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The Semiflow Obtained by Integrating the Projection Onto a Submanifold with Corners of Euclidean Space of a Smooth Vector Field

by

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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The Semiflow Obtained by Integrating the Projection Onto a Submanifold with Corners of Euclidean Space of a Smooth Vector Field

by Timothy John Payne

We study the semiflow on a submanifold with corners M of Euclidean Space R^n obtained as follows. If a smooth vector field X is given on a neighbourhood of M in R^n we project it at each point of M onto the tangent cone to M at the point and integrate the resulting inner vector field X(M) on M: such systems arise in mathematical economics, mathematical biology and in the theory of electrical networks.

We obtain an existence-uniqueness result and construct a device, the iteration, with which to study the local behaviour of trajectories, in particular in relation to the smooth flows obtained by projecting X onto individual strata of M. We investigate the relation between the iteration, right hand time derivatives of the trajectories, and generalisations of the classical tangency sets, establish a canonical form for intersections of the last and establish their generic properties.

We investigate the local geometry of the semiflow and show that in most cases the classical theory has no simple generalisation to these systems, but using an ad hoc equivalence relation which respects the natural stratification of M we show that some significant local geometric results can be established. We show that if a condition involving the absence of infinite order tangencies is satisfied at a point then the number of stratum jumps made by the trajectories on a neighbourhood of this point is uniformly bounded, and we use this to show that the semiflow obtained from a residual subset of polynomial vector fields with M an orthant (this context includes the biological models) is in our strong stratum preserving sense locally stable near points x where X(M)(x) is non-vanishing.

We consider briefly the global geometry of these systems, and in particular obtain a result with significant implications for the piece-wise linear systems occurring in mathematical biology which inspired the study.

Contents

Preface

In this thesis we study a class of semidynamical system which arises in models in mathematical economics and biology, and in the theory of electrical networks. The author thanks the S.E.R.C. which provided funding, The Faculty of Maths in Southampton where the author was a research student when the work was carried out, Dr D.R.J. Chillingworth who suggested [60] as inspiration for a thesis and dealt with all Faculty matters, Professor S.A. Robertson, an idea of whose led to some improvement in the material in the first half of Chapter Eight, and Dr A.K. Manning (of Warwick University) who made a number of suggestions regarding the writing up.

Notational Conventions

Whenever the symbols $\to$, $\uparrow$, $\downarrow$ are used the existence of the appropriate sequence is implied whether explicitly stated or not. Thus "suppose there exists $h_j \downarrow 0$" is short for "suppose there exists a sequence of positives reals $\{h_j\}_{j\in Z^+}$ such that $h_j \downarrow 0$ as $j \to \infty$",

and

"supppose K is such that there exists $h_j(K) \uparrow 0$" is short for "suppose K is such that there exists a sequence of negative reals $\{h_j(K)\}_{j\in Z^+}$ such that $h_j(K) \uparrow 0$ as $j \to \infty$".

The ith time derivatve of f evaluated at t=0 is denoted $D_t^i f(t=0)$.

Sets of indices are enclosed within round brakets, eg (1,2,3), instead of the usual { }.

The reader's attention is drawn to the existence of an index to symbols and notation, beginning on page 222.

Introduction

In this thesis we shall study the geometric properties of a set of differential equations subject to a particular type of constraint, or equivalently a particular kind of semiflow on a submanifold with corners of R^n, which arise in mathematical economics, mathematical biology, in the theory of electrical networks, and elsewhere.

If we are given a submanifold with corners (these terms will be defined formally in Chapter One) M of R^n and a smooth vector field X on R^n (or at least on a neighbourhood of M) we construct a new vector field X(M) on M by at each point x of M projecting X(x) onto the tangent cone to M at x and calling the result X(M)(x). We will show we can integrate the resulting (inner) vector field X(M) on M to form trajectories on M. The trajectories of our system (M,X) will then be just those of X as long as the trajectory x(t) remains in the interior of M, but on intersecting ∂M x(t) will crudely speaking "slide" along the boundary of M until the vector field lifts it off again.

<u>Example 0.1</u> Suppose M is the cube in R^2 $M=\{x\in R^2: |x_i| \leq 1\}$, then if $X(x_1,x_2)=(-x_2,x_1)$ we obtain by this construction X(M)(x)=X(x) unless $x\in \ell_1,\ell_2,\ell_3$, or ℓ_4 (see Figure 0.1a) where $X(M)(x)=(-x_2,0)$ or $(0,x_1)$:

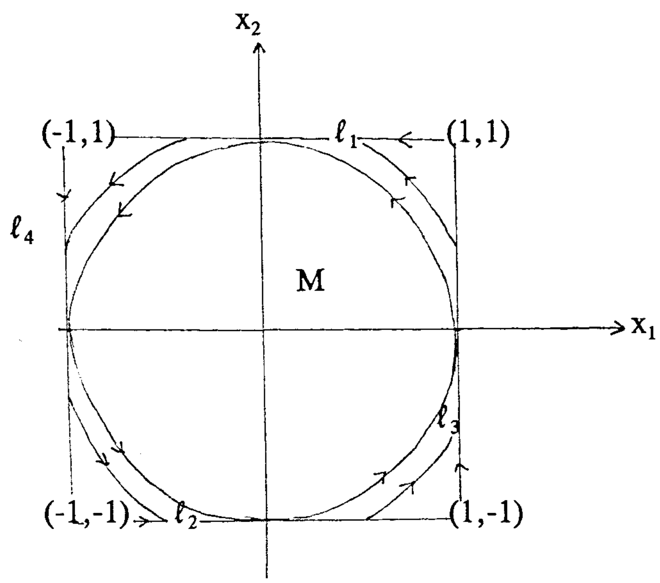


Figure 0.1a

If we take M as above but with $X(x_1,x_2) = (1,1)$ we obtain $X(M)(x)=X(x)$ unless $x\in \ell_1,\ell_2$,or ℓ_{12}, where $X(M)(x)=(1,0)$ or $(0,1)$ or $(0,0)$:

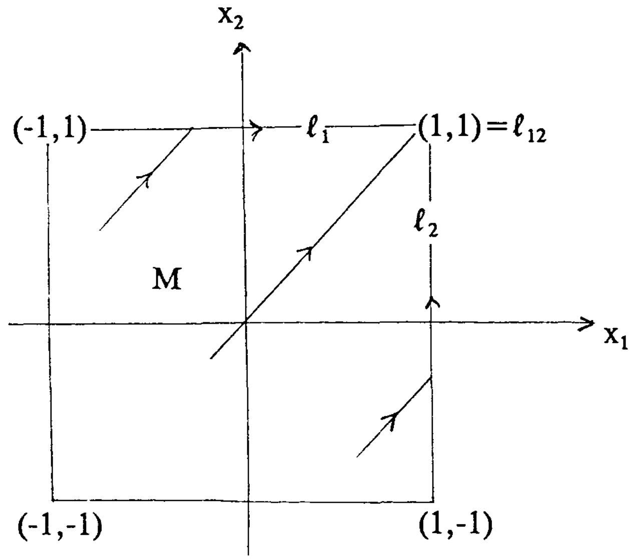
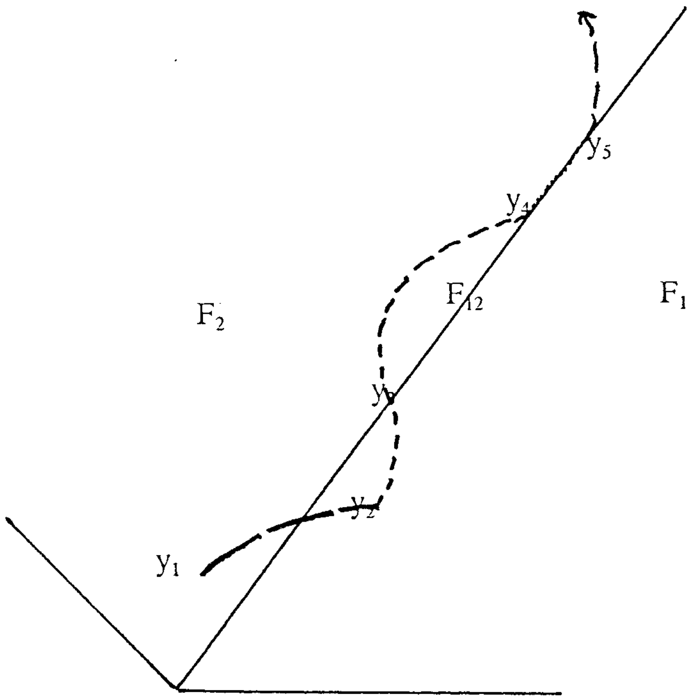


Figure 0.1b

We see that by making X(M) discontinuous around part of ∂M we are able to confine the trajectories starting in M to M for all $t\geq 0$.

