_l-modules.

Lemma 3.5.1. *The actions of symmetrisation and permutation on a Young tableau commute. That is:*

$$\pi\{T^{\lambda}\} = \{\pi T^{\lambda}\}. \tag{3.5.1}$$

Proof. Let $T_1^\lambda = \pi T^\lambda$. Then, by (3.3.8) and (3.3.9),

$$Y_{T_1^\lambda} = \pi Y_{T^\lambda} \pi^{-1},$$

whereupon (3.3.5) gives:

$$\{T_1^\lambda\} = Y_{T_1^\lambda} T_1^\lambda = \pi Y_{T^\lambda} \pi^{-1} \pi T^\lambda = \pi Y_{T^\lambda} T^\lambda = \pi \{T^\lambda\},$$

which proves the lemma.

For $\lambda \in P(l)$, let the f^λ S_l-standard tableaux be denoted $T_1^\lambda, T_2^\lambda, \ldots, T_{f^\lambda}^\lambda$. By Lemma 3.5.1, the action of $\pi \in S_l$ on $\{T^\lambda\}$ yields $\{\pi T^\lambda\}$ which, through the use of the techniques of Section 3.4, may be re-expressed as a linear combination of the symmetrised S_l-standard tableaux:

$$\pi\{T_i^\lambda\} = \{\pi T_i^\lambda\} = \sum_{j=1}^{f^\lambda} \Gamma^\lambda(\pi)_{ji} \{T_j^\lambda\}, \tag{3.5.2}$$

where $\Gamma^\lambda(\pi)_{ji} \in \mathbf{Z}$. This construction therefore defines an S_l-module with a basis consisting of the symmetrised S_l-standard tableaux of shape λ. It is known as the Specht module and denoted S^λ. In order to show that S^λ is the irreducible S_l-module desired, it is necessary to make the connection with the minimal left ideals of $\mathsf{F}\mathcal{G}$. To this end, let τ_{ji} be defined by (3.3.8) for the S_l-standard tableaux T_i^λ and T_j^λ. Then, from (3.5.1) and (3.5.2):

$$\pi\{T_i^\lambda\} = \sum_{j=1}^{f^\lambda} \Gamma^\lambda(\pi)_{ji} \tau_{ji} \{T_i^\lambda\}, \tag{3.5.3a}$$

$$\pi \tau_{ik}\{T_k^\lambda\} = \sum_{j=1}^{f^\lambda} \Gamma^\lambda(\pi)_{ji} \tau_{jk} \{T_k^\lambda\}, \tag{3.5.3b}$$

for any k for which $1 \le k \le f^\lambda$. Then:

$$\pi \tau_{ik} Y_{T_k^\lambda} T_k^\lambda = \sum_{j=1}^{f^\lambda} \Gamma^\lambda(\pi)_{ji} \tau_{jk} Y_{T_k^\lambda} T_k^\lambda, \tag{3.5.3c}$$

and therefore, since the entries of T_k^λ are distinct:

$$\pi \tau_{ik} Y_{T_k^\lambda} = \sum_{j=1}^{f^\lambda} \Gamma^\lambda(\pi)_{ji} \tau_{jk} Y_{T_k^\lambda}. \tag{3.5.3d}$$

This proves the following:

Theorem 3.5.4. *If $\lambda \in P(l)$ and $T_1^\lambda, T_2^\lambda, \ldots, T_{f^\lambda}^\lambda$ are the S_l-standard tableaux of shape λ, for which $T_j^\lambda = \tau_{ji} T_i^\lambda$, then the minimal left ideal generated by $Y_{T_k^\lambda}$ has a basis $\{\tau_{ik} Y_{T_k^\lambda} : 1 \le i \le f^\lambda\}$.*

Since, from (3.5.2) and (3.5.3d), the Specht module S^λ is equivalent to the left ideal generated by $Y_{T_k^\lambda}$, S^λ is irreducible. Theorem 3.3.21 shows that the left ideals generated by each T_k^λ are linearly independent. Despite this, the matrices of the irreducible representations obtained via (3.5.3d) are identical and thus their equivalence is demonstrated explicitly.

Quintessentially, the structure of the Specht module is as follows:

Definition 3.5.5. *Let $\lambda \in P(l)$. The Specht module S^λ is the irreducible S_l-module spanned by $\{T^\lambda\}$ for all T^λ with distinct entries from the set $\mathbb{N}_l$, modulo relations* (3.4.2) *and* (3.4.3), *and on which $\pi \in S_l$ acts according to* (3.5.1).

As an illustration, let $\lambda = (3,2)$ and consider the S_5-module $S^{(3,2)}$. The S_5-standard tableaux are:

$$T_1^\lambda = \begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix}, \quad T_2^\lambda = \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \end{matrix}, \quad T_3^\lambda = \begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix},$$
$$T_4^\lambda = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}, \quad T_5^\lambda = \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}. \tag{3.5.6}$$

Acting with the permutation (34) on each of the symmetrised S_5-standard tableaux in turn and standardising the results, produces the following sequence of calculations:

$$(34)\{T_1^\lambda\} = \left\{\begin{matrix} 1 & 4 & 5 \\ 2 & 3 & \end{matrix}\right\} = -\left\{\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix}\right\} = -\{T_1^\lambda\}; \tag{3.5.7a}$$

$$(34)\{T_2^\lambda\} = \left\{\begin{matrix} 1 & 2 & 5 \\ 4 & 3 & \end{matrix}\right\} = \left\{\begin{matrix} 1 & 2 & 5 \\ 3 & 4 & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix}\right\} = \{T_2^\lambda\} - \{T_1^\lambda\}; \tag{3.5.7b}$$

$$(34)\{T_3^\lambda\} = \left\{\begin{matrix} 1 & 4 & 3 \\ 2 & 5 & \end{matrix}\right\} = \left\{\begin{matrix} 1 & 3 & 4 \\ 2 & 5 & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 3 & 5 \\ 2 & 4 & \end{matrix}\right\} = \{T_3^\lambda\} - \{T_1^\lambda\}; \tag{3.5.7c}$$

$$(34)\{T_4^\lambda\} = \left\{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & \end{matrix}\right\} = \{T_5^\lambda\}; \tag{3.5.7d}$$

$$(34)\{T_5^\lambda\} = \left\{\begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}\right\} = \{T_4^\lambda\}; \tag{3.5.7e}$$

where (3.4.2) has been used in (3.5.7a), and (3.4.3) has been used in (3.5.7b) and (3.5.7c). In accordance with (3.5.2), these calculations show that in the representation labelled by $\lambda = (3,2)$, the permutation (34) is represented by

$$\Gamma^\lambda(34) = \begin{pmatrix} -1 & -1 & -1 & . & . \\ . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & . & 1 \\ . & . & . & 1 & . \end{pmatrix}, \tag{3.5.8a}$$

where each zero has been replaced by a dot. Similar calculations in $S^{(3,2)}$ yield the representation matrices:

$$\Gamma^\lambda(12) = \begin{pmatrix} -1 & -1 & . & . & 1 \\ . & 1 & . & . & . \\ . & . & -1 & -1 & -1 \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{pmatrix}, \quad \Gamma^\lambda(23) = \begin{pmatrix} . & 1 & . & . & 1 \\ 1 & . & . & . & -1 \\ . & . & . & 1 & . \\ . & . & 1 & . & . \\ . & . & . & . & 1 \end{pmatrix},$$

$$\Gamma^\lambda(45) = \begin{pmatrix} . & . & 1 & . & 1 \\ . & . & . & 1 & . \\ 1 & . & . & . & -1 \\ . & 1 & . & . & . \\ . & . & . & . & 1 \end{pmatrix}. \tag{3.5.8b}$$

Since, as was shown in Section 3.1, the permutations (12), (23), (34) and (45) may be used to generate any element of S_5, the representation matrices given in (3.5.8) may be used to generate the matrix representing any such element. Consider the permutation $\pi = (1352)$. Its action on each of tableaux given in (3.5.6) results in:

$$\pi\{T_1^\lambda\} = \left\{ \begin{matrix} 3 & 5 & 2 \\ 1 & 4 & \end{matrix} \right\}, \quad \pi\{T_2^\lambda\} = \left\{ \begin{matrix} 3 & 1 & 2 \\ 5 & 4 & \end{matrix} \right\}, \quad \pi\{T_3^\lambda\} = \left\{ \begin{matrix} 3 & 5 & 4 \\ 1 & 2 & \end{matrix} \right\},$$

$$\pi\{T_4^\lambda\} = \left\{ \begin{matrix} 3 & 1 & 4 \\ 5 & 2 & \end{matrix} \right\}, \quad \pi\{T_5^\lambda\} = \left\{ \begin{matrix} 3 & 1 & 5 \\ 4 & 2 & \end{matrix} \right\}. \tag{3.5.9}$$

Upon standardisation, these give, according to (3.5.2):

$$\Gamma^\lambda(1352) = \begin{pmatrix} . & 1 & . & . & 1 \\ -1 & -1 & . & . & . \\ . & . & . & 1 & . \\ 1 & . & 1 & . & . \\ . & 1 & . & . & . \end{pmatrix}. \tag{3.5.10}$$

Lemma 3.1.10 results in the identity (1352) = (23)(12)(34)(45)(34). Therefore, since the matrices obtained in (3.5.8) are representation matrices:

$$\Gamma^\lambda(1352) = \Gamma^\lambda(23)\Gamma^\lambda(12)\Gamma^\lambda(34)\Gamma^\lambda(45)\Gamma^\lambda(34), \tag{3.5.11}$$

as may be confirmed by direct multiplication.

All the techniques involved in the generation of matrices of the irreducible representations of S_l through the Specht modules and Young tableaux have been implemented as a computer program. In each case tested, each representation matrix that is not that of a simple transposition has been verified as in (3.5.11).

The Specht modules S^λ were first obtained by Specht in [**Sp35**]. Here, each tableau of shape $\lambda \in P(l)$ was associated with a certain polynomial in l indeterminates. On showing that the polynomials associated with the S_l-standard tableaux are linearly independent and knowing the dimension of the S_l-module so obtained, it was concluded that these polynomials form a basis for that S_l-module.

Garnir [**Ga50**] adopted the same construction. Having obtained the specific Garnir relations dealing with non-standard tableaux of the type given by (3.4.9), the standardisation algorithm given in Section 3.4 was developed, and the irreducible S_l-module S^λ thus constructed explicitly.

Chapter 4
Linear and Symplectic Group Modules

§4.1. The double centraliser technique

Throughout this section $\mathcal{G}$ will be an arbitrary finite group and M will be a left $\mathsf{F}\mathcal{G}$-module. Let $C = \mathrm{Hom}_{\mathsf{F}\mathcal{G}}(M, M)$ be the algebra of endomorphisms of M which centralise the action of $\mathsf{F}\mathcal{G}$. It is the purpose of this section to derive the relationship between the irreducible $\mathsf{F}\mathcal{G}$-modules which occur as submodules of M and the structure of M as a C-module. This problem was considered by Schur [**Sc01**] and Weyl [**We39**] in constructing the irreducible representations of $GL(m)$. A more general treatment of 'symmetric algebras' is overviewed in [**CR62**]. The exposition given below follows this treatment but deals only with the special case of the Frobenius algebra $\mathsf{F}\mathcal{G}$, this being sufficient for the purposes of this thesis.

Many of the results of this section require a knowledge of the composition of the Frobenius algebra $\mathsf{F}\mathcal{G}$ in terms of its *right* ideals. By using a particular map, which is defined below, it is straightforward to show that each of the lemmas and theorems of Section 3.2 hold when the word 'left' is replaced by the word 'right' and, for Lemmas 3.2.10 and 3.2.11, vice-versa. In the next section, this map will be required to obtain the right ideals of the Frobenius algebra of the symmetric group from its left ideals.

Definition 4.1.1. *Let $\vartheta : \mathsf{F}\mathcal{G} \to \mathsf{F}\mathcal{G}$ be the involution defined by:*

$$\vartheta\left(\sum_{\pi\in\mathcal{G}} x(\pi)\pi\right) = \sum_{\pi\in\mathcal{G}} x(\pi)\pi^{-1}. \tag{4.1.1}$$

Lemma 4.1.2. *ϑ is such that:*

(i) $\vartheta(xy) = \vartheta(y)\vartheta(x)$ for all $x, y \in \mathsf{F}\mathcal{G}$;

(ii) $e \in \mathsf{F}\mathcal{G}$ is an idempotent if and only if $\vartheta(e)$ is an idempotent;

(iii) $e \in \mathsf{F}\mathcal{G}$ is a primitive idempotent if and only if $\vartheta(e)$ is a primitive idempotent;

(iv) If $U \subset \mathsf{F}\mathcal{G}$ is a left ideal, then $\vartheta(U)$ is a right ideal and vice-versa;

(v) If $U \subset \mathsf{F}\mathcal{G}$ is a minimal left ideal, then $\vartheta(U)$ is a minimal right ideal and vice-versa;

(vi) If $e \in \mathsf{R}\mathcal{G}$ is a primitive idempotent, then e and $\vartheta(e)$ are elements of the same minimal two-sided ideal.

Proof. (i). For $x = \sum_{\pi\in\mathcal{G}} x(\pi)\pi$ and $y = \sum_{\pi\in\mathcal{G}} y(\pi)\pi$,

$$\vartheta(xy) = \vartheta\left(\sum_{\pi\in\mathcal{G}}\sum_{\tau\in\mathcal{G}} x(\pi)y(\tau)\pi\tau\right) = \left(\sum_{\pi\in\mathcal{G}}\sum_{\tau\in\mathcal{G}} x(\pi)y(\tau)\tau^{-1}\pi^{-1}\right) = \vartheta(y)\vartheta(x).$$

(ii). If $e^2 = e$ then, by (i),

$$\vartheta(e)\vartheta(e) = \vartheta(ee) = \vartheta(e).$$

(iii). If $\vartheta(e) = e_1' + e_2'$ where $e_1'^2 = e_1'$, $e_2'^2 = e_2'$, and $e_1'e_2' = e_2'e_1' = 0$, then $e = \vartheta(e_1') + \vartheta(e_2')$ and, by (i) and (ii), $\vartheta(e_1')^2 = \vartheta(e_1')$, $\vartheta(e_2')^2 = \vartheta(e_2')$, $\vartheta(e_1')\vartheta(e_2') = \vartheta(e_2'e_1') = 0$ and likewise $\vartheta(e_2')\vartheta(e_1') = 0$. Since e is primitive, either $\vartheta(e_1') = 0$ or $\vartheta(e_2') = 0$ whereupon either $e_1' = 0$ or $e_2' = 0$ implying that $\vartheta(e)$ is primitive. (iv). Let $x \in U$. It is required to prove that $\vartheta(x)y \in \vartheta(U)$ for all $y \in \mathsf{F}\mathcal{G}$. From

$$\vartheta(x)y = \vartheta(x)\vartheta(\vartheta(y)) = \vartheta(\vartheta(y)x),$$

this immediately follows since $\vartheta(y)x \in U$ for U a left ideal. The other case follows in the same manner. (v). If U is a minimal left ideal, let $V = \vartheta(U)$ be the corresponding right ideal. Let $V' \subset V$ be a proper right ideal of V with $v \in V\backslash V'$ and $v \neq 0$. By (iv), $\vartheta(V')$ is a left ideal within U which is proper since $\vartheta(v) \neq 0$ and $\vartheta(v) \in U\backslash\vartheta(V')$. This contradiction implies that $V' = V$ and hence that V is a minimal right ideal. (vi). Since elements from different minimal two-sided ideals annihilate one another, it is sufficient to show that $e\vartheta(e) \neq 0$. This is so since if $e = \sum_{\pi\in\mathcal{G}} e(\pi)\pi$, then $\theta(e) = \sum_{\pi\in\mathcal{G}} e(\pi)\pi^{-1}$ and the coefficient of I in $e\vartheta(e)$ is $\sum_{\pi\in\mathcal{G}} e(\pi)^2$ which is non-zero if $e \in \mathsf{R}\mathcal{G}$ is non-zero.

The following lemma will be required below.

Lemma 4.1.3. *Let $x = \sum_{\pi\in\mathcal{G}} x(\pi)\pi$. If $\zeta \in \mathcal{G}$ then:*

$$\zeta x = \sum_{\pi\in\mathcal{G}} x(\zeta^{-1}\pi)\pi; \qquad (4.1.3a)$$

$$x\zeta = \sum_{\pi\in\mathcal{G}} x(\pi\zeta^{-1})\pi; \qquad (4.1.3b)$$

$$\zeta^{-1}x = \sum_{\pi\in\mathcal{G}} x(\zeta\pi^{-1})\pi^{-1}; \qquad (4.1.3c)$$

and

$$x\zeta^{-1} = \sum_{\pi\in\mathcal{G}} x(\pi^{-1}\zeta)\pi^{-1}. \qquad (4.1.3d)$$

Proof.

$$\zeta x = \sum_{\pi' \in \mathcal{G}} x(\pi')\zeta\pi' = \sum_{\pi \in \mathcal{G}} x(\zeta^{-1}\pi)\pi,$$

gives (4.1.3a) whereas:

$$x\zeta = \sum_{\pi' \in \mathcal{G}} x(\pi')\pi'\zeta = \sum_{\pi \in \mathcal{G}} x(\pi\zeta^{-1})\pi,$$

gives (4.1.3b). (4.1.3c) and (4.1.3d) follow from (4.1.3a) and (4.1.3b) respectively by, in each case, substituting ζ^{-1} for ζ and π^{-1} for π.

Definition 4.1.4. *Let $M^* = \mathrm{Hom}_{\mathsf{F}}(M, \mathsf{F})$ be dual to M. M^* is a right $\mathsf{F}\mathcal{G}$-module upon defining:*

$$(vx)u = v(xu), \tag{4.1.4}$$

for all $x \in \mathsf{F}\mathcal{G}$, $u \in M$ and $v \in M^*$.

Definition 4.1.5. *Let the map $\Omega : M \otimes M^* \to \mathsf{F}\mathcal{G}$ be defined by:*

$$\Omega(u, v) = \sum_{\pi \in \mathcal{G}} v(\pi^{-1}u)\pi, \tag{4.1.5}$$

for all $u \in M$ and $v \in M^*$.

Lemma 4.1.6. *The map Ω is bilinear over $\mathsf{F}\mathcal{G}$ in that:*

$$\Omega(x_1u_1 + x_2u_2, v) = x_1\Omega(u_1, v) + x_2\Omega(u_2, v) \tag{4.1.6a}$$

and

$$\Omega(u, v_1y_1 + v_2y_2) = \Omega(u, v_1)y_1 + \Omega(u, v_2)y_2. \tag{4.1.6b}$$

for all $x_1, x_2, y_1, y_2 \in \mathsf{F}\mathcal{G}$, $u, u_1, u_2 \in M$ and $v, v_1, v_2 \in M^*$. In addition Ω is non-degenerate.

Proof. Let $x_1 = \sum_{\tau \in \mathcal{G}} x_1(\tau)\tau$ and $x_2 = \sum_{\tau \in \mathcal{G}} x_2(\tau)\tau$. Then, for the first argument:

$$\begin{aligned}
\Omega(x_1u_1 + x_2u_2, v) &= \sum_{\pi \in \mathcal{G}} v(\pi^{-1}(x_1u_1 + x_2u_2))\pi \\
&= \sum_{\pi \in \mathcal{G}} v(\pi^{-1}x_1u_1)\pi + \sum_{\pi \in \mathcal{G}} v(\pi^{-1}x_2u_2)\pi \\
&= \sum_{\pi \in \mathcal{G}} \sum_{\tau \in \mathcal{G}} v(x_1(\pi\tau^{-1})\tau^{-1}u_1)\pi + \sum_{\pi \in \mathcal{G}} \sum_{\tau \in \mathcal{G}} v(x_2(\pi\tau^{-1})\tau^{-1}u_2)\pi \\
&\qquad \text{(on using (4.1.3}c\text{))} \\
&= \sum_{\pi \in \mathcal{G}} \sum_{\tau \in \mathcal{G}} v(\tau^{-1}u_1)x_1(\pi\tau^{-1})\pi + \sum_{\pi \in \mathcal{G}} \sum_{\tau \in \mathcal{G}} v(\tau^{-1}u_2)x_2(\pi\tau^{-1})\pi \\
&= \sum_{\tau \in \mathcal{G}} v(\tau^{-1}u_1)x_1\tau + \sum_{\tau \in \mathcal{G}} v(\tau^{-1}u_2)x_2\tau \\
&\qquad \text{(on using (4.1.3}b\text{))}
\end{aligned}$$

$$= x_1\Omega(u_1, v) + x_2\Omega(u_2, v).$$

For the second argument,

$$\begin{aligned}\Omega(u, v_1y_1 + v_2y_2) &= \sum_{\pi\in\mathcal{G}}(v_1y_1 + v_2y_2)(\pi^{-1}u)\pi \\ &= \sum_{\pi\in\mathcal{G}}(v_1y_1)(\pi^{-1}u)\pi + \sum_{\pi\in\mathcal{G}}(v_2y_2)(\pi^{-1}u)\pi \\ &= \sum_{\pi\in\mathcal{G}} v_1(y_1\pi^{-1}u)\pi + \sum_{\pi\in\mathcal{G}} v_2(y_2\pi^{-1}u)\pi \\ &= \sum_{\pi\in\mathcal{G}}\sum_{\tau\in\mathcal{G}} v_1(y_1(\tau^{-1}\pi)\tau^{-1}u)\pi + \sum_{\pi\in\mathcal{G}}\sum_{\tau\in\mathcal{G}} v_2(y_2(\tau^{-1}\pi)\tau^{-1}u)\pi \\ &= \sum_{\pi\in\mathcal{G}}\sum_{\tau\in\mathcal{G}} v_1(\tau^{-1}u)y_1(\tau^{-1}\pi)\pi + \sum_{\pi\in\mathcal{G}}\sum_{\tau\in\mathcal{G}} v_2(\tau^{-1}u)y_2(\tau^{-1}\pi)\pi \\ &= \sum_{\tau\in\mathcal{G}} v_1(\tau^{-1}u)\tau y_1 + \sum_{\tau\in\mathcal{G}} v_2(\tau^{-1}u)\tau y_2 \\ &= \Omega(u, v_1)y_1 + \Omega(u, v_2)y_2.\end{aligned}$$

Suppose that $\Omega(u, v) = 0$ for all $u \in M$. Then $\sum_{\pi\in\mathcal{G}} v(\pi^{-1}u)\pi = 0$ and $v(\pi^{-1}u) = 0$ for each $\pi \in \mathcal{G}$ since the group elements are linearly independent in $\mathsf{F}\mathcal{G}$. For $\pi = I$, this gives $v(u) = 0$ and hence $v = 0$. By a similar argument, $\Omega(u, v) = 0$ for all $v \in M^*$ implies that $u = 0$. Therefore Ω is non-degenerate.

Definition 4.1.7. *The nucleus $\mathsf{F}\mathcal{G}_N$ of $\mathsf{F}\mathcal{G}$ is defined to be the set of all finite sums of the form*

$$\sum_i \Omega(u_i, v_i),$$

where each $u_i \in M$ and each $v_i \in M^*$.

The bilinearity of Ω with respect to $\mathsf{F}\mathcal{G}$, as determined by Lemma 4.1.6, implies that $\mathsf{F}\mathcal{G}_N$ is a two-sided ideal in $\mathsf{F}\mathcal{G}$.

Lemma 4.1.8. *The nucleus $\mathsf{F}\mathcal{G}_N$ possesses an idempotent e_N which is central in $\mathsf{F}\mathcal{G}$ and generates $\mathsf{F}\mathcal{G}_N$ through:*

$$\mathsf{F}\mathcal{G}_N = e_N\mathsf{F}\mathcal{G} = \mathsf{F}\mathcal{G}e_N. \tag{4.1.8}$$

Proof. By Lemma 3.2.14, $\mathsf{F}\mathcal{G}_N$, being a two-sided ideal, may be uniquely written as a direct sum of a subset of the minimal two-sided ideals of $\mathsf{F}\mathcal{G}$:

$$\mathsf{F}\mathcal{G}_N = W_{i_1} \oplus W_{i_2} \oplus \cdots \oplus W_{i_s}.$$

Then $e_N = e_{i_1} + e_{i_2} + \cdots + e_{i_s}$ where each e_i, as given by 3.2.14, is the unique idempotent of the minimal two-sided ideal W_i. Since each e_i commutes with all the elements of $\mathsf{F}\mathcal{G}$, e_N belongs to its centre. Since e_i generates W_i, e_N generates $\mathsf{F}\mathcal{G}_N$ according to (4.1.8).

Lemma 4.1.9. *The idempotent $e_N \in \mathsf{F}\mathcal{G}_N$ is such that $e_N u = u$ for all $u \in M$ and $v e_N = v$ for all $v \in M^*$.*

Proof. For all $u \in M$ and $v \in M^*$,

$$\Omega(e_N u - u, v) = e_N \Omega(u, v) - \Omega(u, v) = 0,$$

where the linearity of Ω has been used. Since Ω is non-degenerate, it follows that $e_N u = u$ for all $u \in M$. Similarly $v e_N = v$ for all $v \in M^*$.

As above, let $C = \mathrm{Hom}_{\mathsf{F}\mathcal{G}}(M, M)$ be the ring of endomorphisms of M which centralise the action of $\mathsf{F}\mathcal{G}$. In this way M is also viewed as a left C-module. Define the map $\Psi : M \otimes M^* \to \mathrm{End}\, M$ by

$$\Psi(u, v)u' = \Omega(u', v)u, \tag{4.1.10}$$

for all $u, u' \in M$ and $v \in M^*$. Then, for all $x \in \mathsf{F}\mathcal{G}$,

$$\Psi(u, v)(xu') = \Omega(xu', v)u = x\Omega(u', v)u = x\Psi(u, v)u', \tag{4.1.11}$$

where the linearity of Ω has been used. This shows that $\Psi(u, v) \in C$ for all $u \in M$ and $v \in M^*$.

Since $e_N \in \mathsf{F}\mathcal{G}_N$, it can be expressed as some finite sum:

$$e_N = \sum_i \Omega(u_i^0, v_i^0), \tag{4.1.12}$$

where each $u_i^0 \in M$ and each $v_i^0 \in M^*$.

Lemma 4.1.13. *If $y_c \in \mathrm{Hom}_C(M, M)$ then there exists an element $y \in \mathsf{F}\mathcal{G}_N$ such that $yu = y_c u$ for all $u \in M$. One such y is given by:*

$$y = \sum_i \Omega(y_c u_i^0, v_i^0). \tag{4.1.13}$$

Proof. Since $y_c \in \mathrm{Hom}_C(M, M)$,

$$\Psi(u, v)(y_c u') = y_c(\Psi(u, v)u'),$$

which, from (4.1.10), implies that:

$$\Omega(y_c u', v)u = y_c \Omega(u', v)u,$$

for all $u, u' \in M$ and $v \in M^*$. With y as given by (4.1.13),

$$yu = \sum_i \Omega(y_c u_i^0, v_i^0)u = y_c(\sum_i \Omega(u_i^0, v_i^0)u)$$
$$= y_c(e_N u) = y_c u,$$

since $e_N u = u$ from Lemma 4.1.9. This proves Lemma 4.1.13.

In what follows R will be a C-submodule of M. Then

$$\Omega(R, M^*) = \left\{ \sum_i (u_i^r, v_i) : u_i^r \in R, v_i \in M^* \right\} \tag{4.1.14}$$

is a right ideal of $\mathsf{F}\mathcal{G}$ contained in $\mathsf{F}\mathcal{G}_N$ because of (4.1.6b). In particular $\Omega(M, M^*) = \mathsf{F}\mathcal{G}_N$.

Lemma 4.1.15. *If R is a C-submodule of M then there exists an idempotent $e_R \in \mathsf{F}\mathcal{G}_N$ such that $\Omega(R, M^*) = e_R \mathsf{F}\mathcal{G}$. In addition, $\Omega(R, M^*)M = R$.*

Proof. Let $p \in \mathrm{Hom}_C(M, R)$ project M onto R. By Lemma 4.1.13, there exists an e_R such that $pu = e_R u$ for all $u \in M$, given by:

$$e_R = \sum_i \Omega(pu_i^0, v_i^0).$$

Note that $e_R \in \Omega(R, M^*)$. If $x^r = \sum_i \Omega(u_i^r, v_i)$ is an arbitrary element of $\Omega(R, M^*)$, then

$$e_R x^r = \sum_i \Omega(e_R u_i^r, v_i) = \sum_i \Omega(pu_i^r, v_i) = \sum_i \Omega(u_i^r, v_i) = x^r.$$

This implies that $\Omega(R, M^*) = e_R \mathsf{F}\mathcal{G}$ since $\Omega(R, M^*)$ is a right ideal of $\mathsf{F}\mathcal{G}$. Putting $x^r = e_R$ shows that $e_R^2 = e_R$.

Since $\Omega(u^r, v)u' = \Psi(u', v)u^r$ for all $u^r \in R$, $u \in M$ and $v \in M^*$, $\Psi(u', v) \in C$, and R is a C-submodule of M, it follows that $\Omega(R, M^*)M \subset R$. Also $u^r = e_R u^r \in \Omega(R, M^*)M$, showing that $\Omega(R, M^*)M = R$.

Lemma 4.1.16. *If $U = e\mathsf{F}\mathcal{G}$ where $e^2 = e \in \mathsf{F}\mathcal{G}_N$, then $UM = eM$ is a direct sum C-submodule of M. In addition $\Omega(UM, M^*) = U$.*

Proof. The first part follows directly from noting that $UM = e\mathsf{F}\mathcal{G}M = eM$ and that $M = eM \oplus (1-e)M$ where eM and $(1-e)M$ are C-submodules of M since, for each $A \in C$ and $u \in M$, $Aeu = eAu \in eM$ and similarly $A(1-e)u \in (1-e)M$.

Since U is a right ideal in $\mathsf{F}\mathcal{G}$ and $\Omega(UM, M^*) = U\Omega(M, M^*)$, it follows that $\Omega(UM, M^*) \subset U$. Now, for $x \in U$,

$$x = xe_N = \sum_i \Omega(xu_i^0, v_i^0) \in \Omega(UM, M^*),$$

implying that $\Omega(UM, M^*) = U$.

Theorem 4.1.17. *There exists a bijection between the set of right ideals of* $\mathsf{F}\mathcal{G}_N$ *and the direct sum* C*-submodules of* M*. Two right ideals* $U_1 = e_1\mathsf{F}\mathcal{G}$ *and* $U_2 = e_2\mathsf{F}\mathcal{G}$ *generated by idempotents* $e_1, e_2 \in \mathsf{F}\mathcal{G}_N$*, are equivalent if and only if the* C*-modules* e_1M *and* e_2M *are equivalent.*

Proof. From Lemma 4.1.16, the right ideal $U = e\mathsf{F}\mathcal{G}$ in $\mathsf{F}\mathcal{G}_N$, with $e^2 = e \in \mathsf{F}\mathcal{G}_N$, maps into the direct sum C-submodule $UM = eM$ of M. From Lemma 4.1.15, the C-submodule R of M maps into the right ideal $\Omega(R, M^*) = e_R\mathsf{F}\mathcal{G}$, where $e_R^2 = e_R$. Since $\Omega(UM, M^*) = U$ and $\Omega(R, M^*)M = R$, these maps are inverse to one another and therefore they define a bijection.

Let $\theta : U_1 \to U_2$ be an equivalence map between the right ideals $U_1 = e_1\mathsf{F}\mathcal{G}$ and $U_2 = e_2\mathsf{F}\mathcal{G}$. Let $\theta(e_1) = a$ and $\theta^{-1}(e_2) = b$. Then, for $c, d \in \mathsf{F}\mathcal{G}$,

$$\theta(e_1c) = \theta(e_1)c = \theta(e_1^2)c = \theta(e_1)e_1c = ae_1c \in U_2 \tag{4.1.17a}$$

and

$$\theta^{-1}(e_2d) = \theta^{-1}(e_2)d = \theta^{-1}(e_2^2)d = \theta^{-1}(e_2)e_2d = be_2d \in U_1. \tag{4.1.17b}$$

Therefore $e_2\mathsf{F}\mathcal{G} = ae_1\mathsf{F}\mathcal{G}$ and $e_1\mathsf{F}\mathcal{G} = be_2\mathsf{F}\mathcal{G}$. In addition, for all $c \in U_1 = e_1\mathsf{F}\mathcal{G}$,

$$bac = bae_1c = b\theta(e_1c) = \theta^{-1}(e_2)\theta(e_1c) = \theta^{-1}(e_2\theta(e_1c)) = \theta^{-1}(\theta(e_1c)) = e_1c = c,$$

where (4.1.17a) and (4.1.17b) have both been used, and also $c = e_1c$ since $c \in U_1$, and $\theta(e_1c) = e_2\theta(e_1c)$ since $\theta(e_1c) \in U_2$. It is shown in a similar way that $abd = d$ for all $d \in U_2 = e_2\mathsf{F}\mathcal{G}$.

Define the maps $\bar{\theta}$ and $\bar{\theta}'$ between e_1M and e_2M by:

$$\bar{\theta}(e_1u_1) = ae_1u_1$$

and

$$\bar{\theta}'(e_2u_2) = be_2u_2,$$

for all $u_1, u_2 \in M$. These are clearly C-homomorphisms. Combining them gives:

$$\begin{aligned}\bar{\theta}'\bar{\theta}(e_1u_1) &= \bar{\theta}'(ae_1u_1) = \bar{\theta}'(e_2ae_1u_1)\\ &= be_2ae_1u_1 = bae_1u_1 = e_1u_1,\end{aligned}$$

where $e_2a = a$ (since $a \in U_2$) has been used. In a similar way, it can be shown that $\bar{\theta}\bar{\theta}'(e_2u_2) = e_2u_2$. It then follows that the C-modules e_1M and e_2M are equivalent.

Now suppose that there is an equivalence map ϕ between the C-modules e_1M and e_2M. Define:

$$\bar{\phi} : \sum_i \Omega(u_i^{(1)}, v_i) = \sum_i \Omega(\phi u_i^{(1)}, v_i)$$

for all $u_i^{(1)} \in e_1M$ and $v_i \in M^*$. Then $\bar{\phi}$ provides a map from $\Omega(e_1M, M^*) = e_1\mathsf{F}\mathcal{G}$ into $\Omega(e_2M, M^*) = e_2\mathsf{F}\mathcal{G}$. If $\bar{\phi}'$ is defined by:

$$\bar{\phi}' : \sum_i \Omega(u_i^{(2)}, v_i) = \sum_i \Omega(\phi^{-1}u_i^{(2)}, v_i),$$

for all $u_i^{(2)} \in e_2M$ and $v_i \in M^*$, then $\bar{\phi}'$ maps $e_2\mathsf{F}\mathcal{G}$ into $e_1\mathsf{F}\mathcal{G}$ such that $\bar{\phi}'\bar{\phi}$ and $\bar{\phi}\bar{\phi}'$ are the identity maps on $e_1\mathsf{F}\mathcal{G}$ and $e_2\mathsf{F}\mathcal{G}$ respectively. Since the right action of $\mathsf{F}\mathcal{G}$ commutes with the left action of both $\bar{\phi}$ and $\bar{\phi}'$, then $\bar{\phi}$ and $\bar{\phi}'$ are equivalence maps between the right ideals $\Omega(e_1M, M^*) = e_1\mathsf{F}\mathcal{G}$ and $\Omega(e_2M, M^*) = e_2\mathsf{F}\mathcal{G}$. Therefore $e_1\mathsf{F}\mathcal{G}$ and $e_2\mathsf{F}\mathcal{G}$ are equivalent right ideals.

Theorem 4.1.18. *If $e \in \mathsf{F}\mathcal{G}_N$ is an idempotent then the C-module eM is irreducible if and only if e is primitive. M is a completely reducible C-module.*

Proof. Let $e \in \mathsf{F}\mathcal{G}_N$ be a primitive idempotent and let R be a non-zero C-submodule of eM. Lemma 4.1.16 shows that $e\mathsf{F}\mathcal{G} = \Omega(eM, M^*)$ whereupon $\Omega(R, M^*) \subset e\mathsf{F}\mathcal{G}$. Since Ω is non-degenerate, $\Omega(R, M^*) \neq 0$. Since $\Omega(R, M^*)$ is a right ideal and $e\mathsf{F}\mathcal{G}$ is minimal, it follows that $e\mathsf{F}\mathcal{G} = \Omega(R, M^*)$. Then, by Lemma 4.1.15, $R = \Omega(R, M^*)M = eM$ so that eM is an irreducible C-module.

Conversely, let $e \in \mathsf{F}\mathcal{G}_N$ be an idempotent and eM an irreducible C-module. Let $e = e_1 + e_2$ where e_1 and e_2 are each idempotents. By Lemma 4.1.16, both e_1M and e_2M are C-submodules of eM. Since eM is irreducible either $e_1M = 0$ or e_2M=0. In the first case Lemma 4.1.16 implies that $e_1\mathsf{F}\mathcal{G} = \Omega(e_1M, M^*) = 0$ whereupon $e_1 = 0$. The second case is similar. Therefore either $e_1 = 0$ or $e_2 = 0$, implying that e is primitive.

Because $\mathsf{F}\mathcal{G}_N$ is a completely reducible right $\mathsf{F}\mathcal{G}$-module,

$$\mathsf{F}\mathcal{G}_N = \bigoplus_i e_i\mathsf{F}\mathcal{G},$$

where each e_i is a primitive idempotent and each $e_i\mathsf{F}\mathcal{G}$ is a minimal right ideal. From Lemma 4.1.16, it follows that $M = \bigoplus_i e_iM$, where each e_iM is an irreducible C-submodule of M. M is thus completely reducible.

It has been determined that for each primitive idempotent $e_i \in \mathsf{F}\mathcal{G}_N$, e_iM is an irreducible C-module. However, it is often difficult to determine whether an idempotent e_i actually belongs to $\mathsf{F}\mathcal{G}_N$. For this purpose the following lemma is useful.

Lemma 4.1.19. *If $e \in \mathsf{F}\mathcal{G}$ is a non-zero primitive idempotent of $\mathsf{F}\mathcal{G}$ then either $e \in \mathsf{F}\mathcal{G}_N$ whereupon eM is an irreducible C-submodule of M, or $e \notin \mathsf{F}\mathcal{G}_N$ whereupon $eM = 0$.*

Proof. From Lemma 4.1.8, $xe_N = e_N x$ for all $x \in \mathsf{F}\mathcal{G}$. For any idempotent $e \in \mathsf{F}\mathcal{G}$,

$$e = ee_N + e(1 - e_N)$$

for which $(ee_N)(e(1-e_N)) = ee_Ne - ee_Nee_N = 0$ and similarly $(e(1-e_N))(ee_N) = 0$. Then, since e is primitive, either $ee_N = e$ or $e(1-e_N) = e$. In the first case, it follows that $e \in \mathsf{F}\mathcal{G}_N$ and that, since $e\mathsf{F}\mathcal{G} = \Omega(eM, M^*)$ is non-zero, eM is a proper irreducible C-module. In the second case, for all $u \in M$, $eu = eu - ee_Nu = eu - eu = 0$ by Lemma 4.1.9, so that $eM = 0$.

Theorem 4.1.20. *If the nucleus $\mathsf{F}\mathcal{G}_N$ is the direct sum of the minimal two-sided ideals $W_1, W_2, \ldots, W_s$, then M decomposes into s inequivalent irreducible $\mathsf{F}\mathcal{G} \otimes C$-modules:*

$$M = \bigoplus_{i=1}^{s} e_i M, \tag{4.1.20}$$

where each e_i is the unique central idempotent of W_i. The dimension of e_iM is equal to f_is_i where f_i is the dimension of any minimal right ideal $e_i'\mathsf{F}\mathcal{G} \subset e_i\mathsf{F}\mathcal{G}$, for e_i' a primitive idempotent, and s_i is the dimension of the irreducible C-module $e_i'M$. The dimension of M is $\sum_i f_is_i$.

Proof. By Theorem 3.2.17 and Lemma 4.1.2, each W_i is the direct sum of f_i linearly independent right ideals. Thus e_i may be written as the sum:

$$e_i = e_i^{(1)} + e_i^{(2)} + \cdots + e_i^{(f_i)},$$

where $e_i^{(j)}$ is a primitive idempotent and $V_j = e_i^{(j)}\mathsf{F}\mathcal{G}$ is a minimal right ideal for $j = 1, 2, \ldots, f_i$. Since the right ideals V_j are mutually equivalent, so are the irreducible C-modules $e_i^{(j)}M = V_jM$ by Theorem 4.1.17. The left action of $\mathsf{F}\mathcal{G}$ on any one $e_i^{(k)}M$ generates all the C-modules $e_i^{(j)}M$, and hence e_iM, through the right ideal analogue of Lemma 3.2.10. Therefore e_iM is an irreducible $\mathsf{F}\mathcal{G} \otimes C$-module. It remains to prove that the C-modules $e_i^{(j)}M$ are linearly independent. For $j = 1, 2, \ldots, f_i$, let $u^{(j)} \in e_i^{(j)}M$ with $u^{(j)} \neq 0$. Then if $\kappa_1u^{(1)} + \kappa_2u^{(2)} + \cdots + \kappa_{f_i}u^{(f_i)} = 0$, for all $v \in M^*$,

$$\begin{aligned} 0 &= \Omega(\kappa_1u^{(1)} + \kappa_2u^{(2)} + \cdots + \kappa_{f_i}u^{(f_i)}, v) \\ &= \kappa_1\Omega(u^{(1)}, v) + \kappa_2\Omega(u^{(2)}, v) + \cdots + \kappa_{f_i}\Omega(u^{(f_i)}, v) \end{aligned}$$

By Lemma 4.1.16, $\Omega(u_i^{(j)}, v) \in e_i^{(j)}\mathsf{F}\mathcal{G}$. The linear independence of the right ideals $e_i^{(j)}\mathsf{F}\mathcal{G}$ implies that $\kappa_1 = \kappa_2 = \cdots = \kappa_{f_i} = 0$. Thus the C-modules $e_i^{(j)}M$ are linearly independent and, if s_i is the dimension of any one of them, then the dimension of the irreducible $\mathsf{F}\mathcal{G} \otimes C$-module e_iM is f_is_i.

§4.2. The Weyl module and covariant tensor representations of $GL(m)$

This section uses the results of the previous to obtain the irreducible $GL(m)$-modules which arise as submodules of tensor powers of the defining $GL(m)$-module.

Let V be the m-dimensional $GL(m,\mathsf{F})$-defining module with basis $\{e_i : i = 1,2,\ldots,m\}$. $G \in GL(m)$ acts on V by linear extension of the action:

$$Ge_i = \sum_{j=1}^{m} G^j{}_i e_j, \tag{4.2.1}$$

to the whole of V.

The l-fold tensor power $GL(m)$-module $V^{\otimes l}$ has a basis $\{e_{i_1i_2\cdots i_l} : 1 \leq i_k \leq m \text{ for } k = 1,2,\ldots,l\}$ where $e_{i_1i_2\cdots i_l}$ denotes $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_l}$. If $G \in GL(m)$ then, from (1.5.7), the induced action, $G \in \mathrm{End}(V^{\otimes l})$, on this basis is given by:

$$Ge_{i_1i_2\cdots i_l} = \sum_{1\leq j_1,j_2,\ldots,j_l\leq m} G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l} e_{j_1j_2\cdots j_l}, \tag{4.2.2}$$

which extends linearly to the whole of $V^{\otimes l}$, making $V^{\otimes l}$ a $GL(m)$-module.

Definition 4.2.3. *The symmetric group S_l is defined to act on $V^{\otimes l}$ by:*

$$\pi e_{i_1i_2\cdots i_l} = e_{i_{\pi^{-1}(1)}i_{\pi^{-1}(2)}\cdots i_{\pi^{-1}(l)}}, \tag{4.2.3}$$

for $\pi \in S_l$, with linear extension to the whole of $V^{\otimes l}$. In addition, $V^{\otimes l}$ is made into an $\mathsf{F}S_l$-module by linearly extending this S_l action.

Once it has been determined that $\mathrm{End}_{\mathsf{F}S_l}(V^{\otimes l})$ is actually the enveloping algebra (linear hull) of the induced action of $GL(m)$ on $V^{\otimes l}$, the results of the last section will enable the irreducible $GL(m)$-modules occurring as submodules of $V^{\otimes l}$ to be obtained from the analysis of the Frobenius algebra of the symmetric group presented in Chapter 3.

A general transformation, $A \in \mathrm{End}(V^{\otimes l})$, of $V^{\otimes l}$ takes the form:

$$Ae_{i_1i_2\cdots i_l} = \sum_{1\leq j_1,\ldots,j_l\leq m} A^{j_1j_2\cdots j_l}_{i_1i_2\cdots i_l} e_{j_1j_2\cdots j_l}. \tag{4.2.4}$$

Therefore $\mathrm{End}(V^{\otimes l})$ has dimension m^{2l}. If $A \in \mathrm{End}(V^{\otimes l})$ commutes with the action of $\mathsf{F}S_l$, then from (4.2.3) and (4.2.4), $\pi A e_{i_1 i_2 \cdots i_l} = A \pi e_{i_1 i_2 \cdots i_l}$ if and only if

$$A^{j_1 j_2 \cdots j_l}_{i_1 i_2 \cdots i_l} = A^{j_{\pi(1)} \cdots j_{\pi(l)}}_{i_{\pi(1)} \cdots i_{\pi(l)}}, \tag{4.2.5}$$

for all $\pi \in S_l$. This property characterises the elements of $\mathrm{End}_{\mathsf{F}S_l}(V^{\otimes l})$ which are known as bisymmetric. If $G \in \mathrm{End}(V)$, the induced action, $G \in \mathrm{End}(V^{\otimes l})$, is given by (4.2.2):

$$G^{j_1 j_2 \cdots j_l}_{i_1 i_2 \cdots i_l} = G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l}. \tag{4.2.6}$$

It is immediately clear that G is bisymmetric and therefore that the actions of $GL(m)$ and $\mathsf{F}S_l$ on $V^{\otimes l}$ commute. Let $\mathrm{End}'_{\mathsf{F}S_l}(V^{\otimes l})$ denote the enveloping algebra of the induced actions of $G \in GL(m)$ on $V^{\otimes l}$. This makes $\mathrm{End}'_{\mathsf{F}S_l}(V^{\otimes l})$ a vector space for which, if $G_{(1)}, G_{(2)} \in GL(m)$, then $(G_{(1)} + G_{(2)}) \in \mathrm{End}'_{\mathsf{F}S_l}(V^{\otimes l})$ is defined by:

$$(G_{(1)} + G_{(2)}) e_{i_1 i_2 \cdots i_l} = G_{(1)} e_{i_1 i_2 \cdots i_l} + G_{(2)} e_{i_1 i_2 \cdots i_l}. \tag{4.2.7}$$

The following lemma gives the desired result that the ring of endomorphisms of $V^{\otimes l}$, commuting with $\mathsf{F}S_l$ is the enveloping algebra of $GL(m)$. The proof is a reworking of that given in [**Bo63**].

Lemma 4.2.8. $\mathrm{End}'_{\mathsf{F}S_l}(V^{\otimes l}) = \mathrm{End}_{\mathsf{F}S_l}(V^{\otimes l})$.

Proof. It has already been determined that $\mathrm{End}'_{\mathsf{F}S_l}(V^{\otimes l}) \subset \mathrm{End}_{\mathsf{F}S_l}(V^{\otimes l})$. If $A \in \mathrm{End}_{\mathsf{F}S_l}(V^{\otimes l})$ is given by (4.2.4), then (4.2.5) implies that A is completely specified by those components $A^{j_1 j_2 \cdots j_l}_{i_1 i_2 \cdots i_l}$ for which:

$$(j_1, i_1) \leq (j_2, i_2) \leq \cdots \leq (j_l, i_l), \tag{4.2.8a}$$

where $(a, b) < (c, d)$ if and only if either $a < c$, or $a = c$ and $b < d$; and $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. Furthermore, these components may be varied independently. Therefore $\mathrm{End}_{\mathsf{F}S_l}(V^{\otimes l})$ is a vector space of dimension $\binom{m^2+l-1}{l}$ since (4.2.8a) implies that l choices are to be made from m^2 pairs of indices with repetitions permitted. By a similar argument, $G^{j_1 j_2 \cdots j_l}_{i_1 i_2 \cdots i_l} = G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l}$, has $\binom{m^2+l-1}{l}$ representative elements which satisfy (4.2.8a). Since $\mathrm{End}'_{\mathsf{F}S_l}(V^{\otimes l})$ is a vector space, if it can be shown that these are linearly independent then the lemma is proved. To this end, let

$$\sum_{(j_1, i_1) \leq \cdots \leq (j_l, i_l)} g_{j_1 j_2 \cdots j_l i_1 i_2 \cdots i_l} G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l} = 0, \tag{4.2.8b}$$

where the sum is over all indices which satisfy (4.2.8a). In this expression, each term $G^{j_1}{}_{i_1}G^{j_2}{}_{i_2}\cdots G^{j_l}{}_{i_l}$ may be uniquely written:

$$(G^1{}_1)^{k_{11}}(G^1{}_2)^{k_{12}}\cdots(G^1{}_m)^{k_{1m}}(G^2{}_1)^{k_{21}}\cdots(G^2{}_m)^{k_{2m}}\cdots(G^m{}_m)^{k_{mm}}, \qquad (4.2.8c)$$

where $k_{ab} = \#\{k : (a,b) = (j_k, i_k), 1 \le k \le l\}$, $0 \le k_{ab} \le l$ and $\sum_{a,b=1}^{m} k_{ab} = l$. Since each term satisfying these criteria corresponds to a unique term of (4.2.8b), that expression may be written:

$$\sum_{\substack{0\le k_{ab}\le l \\ k_{11}+k_{12}+\cdots+k_{mm}=l}} g_{k_{11}k_{12}\cdots k_{mm}}(G^1{}_1)^{k_{11}}(G^1{}_2)^{k_{12}}\cdots(G^m{}_m)^{k_{mm}} = 0 \qquad (4.2.8d)$$

Here, the left side is a homogeneous polynomial of degree l in the m^2 elements of the matrix G. If each element is permitted an arbitrary value this would imply that each coefficient in (4.2.8d) is zero. However, if $m \ge 2$, those elements G with non-zero determinant have a co-dimension of one in the m^2-dimensional space of all $m \times m$ matrices. Thus the conclusion remains valid for $G \in GL(m)$ and the lemma is proved for $m \ge 2$. If $m = 1$ the same conclusion follows directly from (4.2.8d).

The irreducible $GL(m)$-submodules of $V^{\otimes l}$ are now obtained via the right ideals of $\mathsf{F}S_l$ which, in turn, are obtained from the Young symmetrisers Y_{T^λ} and the map ϑ.

Theorem 4.2.9. *The $GL(m)$-module $V^{\otimes l}$ is completely reducible. Let $\lambda \in P(l)$ and $\{T_i^\lambda : i = 1,2,\ldots,f^\lambda\}$ be the set of S_l-standard tableaux of shape λ. Then, for $i = 1,2,\ldots,f^\lambda$, $Y'_{T_i^\lambda} = P_{T_i^\lambda}Q_{T_i^\lambda}$ generates a set of f^λ linearly independent minimal right ideals. The $GL(m)$-modules $Y'_{T_i^\lambda}V^{\otimes l}$ are linearly independent and equivalent.*

Proof. Theorem 3.3.19 shows that, for each $i = 1,2,\ldots,f^\lambda$, $Y_{T_i^\lambda}$ is a primitive idempotent upon normalisation, and generates a minimal left ideal. By Theorem 3.3.22, these are linearly independent. By Lemma 4.1.2, $\vartheta(Y_{T_i^\lambda})$ is a primitive idempotent upon normalisation, and generates a minimal right ideal for each $i = 1,2,\ldots,f^\lambda$. These are linearly independent. Let $Y'_{T_i^\lambda} = \vartheta(Y_{T_i^\lambda})$, whereupon

$$\begin{aligned} Y'_{T_i^\lambda} &= \vartheta\Big(\sum_{\sigma\in\mathcal{C}_{T^\lambda}}\sum_{\rho\in\mathcal{R}_{T^\lambda}}(-1)^\sigma\sigma\rho\Big) = \sum_{\sigma\in\mathcal{C}_{T^\lambda}}\sum_{\rho\in\mathcal{R}_{T^\lambda}}(-1)^\sigma\rho^{-1}\sigma^{-1} \\ &= \sum_{\sigma\in\mathcal{C}_{T^\lambda}}\sum_{\rho\in\mathcal{R}_{T^\lambda}}(-1)^\sigma\rho\sigma = P_{T_i^\lambda}Q_{T_i^\lambda}\,. \end{aligned}$$

The second part of the theorem now follows directly from Theorem 4.1.18.

Theorem 4.1.17 and Lemma 4.1.19 now imply that for $\lambda \in P(l)$, those $Y'_{t\lambda} V^{\otimes l}$ which are non-zero provide a set of inequivalent irreducible $GL(m)$-modules present as submodules of $V^{\otimes l}$.

The connection with Young tableaux will now be made. For each $\lambda \in P(l)$ the tableau T^λ with $T^\lambda_{(k)} = i_k$ for $k = 1, 2, \ldots, l$, is conveniently identified with the basis element $w = e_{i_1 i_2 \cdots i_l}$ of $V^{\otimes l}$. Therefore, among others, both

$$\begin{matrix} 1 & 3 & 2 & 6 \\ 1 & 9 & & \\ 8 & & & \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 9 \\ 1 & 2 \\ 8 & 6 \\ 3 & \end{matrix} \tag{4.2.10}$$

are identified with $e_{1183926} \in V^{\otimes 7}$. The place permutation action of $\pi \in S_l$ on $V^{\otimes l}$ as given by Definition 4.2.3 then corresponds to the place permutation action of π_* on T^λ as given by Definition 3.3.11. Then for $w \in V^{\otimes l}$, the tensor $Y'_{t\lambda} w \in Y'_{t\lambda} V^{\otimes l}$ is identified with the Young symmetrised tableau $\{T^\lambda\} = Y^\lambda_* T^\lambda$ as in (3.3.13d).

The Weyl module W^λ is defined to be the span of all $\{T^\lambda\}$ where the entries of T^λ are all from the set $\mathcal{I}^{GL(m)} = \mathbb{N}_m$. However, despite there being m^l such tableaux, the $\{T^\lambda\}$ are not linearly independent since the Column relations (3.4.2) and the Garnir relations (3.4.3) apply. In particular, if T'^λ has an entry repeated in any column, then $\{T'^\lambda\}$ is zero. In those cases for which $\tilde{\lambda}_1 > m$ this situation must necessarily arise in the first column of every tableau T^λ. This implies that the $GL(m)$-module $Y'_{t\lambda} V^{\otimes l}$ is zero. Conversely, if $\tilde{\lambda}_1 \leq m$, there exists a T^λ for which $\{T^\lambda\}$ is non-zero (consider, for example, $T^\lambda_>$ of Definition 2.6.6). Lemma 4.1.19 then yields the following theorem.

Theorem 4.2.11. [**We39**]. *The set*

$$\{W^\lambda = Y'_{t\lambda} V^{\otimes l} : \lambda \in P(l; m)\}$$

provides a complete list of inequivalent irreducible $GL(m)$-modules occurring as submodules of $V^{\otimes l}$.

Therefore, since every irreducible covariant $GL(m)$-module occurs in $V^{\otimes l}$ for some l [**Li44**], each irreducible covariant $GL(m)$-module is equivalent to W^λ for some $\lambda \in P(l; m)$ for some l.

For each W^λ a set of favoured tableaux are provided by the following.

Definition 4.2.12. *The tableau T^λ is $GL(m)$-standard if and only if:*

(i) the entries are taken from the set $\mathcal{I}^{GL(m)} = \mathbb{N}_m$;

(ii) the entries are strictly increasing from top to bottom down each column;

(iii) the entries are non-decreasing from left to right across each row.

For example, of the tableaux:

$$\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & & & \\ 4 & & & \end{matrix} \; , \qquad \begin{matrix} 1 & 3 & 3 & 4 \\ 2 & 4 & 5 & 5 \\ 4 & & & \end{matrix} \; , \qquad \text{and} \qquad \begin{matrix} 1 & 3 & 3 & 2 \\ 4 & 4 & & \\ 5 & & & \end{matrix} \; ; \tag{4.2.13}$$

the first two are $GL(m)$-standard for $m \geq 5$ and the last is not $GL(m)$-standard for any m. In addition, neither of the tableaux of (4.2.10) are $GL(m)$-standard.

The techniques that were employed in Section 3.4 to write a symmetrised tableau with distinct entries in terms of S_l-standard tableaux, may also be used to write an arbitrary symmetrised tableau with entries from the set $\mathcal{I}^{GL(m)}$, in terms of $GL(m)$-standard tableaux. Once more, the Column relations (3.4.2) enable $\{T'^\lambda\}$ to be expressed $\{T^\lambda\}$ for some column strict T^λ. Then, if T^λ is not $GL(m)$-standard, condition (iii) of Definition 4.2.12 implies the existence of a neighbouring pair of entries $T^\lambda_{(a,b)}$ and $T^\lambda_{(a,b+1)}$ for which $T^\lambda_{(a,b)} > T^\lambda_{(a,b+1)}$ as in (3.4.9). On selecting $\mathcal{X}$ to be the set of positions below and including that of $T^\lambda_{(a,b)}$ in the bth column, and $\mathcal{Y}$ to be the set of positions above and including that of $T^\lambda_{(a,b+1)}$ in the $(b+1)$th column, the Garnir relations (3.4.3) enable $\{T^\lambda\}$ to be written in terms of higher tableaux. This process can be iterated until just $GL(m)$-standard tableaux remain. Again the termination of this iterative process is guaranteed by the finite number of tableaux and the order on the tableaux given by Definition 2.6.8.

As an example of this standardisation procedure, consider the $GL(5)$-module W^λ where $\lambda = (3,3,2)$, and the non-standard tableau:

$$T^\lambda = \begin{matrix} 5 & 2 & 3 \\ 4 & 1 & 2 \\ 3 & 4 & \end{matrix} \, . \tag{4.2.14}$$

The Column relations enable $\{T^\lambda\}$ to be written in terms of a column strict tableau:

$$\{T^\lambda\} = -\left\{ \begin{matrix} 3 & 1 & 2 \\ 4 & 2 & 3 \\ 5 & 4 & \end{matrix} \right\} . \tag{4.2.15a}$$

Let $\mathcal{X} = \{1,2,3\}$ and $\mathcal{Y} = \{4\}$. With an appropriate set of coset representatives, the Garnir relations then give:

$$\left\{\begin{array}{ccc} 3 & 1 & 2 \\ 4 & 2 & 3 \\ 5 & 4 & \end{array}\right\} - \left\{\begin{array}{ccc} 1 & 3 & 2 \\ 4 & 2 & 3 \\ 5 & 4 & \end{array}\right\} + \left\{\begin{array}{ccc} 1 & 4 & 2 \\ 3 & 2 & 3 \\ 5 & 4 & \end{array}\right\} - \left\{\begin{array}{ccc} 1 & 5 & 2 \\ 3 & 2 & 3 \\ 4 & 4 & \end{array}\right\} = 0. \qquad (4.2.15b)$$

The Column relations imply that the third term here is identically zero. In addition, they permit the rearrangement of the entries in the second columns of the second and fourth terms to give column strict tableaux. Thereupon, (4.2.15a) and (4.2.15b) imply that:

$$\{T^\lambda\} = \left\{\begin{array}{ccc} 1 & 2 & 2 \\ 4 & 3 & 3 \\ 5 & 4 & \end{array}\right\} - \left\{\begin{array}{ccc} 1 & 2 & 2 \\ 3 & 4 & 3 \\ 4 & 5 & \end{array}\right\}. \qquad (4.2.15c)$$

Consider the first term on the right side of this identity and let $\mathcal{X} = \{2,3\}$ and $\mathcal{Y} = \{4,5\}$. The Garnir relations then yield the identity:

$$\begin{aligned} &\left\{\begin{array}{ccc} 1 & 2 & 2 \\ 4 & 3 & 3 \\ 5 & 4 & \end{array}\right\} + \left\{\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 5 & 4 & \end{array}\right\} - \left\{\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 5 & 3 \\ 4 & 4 & \end{array}\right\} \\ &\quad - \left\{\begin{array}{ccc} 1 & 2 & 2 \\ 3 & 4 & 3 \\ 5 & 4 & \end{array}\right\} + \left\{\begin{array}{ccc} 1 & 2 & 2 \\ 3 & 5 & 3 \\ 4 & 4 & \end{array}\right\} + \left\{\begin{array}{ccc} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & 4 & \end{array}\right\} = 0, \end{aligned} \qquad (4.2.15d)$$

from which the Column relations give:

$$\left\{\begin{array}{ccc} 1 & 2 & 2 \\ 4 & 3 & 3 \\ 5 & 4 & \end{array}\right\} = - \left\{\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 4 & 5 & \end{array}\right\} + \left\{\begin{array}{ccc} 1 & 2 & 2 \\ 3 & 4 & 3 \\ 4 & 5 & \end{array}\right\}. \qquad (4.2.15e)$$

Therefore, from (4.2.15c):

$$\{T^\lambda\} = - \left\{\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 4 & 5 & \end{array}\right\}. \qquad (4.2.15f)$$

Note that this identity differs from (4.2.15a) only in that two columns have been interchanged. Although this column interchange relationship between symmetrised tableaux is easily obtained from the definition of $\{T^\lambda\}$, it is superfluous to requirements since the Column and Garnir relations are sufficient to obtain the required expansion in terms of $GL(m)$-standard tableaux. It may, however, reduce the number of iterations required to produce that expression.

Now consider the term on the right side of (4.2.15f). Let $\mathcal{X} = \{5,6\}$ and $\mathcal{Y} = \{7,8\}$. The Garnir relations then give, on ignoring those terms with entries repeated within a column:

$$\left\{\begin{matrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 4 & 5 & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 3 & 5 \\ 2 & 2 & 3 \\ 4 & 4 & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 3 & 4 \\ 2 & 2 & 3 \\ 4 & 5 & \end{matrix}\right\} = 0, \tag{4.2.15g}$$

whereupon, from (4.2.15f):

$$\{T^\lambda\} = \left\{\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 4 & \end{matrix}\right\} - \left\{\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & \end{matrix}\right\}, \tag{4.2.15h}$$

the required expansion in terms of $GL(5)$-tableaux.

Theorem 4.2.16. [**JK81**]. *The set*

$$\{\,\{T^\lambda\}\ : T^\lambda \textit{ is } GL(m)\textit{-standard}\}$$

constitutes a basis for the irreducible $GL(m)$-module W^λ.

Proof. The existence of the standardisation algorithm given above implies that this set spans W^λ. Thus it is sufficient to demonstrate linear independence. To do this, the following order, which differs from that given by Definition 2.6.8, is introduced on the set of all tableaux. Let t_u^b be the sum of the entries in the bth row of T_u^λ for $b = 1, 2, \ldots, q$, where $q = \tilde{\lambda}_1$. Let $|T_u^\lambda|'$ be the equivalence class of all tableaux which have their sequences of row sums identical to that of T_u^λ; that is $T_v^\lambda \in |T_u^\lambda|'$ if $t_v^b = t_u^b$ for $b = 1, 2, \ldots, q$. A total order on the set of these equivalence classes of tableaux is defined by $|T_u^\lambda|' > |T_v^\lambda|'$ if for some $k \le q$, $t_u^k > t_v^k$ with $t_u^b = t_v^b$ for each $b = k+1, k+2, \ldots, q$.

Let $\rho \in \mathcal{R}^\lambda$ and $\sigma \in \mathcal{C}^\lambda$. Since the action of ρ_* on T^λ leaves the elements of T^λ in their original rows, $\rho_* T^\lambda \in |T^\lambda|'$. If T^λ is $GL(m)$-standard then $|\sigma_* T^\lambda|' \le |T^\lambda|'$ since the action of σ_* only serves to move smaller entries down the columns. The inequality here is strict if $\sigma \ne I$. Let the $GL(m)$-standard tableaux be labelled

$$T^\lambda_{1,1}, T^\lambda_{1,2}, \ldots, T^\lambda_{1,\kappa_1}, T^\lambda_{2,1}, T^\lambda_{2,2}, \ldots, T^\lambda_{2,\kappa_2}, \ldots, T^\lambda_{r,\kappa_r}, \tag{4.2.16a}$$

such that:

$$\left|T^\lambda_{s,1}\right|' = \left|T^\lambda_{s,2}\right|' = \cdots = \left|T^\lambda_{s,\kappa_s}\right|', \tag{4.2.16b}$$

for $1 \le s \le r$, and such that:

$$\left|T^\lambda_{1,1}\right|' < \left|T^\lambda_{2,1}\right|' < \left|T^\lambda_{3,1}\right|' < \cdots < \left|T^\lambda_{r,1}\right|'. \tag{4.2.16c}$$

It is required to show that if:

$$\sum_{i=1}^{r}\sum_{j=1}^{\kappa_i} k_{i,j}\{T^\lambda_{i,j}\} = 0, \tag{4.2.16d}$$

where each $k_{i,j} \in \mathsf{F}$, then each $k_{i,j} = 0$. If this is not the case, there exist a and b such that $k_{a,b} \neq 0$ with $k_{a,j} = 0$ for $1 \leq j < b$ and each $k_{i,j} = 0$ for $i < a$. Thus:

$$\begin{aligned} 0 &= \sum_{j=b}^{\kappa_a} k_{a,j} P^\lambda_* Q^\lambda_* T^\lambda_{a,j} + \sum_{i=a+1}^{r}\sum_{j=1}^{\kappa_i} k_{i,j} P^\lambda_* Q^\lambda_* T^\lambda_{i,j} \\ &= \sum_{j=b}^{\kappa_a} k_{a,j} P^\lambda_* T^\lambda_{a,j} + \sum_{j=b}^{\kappa_a}\sum_{\sigma\in\mathcal{C}\backslash\{I\}} (-1)^\sigma k_{a,j} P^\lambda_* \sigma_* T^\lambda_{a,j} + \sum_{i=a+1}^{r}\sum_{j=1}^{\kappa_i} k_{i,j} P^\lambda_* Q^\lambda_* T^\lambda_{i,j} \end{aligned}$$

In view of (4.2.16b) and (4.2.16c), all the tableaux T^λ comprising the third term are such that $|T^\lambda|' > \left|T^\lambda_{a,b}\right|'$. In addition, since $\left|\sigma_* T^\lambda_{a,j}\right|' > \left|T^\lambda_{a,j}\right|'$ for each $\sigma \in \mathcal{C}\backslash\{I\}$, all the tableaux T^λ comprising the second term are such that $|T^\lambda|' > \left|T^\lambda_{a,b}\right|'$. Therefore, since each tableau is uniquely identified with a basis element of $V^{\otimes l}$, it follows that:

$$\sum_{j=b}^{\kappa_a} k_{a,j} P^\lambda_* T^\lambda_{a,j} = 0. \tag{4.2.16e}$$

Since the tableaux $T^\lambda_{a,b}, T^\lambda_{a,b+1}, \ldots, T^\lambda_{a,\kappa_a}$ are $GL(m)$-standard and distinct, it follows that the sets $\{\rho_* T^\lambda_{a,c} : \rho \in \mathcal{R}^\lambda\}$ each contain a single unique $GL(m)$-standard tableau $c = b, b+1, \ldots, \kappa_a$. It then follows from (4.2.16e) that $k_{a,b} = k_{a,b+1} = \cdots = k_{a,\kappa_a} = 0$. This contradicts $k_{a,b} \neq 0$ whereupon all the $k_{i,j}$ of (4.2.16d) are zero and the theorem is proved.

Let $\lambda \in P(l)$. From (4.2.2) and Lemma 4.2.8, the element $G \in GL(m)$ acts on $\{T^\lambda\} \in W^\lambda$ according to:

$$G\{T^\lambda\} = \sum_{T'^\lambda} G^{T'^\lambda_{(1)}}{}_{T^\lambda_{(1)}} G^{T'^\lambda_{(2)}}{}_{T^\lambda_{(2)}} \cdots G^{T'^\lambda_{(l)}}{}_{T^\lambda_{(l)}} \{T'^\lambda\}, \tag{4.2.17}$$

the sum being over all tableaux T'^λ with entries from the set $\mathcal{I}^{GL(m)}$. In order to determine the action of $E_a{}^b \in gl(m)$ on $\{T^\lambda\}$, let p be the number of times that the index b occurs in T^λ and form the set of p tableaux $\{T^\lambda_1, T^\lambda_2, \ldots, T^\lambda_p\}$ by, in each case, replacing a single index b in T^λ with a. Then, using (1.5.9) and Lemma 4.2.8,

$$E_a{}^b\{T^\lambda\} = \sum_{i=1}^{p}\{T^\lambda_i\}. \tag{4.2.18}$$

For example:

$$E_2{}^5 \left\{\begin{matrix} 1 & 5 & 6 \\ 2 & 4 & \\ 5 & 3 & \end{matrix}\right\} = \left\{\begin{matrix} 1 & 5 & 6 \\ 2 & 4 & \\ 2 & 3 & \end{matrix}\right\} + \left\{\begin{matrix} 1 & 2 & 6 \\ 2 & 4 & \\ 5 & 3 & \end{matrix}\right\}. \tag{4.2.19}$$

Note that unless the index b appears in T^λ, then $E_a{}^b\{T^\lambda\} = 0$.

Quintessentially, the Weyl module is as follows.

Theorem 4.2.20. *Let $\lambda \in P(l)$. The Weyl module W^λ is the irreducible $GL(m)$-module spanned by $\{T^\lambda\}$ for all T^λ with entries from the set $\mathcal{I}^{GL(m)}$, modulo relations* (3.4.2) *and* (3.4.3), *and on which $GL(m)$ and $gl(m)$ act according to* (4.2.17) *and* (4.2.18) *respectively.*

This theorem effectively provides a definition for W^λ.

Since the symmetrised $GL(m)$-standard tableaux constitute a basis for W^λ, explicit representation matrices are readily obtained from the actions of $GL(m)$ and $gl(m)$ on these tableaux. Let s^λ be the dimension of W^λ and $T_1^\lambda, T_2^\lambda, \ldots, T_{s^\lambda}^\lambda$ the $GL(m)$-standard tableaux. The action of $G \in GL(m)$ on each $\{T_i^\lambda\}$ yields, according to (4.2.17), a linear combination of, in general, non-standard tableaux. By using the techniques of this section, each may be written in terms of the $GL(m)$-standard tableaux so that:

$$G\{T_i^\lambda\} = \sum_{j=1}^{s^\lambda} \Gamma^{\{\lambda\}}(G)_{ji}\{T_j^\lambda\}, \tag{4.2.21}$$

for some set of numbers $\Gamma^{\{\lambda\}}(G)_{ji} \in \mathsf{F}$. These are the elements of the matrix $\Gamma^{\{\lambda\}}(G)$ which represents G in the representation $\{\lambda\}$. In a similar way, the representation matrix $\Gamma^{\{\lambda\}}(E)$ of $E \in gl(m)$ is given, via (4.2.18), by

$$E\{T_i^\lambda\} = \sum_{j=1}^{s^\lambda} \Gamma^{\{\lambda\}}(E)_{ji}\{T_j^\lambda\}. \tag{4.2.22}$$

As an example, consider the 15-dimensional representation $\{3,1\}$ of $GL(3)$. Here, the $GL(3)$-standard tableaux are given by:

$$\begin{array}{lllll}
\begin{array}{lll} 2&3&3\\ 3&& \end{array}, & \begin{array}{lll} 1&3&3\\ 3&& \end{array}, & \begin{array}{lll} 2&2&3\\ 3&& \end{array}, & \begin{array}{lll} 1&2&3\\ 3&& \end{array}, & \begin{array}{lll} 1&1&3\\ 3&& \end{array}, \\
\begin{array}{lll} 2&2&2\\ 3&& \end{array}, & \begin{array}{lll} 1&2&2\\ 3&& \end{array}, & \begin{array}{lll} 1&1&2\\ 3&& \end{array}, & \begin{array}{lll} 1&1&1\\ 3&& \end{array}, & \begin{array}{lll} 1&3&3\\ 2&& \end{array}, \\
\begin{array}{lll} 1&2&3\\ 2&& \end{array}, & \begin{array}{lll} 1&1&3\\ 2&& \end{array}, & \begin{array}{lll} 1&2&2\\ 2&& \end{array}, & \begin{array}{lll} 1&1&2\\ 2&& \end{array}, & \begin{array}{lll} 1&1&1\\ 2&& \end{array}.
\end{array} \tag{4.2.23}$$

From (4.2.18), the element $E_1{}^3$ acts on the symmetrised counterpart of the first of these according to:

$$E_1{}^3\left\{\begin{array}{lll} 2&3&3\\ 3&& \end{array}\right\} = \left\{\begin{array}{lll} 2&3&3\\ 1&& \end{array}\right\} + \left\{\begin{array}{lll} 2&1&3\\ 3&& \end{array}\right\} + \left\{\begin{array}{lll} 2&3&1\\ 3&& \end{array}\right\}. \tag{4.2.24}$$

Then:

$$\left\{\begin{matrix}2&3&3\\1&&\end{matrix}\right\}=-\left\{\begin{matrix}1&3&3\\2&&\end{matrix}\right\},\quad\text{from (3.4.2),}$$

$$\left\{\begin{matrix}2&1&3\\3&&\end{matrix}\right\}=\left\{\begin{matrix}1&2&3\\3&&\end{matrix}\right\}-\left\{\begin{matrix}1&3&3\\2&&\end{matrix}\right\},\quad\text{from (3.4.3), and}$$

$$\left\{\begin{matrix}2&3&1\\3&&\end{matrix}\right\}=\left\{\begin{matrix}2&1&3\\3&&\end{matrix}\right\}=\left\{\begin{matrix}1&2&3\\3&&\end{matrix}\right\}-\left\{\begin{matrix}1&3&3\\2&&\end{matrix}\right\},\quad\text{from (3.4.3).}$$

Thereupon, (4.2.24) implies:

$$E_1{}^3\left\{\begin{matrix}2&3&3\\3&&\end{matrix}\right\}=2\left\{\begin{matrix}1&2&3\\3&&\end{matrix}\right\}-3\left\{\begin{matrix}1&3&3\\2&&\end{matrix}\right\}. \tag{4.2.25}$$

Similar calculations, when carried out for each of the tableaux of (4.2.23), yield the representation matrix:

$$\Gamma^{\{3,1\}}(E_a{}^b)=\begin{pmatrix}
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
-3&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&-2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot
\end{pmatrix},$$

where each zero has been replaced by a dot. The identity (4.2.25) is manifest as the first column of this matrix.

By using a computer implementation of the techniques presented in this section, representation matrices have been obtained for each of the basis elements of $gl(m)$ in a number of $GL(m)$-modules W^λ. It has been checked that the commutation relations satisfied by the basis elements of $gl(m)$ are also satisfied by the matrices obtained through the methods of this section. That is, from (2.2.2), that:

$$[\Gamma^{\{\lambda\}}(E_a{}^b),\Gamma^{\{\lambda\}}(E_c{}^d)]=\delta^b_c\,\Gamma^{\{\lambda\}}(E_a{}^d)+\delta^d_a\,\Gamma^{\{\lambda\}}(E_c{}^b). \tag{4.2.26}$$

This provides a verification that the matrices produced actually constitute a representation of $gl(m)$.

Definition 4.2.27. *$GL(m)$-weight. Let $n_i^{GL(m)}(T^\lambda)$ be the number of appearances of the index i in T^λ. The vector $n^{GL(m)}(T^\lambda) = (n_1^{GL(m)}(T^\lambda), n_2^{GL(m)}(T^\lambda), \ldots, n_m^{GL(m)}(T^\lambda))$ is known as the $GL(m)$-weight of T^λ.*

Note that the tableaux present in each Column relation and each Garnir relation have identical $GL(m)$-weights. This implies that a symmetrised tableaux is a linear combination of $GL(m)$-standard tableaux of the same $GL(m)$-weight. This observation is used below.

From (4.2.18) the action of the elements ${E_a}^a$ on $\{T^\lambda\}$ for $a = 1, 2, \ldots, m$, give:

$$ {E_a}^a\{T^\lambda\} = n_a^{GL(m)}(T^\lambda)\{T^\lambda\}. \qquad (4.2.28) $$

Since the elements ${E_a}^a$ for $a = 1, 2, \ldots, m$, form a basis for the Cartan subalgebra of $gl(m)$, the $GL(m)$-weight $n^{GL(m)}(T^\lambda)$ of T^λ determines the weight of the element $\{T^\lambda\} \in W^\lambda$ in this basis.

With $T^\lambda_>$ as in Definition 2.6.6, $n^{GL(m)}(T^\lambda_>) = (\lambda_1, \lambda_2, \ldots, \lambda_m) = \lambda$ and is the unique $GL(m)$-standard tableau of shape F^λ for which this is so. If $a < b$ then

$$ {E_a}^b\{T^\lambda_>\} = 0, \qquad (4.2.29) $$

since each term resulting from the right side of (4.2.18) will necessarily have an entry repeated in some column. $T^\lambda_>$ is the only $GL(m)$-standard tableau with this property. The set $\{{E_a}^b : a, b \in \mathcal{I}^{GL(m)}, a < b\}$ is a basis for $B_+^{GL(m)}$. Thus (4.2.29) shows that $\{T^\lambda_>\}$ is the unique highest weight of the $GL(m)$-module W^λ.

Theorem 4.2.30. *The character of the irreducible representation $\{\lambda\}$ of $GL(m)$ derived from W^λ is:*

$$ \{\lambda\}(x) = \sum_{T^\lambda:GL(m)-\text{standard}} x^{T^\lambda}, \qquad (4.2.30) $$

for the class(es) of $GL(m)$ with eigenvalues $x_1, x_2, \ldots, x_m$, where (x) denotes the vector $(x_1, x_2, \ldots, x_m)$ and $x^{T^\lambda} = x_1^{n_1^{GL(m)}(T^\lambda)} x_2^{n_2^{GL(m)}(T^\lambda)} \cdots x_m^{n_m^{GL(m)}(T^\lambda)}$.

Proof. This theorem is proved using the Jordan normal form G' of the matrix $G \in GL(m)$. Since G' is equivalent to G, the representation matrices $\Gamma^{\{\lambda\}}(G)$ and $\Gamma^{\{\lambda\}}(G')$ have the same trace. Let the eigenvalues of G be labelled $x_1, x_2, \ldots, x_m$. These appear along the diagonal of G' and, if distinct, G' has zeros elsewhere. In such a case the set of eigenvalues specifies a unique class of $GL(m)$, to which both G and G' belong. If G has a repeated eigenvalue, then G' may, in addition to the eigenvalues on the main diagonal, possess non-zero entries on the diagonal immediately above. Therefore, in this case, the set of eigenvalues do not determine a unique class within $GL(m)$. Consider the action of G' on a symmetrised $GL(m)$-standard tableau $\{T^\lambda\}$. When G' is purely diagonal, (4.2.17) implies that:

$$G' : \{T^\lambda\} = x_1^{n_1^{GL(m)}(T^\lambda)} x_2^{n_2^{GL(m)}(T^\lambda)} \cdots x_m^{n_m^{GL(m)}(T^\lambda)} \{T^\lambda\}, \tag{4.2.30a}$$

In the case where G' is not diagonal, other symmetrised tableaux will appear on the right side of (4.2.17). These tableaux will have weights different to that of T^λ and therefore, upon standardisation, will not alter the coefficient of $\{T^\lambda\}$. Thus in both cases the $GL(m)$-standard tableau T^λ contributes $x_1^{n_1^{GL(m)}(T^\lambda)} x_2^{n_2^{GL(m)}(T^\lambda)} \cdots x_m^{n_m^{GL(m)}(T^\lambda)}$ to the trace of the representation matrix $\Gamma^{\{\lambda\}}(G')$. Summing over all $GL(m)$-standard tableaux then yields (4.2.30).

As in the statement of Theorem 4.2.30, it is conventional to use the same symbol, in this case $\{\lambda\}$, to denote both the representation and its character. This theorem implies that s^λ, the number of $GL(m)$-standard tableaux of shape F^λ, is given by $D_m\{\lambda\}$ as in (2.5.5a).

The function $\{\lambda\}(x)$ defined by (4.2.30) is known as a Schur function or S-function (see **[Sta71]**, where various ways of defining $\{\lambda\}$ are considered). Each S-function is a symmetric function in its arguments and the ring of symmetric functions has a basis comprising all S-functions **[Ma79]**. The S-functions feature prominently in the representation theory of the classical groups (see **[Li50,Ro61,Ki75, BK83,Ki89]**, for example).