In addition to scrutinizing figure A.7 of the appendix the reader may consider the following:

<u>Example 0.2</u> Take for M $\{x\in R^3:2x_2-x_1\geq 0,2x_2+x_1\geq 0\}$ where in cross-section the angle between the two faces is greater than a right angle (this is important): then for a suitable vector field X we may find a trajectory of X(M) beginning at $y_1\in int(M)$, hitting the face $F_1=\{x\in R^3:2x_2-x_1=0,2x_2+x_1>0\}$ at y_2, sliding along F_1 until meeting $F_{12}=\{x\in R^3:2x_2-x_1=0,2x_2+x_1=0\}$ at y_3, crossing straight over to $F_2=\{x\in R^3:2x_2-x_1>0,2x_2+x_1=0\}$, returning to F_{12} at y_4, and sliding along F_{12} until re-entering F_2 at y_5 (Figure 0.2).



Trajectory in int(M) ———————

Trajectory in F_1 or F_2 - - - - - - - -

Trajectory in F_{12} ················

Figure 0.2

We now consider situations where such systems arise.

One class of examples occurs in economics. Suppose we have N consumers, m public goods (this is an economics term for a commodity with the property that the total amount produced may in its entirety be "consumed" simultaneously by every one of the consumers, such as broadcasting), and one private good. Let us denote by x_i the amount of the ith public good produced, and by y_j the amount of the private good consumed by the jth consumer. One objective of [16,40] is to exhibit a set of differential equations involving $x_1,..,x_m$, $y_1,..,y_N$ satisfying various properties, such as that the integral curve starting from any initial condition converges to a pseudo-equilibrium ([16]). This is done by finding suitable differentiable functions $f,g:R^{N+m} \to R^m, R^N$ (these are non-trivial functions of marginal costs, rates of substitution, etc) and setting

$$\left.\begin{array}{l} \dot{x}_k = \begin{cases} f_k(x,y) \text{ if } x_k > 0 \\ \max(0, f_k(x,y)) \text{ if } x_k = 0 \end{cases} \\ \dot{y} = g(x,y) \end{array}\right\} \quad (*)$$

(for k=1,..,m, where $x=(x_1,..,x_m)$ and $y=(y_1,..,y_N)$). The form of the right hand side of (*) arises to avoid generating negative output levels. We see that this is of our form with $M=\{x\in R^{m+N}:x_i\geq 0\ \forall i=1,..,m\}$ and X(x,y)=(f(x,y),g(x,y)), f,g as given above; in the case where M is an orthant (ie, a set of the form $\{x\in R^n:x_i\geq 0,\ i=1,..,k\}$, some $k\leq n$) projection takes the form (*) (see eg Remarks 2.5).

A second example comes from a model for the levels of activity in coupled neural populations [60]: the populations are of four types, each subject to constant excitatory input, and each emitting a signal which inhibits activity levels in each of the other populations (including in itself), the strength of the inhibition being different for different receiving populations but rising with the activity level in the population emitting the signal. A simple set of equations exhibiting such behaviour is (see [60])

$$(**) \qquad \dot{y}_i=\begin{cases} f_i(y) \text{ if } y_i>0 \\ \max(0,f_i(y)) \text{ if } y_i=0 \end{cases}$$

where y_i is the activity level of the ith population, i=1,..,4, or for more general systems of this type i=1,..,n, f(y)=k-Ay, k,y are n-vectors with each $k_i>0$, and A is an $(n\times n)$ matrix with $A_{ij}\geq 0$ for all $1\leq i,j\leq n$. The form of (**) arises to prevent negative activity levels. The author of [60] considers systems of this type both with n=4 and (coupling four such systems together) n=16; in both cases we see we have a system of our form with $M=(R^+\cup\{0\})^n$, n=4 or 16, X(x)=k-Ax.

We may stratify $(R^+\cup\{0\})^n$ into 2^n "strata" - sets of the form $\sigma^k=\{x\in R^n:x_i=0,\ i=1,..,k,\ x_i>0,\ i=k+1,..,n\}$ - and we see that a solution trajectory starting in any given stratum will travel along it until lifting off to a higher dimensional stratum or hitting a lower dimensional one; denoting the set of points where the flow on stratum σ^k_i lifts off to σ^{k-1}_j by $l_k(i,j)$ an interesting consequence (Chapter 8) of positivity of the coefficients A_{ij} and of the fact $M=(R^+\cup\{0\})^n$ is that subject to mild extra restrictions on A each of the iterated maps induced by the flow of the form $l_k(0,1)\rightarrow\sigma^{k-1}_1\rightarrow\sigma^k_2\rightarrow l_k(2,3)\rightarrow..\rightarrow l_k(0,1)$ is invertible.

This is an interesting result, although one would only regard it as amongst the most important results in the thesis if one's interest in this thesis was exclusively in its implications for the global dynamics of Willis Models (Willis being the author of [60]). Willis Models and other specific models bear the same relation to this thesis that any specific set of differential equations do to the classical geometric theory of differential equations, as expounded in [37,42]: they inspired our interest and occasionally influence our choice of topic, as when for example in Chapter Seven we treat polynomial systems, a case motivated by the structure of Willis models, but the main aim of this thesis is to develop the general theory of these systems in a way analogous to that for classical systems found in [37,42]. Nothing like this has been attempted before: scrutiny of the books [4] and [20], which present respectively the Western and Russian Schools of the group of subjects to which this thesis is most closely related- differential equations with discontinuous right hand sides, differential inclusions (which arise as the regularisation of differential equations with discontinuous right hand sides), and viability theory (concerning "viable" trajectories: if K is a closed subset of R^n a trajectory ϕ_x is viable if for all $t \geq 0$ $\phi_x(t) \in K$) - will give the reader a clear picture of how this subject has been treated hitherto: most of the work has been concerned with establishing minimum conditions to guarantee existence, uniqueness or viability of solutions, and when qualitative theory is discussed (in [20]) it is in the general context of differential equations with discontinuous right hand side (and then mainly in the plane) where little of consequence can be proved.

The context in which we shall develop our theory is as follows: we shall begin with a submanifold with corners M of Euclidean space R^n and a smooth vector field X given on M, and at each $x \in M$ we project X(x) onto the tangent cone to M at x. The result will be a vector field X(M) equal to X in int(M) but in general discontinuous on part of ∂M. In the theory of differential equations with discontinuous right hand side an often used definition of "solution" (see [20]) is a continuous almost everywhere differentiable curve with derivative, where it exists, equal to the right hand side; we adopt this definition and show that for any $x \in M$ there exists a right-hand interval $[0,\delta)$ of $t=0$ (which may if M is compact be taken as $[0,\infty)$) and a continuous a.e. differentiable map $\phi(M,X)(x):[0,\delta) \rightarrow M$ such that $D_t\phi(M,X)(x)(t) = X(M)\phi(M,X)(x)(t)$ for almost all $t \in [0,\delta)$, and that $\phi(M,X)(x)$ is unique and depends continuously on x.

A semidynamical system is a continuous map $\psi: G \times M \rightarrow M$, where G is the set of non-negative integers under addition or non-negative reals under addition (in which case ψ may be termed a semiflow) satisfying for all $x \in M$, for all $s,t \geq 0$, $\psi(t+s,x) = \psi(t,\psi(s,x))$ and $\psi(0,x) = x$. Hence at least for compact M $\phi(M,X)$ is a semiflow, and this thesis is a study of a class of semiflow or semidynamical system.

The theory of trajectories is much richer than is the case for unconstrained systems and Chapters 1-5 are mainly devoted to it. In Chapter 6 we consider a few questions concerning the local geometry of the semiflow. In chapter 7 we establish a local stability theorem for polynomial systems, and in chapter 8 make a brief study of linear systems and prove the result concerning Willis models mentioned above. In an appendix we consider aspects of global theory.

All the material in this thesis is new, subject to two qualifications:

(1) Theorem 1.1 does not go far beyond results of Cornet [12] or Chikin [10] (roughly speaking parts 1 and 2 of theorem 1.1 extend to submanifolds with corners of arbitrary codimension what Chikin establishes for submanifolds with corners of codimension 0) and part 1 of theorem 3.1 - the part which says that the right hand derivative of $\phi(M)(x)$ at $t=0$ is $X(M)(x)$ - has been done for M an orthant by Henry [31].

(2) Versions of those preparatory results which are of a very general nature will clearly have been obtained elewhere already. Into this category will certainly come Lemmas 1.1 and 2.1 and Remarks 2.1, and probably Lemmas 2.2, 3.1 and part (1) of the proof of Lemma 5.9, and Remarks 2.5(1) and 4.2.

The questions we consider in this thesis are ones which we consider to be basic to this class of system, and it will be seen that the character of the theory we develop, based upon convexity, stratifications and ideas such as the iteration which make their appearance here for the first time, is unlike that of the classical theory. The local stability result in Chapter 7 mentioned above for example is established using the locally finite stratifiability of subanalytic sets and our uniform bound theorem of Chapter 5, which is well removed from the methods of [37,42]. In chapter 6 we consider local theory in a classical way and find that most of the classical results have no straightforward generalisations to these systems. Additionally because of the way our trajectories jump about between strata the division between the theory of trajectories and local theory is not as clear-cut as it is for unconstrained systems - it is for example no easier to show Theorem 5.1 for points on an individual trajectory than it is for any convergent sequence of points on M.

Chapter One
Preliminaries

In this chapter we shall formalize the concepts mentioned in the introduction and we shall show that projecting a vector field onto a submanifold with corners in the way described does yield a unique semiflow. We establish some results which are needed later in this thesis, such as lemma 1.2 which is a critical result in constructing the iteration of Chapter Two, in a stronger form than is necessary for this chapter.

Corners and Stratifications

R^n denotes n-dimensional Euclidean space with, for $x,y \in R^n$, inner product $\langle x,y\rangle = \Sigma_{i=1}^n x_i y_i$ and norm $|x| = \sqrt{\langle x,x\rangle}$. A closed corner is a subset of R^n of the form $\{x \in R^n : \langle x,n_i\rangle = 0\ i=1,..,k, \langle x,n_i\rangle \geq 0\ i=k+1,..,k+m\}$ for independent vectors $\{n_i : i=1,..,k+m\}$ and generalising the notation $L(n_1,..,n_k)$ for $\{x \in R^n : \langle x,n_i\rangle = 0\ i=1,..,k\}$ is denoted $LC(n_1,..,n_k;n_{k+1},..,n_{k+m})$. Similarly a relatively open corner is a subset of the form (again with $\{n_i\}_{i=1,...,k+m}$ an independent set) $\{x \in R^n : \langle x,n_i\rangle = 0\ i=1,..,k, \langle x,n_i\rangle > 0\ i=k+1,..,k+m\}$ and is denoted by $LO(n_1,..,n_k;n_{k+1},..,n_{k+m})$. If for the purposes of the discussion the vectors are already prescribed these corners may be denoted respectively LC(I;J) and LO(I;J) (the C is for closed, the O for open) where $I=(1,..,k)$ and $J=(k+1,..,k+m)$ are sets of indices.

For our purposes a C^r stratification of a subset M of R^n may be defined as follows (see [28-30,57,59] for a fuller treatment). A C^r stratum in M is a connected boundaryless C^r manifold contained in M. A partition of M into strata is locally finite if for each $x \in M$ there exists a neighbourhood of x intersecting only finitely many members of the partition. A locally finite partition $\mathfrak{C}$ of M into C^r strata is called a C^r stratification of M if whenever $\sigma_1, \sigma_2 \in \mathfrak{C}$ with $\sigma_1 \cap (\text{clos}\sigma_2 \backslash \text{int}\sigma_2) \neq \varnothing$ then $\sigma_1 \subset \text{clos}\sigma_2 \backslash \text{int}\sigma_2$ and $\dim\sigma_1 < \dim\sigma_2$. As an example we may stratify the closed corner LC(I;J) into relatively open corners $LO(K;J\backslash K)$ for $I \subset K \subset I \cup J$; when we refer to the strata of LC(I;J) we always mean these relatively open corners.

A subcorner of LC(I;J) is a subset of the form $\{x \in R^n : \langle x,n_i\rangle = 0\ \forall i \in K_1, \langle x,n_i\rangle \geq 0\ \forall i \in K_2, \langle x,n_i\rangle > 0\ \forall i \in K_3\}$, for K_1,K_2,K_3 satisfying $I \subset K_1 \subset K_1 \cup K_2 \subset K_1 \cup K_2 \cup K_3 = I \cup J$ ($\{n_i\}_{I \cup J}$ as above a set of independent vectors) and may be denoted $LCO(K_1;K_2;K_3)$. We see that any stratum $LO(K;J\backslash K)$ of $LC(I;J)$ is a subcorner of $LC(I;J)$ as is its closure $LC(K;J\backslash K)$. We may decompose any closed corner $LC(I;J)$ as $\cup_{I\subset K\subset I\cup J} LO(K;J\backslash K)$ and any subcorner $LCO(K_1;K_2;K_3)$ as $\cup_{K_1\subset K'\subset K_1\cup K_2\subset K'\cup K''=K_1\cup K_2\cup K_3} LO(K';K'')$; if the number of elements of J is denoted $|J|$ then a closed corner $LC(I;J)$ may be decomposed into $2^{|J|}$ strata and contains $3^{|J|}$ subcorners.

For example, Figure 1.1 shows the closed corner $LC(I;J)$ with $I=\varnothing$ and $J=(1,2)$:

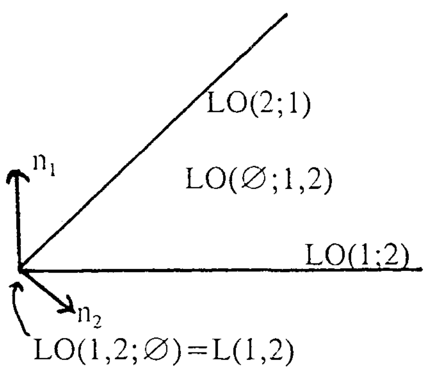


Figure 1.1

- the strata are $LO(\varnothing;1,2), LO(1;2), LO(2;1), LO(1,2;\varnothing)$ (respectively the interior of the closed corner, the two open half-lines, and the vertex); the subcorners of $LC(I;J)$ are all these strata, their closures, and $LCO(\varnothing;1;2)$ and $LCO(\varnothing;2;1)$ which are respectively the unions of the first and second and of the first and third strata in the above list (making 9 subcorners in total, since $\text{clos}LO(1,2;\varnothing)=LO(1,2;\varnothing)$).

<u>Projections Onto Convex Sets and Onto Corners</u>

A good general reference on convex sets is Bazaraa and Shetty [5]. It is shown in [4] that for any closed convex subset C of R^n and $y\in R^n$ there exists a unique x_0 such that $|y-x_0| = \min\{|y-x| : x\in C\}$. We define the <u>projection operator</u> $P(C):R^n\to C$ by $P(C)y=x_0$ and say x_0 is the <u>projection</u> of y onto C. The following <u>Characterisation of Projection</u> is also established in [4]: $x_0=P(C)y$ iff $x_0\in C$ and $\langle x-x_0, y-x_0\rangle \le 0$ for each $x\in C$.

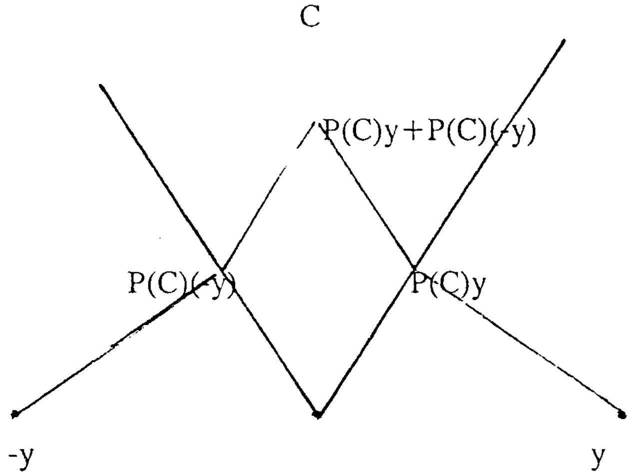


Fig 1.2 P(C) is non-linear

The reader will of course recall (see eg [14]) that if the convex set is a linear subspace L then P(L) is linear, is self-adjoint (ie $\langle P(L)x,y\rangle=\langle x,P(L)y\rangle$ for all x,y) and idempotent (ie $P(L)^2=P(L)$), and the Characterisation of Projection takes the form $x_0=P(L)y$ iff $x_0\in L$ and $\langle x,y-x_0\rangle=0$ for all $x\in L$. We also recall the idea of the convex hull of a set ([5 p.16]): the convex hull of S is $\{x\in R^n:x=\Sigma_{i=1}^k\lambda_ix_i,\ x_i\in S,\lambda_i\geq 0,i=1,..,k,\Sigma\lambda_i=1,k\geq 1\}$; it is the smallest convex set containing S. We shall denote its closure by conv(S) (which is convex by [5, p.35]). If C is a closed convex set in R^n then for $y\in C$ we set $P(C)^{-1}y=\{x\in R^n:P(C)x=y\}$. This notion is related to that of polar cone, defined for any subset C of R^n as $C^*=\{p\in R^n:\langle x,p\rangle\leq 0$ for all $x\in C\}$;if for example C is a closed convex cone, ie a set of points in R^n invariant under multiplication by non-negative scalars and under addition, then by the Characterisation of Projection it follows that $C^*=P(C)^{-1}$(origin).