§4.3. Symplectic group modules and trace tensors

This section expounds the techniques used by Berele **[Be86]** in using Young tableaux to construct irreducible $Sp(2r)$-modules. However, the presentation given here differs substantially from that given in **[Be86]**. This is so that when these techniques are extended and applied to obtain the irreducible modules of other classical groups, the parallels between them are readily apparent.

Since $Sp(2r)$ is a subgroup of $GL(2r)$, the $GL(2r)$-module W^λ also serves as an $Sp(2r)$-module. However, W^λ is not, in general, an irreducible $Sp(2r)$-module. This is due to the existence of trace tensors (defined below) which need to be removed in order to obtain the irreducible $Sp(2r)$-modules present in $V^{\otimes l}$.

Fix r and let $J = J_{2r}^-$, as given by (2.1.1a). Let V be the defining $Sp(2r)$-module with basis $\{e_i : i \in \mathcal{I}^{Sp(2r)}\}$. Then for all $G \in Sp(2r)$, $\tilde{G}JG = J$ whereupon the tensor:

$$\sum_{j,k \in \mathcal{I}^{Sp(2r)}} J_{jk} e_j \otimes e_k = \sum_{i=1}^{r} (e_i \otimes e_{\bar{i}} - e_{\bar{i}} \otimes e_i), \tag{4.3.1}$$

is preserved.

Definition 4.3.2. *A trace tensor of $V^{\otimes l}$ is any linear combination of terms of the form:*

$$\sum_{i=1}^{r}(x \otimes e_i \otimes y \otimes e_{\bar{\imath}} \otimes z - x \otimes e_{\bar{\imath}} \otimes y \otimes e_i \otimes z), \qquad (4.3.2)$$

where x, y and z are elements of some (possibly zero) tensor power of V and $x \otimes y \otimes z \in V^{\otimes(l-2)}$. Define $U^{Sp(2r)} \subset V^{\otimes l}$ to be the span of all such trace tensors.

The preservation of (4.3.1) under the action of $Sp(2r)$ implies that $U^{Sp(2r)}$ is invariant under the action of $Sp(2r)$. Since $V^{\otimes l}$ is completely reducible [**We39**], it follows that $V^{\otimes l}/U^{Sp(2r)}$ is isomorphic to a subspace of $V^{\otimes l}$ which is invariant under the action of $Sp(2r)$. Therefore $B^\lambda = W^\lambda/(W^\lambda \cap U^{Sp(2r)})$ is an $Sp(2r)$-submodule of W^λ.

Let $\langle T^\lambda \rangle$ denote the traceless symmetrised tableau resulting from the removal of all trace terms (4.3.2) from the symmetrised tableau $\{T^\lambda\}$, by forming its quotient with respect to the elements of $U^{Sp(2r)}$. B^λ is therefore spanned by all $\langle T^\lambda \rangle$ where the entries of each T^λ are from the set $\mathcal{I}^{Sp(2r)}$.

Lemma 4.3.3. *Let T_i^λ, for $i = 1, 2, \ldots, r$, be r tableaux, identical except for the entries in two positions in the cth column where $T^\lambda_{i(a,c)} = \bar{\imath}$ and $T^\lambda_{i(b,c)} = i$ for some fixed a, b and $c \leq \lambda_1$ with $1 \leq a, b \leq \tilde{\lambda}_c$. Then:*

$$\sum_{i=1}^{r} \langle T_i^\lambda \rangle = 0. \qquad (4.3.3)$$

Proof. For $i = 1, 2, \ldots, r$, let $T_{\bar{\imath}}^\lambda$ be identical to T_i^λ apart from the transposition of the entries i and $\bar{\imath}$ in the cth column. Since $\sum_{i=1}^{r}(T_i^\lambda - T_{\bar{\imath}}^\lambda) \in U^{Sp(2r)}$ and the action by place permutation of each summand of the Young symmetriser, Y_*^λ (3.3.13c), only serves to give similar terms in $U^{Sp(2r)}$ with appropriate changes of the positions (a, c) and (b, c), it follows that $\sum_{i=1}^{r}(\{T_i^\lambda\} - \{T_{\bar{\imath}}^\lambda\}) \in U^{Sp(2r)}$. The identity $\{T_i^\lambda\} = -\{T_{\bar{\imath}}^\lambda\}$, then implies that $\sum_{i=1}^{r}\{T_i^\lambda\} \in U^{Sp(2r)}$, whereupon (4.3.3) follows from the definition of $\langle T^\lambda \rangle$ as a quotient.

The following lemma, despite its technical appearance, is a straightforward consequence of Lemma 4.3.3, being obtained by the simultaneous application of the trace condition over a number of index pairs. In the context of trace removal techniques, it is a generalisation of a result that appears in [**Be86**], albeit in a vastly different form. The proof is based on the techniques used in [**Be86**]. The elements of the index set $\mathcal{I}^{Sp(2r)}$ are ordered according to: $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r$.

Lemma 4.3.4. *Let k be such that $1 \leq k \leq \lambda_1$. Let $\mathsf{N}_r = \mathcal{B}^\alpha \cup \mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$ be a union of disjoint sets such that, with $b^\alpha = \#\mathcal{B}^\alpha$, $b^\beta = \#\mathcal{B}^\beta$, $e = \#\mathcal{E}$, $g = \#\mathcal{G}$, $h = \#\mathcal{H}$ and $d > g$, $\tilde{\lambda}_k = b^\alpha + b^\beta + 2e + 2d$. Let $\mathcal{D}_w$, for various w, run over all distinct $\binom{h}{d}$ subsets of $\mathcal{H}$ of cardinality d, and let the tableaux T^λ_w be identical apart from column k which contains entries from the set $\mathcal{B}^\alpha \cup \mathcal{E} \cup \mathcal{D}_w \cup \overline{\mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{D}_w}$. If the indices from the set $\mathcal{B}^\alpha \cup \mathcal{E} \cup \overline{\mathcal{B}^\beta \cup \mathcal{E}}$ are in the same positions in each T^λ_w and the indices from $\overline{\mathcal{D}_w} \cup \mathcal{D}_w$ are in column strict order, then:*

$$\sum_w \langle T^\lambda_w \rangle = 0. \tag{4.3.4}$$

Proof. For $\langle T^\lambda_w \rangle$ write the column k of T^λ_w as a product, θ_w, of elements of $\mathcal{I}^{Sp(2r)}$. For example, if $k = 2$ and

$$T^\lambda = \begin{array}{cccc} 1 & \bar{1} & \bar{2} & 3 \\ \bar{2} & \bar{2} & \bar{3} & \\ \bar{3} & 2 & & \\ 3 & \bar{3} & & \end{array},$$

then $\langle T^\lambda \rangle$ gives rise to $\theta = \bar{1}\bar{2}23$. By virtue of (3.4.2), interchanging elements of θ changes the sign of θ, and the presence of an identical pair of elements implies that $\theta = 0$. In this notation, (4.3.4) may be proved by showing that:

$$\sum_w \theta_w = 0. \tag{4.3.4a}$$

Let $\omega_i = \bar{i}i$. The trace equation, (4.3.3), implies that:

$$\sum_{i \in \mathsf{N}_r} \omega_i = 0. \tag{4.3.4b}$$

With $\mathcal{B} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta$, split this identity according to:

$$\sum_{i \in \mathcal{H} \cup \mathcal{B} \cup \mathcal{E}} \omega_i = -\sum_{i \in \mathcal{G}} \omega_i. \tag{4.3.4c}$$

Since $d > g$, on raising each side of this identity to the power of d, the right side is annihilated by virtue of repeated terms, giving:

$$\left(\sum_{i \in \mathcal{H} \cup \mathcal{B} \cup \mathcal{E}} \omega_i \right)^d = 0. \tag{4.3.4d}$$

This implies that:

$$\sum_{\substack{\gamma_1 < \gamma_2 < \cdots < \gamma_d \\ \gamma_1, \gamma_2, \ldots, \gamma_d \in \mathcal{H} \cup \mathcal{B} \cup \mathcal{E}}} \omega_{\gamma_1} \omega_{\gamma_2} \cdots \omega_{\gamma_d} = 0, \tag{4.3.4e}$$

whereupon, on setting $\theta^{\mathcal{B}} = \prod_{i\in\mathcal{B}^\beta}\bar{i}\prod_{i\in\mathcal{B}^\alpha} i$ and $\theta^{\mathcal{E}} = \prod_{i\in\mathcal{E}}\omega_i$, multiplication by $\theta^{\mathcal{B}}\theta^{\mathcal{E}}$ annihilates those terms featuring ω_i for $i \in \mathcal{B}\cup\mathcal{E}$ due to a repeated index. Therefore:

$$\theta^{\mathcal{B}}\theta^{\mathcal{E}} \sum_{\substack{\gamma_1<\gamma_2<\cdots<\gamma_d \\ \gamma_1,\gamma_2,\ldots,\gamma_d\in\mathcal{H}}} \omega_{\gamma_1}\omega_{\gamma_2}\cdots\omega_{\gamma_d} = 0, \tag{4.3.4f}$$

and hence:

$$\theta^{\mathcal{B}}\theta^{\mathcal{E}} \sum_w \theta^{\mathcal{D}}_w = 0, \tag{4.3.4g}$$

where $\theta^{\mathcal{D}}_w = \prod_{i\in\mathcal{D}_w}\omega_i$. Let $\theta'_w = \theta^{\mathcal{B}}\theta^{\mathcal{E}}\theta^{\mathcal{D}}_w$, so that then $\sum_w \theta'_w = 0$. With the indices as specified in the statement of the Lemma, the application of an identical permutation to the factors of each θ'_w produces θ_w. Therefore $\theta'_w = \pm\theta_w$ with the sign being independent of w. Thus (4.3.4g) is equivalent to (4.3.4a) and the Lemma is proved.

As a simple example, consider the case where $r = 4$ and $\tilde{\lambda}_k = 4$ for some k. Let $\mathcal{B}^\alpha = \mathcal{B}^\beta = \mathcal{E} = \emptyset$, $\mathcal{G} = \{1\}$, $\mathcal{H} = \{2,3,4\}$ and $d = 2$. Then $b^\alpha = b^\beta = e = 0$, $g = 1$ and $h = 3$ so that $d > g$, and $\tilde{\lambda}_k = b^\alpha + b^\beta + 2e + 2d$ as required by Lemma 4.3.4. In this particular case (4.3.4c) becomes:

$$\omega_2 + \omega_3 + \omega_4 = -\omega_1. \tag{4.3.5a}$$

As in (4.3.4d), raising this expression to the power of $d = 2$ annihilates the right side, whereupon, as in (4.3.4e):

$$\omega_2\omega_3 + \omega_2\omega_4 + \omega_3\omega_4 = 0, \tag{4.3.5b}$$

with all other terms zero due to repeated factors. Since $\mathcal{B} = \mathcal{E} = \emptyset$, $\theta^{\mathcal{B}}\theta^{\mathcal{E}}$ is the unit element, and this expression is, as in (4.3.4g):

$$\theta^{\mathcal{B}}\theta^{\mathcal{E}} \sum_{w=1}^{3} \theta^{\mathcal{D}}_w = 0, \tag{4.3.5c}$$

where the $\binom{h}{d} = 3$ terms $\theta^{\mathcal{D}}_1 = \omega_2\omega_3$, $\theta^{\mathcal{D}}_2 = \omega_2\omega_4$ and $\theta^{\mathcal{D}}_3 = \omega_3\omega_4$ respectively correspond to the subsets $\mathcal{D}_1 = \{2,3\}$, $\mathcal{D}_2 = \{2,4\}$ and $\mathcal{D}_3 = \{3,4\}$ of $\mathcal{H}$. Setting $\theta'_w = \theta^{\mathcal{B}}\theta^{\mathcal{E}}\theta^{\mathcal{D}}_w$ gives $\sum_w \theta'_w = 0$. In this example the terms are explicitly $\theta'_1 = \theta^{\mathcal{B}}\theta^{\mathcal{E}}\theta^{\mathcal{D}}_1 = \omega_2\omega_3 = \bar{2}2\bar{3}3$, $\theta'_2 = \bar{2}2\bar{4}4$ and $\theta'_3 = \bar{3}3\bar{4}4$ so that:

$$\bar{2}2\bar{3}3 + \bar{2}2\bar{4}4 + \bar{3}3\bar{4}4 = 0. \tag{4.3.5d}$$

This leads to, for instance, the following tableaux identity in which each term θ'_w in the above expression is identified with θ_w arising from the corresponding traceless symmetrised tableaux:

$$\left\langle\begin{array}{ll}\bar{2} & 2\\ 2 & \\ \bar{3} & \\ 3 & \end{array}\right\rangle+\left\langle\begin{array}{ll}\bar{2} & 2\\ 2 & \\ \bar{4} & \\ 4 & \end{array}\right\rangle+\left\langle\begin{array}{ll}\bar{3} & 2\\ 3 & \\ \bar{4} & \\ 4 & \end{array}\right\rangle=0. \qquad (4.3.5e)$$

For a more extensive example, consider the case where $r=9$ and $\tilde{\lambda}_k=9$ for some k. Let $\mathcal{B}^\alpha=\emptyset$, $\mathcal{B}^\beta=\{4\}$, $\mathcal{E}=\{7\}$, $\mathcal{G}=\{1,3\}$, $\mathcal{H}=\{2,5,6,8,9\}$ and $d=3$. Then $b^\alpha=0$, $b^\beta=1$, $e=1$, $g=2$ and $h=5$ so that $d>g$, and $\tilde{\lambda}_k=b^\alpha+b^\beta+2e+2d$ as required by Lemma 4.3.4. In this particular case (4.3.4c) becomes:

$$\omega_2+\omega_4+\omega_5+\omega_6+\omega_7+\omega_8+\omega_9=\;-\omega_1-\omega_3. \qquad (4.3.6a)$$

Raising this expression to the power of $d=3$ annihilates the right side, while the left side yields $\binom{7}{3}=35$ non-zero terms as in (4.3.4e). Of these, all but $\binom{5}{3}=10$ are annihilated on mutliplication by $\theta^{\mathcal{B}}\theta^{\mathcal{E}}=\bar{4}\bar{7}7$, whereupon, as in (4.3.4f):

$$\begin{aligned}&\bar{4}\bar{7}7\omega_2\omega_5\omega_6+\bar{4}\bar{7}7\omega_2\omega_5\omega_8+\bar{4}\bar{7}7\omega_2\omega_5\omega_9+\bar{4}\bar{7}7\omega_2\omega_6\omega_8+\bar{4}\bar{7}7\omega_2\omega_6\omega_9\\&\quad+\bar{4}\bar{7}7\omega_2\omega_8\omega_9+\bar{4}\bar{7}7\omega_5\omega_6\omega_8+\bar{4}\bar{7}7\omega_5\omega_6\omega_9+\bar{4}\bar{7}7\omega_5\omega_8\omega_9+\bar{4}\bar{7}7\omega_6\omega_8\omega_9=0.\end{aligned} \qquad (4.3.6b)$$

Here the ten terms correspond to the ten subsets $\mathcal{D}_w$ of $\mathcal{H}$ of cardinality 3. The first corresponds to $\mathcal{D}_1=\{2,5,6\}$ while the last corresponds to $\mathcal{D}_{10}=\{6,8,9\}$. Expanding $w_i=\bar{i}i$ for each term and rearranging gives:

$$\begin{aligned}&\bar{2}2\bar{4}\bar{5}5\bar{6}6\bar{7}7+\bar{2}2\bar{4}\bar{5}5\bar{7}7\bar{8}8+\bar{2}2\bar{4}\bar{5}5\bar{7}7\bar{9}9+\bar{2}2\bar{4}\bar{6}6\bar{7}7\bar{8}8+\bar{2}2\bar{4}\bar{6}6\bar{7}7\bar{9}9\\&\quad+\bar{2}2\bar{4}\bar{7}7\bar{8}8\bar{9}9+\bar{4}\bar{5}5\bar{6}6\bar{7}7\bar{8}8+\bar{4}\bar{5}5\bar{6}6\bar{7}7\bar{9}9+\bar{4}\bar{5}5\bar{7}7\bar{8}8\bar{9}9+\bar{4}\bar{6}6\bar{7}7\bar{8}8\bar{9}9=0.\end{aligned} \qquad (4.3.6c)$$

If $\lambda=(1^9)$, this results in the following tableaux identity:

$$\left\langle\begin{array}{c}\bar{2}\\2\\\bar{4}\\\bar{5}\\5\\\bar{6}\\6\\\bar{7}\\7\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{2}\\2\\\bar{4}\\\bar{5}\\5\\\bar{7}\\7\\\bar{8}\\8\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{2}\\2\\\bar{4}\\\bar{5}\\5\\\bar{7}\\7\\\bar{9}\\9\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{2}\\2\\\bar{4}\\\bar{6}\\6\\\bar{7}\\7\\\bar{8}\\8\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{2}\\2\\\bar{4}\\\bar{6}\\6\\\bar{7}\\7\\\bar{9}\\9\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{2}\\2\\\bar{4}\\\bar{7}\\7\\\bar{8}\\8\\\bar{9}\\9\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{4}\\\bar{5}\\5\\\bar{6}\\6\\\bar{7}\\7\\\bar{8}\\8\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{4}\\\bar{5}\\5\\\bar{6}\\6\\\bar{7}\\7\\\bar{9}\\9\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{4}\\\bar{5}\\5\\\bar{7}\\7\\\bar{8}\\8\\\bar{9}\\9\end{array}\right\rangle+\left\langle\begin{array}{c}\bar{4}\\\bar{6}\\6\\\bar{7}\\7\\\bar{8}\\8\\\bar{9}\\9\end{array}\right\rangle=0. \qquad (4.3.6d)$$

Standard tableaux for representations of the symplectic group were first obtained by King **[Ki76]** to provide a convenient means of obtaining weights and characters of these representations.

Definition 4.3.7. [Ki76] *The tableau T^λ is $Sp(2r)$-standard if and only if:*

(i) the entries are taken from the set $\mathcal{I}^{Sp(2r)}$;

(ii) the entries are strictly increasing from top to bottom down each column;

(iii) the entries are non-decreasing from left to right across each row;

(iv) $T^\lambda_{(i)} \geq \bar{i}$ for $i = 1, 2, \ldots, \tilde{\lambda}_1$.

Note that each $Sp(2r)$-standard tableau is $GL(2r)$-standard if each entry $\bar{a}$ is exchanged for $2a-1$ and a is exchanged for $2a$. Also note that since $T^\lambda_{(i)}$ is the first entry in the ith row for $i = 1, 2, \ldots, \tilde{\lambda}_1$, it follows from condition (iii) that $T^\lambda_{(i,j)} \geq \bar{i}$ for each $j = 1, 2, \ldots, \lambda_i$. Finally note that if $\tilde{\lambda}_1 > r$, conditions (i) and (iv) imply that there exist no $Sp(2r)$-standard tableaux of shape F^λ.

The techniques of Section 4.2 may be applied to the case of the $Sp(2r)$-module B^λ, once an appropriate order is provided on tableaux with entries from the set $\mathcal{I}^{Sp(2r)}$. This is given by, once more, mapping $\bar{a} \in \mathcal{I}^{Sp(2r)}$ to $2a-1$ and $a \in \mathcal{I}^{Sp(2r)}$ to $2a$, and then using Definition 2.6.8. The column relations can then be used to write any traceless symmetrised tableau in terms of a column strict tableau while, if the column strict tableau T^λ violates condition (iii) of Definition 4.3.7, then the Garnir relations enable $\langle T^\lambda \rangle$ to be written in terms of higher tableaux. Violations of condition (iv) of Definition 4.3.7 are dealt with by the following lemma.

Lemma 4.3.8. *Let T^λ be a column strict tableau which is not $Sp(2r)$-standard in that $T^\lambda_{(j^+)} < \bar{j^+}$ for some j^+. Then $\langle T^\lambda \rangle$ may be expressed as a signed sum of traceless symmetrised tableaux $\langle T^\lambda_w \rangle$, where for each w, $T^\lambda_w > T^\lambda$.*

Proof. Let $k = 1$ and $\mathcal{Q} \subset \mathcal{I}^{Sp(2r)}$ be the set of indices in the first column of T^λ. Let $\mathcal{A} = \{i \in \mathsf{N}_r : \bar{i} \in \mathcal{Q}, i \in \mathcal{Q}\}$, $\mathcal{B}^\alpha = \{i \in \mathsf{N}_r : i \in \mathcal{Q}, \bar{i} \notin \mathcal{Q}\}$, $\mathcal{B}^\beta = \{i \in \mathsf{N}_r : \bar{i} \in \mathcal{Q}, i \notin \mathcal{Q}\}$, $\mathcal{C} = \{i \in \mathsf{N}_r : \bar{i} \notin \mathcal{Q}, i \notin \mathcal{Q}\}$ and $\mathcal{B} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta$. Then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are distinct with $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \mathsf{N}_r$, and if $a = \#\mathcal{A}$, $b = \#\mathcal{B}$ and $c = \#\mathcal{C}$, then $a + b + c = r$ and $\tilde{\lambda}_1 = 2a + b$. Let $j = j^+ - 1$ and $\mathcal{J} = \mathsf{N}_j$ so that $\#\mathcal{J} = j$. The sets created above are now split with respect to $\mathcal{J}$: $\mathcal{D} = \mathcal{A} \cap \mathcal{J}$, $\mathcal{E} = \mathcal{A} \backslash \mathcal{D}$, $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{J}$, $\mathcal{B}_1 = \mathcal{B} \backslash \mathcal{B}_0$, $\mathcal{G} = \mathcal{C} \cap \mathcal{J}$ and $\mathcal{F} = \mathcal{C} \backslash \mathcal{G}$. In addition let $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$. With the cardinalities of the sets just created d, e, b_0, b_1, g, f and h respectively, and the cardinalities of $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$, b^α and b^β respectively, then $d + e + b_0 + b_1 + g + f = r$, $h = d + f$, $\tilde{\lambda}_1 = b^\alpha + b^\beta + 2d + 2e$ and $d + b_0 + g = j$. The condition $T^\lambda_{(j^+)} < \bar{j^+}$ implies that $2d + b_0 > j$ since at least the first j^+ positions are filled with entries from the set $\overline{\mathcal{J}} \cup \mathcal{J}$. This implies that $d > g$. Therefore the conditions of Lemma 4.3.4 are satisfied and the identity:

$$\sum_{\mathcal{D}_w \subset \mathcal{H}} \langle T_w^\lambda \rangle = 0, \tag{4.3.8a}$$

follows, where the sum is over all $\binom{h}{d}$ distinct subsets $\mathcal{D}_w$ of $\mathcal{H}$ and T_w^λ is identical to T^λ apart from the indices from the set $\overline{\mathcal{D}} \cup \mathcal{D}$ in the first column of each portion, having been replaced by those from the set $\overline{\mathcal{D}_w} \cup \mathcal{D}_w$. Therefore:

$$\langle T^\lambda \rangle = - \sum_{\substack{\mathcal{D}_w \subset \mathcal{H} \\ \mathcal{D}_w \neq \mathcal{D}}} \langle T_w^\lambda \rangle. \tag{4.3.8b}$$

Since $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$, $\mathcal{D} \subset \mathcal{J}$ and $\mathcal{F} \cap \mathcal{J} = \emptyset$, each of the terms from the set $\mathcal{F}$ is higher than those from the set $\mathcal{D}$ and it follows that for each term on the right of (4.3.8*b*), $T_w^\lambda > T^\lambda$, thereby proving Lemma 4.3.8.

To illustrate this lemma, let $\lambda = (2, 1^3)$ and consider the $Sp(8)$-module B^λ. The tableau:

$$T^\lambda = \begin{matrix} \bar{2} & 2 \\ 2 & \\ \bar{3} & \\ 3 & \end{matrix} \quad , \tag{4.3.9a}$$

is not $Sp(8)$-standard since for $j^+ = 4$, $T^\lambda_{(j^+)} < \bar{j^+}$. For this case, the proof of Lemma 4.3.8 specifies the following sets: $\mathcal{Q} = \{\bar{2}, 2, \bar{3}, 3\}$ and hence $\mathcal{A} = \{2, 3\}$, $\mathcal{B}^\alpha = \emptyset$, $\mathcal{B}^\beta = \emptyset$, $\mathcal{B} = \emptyset$ and $\mathcal{C} = \{1, 4\}$. With $j = j^+ - 1 = 3$, $\mathcal{J} = \{1, 2, 3\}$. Then splitting the sets $\mathcal{A}$ and $\mathcal{C}$ with respect to $\mathcal{J}$, produces the sets $\mathcal{D} = \{2, 3\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{1\}$ and $\mathcal{F} = \{4\}$. Additionally $\mathcal{H} = \{2, 3, 4\}$. Note that since $d = 2$ and $g = 1$ then $d > g$ and that, since $h = 3$, an expression involving $\binom{3}{2} = 3$ terms is expected. In fact, the sets $\mathcal{B}^\alpha$, $\mathcal{B}^\beta$, $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{H}$ are precisely those in the example immediately following Lemma 4.3.4, and thus (4.3.5*e*), the result of that example is, in this particular case, expression (4.3.8*a*). From this, the required expression (4.3.8*b*), with each tableau on the right higher than the original tableau, follows immediately:

$$\left\langle \begin{matrix} \bar{2} & 2 \\ 2 & \\ \bar{3} & \\ 3 & \end{matrix} \right\rangle = - \left\langle \begin{matrix} \bar{2} & 2 \\ 2 & \\ \bar{4} & \\ 4 & \end{matrix} \right\rangle - \left\langle \begin{matrix} \bar{3} & 2 \\ 3 & \\ \bar{4} & \\ 4 & \end{matrix} \right\rangle. \tag{4.3.9b}$$

The second term on the right here is not $Sp(8)$-standard. However, it can written in terms of such by using a single Garnir relation.

As a further example, let $\lambda = (1^9)$ and consider the $Sp(18)$-module B^λ. The tableau:

$$T^\lambda = \begin{matrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{5} \\ 5 \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \end{matrix}\ , \qquad (4.3.10a)$$

is not $Sp(18)$-standard since for $j^+ = 7$, $T^\lambda_{(j+)} < \bar{j}^+$. T^λ gives rise to the following sets: $\mathcal{Q} = \{\bar{2}, 2, \bar{4}, \bar{5}, 5, \bar{6}, 6, \bar{7}, 7\}$, $\mathcal{A} = \{2, 5, 6, 7\}$, $\mathcal{B}^\alpha = \emptyset$, $\mathcal{B}^\beta = \{4\}$, $\mathcal{B} = \{4\}$ and $\mathcal{C} = \{1, 3, 8, 9\}$. With $j = j^+ - 1 = 6$, splitting the sets $\mathcal{A}$ and $\mathcal{C}$ with respect to $\mathcal{J} = \{1, 2, 3, 4, 5, 6\}$ produces the sets $\mathcal{D} = \{2, 5, 6\}$, $\mathcal{E} = \{7\}$, $\mathcal{G} = \{1, 3\}$ and $\mathcal{F} = \{8, 9\}$. Additionally $\mathcal{H} = \{2, 5, 6, 8, 9\}$. Then $d > g$ since $d = 3$ and $g = 2$, and $h = 5$ implies that an expression involving $\binom{5}{3} = 10$ terms is expected. In this particular case, the sets $\mathcal{B}^\alpha$, $\mathcal{B}^\beta$, $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{H}$ are precisely those in the second example following Lemma 4.3.4, and thus (4.3.6d), the result of that example is, in this case, expression (4.3.8a). This yields the following expression with each tableau on the right higher than the original tableau:

$$\begin{pmatrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{5} \\ 5 \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \end{pmatrix} = -\begin{pmatrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{5} \\ 5 \\ \bar{7} \\ 7 \\ \bar{8} \\ 8 \end{pmatrix} - \begin{pmatrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{5} \\ 5 \\ \bar{7} \\ 7 \\ \bar{9} \\ 9 \end{pmatrix} - \begin{pmatrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \\ \bar{8} \\ 8 \end{pmatrix} - \begin{pmatrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \\ \bar{9} \\ 9 \end{pmatrix} - \begin{pmatrix} \bar{2} \\ 2 \\ \bar{4} \\ \bar{7} \\ 7 \\ \bar{8} \\ 8 \\ \bar{9} \\ 9 \end{pmatrix} - \begin{pmatrix} \bar{4} \\ \bar{5} \\ 5 \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \\ \bar{8} \\ 8 \end{pmatrix} - \begin{pmatrix} \bar{4} \\ \bar{5} \\ 5 \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \\ \bar{9} \\ 9 \end{pmatrix} - \begin{pmatrix} \bar{4} \\ \bar{5} \\ 5 \\ \bar{7} \\ 7 \\ \bar{8} \\ 8 \\ \bar{9} \\ 9 \end{pmatrix} - \begin{pmatrix} \bar{4} \\ \bar{6} \\ 6 \\ \bar{7} \\ 7 \\ \bar{8} \\ 8 \\ \bar{9} \\ 9 \end{pmatrix}. \qquad (4.3.10b)$$

In this identity, a number of the tableaux on the right side are not $Sp(2r)$-standard: the 1st, 3rd and 6th each violate condition (iv) of Definition 4.3.7 for $i = 9$. However, the procedure given by Lemma 4.3.8 enables each of these terms to be written in terms of $Sp(18)$-standard tableaux in one more step.

Lemma 4.3.11. *The set*

$$\{\langle T^\lambda \rangle : T^\lambda \text{ is } Sp(2r)\text{-standard}\}$$

spans the $Sp(2r)$*-module* B^λ.

Proof. If the column strict T^λ is not $Sp(2r)$-standard due to a violation of condition (iii) of Definition 4.3.7 then the techniques of Section 3.4 enable the Garnir relations to be used to write $\langle T^\lambda \rangle$ is terms of higher tableaux. If the column strict T^λ violates condition (iv) of Definition 4.3.7 then Lemma 4.3.8 shows that $\langle T^\lambda \rangle$ can be written in terms of higher tableaux. Therefore, by iterating these two procedures, $\langle T^\lambda \rangle$ may be written in terms of $Sp(2r)$-standard tableaux due to the ordering on the set of all tableaux and their finite number.

This lemma has the direct implication that if $\tilde{\lambda}_1 > r$, then the $Sp(2r)$-module B^λ is zero since there exist no $Sp(2r)$-standard tableaux and therefore such a B^λ is zero-dimensional.

Let $\lambda \in P(l)$. Since $U^{Sp(2r)} \subset V^{\otimes l}$ is invariant under $Sp(2r)$, (4.2.17) implies that the element $G \in Sp(2r)$ acts on $\langle T^\lambda \rangle \in B^\lambda$ according to:

$$G\langle T^\lambda \rangle = \sum_{T'^\lambda} G_{T'^\lambda_{(1)} T^\lambda_{(1)}} G_{T'^\lambda_{(2)} T^\lambda_{(2)}} \cdots G_{T'^\lambda_{(l)} T^\lambda_{(l)}} \langle T'^\lambda \rangle, \tag{4.3.12}$$

the sum being over all tableaux T'^λ with entries from the set $\mathcal{I}^{Sp(2r)}$. In order to determine the action of ${C_a}^b \in sp(2r)$ on $\langle T^\lambda \rangle$, let p and q be the number of times that the indices b and $\bar{a}$ respectively occur in T^λ. Form the set of p tableaux $\{T^\lambda_{1,1}, T^\lambda_{1,2}, \ldots, T^\lambda_{1,p}\}$ by, in each case, replacing a single index b in T^λ with a, and the set of q tableaux $\{T^\lambda_{2,1}, T^\lambda_{2,2}, \ldots, T^\lambda_{2,q}\}$ by, in each case, replacing a single index $\bar{a}$ in T^λ with $\bar{b}$. Then, it follows from (4.2.18), (2.2.13) and the definition of $\langle T^\lambda \rangle$ that:

$$\begin{aligned} {C_a}^b \langle T^\lambda \rangle &= {E_a}^b \langle T^\lambda \rangle - \operatorname{sgn}(ab) {E_{\bar{b}}}^{\bar{a}} \langle T^\lambda \rangle \\ &= \sum_{i=1}^{p} \langle T^\lambda_{1,i} \rangle - \operatorname{sgn}(ab) \sum_{i=1}^{q} \langle T^\lambda_{2,i} \rangle. \end{aligned} \tag{4.3.13}$$

For example:

$${C_{\bar{3}}}^2 \left\langle \begin{matrix} \bar{1} & \bar{3} & 2 \\ \bar{2} & 3 & \\ 3 & 1 & \end{matrix} \right\rangle = \left\langle \begin{matrix} \bar{1} & \bar{3} & \bar{3} \\ \bar{2} & 3 & \\ 3 & 1 & \end{matrix} \right\rangle + \left\langle \begin{matrix} \bar{1} & \bar{3} & 2 \\ \bar{2} & 3 & \\ \bar{2} & 1 & \end{matrix} \right\rangle + \left\langle \begin{matrix} \bar{1} & \bar{3} & 2 \\ \bar{2} & \bar{2} & \\ 3 & 1 & \end{matrix} \right\rangle, \tag{4.3.14}$$

where, of course, the second term on the right side is identically zero.

Definition 4.3.15. *$Sp(2r)$-weight. For $i = 1, 2, \ldots, r$, let*

$$n_i^{Sp(2r)}(T^\lambda) = n_i(T^\lambda) - n_{\bar{i}}(T^\lambda),$$

where $n_j(T^\lambda)$ is the number of appearances of the index $j \in \mathcal{I}^{Sp(2r)}$ in T^λ. The vector $n^{Sp(2r)}(T^\lambda) = (n_1^{Sp(2r)}(T^\lambda), n_2^{Sp(2r)}(T^\lambda), \ldots, n_r^{Sp(2r)}(T^\lambda))$ is known as the $Sp(2r)$-weight of T^λ.

By (4.3.13), ${C_a}^a$ acts on $\langle T^\lambda \rangle$ to give:

$$ {C_a}^a \langle T^\lambda \rangle = n_a^{Sp(2r)}(T^\lambda) \langle T^\lambda \rangle, \tag{4.3.16} $$

for $a = 1, 2, \ldots, r$. Since the elements ${C_a}^a$ for $a = 1, 2, \ldots, r$, form a basis for the Cartan subalgebra of $sp(2r)$, the $Sp(2r)$-weight $n^{Sp(2r)}(T^\lambda)$ of T^λ determines the weight of the element $\langle T^\lambda \rangle \in B^\lambda$ in this basis.

With $T^\lambda_>$ given by Definition 2.6.6, $n^{Sp(2r)}(T^\lambda_>) = (\lambda_1, \lambda_2, \ldots, \lambda_r) = \lambda$ and $T^\lambda_>$ is the unique $Sp(2r)$-standard tableau of shape F^λ for which this is so. If $a, b \in \mathbb{N}_r$ and $a < b$ then:

$$ {C_a}^b \langle T^\lambda_> \rangle = 0, \tag{4.3.17a} $$

and, if $a \leq b$ then:

$$ {C_a}^{\bar{b}} \langle T^\lambda_> \rangle = 0. \tag{4.3.17b} $$

$T^\lambda_>$ is the only $Sp(2r)$-standard tableau with this property. Since $\{{C_a}^b : a, b \in \mathbb{N}_r, a < b\} \cup \{{C_a}^{\bar{b}} : a, b \in \mathbb{N}_r, a \leq b\}$ is a basis for $B_+^{Sp(2r)}$, (4.3.17) shows that $\langle T^\lambda_> \rangle$ is the unique highest weight of the $Sp(2r)$-module B^λ.

Theorem 4.3.18. [**Ki76**] *The dimension of the irreducible representation of the compact simple group $Sp(2r, \mathbb{R})$ of highest weight λ is equal to the number of $Sp(2r)$-standard tableaux of shape λ.*

This leads to the following theorem.

Theorem 4.3.19. *The $Sp(2r)$-module B^λ is irreducible with basis:*

$$ \{ \langle T^\lambda \rangle \ : T^\lambda \text{ is } Sp(2r)\text{-standard}\}. $$

Moreover [**We39**], *the set $\{B^\lambda : \lambda \in P(l; r)\}$ provides a complete list of inequivalent irreducible $Sp(2r)$-modules.*

Proof. Since B^λ has highest weight λ, and from Lemma 4.3.11, and Theorem 4.3.18, a dimension less than or equal to that of the irreducible representation $\langle \lambda \rangle$ of $Sp(2r, \mathbb{R})$, it is the $Sp(2r, \mathbb{R})$-module corresponding to the irreducible representation $\langle \lambda \rangle$ of $Sp(2r, \mathbb{R})$. It also holds for $Sp(2r, \mathbb{C})$ since Lemma 4.3.11 is equally valid for this case, and $Sp(2r, \mathbb{R})$ is a subgroup of $Sp(2r, \mathbb{C})$. The second part of the theorem follows because firstly every $Sp(2r)$-module occurs in $V^{\otimes l}$ for some l [**Li44**]; secondly, $Sp(2r)$-standard tableaux of shape λ exist if and only if $\tilde{\lambda}_1 \leq r$; and thirdly, λ is the highest weight of B^λ.

The quintessential structure of B^λ may now be stated.

Theorem 4.3.20. *Let $\lambda \in P(l;r)$. B^λ is the irreducible $Sp(2r)$-module spanned by $\langle T^\lambda \rangle$ for all T^λ with entries from the set $\mathcal{I}^{Sp(2r)}$, modulo relations* (3.4.2), (3.4.3) *and* (4.3.3), *and on which $Sp(2r)$ and $sp(2r)$ act according to* (4.3.12) *and* (4.3.13) *respectively.*

This theorem effectively provides a definition for B^λ.

Through the techniques of this section, explicit representation matrices for elements of $Sp(2r)$ and $sp(2r)$ are readily obtained in the representation $\langle \lambda \rangle$. Let $b^\lambda = D_{2r}\langle \lambda \rangle$ be the dimension of B^λ and $T_1^\lambda, T_2^\lambda, \ldots, T_{b^\lambda}^\lambda$, the $Sp(2r)$-standard tableaux. The action of $G \in Sp(2r)$ on each $\langle T_i^\lambda \rangle$ yields, through (4.3.12), a linear combination of, in general, non-standard tableaux. The techniques of this section enable each to be written in terms of $Sp(2r)$-standard tableaux, so that:

$$G\,\langle T_i^\lambda \rangle = \sum_{j=1}^{b^\lambda} \Gamma^{(\lambda)}(G)_{ji} \langle T_j^\lambda \rangle, \tag{4.3.21}$$

where each $\Gamma^{(\lambda)}(G)_{ji} \in \mathsf{F}$. These are the matrix elements of G in the representation $\langle \lambda \rangle$. In a similar way, the representation matrix $\Gamma^{(\lambda)}(C)$ of $C \in sp(2r)$ is given, via (4.3.13), by:

$$C\,\langle T_i^\lambda \rangle = \sum_{j=1}^{b^\lambda} \Gamma^{(\lambda)}(C)_{ji} \langle T_j^\lambda \rangle. \tag{4.3.22}$$

As an example, consider the 16-dimensional $Sp(4)$-module B^λ where $\lambda = (2,1)$. In this case, the $Sp(4)$-standard tableaux are:

$$\begin{array}{cccccccc}
\begin{smallmatrix}\bar 2 & 2\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}1 & 2\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & 2\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 2 & \bar 2\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}1 & \bar 2\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & \bar 2\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}1 & 1\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & 1\\ 2 & \end{smallmatrix}, \\
\begin{smallmatrix}\bar 1 & \bar 1\\ 2 & \end{smallmatrix}, &
\begin{smallmatrix}1 & 2\\ \bar 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & 2\\ \bar 2 & \end{smallmatrix}, &
\begin{smallmatrix}1 & \bar 2\\ \bar 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & \bar 2\\ \bar 2 & \end{smallmatrix}, &
\begin{smallmatrix}1 & 1\\ \bar 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & 1\\ \bar 2 & \end{smallmatrix}, &
\begin{smallmatrix}\bar 1 & \bar 1\\ \bar 2 & \end{smallmatrix}.
\end{array} \tag{4.3.23}$$

With these tableaux denoted $T_1^\lambda, T_2^\lambda, \ldots, T_{16}^\lambda$ respectively, then by (4.3.13), ${C_1}^2$ acts on $\langle T_8^\lambda \rangle$ according to:

$$\begin{aligned}
{C_1}^2 \left\langle \begin{smallmatrix}\bar 1 & 1\\ 2 & \end{smallmatrix} \right\rangle &= {E_1}^2 \left\langle \begin{smallmatrix}\bar 1 & 1\\ 2 & \end{smallmatrix} \right\rangle - {E_{\bar 2}}^{\bar 1} \left\langle \begin{smallmatrix}\bar 1 & 1\\ 2 & \end{smallmatrix} \right\rangle \\
&= + \left\langle \begin{smallmatrix}\bar 1 & 1\\ 1 & \end{smallmatrix} \right\rangle - \left\langle \begin{smallmatrix}\bar 2 & 1\\ 2 & \end{smallmatrix} \right\rangle \\
&= - \left\langle \begin{smallmatrix}\bar 2 & 1\\ 2 & \end{smallmatrix} \right\rangle - \left\langle \begin{smallmatrix}\bar 2 & 1\\ 2 & \end{smallmatrix} \right\rangle \quad \text{(by (4.3.3))} \\
&= 2 \left\langle \begin{smallmatrix}1 & 2\\ \bar 2 & \end{smallmatrix} \right\rangle - 2 \left\langle \begin{smallmatrix}1 & \bar 2\\ 2 & \end{smallmatrix} \right\rangle \quad \text{(by (3.4.3)).}
\end{aligned}$$

For $\langle T_{16}^\lambda \rangle$:

$$C_1{}^2\left\langle \begin{array}{ll} \bar{1} & \bar{1} \\ \bar{2} & \end{array} \right\rangle = -\left\langle \begin{array}{ll} \bar{2} & \bar{1} \\ \bar{2} & \end{array} \right\rangle - \left\langle \begin{array}{ll} \bar{1} & \bar{2} \\ \bar{2} & \end{array} \right\rangle$$

$$= -\left\langle \begin{array}{ll} \bar{1} & \bar{2} \\ \bar{2} & \end{array} \right\rangle \quad \text{(by (3.4.2))},$$

for $\langle T^{\lambda}_{12} \rangle$:

$$C_1{}^2\left\langle \begin{array}{ll} 1 & \bar{2} \\ \bar{2} & \end{array} \right\rangle = 0,$$

whereas for $\langle T^{\lambda}_{15} \rangle$:

$$C_1{}^2\left\langle \begin{array}{ll} \bar{1} & 1 \\ \bar{2} & \end{array} \right\rangle = -\left\langle \begin{array}{ll} \bar{2} & 1 \\ \bar{2} & \end{array} \right\rangle = 0 \quad \text{(by (3.4.2))}.$$

Similar calculations, when carried out for the other $Sp(4)$-standard tableaux of (4.3.23), give rise to the following explicit representation of $C_1{}^2$:

$$\Gamma^{(2,1)}(C_1{}^2) = \begin{pmatrix}
\cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix},$$

where each zero has been replaced by a dot. The identities obtained above result in columns 8, 16, 12 and 15 of this matrix respectively.

A computer program has been written dealing with the construction of B^{λ} as elucidated in this section. This program produced the above matrix, together with those for the other basis elements of $sp(4)$ in the same irreducible representation $\langle 2, 1 \rangle$. The construction algorithm has been checked by confirming that these representation matrices satisfy the commutation relations given by (2.2.14). In addition, representation matrices for $sp(2r)$ in a number of other modules, in particular those requiring the use of Lemma 4.3.8 in their construction, have been calculated and validated.

§4.4. Mixed tensor $GL(m)$-modules

As in Section 4.2, let V be the defining m-dimensional $GL(m)$-module with basis $\{e_1, e_2, \ldots, e_m\}$. Let V^* be the m-dimensional vector space dual to V with basis $\{e^1, e^2, \ldots, e^m\}$, the dual action $e^i(e_j) = J^i{}_j$ being encoded in the non-degenerate $m \times m$ matrix J. The covariant action of $G \in GL(m)$ on the basis of V,

$$Ge_i = \sum_{j=1}^{m} G^j{}_i e_j, \tag{4.4.1}$$

induces the contravariant action on the basis of V^* given by:

$$Ge^i = \sum_{j=1}^{m} (JG^{-1}J^{-1})^i{}_j e^j. \tag{4.4.2}$$

Naturally, these actions extend linearly to the whole of V and V^* respectively. In particular V^* is a $GL(m)$-module. Incidentally, by (4.4.2), it gives rise to a representation which is equivalent to the contragredient of that corresponding to V. The actions of $GL(m)$ on V and V^* imply that the mixed tensor $\sum_{i,j=1}^{m}(J^{-1})^j{}_i e^i \otimes e_j$ is invariant under the action of $GL(m)$.

In order to avoid unnecessary complications, the specialisation $J = I_m$, the identity matrix, will be made, whereupon the results of this section will pertain solely to this choice. It will be discussed later as to how a different choice would have affected the developments of this section.

The mixed tensor space $(V^*)^{\otimes v} \otimes V^{\otimes u}$ has a basis:

$$\{e^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u} : 1 \le b_j \le m, j = 1, 2, \ldots, v;\ 1 \le a_i \le m, i = 1, 2, \ldots, u\} \tag{4.4.3}$$

where $e^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u} = e^{b_1} \otimes e^{b_2} \otimes \cdots \otimes e^{b_v} \otimes e_{a_1} \otimes e_{a_2} \otimes \cdots \otimes e_{a_u}$. By (4.4.1), (4.4.2) and (1.5.5), $G \in GL(m)$ acts upon these basis elements according to:

$$Ge^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u} = (G^{-1})^{b_1}{}_{d_1} \cdots (G^{-1})^{b_v}{}_{d_v} G^{c_1}{}_{a_1} \cdots G^{c_u}{}_{a_u} e^{d_1 d_2 \cdots d_v}{}_{c_1 c_2 \cdots c_u}, \tag{4.4.4}$$

where repeated indices are summed over.

Definition 4.4.5. *The direct product group $S_v \otimes S_u$ is defined to act on the basis elements of $(V^*)^{\otimes v} \otimes V^{\otimes u}$ according to:*

$$\pi \otimes \tau : e^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u} = e^{b_{\pi^{-1}(1)} b_{\pi^{-1}(2)} \cdots b_{\pi^{-1}(v)}}{}_{a_{\tau^{-1}(1)} a_{\tau^{-1}(2)} \cdots a_{\tau^{-1}(u)}}, \tag{4.4.5}$$

where each $\pi \in S_v$ and $\tau \in S_u$. This action is extended linearly to make $(V^*)^{\otimes v} \otimes V^{\otimes u}$ an $S_v \otimes S_u$-module and thence a $\mathbb{C}(S_v \otimes S_u)$-module.

The following notation will be employed. For $y \in \mathbb{C}S_v$, $\bar{y} : (V^*)^{\otimes v} \otimes V^{\otimes u}$ denotes $y \otimes 1 : (V^*)^{\otimes v} \otimes V^{\otimes u}$, while for $x \in \mathbb{C}S_u$, $x : (V^*)^{\otimes v} \otimes V^{\otimes u}$ denotes $1 \otimes x : (V^*)^{\otimes v} \otimes V^{\otimes u}$.

Since if $G \in GL(m)$ then $\tilde{G}^{-1} \in GL(m)$, it follows from (4.4.3) and Lemma 4.2.8, that $S_v \otimes S_u$ and $GL(m)$ commute in their actions on $(V^*)^{\otimes v} \otimes V^{\otimes u}$. If $\mu \in P(u)$ and $\nu \in P(v)$, it then follows that $(Y'_{t^\nu} \otimes Y'_{t^\mu})(V^*)^{\otimes v} \otimes V^{\otimes u}$ is a $GL(m)$-submodule of $(V^*)^{\otimes v} \otimes V^{\otimes u}$, where Y'_{t^μ} and Y'_{t^ν} are Young symmetrisers given by Theorem 4.2.9. As will be seen, $(Y'_{t^\nu} \otimes Y'_{t^\mu})(V^*)^{\otimes v} \otimes V^{\otimes u}$ is not irreducible in general, and unless $\tilde{\mu}_1 + \tilde{\nu}_1 \leq m$, it is zero.

Let $\mu \in P(u)$ and $\nu \in P(v)$. Each basis element $w = e^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u}$ of $(V^*)^{\otimes v} \otimes V^{\otimes u}$ is identified with the composite tableau $T_w^{\bar{\nu};\mu}$ obtained from $t^{\bar{\nu};\mu}$ by replacing each integer i by a_i for $i = 1, \ldots, u$, and each barred integer $\bar{j}$ by $\bar{b}_j$ for $j = 1, \ldots, v$. The barred and unbarred entries of $T^{\bar{\nu};\mu}$ therefore represent contravariant and covariant indices respectively. For example, if $(\bar{\nu}; \mu) = (\overline{3,2}; 4, 3, 1)$ then $w = e^{21263}{}_{41234325}$ is identified with:

$$T_w^{\overline{32};431} = \begin{array}{ccccccc} & \bar{6} & \bar{1} & & & & \\ \bar{3} & \bar{2} & \bar{2} & & & & \\ & & & 4 & 3 & 3 & 5 \\ & & & 1 & 4 & 2 & \\ & & & 2 & & & \end{array} . \tag{4.4.6}$$

Following from the action of $S_v \otimes S_u$ on w, $S_v \otimes S_u$ acts on $T_w^{\bar{\nu};\mu}$ by place permutation. This place permutation action is given by the following.

Definition 4.4.7. *If $\mu \in P(u)$, $\nu \in P(v)$, then $S_v \otimes S_u$ acts by place permutation on $T^{\bar{\nu};\mu}$ to give $T'^{\bar{\nu};\mu} = (\pi_* \otimes \tau_*) T^{\bar{\nu};\mu}$, where $T'^{\bar{\nu};\mu}_{(\bar{a})} = T^{\bar{\nu};\mu}_{\overline{(\pi^{-1}(a))}}$ for $a \in \mathbb{N}_v$ and $T'^{\bar{\nu};\mu}_{(b)} = T^{\bar{\nu};\mu}_{(\tau^{-1}(b))}$ for $b \in \mathbb{N}_u$. This action extends linearly to $\mathbb{C}S_v \otimes S_u$.*

This definition is a direct generalisation of Definition 3.3.11. Here $S_v \otimes S_u$ acts to permute the barred (contravariant) entries amongst themselves and to permute the unbarred (covariant) entries amongst themselves.

Let $T^{\bar{\nu};\mu}$ be a composite tableau with entries from the set $\overline{\mathcal{I}^{GL(m)}}$ in the $F^{\bar{\nu}}$ portion and entries from the set $\mathcal{I}^{GL(m)}$ in the F^μ portion, and let $(T^{\bar{\nu};\mu})$ denote the symmetrised composite tableau:

$$(T^{\bar{\nu};\mu}) = (Y_*^\nu \otimes Y_*^\mu) T^{\bar{\nu};\mu}. \tag{4.4.8}$$

The symmetrised composite tableau $(T_w^{\bar{\nu};\mu})$ is thus identified with $(Y'_{t^\nu} \otimes Y'_{t^\mu})w \in (V^*)^{\otimes v} \otimes V^{\otimes u}$ where $T_w^{\bar{\nu};\mu}$ is identified with the basis element $w \in (V^*)^{\otimes v} \otimes V^{\otimes u}$. Let $M^{\bar{\nu};\mu}$ denote the span of all such $(T^{\bar{\nu};\mu})$. As indicated above, the $GL(m)$-module $M^{\bar{\nu};\mu}$ is reducible, in general.

Since the Young symmetrisers act independently on the two portions of a composite tableau, the generalisation of Lemmas 3.4.2 and 3.4.3 to this case is straightforward.

Lemma 4.4.9. *Let $T^{\bar{\nu};\mu}$ be a composite tableau. If $\tau \in \mathcal{C}^{\mu}$ and $\phi \in \mathcal{C}^{\nu}$, then:*

$$(T^{\bar{\nu};\mu}) = (-1)^{\tau}(\tau T^{\bar{\nu};\mu}) \tag{4.4.9a}$$

and

$$(T^{\bar{\nu};\mu}) = (-1)^{\phi}(\bar{\phi} T^{\bar{\nu};\mu}). \tag{4.4.9b}$$

As for Lemma 3.4.2, this Lemma has the consequence that if $T^{\bar{\nu};\mu}$ has an entry repeated in any column then $(T^{\bar{\nu};\mu})$ vanishes, and that any $(T^{\bar{\nu};\mu})$ may be expressed as $\pm(T'^{\bar{\nu};\mu})$ for some composite tableau $T'^{\bar{\nu};\mu}$ which is column strict, where the indices from $\mathcal{I}^{GL(m)}$ are ordered $1 < 2 < \cdots < m$ and those from $\overline{\mathcal{I}^{GL(m)}}$ are ordered $\bar{1} > \bar{2} > \cdots > \bar{m}$.

The Garnir relations take the form:

Lemma 4.4.10. *For $i < j$, let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the entries in the ith and jth columns, respectively, of either a) t^{μ} such that $\#(\mathcal{X} \cup \mathcal{Y}) > \tilde{\mu}_i$, or b) t^{ν} such that $\#(\mathcal{X} \cup \mathcal{Y}) > \tilde{\nu}_i$. Let $S(\mathcal{X})$, $S(\mathcal{Y})$ and $S(\mathcal{X} \cup \mathcal{Y})$ be the subgroups of a) S_u, or b) S_v, preserving $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Y}$, respectively. Then if $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ is a set of right coset representatives for $S(\mathcal{X}) \otimes S(\mathcal{Y})$ in $S(\mathcal{X} \cup \mathcal{Y})$, either:*

$$\sum_{\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^{\eta}(\eta T^{\bar{\nu};\mu}) = 0, \tag{4.4.10a}$$

or

$$\sum_{\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^{\eta}(\bar{\eta} T^{\bar{\nu};\mu}) = 0. \tag{4.4.10b}$$

To illustrate Lemma 4.4.10, consider the module $M^{\overline{21};211}$. On using $\mathcal{X} = \{1, 2, 3\}$, $\mathcal{Y} = \{4\}$ and an appropriate set of coset representatives, (4.4.10a) gives the identity:

$$\left(\begin{array}{cccc} & \bar{3} & & \\ \bar{1} & \bar{2} & & \\ & & 3 & 1 \\ & & 4 & \\ & & 5 & \end{array}\right) = \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{1} & \bar{2} & & \\ & & 1 & 3 \\ & & 4 & \\ & & 5 & \end{array}\right) - \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{1} & \bar{2} & & \\ & & 1 & 4 \\ & & 3 & \\ & & 5 & \end{array}\right) + \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{1} & \bar{2} & & \\ & & 1 & 5 \\ & & 3 & \\ & & 4 & \end{array}\right). \tag{4.4.11a}$$

The use of (4.4.10b) with $\mathcal{X} = \{1,2\}$, $\mathcal{Y} = \{3\}$ and an appropriate set of coset representatives, allows the first term on the right side of (4.4.11a) to be re-expressed:

$$\left(\begin{array}{cccc} & \bar{3} & & \\ \bar{1} & \bar{2} & & \\ & & 1 & 3 \\ & & 4 & \\ & & 5 & \end{array}\right) = \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{2} & \bar{1} & & \\ & & 1 & 3 \\ & & 4 & \\ & & 5 & \end{array}\right) - \left(\begin{array}{cccc} & \bar{2} & & \\ \bar{3} & \bar{1} & & \\ & & 1 & 3 \\ & & 4 & \\ & & 5 & \end{array}\right), \tag{4.4.11b}$$

with similar identities arising from the application of the same set of coset representatives to the other two terms on the right side of (4.4.11a). These four expressions may be combined to yield:

$$\begin{aligned} \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{1} & \bar{2} & & \\ & & 3 & 1 \\ & & 4 & \\ & & 5 & \end{array}\right) = {} & \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{2} & \bar{1} & & \\ & & 1 & 3 \\ & & 4 & \\ & & 5 & \end{array}\right) - \left(\begin{array}{cccc} & \bar{2} & & \\ \bar{3} & \bar{1} & & \\ & & 1 & 3 \\ & & 4 & \\ & & 5 & \end{array}\right) - \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{2} & \bar{1} & & \\ & & 1 & 4 \\ & & 3 & \\ & & 5 & \end{array}\right) \\ & + \left(\begin{array}{cccc} & \bar{2} & & \\ \bar{3} & \bar{1} & & \\ & & 1 & 4 \\ & & 3 & \\ & & 5 & \end{array}\right) + \left(\begin{array}{cccc} & \bar{3} & & \\ \bar{2} & \bar{1} & & \\ & & 1 & 5 \\ & & 3 & \\ & & 4 & \end{array}\right) - \left(\begin{array}{cccc} & \bar{2} & & \\ \bar{3} & \bar{1} & & \\ & & 1 & 5 \\ & & 3 & \\ & & 4 & \end{array}\right). \end{aligned} \tag{4.4.11c}$$

The reducibility of the $GL(m)$-module $M^{\nu;\mu}$ is implied by the existence of trace tensors. It will be shown that the removal of all trace tensors results in an irreducible $GL(m)$-module.

Definition 4.4.12. *A trace tensor of $(V^*)^{\otimes v} \otimes V^{\otimes u}$ is any linear combination of terms of the form:*

$$\sum_{i \in \mathcal{I}^{GL(m)}} w \otimes e^i \otimes x \otimes y \otimes e_i \otimes z, \tag{4.4.12}$$

where w and x are elements of some (possibly zero) tensor power of V^*, $w \otimes x \in (V^*)^{\otimes(v-1)}$, y and z are elements of some (possibly zero) tensor power of V, and $y \otimes z \in V^{\otimes(u-1)}$. Define $U^{GL(m)} \subset (V^*)^{\otimes v} \otimes V^{\otimes u}$ to be the span of all such trace tensors.

The invariance of the mixed tensor $\sum_{i=1}^m e^i \otimes e_i$ under the action of $GL(m)$ implies that $U^{GL(m)}$ is likewise invariant. Let $\{T^{\nu;\mu}\}$ denote the traceless symmetrised composite tableau resulting from the removal of all trace terms (4.4.12) from the symmetrised composite tableau $(T^{\nu;\mu})$ by forming its quotient with respect to the elements of $U^{GL(m)}$.

Lemma 4.4.13. *For $i \in \mathcal{I}^{GL(m)}$, let $T_i^{\bar{\nu};\mu}$ be m composite tableaux identical except for the entries in the two positions corresponding to a of t^μ and $\bar{b}$ of $t^{\bar{\nu}}$ where $T_i^{\bar{\nu};\mu}$ has the entries i and $\bar{i}$, respectively. Then*

$$\sum_{i \in \mathcal{I}^{GL(m)}} \{T_i^{\bar{\nu};\mu}\} = 0. \tag{4.4.13}$$

Proof. Since $\sum_{i\in\mathcal{I}^{GL(m)}} T_i^{\bar{\nu};\mu} \in U^{GL(m)}$, and the action by place permutation of each summand of the Young symmetrisers Y_*^μ and $Y_*^{\bar{\nu}}$ as in (4.4.8), only serves to give similar terms in $U^{GL(m)}$ with appropriate changes of a and $\bar{b}$, $\sum_{i\in\mathcal{I}^{GL(m)}} \{T_i^{\bar{\nu};\mu}\} \in U^{GL(m)}$, whereupon (4.4.13) follows from the definition of $\{T^{\bar{\nu};\mu}\}$ as a quotient.

The following lemma is the mixed tensor analogue of Lemma 4.3.4 in that the trace condition is simultaneously applied over a number of index pairs, in this case covariant-contravariant index pairs. The proof is virtually identical to that of Lemma 4.3.4, but is reformulated here to clarify the distinct roles of the unbarred (covariant) and barred (contravariant) indices.

Lemma 4.4.14. *Let k_1 and k_2 be such that $1 \leq k_1 \leq \nu_1$ and $1 \leq k_2 \leq \mu_1$. Let $\mathcal{I}^{GL(m)} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$ be a union of disjoint sets such that, with $b^\alpha = \#\mathcal{B}^\alpha$, $b^\beta = \#\mathcal{B}^\beta$, $e = \#\mathcal{E}$, $g = \#\mathcal{G}$, $h = \#\mathcal{H}$ and $d > g$, $\tilde{\nu}_{k_1} = b^\beta + e + d$ and $\tilde{\mu}_{k_2} = b^\alpha + e + d$. Let $\mathcal{D}_w$, for various w, run over all distinct $\binom{h}{d}$ subsets of $\mathcal{H}$ of cardinality d, and let the composite tableaux $T_w^{\bar{\nu};\mu}$, be identical apart from column k_1 of the $F^{\bar{\nu}}$ portion which contains entries from the set $\overline{\mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{D}_w}$ and column k_2 of the F^μ portion which contains indices from the set $\mathcal{B}^\alpha \cup \mathcal{E} \cup \mathcal{D}_w$. If the indices from the sets $\overline{\mathcal{E}}$, $\overline{\mathcal{B}^\beta}$, $\mathcal{E}$ and $\mathcal{B}^\alpha$ are in the same positions in each $T_w^{\bar{\nu};\mu}$ and the indices from $\overline{\mathcal{D}_w}$ and $\mathcal{D}_w$ are in column strict order, then:*

$$\sum_w \{T_w^{\bar{\nu};\mu}\} = 0. \tag{4.4.14}$$

Proof. The entries from the two relevant columns of $T_w^{\bar{\nu};\mu}$ may be schematically represented thus:

$$\begin{array}{|c|c}\hline \overline{\mathcal{D}_w} & \\ \hline \overline{\mathcal{E}} & \\ \hline \overline{\mathcal{B}^\beta} & \\ \hline \multicolumn{1}{c|}{} & \mathcal{B}^\alpha \\ \cline{2-2} \multicolumn{1}{c|}{} & \mathcal{E} \\ \cline{2-2} \multicolumn{1}{c|}{} & \mathcal{D}_w \\ \cline{2-2} \end{array} \; . \tag{4.4.14a}$$

Write these two columns as a product, θ_w, of elements of $\overline{\mathcal{I}^{GL(m)}} \cup \mathcal{I}^{GL(m)}$. By virtue of (4.4.9), interchanging elements of θ_w which are either both barred or both unbarred changes the sign of θ_w, and the presence of an identical pair of elements implies that $\theta_w = 0$. In this notation, (4.4.14) will be proved if it can be shown that:

$$\sum_w \theta_w = 0. \tag{4.4.14b}$$

Let $\omega_i = \bar{i}i$. The trace equation, (4.4.13), implies that:

$$\sum_{i \in \mathcal{I}} \omega_i = 0. \tag{4.4.14c}$$

With $\mathcal{B} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta$, split this identity according to:

$$\sum_{i \in \mathcal{H} \cup \mathcal{B} \cup \mathcal{E}} \omega_i = -\sum_{i \in \mathcal{G}} \omega_i. \tag{4.4.14d}$$

Since $d > g$, on raising each side of this identity to the power of d, the right side is annihilated due to repeated indices, whereupon

$$\left(\sum_{i \in \mathcal{H} \cup \mathcal{B} \cup \mathcal{E}} \omega_i \right)^d = 0. \tag{4.4.14e}$$

This implies that:

$$\sum_{\substack{\gamma_1 < \gamma_2 < \cdots < \gamma_d \\ \gamma_1, \gamma_2, \ldots, \gamma_d \in \mathcal{H} \cup \mathcal{B} \cup \mathcal{E}}} \omega_{\gamma_1} \omega_{\gamma_2} \cdots \omega_{\gamma_d} = 0, \tag{4.4.14f}$$

whereupon, on setting $\theta^{\mathcal{B}} = \prod_{i \in \mathcal{B}^\beta} \bar{i} \prod_{i \in \mathcal{B}^\alpha} i$ and $\theta^{\mathcal{E}} = \prod_{i \in \mathcal{E}} \omega_i$, multiplication by $\theta^{\mathcal{B}} \theta^{\mathcal{E}}$ annihilates those terms featuring ω_i for $i \in \mathcal{B} \cup \mathcal{E}$ due to a repeated index. Therefore:

$$\theta^{\mathcal{B}} \theta^{\mathcal{E}} \sum_{\substack{\gamma_1 < \gamma_2 < \cdots < \gamma_d \\ \gamma_1, \gamma_2, \ldots, \gamma_d \in \mathcal{H}}} \omega_{\gamma_1} \omega_{\gamma_2} \cdots \omega_{\gamma_d} = 0, \tag{4.4.14g}$$

and hence:

$$\theta^{\mathcal{B}} \theta^{\mathcal{E}} \sum_w \theta_w^{\mathcal{D}} = 0, \tag{4.4.14h}$$

where $\theta_w^{\mathcal{D}} = \prod_{i \in \mathcal{D}_w} \omega_i$. Let $\theta'_w = \theta^{\mathcal{B}} \theta^{\mathcal{E}} \theta_w^{\mathcal{D}}$, so that then $\sum_w \theta'_w = 0$. With the indices as specified in the statement of the Lemma, the application of an identical permutation to the factors of each θ'_w produces θ_w. Therefore $\theta'_w = \pm \theta_w$ with the sign being independent of w. Thus (4.4.14h) is equivalent to (4.4.14b) and the Lemma is proved.

As an example, consider the case where $m = 6$, $\mu = (2, 2, 1)$ and $\nu = (1, 1)$, and deal with the first column of each portion of $F^{\bar{\nu};\mu}$ so that $k_1 = 1$, $k_2 = 1$, $\tilde{\mu}_1 = 3$ and $\tilde{\nu}_1 = 2$. Let $\mathcal{B}^\alpha = \{1\}$, $\mathcal{B}^\beta = \emptyset$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{2\}$, $\mathcal{H} = \{3, 4, 5, 6\}$ and $d = 2$.

Then $b^\alpha = 1$, $b^\beta = 0$, $e = 0$, $g = 1$ and $h = 4$ so that $d > g$, $\tilde{\nu}_1 = b^\beta + e + d$ and $\tilde{\mu}_1 = b^\alpha + e + d$, as required by Lemma 4.4.14. In this case (4.4.14d) becomes

$$\omega_1 + \omega_3 + \omega_4 + \omega_5 + \omega_6 = -\omega_2. \qquad (4.4.15a)$$

As in (4.4.14e) raising this expression to the power of $d = 2$ annihilates the right side, whereupon, as in (4.4.14f):

$$\omega_1\omega_3 + \omega_1\omega_4 + \omega_1\omega_5 + \omega_1\omega_6 + \omega_3\omega_4 + \omega_3\omega_5 + \omega_3\omega_6 + \omega_4\omega_5 + \omega_4\omega_6 + \omega_5\omega_6 = 0, \qquad (4.4.15b)$$

with all the other terms zero due to repeated factors. Since $\mathcal{B} = \{1\}$ and $\mathcal{E} = \emptyset$, $\theta^{\mathcal{B}}\theta^{\mathcal{E}} = 1$. The multiplication of the above expression by this term annihilates those terms featuring ω_1. Therefore, as in (4.4.14g):

$$\theta^{\mathcal{B}}\theta^{\mathcal{E}}\omega_3\omega_4 + \theta^{\mathcal{B}}\theta^{\mathcal{E}}\omega_3\omega_5 + \theta^{\mathcal{B}}\theta^{\mathcal{E}}\omega_3\omega_6 + \theta^{\mathcal{B}}\theta^{\mathcal{E}}\omega_4\omega_5 + \theta^{\mathcal{B}}\theta^{\mathcal{E}}\omega_4\omega_6 + \theta^{\mathcal{B}}\theta^{\mathcal{E}}\omega_5\omega_6 = 0, \qquad (4.4.15c)$$

or, as in (4.4.14h):

$$\theta^{\mathcal{B}}\theta^{\mathcal{E}} \sum_{w=1}^{6} \theta^{\mathcal{D}}_w = 0, \qquad (4.4.15d)$$

where the $\binom{h}{d} = 6$ terms $\theta^{\mathcal{D}}_1 = \omega_3\omega_4$, $\theta^{\mathcal{D}}_2 = \omega_3\omega_5$, $\theta^{\mathcal{D}}_3 = \omega_3\omega_6$, $\theta^{\mathcal{D}}_4 = \omega_4\omega_5$, $\theta^{\mathcal{D}}_5 = \omega_4\omega_6$ and $\theta^{\mathcal{D}}_6 = \omega_5\omega_6$ respectively correspond to the subsets $\mathcal{D}_1 = \{3,4\}$, $\mathcal{D}_2 = \{3,5\}$, $\mathcal{D}_3 = \{3,6\}$, $\mathcal{D}_4 = \{4,5\}$, $\mathcal{D}_5 = \{4,6\}$ and $\mathcal{D}_6 = \{5,6\}$ of $\mathcal{H}$. Setting $\theta'_w = \theta^{\mathcal{B}}\theta^{\mathcal{E}}\theta^{\mathcal{D}}_w$ gives $\sum_w \theta'_w = 0$. In this example the terms are explicitly $\theta'_1 = \theta^{\mathcal{B}}\theta^{\mathcal{E}}\theta^{\mathcal{D}}_1 = 1\omega_3\omega_4 = 1\bar{3}3\bar{4}4$, $\theta'_2 = 1\bar{3}3\bar{5}5$, $\theta'_3 = 1\bar{3}3\bar{6}6$, $\theta'_4 = 1\bar{4}4\bar{5}5$, $\theta'_5 = 1\bar{4}4\bar{6}6$ and $\theta'_6 = 1\bar{5}5\bar{6}6$ so that:

$$1\bar{3}3\bar{4}4 + 1\bar{3}3\bar{5}5 + 1\bar{3}3\bar{6}6 + 1\bar{4}4\bar{5}5 + 1\bar{4}4\bar{6}6 + 1\bar{5}5\bar{6}6 = 0, \qquad (4.4.15e)$$

and, by rearranging the factors:

$$\bar{3}\bar{4}134 + \bar{3}\bar{5}135 + \bar{3}\bar{6}136 + \bar{4}\bar{5}145 + \bar{4}\bar{6}146 + \bar{5}\bar{6}156 = 0. \qquad (4.4.15f)$$

This leads to, for instance, the following tableaux identity in $W^{\overline{1,1};2,2,1}$, in which each term in the above expression is identified with the θ_w arising from the corresponding composite tableau:

$$\left\{\begin{array}{ccc} \bar{4} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 4 & \end{array}\right\} + \left\{\begin{array}{ccc} \bar{5} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 5 & \end{array}\right\} + \left\{\begin{array}{ccc} \bar{6} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 6 & \end{array}\right\}$$
$$+ \left\{\begin{array}{ccc} \bar{5} & & \\ \bar{4} & & \\ & 1 & 2 \\ & 4 & 4 \\ & 5 & \end{array}\right\} + \left\{\begin{array}{ccc} \bar{6} & & \\ \bar{4} & & \\ & 1 & 2 \\ & 4 & 4 \\ & 6 & \end{array}\right\} + \left\{\begin{array}{ccc} \bar{6} & & \\ \bar{5} & & \\ & 1 & 2 \\ & 5 & 4 \\ & 6 & \end{array}\right\} = 0. \qquad (4.4.15g)$$

The $GL(m)$-module $W^{\bar{\nu};\mu}$ is defined to be the span of all $\{T^{\bar{\nu};\mu}\}$ where each $T^{\bar{\nu};\mu}$ is a composite tableau whose F^{μ} portion contains entries from the set $\mathcal{I}^{GL(m)}$ and whose $F^{\bar{\nu}}$ portion contains entries from the set $\overline{\mathcal{I}^{GL(m)}}$. Then $W^{\bar{\nu};\mu} = M^{\bar{\nu};\mu}/(M^{\bar{\nu};\mu} \cap U^{GL(m)})$.

Standard composite tableaux for the mixed tensor $GL(m)$-modules are provided by the following definition which generalises Definition 4.2.12.

Definition 4.4.16. [Ki76] *Let $\mu \in P(u)$ and $\nu \in P(v)$. Let $T^{\bar{\nu};\mu}$ be a composite tableau for which, for $i = 1, 2, \ldots, m$, α_i is the number of entries less than or equal to i in the first column of the F^{μ} portion of $T^{\bar{\nu};\mu}$, and β_i is the number of entries greater than or equal to $\bar{i}$ in the rightmost column of the $F^{\bar{\nu}}$ portion of $T^{\bar{\nu};\mu}$. $T^{\bar{\nu};\mu}$ is $GL(m)$-standard if:*

(i) each entry in the F^{μ} portion is from the set $\mathcal{I}^{GL(m)} = \{1, 2, \ldots, m\}$;

(ii) each entry in the $F^{\bar{\nu}}$ portion is from the set $\overline{\mathcal{I}^{GL(m)}} = \{\bar{1}, \bar{2}, \ldots, \bar{m}\}$;

(iii) the entries are strictly increasing from top to bottom down each column;

(iv) the entries are non-decreasing from left to right across each row;

(v) $\alpha_i + \beta_i \leq i$ for $i = 1, 2, \ldots, m$.

Lemma 4.4.17. *Let $T^{\bar{\nu};\mu}$ be a composite tableau which is not $GL(m)$-standard in that $\alpha_j + \beta_j > j$ for some j. Then $\{T^{\bar{\nu};\mu}\}$ may be expressed as a signed sum of traceless symmetrised composite tableaux $\{T_w^{\bar{\nu};\mu}\}$, where for each w, $T_w^{\bar{\nu};\mu} > T^{\bar{\nu};\mu}$.*

Proof. The procedure exhibited here makes use of Lemma 4.4.14 and is similar to that used for the proof of Lemma 4.3.8 in the construction of irreducible $Sp(2r)$-modules.

Let $\mathcal{Q} \subset \overline{\mathcal{I}^{GL(m)}} \cup \mathcal{I}^{GL(m)}$ be the set of entries in the first column of each of the two portions of $T^{\bar{\nu};\mu}$. Let $\mathcal{A} = \{i \in \mathcal{I}^{GL(m)} : \bar{i} \in \mathcal{Q},\, i \in \mathcal{Q}\}$, $\mathcal{B}^{\alpha} = \{i \in \mathcal{I}^{GL(m)} :$

$i \in \mathcal{Q}, \bar{i} \notin \mathcal{Q}\}$, $\mathcal{B}^\beta = \{i \in \mathcal{I}^{GL(m)} : \bar{i} \in \mathcal{Q}, i \notin \mathcal{Q}\}$, $\mathcal{C} = \{i \in \mathcal{I}^{GL(m)} : \bar{i} \notin \mathcal{Q}, i \notin \mathcal{Q}\}$ and $\mathcal{B} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta$. Then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are distinct with $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \mathcal{I}^{GL(m)}$, and if $a = \#\mathcal{A}$, $b^\alpha = \#\mathcal{B}^\alpha$, $b^\beta = \#\mathcal{B}^\beta$, $b = \#\mathcal{B}$, and $c = \#\mathcal{C}$, then $a+b+c = m$, $\tilde{\mu}_1 = a + b^\alpha$ and $\tilde{\nu}_1 = a + b^\beta$. Let $\mathcal{J} = \mathsf{N}_j$ so that $\#\mathcal{J} = j$. The sets created above are now split with respect to $\mathcal{J}$: $\mathcal{D} = \mathcal{A} \cap \mathcal{J}$, $\mathcal{E} = \mathcal{A} \backslash \mathcal{D}$, $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{J}$, $\mathcal{B}_1 = \mathcal{B} \backslash \mathcal{B}_0$, $\mathcal{G} = \mathcal{C} \cap \mathcal{J}$ and $\mathcal{F} = \mathcal{C} \backslash \mathcal{G}$. In addition let $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$. With the cardinalities of the sets just created d, e, b_0, b_1, g, f and h respectively, then $d + e + b_0 + b_1 + g + f = m$, $h = d + f$, $\tilde{\mu}_1 = d + e + b^\alpha$, $\tilde{\nu}_1 = d + e + b^\beta$, and $d + b_0 + g = j$. The condition $\alpha_j + \beta_j > j$ implies that $2d + b_0 > j$, so that $d > g$. Therefore the conditions of Lemma 4.4.14 are satisfied and the identity:

$$\sum_{\mathcal{D}_w \subset \mathcal{H}} \{T_w^{\nu;\mu}\} = 0, \qquad (4.4.17a)$$

follows, where the sum is over all $\binom{h}{d}$ distinct subsets $\mathcal{D}_w$ of $\mathcal{H}$, and $T_w^{\nu;\mu}$ is identical to $T^{\nu;\mu}$ apart from the indices from the set $\overline{\mathcal{D}} \cup \mathcal{D}$ in the first column of each portion, having been replaced by those from the set $\overline{\mathcal{D}_w} \cup \mathcal{D}_w$. Therefore:

$$\{T^{\nu;\mu}\} = - \sum_{\substack{\mathcal{D}_w \subset \mathcal{H} \\ \mathcal{D}_w \neq \mathcal{D}}} \{T_w^{\nu;\mu}\}. \qquad (4.4.17b)$$

Since $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$ and each of the terms from the set $\mathcal{F}$ is higher than those from the set $\mathcal{D}$, it follows that for each term on the right of (4.4.17b), $T_w^{\nu;\mu} > T^{\nu;\mu}$, thereby proving Lemma 4.4.17.

As an illustration of the algorithm described in the above proof, consider the composite tableau:

$$\begin{array}{ccc} \bar{4} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 4 & \end{array}, \qquad (4.4.18a)$$

and the $GL(6)$-module $W^{(\bar{1},\bar{1};2,2,1)}$. Here $\alpha_1 = 1$, $\beta_1 = 0$, so that $\alpha_1 + \beta_1 = 1 \leq 1$; $\alpha_2 = 1$, $\beta_2 = 0$, so that $\alpha_2 + \beta_2 = 1 \leq 2$; $\alpha_3 = 2$, $\beta_3 = 1$, so that $\alpha_3 + \beta_3 = 3 \leq 3$; but $\alpha_4 = 3$, $\beta_4 = 2$, so that $\alpha_4 + \beta_4 = 5 > 4$ implying that the tableau is not $GL(6)$-standard. Thus $j = 4$ satisfies the condition of Lemma 4.4.17, so that $\mathcal{J} = \{1, 2, 3, 4\}$. Reading the entries from the first columns of the above tableau gives $\mathcal{Q} = \{1, 3, 4, \bar{3}, \bar{4}\}$ and hence $\mathcal{A} = \{3, 4\}$, $\mathcal{B}^\alpha = \{1\}$, $\mathcal{B}^\beta = \emptyset$, $\mathcal{B} = \{1\}$ and $\mathcal{C} = \{2, 5, 6\}$. Splitting the sets $\mathcal{A}$ and $\mathcal{C}$ with respect to $\mathcal{J}$, as in the proof of Lemma 4.4.17, produces the sets $\mathcal{D} = \{3, 4\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{2\}$ and $\mathcal{F} = \{5, 6\}$. Additionally $\mathcal{H} = \{3, 4, 5, 6\}$. Note that since $d = 2$ and $g = 1$ then $d > g$, and that since $h = 4$, an expression involving $\binom{4}{2} = 6$ terms is expected. In fact, the sets $\mathcal{B}^{\alpha}$, $\mathcal{B}^{\beta}$, $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{H}$ are precisely those in the example (4.4.15). The result of that example, (4.4.15g), is in this case, expression (4.4.17a). From this, the required expression (4.4.17b), with each tableau on the right side higher than the original tableau, follows immediately:

$$\begin{aligned}
\left\{\begin{array}{ccc} \bar{4} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 4 & \end{array}\right\} = &-\left\{\begin{array}{ccc} \bar{5} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 5 & \end{array}\right\} - \left\{\begin{array}{ccc} \bar{6} & & \\ \bar{3} & & \\ & 1 & 2 \\ & 3 & 4 \\ & 6 & \end{array}\right\} \\
&- \left\{\begin{array}{ccc} \bar{5} & & \\ \bar{4} & & \\ & 1 & 2 \\ & 4 & 4 \\ & 5 & \end{array}\right\} - \left\{\begin{array}{ccc} \bar{6} & & \\ \bar{4} & & \\ & 1 & 2 \\ & 4 & 4 \\ & 6 & \end{array}\right\} - \left\{\begin{array}{ccc} \bar{6} & & \\ \bar{5} & & \\ & 1 & 2 \\ & 5 & 4 \\ & 6 & \end{array}\right\}.
\end{aligned} \qquad (4.4.18b)$$

Note that, in this case, all but the last of these terms is $GL(m)$-standard. A single application of a specific Garnir relation to this term will produce an expression involving only $GL(m)$-standard tableaux.

Lemma 4.4.19. *The set*

$$\{\{T^{\bar{\nu};\mu}\} : T^{\bar{\nu};\mu} \textit{ is } GL(m)\textit{-standard}\}$$

spans the $GL(m)$-module $W^{\bar{\nu};\mu}$.