Lemma 1.1 If C_1, C_2 are a pair of closed convex sets in Euclidean space then
(1) If $C_1\subset C_2$ and $P(C_2)x\in C_1$, then $P(C_1)x=P(C_2)x$
(2) If $y\in C_1\cap C_2$, then $P(C_1)^{-1}y\cap P(C_2)^{-1}y = P(\mathrm{conv}(C_1\cup C_2))^{-1}y$

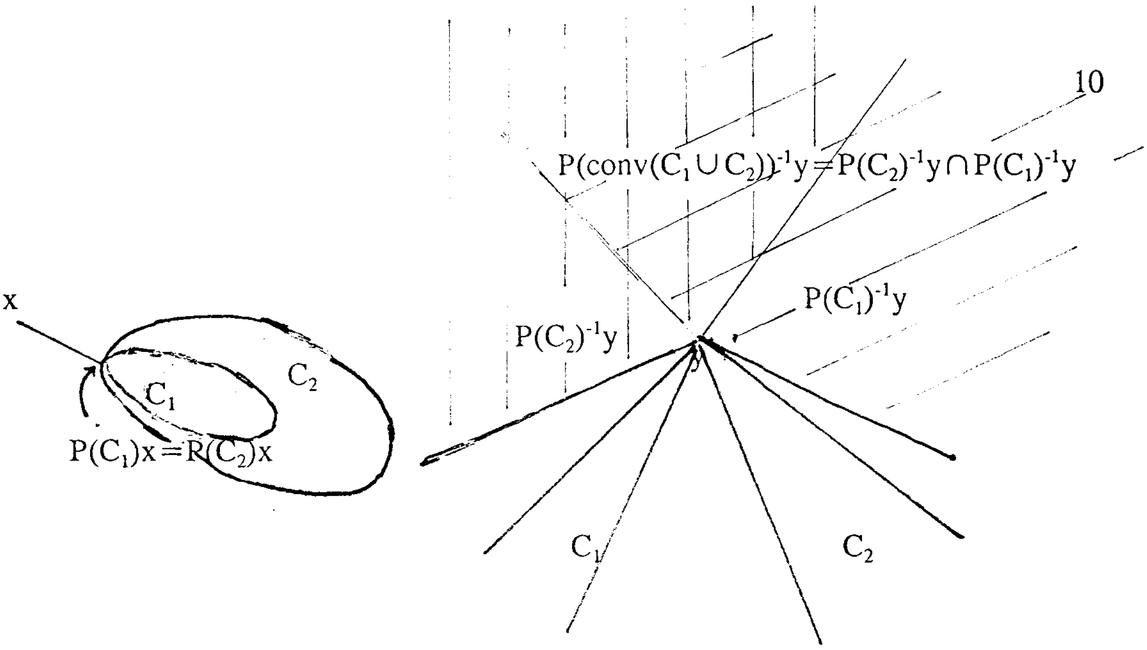


Figure 1.3

Proof

(1) Setting $P(C_2)x=x_2$ we know by the Characterisation of Projection above that $\langle x-x_2,x_2-y\rangle\geq 0$ for all $y\in C_2$ and so a fortiori for all $y\in C_1$, hence by the Characterisation of Projection $x_2=P(C_1)x$.

(2) Suppose $P(C_1)x=P(C_2)x=y$.

Set $H=\{z:\langle x-y,y-z\rangle\geq 0\}$; as above we know for each $x_i\in C_i$ that $\langle x-y,y-x_i\rangle\geq 0$ and hence $C_i\subset H$, and hence (since $conv(C_1\cup C_2)$ is the closure of the smallest convex set containing C_1 and C_2) that $conv(C_1\cup C_2)\subset H$. But $P(H)x=y$ by definition of H and Characterisation of Projection again, and since $P(H)x=y\in C_1\cap C_2$ and hence $\in conv(C_1\cup C_2)$, by (1) $y=P(H)x=P(conv(C_1\cup C_2))x$. Conversely if $P(conv(C_1\cup C_2))x =y\in C_1\cap C_2$ we get by (1) that $y=P(C_1)x=P(C_2)x$. ∎

We now specialise to the case where the convex set is a closed corner. It is fairly clear that for any given $y\in R^n$ there will exist at least one stratum of LC(I;J) such that projecting onto LC(I;J) will give the same result as projecting onto the affine span of this stratum (=smallest linear subspace containing this stratum). We shall need the more subtle fact that the set of strata for which this is true together form a subcorner; we have observed that the subcorner $LCO(P;Q;I\cup J\backslash(P\cup Q))$ of LC(I;J) equals the union of strata $\cup LO(H;I\cup J\backslash H)$ with the union taken over those H satisfying $P\subset H\subset P\cup Q$, so this is equivalent to saying

Lemma 1.2 For any closed corner LC(I;J) as above and $x \in R^n$ there exist I',J' with $I' \supset I$ and $I' \cup J' \subset I \cup J$ such that P(LC(I;J))x=P(K)x if and only if $I' \subset K \subset I' \cup J'$.

Proof

We have above stratified LC(I;J) $= \cup_{I \cup J = K \cup K' \supset K \supset I}$ LO(K;K')
and for any given $x \in R^n$ we must have P(LC(I;J))x lying in one of these strata, say P(LC(I;J))x=$y \in$ LO(K;I∪J\K) some $I \subset K \subset I \cup J$, (which equals LO(K;J\K) since $K \supset I$). Since LO(K;J\K) is relatively open in L(K) there exists a convex compact neighbourhood N_y of y in LO(K;J\K). Then by part 1 of Lemma 1.1 P(LC(I;J))x =P(N_y)x. Since N_y is a neighbourhood of y in L(K) for each $z \in$ L(K) there exists $z' \in N_y$ with y-z=λ(y-z') some $\lambda \geq 0$ (See Figure 1.4).

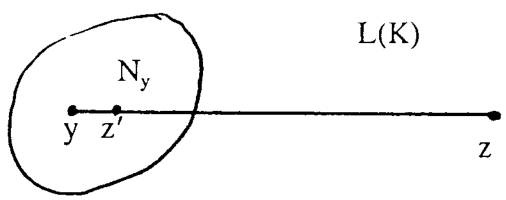


Figure 1.4

Hence ⟨x-y,y-z⟩ = λ⟨x-y,y-z'⟩ and this last quantity is ≥ 0 for all $z' \in N_y$ since P(N_y)x=y and using the Characterisation of Projection for P(N_y). Hence by the Characterisation of Projection for P(L(K)) (which we henceforth abbreviate to P(K)), P(K)x= P(LC(I;J))x and there exists K such that P(LC(I;J))x=P(K)x.

Suppose now K_1,K_2 with $I \subset K_1,K_2 \subset I \cup J$ satisfy this condition, ie P(K_1)x=P(K_2)x=y. By Lemma1.1 part 2 P(conv(L(K_1)∪L(K_2)))x=y. The result will follow if we can show that $K_1 \cap K_2$ and $K_1 \cup K_2$ satisfy the condition too, ie that P($K_1 \cap K_2$)x=y and P($K_1 \cup K_2$)x=y.

By [5, Section 3.1] conv(L(K_1)∪L(K_2))=(L(K_1)*∩L(K_2)*)*= (span$\{n_i, i \in K_1\}$∩span$\{n_i, i \in K_2\}$)* =(span$\{n_i, i \in K_1 \cap K_2\}$)*=L($K_1 \cap K_2$) (where C^* is the polar cone of C as above).

Hence y=P(L($K_1 \cap K_2$))x=P($K_1 \cap K_2$)x .

If y=P(K_1)x=P(K_2)x clearly $y \in$ L(K_1UK_2), hence by part one of Lemma 1.1 P($K_1 \cup K_2$)x=y. Hence result. ∎

Eg. With I=∅, J=(1,2) we see that for each x_i, i=1,..,5 in Figure 1.5 below there exists a unique pair I_i', J_i' with $I \subset I_i' \subset I_i' \cup J_i' \subset I \cup J$ such that

$\{K:I\subset K\subset I\cup J$ and $P(C)x_i=P(K)x_i\}=\{K:I_i'\subset K\subset I_i'\cup J_i'\}$ (where $C=LC(\varnothing;1,2)$)