Proof. As for covariant $GL(m)$-modules, this lemma is proved by demonstrating that a standardisation algorithm exists by which non-standard terms may be written in terms of higher tableaux. If the column strict $T^{\bar{\nu};\mu}$ is not $GL(m)$-standard due to a violation of condition (iv) of Definition 4.4.16, then this violation will occur in either the F^{μ} or the $F^{\bar{\nu}}$ portion. If the former, then the Garnir relations (4.4.10a) may be used on the F^{μ} portion precisely as in Section 3.4 to write $\{T^{\bar{\nu};\mu}\}$ in terms of higher tableaux, as given by the order of Definition 2.6.19. An example is provided by (4.4.11a). For the $F^{\bar{\nu}}$ portion, violations are dealt with by locating the offending neighbouring pair of entries and applying the procedure as if this portion of the tableau were the 'correct' way up. Here, the barred entries, interpreted as negative integers, ensure that non-standard tableaux are written in terms of higher tableaux. For example, identity (4.4.11b) results from the use of this procedure.

If $T^{\bar{\nu};\mu}$ is not $GL(m)$-standard through a violation of condition (v) of Definition 4.4.16, then Lemma 4.4.17 shows how to write $\{T^{\bar{\nu};\mu}\}$ in terms of higher tableaux.

By iterating these three procedures, any $\{T^{\bar{\nu};\mu}\}$ may be written in terms of $GL(m)$-standard tableaux due to the ordering on the set of all composite tableaux and their finite number.

Note that, in general, the trace relations and the Garnir relations will have to be applied many times in order to elicit an expression solely in terms of $GL(m)$-standard tableaux. Lemma 4.4.19 has the direct implication that if $\tilde{\mu}_1 + \tilde{\nu}_1 > m$, then the $GL(m)$-module $W^{\bar{\nu};\mu}$ is zero since no $GL(m)$-standard tableaux exist for such modules.

Since $GL(m)$ commutes with the action of $S_v \otimes S_u$, (4.4.2) implies that the element $G \in GL(m)$ acts on $\{T^{\bar{\nu};\mu}\} \in W^{\bar{\nu};\mu}$ according to:

$$G\{T^{\bar{\nu};\mu}\} = \sum_{T'^{\bar{\nu};\mu}} (G^{-1})^{T^{\bar{\nu};\mu}_{(\bar{1})}}{}_{T'^{\bar{\nu};\mu}_{(\bar{1})}} \cdots (G^{-1})^{T^{\bar{\nu};\mu}_{(\bar{v})}}{}_{T'^{\bar{\nu};\mu}_{(\bar{v})}} G^{T'^{\bar{\nu};\mu}_{(1)}}{}_{T^{\bar{\nu};\mu}_{(1)}} \cdots G^{T'^{\bar{\nu};\mu}_{(u)}}{}_{T^{\bar{\nu};\mu}_{(u)}} \{T'^{\bar{\nu};\mu}\}, \tag{4.4.20}$$

the sum being over all tableaux $T'^{\bar{\nu};\mu}$ with entries in the F^{μ} portion from the set $\overline{\mathcal{I}^{GL(m)}}$ and in the $F^{\bar{\nu}}$ portion from the set $\mathcal{I}^{GL(m)}$. Since the $GL(m)$-module V^* is contragredient to V, it follows from (1.5.2) that the element $E_a{}^b$ acts on the basis element e^i of V^* according to:

$$E_a{}^b e^i = -\delta^i_a e^b. \tag{4.4.21}$$

Consequently the action on the basis element $e^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u}$ of $(V^*)^{\otimes v} \otimes V^{\otimes u}$ is given by:

$$\begin{aligned} &E_a{}^b e^{b_1 b_2 \cdots b_v}{}_{a_1 a_2 \cdots a_u} = \\ &\qquad \sum_{i=1}^{u} \delta^b_{a_i} e^{b_1 b_2 \cdots b_v}{}_{a_1 \cdots a_{i-1} a a_{i+1} \cdots a_u} - \sum_{i=1}^{v} \delta^{b_i}_a e^{b_1 \cdots b_{i-1} b b_{i+1} b_v}{}_{a_1 a_2 \cdots a_u}. \end{aligned} \tag{4.4.22}$$

Let p and q be the number of times that the indices b and $\bar{a}$ respectively occur in $T^{\bar{\nu};\mu}$. Form the set of p tableaux $\{T^{\bar{\nu};\mu}_{1,1}, T^{\bar{\nu};\mu}_{1,2}, \ldots, T^{\bar{\nu};\mu}_{1,p}\}$ by, in each case, replacing a single index b in $T^{\bar{\nu};\mu}$ with a, and the set of q tableaux $\{T^{\bar{\nu};\mu}_{2,1}, T^{\bar{\nu};\mu}_{2,2}, \ldots, T^{\bar{\nu};\mu}_{2,q}\}$ by, in each case, replacing a single index $\bar{a}$ in $T^{\bar{\nu};\mu}$ with $\bar{b}$. It then follows from (4.4.22) and the definition of $\{T^{\bar{\nu};\mu}\}$ that:

$$E_a{}^b\{T^{\bar{\nu};\mu}\} = \sum_{i=1}^{p}\{T^{\bar{\nu};\mu}_{1,i}\} - \sum_{i=1}^{q}\{T^{\bar{\nu};\mu}_{2,i}\}. \tag{4.4.23}$$

The following generalises Definition 4.2.27.

Definition 4.4.24. *$GL(m)$-weight. For $i = 1, 2, \ldots, m$, let*

$$n_i^{GL(m)}(T^{\bar{\nu};\mu}) = n_i(T^{\bar{\nu};\mu}) - n_{\bar{i}}(T^{\bar{\nu};\mu}),$$

where $n_j(T^{\bar{\nu};\mu})$ *is the number of appearances of the index* $j \in \overline{\mathcal{I}^{GL(m)}} \cup \mathcal{I}^{GL(m)}$ *in* $T^{\bar{\nu};\mu}$. *The vector* $n^{GL(m)}(T^{\bar{\nu};\mu}) = (n_1^{GL(m)}(T^{\bar{\nu};\mu}), n_2^{GL(m)}(T^{\bar{\nu};\mu}), \ldots, n_m^{GL(m)}(T^{\bar{\nu};\mu}))$ *is known as the* $GL(m)$*-weight of* $T^{\bar{\nu};\mu}$.

By (4.4.23), $E_a{}^a$ acts on $\{T^{\bar{\nu};\mu}\}$ to give:

$$E_a{}^a\{T^{\bar{\nu};\mu}\} = n_a^{GL(m)}(T^{\bar{\nu};\mu})\{T^{\bar{\nu};\mu}\}, \tag{4.4.25}$$

for $a = 1, 2, \ldots, m$. Since the elements $E_a{}^a$ for $a = 1, 2, \ldots, m$, form a basis for the Cartan subalgebra of $gl(m)$, the $GL(m)$-weight $n^{GL(m)}(T^{\bar{\nu};\mu})$ of $T^{\bar{\nu};\mu}$ determines the weight of the element $\{T^{\bar{\nu};\mu}\} \in W^{\bar{\nu};\mu}$ in this basis.

Definition 4.4.26. *On fixing* m, *define* $T_>^{\bar{\nu};\mu}$ *to be the composite tableau for which* $T^{\bar{\nu};\mu}_{>(i,j)} = i$ *for* $1 \le i \le \tilde{\mu}_1$ *and* $1 \le j \le \mu_i$, *and for which* $T^{\bar{\nu};\mu}_{>(\bar{i},\bar{j})} = \overline{m - \tilde{\nu}_j + i}$ *for* $1 \le i \le \tilde{\nu}_1$ *and* $1 \le j \le \nu_i$.

When $m = 8$, this definition implies, for example, that:

$$T_>^{\overline{32};431} = \begin{array}{ccccccc} & \bar{8} & \bar{8} & & & & \\ \bar{8} & \bar{7} & \bar{7} & & & & \\ & & & 1 & 1 & 1 & 1 \\ & & & 2 & 2 & 2 & \\ & & & 3 & & & \end{array}. \tag{4.4.27}$$

As was noted earlier, only those $W^{\bar{\nu};\mu}$ need be considered for which $\tilde{\mu}_1 + \tilde{\nu}_1 \le m$. In such a case, the $GL(m)$-weight of $T_>^{\bar{\nu};\mu}$ is given by $(\bar{\nu};\mu)_m$ (see Definition 2.3.11). Then (4.4.23) implies that $E_a{}^b\{T_>^{\bar{\nu};\mu}\} = 0$ for all $a, b \in \mathcal{I}^{GL(m)}$ with $a < b$. Moreover, $T_>^{\bar{\nu};\mu}$ is the only $GL(m)$-standard tableau of shape $F^{\bar{\nu};\mu}$ for which this is true. This shows that $\{T^{\bar{\nu};\mu}\}$ is the highest weight vector of $W^{\bar{\nu};\mu}$.

Theorem 4.4.28. [**Ki76**] *The dimension of the irreducible representation of the compact simple group* $U(m)$ *of highest weight* $(\bar{\nu};\mu)$ *is equal to the number of* $GL(m)$*-standard tableaux of shape* $F^{\bar{\nu};\mu}$.

This leads to the following theorem.

Theorem 4.4.29. *The* $GL(m)$*-module* $W^{\bar{\nu};\mu}$ *is irreducible with basis:*

$$\{\, \{T^{\bar{\nu};\mu}\} : T^{\bar{\nu};\mu} \text{ is } GL(m)\text{-standard}\}$$

Moreover [**We39**], *the set* $\{W^{\bar{\nu};\mu} : \mu \in P(u;s), \nu \in P(v;t), s + t \le m\}$ *provides a complete list of inequivalent irreducible* $GL(m)$*-modules.*

Proof. Since $W^{\bar{\nu};\mu}$ has highest weight $(\bar{\nu};\mu)$, it contains the $U(m)$-module corresponding to the irreducible representation $\{\bar{\nu};\mu\}$ of $U(m)$. Then, for $U(m)$, the first part of the theorem follows from Theorem 4.4.28 and Lemma 4.4.19. It also holds for $GL(m, \mathbb{C})$ since Lemma 4.4.19 is equally valid for this case and $U(m)$ is a subgroup of $GL(m, \mathbb{C})$. The second part of the theorem follows because firstly every $GL(m)$-module occurs in $(V^*)^{\otimes v} \otimes V^{\otimes u}$ for some pair v and u **[We39]**; secondly, $GL(m)$-standard tableaux of shape $F^{\bar{\nu};\mu}$ exist if and only if $\tilde{\nu}_1 + \tilde{\mu}_1 \leq m$; and thirdly, $(\bar{\nu}; \mu)$ is the highest weight of $W^{\bar{\nu};\mu}$.

The quintessential structure of $W^{\bar{\nu};\mu}$ may now be stated.

Theorem 4.4.30. *Let $\mu \in P(u; s)$ and $\nu \in P(v; t)$ with $s + t \leq m$. The $GL(m)$-module $W^{\bar{\nu};\mu}$ is the irreducible $GL(m)$-module spanned by $\{T^{\bar{\nu};\mu}\}$ for all $T^{\bar{\nu};\mu}$ with entries in the F^μ portion from the set $\mathcal{I}^{GL(m)}$ and entries from the $F^{\bar{\nu}}$ portion from the set $\overline{\mathcal{I}^{GL(m)}}$, modulo relations* (4.4.9), (4.4.10) *and* (4.4.13) *and on which $GL(m)$ and $gl(m)$ act according to* (4.4.20) *and* (4.4.23) *respectively.*

This theorem effectively provides a definition for $W^{\bar{\nu};\mu}$.

The techniques of this section enable explicit representation matrices for elements of $GL(m)$ and $gl(m)$ to be readily obtained in the representation $\{\bar{\nu}; \mu\}$, in a direct extension to the techniques presented in Section 4.2 for the covariant $GL(m)$-module W^λ.

As indicated earlier, the techniques of this section depend, to a large extent on the choice of J. In fact, for certain choices of J, for example $J = J_m^+$, as given by (2.1.1b/c), the $GL(m)$-standard composite tableaux of Definition 4.4.16 do not provide a basis for the corresponding irreducible $GL(m)$-module. Nevertheless, in this particular case, it is possible to define an alternative set of standard tableaux and to devise a standardisation procedure analogous to that used in this section to write an arbitrary symmetrised traceless composite tableau in terms of these alternative standard tableaux. However, such standard tableaux do not readily yield the weights and characters of the representation $\{\bar{\nu}; \mu\}$ of $GL(m)$.

§4.5. Duality between Garnir and Trace relations

In this section a most intriguing result is presented. It is that any Trace relation of the type considered in Lemma 4.4.14, implies the validity of a Garnir relation in the canonical (amongst others) associate module and vice-versa.

In order to demonstrate the duality, a bijection is required between the set of column strict composite tableaux in an irreducible mixed tensor $GL(m)$-module and the set of column strict tableaux, ordinary or composite, in one of its associate modules.

Definition 4.5.1. *Let $\mu \in P(u;s)$ and $\nu \in P(v;t)$ with $s+t \leq m$. Let the generalised partition $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ be that corresponding to $(\bar{\nu};\mu)_m$. For $k \in \mathbf{Z}$ let $(\bar{\nu}^k;\mu^k)_m$ correspond to the generalised partition $\Lambda^k = (\Lambda_1+k, \Lambda_2+k, \ldots, \Lambda_m+k)$. The irreducible representations $\{\bar{\nu}^k;\mu^k\}$ for all $k \in \mathbf{Z}$ are said to be mutually associate. The canonical associate of $\{\bar{\nu};\mu\}$ is defined to be $\{\lambda\}$ where $\lambda = (\bar{\nu}^q;\mu^q)_m$ with $q = \nu_1$ so that $\nu^q = (0)$ and $\mu^q = \lambda$.*

Correspondingly, the associates of a column strict composite tableau are defined by the following.

Definition 4.5.2. *For $i = 1, 2, \ldots, \nu_1, \ldots$, let $\overline{\mathcal{J}_i} \subset \overline{\mathcal{I}^{GL(m)}}$ be the set of indices in the ith column (from the right) of the $F^{\bar{\nu}}$ portion of the column strict composite tableau $T^{\bar{\nu};\mu} = T_{*0}^{\bar{\nu}^0;\mu^0}$. Then $\#\mathcal{J}_i = \tilde{\nu}_i$. Let $\mathcal{K}_i = \mathcal{I}^{GL(m)} \backslash \mathcal{J}_i$ whereupon $\#\mathcal{K}_i = m - \tilde{\nu}_i$. The composite tableau $T_{*i}^{\bar{\nu}^i;\mu^i}$ is formed from $T_{*i-1}^{\bar{\nu}^{i-1};\mu^{i-1}}$ by removing the first column of the $F^{\bar{\nu}^{i-1}}$ portion containing the entries from $\overline{\mathcal{J}_i}$, and creating, immediately below it, the first column of the F^{μ^i} portion of $T_{*i}^{\bar{\nu}^i;\mu^i}$ filled, in column strict order, with entries from the set $\mathcal{K}_i$. The tableaux $T_{*i}^{\bar{\nu}^i;\mu^i}$, for $i = 1, 2, \ldots, \nu_1, \ldots$, are said to be mutually associate. The canonical associate of the composite column strict tableau $T^{\bar{\nu};\mu}$ is defined to be the ordinary column strict tableau $T_*^{\lambda} = T_{*q}^{\bar{\nu}^q;\mu^q}$ with $q = \nu_1$ so that $\nu^q = (0)$ and $\mu^q = \lambda$. For $i < 0$, the composite tableaux $T_{*i}^{\bar{\nu}^i;\mu^i}$ are obtained through the reverse of the above process.*

The following illustrates these definitions when $m = 6$:

$$T^{\overline{21};31} = \begin{array}{ccccc} & \bar{2} & & & \\ \bar{3} & \bar{1} & & & \\ & & 2 & 3 & 5 \\ & & 4 & & \end{array} \quad \Longrightarrow \quad T_{*1}^{\bar{1};4211} = \begin{array}{ccccc} \bar{3} & & & & \\ & 3 & 2 & 3 & 5 \\ & 4 & 4 & & \\ & 5 & & & \\ & 6 & & & \end{array}$$

$$\Longrightarrow \quad T_*^{53221} = \begin{array}{ccccc} 1 & 3 & 2 & 3 & 5 \\ 2 & 4 & 4 & & \\ 4 & 5 & & & \\ 5 & 6 & & & \\ 6 & & & & \end{array} \quad ,$$

$$T^{\overline{21};31} = \begin{array}{ccccc} & \bar{2} & & & \\ \bar{3} & \bar{1} & & & \\ & & 2 & 3 & 5 \\ & & 4 & & \end{array} \quad \Longrightarrow \quad T_{*-1}^{\overline{3211};2} = \begin{array}{ccccc} & & \bar{6} & & \\ & & \bar{5} & & \\ & \bar{2} & \bar{3} & & \\ \bar{3} & \bar{1} & \bar{1} & & \\ & & & 3 & 5 \end{array} \quad ,$$

$$
T^{\overline{331};221} = \begin{array}{ccccc} & & \bar{6} & & \\ \bar{5} & \bar{4} & \bar{3} & & \\ \bar{3} & \bar{3} & \bar{2} & & \\ & & & 2 & 2 \\ & & & 4 & 5 \\ & & & 5 & \end{array} \implies T_{*1}^{\overline{22};332} = \begin{array}{ccccc} \bar{5} & \bar{4} & & & \\ \bar{3} & \bar{3} & & & \\ & & 1 & 2 & 2 \\ & & 4 & 4 & 5 \\ & & 5 & 5 & \end{array}
$$

$$
\implies T_{*2}^{\overline{11};4431} = \begin{array}{ccccc} \bar{5} & & & & \\ \bar{3} & & & & \\ & 1 & 1 & 2 & 2 \\ & 2 & 4 & 4 & 5 \\ & 5 & 5 & 5 & \\ & 6 & & & \end{array} \implies T_{*}^{5542} = \begin{array}{ccccc} 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 & 5 \\ 4 & 5 & 5 & 5 & \\ 6 & 6 & & & \end{array} .
$$

As exemplified by this last example, the map from $T^{\bar{\nu};\mu}$ to T_*^{λ} defined by 4.5.2 does, in fact, provide a bijection between sets of $GL(m)$-standard tableaux.

Lemma 4.5.3. *If the composite tableau $T^{\bar{\nu};\mu}$ is $GL(m)$-standard then $T_{*i}^{\bar{\nu}^i;\mu^i}$ is $GL(m)$-standard for each $i \in \mathbf{Z}$. T_*^{λ} is $GL(m)$-standard.*

Proof. Define the sets $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{K}_1$ as for Definition 4.5.2. In addition, let $\mathcal{K}_0$ be the set of the $\tilde{\mu}_1$ indices in the first column of the F^{μ} portion of $T^{\bar{\nu};\mu}$. For $j = 1, 2, \ldots, m$, let $\mathcal{J}_k^j = \{i \in \mathcal{J}_k : i \leq j\}$ for $k = 1, 2$, and $\mathcal{K}_k^j = \{i \in \mathcal{K}_k : i \leq j\}$ for $k = 0, 1$. Conditions (iv) and (v) of Definition 4.4.16 respectively imply that $\#\mathcal{J}_2^j \leq \#\mathcal{J}_1^j$ and $\#\mathcal{J}_1^j + \#\mathcal{K}_0^j \leq j$ for each $j = 1, 2, \ldots, m$. Since $\#\mathcal{J}_1^j + \#\mathcal{K}_1^j = j$, the conditions $\#\mathcal{J}_2^j + \#\mathcal{K}_1^j \leq j$ and $\#\mathcal{K}_0^j \leq \#\mathcal{K}_1^j$ are satisfied for each $j = 1, 2, \ldots, m$. Since the remaining entries of $T_{*1}^{\bar{\nu}^1;\mu^1}$ are exactly as for $T^{\bar{\nu};\mu}$, it follows that $T_{*1}^{\bar{\nu}^1;\mu^1}$ is $GL(m)$-standard. It then follows, by induction, that $T_{*i}^{\bar{\nu}^i;\mu^i}$ is $GL(m)$-standard for all $i \in \mathbf{N}$. Conversely, with the above definitions, $\#\mathcal{J}_2^j + \#\mathcal{K}_1^j \leq j$ and $\#\mathcal{K}_0^j \leq \#\mathcal{K}_1^j$ imply that $\#\mathcal{J}_2^j \leq \#\mathcal{J}_1^j$ and $\#\mathcal{J}_1^j + \#\mathcal{K}_0^j \leq j$ for each $j = 1, 2, \ldots, m$. Thus, if $T_{*1}^{\bar{\nu}^1;\mu^1}$ is $GL(m)$-standard, then $T^{\bar{\nu};\mu}$ is $GL(m)$-standard and, by induction, $T_{*i}^{\bar{\nu}^i;\mu^i}$ is $GL(m)$-standard for all $i < 0$. This completes the proof.

If $\mathcal{J} = \mathcal{J}_1$ and $\mathcal{K} = \mathcal{K}_1$ are as above, then associated with each transition from $T^{\bar{\nu};\mu}$ to $T_{*1}^{\bar{\nu}^1;\mu^1}$ is a sign factor given by

$$
\epsilon^{(\mathcal{K},\mathcal{J})} = \epsilon^{k_1 k_2 \cdots k_{m-\tilde{\nu}_1} j_1 j_2 \cdots j_{\tilde{\nu}_1}} \tag{4.5.4}
$$

where $k_i \in \mathcal{K}$, $j_i \in \mathcal{J}$, $k_1 < k_2 < \cdots < k_{m-\tilde{\nu}_1}$ and $j_1 < j_2 < \cdots < j_{\tilde{\nu}_1}$. At this point, it is convenient to define the map $L_* : W^{\bar{\nu};\mu} \to W^{\bar{\nu}^1;\mu^1}$ given by

$$
L_* : \{T^{\bar{\nu};\mu}\} = \epsilon^{(\mathcal{K},\mathcal{J})}\{T_{*1}^{\bar{\nu}^1;\mu^1}\}, \tag{4.5.5}
$$

for all column strict $T^{\nu;\mu}$, extending this map linearly to the whole of $W^{\nu;\mu}$. The following, seemingly mysterious, result is observed.

Theorem 4.5.6. *Let $\mu \in P(u;s)$ and $\nu \in P(v;t)$ with $s+t \leq m$. If:*

$$\sum_w \{T_w^{\nu;\mu}\} = 0 \tag{4.5.6a}$$

is a Trace relation in the $GL(m)$-module $W^{\nu;\mu}$ of the type specified in Lemma 4.4.14, with all the composite tableaux $T_w^{\nu;\mu}$ differing only in entries in the lth column of the F^ν portion and the jth column of the F^μ portion, and $\mathcal{J}_w$ and $\mathcal{K}_w$ are defined by analogy with $\mathcal{J}$ and $\mathcal{K}$ in Definition 4.5.2, then for $k \geq l$:

$$\sum_w \epsilon^{(\mathcal{K}_w,\mathcal{J}_w)} \left\{T_{w*k}^{\nu^k;\mu^k}\right\} = 0 \tag{4.5.6b}$$

is a Garnir relation of the type (4.4.10a):

$$\sum_{\eta\in\mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \left\{\eta T_{*k}^{\nu^k;\mu^k}\right\} = 0, \tag{4.5.6c}$$

involving entries in positions $\mathcal{X}$ of the $(k-l+1)$th column and positions $\mathcal{Y}$ of the $(k+j)$th column of the F^{μ^k} portion. Conversely a Garnir relation of type (4.5.6c) gives rise to a Trace relation of type (4.5.6a).

Proof. Form θ_w as for the proof of Lemma 4.4.14 from the indices of the relevant two columns of $T_w^{\nu;\mu}$. Let $\mathcal{A}_w \subset \mathcal{I}^{GL(m)}$ be the set of all $i \in \mathcal{A}_w$ such that $\bar{i}$ and i are both present in θ_w. Let $\mathcal{B}_w \subset \mathcal{I}^{GL(m)}$ be the set of all $i \in \mathcal{B}_w$ such that one only of $\bar{i}$ and i is present in θ_w. Let $\mathcal{C}_w \subset \mathcal{I}^{GL(m)}$ be the set of all $i \in \mathcal{C}_w$ such that neither $\bar{i}$ nor i are present in θ_w. Then, if $a_w = \#\mathcal{A}_w$, $b_w = \#\mathcal{B}_w$ and $c_w = \#\mathcal{C}$, $a_w + b_w + c_w = m$ for each w. Factorise $\theta_w = \theta_w^{\mathcal{B}}\theta_w^{\mathcal{A}}$ where $\theta_w^{\mathcal{B}}$ and $\theta_w^{\mathcal{A}}$ are formed solely of barred and unbarred indices from $\mathcal{B}$ and $\mathcal{A}$ respectively. Since a Trace relation involves expressing a number of barred-unbarred index pairs in terms of other such pairs, $\mathcal{B}_w$, b_w and $\theta_w^{\mathcal{B}}$ are constant and their subscripts may be dropped. In addition, since $\theta_w = \theta^{\mathcal{B}}\theta_w^{\mathcal{A}}$, it follows that $a = a_w$ and $c = c_w$ are also constant. Split $\mathcal{B}$ into $\mathcal{B}^\alpha \cup \mathcal{B}^\beta$, such that $i \in \mathcal{B}^\alpha$ if i is present in $\theta^{\mathcal{B}}$ and $i \in \mathcal{B}^\beta$ if $\bar{i}$ is present. If $b^\alpha = \#\mathcal{B}^\alpha$ and $b^\beta = \#\mathcal{B}^\beta$, $b = b^\alpha + b^\beta$.

Let $\mathcal{E} = \cap_w \mathcal{A}_w$, $\mathcal{D}_w = \mathcal{A}_w \backslash \mathcal{E}$, $\mathcal{H} = \cup_w \mathcal{D}_w$, $\mathcal{F}_w = \mathcal{H}\backslash\mathcal{D}_w$ and $\mathcal{G} = \mathcal{I}^{GL(m)}\backslash(\mathcal{H} \cup \mathcal{E} \cup \mathcal{B})$ with $e = \#\mathcal{E}$, $d = \#\mathcal{D}_w$, $h = \#\mathcal{H}$, $f = \#\mathcal{F}_w$, $g = \#\mathcal{G}$, whereupon $h = d + f$, $a = d + e$ and $c = f + g$. Since the Trace equation is of the type specified in Lemma 4.4.14 then $d > g$. Note that $\mathcal{H} = \mathcal{D}_w \cup \mathcal{F}_w$ for each w. With the sets defined in this way, the Trace relation (4.5.6a) may be obtained exactly as in the proof of Lemma 4.4.14. In particular, from (4.4.14h):

$$\theta^{\mathcal{B}}\theta^{\mathcal{E}}\sum_{w}\theta^{\mathcal{D}}_{w}=0, \tag{4.5.7}$$

where $\theta^{\mathcal{D}}_{w}=\prod_{i\in\mathcal{D}_w}\omega_i$, $\theta^{\mathcal{E}}=\prod_{i\in\mathcal{E}}\omega_i$ and $\omega_i=\bar{i}i$. Then, setting $\theta^{\mathcal{A}}_{w}=\theta^{\mathcal{E}}\theta^{\mathcal{D}}_{w}$ gives:

$$\theta^{\mathcal{B}}\sum_{w}\theta^{\mathcal{A}}_{w}=0$$

and then:

$$\sum_{w}\theta_w=0.$$

This final equation is that giving rise to (4.5.6a).

Let θ^*_w be the term of (4.5.6b) corresponding to θ_w of (4.5.6a). θ^*_w is a product of the unbarred terms of θ_w together with starred terms from the complement in $\overline{\mathcal{I}^{GL(m)}}$ of the barred terms of θ_w. These latter terms are starred in order to distinguish them from the unbarred terms of θ^*_w. In fact, they are dopplegangers which when unstarred, will form the $(k-l+1)$th column of the F^{μ^k} portion of $T^{\mathcal{D}^k;\mu^k}_{w*k}$. θ^*_w may be specified simply: $i\in\mathcal{I}^{GL(m)}$ is present in θ^*_w if and only if i is present in θ_w, that is $i\in\mathcal{D}_w\cup\mathcal{E}\cup\mathcal{B}^\alpha$; and $\bar{i}\in\overline{\mathcal{I}^{GL(m)}}$ is present in θ^*_w if and only if $\bar{i}$ is not present in θ_w, that is $i\notin\mathcal{D}_w\cup\mathcal{E}\cup\mathcal{B}^\beta$ and hence $i\in\mathcal{F}_w\cup\mathcal{G}\cup\mathcal{B}^\alpha$. The situation in the corresponding tableaux may be schematically represented thus:

$$T^{\mathcal{D};\mu}_{w}=\begin{array}{|c|c}\hline \overline{\mathcal{D}_w} & \\ \hline \overline{\mathcal{E}} & \\ \hline \overline{\mathcal{B}^\beta} & \\ \hline & \multicolumn{1}{|c|}{\mathcal{B}^\alpha} \\ \cline{2-2} & \multicolumn{1}{|c|}{\mathcal{E}} \\ \cline{2-2} & \multicolumn{1}{|c|}{\mathcal{D}_w} \\ \cline{2-2}\end{array}\quad\Longrightarrow\quad T^{\mathcal{D}^k;\mu^k}_{w*k}=\begin{array}{|c|c|}\hline \mathcal{B}^\alpha & \mathcal{B}^\alpha \\ \hline \mathcal{G} & \mathcal{E} \\ \hline \mathcal{F}_w & \mathcal{D}_w \\ \hline\end{array}\,. \tag{4.5.8}$$

On varying w, the terms θ^*_w run through all partitionings of the set $\mathcal{H}$ into f starred entries and d unbarred entries. Since these are to be respectively placed in the $(k-l+1)$th and the $(k+j)$th column of the F^{μ^k} portion of $T^{\mathcal{D}^k;\mu^k}_{w*k}$, this is an expression of Garnir type. It is necessary to check that sufficient indices from the two columns are involved in this expression. Consider a Garnir relation involving the set of indices $\mathcal{F}_w\cup\mathcal{B}^\alpha$ from the $(k-l+1)$th column and $\mathcal{D}_w$ from the other. This gives the same expression as the above since each migration of an index from $\mathcal{B}^\alpha$ to the other column results in a repeated entry in that column and thus a zero term. This expression involves $f+b^\alpha+d$ terms from the two columns which is a greater number than the length of the $(k-l+1)$th column, $f+g+b^\alpha$, since $d>g$. It remains to show that the sign of each term is as required. By fixing the positions of the elements from the sets $\mathcal{E}$, $\overline{\mathcal{E}}$, $\mathcal{B}_\alpha$ and $\overline{\mathcal{B}_\beta}$ in $T_w^{\rho;\mu}$ and $\mathcal{B}^\alpha$, $\mathcal{G}$ and $\mathcal{E}$ in $T_{w*k}^{\rho^k;\mu^k}$, it can be seen that $\epsilon^{(\mathcal{K}_w,\mathcal{J}_w)} = \pm\epsilon^{(\mathcal{F}_w,\mathcal{D}_w)}$, the sign being independent of w. For each w, the factor $\epsilon^{(\mathcal{F}_w,\mathcal{D}_w)}$ is precisely that required for the appropriate coset representative of the Garnir element giving rise to $T_{w*k}^{\rho^k;\mu^k}$. Thus the sign factor of $\epsilon^{(\mathcal{K}_w,\mathcal{J}_w)}$ given in (4.5.6b) is precisely that required to make (4.5.6b) a Garnir relation.

By partitioning the two relevant columns of the F^{μ^k} portions of a set of tableaux satisfying (4.5.6b) into the sets $\mathcal{B}^\alpha$, $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{H}=\mathcal{F}_w\cup\mathcal{D}_w$ as in (4.5.8), and setting $\mathcal{B}^\beta$ to be the remaining indices, the Trace relation corresponding to any Garnir relation may be found by reversing the above construction.

In order to illustrate Theorem 4.5.8, consider the mutually associate $GL(7)$-modules $W^{2,2,2,1}$ and $W^{\overline{1,1,1};1,1,1}$, and consider the Garnir relation, in the former of these modules, which takes the form:

$$\left\{\begin{matrix}1&1\\3&4\\6&5\\7&\end{matrix}\right\}-\left\{\begin{matrix}1&1\\3&4\\5&6\\7&\end{matrix}\right\}+\left\{\begin{matrix}1&1\\3&4\\5&7\\6&\end{matrix}\right\}+\left\{\begin{matrix}1&1\\3&5\\4&6\\7&\end{matrix}\right\}-\left\{\begin{matrix}1&1\\3&5\\4&7\\6&\end{matrix}\right\}+\left\{\begin{matrix}1&1\\3&6\\4&7\\5&\end{matrix}\right\}=0. \quad (4.5.9a)$$

This is the Garnir relation resulting from permuting the sets of indices from the positions given by the sets $\mathcal{X}=\{1,3,4\}$ and $\mathcal{Y}=\{6,7\}$ as in Lemma 4.4.10a. Notice that the anticipated $5!/3!2!=10$ terms is reduced to 6 in this case, since the 4 terms in which two identical indices 1 appear in the second column have been excluded since they are zero by Lemma 4.4.9.

The corresponding Trace relation may be constructed by noting that, in this example, the w-independent sets encountered in the proof of Theorem 4.5.6 are, as indicated in the final paragraph of the proof:

$$\mathcal{B}^\alpha=\{1\},\quad \mathcal{G}=\{3\},\quad \mathcal{E}=\emptyset,\quad \mathcal{H}=\{4,5,6,7\},\quad \mathcal{B}^\beta=\{2\},$$

and that $d=f=2$. Then $\theta^{\mathcal{B}}\theta^{\mathcal{E}}=1\bar{2}$ and (4.5.7) becomes:

$$1\bar{2}(\omega_4\omega_5+\omega_4\omega_6+\omega_4\omega_7+\omega_5\omega_6+\omega_5\omega_7+\omega_6\omega_7)=0,$$

the sets $\mathcal{D}_w\subset\mathcal{H}$ being respectively $\mathcal{D}_1=\{4,5\}$, $\mathcal{D}_2=\{4,6\}$, $\mathcal{D}_3=\{4,7\}$, $\mathcal{D}_4=\{5,6\}$, $\mathcal{D}_5=\{5,7\}$ and $\mathcal{D}_6=\{6,7\}$. Hereupon, $\theta_1=1\bar{2}\bar{4}4\bar{5}5$, $\theta_2=1\bar{2}\bar{4}4\bar{6}6$, $\theta_3=1\bar{2}\bar{4}4\bar{7}7$, $\theta_4=1\bar{2}\bar{5}5\bar{6}6$, $\theta_5=1\bar{2}\bar{5}5\bar{7}7$, $\theta_6=1\bar{2}\bar{6}6\bar{7}7$ and $\sum_{w=1}^{6}\theta_w=0$. In terms of the composite tableaux of $W^{\bar{1},\bar{1},\bar{1};1,1,1}$, this expression takes the form:

$$\left\{\begin{array}{cc} \bar{5} & \\ \bar{4} & \\ \bar{2} & \\ & 1 \\ & 4 \\ & 5 \end{array}\right\} + \left\{\begin{array}{cc} \bar{6} & \\ \bar{4} & \\ \bar{2} & \\ & 1 \\ & 4 \\ & 6 \end{array}\right\} + \left\{\begin{array}{cc} \bar{7} & \\ \bar{4} & \\ \bar{2} & \\ & 1 \\ & 4 \\ & 7 \end{array}\right\} + \left\{\begin{array}{cc} \bar{6} & \\ \bar{5} & \\ \bar{2} & \\ & 1 \\ & 5 \\ & 6 \end{array}\right\} + \left\{\begin{array}{cc} \bar{7} & \\ \bar{5} & \\ \bar{2} & \\ & 1 \\ & 5 \\ & 7 \end{array}\right\} + \left\{\begin{array}{cc} \bar{7} & \\ \bar{6} & \\ \bar{2} & \\ & 1 \\ & 6 \\ & 7 \end{array}\right\} = 0. \quad (4.5.9b)$$

These terms are in one-to-one correspondence with those of the Garnir relation (4.5.9a), each one being an associate of the corresponding term in (4.5.9a). Note that the signs in (4.5.9b) map to those in (4.5.9a) on multiplication by the respective $\epsilon^{(\mathcal{K}_w,\mathcal{J}_w)}$.

Corresponding to Theorem 4.5.6, there is an analogous result concerning a Garnir relation involving the first and another column of the F^{ν} portion of a composite tableau $T^{\nu;\mu}$ and a Trace relation involving the first columns of the two portions of $T_{*1}^{\nu^1;\mu^1}$. The proof of this result procedes along lines similar to that of Theorem 4.5.6. Although mysterious at first, these two results enable it to be proved that the $GL(m)$-modules $W^{\nu;\mu}$ and $W^{\nu^1;\mu^1}$, and hence $W^{\nu^k;\mu^k}$, are isomorphic upon restriction to $SL(m)$. For the moment consider the $GL(m)$-modules $W^{\nu;\mu}$ and $W^{\nu^1;\mu^1}$ solely as vector spaces. By Theorem 4.4.30 these are spanned by all $\{T^{\nu;\mu}\}$ and $\{T^{\nu^1;\mu^1}\}$ respectively modulo relations 4.4.9, 4.4.10 and 4.4.13.

Lemma 4.5.10. *The linear map* $L_* : W^{\nu;\mu} \longrightarrow W^{\nu^1;\mu^1}$ *is a well defined isomorphism between the vector spaces* $W^{\nu;\mu}$ *and* $W^{\nu^1;\mu^1}$.

Proof. Let $T^{\nu;\mu}$ be column strict. By using the Column relations, Garnir relations and Trace relations, $\{T^{\nu;\mu}\}$ is uniquely expressible in terms of the $GL(m)$-standard tableaux:

$$\{T^{\nu;\mu}\} = \sum_i \zeta_i \{T_i^{\nu;\mu}\}, \qquad (4.5.10a)$$

where each $T_i^{\nu;\mu}$ is $GL(m)$-standard. Theorem 4.5.6 shows that to each Trace relation involving the first columns of each portion of some $\{T^{\nu;\mu}\}$, there is a Garnir relation resulting from the action of L_*, as given by (4.5.5), on each term. This Garnir relation necessarily involves $\{T_{*1}^{\nu^1;\mu^1}\}$. Likewise, every Garnir relation involving the first column of the F^{ν} portion of some $T^{\nu;\mu}$ corresponds, through the action of L_*, to a Trace relation, necessarily involving $\{T_{*1}^{\nu^1;\mu^1}\}$. Garnir relations involving other columns remain as they are under the action of L_*. Thus, since if

$T_i^{\nu;\mu}$ is $GL(m)$-standard then $T_{i*1}^{\nu^1;\mu^1}$ is $GL(m)$-standard, the standardisation of $T_{*1}^{\nu^1;\mu^1}$ mirrors, under the action of L_*, the standardisation producing (4.5.10). Therefore:

$$\{T_{*1}^{\nu^1;\mu^1}\} = \sum_i \zeta_i \{T_{i*1}^{\nu^1;\mu^1}\}. \tag{4.5.10b}$$

Since this is the result of the application of L_* to (4.5.10a) and the two expressions in terms of $GL(m)$-standard tableaux are unique, the lemma follows.

In order to utilise these results to deal with $W^{\nu;\mu}$ and $W^{\nu^k;\mu^k}$ as $GL(m)$-modules, it is necessary to define the raising and lowering operators which formally perform the transition from $\{T^{\nu;\mu}\}$ to $\{T_{*1}^{\nu^1;\mu^1}\}$. Throughout the remainder of this section the convention of summing over all repeated indices will be used unless otherwise indicated.

Let $m = s + t$,

$$L_{a_1\cdots a_s b_1\cdots b_t} = \frac{1}{t!}\epsilon_{a_1\cdots a_s b_1\cdots b_t}, \tag{4.5.11a}$$

$$L^{a_1\cdots a_s b_1\cdots b_t} = \frac{1}{t!}\epsilon^{a_1\cdots a_s b_1\cdots b_t}, \tag{4.5.11b}$$

and

$$K_{a_1\cdots a_s}^{c_1\cdots c_s} = (-1)^{st} L_{a_1\cdots a_s b_1\cdots b_t} L^{b_1\cdots b_t c_1\cdots c_s} = (-1)^{st} L^{c_1\cdots c_s b_1\cdots b_t} L_{b_1\cdots b_t a_1\cdots a_s}. \tag{4.5.12}$$

Lemma 4.5.13.

$$K_{a_1\cdots a_s}^{c_1\cdots c_s} = \frac{1}{s!}\left(\sum_{\pi\in S_s}(-1)^\pi \delta_{a_1}^{c_{\pi(1)}}\cdots\delta_{a_s}^{c_{\pi(s)}}\right). \tag{4.5.13}$$

Proof. $\epsilon_{a_1\cdots a_s b_1\cdots b_t} = (-1)^{st}\epsilon_{b_1\cdots b_t a_1\cdots a_s}$ since the order of the subscripts may be changed from one to the other by 'passing' each a subscript through each b subscript and this requires st transpositions altogether. Then, from (4.5.11) and (4.5.12),

$$K_{a_1\cdots a_s}^{c_1\cdots c_s} = \frac{1}{t!s!}\epsilon_{a_1\cdots a_s b_1\cdots b_t}\epsilon^{c_1\cdots c_s b_1\cdots b_t}.$$

Since $s+t = m$, all indices from $\mathcal{I}^{GL(m)}$ must appear as both a subscript and a superscript on the right side for non-zero contributions, and thus only if $\{a_1, a_2, \ldots, a_s\} = \{c_1, c_2, \ldots, c_s\}$. Then, for fixed distinct $b_1, b_2, \ldots b_t$,

$$\epsilon_{a_1\cdots a_s b_1\cdots b_t}\epsilon^{c_1\cdots c_s b_1\cdots b_t} = \sum_{\pi\in S_s}(-1)^\pi \delta_{a_1}^{c_{\pi(1)}}\cdots\delta_{a_n}^{c_{\pi(m)}}.$$

The lemma then follows from summing the bs over all $t!$ permutations of the set $\mathcal{I}^{GL(m)}\backslash\{a_1, a_2, \ldots, a_s\}$.

This lemma shows that K is an antisymmetriser, prompting the definitions:

$$e_{[a_1\cdots a_s]} = K^{b_1\cdots b_s}_{a_1\cdots a_s} e_{b_1\cdots b_s} \tag{4.5.14a}$$

and

$$e^{[a_1\cdots a_s]} = K^{a_1\cdots a_s}_{b_1\cdots b_s} e^{b_1\cdots b_s}. \tag{4.5.14b}$$

However, the tensor $e^{[j_1\cdots j_s]}$, antisymmetric in the indices $\mathcal{J} = \{j_1,\ldots,j_s\}$ is equal to the tensor:

$$\frac{1}{s!}\epsilon^{(\mathcal{K},\mathcal{J})} \sum_{1\le b_1,\ldots,b_s\le m} \epsilon_{k_1\cdots k_t b_1\cdots b_s} e^{b_1\cdots b_s} = \epsilon^{(\mathcal{K},\mathcal{J})} \sum_{1\le b_1,\ldots,b_s\le m} L_{k_1\cdots k_t b_1\cdots b_s} e^{b_1\cdots b_s}, \tag{4.5.15}$$

antisymmetric in the set of t contravariant indices $\mathcal{K} = \{k_1,\ldots,k_t\} = \mathcal{I}^{GL(m)}\backslash\mathcal{J}$. Here, for clarity, the summations are shown explicitly. Thus the operator $L_{a_1\cdots a_s b_1\cdots b_t}$ may be used to lower an antisymmetric set of contravariant indices. Similarly, the operator $L^{a_1\cdots a_s b_1\cdots b_t}$ may be used to raise an antisymmetric set of covariant indices. In each case the tensor is antisymmetric in the new indices. Note that $GL(m)$ does not act on these new lowered or raised indices directly since, in general, $GL(m)$ does not commute with the raising and lowering operators:

$$\begin{aligned} &L^{a_1\cdots a_s b_1\cdots b_t} G^{c_1}{}_{b_1}\cdots G^{c_t}{}_{b_t} \\ &\quad = (\det G) L^{d_1\cdots d_s e_1\cdots e_t} (G^{-1})^{a_1}{}_{d_1}\cdots (G^{-1})^{a_s}{}_{d_s} (G^{-1})^{b_1}{}_{e_1}\cdots (G^{-1})^{b_t}{}_{e_t} G^{c_1}{}_{b_1}\cdots G^{c_t}{}_{b_t} \\ &\quad = (\det G)(G^{-1})^{a_1}{}_{d_1}\cdots (G^{-1})^{a_s}{}_{d_s} L^{d_1\cdots d_s c_1\cdots c_t}, \end{aligned} \tag{4.5.16a}$$

and similarly,

$$L_{a_1\cdots a_s b_1\cdots b_t} (G^{-1})^{b_1}{}_{c_1}\cdots (G^{-1})^{b_t}{}_{c_t} = (\det G)^{-1} G^{d_1}{}_{a_1}\cdots G^{d_s}{}_{a_s} L_{d_1\cdots d_s c_1\cdots c_t}. \tag{4.5.16b}$$

The upshot of this analysis is that an antisymmetrised column of a composite tableau $T^{\bar{\nu};\mu}$ containing the barred indices from the set $\overline{\mathcal{J}}$ may be replaced by a column of unbarred indices formed from the complement of $\mathcal{J}$ in $\mathcal{I}^{GL(m)}$, provided that the appropriate factors of $(\det G)$ are included for each module action of $G \in GL(m)$. If $\tilde{\nu}_1 = s$ and $e^{j_1\cdots j_s b_{s+1}\cdots b_v}{}_{a_1\cdots a_u}$ is the tensor corresponding to $T^{\bar{\nu};\mu}$, then the action of $L_{k_1\cdots k_t j_1\cdots j_s}$ is to produce the tensor $\epsilon^{(\mathcal{K},\mathcal{J})} e^{b_{s+1}\cdots b_v}{}_{[k_1\cdots k_t]a_1\cdots a_u}$, antisymmetric in the indices from the set $\mathcal{K} = \mathcal{I}^{GL(m)}\backslash\mathcal{J}$. This may naturally be replaced by a sum over $t!$ composite tableaux of shape $F^{\bar{\nu}^1;\mu^1}$. A column strict representative of these tableaux may be selected. This tableau is $T^{\bar{\nu}^1;\mu^1}_{*1}$. By the foregoing argument it may be assumed to be antisymmetric in the indices of that first column. Similarly, the image of $\{T^{\bar{\nu};\mu}\}$ under lowering the indices of the first column of the $F^{\bar{\nu}}$ portion may be denoted $\epsilon^{(\mathcal{K},\mathcal{J})}\{T^{\bar{\nu}^1;\mu^1}_{*1}\}^*$, the extra asterisk indicating that it is yet to be shown that these objects have all the properties implied by the Young operators Y^{μ^1} and Y^{ν^1}. That they are antisymmetric in each column follows immediately from the foregoing argument. Since the lowering operator acts linearly, the space which they occupy is isomorphic to $W^{\bar{\nu};\mu}$ and hence, by Lemma 4.5.10, isomorphic to $W^{\bar{\nu}^1;\mu^1}$. Therefore $\epsilon^{(\mathcal{K},\mathcal{J})}\{T_{*1}^{\bar{\nu}^1;\mu^1}\}^*$ may be identified with $\epsilon^{(\mathcal{K},\mathcal{J})}\{T_{*1}^{\bar{\nu}^1;\mu^1}\}$ and the lowering operator given by (4.5.11a) may be identified with L_* from (4.5.5). The main point here is that the Garnir relations, newly discovered by virtue of Theorem 4.5.6, enable the 'new' column of the composite tableau $T_{*1}^{\bar{\nu}^1;\mu^1}$ to participate in the symmetry implied by the composite Young operator $Y_*^{\nu^1} \otimes Y_*^{\mu^1}$. From (4.5.16$b$):

$$\begin{aligned} G\,(\epsilon^{(\mathcal{K},\mathcal{J})}\{T_{*1}^{\bar{\nu}^1;\mu^1}\}) &= G\,(L_* : \{T^{\bar{\nu};\mu}\}) \\ &= (\det G)L_*\,(G\,\{T^{\bar{\nu};\mu}\}), \end{aligned} \tag{4.5.17}$$

for all $G \in GL(m)$. Therefore, the following theorem has been proved.

Theorem 4.5.18. *Let $\mu \in P(u;s)$ and $\nu \in P(v;t)$ with $s+t \leq m$. Under restriction of $GL(m)$ to $SL(m)$, the $GL(m)$-module $W^{\bar{\nu};\mu}$ is isomorphic to $W^{\bar{\nu}^1;\mu^1}$. Each of these modules is isomorphic to W^λ where $\lambda = (\bar{\nu}^q;\mu^q)$ is the partition canonically associate to $(\bar{\nu};\mu)$ where $q = \nu_1$. The representation $\{\lambda\}$ of $SL(m)$ is equivalent to each representation $\{\bar{\nu}^i;\mu^i\}$ of $SL(m)$. The representation $\{\lambda\} = \{\bar{\nu}^q;\mu^q\}$ of $GL(m)$ is equivalent to each representation* $(\det G)^{q-i}\{\bar{\nu}^i;\mu^i\}$ *of $GL(m)$.*

Conversely, the following theorem holds.

Theorem 4.5.19. *Let $\mu \in P(u;s)$ and $\nu \in P(v;t)$ with $s+t \leq m$. The isomorphism of the $GL(m)$-modules $W^{\bar{\nu};\mu}$ and $W^{\bar{\nu}^1;\mu^1}$ under restriction to $SL(m)$ implies that the Trace relations and the Garnir relations are equivalent statements; that is, one implies the other.*

Proof. This follows since the operators given by (4.5.16a) and (4.5.16b), which define the isomorphism, may be used to convert between the Trace relations (4.5.6a) and the Garnir relations (4.5.6b).

Chapter 5
Orthogonal Group Modules

§5.1. Orthogonal standard tableaux

This chapter introduces the Young tableaux techniques used to construct irreducible modules of the orthogonal groups $O(m)$ and $SO(m)$, defined in Definition 2.1.1. As for the classical groups considered in the previous chapter, these modules are constructed as submodules of $V^{\otimes l}$ where V is the defining $O(m)$- or $SO(m)$-module. Since $O(m)$ and $SO(m)$ are subgroups of $GL(m)$, the $GL(m)$-module W^λ also serves as a module for these orthogonal groups. As in the symplectic case, W^λ is, in general, reducible due to the existence of trace tensors. The primary objective of this chapter is to extract these trace tensors in a systematic way and thence to project the irreducible $O(m)$- or $SO(m)$-modules out of W^λ. However, as a consequence of the invariant form being symmetric, the situation is more complicated for the orthogonal groups than for $Sp(2r)$, and trace terms will need to be simultaneously extracted from two columns of the symmetrised tableaux.

One major difficulty in the orthogonal case is the specification of a set of suitable standard tableaux. In recent years, a number of authors have derived various such sets to facilitate the calculation of dimensions, weights and characters of the irreducible representations. The first such set [**KE83**] employed indices from the set $\mathcal{I}^{O(m)}$. However, it was necessary to count tableaux having certain entry configurations more than once. This is clearly inconvenient in specifying basis elements for the irreducible $O(m)$-modules. Furthermore, those $O(m)$-modules that are mutually associate use the same set of tableaux. The first of these problems was dealt with in [**KT90**] by introducing extra indices, each of which could only appear in a particular position in a tableau. Here the extra indices and the rules associated with them obviate the need to count any tableau more than once. For $O(2r)$ it is easy to see how these tableaux are equivalent to those of [**KE83**]. However, with a view to the present problem, it is not clear how these extra indices could arise from the $GL(m)$-standard tableaux of W^λ, and in particular, how $O(m)$ would act on these indices.

A further set of tableaux for the $O(2r+1)$ case was proposed in [**Su90**]. These employ the index ∞ with the seemingly extraordinary properties that it may occur in certain columns more than once, and may not occur more than once in any one row. Thus, once more, these tableaux seem to offer no hope of effecting the reduction of W^λ as an $O(m)$-module.

One feature of the sets of tableaux just mentioned is that if $\tilde{\lambda}_1 > m/2$ then each is empty for the case of $O(m)$. This appears to be at odds with Weyl's reasoning **[We39]** that inequivalent irreducible representations of $O(m)$ are labelled by those partitions λ for which $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$. In answer to this objection, Proctor **[Pr89]** derived two important sets of tableaux. In the definitions that follow $\mathcal{I}^{O(2r)} = \mathsf{N}_r \cup \overline{\mathsf{N}}_r$ and $\mathcal{I}^{O(2r+1)} = \mathsf{N}_r \cup \overline{\mathsf{N}}_r \cup \{0\}$, these indices being ordered such that $\bar{1} < 1 < \bar{2} < 2 < \cdots < \bar{r} < r < 0$. Then in the tableau T^λ, α_i is the number of indices less than or equal to i in the first column of T^λ, and likewise β_i for the second column of T^λ, for each $i \in \mathcal{I}^{O(m)}$.

Definition 5.1.1. **[Pr89]** *With $\lambda \in P(l)$ and $m = 2r$ or $m = 2r + 1$, let $\mathcal{P}^\lambda_m$ be the set of tableaux such that $T^\lambda \in \mathcal{P}^\lambda_m$ if and only if:*

(i) the entries are taken from the set $\mathcal{I}^{O(m)}$;

(ii) the entries are strictly increasing from top to bottom down each column;

(iii) the entries are non-decreasing from left to right across each row;

(iv) $\alpha_i + \beta_i \leq 2i$ for each $i \in \mathsf{N}_r$, and $\alpha_0 + \beta_0 \leq m$;

(v) if, for some $i \in \mathsf{N}_r$, $\alpha_i + \beta_i = 2i$ with $\alpha_i > \beta_i$ and $T^\lambda_{(\alpha_i,1)} = \bar{i}$ and $T^\lambda_{(\beta_i,b)} = i$ for some b then $T^\lambda_{(\beta_i-1,b)} = \bar{i}$;

(vi) if, for some $i \in \mathsf{N}_r$, $\alpha_i + \beta_i = 2i$ with $\alpha_i = \beta_i (= i)$ and $T^\lambda_{(i,1)} = \bar{i}$ and $T^\lambda_{(i,b)} = i$ for some b then $T^\lambda_{(i-1,b)} = \bar{i}$.

Conditions (v) and (vi) of this definition may be combined, but distinguishing the two will prove convenient later. These two conditions are known as protection conditions since in each case, certain combinations of i and $\bar{i}$ require the i to be protected by an $\bar{i}$ immediately above it. Protection conditions similar to these were first encountered in the tableaux introduced in **[KE83]** for $O(m)$.

Definition 5.1.2. **[Pr89]** *With $\lambda \in P(l)$ let $\mathcal{Q}^\lambda_m$ be the set of tableaux such that $T^\lambda \in \mathcal{Q}^\lambda_m$ if and only if:*

(i) the entries are taken from the set $\mathcal{I}^{O(m)}$;

(ii) the entries are strictly increasing from top to bottom down each column;

(iii) the entries are non-decreasing from left to right across each row;

(iv) $\alpha_i + \beta_i \leq 2i$ for each $i \in \mathsf{N}_r$, $\alpha_{\bar{i}} + \beta_{\bar{i}} \leq 2i - 1$ for each $i \in \mathsf{N}_r$ and $\alpha_0 + \beta_0 \leq m$.

The sets of tableaux $\mathcal{P}_m^\lambda$ and $\mathcal{Q}_m^\lambda$ were referred to as fine and coarse tableaux respectively in [**Pr90**] where each was used to develop a Robinson-Schensted correspondence (see [**Kn70**]) for $O(m)$. In order to illustrate these definitions, consider the tableaux:

$$\begin{array}{llll} \begin{array}{lll} \bar{1} & \bar{1} & 3 \\ 1 & \bar{2} & \\ 3 & 3 & \end{array}, & \begin{array}{lllll} \bar{1} & 2 & 2 & \bar{3} & 3 \\ 1 & \bar{3} & \bar{3} & 3 & \\ \bar{2} & \bar{4} & 4 & & \\ \bar{3} & & & & \end{array}, & \begin{array}{lll} \bar{1} & \bar{2} & \bar{2} \\ 1 & \bar{3} & \bar{3} \\ \bar{2} & \bar{4} & \\ 3 & 4 & \end{array}, & \begin{array}{lll} \bar{1} & 1 & \bar{2} \\ 2 & 2 & \\ \bar{3} & & \\ 3 & & \end{array}, \\ \begin{array}{llll} \bar{1} & \bar{1} & \bar{1} & 3 \\ 2 & 2 & \bar{3} & \\ \bar{3} & \bar{3} & 3 & \end{array}, & \begin{array}{lll} 1 & 1 & \bar{3} \\ \bar{2} & 3 & 3 \\ \bar{3} & & \\ 3 & & \end{array}, & \begin{array}{lll} \bar{1} & 2 & 3 \\ 1 & \bar{3} & 0 \\ 2 & 0 & \\ \bar{3} & & \end{array}, & \begin{array}{lll} \bar{1} & 2 & 2 \\ 1 & \bar{3} & 0 \\ 2 & 0 & \\ \bar{3} & & \\ 0 & & \end{array}, \end{array} \tag{5.1.3}$$

labelled $T_1^{3,2,2}$, $T_2^{5,4,3,1}$, $T_3^{3,3,2,2}$, $T_4^{3,2,1,1}$, $T_5^{4,3,3}$, $T_6^{3,3,1,1}$, $T_7^{3,3,2,1}$ and $T_8^{3,3,2,1,1}$ respectively. Then only $T_3^{3,3,2,2}$, $T_5^{4,3,3}$, $T_6^{3,3,1,1}$ and $T_7^{3,3,2,1}$ are respectively members of some $\mathcal{P}_m^\lambda$, and only $T_4^{3,2,1,1}$ and $T_6^{3,3,1,1}$ are respectively members of some $\mathcal{Q}_m^\lambda$. In particular $T_1^{3,2,2}$ violates the $i = 1$ case of condition (iv) in each case, and $T_2^{5,4,3,1}$ violates the $i = 2$ case of protection condition (v) of Definition 5.1.1 and the $i = \bar{3}$ case of condition (iv) of Definition 5.1.2.

Note that in order for $\mathcal{P}_m^\lambda$ and $\mathcal{Q}_m^\lambda$ to be non-empty, conditions (ii) and (iv) of Definitions 5.1.1 and 5.1.2 require that $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$ in each case. In fact, there is a straightforward bijection between $\mathcal{P}_m^\lambda$ and $\mathcal{Q}_m^\lambda$, given in [**Pr90**], demonstrating that these two sets are of the same cardinality. Their derivation [**Pr89**], shows that this number is the dimension of the irreducible representation $[\lambda]$ of $O(m)$. This is also given by (2.5.5c). The set $\mathcal{P}_m^\lambda$ also yields the weights and the character of this representation.

Definition 5.1.4. *$O(m)$-weight. Let $m = 2r$ or $m = 2r + 1$ and for $i = 1, 2, \ldots, r$, let*

$$n_i^{O(m)}(T^\lambda) = n_i(T^\lambda) - n_{\bar{i}}(T^\lambda),$$

where $n_j(T^\lambda)$ is the number of appearances of the index $j \in \mathcal{I}^{O(m)}$ in T^λ. The vector $n^{O(m)}(T^\lambda) = (n_1^{O(m)}(T^\lambda), n_2^{O(m)}(T^\lambda), \ldots, n_r^{O(m)}(T^\lambda))$ is known as the $O(m)$-weight of T^λ.

Theorem 5.1.5. [**Pr89**] *Let $m = 2r$ or $m = 2r + 1$. The multiplicity of the weight $(n_1, n_2, \ldots, n_r)$ in the irreducible representation $[\lambda]$ of $O(m)$ is given by the number* *of tableaux* $T^\lambda \in \mathcal{P}_m^\lambda$ *such that* $n^{O(m)}(T^\lambda) = (n_1, n_2, \ldots, n_r)$. *The character of this representation is given by:*

$$[\lambda](y) = \sum_{T^\lambda \in \mathcal{P}_m^\lambda} y^{T^\lambda}, \tag{5.1.5}$$

where (y) *denotes the vector* $(y_1, y_2, \ldots, y_r)$ *and* $y^{T^\lambda} = y_1^{n_1^{O(m)}(T^\lambda)} y_2^{n_2^{O(m)}(T^\lambda)} \cdots y_r^{n_r^{O(m)}(T^\lambda)}$, *for those elements of* $O(m)$ *with positive determinant and if* $m = 2r+1$, *eigenvalues* $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r, 1$, *and if* $m = 2r$, *eigenvalues* $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r$.

Although $\mathcal{P}_m^\lambda$ goes a long way towards fulfilling all the desired properties of a definitive set of $O(m)$-standard tableaux, the construction of a standardisation procedure proves tricky. In view of this, a different set of tableaux, closely related to $\mathcal{P}_m^\lambda$, are used. These new tableaux are especially convenient when the reduction to $SO(m)$ is made.

Definition 5.1.6. *With* $\lambda \in P(l)$ *and* $m = 2r$ *or* $m = 2r+1$, *the tableau* T^λ *is* $O(m)$*-standard if and only if:*

(i) the entries are taken from the set $\mathcal{I}^{O(m)}$;

(ii) the entries are strictly increasing from top to bottom down each column;

(iii) the entries are non-decreasing from left to right across each row;

(iv) $\alpha_i + \beta_i \leq 2i$ *for each* $i \in \mathbb{N}_r$, *and* $\alpha_0 + \beta_0 \leq m$;

(v) if, for some $i \in \mathbb{N}_r$, $\alpha_i + \beta_i = 2i$ *with* $\alpha_i > \beta_i$ *and* $T^\lambda_{(\alpha_i,1)} = i$ *and* $T^\lambda_{(\beta_i,2)} = \bar{i}$ *then* $T^\lambda_{(\alpha_i-1,1)} = \bar{i}$;

(vi) if, for some $i \in \mathbb{N}_r$, $\alpha_i + \beta_i = 2i$ *with* $\alpha_i = \beta_i (= i)$ *and* $T^\lambda_{(i,1)} = \bar{i}$ *and* $T^\lambda_{(i,b)} = i$ *for some* b *then* $T^\lambda_{(i-1,b)} = \bar{i}$.

The set of all $O(m)$*-standard tableaux* T^λ *of shape* λ *is denoted by* $\mathcal{O}_m^\lambda$.

This definition implies that of those tableaux given in (5.1.3), only $T_2^{5,4,3,1}$, $T_5^{4,3,3}$, $T_6^{3,3,1,1}$ and $T_7^{3,3,2,1}$ are $O(m)$-standard for some m.

Note that Definition 5.1.6 differs from Definition 5.1.1 only through their conditions (v). In order to show that $\mathcal{O}_m^\lambda$ has the desired properties of $\mathcal{P}_m^\lambda$, in particular that $\mathcal{O}_m^\lambda$ satisfies the analogue to Theorem 5.1.5, it is necessary to construct an $O(m)$-weight preserving bijection between $\mathcal{P}_m^\lambda$ and $\mathcal{O}_m^\lambda$.

Let $m = 2r$ or $m = 2r+1$. For $i = 1, 2, \ldots, r$, let $\mathcal{P}_{m,i}^\lambda \subset \mathcal{P}_m^\lambda$ be such that $T^\lambda \in \mathcal{P}_{m,i}^\lambda$ if $\alpha_i + \beta_i = 2i$, $\alpha_i > \beta_i$, $T^\lambda_{(\alpha_i,1)} = i$, $T^\lambda_{(\alpha_i-1,1)} \neq \bar{i}$ and $T^\lambda_{(\beta_i,2)} = \bar{i}$. For example, from (5.1.3), $T_3^{3,3,2,2} \in \mathcal{P}_{m,3}^{3,3,2,2}$ for each $m \geq 8$. Then, if $T^\lambda \in \mathcal{P}_{m,i}^\lambda$, $T^\lambda \notin \mathcal{O}_m^\lambda$ since it violates condition (v) of Definition 5.1.6. Let $\mathcal{P}_{m,0}^\lambda = \bigcup_{i=1}^r \mathcal{P}_{m,i}^\lambda$. In general this union is not disjoint. Each element of $\mathcal{P}_m^\lambda \backslash \mathcal{P}_{m,0}^\lambda$ satisfies all the conditions of Definition 5.1.6 so that $\mathcal{P}_m^\lambda \backslash \mathcal{P}_{m,0}^\lambda \subset \mathcal{O}_m^\lambda$. Since $\mathcal{P}_{m,0}^\lambda \cap \mathcal{O}_m^\lambda = \emptyset$, $\mathcal{P}_m^\lambda \cap \mathcal{O}_m^\lambda = \mathcal{P}_m^\lambda \backslash \mathcal{P}_{m,0}^\lambda$.

Let ϕ_i be an operator such that if $T^\lambda \in \mathcal{P}_m^\lambda \backslash \mathcal{P}_{m,i}^\lambda$ then $\phi_i T^\lambda = T^\lambda$, and if $T^\lambda \in \mathcal{P}_{m,i}^\lambda$ then $\phi_i T^\lambda$ is obtained from T^λ by interchanging the entry i in the first column and the rightmost entry $\bar{\imath}$ in the β_ith row:

$$
\begin{array}{cllllllll}
 & \cdot & \cdot & \cdot & k & \cdot & \cdot & \bar{\imath} & \bar{\imath} \\
\beta_i & \cdot & \bar{\imath} & \bar{\imath} & \bar{\imath} & i & i & i & i \\
 & \vdots & & & & & & & \\
 & j & & & & & & & \\
\alpha_i & i & & & & & & &
\end{array}
\quad \xrightarrow{\phi_i} \quad
\begin{array}{cllllllll}
 & \cdot & \cdot & \cdot & k & \cdot & \cdot & \bar{\imath} & \bar{\imath} \\
\beta_i & \cdot & \bar{\imath} & \bar{\imath} & i & i & i & i & i \\
 & \vdots & & & & & & & \\
 & j & & & & & & & \\
\alpha_i & \bar{\imath} & & & & & & &
\end{array}
\quad ,
$$

with $j, k < \bar{\imath}$.

It follows that $\phi_i T^\lambda$ satisfies $\alpha_i + \beta_i = 2i$, $\alpha_i > \beta_i$, $T_{(\alpha_i,1)}^\lambda = \bar{\imath}$, $T_{(\beta_i,b)}^\lambda = i$ for some b such that $T_{(\beta_i-1,b)}^\lambda \neq \bar{\imath}$, thereby violating condition (v) of Definition 5.1.1. Hence $\phi_i T^\lambda \notin \mathcal{P}_m^\lambda$. However, $\phi_i T^\lambda$ does not violate, for the given i, condition (v) of Definition 5.1.6.

Let $\phi = \prod_{i=1}^r \phi_i$. If $T^\lambda \in \mathcal{P}_m^\lambda \backslash \mathcal{P}_{m,0}^\lambda = \mathcal{P}_m^\lambda \cap \mathcal{O}_m^\lambda$ then $\phi T^\lambda = T^\lambda \in \mathcal{O}_m^\lambda$. On the other hand if $T^\lambda \in \mathcal{P}_{m,0}^\lambda$ so that $T^\lambda \notin \mathcal{O}_m^\lambda$ then $\phi T^\lambda \notin \mathcal{P}_{m,0}^\lambda$ but $\phi T^\lambda \in \mathcal{O}_m^\lambda$. Hence $\phi \mathcal{P}_m^\lambda \subseteq \mathcal{O}_m^\lambda$, and more precisely $\phi \mathcal{P}_m^\lambda = \mathcal{O}_m^\lambda$ since $\mathcal{O}_m^\lambda \backslash (\mathcal{P}_m^\lambda \cap \mathcal{O}_m^\lambda) = \phi \mathcal{P}_{m,0}^\lambda$, as can be seen by comparing conditions (v) of Definitions 5.1.1 and 5.1.6. Finally, the nature of ϕ_i illustrated above implies that ϕ is one-to-one and thus a bijection. In addition ϕ preserves weights since under each map ϕ_i, the list of entries in any tableau remains fixed. As an example, let $\lambda = (5, 4, 1^4)$ and consider:

$$
\begin{array}{lllll}
\bar{1} & \bar{2} & \bar{2} & 2 & 2 \\
1 & \bar{4} & 4 & 4 & \\
2 & & & & \\
\bar{3} & & & & \\
3 & & & & \\
4 & & & &
\end{array}
\quad \xrightarrow{\phi_2} \quad
\begin{array}{lllll}
\bar{1} & \bar{2} & 2 & 2 & 2 \\
1 & \bar{4} & 4 & 4 & \\
\bar{2} & & & & \\
\bar{3} & & & & \\
3 & & & & \\
4 & & & &
\end{array}
\quad \xrightarrow{\phi_4} \quad
\begin{array}{lllll}
\bar{1} & \bar{2} & 2 & 2 & 2 \\
1 & 4 & 4 & 4 & \\
\bar{2} & & & & \\
\bar{3} & & & & \\
3 & & & & \\
\bar{4} & & & &
\end{array}
\quad . \qquad (5.1.7)
$$

Here ϕ_1 and ϕ_3 have no effect. Thus the tableau on the left, which is a member of $\mathcal{P}_8^\lambda$ and not of $\mathcal{O}_8^\lambda$, is mapped, under ϕ, to that $O(8)$-standard tableau on the right. This tableau is not a member of $\mathcal{P}_8^\lambda$.

Theorem 5.1.5 can now be stated in terms of the $O(m)$-standard tableaux.

Theorem 5.1.8. *Let $m = 2r$ or $m = 2r + 1$. The multiplicity of the weight $(n_1, n_2, \ldots, n_r)$ in the irreducible representation $[\lambda]$ of $O(m)$ is given by the number* *of $O(m)$-standard tableaux T^λ for which $n^{O(m)}(T^\lambda) = (n_1, n_2, \ldots, n_r)$. The character of this representation is given by:*

$$[\lambda](y) = \sum_{T^\lambda : T^\lambda\, O(m)-standard} y^{T^\lambda}, \tag{5.1.8}$$

where (y) denotes the vector $(y_1, y_2, \ldots, y_r)$ and $y^{T^\lambda} = y_1^{n_1^{O(m)}(T^\lambda)} y_2^{n_2^{O(m)}(T^\lambda)} \cdots y_r^{n_r^{O(m)}(T^\lambda)}$, for those elements of $O(m)$ with positive determinant and if $m = 2r+1$, eigenvalues $y_1^{-1}, y_1,\ y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r, 1$, and if $m = 2r$, eigenvalues $y_1^{-1}, y_1,\ y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r$.

§5.2. Irreducible $O(m)$-modules

Let V be the defining $GL(m)$-module with basis $\{e_i : i \in \mathcal{I}^{O(m)}\}$. Then, since $O(m)$ is a subgroup of $GL(m)$, V and the $GL(m)$-module $W^\lambda \subset V^{\otimes l}$ also serve as $O(m)$-modules. As for the symplectic groups, the $O(m)$-module W^λ is, in general, reducible due to the presence of trace tensors. As will transpire, the irreducible $O(m)$-modules are also obtained on extracting all appropriate trace tensors from W^λ.

With m fixed and $J = J_m^+$ as given by (2.1.1b) or (2.1.1c), $\tilde{G}JG = J$ for all $G \in O(m)$, whereupon $O(m)$ preserves the tensor

$$\sum_{j,k \in \mathcal{I}^{O(m)}} J_{jk} e_j \otimes e_k = \sum_{i \in \mathcal{I}^{O(m)}} e_i \otimes e_{\bar{i}}, \tag{5.2.1}$$

Definition 5.2.2. *With respect to $O(m)$, a trace tensor of $V^{\otimes l}$ is any linear combination of terms of the form:*

$$\sum_{i \in \mathcal{I}^{O(m)}} x \otimes e_i \otimes y \otimes e_{\bar{i}} \otimes z, \tag{5.2.2}$$

where x, y and z are elements of some (possibly zero) tensor power of V and $x \otimes y \otimes z \in V^{\otimes(l-2)}$. Define $U^{O(m)} \subset V^{\otimes l}$ to be the span of all such trace tensors.

The preservation of (5.2.1) under the action of $O(m)$ implies that $U^{O(m)}$ is invariant under the action of $O(m)$. Since $V^{\otimes l}$ is completely reducible [**We39**], it follows that $V^{\otimes l}/U^{O(m)}$ is isomorphic to a subspace of $V^{\otimes l}$, which is invariant under the action of $O(m)$. Therefore $O^\lambda = W^\lambda/(W^\lambda \cap U^{O(m)})$ is an $O(m)$-submodule of W^λ.

Let $[T^\lambda]$ denote the traceless symmetrised tableau resulting from the removal of all trace terms (5.2.2) from the symmetrised tableau $\{T^\lambda\}$, by forming its quotient with respect to the elements of $U^{O(m)}$. O^λ is therefore spanned by all $[T^\lambda]$ where the entries of each T^λ are from the set $\mathcal{I}^{O(m)}$.

Lemma 5.2.3. *Let T_i^λ, for $i \in \mathcal{I}^{O(m)}$, be m tableaux, identical except for the entries in two positions where $T_{i(a,b)}^\lambda = i$ and $T_{i(c,d)}^\lambda = \bar{i}$ for some fixed a, b, c and d with $a, c \leq \tilde{\lambda}_1$, $b \leq \lambda_a$ and $d \leq \lambda_b$. Then:*

$$\sum_{i \in \mathcal{I}^{O(m)}} [T_i^\lambda] = 0. \tag{5.2.3}$$

Proof. Since $\sum_{i \in \mathcal{I}^{O(m)}} T_i^\lambda \in U^{O(m)}$ and the place permutation action of each summand of the Young symmetriser Y_*^λ, defined by (3.3.13c), only serves to give similar terms in $U^{O(m)}$ with appropriate changes of the positions (a, b) and (c, d), it follows that $\sum_{i \in \mathcal{I}^{O(m)}} \{T_i^\lambda\} \in U^{O(m)}$, whereupon (5.2.3) follows from the definition of $[T^\lambda]$ as a quotient.

The identity (5.2.3) is known as the orthogonal Trace relation.

Once more, it is appropriate to proceed via a rather technical result which facilitates the simultaneous application of the orthogonal Trace condition over a number of index pairs.

Lemma 5.2.4. *Let k_1, k_2 be such that $1 \leq k_1 < k_2 \leq \lambda_1$. Let $\mathcal{I}^{O(m)} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$ be a union of disjoint sets such that, with $b^\alpha = \#\mathcal{B}^\alpha$, $b^\beta = \#\mathcal{B}^\beta$, $e = \#\mathcal{E}$, $g = \#\mathcal{G}$, $h = \#\mathcal{H}$ and $d > g$, $\tilde{\lambda}_{k_1} = b^\alpha + e + d$ and $\tilde{\lambda}_{k_2} = b^\beta + e + d$. Let $\mathcal{D}_w$, for various w, run over all distinct $\binom{h}{d}$ subsets of $\mathcal{H}$ of cardinality d and let the tableaux T_w^λ, be identical apart from column k_1 which contains entries from the set $\mathcal{B}^\alpha \cup \mathcal{E} \cup \mathcal{D}_w$ and column k_2 which contains entries from the set $\overline{\mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{D}_w}$. If the indices from the set $\mathcal{B}^\alpha \cup \mathcal{E} \cup \overline{\mathcal{B}^\beta \cup \mathcal{E}}$ are in the same positions in each T_w^λ and if, with $\mathcal{D}_w = \{\gamma_{w,1}, \gamma_{w,2}, \ldots, \gamma_{w,d}\}$, for fixed i with $1 \leq i \leq d$, the indices $\gamma_{w,i}$ occur in the same position of the k_1th column of each T_w^λ and the indices $\bar{\gamma}_{w,i}$ occur in the same position in the k_2th column of each T_w^λ, then:*

$$\sum_w [T_w^\lambda] = 0. \tag{5.2.4}$$

Proof. For $[T_w^\lambda]$ write the columns k_1 and k_2 of T_w^λ as a product, θ_w, of elements of $\mathcal{I}^{O(m)}$ with each element superscripted either α or β to indicate that it arose from column k_1 or k_2, respectively. For example, if $k_1 = 1$ and $k_2 = 2$ then:

$$\begin{bmatrix} 1 & \bar{1} & \bar{2} & 3 \\ \bar{2} & 2 & \bar{3} & \\ \bar{3} & 3 & & \\ 3 & & & \end{bmatrix}$$

gives rise to $\theta = 1^\alpha\bar{2}^\alpha\bar{3}^\alpha 3^\alpha \bar{1}^\beta 2^\beta 3^\beta$. By virtue of the Column relations (3.4.2), interchanging elements of θ with the same superscript changes the sign of θ, and the presence of an identical pair of elements with the same superscripts implies that $\theta = 0$. In this notation, (5.2.4) may be proved by showing that:

$$\sum_w \theta_w = 0. \tag{5.2.4a}$$

Let $\omega_i = i^\alpha \bar{i}^\beta$ and thence $\omega_{\bar{i}} = \bar{i}^\alpha i^\beta$. The trace condition, (5.2.3), implies that:

$$\sum_{i \in \mathcal{I}^{O(m)}} \omega_i = 0. \tag{5.2.4b}$$

With $\mathcal{B} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta$, split this identity according to:

$$\sum_{i \in \mathcal{H}\cup\mathcal{B}\cup\mathcal{E}} \omega_i = -\sum_{i\in\mathcal{G}} \omega_i. \tag{5.2.4c}$$

Since $d > g$, on raising each side of this identity to the power of d, the right side is annihilated, giving:

$$\left(\sum_{i \in \mathcal{H}\cup\mathcal{B}\cup\mathcal{E}} \omega_i\right)^d = 0. \tag{5.2.4d}$$

This implies that:

$$\sum_{\substack{\gamma_1<\gamma_2<\cdots<\gamma_d \\ \gamma_1,\gamma_2,\ldots,\gamma_d \in \mathcal{H}\cup\mathcal{B}\cup\mathcal{E}}} \omega_{\gamma_1}\omega_{\gamma_2}\cdots\omega_{\gamma_d} = 0, \tag{5.2.4e}$$

whereupon, on setting $\theta^\mathcal{B} = \prod_{i\in\mathcal{B}^\alpha} i^\alpha \prod_{i\in\mathcal{B}^\beta} \bar{i}^\beta$ and $\theta^\mathcal{E} = \prod_{i\in\mathcal{E}} \omega_i$, multiplication by $\theta^\mathcal{B}\theta^\mathcal{E}$ annihilates those terms featuring ω_i for $i \in \mathcal{B}\cup\mathcal{E}$ due to a repeated index. Therefore:

$$\theta^\mathcal{B}\theta^\mathcal{E} \sum_{\substack{\gamma_1<\gamma_2<\cdots<\gamma_d \\ \gamma_1,\gamma_2,\ldots,\gamma_d \in \mathcal{H}}} \omega_{\gamma_1}\omega_{\gamma_2}\cdots\omega_{\gamma_d} = 0, \tag{5.2.4f}$$

and hence,

$$\theta^\mathcal{B}\theta^\mathcal{E} \sum_w \theta_w^\mathcal{D} = 0, \tag{5.2.4g}$$

where $\theta_w^\mathcal{D} = \prod_{i\in\mathcal{D}_w} \omega_i$. Let $\theta'_w = \theta^\mathcal{B}\theta^\mathcal{E}\theta_w^\mathcal{D}$, so that, then $\sum_w \theta'_w = 0$. With the indices as specified in the statement of the lemma, the application of an identical permutation to the factors of each θ'_w produces θ_w. Therefore $\theta'_w = \pm\theta_w$ with the sign being independent of w. Thus (5.2.4g) is equivalent to (5.2.4a) and the lemma is proved.

To illustrate the algorithm described in the above proof, let $m = 6$, $\tilde{\lambda}_{k_1} = 3$, $\tilde{\lambda}_{k_2} = 2$, $\mathcal{B}^\alpha = \{\bar{1}\}$, $\mathcal{B}^\beta = \mathcal{E} = \emptyset$, $\mathcal{G} = \{1\}$, $\mathcal{H} = \{\bar{2}, 2, \bar{3}, 3\}$ and $d = 2$. Then $d > g$, $\tilde{\lambda}_{k_1} = b^\alpha + e + d$ and $\tilde{\lambda}_{k_2} = b^\beta + e + d$ as required by the premise of Lemma 5.2.4.

For this example, the trace condition, as in (5.2.4c), is written:

$$\omega_{\bar{1}} + \omega_{\bar{2}} + \omega_2 + \omega_{\bar{3}} + \omega_3 = -\omega_1, \tag{5.2.5a}$$

where the term from $\mathcal{G}$ has been placed on the right side of this equation. Raising the two sides of this equation to the power of $d = 2$ yields, corresponding to (5.2.4e):

$$\omega_{\bar{1}}\omega_{\bar{2}} + \omega_{\bar{1}}\omega_2 + \omega_{\bar{1}}\omega_{\bar{3}} + \omega_{\bar{1}}\omega_3 + \omega_{\bar{2}}\omega_2 + \omega_{\bar{2}}\omega_{\bar{3}} + \omega_{\bar{2}}\omega_3 + \omega_2\omega_{\bar{3}} + \omega_2\omega_3 + \omega_{\bar{3}}\omega_3 = 0, \tag{5.2.5b}$$

with all other terms zero due to repeated factors. Multiplying this identity by $\theta^{\mathcal{B}}\theta^{\mathcal{E}} = \bar{1}^\alpha$ annihilates those terms featuring $\omega_{\bar{1}}$ whereupon:

$$\bar{1}^\alpha\omega_{\bar{2}}\omega_2 + \bar{1}^\alpha\omega_{\bar{2}}\omega_{\bar{3}} + \bar{1}^\alpha\omega_{\bar{2}}\omega_3 + \bar{1}^\alpha\omega_2\omega_{\bar{3}} + \bar{1}^\alpha\omega_2\omega_3 + \bar{1}^\alpha\omega_{\bar{3}}\omega_3 = 0, \tag{5.2.5c}$$

and

$$\bar{1}^\alpha\bar{2}^\alpha 2^\beta 2^\alpha\bar{2}^\beta + \bar{1}^\alpha\bar{2}^\alpha 2^\beta\bar{3}^\alpha 3^\beta + \bar{1}^\alpha\bar{2}^\alpha 2^\beta 3^\alpha\bar{3}^\beta + \bar{1}^\alpha 2^\alpha\bar{2}^\beta\bar{3}^\alpha 3^\beta + \bar{1}^\alpha 2^\alpha\bar{2}^\beta 3^\alpha\bar{3}^\beta + \bar{1}^\alpha\bar{3}^\alpha 3^\beta 3^\alpha\bar{3}^\beta = 0. \tag{5.2.5d}$$

If $\lambda = (2,2,1)$ and $k_1 = 1$ and $k_2 = 2$, then this identity transfers back into the language of tableaux to give:

$$\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{2} \\ 2 & \end{bmatrix} + \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix} + \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix} + \begin{bmatrix} \bar{1} & 3 \\ \bar{3} & \bar{3} \\ 3 & \end{bmatrix} = 0, \tag{5.2.5e}$$

corresponding to (5.2.4). The column relations can now be applied to the first and last of these terms to give an expression solely in terms of column strict tableaux.

Lemma 5.2.4 is now used in each of a sequence of three lemmas dealing with violations of conditions (iv), (v) and (vi) of Definition 5.1.6. In each case, the non-standard tableau is written in terms of higher tableaux. Once more, the order on the set of tableaux is provided by Definition 2.6.6, after the entries from $\mathcal{I}^{O(m)}$ have been mapped into $\mathcal{I}^{GL(m)}$ through $a \to 2a$, $\bar{a} \to 2a-1$, $0 \to m$.

Lemma 5.2.6. *Let T^λ be a column strict tableau which is not $O(m)$-standard in that $\alpha_j + \beta_j > 2j$ for some j. Then $[T^\lambda]$ may be expressed as a signed sum of traceless, symmetrised tableaux $[T^\lambda_w]$, where for each w, $T^\lambda_w > T^\lambda$.*

Proof. Let $k_1 = 1$, $k_2 = 2$, $\mathcal{Q}^\alpha \subset \mathcal{I}^{O(m)}$ be the set of indices in the first column of T^λ, and $\mathcal{Q}^\beta \subset \mathcal{I}^{O(m)}$ the set of indices in the second column of T^λ. Let $\mathcal{A} = \{i \in \mathcal{I}^{O(m)} : i \in \mathcal{Q}^\alpha, \bar{i} \in \mathcal{Q}^\beta\}$, $\mathcal{B}^\alpha = \{i \in \mathcal{I}^{O(m)} : i \in \mathcal{Q}^\alpha, \bar{i} \notin \mathcal{Q}^\beta\}$, $\mathcal{B}^\beta = \{i \in \mathcal{I}^{O(m)} : i \notin \mathcal{Q}^\alpha, \bar{i} \in \mathcal{Q}^\beta\}$, $\mathcal{C} = \{i \in \mathcal{I}^{O(m)} : i \notin \mathcal{Q}^\alpha, \bar{i} \notin \mathcal{Q}^\beta\}$ and $\mathcal{B} = \mathcal{B}^\alpha \cup \mathcal{B}^\beta$. Then $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are distinct with $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} = \mathcal{I}^{O(m)}$, and if $a = \#\mathcal{A}$, $b = \#\mathcal{B}$ and $c = \#\mathcal{C}$, then $a+b+c = m$. Let $\mathcal{J} = \{i \in \mathcal{I}^{O(m)} : i \leq j\}$ so that $\#\mathcal{J} = 2j$. The sets created above are now split with respect to $\mathcal{J}$: $\mathcal{D} = \mathcal{A} \cap \mathcal{J}$, $\mathcal{E} = \mathcal{A} \backslash \mathcal{D}$, $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{J}$, $\mathcal{B}_1 = \mathcal{B} \backslash \mathcal{B}_0$, $\mathcal{G} = \mathcal{C} \cap \mathcal{J}$ and $\mathcal{F} = \mathcal{C} \backslash \mathcal{G}$. In addition let $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$. With the cardinalities of the sets just created d, e, b_0, b_1, g, f and h respectively, and the cardinalities of $\mathcal{B}^\alpha$ and $\mathcal{B}^\beta$, b^α and b^β respectively, then $d + e + b_0 + b_1 + g + f = m$, $h = d + f$, $\tilde{\lambda}_1 = b^\alpha + d + e$, $\tilde{\lambda}_2 = b^\beta + d + e$ and $d + b_0 + g = 2j$. The condition $\alpha_j + \beta_j > 2j$ implies that $2d + b_0 > 2j$, and therefore $d > g$. Thus the conditions of Lemma 5.2.4 are satisfied and the identity:

$$\sum_{\mathcal{D}_w \subset \mathcal{H}} [T_w^\lambda] = 0, \tag{5.2.6a}$$

follows, where the sum is over all $\binom{h}{d}$ distinct subsets $\mathcal{D}_w$ of $\mathcal{H}$, and T_w^λ is identical to T^λ apart from, if $\mathcal{D} = \{\delta_1, \ldots, \delta_d\}$ and $\mathcal{D}_w = \{\gamma_{w,1}, \ldots, \gamma_{w,d}\}$, the pair δ_i and $\bar{\delta}_i$ in the first and second columns respectively of T^λ, having been replaced by the pair $\gamma_{w,i}$ and $\bar{\gamma}_{w,i}$ respectively, for each $i = 1, 2, \ldots, d$. Thereupon:

$$[T^\lambda] = - \sum_{\substack{\mathcal{D}_w \subset \mathcal{H} \\ \mathcal{D}_w \neq \mathcal{D}}} [T_w^\lambda]. \tag{5.2.6b}$$

Since $\mathcal{D} \subset \mathcal{J}$, $\mathcal{F} \cap \mathcal{J} = \emptyset$ and $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$, each $\mathcal{D}_w \neq \mathcal{D}$ must include at least one element from the set $\mathcal{F}$. Thus, if $\delta_1 < \delta_2 < \cdots < \delta_d$ and for each $\mathcal{D}_w \neq D$, $\gamma_{w,1} < \gamma_{w,2} < \cdots < \gamma_{w,d}$, then $\gamma_{w,d} > \delta_d$ with $\bar{\delta}_d \in \mathcal{J}$ and $\bar{\gamma}_{w,d} \notin \mathcal{J}$. Consequently, as these appear as entries in the second columns of T^λ and T_w^λ respectively, it follows from Definition 2.6.6 that $T_w^\lambda > T^\lambda$, thereby proving Lemma 5.2.6.

As an illustration of the algorithm described in the above proof, let $\lambda = (2, 2, 1)$ and consider the $O(6)$-module O^λ and the tableau:

$$T^\lambda = \begin{matrix} \bar{1} & \bar{2} \\ \bar{2} & 2 \\ 2 & \end{matrix}. \tag{5.2.7a}$$

Here $\alpha_1 = 1$, $\beta_1 = 0$, so that $\alpha_1 + \beta_1 = 1 < 2$; but $\alpha_2 = 3$, $\beta_2 = 2$, so that $\alpha_2 + \beta_2 = 5 > 4$ and the tableau is not $O(6)$-standard. Thus $j = 2$. The first two columns of T^λ yield the sets $\mathcal{Q}^\alpha = \{\bar{1}, \bar{2}, 2\}$ and $\mathcal{Q}^\beta = \{\bar{2}, 2\}$, whereupon $\mathcal{A} = \{\bar{2}, 2\}$, $\mathcal{B}^\alpha = \{\bar{1}\}$, $\mathcal{B}^\beta = \emptyset$, $\mathcal{C} = \{1, \bar{3}, 3\}$ and $\mathcal{B} = \{\bar{1}\}$. Splitting $\mathcal{A}$ and $\mathcal{C}$ with respect to $\mathcal{J} = \{\bar{1}, 1, \bar{2}, 2\}$, yields $\mathcal{D} = \{\bar{2}, 2\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{1\}$ and $\mathcal{F} = \{\bar{3}, 3\}$ and thence $\mathcal{H} = \{\bar{2}, 2, \bar{3}, 3\}$. Since $d = 2$ and $g = 1$, $d > g$ as required by Lemma 5.2.4. Note that since $h = 4$ and $d = 2$ then an expression involving $\binom{4}{2} = 6$ terms is expected. In fact, the sets $\mathcal{B}^\alpha$, $\mathcal{B}^\beta$, $\mathcal{E}$, $\mathcal{G}$ and $\mathcal{H}$ are precisely those in the example following Lemma 5.2.4. Identity (5.2.5e) is, for this particular case, expression (5.2.6a). From this, the required expression (5.2.6b), with each tableau on the right higher than the original tableau, follows immediately:

$$\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{2} \\ 2 & \end{bmatrix} = - \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix} - \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix} - \begin{bmatrix} \bar{1} & 3 \\ \bar{3} & \bar{3} \\ 3 & \end{bmatrix}. \tag{5.2.7b}$$

This yields:

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{2} & 2 \\ 2 & \end{bmatrix} = \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix} + \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{3} & 3 \\ 3 & \end{bmatrix}, \tag{5.2.7c}$$

where, incidentally each term on the right is $O(6)$-standard. This will not be the case, in general.

Violations of the protection condition (v) of Definition 5.1.6 are dealt with by using the following lemma.

Lemma 5.2.8. *Let T^λ be a column strict tableau which is not $O(m)$-standard in that $\alpha_j + \beta_j = 2j$ for some j with $\alpha_j > \beta_j$ and an unprotected j occurs in the first column, in that $T^\lambda_{\alpha_j,1} = j$, $T^\lambda_{\beta_j,2} = \bar{j}$ and $T^\lambda_{\alpha_j-1,1} \neq \bar{j}$. Then $[T^\lambda]$ may be expressed as a signed sum of traceless, symmetrised tableaux $[T^\lambda_w]$, where for each w, $T^\lambda_w > T^\lambda$.*

Proof. Define $\mathcal{Q}^\alpha$, $\mathcal{Q}^\beta$, $\mathcal{A}$, $\mathcal{B}^\alpha$, $\mathcal{B}^\beta$, $\mathcal{B}$, $\mathcal{C}$, a, b^α, b^β, b and c as for the proof of Lemma 5.2.6. Note that here $j \in \mathcal{A}$ (since j is in the first column of T^λ and $\bar{j}$ is in the second) and $\bar{j} \in \mathcal{C}$. Let $\mathcal{J} = \{i \in \mathcal{I}^{O(m)} : i \leq j, i \neq \bar{j}\}$ so that $\#\mathcal{J} = 2j - 1$. The sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are now split with respect to $\mathcal{J}$: $\mathcal{D} = \mathcal{A} \cap \mathcal{J}$, $\mathcal{E} = \mathcal{A} \backslash \mathcal{D}$, $\mathcal{B}_0 = \mathcal{B} \cap \mathcal{J}$, $\mathcal{B}_1 = \mathcal{B} \backslash \mathcal{B}_0$, $\mathcal{G} = \mathcal{C} \cap \mathcal{J}$ and $\mathcal{F} = \mathcal{C} \backslash \mathcal{G}$. In addition let $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$. Note now that $\bar{j} \in \mathcal{H}$ and $\bar{j} \notin \mathcal{D}$. With the cardinalities of the sets just created d, e, b_0, b_1, g, f and h respectively, then $d + e + b_0 + b_1 + g + f = a + b + c = m$, $h = d + f$, $\tilde{\lambda}_1 = b^\alpha + d + e$, $\tilde{\lambda}_2 = b^\beta + d + e$ and $d + b_0 + g = 2j - 1$. The condition $\alpha_j + \beta_j = 2j$ implies that $2d + b_0 = 2j$, and therefore $d > g$. Thus the conditions of Lemma 5.2.4 are satisfied and the identity:

$$\sum_{\mathcal{D}_w \subset \mathcal{H}} [T^\lambda_w] = 0, \tag{5.2.8a}$$

follows, where the sum is over all $\binom{h}{d}$ distinct subsets $\mathcal{D}_w$ of $\mathcal{H}$, and T^λ_w is identical to T^λ apart from, if $\mathcal{D} = \{\delta_1, \ldots, \delta_d\}$ and $\mathcal{D}_w = \{\gamma_{w,1}, \ldots, \gamma_{w,d}\}$, the pair δ_i and $\bar{\delta}_i$ in the first and second columns respectively of T^λ, having been replaced by the pair $\gamma_{w,i}$ and $\bar{\gamma}_{w,i}$ respectively, for each $i = 1, 2, \ldots, d$. Thereupon:

$$[T^\lambda] = - \sum_{\substack{\mathcal{D}_w \subset \mathcal{H} \\ \mathcal{D}_w \neq \mathcal{D}}} [T^\lambda_w]. \tag{5.2.8b}$$

Since $\mathcal{D} \subset \mathcal{J}$, $\mathcal{F} \cap \mathcal{J} = \emptyset$ and $\mathcal{H} = \mathcal{D} \cup \mathcal{F}$, each $\mathcal{D}_w \neq \mathcal{D}$ must include at least one element from the set $\mathcal{F}$. Apart from the one case T^λ_x arising from the set $\mathcal{D}_x = \{\delta_1, \ldots, \delta_{d-1}, \bar{j}\}$, the argument given in the proof of Lemma 5.2.6 shows that $T^\lambda_w > T^\lambda$. However, even in the exceptional case, $T^\lambda_x > T^\lambda$ since the entry $\bar{j}$ in the second column of T^λ has been replaced by the greater entry j. Thus expression (5.2.8b) is that required and the lemma is proved.

As an illustration of the algorithm in the above proof, let $\lambda = (2,2,1,1)$ and consider the tableau:

$$T^\lambda = \begin{array}{cc} \bar{1} & 2 \\ 1 & \bar{3} \\ \bar{2} & \\ 3 & \end{array}, \qquad (5.2.9a)$$

and the $O(6)$-module $O^{(2,2,1,1)}$. Here $\alpha_i + \beta_i = 2i$ for each of $i = 1,2,3$. However, the entry 3 in the first column is not protected. This implies that T^λ is not $O(6)$-standard. The above proof specifies that $j = 3$ and $\mathcal{J} = \{\bar{1}, 1, \bar{2}, 2, 3\}$. T^λ gives rise to the sets $\mathcal{A} = \{\bar{2}, 3\}$, $\mathcal{B} = \mathcal{B}^\alpha = \{\bar{1}, 1\}$, $\mathcal{B}^\beta = \emptyset$, $\mathcal{C} = \{2, \bar{3}\}$ and thence the sets $\mathcal{D} = \{\bar{2}, 3\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{2\}$, $\mathcal{F} = \{\bar{3}\}$ and $\mathcal{H} = \{\bar{2}, \bar{3}, 3\}$. Then Lemma 5.2.4 yields the following expression involving $\binom{h}{d} = \binom{3}{2} = 3$ terms:

$$\left[\begin{array}{cc} \bar{1} & 2 \\ 1 & \bar{3} \\ \bar{2} & \\ 3 & \end{array}\right] + \left[\begin{array}{cc} \bar{1} & 2 \\ 1 & 3 \\ \bar{2} & \\ \bar{3} & \end{array}\right] + \left[\begin{array}{cc} \bar{1} & 3 \\ 1 & \bar{3} \\ \bar{3} & \\ 3 & \end{array}\right] = 0. \qquad (5.2.9b)$$

This corresponds to (5.2.8a). The second term here is the exceptional term $[T^\lambda_x]$ arising from the set $\mathcal{D}_x = \{\bar{2}, \bar{3}\} \subset \mathcal{H}$. Rearranging and reordering the column entries of (5.2.9b) gives:

$$\left[\begin{array}{cc} \bar{1} & 2 \\ 1 & \bar{3} \\ \bar{2} & \\ 3 & \end{array}\right] = - \left[\begin{array}{cc} \bar{1} & 2 \\ 1 & 3 \\ \bar{2} & \\ \bar{3} & \end{array}\right] + \left[\begin{array}{cc} \bar{1} & \bar{3} \\ 1 & 3 \\ \bar{3} & \\ 3 & \end{array}\right], \qquad (5.2.9c)$$

where, in this case, the two terms on the right side are $O(6)$-standard.

Violations of the second protection condition (vi) of Definition 5.1.6 are dealt with by the following lemma.

Lemma 5.2.10. *Let T^λ be a column strict tableau which is not $O(m)$-standard in that, for some j, $\alpha_j = \beta_j = j$ and an unprotected j occurs in the bth column for some $b \geq 2$, in that $T^\lambda_{j,1} = \bar{j}$, $T^\lambda_{j,b} = j$ and $T^\lambda_{j-1,b} \neq \bar{j}$. Then $[T^\lambda]$ may be expressed as a signed sum of traceless, symmetrised tableaux $[T^\lambda_w]$, where for each w, $T^\lambda_w > T^\lambda$.*

Proof. Proceed as for the proof of Lemma 5.2.8 but instead of working with the second column, set $k_2 = b$ and work with the bth column, and instead of using the $\mathcal{J}$ defined there, substitute it for $\mathcal{J} = \{i \in \mathcal{I}^{O(m)} : i \leq \bar{j}\}$. Consequently, $\bar{j} \in \mathcal{D}$, $j \in \mathcal{F}$ and $\{\bar{j}, j\} \subset \mathcal{H}$. With $\mathcal{D} = \{\delta_1, \ldots, \delta_{d-1}, \delta_d = \bar{j}\}$ let $\mathcal{D}_x = \{\delta_1, \ldots, \delta_{d-1}, j\}$. Then expression (5.2.8$a$) has the analogue

$$[T^\lambda] + [T^\lambda_x] + \sum_{\substack{\mathcal{D}_w \subset \mathcal{H} \\ \mathcal{D}_w \neq \mathcal{D} \\ \mathcal{D}_w \neq \mathcal{D}_x}} \eta_u [T^\lambda_u] = 0, \tag{5.2.10a}$$

where each tableau is column strict, each $\eta_u = \pm 1$, and for each of the terms under the summation, $T^\lambda_u > T^\lambda$, In this case, for the exceptional term, T^λ_x may be obtained from T^λ by transposing the $\bar{j}$ in the first column with the j in the bth column of the same row. Incidentally, $T^\lambda_x < T^\lambda$. Consider a Garnir element involving those positions below and including j in the first column and those above and including $\bar{j}$ in the bth column of T^λ_x which yields a Garnir relation (3.4.3), solely in terms of column strict tableaux. Such a Garnir relation involves $[T^\lambda_x]$, $[T^\lambda]$ and various $[T^\lambda_v]$ for which $T^\lambda_v \neq T^\lambda_x$ and $T^\lambda_v \neq T^\lambda$ for all v:

$$[T^\lambda_x] - [T^\lambda] + \sum_{v \neq x} \eta_v [T^\lambda_v] = 0. \tag{5.2.10b}$$

Note that T^λ arises from T^λ_x, through the transposition of j and $\bar{j}$ and consequently $[T^\lambda]$ has a coefficient of -1 in this expression. Just as $T^\lambda_u > T^\lambda$ for each u, it can be seen that $T^\lambda_v > T^\lambda$ for each v via an argument similar to that following (3.4.9). Combining (5.2.10a) and (5.2.10b) gives:

$$[T^\lambda] = \frac{1}{2}\left(\sum_{v \neq x} \eta_v [T^\lambda_v] - \sum_{u \neq x} \eta_u [T^\lambda_u]\right), \tag{5.2.10c}$$

an expression in terms of tableaux all greater than T^λ.

In order to illustrate the above proof, consider the $O(7)$-module $O^{(3,3,1)}$ and the tableau:

$$T^\lambda = \begin{array}{ccc} 1 & \bar{2} & 1 \\ \bar{2} & 2 & 2 \\ 3 & & \end{array}. \tag{5.2.11a}$$

Here a protection violation occurs in the second row since $\alpha_2 = \beta_2 = 2$, the first column contains a $\bar{2}$ but no 2, and the third column contains a 2 but no $\bar{2}$. Thus $j = 2$ and the above proof requires that $\mathcal{J} = \{\bar{1}, 1, \bar{2}\}$. T^λ gives rise to the sets $\mathcal{A} = \{\bar{2}\}$, $\mathcal{B} = \mathcal{B}^\alpha = \{1, 3\}$, $\mathcal{B}^\beta = \{\bar{1}\}$, $\mathcal{B} = \{\bar{1}, 1, 3\}$, $\mathcal{C} = \{2, \bar{3}, 0\}$ and thence the sets $\mathcal{D} = \{\bar{2}\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \emptyset$, $\mathcal{F} = \{2, \bar{3}, 0\}$ and $\mathcal{H} = \{\bar{2}, 2, \bar{3}, 0\}$. Lemma 5.2.4 then yields the following expression:

$$\begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{2} & 2 & 2 \\ 3 & & \end{bmatrix} + \begin{bmatrix} 1 & \bar{2} & 1 \\ 2 & 2 & \bar{2} \\ 3 & & \end{bmatrix} + \begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{3} & 2 & 3 \\ 3 & & \end{bmatrix} - \begin{bmatrix} 1 & \bar{2} & 1 \\ 3 & 2 & 0 \\ 0 & & \end{bmatrix} = 0. \qquad (5.2.11b)$$

The second term is the exceptional term $[T_x^\lambda]$ on which the action of the specific Garnir element is required. For this case $\mathcal{X} = \{2, 3\}$ and $\mathcal{Y} = \{6, 7\}$, whereupon (3.4.3) yields:

$$\begin{bmatrix} 1 & \bar{2} & 1 \\ 2 & 2 & \bar{2} \\ 3 & & \end{bmatrix} - \begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{2} & 2 & 2 \\ 3 & & \end{bmatrix} + \begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{2} & 2 & 3 \\ 2 & & \end{bmatrix} = 0, \qquad (5.2.11c)$$

where a number of terms with a pair of identical entries in a column have been omitted since they are zero. Expressions (5.2.11b) and (5.2.11c) imply that:

$$\begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{2} & 2 & 2 \\ 3 & & \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{2} & 2 & 3 \\ 2 & & \end{bmatrix} - \begin{bmatrix} 1 & \bar{2} & 1 \\ \bar{3} & 2 & 3 \\ 3 & & \end{bmatrix} + \begin{bmatrix} 1 & \bar{2} & 1 \\ 3 & 2 & 0 \\ 0 & & \end{bmatrix} \right). \qquad (5.2.11d)$$

This expression, with each tableau on the right higher than T^λ, corresponds to (5.2.10c). However, each tableaux here is not $O(7)$-standard. The techniques of at least one of the Lemmas 3.4.3, 5.2.6, 5.2.8 and 5.2.10 will need to be reapplied to each of these terms in order to elicit an expression for $[T^\lambda]$ solely in terms of $O(7)$-standard tableaux.

With the above standardisation lemmas established, the argument now closely follows that of Section 4.3 where the symplectic group modules were obtained.

Lemma 5.2.12. *The set*

$$\{\, [T^\lambda] \,:\, T^\lambda \text{ is } O(m)\text{-standard}\}$$

spans the $O(m)$-module O^λ.

Proof. If the column strict T^λ is not $O(m)$-standard due to a violation of condition (iii) of Definition 5.1.6 then the techniques of Section 3.4 enable the Garnir relations to be used to write $[T^\lambda]$ is terms of higher column strict tableaux. If the column strict T^λ violates conditions (iv), (v) or (vi) of Definition 5.1.6 then Lemma 5.2.6, Lemma 5.2.8 or Lemma 5.2.10 shows that $[T^\lambda]$ can be written in terms of higher column strict tableaux. Therefore, by iterating these procedures, $[T^\lambda]$ may be written in terms of $O(m)$-standard tableaux by virtue of the ordering on the set of all tableaux and their finite number.

This lemma has the direct implication that if $\tilde{\lambda}_1 + \tilde{\lambda}_2 > m$, then the $O(m)$-module O^λ is zero since in such a case there exist no $O(m)$-standard tableaux.

Let $\lambda \in P(l)$. Since $U^{O(m)}$, specified by Definition 5.2.2, and hence $U^{O(m)} \cap W^\lambda \subset V^{\otimes l}$ are invariant under $O(m)$, (4.2.17) implies that the element $G \in O(m)$ acts on $[T^\lambda] \in O^\lambda$ according to:

$$G\,[T^\lambda] = \sum_{T'^\lambda} G_{T'^\lambda_{(1)} T^\lambda_{(1)}} G_{T'^\lambda_{(2)} T^\lambda_{(2)}} \cdots G_{T'^\lambda_{(l)} T^\lambda_{(l)}} [T'^\lambda], \tag{5.2.13}$$

the sum being over all tableaux T'^λ with entries from the set $\mathcal{I}^{O(m)}$. In order to determine the action of $B_a{}^b \in so(2r+1)$ or $D_a{}^b \in so(2r)$ on $[T^\lambda]$, let p and q be the number of times that the indices b and $\bar{a}$ respectively occur in T^λ. Form the set of p tableaux $\{T^\lambda_{1,1}, T^\lambda_{1,2}, \ldots, T^\lambda_{1,p}\}$ by, in each case, replacing a single index b in T^λ with a, and the set of q tableaux $\{T^\lambda_{2,1}, T^\lambda_{2,2}, \ldots, T^\lambda_{2,q}\}$ by, in each case, replacing a single index $\bar{a}$ in T^λ with $\bar{b}$. It then follows from (4.2.18), (2.2.21) and the definition of $[T^\lambda]$ that, for $o(2r+1)$:

$$\begin{aligned} B_a{}^b[T^\lambda] &= E_a{}^b[T^\lambda] - E_{\bar{b}}{}^{\bar{a}}[T^\lambda] \\ &= \sum_{i=1}^{p}[T^\lambda_{1,i}] - \sum_{i=1}^{q}[T^\lambda_{2,i}], \end{aligned} \tag{5.2.14a}$$

and similarly, for $O(2r)$:

$$D_a{}^b[T^\lambda] = \sum_{i=1}^{p}[T^\lambda_{1,i}] - \sum_{i=1}^{q}[T^\lambda_{2,i}]. \tag{5.2.14b}$$

These imply that:

$$B_a{}^a[T^\lambda] = n_a^{O(2r+1)}(T^\lambda)[T^\lambda] \tag{5.2.15a}$$

$$\text{and} \qquad D_a{}^a[T^\lambda] = n_a^{O(2r)}(T^\lambda)[T^\lambda]. \tag{5.2.15b}$$

Since bases for the Cartan subalgebras of $so(2r+1)$ and $so(2r)$ are provided by the elements $B_a{}^a$ and $D_a{}^a$ respectively for $a = 1, 2, \ldots, r$, the $O(m)$-weight $n^{O(m)}(T^\lambda)$ of T^λ determines the weight of the element $[T^\lambda] \in O^\lambda$ in this basis.

If $m = 2r+1$ and $\tilde{\lambda}_1 \le r$, let $T^\lambda_>$ be given by Definition 2.6.6. Then $n^{O(m)}(T^\lambda_>) = (\lambda_1, \lambda_2, \ldots, \lambda_r) = \lambda$ and $T^\lambda_>$ is the unique $O(m)$-standard tableau of shape F^λ for which this is so. If $a, b \in \mathbb{N}_r$ and $a < b$ then:

$$B_a{}^b[T^\lambda_>] = 0, \tag{5.2.16a}$$

$$B_a{}^{\bar{b}}[T^\lambda_>] = 0, \tag{5.2.16b}$$

$$\text{and} \qquad B_a{}^0[T^\lambda_>] = 0. \tag{5.2.16c}$$

It is readily apparent that $T^\lambda_>$ is the only $O(2r+1)$-standard tableau with this property. Since $\{B_a{}^b : a, b \in \mathbb{N}_r, a < b\} \cup \{B_a{}^{\bar{b}} : a, b \in \mathbb{N}_r, a < b\} \cup \{B_a{}^0 : a \in \mathbb{N}_r\}$ is a basis for $B_+^{O(2r+1)}$, (5.2.16) shows that $[T^\lambda_>]$ is the unique highest weight vector of the $O(2r+1)$-module O^λ.

If $\tilde{\lambda}_1 > r$, let $T^\lambda_{>(i,j)} = i$ as before, but only for $1 \le i \le m - \tilde{\lambda}_1$ and $1 \le j \le \lambda_i$ (this deals with all but part of the first column). In addition, for $m - \tilde{\lambda}_1 < i \le r$ let $T^\lambda_{>(\tilde{\lambda}_1+2i-m-1,1)} = \bar{i}$ and $T^\lambda_{>(\tilde{\lambda}_1+2i-m,1)} = i$. Finally, let $T^\lambda_{>(\tilde{\lambda}_1,1)} = 0$. For instance, if $m = 9$ and $\lambda = (3,2,1,1,1,1)$ then:

$$T^\lambda_> = \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & & \\ \bar{4} & & \\ 4 & & \\ 0 & & \end{matrix} \quad . \tag{5.2.17}$$

Once more each case of (5.2.16) holds. However $n^{O(m)}(T^\lambda_>) = (\lambda^*_1, \lambda^*_2, \ldots, \lambda^*_r) = \lambda^*$ where $\tilde{\lambda}^*_1 = m - \tilde{\lambda}_1$ and $\tilde{\lambda}^*_i = \tilde{\lambda}_i$ for $i > 1$. Thus $n^{O(m)}(T^\lambda_>) = n^{O(m)}(T^{\lambda^*}_>)$ and O^λ and O^{λ^*} are not distinguished as $so(m)$-modules. As will be seen later, they are distinct as $O(m)$-modules. For now this will be assumed.

If $m = 2r$ and $\tilde{\lambda}_1 \le r$, then $T^\lambda_>$ is again provided by Definition 2.6.6. If $m = 2r$ and $\tilde{\lambda}_1 > r$ then $T^\lambda_>$ is given by the same prescription as for the $m = 2r+1$ case described above except that the index 0 is not entered. Thereupon, the $so(2r)$-analogues of (5.2.16a) and (5.2.16b), with 'B' replaced by 'D', result in the same conclusions for $O(2r)$ as for $O(2r+1)$. This leads to the following theorem.

Theorem 5.2.18. *The $O(m)$-module O^λ is irreducible with basis:*

$$\{\,[T^\lambda] \,:\, T^\lambda \text{ is } O(m)\text{-standard}\}.$$

Moreover [**We39**], *the set $\{O^\lambda : \tilde{\lambda}_1 + \tilde{\lambda}_2 \le m\}$ provides a complete list of inequivalent irreducible $O(m)$-modules.*

Proof. Since O^λ has highest weight λ, it contains the $O(m)$-module corresponding to the irreducible representation $[\lambda]$ of $O(m,\mathbb{C})$. That first part of the theorem then follows from Theorem 5.1.8 and Lemma 5.2.12. The second part of the theorem follows because firstly every $O(m)$-module occurs in $V^{\otimes l}$ for some l [**Li44**]; secondly, $O(m)$-standard tableaux of shape F^λ exist if and only if $\tilde{\lambda}_1 + \tilde{\lambda}_2 \le m$; and thirdly, if $\tilde{\lambda}_1 \le m/2$ then λ is the highest weight of O^λ and if $\tilde{\lambda}_1 > m/2$ then λ^* is the highest weight of O^λ, but O^λ and O^{λ^*} are inequivalent $O(m)$-modules.

The quintessential structure of O^λ may now be stated.

Theorem 5.2.19. *Let $\lambda \in P(l)$ with $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$. O^λ is the irreducible $O(m)$-module spanned by $[T^\lambda]$ for all T^λ with entries from the set $\mathcal{I}^{O(m)}$, modulo relations* (3.4.2), (3.4.3) *and* (5.2.3)*, and on which $O(m)$ and $so(m)$ act according to* (5.2.13) *and* (5.2.14) *respectively.*

This theorem effectively provides a definition for O^λ.

The techniques of this section enable explicit representation matrices for elements of $O(m)$ and $so(m)$ to be obtained in the representation $[\lambda]$. Let $o^\lambda = D_m[\lambda]$ be the dimension of O^λ and let $T_1^\lambda, T_2^\lambda, \ldots, T_{o^\lambda}^\lambda$ be the $O(m)$-standard tableaux. The action of $G \in O(m)$ on each $[T_i^\lambda]$ yields, through (5.2.13), a linear combination of, in general, non-standard tableaux. The techniques of this section enable each to be written in terms of $O(m)$-standard tableaux, so that:

$$G\,[T_i^\lambda] = \sum_{j=1}^{o^\lambda} \Gamma^{[\lambda]}(G)_{ji}[T_j^\lambda], \tag{5.2.20}$$

where the $\Gamma^{[\lambda]}(G)_{ji} \in \mathsf{F}$ are the matrix elements of G in the representation $[\lambda]$. In a similar way, the representation matrix $\Gamma^{[\lambda]}(B)$ of $B \in so(m)$ is given, via (5.2.14), by:

$$B\,[T_i^\lambda] = \sum_{j=1}^{o^\lambda} \Gamma^{[\lambda]}(B)_{ji}[T_j^\lambda]. \tag{5.2.21}$$

Note that in the reduction of an arbitrary traceless symmetrised tableau to a linear combination over the $O(m)$-standard tableaux, the coefficients are integral apart from those arising from using the algorithm of Lemma 5.2.10. In this case, factors of $1/2$ may occur. Consequently, for the basis elements, ${B_a}^b$ or ${D_a}^b$ of $so(2r+1)$ or $so(2r)$ respectively, the matrix elements, $\Gamma^{[\lambda]}({B_a}^b)_{ji}$ or $\Gamma^{[\lambda]}({D_a}^b)_{ji}$, are rational numbers whose denominators are integral powers of 2. This situation contrasts with that of the $GL(m)$-modules and $Sp(2r)$-modules considered in Chapter 4, where the coefficients of the standard terms and the representation matrix elements are all guaranteed to be integral.

As an example, let $\lambda = (2,1)$ and consider the element ${B_2}^1 \in so(5)$ in the 35-dimensional $O(5)$-module O^λ. Then if:

$$T^\lambda = \begin{matrix} 1 & \bar{2} \\ \bar{2} & \end{matrix},$$

the action of ${B_2}^1$ on $[T^\lambda]$ is given, via (5.2.14), by:

$$\begin{aligned}
{B_2}^1 \begin{bmatrix} 1 & \bar{2} \\ \bar{2} & \end{bmatrix} &= {E_2}^1 \begin{bmatrix} 1 & \bar{2} \\ \bar{2} & \end{bmatrix} - {E_{\bar{1}}}^{\bar{2}} \begin{bmatrix} 1 & \bar{2} \\ \bar{2} & \end{bmatrix} \\
&= + \begin{bmatrix} 2 & \bar{2} \\ \bar{2} & \end{bmatrix} - \begin{bmatrix} 1 & \bar{2} \\ \bar{1} & \end{bmatrix} - \begin{bmatrix} 1 & \bar{1} \\ \bar{2} & \end{bmatrix} \\
&= - \begin{bmatrix} \bar{2} & 2 \\ 2 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 1 & \end{bmatrix} - \begin{bmatrix} \bar{1} & 1 \\ \bar{2} & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 1 & \end{bmatrix} \quad \text{(from (3.4.2) and (3.4.3))}.
\end{aligned}$$

However:

$$\begin{aligned}
\begin{bmatrix} \bar{1} & 1 \\ \bar{2} & \end{bmatrix} &= - \begin{bmatrix} 1 & \bar{1} \\ \bar{2} & \end{bmatrix} - \begin{bmatrix} \bar{2} & 2 \\ \bar{2} & \end{bmatrix} - \begin{bmatrix} 2 & \bar{2} \\ \bar{2} & \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \bar{2} & \end{bmatrix} \quad \text{(from (5.2.3) or (5.2.10}a\text{))} \\
&= - \begin{bmatrix} 1 & \bar{1} \\ \bar{2} & \end{bmatrix} + \begin{bmatrix} \bar{2} & \bar{2} \\ 2 & \end{bmatrix} + \begin{bmatrix} \bar{2} & 0 \\ 0 & \end{bmatrix} \quad \text{(from (3.4.2))},
\end{aligned}$$

and:

$$\begin{bmatrix} 1 & \bar{1} \\ \bar{2} & \end{bmatrix} = + \begin{bmatrix} \bar{1} & 1 \\ \bar{2} & \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{2} \\ 1 & \end{bmatrix} \quad \text{(from (3.4.3) or (5.2.10}b\text{))},$$

so that:

$$\begin{bmatrix} \bar{1} & 1 \\ \bar{2} & \end{bmatrix} = \frac{1}{2}\left(+ \begin{bmatrix} \bar{1} & \bar{2} \\ 1 & \end{bmatrix} + \begin{bmatrix} \bar{2} & \bar{2} \\ 2 & \end{bmatrix} + \begin{bmatrix} \bar{2} & 0 \\ 0 & \end{bmatrix} \right) \quad \text{(as in (5.2.10}c\text{))}.$$

Hence:

$${B_2}^1 \begin{bmatrix} 1 & \bar{2} \\ \bar{2} & \end{bmatrix} = \frac{3}{2} \begin{bmatrix} \bar{1} & \bar{2} \\ 1 & \end{bmatrix} - \frac{3}{2} \begin{bmatrix} \bar{2} & \bar{2} \\ 2 & \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \bar{2} & 0 \\ 0 & \end{bmatrix}.$$

The calculation need not be so complex, as the following example makes clear:

$${B_2}^1 \begin{bmatrix} \bar{2} & 2 \\ 0 & \end{bmatrix} = {E_2}^1 \begin{bmatrix} \bar{2} & 2 \\ 0 & \end{bmatrix} - {E_{\bar{1}}}^{\bar{2}} \begin{bmatrix} \bar{2} & 2 \\ 0 & \end{bmatrix} = - \begin{bmatrix} \bar{1} & 2 \\ 0 & \end{bmatrix}.$$

Similar calculations, when carried out for each of the thirty-five $O(5)$-standard tableaux in O^λ, yield the following explicit representation matrix $\Gamma^{[2,1]}({B_2}^1)$ for ${B_2}^1$:

$$\left(\begin{array}{ccccccccccccccccccccccccccccccccccc}
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}\right),$$

where each zero has been replaced by a dot. The two calculations carried out above give the entries in the 31st and the 6th columns of this matrix, respectively.

The algorithmic nature of the process lends itself to computer implementation. The above matrix has been produced in this way, together with similar matrices for the remaining generators of $so(5)$ in the same irreducible representation $[2,1]$. As a check on the calculations it has been confirmed that the resulting matrices satisfy the commutation relations (2.2.22). A large number of $O(m)$-modules have been constructed and verified in a similar way. As an additional check, representation matrices for arbitrary elements of the Lie algebras $so(m)$ have been generated from the representation matrices of the simple root vectors via (2.2.19) and (2.2.23), these simple root vector representations having been obtained through the techniques of this section. These agree with the matrices obtained directly from (5.2.21).

The techniques of this chapter now enable the $O(m)$-modules O^λ to be used to yield the characters of the elements with determinant -1 directly.

Theorem 5.2.22. *If $m = 2r+1$ is odd, then the character of the representation $[\lambda]$ is given by:*

$$[\lambda](y) = \sum_{T^\lambda : T^\lambda\, O(m)-standard} (-1)^{n_0(T^\lambda)} y^{T^\lambda}, \tag{5.2.22a}$$

where (y) denotes the vector $(y_1, y_2, \ldots, y_r)$ and $y^{T^\lambda} = y_1^{n_1^{O(m)}(T^\lambda)} y_2^{n_2^{O(m)}(T^\lambda)} \cdots y_r^{n_r^{O(m)}(T^\lambda)}$, for those elements of $O(2r+1)$ with eigenvalues $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r, -1$, and thus determinant -1. If $m = 2r$ is even and:

$$z^{T^\lambda} = \begin{cases} y^{T^\lambda} & \text{if neither } \bar{1} \text{ nor } 1 \text{ is present in } T^\lambda; \\ -y^{T^\lambda} & \text{if both } \bar{1} \text{ and } 1 \text{ are present in } T^\lambda; \\ 0 & \text{otherwise,} \end{cases} \tag{5.2.22b}$$

then the character of the representation $[\lambda]$ is:

$$[\lambda](y) = \sum_{T^\lambda : T^\lambda O(m)-standard} z^{T^\lambda}, \tag{5.2.22c}$$

for those elements of $O(2r)$ with eigenvalues $-1, 1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r$, and thus determinant -1, where $y_1 = 1$.

Proof. If $m = 2r+1$, consider the following generic element of $O(2r+1)$:

$$\begin{pmatrix} y_1^{-1} & 0 \\ 0 & y_1 \end{pmatrix} \oplus \begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} y_r^{-1} & 0 \\ 0 & y_r \end{pmatrix} \oplus -1. \tag{5.2.22d}$$

By (5.2.13), its action on $[T^\lambda]$ yields $(-1)^{n_0(T^\lambda)} y^{T^\lambda} [T^\lambda]$. Summing the coefficients over the set of $O(m)$-standard tableaux which provide a basis for O^λ, then yields (5.2.22a) as the trace of the matrix representing (5.2.22d).

For $m = 2r$, consider the following generic element of $O(2r)$:

$$G = \begin{pmatrix} 0 & y_1 \\ y_1^{-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} y_r^{-1} & 0 \\ 0 & y_r \end{pmatrix}. \tag{5.2.22e}$$

Let T^λ be $O(m)$-standard. By (5.2.13), the action of G on $[T^\lambda]$ yields $y^{T^\lambda}[T'^\lambda]$, where T'^λ is identical to T^λ except that each $\bar{1}$ has been changed to a 1 and vice-versa. If T^λ contains neither, then $T'^\lambda = T^\lambda$ and y^{T^λ} appears on the diagonal of the matrix representing G. If T^λ contains $\bar{1}$s or 1s, but not both, then T'^λ is also $O(m)$-standard. Therefore, since $T'^\lambda \neq T^\lambda$, this case contributes nothing to the trace. If T^λ contains both $\bar{1}$s and 1s, then Definition 5.1.6 implies that both occur in the first column and neither occur elsewhere. T'^λ thus has the two entries reversed and therefore, by the Column relations, $[T'^\lambda] = -[T^\lambda]$. This case thus contributes $-y^{T^\lambda}$ to the trace of the matrix representing G. Summing over the $O(m)$-standard tableaux, as above, proves (5.2.22c).

Theorem 5.2.22 has the straightforward corollary that if $\tilde{\lambda}_1 = r$ then $[\lambda](y) = 0$ for elements of $O(2r)$ of determinant -1. In **[Pr89]** a result similar to Theorem 5.2.22 is obtained. Although the contribution from each standard tableau differs from that given here, the overall characters are in agreement.

§5.3. Irreducible $SO(m)$-modules

Let S^λ be the $SO(m)$-module arising from the restriction of the $O(m)$-module O^λ to the subgroup $SO(m)$ of $O(m)$. As was stated in Section 2.5, the representations $[\lambda]$ of $O(m)$ remain irreducible on restriction to $SO(m)$ if and only if $\tilde{\lambda}_1 \neq m/2$. Thus, if $\tilde{\lambda}_1 \neq m/2$, then S^λ is an irreducible $SO(m)$-module and the construction procedures of the previous section can be used to obtain these irreducible $SO(m)$-modules. In particular, if $\tilde{\lambda}_1 \neq m/2$ then the $O(m)$-standard tableaux will also be referred to as $SO(m)$-standard tableaux. It is the purpose of this section to elucidate the decomposition of the $SO(2r)$-module S^λ when $\tilde{\lambda}_1 = r$ and to use this analysis to derive a set of $SO(2r)$-standard tableaux and devise a standardisation algorithm. It will be also be shown that certain pairs of the $SO(m)$-modules S^λ are equivalent.

The analysis of this section borrows a number of the notions employed in Section 4.5. Although the two are related, the notions of this section are distinct and should not be confused with the corresponding notions of Section 4.5.

Definition 5.3.1. *Associate partition. Fix m and let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$. Define λ^*, the partition associate to λ, to be such that $\tilde{\lambda}^*_1 = m - \tilde{\lambda}_1$ and $\tilde{\lambda}^*_i = \tilde{\lambda}_i$ for $i > 1$.*

Since $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$, it follows from $\tilde{\lambda}^*_1 = m - \tilde{\lambda}_1$ and $\tilde{\lambda}^*_2 = \tilde{\lambda}_2$ that $\tilde{\lambda}^*_1 \geq \tilde{\lambda}^*_2$. This verifies that λ^* is indeed a partition. Furthermore, from $\tilde{\lambda}_1 \geq \tilde{\lambda}_2$, it follows that $\tilde{\lambda}^*_1 + \tilde{\lambda}^*_2 \leq m$. Thus O^{λ^*} is also an irreducible $O(m)$-module. The two $O(m)$-modules O^λ and O^{λ^*} are intimately related and are said to be associate. If $\tilde{\lambda}_1 = m/2$ then $\lambda^* = \lambda$ and both λ and O^λ are said to be self-associate.

Definition 5.3.2. *Associate tableau. Fix m, let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$ and let $s = \tilde{\lambda}_1$. Let T^λ be a column strict tableau with entries in the first column from the set $\mathcal{J} = \{j_1, j_2, \ldots, j_s\}$. The associate of T^λ is defined to be that column strict tableau $T^{\lambda^*}_*$ which differs from T^λ only in its first column which has length $t = m - s$, and the entries in that column constitute the set $\mathcal{K} = \{k_1, k_2, \ldots, k_t\} = \{i \in \mathcal{I}^{O(m)} : \bar{i} \notin \mathcal{J}\}$.*

To illustrate the above two definitions let $m = 6$. Then:

$$T^{(2,2,1)} = \begin{matrix} \bar{1} & \bar{1} \\ \bar{2} & 3 \\ 2 & \end{matrix} \qquad \Longrightarrow \qquad T^{(2,2,1)}_* = \begin{matrix} \bar{1} & \bar{1} \\ \bar{3} & 3 \\ 3 & \end{matrix} .$$

If $m = 7$:

$$T^{(2,2,1,1)} = \begin{matrix} \bar{1} & 2 \\ 1 & \bar{3} \\ \bar{2} & \\ 0 & \end{matrix} \qquad \Longrightarrow \qquad T_*^{(2,2,1)} = \begin{matrix} \bar{2} & 2 \\ \bar{3} & \bar{3} \\ 3 & \end{matrix}\ .$$

In these two examples the associate of each $O(m)$-standard tableau is itself $O(m)$-standard. In addition, as may be easily verified, $T^{\lambda}_{>*} = T^{\lambda^*}_{>}$. This illustrates the following lemma.

Lemma 5.3.3. *If T^λ is an $O(m)$-standard tableau then the tableau $T_*^{\lambda^*}$ is also $O(m)$-standard.*

Proof. For all $i \in \mathcal{I}^{O(m)}$ let α_i and β_i be the number of entries less than or equal to i in the first and second columns of T^λ, respectively. Let α_i^* and β_i^* be defined likewise for $T_*^{\lambda^*}$. Then $\beta_i^* = \beta_i$ for each $i \in \mathcal{I}$. For the purposes of this proof let $\alpha_0 = \beta_0 = \alpha_0^* = \beta_0^* = 0$.

Let even $m = 2r$ and odd $m = 2r+1$. If T^λ is $O(m)$-standard then Definition 5.1.6 implies that for each $i = 1, 2, \ldots, r$:

(i) $\alpha_i \geq \beta_i$,

(ii) $\alpha_{\bar{i}} \geq \beta_{\bar{i}}$,

(iii) $\alpha_i + \beta_i \leq 2i$,

(iv) if $\alpha_i + \beta_i = 2i$ and $\alpha_i > \beta_i$ and $\alpha_{i-1} + 1 = \alpha_{\bar{i}} + 1 = \alpha_i$ and $\beta_{\bar{i}} = \beta_{i-1} + 1$ then $\beta_i = \beta_{\bar{i}} + 1$,

(v) if $\alpha_i = \beta_i = i$ and $\alpha_{i-1} + 1 = \alpha_{\bar{i}} = \alpha_i$ and $T^\lambda_{i,b} = i$ for some b then $T^\lambda_{i-1,b} = \bar{i}$.

It is required to demonstrate that each of these five conditions hold when α_i^* and β_i^* are exchanged for α_i and β_i respectively.

Let $\mathcal{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathcal{I}^{O(m)}$ be the set of indices in the first column of T^λ with $j_1 < j_2 < \cdots < j_s$. Further, for $i = 1, 2, \ldots, r$, let $\mathcal{J}_i = \{j \in \mathcal{J} : j \leq i\}$, so that $\#\mathcal{J}_i = \alpha_i$. Then the set of indices in the first column of T_*^λ is $\mathcal{K} = \{k_1, k_2, \ldots, k_t : \bar{k}_i \notin \mathcal{J}\} \subset \mathcal{I}^{O(m)}$. Let $k_1 < k_2 < \cdots < k_t$ and let $\mathcal{K}_i = \{k \in \mathcal{K} : k \leq i\}$ so that $\alpha_i^* = \#\mathcal{K}_i$. Then $\mathcal{K}_i = \{k \leq i : \bar{k} \notin \mathcal{J}\} = \{k \leq i : \bar{k} \notin \mathcal{J}_i\}$ so that $\#\mathcal{K}_i = 2i - \#\mathcal{J}_i$ and hence $\alpha_i^* = 2i - \alpha_i$. Consequently (i) $\alpha_i^* = 2i - \alpha_i \geq 2i - (2i - \beta_i) = \beta_i = \beta_i^*$, as required since $\alpha_i \leq 2i - \beta_i$ and (iii) $\alpha_i^* + \beta_i^* = \alpha_i^* + \beta_i = 2i - \alpha_i + \beta_i \leq 2i$, as required since $\alpha_i \geq \beta_i$.

Since $\alpha_i^* - 1 \leq \alpha_{\bar{i}}^* \leq \alpha_i^*$, $\beta_i^* - 1 \leq \beta_{\bar{i}}^* \leq \beta_i^*$ and $\alpha_i^* \geq \beta_i^*$, the condition $\alpha_{\bar{i}}^* \geq \beta_{\bar{i}}^*$ may only be violated if $\alpha_i^* = \beta_i^* = \beta_{\bar{i}}^* = \alpha_{\bar{i}}^* + 1$ whereupon $\alpha_i + \beta_i = 2i$ and $i \in \mathcal{K}$. Similarly $\alpha_{i-1}^* \leq \alpha_{\bar{i}}^* \leq \alpha_{i-1}^* + 1$, $\beta_{i-1}^* \leq \beta_{\bar{i}}^* \leq \beta_{i-1}^* + 1$, $\alpha_i^* \geq \beta_i^*$ and $\alpha_{\bar{i}}^* < \beta_{\bar{i}}^*$ imply that $\beta^*_{i-1} = \alpha^*_{i-1} = \alpha^*_{\bar{i}} = \beta^*_{\bar{i}} - 1$ and thus $\bar{i} \notin \mathcal{K}$. Thus $i \in \mathcal{J}$ and $\bar{i} \notin \mathcal{J}$ whereupon $\alpha_i = \alpha_{\bar{i}} + 1 = \alpha_{i-1} + 1$. Two cases now need to be considered. In the first $\alpha_i = \beta_i = i$ whereupon $\alpha^*_i = 2i - \alpha_i = \alpha_i = \alpha_{\bar{i}} + 1$ so that $\alpha_{\bar{i}} = \alpha^*_i - 1 = \beta^*_{\bar{i}} - 1 = \beta_{\bar{i}} - 1$ and $\alpha_{\bar{i}} < \beta_{\bar{i}}$ contradicting the assumption that T^λ is standard. In the other case where $\alpha_i > \beta_i$, since $\beta^*_i = \beta^*_{\bar{i}} = \beta^*_{i-1} + 1$, $\bar{i}$ would occur in the second column of T^λ but no i. This, with $i \in \mathcal{J}$, $\bar{i} \notin \mathcal{J}$ implies that T^λ would suffer a protection violation, again contradicting the assumption that T^λ is standard. Thus $\alpha^*_{\bar{i}} < \beta^*_{\bar{i}}$ cannot occur, giving the conclusion (ii) $\alpha^*_{\bar{i}} \geq \beta^*_{\bar{i}}$.

Now assume that a protection violation occurs in $T^{\lambda^*}_*$. This requires $\alpha^*_i + \beta^*_i = 2i$ and hence $\alpha_i = \beta_i$. Two cases need to be considered. If $\alpha^*_i = \beta^*_i = i$ then the protection violation insists that $\bar{i} \in \mathcal{K}$, $i \notin \mathcal{K}$ whereupon $i \notin \mathcal{J}$, $\bar{i} \in \mathcal{J}$. Thus, if an unprotected i occurs somewhere to the right of the $\bar{i}$ in $T^{\lambda^*}_*$, it would also do so in T^λ so that a violation of (v) can be excluded. In the other case, $\alpha^*_i > \beta^*_i$, and a protection violation of (iv) requires that $i \in \mathcal{K}$, $\bar{i} \notin \mathcal{K}$ whereupon $\bar{i} \notin \mathcal{J}$, $i \in \mathcal{J}$ and $\alpha_i = \alpha_{\bar{i}} + 1$. Also required is an $\bar{i}$ in the second column but no i. This would imply that $\beta_i = \beta_{\bar{i}} = \beta_{i-1} + 1$, and since $\alpha_i = \beta_i$, $\alpha_{\bar{i}} = \alpha_i - 1 = \beta_i - 1 = \beta_{\bar{i}} - 1$, once more contradicting the assumption that T^λ is standard. This completes the proof.

Associated with each transition from $[T^\lambda]$ to $[T^{\lambda^*}_*]$ is a sign factor given by

$$\epsilon^{(\mathcal{J},\bar{\mathcal{K}})} = \epsilon_{j_1 \cdots j_s \bar{k}_1 \cdots \bar{k}_t}, \tag{5.3.4}$$

where $\mathcal{J} = \{j_1, \ldots, j_s\}$ are the entries in the first column of T^λ with each $j_i = T^\lambda_{(i)}$ and $\mathcal{K} = \{k_1, \ldots, k_s\}$ are the entries in the first column of $T^{\lambda^*}_*$ with each $k_i = T^{\lambda^*}_{*(i)}$. Throughout this section, the convention of summing over all repeated indices which are displayed explicitly will be used. Let $m = s + t$ and define:

$$\begin{aligned} L_{a_1 \cdots a_s b_1 \cdots b_t} &= \frac{1}{t!} \epsilon_{a_1 \cdots a_s c_1 \cdots c_t} J_{c_1 b_1} \cdots J_{c_t b_t} \\ &= \frac{1}{t!} \epsilon_{a_1 \cdots a_s \bar{b}_1 \cdots \bar{b}_t}, \end{aligned} \tag{5.3.5}$$

so that:

$$L_{j_1 \cdots j_s k_1 \cdots k_t} = \frac{1}{t!} \epsilon^{(\mathcal{J},\bar{\mathcal{K}})}. \tag{5.3.6}$$

Define:

$$K_{a_1 \cdots a_s c_1 \cdots c_s} = (-1)^{r+st} L_{a_1 \cdots a_s b_1 \cdots b_t} L_{b_1 \cdots b_t c_1 \cdots c_s} \tag{5.3.7}$$

Lemma 5.3.8.

$$K_{a_1 \cdots a_s c_1 \cdots c_s} = \frac{1}{s!} \left(\sum_{\pi \in S_s} (-1)^\pi \delta_{a_1 c_{\pi(1)}} \cdots \delta_{a_s c_{\pi(s)}} \right). \tag{5.3.8}$$

Proof. Substituting (5.3.5) into (5.3.7) gives:

$$K_{a_1\cdots a_s c_1\cdots c_s} = (-1)^{r+st}\frac{1}{s!t!}\epsilon_{a_1\cdots a_s\bar{b}_1\cdots\bar{b}_t}\epsilon_{b_1\cdots b_t\bar{c}_1\cdots\bar{c}_s}.$$

It may be seen that $\epsilon_{b_1\cdots b_t\bar{c}_1\cdots\bar{c}_s} = (-1)^r\epsilon_{\bar{b}_1\cdots\bar{b}_t c_1\cdots c_s}$ by transposing i and $\bar{i}$ for each $i = 1, 2, \ldots, r$, and $\bar{0} = 0$ where $m = 2r+1$. Thus:

$$K_{a_1\cdots a_s c_1\cdots c_s} = (-1)^{st}\frac{1}{s!t!}\epsilon_{a_1\cdots a_s b_1\cdots b_t}\epsilon_{b_1\cdots b_t c_1\cdots c_s},$$

whereupon the lemma is proved by precisely the same reasoning as for the proof of Lemma 4.5.13.

This lemma shows that K is an antisymmetriser so that:

$$e_{[a_1\cdots a_s]} = K_{a_1\cdots a_s b_1\cdots b_s}e_{b_1\cdots b_s}. \tag{5.3.9}$$

In addition, it shows that:

$$L_{a_1\cdots a_s b_1\cdots b_t}K_{b_1\cdots b_t c_1\cdots c_t} = K_{a_1\cdots a_s b_1\cdots b_s}L_{b_1\cdots b_s c_1\cdots c_t} = L_{a_1\cdots a_s c_1\cdots c_t}, \tag{5.3.10}$$

and hence:

$$L_{a_1\cdots a_s b_1\cdots b_t}e_{[b_1\cdots b_t]} = L_{a_1\cdots a_s b_1\cdots b_t}e_{b_1\cdots b_t}. \tag{5.3.11}$$

The developments of this section and the next depend on how L commutes with elements of $O(m)$. If $G \in GL(m)$ then:

$$(\det G)\epsilon_{a_1\cdots a_s c_1\cdots c_t} = \epsilon_{e_1\cdots e_s f_1\cdots f_t}G_{a_1e_1}\cdots G_{a_se_s}G_{c_1f_1}\cdots G_{c_tf_t}. \tag{5.3.12}$$

If $G \in GL(m)$ preserves the form given by the matrix J, then $G_{cf}J_{cb}G_{bd} = J_{fd}$ and $\det G = (\det G)^{-1} = \pm 1$, whereupon:

$$\begin{aligned}
&L_{a_1\cdots a_s b_1\cdots b_t}G_{b_1d_1}\cdots G_{b_td_t}\\
&\quad= \epsilon_{a_1\cdots a_s c_1\cdots c_t}J_{c_1b_1}\cdots J_{c_tb_t}G_{b_1d_1}\cdots G_{b_td_t}\\
&\quad= (\det G)\epsilon_{e_1\cdots e_s f_1\cdots f_t}G_{a_1e_1}\cdots G_{a_se_s}G_{c_1f_1}\cdots G_{c_tf_t}J_{c_1b_1}\cdots J_{c_tb_t}G_{b_1d_1}\cdots G_{b_td_t}\\
&\quad= (\det G)\epsilon_{e_1\cdots e_s f_1\cdots f_t}G_{a_1e_1}\cdots G_{a_se_s}J_{f_1d_1}\cdots J_{f_td_t}\\
&\quad= (\det G)G_{a_1e_1}\cdots G_{a_se_s}L_{e_1\cdots e_s d_1\cdots d_t}.
\end{aligned} \tag{5.3.13}$$

Thus if $G \in SO(m)$, the operator L commutes with G. This implies that if T^{λ}_{w} has the indices from the set $\mathcal{J}_w$ comprising its first column and $T^{\lambda^*}_{w*}$ has indices from the set $\mathcal{K}_w = \{k \in \mathcal{I}^{O(m)} : \bar{k} \in \mathcal{J}_w\}$ comprising its first column, then the transformations of the $[T^{\lambda}_{w}]$ under $SO(m)$ are identical to those of $\epsilon^{(\mathcal{J},\mathcal{K})}[T^{\lambda^*}_{w*}]$.

Now restrict attention to those $SO(2r)$-modules S^λ for which $\tilde{\lambda}_1 = r$, and therefore $\lambda^* = \lambda$. Let $s = t = r$ and define:

$$L^{\pm}_{a_1\cdots a_r b_1\cdots b_r} = \frac{1}{2}\left(K_{a_1\cdots a_r b_1\cdots b_r} \pm (-1)^{r(r+1)/2} L_{a_1\cdots a_r b_1\cdots b_r}\right). \qquad (5.3.14)$$

The sign factor $(-1)^{r(r+1)/2}$ is simply a matter of convention in determining which of these operators is to be known as L^+ and which L^-. What is important is that they are both idempotent and commute with the action of all $G \in SO(2r)$. That they commute with G follows from (5.3.7) and (5.3.13) with $s = t = r$ and $\det G = 1$. That they are idempotent follows from the identities $LL = K$, $LK = KL = L$ and $KK = LLK = LL = K$, where all indices have been suppressed for typographical convenience, and use has been made of (5.3.7) and (5.3.10) with $s = t = r$. In particular:

$$L^{\pm}_{j_1\cdots j_r i_1\cdots i_r} e_{i_1\cdots i_r} = \frac{1}{2}\left(e_{[j_1\cdots j_r]} \pm (-1)^{r(r+1)/2} \epsilon^{(\mathcal{J},\mathcal{K})} e_{[k_1\cdots k_r]}\right), \qquad (5.3.15)$$

where $\mathcal{J} = \{j_1, \ldots, j_r\}$ and $\mathcal{K} = \{k_1, \ldots, k_r\} = \{k \in \mathcal{I}^{O(m)} : \bar{k} \notin \mathcal{J}\}$. It follows from the above that the subspace of $V^{\otimes r}$ spanned by all tensors of the form $L^{+}_{j_1\cdots j_r i_1\cdots i_r} e_{i_1\cdots i_r}$ is invariant under $SO(2r)$. This is likewise true for all tensors of the form $L^{-}_{j_1\cdots j_r i_1\cdots i_r} e_{i_1\cdots i_r}$. Similarly the subspaces $U^\pm \in V^{\otimes l}$ spanned by all tensors of the form:

$$\sum_{\sigma \in S_r} (-1)^\sigma (x_{i_0} \otimes w_{j_{\sigma(1)}} \otimes x_{i_1} \otimes w_{j_{\sigma(2)}} \otimes x_{i_2} \otimes \cdots \otimes w_{j_{\sigma(r)}} \otimes x_{i_r}$$
$$\pm (-1)^{r(r+1)/2} \epsilon^{(\mathcal{J},\mathcal{K})} x_{i_0} \otimes w_{k_{\sigma(1)}} \otimes x_{i_1} \otimes w_{k_{\sigma(2)}} \otimes x_{i_2} \otimes \cdots \otimes w_{k_{\sigma(r)}} \otimes x_{i_r}), \qquad (5.3.16)$$

where $x_{i_0}, x_{i_1}, \ldots, x_{i_r}$ are each an element of some, possibly zero, tensor power of V, are each invariant under $SO(m)$. Now let $S^{\lambda+} = S^\lambda/(S^\lambda \cap U^-)$ and $S^{\lambda-} = S^\lambda/(S^\lambda \cap U^+)$. Weyl proved the following theorem.

Theorem 5.3.17. [**We39**] *If $\lambda \in P(l)$ is such that $\tilde{\lambda}_1 = r$, then the $O(2r)$-module O^λ is decomposable on restriction to $SO(2r)$ into the direct sum of two inequivalent irreducible $SO(2r)$-modules, $S^{\lambda+}$ and $S^{\lambda-}$, the dimension of each being half that of O^λ.*

To exploit Theorem 5.3.17 in a constructive manner, define:

$$[T^\lambda]^\pm = [T^\lambda] \pm (-1)^{r(r+1)/2} \epsilon^{(\mathcal{J},\mathcal{K})} [T^\lambda_*], \qquad (5.3.18)$$

where $\mathcal{J} = \{j_1, \ldots, j_r\}$ are the entries in the first column of T^λ with each $j_i = T^\lambda_{(i)}$ and $\mathcal{K} = \{k_1, \ldots, k_r\}$ are the entries in the first column of $T^{\lambda^*}_*$ with each $k_i = T^{\lambda^*}_{*(i)}$.

This implies, since

$$\begin{aligned}\epsilon^{(\mathcal{K},\bar{\mathcal{J}})} &= \epsilon_{k_1\cdots k_r\bar{j}_1\cdots\bar{j}_r} = (-1)^r \epsilon_{\bar{k}_1\cdots\bar{k}_r j_1\cdots j_r} \\ &= \epsilon_{j_1\cdots j_r\bar{k}_1\cdots\bar{k}_r} = \epsilon^{(\mathcal{J},\bar{\mathcal{K}})},\end{aligned} \tag{5.3.19}$$

that:

$$[T^\lambda]^+ = +(-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\bar{\mathcal{K}})}[T^\lambda_*]^+ \tag{5.3.20a}$$

and

$$[T^\lambda]^- = -(-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\bar{\mathcal{K}})}[T^\lambda_*]^-. \tag{5.3.20b}$$

This leads to the following lemma.

Lemma 5.3.21. *Let $m = 2r$ and $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 = r$, whereupon $\lambda^* = \lambda$. Then $[T^\lambda]^\pm \in U^\pm$ and $[T^\lambda]^\pm \in S^{\lambda\pm}$. Moreover, in $S^{\lambda\pm}$, $[T^\lambda]^\mp = 0$.*

Proof. Since $\{T^\lambda\}$ is antisymmetric in the r indices of its first column, it follows from (5.3.16) that $\{T^\lambda\} \pm (-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\bar{\mathcal{K}})}\{T^\lambda_*\} \in U^\pm$. On removing the trace terms this gives $[T^\lambda] \pm (-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\bar{\mathcal{K}})}[T^\lambda_*] \in U^\pm$. Therefore, from (5.3.18), $[T^\lambda]^\pm \in U^\pm$. The remainder of the lemma then follows directly from the definitions of $S^{\lambda\pm}$.

As an illustration of (5.3.18), let $r = 3$ and $\lambda = (2,2,1)$. Then:

$$T^\lambda = \begin{matrix} 1 & \bar{1} \\ \bar{2} & 3 \\ 2 & \end{matrix} \qquad \Longrightarrow \qquad T^\lambda_* = \begin{matrix} 1 & \bar{1} \\ \bar{3} & 3 \\ 3 & \end{matrix},$$

where $\epsilon^{(\mathcal{J},\bar{\mathcal{K}})} = +1$, so that:

$$\begin{bmatrix} 1 & \bar{1} \\ \bar{2} & 3 \\ 2 & \end{bmatrix}^+ = \begin{bmatrix} 1 & \bar{1} \\ \bar{2} & 3 \\ 2 & \end{bmatrix} + \begin{bmatrix} 1 & \bar{1} \\ \bar{3} & 3 \\ 3 & \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \bar{1} \\ \bar{2} & 3 \\ 2 & \end{bmatrix}^- = \begin{bmatrix} 1 & \bar{1} \\ \bar{2} & 3 \\ 2 & \end{bmatrix} - \begin{bmatrix} 1 & \bar{1} \\ \bar{3} & 3 \\ 3 & \end{bmatrix}.$$

It may occur that one of the resulting terms is zero, for example:

$$T^\lambda = \begin{matrix} \bar{1} & 1 \\ 2 & 3 \\ 3 & \end{matrix} \qquad \Longrightarrow \qquad T^\lambda_* = \begin{matrix} \bar{1} & 1 \\ 2 & 3 \\ 3 & \end{matrix},$$

where $\epsilon^{(\mathcal{J},\bar{\mathcal{K}})} = -1$, so that:

$$\begin{bmatrix} \bar{1} & 1 \\ 2 & 3 \\ 3 & \end{bmatrix}^- = 2\begin{bmatrix} \bar{1} & 1 \\ 2 & 3 \\ 3 & \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} \bar{1} & 1 \\ 2 & 3 \\ 3 & \end{bmatrix}^+ = 0.$$

The subsequent definition acts as a preliminary to obtaining a set of standard tableaux for each of the modules $S^{\lambda\pm}$ with $\tilde{\lambda}_1 = r$.

Definition 5.3.22. *Let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 = r$. For each $O(2r)$-standard tableau $T^\lambda \in \mathcal{O}^\lambda_{2r}$ let α_i be the number of entries in the first column less than or equal to i, for $i = 1, 2, \ldots r$. Then define:*

$$\begin{aligned}
\mathcal{S}_0^{\lambda+} &= \{T^\lambda \in \mathcal{O}^\lambda_{2r} : \alpha_i = i \text{ for } i = 1,2,\ldots,r;\ [T^\lambda]^+ \neq 0\};\\
\mathcal{S}_0^{\lambda-} &= \{T^\lambda \in \mathcal{O}^\lambda_{2r} : \alpha_i = i \text{ for } i = 1,2,\ldots,r;\ [T^\lambda]^- \neq 0\};\\
\mathcal{S}_1^{\lambda+} &= \{T^\lambda \in \mathcal{O}^\lambda_{2r} : \alpha_i = i \text{ for } i = 1,2,\ldots,j-1;\ \alpha_j < j \text{ for some } j < r\};\\
\mathcal{S}_1^{\lambda-} &= \{T^\lambda \in \mathcal{O}^\lambda_{2r} : \alpha_i = i \text{ for } i = 1,2,\ldots,j-1;\ \alpha_j > j \text{ for some } j < r\}.
\end{aligned} \tag{5.3.22}$$

Note that if $T^\lambda \in \mathcal{O}^\lambda_{2r}$ and $\alpha_i = i$ for each $i = 1, 2, \ldots, r$, then $T^\lambda = T^\lambda_*$ whereupon, from (5.3.18) or (5.3.20), exactly one of $[T^\lambda]^+$ and $[T^\lambda]^-$ is zero. It then follows that $\mathcal{O}^\lambda_{2r}$ is the disjoint union of $\mathcal{S}_0^{\lambda+}$, $\mathcal{S}_0^{\lambda-}$, $\mathcal{S}_1^{\lambda+}$, and $\mathcal{S}_1^{\lambda-}$.

Lemma 5.3.23. *If $T^\lambda \in \mathcal{S}_0^{\lambda\mp}$ then:*

$$[T^\lambda]^\pm = 0. \tag{5.3.23a}$$

If $T^\lambda \in \mathcal{S}_1^{\lambda-}$ then $T^\lambda_* \in \mathcal{S}_1^{\lambda+}$ and:

$$[T^\lambda]^\pm = \pm(-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\mathcal{K})}[T^\lambda_*]^\pm. \tag{5.3.23b}$$

where $\mathcal{J} = \{j_1, \ldots, j_r\}$ are the entries in the first column of T^λ with each $j_i = T^\lambda_{(i)}$ and $\mathcal{K} = \{k_1, \ldots, k_r\}$ are the entries in the first column of $T^{\lambda^*}_*$ with each $k_i = T^{\lambda^*}_{*(i)}$.

Proof. If $T^\lambda \in \mathcal{S}_0^{\lambda\mp}$ then $T^\lambda = T^\lambda_*$ whereupon, since exactly one of $[T^\lambda]^+$ and $[T^\lambda]^-$ is zero, (5.3.23a) follows from the definitions of $\mathcal{S}_0^{\lambda\pm}$. With α_i as in Definition 5.3.22, if $T^\lambda \in \mathcal{S}_1^{\lambda-}$ then $j < \alpha_j$ for some j with $i = \alpha_i$ for $i = 1, 2, \ldots, j-1$. Then $\alpha_i^* = 2i - \alpha_i$ implies that $i = \alpha_i^*$ for $i = 1, 2, \ldots, j-1$ and also that $j > \alpha_j^*$. Therefore $T^\lambda_* \in \mathcal{S}_1^{\lambda+}$ since $T^\lambda_* \in \mathcal{O}^\lambda_{2r}$ by Lemma 5.3.3. Identity (5.3.23b) combines (5.3.20a) and (5.3.20b).

Lemma 5.3.24. *The cardinality of the set $\mathcal{S}_0^{\lambda+}$ equals the cardinality of the set $\mathcal{S}_0^{\lambda-}$, and the cardinality of the set $\mathcal{S}_1^{\lambda+}$ equals the cardinality of the set $\mathcal{S}_1^{\lambda-}$.*

Proof. For the tableaux T^λ and T^λ_* define $\mathcal{J}$, $\mathcal{K}$, $\mathcal{J}_i$, $\mathcal{K}_i$, α_i and α_i^* exactly as for the proof of Lemma 5.3.3.

If $T^\lambda \in \mathcal{S}_0^{\lambda+}$ then $\mathcal{K} = \mathcal{J}$, $T^\lambda = T^\lambda_*$ and

$$(-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\mathcal{K})} = (-1)^{r(r+1)/2}\epsilon_{j_1\cdots j_r\bar{j}_1\cdots\bar{j}_r} = 1.$$

Since $\alpha_1 = 1$, $j_2 \geq \bar{2}$ (if $r > 1$) and either $j_1 = \bar{1}$ or $j_1 = 1$. Since T^λ is $O(2r)$-standard this entry may only occur in the first row of T^λ and $\bar{j}_1$ may not occur at all. Form the $O(2r)$-standard tableau T'^{λ} from T^{λ} by replacing each j_1 with $\bar{j}_1$. It is easily seen that $T'^{\lambda}_* = T'^{\lambda}$. However, from (5.3.20$a$):

$$\begin{aligned}
\left[T'^{\lambda}\right]^+ &= +(-1)^{r(r+1)/2}\epsilon_{\bar{j}_1 j_2 \cdots j_r j_1 \bar{j}_2 \cdots \bar{j}_r}\left[T'^{\lambda}\right]^+ \\
&= -(-1)^{r(r+1)/2}\epsilon_{j_1 j_2 \cdots j_r \bar{j}_1 \bar{j}_2 \cdots \bar{j}_r}\left[T'^{\lambda}\right]^+ \\
&= -\left[T'^{\lambda}\right]^+ .
\end{aligned}$$

Therefore $[T'^{\lambda}]^+ = 0$, implying that $T'^{\lambda} \in \mathcal{S}_0^{\lambda -}$. The map $T^{\lambda} \to T'^{\lambda}$ is clearly a bijection between $\mathcal{S}_0^{\lambda +}$ and $\mathcal{S}_0^{\lambda -}$ implying that $\#\mathcal{S}_0^{\lambda +} = \#\mathcal{S}_0^{\lambda -}$.

If $T^{\lambda} \in \mathcal{S}_1^{\lambda -}$ then Lemma 5.3.23 shows that $T^{\lambda}_* \in \mathcal{S}_1^{\lambda +}$. In a similar way it is shown that $T^{\lambda} \in \mathcal{S}_1^{\lambda +}$ implies that $T^{\lambda}_* \in \mathcal{S}_1^{\lambda -}$. These maps are inverse to one another, demonstrating that the map $T^{\lambda} \to T^{\lambda}_*$ is a bijection between the sets $\mathcal{S}_1^{\lambda -}$ and $\mathcal{S}_1^{\lambda +}$. This shows that $\#\mathcal{S}_1^{\lambda +} = \#\mathcal{S}_1^{\lambda -}$.

This lemma enables appropriate sets of $SO(2r)$-standard tableaux to be defined.

Definition 5.3.25. *Let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 = r$, and the sets $\mathcal{S}_0^{\lambda +}$, $\mathcal{S}_0^{\lambda -}$, $\mathcal{S}_1^{\lambda +}$ and $\mathcal{S}_1^{\lambda -}$ are as in Definition* 5.3.22. *Then:*

(i) $\mathcal{S}_0^{\lambda +} \cup \mathcal{S}_1^{\lambda +}$ is the set of $SO(2r)$-standard tableaux in the module $S^{\lambda +}$; and

(ii) $\mathcal{S}_0^{\lambda -} \cup \mathcal{S}_1^{\lambda +}$ is the set of $SO(2r)$-standard tableaux in the module $S^{\lambda -}$.

The significance of these standard tableaux lies in the following two theorems.

Theorem 5.3.26. *If $\lambda \in P(l)$ is such that $\tilde{\lambda}_1 = r$, then the multiplicity of the weight $(n_1, n_2, \ldots, n_r)$ in the irreducible representations $[\lambda]_{\pm}$ of $O(m)$ is given by the number of appropriate $SO(m)$-standard tableaux T^{λ} for which $n^{O(m)}(T^{\lambda}) = (n_1, n_2, \ldots, n_r)$. The characters of these representations are given by:*

$$[\lambda]_{\pm}(y) = \sum_{T^{\lambda} \in \mathcal{S}_0^{\lambda \pm} \cup \mathcal{S}_1^{\lambda +}} y^{T^{\lambda}}, \tag{5.3.26}$$

where (y) denotes the vector $(y_1, y_2, \ldots, y_r)$ and $y^{T^{\lambda}} = y_1^{n_1^{O(m)}(T^{\lambda})} y_2^{n_2^{O(m)}(T^{\lambda})} \cdots y_r^{n_r^{O(m)}(T^{\lambda})}$, for the class of $SO(m)$ with eigenvalues $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r$, if $m = 2r$; or eigenvalues $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r, 1$, if $m = 2r + 1$.

Proof. If $T^{\lambda} \in \mathcal{S}_0^{\lambda +}$ then $[T^{\lambda}]^+ = 2[T^{\lambda}] \neq 0$ and $[T^{\lambda}] \in S^{\lambda +}$, whereupon, since $n^{O(m)}(T^{\lambda})$ is a weight of the $SO(m)$-module S^{λ}, it follows that $n^{O(m)}(T^{\lambda})$ is a weight of the $SO(m)$-module $S^{\lambda +}$. If $T^{\lambda} \in \mathcal{S}_1^{\lambda +}$ then $[T^{\lambda}]^+ = [T^{\lambda}] + (-1)^{r(r+1)/2}\epsilon^{(\mathcal{J},\mathcal{K})}[T^{\lambda}_*] \neq 0$ and $[T^{\lambda}]^+ \in S^{\lambda +}$. Then, since the $O(m)$-weights of T^{λ} and T^{λ}_* are equal, $n^{O(m)}(T^{\lambda})$ is a weight of $S^{\lambda +}$. By similar reasoning, if $T^{\lambda} \in \mathcal{S}_0^{\lambda -}$ or $T^{\lambda} \in \mathcal{S}_1^{\lambda +}$ then $n^{O(m)}(T^{\lambda})$

is a weight of $S^{\lambda-}$. Since $\#\mathcal{S}_1^{\lambda+} = \#\mathcal{S}_1^{\lambda-}$, this exhausts all the weights of S^λ. The result then follows.

Theorem 5.3.27. *If $\lambda \in P(l)$ is such that $\tilde{\lambda}_1 = r$, then the $SO(2r)$-standard tableaux form bases for the irreducible $SO(2r)$-modules $S^{\lambda\pm}$; that is:*

$$\left\{ \left[T^\lambda\right]^+ : T^\lambda \in \mathcal{S}_0^{\lambda+} \cup \mathcal{S}_1^{\lambda+} \right\}$$

is a basis for the $SO(2r)$-module $S^{\lambda+}$, and:

$$\left\{ \left[T^\lambda\right]^- : T^\lambda \in \mathcal{S}_0^{\lambda-} \cup \mathcal{S}_1^{\lambda+} \right\}$$

is a basis for the $SO(2r)$-module $S^{\lambda-}$. Moreover $S^{\lambda+}$ is isomorphic to $V^{\otimes l}$ modulo the relations (3.4.2), (3.4.3), (5.2.3) *and* (5.3.20*a*), *and $S^{\lambda-}$ is isomorphic to $V^{\otimes l}$ modulo the relations* (3.4.2), (3.4.3), (5.2.3) *and* (5.3.20*b*).