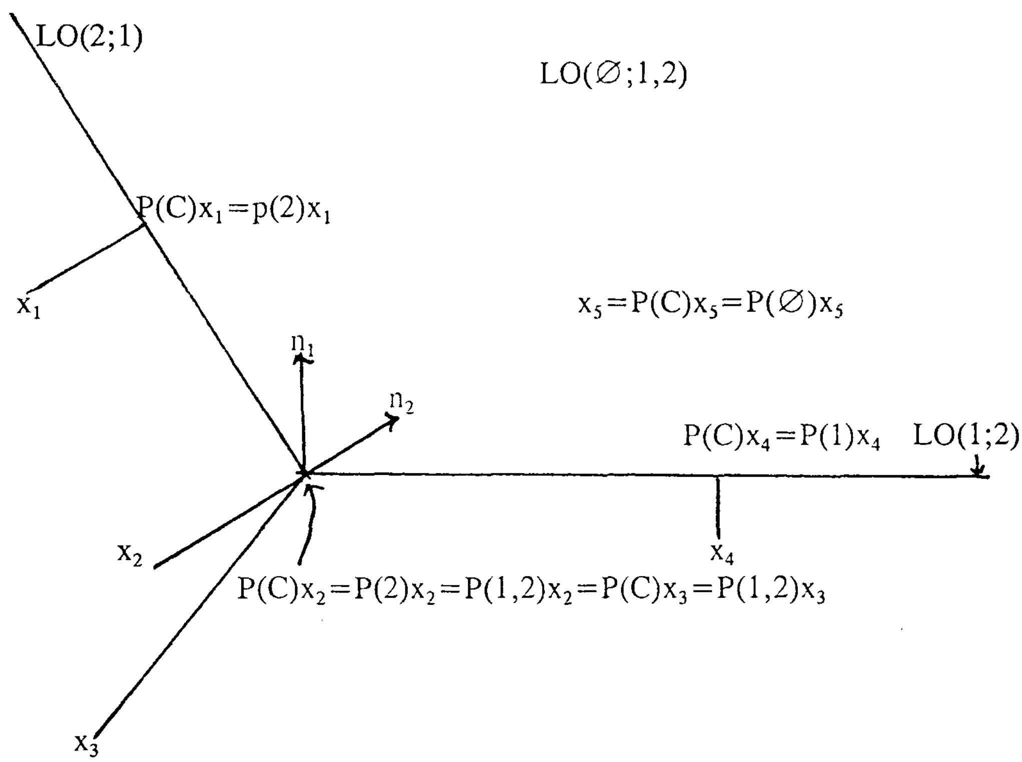


Figure 1.5

Submanifolds with corners

A subset M of R^n is a C^r submanifold with corners of dimension n-k if for each $x\in M$ there exists a closed corner $LC(I;J(x))$ in R^n, with $I=(1,..,k)$ and $J(x)=(k+1,..,k+m(x))$, a C^r map $\beta:R^n\rightarrow R^n$ mapping the origin to x which is a C^r diffeomorphism on a neighbourhood U of the origin, and such that $\beta(U\cap LC(n_1,..,n_k;n_{k+1},..,n_{k+m(x)}))$ is a neighbourhood of x in M (See [41] or [15] for a more general treatment). With $r=0$ we get a topological submanifold with boundary. The interesting cases from our point of view are C^∞ and C^ω and henceforth C^r will mean one of these two (ie consistently).

All submanifolds with corners appearing in this thesis are assumed connected. We define the tangent cone to M at x as $T_xM=(D\beta(0))LC(I;J)$ and the tangent space to M at x by $(D\beta(0))L(I)$. Of course, if M is a smooth submanifold the two coincide at every point. Setting $h_i(x)=\langle x,n_i\rangle$ (so $\text{grad}h_i(x)=n_i$) we have that our neighbourhood of x in M is $\beta(U \cap LC(I;J)) = \beta U \cap \beta LC(I;J) = \beta U \cap \{x \in R^n : h_i\beta^{-1}(x) = 0$ $i \in I, h_i\beta^{-1}(x) \geq 0 \ i \in J\}$. Writing vectors in R^n as columns of reals and representing the linear map $Df(x)$ as a matrix in the usual way we have for $f: R^n \rightarrow R$ that $\text{grad}f(x) = Df(x)^T$ (= transpose of $Df(x)$), hence $\text{grad}(h_i\beta^{-1})(x) = (D\beta^{-1}(x))^T n_i$, so if β is a diffeomorphism $\{\text{grad}(h_i\beta^{-1})(x) : i \in I \cup J\}$ is a set of independent vectors and if y is such that $h_i\beta^{-1}y = 0$ for $i \in I \cup J'$ with $J' = (k+1,..,k+m') \subset J$ (see Figure 1.6) we get T_yM $= D\beta(0)LC(I;J') = D\beta(0)\{D\beta^{-1}(y)z \in R^n : \langle D\beta^{-1}(y)z, n_i\rangle = 0 \ i \in I, \langle D\beta^{-1}(y)z, n_i\rangle \geq 0, i \in J'\} =$ $\{z \in R^n : \langle D\beta^{-1}(y)z, n_i\rangle = 0 \ i \in I, \langle D\beta^{-1}(y)z, n_i\rangle \geq 0, i \in J'\} =$ $\{z \in R^n : \langle z, \text{grad}(h_i\beta^{-1})(y)\rangle = 0 \ i \in I, \langle z, \text{grad}(h_i\beta^{-1})(y)\rangle \geq 0 \ i \in J'\}$ $=$ $LC(\text{grad}(h_1\beta^{-1})(y),..,\text{grad}(h_k\beta^{-1})(y); \text{grad}(h_{k+1}\beta^{-1})(y),..,\text{grad}(h_{k+m'}\beta^{-1})(y)\}$ and similarly the tangent space to M at y is $L(\text{grad}(h_1\beta^{-1})(y),..,\text{grad}(h_k\beta^{-1})(y))$.

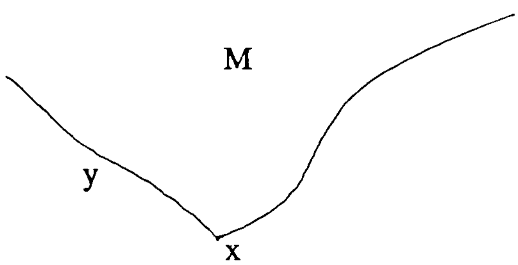


Figure 1.6
If M near x is $\beta(U \cap LC(I;J))$ and y is near x, M near y is $\beta(U \cap LC(I;J'))$ some $J' \subset J$.

One pictures the tangent cone and tangent space this way-

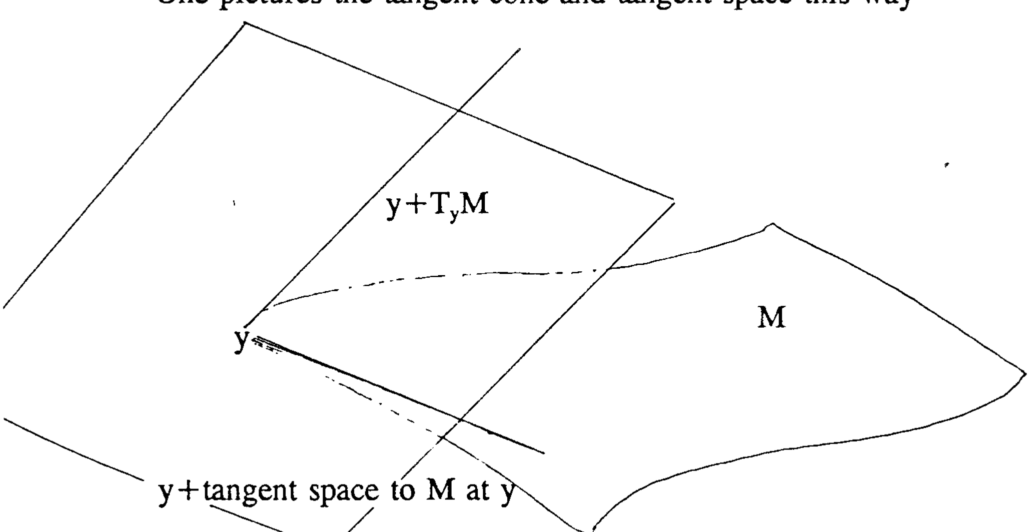


Figure 1.7

A <u>vector field</u> on M is a map $X: M \rightarrow R^n$ such that for each $x \in M$ $X(x)$ is in the tangent space to M at x. We shall say a vector field on M is C^r if there exists a neighbourhood U of the origin in R^n and a C^r vector field Y on $U \cap L(I)$ such that

$X \mid \beta(U\cap LC(I;J)) = \beta_* Y \mid \beta(U\cap LC(I;J))$ where β,I,LC(I;J) are as in the definition above and β is C^r. As usual $X \mid V$ means X restricted to V and $\beta_* Y$ means the push forward of Y by β (see eg [1 section 4.2]).

From the definition of submanifold with corners we have observed above that we can represent M near any $x_0 \in M$ locally as $\{x\in R^n : h_i\beta^{-1}x=0\ \forall i\in I, h_i\beta^{-1}x\geq 0\ \forall i\in J\}$ (where the functions $h_i\beta^{-1}:R^n\to R$, $i\in I\cup J$, are independent near x_0 ie their gradients are independent near x_0) but it is convenient to choose a slight refinement of this. For independent functions $f_1,..,f_k$ define $Z(f_1,..,f_k)=\{x\in R^n : f_1(x)=..=f_k(x)=0\}$ which if the functions are prescribed in advance is abbreviated to Z(I) where I=(1,..,k). Setting $f_1,..,f_k = h_1\beta^{-1},..,h_k\beta^{-1}$, if for $j\in J=(k+1,..,k+m)$ we act on the codimension 1 submanifolds $(h_j\beta^{-1})^{-1}(0)\cap Z(I)$ of Z(I) with the vector fields $\{\mathrm{grad} f_i\}_{i\in I}$ to form local hypersurfaces S_j (see Figure 1.8) we may find independent C^r functions $f_j:V\to R$ for V a neighbourhood of x_0 in R^n cutting out the S_j, that is , $S_j=f_j^{-1}(0)\cap V$.