Proof. Lemma 5.3.21 shows that $S^{\lambda\pm}$ are spanned by the sets of terms of the form $[T^\lambda]^\pm$. Theorem 5.3.17 and Lemma 5.3.24 show that the sets of $SO(2r)$-standard tableaux are of the correct cardinality. The theorem is thus proved if it can be shown that for every tableau T^λ, $[T^\lambda]^\pm$ can be expressed as a linear combination of $SO(2r)$-standard terms in the $SO(2r)$-modules $S^{\lambda\pm}$. Theorem 5.2.19 indicates how to express $[T^\lambda]$ as a linear combination of $O(2r)$-standard terms. The analogous result is achieved for $[T^\lambda]^\pm$ in the same way by the definition of $S^{\lambda\pm}$ as a quotient. Any term $[T_w^\lambda]^\pm$ for which T_w^λ is $O(2r)$-standard but not $SO(2r)$-standard is dealt with using one or other of the identities given by Lemma 5.3.23. These immediately produce either an $SO(2r)$-standard term or zero. Thus, every $[T^\lambda]^\pm$ can be reduced to a linear combination of $SO(2r)$-standard terms in each of the $SO(2r)$-modules $S^{\lambda\pm}$.

With the module actions of $SO(2r)$ and $so(2r)$ analogous to those given by (5.2.13) and (5.2.14) respectively, this theorem effectively provides definitions for $S^{\lambda\pm}$.

As an example of the way in which the reduction to a linear combination of standard tableaux is achieved in the different modules, consider the tableau:

$$T^{(2,2,1)} \quad = \quad \begin{matrix} \bar{1} & \bar{2} \\ \bar{3} & 2 \\ 3 & \end{matrix} \; ,$$

the irreducible $O(6)$-module $O^{(2,2,1)}$, and the irreducible $SO(6)$-modules $S^{(2,2,1)+}$ and $S^{(2,2,1)-}$. In $O^{(2,2,1)}$ the reduction of $[T^{(2,2,1)}]$ to terms involving $O(6)$-standard tableaux results from the application of a single Garnir relation (3.4.3):

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{3} & 2 \\ 3 & \end{bmatrix} = -\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix} + \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{2} & 3 \\ 2 & \end{bmatrix}.$$

The corresponding identities in the $SO(6)$-modules $S^{(2,2,1)\pm}$ are

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{3} & 2 \\ 3 & \end{bmatrix}^{\pm} = -\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix}^{\pm} + \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix}^{\pm} + \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix}^{\pm} - \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix}^{\pm} - \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{2} & 3 \\ 2 & \end{bmatrix}^{\pm}.$$

However, the 2nd, 3th and 5th terms on the right side are not $SO(6)$-standard in the module $S^{(2,2,1)+}$ and the 1st, 4th and 5th terms on the right side are not $SO(6)$-standard in the module $S^{(2,2,1)-}$. In $S^{(2,2,1)+}$ the identities:

$$\begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix}^{+} = 0, \qquad \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix}^{+} = 0 \quad \text{and} \quad \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{2} & 3 \\ 2 & \end{bmatrix}^{+} = -\begin{bmatrix} \bar{1} & \bar{3} \\ \bar{3} & 3 \\ 3 & \end{bmatrix}^{+},$$

effect the standardisation:

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{3} & 2 \\ 3 & \end{bmatrix}^{+} = -\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix}^{+} - \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix}^{+} + \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{3} & 3 \\ 3 & \end{bmatrix}^{+}.$$

In $S^{(2,2,1)-}$ the identities:

$$\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \bar{3} \\ 3 & \end{bmatrix}^{-} = 0, \qquad \begin{bmatrix} \bar{1} & \bar{2} \\ 2 & 3 \\ \bar{3} & \end{bmatrix}^{-} = 0 \quad \text{and} \quad \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{2} & 3 \\ 2 & \end{bmatrix}^{-} = +\begin{bmatrix} \bar{1} & \bar{3} \\ \bar{3} & 3 \\ 3 & \end{bmatrix}^{-},$$

give rise to the standardisation:

$$\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{3} & 2 \\ 3 & \end{bmatrix}^{-} = +\begin{bmatrix} \bar{1} & \bar{2} \\ 2 & \bar{3} \\ 3 & \end{bmatrix}^{-} + \begin{bmatrix} \bar{1} & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{bmatrix}^{-} - \begin{bmatrix} \bar{1} & \bar{3} \\ \bar{3} & 3 \\ 3 & \end{bmatrix}^{-}.$$

To illustrate the construction of explicit representation matrices, let $\lambda = (2,1)$ and consider the eight dimensional $SO(4)$-module $S^{(2,1)-}$. For this case, Definitions 5.1.6 and 5.3.25 specify the following $SO(4)$-standard tableaux:

$$\begin{matrix} \bar{2} & 2 \\ 2 & \end{matrix}, \quad \begin{matrix} \bar{1} & 2 \\ 2 & \end{matrix}, \quad \begin{matrix} \bar{2} & \bar{2} \\ 2 & \end{matrix}, \quad \begin{matrix} \bar{1} & \bar{2} \\ 2 & \end{matrix}, \quad \begin{matrix} \bar{1} & \bar{1} \\ 2 & \end{matrix}, \quad \begin{matrix} 1 & 2 \\ \bar{2} & \end{matrix}, \quad \begin{matrix} 1 & \bar{2} \\ \bar{2} & \end{matrix}, \quad \begin{matrix} 1 & 1 \\ \bar{2} & \end{matrix}.$$

The eight remaining $O(4)$-standard tableaux from $\mathcal{O}_4^{(2,1)}$ satisfy the relations:

$$\begin{bmatrix} 1 & 2 \\ 2 & \end{bmatrix}^- = \begin{bmatrix} 1 & \bar{2} \\ 2 & \end{bmatrix}^- = \begin{bmatrix} 1 & 1 \\ 2 & \end{bmatrix}^- = 0,$$

$$\begin{bmatrix} \bar{1} & 2 \\ \bar{2} & \end{bmatrix}^- = \begin{bmatrix} \bar{1} & \bar{2} \\ \bar{2} & \end{bmatrix}^- = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & \end{bmatrix}^- = 0,$$

$$\begin{bmatrix} \bar{1} & 2 \\ 1 & \end{bmatrix}^- = -\begin{bmatrix} \bar{2} & 2 \\ 2 & \end{bmatrix}^- \quad \text{and} \quad \begin{bmatrix} \bar{1} & \bar{2} \\ 1 & \end{bmatrix}^- = -\begin{bmatrix} \bar{2} & \bar{2} \\ 2 & \end{bmatrix}^-,$$

each of which is obtained via Lemma 5.3.23. Calculations involving, in addition to these identities, the use of the Column relations (3.4.2), the Garnir relations (3.4.3), and the orthogonal Trace relations (5.2.3), give:

$$\Gamma^{[\lambda]^-}(B_1{}^2) = \begin{pmatrix} \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}, \quad \Gamma^{[\lambda]^-}(B_2{}^1) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot \\ \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$\Gamma^{[\lambda]^-}(B_{\bar{1}}{}^2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad \Gamma^{[\lambda]^-}(B_2{}^{\bar{1}}) = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{pmatrix},$$

along with the diagonal generators belonging to the Cartan subalgebra:

$$\Gamma^{[\lambda]^-}(B_1{}^1) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 \end{pmatrix}, \quad \Gamma^{[\lambda]^-}(B_2{}^2) = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{pmatrix}.$$

Notice that the highest weight vector of this $so(4)$-module $S^{(2,1)-}$ is $\begin{bmatrix} 1 & 1 \\ \bar{2} & \end{bmatrix}^-$, for which

$$B_1{}^1 \begin{bmatrix} 1 & 1 \\ \bar{2} & \end{bmatrix}^- = 2 \begin{bmatrix} 1 & 1 \\ \bar{2} & \end{bmatrix}^- \quad \text{and} \quad B_2{}^2 \begin{bmatrix} 1 & 1 \\ \bar{2} & \end{bmatrix}^- = - \begin{bmatrix} 1 & 1 \\ \bar{2} & \end{bmatrix}^-$$

confirming that its highest weight is $(2, -1)$. In fact the rather unexpected incorporation of the factor $(-1)^{r(r+1)/2}$ in (5.3.14) has been adopted precisely so as to ensure that the highest weights of the $so(2r)$-modules $S^{\lambda+}$ and $S^{\lambda-}$ are $(\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, \lambda_r)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{r-1}, -\lambda_r)$, respectively.

The techniques of this section have been implemented on a computer. In this way they have been used to construct various explicit representation matrices for the irreducible representations $[\lambda]_\pm$ of $SO(2r)$ with $\tilde{\lambda}_1 = r$. In all cases, including that of $[2,1]_-$ given above, the matrices obtained satisfy (2.2.18) and (2.2.22) in place of the elements they represent, thus verifying the techniques of this section.

§5.4. Duality between associate $O(m)$-modules

This section demonstrates and examines a duality between associate $O(m)$-modules which is analogous to that found in Section 4.5 for associate $GL(m)$-modules.

Define the linear map $L_* : O^\lambda \to O^{\lambda^*}$ by:

$$L_*[T^\lambda] = \epsilon^{(\mathcal{J},\bar{\mathcal{K}})}[T_*^{\lambda^*}], \tag{5.4.1}$$

where $\mathcal{J} = \{j_1, \ldots, j_s\}$ are the entries in the first column of T^λ with each $j_i = T^\lambda_{(i)}$, $\mathcal{K} = \{k_1, \ldots, k_t\}$ are the entries in the first column of $T_*^{\lambda^*}$ with each $k_i = T^{\lambda^*}_{*(i)}$ and $\epsilon^{(\mathcal{J},\bar{\mathcal{K}})}$ is given by (5.3.4). Theorem 4.5.6 has the following analogue.

Theorem 5.4.2. *Let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$. If*

$$\sum_w [T_w^\lambda] = 0 \tag{5.4.2a}$$

is a Trace relation in the $O(m)$-module O^λ, of the type specified in Lemma 5.2.4 *with $k_1 = 1$ and $k_2 > 1$, so that all the tableaux T_w^λ differ only in entries in the* 1*st and k_2th columns, then:*

$$\sum_w \epsilon^{(\mathcal{J}_w,\bar{\mathcal{K}}_w)}[T_{w*}^{\lambda^*}] = 0, \tag{5.4.2b}$$

with $\mathcal{J}_w$ and $\mathcal{K}_w$ defined by analogy with $\mathcal{J}$ and $\mathcal{K}$, is a Garnir relation of the type (3.4.3)*:*

$$\sum_{\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta [\eta T_*^{\lambda^*}] = 0, \tag{5.4.2c}$$

involving entries in positions $\mathcal{X}$ of the 1*st column and positions $\mathcal{Y}$ of the k_2th column. Conversely a Garnir relation of type* (5.4.2*c*) *gives rise to a Trace relation of type* (5.4.2*a*).

Proof. Form θ_w as for the proof of Lemma 5.2.4 from the indices of the relevant two columns of T^λ_w. Let $\mathcal{A}_w \subset \mathcal{I}^{O(m)}$ be the set of all $i \in \mathcal{A}_w$ such that i^α and $\bar{i}^\beta$ are both present in θ_w. Let $\mathcal{B}_w \subset \mathcal{I}^{O(m)}$ be the set of all $i \in \mathcal{B}_w$ such that one only of i^α and $\bar{i}^\beta$ is present in θ_w. Let $\mathcal{C}_w \subset \mathcal{I}^{O(m)}$ be the set of all $i \in \mathcal{C}_w$ such that neither i^α nor $\bar{i}^\beta$ is present in θ_w. Then, if $a_w = \#\mathcal{A}_w$, $b_w = \#\mathcal{B}_w$ and $c_w = \#\mathcal{C}$, $a_w + b_w + c_w = m$ for each w. Factorise $\theta_w = \theta^{\mathcal{B}}_w \theta^{\mathcal{A}}_w$ where $\theta^{\mathcal{B}}_w$ and $\theta^{\mathcal{A}}_w$ are formed solely of superscripted indices from $\mathcal{B}$ and $\mathcal{A}$ respectively. Since a Trace relation involves expressing a number of barred-unbarred index pairs in terms of other such pairs, $\mathcal{B}_w$, b_w and $\theta^{\mathcal{B}}_w$ are constant and their subscripts may be dropped. In addition, since $\theta_w = \theta^{\mathcal{B}} \theta^{\mathcal{A}}_w$, it follows that $a = a_w$ and $c = c_w$ are also constant. Split $\mathcal{B}$ into $\mathcal{B}^\alpha \cup \mathcal{B}^\beta$, such that $i \in \mathcal{B}^\alpha$ if i^α is present in $\theta^{\mathcal{B}}$ and $i \in \mathcal{B}^\beta$ if $\bar{i}^\beta$ is present. If $b^\alpha = \#\mathcal{B}^\alpha$ and $b^\beta = \#\mathcal{B}^\beta$, $b = b^\alpha + b^\beta$.

Let $\mathcal{E} = \cap_w \mathcal{A}_w$, $\mathcal{D}_w = \mathcal{A}_w \backslash \mathcal{E}$, $\mathcal{H} = \cup_w \mathcal{D}_w$, $\mathcal{F}_w = \mathcal{H} \backslash \mathcal{D}_w$ and $\mathcal{G} = \mathcal{I}^{O(m)} \backslash (\mathcal{H} \cup \mathcal{E} \cup \mathcal{B})$ with $e = \#\mathcal{E}$, $d = \#\mathcal{D}_w$, $h = \#\mathcal{H}$, $f = \#\mathcal{F}_w$, $g = \#\mathcal{G}$, whereupon $h = d + f$, $a = d + e$ and $c = f + g$. Since the Trace relation is of the type specified in Lemma 5.2.4 then $d > g$. Note that $\mathcal{H} = \mathcal{D}_w \cup \mathcal{F}_w$ for each w. With the sets defined in this way, the Trace relation (5.4.2*a*) may be obtained exactly as in the proof of Lemma 5.2.4. In particular, from (5.2.4*g*):

$$\theta^{\mathcal{B}} \theta^{\mathcal{E}} \sum_w \theta^{\mathcal{D}}_w = 0, \tag{5.4.7}$$

where $\theta^{\mathcal{D}}_w = \prod_{i \in \mathcal{D}_w} \omega_i$, $\theta^{\mathcal{E}} = \prod_{i \in \mathcal{E}} \omega_i$ and $\omega_i = i^\alpha \bar{i}^\beta$. Then, setting $\theta^{\mathcal{A}}_w = \theta^{\mathcal{E}} \theta^{\mathcal{D}}_w$ gives:

$$\theta^{\mathcal{B}} \sum_w \theta^{\mathcal{A}}_w = 0$$

and then:

$$\sum_w \theta_w = 0.$$

This final equation is that giving rise to (5.4.2*a*).

Let θ^*_w be the term of (5.4.2*b*) corresponding to θ_w of (5.4.2*a*). Corresponding to the way in which $T^{\lambda^*}_*$ is formed from T^λ, i^α is present in θ^*_w if and only if $\bar{i}^\alpha$ is not present in θ_w, that is $\bar{i} \notin \mathcal{D}_w \cup \mathcal{E} \cup \mathcal{B}^\alpha$ and hence $i \in \overline{\mathcal{F}_w \cup \mathcal{G} \cup \mathcal{B}^\beta}$; and i^β is present in θ^*_w if and only if i^β is present in θ_w, that is $i \in \overline{\mathcal{D}_w \cup \mathcal{E} \cup \mathcal{B}^\beta}$. The situation in the corresponding tableaux may be schematically represented thus:

$$T_w^\lambda = \begin{array}{|c|c|}\hline \mathcal{B}^\alpha & \overline{\mathcal{B}^\beta} \\ \hline \mathcal{E} & \overline{\mathcal{E}} \\ \hline \mathcal{D}_w & \overline{\mathcal{D}_w} \\ \hline \end{array} \quad \Longrightarrow \quad T_{w*}^{\lambda^*} = \begin{array}{|c|c|}\hline \overline{\mathcal{B}^\beta} & \overline{\mathcal{B}^\beta} \\ \hline \overline{\mathcal{G}} & \overline{\mathcal{E}} \\ \hline \overline{\mathcal{F}_w} & \overline{\mathcal{D}_w} \\ \hline \end{array} \tag{5.4.3}$$

On varying w, the terms θ_w^* run through all partitionings of the set $\overline{\mathcal{H}}$ into f entries superscripted with α and d entries superscripted with β. Since these are to be respectively placed in the 1st and the k_2th column of $T_{w*}^{\lambda^*}$, this is an expression of Garnir type. It is necessary to check that sufficient indices from the two columns are involved in this expression. Consider a Garnir relation involving the set of indices $\overline{\mathcal{F}_w} \cup \overline{\mathcal{B}^\beta}$ from the 1st column and $\overline{\mathcal{D}_w}$ from the other. This results in the same expression as the above since each migration of an index from $\overline{\mathcal{B}^\beta}$ to the k_2th column results in a repeated entry in that column and thus a zero term. This expression involves $f + b^\beta + d$ terms from the two columns which is a greater number than the length of the 1st column, $f + g + b^\beta$, since $d > g$. It remains to show that the sign of each term is as required. The sign $\epsilon^{(\mathcal{J}_w, \overline{\mathcal{K}_w})}$ required for the transition from $[T^\lambda]$ to $[T_{w*}^{\lambda^*}]$ may be expressed $\epsilon^{(\mathcal{B}^\alpha, \mathcal{E}, \mathcal{D}_w, \mathcal{B}^\beta, \mathcal{G}, \mathcal{F}_w)}$. By fixing the positions of the elements from the sets $\mathcal{E}$, $\overline{\mathcal{E}}$, $\mathcal{B}_\alpha$ and $\overline{\mathcal{B}_\beta}$ in each T_w^λ and $\overline{\mathcal{B}^\beta}$, $\overline{\mathcal{G}}$ and $\overline{\mathcal{E}}$ in $T_{w*}^{\lambda^*}$, it can be seen that $\epsilon^{(\mathcal{J}_w, \overline{\mathcal{K}_w})} = \pm \epsilon^{(\mathcal{F}_w, \mathcal{D}_w)}$, the sign being independent of w. For each w, the factor $\epsilon^{(\mathcal{F}_w, \mathcal{D}_w)}$ is precisely that required for the appropriate coset representative of the Garnir element giving rise to $T_{w*}^{\lambda^*}$. Thus the sign factor of $\epsilon^{(\mathcal{J}_w, \overline{\mathcal{K}_w})}$ given in (5.4.2b) is precisely that required to make (5.4.2b) a Garnir relation.

By partitioning the two relevant columns of the set of tableaux satisfying (5.4.2c) into the sets $\overline{\mathcal{B}^\beta}$, $\overline{\mathcal{G}}$, $\overline{\mathcal{E}}$ and $\overline{\mathcal{H}} = \overline{\mathcal{F}_w \cup \mathcal{D}_w}$ as in (5.4.3), and setting $\overline{\mathcal{B}^\alpha}$ to be the remaining indices, the Trace relation corresponding to any Garnir relation may be found by reversing the above construction.

As an example consider the mutually associate $O(7)$-modules $O^{(2,2,2)}$ and $O^{(2,2,2,1)}$. Let $\lambda = (2,2,2)$, so that $\lambda^* = (2,2,2,1)$ and let $\mathcal{B}^\alpha = \{\bar{1}\}$, $\mathcal{B}^\beta = \{1\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{2\}$, $\mathcal{H} = \{\bar{2}, \bar{3}, 3, 0\}$ and $d = 2$. Then, with $k_1 = 1$ and $k_2 = 2$, Lemma 5.2.4 yields the Trace relation:

$$\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \\ \bar{3} & 3 \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \\ 3 & \bar{3} \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{3} & 3 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{3} & 3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ 3 & \bar{3} \\ 0 & 0 \end{bmatrix} = 0. \tag{5.4.4a}$$

For this expression $\mathcal{D}_1 = \{\bar{2}, \bar{3}\}$, $\mathcal{D}_2 = \{\bar{2}, 3\}$, $\mathcal{D}_3 = \{\bar{2}, 0\}$, $\mathcal{D}_4 = \{\bar{3}, 3\}$, $\mathcal{D}_5 = \{\bar{3}, 0\}$ and $\mathcal{D}_6 = \{3, 0\}$ respectively. The taking of associates of the terms in (5.4.4a) requires the first columns to be replaced with entries from the respective sets

$\overline{\mathcal{B}^\beta \cup \mathcal{G} \cup \mathcal{F}_w}$. Since $\overline{\mathcal{B}^\beta} = \{\bar{1}\}$, $\overline{\mathcal{G}} = \{\bar{2}\}$, $\overline{\mathcal{F}_1} = \{\bar{3},0\}$, $\overline{\mathcal{F}_2} = \{3,0\}$, $\overline{\mathcal{F}_3} = \{\bar{3},3\}$, $\overline{\mathcal{F}_4} = \{2,0\}$, $\overline{\mathcal{F}_5} = \{2,\bar{3}\}$ and $\overline{\mathcal{F}_6} = \{2,3\}$, expression (5.4.2$b$) is, in this case,

$$-\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \\ \bar{3} & 3 \\ 0 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \\ 3 & \bar{3} \\ 0 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 2 \\ \bar{3} & 0 \\ 3 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & \bar{3} \\ 2 & 3 \\ 0 & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & 3 \\ 2 & 0 \\ \bar{3} & \end{bmatrix} + \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & \bar{3} \\ 2 & 0 \\ 3 & \end{bmatrix} = 0, \quad (5.4.4b)$$

the signs having been obtained by multiplying the sign of the corresponding term in (5.4.4a) by the respective $\epsilon^{(\mathcal{J}_w,\overline{\mathcal{K}_w})}$. This is the Garnir relation that arises on using the sets of positions $\mathcal{X} = \{1,3,4\}$ and $\mathcal{Y} = \{6,7\}$.

Theorem 5.4.3 will enable it to be proved that the $SO(m)$-modules S^λ and S^{λ^*} are isomorphic. For the moment, ignore the group action and consider O^λ and O^{λ^*} solely as vector spaces.

Lemma 5.4.5. *The linear map $L_* : O^\lambda \to O^{\lambda^*}$ is a well defined isomorphism between the vector spaces O^λ and O^{λ^*}.*

Proof. Let T^λ be column strict. By using the Column relations, Garnir relations and Trace relations, $[T^\lambda]$ is uniquely expressible:

$$[T^\lambda] = \sum_i \zeta_i [T_i^\lambda], \quad (5.4.5a)$$

where each T_i^λ is $O(m)$-standard. Theorem 5.4.3 shows that to each Trace relation involving the first column of some $[T^\lambda]$, there is a Garnir relation resulting from the action of L_* on each term. This Garnir relation necessarily involves $[T_*^{\lambda^*}]$. Likewise, every Garnir relation involving the first column of some $[T^\lambda]$ corresponds, through the action of L_*, to a Trace relation, necessarily involving $[T_*^{\lambda^*}]$. Garnir relations involving other columns remain as they are under the action of L_*. Thus, since if T_i^λ is $O(m)$-standard then $T_{i*}^{\lambda^*}$ is $O(m)$-standard, the standardisation of $[T_*^{\lambda^*}]$ mirrors, under the action of L_*, the standardisation producing (5.4.5a). Therefore,

$$[T_*^{\lambda^*}] = \sum_i \zeta_i [T_{i*}^{\lambda^*}]. \quad (5.4.5b)$$

Since this is the result of the direct application of L_* to (5.4.5a) and the two expressions in terms of $O(m)$-standard tableaux are unique, the lemma follows.

On considering O^λ and O^{λ^*} as $O(m)$-modules once more, this Lemma together with (5.3.13), shows that the traceless symmetrised tableaux $[T^\lambda]$ of O^λ may be replaced by the signed traceless symmetrised tableaux $\epsilon^{(\mathcal{J},\mathcal{K})}[T_*^{\lambda^*}]$ of O^{λ^*} provided that factors of $(\det G)$ are included for each module action of $G \in O(m)$. In particular, when $G \in SO(m)$, $\det G = 1$ and the $SO(m)$-modules S^λ and S^{λ^*} are isomorphic.

Using an argument similar to that given in Section 4.5, let the operator $L_{k_1\cdots k_t j_1\cdots j_s}$ act on the indices of the first column of $[T^\lambda]$ to give $[T_*^{\lambda^*}]^*$ where the extra asterisk indicates that it is yet to be determined that $[T_*^{\lambda^*}]^*$ has the properties implied by the Young operator Y^{λ^*}, or the tracelessness. That this is so is due to the vector space they inhabit being isomorphic to O^λ and hence, by Lemma 5.4.5, isomorphic to O^{λ^*}. This implies that the Garnir relations and orthogonal Trace relations involving the first column, obtained as a result of Lemma 5.4.2, enable the first column to participate in the symmetry implied by the Young operator Y^{λ^*} and the tracelessness resulting from the extraction of all trace terms of the form (5.2.2). From (5.3.13):

$$\begin{aligned} G(\epsilon^{(\mathcal{J},\mathcal{K})}[T_*^{\lambda^*}]) &= G(L_*[T^\lambda]) \\ &= (\det G)L_*(G[T^\lambda]), \end{aligned} \tag{5.4.6}$$

for all $G \in O(m)$. Therefore, the following theorem has been proved.

Theorem 5.4.7. *Let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$. Under restriction of $O(m)$ to $SO(m)$, the $O(m)$-module O^λ is isomorphic to O^{λ^*}. The representation $[\lambda]$ of $SO(m)$ is equivalent to the representation $[\lambda^*]$ of $SO(m)$. The representation $[\lambda]$ of $O(m)$ is equivalent to the representation $(\det G)[\lambda^*]$ of $O(m)$.*

Conversely, the following theorem holds.

Theorem 5.4.8. *Let $\lambda \in P(l)$ be such that $\tilde{\lambda}_1 + \tilde{\lambda}_2 \leq m$. The isomorphism of the $SO(m)$-modules S^λ and S^{λ^*} implies that the Trace relations and the Garnir relations are equivalent statements; that is, one implies the other.*

Proof. This follows since the operator L given by (5.3.5) which defines the isomorphism, may be used to convert between the Trace relations (5.4.2*a*) and the Garnir relations (5.4.2*b*).

§5.5. The diagonal form

In this section, consideration is given to the orthogonal groups $O'(m)$ and $SO'(m)$ which as specified by Definition 2.1.1, preserve a diagonal form. Although the form given by the identity matrix will be considered here, the comments and results apply equally to a diagonal form with arbitrary signature. In particular this enables Young tableaux to be used in the explicit construction of representations of the Lorentz $O(3,1)$ and proper Lorentz $SO(3,1)$ groups. However, as will transpire, inconveniences arise in developing the techniques here as for the previous sections of this chapter. Since the arguments proceed very much as for those previous sections, the material of this section will only be considered in outline and proofs will be sketched only where there is substantial deviation from the analogous proofs for $O(m)$ and $SO(m)$. For this section the index set is $\mathcal{I}^{O'(m)} = \mathsf{N}_m$.

The trace tensors in this 'diagonal' case are all linear combinations of terms of the form

$$\sum_{i\in\mathcal{I}^{O'(m)}} x\otimes e_i\otimes y\otimes e_i\otimes z. \tag{5.5.1}$$

Their removal from $\{T^\lambda\}$ defines $[T^\lambda]'$. These are defined to span the $O(m)$-modules O'^λ. The Trace relation takes the following form.

Lemma 5.5.2. *Let T_i^λ, for $i\in\mathcal{I}^{O'(m)}$, be m tableaux, identical except for the entries in two positions where $T^\lambda_{i(a,b)} = i$ and $T^\lambda_{i(c,d)} = i$ for some fixed a, b, c and d with $a,c\leq\tilde{\lambda}_1$, $b\leq\lambda_a$ and $d\leq\lambda_b$. Then*

$$\sum_{i\in\mathcal{I}^{O'(m)}} [T_i^\lambda] = 0. \tag{5.5.2}$$

If the form being used is of a signature other than m, the only modification that needs to be made to (5.5.1) and (5.5.2) is a switch of sign for particular summands.

For $O'(m)$, the set of standard tableaux are provided by the set $\mathcal{Q}^\lambda_m$ of Definition 5.1.2. With the index set $\mathcal{I}^{O'(m)}$ in place of $\mathcal{I}^{O(m)}$, condition (iv) of Definition 5.1.2 takes the form:

$$\alpha_i+\beta_i\leq i \text{ for each } i\in\mathcal{I}^{O'(m)}, \tag{5.5.3}$$

where α_i and β_i are the number of entries less than or equal to i in the first and second columns respectively of T^λ. These tableaux do not, in fact, readily yield weights and characters of the irreducible representations of $O'(m)$. This inconvenience is a direct consequence of the Cartan subalgebra of $so(m)'$ not comprising diagonal elements. Nonetheless, standardisation is straightforward compared with the $O(m)$ case since there are no protection conditions to account for. In addition, violations of (5.5.3) may be dealt with by using the following 'diagonal' analogue of Lemma 5.2.6.

Lemma 5.5.4. *Let the column strict tableau $T^\lambda\notin\mathcal{Q}^\lambda_m$ be such that $\alpha_j+\beta_j>j$ for some $j\in\mathcal{I}^{O'(m)}$. Then $[T^\lambda]'$ may be expressed as a signed sum of traceless, symmetrised tableaux $[T^\lambda_w]'$, where for each w, $T^\lambda_w > T^\lambda$.*

This result may be obtained by using means similar to those used in the proofs of Lemmas 5.2.4 and 5.2.6. The essential difference is that, as a consequence of (5.5.2), the trace identity:

$$\sum_{i \in \mathcal{I}^{O'(m)}} \omega_i = 0, \tag{5.5.5}$$

where $\omega_i = i^\alpha i^\beta$, is used in place of (5.2.4b). Again, particular summands switch their signs when the signature of the diagonal form being used differs from m. The following expression from the $O'(7)$-module $O'^{(2,2,1,1)}$ is typical of that resulting from Lemma 5.5.4 and its proof:

$$\begin{bmatrix} 1 & 3 \\ 3 & 5 \\ 4 & \\ 5 & \end{bmatrix}' = -\begin{bmatrix} 1 & 3 \\ 3 & 6 \\ 4 & \\ 6 & \end{bmatrix}' - \begin{bmatrix} 1 & 3 \\ 3 & 7 \\ 4 & \\ 7 & \end{bmatrix}' + \begin{bmatrix} 1 & 5 \\ 4 & 6 \\ 5 & \\ 6 & \end{bmatrix}' + \begin{bmatrix} 1 & 5 \\ 4 & 7 \\ 5 & \\ 7 & \end{bmatrix}' + \begin{bmatrix} 1 & 6 \\ 4 & 7 \\ 6 & \\ 7 & \end{bmatrix}' . \tag{5.5.6}$$

Here, the non-standard term on the left has been expressed in terms of higher terms, each of which is standard, in this case. Note that the use of Lemma 5.5.4 necessarily results in an expression for $[T^\lambda]'$ with integral coefficients.

The combination of Column relations, Garnir relations and Trace relations, via in this case Lemma 5.5.4, once more enables an arbitrary non-standard term to be written as a linear combination of standard terms. This leads to the following analogue of Theorem 5.2.18.

Theorem 5.5.7. *The $O'(m)$-module O'^λ is irreducible with basis:*

$$\{ [T^\lambda]' : T^\lambda \in \mathcal{Q}^\lambda_m \}.$$

Moreover, the set $\{O'^\lambda : \tilde{\lambda}_1 + \tilde{\lambda}_2 \le m\}$ provides a complete list of inequivalent irreducible $O'(m)$-modules.

Explicit representation matrices for elements of $O'(m)$ and $so'(m)$ may now be generated in the representation $[\lambda]$ using the set of tableaux $\mathcal{Q}^\lambda_m$ and the standardisation techniques outlined above, by precisely the same means as in Section 5.2. However, in contrast to Section 5.2, standardisation of an arbitrary traceless symmetrised tableau cannot introduce any non-integral factors. Therefore, the representation matrices for the basis elements of $so'(m)$ will necessarily be integral.

In the reduction from $O'(2r)$ to $SO'(2r)$, a major disadvantage arises for those cases where r is odd. The definition of (5.3.5) has the analogue:

$$L'_{a_1 \cdots a_s b_1 \cdots b_t} = \frac{1}{t!} \epsilon_{a_1 \cdots a_s b_1 \cdots b_t} . \tag{5.5.8}$$

This leads to the following definition of a diagonal associate.

Definition 5.5.9. *Let $\lambda \in P(l)$ be such that $s = \tilde{\lambda}_1$, let $t = m - s$ and let T^λ be column strict. The diagonal associate of T^λ is defined to be that tableau, denoted $T^\lambda_{*'}$, identical to T^λ apart from the first column which contains entries from the set $\mathcal{K} = \mathcal{I}^{O'(m)} \backslash \mathcal{J}$ where the entries from the first column of T^λ constitute the set $\mathcal{J}$.*

Lemma 5.5.10. *If $T^\lambda \in \mathcal{Q}^\lambda_m$ then $T^{\lambda^*}_{*'} \in \mathcal{Q}^\lambda_m$.*

The proof of this result follows the same lines as that of Lemma 5.3.3, but is more straightforward since protection conditions need not be considered.

With L' given by (5.5.8), in order that (5.3.8), (5.3.9), (5.3.10), (5.3.11) and (5.3.13) should each hold with L replaced by L' and K replaced by K', it is necessary to define:

$$K'_{a_1 \cdots a_s c_1 \cdots c_s} = (-1)^{st} L'_{a_1 \cdots a_s b_1 \cdots b_t} L'_{b_1 \cdots b_t c_1 \cdots c_s}, \tag{5.5.11}$$

corresponding to (5.3.7). Then when r is odd and $r = s = t$, this implies that $K' = -L'L'$, whereupon the direct analogues of (5.3.14) are not idempotent. The appropriate expression is:

$$L'^{\pm}_{a_1 \cdots a_r b_1 \cdots b_r} = \frac{1}{2} \left(K'_{a_1 \cdots a_r b_1 \cdots b_r} \pm i^r (-1)^{r(r+1)/2} L'_{a_1 \cdots a_r b_1 \cdots b_r} \right), \tag{5.5.12}$$

where $i = \sqrt{-1}$. If $\tilde{\lambda}_1 = r$ so that the $O'(2r)$-module O'^λ is self-associate, then through arguments similar similar to those of Section 5.3, the $SO'(2r)$-modules $S'^{\lambda\pm}$ may be defined to be the span of all $[T^\lambda]'^{\pm}$ where:

$$[T^\lambda]'^{\pm} = [T^\lambda]' \pm i^r (-1)^{r(r+1)/2} \epsilon^{(\mathcal{J},\mathcal{K})} [T^\lambda_{*'}]', \tag{5.5.13}$$

$\epsilon^{(\mathcal{J},\mathcal{K})} = \epsilon_{j_1 \cdots j_r k_1 \cdots k_r}$, $\mathcal{J} = \{j_1, \ldots, j_r\}$ are the entries in the first column of T^λ with each $j_a = T^\lambda_{(a)}$ and $\mathcal{K} = \{k_1, \ldots, k_r\}$ are the entries in the first column of $T^{\lambda^*}_{*'}$ with each $k_a = T^{\lambda^*}_{*(a)}$. This implies that:

$$[T^\lambda]'^{\pm} = \pm i^r (-1)^{r(r+1)/2} \epsilon^{(\mathcal{J},\mathcal{K})} [T^\lambda_{*'}]'^{\pm}. \tag{5.5.14}$$

For example, if $m = 6$ and $\lambda = (2,2,2)$ then:

$$\begin{bmatrix} 1 & 2 \\ 4 & 4 \\ 5 & 5 \end{bmatrix}'^{+} = -i \begin{bmatrix} 2 & 2 \\ 3 & 4 \\ 6 & 5 \end{bmatrix}'^{+}. \tag{5.5.15}$$

Definition 5.5.16. *If $\lambda \in P(l)$ is such that $\tilde{\lambda}_1 = r$ then let $\mathcal{S}'^\lambda_{2r} \subset \mathcal{Q}^\lambda_{2r}$ be such that $T^\lambda \in \mathcal{S}'^\lambda_{2r}$ if $T^\lambda_{(1)} \neq 1$.*

This definition implies that $\mathcal{S}'^{\lambda}_{2r}$ consists of those tableaux of $\mathcal{Q}^{\lambda}_{2r}$ which contain no 1s. The following lemma may be proved using a simple bijection argument.

Lemma 5.5.17. *The cardinality of $\mathcal{Q}^{\lambda}_{2r}$ is precisely twice that of $\mathcal{S}'^{\lambda}_{2r}$.*

Theorem 5.3.17 now implies the following theorem (see [**We39**], theorem 5.9A).

Theorem 5.5.18. *If r is even, the $O'(2r)$ module O'^{λ} decomposes on restriction to $SO'(2r)$ into the direct sum of two inequivalent $SO'(2r)$-modules $S'^{\lambda+}$ and $S'^{\lambda-}$ having bases:*

$$\left\{\left[T^{\lambda}\right]'^{+} : T^{\lambda} \in \mathcal{S}'^{\lambda}_{2r}\right\}$$

and

$$\left\{\left[T^{\lambda}\right]'^{-} : T^{\lambda} \in \mathcal{S}'^{\lambda}_{2r}\right\}$$

respectively. If r is odd, the same is true over the field of complex numbers. However, over the field of real numbers O'^{λ} remains irreducible on restriction to $SO'(2r)$.

To show that the sets given do actually provide bases, note that the Column relations, Garnir relations and Trace relations enable an arbitrary $[T^{\lambda}]'^{\pm}$ to be expressed in terms of tableaux from $\mathcal{Q}^{\lambda}_{2r}$. Then identity (5.5.14), with the appropriate sign, is used for those $T^{\lambda} \notin \mathcal{S}'^{\lambda}_{2r}$. It is this final reduction that necessitates the use of complex numbers when r is odd. In such cases, the matrix elements of the basis elements of $so(m)'$ are integral complex numbers.

Chapter 6
Spinor Modules of the Orthogonal Groups

§6.1. Clifford algebras

In this chapter, the irreducible spinor modules of the orthogonal groups $O(m)$ and $SO(m)$, and the Lie algebras $so(m)$ are constructed using Young tableaux. Here, the appropriate Young tableaux are the 'half' tableaux of Definition 2.6.20. The construction proceeds via a generalised Clifford algebra based on that employed in [**BW35**] in studying the basic spin representations of $O'(m)$ and $SO'(m)$; this is itself, a generalisation of that used in Dirac's account [**Di27**] of the 'spinning' electron.

The Clifford algebra is only usually defined for those orthogonal groups which preserve a diagonal form. The following provides the requisite generalisation to an arbitrary non-degenerate bilinear form J.

Definition 6.1.1. *The Clifford algebra in m dimensions is generated by the m elements $\alpha_1, \alpha_2, \ldots, \alpha_m$, subject to the constraints:*

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2J_{ij}, \tag{6.1.1}$$

for $1 \leq i, j \leq m$.

In the usual definition $J = I_m$ and consequently the defining relation is $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ [**BW35**]. However, here as in the previous chapter, it will be appropriate to use the index set $\mathcal{I}^{O(m)}$ and to take $J_{ij} = \delta_{i\bar{j}}$, whereupon:

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{i\bar{j}}, \tag{6.1.2}$$

for $i, j \in \mathcal{I}^{O(m)}$. This particular Clifford algebra will be denoted $\mathcal{N}_m$. It is sometimes known as a Heisenberg superalgebra. When $m = 2r$ is even, $\mathcal{N}_m$ has a basis $\{\alpha_{\bar{1}}^{a_{\bar{1}}}\alpha_1^{a_1}\alpha_{\bar{2}}^{a_{\bar{2}}}\alpha_2^{a_2}\cdots\alpha_r^{a_r} : a_i \in \{0,1\}, i \in \mathcal{I}^{O(2r)}\}$, and when $m = 2r+1$ is odd, it has a basis $\{\alpha_{\bar{1}}^{a_{\bar{1}}}\alpha_1^{a_1}\alpha_{\bar{2}}^{a_{\bar{2}}}\alpha_2^{a_2}\cdots\alpha_r^{a_r}\alpha_0^{a_0} : a_i \in \{0,1\}, i \in \mathcal{I}^{O(2r+1)}\}$. Consequently, $\mathcal{N}_m$ has dimension 2^m.

In what follows, a representation of $\mathcal{N}_m$ will be constructed. It is useful to note, at this point, that no one-dimensional representations of $\mathcal{N}_m$ exist, as a brief consideration of (6.1.2) will show. Define the four 2×2 matrices I_2, σ_1, σ_2 and σ_3, by:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6.1.3}$$

These play a role here analogous to that of the Pauli matrices in [**Di27,BW35**]. Now for $a \in \mathcal{I}^{O(m)}$, the $2^r \times 2^r$ matrices γ_a are constructed by taking the Kronecker product of r of the matrices of (6.1.3). For $a \in \mathsf{N}_r$, define:

$$\gamma_a = \sqrt{2}.\sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_2 \otimes I_2 \otimes \cdots \otimes I_2, \tag{6.1.4a}$$

where σ_2 occurs in the ath position; and define:

$$\gamma_{\bar{a}} = \sqrt{2}.\sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_3 \otimes I_2 \otimes \cdots \otimes I_2, \tag{6.1.4b}$$

where σ_3 occurs in the ath position. In addition, for all m, define:

$$\gamma_0 = \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \sigma_1. \tag{6.1.4c}$$

For later convenience, a list of all possible two fold products of these matrices will now be compiled. Let $a, b \in \mathsf{N}_r$ and $a < b$. Then, since the product of two matrices of the form (6.1.4) may be obtained by multiplying the factor matrices componentwise:

$$\gamma_a\gamma_b = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \tag{6.1.5a}$$

where the first explicit matrix is the ath factor and the second is the bth factor. The same will be implicit in each of the products that follow. Thus:

$$\gamma_b\gamma_a = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2,$$

so that $\gamma_b\gamma_a = -\gamma_b\gamma_a$. Likewise

$$\gamma_{\bar{a}}\gamma_{\bar{b}} = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2. \tag{6.1.5b}$$

Similarly, direct multiplication shows that $\gamma_{\bar{b}}\gamma_{\bar{a}} = -\gamma_{\bar{a}}\gamma_{\bar{b}}$. Continuing:

$$\gamma_a\gamma_{\bar{b}} = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \tag{6.1.5c}$$

and similarly $\gamma_{\bar{b}}\gamma_a = -\gamma_a\gamma_{\bar{b}}$;

$$\gamma_{\bar{a}}\gamma_b = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \quad (6.1.5d)$$

and similarly $\gamma_b\gamma_{\bar{a}} = -\gamma_{\bar{a}}\gamma_b$. Since $\sigma_2^2 = \sigma_3^2 = 0$, it follows that:

$$\gamma_a^2 = \gamma_{\bar{a}}^2 = 0. \quad (6.1.5e)$$

Also:

$$\gamma_a\gamma_{\bar{a}} = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \quad (6.1.5f)$$

and similarly:

$$\gamma_{\bar{a}}\gamma_a = 2.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes I_2 \otimes \cdots \otimes I_2, \quad (6.1.5g)$$

so that $\gamma_a\gamma_{\bar{a}} + \gamma_{\bar{a}}\gamma_a = 2.I_{2^r}$, because $I_{2^r} = I_2 \otimes \cdots \otimes I_2$ (r factors). For products involving γ_0:

$$\gamma_a\gamma_0 = \sqrt{2}.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1, \quad (6.1.5h)$$

and $\gamma_0\gamma_a = -\gamma_a\gamma_0$;

$$\gamma_{\bar{a}}\gamma_0 = \sqrt{2}.I_2 \otimes \cdots \otimes I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \sigma_1 \otimes \cdots \otimes \sigma_1, \quad (6.1.5i)$$

and $\gamma_0\gamma_{\bar{a}} = -\gamma_{\bar{a}}\gamma_0$; and finally:

$$\gamma_0^2 = I_{2^r}. \quad (6.1.5j)$$

Comparing these products with (6.1.2) proves the following lemma.

Lemma 6.1.6. *Through the map $\alpha_a \to \gamma_a$, the matrices γ_a, for $a \in \mathcal{I}^{O(m)}$, provide a representation of $\mathcal{N}_m$. In addition, the map $\alpha_a \to -\gamma_a$ for $a \in \mathcal{I}^{O(m)}$, provides a further representation of $\mathcal{N}_m$.*

It should be noted that the two representations indicated here are not necessarily inequivalent. This question, together with that of the reducibility of these representations, is addressed below. As a preliminary, define the 2×2 matrices:

$$\sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (6.1.7a)$$

and for $a \in \mathbf{N}_r$ and $b = 1, 2, \ldots, 5$, the $2^r \times 2^r$ matrices:

$$\hat{\gamma}_a^{(b)} = I_2 \otimes \cdots \otimes I_2 \otimes \sigma_b \otimes I_2 \otimes \cdots \otimes I_2, \quad (6.1.7b)$$

where the exceptional factor σ_b occurs in the ath position. Then, from (6.1.5f) and (6.1.5g):

$$\hat{\gamma}_a^{(4)} = \frac{1}{2}\gamma_{\bar{a}}\gamma_a \quad \text{and} \quad \hat{\gamma}_a^{(5)} = \frac{1}{2}\gamma_a\gamma_{\bar{a}}, \tag{6.1.7c}$$

whereupon:

$$\hat{\gamma}_a^{(1)} = \frac{1}{2}(\gamma_a\gamma_{\bar{a}} - \gamma_{\bar{a}}\gamma_a). \tag{6.1.7d}$$

Note that:

$$\hat{\gamma}_a^{(2)} = \frac{1}{\sqrt{2}}\hat{\gamma}_1^{(1)}\hat{\gamma}_2^{(1)}\cdots\hat{\gamma}_{a-1}^{(1)}\gamma_a \tag{6.1.7e}$$

$$\text{and} \qquad \hat{\gamma}_a^{(3)} = \frac{1}{\sqrt{2}}\hat{\gamma}_1^{(1)}\hat{\gamma}_2^{(1)}\cdots\hat{\gamma}_{a-1}^{(1)}\gamma_{\bar{a}}. \tag{6.1.7f}$$

Therefore, under either of the maps given in Lemma 6.1.6, the matrices $\hat{\gamma}_a^{(b)}$ each represent some element of $\mathcal{N}_m$.

Let $\eta_{2r} \in \mathcal{N}_{2r}$ be defined by:

$$\begin{aligned}\eta_{2r} &= \frac{1}{2^r}(\alpha_1\alpha_{\bar{1}} - \alpha_{\bar{1}}\alpha_1)(\alpha_2\alpha_{\bar{2}} - \alpha_{\bar{2}}\alpha_2)\cdots(\alpha_r\alpha_{\bar{r}} - \alpha_{\bar{r}}\alpha_r) \\ &= \frac{(-1)^r}{2^r}\sum_{\rho\in S_2\otimes\cdots\otimes S_2}(-1)^\rho\alpha_{\rho(\bar{1})}\alpha_{\rho(1)}\cdots\alpha_{\rho(\bar{r})}\alpha_{\rho(r)},\end{aligned} \tag{6.1.8a}$$

where the sum is over all 2^r elements of the group $S_2\otimes\cdots\otimes S_2$ (r factors), for which $\rho(a) = a$ or $\rho(a) = \bar{a}$ for each $a \in \mathcal{I}^{O(2r)}$. In a similar way, let $\eta_{2r+1} \in \mathcal{N}_{2r+1}$ be defined by:

$$\begin{aligned}\eta_{2r+1} &= \frac{1}{2^r}(\alpha_1\alpha_{\bar{1}} - \alpha_{\bar{1}}\alpha_1)(\alpha_2\alpha_{\bar{2}} - \alpha_{\bar{2}}\alpha_2)\cdots(\alpha_r\alpha_{\bar{r}} - \alpha_{\bar{r}}\alpha_r)\alpha_0 \\ &= \frac{(-1)^r}{2^r}\sum_{\rho\in S_2\otimes\cdots\otimes S_2}(-1)^\rho\alpha_{\rho(\bar{1})}\alpha_{\rho(1)}\cdots\alpha_{\rho(\bar{r})}\alpha_{\rho(r)}\alpha_0.\end{aligned} \tag{6.1.8b}$$

If γ_{η_m} denotes the image of η_m in the representation $\alpha_a \to \gamma_a$ then, from (6.1.7d),

$$\gamma_{\eta_{2r}} = \hat{\gamma}_1^{(1)}\hat{\gamma}_2^{(1)}\cdots\hat{\gamma}_r^{(1)} = \gamma_0, \tag{6.1.9a}$$

for even $m = 2r$, and

$$\gamma_{\eta_{2r+1}} = \hat{\gamma}_1^{(1)}\hat{\gamma}_2^{(1)}\cdots\hat{\gamma}_r^{(1)}\gamma_0 = I_{2^r}, \tag{6.1.9b}$$

for odd $m = 2r+1$. It then follows from (6.1.8b) that in the representation of $\mathcal{N}_{2r+1}$ generated by the map $\alpha_a \to -\gamma_a$, the image of η_{2r+1} is $-I_{2^r}$. This shows that for $m = 2r+1$, the two representations of Lemma 6.1.6 are inequivalent.

Lemma 6.1.10. *For even $m = 2r$, $\mathcal{N}_{2r}$ is isomorphic to $\mathcal{M}_{2^r}$, the full ring of $2^r \times 2^r$ matrices. For odd $m = 2r+1$, $\mathcal{N}_{2r+1}$ is isomorphic to the direct sum $\mathcal{M}_{2^r} \oplus \mathcal{M}_{2^r}$.*

Proof. In the notation of this section, the usual basis elements of $\mathcal{M}_{2^r}$ may be expressed thus:

$$\sigma_{a_1} \otimes \sigma_{a_2} \otimes \cdots \otimes \sigma_{a_r}, \tag{6.1.10a}$$

where $a_i \in \{2,3,4,5\}$ for $i = 1,2,\ldots,r$. The representation $\alpha_a \to \gamma_a$ of $\mathcal{N}_m$ then yields the basis elements (6.1.10a) through:

$$\hat{\gamma}_1^{(a_1)} \otimes \hat{\gamma}_2^{(a_2)} \otimes \cdots \otimes \hat{\gamma}_r^{(a_r)}. \tag{6.1.10b}$$

Thus the representation in terms of the γ matrices, is realised on the complete space of $2^r \times 2^r$ matrices. In the case of even $m = 2r$, since $\mathcal{N}_{2r}$ is of the same dimension as $\mathcal{M}_{2^r}$, it follows that this representation is faithful and thus that $\mathcal{N}_{2r}$ is isomorphic to the complete ring of $2^r \times 2^r$ matrices. For later convenience, let $\beta_{a_1 a_2 \cdots a_r}$ denote the element of $\mathcal{N}_{2r}$ having the image (6.1.10b) where $a_i \in \{2,3,4,5\}$ for $i = 1,2,\ldots,r$.

Now consider the case of odd $m = 2r + 1$. As above $\mathcal{M}_{2^r}$ provides a representation of $\mathcal{N}_{2r+1}$. However, since $\gamma_{\eta_{2r+1}} = I_{2r+1}$, it is not a faithful representation. Now map each element $\alpha_a \in \mathcal{N}_{2r+1}$ to a $2^{r+1} \times 2^{r+1}$ matrix in which γ_a appears as the top left $2^r \times 2^r$ submatrix, $-\gamma_a$ appears as the bottom right $2^r \times 2^r$ submatrix and zeros are elsewhere. This map may be denoted:

$$\alpha_a \to \gamma_a \oplus (-\gamma_a). \tag{6.1.10c}$$

Such matrices comprise a reducible representation of $\mathcal{N}_{2r+1}$. It is required to show that each of the 2^{2r+1} usual basis elements of $\mathcal{M}_{2^r} \oplus \mathcal{M}_{2^r}$ can be expressed in terms of the images under (6.1.10c). The image of $\beta_{a_1 a_2 \cdots a_r}$, now considered as an element of $\mathcal{N}_{2r+1}$, is:

$$(\hat{\gamma}_1^{(a_1)} \otimes \hat{\gamma}_2^{(a_2)} \otimes \cdots \otimes \hat{\gamma}_r^{(a_r)}) \oplus (-\hat{\gamma}_1^{(a_1)} \otimes \hat{\gamma}_2^{(a_2)} \otimes \cdots \otimes \hat{\gamma}_r^{(a_r)}). \tag{6.1.10d}$$

However, from (6.1.9b), the image of $\beta_{a_1 a_2 \cdots a_r} \eta_{2r+1}$ is:

$$(\hat{\gamma}_1^{(a_1)} \otimes \hat{\gamma}_2^{(a_2)} \otimes \cdots \otimes \hat{\gamma}_r^{(a_r)}) \oplus (\hat{\gamma}_1^{(a_1)} \otimes \hat{\gamma}_2^{(a_2)} \otimes \cdots \otimes \hat{\gamma}_r^{(a_r)}). \tag{6.1.10e}$$

Thus $(\beta_{a_1 a_2 \cdots a_r} + \beta_{a_1 a_2 \cdots a_r} \eta_{2r+1})/2 \in \mathcal{N}_{2r+1}$ maps to the basis elements of $\mathcal{M}_{2^r} \oplus 0$ and $(\beta_{a_1 a_2 \cdots a_r} \eta_{2r+1} - \beta_{a_1 a_2 \cdots a_r})/2$ maps to the basis elements of $0 \oplus \mathcal{M}_{2^r}$. Comparison of dimensions then shows that $\mathcal{N}_{2r+1}$ is isomorphic to $\mathcal{M}_{2^r} \oplus \mathcal{M}_{2^r}$.

Proofs of the following lemma may be found in **[Bo63,CR62]**.

Lemma 6.1.11. *If an algebra A is a direct sum of full matrix rings, then every representation of A is completely reducible and every irreducible representation is* *equivalent to one of the summands. A representation is faithful if and only if it contains each summand at least once.*

The following lemma is a corollary of the previous two.

Lemma 6.1.12. *Every representation of $\mathcal{N}_m$ is completely reducible. For even $m = 2r$, there exists just one irreducible representation of $\mathcal{N}_{2r}$ up to equivalence. Its dimension is 2^r and it is faithful. For odd $m = 2r+1$, there exists just two inequivalent irreducible representations of $\mathcal{N}_{2r+1}$. Each has dimension 2^r and neither is faithful. Their direct sum is faithful.*

By virtue of the above construction, the maps $\alpha_a \to \gamma_a$ and $\alpha_a \to -\gamma_a$ for $a \in \mathcal{I}^{O(m)}$ provide two irreducible representations of $\mathcal{N}_m$ which are equivalent if and only if m is even. As indicated above, the image of η_{2r+1} which, being a multiple of the identity is invariant under similarity transformations, serves to distinguish between the two representations of $\mathcal{N}_{2r+1}$ for odd $m = 2r+1$.

The following lemma will be required below.

Lemma 6.1.13. *If $\eta_m \in \mathcal{N}_m$ is as defined by* (6.1.8), *then:*

$$\eta_{2r} = \frac{(-1)^r}{2^r(2r)!} \sum_{\pi \in S_{2r}} (-1)^\pi \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^\rho \alpha_{\rho\pi(\bar{1})}\alpha_{\rho\pi(1)} \cdots \alpha_{\rho\pi(\bar{r})}\alpha_{\rho\pi(r)}, \tag{6.1.13a}$$

for even $m = 2r$, and:

$$\eta_{2r+1} = \frac{(-1)^r}{2^r(2r+1)!} \sum_{\pi \in S_{2r+1}} (-1)^\pi \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^\rho \alpha_{\rho\pi(\bar{1})}\alpha_{\rho\pi(1)} \cdots \alpha_{\rho\pi(\bar{r})}\alpha_{\rho\pi(r)}\alpha_{\rho\pi(0)}, \tag{6.1.13b}$$

for odd $m = 2r+1$.

Proof. In this proof, the case of odd $m = 2r+1$ only will be considered. The proof for $m = 2r$ is obtained simply by excluding the 0 index at each stage.

For $\rho, \pi \in S_m$ define:

$$\eta(\rho, \pi) = \alpha_{\rho\pi(\bar{1})}\alpha_{\rho\pi(1)} \cdots \alpha_{\rho\pi(\bar{r})}\alpha_{\rho\pi(r)}\alpha_{\rho\pi(0)}, \tag{6.1.13c}$$

so that, by (6.1.8),

$$\eta_m = \frac{(-1)^r}{2^r} \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^\rho \eta(\rho, I). \tag{6.1.13d}$$

If $\rho' \in S_2 \otimes \cdots \otimes S_2$, then:

$$\frac{(-1)^r}{2^r} \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^\rho \eta(\rho, \rho') = \frac{(-1)^r}{2^r} \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^\rho \eta(\rho\rho', I)$$

$$= \frac{(-1)^r}{2^r}(-1)^{\rho'} \sum_{\rho\rho' \in S_2 \otimes \cdots \otimes S_2} (-1)^{\rho\rho'} \eta(\rho\rho', I)$$

$$= (-1)^{\rho'} \eta_m. \tag{6.1.13e}$$

Now fix $\pi \in S_m$ and let $\tau \in S_m$ be such that $\tau\pi \in S_2 \otimes \cdots \otimes S_2$ and if $\pi(\bar{a}) < \pi(a)$ then $\tau\pi(\bar{a}) < \tau\pi(a)$ and vice-versa, for $a = 1, 2, \ldots, r$. That is, τ returns $\bar{a}$ and a to their original two positions but maintains their order as that given by π. This determines τ uniquely. For instance, if $m = 7$ and

$$\pi = \begin{pmatrix} \bar{1} & 1 & \bar{2} & 2 & \bar{3} & 3 & 0 \\ \bar{2} & 1 & 3 & 0 & 2 & \bar{1} & \bar{3} \end{pmatrix}, \quad \text{then} \quad \tau = \begin{pmatrix} \bar{2} & 1 & 3 & 0 & 2 & \bar{1} & \bar{3} \\ 1 & \bar{1} & \bar{2} & 2 & 3 & \bar{3} & 0 \end{pmatrix}. \tag{6.1.13f}$$

Consider the single term $\eta(\rho^{-1}, \pi^{-1})$. By (3.1.4), the factor α_a occurs in position $\pi\rho(a)$ of $\eta(\rho^{-1}, \pi^{-1})$ for each $a \in \mathcal{I}^{O(m)}$. Now consider the term $\eta(\rho^{-1}, \pi^{-1}\tau^{-1})$. By (3.1.4), each factor α_a occurs in position $\tau\pi\rho(a)$ of $\eta(\rho^{-1}, \pi^{-1}\tau^{-1})$. Therefore, $\eta(\rho^{-1}, \pi^{-1}\tau^{-1})$ may be obtained from $\eta(\rho^{-1}, \pi^{-1})$ by a sequence of transpositions, none of which is the transposition of $\alpha_{\bar{a}}$ and α_a for all $a = 1, 2, \ldots, r$, by virtue of the above construction of τ. Therefore, by (6.1.2):

$$\eta(\rho^{-1}, \pi^{-1}\tau^{-1}) = (-1)^{\tau} \eta(\rho^{-1}, \pi^{-1}). \tag{6.1.13g}$$

Then since $\pi^{-1}\tau^{-1} \in S_2 \otimes \cdots \otimes S_2$, it follows from (6.1.13e) that:

$$\frac{(-1)^r}{2^r}(-1)^{\tau} \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^{\rho} \eta(\rho^{-1}, \pi^{-1}) = (-1)^{\pi\tau} \eta_m, \tag{6.1.13h}$$

whereupon:

$$\eta_m = \frac{(-1)^r}{2^r}(-1)^{\pi} \sum_{\rho \in S_2 \otimes \cdots \otimes S_2} (-1)^{\rho} \eta(\rho, \pi). \tag{6.1.13i}$$

The lemma then follows by summing over all $\pi \in S_m$ and dividing by $m!$.

§6.2. The basic spin representations

In this section, the basic spin representations of $O(m)$, $SO(m)$ and $so(m)$ are obtained by means of the irreducible representations of the generalised Clifford algebras $\mathcal{N}_m$ determined in Section 6.1.

Lemma 6.1.6 states that the matrices γ_a form a representation of $\mathcal{N}_m$. Therefore:

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\delta_{a\bar{b}} I_{2^r}. \tag{6.2.1}$$

Consider an element $G \in O(m)$ and let:

$$\gamma'_a = \sum_c G_{ca}\gamma_c. \tag{6.2.2}$$

Then:

$$\begin{aligned}\gamma'_a\gamma'_b + \gamma'_b\gamma'_a &= \sum_c\sum_d G_{ca}G_{db}(\gamma_c\gamma_d + \gamma_d\gamma_c) \\ &= 2\sum_c\sum_d G_{ca}G_{db}J_{cd}I_{2^r} \\ &= 2J_{ab}I_{2^r}, \end{aligned} \tag{6.2.3}$$

so that the matrices γ' also constitute a representation of $\mathcal{N}_m$ of the same dimension. It is now necessary to treat separately the cases of m even and odd. First consider even $m = 2r$, where the existence of only one representation of $\mathcal{N}_{2r}$ up to equivalence implies that there exists a $2^r \times 2^r$ matrix $\Delta(G)$ such that:

$$\gamma'_a = \Delta(G)\gamma_a\Delta(G)^{-1}, \tag{6.2.4}$$

for all $a \in \mathcal{I}^{O(m)}$. The matrix $\Delta(G)$ is not defined uniquely by (6.2.4) for any non-zero multiple also suffices. Conversely, if there exists a second matrix $\Delta'(G)$ satisfying (6.2.4) in place of $\Delta(G)$, then:

$$\gamma_a = \Delta'(G)^{-1}\Delta(G)\gamma_a\Delta(G)^{-1}\Delta'(G),$$

so that the matrix $\Delta'(G)^{-1}\Delta(G)$ commutes with every element of the irreducible representation of $\mathcal{N}_{2r}$ Then by Schur's lemma (1.4.11), $\Delta'(G)^{-1}\Delta(G) = g.I_{2^r}$ for some $g \in \mathbb{C}$, implying that $\Delta'(G)$ is a multiple of $\Delta(G)$.

If $G', G'' \in O(2r)$, then the above analysis yields:

$$\begin{aligned}\sum_e (G'G'')_{ea}\gamma_e &= \sum_a\sum_c G''_{ca}G'_{ec}\gamma_e \\ &= \sum_c G''_{ca}\Delta(G')\gamma_c\Delta(G')^{-1} \\ &= \Delta(G')\Delta(G'')\gamma_a\Delta(G'')^{-1}\Delta(G')^{-1}. \end{aligned} \tag{6.2.5a}$$

for some matrices $\Delta(G')$ and $\Delta(G'')$ and all $a \in \mathcal{I}^{O(m)}$. However, in addition,

$$\sum_e (G'G'')_{ea}\gamma_e = \Delta(G'G'')\gamma_a\Delta(G'G'')^{-1}, \tag{6.2.5b}$$

for some matrix $\Delta(G'G'')$ and all $a \in \mathcal{I}^{O(m)}$. Comparison of (6.2.5a) and (6.2.5b) implies that:

$$\Delta(G'G'') = \kappa\Delta(G')\Delta(G''), \tag{6.2.6}$$

for some $\kappa \in \mathbb{C}$. It will now be shown that it is possible to choose the matrices $\Delta(G')$, $\Delta(G'')$ and $\Delta(G'G'')$ such that $\kappa = \pm 1$ in all instances.

The transpose of (6.2.1):

$$\tilde{\gamma}_a\tilde{\gamma}_b + \tilde{\gamma}_b\tilde{\gamma}_a = 2\delta_{a\bar{b}}I_{2^r}, \tag{6.2.7}$$

shows that the matrices $\tilde{\gamma}_a$ also generate an irreducible representation of $\mathcal{N}_{2r}$. Therefore there exists a $2^r \times 2^r$ matrix C such that:

$$\tilde{\gamma}_a = C\gamma_a C^{-1}, \tag{6.2.8}$$

for all $a \in \mathcal{I}^{O(m)}$. The transposes of (6.2.2) and (6.2.4) give:

$$\sum_c G_{ca}\tilde{\gamma}_c = \widetilde{\Delta(G)}^{-1}\tilde{\gamma}_a\widetilde{\Delta(G)}$$

whereupon, using (6.2.8):

$$\begin{aligned}\sum_c G_{ca}\gamma_c &= C^{-1}\widetilde{\Delta(G)}^{-1}C\gamma_a C^{-1}\widetilde{\Delta(G)}C\\ &= \overline{\Delta}(G)\gamma_a\overline{\Delta}(G)^{-1},\end{aligned} \tag{6.2.9}$$

for all $a \in \mathcal{I}^{O(m)}$, where $\overline{\Delta}(G) = C^{-1}\widetilde{\Delta(G)}^{-1}C$. Comparing (6.2.9) and (6.2.4) then implies that:

$$\overline{\Delta}(G) = \phi\Delta(G), \tag{6.2.10}$$

for some non-zero $\phi \in \mathbb{C}$. Let $\Delta'(G) = \sqrt{\phi}\Delta(G)$. Then, using $\Delta'(G)$ in place of $\Delta(G)$ in (6.2.9), results in:

$$\overline{\Delta'}(G) = \Delta'(G). \tag{6.2.11}$$

Now assume that the arbitrary factors in the original matrices $\Delta(G')$, $\Delta(G'')$ and $\Delta(G'G'')$ have been chosen so that each of these matrices satisfy (the unprimed version of) expression (6.2.11). From (6.2.6):

$$\widetilde{\Delta(G'G'')}^{-1} = \frac{1}{\kappa}\widetilde{\Delta(G')}^{-1}\widetilde{\Delta(G'')}^{-1}, \tag{6.2.12}$$

whereupon:

$$\begin{aligned}C\Delta(G'G'')C^{-1} &= \frac{1}{\kappa}C\Delta(G')C^{-1}C\Delta(G'')C^{-1}\\ &= \frac{1}{\kappa^2}C\Delta(G'G'')C^{-1}.\end{aligned} \tag{6.2.13}$$

Therefore $\kappa = \pm 1$. These two values are essential and reflect the fact that the representation $G \to \pm\Delta(G)$ is necessarily two-valued. This representation is known as the basic spin representation of $O(2r)$ and, as will be seen later, is irreducible.

Lemma 6.2.14. *Let $G \in O(m)$. In the representation of $\mathcal{N}_m$ generated by $\alpha_a \to \gamma'_a$ through* (6.2.2)*, the image γ'_{η_m} of η_m is such that:*

$$\gamma'_{\eta_m} = \det G.\gamma_{\eta_m}. \tag{6.2.14}$$

Proof. From Lemma 6.1.13,

$$\gamma_{\eta_m} = \frac{(-1)^r}{2^r m!}\epsilon_{i_1 i_2 \cdots i_m} \sum_{\rho\in S_2\otimes\cdots\otimes S_2} (-1)^\rho \gamma_{\rho(i_1)} \cdots \gamma_{\rho(i_m)}, \tag{6.2.14a}$$

where there is an implied summation over all $i_k \in \mathcal{I}^{O(m)}$ for $k = 1, 2, \ldots, m$. In addition, in the representation generated by the map $\alpha_a \to \gamma'_a$, the image γ'_{η_m} of η_m is:

$$\gamma'_{\eta_m} = \frac{(-1)^r}{2^r m!}\epsilon_{i_1 i_2 \cdots i_m} \sum_{\rho\in S_2\otimes\cdots\otimes S_2} (-1)^\rho \gamma'_{\rho(i_1)} \cdots \gamma'_{\rho(i_m)}, \tag{6.2.14b}$$

where γ'_a is given by (6.2.2), so that:

$$\begin{aligned}
\gamma'_{\eta_m} &= \frac{(-1)^r}{2^r m!}\epsilon_{i_1 i_2 \cdots i_m} \sum_{\rho\in S_2\otimes\cdots\otimes S_2} (-1)^\rho G_{\rho(j_1)\rho(i_1)} \cdots G_{\rho(j_m)\rho(i_m)} \gamma_{\rho(j_1)} \cdots \gamma_{\rho(j_m)} \\
&= \frac{(-1)^r}{2^r m!}\epsilon_{i_1 i_2 \cdots i_m} G_{j_1 i_1} \cdots G_{j_m i_m} \sum_{\rho\in S_2\otimes\cdots\otimes S_2} (-1)^\rho \gamma_{\rho(j_1)} \cdots \gamma_{\rho(j_m)} \\
&= \frac{(-1)^r}{2^r m!} \det G \, \epsilon_{j_1 j_2 \cdots j_m} \sum_{\rho\in S_2\otimes\cdots\otimes S_2} (-1)^\rho \gamma_{\rho(j_1)} \cdots \gamma_{\rho(j_m)} \\
&= \det G . \gamma_{\eta_m}.
\end{aligned} \tag{6.2.14c}$$

In the case of even $m = 2r$, the matrices γ'_a satisfy (6.2.4) for some $\Delta(G)$. Therefore, from (6.2.14b):

$$\begin{aligned}
\gamma'_{\eta_{2r}} &= \frac{(-1)^r}{2^r (2r)!}\epsilon_{i_1 i_2 \cdots i_m} \sum_{\rho\in S_2\otimes\cdots\otimes S_2} (-1)^\rho \Delta(G)\gamma_{\rho(i_1)}\Delta(G)^{-1} \cdots \Delta(G)\gamma_{\rho(i_m)}\Delta(G)^{-1} \\
&= \Delta(G)\gamma_{\eta_{2r}}\Delta(G)^{-1},
\end{aligned}$$

whereupon, from (6.2.14):

$$\Delta(G)\gamma_{\eta_{2r}}\Delta(G)^{-1} = \det G . \gamma_{\eta_{2r}}. \tag{6.2.15}$$

From (6.1.9a) and (6.1.4c), $\gamma_{\eta_{2r}}$ is a $2^r \times 2^r$ diagonal matrix with 1 and -1 each occurring on the diagonal with a multiplicity of 2^{r-1}. Thus the basis elements may be arranged so that $\gamma_{\eta_{2r}}$ is expressed in the block diagonal form:

$$\gamma_{\eta_{2r}} = \begin{pmatrix} I_{2^{r-1}} & 0 \\ 0 & -I_{2^{r-1}} \end{pmatrix}. \tag{6.2.16}$$

In the same basis, let:

$$\Delta(G) = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{6.2.17}$$

where P, Q, R and S are each $2^{r-1} \times 2^{r-1}$ submatrices. If $G \in SO(2r)$ then $\det G = 1$, and (6.2.15) implies that $\Delta(G)\gamma_{\eta_{2r}} = \gamma_{\eta_{2r}}\Delta(G)$. Then direct multiplication of (6.2.16) and (6.2.17) shows that the submatrices R and Q are both identically zero. Therefore $\Delta(G)$ takes the form:

$$\Delta(G) = \begin{pmatrix} \Delta^+(G) & 0 \\ 0 & \Delta^-(G) \end{pmatrix}. \tag{6.2.18}$$

This demonstrates that on restriction from $O(2r)$ to $SO(2r)$, the basic spin representation Δ of $O(2r)$ decomposes into the direct sum of two representations. These are denoted Δ^+ and Δ^-. As will be seen later, each is irreducible. Incidentally, if $G \in O(2r)$ and $\det G = -1$, then (6.2.15) implies that $\Delta(G)$ takes the form:

$$\Delta(G) = \begin{pmatrix} 0 & Q(G) \\ R(G) & 0 \end{pmatrix}. \tag{6.2.19}$$

In the case of odd $m = 2r+1$, the matrices γ'_a given by (6.2.2) still generate a 2^r-dimensional representation of $\mathcal{N}_{2r+1}$. However, since there are two such representations, expression (6.2.4) does not follow. However, if $\det G = 1$, then Lemma 6.2.14 shows that $\gamma'_{\eta_{2r+1}}$, the image of η_{2r+1} is $\gamma_{\eta_{2r+1}} = I_{2^r}$. This shows that the representation of $\mathcal{N}_{2r+1}$ generated by the matrices γ'_a is equivalent to that generated by the matrices γ_a and therefore that there exists a $\Delta(G)$ such that (6.2.4) does hold for the case of $G \in SO(2r+1)$. As in (6.2.7), the matrices $\tilde{\gamma}_a$ also generate a representation of $\mathcal{N}_{2r+1}$. Here, from (6.1.13b), the image of η_{2r+1} is given by:

$$\begin{aligned}
&\frac{(-1)^r}{2^r(2r+1)!}\sum_{\pi\in S_{2r+1}}(-1)^\pi\sum_{\rho\in S_2\otimes\cdots\otimes S_2}(-1)^\rho\tilde{\gamma}_{\rho\pi(\bar{1})}\tilde{\gamma}_{\rho\pi(1)}\cdots\tilde{\gamma}_{\rho\pi(\bar{r})}\tilde{\gamma}_{\rho\pi(r)}\tilde{\gamma}_{\rho\pi(0)}\\
=&\frac{(-1)^r}{2^r(2r+1)!}(-1)^r\sum_{\pi\in S_{2r+1}}(-1)^\pi\sum_{\rho\in S_2\otimes\cdots\otimes S_2}(-1)^\rho\tilde{\gamma}_{\rho\pi(0)}\tilde{\gamma}_{\rho\pi(r)}\cdots\tilde{\gamma}_{\rho\pi(\bar{2})}\tilde{\gamma}_{\rho\pi(1)}\tilde{\gamma}_{\rho\pi(\bar{1})}\\
=&(-1)^r\tilde{\gamma}_{\eta_{2r+1}}\\
=&(-1)^r I_{2^r}.
\end{aligned} \tag{6.2.20}$$

Therefore, the map $\alpha_a \to \tilde{\gamma}_a$ generates a representation of $\mathcal{N}_{2r+1}$ equivalent to that generated by $\alpha_a \to \gamma_a$ if r is even, and equivalent to that generated by $\alpha_a \to -\gamma_a$ if r is odd. Consequently, there exists a $2^r \times 2^r$ matrix C such that:

$$\tilde{\gamma}_a = (-1)^r C\gamma_a C^{-1}. \tag{6.2.21}$$

This matrix C enables the matrices $\Delta(G)$ to be normalised in precisely the same way as previously. Then, as before,

$$\Delta(G'G'') = \pm\Delta(G')\Delta(G''), \tag{6.2.22}$$

for all $G', G'' \in SO(2r+1)$.

In the case where $\det G = -1$, Lemma 6.2.14 shows that $\gamma'_{\eta_{2r+1}} = -\gamma_{\eta_{2r+1}} = -I_{2^r}$. This shows that the representation of $\mathcal{N}_{2r+1}$ generated by the matrices γ'_a is equivalent to that generated through the map $\alpha_a \to -\gamma_a$. Therefore there exists a $\Delta(G)$ such that:

$$\gamma'_a = -\Delta(G)\gamma_a\Delta(G)^{-1}, \tag{6.2.23}$$

for all $a \in \mathcal{I}^{O(m)}$. With the matrix C as given by (6.2.21), $\Delta(G)$ may be selected as before, so that if $\overline{\Delta}(G) = C^{-1}\widetilde{\Delta(G)}^{-1}C$ then $\overline{\Delta}(G) = \Delta(G)$. Then for $G', G'' \in O(2r+1)$ ($\det G' = \pm 1, \det G'' = \pm 1$), identities (6.2.13) follow as before as hence also (6.2.22).

Thus the two-valued, 2^r-dimensional basic 'spin' representations of $O(2r+1)$ have also been constructed. As will be shown later, they are irreducible and remain irreducible on restriction to the subgroup $SO(2r+1)$.

For the Lie algebra $so(m)$, it is possible to give an explicit description of the representation Δ. Let $m = 2r$ or $m = 2r+1$, and define the $2^r \times 2^r$ matrices $\Lambda_a{}^b$ for $a, b \in \mathcal{I}^{O(2r+1)}$ by:

$$\Lambda_a{}^b = \frac{1}{4}[\gamma_a, \gamma_{\bar{b}}] = \frac{1}{4}(\gamma_a\gamma_{\bar{b}} - \gamma_{\bar{b}}\gamma_a) \tag{6.2.24a}$$

$$= \frac{1}{2}(\gamma_a\gamma_{\bar{b}} - \delta_{ab}.I_{2^r}), \tag{6.2.24b}$$

where (6.2.24b) has been obtained from (6.2.24a) by using (6.2.1).

Lemma 6.2.25. *Let $m = 2r$ or $m = 2r+1$. In the basic spin representation Δ of the Lie algebra $so(m)$, the matrices $\Lambda_a{}^b$ represent the elements $D_a{}^b \in so(2r)$ or the elements $B_a{}^b \in so(2r+1)$ for all $a, b \in \mathcal{I}^{O(m)}$.*

Proof. Consider first the case $m = 2r$. Let $G(t)$ be a one parameter subgroup of $SO(m)$ for which:

$$D_a{}^b = \left.\frac{d}{dt}G(t)\right|_{t=0}. \tag{6.2.25a}$$

Let $\Delta(t) = \Delta(G(t))$ be one of the pair of $2^r \times 2^r$ matrices representing $G(t)$ in the representation Δ, satisfying (6.2.4) and the unprimed version of (6.2.11). Expression (6.2.11) implies that $\det\Delta(t) = \pm 1$. For t in a sufficiently small neighbourhood of

0, $\Delta(t)$ may be selected to be close to I_{2^r}, ensuring that $\det \Delta(t) = 1$. Combining (6.2.2) and (6.2.4), and differentiating with respect to t, yields:

$$\sum_q (D_a{}^b)_{qp}\gamma_q = \Delta(D_a{}^b)\gamma_p - \gamma_p \Delta(D_a{}^b), \tag{6.2.25b}$$

where $\Delta(D_a{}^b) = d/dt(\Delta(t))|_{t=0}$ represents $D_a{}^b$ in the representation Δ. From (2.2.17), $(D_a{}^b)_{qp} = \delta_{aq}\delta_{bp} - \delta_{\bar{b}q}\delta_{\bar{a}p}$, whereupon (6.2.25$b$) gives:

$$\delta_{bp}\gamma_a - \delta_{a\bar{p}}\gamma_{\bar{b}} = \Delta(D_a{}^b)\gamma_p - \gamma_p\Delta(D_a{}^b). \tag{6.2.25c}$$

It will now be confirmed that for all $a \in \mathcal{I}^{O(m)}$, this expression is satisfied by $\Delta(D_a{}^b) = \Lambda_a{}^b$. Substituting (6.2.24$b$) into the right side of (6.2.25c) yields:

$$\begin{aligned}
\Lambda_a{}^b\gamma_p - \gamma_p\Lambda_a{}^b &= \frac{1}{2}(\gamma_a\gamma_{\bar{b}}\gamma_p - \delta_{ab}\gamma_p - \gamma_p\gamma_a\gamma_{\bar{b}} + \delta_{ab}\gamma_p) \\
&= \frac{1}{2}(\gamma_a\gamma_{\bar{b}}\gamma_p - \gamma_p\gamma_a\gamma_{\bar{b}}) \\
&= \frac{1}{2}(-\gamma_a\gamma_p\gamma_{\bar{b}} + 2\delta_{bp}\gamma_a - \gamma_p\gamma_a\gamma_{\bar{b}}) \\
&= \frac{1}{2}(\gamma_p\gamma_a\gamma_{\bar{b}} - 2\delta_{a\bar{p}}\gamma_{\bar{b}} + 2\delta_{bp}\gamma_a - \gamma_p\gamma_a\gamma_{\bar{b}}) \\
&= \delta_{bp}\gamma_a - \delta_{a\bar{p}}\gamma_{\bar{b}},
\end{aligned} \tag{6.2.25d}$$

where (6.2.1) has been used twice. Since the right side of (6.2.25d) is the left side of (6.2.25c), the required confirmation is achieved. However, $\Delta(D_a{}^b) = \Lambda_a{}^b$ is not the unique solution of (6.2.25c). Let $\Delta'(D_a{}^b)$ be another solution so that:

$$\Delta'(D_a{}^b)\gamma_p - \gamma_p\Delta'(D_a{}^b) = \Lambda_a{}^b\gamma_p - \gamma_p\Lambda_a{}^b,$$

implying that:

$$(\Lambda_a{}^b - \Delta'(D_a{}^b))\gamma_p = \gamma_p(\Lambda_a{}^b - \Delta'(D_a{}^b)), \tag{6.2.25e}$$

for all $p \in \mathcal{I}^{O(m)}$. Thus $\Lambda_a{}^b - \Delta'(D_a{}^b)$ commutes with every element of an irreducible representation of $\mathcal{N}_m$. Thus by Schur's lemma, $\Delta'(D_a{}^b) = \Lambda_a{}^b + gI_{2^r}$ for some $g \in \mathbf{C}$. It is easily seen that all such $\Delta'(D_a{}^b)$ are solutions of (6.2.25c). However, since for small t, $\det \Delta(t) = 1$, it follows from Lemma 2.2.3 that $\operatorname{tr}\Delta(D_a{}^b) = 0$. Then, since $\operatorname{tr}\Lambda_a{}^b = 0$, this matrix is the unique representative of $D_a{}^b$ in the representation Δ of $so(2r)$. The argument is precisely the same for $B_a{}^b \in so(2r+1)$.

This lemma implies that the matrices $\Lambda_a{}^b$ satisfy the commutation relations (2.2.18) and (2.2.22), as may be confirmed directly by repeated use of (6.2.1).

§6.3. The Spinor relations

In this section, the appropriate Young tableaux are defined for the $O(m)$-modules associated with the representations $[\Delta;\lambda]$ of $O(m)$. Identities linking these tableaux are then derived for the irreducible $O(m)$-modules.

For the moment, consider the basic spin representation Δ of $O(m)$. Let Ψ denote the 2^r-dimensional module on which the elements α_a of the Clifford algebras $\mathcal{N}_m$ for $m = 2r$ and $m = 2r+1$, act through their irreducible matrix representatives γ_a. Thus, if $\psi \in \Psi$, then

$$\alpha_a \psi = \gamma_a \psi, \tag{6.3.1}$$

for $a \in \mathcal{I}^{O(m)}$. A convenient basis for Ψ is provided by the set:

$$\{\psi_{s_1 \cdots s_r} : s_j \in \{\bar{j}, j\}, j = 1, 2, \ldots, r\}. \tag{6.3.2}$$

Any $2^r \times 2^r$ matrix γ which can be expressed as the Kroneker product of r 2×2 matrices,

$$\gamma = \sigma^{(1)} \otimes \sigma^{(2)} \otimes \cdots \otimes \sigma^{(r)}, \tag{6.3.3}$$

is defined to act upon the basis elements of Ψ according to:

$$\gamma \psi_{s_1 \cdots s_r} = \sum_{t_1,\ldots,t_r} \sigma^{(1)}_{t_1 s_1} \sigma^{(2)}_{t_2 s_2} \cdots \sigma^{(r)}_{t_r s_r} \psi_{t_1 \cdots t_r}. \tag{6.3.4}$$

By defining this action to be linear, it provides a module action for the ring $\mathcal{M}_{2^r}$ since $(\gamma'\gamma'')\psi = \gamma'(\gamma''\psi)$ for all γ', γ'' of the form (6.3.3), and, as shown in the proof of Lemma 6.1.10, any element of $\mathcal{M}_{2^r}$ may be expressed in terms of elements of the form (6.3.3).

The basis elements $\psi_{s_1,\ldots,s_r}$ may be identified with the half tableaux of Definition 2.6.20. The tableau T^{Δ_r} corresponding to $\psi_{s_1 \cdots s_r}$ is obtained simply by setting $T^{\Delta_r}_{(j,0)} = s_j$ for $j = 1, 2, \ldots, r$. Thus for example, $\psi_{\bar{1}2\bar{3}\bar{4}\bar{5}}$ corresponds to the tableau:

$$T^{\Delta_r} = \begin{matrix} \bar{1} \,\cdot \\ 2 \,\cdot \\ \bar{3} \,\cdot \\ \bar{4} \,\cdot \\ \bar{5} \,\cdot \end{matrix} \tag{6.3.5}$$

Note that here, the element in the jth row is either j or $\bar{j}$ for each $j = 1, 2, \ldots, r$. This will be true for the half boxes (to the left of the dots) in all the tableaux $T^{\Delta_r;\lambda}$ that arise in this chapter. Note also that the index 0 does not appear in the half boxes at all. However, it will sometimes be convenient to create an extra half box below the rth, with only 0 permitted as an entry.

It will also prove convenient to introduce notation concerning sign factors related to the indices in the half boxes. Define:

$$(s) = \begin{cases} 0 & \text{if } s \in \mathbf{N}_r; \\ 1 & \text{if } s \in \overline{\mathbf{N}}_r, \end{cases} \tag{6.3.6}$$

and

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_k \end{bmatrix} = (-1)^{((s_1)+(s_2)+\cdots+(s_k))}. \tag{6.3.7}$$

As above, let $T^{\Delta_r}_{(j,0)} = s_j \in \{\bar{j}, j\}$ for $j = 1, 2, \ldots, r$ and let $a \in \mathbf{N}_r$. Through (6.3.4), it follows from (6.1.4a) that γ_a acts on T^{Δ_r} according to:

$$\gamma_a T^{\Delta_r} = \gamma_a : \begin{array}{l} s_1 \cdot \\ \vdots \\ s_{a-1} \\ s_a \cdot \\ s_{a+1} \\ \vdots \\ s_r \cdot \end{array} = \sqrt{2}\, \delta^{\bar{a}}_{s_a} \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \begin{array}{l} s_1 \cdot \\ \vdots \\ s_{a-1} \\ a \cdot \\ s_{a+1} \\ \vdots \\ s_r \cdot \end{array} \quad , \tag{6.3.8a}$$

where s_{a-1} means $s_{\bar{a}-1}$ if a is barred and similarly for s_{a+1}. From (6.1.4b), $\gamma_{\bar{a}}$ acts on T^{Δ_r} according to:

$$\gamma_{\bar{a}} T^{\Delta_r} = \gamma_{\bar{a}} : \begin{array}{l} s_1 \cdot \\ \vdots \\ s_{a-1} \\ s_a \cdot \\ s_{a+1} \\ \vdots \\ s_r \cdot \end{array} = \sqrt{2}\, \delta^{a}_{s_a} \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \begin{array}{l} s_1 \cdot \\ \vdots \\ s_{a-1} \\ \bar{a} \cdot \\ s_{a+1} \\ \vdots \\ s_r \cdot \end{array} \quad . \tag{6.3.8b}$$

Similarly, from (6.1.4c):

$$\gamma_0 T^{\Delta_r} = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} T^{\Delta_r}. \tag{6.3.8c}$$

In order to avoid a proliferation of cases later, these three expressions will be combined into one single expression. In order to achieve this, let $s_{\bar{j}} = s_j$ for $j = 1, 2, \ldots, r$, let:

$$\phi(a) = \begin{cases} 1 & \text{if } a \in \mathbf{N}_r \cup \overline{\mathbf{N}}_r; \\ 1/\sqrt{2} & \text{if } a = 0, \end{cases} \tag{6.3.9}$$

and append the extra (redundant) index $s_0 = 0$ to the bottom of the column of half boxes of T^{Δ_r}, so that:

$$T^{\Delta_r} = \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_{r-1}\,\cdot \\ s_r\,\cdot \end{array} \equiv \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_{r-1}\,\cdot \\ s_r\,\cdot \\ s_0\,\cdot \end{array} \quad . \tag{6.3.10}$$

Expressions (6.3.8) then combine to yield:

$$\gamma_a T^{\Delta_r} = \gamma_a : \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_{a-1}\,\cdot \\ s_a\,\cdot \\ s_{a+1}\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{array} = \sqrt{2}\phi(a)\,\delta^{\bar{a}}_{s_a} \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_{a-1}\,\cdot \\ \bar{s}_a\,\cdot \\ s_{a+1}\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{array} \quad , \tag{6.3.11}$$

for all $a \in \mathcal{I}^{O(m)}$, where if $a = 0$, it is implied that $a - 1 = r$.

By means of the construction of the basic spin representations presented in Section 6.2, the tableaux T^{Δ_r} constitute a basis for an $O(m)$-module. This $O(m)$-module will be denoted O^{Δ_r}. The action of $G \in O(m)$ on O^{Δ_r} is provided by the matrix $\Delta(G)$. This action therefore has an ambiguity in sign. Through Lemma 6.2.25, O^{Δ_r} also serves as a module for the Lie algebra $so(m)$. An explicit description of the action of $so(m)$ on the tableaux T^{Δ_r} will now be given.

Let $a, b \in \mathcal{I}^{O(m)}$ be such that $a < \bar{b}, b$. Then $\Lambda_a{}^b = \frac{1}{2}\gamma_a\gamma_{\bar{b}}$ from (6.2.24*b*), whereupon:

$$\Lambda_a{}^b : \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_b\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{array} = \frac{1}{2}\gamma_a\gamma_{\bar{b}} \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_b\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{array} = \frac{1}{\sqrt{2}}\phi(b)\,\delta^{b}_{s_b} \begin{bmatrix} s_1 \\ \vdots \\ s_{b-1} \end{bmatrix} \gamma_a : \begin{array}{c} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ \bar{s}_b\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{array}$$

$$= \phi(a)\phi(b)\,\delta^{b}_{s_b}\delta^{\bar a}_{s_a} \begin{bmatrix} s_1 \\ \vdots \\ s_{b-1} \end{bmatrix} \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \begin{matrix} s_1\,\cdot \\ \vdots \\ \bar s_a\,\cdot \\ \vdots \\ \bar s_b\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} = \phi(a)\phi(b)\,\delta^{b}_{s_b}\delta^{\bar a}_{s_a} \begin{bmatrix} s_a \\ \vdots \\ s_{b-1} \end{bmatrix} \begin{matrix} s_1\,\cdot \\ \vdots \\ \bar s_a\,\cdot \\ \vdots \\ \bar s_b\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} \tag{6.3.12a}$$

Similarly, if $a > \bar b, b$ then:

$${\Lambda_a}^{b} : \begin{matrix} s_1\,\cdot \\ \vdots \\ s_b\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} = -\phi(a)\phi(b)\,\delta^{b}_{s_b}\delta^{\bar a}_{s_a} \begin{bmatrix} s_b \\ \vdots \\ s_{a-1} \end{bmatrix} \begin{matrix} s_1\,\cdot \\ \vdots \\ \bar s_b\,\cdot \\ \vdots \\ \bar s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix}\ , \tag{6.3.12b}$$

where the minus sign appears because s_b is used to calculate the sign and not $\bar s_b$. If $a \in \mathcal{I}^{O(m)}$, then ${\Lambda_a}^{\bar a} = 0$ from (6.2.24), (6.1.5e) and (6.1.5j), whereupon:

$${\Lambda_a}^{\bar a} T^{\Delta_r} = 0. \tag{6.3.12c}$$

If $a \in \mathcal{I}^{O(m)}$ and $a \neq 0$, then ${\Lambda_a}^{a} = \frac{1}{2}(\gamma_a\gamma_{\bar a} - I_{2^r})$, whereupon:

$${\Lambda_a}^{a} : \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} = \frac{1}{2}\gamma_a\gamma_{\bar a} : \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} - \frac{1}{2} \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} = \frac{1}{\sqrt{2}}\phi(a)\,\delta^{a}_{s_a} \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \gamma_a : \begin{matrix} s_1\,\cdot \\ \vdots \\ \bar s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} - \frac{1}{2} \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix}$$

$$= \phi(a)^2\,\delta^{a}_{s_a}\delta^{\bar a}_{\bar s_a} \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} - \frac{1}{2} \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix} = \left(\delta^{a}_{s_a} - \frac{1}{2}\right) \begin{matrix} s_1\,\cdot \\ \vdots \\ s_a\,\cdot \\ \vdots \\ s_r\,\cdot \\ s_0\,\cdot \end{matrix}\ . \tag{6.3.12d}$$

As in Chapter 5, let V be the defining $O(m)$-module with basis $\{e_a : a \in \mathcal{I}^{O(m)}\}$. If $m = 2r$ or $m = 2r+1$, then the vector space $\Psi \otimes V^{\otimes l}$ has dimension $2^r m^l$

and a basis:

$$\{\psi_{s_1s_2\cdots s_r}\otimes e_{a_1a_2\cdots a_l} : s_j\in\{\bar{j},j\}, j=1,2,\ldots,r;\ a_i\in\mathcal{I}^{O(m)}, i=1,2,\ldots,l\}. \tag{6.3.13}$$

The action of $G\in O(m)$ on $\Psi\otimes V^{\otimes l}$ is defined by the linear extension of the following action on the basis elements:

$$\begin{aligned} G : \psi_{s_1s_2\cdots s_r}&\otimes e_{a_1a_2\cdots a_l} \\ &= \pm\sum_{s_1,\ldots,s_r}\sum_{b_1,\ldots,b_l}\Delta(G)_{t_1t_2\cdots t_r s_1s_2\cdots s_r}G_{b_1a_1}G_{b_2a_2}\cdots G_{b_la_l}\psi_{t_1t_2\cdots t_r}\otimes e_{b_1b_2\cdots b_l}. \end{aligned} \tag{6.3.14}$$

Note that the arbitrary sign of $\Delta(G)$ is only arbitrary overall. Therefore the '$\pm$' is written here before the summations and the action is two-valued. Moreover, the arbitrary sign is an overall sign for the linear extension of (6.3.14). Fix m, let $J=J_m^+$ and consider the tensor:

$$\hat{\psi}_{s_1\cdots s_r}=\sum_{a,b}\sum_{t_1,\ldots,t_r}J_{ab}(\gamma_b)_{t_1\cdots t_r s_1\cdots s_r}\psi_{t_1\cdots t_r}\otimes e_a. \tag{6.3.15}$$

The action of $G\in O(m)$ results in:

$$G\hat{\psi}_{s_1\cdots s_r}=\pm\sum_{a,b}\sum_{t_1,\ldots,t_r}\sum_{c}\sum_{u_1,\ldots,u_r}J_{ab}(\gamma_b)_{t_1\cdots t_r s_1\cdots s_r}\Delta(G)_{u_1\cdots u_r t_1\cdots t_r}G_{ca}\psi_{u_1\cdots u_r}\otimes e_c. \tag{6.3.16a}$$

Then (6.2.2) and (6.2.4) imply that $\Delta(G)\gamma_b=\sum_d G_{db}\gamma_d\Delta(G)$, whereupon:

$$G\hat{\psi}_{s_1\cdots s_r}=\pm\sum_{a,b,c,d}\sum_{v_1,\ldots,v_r}\sum_{u_1,\ldots,u_r}J_{ab}G_{db}G_{ca}(\gamma_d)_{u_1\cdots u_r v_1\cdots v_r}\Delta(G)_{v_1\cdots v_r s_1\cdots s_r}\psi_{u_1\cdots u_r}\otimes e_c. \tag{6.3.16b}$$

From Lemma 2.1.2, $GJ\tilde{G}=J$ for all $G\in O(m)$. Therefore:

$$G\hat{\psi}_{s_1\cdots s_r}=\pm\sum_{c,d}\sum_{v_1,\ldots,v_r}\sum_{u_1,\ldots,u_r}J_{cd}(\gamma_d)_{u_1\cdots u_r v_1\cdots v_r}\Delta(G)_{v_1\cdots v_r s_1\cdots s_r}\psi_{u_1\cdots u_r}\otimes e_c, \tag{6.3.16c}$$

which is a linear combination of terms $\hat{\psi}_{v_1\cdots v_r}$. The space of all such tensors is thus invariant. For this to occur it is necessary that the sign is fixed to be an overall sign as in (6.3.16a).

Definition 6.3.17. *For $O(m)$, a trace tensor of $\Psi\otimes V^{\otimes l}$ is any linear combination of terms of the form:*

$$\sum_{a,b}\sum_{t_1,\ldots,t_r}J_{ab}(\gamma_b)_{t_1\cdots t_r s_1\cdots s_r}\psi_{t_1\cdots t_r}\otimes x\otimes e_a\otimes y, \tag{6.3.17}$$

where x and y are elements of some (possibly zero) tensor power of V and $x\otimes y\in V^{\otimes(l-1)}$. The vector space $\hat{U}\subset\Psi\otimes V^{\otimes l}$ is defined to be the span of all such trace tensors.

It follows from (6.3.15) and (6.3.16), that $\hat{U}$ is invariant under the action of $O(m)$. The complete reducibility of $\Psi \otimes V^{\otimes l}$ implies that $(\Psi \otimes V^{\otimes l})/\hat{U}$ is isomorphic to a subspace of $\Psi \otimes V^{\otimes l}$, which is invariant under the action of $O(m)$. Thereupon $O^{\Delta_r;\lambda} = (\Psi \otimes W^\lambda)/(\Psi \otimes W^\lambda \cap \hat{U})$ is an $O(m)$-submodule of $\Psi \otimes W^\lambda$.

Let $T^{\Delta_r;\lambda}$ be formed by adjoining T^{Δ_r} and T^λ, and let $\{T^{\Delta_r;\lambda}\}$ denote the symmetrised element $T^{\Delta_r} \otimes \{T^\lambda\} \in \Psi \otimes W^\lambda$. Now let $[T^{\Delta_r;\lambda}]$ denote the traceless symmetrised tableau resulting from the removal of all trace terms (6.3.17) from the symmetrised tableau $\{T^{\Delta_r;\lambda}\}$, by forming its quotient with respect to the elements of $\hat{U}$. $O^{\Delta_r;\lambda}$ is therefore spanned by all $[T^{\Delta_r;\lambda}]$ where the entries of each $T^{\Delta_r;\lambda}$ are from the set $\mathcal{I}^{O(m)}$ and $T^{\Delta_r;\lambda}_{(j,0)} \in \{\bar{j}, j\}$ for $j = 1, 2, \ldots, r$.

Lemma 6.3.18. *Let $T_0^{\Delta_r}$ be a tableau for which $T^{\Delta_r}_{0(j,0)} = s_j$ for $j = 1, 2, \ldots, r, 0$, and let $T_i^{\Delta_r}$ be r tableaux, each identical to $T_0^{\Delta_r}$ apart from one position where $T^{\Delta_r}_{i(i,0)} = \bar{s}_i$. Now for $i = 1, 2, \ldots, r, 0$, let T_i^λ be $r+1$ tableaux identical apart for the entry in one fixed position for which $T^\lambda_{i(a,b)} = s_i$, for which $a \le \tilde{\lambda}_1$ and $b \le \lambda_a$. Let $T_i^{\Delta_r;\lambda}$ be formed by adjoining $T_i^{\Delta_r}$ and T_i^λ. If $m = 2r$ is even then:*

$$\sum_{i \in \mathbf{N}_r} \begin{bmatrix} s_1 \\ \vdots \\ s_{i-1} \end{bmatrix} [T_i^{\Delta_r;\lambda}] = 0, \tag{6.3.18a}$$

and if $m = 2r + 1$ is odd then:

$$\sum_{i \in \mathbf{N}_r \cup \{0\}} \phi(i) \begin{bmatrix} s_1 \\ \vdots \\ s_{i-1} \end{bmatrix} [T_i^{\Delta_r;\lambda}] = 0. \tag{6.3.18b}$$

Proof. From (6.1.4), γ_b may be expressed:

$$(\gamma_b)_{t_1\cdots t_r s_1 \cdots s_r} = \sqrt{2}\,\phi(b) \begin{bmatrix} s_1 \\ \vdots \\ s_{b-1} \end{bmatrix} \delta^{t_1}_{s_1} \cdots \delta^{t_{b-1}}_{s_{b-1}} \delta^{t_b}_{b} \delta^{\bar{b}}_{s_b} \delta^{t_{b+1}}_{s_{b+1}} \cdots \delta^{t_r}_{s_r}, \tag{6.3.18c}$$

for each $b \in \mathcal{I}^{O(m)}$. Then since $J_{ab} = \delta_{a\bar{b}}$, it follows that:

$$\sum_{a,b} \sum_{t_1,\ldots,t_r} J_{ab} (\gamma_b)_{t_1\cdots t_r s_1 \cdots s_r} \psi_{t_1\cdots t_r} \otimes x \otimes e_a \otimes y$$

$$= \sum_{a,b} \sum_{t_1,\ldots,t_r} \sqrt{2}\,\phi(b) \begin{bmatrix} s_1 \\ \vdots \\ s_{b-1} \end{bmatrix} \delta_{a\bar{b}} \delta^{t_1}_{s_1} \cdots \delta^{t_{b-1}}_{s_{b-1}} \delta^{t_b}_{b} \delta^{\bar{b}}_{s_b} \delta^{t_{b+1}}_{s_{b+1}} \cdots \delta^{t_r}_{s_r} \psi_{t_1\cdots t_r} \otimes x \otimes e_a \otimes y$$

$$= \sum_{a} \sum_{t_1,\ldots,t_r} \sqrt{2}\,\phi(a) \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \delta^{t_1}_{s_1} \cdots \delta^{t_{a-1}}_{s_{a-1}} \delta^{t_a}_{\bar{a}} \delta^{a}_{s_a} \delta^{t_{a+1}}_{s_{a+1}} \cdots \delta^{t_r}_{s_r}, \psi_{t_1\cdots t_r} \otimes x \otimes e_a \otimes y$$

$$= \sum_{a \in \mathcal{I}^{O(m)}} \sum_{t_1,\ldots,t_r} \sqrt{2}\,\phi(a) \begin{bmatrix} s_1 \\ \vdots \\ s_{a-1} \end{bmatrix} \delta_a^{s_a}\, \psi_{s_1 \cdots s_{a-1} \overline{s_a} s_{a+1} \cdots s_r} \otimes x \otimes e_{s_a} \otimes y. \quad (6.3.18d)$$

This shows that:

$$\sum_i \phi(i) \begin{bmatrix} s_1 \\ \vdots \\ s_{i-1} \end{bmatrix} T_i^{\Delta_r;\lambda} \in \hat{U},$$

where the sum is over N_r or $\mathsf{N}_r \cup \{0\}$ as appropriate. Then, in each case, the place permutation action on the T_i^λ portion by each summand of the Young symmetriser Y_*^λ, produces a similar term in $\hat{U}$ with appropriate changes of the position (a, b). In both cases it thus follows that:

$$\sum_i \phi(i) \begin{bmatrix} s_1 \\ \vdots \\ s_{i-1} \end{bmatrix} \left\{T_i^{\Delta_r;\lambda}\right\} \in \hat{U}.$$

Thereupon (6.3.18a) and (6.3.18b) follow from the definition of $[T^{\Delta_r;\lambda}]$ as a quotient.

In this chapter, each identity of the type (6.3.18a) or (6.3.18b) will be known as a Spinor relation. To demonstrate such a relation, let $m = 11$, $\lambda = (1^3)$ and $(s_1, \ldots, s_r) = (1, \bar{2}, 3, \bar{4}, \bar{5})$ Lemma (6.3.18) then implies that:

$$\begin{bmatrix} \bar{1} \cdot \bar{2} \\ \bar{2} \cdot 5 \\ 3 \cdot 1 \\ \bar{4} \cdot \\ \bar{5} \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{2} \\ 2 \cdot 5 \\ 3 \cdot \bar{2} \\ \bar{4} \cdot \\ \bar{5} \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 5 \\ \bar{3} \cdot 3 \\ \bar{4} \cdot \\ \bar{5} \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 5 \\ 3 \cdot \bar{4} \\ 4 \cdot \\ \bar{5} \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 5 \\ 3 \cdot \bar{5} \\ \bar{4} \cdot \\ 5 \cdot \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 5 \\ 3 \cdot 0 \\ \bar{4} \cdot \\ \bar{5} \cdot \end{bmatrix} = 0. \quad (6.3.19)$$

Note that the second term here is identically zero by virtue of the Column relations (3.4.2). In addition, note that the removal of the sixth term results in a valid Spinor relation for the case $m = 10$.

It will prove useful at this stage to generalise Definition 5.1.4 to the half tableaux appearing in this chapter.

Definition 6.3.20. *$O(m)$-weight. Let $m = 2r$ or $m = 2r + 1$ and let $T^{\Delta_r;\lambda}$ be formed by adjoining T^{Δ_r} and T^λ. For $i = 1, 2, \ldots, r$, define*

$$n_i^{O(m)}(T^{\Delta_r;\lambda}) = n_i(T^\lambda) - n_{\bar{i}}(T^\lambda) + \frac{1}{2}(n_i(T^{\Delta_r}) - n_{\bar{i}}(T^{\Delta_r})),$$

where $n_j(T^\lambda)$ is the number of appearances of the index $j \in \mathcal{I}^{O(m)}$ in T^λ and $n_j(T^{\Delta_r})$ ($\in \{0, 1\}$) is the number of appearances of the index $j \in \mathcal{I}^{O(m)}$ in T^{Δ_r}. The vector

$n^{O(m)}(T^{\Delta_r;\lambda}) = (n_1^{O(m)}(T^{\Delta_r;\lambda}), n_2^{O(m)}(T^{\Delta_r;\lambda}), \ldots, n_r^{O(m)}(T^{\Delta_r;\lambda}))$ *is known as the* $O(m)$-*weight of* $T^{\Delta_r;\lambda}$.

Notice that each of the tableaux appearing in (6.3.18a) and (6.3.18b) have the same $O(m)$-weight. This fact will be useful in developing the analogy of Lemma 5.2.4.

§6.4. Standardisation in the irreducible spinor modules

In this section, the Spinor relation is applied simultaneously over a number of positions in a column of a tableau to provide an analogue of Lemma 5.2.4. Having defined suitable sets of standard tableaux, a standardisation algorithm is developed.

Let Q_q^p denote the algebra generated by q elements $\mathring{\zeta}_1, \mathring{\zeta}_2, \ldots, \mathring{\zeta}_q$, for which $\mathring{\zeta}_1 + \mathring{\zeta}_2 + \cdots + \mathring{\zeta}_q = 0$ and $\mathring{\zeta}_i^p = 0$ for $1 \leq i \leq q$. As demonstrated in Section 4.3 and first noted by Berele [**Be86**], there is an intimate association between the construction of Q_q^2 and the construction of the irreducible $Sp(2r)$-modules B^λ. The lemma that follows shows that there exists a similar association between the algebras Q_q^3 and the irreducible spinor $O(m)$-modules $O^{\Delta_r;\lambda}$. Here, the expressions of greatest interest are those of the form:

$$\mathring{y}\left(\mathring{\zeta}_1 + \mathring{\zeta}_2 + \cdots + \mathring{\zeta}_{q-t}\right)^{2t+1} = 0, \tag{6.4.1}$$

where $\mathring{y}$ is a homogeneous polynomial in Q_q^3 of a specific form.

In this section, it will be convenient to define even $m = 2r$, odd $m = 2r+1$, and $r' = m - r$. $\mathcal{I}'_m$ is defined by $\mathcal{I}'_m = \mathsf{N}_r$ if m is even and $\mathcal{I}'_m = \mathsf{N}_r \cup \{0\}$ if m is odd. Thus $\#\mathcal{I}'_m = r'$ and $\mathcal{I}^{O(m)} = \mathcal{I}'_m \cup \overline{\mathsf{N}}_r$.

Lemma 6.4.2. *Let k be such that $1 \leq k \leq \lambda_1$. Let $\mathcal{I}'_m = \mathcal{B}_1^\alpha \cup \mathcal{B}^\beta \cup \mathcal{E} \cup \mathcal{G} \cup \mathcal{H}$ be a union of disjoint sets such that $0 \in \mathcal{B}_1^\alpha \cup \mathcal{H}$ if $0 \in \mathcal{I}'_m$, and if $b_1^\alpha = \#\mathcal{B}_1^\alpha$, $b^\beta = \#\mathcal{B}^\beta$, $e = \#\mathcal{E}$, $g = \#\mathcal{G}$ and $h = \#\mathcal{H}$ then $\tilde{\lambda}_k = b_1^\alpha + b^\beta + 2e + u + v$ where $u > 2g$. Fix the indices $s_1, s_2, \ldots, s_r, s_0$ where $s_p \in \{\bar{p}, p\}$ for each $p \in \mathcal{I}'_m$. Let $\mathcal{B}_1^\alpha \cup \mathcal{H} = \{p_1, p_2, \ldots, p_{b_1^\alpha+h}\}$, let $\mathring{y}$ be a homogeneous polynomial of degree v in the elements of $\{\mathring{p} : p \in \mathcal{B}_1^\alpha \cup \mathcal{H}\}$, and consider the following homogeneous polynomial:*

$$\mathring{x} = \mathring{y}\left(\sum_{p \in \mathcal{B}_1^\alpha \cup \mathcal{H}} \mathring{p}\right)^u. \tag{6.4.2a}$$

Let $\mathring{x} = \sum_w \mathring{x}_w$, the sum being over various w with $\mathring{x}_w = \eta_w \mathring{p}_1^{\kappa_w(p_1)} \mathring{p}_2^{\kappa_w(p_2)} \cdots \mathring{p}_{b_1^\alpha+h}^{\kappa_w(p_{b_1^\alpha+h})}$ where $\eta_w \in \mathbf{Z}$ is non-zero, $\kappa_w(p) \in \{0,1,2\}$ if $p \in \mathcal{H}$, $\kappa_w(p) \in \{0,1\}$ if $p \in \mathcal{B}_1^\alpha \cup \{0\}$, *and* $\sum_{p\in\mathcal{B}_1^\alpha\cup\mathcal{H}}\kappa_w(p)=u+v$. *Let the tableaux* T_w^λ *be identical except for* $u+v$ *positions in the* k*th column for which* T_w^λ *contains the indices*

$$\mathcal{Q}_w=\{\bar{p},p:p\in\mathcal{B}_1^\alpha\cup\mathcal{H},\kappa_w(p)=2\}\cup\{s_p:p\in\mathcal{B}_1^\alpha\cup\mathcal{H},\kappa_w(p)=1\}, \qquad (6.4.2b)$$

in column strict order. In the other $b_1^\alpha+b^\beta+2e$ *positions reside the indices* $\{\bar{p},p:p\in\mathcal{E}\}\cup\{\bar{s}_p:p\in\mathcal{B}_1^\alpha\}\cup\{s_p:p\in\mathcal{B}^\beta\}$. *For each* w, *form the tableau* $T_w^{\Delta_r}$ *by setting* $T_{w(p,0)}^{\Delta_r}=\bar{s}_p$ *if* $p\in\mathcal{B}_1^\alpha\cup\mathcal{H}$ *and* $\kappa_w(p)=1$, *and setting* $T_{w(p,0)}^{\Delta_r}=s_p$ *otherwise. If each* $T_w^{\Delta_r;\lambda}$ *is obtained by adjoining* $T_w^{\Delta_r}$ *and* T_w^λ, *then:*

$$\sum_w\eta_w\left(\prod_{\substack{p\in\mathcal{B}_1^\alpha\cup\mathcal{H}\\ \kappa_w(p)=2}}[s_p]\right)\left(\prod_{\substack{p\in\mathcal{B}_1^\alpha\cup\mathcal{H}\\ \kappa_w(p)=1}}\phi(p)\begin{bmatrix}s_1\\ \vdots\\ s_{p-1}\end{bmatrix}\right)[T_w^{\Delta_r;\lambda}]=0. \qquad (6.4.2c)$$

Proof. The following proof deals with the case of odd $m=2r+1$. The proof for even $m=2r$ may be obtained by ignoring all reference to the index '0'.

The proof follows the strategy of applying the Spinor relation repeatedly over the $u+v$ positions in the kth column. Although the order in which these are applied is irrelevant, it is useful to consider them from the bottom up.

For the moment ignore signs. The Spinor relation may then be represented by:

$$\mathring{1}+\mathring{2}+\mathring{3}+\cdots+\mathring{r}+\mathring{0}=0, \qquad (6.4.2d)$$

where each of the distinguished integers denotes an index which appears in some fixed position in the kth column of the F^λ portions of the respective half tableaux, as in (6.3.19) for example. Each $\mathring{p}$ appearing in this position is to be replaced by either the index p or the index $\bar{p}$ depending on whether the index s_p is unbarred or barred. However, repeated application of (6.4.2d) over different positions within the same kth column, results in the indices in the half box positions being altered. Then a subsequent appearance of $\mathring{p}$ will correspond to the opposite (barred or unbarred) index to that which first appeared. A further subsequent appearance of $\mathring{p}$ will result in the reappearance of the first index. Since these occur in the same column of a symmetrised tableau, the term vanishes. Thus the rule, in taking powers of (6.4.2d), is that terms containing cubed factors are annihilated, those containing squared factors correspond to barred/unbarred pairs, whereas the single power factors correspond to the indices $s_1,s_2,\ldots,s_r,s_0$. In addition, note that $\mathring{0}^2$ is annihilated.

Let $\mathcal{H}' = \mathcal{H} \cup \mathcal{B}_1^{\alpha} \cup \mathcal{B}^{\beta} \cup \mathcal{E}$ and $h' = \#\mathcal{H}'$. Then splitting (6.4.2d) with respect to $\mathcal{I}'_m = \mathcal{H}' \cup \mathcal{G}$ yields:

$$\sum_{p \in \mathcal{H}'} \mathring{p} = -\sum_{p \in \mathcal{G}} \mathring{p}. \tag{6.4.2e}$$

Raising each side to the power of u annihilates the right side because $\mathcal{G}$ has g elements and, since $u > 2g$, one of these must be cubed in every term. Therefore:

$$\left(\sum_{p \in \mathcal{H}'} \mathring{p} \right)^u = 0, \tag{6.4.2f}$$

and thus

$$\mathring{y} \left(\sum_{p \in \mathcal{H}'} \mathring{p} \right)^u = 0. \tag{6.4.2g}$$

If it is assumed that the indices $\mathring{p}$ commute with one another, then each term arising from (6.4.2g) is of the form:

$$\eta_w \, \mathring{p}_1^{\kappa_w(p_1)} \mathring{p}_2^{\kappa_w(p_2)} \cdots \mathring{p}_{h'}^{\kappa_w(p_{h'})}, \tag{6.4.2h}$$

where $\eta_w \in \mathbb{Z}$, $\kappa_w(p) \in \{0,1,2\}$ for each $p \in \mathcal{H}'$, $\kappa_w(0) \in \{0,1\}$ and $\sum_{p \in \mathcal{H}'} \kappa_w(p) = u + v$. In order to show that the terms $\mathring{p}$ do commute, it will be shown that the η_w symmetrised tableaux corresponding to the term (6.4.2h) in (6.4.2g) are equal. In addition, the sign associated with this term will be calculated.

On defining $\mathcal{D}_w = \{p \in \mathcal{H}' : \kappa_w(p) = 2\}$ and $\mathcal{B}_w = \{p \in \mathcal{H}' : \kappa_w(p) = 1\}$, it may be seen from (6.3.18d) that the tableau corresponding to the term (6.4.2h) possesses the indices

$$\mathcal{Q}_w = \{p, \bar{p} : p \in \mathcal{D}_w\} \cup \{s_p : p \in \mathcal{B}_w\} \tag{6.4.2i}$$

in the kth column of the F^{λ} portion. Then those in the F^{Δ_r} portion are given by $\{\bar{s}_p : p \in \mathcal{B}_w\} \cup \{s_p : p \notin \mathcal{B}_w\}$, as may be inferred through the constant $O(m)$-weight. Assume that the indices from $\mathcal{Q}_w$ have been generated by applying the Spinor relation, in the guise of (6.4.2g), over $u + v$ boxes, one box at a time, beginning with the lowest relevant box. Consider the one tableau $T_w^{\Delta_r;\lambda}$ arising from (6.4.2h) with the indices from $\mathcal{Q}_w$ in column strict order. This tableau may be generated by first choosing the largest factor from those of (6.4.2h) remaining at each Spinor relation. From (6.3.18a/b), each index $\mathring{p}$ has a sign factor

$$\begin{bmatrix} s_1 \\ \vdots \\ s_{p-1} \end{bmatrix} \tag{6.4.2j}$$

associated with it. Dealing with the largest indices first, ensures that these sign factors are not interfered with by earlier indices. If $\kappa_w(p) = 2$ then the same sign factor occurs twice and thus a cancellation occurs in this case. The sign factors for the $\kappa_w(p) = 1$ cases remain. By virtue of the order in which the indices were selected, the resultant tableau is column strict on $\mathcal{Q}_w$, apart from those indices $s_p, \bar{s}_p$ for which $\kappa_w(p) = 2$ and $s_p = \bar{p}$. Then $\bar{p}$ will be below p. Accounting for this in each case, through the Column relations, generates the sign factor:

$$\prod_{p \in \mathcal{D}_w} [s_p]. \tag{6.4.2k}$$

By Lemma 6.3.18, a factor of $1/\sqrt{2}$ is also associated with the index '0'. Provided that it can be shown that if the tableau arising from selecting the indices of (6.4.2h) in an order different to that above, gives rise to a column strict tableau of the same sign, then the coefficients appearing in (6.4.2c) have been explained.

Now consider the indices of (6.4.2h) taken in an arbitrary order. However, when $v(p) = 2$ and $\mathring{p}^2$ is substituted by s_p and $\bar{s}_p$ (not necessarily consecutively), the former will once more precede the latter and require the factor (6.4.22k). With this in mind, let the order be:

$$\mathring{p}_{\pi(1)}, \mathring{p}_{\pi(2)}, \ldots, \mathring{p}_{\pi(q)}, \mathring{p}_{\pi(q+1)}, \ldots, \mathring{p}_{\pi(t)}, \tag{6.4.2l}$$

where $p_{\pi(q)} \neq p_{\pi(q+1)}$. The sign associated with this term is:

$$\begin{bmatrix} s_1^{(1)} \\ \vdots \\ s_{p_{\pi(1)}-1}^{(1)} \end{bmatrix} \begin{bmatrix} s_1^{(2)} \\ \vdots \\ s_{p_{\pi(2)}-1}^{(2)} \end{bmatrix} \cdots \begin{bmatrix} s_1^{(q)} \\ \vdots \\ s_{p_{\pi(q)}-1}^{(q)} \end{bmatrix} \begin{bmatrix} s_1^{(q+1)} \\ \vdots \\ s_{p_{\pi(q+1)}-1}^{(q+1)} \end{bmatrix} \cdots \begin{bmatrix} s_1^{(t)} \\ \vdots \\ s_{p_{\pi(t)}-1}^{(t)} \end{bmatrix}, \tag{6.4.2m}$$

where $s_b^{(1)} = s_b$ for $b = 1, 2, \ldots, r$, and each vector $(s_1^{(c+1)}, \ldots, s_r^{(c+1)})$ differs from $(s_1^{(c)}, \ldots, s_r^{(c)})$ only in the one component for which $s_{p_{\pi(c)}}^{(c+1)} = \bar{s}_{p_{\pi(c)}}^{(c)}$. Now consider the order:

$$\mathring{p}_{\pi(1)}, \mathring{p}_{\pi(2)}, \ldots, \mathring{p}_{\pi(q+1)}, \mathring{p}_{\pi(q)}, \ldots, \mathring{p}_{\pi(t)}, \tag{6.4.2n}$$

where, with respect to (6.4.2l), $\mathring{p}_{\pi(q)}$ and $\mathring{p}_{\pi(q+1)}$ have swapped places. The sign associated with this term is:

$$\begin{bmatrix} s_1^{(1)} \\ \vdots \\ s_{p_{\pi(1)}-1}^{(1)} \end{bmatrix} \begin{bmatrix} s_1^{(2)} \\ \vdots \\ s_{p_{\pi(2)}-1}^{(2)} \end{bmatrix} \cdots \begin{bmatrix} s_1^{(q)'} \\ \vdots \\ s_{p_{\pi(q+1)}-1}^{(q)'} \end{bmatrix} \begin{bmatrix} s_1^{(q+1)'} \\ \vdots \\ s_{p_{\pi(q)}-1}^{(q+1)'} \end{bmatrix} \cdots \begin{bmatrix} s_1^{(t)} \\ \vdots \\ s_{p_{\pi(t)}-1}^{(t)} \end{bmatrix}, \tag{6.4.2o}$$

where $(s_1^{(q+1)'}, \ldots, s_r^{(q+1)'})$ differs from $(s_1^{(q)}, \ldots, s_r^{(q)})$ only in the one component for which $s_{p_{\pi(q+1)}}^{(q+1)'} = \bar{s}_{p_{\pi(q+1)}}^{(q)}$ and $(s_1^{(q)'}, \ldots, s_r^{(q)'})$ differs from $(s_1^{(q+1)}, \ldots, s_r^{(q+1)})$ only in the one component for which $s^{(q)'}_{p_{\pi(q)}} = \bar{s}^{(q+1)}_{p_{\pi(q)}}$. Since either $p_{\pi(q)} > p_{\pi(q+1)}$ or $p_{\pi(q)} < p_{\pi(q+1)}$, only one of these differences manifests itself as a difference between (6.4.2m) and (6.4.2o). Consequently, these two sign factors differ by a factor of -1. Therefore, since the tableaux resulting from (6.4.2l) and (6.4.2n) differ by a simple transposition, the two resultant symmetrised tableaux are equal. Since the transpositions of adjacent unequal elements generate the whole set of terms from (6.4.2h), it follows that the factors of (6.4.2h) commute with one another and that the resulting symmetrised column strict tableau has a multiplicity η_w.

The lemma is now proved by noting that, apart from those from the set $\mathcal{Q}_w$, the indices that reside in the kth column are given by $\theta^{\mathcal{B}_{\bar{1}}^{\alpha}} \theta^{\mathcal{B}^{\beta}} \theta^{\mathcal{E}}$, where $\theta^{\mathcal{B}_{\bar{1}}^{\alpha}} = \prod_{q \in \mathcal{B}_{\bar{1}}^{\alpha}} \bar{s}_q$, $\theta^{\mathcal{B}^{\beta}} = \prod_{q \in \mathcal{B}^{\beta}} s_q$ and $\theta^{\mathcal{E}} = \prod_{q \in \mathcal{E}} \bar{q}q$. Therefore, if $\kappa_w(p) \geq 1$ for any $p \in \mathcal{B}^{\beta} \cup \mathcal{E}$ or $\kappa_w(p) = 2$ for any $p \in \mathcal{B}_{\bar{1}}^{\alpha}$, then $T_w^{\Delta_r;\lambda}$ is zero by virtue of a repeated index in the kth column. Then, since $\mathcal{H}' = \mathcal{H} \cup \mathcal{B}_{\bar{1}}^{\alpha} \cup \mathcal{B}^{\beta} \cup \mathcal{E}$, the remaining terms are those in the statement of the lemma.

The tableaux identity resulting from this lemma may be conveniently expressed thus:

$$\theta^{\mathcal{B}_{\bar{1}}^{\alpha}} \theta^{\mathcal{B}^{\beta}} \theta^{\mathcal{E}} \mathring{y} \left(\sum_{p \in \mathcal{B}_{\bar{1}}^{\alpha} \cup \mathcal{H}} \mathring{p} \right)^u = 0, \tag{6.4.3}$$

each non-zero term of which yields the appropriate term of (6.4.2c). The tableaux identities are readily obtained from the algebras Q_q^3 with various sign factors introduced. In addition, for those tableaux in which the index '0' appears, an extra factor of $1/\sqrt{2}$ is necessary. The identity $\mathring{0}^2 = 0$ is also used. The following example exhibits this construction.

Let $m = 12$ so that $r = r' = 6$, let $u = 5$, $v = 0$, $\mathring{y} = 1$, $\lambda = (1,1,1,1,1)$, $(s_1, s_2, \ldots, s_6) = (1, \bar{2}, \bar{3}, 4, \bar{5}, 6)$, $\mathcal{B}_{\bar{1}}^{\alpha} = \mathcal{B}^{\beta} = \mathcal{E} = \emptyset$, $\mathcal{G} = \{1,2\}$ and $\mathcal{H}' = \mathcal{H} = \{3,4,5,6\}$. In accordance with (6.4.2e), the Spinor relation is split:

$$-\mathring{1} - \mathring{2} = \mathring{3} + \mathring{4} + \mathring{5} + \mathring{6}. \tag{6.4.4a}$$

Raising this to the power of $u = 5$ annihilates the left side by virtue of cubed terms. Thereupon:

$$\begin{aligned} 30.(&\mathring{3}^2\mathring{4}^2\mathring{5} + \mathring{3}^2\mathring{4}^2\mathring{6} + \mathring{3}^2\mathring{4}\mathring{5}^2 + \mathring{3}^2\mathring{5}^2\mathring{6} + \mathring{3}^2\mathring{4}\mathring{6}^2 + \mathring{3}^2\mathring{5}\mathring{6}^2 + \mathring{3}\mathring{4}^2\mathring{5}^2 + \mathring{3}\mathring{4}^2\mathring{6}^2 + \mathring{3}\mathring{5}^2\mathring{6}^2 \\ &+ \mathring{4}^2\mathring{5}^2\mathring{6} + \mathring{4}^2\mathring{5}\mathring{6}^2 + \mathring{4}\mathring{5}^2\mathring{6}^2) + 60.(\mathring{3}^2\mathring{4}\mathring{5}\mathring{6} + \mathring{3}\mathring{4}^2\mathring{5}\mathring{6} + \mathring{3}\mathring{4}\mathring{5}^2\mathring{6} + \mathring{3}\mathring{4}\mathring{5}\mathring{6}^2) = 0. \end{aligned} \tag{6.4.4b}$$

Now consider the term $\mathring{3}^2\mathring{5}\mathring{6}^2$. Applying the Spinor relation once to the lowest box of T^{λ} deposits the index 6 there together with the sign factor -1. The indices in the half boxes are now $(1, \bar{2}, \bar{3}, 4, \bar{5}, \bar{6})$. Repeating this process using the next box up and then subsequent boxes, generates the following sequence of traceless symmetrised tableaux:

$$\begin{bmatrix} 1 \cdot \times \\ \bar{2} \cdot \times \\ \bar{3} \cdot \times \\ 4 \cdot \times \\ \bar{5} \cdot \times \\ 6 \cdot \end{bmatrix} \to - \begin{bmatrix} 1 \cdot \times \\ \bar{2} \cdot \times \\ \bar{3} \cdot \times \\ 4 \cdot \times \\ \bar{5} \cdot 6 \\ \bar{6} \cdot \end{bmatrix} \to \begin{bmatrix} 1 \cdot \times \\ \bar{2} \cdot \times \\ \bar{3} \cdot \times \\ 4 \cdot \bar{6} \\ \bar{5} \cdot 6 \\ 6 \cdot \end{bmatrix} \to \begin{bmatrix} 1 \cdot \times \\ \bar{2} \cdot \times \\ \bar{3} \cdot \bar{5} \\ 4 \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix} \to - \begin{bmatrix} 1 \cdot \times \\ \bar{2} \cdot \bar{3} \\ 3 \cdot \bar{5} \\ 4 \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix} \to$$

$$\begin{bmatrix} 1 \cdot 3 \\ \bar{2} \cdot \bar{3} \\ \bar{3} \cdot \bar{5} \\ \bar{4} \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix} = - \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot \bar{5} \\ \bar{4} \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix}. \qquad (6.4.4c)$$

The sign resulting here corresponds to that given by (6.4.2*c*). It may be confirmed that the same term results on taking any of the other 29 permutations of the indices $\mathring{3}\mathring{3}\mathring{5}\mathring{6}\mathring{6}$. Calculating the sign for each term of (6.4.4*b*) yields:

$$\begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot \bar{4} \\ 4 \cdot 4 \\ 5 \cdot \bar{5} \\ 6 \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot \bar{4} \\ 4 \cdot 4 \\ \bar{5} \cdot 6 \\ \bar{6} \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot 4 \\ \bar{4} \cdot \bar{5} \\ \bar{5} \cdot 5 \\ 6 \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot \bar{5} \\ 4 \cdot 5 \\ \bar{5} \cdot 6 \\ \bar{6} \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot 4 \\ \bar{4} \cdot \bar{6} \\ \bar{5} \cdot 6 \\ 6 \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot \bar{5} \\ 4 \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix}$$

$$- \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot \bar{4} \\ 3 \cdot 4 \\ 4 \cdot \bar{5} \\ \bar{5} \cdot 5 \\ 6 \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot \bar{4} \\ 3 \cdot 4 \\ 4 \cdot \bar{6} \\ \bar{5} \cdot 6 \\ 6 \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot \bar{5} \\ 3 \cdot 5 \\ 4 \cdot \bar{6} \\ \bar{5} \cdot 6 \\ 6 \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{4} \\ \bar{2} \cdot 4 \\ \bar{3} \cdot \bar{5} \\ 4 \cdot 5 \\ \bar{5} \cdot 6 \\ \bar{6} \cdot \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{4} \\ \bar{2} \cdot 4 \\ \bar{3} \cdot \bar{5} \\ 4 \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix} + \begin{bmatrix} 1 \cdot 4 \\ \bar{2} \cdot \bar{5} \\ \bar{3} \cdot 5 \\ \bar{4} \cdot \bar{6} \\ \bar{5} \cdot 6 \\ 6 \cdot \end{bmatrix}$$

$$- 2 \begin{bmatrix} 1 \cdot \bar{3} \\ \bar{2} \cdot 3 \\ \bar{3} \cdot 4 \\ \bar{4} \cdot \bar{5} \\ 5 \cdot 6 \\ \bar{6} \cdot \end{bmatrix} - 2 \begin{bmatrix} 1 \cdot 3 \\ \bar{2} \cdot \bar{4} \\ 3 \cdot 4 \\ 4 \cdot \bar{5} \\ 5 \cdot 6 \\ \bar{6} \cdot \end{bmatrix} + 2 \begin{bmatrix} 1 \cdot 3 \\ \bar{2} \cdot 4 \\ 3 \cdot \bar{5} \\ \bar{4} \cdot 5 \\ \bar{5} \cdot 6 \\ \bar{6} \cdot \end{bmatrix} + 2 \begin{bmatrix} 1 \cdot 3 \\ \bar{2} \cdot 4 \\ 3 \cdot \bar{5} \\ \bar{4} \cdot \bar{6} \\ 5 \cdot 6 \\ 6 \cdot \end{bmatrix} = 0. \qquad (6.4.4d)$$

As a further example, let $m = 13$ so that $r = 6$ and $r' = 7$, let $\lambda = (1,1,1,1,1,1)$, $u = 3$, $v = 1$, $\mathring{y} = 2.\mathring{5} - \mathring{6} + \mathring{0}$, $(s_1, s_2, \ldots, s_6) = (1, 2, 3, \bar{4}, \bar{5}, \bar{6})$, $\mathcal{B}_1^\alpha =$

$\{5\}$, $\mathcal{B}^{\beta} = \{4\}$, $\mathcal{E} = \emptyset$, $\mathcal{G} = \{2\}$ and $\mathcal{H} = \{1,3,6,0\}$ so that $\mathcal{H}' = \{1,3,4,5,6,0\}$. Then raising the Spinor relation,

$$-\mathring{2} = \mathring{1} + \mathring{3} + \mathring{4} + \mathring{5} + \mathring{6} + \mathring{0}, \tag{6.4.5a}$$

to the power of $u = 3$ annihilates the left side. Since $\mathring{0}^2 = 0$ and each $\mathring{p}^3 = 0$, the resulting right side consists of 25 distinct terms of the form $3\mathring{p}_1^2\mathring{p}_2$ with $p_1 \neq p_2$, and 20 distinct terms of the form $6\mathring{p}_1\mathring{p}_2\mathring{p}_3$ with $p_1 < p_2 < p_3$. However, in addition to the indices arising from these terms, each tableau is to contain the index $\bar{4}$ in the first column. Since $s_4 = \bar{4}$, those terms arising from (6.4.5a) which contain $\mathring{4}$ may be ignored. Furthermore, each tableau is to contain the index 5 in the first column. Since $s_5 = \bar{5}$, those terms containing $\mathring{5}^2$ may be ignored. This leaves 12 distinct terms of the form $3\mathring{p}_1^2\mathring{p}_2$ with $p_1 \neq p_2$, and 10 distinct terms of the form $6\mathring{p}_1\mathring{p}_2\mathring{p}_3$ with $p_1 < p_2 < p_3$:

$$\begin{aligned} &3.(\mathring{1}^2\mathring{3} + \mathring{1}^2\mathring{5} + \mathring{1}^2\mathring{6} + \mathring{1}^2\mathring{0} + \mathring{1}\mathring{3}^2 + \mathring{3}^2\mathring{5} + \mathring{3}^2\mathring{6} + \mathring{3}^2\mathring{0} + \mathring{1}\mathring{6}^2 + \mathring{3}\mathring{6}^2 + \mathring{5}\mathring{6}^2 + \mathring{6}^2\mathring{0}) \\ &\quad + 6.(\mathring{1}\mathring{3}\mathring{5} + \mathring{1}\mathring{3}\mathring{6} + \mathring{1}\mathring{3}\mathring{0} + \mathring{1}\mathring{5}\mathring{6} + \mathring{1}\mathring{5}\mathring{0} + \mathring{1}\mathring{6}\mathring{0} + \mathring{3}\mathring{5}\mathring{6} + \mathring{3}\mathring{5}\mathring{0} + \mathring{3}\mathring{6}\mathring{0} + \mathring{5}\mathring{6}\mathring{0}) = 0. \end{aligned} \tag{6.4.5b}$$

Multiplying this by $\mathring{y} = 2.\mathring{5} - \mathring{6} + \mathring{0}$, and discarding terms containing $\mathring{0}^2$, $\mathring{5}^2$ or $\mathring{6}^3$ as above, results in:

$$\begin{aligned} &3.(\mathring{1}^2\mathring{5}\mathring{6} + \mathring{3}^2\mathring{5}\mathring{6} + \mathring{5}\mathring{6}^2\mathring{0} - \mathring{1}^2\mathring{3}\mathring{6} - \mathring{1}^2\mathring{6}^2 - \mathring{1}\mathring{3}^2\mathring{6} - \mathring{3}^2\mathring{6}^2 - \mathring{1}\mathring{6}^2\mathring{0} \\ &\quad - \mathring{3}\mathring{6}^2\mathring{0} + \mathring{1}^2\mathring{3}\mathring{0} + \mathring{1}\mathring{3}^2\mathring{0}) + 6.(\mathring{1}^2\mathring{3}\mathring{5} + \mathring{1}\mathring{3}^2\mathring{5} + \mathring{1}\mathring{3}\mathring{5}\mathring{6} - \mathring{1}\mathring{3}\mathring{6}^2) \\ &\qquad + 9.(\mathring{1}^2\mathring{5}\mathring{0} + \mathring{3}^2\mathring{5}\mathring{0}) + 12.(\mathring{1}\mathring{5}\mathring{6}\mathring{0} + \mathring{3}\mathring{5}\mathring{6}\mathring{0}) + 18.\mathring{1}\mathring{3}\mathring{5}\mathring{0} = 0. \end{aligned} \tag{6.4.5c}$$

From this identity, a tableau results from each term by replacing each $\mathring{p}$ by s_p, each $\mathring{p}^2$ by $\bar{s}_p s_p$, and appending $\theta^{\mathcal{B}^{\alpha}_{1}}\theta^{\mathcal{B}^{\beta}}\theta^{\mathcal{E}} = \bar{4}5$ to form the first column. The indices in the half boxes and the coefficients are calculated as before. The result is:

$$
\begin{aligned}
&3\begin{bmatrix} 1\cdot\bar{1} \\ 2\cdot 1 \\ 3\cdot\bar{6} \\ \bar{4}\cdot 6 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-3\begin{bmatrix} 1\cdot\bar{1} \\ 2\cdot 1 \\ 3\cdot\bar{5} \\ \bar{4}\cdot\bar{6} \\ 5\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix}
-3\begin{bmatrix} 1\cdot\bar{3} \\ 2\cdot 3 \\ 3\cdot\bar{5} \\ \bar{4}\cdot\bar{6} \\ 5\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix}
-\frac{3}{\sqrt{2}}\begin{bmatrix} 1\cdot\bar{5} \\ 2\cdot\bar{6} \\ 3\cdot 6 \\ \bar{4}\cdot 0 \\ 5\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-3\begin{bmatrix} 1\cdot\bar{1} \\ 2\cdot 1 \\ \bar{3}\cdot 3 \\ \bar{4}\cdot\bar{6} \\ \bar{5}\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix}
-3\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot\bar{3} \\ 3\cdot 3 \\ \bar{4}\cdot\bar{6} \\ \bar{5}\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix} \\
&+3\begin{bmatrix} 1\cdot\bar{3} \\ 2\cdot 3 \\ 3\cdot\bar{6} \\ \bar{4}\cdot 6 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-\frac{3}{\sqrt{2}}\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot\bar{6} \\ 3\cdot 6 \\ \bar{4}\cdot 0 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-\frac{3}{\sqrt{2}}\begin{bmatrix} 1\cdot 3 \\ 2\cdot\bar{6} \\ \bar{3}\cdot 6 \\ \bar{4}\cdot 0 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-\frac{3}{\sqrt{2}}\begin{bmatrix} 1\cdot\bar{1} \\ 2\cdot 1 \\ \bar{3}\cdot 3 \\ \bar{4}\cdot 0 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-\frac{3}{\sqrt{2}}\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot\bar{3} \\ 3\cdot 3 \\ \bar{4}\cdot 0 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix} \\
&-6\begin{bmatrix} 1\cdot\bar{1} \\ 2\cdot 1 \\ \bar{3}\cdot 3 \\ \bar{4}\cdot\bar{5} \\ 5\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-6\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot\bar{3} \\ 3\cdot 3 \\ \bar{4}\cdot\bar{5} \\ 5\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
-6\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot 3 \\ \bar{3}\cdot\bar{5} \\ \bar{4}\cdot\bar{6} \\ 5\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix}
+6\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot 3 \\ \bar{3}\cdot\bar{6} \\ \bar{4}\cdot 6 \\ \bar{5}\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
+\frac{9}{\sqrt{2}}\begin{bmatrix} 1\cdot\bar{1} \\ 2\cdot 1 \\ 3\cdot\bar{5} \\ \bar{4}\cdot 0 \\ 5\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix} \\
&+\frac{9}{\sqrt{2}}\begin{bmatrix} 1\cdot\bar{3} \\ 2\cdot 3 \\ 3\cdot\bar{5} \\ \bar{4}\cdot 0 \\ 5\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix}
+\frac{12}{\sqrt{2}}\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot\bar{5} \\ 3\cdot\bar{6} \\ \bar{4}\cdot 0 \\ 5\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix}
+\frac{12}{\sqrt{2}}\begin{bmatrix} 1\cdot 3 \\ 2\cdot\bar{5} \\ \bar{3}\cdot\bar{6} \\ \bar{4}\cdot 0 \\ 5\cdot\bar{4} \\ 6\cdot 5 \end{bmatrix}
+\frac{18}{\sqrt{2}}\begin{bmatrix} \bar{1}\cdot 1 \\ 2\cdot 3 \\ \bar{3}\cdot\bar{5} \\ \bar{4}\cdot 0 \\ 5\cdot\bar{4} \\ \bar{6}\cdot 5 \end{bmatrix} = 0.
\end{aligned}
\tag{6.4.5d}
$$

Of course, the Column relations may now be used to make each term column strict.

Lemma (6.4.2) now enables a standardisation algorithm to be developed. This is based on the following favoured sets of tableaux first obtained by King and El-Sharkaway. Here the elements of $\mathcal{I}^{O(m)}$ are ordered as in Chapter 5.

Definition 6.4.6. [**KE83**] *Let m be such that even $m = 2r$ and odd $m = 2r+1$, and let $T^{\Delta_r;\lambda}$ be obtained by adjoining T^{Δ_r} to T^{λ}. For $j \in \mathcal{I}^{O(m)}$, let α_j be the number of entries in the first column of T^{λ} less than or equal to j. Let $s_i = T^{\Delta_r}_{(i,0)}$ for $i = 1, 2, \ldots, r$, and let $s_{r+1} = 0$. The tableau $T^{\Delta_r;\lambda}$ is $O(m)$-standard if:*

(i) the entries are taken from the set $\mathcal{I}^{O(m)}$;

(ii) the entries are strictly increasing from top to bottom down each column of T^{λ};

(iii) the entries are non-decreasing from left to right across each row of T^{λ};

(iv) $s_j \in \{\bar{j}, j\}$ for $j = 1, 2, \ldots, r$;

(v) $\alpha_j \leq j$ *for* $j = 1, 2, \ldots, r$;

(vi) if $T^{\lambda}_{(j,k)} = \bar{s}_j$ *for* j *and* k *satisfying* $1 \leq k \leq \lambda_j$ *and* $1 \leq j \leq \tilde{\lambda}_1$, *then* $j > 1$ *and* $T^{\lambda}_{(j-1,k)} = s_j$.

Note that condition (vi) implies that if $T^{\Delta_r;\lambda}$ is $O(m)$-standard then $T^{\lambda}_{(r+1,1)} \neq 0$. This, together with conditions (ii), (iii) and (v), implies that there exist no $O(m)$-standard tableaux $T^{\Delta_r;\lambda}$ when $\tilde{\lambda}_1 > r$. Also note that if $j > 1$, $T^{\lambda}_{(j,0)} = j$, $T^{\lambda}_{(j,k)} = \bar{j}$ and $T^{\lambda}_{(j-1,k)} = j$ then condition (vi) is satisfied for that particular j and k, but however, condition (ii) is violated. It is also interesting to note that the first three conditions together with the 5th imply that if $T^{\Delta_r;\lambda}$ is $O(2r)$-standard then T^{λ} is $Sp(2r)$-standard. Definition 6.4.6 implies that of the tableaux:

$$\begin{array}{llll} \bar{1} \cdot 1 & 1 & 2 \\ 2 \cdot \bar{3} \\ 3 \cdot 3 \end{array}, \quad \begin{array}{llll} \bar{1} \cdot \bar{1} & \bar{3} & 3 \\ \bar{2} \cdot \bar{2} \\ 3 \cdot 2 \end{array} \quad \begin{array}{llll} 1 \cdot 1 & 2 & \bar{3} \\ 2 \cdot 2 \\ \bar{3} \cdot 3 \end{array}, \quad \text{and} \quad \begin{array}{llll} 1 \cdot 1 & 0 & 0 \\ 2 \cdot \bar{3} \\ \bar{3} \cdot 3 \end{array}, \tag{6.4.7}$$

only the last is $O(7)$-standard.

Theorem 6.4.8. [KE83] *Let* m *be such that even* $m = 2r$ *and odd* $m = 2r + 1$. *The multiplicity of the weight* $(n_1, n_2, \ldots, n_r)$ *in the irreducible representation* $[\Delta_r; \lambda]$ *of* $O(m)$ *is given by the number of* $O(m)$*-standard tableaux* $T^{\Delta_r;\lambda}$ *such that* $n^{O(m)}(T^{\Delta_r;\lambda}) = (n_1, n_2, \ldots, n_r)$. *The character of this representation is given by:*

$$[\Delta_r; \lambda](y) = \sum_{T^{\Delta_r;\lambda} : T^{\Delta_r;\lambda}\, O(m)-standard} y^{T^{\Delta_r;\lambda}}, \tag{6.4.8}$$

where $(y) = (y_1, y_2, \ldots, y_r)$ *and* $y^{T^{\Delta_r;\lambda}} = y_1^{n_1^{O(m)}(T^{\Delta_r;\lambda})} y_2^{n_2^{O(m)}(T^{\Delta_r;\lambda})} \cdots y_r^{n_r^{O(m)}(T^{\Delta_r;\lambda})}$, *for those elements of* $O(m)$ *with positive determinant and, if* $m = 2r$, *eigenvalues* $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r$, *and if* $m = 2r + 1$, *eigenvalues* $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r, 1$.

It is important to realise that here, each $y^{T^{\Delta_r;\lambda}}$ is a function of $y_1^{\frac{1}{2}}, y_2^{\frac{1}{2}}, \ldots, y_r^{\frac{1}{2}}$. Hence it is two valued and so is $[\Delta_r; \lambda]$. This reflects the fact that $[\Delta_r; \lambda]$ is a two-valued representation of $O(m)$.

As in previous chapters, a standardisation procedure will enable each non-standard tableau in to be written terms of tableaux which are higher; in the sense of Definition 2.6.23 in this instance. Once more, the Garnir relations and the Column relations may be applied to the F^{λ} portion to yield a linear combination of higher tableaux $T^{\Delta_r;\lambda}$ which satisfy conditions (i), (ii) and (iii) of Definition 6.4.6. It is thus only necessary to concentrate on violations of the remaining conditions. In order to use the techniques described below, it is necessary to determine certain polynomials.

Definition 6.4.9. *For positive odd u and non-negative integer v, define $\hat{t}_{(u,v)}$ to be a homogeneous polynomial of degree v in the $v+1$ variables $x_0, x_1, x_2, \ldots, x_v$ such that if $x_1^3 = x_2^3 = \cdots = x_v^3 = 0$ then the coefficients of each term in*

$$(x_0 + x_1 + x_2 + \cdots + x_v)^u \hat{t}_{(u,v)}(x_0; x_1, x_2, \ldots, x_v) \tag{6.4.9a}$$

having an exponent of any of $x_1, x_2, \ldots, x_v$ equal to 2 is zero, and in which the coefficient of $x_0^u x_1 x_2 \cdots x_v$ is positive.

Note that if $\hat{t}_{(u,v)}$ exists then any positive multiple also satisfies this definition. It is not yet known whether $\hat{t}_{(u,v)}$ exists for all u and v. However, Table 6.4.10 shows that $\hat{t}_{(u,v)}$ certainly exists for all $u + v \leq 8$.

u	v	$\hat{t}_{(u,v)}$
u	0	1
u	1	$2x_0 - (u-1)\square$
u	2	$4x_0^2 - 2(u-2)x_0\square - 2u\square_{(2)} + (u^2-u+2)\square_{(1,1)}$
u	3	$8x_0^3 - 4(u-3)x_0^2\square - 8ux_0\square_{(2)} + 2(u^2-3u+6)x_0\square_{(1,1)}$ $+2u(u-1)\square_{(2,1)} - (u-1)(u^2+u+6)\square_{(1,1,1)}$
u	4	$16x_0^4 - 8(u-4)x_0^3\square - 24ux_0^2\square_{(2)} + 4(u^2-5u+12)x_0^2\square_{(1,1)}$ $+8u(u-2)x_0\square_{(2,1)} - 2(u-2)(u^2-u+12)x_0\square_{(1,1,1)} + 8u(u+1)\square_{(2,2)}$ $-2u(u^2-u+4)\square_{(2,1,1)} + (u^4+2u^3+11u^2-14u+24)\square_{(1,1,1,1)}$
1	5	$x_0^3\square_{(1,1)} - x_0^2\square_{(2,1)} + 3x_0^2\square_{(1,1,1)} + 2x_0\square_{(2,2)} - x_0\square_{(2,1,1)} + 3x_0\square_{(1,1,1,1)}$
1	6	$x_0^3\square_{(1,1,1)} - x_0^2\square_{(2,1,1)} + 2x_0\square_{(2,2,1)} - 3x_0\square_{(2,1,1,1)} - 9x_0\square_{(1,1,1,1,1)}$ $-6\square_{(2,2,2)} + 2\square_{(2,2,1,1)} - 3\square_{(2,1,1,1,1)} + \square_{(1,1,1,1,1,1)}$
1	7	$x_0^4\square_{(1,1,1)} - x_0^3\square_{(2,1,1)} + 2x_0^2\square_{(2,2,1)} - 3x_0^2\square_{(2,1,1,1)} - 9x_0^2\square_{(1,1,1,1,1)}$ $-6x_0\square_{(2,2,2)} + 2x_0\square_{(2,2,1,1)} - 3x_0\square_{(2,1,1,1,1)} + x_0\square_{(1,1,1,1,1,1)}$

Table 6.4.10

In this table, each Young diagram F^λ represents the corresponding monomial symmetric function m_λ [Ma79] in the appropriate v variables $x_1, x_2, \ldots, x_v$. m_λ is defined to be the sum of all distinct terms $x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_p}^{\lambda_p}$, where p is the number of parts of λ. For example, if $v = 3$ then $\boxplus = m_{(2,1)} = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2$. The notation is especially convenient here since $\hat{t}_{(1,7)}$ comprises 763 terms!

If the tableau $T^{\Delta_r;\lambda}$ violates condition (v) of Definition 6.4.6 then, in order to invoke the following standardisation procedure, it is necessary to identify a specific j which violates condition (v) *and* for which both $\bar{j}$ and j ($j \neq 0$) are present in the offending column. A straightforward induction argument shows that if $\alpha_{j'} > j'$ then there necessarily exists $j \leq j'$ for which both $\bar{j}$ and j are present and $\alpha_j > j$.

Lemma 6.4.11. *Let $T^{\Delta_r;\lambda}$ be column strict but non-standard in that there exists j and k such that $\alpha_j > j$, where α_j is the number of entries less than j in the kth column of $T^{\Delta_r;\lambda}$, and such that both $\bar{j}$ and j are present in the kth column of $T^{\Delta_r;\lambda}$. If $\hat{t}_{(u,v)}$ exists for all $u + v < \tilde{\lambda}_1$, then $[T^{\Delta_r;\lambda}]$ may be expressed as a linear combination of traceless symmetrised tableaux $[T_w^{\Delta_r;\lambda}]$, where for each w, $T_w^{\Delta_r;\lambda} > T^{\Delta_r;\lambda}$.*

Proof. Let $s'_i = T^{\Delta_r;\lambda}_{(i,0)}$ for $i = 1, 2, \ldots, r$, and let $\mathcal{Q} \subset \mathcal{I}^{O(m)}$ be the set of indices in the kth column of $T^{\Delta_r;\lambda}$. Let $\mathcal{A} = \{p \in \mathbb{N}_r : \bar{p}, p \in \mathcal{Q}\}$, $\mathcal{B}^\alpha = \{p \in \mathbb{N}_r : \bar{s}'_p \in \mathcal{Q},\ s'_p \notin \mathcal{Q}\} \cup (\{0\} \cap \mathcal{Q})$, $\mathcal{B}^\beta = \{p \in \mathbb{N}_r : \bar{s}'_p \notin \mathcal{Q},\ s'_p \in \mathcal{Q}\}$ and $\mathcal{C} = \{p \in \mathbb{N}_r \cup \{0\} : p \notin \mathcal{Q},\ \bar{p} \notin \mathcal{Q}\}$. Then $\mathcal{A}$, $\mathcal{B}^\alpha$, $\mathcal{B}^\beta$ and $\mathcal{C}$ are distinct with $\mathcal{A} \cup \mathcal{B}^\alpha \cup \mathcal{B}^\beta \cup \mathcal{C} = \mathcal{I}'_m$ and, if $a = \#\mathcal{A}$, $b^\alpha = \#\mathcal{B}^\alpha$, $b^\beta = \#\mathcal{B}^\beta$ and $c = \#\mathcal{C}$, then $a + b^\alpha + b^\beta + c = r'$ and $\tilde{\lambda}_k = 2a + b^\alpha + b^\beta$. Let $\mathcal{J} = \{1, 2, \ldots, j\}$ so that $\#\mathcal{J} = j$. Now create the sets $\mathcal{D} = (\mathcal{A} \cap \mathcal{J}) \backslash \{j\}$, $\mathcal{E} = \mathcal{A} \backslash (\mathcal{D} \cup \{j\})$, $\mathcal{B}_0^\alpha = \mathcal{B}^\alpha \cap \mathcal{J}$, $\mathcal{B}_1^\alpha = (\mathcal{B}^\alpha \backslash \mathcal{B}_0^\alpha) \cup \{j\}$, $\mathcal{B}_0^\beta = \mathcal{B}^\beta \cap \mathcal{J}$, $\mathcal{B}_1^\beta = \mathcal{B}^\beta \backslash \mathcal{B}_0^\beta$, $\mathcal{G} = \mathcal{C} \cap \mathcal{J}$ and $\mathcal{F} = \mathcal{C} \backslash \mathcal{G}$. In addition, let $\mathcal{H} = \mathcal{D} \cup \mathcal{B}_0^\alpha \cup \mathcal{F}$ so that $\mathcal{I}'_m = \mathcal{E} \cup \mathcal{B}_1^\alpha \cup \mathcal{B}^\beta \cup \mathcal{G} \cup \mathcal{H}$. Define $(s_1, \ldots, s_r)$ by $s_i = \bar{s}'_i$ if $i \in \mathcal{B}_0^\alpha \cup \{j\}$, and $s_i = s'_i$ if $i \notin \mathcal{B}_0^\alpha \cup \{j\}$. Note that $j \in \mathcal{A}$ and $j \in \mathcal{B}_1^\alpha$ but $j \notin \mathcal{H}$, $j \notin \mathcal{D}$ and $j \notin \mathcal{B}_0^\alpha$. Let the cardinalities of the sets just created be d, e, b_0^α, b_1^α, b_0^β, b_1^β, g, f, and h respectively. Then $j = d + b_0^\alpha + b_0^\beta + g + 1$, $a = d + e + 1$, $b^\alpha = b_0^\alpha + b_1^\alpha - 1$, $b^\beta = b_0^\beta + b_1^\beta$, $c = g + f$, $h = d + f + b_0^\alpha$ and $\tilde{\lambda}_k = 2d + 2e + b_0^\alpha + b_1^\alpha + b^\beta + 1$. In addition, $\alpha_j = 2(d+1) + b_0^\alpha + b_0^\beta = 2d + b_0^\alpha + b_0^\beta + 2$. From $\alpha_j > j$ then follows $d \geq g$. Thereupon, if $u = 2d + 1$ and $v = b_0^\alpha$, the conditions of Lemma 6.4.2 are satisfied and if $\mathcal{B}_0^\alpha = \{p'_1, p'_2, \ldots, p'_{b_0^\alpha}\}$ and

$$\mathring{y} = \hat{t}_{(u,v)} \left(\sum_{p \in \mathcal{D} \cup \mathcal{B}_1^\alpha \cup F} \mathring{p}; \mathring{p}'_1, \mathring{p}'_2, \ldots, \mathring{p}'_{b_0^\alpha} \right), \tag{6.4.11a}$$

the expression:

$$\theta^{\mathcal{B}_1^\alpha}\theta^{\mathcal{B}^\beta}\theta^{\mathcal{E}}\left(\sum_{p\in\mathcal{B}_1^\alpha\cup\mathcal{H}}\mathring{p}\right)^u\hat{t}_{(u,v)}\left(\sum_{p\in\mathcal{D}\cup\mathcal{B}_1^\alpha\cup F}\mathring{p};\mathring{p}'_1,\mathring{p}'_2,\ldots,\mathring{p}'_{b_0^\alpha}\right)=0 \qquad (6.4.11b)$$

results, where $\theta^{\mathcal{B}_1^\alpha}=\prod_{q\in\mathcal{B}_1^\alpha}\bar{s}_q$, $\theta^{\mathcal{B}^\beta}=\prod_{q\in\mathcal{B}^\beta}s_q$, and $\theta^{\mathcal{E}}=\prod_{q\in\mathcal{E}}\bar{q}q$. Of the tableaux resulting from this expression, the construction ensures that:

$$\theta^{\mathcal{B}_1^\alpha}\theta^{\mathcal{B}^\beta}\theta^{\mathcal{E}}\prod_{p\in\mathcal{D}}\bar{p}p\prod_{p\in\mathcal{B}_0^\alpha\cup\{j\}}s_p \qquad (6.4.11c)$$

is the lowest in that in all other terms, the corresponding factor is greater or equal. Note that the indices s'_j and $\bar{s}'_j$ both appear in this term. The indices in this term are those in the kth column of $T^{\Delta_r;\lambda}$ and therefore, under the appropriate substitutions and coefficient impositions as in (6.4.2c), (6.4.11b) yields an expression for $[T^{\Delta_r;\lambda}]$ in terms of higher tableaux.

To illustrate the algorithm used in this proof, let $m=10$, $\lambda=(1^5)$ and consider the tableau:

$$T^{\Delta_5;1^5}=\begin{matrix}1\cdot 2\\ \bar{2}\cdot\bar{3}\\ \bar{3}\cdot 3\\ \bar{4}\cdot\bar{4}\\ 5\cdot 4\end{matrix}\,, \qquad (6.4.12a)$$

which is non-standard since $\alpha_4=5>4$. With $k=1$ and $j=4$, the proof of Lemma 6.4.11 involves the sets $\mathcal{A}=\{3,4\}$, $\mathcal{B}^\alpha=\{2\}$, $\mathcal{B}^\beta=\emptyset$ and $\mathcal{C}=\{1,5\}$. In addition $(s'_1,\ldots,s'_5)=(1,\bar{2},\bar{3},\bar{4},5)$. With $\mathcal{J}=\{1,2,3,4\}$, it follows that $\mathcal{D}=\{3\}$, $\mathcal{E}=\emptyset$, $\mathcal{B}_0^\alpha=\{2\}$, $\mathcal{B}_1^\alpha=\{4\}$, $\mathcal{B}_0^\beta=\mathcal{B}_1^\beta=\emptyset$, $\mathcal{F}=\{5\}$, $\mathcal{G}=\{1\}$ and $\mathcal{H}=\{2,3,5\}$. Also $(s_1,\ldots,s_5)=(1,2,\bar{3},4,5)$. The Spinor relation is now written:

$$-\mathring{1}=\mathring{2}+\mathring{3}+\mathring{4}+\mathring{5}, \qquad (6.4.12b)$$

which, on being raised to the power of $u=2d+1=3$, yields:

$$(\mathring{2}+\mathring{3}+\mathring{4}+\mathring{5})^3=0. \qquad (6.4.12c)$$

Since $v=b_0^\alpha=1$, according to (6.4.11b), this should be multiplied by $\hat{t}_{(3,1)}(\mathring{3}+\mathring{4}+\mathring{5};\mathring{2})=\mathring{3}+\mathring{4}+\mathring{5}-\mathring{2}$ (a factor of 2 having been removed, for convenience, from the polynomial given by Table 6.4.10). The result is:

$$\begin{aligned}0&=(\mathring{2}+\mathring{3}+\mathring{4}+\mathring{5})^3(\mathring{3}+\mathring{4}+\mathring{5}-\mathring{2})\\&=(\mathring{2}+\mathring{3}+\mathring{4}+\mathring{5})^2((\mathring{3}+\mathring{4}+\mathring{5})^2-\mathring{2}^2)\\&=(\mathring{3}+\mathring{4}+\mathring{5})^4+2.\mathring{2}(\mathring{3}+\mathring{4}+\mathring{5})^3-2.\mathring{2}^3(\mathring{3}+\mathring{4}+\mathring{5})-\mathring{2}^4\end{aligned}$$

$$= (\mathring{3}+\mathring{4}+\mathring{5})^4 + 2.\mathring{2}(\mathring{3}+\mathring{4}+\mathring{5})^3, \tag{6.4.12d}$$

because $\mathring{2}^3 = \mathring{2}^4 = 0$. Notice that no terms containing $\mathring{2}^2$ appear and that the coefficient of $(\mathring{3}+\mathring{4}+\mathring{5})^3\mathring{2}$ is positive. Expanding (6.4.12d) and dividing by 6 results in:

$$\begin{aligned} &\mathring{2}\mathring{3}^2\mathring{4} + \mathring{2}\mathring{3}^2\mathring{5} + \mathring{2}\mathring{3}\mathring{4}^2 + \mathring{2}\mathring{4}^2\mathring{5} + \mathring{2}\mathring{3}\mathring{5}^2 + \mathring{2}\mathring{4}\mathring{5}^2 + 2.\mathring{2}\mathring{3}\mathring{4}\mathring{5} \\ &\quad + \mathring{3}^2\mathring{4}^2 + \mathring{3}^2\mathring{5}^2 + \mathring{4}^2\mathring{5}^2 + 2.(\mathring{3}^2\mathring{4}\mathring{5} + \mathring{3}\mathring{4}^2\mathring{5} + \mathring{3}\mathring{4}\mathring{5}^2) = 0. \end{aligned} \tag{6.4.12e}$$

In this expression, the first term is the lowest — it is that giving rise to $T^{\Delta_5;1^5}$. Substituting $\bar{p}p$ for $\mathring{p}^2$, and s_p for $\mathring{p}$, calculating the coefficients as in (6.4.2c) and multiplying the resultant expression by $\theta^{\mathcal{B}^\alpha_1}\theta^{\mathcal{B}^\beta}\theta^{\mathcal{E}} = \bar{4}$, yields:

$$\begin{aligned} &2\bar{3}34\bar{4} + 2\bar{3}35\bar{4} + 2\bar{3}\bar{4}4\bar{4} - 2\bar{4}45\bar{4} + 23\bar{5}5\bar{4} - 24\bar{5}5\bar{4} - \bar{3}3\bar{4}4\bar{4} \\ &\quad -\bar{3}3\bar{5}5\bar{4} + \bar{4}4\bar{5}5\bar{4} + 2.(2\bar{3}45\bar{4} - \bar{3}345\bar{4} - \bar{3}\bar{4}45\bar{4} - \bar{3}4\bar{5}5\bar{4}) = 0. \end{aligned} \tag{6.4.12f}$$

The tableau corresponding to each term is obtained by forming a column from the indices indicated and selecting indices for the half boxes so that a constant $O(m)$-weight is obtained. Rearrangement and use of the Column relations then yields the following expression for $T^{\Delta_5;1^5}$ in terms of higher tableaux:

$$\begin{aligned} \begin{bmatrix} 1\cdot 2 \\ \bar{2}\cdot\bar{3} \\ \bar{3}\cdot 3 \\ \bar{4}\cdot\bar{4} \\ 5\cdot 4 \end{bmatrix} = &- \begin{bmatrix} 1\cdot 2 \\ \bar{2}\cdot\bar{3} \\ \bar{3}\cdot 3 \\ 4\cdot\bar{4} \\ \bar{5}\cdot 5 \end{bmatrix} + \begin{bmatrix} 1\cdot 2 \\ \bar{2}\cdot\bar{3} \\ 3\cdot\bar{4} \\ 4\cdot\bar{5} \\ 5\cdot 5 \end{bmatrix} + \begin{bmatrix} 1\cdot 2 \\ \bar{2}\cdot\bar{4} \\ \bar{3}\cdot 4 \\ \bar{4}\cdot\bar{5} \\ 5\cdot 5 \end{bmatrix} - \begin{bmatrix} 1\cdot\bar{3} \\ 2\cdot 3 \\ \bar{3}\cdot\bar{4} \\ 4\cdot\bar{5} \\ 5\cdot 5 \end{bmatrix} \\ &+2 \begin{bmatrix} 1\cdot 2 \\ \bar{2}\cdot\bar{3} \\ 3\cdot\bar{4} \\ \bar{4}\cdot 4 \\ \bar{5}\cdot 5 \end{bmatrix} - 2 \begin{bmatrix} 1\cdot\bar{3} \\ 2\cdot 3 \\ \bar{3}\cdot\bar{4} \\ \bar{4}\cdot 4 \\ \bar{5}\cdot 5 \end{bmatrix} + 2 \begin{bmatrix} 1\cdot\bar{3} \\ 2\cdot\bar{4} \\ 3\cdot 4 \\ \bar{4}\cdot\bar{5} \\ 5\cdot 5 \end{bmatrix}. \end{aligned} \tag{6.4.12g}$$

Note that of the tableaux on the right side, the 1st, 5th and 6th are non-standard in that they each violate condition (vi) of Definition 6.4.6 for $j = 5$. The procedure required to standardise either of these is presented in the following lemma where techniques very similar to those of the previous lemma are applied.