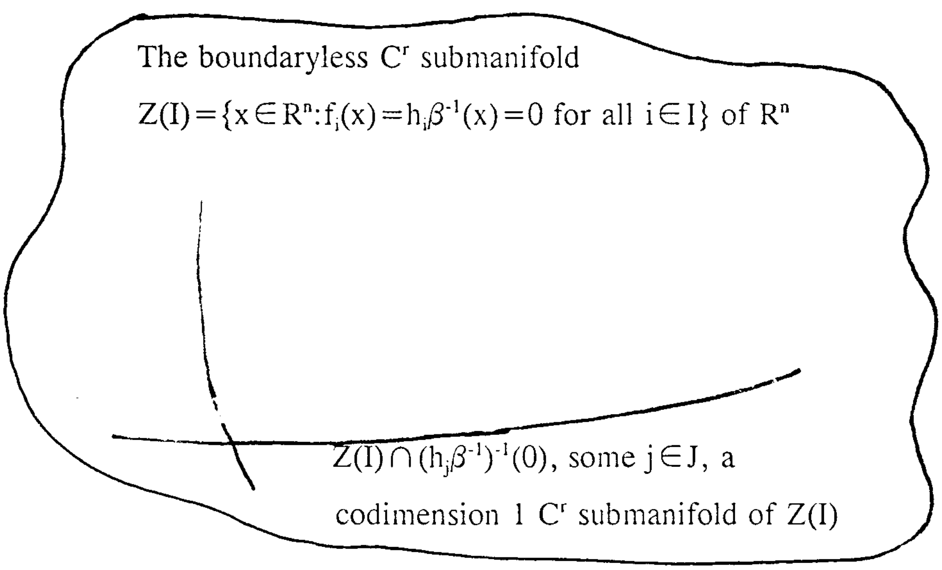


Figure 1.8. Acting on $Z(I)\cap(h_j\beta^{-1})^{-1}(0)$ with each of the flows of $\mathrm{grad} f_1,..,\mathrm{grad} f_k$ in turn yields (locally) a codimension 1 C^r submanifold S_j of R^n.

These f_j satisfy $f_j^{-1}(0)\cap Z(I) = (h_j\beta^{-1})^{-1}(0)\cap Z(I)$ for $j=k+1,..,k+m$, and have the additional property that for each $x\in(h_j\beta^{-1})^{-1}(0)\cap Z(I)$ and any $i\in I$, $j\in J$ $\langle \mathrm{grad} f_i(x), \mathrm{grad} f_j(x)\rangle=0$ (which would not necessarily have been the case if we had set $f_i=h_i\beta^{-1}$ for $i\in J$ as well as for $i\in I$). M near x_0 is locally $\{x\in R^n : f_i(x)=0$ for all $i\in I$,

$f_i(x) \geq 0$ for all $i \in J\}$. The results concerning the tangent cone and space are as above, replacing each $\text{grad}h_i\beta^{-1}(x)$ with $\text{grad}f_i(x)$, $i=1,..,k+m$, that is, with x_0 as above $T_{x_0}M = \{y \in R^n : \langle \text{grad}f_i(x_0), y\rangle = 0$ for all $i \in I$, $\langle \text{grad}f_i(x_0), y\rangle \geq 0$ for all $i \in J\}$ and the tangent space to M at x_0 is $\{y \in R^n : \langle \text{grad}f_i(x_0), y\rangle = 0$ for all $i \in I\}$.

By analogy with the closed (linear) corner we shall use the notation $ZN(f_1,..,f_k;f_{k+1},..,f_{k+m})$ abbreviated to $ZN(I;J)$ (where $I=(1,..,k)$, $J=(k+1,..,k+m)$) for $\{x \in R^n : f_i(x)=0\ \forall i \in I, f_i(x) \geq 0\ \forall i \in J\}$ and $ZP(f_1,..,f_k;f_{k+1},..,f_{k+m})$ abbreviated to $ZP(I;J)$ for $\{x \in R^n : f_i(x)=0\ \forall i \in I, f_i(x) > 0\ \forall i \in J\}$ (where Z is for zero, P for positive, N for non-negative and are of course the non-linear analogues of respectively L,C,O. To stress its construction from linear functions $f_i(x) = \langle x, n_i\rangle$ we may refer to $LC(I;J)$ as a linear corner). Thus near x_0 we are representing M locally in the form $ZN(f_1,..,f_k;f_{k+1},..,f_{k+m}) = ZN(I;J)$, which may be stratified into $2^{|J|}$ strata $ZP(K;J\backslash K)$ for $I \subset K \subset I \cup J$. Of course if $x \in Z(I \cup J)$ then $T_xZN(I;J) = LC(\text{grad}f_1(x),..,\text{grad}f_k(x);\text{grad}f_{k+1}(x),..,\text{grad}f_{k+m}(x)) = LC(I;J)$, if $I \subset K \subset I \cup J$ that $T_xZ(K) = L(\text{grad}f_i(x) : i \in K)$ etc - on this basis, an expression such as $P(T_xZ(K))\text{grad}f_j(x)$ may occasionally get abbreviated to $P(K)\text{grad}f_j(x)$. When we say M is represented near x as $ZN(I;J)$ we shall always suppose that x itself is in $Z(I \cup J)$. Note incidentally that the vector field X we begin with is supposed defined on M and hence that where M is represented as $ZN(I;J)$ is equal to $X(I)$. We shall call the region of M which may be represented by a particular representation $ZN(f_1,..,f_k;f_{k+1},..,f_{k+m})$ the domain of the representation. We shall as with the linear case call $ZNP(K_1;K_2;K_3) = \{x \in R^n : f_i(x)=0\ \forall i \in K_1, f_i(x) \geq 0\ \forall i \in K_2, f_i(x) > 0\ \forall i \in K_3\}$, for K_1, K_2, K_3 satisfying $I \subset K_1 \subset K_1 \cup K_2 \subset K_1 \cup K_2 \cup K_3 = I \cup J$, a subcorner of $ZN(I;J)$. Henceforth whenever notation of this kind is used it will always be supposed that the functions involved are independent.

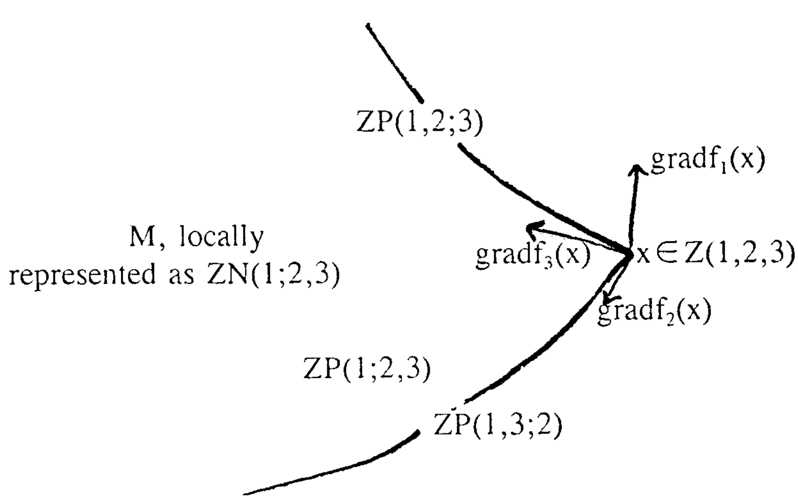


Figure 1.9 A local representation of a submanifold with corners

In Figure 1.9 the tangent cone to M at x is $T_xZN(1;2,3)=$
$\{y\in R^n:\langle y,\text{grad}f_1(x)\rangle=0$ and $\langle y,\text{grad}f_i(x)\rangle\geq 0$ for $i=2,3\}$, and the tangent space to M at x is $\{y\in R^n:\langle y,\text{grad}f_1(x)\rangle=0\}$.

The Projection of a Vector Field onto a Submanifold With Corners and the Semiflow of this Projection

Since the tangent cone to M at x is a closed corner in the tangent space any vector X(x) in the tangent space to M at x may be uniquely projected onto it; we set $X(M)(x)=P(T_xM)X(x)$. If X(x) points into the tangent cone to M at x $X(M)(x)=X(x)$; if it doesn't X(M)(x) is the unique vector in the tangent cone closest to X(x) (hence this kind of projection is called in [4] "projection of best approximation"). It is usual (as in [41]) to call a vector field Y on M with every $Y(x)\in T_xM$ an inner vector field; thus our vector field X(M), as defined pointwise above, is inner. We gave some examples in the Introduction (Examples 0.1 and 0.2). Another is

Example 1.1 M is the half space $\{(x,y)\in R^2:y\geq 0\}$ and X is the vector field $X(x,y)=(1,x)$. Then X(M)(x,y) is the vector field

$$X(M)(x,y)=\begin{cases} X(x,y) \text{ if } x\geq 0 \text{ or } y>0 \\ (1,0) \text{ if } x<0 \text{ and } y=0 \end{cases}$$

and so is discontinuous on the half line $(x<0,y=0)$.

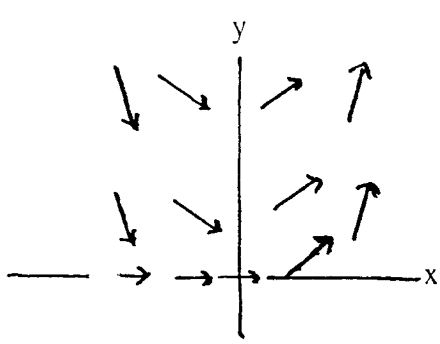


Figure 1.10

We want next to generalize the classical notion of trajectory. A definition suited to our needs (which is for example that used by Chikin in his paper [10] which forms the basis for our Theorem 1.1) is as follows.

A function x: $[a,b] \to R^n$ is <u>absolutely continuous</u> [4] if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any countable collection of disjoint subintervals $[a_k, b_k]$ of [a,b] such that

$\Sigma(b_k - a_k) < \delta$ we have $\Sigma \mid x(b_k) - x(a_k) \mid < \epsilon$.

It is known (eg [4, section 0] or [48]) that an absolutely continuous function is a.e. differentiable and satisfies $x(t)-x(s) = \int_s^t \dot{x}(u)du$ (in fact a continuous function is absolutely continuous iff it satisfies this condition). If for $x \in M$ there exists $t_x > 0$ and absolutely continuous $\phi(M,X)(x):[0,t_x) \to M$ satisfying $\phi(M,X)(x)(0) = x$ and $D_t\phi(M,X)(x)(t) = X(M)\phi(M,X)(x)(t)$ for a.a $t \in [0,t_x)$ (where D_t denotes differentiation with respect to t) we say $\phi(M,X)(x)$ is a <u>trajectory of X(M) at x</u>. $\phi(M,X)$ will usually be abbreviated to $\phi(M)$ and $\phi(M,X)(x)(t)$ written as $\phi(M,X)(x,t)$.

For our Example 1.1 the curves sketched in Figure 1.11 are certainly absolutely continuous and satisfy the condition to be trajectories of X(M) (with $t_x = \infty$ for every point).

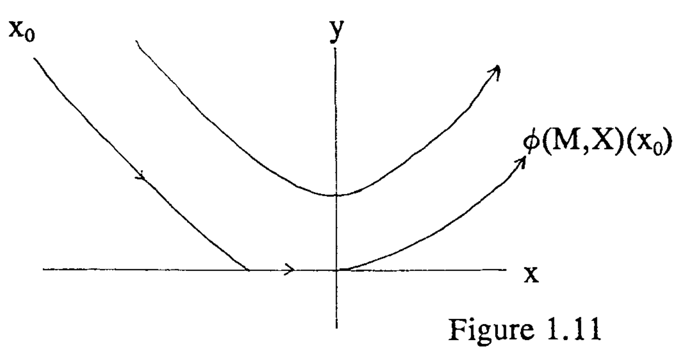


Figure 1.11

<u>Theorem 1.1</u> If X is a smooth vector field on a submanifold with corners M of R^n then at each point x of M

1. There exists a unique trajectory $\phi(M,X)(x)$ of X(M)
2. If M is compact we may take for each x $t_x = \infty$
3. For any $x \in M$ and $t < t_x$ $\phi(M,X)(x,t)$ is continuous in x.

This will be proved after Lemma 1.3. If for all x $t_x = \infty$ Parts 1 and 3 of Theorem 1.1 tell us (cf definition in the Introduction) that the map $\phi(M,X): M \times [0,\infty) \to M$ is a semiflow. In this thesis on the only occasion M is non-compact we have M an orthant and X linear, where it is straightforward to check that $t_x = \infty$ for all x, hence $\phi(M,X)$ is a semiflow throughout. Cornet proves an existence-uniqueness theorem nearly equivalent to Theorem 1.1 in [12] (see also Remark 3.1(4)). Parts 1 and 2 of this result have been established by Chikin ([10]) in the case M is an admissable subset, which is a bounded connected subset of R^n of the form $\{x\in R^n: f_i(x)\geq 0 \text{ for all } i\in I\}$ where the $f_i:R^n\to R$ are C^2 and satisfy the condition that if at $x\in M$ $f_i(x)=0$ for all $i\in I_0$ then $\{\text{grad}f_i(x), i\in I_0\}$ is an independent set. Locally a codimension 0 submanifold with corners is of this form, and we can establish parts 1 and 2 of Theorem 1.1 for an arbitrary submanifold with corners if we can extend it locally to a codimension 0 submanifold with corners A and find a vector field Y on A such that the trajectories produced by Chikin's Theorem applied to A,Y are, for a starting point on M, trajectories of X(M) - the technical aspects of this are done in Lemma 1.3.

We have shown that locally M may be represented as ZN(I;J) and we extend this codimension $|I|$ submanifold with corners to the codimension 0 submanifold with corners $ZN(\varnothing;I\cup J)$ which we denote A; it has ZN(I;J) as the closure of one of its strata. We recall that the functions f_i in the local representation of M as ZN(I;J) were constructed in such a way that if $y\in ZN(I;J)$ with $f_j(y)=0$ for all $j\in J'\subset J$, then $\langle \text{grad}f_i(y),\text{grad}f_j(y)\rangle=0$ for all $i\in I$, $j\in J'$. We extend the C^r vector field X on M to a C^r vector field X_e on a neighbourhood of y in R^n by pushing forward X by the flows of $\text{grad}f_1,\text{grad}f_2,..,\text{grad}f_k$ in turn - this will leave X on M unchanged, ie$X_e|M=X|M$ - and set $Y=X_e-\Sigma_{i\in I}\text{grad}f_i$ on this neighbourhood (Figure 1.12).

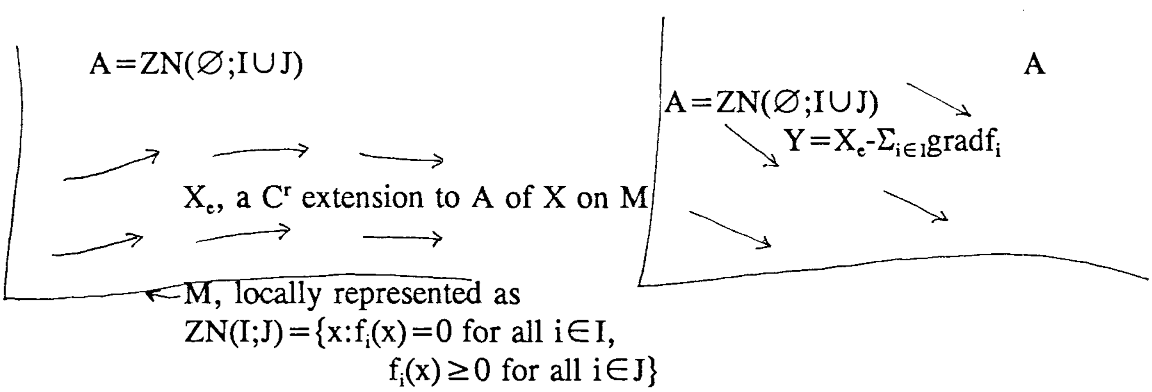


Figure 1.12

<u>Lemma 1.3</u> If $x_0\in M$ with M locally represented as ZN(I;J), and A,Y are as above, there exists a neighbourhood V of x_0 in R^n such that

(1) $P(T_xA)Y(x) = P(T_xM)X(x)$ for all $x\in M\cap V$ (and hence $\langle P(T_xA)Y(x),\text{grad}f_i(x)\rangle=0$ for all $x\in M\cap V$, $i\in I$), and

(2) $\langle P(T_xA)Y(x), \Sigma_{i\in I}\text{grad}f_i(x)\rangle<0$ for all $x\in A\cap V\setminus M$

Proof We are supposing that M is represented locally as ZN(I;J).

(1) At $x \in M \cap V$ we have $x \in Z(I \cup J')$ some $J' \subset J$ and $T_xM = LC(I;J')$, $T_xA = LC(\varnothing, I \cup J')$. If X is a C^r vector field on M we have of course $\langle X(x), \mathrm{grad} f_i(x) \rangle = 0$ for all $i \in I$, any $x \in M$. Fixing x, set $e_i = \mathrm{grad} f_i(x)$ for each $i \in I \cup J'$, $v = X(x)$, $C = LC(\varnothing; I \cup J')$ where $J' \subset J$ as above, and $C' = LC(I;J')$. For example with $x_0 = x \in Z(1,2,3)$ in Figure 1.9 where $I=(1)$, $J=J'=(2,3)$ we have C, C′ as illustrated in Figure 1.13 below.