Lemma 6.4.13. *Let $T^{\Delta_r;\lambda}$ be column strict but non-standard in that there exists j and k such that either $\alpha_j \geq j$, where α_j is the number of entries less than j in the kth column of $T^{\Delta_r;\lambda}$, and such that $\bar{s}'_j$ is present and s'_j is not present in the kth column of $T^{\Delta_r;\lambda}$ where $s'_j = T^{\Delta_r;\lambda}_{(j,0)}$, or $\alpha_0 \geq r+1$ and 0 is present in the kth column of $T^{\Delta_r;\lambda}$. If $\hat{t}_{(u,v)}$ exists for all $u+v \leq \tilde{\lambda}_1$, then $[T^{\Delta_r;\lambda}]$ may be expressed as a linear combination of traceless symmetrised tableaux $[T^{\Delta_r;\lambda}_w]$, where for each w, $T^{\Delta_r;\lambda}_w > T^{\Delta_r;\lambda}$.*

Proof. Let $s'_i = T^{\Delta_r;\lambda}_{(i,0)}$, for $i = 1, 2, \ldots, r$, and define $\mathcal{Q}$, $\mathcal{A}$, $\mathcal{B}^\alpha$, $\mathcal{B}^\beta$, $\mathcal{C}$, a, b^α, b^β and c precisely as for the proof of Lemma 6.4.11. Now let $\mathcal{J} = \{1, 2, \ldots, j-1\}$ if $j \le r$ and $\mathcal{J} = \{1, 2, \ldots, r\}$ if $j = 0$. Let $\mathcal{D} = \mathcal{A} \cap \mathcal{J}$, $\mathcal{E} = \mathcal{A} \backslash \mathcal{D}$, $\mathcal{B}^\alpha_0 = \mathcal{B}^\alpha \cap \mathcal{J}$, $\mathcal{B}^\alpha_1 = \mathcal{B}^\alpha \backslash (\mathcal{B}^\alpha_0 \cup \{j\})$, $\mathcal{B}^\beta_0 = \mathcal{B}^\beta \cap \mathcal{J}$, $\mathcal{B}^\beta_1 = \mathcal{B}^\beta \backslash \mathcal{B}^\beta_0$, $\mathcal{G} = \mathcal{C} \cap \mathcal{J}$ and $\mathcal{F} = \mathcal{C} \backslash \mathcal{G}$. In addition, let $\mathcal{H} = \mathcal{D} \cup \mathcal{B}^\alpha_0 \cup \mathcal{F} \cup \{j\}$. Note that of the sets just created, $j \in \mathcal{H}$ but $j \notin \mathcal{B}^\alpha_0$ and $j \notin \mathcal{B}^\alpha_1$. Let the cardinalities of the sets just created be d, e, b^α_0, b^α_1, b^β_0, b^β_1, g, f, and h respectively. Then $j = d + b^\alpha_0 + b^\beta_0 + g + 1$, $a = d + e$, $b^\alpha = b^\alpha_0 + b^\alpha_1 + 1$, $b^\beta = b^\beta_0 + b^\beta_1$, $c = g + f$, $h = d + f + b^\alpha_0 + 1$ and $\lambda_k = 2d + 2e + b^\alpha_0 + b^\alpha_1 + b^\beta + 1$. In addition, $\alpha_j = 2d + b^\alpha_0 + b^\beta_0 + 1$. From $\alpha_j \ge j$ then follows $d \ge g$. $(s_1, \ldots, s_r)$ is defined by $s_i = \bar{s}'_i$ if $i \in \mathcal{B}^\alpha_0 \cup \{j\}$ and $s_i = s'_i$ if $i \notin \mathcal{B}^\alpha_0 \cup \{j\}$. If $u = 2d + 1$ and $v = b^\alpha_0$, then as in Lemma 6.4.11, the conditions of Lemma 6.4.2 are satisfied and the expression:

$$\theta^{\mathcal{B}^\alpha_1} \theta^{\mathcal{B}^\beta} \theta^{\mathcal{E}} \left(\sum_{p \in \mathcal{B}^\alpha_1 \cup \mathcal{H}} \mathring{p} \right)^u \hat{t}_{(u,v)} \left(\sum_{p \in \mathcal{D} \cup \mathcal{B}^\alpha_1 \cup F} \mathring{p}; \mathring{p}'_1, \mathring{p}'_2, \ldots, \mathring{p}'_{b^\alpha_0} \right) = 0 \qquad (6.4.13a)$$

results, where $\mathcal{B}^\alpha_0 = \{p'_1, p'_2, \ldots, p'_{b^\alpha_0}\}$, $\theta^{\mathcal{B}^\alpha_1} = \prod_{q \in \mathcal{B}^\alpha_1} \bar{s}_q$, $\theta^{\mathcal{B}^\beta} = \prod_{q \in \mathcal{B}^\beta} s_q$, and $\theta^{\mathcal{E}} = \prod_{q \in \mathcal{E}} \bar{q} q$. Of the tableaux resulting from this expression, the construction ensures that:

$$\theta^{\mathcal{B}^\alpha_1} \theta^{\mathcal{B}^\beta} \theta^{\mathcal{E}} \prod_{p \in \mathcal{D}} \bar{p} p \prod_{p \in \mathcal{B}^\alpha_0 \cup \{j\}} s_p \qquad (6.4.13b)$$

is the lowest. Note that the index $\bar{s}'_j$ appears in this term but not s'_j. The term (6.4.13b) corresponds to the kth column of $T^{\Delta_r;\lambda}$ and therefore, under the appropriate substitutions and coefficient impositions as in (6.4.2c), (6.4.13a) yields an expression for $[T^{\Delta_r;\lambda}]$ in terms of higher tableaux.

To illustrate this lemma, consider the 6th term on the right side of (6.4.12g). However, instead of $m = 10$, let $m = 11$. The tableau in question is non-standard by virtue of a $j = 5$ violation of condition (vi) of Definition 6.4.6 since $T^{\Delta_5;\lambda}_{(5,1)} = \bar{s}'_5$ and $T^{\Delta_5;\lambda}_{(4,1)} \ne s'_5$ where $(s'_1, \ldots, s'_5) = (1, 2, \bar{3}, \bar{4}, \bar{5})$. Here $\mathcal{A} = \{3, 4\}$, $\mathcal{B}^\alpha = \{5\}$, $\mathcal{B}^\beta = \emptyset$ and $\mathcal{C} = \{1, 2, 0\}$. With $\mathcal{J} = \{1, 2, 3, 4\}$, it follows that $\mathcal{D} = \{3, 4\}$, $\mathcal{E} = \emptyset$, $\mathcal{B}^\alpha_0 = \mathcal{B}^\alpha_1 = \mathcal{B}^\beta_0 = \mathcal{B}^\beta_1 = \emptyset$, $\mathcal{F} = \{0\}$, $\mathcal{G} = \{1, 2\}$ and $\mathcal{H} = \{3, 4, 5, 0\}$. Also $(s_1, \ldots, s_5) = (1, 2, \bar{3}, \bar{4}, 5)$. With $u = 2d + 1 = 5$ and $v = b^\alpha_0 = 0$, $\hat{t}_{(u,v)} = 1$ from Table 6.4.10, whereupon (6.4.13a) yields:

$$\begin{aligned} 0 &= (\mathring{3} + \mathring{4} + \mathring{5} + \mathring{0})^5 \\ &= \mathring{3}^2\mathring{4}^2\mathring{5} + \mathring{3}^2\mathring{4}\mathring{5}^2 + \mathring{3}\mathring{4}^2\mathring{5}^2 + \mathring{3}^2\mathring{4}^2\mathring{0} + \mathring{3}^2\mathring{5}^2\mathring{0} + \mathring{4}^2\mathring{5}^2\mathring{0} \\ &\qquad + 2.(\mathring{3}^2\mathring{4}\mathring{5}\mathring{0} + \mathring{3}\mathring{4}^2\mathring{5}\mathring{0} + \mathring{3}\mathring{4}\mathring{5}^2\mathring{0}). \end{aligned} \qquad (6.4.14a)$$

Using (6.4.2c), this expression directly yields the following tableaux identity:

$$\begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot 3 \\ \bar{3} \cdot \bar{4} \\ \bar{4} \cdot 4 \\ \bar{5} \cdot 5 \end{bmatrix} + \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot 3 \\ \bar{3} \cdot \bar{4} \\ 4 \cdot \bar{5} \\ 5 \cdot 5 \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot \bar{4} \\ 3 \cdot 4 \\ \bar{4} \cdot \bar{5} \\ 5 \cdot 5 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot 3 \\ \bar{3} \cdot \bar{4} \\ \bar{4} \cdot 4 \\ 5 \cdot 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot 3 \\ \bar{3} \cdot \bar{5} \\ \bar{4} \cdot 5 \\ 5 \cdot 0 \end{bmatrix}$$

$$- \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot \bar{4} \\ 2 \cdot 4 \\ \bar{3} \cdot \bar{5} \\ \bar{4} \cdot 5 \\ 5 \cdot 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot 3 \\ \bar{3} \cdot \bar{4} \\ 4 \cdot 5 \\ \bar{5} \cdot 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot \bar{4} \\ 3 \cdot 4 \\ \bar{4} \cdot 5 \\ \bar{5} \cdot 0 \end{bmatrix} - \sqrt{2} \begin{bmatrix} 1 \cdot \bar{3} \\ 2 \cdot \bar{4} \\ 3 \cdot \bar{5} \\ 4 \cdot 5 \\ 5 \cdot 0 \end{bmatrix} = 0. \tag{6.4.14b}$$

This is easily rearranged to enable the first term to be written in terms of higher tableaux, each of which, in this case, are $O(11)$-standard. If the last six terms are omitted then the resulting identity is that obtained for $O(10)$. Incidentally, standardisation of the 5th term on the right side of (6.4.12g) using this technique, requires $\hat{t}_{(3,2)}$.

Lemma 6.4.15. *If* $\hat{t}_{(u,v)}$ *exists for all* $u + v \leq \tilde{\lambda}_1$ *then the set*

$$\{\, [T^{\Delta_r;\lambda}] \,:\, T^{\Delta_r;\lambda} \textit{ is } O(m)\textit{-standard}\}$$

spans the $O(m)$*-module* $O^{\Delta_r;\lambda}$.

Proof. If the column strict $T^{\Delta;\lambda}$ is not $O(m)$-standard due to a violation of condition (*iii*) of Definition 6.4.6, then the techniques of Section 3.4 enable the Garnir relations, acting on the F^λ portion, to be used to write $[T^{\Delta_r;\lambda}]$ in terms of higher column strict tableaux. If the column strict $T^{\Delta_r;\lambda}$ violates conditions (v) or (vi) of Definition 6.4.6 then either Lemma 6.4.11 or Lemma 6.4.13 can be used to express $[T^{\Delta_r;\lambda}]$ in terms of higher column strict tableaux. Therefore, by iterating these procedures, $[T^{\Delta_r;\lambda}]$ may be written in terms of $O(m)$-standard tableaux by virtue of the ordering on the set of all tableaux and their finite number.

In addition to the techniques of standardisation employed in this section, those of the orthogonal Trace relation may also be used on any two columns of the F^λ portion. However, as indicated by Lemma 6.4.15, this Trace relation is not necessary to effect a standardisation. Nonetheless, it may enable standardisation to be achieved more efficiently.

Lemma 6.4.16. *If* $\hat{t}_{(u,v)}$ *exists for all* $u + v \leq r'$ *and* $\tilde{\lambda}_1 > r$ *then the* $O(m)$*-module* $O^{\Delta_r;\lambda}$ *is zero.*

Proof. Consider the case of even $m = 2r = 2r'$ first. If $\tilde{\lambda}_1 > r$ there necessarily exists a $j \leq r$ such that both $\bar{j}$ and j occur in the first column of each column strict $T^{\Delta_r;\lambda}$ and for which $\alpha_j > j$ and $\alpha_{\bar{j}} \leq r$. Thus Lemma 6.4.11 can be used to write $[T^{\Delta_r;\lambda}]$ in terms of higher tableaux. These tableaux are also necessarily non-standard and thus iterating this process must eventually result in $[T^{\Delta_r;\lambda}] = 0$ since the total number of tableaux of shape $F^{\Delta_r;\lambda}$ is finite.

For the case $m = 2r + 1$, $r' = r + 1$, if the index 0 does not appear in the first column or appears below the $(r + 1)$th row then the argument above is used for $j \neq 0$. This leaves the case for which $T^{\Delta_r;\lambda}_{(r+1,0)} = 0$. This is a violation of condition (vi) of 6.4.6, and thence Lemma 6.4.13 can be invoked if $\hat{t}_{(u,v)}$ exists for all $u + v \leq r + 1 = r'$.

§6.5. The irreducible spinor modules of $O(m)$

Armed with the $O(m)$-standard tableaux for the spinor modules and the standardisation techniques developed in Section 6.4, the $O(m)$-modules $O^{\Delta_r;\lambda}$ are defined explicitly in this section.

Let $\lambda \in P(l)$. Since $\hat{U}$, specified by Definition 6.3.17, is invariant under the action of $G \in O(m)$, it follows that $\hat{U} \cap (\Psi \otimes W^{\lambda})$ is also invariant under the same action, and thence from (6.3.14) that:

$$G\,[T^{\Delta_r;\lambda}] = \pm \sum_{T'^{\Delta_r;\lambda}} \Delta(G)_{s'_1 \cdots s'_r s_1 \cdots s_r} G_{T'^{\lambda}_{(1)} T^{\lambda}_{(1)}} G_{T'^{\lambda}_{(2)} T^{\lambda}_{(2)}} \cdots G_{T'^{\lambda}_{(l)} T^{\lambda}_{(l)}} [T'^{\Delta;\lambda}], \qquad (6.5.1)$$

where the sum is over all $T'^{\Delta_r;\lambda}$ with entries from the set $\mathcal{I}^{O(m)}$, and for which $T'^{\Delta;\lambda}_{(j,0)} \in \{\bar{j}, j\}$ for $j = 1, 2, \ldots, r$. Here $T^{\Delta_r;\lambda}$ is T^{Δ_r} adjoined to T^{λ}, $T'^{\Delta_r;\lambda}$ is T'^{Δ_r} adjoined to T'^{λ}, and $s_j = T^{\Delta_r;\lambda}_{(j,0)}$ and each $s'_j = T'^{\Delta_r;\lambda}_{(j,0)}$ for $j = 1, 2, \ldots, r$.

The action of $B_a{}^b \in so(2r+1)$ or $D_a{}^b \in so(2r)$ on $[T^{\Delta_r;\lambda}]$ derives simply from (5.2.14) and (6.3.12). As above, let $T^{\Delta_r;\lambda}$ be T^{Δ_r} adjoined to T^{λ}, and $s_j = T^{\Delta_r;\lambda}_{(j,0)}$ for $j = 1, 2, \ldots, r$. In addition, let $s_0 = 0$. In order to specify the action of $\Lambda_a{}^b$ on T^{Δ_r}, let $T_0^{\Delta_r}$ be identical to T^{Δ_r} if $a = b$, and if $a \neq b$ let $T_0^{\Delta_r}$ be identical to T^{Δ_r} except for two positions for which $T^{\Delta_r}_{(a,0)} = \bar{s}_a$ and $T^{\Delta_r}_{(b,0)} = \bar{s}_b$. If:

$$\phi(a,b,T^{\Delta_r}) = \begin{cases} 0 & \text{if } a = \bar{b}; \\ \delta^a_{s_a} - \frac{1}{2} & \text{if } a = b \neq 0; \\ \phi(a)\phi(b)\delta^b_{s_a}\delta^{\bar{b}}_{s_b} \begin{bmatrix} s_a \\ \vdots \\ s_{b-1} \end{bmatrix} & \text{if } a < \bar{b}, b; \quad \text{and} \\ -\phi(a)\phi(b)\delta^b_{s_a}\delta^{\bar{b}}_{s_b} \begin{bmatrix} s_b \\ \vdots \\ s_{a-1} \end{bmatrix} & \text{if } a > \bar{b}, b, \end{cases} \tag{6.5.2}$$

then it follows from (6.3.12) that:

$$\Lambda_a{}^b\,[T^{\Delta_r}] = \phi(a,b,T^{\Delta_r})[T^{\Delta_r}_0]. \tag{6.5.3}$$

Now let $T_0^{\Delta_r;\lambda}$ denote the tableau formed by adjoining $T_0^{\Delta_r}$ to T^λ and define:

$$\Lambda_a{}^b\,[T^{\Delta_r;\lambda}] = \phi(a,b,T^{\Delta_r})[T_0^{\Delta_r;\lambda}]. \tag{6.5.4}$$

As in Section 5.2, let p and q be the number of times that the indices b and $\bar{a}$ respectively occur in T^λ. Form the set of p tableaux $\{T_{1,1}^{\Delta_r;\lambda}, T_{1,2}^{\Delta_r;\lambda}, \dots, T_{1,p}^{\Delta_r;\lambda}\}$ by, in each case, replacing a single index b in the F^λ portion of $T^{\Delta_r;\lambda}$ with a, and the set of q tableaux $\{T_{2,1}^{\Delta_r;\lambda}, T_{2,2}^{\Delta_r;\lambda}, \dots, T_{2,q}^{\Delta_r;\lambda}\}$ by, in each case, replacing a single index $\bar{a}$ in the F^λ portion of $T^{\Delta_r;\lambda}$ with $\bar{b}$. Then, it follows from (5.2.14a), (6.5.3), and the definition of $[T^{\Delta_r;\lambda}]$ that, for $B_a{}^b \in so(2r+1)$:

$$\begin{aligned} B_a{}^b\,[T^{\Delta_r;\lambda}] &= \Lambda_a{}^b\,[T^{\Delta_r;\lambda}] + E_a{}^b\,[T^{\Delta_r;\lambda}] - E_{\bar{b}}{}^{\bar{a}}\,[T^{\Delta_r;\lambda}] \\ &= \phi(a,b,T^{\Delta_r})[T_0^{\Delta_r;\lambda}] + \sum_{i=1}^{p}[T_{1,i}^{\Delta_r;\lambda}] - \sum_{i=1}^{q}[T_{2,i}^{\Delta_r;\lambda}], \end{aligned} \tag{6.5.5a}$$

and similarly, for $D_a{}^b \in so(2r)$:

$$D_a{}^b\,[T^{\Delta_r;\lambda}] = \phi(a,b,T^{\Delta_r})[T_0^{\Delta_r;\lambda}] + \sum_{i=1}^{p}[T_{1,i}^{\Delta_r;\lambda}] - \sum_{i=1}^{q}[T_{2,i}^{\Delta_r;\lambda}]. \tag{6.5.5b}$$

These imply that:

$$B_a{}^a\,[T^{\Delta_r;\lambda}] = n_a^{O(2r+1)}(T^{\Delta_r;\lambda})[T^{\Delta_r;\lambda}] \tag{6.5.6a}$$

$$\text{and} \qquad D_a{}^a\,[T^{\Delta_r;\lambda}] = n_a^{O(2r)}(T^{\Delta_r;\lambda})[T^{\Delta_r;\lambda}]. \tag{6.5.6b}$$

Since a basis for the Cartan subalgebras of $so(2r+1)$ and $so(2r)$ are provided by the elements $B_a{}^a$ and $D_a{}^a$ respectively for $a = 1, 2, \dots, r$, the $O(m)$-weight $n^{O(m)}(T^{\Delta_r;\lambda})$ of $T^{\Delta_r;\lambda}$ determines the weight of the element $[T^{\Delta_r;\lambda}] \in O^{\Delta_r;\lambda}$ in this basis.

Let the tableau $T_{>}^{\Delta_r;\lambda}$ be defined by $T_{>(j,k)}^{\Delta_r;\lambda} = j$ for $1 \le j \le r$ and $0 \le k \le \lambda_j$. Then $n^{O(m)}(T_{>(j,k)}^{\Delta_r;\lambda}) = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \ldots, \lambda_r + \frac{1}{2}) = (\Delta_r; \lambda)$. $T_{>}^{\Delta_r;\lambda}$ is the only tableau of shape $F^{\Delta_r;\lambda}$ with this property. It is easily shown that $B_a{}^b\,[T_{>}^{\Delta_r;\lambda}] = 0$ for all $B_a{}^b \in \mathcal{B}_+^{O(2r+1)}$ and $D_a{}^b\,[T_{>}^{\Delta_r;\lambda}] = 0$ for all $D_a{}^b \in \mathcal{B}_+^{O(2r)}$. This leads to the following theorem.

Theorem 6.5.7. *Let m be such that even $m = 2r$ and odd $m = 2r+1$, and let $r' = m - r$. If $\hat{t}_{(u,v)}$ exists for all $u + v \le \tilde{\lambda}_1$ then the $O(m)$-module $O^{\Delta_r;\lambda}$ is irreducible with basis:*

$$\{\,[T^{\Delta_r;\lambda}] : T^{\Delta_r;\lambda} \text{ is } O(m)\text{-standard}\}.$$

Moreover, if $\hat{t}_{(u,v)}$ exists for all $u + v \le r'$ then the set $\{O^{\Delta_r;\lambda} : \tilde{\lambda}_1 \le r\}$ provides a complete list of inequivalent irreducible spinor $O(m)$-modules.

Proof. By virtue of Lemma 6.4.15, the dimension of $O^{\Delta_r;\lambda}$ is not greater than the number of $O(m)$-standard tableaux. From Theorem 6.4.8, this number is equal to dimension of the irreducible representation $[\Delta_r; \lambda]$ of $O(m, \mathbb{C})$. Then, since $O^{\Delta_r;\lambda}$ has highest weight $(\Delta_r; \lambda)$, $O^{\Delta_r;\lambda}$ is the $O(m)$-module corresponding to that irreducible representation. This proves the first part of the Theorem. The second part follows because firstly every spinor $O(m)$-module occurs in $\Psi \otimes V^{\otimes l}$ for some l [**Li50**]; secondly, $O(m)$-standard tableaux of shape $F^{\Delta_r;\lambda}$ exist if and only if $\tilde{\lambda}_1 \le r$; and thirdly, $(\Delta_r; \lambda)$ is the highest weight of $O^{\Delta_r;\lambda}$.

The quintessential structure of $O^{\Delta;\lambda}$ may now be stated.

Theorem 6.5.8. *Let m be such that even $m = 2r$ and odd $m = 2r+1$ and let $\lambda \in P(l; r)$. If $\hat{t}_{(u,v)}$ exists for all $u + v \le r$ then $O^{\Delta_r;\lambda}$ is the irreducible $O(m)$-module spanned by $[T^{\Delta_r;\lambda}]$ for all $T^{\Delta_r;\lambda}$ with entries from the set $\mathcal{I}^{O(m)}$ and for which $T_{(j,0)}^{\Delta;\lambda} \in \{\bar{j}, j\}$ for $j = 1, 2, \ldots, r$; modulo relations* (3.4.2), (3.4.3), *and* (6.3.18*a*) *if $m = 2r$, or* (6.3.18*b*) *if $m = 2r+1$; and on which $O(m)$ and $so(m)$ act according to* (6.5.1) *and* (6.5.5) *respectively.*

The techniques of this section enable explicit representation matrices for elements of $O(m)$ and $so(m)$ to be obtained in the representation $[\Delta_r; \lambda]$. Let $D_m[\Delta_r; \lambda]$ be the dimension of $O^{\Delta_r;\lambda}$ and let $T_1^{\Delta_r;\lambda}, T_2^{\Delta_r;\lambda}, \ldots, T_{D_m[\Delta_r;\lambda]}^{\Delta_r;\lambda}$ the $O(m)$-standard tableaux. The action of $G \in O(m)$ on each $[T_i^{\Delta_r;\lambda}]$ yields, via (6.5.1), a linear combination of traceless symmetrised tableaux which are, in general, non-standard. If $\hat{t}_{(u,v)}$ exists for all $u + v \le r$, then the techniques of this section enable each to be written in terms of $O(m)$-standard tableaux, so that:

$$G\,[T_i^{\Delta_r;\lambda}] = \sum_{j=1}^{D_m[\Delta_r;\lambda]} \Gamma^{[\Delta_r;\lambda]}(G)_{ji}[T_j^{\Delta_r;\lambda}], \tag{6.5.9}$$

where the $\Gamma^{[\Delta_r;\lambda]}(G)_{ji} \in \mathsf{F}$ are the matrix elements of G in the representation $[\Delta_r;\lambda]$. In a similar way, the representation matrix $\Gamma^{[\Delta_r;\lambda]}(B)$ of $B \in so(m)$ is given, via (6.5.5), by:

$$B\,[T_i^{\Delta_r;\lambda}] = \sum_{j=1}^{D_m[\Delta_r;\lambda]} \Gamma^{[\Delta_r;\lambda]}(B)_{ji}[T_j^{\Delta_r;\lambda}]. \tag{6.5.10}$$

Note that in the reduction of an arbitrary traceless symmetrised tableau to a linear combination over the $O(m)$-standard tableaux, the coefficients are rational if m is even. However, if m is odd then in general, factors of $1/\sqrt{2}$ arise either through the Spinor relation or through the action of $\Gamma_a{}^b$ if either $a = 0$ or $b = 0$. Consequently, if $D_a{}^b \in so(2r)$ then the matrix elements, $\Gamma^{[\Delta_r;\lambda]}(D_a{}^b)_{ji}$ are all rational numbers, whereas if $B_a{}^b \in so(2r+1)$ then the matrix elements, $\Gamma^{[\Delta_r;\lambda]}(B_a{}^b)_{ji}$, are each a linear combination of rational numbers multiplied by an integral power of $\sqrt{2}$.

The techniques developed above will now be applied to the particular case of the representation $[\Delta_2;1,1]$ of $O(5)$. Although, for such small rank, the standardisation techniques are relatively straightforward, this case exhibits all other aspects peculiar to obtaining explicit matrix spinor representations.

The following $O(5)$-standard tableaux provide a basis for the 20-dimensional $O(5)$-module $O^{\Delta_2;1,1}$:

$$\begin{array}{ccccccc}
\begin{matrix}\bar1\cdot\bar1\\ \bar2\cdot\bar2\end{matrix}, & \begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot2\end{matrix}, & \begin{matrix}\bar1\cdot\bar1\\ \bar2\cdot0\end{matrix}, & \begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot0\end{matrix}, & \begin{matrix}\bar1\cdot2\\ \bar2\cdot0\end{matrix}, & \begin{matrix}\bar1\cdot\bar1\\ 2\cdot2\end{matrix}, & \begin{matrix}\bar1\cdot\bar2\\ 2\cdot2\end{matrix}, \\
\begin{matrix}\bar1\cdot\bar1\\ 2\cdot0\end{matrix}, & \begin{matrix}\bar1\cdot\bar2\\ 2\cdot0\end{matrix}, & \begin{matrix}\bar1\cdot2\\ 2\cdot0\end{matrix}, & \begin{matrix}1\cdot1\\ \bar2\cdot\bar2\end{matrix}, & \begin{matrix}1\cdot\bar2\\ \bar2\cdot2\end{matrix}, & \begin{matrix}1\cdot1\\ \bar2\cdot0\end{matrix}, & \begin{matrix}1\cdot\bar2\\ \bar2\cdot0\end{matrix}, \\
\begin{matrix}1\cdot2\\ \bar2\cdot0\end{matrix}, & \begin{matrix}1\cdot1\\ 2\cdot2\end{matrix}, & \begin{matrix}1\cdot\bar2\\ 2\cdot2\end{matrix}, & \begin{matrix}1\cdot1\\ 2\cdot0\end{matrix}, & \begin{matrix}1\cdot\bar2\\ 2\cdot0\end{matrix}, & \begin{matrix}1\cdot2\\ 2\cdot0\end{matrix}. &
\end{array} \tag{6.5.11}$$

Denote these by $T_1^{\Delta_2;1,1}, T_2^{\Delta_2;1,1}, \ldots, T_{20}^{\Delta_2;1,1}$ respectively. According to (6.5.5), $B_1{}^2 \in so(5)$ acts on $[T_2^{\Delta_2;1,1}]$ by:

$$\begin{aligned}
B_1{}^2\left[\begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot2\end{matrix}\right] &= \Lambda_1{}^2\left[\begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot2\end{matrix}\right] + E_1{}^2\left[\begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot2\end{matrix}\right] - E_{\bar2}{}^{\bar1}\left[\begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot2\end{matrix}\right] \\
&= 0 + \left[\begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot1\end{matrix}\right] - 0 \\
&= -\left[\begin{matrix}1\cdot\bar2\\ 2\cdot\bar2\end{matrix}\right] + \frac{1}{\sqrt{2}}\left[\begin{matrix}\bar1\cdot\bar2\\ \bar2\cdot2\end{matrix}\right] \quad \text{(by (6.3.18))}
\end{aligned}$$

$$= +\frac{1}{\sqrt{2}} \begin{bmatrix} \bar{1} \cdot \bar{2} \\ \bar{2} \cdot 2 \end{bmatrix} \tag{6.5.12a}$$

where the final line is obtained using the Column relations (3.4.2). For $[T_9^{\Delta_2;1,1}]$, the action of ${B_1}^2$ yields:

$$\begin{aligned} {B_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 0 \end{bmatrix} &= {\Lambda_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 0 \end{bmatrix} + {E_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 0 \end{bmatrix} - E_{\bar{2}}{}^{\bar{1}} \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 0 \end{bmatrix} \\ &= -\begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 0 \end{bmatrix} + 0 - 0. \end{aligned} \tag{6.5.12b}$$

For $[T_6^{\Delta_2;1,1}]$, the action of ${B_1}^2$ yields:

$$\begin{aligned} {B_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 2 \end{bmatrix} &= {\Lambda_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 2 \end{bmatrix} + {E_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 2 \end{bmatrix} - E_{\bar{2}}{}^{\bar{1}} \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 2 \end{bmatrix} \\ &= -\begin{bmatrix} 1 \cdot \bar{1} \\ \bar{2} \cdot 2 \end{bmatrix} + \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 1 \end{bmatrix} + \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix}. \end{aligned} \tag{6.5.12c}$$

The Spinor relations then yield:

$$\begin{bmatrix} 1 \cdot \bar{1} \\ \bar{2} \cdot 2 \end{bmatrix} = \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{1} \cdot 0 \\ \bar{2} \cdot 2 \end{bmatrix} = \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{1} \cdot 2 \\ \bar{2} \cdot 0 \end{bmatrix}, \tag{6.5.12d}$$

and their consecutive use over both 'whole' boxes yields:

$$\begin{aligned} \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 1 \end{bmatrix} &= -\begin{bmatrix} 1 \cdot \bar{1} \\ \bar{2} \cdot 2 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \cdot \bar{1} \\ 2 \cdot 0 \end{bmatrix} \\ &= -\begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix} + \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{1} \cdot 0 \\ \bar{2} \cdot 2 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \bar{1} \cdot 2 \\ \bar{2} \cdot 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \bar{1} \cdot 0 \\ 2 \cdot 0 \end{bmatrix} \\ &= -\begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix} - \sqrt{2} \begin{bmatrix} \bar{1} \cdot 2 \\ \bar{2} \cdot 0 \end{bmatrix}. \end{aligned} \tag{6.5.12e}$$

Incidentally, this expression may be obtained from the Spinor relation written $-\mathring{1} = \mathring{2} + \mathring{0}$, and squared to give $\mathring{1}^2 = \mathring{2}^2 + 2.\mathring{2}\mathring{0}$. Combining (6.5.12$c$), (6.5.12$d$) and (6.5.12$e$) gives:

$${B_1}^2 \begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 2 \end{bmatrix} = -3 \begin{bmatrix} \bar{1} \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix} - \frac{3}{\sqrt{2}} \begin{bmatrix} \bar{1} \cdot 2 \\ \bar{2} \cdot 0 \end{bmatrix}. \tag{6.5.12f}$$

A simpler example is provided by:

$${B_1}^2 \begin{bmatrix} 1 \cdot 1 \\ \bar{2} \cdot \bar{2} \end{bmatrix} = 0. \tag{6.5.12g}$$

The action of ${B_1}^2$ on each of the twenty $O(5)$-standard tableaux of (6.5.11) produces, via (6.5.10), the following explicit representation matrix $\Gamma^{[\Delta_2;1,1]}({B_1}^2)$ for ${B_1}^2$:

$$
\left(\begin{array}{cccccccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\frac{3}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right),
$$

where each zero has been replaced by a dot. The four calculations carried out above give rise to the entries in the 2nd, 9th, 6th and 11th columns of this matrix, respectively.

Now consider $B_0{}^{\bar{2}} \in so(5)$. Its action on the third element of the basis is as follows:

$$
\begin{aligned}
B_0{}^{\bar{2}}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ \bar{2} \cdot 0 \end{bmatrix} &= \Lambda_0{}^{\bar{2}}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ \bar{2} \cdot 0 \end{bmatrix} + E_0{}^{\bar{2}}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ \bar{2} \cdot 0 \end{bmatrix} - E_2{}^{0}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ \bar{2} \cdot 0 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 0 \end{bmatrix} - \begin{bmatrix} \bar{1} \cdot \bar{1} \\ \bar{2} \cdot 2 \end{bmatrix} \quad \text{using (6.4.5}a\text{)} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 0 \end{bmatrix} - \begin{bmatrix} 1 \cdot \bar{1} \\ 2 \cdot \bar{1} \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 0 \end{bmatrix} \quad \text{from (6.3.18)} \\
&= \sqrt{2}\begin{bmatrix} \bar{1} \cdot \bar{1} \\ 2 \cdot 0 \end{bmatrix}.
\end{aligned} \qquad (6.5.13a)
$$

As a further example:

$$
\begin{aligned}
B_0{}^{\bar{2}}\begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 2 \end{bmatrix} &= \Lambda_0{}^{\bar{2}}\begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 2 \end{bmatrix} + E_0{}^{\bar{2}}\begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 2 \end{bmatrix} - E_2{}^{0}\begin{bmatrix} 1 \cdot \bar{2} \\ \bar{2} \cdot 2 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \cdot \bar{2} \\ 2 \cdot 2 \end{bmatrix} - \begin{bmatrix} 1 \cdot 2 \\ \bar{2} \cdot 0 \end{bmatrix}.
\end{aligned} \qquad (6.5.13b)
$$

These two calculations provide the 3th and 12th columns of the following explicit representation matrix $\Gamma^{[\Delta_2;1,1]}(B_0{}^{\bar{2}})$ for $B_0{}^{\bar{2}}$:

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{3}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{3}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot & -1 & \cdot & \cdot & \cdot
\end{pmatrix}.
$$

Similarly, the action of $B_1{}^0$ on each of the twenty basis elements of $O^{\Delta_2;1,1}$ yields the following explicit representation matrix $\Gamma^{[\Delta_2;1,1]}(B_1{}^0)$ for $B_1{}^0$:

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\frac{3}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{3}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\frac{3}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\frac{3}{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{2}} & \cdot \\
\cdot & \cdot & \cdot & -\sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & -1 & \cdot & \frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{2}} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix},
$$

It may be verified that these matrices satisfy the commutation relation:

$$[\Gamma^{[\Delta_2;1,1]}(B_1{}^2), \Gamma^{[\Delta_2;1,1]}(B_0{}^{\bar{2}})] = -\,\Gamma^{[\Delta_2;1,1]}(B_1{}^0). \qquad (6.5.14)$$

Since this is the representation analogue of (2.2.22), this provides a verification of the techniques presented in this chapter.

Using the techniques of this chapter, the characters of the elements of $O(m)$ with determinant -1 may be obtained directly.

Theorem 6.5.15. *If $m = 2r+1$ is odd, then the character of the representation $[\Delta_r;\lambda]$ is given by:*

$$[\Delta_r;\lambda](y) = \pm i^r \sum_{T^{\Delta_r;\lambda}:T^{\Delta_r;\lambda}\, O(m)-standard} (-1)^{n_0(T^{\Delta_r;\lambda})} \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} y^{T^{\Delta_r;\lambda}}, \qquad (6.5.15a)$$

for those elements of $O(2r+1)$ with eigenvalues $y_1^{-1}, y_1, y_2^{-1}, y_2, \ldots, y_r^{-1}, y_r, -1$ and hence determinant -1, where $(y) = (y_1, y_2, \ldots, y_r)$, $s_j = T^{\Delta_r;\lambda}_{(j,0)}$ for each $T^{\Delta_r;\lambda}$ and $y^{T^{\Delta_r;\lambda}} = y_1^{n_1^{O(m)}(T^{\Delta_r;\lambda})} y_2^{n_2^{O(m)}(T^{\Delta_r;\lambda})} \cdots y_r^{n_r^{O(m)}(T^{\Delta_r;\lambda})}$. If $m = 2r$ is even, then the character of the representation $[\Delta_r;\lambda]$ is:

$$[\Delta_r;\lambda](y) = 0, \qquad (6.5.15b)$$

for those elements of $O(2r)$ with negative determinant.

Proof. If $m = 2r+1$ consider the following generic element of $O(2r+1)$:

$$G = \begin{pmatrix} y_1^{-1} & 0 \\ 0 & y_1 \end{pmatrix} \oplus \begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} y_r^{-1} & 0 \\ 0 & y_r \end{pmatrix} \oplus -1. \qquad (6.5.15c)$$

It may be verified that

$$\Delta(G) = i^r \begin{pmatrix} -y_1^{-\frac{1}{2}} & 0 \\ 0 & y_1^{\frac{1}{2}} \end{pmatrix} \otimes \begin{pmatrix} -y_2^{-\frac{1}{2}} & 0 \\ 0 & y_2^{\frac{1}{2}} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} -y_r^{-\frac{1}{2}} & 0 \\ 0 & y_r^{\frac{1}{2}} \end{pmatrix} \qquad (6.5.15d)$$

satisfies (6.2.23) and the unprimed (6.2.11) when C satisfies (6.2.21). The action of $\Delta(G)$ on T^{Δ_r} yields: $i^r \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} y^{T^{\Delta_r}} T^{\Delta_r}$ and hence, by (6.5.1), on $[T^{\Delta_r;\lambda}]$, to yield:

$$G[T^{\Delta_r;\lambda}] = i^r \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} y^{T^{\Delta_r;\lambda}} T^{\Delta_r;\lambda}. \qquad (6.5.15e)$$

These coefficients thus appear on the diagonal of the matrix representing G. Summing over the basis of traceless symmetrised $O(m)$-standard tableaux then proves (6.5.15a).

For $m = 2r$, consider the following generic element of $O(2r)$:

$$G = \begin{pmatrix} 0 & y_1 \\ y_1^{-1} & 0 \end{pmatrix} \oplus \begin{pmatrix} y_2^{-1} & 0 \\ 0 & y_2 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} y_r^{-1} & 0 \\ 0 & y_r \end{pmatrix}. \qquad (6.5.15f)$$

It may be verified that

$$\Delta(G) = i^{r-1}\begin{pmatrix} 0 & y_1^{\frac{1}{2}} \\ y_1^{-\frac{1}{2}} & 0 \end{pmatrix} \otimes \begin{pmatrix} -y_2^{-\frac{1}{2}} & 0 \\ 0 & y_2^{\frac{1}{2}} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} -y_r^{-\frac{1}{2}} & 0 \\ 0 & y_r^{\frac{1}{2}} \end{pmatrix} \tag{6.5.15g}$$

satisfies (6.2.4) and the unprimed (6.2.11) when C satisfies (6.2.8). Then, by (6.5.1), the action of G on $[T^{\Delta_r;\lambda}]$ yields a multiple of $[T'^{\Delta_r;\lambda}]$ where $T'^{\Delta_r;\lambda}$ is identical to $T^{\Delta_r;\lambda}$ except that each 1 is changed to a $\bar{1}$ and vice-versa. If $T^{\Delta_r;\lambda}$ is $O(m)$-standard then $T'^{\Delta_r;\lambda}$ is also $O(m)$-standard. Thus, in the basis of traceless symmetrised $O(m)$-standard tableaux, each diagonal entry of the representation matrix is zero. This proves (6.5.15b).

§6.6. The irreducible spinor modules of $SO(m)$

In this section, the reducibility of the $O(2r)$-modules $O^{\Delta_r;\lambda}$ on restriction to $SO(2r)$ is demonstrated and, once more, bases for the irreducible components are derived in terms of Young tableaux.

In this section, $m = 2r$ will be even. The element $\eta_{2r} \in \mathcal{N}_{2r}$ defined by (6.1.8a) is, as shown by (6.1.9a), represented by $\gamma_{\eta_{2r}} = \gamma_0$, itself defined by (6.1.4c). By (6.3.4), the action of γ_0 on each basis element $\psi_{s_1\cdots s_r} \in \Psi$ is given by:

$$\gamma_0\psi_{s_1\cdots s_r} = \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} \psi_{s_1\cdots s_r}. \tag{6.6.1}$$

Then, by virtue of (6.2.16) and (6.2.18),

$$\left\{ \psi_{s_1\cdots s_r} : \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} = +1 \right\} \tag{6.6.2a}$$

is a basis for $O^{\Delta_r^+}$, and

$$\left\{ \psi_{s_1\cdots s_r} : \begin{bmatrix} s_1 \\ \vdots \\ s_r \end{bmatrix} = -1 \right\} \tag{6.6.2b}$$

is a basis for $O^{\Delta_r^-}$, where $O^{\Delta_r^+}$ and $O^{\Delta_r^-}$ are both $SO(2r)$-submodules of O^{Δ_r}. The following definition reflects this observation.

Definition 6.6.3. *Let $m = 2r$, $\lambda \in P(l; r)$ and let $\mathcal{O}_m^{\Delta_r;\lambda}$ denote the set of all $O(m)$-standard tableaux of shape $F^{\Delta_r;\lambda}$. Then define*

$$\begin{aligned} \mathcal{S}^{\Delta_r;\lambda+} &= \left\{ T^{\Delta_r;\lambda} \in \mathcal{O}^{\Delta_r;\lambda} : \#\{j : 1 \le j \le r, T^{\Delta_r;\lambda}_{(j,0)} = \bar{j}\} \in 2\mathbf{Z} \right\} \\ \textit{and} \quad \mathcal{S}^{\Delta_r;\lambda-} &= \left\{ T^{\Delta_r;\lambda} \in \mathcal{O}^{\Delta_r;\lambda} : \#\{j : 1 \le j \le r, T^{\Delta_r;\lambda}_{(j,0)} = \bar{j}\} \in 2\mathbf{Z}+1 \right\}. \end{aligned} \tag{6.6.3}$$

Thus there are an even number of barred indices in the half boxes of each $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda+}$ and an odd number of barred indices in the half boxes of each $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda-}$.

Lemma 6.6.4. $\#\mathcal{S}^{\Delta_r;\lambda+} = \#\mathcal{S}^{\Delta_r;\lambda-}$.

Proof. Let $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda+}$ and let $T^{\Delta_r;\lambda*}$ be identical to $T^{\Delta_r;\lambda}$ except that each index $\bar{1}$ is changed to 1 and vice-versa. By Definition 6.4.6, $T^{\Delta_r;\lambda}$ contains at least one $\bar{1}$ or 1 but it may not contain both. It is then straightforward to see that $T^{\Delta_r;\lambda*}$ is $O(m)$-standard, and thus $T^{\Delta_r;\lambda*} \in \mathcal{S}^{\Delta_r;\lambda-}$. This demonstrates a bijection between $\mathcal{S}^{\Delta_r;\lambda-}$ and $\mathcal{S}^{\Delta_r;\lambda+}$, thus proving the lemma.

Now let $S^{\Delta_r;\lambda+}$ and $S^{\Delta_r;\lambda-}$ denote the vector subspaces of the $O(m)$-module $O^{\Delta_r;\lambda}$ spanned by $[T^{\Delta_r;\lambda}]$ for $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda+}$ and $[T^{\Delta_r;\lambda}]$ for $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda-}$ respectively.

Theorem 6.6.5. *If $m = 2r$, $\lambda \in P(l;r)$ and $\hat{t}_{(u,v)}$ exists for all $u + v \leq \tilde{\lambda}_1$ then $S^{\Delta_r;\lambda+}$ and $S^{\Delta_r;\lambda-}$ are inequivalent irreducible $SO(2r)$-submodules of $O^{\Delta_r;\lambda}$ under the induced action of* (6.5.1).

Proof. To prove that they are $SO(2r)$-modules, it is sufficient to demonstrate closure. If $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda+}$ then, through (6.5.1), the action of $G \in SO(2r)$ on $[T^{\Delta_r;\lambda}]$ results in a linear combination of traceless symmetrised tableaux each of which, by virtue of (6.2.18), contains an even number of barred indices in the half boxes. Since the polynomials used in the standardisation procedures are homogeneous, the number of indices that are raised to the power of exactly 1, is even or odd for all the terms. Thus, within each tableaux identity, the number of barred indices in the half boxes of each of the tableaux appearing differ from each other by an even number. Thus, standardisation results in an expression involving tableaux from $\mathcal{S}^{\Delta_r;\lambda+}$ solely. A similar argument holds for $T^{\Delta_r;\lambda} \in \mathcal{S}^{\Delta_r;\lambda-}$ and closure is proved.

On restriction to $SO(2r)$, the irreducible representation $[\Delta_r;\lambda]$ of $O(2r)$ decomposes into the direct sum of two irreducible representations of the same dimension. Thus the $SO(2r)$-modules $S^{\Delta_r;\lambda+}$ and $S^{\Delta_r;\lambda-}$ are irreducible. They are inequivalent by virtue of differing highest weights: $(\Delta_r;\lambda)_+$ for $[T^{\Delta_r;\lambda}_{>}] \in S^{\Delta_r;\lambda+}$ and $(\Delta_r;\lambda)_-$ for $[T^{\Delta_r;\lambda}_{\underline{>}}] \in S^{\Delta_r;\lambda+}$, where $T^{\Delta_r;\lambda}_{\underline{>}}$ is obtained from $T^{\Delta_r;\lambda}_{>}$ by exchanging each entry r for $\bar{r}$.

Although an analogue to Theorem 5.3.27 could now be stated, this is not necessary since, on restriction to $SO(2r)$, the $O(2r)$-module $O^{\Delta_r;\lambda}$ decomposes naturally with respect to the basis of traceless symmetrised $O(2r)$-standard tableaux; there being no need to quotient out the invariant subspaces.

To illustrate the above, consider the 10-dimensional $SO(4)$-module $S^{\Delta_2;2,1-}$, for which the $SO(4)$-standard tableaux are:

$$\begin{array}{l} \begin{matrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 2 & \end{matrix}\,, \quad \begin{matrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{matrix}\,, \quad \begin{matrix} \bar{1}\cdot\bar{1} & 2 \\ 2\cdot 2 & \end{matrix}\,, \quad \begin{matrix} \bar{1}\cdot\bar{2} & \bar{2} \\ 2\cdot 2 & \end{matrix}\,, \quad \begin{matrix} \bar{1}\cdot\bar{2} & 2 \\ 2\cdot 2 & \end{matrix}\,, \\ \begin{matrix} 1\cdot 1 & 1 \\ \bar{2}\cdot\bar{2} & \end{matrix}\,, \quad \begin{matrix} 1\cdot 1 & \bar{2} \\ \bar{2}\cdot\bar{2} & \end{matrix}\,, \quad \begin{matrix} 1\cdot 1 & 2 \\ \bar{2}\cdot\bar{2} & \end{matrix}\,, \quad \begin{matrix} 1\cdot\bar{2} & \bar{2} \\ \bar{2}\cdot 2 & \end{matrix}\,, \quad \begin{matrix} 1\cdot\bar{2} & 2 \\ \bar{2}\cdot 2 & \end{matrix}\,. \end{array} \tag{6.6.20}$$

Then, by (6.6.14),

$$\begin{aligned} {D_1}^2 \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} &= {\Lambda_1}^2 \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} + {E_1}^2 \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} - E_{\bar{2}}^{\bar{1}} \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} \\ &= - \begin{bmatrix} 1\cdot\bar{1} & \bar{1} \\ \bar{2}\cdot 2 & \end{bmatrix} + \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 1 & \end{bmatrix} \\ &\qquad - \begin{bmatrix} \bar{1}\cdot\bar{2} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} - \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix}. \end{aligned} \tag{6.6.21a}$$

The Spinor relations yield the identities:

$$- \begin{bmatrix} 1\cdot\bar{1} & \bar{1} \\ \bar{2}\cdot 2 & \end{bmatrix} = - \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix}, \tag{6.6.21b}$$

$$\begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 1 & \end{bmatrix} = - \begin{bmatrix} 1\cdot\bar{1} & \bar{1} \\ \bar{2}\cdot 2 & \end{bmatrix} = - \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix}, \tag{6.6.21c}$$

and, together with a Garnir relation,

$$\begin{aligned} - \begin{bmatrix} \bar{1}\cdot\bar{2} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} &= \begin{bmatrix} \bar{1}\cdot\bar{1} & 2 \\ 2\cdot\bar{2} & \end{bmatrix} - \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot\bar{1} & \bar{2} \\ \bar{2}\cdot\bar{1} & \end{bmatrix} - \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix} = - \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix}. \end{aligned} \tag{6.6.21d}$$

Combining these, results in:

$${D_1}^2 \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{1} \\ 2\cdot 2 & \end{bmatrix} = -4 \begin{bmatrix} \bar{1}\cdot\bar{1} & \bar{2} \\ 2\cdot 2 & \end{bmatrix}. \tag{6.6.21e}$$

As a second example, consider:

$$\begin{aligned} {D_1}^2 \begin{bmatrix} 1\cdot\bar{2} & 2 \\ \bar{2}\cdot 2 & \end{bmatrix} &= \begin{bmatrix} 1\cdot\bar{2} & 1 \\ \bar{2}\cdot 2 & \end{bmatrix} + \begin{bmatrix} 1\cdot\bar{2} & 2 \\ \bar{2}\cdot 1 & \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot 1 & \bar{2} \\ \bar{2}\cdot 2 & \end{bmatrix} - \begin{bmatrix} 1\cdot 1 & 2 \\ \bar{2}\cdot\bar{2} & \end{bmatrix} - \begin{bmatrix} 1\cdot 1 & 2 \\ \bar{2}\cdot\bar{2} & \end{bmatrix} \\ &= - \begin{bmatrix} \bar{1}\cdot 1 & \bar{2} \\ 2\cdot 1 & \end{bmatrix} - 2 \begin{bmatrix} 1\cdot 1 & 2 \\ \bar{2}\cdot\bar{2} & \end{bmatrix} = -2 \begin{bmatrix} 1\cdot 1 & 2 \\ \bar{2}\cdot\bar{2} & \end{bmatrix}, \end{aligned} \tag{6.6.22}$$

where the standardisation has involved the Garnir relations (3.4.3), the Column relations (3.4.2) and the Spinor relations (6.3.18). The above two calculation gives rise to the 1st and 10th columns of the following explicit representation matrix for $D_1{}^2$:

$$\Gamma^{[\Delta_2;2,1-]}(D_1{}^2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 \\ \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

The highest weight vector of the irreducible $so(4)$-module $S^{\Delta;2,1-}$ considered above is $\begin{bmatrix} 1 & \cdot & 1 & 1 \\ \bar{2} & \cdot & \bar{2} & \end{bmatrix}$, for which:

$$B_1{}^1 \begin{bmatrix} 1 & \cdot & 1 & 1 \\ \bar{2} & \cdot & \bar{2} & \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 & \cdot & 1 & 1 \\ \bar{2} & \cdot & \bar{2} & \end{bmatrix} \quad \text{and} \quad B_2{}^2 \begin{bmatrix} 1 & \cdot & 1 & 1 \\ \bar{2} & \cdot & \bar{2} & \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} 1 & \cdot & 1 & 1 \\ \bar{2} & \cdot & \bar{2} & \end{bmatrix},$$

confirming that its highest weight is $(\frac{5}{2}, -\frac{3}{2})$.

Recall that for the $O(m)$-modules $O^{\Delta_r;\lambda}$ considered in Section 6.5, the matrix elements obtained for the representations $[\Delta_r; \lambda]$ of $O(2r)$ are always rational. In view of the natural decomposition on restriction to $SO(2r)$, this also holds for the representations $[\Delta_r; \lambda]_\pm$ of $SO(2r)$.

Chapter 7
Modules of Lie supergroups and Lie superalgebras

§7.1. Grassmann algebras and Lie supergroups

In this chapter, Young tableaux techniques, similar in flavour to those of the previous chapters, are developed to deal with modules of Lie supergroups and Lie superalgebras. In particular, application of the double centraliser technique enables the irreducible covariant tensor modules of $GL(m/n)$ to be constructed. This involves the generalisation of the Column and Garnir relations to take account of the $\mathbb{Z}_2$-grading of the Grassmann parameters which occur here as matrix elements. These techniques extend, in a fairly straightforward way, to the Lie superalgebras $gl(m/n)$ and $sl(m/n)$. The following account of Grassmann algebras and Lie supergroups is based on that given in [Co89].

Define $B = \mathsf{F}B_L$ to be the exterior algebra of $\{\zeta_1, \zeta_2, \ldots, \zeta_L\}$ over the field F, where L is arbitrary. B is known as a Grassmann algebra and its elements are referred to as Grassmann parameters. Let the exterior product $\zeta_{i_1} \wedge \zeta_{i_2} \wedge \ldots \wedge \zeta_{i_l} \in B$ be denoted by $\zeta_{i_1 i_2 \cdots i_l}$. Then $\zeta_{i_2 i_1} = -\zeta_{i_1 i_2}$, and more generally if $\sigma \in S_l$ then:

$$\zeta_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(l)}} = (-1)^{\sigma} \zeta_{i_1 i_2 \cdots i_l}. \tag{7.1.1}$$

The product of the elements $\zeta_{i_1 i_2 \cdots i_s}, \zeta_{j_1 j_2 \cdots j_t} \in B$ is given by:

$$\zeta_{i_1 i_2 \cdots i_s} \zeta_{j_1 j_2 \cdots j_t} = \zeta_{i_1 i_2 \cdots i_s j_1 j_2 \cdots j_t}, \tag{7.1.2}$$

which extends linearly over F in both factors. In general this product does not commute. If $s + t > L$ then (7.1.1) necessarily implies that $\zeta_{i_1 i_2 \cdots i_s j_1 j_2 \cdots j_t} = 0$.

Let $\mu = \{i_1, i_2, \ldots, i_l\}$ where $1 \leq i_1 < i_2 < \cdots < i_l \leq L$ and define $\zeta_\mu = \zeta_{i_1 i_2 \cdots i_l}$. Denote by B_l the subspace of B whose basis is the set of all l-fold exterior products of the generators $\{\zeta_1, \zeta_2, \ldots, \zeta_L\}$. Thus B_l has a basis $\{\zeta_{i_1 i_2 \cdots i_l} : 1 \leq i_1 < i_2 < \cdots < i_l \leq L\} = \{\zeta_\mu : \mu \subset \mathsf{N}_L, \#\mu = l\}$ and B_0 has basis $\{\zeta_\emptyset = 1 \in \mathsf{F}\}$. Then $b \in B$ may be written $\sum_\mu b_\mu \zeta_\mu$ where each $b_\mu \in \mathsf{F}$ and the sum is over all subsets μ of N_L. Let $B_{\bar{0}} = B_0 \oplus B_2 \oplus B_4 \oplus \cdots$ and $B_{\bar{1}} = B_1 \oplus B_3 \oplus B_5 \oplus \cdots$. Then $B = B_{\bar{0}} \oplus B_{\bar{1}}$ with both $B_{\bar{0}}$ and $B_{\bar{1}}$ having dimension 2^{L-1}, whereupon B has dimension 2^L. It then follows immediately from (7.1.1) and (7.1.2) that:

$$f^0 \wedge g^0 = g^0 \wedge f^0, \quad f^0 \wedge g^1 = g^1 \wedge f^0 \quad \text{and} \quad f^1 \wedge g^1 = -g^1 \wedge f^1, \tag{7.1.3}$$

for all $f^0, g^0 \in B_{\bar{0}}$ and $f^1, g^1 \in B_{\bar{1}}$.

The properties of the Grassmann algebra are typical of a structure with a $\mathbf{Z}_2$-grading. In view of (7.1.3), the $\mathbf{Z}_2$-graded subspaces, $B_{\bar{0}}$ and $B_{\bar{1}}$, of B, are known as the even and odd subspaces respectively. Their elements are known as even or odd Grassmann parameters respectively. Each element $b \in B$ may be written $b = b_{\bar{0}} + b_{\bar{1}}$ where $b_{\bar{0}} \in B_{\bar{0}}$ is even and $b_{\bar{1}} \in B_{\bar{1}}$ is odd. If $b \neq 0$ and either $b_{\bar{0}} = 0$ or $b_{\bar{1}} = 0$ then b is said to be an homogeneous element of B. In such a case the degree of b, denoted $\deg b$, is defined to be:

$$\deg b = \begin{cases} 0, & \text{if } b_{\bar{1}} = 0; \\ 1, & \text{if } b_{\bar{0}} = 0. \end{cases} \tag{7.1.4}$$

Let $B^{m,n}$ be the vector space $B_{\bar{0}}^{\oplus m} \oplus B_{\bar{1}}^{\oplus n}$. A typical element of $B^{m,n}$ is then $X = (X^{\bar{0}}; X^{\bar{1}}) = (X_1^{\bar{0}}, X_2^{\bar{0}}, \ldots, X_m^{\bar{0}}; X_{m+1}^{\bar{1}}, \ldots, X_{m+n}^{\bar{1}})$ where each $X_i^{\bar{0}} \in B_{\bar{0}}$ and each $X_i^{\bar{1}} \in B_{\bar{1}}$. It is convenient to define the index sets $\mathcal{I}_{\bar{0}} = \{1, 2, \ldots, m\}$, $\mathcal{I}_{\bar{1}} = \{m+1, m+2, \ldots, m+n\}$ and $\mathcal{I}^{GL(m/n)} = \mathcal{I}_{\bar{0}} \cup \mathcal{I}_{\bar{1}}$. $B^{m,n}$ is naturally $\mathbf{Z}_2$-graded, having a $\mathbf{Z}_2$-graded basis $\{e_i : i \in \mathcal{I}^{GL(m/n)}\}$. A typical element of $B^{m,n}$ may thus be expressed as $X = (X^{\bar{0}}, X^{\bar{1}}) = \sum_{i \in \mathcal{I}^{GL(m/n)}} X_i e_i$, where $\deg X_i = 0$ if $i \in \mathcal{I}_{\bar{0}}$ and $\deg X_i = 1$ if $i \in \mathcal{I}_{\bar{1}}$.

In view of the above, the notion of graded indices is useful and may be employed via the notation:

$$\operatorname{grad} i = (i) = \begin{cases} 0, & \text{if } i \in \mathcal{I}_{\bar{0}}; \\ 1, & \text{if } i \in \mathcal{I}_{\bar{1}}. \end{cases} \tag{7.1.5}$$

Thus, if $X = \sum_{i \in \mathcal{I}^{GL(m/n)}} X_i e_i \in B^{m,n}$ then $X_i \in B_{\overline{(i)}}$. A further useful notation assigns to the symbol:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\lambda_1} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ a_{\bar{\lambda}_1 1} & & & \end{bmatrix}, \tag{7.1.6}$$

the value $(-1)^{((a_{11})+(a_{21})+\cdots+(a_{\bar{\lambda}_1 1}))((a_{12})+\cdots+(a_{\bar{\lambda}_2 2}))\cdots((a_{1\lambda_1})\cdots)}$, that is -1 to the power of the product of the column sums of the respective gradings (which may all be taken mod2). With this notation (7.1.3) may be expressed $f^a \wedge g^b = [a\, b]\, g^b \wedge f^a$ where f^a and g^b are homogeneous, $f^a \in B_{\overline{(a)}}$ and $g^b \in B_{\overline{(b)}}$.

Definition 7.1.7. *A square even (m/n)-supermatrix is a matrix of the form:*

$$G = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{7.1.7}$$

where P, Q, R and S are submatrices of sizes $m \times m$, $m \times n$, $n \times m$ and $n \times n$ respectively, with $P_{ij} \in B_{\bar{0}}$, $Q_{ik} \in B_{\bar{1}}$, $R_{lj} \in B_{\bar{1}}$ and $S_{lk} \in B_{\bar{0}}$ for $1 \leq i,j \leq m$ and $1 \leq k,l \leq n$. This may be expressed by $G_{ij} \in B_{\overline{(i)+(j)}}$ where the sum is taken mod2. *In a similar way, an odd (m/n)-supermatrix G is defined to be such that $G_{ij} \in B_{\overline{(i)+(j)+1}}$ for $i,j \in \mathcal{I}^{GL(m/n)}$.*

If G, G' are supermatrices then the product $G'' = GG'$ is defined in the usual way:

$$G''_{ij} = \sum_k G_{ik} G'_{kj}. \tag{7.1.8}$$

Since multiplication of Grassmann parameters is associative, this definition immediately implies that supermatrix multiplication is associative, for if G, G', G'' are all (m/n)-supermatrices then $(GG')G'' = G(G'G'')$. The identity (m/n)-supermatrix is provided by I_{m+n} for which the only non-zero entries are 1s, appearing in each diagonal position, where $1 \in B_0$ is the identity element of B. I_{m+n} is clearly an even supermatrix. An (m/n)-supermatrix G is said to be invertible if and only if there exists a supermatrix G^{-1} such that $GG^{-1} = G^{-1}G = I_{m+n}$. The following lemma concerning invertible square even supermatrices is proved in [**Co89**].

Lemma 7.1.9. *(i) Let G be an $(m/0)$-supermatrix with $G = \sum_\mu \zeta_\mu G_\mu$ where each G_μ is an $m \times m$ matrix with entries from* F. *G is invertible if and only if $G_\emptyset$, corresponding to the Grassmann identity $\zeta_\emptyset = 1$, is invertible.*

(ii) Let G be an even (m/n)-supermatrix, partitioned as in (7.1.7). *G is invertible if and only if the submatrices P and S are invertible.*

(iii) If G and G' are invertible even (m/n)-supermatrices, then GG' is also an invertible even (m/n)-supermatrix.

The general definition of a Lie supergroup $\mathcal{G}_s$ of even dimension p and odd dimension q states that the elements form an superanalytic supermanifold locally isomorphic to the real superspace $\mathbb{R}B^{p,q}$. Let $(X;Y)$ denote an element of $B^{p,q}$ where $(X;Y) = (X_1, X_2, \ldots, X_p; Y_1, Y_2, \ldots, Y_q)$ with each $X_h \in B_{\bar{0}}$ and each $Y_k \in B_{\bar{1}}$. Then in a neighbourhood of the identity, a matrix realisation of $\mathcal{G}_s$, consisting of even (m/n)-supermatrices $G(X;Y)$, is parameterised by neighbourhood of $(0;0) \in \mathbb{R}B^{p,q}$. However, it is not required that each matrix element $G(X;Y)_{ij}$ is a function on the whole of $\mathbb{R}B^{p,q}$, but that the even elements, where $(i) = (j)$, are functions on $\mathbb{R}B^{p,0}$ so that in this case $G(X;Y)_{ij} = G(X)_{ij}$; and similarly the odd elements, where $(i) \neq (j)$, are functions on $\mathbb{R}B^{0,q}$ so that in this case $G(X;Y)_{ij} = G(Y)_{ij}$. There arise special difficulties when attempting to define the derivative of a function with respect to a Grassmann variable. These were overcome by Rogers (see [**Co89**]) who was able to obtain derivatives having standard properties. In connection with Lie supergroups, she found it especially convenient to use $L \geq 2q$. Here, it suffices to say that the analyticity ensures that the derivatives:

$$\left.\frac{\partial G_{ij}}{\partial X_h}\right|_{(0;0)} \quad \text{and} \quad \left.\frac{\partial G_{ij}}{\partial Y_k}\right|_{(0;0)} \tag{7.1.10}$$

exist for $i, j \in \mathcal{I}^{GL(m/n)}$, $h = 1, \ldots, p$ and $k = 1, \ldots, q$. Incidentally, each of these derivatives is an element of $B_{\bar{0}}$.

The following definition generalises that of $GL(m, \mathsf{F})$.

Definition 7.1.11. *The General Linear Lie supergroup $GL(m/n, B)$ is the set of all invertible even (m/n)-supermatrices whose entries are members of the Grassmann algebra $B = \mathsf{F}B_L$.*

It may be shown that $GL(m/n, B)$ has even dimension $m^2 + n^2$ and odd dimension $2mn$. Thus L will be taken to be such that $L \geq 4mn$.

§7.2. Covariant tensor $GL(m/n)$-modules

In this section, the tensor product space of the defining $GL(m/n)$-module is defined. This is decomposed by means similar to those used in Section 4.2, yielding the irreducible covariant $GL(m/n)$-modules, based on Young tableaux.

Let $V_s = B^{m,n}$ be the $\mathbf{Z}_2$-graded defining $GL(m/n)$-module. The l-fold tensor product space $V_s^{\otimes l}$ has a $\mathbf{Z}_2$-graded basis $\{e_{i_1 i_2 \cdots i_l} : i_k \in \mathcal{I}^{GL(m/n)} \text{ for } k = 1, 2, \ldots, l\}$ where $e_{i_1 i_2 \cdots i_l} = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_l}$ and for which:

$$\deg e_{i_1 i_2 \cdots i_l} = (i_1) + (i_2) + \cdots + (i_l) \bmod 2. \tag{7.2.1}$$

Define the 'diagonal' action of $G \in GL(m/n)$ on $V_s^{\otimes l}$ by linearly extending the following action on its basis elements:

$$G e_{i_1 i_2 \cdots i_l} = \sum_{j_1, j_2, \ldots, j_l} \prod_{1 \leq a < b \leq l} [j_a\, j_b][j_a\, i_b]\, G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l}\, e_{j_1 j_2 \cdots j_l}. \tag{7.2.2a}$$

Note that this may be expressed in the form:

$$G e_{i_1 i_2 \cdots i_l} = \sum_{j_1, j_2, \ldots, j_l} e_{j_1} G^{j_1}{}_{i_1} \otimes e_{j_2} G^{j_2}{}_{i_2} \otimes \cdots \otimes e_{j_l} G^{j_l}{}_{i_l}, \tag{7.2.2b}$$

where rearrangement to give (7.2.2a) involves consideration of both the degree of the Grassmann parameters $G^j{}_i$ and the degree of the basis elements e_i of V_s [**Ba85**].

Lemma 7.2.3. *If $V_s = B^{m,n}$ then $V_s^{\otimes l}$ is a $GL(m/n)$-module.*

Proof. It is only necessary to check that $(G'G)e_{i_1 i_2 \cdots i_l} = G'(Ge_{i_1 i_2 \cdots i_l})$. With the summation convention adopted, (7.2.2a) gives:

$$\begin{aligned}
G'(Ge_{i_1 i_2 \cdots i_l}) &= G'\left(\prod_{a\le b}[j_a\, j_b][j_a\, i_b]\, G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l} e_{j_1 j_2 \cdots j_l}\right) \\
&= \prod_{a\le b}[j_a\, j_b][j_a\, i_b][k_a\, k_b][k_a\, j_b] \\
&\qquad G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l} G'^{k_1}{}_{j_1} G'^{k_2}{}_{j_2} \cdots G'^{k_l}{}_{j_l} e_{k_1 k_2 \cdots k_l} \\
&= \prod_{a\le b}[j_a\, j_b][j_a\, i_b][k_a\, k_b][k_a\, j_b] \begin{bmatrix} j_a & i_b \\ k_a & j_b \end{bmatrix} \\
&\qquad G'^{k_1}{}_{j_1} G^{j_1}{}_{i_1} G'^{k_2}{}_{j_2} G^{j_2}{}_{i_2} \cdots G'^{k_l}{}_{j_l} G^{j_l}{}_{i_l} e_{k_1 k_2 \cdots k_l} \\
&= \prod_{a\le b}[k_a\, k_b][k_a\, i_b]\, (G'G)^{k_1}{}_{i_1} (G'G)^{k_2}{}_{i_2} \cdots (G'G)^{k_l}{}_{i_l} e_{k_1 k_2 \cdots k_l} \\
&= (G'G)e_{i_1 i_2 \cdots i_l}.
\end{aligned}$$

This proves the lemma.

Definition 7.2.4. **[DJ81,BR87]** *If $\pi \in S_l$, then its graded action $\tilde{\pi}$ on $V_s^{\otimes l}$ is defined by:*

$$\begin{aligned}
\tilde{\pi} e_{i_1 i_2 \cdots i_l} &= \prod_{\substack{1\le a<b\le l \\ \pi(a)>\pi(b)}} [i_a\, i_b]\, (\pi e_{i_1 i_2 \cdots i_l}) \\
&= \prod_{\substack{1\le a<b\le l \\ \pi(a)>\pi(b)}} [i_a\, i_b]\, e_{i_{\pi^{-1}(1)} i_{\pi^{-1}(2)} \cdots i_{\pi^{-1}(l)}}.
\end{aligned} \tag{7.2.4}$$

for $\pi \in S_l$ with linear extension to the whole of $V_s^{\otimes l}$. The set of all actions $\tilde{\pi}$ for $\pi \in S_l$ is denoted $\tilde{S}_l$ and known as the graded symmetric group.

Lemma 7.2.5. **[DJ81]** *The actions of $\tilde{S}_l$ on $V_s^{\otimes l}$ form a group isomorphic to S_l. Moreover, if $\rho, \sigma, \tau \in S_l$ with $\rho\sigma = \tau$ then $\tilde{\rho}\tilde{\sigma} = \tilde{\tau}$.*

Proof. Consider the action on the element $e_{i_1i_2\cdots i_l} \in V_s^{\otimes l}$:

$$\begin{aligned}
\tilde{\rho}\tilde{\sigma} e_{i_1i_2\cdots i_l} &= \tilde{\rho} \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a\, i_b]\, e_{i_{\sigma^{-1}(1)} i_{\sigma^{-1}(2)} \cdots i_{\sigma^{-1}(l)}} \\
&= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a\, i_b] \prod_{\substack{a<b \\ \rho(a)>\rho(b)}} [i_{\sigma^{-1}(a)}\, i_{\sigma^{-1}(b)}]\, e_{i_{\sigma^{-1}\rho^{-1}(1)} i_{\sigma^{-1}\rho^{-1}(2)} \cdots i_{\sigma^{-1}\rho^{-1}(l)}} \\
&= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b)}} [i_a\, i_b] \prod_{\substack{\sigma(c)<\sigma(d) \\ \rho\sigma(c)>\rho\sigma(d)}} [i_c\, i_d]\, e_{i_{\tau^{-1}(1)} i_{\tau^{-1}(2)} \cdots i_{\tau^{-1}(l)}} \\
&= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b) \\ \rho\sigma(a)<\rho\sigma(b)}} [i_a\, i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b) \\ \rho\sigma(a)>\rho\sigma(b)}} [i_a\, i_b] \prod_{\substack{c<d \\ \sigma(c)<\sigma(d) \\ \rho\sigma(c)>\rho\sigma(d)}} [i_c\, i_d] \prod_{\substack{c>d \\ \sigma(c)<\sigma(d) \\ \rho\sigma(c)>\rho\sigma(d)}} [i_c\, i_d]\, \tau e_{i_1i_2\cdots i_l} \\
&= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b) \\ \rho\sigma(a)<\rho\sigma(b)}} [i_a\, i_b] \prod_{\substack{a<b \\ \sigma(a)>\sigma(b) \\ \rho\sigma(a)>\rho\sigma(b)}} [i_a\, i_b] \prod_{\substack{c<d \\ \sigma(c)<\sigma(d) \\ \rho\sigma(c)>\rho\sigma(d)}} [i_c\, i_d] \prod_{\substack{c<d \\ \sigma(c)>\sigma(d) \\ \rho\sigma(c)<\rho\sigma(d)}} [i_c\, i_d]\, \tau e_{i_1i_2\cdots i_l} \\
&= \prod_{\substack{a<b \\ \sigma(a)>\sigma(b) \\ \rho\sigma(a)>\rho\sigma(b)}} [i_a\, i_b] \prod_{\substack{c<d \\ \sigma(c)<\sigma(d) \\ \rho\sigma(c)>\rho\sigma(d)}} [i_c\, i_d]\, \tau e_{i_1i_2\cdots i_l} \\
&= \prod_{\substack{a<b \\ \tau(a)>\tau(b)}} [i_a\, i_b] \tau e_{i_1i_2\cdots i_l} = \tilde{\tau} e_{i_1i_2\cdots i_l}.
\end{aligned}$$

This extends linearly to the whole of $V_s^{\otimes l}$ and the lemma is proved.

A general transformation, $A \in \mathrm{End}(V_s^{\otimes l})$, of the $\mathbf{Z}_2$-graded tensor space $V_s^{\otimes l}$ takes the form:

$$Ae_{i_1i_2\cdots i_l} = \sum_{1\le j_1,\ldots,j_l\le m+n} A^{j_1j_2\cdots j_l}_{i_1i_2\cdots i_l} e_{j_1j_2\cdots j_l}, \tag{7.2.6}$$

where if $g = (j_1) + (j_2) + \cdots + (j_l) + (i_1) + \cdots + (i_l) \bmod 2$, then $A^{j_1j_2\cdots j_l}_{i_1i_2\cdots i_l} \in B_{\bar{g}}$. If $A \in \mathrm{End}(V^{\otimes l})$ commutes with the action of $\mathsf{F}\tilde{S}_l$, so that $\tilde{\pi}Ae_{i_1i_2\cdots i_l} = A\tilde{\pi}e_{i_1i_2\cdots i_l}$ for all $\tilde{\pi} \in \tilde{S}_l$, then (7.2.4) and (7.2.6) imply that:

$$A^{j_1j_2\cdots j_l}_{i_1i_2\cdots i_l} = A^{j_{\pi^{-1}(1)}\cdots j_{\pi^{-1}(l)}}_{i_{\pi^{-1}(1)}\cdots i_{\pi^{-1}(l)}} \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} [j_a\, j_b][i_a\, i_b], \tag{7.2.7}$$

for all $\pi \in S_l$. Thus the elements of $\mathrm{End}_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$ are characterised by (7.2.7). Such elements are known as bisymmetric. It is important to note that if $1 \le c < d \le l$, $j_c = j_d$, $i_c = i_d$ and $(i_c) \ne (j_c)$ then $A^{j_1j_2\cdots j_l}_{i_1i_2\cdots i_l} = 0$ since with $\pi = (c, d)$, the sign factor on the right side of (7.2.7) consists of the one term $[i_c\, i_d][j_c\, j_d]$ which is -1 since $(i_c) = (i_d) \ne (j_c) = (j_d)$.

If $G \in \mathrm{End}(V_s)$, the induced action, $G \in \mathrm{End}(V_s^{\otimes l})$, is given by (7.2.2$a$):

$$G^{j_1j_2\cdots j_l}_{i_1i_2\cdots i_l} = G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l} \prod_{a\le b} [j_a\, j_b][j_a\, i_b]. \tag{7.2.8}$$

Thereupon:

$$G^{j_{\pi^{-1}(1)}\cdots j_{\pi^{-1}(l)}}_{i_{\pi^{-1}(1)}\cdots i_{\pi^{-1}(l)}} = G^{j_{\pi^{-1}(1)}}{}_{i_{\pi^{-1}(1)}} G^{j_{\pi^{-1}(2)}}{}_{i_{\pi^{-1}(2)}} \cdots G^{j_{\pi^{-1}(l)}}{}_{i_{\pi^{-1}(l)}} \prod_{a\leq b} [j_{\pi^{-1}(a)}\, j_{\pi^{-1}(b)}][j_{\pi^{-1}(a)}\, i_{\pi^{-1}(b)}]$$

$$= G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l} \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} \begin{bmatrix} i_a & i_b \\ j_a & j_b \end{bmatrix} \prod_{\pi(a)\leq\pi(b)} [j_a\, j_b][j_a\, i_b]. \tag{7.2.9}$$

Since

$$\prod_{a\leq b} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} [j_a\, j_b][i_a\, i_b] = \prod_{\substack{a\leq b \\ \pi(a)\lesseqgtr\pi(b)}} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} [i_a\, i_b][j_a\, i_b]$$

$$= \prod_{\pi(a)\leq\pi(b)} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a>b \\ \pi(a)<\pi(b)}} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} [i_a\, i_b][j_a\, i_b]$$

$$= \prod_{\pi(a)\leq\pi(b)} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} [j_a\, j_b][i_a\, j_b] \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} [i_a\, i_b][j_a\, i_b]$$

$$= \prod_{\pi(a)\leq\pi(b)} [j_a\, j_b][j_a\, i_b] \prod_{\substack{a<b \\ \pi(a)>\pi(b)}} \begin{bmatrix} i_a & i_b \\ j_a & j_b \end{bmatrix},$$

it follows from (7.2.7), (7.2.8) and (7.2.9) that G is bisymmetric and that the actions of $GL(m/n)$ and $\mathsf{F}\tilde{S}_l$ on $V_s^{\otimes l}$ commute. Let $\mathrm{End}'_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$ denote the enveloping algebra of the induced actions of $G \in GL(m/n)$ on $V_s^{\otimes l}$. This makes $\mathrm{End}'_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$ a vector space as in (4.2.7). The following is the super-analogue of Lemma 4.2.8, the proof being of a similar form.

Lemma 7.2.10. $\mathrm{End}'_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l}) = \mathrm{End}_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$.

Proof. It has already been determined that $\mathrm{End}'_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l}) \subset \mathrm{End}_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$. If $A \in \mathrm{End}_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$ is given by (7.2.6), then (7.2.7) implies that A is completely specified by those components $A^{j_1 j_2\cdots j_l}_{i_1 i_2\cdots i_l}$ for which:

$$(j_1, i_1) \leq (j_2, i_2) \leq \cdots \leq (j_l, i_l), \tag{7.2.10a}$$

where $(a,b) < (c,d)$ if and only either $a < c$ or $a = c$ and $b < d$, and $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. Apart from those components $A^{j_1 j_2\cdots j_l}_{i_1 i_2\cdots i_l}$ for which any pair (j_c, i_c) with $1 \leq c \leq l$ and $(j_c) \neq (i_c)$ occurs twice, thus implying, as in the note following (7.2.7), that $A^{j_1 j_2\cdots j_l}_{i_1 i_2\cdots i_l}$ is identically zero, the components $A^{j_1 j_2\cdots j_l}_{i_1 i_2\cdots i_l}$ satisfying (7.2.10a) may be varied independently. Therefore $\mathrm{End}_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$ is a vector space of dimension equal to the number of ways of choosing l from $m^2 + n^2$ items with replacement and $2mn$ without replacement. Let this number be denoted $D_{(m^2+n^2)/2mn}\{l\}$. The reason for this notation will become clear later. By a similar argument, $G^{j_1 j_2\cdots j_l}_{i_1 i_2\cdots i_l} = G^{j_1}{}_{i_1} G^{j_2}{}_{i_2} \cdots G^{j_l}{}_{i_l}$, has $D_{(m^2+n^2)/2mn}\{l\}$ representative elements which satisfy (7.2.10a) and are not identically zero. Since $\mathrm{End}'_{\mathsf{F}\tilde{S}_l}(V_s^{\otimes l})$ is a vector space, if it can be shown that these are linearly independent then the lemma is proved.

Each term $G^{j_1}{}_{i_1}G^{j_2}{}_{i_2}\cdots G^{j_l}{}_{i_l}$, which is not identically zero and for which the indices satisfy (7.2.10a), may be uniquely written:

$$(G^1{}_1)^{k_{11}}(G^1{}_2)^{k_{12}}\cdots(G^1{}_m)^{k_{1m}}(G^2{}_1)^{k_{21}}\cdots(G^2{}_m)^{k_{2m}}\cdots(G^m{}_m)^{k_{mm}}, \qquad (7.2.10b)$$

where $k_{ab} = \#\{k : (a,b) = (j_k, i_k), 1 \le k \le l\}$, $\sum_{a,b=1}^{m+n} k_{ab} = l$, $0 \le k_{ab} \le l$ if $(a) = (b)$, and $0 \le k_{ab} \le 1$ if $(a) \ne (b)$. Thus it is required to show that if:

$$\sum_{\substack{0\le k_{ab}\le l \text{ if } (a)=(b)\\ 0\le k_{ab}\le 1 \text{ if } (a)\ne(b)\\ k_{11}+k_{12}+\cdots+k_{mm}=l}} g_{k_{11}k_{12}\cdots k_{mm}}(G^1{}_1)^{k_{11}}(G^1{}_2)^{k_{12}}\cdots(G^m{}_m)^{k_{mm}} = 0 \qquad (7.2.10c)$$

where $g_{k_{11}k_{12}\cdots k_{mm}} \in \mathsf{F}$, then it is necessary that each $g_{k_{11}k_{12}\cdots k_{mm}} = 0$. To do this, restrict attention to the case where each even element, $G^j{}_i$ for $(j) = (i)$, is an F-multiple of $\zeta_\emptyset = 1$ and each odd element $G^j{}_i$ for $(j) \ne (i)$ is an F-multiple of a unique $\zeta_a \in B_1$. This requires $L \ge 2mn$ which, since $L \ge 4mn$, is always true. Thereupon the left side of (7.2.10c) is a homogeneous polynomial of degree l in $(m+n)^2$ variables in F, each term multiplied by a non-zero element of B. If each element is permitted an arbitrary value this would imply that each coefficient in (7.2.10c) is zero. However, by Lemma 7.1.9, those elements G that are not invertible form the union of two subspaces of the $(m+n)^2$-dimensional space of all (m/n) supermatrices; one of dimension $m^2 + 2mn + 1$ if $n > 1$, $m^2 + 2m$ if $n = 1$ and 0 if $n = 0$, and the other of dimension $n^2 + 2mn + 1$ if $m > 1$, $n^2 + 2n$ if $m = 1$ and 0 if $m = 0$. Thus the elements that are not invertible form a proper subspace of dimension less than $(m+n)^2$, and the conclusion that each coefficient in (7.2.10c) is zero remains valid for $G \in GL(m/n)$. Thus the lemma is proved.

Lemma 7.2.10 implies, via Theorem 4.1.18 and Lemma 4.1.19 that $V_s^{\otimes l}$ is a completely reducible $GL(m/n)$-module, the constituent irreducible modules being obtained from the minimal right ideals of $\mathsf{F}\tilde{S}_l$. If $x \in \mathsf{F}S_l$ and $x = \sum_{\pi\in S_l} x(\pi)\pi$ where each $x(\pi) \in \mathsf{F}$, define:

$$\tilde{x} = \sum_{\pi\in S_l} x(\pi)\tilde{\pi}. \qquad (7.2.11)$$

By virtue of Lemma 7.2.5, the map $x \to \tilde{x}$ defines an isomorphism between $\mathsf{F}S_l$ and $\mathsf{F}\tilde{S}_l$. Thus with $Y'_{T_i^\lambda}$ as defined in Theorem 4.2.9, a set of minimal right ideals of $\mathsf{F}\tilde{S}_l$ are provided by:

$$\tilde{Y}'_{T_i^\lambda} = \sum_{\sigma\in\mathcal{C}_{T_i^\lambda}} \sum_{\rho\in\mathcal{R}_{T_i^\lambda}} (-1)^\sigma \tilde{\rho}\tilde{\sigma}. \qquad (7.2.12)$$

The following two theorems now follow directly from Theorem 4.1.18 and Lemma 4.1.19.

Theorem 7.2.13. *The $GL(m/n)$-module $V_s^{\otimes l}$ is completely reducible. Let $\lambda \in P(l)$ and $\{T_i^\lambda : i = 1, 2, \ldots, f^\lambda\}$ be the set of S_l-standard tableaux of shape λ. Then, for $i = 1, 2, \ldots, f^\lambda$, $\tilde{Y}'_{T_i^\lambda}$ generates a set of f^λ linearly independent minimal right ideals. The $GL(m/n)$-modules $\tilde{Y}'_{T_i^\lambda} V_s^{\otimes l}$ are linearly independent and isomorphic.*

Theorem 7.2.14. *Those non-zero $\tilde{Y}'_{t\lambda} V_s^{\otimes l}$, for $\lambda \in P(l)$, provide a complete list of inequivalent irreducible $GL(m/n)$-modules occurring as submodules of $V_s^{\otimes l}$.*

As in Section 4.2, for each $\lambda \in P(l)$ identify the tableau T^λ for which $T^\lambda_{(k)} = i_k$ for $k = 1, 2, \ldots, l$, with the basis element $e_{i_1 i_2 \cdots i_l}$ of $V_s^{\otimes l}$. The graded (signed) place permutation action of $\tilde{\pi}$ on $e_{i_1 i_2 \cdots i_l}$, as given by Definition 7.2.4, then corresponds to the action of $\tilde{\pi}_*$ on T^λ given by:

$$\tilde{\pi}_* T^\lambda = \prod_{\substack{1 \leq a < b \leq l \\ \pi(a) > \pi(b)}} [i_a \, i_b] \, \pi_* T^\lambda, \tag{7.2.15}$$

where $\pi_* T^\lambda$ is given by Definition 3.3.10. Then for $w \in V_s^{\otimes l}$, the element $\tilde{Y}'_{t\lambda} w \in \tilde{Y}'_{t\lambda} V_s^{\otimes l}$ is identified with the grade-symmetrised tableau $\{T^\lambda\}^\sim = \tilde{Y}_*^\lambda T^\lambda$ where Y_*^λ is provided by (3.3.12d).

The graded $GL(m/n)$-module $\tilde{W}^\lambda$ is defined to be the span of all $\{T^\lambda\}^\sim$ where the entries of T^λ are all from the set $\mathcal{I}^{GL(m/n)} = \mathsf{N}_{m+n}$. These objects are not linearly independent since there exist graded versions of the Column and Garnir relations.

Lemma 7.2.16. *For any tableau T^λ and $\tau \in \mathcal{C}^\lambda$,*

$$\{T^\lambda\}^\sim = (-1)^\tau \{\tilde{\tau}_* T^\lambda\}^\sim. \tag{7.2.16}$$

Proof. From (3.3.12c) and (7.2.11),

$$\tilde{Y}_*^\lambda = \sum_{\rho \in \mathcal{R}^\lambda} \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}_* \tilde{\sigma}_*.$$

Then

$$\begin{aligned} \tilde{Y}_*^\lambda \tilde{\tau}_* = \sum_{\rho \in \mathcal{R}^\lambda} \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}_* \tilde{\sigma}_* \tilde{\tau}_* &= \sum_{\rho \in \mathcal{R}^\lambda} \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}_* (\widetilde{\sigma\tau})_* \\ &= (-1)^\tau \sum_{\rho \in \mathcal{R}^\lambda} \sum_{\sigma \in \mathcal{C}^\lambda} (-1)^\sigma \tilde{\rho}_* \tilde{\sigma}_*, \end{aligned}$$

where the isomorphism between $\mathsf{F}\tilde{S}_l$ and $\mathsf{F}S_l$ (Lemma 7.2.5) has been used. Therefore $\tilde{Y}_*^\lambda \tilde{\tau}_* = (-1)^\tau \tilde{Y}_*^\lambda$, which proves the Lemma.

Lemma 7.2.16 implies that if T^λ has an entry from the set $\mathcal{I}_{\bar{0}}$ repeated in any column then $\{T^\lambda\}^\sim$ vanishes. However, due to the grading property, this is not the case for a repeated entry from the set $\mathcal{I}_{\bar{1}}$. Nevertheless, (7.2.16) enables $\{T^\lambda\}^\sim$ to be expressed as $\pm\{T'^\lambda\}$ for some tableau T'^λ in which the entries are non-decreasing, and strictly increasing on the set $\mathcal{I}_{\bar{0}}$, down each column. Such a tableau is termed column superstrict. To illustrate the use of Lemma 7.2.16, consider the $GL(2/2)$-module $\tilde{W}^{(2,2,1)}$ where:

$$\left\{\begin{matrix} 1 & 1 \\ 2 & 4 \\ 2 & \end{matrix}\right\}^\sim = 0, \quad \left\{\begin{matrix} 4 & 1 \\ 3 & 2 \\ 3 & \end{matrix}\right\}^\sim = \left\{\begin{matrix} 3 & 1 \\ 3 & 2 \\ 4 & \end{matrix}\right\}^\sim, \quad \left\{\begin{matrix} 1 & 4 \\ 4 & 2 \\ 3 & \end{matrix}\right\}^\sim = -\left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 4 & \end{matrix}\right\}^\sim. \tag{7.2.17}$$

The Garnir relations have the following graded analogue:

Lemma 7.2.18. *For $i < j$, let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the entries in the ith and jth columns, respectively, of t^λ such that $\#(\mathcal{X} \cup \mathcal{Y}) > \tilde{\lambda}_i$. Let $S(\mathcal{X})$, $S(\mathcal{Y})$ and $S(\mathcal{X} \cup \mathcal{Y})$ be the subgroups of S_l preserving $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X} \cup \mathcal{Y}$, respectively. Then if $\mathcal{G}(\mathcal{X}, \mathcal{Y})$ is a set of right coset representatives for $S(\mathcal{X}) \otimes S(\mathcal{Y})$ in $S(\mathcal{X} \cup \mathcal{Y})$,*

$$\sum_{\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})} (-1)^\eta \{\tilde{\eta}_* T^\lambda\}^\sim = 0. \tag{7.2.18}$$

Proof. With $G^\lambda_{\mathcal{X},\mathcal{Y}}$ given by (3.4.4), the ungraded Garnir relation implies that $Y^\lambda G^\lambda_{\mathcal{X},\mathcal{Y}} = 0$, as in (3.4.5). Therefore $\tilde{Y}^\lambda \tilde{G}^\lambda_{\mathcal{X},\mathcal{Y}} = 0$, on using, once more, the isomorphism between $\mathsf{F}\tilde{S}_l$ and $\mathsf{F}S_l$. This proves the lemma.

To illustrate the graded Garnir relations, consider the $GL(2/3)$-module $\tilde{W}^\lambda$ with $\lambda = (4,3,1)$. Then $i = 1$, $j = 2$, $\mathcal{X} = \{2,3\}$, $\mathcal{Y} = \{4,5\}$ and an appropriate set of coset representatives produces, for example, the identity:

$$\begin{aligned} &\left\{\begin{matrix} 4 & 3 & 3 & 5 \\ 1 & 4 & 2 & \\ 2 & & & \end{matrix}\right\}^\sim - \left\{\begin{matrix} 4 & 1 & 3 & 5 \\ 3 & 4 & 2 & \\ 2 & & & \end{matrix}\right\}^\sim - \left\{\begin{matrix} 4 & 2 & 3 & 5 \\ 1 & 4 & 2 & \\ 3 & & & \end{matrix}\right\}^\sim \\ &\quad + \left\{\begin{matrix} 4 & 3 & 3 & 5 \\ 4 & 1 & 2 & \\ 2 & & & \end{matrix}\right\}^\sim + \left\{\begin{matrix} 4 & 3 & 3 & 5 \\ 1 & 2 & 2 & \\ 4 & & & \end{matrix}\right\}^\sim + \left\{\begin{matrix} 4 & 1 & 3 & 5 \\ 3 & 2 & 2 & \\ 4 & & & \end{matrix}\right\}^\sim = 0. \end{aligned} \tag{7.2.19}$$

It should not be assumed that the occurrence of identical entries in the same column implies that the term vanishes. For example, the fourth term in (7.2.19) is not identically zero.

For each irreducible covariant representation of $GL(m/n)$, Berele and Regev introduced the following favoured set of tableaux.

Definition 7.2.20. **[BR83,BR87]** *$GL(m/n)$-standard tableaux. Define the index sets $\mathcal{I}_{\bar{0}} = \{1, \ldots, m\}$, $\mathcal{I}_{\bar{1}} = \{m+1, \ldots, m+n\}$ and $\mathcal{I}^{GL(m/n)} = \mathcal{I}_{\bar{0}} \cup \mathcal{I}_{\bar{1}}$. The tableau T^λ is $GL(m/n)$-standard if and only if:*

(i) the entries are taken from the set $\mathcal{I}^{GL(m/n)}$;

(ii) the entries from the set $\mathcal{I}_{\bar{0}}$ form a tableau T^μ, for some $\mu \leq \lambda$, within T^λ;

(iii) the entries from the set $\mathcal{I}_{\bar{0}}$ are strictly increasing from top to bottom down each column of T^μ;

(iv) the entries from the set $\mathcal{I}_{\bar{1}}$ are non-decreasing from top to bottom down each column of $T^{\lambda/\mu}$;

(v) the entries from the set $\mathcal{I}_{\bar{0}}$ are non-decreasing from left to right across each row of T^μ;

(vi) the entries from the set $\mathcal{I}_{\bar{1}}$ are strictly increasing from left to right across each row of $T^{\lambda/\mu}$.

This Definition implies that the tableaux:

$$\begin{array}{llll} 1 & 1 & 2 & 6 \\ 2 & 4 & 5 & \\ 3 & 4 & 5 & \\ 6 & & & \end{array} \; , \quad \begin{array}{lll} 4 & 5 & 6 \\ 4 & 5 & 6 \\ 4 & 5 & 6 \\ 4 & & \end{array} \quad \text{and} \quad \begin{array}{llll} 1 & 1 & 2 & 4 \\ 3 & 3 & 4 & \\ 4 & 5 & & \\ 6 & & & \end{array} \tag{7.2.21}$$

are each $GL(3/3)$-standard. Note that if $\lambda_{m+1} > n$ then Definition 7.2.20 implies that no $GL(m/n)$-standard tableaux exist, since below the mth row only entries from the set $\mathcal{I}_{\bar{1}}$ may be present, and these must strictly increase from left to right. Thus $GL(m/n)$-standard tableaux exist if and only if $\lambda \in P(l; m/n)$.

The $GL(m/n)$-standard tableaux were employed in **[Ki83]** to produce a supersymmetric generalisation of the symmetric Schur functions.

A generalisation of the standardisation techniques of Section 3.4 now enable an arbitrary grade-symmetrised tableau $\{T^\lambda\}^\sim$ to be reduced to a linear combination of grade-symmetrised $GL(m/n)$-standard tableaux through the systematic application of the graded Column relations (7.2.16), and the graded Garnir relations (7.2.18). Firstly the graded Column relations enable the entries of a grade-symmetrised tableau to be reordered in their columns to form $\{T^\lambda\}^\sim$, where T^λ is column superstrict. If T^λ is not $GL(m/n)$-standard, then either condition (v) or condition (vi) of Definition 7.2.20 is violated and, in particular, is violated by a neighbouring pair of entries. Let a and b be such that this neighbouring pair is $T^\lambda_{(a,b)}$ and $T^\lambda_{(a,b+1)}$. Then $T^\lambda_{(a,b)} \geq T^\lambda_{(a,b+1)}$ with equality implying that $T^\lambda_{(a,b)} \in \mathcal{I}_{\bar{1}}$. Let $\mathcal{X}$ be the set of positions below and including $T^{\lambda}_{(a,b)}$ in the bth column and let $\mathcal{Y}$ be the set of positions above and including $T^{\lambda}_{(a,b+1)}$ in the $(b+1)$th column. The relevant entries of T^{λ} are then as follows:

$$\begin{array}{ccc} \vdots & & T^{\lambda}_{1,b+1} \\ \vdots & & \wedge\!\mid \\ \vdots & & \vdots \\ \vdots & & \wedge\!\mid \\ \vdots & & T^{\lambda}_{a-1,b+1} \\ \vdots & & \wedge\!\mid \\ T^{\lambda}_{a,b} & \geq & T^{\lambda}_{a,b+1} \\ \wedge\!\mid & & \\ T^{\lambda}_{a+1,b} & & \\ \wedge\!\mid & & \\ \vdots & & \\ \wedge\!\mid & & \\ T^{\lambda}_{\tilde{\lambda}_b,b}. & & \end{array} \tag{7.2.22}$$

This differs from (3.4.9) only in that identical entries are permitted in the same column. With $\mathcal{X}$ and $\mathcal{Y}$ as defined above, $\#(\mathcal{X} \cup \mathcal{Y}) = \tilde{\lambda}_b + 1$, whereupon Lemma 7.2.18 may be used to express $\{T^{\lambda}\}^{\sim}$ in terms of other tableaux.

Consider first the case where $T^{\lambda}_{(a,b)} > T^{\lambda}_{(a,b+1)}$. This case is similar to the situations considered in previous chapters. To recapitulate, with $\eta \in \mathcal{G}(\mathcal{X},\mathcal{Y})$ and $\eta \notin S(\mathcal{X}) \otimes S(\mathcal{Y})$, $T^{\lambda}_{\eta} = \eta_* T^{\lambda}$ has necessarily been formed from T^{λ} by swapping the columns of at least one pair of elements from $\mathcal{X} \cup \mathcal{Y}$. Since the entries at positions $\mathcal{Y}$ are all smaller than those at positions $\mathcal{X}$, $T^{\lambda}_{\eta} > T^{\lambda}$. Hence, in this case, the algorithm enables $\{T^{\lambda}\}^{\sim}$ to be written in terms of higher tableaux (as specified by Definition 2.6.8), the coefficients being all integral. To illustrate this case, let:

$$T^{(2,2,2)} = \begin{matrix} 1 & 2 \\ 5 & 3 \\ 5 & 4 \end{matrix} \,. \tag{7.2.23a}$$

Then, in the $GL(3/2)$-module $\tilde{W}^{(2,2,2)}$, the following identity arises on using the above procedure with $\mathcal{X} = \{2,3\}$ and $\mathcal{Y} = \{4,5\}$:

$$\left\{\begin{matrix} 1 & 2 \\ 5 & 3 \\ 5 & 4 \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 5 \\ 2 & 3 \\ 5 & 4 \end{matrix}\right\}^{\sim} - \left\{\begin{matrix} 1 & 5 \\ 5 & 3 \\ 2 & 4 \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 2 \\ 3 & 5 \\ 5 & 4 \end{matrix}\right\}^{\sim} - \left\{\begin{matrix} 1 & 2 \\ 5 & 5 \\ 3 & 4 \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 5 \\ 2 & 5 \\ 3 & 4 \end{matrix}\right\}^{\sim} = 0. \tag{7.2.23b}$$

On rearrangement, using (7.2.16), and collection of terms, this yields:

$$\left\{\begin{matrix} 1 & 2 \\ 5 & 3 \\ 5 & 4 \end{matrix}\right\}^{\sim} = 2\left\{\begin{matrix} 1 & 3 \\ 2 & 4 \\ 5 & 5 \end{matrix}\right\}^{\sim} - 2\left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & 5 \end{matrix}\right\}^{\sim} - \left\{\begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & 5 \end{matrix}\right\}^{\sim}, \qquad (7.2.23c)$$

where each of the tableaux on the right side is higher than that on the left.

For the case where $T^{\lambda}_{(a,b)} = T^{\lambda}_{(a,b+1)} \in \mathcal{I}_{\bar{1}}$, the same technique produces a similar sum of terms. However, as may be seen by considering the coset containing the permutation which swaps the two neighbouring identical entries, the original grade-symmetrised tableau is repeated in this identity. Since both of these entries are of odd grade, the two terms have the same sign and thus *do not cancel.* The possibility of the entries immediately below $T^{\lambda}_{(a,b)}$ or immediately above $T^{\lambda}_{(a,b+1)}$ being identical to these two is not excluded. If this entry occurs c times in the bth column and d times in the $(b+1)$th column then, by considering coset representatives which permute these entries amongst themselves, it can be seen that the original grade-symmetrised tableau occurs with a multiplicity of $\binom{c+d}{c}$ in the Garnir identity resulting from the selection of $\mathcal{X}$ and $\mathcal{Y}$ given above. Again all of these terms have the same sign. The previous argument shows, once more, that the remaining terms in the expression are higher than the original. Therefore, in this case, $\{T^{\lambda}\}^{\sim}$ may be expressed in terms of higher tableaux, the coefficient of each being rational. This case is exemplified by the following example in the $GL(2/2)$-module $\tilde{W}^{(2,2,1)}$:

$$\left\{\begin{matrix} 1 & 2 \\ 3 & 3 \\ 4 & \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 4 & \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 2 \\ 3 & 3 \\ 4 & \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 3 \\ 2 & 4 \\ 3 & \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & \end{matrix}\right\}^{\sim} + \left\{\begin{matrix} 1 & 3 \\ 2 & 4 \\ 3 & \end{matrix}\right\}^{\sim} = 0, \qquad (7.2.24a)$$

whereupon:

$$\left\{\begin{matrix} 1 & 2 \\ 3 & 3 \\ 4 & \end{matrix}\right\}^{\sim} = -\left\{\begin{matrix} 1 & 3 \\ 2 & 4 \\ 3 & \end{matrix}\right\}^{\sim} - \frac{1}{2}\left\{\begin{matrix} 1 & 2 \\ 3 & 4 \\ 3 & \end{matrix}\right\}^{\sim} - \frac{1}{2}\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 4 & \end{matrix}\right\}^{\sim}. \qquad (7.2.24b)$$

As a further example, consider the $GL(2/2)$-module $\tilde{W}^{(2^3,1^6)}$ where the above process results in:

$$84\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 3 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 4 & \end{matrix}\right\}^{\sim} + 36\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \end{matrix}\right\}^{\sim} = 0, \quad \text{implying} \quad \left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 3 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 4 & \end{matrix}\right\}^{\sim} = -\frac{3}{7}\left\{\begin{matrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \\ 3 & \end{matrix}\right\}^{\sim}, \tag{7.2.25}$$

expressing the non-standard term on the left in terms of a single higher term which, incidentally, is $GL(2/2)$-standard.

As with (7.2.23c), a single application of the above procedure may result in further non-standard terms. However, the process may be iterated until solely $GL(m/n)$-standard tableaux result. That this procedure terminates is guaranteed by the ordering on the set of all tableaux of shape F^λ, given by Definition 2.6.8, and their finite number.

Theorem 7.2.26. (*see* [**BR87**].) *The set*

$$\{\,\{T^\lambda\}^\sim \,:\, T^\lambda \text{ is } GL(m/n)\text{-standard}\}$$

constitutes a basis for the irreducible $GL(m/n)$-module $\tilde{W}^\lambda$.

Proof. The following is a direct generalisation of that used for Theorem 4.2.16 and differs from that of [**BR87**]. The existence of the standardisation algorithm given above implies that this set spans $\tilde{W}^\lambda$. Thus it is sufficient to demonstrate linear independence. As for Theorem 4.2.16, introduce the following order on the set of all tableaux. Let t_u^b be the sum of the entries in the bth row of T_u^λ for $b = 1, 2, \ldots, q$ where $q = \tilde{\lambda}_1$. Let $|T_u^\lambda|'$ be the equivalence class of all tableaux which have their sequences of row sums identical to that of T_u^λ; that is $T_v^\lambda \in |T_u^\lambda|'$ if $t_v^b = t_u^b$ for $b = 1, 2, \ldots, q$. A total order on the set of these equivalence classes of tableaux is defined by $|T_u^\lambda|' > |T_v^\lambda|'$ if for some $k \le q$, $t_u^k > t_v^k$ with $t_u^b = t_v^b$ for each $b = k+1, k+2, \ldots, q$. Let $\rho \in \mathcal{R}^\lambda$ and $\sigma \in \mathcal{C}^\lambda$. Since the action of ρ_* on T^λ leaves the elements of T^λ in their original rows, $\rho_* T^\lambda \in |T^\lambda|'$. If T^λ is $GL(m/n)$-standard then $|\sigma_* T^\lambda|' \le |T^\lambda|'$ since the action of σ_* only serves to move smaller entries down the columns. The inequality here is strict if $\sigma_* T^\lambda \ne T^\lambda$. Since T^λ may possess identical entries in a column $\sigma_* T^\lambda = T^\lambda$ may occur for various $\sigma \in \mathcal{C}^\lambda$. Let the $GL(m/n)$-standard tableaux be labelled:

$$T^\lambda_{1,1}, T^\lambda_{1,2}, \ldots, T^\lambda_{1,\kappa_1}, T^\lambda_{2,1}, T^\lambda_{2,2}, \ldots, T^\lambda_{2,\kappa_2}, \ldots, T^\lambda_{r,\kappa_r}, \qquad (7.2.26a)$$

such that:

$$\left|T^\lambda_{s,1}\right|' = \left|T^\lambda_{s,2}\right|' = \cdots = \left|T^\lambda_{s,\kappa_s}\right|', \qquad (7.2.26b)$$

for $1 \leq s \leq r$, and such that:

$$\left|T^\lambda_{1,1}\right|' < \left|T^\lambda_{2,1}\right|' < \left|T^\lambda_{3,1}\right|' < \cdots < \left|T^\lambda_{r,1}\right|'. \qquad (7.2.26c)$$

It is required to show that if:

$$\sum_{i=1}^{r} \sum_{j=1}^{\kappa_i} k_{i,j} \{T^\lambda_{i,j}\}^\sim = 0, \qquad (7.2.26d)$$

where each $k_{i,j} \in \mathsf{F}$, then each $k_{i,j} = 0$. If this is not the case, there exist a and b such that $k_{a,b} \neq 0$ with $k_{a,j} = 0$ for $1 \leq j < b$ and each $k_{i,j} = 0$ for $i < a$. Thus:

$$\begin{aligned} 0 &= \sum_{j=b}^{\kappa_a} k_{a,j} \tilde{P}^\lambda_* \tilde{Q}^\lambda_* T^\lambda_{a,j} + \sum_{i=a+1}^{r} \sum_{j=1}^{\kappa_i} k_{i,j} \tilde{P}^\lambda_* \tilde{Q}^\lambda_* T^\lambda_{i,j} \\ &= \sum_{j=b}^{\kappa_a} n_{a,j} k_{a,j} \tilde{P}^\lambda_* T^\lambda_{a,j} + \sum_{j=b}^{\kappa_a} \sum_{\substack{\sigma \in \mathcal{C}^\lambda \\ \sigma_* T^\lambda_{a,j} \neq T^\lambda_{a,j}}} (-1)^\sigma k_{a,j} \tilde{P}^\lambda_* \sigma_* T^\lambda_{a,j} + \sum_{i=a+1}^{r} \sum_{j=1}^{\kappa_i} k_{i,j} \tilde{P}^\lambda_* \tilde{Q}^\lambda_* T^\lambda_{i,j}, \end{aligned}$$

where $n_{a,j} = \#\{\sigma \in \mathcal{C}^\lambda : \sigma_* T^\lambda_{a,j} = T^\lambda_{a,j}\} > 0$. In view of (7.2.26*b*) and (7.2.26*c*), all the tableaux T^λ comprising the third term are such that $\left|T^\lambda\right|' > \left|T^\lambda_{a,b}\right|'$. In addition, since $\left|\sigma_* T^\lambda_{a,j}\right|' > \left|T^\lambda_{a,j}\right|'$ whenever $\sigma_* T^\lambda_{a,j} \neq T^\lambda_{a,j}$, all the tableaux T^λ comprising the second term are such that $\left|T^\lambda\right|' > \left|T^\lambda_{a,b}\right|'$. Therefore, since each tableau is uniquely identified with a basis element of $V_s^{\otimes l}$, it follows that:

$$\sum_{j=b}^{\kappa_a} n_{a,j} k_{a,j} \tilde{P}^\lambda_* T^\lambda_{a,j} = 0. \qquad (7.2.26e)$$

Since the tableaux $T^\lambda_{a,b}, T^\lambda_{a,b+1}, \ldots, T^\lambda_{a,\kappa_a}$ are $GL(m/n)$-standard and distinct, it follows that the sets $\{\rho_* T^\lambda_{a,c} : \rho \in \mathcal{R}^\lambda\}$ each contain a single unique $GL(m/n)$-standard tableau for $c = b, b+1, \ldots, \kappa_a$. Since each $n_{a,j} > 0$, it then follows from (7.2.26*e*) that $k_{a,b} = k_{a,b+1} = \cdots = k_{a,\kappa_a} = 0$. This contradicts $k_{a,b} \neq 0$ whereupon all the $k_{i,j}$ of (7.2.26*d*) are zero and the theorem is proved.

Theorem 7.2.27. [**BR87**]. *The set*

$$\{\tilde{W}^\lambda = \tilde{Y}'_{t^\lambda} V_s^{\otimes l} : \lambda \in P(l; m/n)\}$$

provides a complete list of inequivalent irreducible $GL(m/n)$-modules occurring as submodules of $V_s^{\otimes l}$.

Proof. Since $GL(m/n)$-standard tableaux provide a basis for $\tilde{W}^\lambda$ and $GL(m/n)$-standard tableaux exist if and only if $\lambda \in P(l; m/n)$, the theorem follows directly from Theorem 7.2.14.

This theorem is known in [**BR87**] as the 'Hook Theorem' since if $\lambda \in P(l; m/n)$, the Young diagram F^λ lies in a hook with leg width n and arm width m.

Let $\lambda \in P(l; m/n)$. It follows from (7.2.2a) and Lemma 7.2.10, that the element $G \in GL(m/n)$ acts on $\{T^\lambda\}^\sim \in \tilde{W}^\lambda$ according to:

$$G\{T^\lambda\}^\sim = \sum_{T'^\lambda} \left(\prod_{1 \le a \le b \le l} \left[T'^\lambda_{(a)} T'^\lambda_{(b)}\right] \left[T'^\lambda_{(a)} T^\lambda_{(b)}\right] \right) G^{T'^\lambda_{(1)}}{}_{T^\lambda_{(1)}} G^{T'^\lambda_{(2)}}{}_{T^\lambda_{(2)}} \cdots G^{T'^\lambda_{(l)}}{}_{T^\lambda_{(l)}} \{T'^\lambda\}^\sim, \tag{7.2.28}$$

the sum being over all tableaux T'^λ with entries from the set $\mathcal{I}^{GL(m/n)}$. Since the grade-symmetrised $GL(m/n)$-standard tableaux constitute a basis for $\tilde{W}^\lambda$, explicit representation matrices are readily obtained from the action of $GL(m/n)$ on these tableaux. Let $\tilde{s}^\lambda$ be the dimension of $\tilde{W}^\lambda$ and $T_1^\lambda, T_2^\lambda, \ldots, T_{\tilde{s}^\lambda}^\lambda$ the $GL(m/n)$-standard tableaux. The action of $G \in GL(m/n)$ on each $\{T_i^\lambda\}$ yields, according to (7.2.28), a linear combination of, in general, non-standard tableaux with coefficients in B. By using the techniques of this section, each may be written in terms of the $GL(m/n)$-standard tableaux, so that:

$$G\,\{T_i^\lambda\}^\sim = \sum_{j=1}^{\tilde{s}^\lambda} \Gamma^{\{\lambda\}}(G)_{ji} \{T_j^\lambda\}^\sim, \tag{7.2.29}$$

for some set of Grassmann parameters $\Gamma^{\{\lambda\}}(G)_{ji} \in B$. These are the elements of the matrix $\Gamma^{\{\lambda\}}(G)$ which represents G in the representation labelled by λ.

The results of this section show that the quintessential structure of $\tilde{W}^\lambda$ is as follows.

Theorem 7.2.30. *Let $\lambda \in P(l; m/n)$. The module $\tilde{W}^\lambda$ is the irreducible $GL(m/n)$-module spanned by $\{T^\lambda\}^\sim$ for all T^λ with entries from the set $\mathcal{I}^{GL(m/n)}$, modulo relations* (7.2.16) *and* (7.2.18), *and on which $GL(m/n)$ acts according to* (7.2.28).

This theorem effectively provides a definition for $\tilde{W}^\lambda$.

§7.3. Lie superalgebras, gl(m/n) and sl(m/n)

The relationship between a Lie supergroup and a Lie superalgebra is more subtle than that of the ordinary case. Indeed, some Lie supergroups do not possess Lie superalgebras. In addition, the usual definition of a Lie superalgebra makes no reference to Grassmann parameters but nevertheless, the aspect of a graded vector space is retained. It is the purpose of this section to describe the Lie superalgebras $gl(m/n)$ and $sl(m/n)$ and to indicate their position in the general classification of Lie superalgebras. Most of the original results on Lie superalgebras were obtained by Kac (see, for example, [**Ka78**]) and convenient accounts are given in [**Sc79,Co89**].

Definition 7.3.1. *A Lie superalgebra $\mathcal{L}_s$ is a $\mathbf{Z}_2$-graded vector space over F having even subspace $\mathcal{L}_{\bar{0}}$ and odd subspace $\mathcal{L}_{\bar{1}}$ and a generalised product $[a,b] \in \mathcal{L}_s$ for all $a,b \in \mathcal{L}_s$, satisfying:*

(i) $[a,b] \in \mathcal{L}_{\overline{(\deg a + \deg b)}}$;

(ii) $[\alpha a + \beta b, c] = \alpha[a,c] + \beta[b,c]$;

(iii) $[a,b] = -(-1)^{\deg a \deg b}[b,a]$; *and*

(iv) $(-1)^{\deg a \deg c}[a,[b,c]] + (-1)^{\deg a \deg b}[b,[c,a]] + (-1)^{\deg b \deg c}[c,[a,b]] = 0$;

for all homogeneous $a,b,c \in \mathcal{L}_s$ and all $\alpha, \beta \in \mathsf{F}$.

Theorem 7.3.2. *(i) If the even subspace $\mathcal{L}_{\bar{0}}$ of $\mathcal{L}_s$ is non-trivial, then it is an ordinary Lie algebra.*

(ii) If both $\mathcal{L}_{\bar{0}}$ and $\mathcal{L}_{\bar{1}}$ are non-trivial subspaces of $\mathcal{L}_s$, then $\mathcal{L}_{\bar{1}}$ is a carrier space of a representation of $\mathcal{L}_{\bar{1}}$; that is $\mathcal{L}_{\bar{1}}$ is an $\mathcal{L}_{\bar{0}}$-module.

Proof. (i) On restricting a,b,c of Definition 7.3.1 to be elements of $\mathcal{L}_{\bar{0}}$, it is seen that $\mathcal{L}_{\bar{0}}$ satisfies all the requirements of a Lie algebra as in Definition 1.3.4. (ii) It is sufficient to show that $[[a,b],c] = [a,[b,c]] - [b,[a,c]]$ for all $a,b \in \mathcal{L}_{\bar{0}}$ and all $c \in \mathcal{L}_{\bar{1}}$. This follows directly from conditions (iii) and (iv) of Definition 7.3.1 since in each case the exponent of (-1) is 0.

The following definition concerns, for fixed m and n, $(m+n)\times(m+n)$ matrices M partitioned:

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \tag{7.3.3}$$

where P, Q, R and S are submatrices of sizes $m \times m$, $m \times n$, $n \times m$ and $n \times n$ respectively and $M_{ij} \in \mathsf{F}$ for $1 \leq i,j \leq m+n$. M is said to be an even (m/n)-matrix if the submatrices Q and R are zero, whereupon $\deg M = 0$, and an odd

(m/n)-matrix if the submatrices P and S are zero, whereupon $\deg M = 1$. Every matrix M can be uniquely written as the sum $M = M_0 + M_1$ of an even and an odd matrix. If either M_0 or M_1 is zero then M is said to be homogeneous.

Definition 7.3.4. *The Lie superalgebra $gl(m/n, \mathsf{F})$ is the $\mathbf{Z}_2$-graded vector space of all (m/n)-matrices of the form (7.3.3) with generalised Lie product:*

$$[M, N] = MN - (-1)^{(\deg M)(\deg N)} NM, \tag{7.3.4}$$

for all homogeneous $M, N \in gl(m/n, F)$. This is linearly extended to the whole of $gl(m/n, \mathsf{F})$.

Note that unless both M and N are odd, (7.3.4) is the ordinary Lie product. It is easily verified that $gl(m/n)$ is a Lie superalgebra.

If the matrix $E_a{}^b$ is defined as in Section 2.2, then a basis for $gl(m/n)$ is provided by $\{E_a{}^b : a, b \in \mathcal{I}^{GL(m/n)}\}$, for which $\deg E_a{}^b = (a) + (b) \bmod 2$. Thereupon $gl(m/n)$ has even dimension $m^2 + n^2$ and odd dimension $2mn$. (7.3.4) implies that:

$$[E_a{}^b, E_c{}^d] = \delta_c^b E_a{}^d - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \delta_a^d E_c{}^b. \tag{7.3.5}$$

Definition 7.3.6. *A simple Lie superalgebra is a Lie superalgebra that is not abelian and does not possess a proper graded ideal.*

Consider the even element $H \in gl(m/n)$ given by:

$$H = \sum_{i \in \mathcal{I}^{GL(m/n)}} E_i{}^i. \tag{7.3.6}$$

It is easily verified that multiples of H constitute a one-dimensional ideal of $gl(m/n)$. Furthermore, this ideal is trivially graded; all elements are even. Therefore $gl(m/n)$ is not a simple Lie superalgebra.

Definition 7.3.7. *If a matrix $M \in gl(m/n)$ is partitioned as in (7.3.3), then its supertrace, denoted* $\mathrm{str}(A)$, *is defined by:*

$$\mathrm{str}(A) = \mathrm{tr}(P) - \mathrm{tr}(S). \tag{7.3.7}$$

If $M, N \in gl(m/n)$ are such that $\mathrm{str}(M) = \mathrm{str}(N) = 0$ then it is easily verified that $\mathrm{str}([M, N]) = 0$.

Definition 7.3.8. *Let $sl(m/n) = \{M \in gl(m/n) : \mathrm{str}(M) = 0\}$.*

In view of the above $sl(m/n)$ is a Lie superalgebra.

For $a, b \in \mathcal{I}^{GL(m/n)}$, define:

$$A_a{}^b = \begin{cases} E_a{}^b & \text{if } a \neq b; \\ E_a{}^a - E_{a+1}{}^{a+1} & \text{if } a = b < m; \\ E_m{}^m + E_{m+1}{}^{m+1} & \text{if } a = b = m; \\ E_a{}^a - E_{a+1}{}^{a+1} & \text{if } a = b \text{ and } m < a < m+n. \end{cases} \tag{7.3.9}$$

Then a basis for $sl(m/n)$ is provided by $\{A_a{}^b : a, b \in \mathcal{I}^{GL(m/n)}, (a,b) \neq (m+n, m+n)\}$, whereupon $sl(m/n)$ has even dimension $a^2 + b^2 - 1$ and odd dimension $2ab$. If $m \neq n$ and $m, n \geq 1$, it may be shown that $sl(m/n)$ is a simple Lie superalgebra. This simple Lie superalgebra is often denoted $A(m-1/n-1)$. If $m = n$ then since $H \in sl(m/m)$, where H is given by (7.3.6), $sl(m/m)$ is not simple. However, $sl(m/m)/(\mathsf{F}H)$ is simple and is often denoted $A(m-1/m-1)$. Where $m \neq n$, define $C \in sl(m/n)$ by

$$C = \frac{n}{n-m} \sum_{a=1}^{m} E_a{}^a + \frac{m}{n-m} \sum_{a=m+1}^{m+n} E_a{}^a. \tag{7.3.10}$$

The even part $sl(m/n)_{\bar{0}}$ of $sl(m/n)$ consists of matrices of the form (7.3.3) with Q and R both zero and $\text{tr}(P) - \text{tr}(S) = 0$. Thus $sl(m/n)_{\bar{0}}$ is isomorphic to the direct sum of the set of matrices P for which $\text{tr}(P) = 0$, the set of matrices S for which $\text{tr}(S) = 0$, and scalar multiples of C. Therefore:

$$sl(m/n, \mathsf{F})_{\bar{0}} \cong sl(m, \mathsf{F}) \oplus \mathsf{F} \oplus sl(n, \mathsf{F}). \tag{7.3.11}$$

The element C, which is central in $sl(m/n)_{\bar{0}}$, has a very special property. Let $sl(m/n)_{\bar{1}} = sl(m/n)_{+1} \oplus sl(m/n)_{-1}$, where in the form of (7.3.3), the elements of $sl(m/n)_{+1}$ have R equal to zero, and the elements of $sl(m/n)_{-1}$ have Q equal to zero. Then, writing $sl(m/n)_0 = sl(m/n)_{\bar{0}}$, it is easily shown that:

$$[C, M] = kM, \tag{7.3.12}$$

for all $M \in sl(m/n)_{\bar{k}}$ and $k = -1, 0, +1$.

Definition 7.3.13. *A reductive Lie algebra $\mathcal{L}$ is a Lie algebra which is either Abelian, semisimple or a direct sum of an Abelian Lie algebra and a semisimple Lie algebra.*

Definition 7.3.14. *A Lie superalgebra $\mathcal{L}_s$ is said to be classical if $\mathcal{L}_{\bar{0}}$ is a reductive Lie algebra.*

These definitions indicate that $sl(m/n)$ is a classical Lie superalgebra. Furthermore, if $m \neq n$ then $sl(m/n)$ is a classical simple Lie superalgebra. The complex classical simple Lie superalgebras play a role in the theory of superalgebras similar to that of the complex simple Lie algebras in the ordinary theory.

Lemma 7.3.15. (*see* **[Co89]**.) *If $\mathcal{L}_s$ is a complex classical simple Lie superalgebra then the centre (maximal Abelian ideal) of $\mathcal{L}_{\bar{0}}$ is at most one-dimensional.*

This lemma indicates that $\mathcal{L}_{\bar{0}}$ may be written:

$$\mathcal{L}_{\bar{0}} = \mathcal{L}^A \oplus \mathcal{L}^S, \tag{7.3.16}$$

where $\mathcal{L}^A$ is Abelian and at most one-dimensional and $\mathcal{L}^S$ is semisimple.

Definition 7.3.17. *The Cartan subalgebra $\mathcal{H}_s$ of the complex classical Lie superalgebra $\mathcal{L}_s$ is defined by:*

$$\mathcal{H}_s = \mathcal{L}^A \oplus \mathcal{H}^S, \tag{7.3.17}$$

where $\mathcal{H}^S$ is the Cartan subalgebra of $\mathcal{L}^S$. The rank of $\mathcal{L}_s$ is defined to be the dimension of $\mathcal{H}_s$.

If $m \neq n$, this definition implies that $\{A_a{}^a : 1 \leq a < m+n\}$ is a basis for the Cartan subalgebra of $sl(m/n) = A(m-1/n-1)$ and that consequently $A(m-1/n-1)$ is of rank $m+n-1$. If $m = n > 1$, $\{A_a{}^a : 1 \leq a < m+n\}$ is a basis for the Cartan subalgebra of $A(m-1/m-1)$ subject to the constraint:

$$\sum_{a=1}^{m+n} E_a{}^a = \sum_{a=1}^{m} a A_a{}^a - \sum_{a=m+1}^{2m-1} (2m-a) A_a{}^a = 0. \tag{7.3.18}$$

Consequently $A(m-1/m-1)$ has rank $2m-2$.

Since $\mathcal{L}^S$ is semisimple it may be written as the direct sum of root subspaces as in (1.6.1). The zero root space $\mathcal{L}_{s0}$ is equal to $\mathcal{H}_s$. The set of roots α of $\mathcal{L}^S$ is then denoted $\Delta^{\bar{0}}$, these being known as the even roots of $\mathcal{L}_s$. If $\mathcal{L}^A$ is trivial then since $\mathcal{L}_{\bar{1}}$ is an $\mathcal{L}_{\bar{0}}$-module, it may be written as the direct sum of weight subspaces as in (1.7.1). The corresponding set of weights is denoted $\Delta^{\bar{1}}$, these being known as the odd roots of $\mathcal{L}_s$. Then $\Delta = \Delta^{\bar{0}} \cup \Delta^{\bar{1}}$ comprises the complete set of roots for $\mathcal{L}_s$. $\mathcal{L}_s$ may then be written:

$$\mathcal{L}_s = \mathcal{H}^S \oplus (\oplus_{\alpha\in\Delta} \mathbb{C} e_\alpha), \tag{7.3.19}$$

where each root vector e_α of $\mathcal{L}_s$ is either a root vector or weight vector of $\mathcal{L}_{\bar{0}}$. These definitions imply that:

$$[h, e_\alpha] = \alpha(h) e_\alpha, \tag{7.3.20}$$

for all $\alpha \in \Delta$ and all $h \in \mathcal{H}^S$. If $\mathcal{L}^A$ is not trivial then $[c, e_\alpha] = 0$ for all $c \in \mathcal{L}^A$ and $\alpha \in \Delta^{\bar{0}}$. In addition, as may be shown **[Co89]**, there exists $c \in \mathcal{L}^A$ with the property that $[c, e_\alpha] = \pm e_\alpha$ for all $\alpha \in \Delta^{\bar{1}}$. This is the situation in (7.3.12). Thus, in those cases for which $\mathcal{L}^A$ is non-trivial, each $\alpha \in \Delta$ with $\alpha \in \mathcal{H}^{S*}$, the dual of $\mathcal{H}^S$, may be extended so that $\alpha \in \mathcal{H}_s^*$, the dual of $\mathcal{H}_s$.

Not all classical simple Lie superalgebras $\mathcal{L}_s$ possess a non-degenerate invariant bilinear form. Those that do are called basic classical simple Lie superalgebras. The Lie superalgebras $A(r/s)$ belong to this class. In addition, there exist basic classical simple Lie superalgebras denoted $B(r/s)$, $C(s)$, $D(r/s)$, $F(4)$ and $G(3)$, because of the relationships between their even parts with the ordinary Lie algebras denoted using the same letter under the Cartan classification. It may be shown that for the basic classical simple Lie superalgebras, the non-degenerate form is also non-degenerate when restricted to the Cartan subalgebra $\mathcal{H}_s$ of $\mathcal{L}_s$. This leads to a theory of positive roots and simple roots analogous to that of the ordinary case. However, in contrast to the ordinary case, all choices of simple roots are not equivalent, in that they are not related by the Weyl group of inner automorphisms of $\mathcal{L}_s$. However, for each of the basic classical simple Lie superalgebras, the set of simple roots may be selected so as to contain only one odd root. Such a selection is known as the distinguished choice. For such a distinguished choice the (positive) simple root vectors of $sl(m/n)$ are given by:

$$\Pi_+^{sl(m/n)} = \{E_a{}^{a+1} : a = 1, 2, \ldots, m+n-1\}. \tag{7.3.21}$$

Note that of these only $E_m{}^{m+1}$ is an odd element of $sl(m/n)$. These simple root vectors generate the nilpotent Borel subalgebra $\mathcal{B}_+^{sl(m/n)}$ spanned by the set:

$$\Delta_+^{sl(m/n)} = \{E_a{}^b : a, b \in \mathcal{I}^{GL(m/n)}, a < b\}. \tag{7.3.22}$$

The generalisation of the theory of representations to the superalgebra case arises through the following definition.

Definition 7.3.23. *A $\mathbf{Z}_2$-graded representation Γ of $\mathcal{L}_s$ maps each element $a \in \mathcal{L}_s$ to $\Gamma(a)$, an element of the $\mathbf{Z}_2$-graded vector space of square (p/q)-matrices, such that:*

(i) $\deg \Gamma(a) = \deg a$;

(ii) $\Gamma(\alpha a + \beta b) = \alpha\Gamma(a) + \beta\Gamma(b)$; *and*

(iii) $\Gamma([a, b]) = \Gamma(a)\Gamma(b) - (-1)^{\deg a \deg b}\Gamma(b)\Gamma(a)$;

for all homogeneous $a, b \in \mathcal{L}_s$ and all $\alpha, \beta \in \mathsf{F}$. The representation Γ is said to have even dimension p and odd dimension q.

In the usual way, each $\mathbf{Z}_2$-graded representation defines a $\mathbf{Z}_2$-graded module. Let V be a $\mathbf{Z}_2$-graded vector space with even subspace $V_{\bar{0}}$ having basis $\{e_i : 1 \leq i \leq$

$p\}$ and odd subspace $V_{\bar{1}}$ having basis $\{e_i : p < i \leq p+q\}$. Then linearly extending the action:

$$ae_i = \sum_{j=1}^{p+q} \Gamma(a)_{ji} e_j, \qquad (7.3.24)$$

for each $e_i \in V$ and all $a \in \mathcal{L}_s$, defines V as a $\mathbf{Z}_2$-graded $\mathcal{L}_s$-module.

The theory of finite-dimensional graded representations of basic classical Lie superalgebras bears a number of similarities to that of the ordinary Lie algebras, but differs significantly in a number of ways **[Ka78,Sc79,Co89]**. In particular, the highest weight of each finite-dimensional irreducible representation of a rank r basic classical simple Lie superalgebra may be specified by a set of 'Kac-Dynkin' labels $(n_1, n_2, \ldots, n_r)$ associated with the simple roots, with labels associated with odd root(s) non-integral or even complex, in general. Furthermore, certain finite-dimensional representations may be reducible but indecomposable.

In the remainder of this chapter, only covariant representations of $sl(m/n)$ and $gl(m/n)$ will be considered. As will be seen, these representations are fully reducible. As elsewhere in this thesis, partitions will be used to label the corresponding irreducible representations, instead of the 'Kac-Dynkin' labels mentioned above. The relationship between these two sets of labels is a little more involved than in the ordinary case. It is discussed in **[BM83]**.

§7.4. The irreducible covariant tensor modules of $gl(m/n)$ and $sl(m/n)$

This section elucidates the connection between Lie supergroups and Lie superalgebras. This connection is then exploited in the case of $GL(m/n)$ and $gl(m/n)$ in order to use the techniques of Section 7.2 to obtain the irreducible covariant $gl(m/n)$- and $sl(m/n)$-modules based on Young tableaux.

Let $\mathcal{G}_s$ be a Lie supergroup of even (m/n)-supermatrices having even dimension p and odd dimension q. Then for $G \in \mathcal{G}_s$ in a small neighbourhood of the identity, $G = G(X_1, X_2, \ldots, X_p; Y_1, Y_2, \ldots, Y_q)$ with each $X_h \in B_{\bar{0}}$ and each $Y_k \in B_{\bar{1}}$. Since G is an superanalytic function on $\mathbf{R}B^{p,q}$, it follows from (7.1.10) that the supermatrices:

$$M_h = \left.\frac{\partial G}{\partial X_h}\right|_{(0;0)} \qquad \text{and} \qquad N_k = \left.\frac{\partial G}{\partial Y_k}\right|_{(0;0)} \qquad (7.4.1)$$

exist and are non-zero. Each M_h is an even (m/n)-supermatrix and each N_k is an odd (m/n)-supermatrix. If $X_h = \sum_\mu X_h^\mu \zeta_\mu$, where the sum is over all even sets $\mu \subset \mathbf{N}_L$ and each $X_h^\mu \in \mathbf{R}$, then the supermatrix derivatives:

$$M_h^\mu = \left.\frac{\partial G}{\partial X_h^\mu}\right|_{(0;0)} \qquad (7.4.2a)$$

exist and satisfy $M_j^\mu = \zeta_\mu M_j$. Similarly, if $Y_k = \sum_\nu Y_k^\nu \zeta_\nu$, where the sum is over all odd sets $\nu \subset \mathbf{N}_L$ and each $Y_k^\nu \in \mathbf{R}$, then the supermatrix derivatives:

$$N_k^\nu = \left.\frac{\partial G}{\partial Y_k^\nu}\right|_{(0;0)} \tag{7.4.2b}$$

exist and satisfy $N_k^\nu = \zeta_\nu N_k$. Each M_j^μ and each N_k^ν is an even (m/n)-supermatrix.

The elements M_h^μ and N_k^ν provide a basis for the real Lie *algebra* of $\mathcal{G}_s$, for which a general element is an even (m/n)-supermatrix of the form:

$$M = \sum_{h=1}^{p} \sum_{\text{even } \mu} X_h^\mu M_h^\mu + \sum_{k=1}^{q} \sum_{\text{odd } \nu} Y_k^\nu N_k^\nu \tag{7.4.3a}$$

$$= \sum_{h=1}^{p} X_h M_h + \sum_{k=1}^{q} Y_k N_k, \tag{7.4.3b}$$

where each $X_h \in B_{\bar{0}}$ and each $Y_k \in B_{\bar{1}}$. In the usual way, this real Lie algebra is closed under the commutator $[M, M']$ and exponentiation of its elements recovers elements in a small neighbourhood of the identity of $\mathcal{G}_s$.

Now, in order to make the connection with Lie superalgebras, let P be any one of the M_h's or N_k's defined by (7.4.1). Then if $\deg P = 0$, P is one of the M_h's and if $\deg P = 1$, it is one of the N_k's. Let ζ_p be a Grassmann parameter of the same degree. Define P' and $\zeta_{p'}$ in an analogous way. Then $\zeta_p P$ and $\zeta_{p'} P'$ are of the form (7.4.3b), being elements of the real Lie algebra of $\mathcal{G}_s$. Then by straightforward matrix multiplication (7.1.8):

$$\begin{aligned}[\zeta_p P, \zeta_{p'} P'] &= \zeta_p \zeta_{p'} P P' - \zeta_{p'} \zeta_p P' P \\ &= \zeta_p \zeta_{p'} (P P' - (-1)^{\deg P \deg P'} P' P) \\ &= \zeta_p \zeta_{p'} [P, P']_{-(-1)^{\deg P \deg P'}},\end{aligned} \tag{7.4.4}$$

where $[A, B]_-$ denotes the usual commutator: $[A, B]_- = AB - BA$; and $[A, B]_+$ denotes the anti-commutator ([**DJ81**]), defined by:

$$[A, B]_+ = AB + BA. \tag{7.4.5}$$

The closure of the real Lie algebra of $\mathcal{G}_s$, described above, implies that if the entries of each P and P' are solely $\mathbf{C}$-multiples of the Grassmann parameter $\zeta_\emptyset = 1 \in B$, then these matrices form an algebra closed under the super Lie product: $[P, P']_{-(-1)^{\deg P \deg P'}}$. This is the Lie superalgebra $\mathcal{L}_s$ of $\mathcal{G}_s$. If the aforementioned condition is not satisfied then $\mathcal{G}_s$ has no corresponding Lie superalgebra. An example of such a case was first given by Rogers and is reproduced in [**Co89**].

Turning attention to $GL(m/n)$, let $X_{i,j}$, for $i,j \in \mathcal{I}^{GL(m/n)}$ with $(i) = (j)$, be a set of $m^2 + n^2$ even Grassmann variables; and $Y_{i,j}$, for $i,j \in \mathcal{I}^{GL(m/n)}$ with $(i) \neq (j)$, be a set of $2mn$ odd Grassmann variables. A convenient parameterisation for $GL(m/n)$, close to the identity is then provided by:

$$G(X;Y)^i{}_j = \begin{cases} \delta^i_j + X_{i,j} & \text{if } (i) = (j); \\ Y_{i,j} & \text{if } (i) \neq (j). \end{cases} \tag{7.4.6}$$

It then follows from (7.4.1) and (7.4.4) that the m^2+n^2 (m/n)-matrices $E_a{}^b$ for $a,b \in \mathcal{I}^{GL(m/n)}$ comprise the Lie superalgebra of $GL(m/n)$. This is the Lie superalgebra $gl(m/n)$ introduced in Section 7.3. To determine its action on the $\mathbf{Z}_2$-graded vector space $V_s^{\otimes l}$ introduced in Section 7.2, consider the element $\zeta E_a{}^b$ of the real Lie algebra of $GL(m/n)$, where $\zeta \in B_{\overline{(a)+(b)}}$ so that $\deg \zeta = (a) + (b) \pmod 2$. From (7.2.2$a$), it acts on the element $\zeta' e_{i_1 i_2 \cdots i_l} \in V_s^{\otimes l}$, where $\deg \zeta' = (i_1) + (i_2) + \cdots + (i_l) \pmod 2$, according to:

$$\zeta E_a{}^b \zeta' e_{i_1 i_2 \cdots i_l} = \zeta' \sum_{k=1}^{l} \prod_{1 \le c \le d \le l} \begin{bmatrix} i_d & j_c \\ j_d & \end{bmatrix} \delta^{j_1}_{i_1} \cdots \delta^{j_{k-1}}_{i_{k-1}} (\zeta E_a{}^b)^{j_k}{}_{i_k} \delta^{j_{k+1}}_{i_{k+1}} \cdots \delta^{j_l}_{i_l} e_{j_1 j_2 \cdots j_l} \tag{7.4.7a}$$

$$= \zeta' \zeta \sum_{k=1}^{l} \prod_{c \le d} \begin{bmatrix} i_d & j_c \\ j_d & \end{bmatrix} \delta^{j_1}_{i_1} \cdots \delta^{j_{k-1}}_{i_{k-1}} \delta^{j_k}_a \delta^b_{i_k} \delta^{j_{k+1}}_{i_{k+1}} \cdots \delta^{j_l}_{i_l} e_{j_1 j_2 \cdots j_l}, \tag{7.4.7b}$$

where there is an implicit summation over all js. Commuting the ζ and the ζ' and noting that for fixed k, $i_d = j_d$ unless $d = k$, yields:

$$\zeta E_a{}^b \zeta' e_{i_1 i_2 \cdots i_l} = \begin{bmatrix} a & i_1 \\ b & i_2 \\ & \vdots \\ & i_l \end{bmatrix} \zeta \zeta' \sum_{k=1}^{l} \begin{bmatrix} a & j_k \\ b & \end{bmatrix} \prod_{c<k} \begin{bmatrix} a & j_c \\ b & \end{bmatrix} \delta^{j_1}_{i_1} \cdots \delta^{j_{k-1}}_{i_{k-1}} \delta^{j_k}_a \delta^b_{i_k} \delta^{j_{k+1}}_{i_{k+1}} \cdots \delta^{j_l}_{i_l} e_{j_1 j_2 \cdots j_l}. \tag{7.4.7c}$$

Therefore, on removing the common factor of $\zeta\zeta'$:

$$E_a{}^b e_{i_1 i_2 \cdots i_l} = \begin{bmatrix} a & i_1 \\ b & i_2 \\ & \vdots \\ & i_l \end{bmatrix} \begin{bmatrix} a & a \\ b & \end{bmatrix} \sum_{k=1}^{l} \delta^b_{i_k} \prod_{c<k} \begin{bmatrix} a & i_c \\ b & \end{bmatrix} e_{i_1 \cdots i_{k-1} a i_{k+1} \cdots i_l} \tag{7.4.7d}$$

$$= \begin{bmatrix} a & a \\ b & \end{bmatrix} \sum_{k=1}^{l} \delta^b_{i_k} \begin{bmatrix} a & i_k \\ b & i_{k+1} \\ & \vdots \\ & i_l \end{bmatrix} e_{i_1 \cdots i_{k-1} a i_{k+1} \cdots i_l}. \tag{7.4.7e}$$

This defines an action of $gl(m/n)$ on $V_s^{\otimes l}$.

Since $GL(m/n)$ commutes with the action of $\tilde{S}_l$, it follows that $gl(m/n)$ also does so. It then follows from Theorem 4.2.26 that each $\tilde{W}^\lambda$ for $\lambda \in P(l;m/n)$ is an irreducible $gl(m/n)$-module. The action of $E_a{}^b$ on the grade-symmetrised tableau $\{T^\lambda\}^\sim$ now follows directly from (7.4.7*e*). Let s be the number of times that the index b occurs in T^λ, and form s distinct tableaux $T^\lambda_{/k}$ by replacing a single index b in position k of T^λ with a, for all appropriate positions k of T^λ. Then:

$$E_a{}^b\{T^\lambda\}^\sim = \sum_{\{k:T^\lambda_{(k)}=b\}} \begin{bmatrix} a & a \\ b & \end{bmatrix} \begin{bmatrix} a & T^\lambda_{(k)} \\ b & T^\lambda_{(k+1)} \\ & \vdots \\ & T^\lambda_{(l)} \end{bmatrix} \left\{T^\lambda_{/k}\right\}^\sim . \tag{7.4.8}$$

Any non-standard tableau appearing on the right side may then be expressed as a linear combination of $GL(m/n)$-standard tableaux of the same grade, using the techniques described in Section 7.2. In precisely this way the action of $E_a{}^b \in gl(m/n)$ on each grade-symmetrised $GL(m/n)$-standard tableau $\{T^\lambda_u\}^\sim$ results in:

$$E_a{}^b\{T^\lambda_u\}^\sim = \sum_{\{T^\lambda_v:T^\lambda_v\ GL(m/n)\text{-standard}\}} (\Gamma_a{}^b)_{vu}\{T^\lambda_v\}^\sim , \tag{7.4.9}$$

for some rational numbers $(\Gamma_a{}^b)_{vu}$. The matrices $\Gamma_a{}^b$ thus yield a representation of $gl(m/n)$.

In order to make contact with [**DJ81**], it is necessary to transform these representation matrices in a certain way. To do this, it is convenient to grade the indices which refer to the matrix elements of $\Gamma_a{}^b$, so that if $\{\lambda\}$ is a representation of even dimension r and odd dimension s, then:

$$(u) = \begin{cases} 0, & \text{if } 1 \le u \le r; \\ 1, & \text{if } r < u \le r+s, \end{cases} \tag{7.4.10}$$

in analogy with (7.1.5). Only the indices u, v, w will respect this grading, with all others being determined by (7.1.5). The grading of the representation determines that $(\Gamma_a{}^b)_{uv}$ is non-zero only if $(a)+(b) = (u)+(v)$ (mod2). Let $\Gamma_a{}^b$ be an (r/s)-matrix, partitioned as in (7.3.3), and form Γ^{*b}_a from $\Gamma_a{}^b$ by changing the sign of the submatrix R if and only if $E_a{}^b$ is a positive odd root vector (and thus $(a) \neq (b)$ and $a < b$) and changing the sign of the submatrix Q if and only if $E_a{}^b$ is a negative odd root vector (and thus $(a) \neq (b)$ and $a > b$). This prescription may be expressed:

$$\begin{aligned}(\Gamma^{*b}_a)_{uv} &= \begin{bmatrix} a & b & u \\ b & & \end{bmatrix} \begin{bmatrix} a & a & v \\ b & & \end{bmatrix} (\Gamma_a{}^b)_{uv} \\ &= \begin{bmatrix} a & a \\ b & u \end{bmatrix} (\Gamma_a{}^b)_{uv},\end{aligned} \tag{7.4.11}$$

where $(a)+(b)=(u)+(v)\ (\bmod 2)$ has been used. Since the matrices ${\Gamma_a}^b$ represent the elements ${E_a}^b$, it follows from (7.3.4) and (7.3.5) that:

$$({\Gamma_a}^b)_{uw}({\Gamma_c}^d)_{wv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} ({\Gamma_c}^d)_{uw}({\Gamma_a}^b)_{wv} = \delta^b_c({\Gamma_a}^d)_{uv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \delta^d_a({\Gamma_c}^b)_{uv}.$$

Thereupon, recalling that for all non-zero $({\Gamma_c}^d)_{wv}$, $(c)+(w)=(d)+(v)\ (\bmod 2)$:

$$\begin{aligned}
[\Gamma^{*b}_a, \Gamma^{*d}_c]_{uv} &= (\Gamma^{*b}_a)_{uw}(\Gamma^{*d}_c)_{wv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} (\Gamma^{*d}_c)_{uw}(\Gamma^{*b}_a)_{wv} \\
&= \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} c & c \\ d & w \end{bmatrix} ({\Gamma_a}^b)_{uw}({\Gamma_c}^d)_{wv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c & c \\ d & u \end{bmatrix} \begin{bmatrix} a & a \\ b & w \end{bmatrix} ({\Gamma_c}^d)_{uw}({\Gamma_a}^b)_{wv} \\
&= \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} c & d \\ d & v \end{bmatrix} ({\Gamma_a}^b)_{uw}({\Gamma_c}^d)_{wv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c & c \\ d & u \end{bmatrix} \begin{bmatrix} a & b \\ b & v \end{bmatrix} ({\Gamma_c}^d)_{uw}({\Gamma_a}^b)_{wv} \\
&= \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} c & d \\ d & v \end{bmatrix} ({\Gamma_a}^b)_{uw}({\Gamma_c}^d)_{wv} \\
&\qquad - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} c & a \\ d & b \end{bmatrix} \begin{bmatrix} c & d \\ d & v \end{bmatrix} \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} ({\Gamma_c}^d)_{uw}({\Gamma_a}^b)_{wv} \\
&= \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} c & d \\ d & v \end{bmatrix} \left(({\Gamma_a}^b)_{uw}({\Gamma_c}^d)_{wv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} ({\Gamma_c}^d)_{uw}({\Gamma_a}^b)_{wv} \right) \\
&= \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} c & d \\ d & v \end{bmatrix} \left(\delta^b_c({\Gamma_a}^d)_{uv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \delta^d_a({\Gamma_c}^b)_{uv} \right) \\
&= \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} b & a \\ d & u \end{bmatrix} \delta^b_c({\Gamma_a}^d)_{uv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & a \\ b & u \end{bmatrix} \begin{bmatrix} a & a \\ c & v \end{bmatrix} \delta^d_a({\Gamma_c}^b)_{uv} \\
&= \begin{bmatrix} a & a \\ d & u \end{bmatrix} \delta^b_c({\Gamma_a}^d)_{uv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} b & c \\ c & u \end{bmatrix} \delta^d_a({\Gamma_c}^b)_{uv} \\
&= \delta^b_c(\Gamma^{*d}_a)_{uv} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} \delta^d_a(\Gamma^{*b}_c)_{uv},
\end{aligned}$$

where use has been made of the fact that for non-zero $({\Gamma_c}^d)_{uw}({\Gamma_a}^b)_{wv}$, $(a)+(b)+(c)+(d)+(u)+(v)=0\ (\bmod 2)$. Therefore the matrices Γ^{*b}_a also provide a representation of the elements ${E_a}^b \in gl(m/n)$. As will be shown later, the representation Γ^* is equivalent to the representation Γ.

In terms of tableaux, if the tableau T^λ having entries $T^\lambda_{(1)}, T^\lambda_{(2)}, \ldots, T^\lambda_{(l)}$ is indexed by u, then from (7.2.1), $(u)=(T^\lambda_{(1)})+(T^\lambda_{(2)})+\cdots+(T^\lambda_{(l)})$ and consequently:

$$\begin{bmatrix} a & a \\ b & u \end{bmatrix} = \begin{bmatrix} a & a \\ b & \end{bmatrix} \begin{bmatrix} a & T^\lambda_{(1)} \\ b & T^\lambda_{(2)} \\ & \vdots \\ & T^\lambda_{(l)} \end{bmatrix}. \tag{7.4.12}$$

Therefore, if the action of (7.4.8) is modified to:

$$\begin{aligned} E_a{}^b\{T^\lambda\}^\sim &= \sum_{\{k:T^\lambda_{(k)}=b\}} \begin{bmatrix} a & a \\ b & \end{bmatrix} \begin{bmatrix} a & T^\lambda_{(1)} \\ b & T^\lambda_{(2)} \\ & \vdots \\ & T^\lambda_{(l)} \end{bmatrix} \begin{bmatrix} a & a \\ b & \end{bmatrix} \begin{bmatrix} a & T^\lambda_{(k)} \\ b & T^\lambda_{(k+1)} \\ & \vdots \\ & T^\lambda_{(l)} \end{bmatrix} \left\{T^\lambda_{/k}\right\}^\sim \\ &= \sum_{\{k:T^\lambda_{(k)}=b\}} \begin{bmatrix} a & T^\lambda_{(1)} \\ b & T^\lambda_{(2)} \\ & \vdots \\ & T^\lambda_{(k-1)} \end{bmatrix} \left\{T^\lambda_{/k}\right\}^\sim , \end{aligned} \tag{7.4.13}$$

the matrices $\Gamma^{\{\lambda\}}(E_a{}^b)$ which result from standardising all the grade-symmetrised tableaux on the right side:

$$E_a{}^b\{T^\lambda_u\}^\sim = \sum_{\{T^\lambda_v : T^\lambda_v\ GL(m/n)\text{-standard}\}} \Gamma^{\{\lambda\}}(E_a{}^b)_{vu}\{T^\lambda_v\}^\sim, \tag{7.4.14}$$

provide a representation of the basis elements $E_a{}^b \in gl(m/n)$ in the representation labelled by λ.

As an example consider the $gl(2/2)$ odd generator $E_3{}^2$ in the 32-dimensional $gl(2/2)$-module $\tilde{W}^\lambda$ with $\lambda = (2^3, 1)$, and the $gl(2/2)$-standard tableau:

$$T^\lambda = \begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{matrix} . \tag{7.4.15a}$$

By virtue of (7.2.1), T^λ and $\{T^\lambda\}^\sim$ are of even grade. Using (7.4.13), $E_3{}^2$ acts on the basis element $\{T^\lambda\}^\sim$ of M^λ, according to:

$$\begin{aligned} E_3{}^2 \begin{Bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{Bmatrix}^\sim &= \begin{bmatrix} 2 & 1 \\ 3 & \end{bmatrix} \begin{Bmatrix} 1 & 2 \\ 3 & 3 \\ 3 & 4 \\ 4 & \end{Bmatrix}^\sim + \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ & 3 \\ & 4 \end{bmatrix} \begin{Bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{Bmatrix}^\sim \\ &= \begin{Bmatrix} 1 & 2 \\ 3 & 3 \\ 3 & 4 \\ 4 & \end{Bmatrix}^\sim + \begin{Bmatrix} 1 & 3 \\ 2 & 3 \\ 3 & 4 \\ 4 & \end{Bmatrix}^\sim . \end{aligned} \tag{7.4.15b}$$

The graded Garnir relations (7.2.18), with $\mathcal{X} = \{2,3,4\}$ and $\mathcal{Y} = \{5,6\}$, give the identity:

$$\left\{\begin{matrix}1&2\\3&3\\3&4\\4&\end{matrix}\right\}^{\sim}-\left\{\begin{matrix}1&3\\2&3\\3&4\\4&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&3\\3&3\\2&4\\4&\end{matrix}\right\}^{\sim}-\left\{\begin{matrix}1&4\\3&3\\3&4\\2&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&2\\3&3\\3&4\\4&\end{matrix}\right\}^{\sim}$$
$$+\left\{\begin{matrix}1&2\\3&3\\3&4\\4&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&2\\3&4\\3&4\\3&\end{matrix}\right\}^{\sim}-\left\{\begin{matrix}1&3\\2&3\\3&4\\4&\end{matrix}\right\}^{\sim}-\left\{\begin{matrix}1&3\\2&4\\3&4\\3&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&3\\3&4\\2&4\\3&\end{matrix}\right\}^{\sim}=0, \tag{7.4.15c}$$

which, on using the Column relations (7.2.16), and collecting terms, gives:

$$3\left\{\begin{matrix}1&2\\3&3\\3&4\\4&\end{matrix}\right\}^{\sim}-3\left\{\begin{matrix}1&3\\2&3\\3&4\\4&\end{matrix}\right\}^{\sim}-3\left\{\begin{matrix}1&3\\2&4\\3&4\\3&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&2\\3&4\\3&4\\3&\end{matrix}\right\}^{\sim}=0, \tag{7.4.15d}$$

so that:

$$\left\{\begin{matrix}1&2\\3&3\\3&4\\4&\end{matrix}\right\}^{\sim}=\left\{\begin{matrix}1&3\\2&3\\3&4\\4&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&3\\2&4\\3&4\\3&\end{matrix}\right\}^{\sim}-\frac{1}{3}\left\{\begin{matrix}1&2\\3&4\\3&4\\3&\end{matrix}\right\}^{\sim}. \tag{7.4.15e}$$

Hence, from (7.4.15b):

$$E_3{}^2\left\{\begin{matrix}1&2\\2&3\\3&4\\4&\end{matrix}\right\}^{\sim}=2\left\{\begin{matrix}1&3\\2&3\\3&4\\4&\end{matrix}\right\}^{\sim}+\left\{\begin{matrix}1&3\\2&4\\3&4\\3&\end{matrix}\right\}^{\sim}-\frac{1}{3}\left\{\begin{matrix}1&2\\3&4\\3&4\\3&\end{matrix}\right\}^{\sim}. \tag{7.4.15g}$$

The following two examples exhibit shorter calculations:

$$E_3{}^2\left\{\begin{matrix}1&2\\3&4\\3&4\\4&\end{matrix}\right\}^{\sim}=\begin{bmatrix}2&1\\3&3\\&3\\&4\end{bmatrix}\left\{\begin{matrix}1&3\\3&4\\3&4\\4&\end{matrix}\right\}^{\sim}=-\left\{\begin{matrix}1&3\\3&4\\3&4\\4&\end{matrix}\right\}^{\sim}, \tag{7.4.15h}$$

$$E_3{}^2\left\{\begin{matrix}1&1\\3&4\\3&4\\3&\end{matrix}\right\}^{\sim}=0. \tag{7.4.15i}$$

Similar calculations, when carried out for each of the thirty-two $GL(2/2)$-standard tableaux in $\tilde{W}^{\lambda}$, yield, via (7.4.14), the following explicit representation $\Gamma^{\{2^3,1\}}(E_3{}^2)$ of $E_3{}^2$:

$$\left(\begin{array}{cccccccccccccccccccccccccccccccc}
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & -1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & -4 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & -1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & -5 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & -1 & . & -\frac{2}{3} & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 3 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 4 \\
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & -\frac{1}{4} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & -\frac{1}{3} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & -\frac{1}{3} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & \frac{5}{2} & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}\right),$$

where each zero has been replaced by a dot. The three calculations carried out above give rise the entries in the 13th, 23rd and 28th columns of this matrix respectively. Notice that this matrix has the block diagonal nature associated with an odd grading. This structure is ensured by the odd grading of the element ${E_3}^2$ and the adoption of an ordering of the $GL(2/2)$-standard tableaux such that all those of even grade occur first. The above construction process has been implemented on a computer, the above matrix having been produced by this means. In addition similar matrices for the remaining generators of $gl(2/2)$, in the same irreducible representation $\{2^3, 1\}$, have also been produced. As a check on the calculations it has been confirmed that the resulting matrices satisfy the commutation relations given by (7.3.5).

As a second example consider the eight dimensional $GL(2/1)$-module $\tilde{W}^{(2,1)}$ for which the $GL(2/1)$-standard tableaux are:

$$\begin{array}{cc}2&3\\3&\end{array}\,,\quad \begin{array}{cc}1&3\\3&\end{array}\,,\quad \begin{array}{cc}1&2\\2&\end{array}\,,\quad \begin{array}{cc}1&1\\2&\end{array}\,,\quad \begin{array}{cc}1&3\\2&\end{array}\,,\quad \begin{array}{cc}2&2\\3&\end{array}\,,\quad \begin{array}{cc}1&2\\3&\end{array}\,,\quad \begin{array}{cc}1&1\\3&\end{array}\,, \qquad (7.4.16)$$

where they have been ordered in such a way that the four even tableaux occur first. Calculations involving the use of the Column relations (7.2.16) and the Garnir relations (7.2.18) give, via (7.4.14), the following set of explicit representation matrices for the basis elements of $gl(2/1)$:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

are $\Gamma^{\{2,1\}}(E_1{}^2)$, $\Gamma^{\{2,1\}}(E_1{}^3)$ and $\Gamma^{\{2,1\}}(E_2{}^3)$ respectively;

$$\begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & -2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

are $\Gamma^{\{2,1\}}(E_2{}^1)$, $\Gamma^{\{2,1\}}(E_3{}^1)$ and $\Gamma^{\{2,1\}}(E_3{}^2)$ respectively; and for the diagonal elements of the Cartan subalgebra:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2 \end{pmatrix}, \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \begin{pmatrix} 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}.$$

are $\Gamma^{\{2,1\}}(E_1{}^1)$, $\Gamma^{\{2,1\}}(E_2{}^2)$ and $\Gamma^{\{2,1\}}(E_3{}^3)$ respectively. Once more, these matrices possess the block diagonal structure associated with the gradings of the elements of $gl(2/1)$ that they represent.

The $gl(m/n)$-module $\tilde{W}^\lambda$ also serves as a module for the $(m+n)^2-1$ dimensional basic classical Lie superalgebra $sl(m/n)$. Consider $gl(2/1)$ once more. In accordance with (7.3.9), let $A_1{}^1 = E_1{}^1 - E_2{}^2$ and $A_2{}^2 = E_2{}^2 + E_3{}^3$. Then $A_1{}^1$ and $A_2{}^2$ form a basis for the Cartan subalgebra of the oft studied eight dimensional simple basic classical Lie superalgebra $sl(2/1) = A(1/0)$; the other basis elements may be taken to be $E_1{}^2$, $E_2{}^1$, $E_1{}^3$, $E_3{}^1$, $E_2{}^3$ and $E_3{}^2$, as above. $\tilde{W}^{(2,1)}$ then serves as a

$sl(2/1)$-module and the corresponding representation matrices are obtained directly from those given above for $gl(2/1)$.

The highest weight vector of the $sl(2/1)$-module $\tilde{W}^{(2,1)}$ is $\left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\}^{\sim}$, for which

$$A_1{}^1 \left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\}^{\sim} = \left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\}^{\sim} \quad \text{and} \quad A_2{}^2 \left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\}^{\sim} = \left\{ \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right\}^{\sim}. \tag{7.4.17}$$

This yields the 'Kac-Dynkin' label $(1,1)$ for the representation $\{2,1\}$.

In general, the highest weight vector of the $sl(m/n)$-module $\tilde{W}^{\lambda}$ is provided by $\{T^{\lambda}_{s>}\}$ where:

$$T^{\lambda}_{s>(j,k)} = \begin{cases} j & \text{if } 1 \le j \le m, \tilde{\lambda}_1 \text{ and } 1 \le k \le \lambda_j; \\ k+m & \text{if } m < j \le \tilde{\lambda}_1 \text{ and } 1 \le k \le \lambda_j. \end{cases} \tag{7.4.18}$$

For example, if $m = 2$ and $n = 3$ then:

$$T^{(4,3,3,2,2)}_{s>} = \begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & 4 & 5 & \\ 3 & 4 & & \\ 3 & 4 & & \end{matrix} \quad . \tag{7.4.19}$$

It is easily verified that $E_a{}^b\{T^{\lambda}_{s>}\} = 0$ for all $a, b \in \mathcal{I}^{GL(m/n)}$ with $a < b$, confirming that $\{T^{\lambda}_{s>}\}$ is indeed the highest weight vector. This argument holds for either of the actions given by (7.4.8) or (7.4.13). Since $a = b$ in either of these cases implies that the coefficient is $+1$, it follows that the highest weights of the representations Γ^* and Γ are equal. Therefore, by the supersymmetric analogue of the part of Theorem 1.7.7 which states that representations having the same highest weight are equivalent [**Ka78,Co89**], Γ^* and Γ are equivalent representations.

In this chapter, techniques to construct all the irreducible finite-dimensional covariant representations of $gl(m/n)$ and $sl(m/n)$ were demonstrated. In [**DJ81**], it was suggested as to how the generators act on a contravariant basis for V_s^*. In a way similar to that described in this chapter, this action may be generalised to an action on $(V_s^*)^{\otimes l}$ and thence the irreducible contravariant modules are obtained. The use of computer calculations has once again verified this construction. Significantly, these classes of covariant and contravariant irreducible $gl(m/n)$-modules encompass both typical and atypical [**Co89**] cases.

It is expected that a combination of the techniques of this section with those of Section 4.4 will be applicable to the mixed tensor space $(V_s^*)^{\otimes v} \otimes V_s^{\otimes u}$. However, it is known that this $gl(m/n)$-module is not completely reducible [**BM83**]. Thus, it will be of great interest to investigate the extent to which Young tableaux techniques interact with this fact.

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