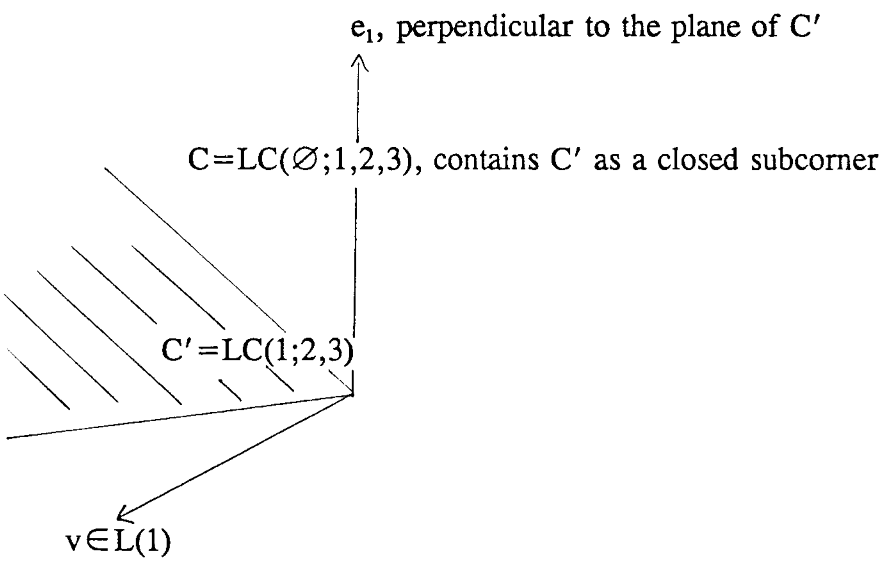


Figure 1.13

Since $\langle e_i, e_j \rangle = 0$ for all $i \in I, j \in J'$ we have the orthogonal direct sum decomposition $R^n = E_I \oplus E_{J'} \oplus N$ where $E_I = \mathrm{span}\{e_i\}_{i \in I}$, $E_{J'} = \mathrm{span}\{e_j\}_{j \in J'}$, and N = orthogonal complement to $\mathrm{span}\{e_i\}_{i \in I \cup J'}$ in R^n, so $E_{J'} \oplus N = L(I)$, and if we decompose any $z \in R^n$ as $z = z_I + z_{J'} + z_N$ we have $P(L(I))z = z_{J'} + z_N$. Furthermore if $z \in C$ then $\langle z, e_j \rangle \geq 0$ for all $j \in J'$, and since $\langle z_I, e_j \rangle = 0$ for all $j \in J'$
$\langle P(L(I))z, e_j \rangle = \langle z_{J'} + z_N, e_j \rangle = \langle z, e_j \rangle \geq 0$ for all $j \in J'$, ie $P(L(I))z \in C'$, and since C′ is a closed convex subset of closed convex L(I) we have by Lemma 1.1 part 1 that $P(L(I))z = P(C')z$.

Hence if $z \in C$ we may decompose z as $z = z_1 + z_2$ where $z_1 = P(L(I))z \in C'$ and $z_2 = z - z_1 \in E_I = \mathrm{span}\{e_i\}_{i \in I}$.

We now show that for $v = X(x)$ as above we get $P(C)(v - \Sigma_{i \in I} e_i) = P(C')v$.

For any $z \in C$ we have $z_1 \in C'$ and we know by the Characterisation of Projection that $\langle v - \Sigma_{i \in I} e_i - P(C')(v - \Sigma_{i \in I} e_i), P(C')(v - \Sigma_{i \in I} e_i) - z_1 \rangle \geq 0$. Furthermore by the definition of C′ and the fact that $\langle v, e_i \rangle = 0$ for all $i \in I$, we have $\langle v - \Sigma_{i \in I} e_i - P(C')(v - \Sigma_{i \in I} e_i), z_2 \rangle =$

$-\Sigma_{i\in I}\langle e_i,z_2\rangle=-\Sigma_{i\in I}\langle e_i,z\rangle$ since $\langle e_i,z_1\rangle=0$ for all $i\in I$, and $-\Sigma_{i\in I}\langle e_i,z\rangle$ is ≤ 0 because $z\in C$. Hence for any z in C $\langle v-\Sigma_{i\in I}e_i-P(C')(v-\Sigma_{i\in I}e_i),P(C')(v-\Sigma_{i\in I}e_i)-z_1\rangle-\langle v-\Sigma_{i\in I}e_i-P(C')(v-\Sigma_{i\in I}e_i),z_2\rangle\geq 0$, ie, $\langle v-\Sigma_{i\in I}e_i-P(C')(v-\Sigma_{i\in I}e_i),P(C')(v-\Sigma_{i\in I}e_i)-z\rangle\geq 0$, which means by the Characterisation of Projection again that $P(C)(v-\Sigma_{i\in I}e_i)=P(C')(v-\Sigma_{i\in I}e_i)$. If $q\in C'$ $\langle v-\Sigma_{i\in I}e_i-P(C')v,P(C')v-q\rangle=\langle v-P(C')v,P(C')v-q\rangle-\Sigma_{i\in I}\langle e_i,P(C')v-q\rangle=$ $\langle v-P(C')v,P(C')v-q\rangle-0$, which is ≥ 0 by the Characterisation of Projection, and so by the Characterisation of projection $P(C')(v-\Sigma e_i)=P(C')v$. This combined with $P(C)(v-\Sigma e_i)=P(C')(v-\Sigma e_i)$ (above) gives us $P(C)(v-\Sigma e_i)=P(C')v$, ie $P(T_xA)(X(x)-\Sigma \mathrm{grad} f_i(x))=P(T_xM)X(x)$, which completes the proof of (1), since if $x\in M$ then $X(x)=X_c(x)$ by construction of X_c.

(2) (a) We show that if $x_0\in Z(I\cup J)$, then there exists a neighbourhood U_0 of x_0 in R^n and $d>0$ such that for any K' satisfying $I\not\subset K'\subset I\cup J$ we have $\langle P(T_xZ(K'))(-\Sigma_{i\in I}\mathrm{grad} f_i(x)),\Sigma_{i\in I}\mathrm{grad} f_i(x)\rangle<-d/2$ for all $x\in U_0\cap Z(K')$ Figure 1.14 below will remind us of what these sets are -

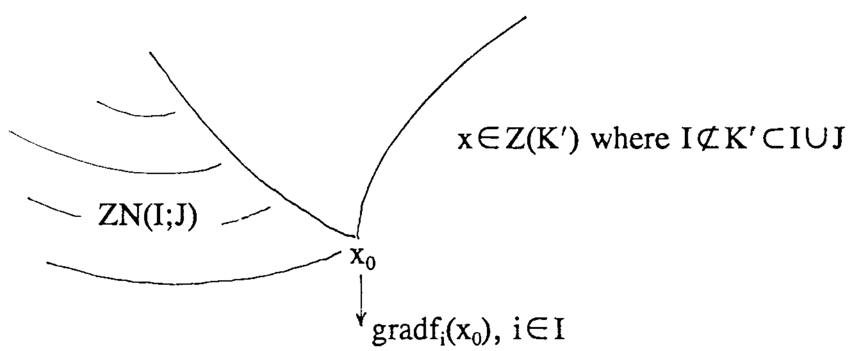


Figure 1.14

Begin by partitioning $K'=(K'\cap I)\cup(K'\cap J)$. We have eg from the Characterisation of Projection that if $v\in T_{x_0}R^n\cong R^n$ then $P(T_{x_0}Z(K'))v=\mathbf{0}$ (the zero vector) iff $v\in \mathrm{span}\{\mathrm{grad} f_i(x_0)\}_{i\in K'}$, so $P(K')\Sigma_{i\in I}\mathrm{grad} f_i(x_0)=P(K')(\Sigma_{i\in I\cap K'}\mathrm{grad} f_i(x_0)+\Sigma_{i\in I\setminus K'}\mathrm{grad} f_i(x_0))=P(K')(\Sigma_{i\in I\setminus K'}\mathrm{grad} f_i(x_0))$ $=\mathbf{0}$ iff $I\setminus K'=\varnothing$, ie iff $K'\supset I$. We have therefore for any $I\not\subset K'\subset I\cup J$ $\langle P(T_{x_0}Z(K'))(-\Sigma_{i\in I}\mathrm{grad} f_i(x_0)),\Sigma_{i\in I}\mathrm{grad} f_i(x_0)\rangle=-\,|\,P(T_{x_0}Z(K'))(\Sigma \mathrm{grad} f_i(x_0))\,|^2 = d(K')$, a real number <0 by the above.

Set $d=\min_{I\not\subset K'\subset I\cup J}|\,d(K')\,|$ and the result follows by continuity.

(b) We show that if $\{e_i\}_{i\in I\cup J}$ are independent vectors with $\langle e_i,e_j\rangle=0$ for all $i\in I, j\in J$ then for any $I'\subset I\cup J$ and any $X\in R^n$ $P(I')X=P(I'\cap I)P(I'\cap J)X=P(I'\cap J)P(I'\cap I)X$

Proof- If $j \in J$ then $e_j \in L(I' \cap I)$ since $\langle e_i, e_j \rangle = 0$ for all $i \in I' \cap I \subset I$, thus $P(I' \cap I)e_j = e_j$. We may verify directly (or use Remarks 2.1) that

$X - P(I' \cap J)X \in \mathrm{span}\{e_i\}_{i \in J}$, and hence applying $P(I' \cap I)$ to this expression (leaving the right hand side unchanged because $P(I' \cap I)e_j = e_j$ for all $j \in J$) we see

$P(I' \cap I)X - P(I' \cap I)P(I' \cap J)X = X - P(I' \cap J)X$

and similarly $P(I' \cap J)X - P(I' \cap J)P(I' \cap I)X = X - P(I' \cap I)X$; hence $P(I' \cap J)P(I' \cap I)X = P(I' \cap I)P(I' \cap J)X$, and hence this quantity is contained in $L(I' \cap I) \cap L(I' \cap J) = L(I')$. The Characterisation of Projection for linear subspaces (ie, that $x_0 = P(L)x$ iff $x_0 \in L$ and $\langle x - x_0, y \rangle = 0$ for all $y \in L$) tells us that $\langle X - P(I' \cap I)X, w \rangle = 0$ for all $w \in L(I' \cap I)$,

$\langle P(I' \cap I)X - P(I' \cap J)P(I' \cap I)X, w \rangle = 0$ for all $w \in L(I' \cap J)$, hence

$\langle X - P(I' \cap I)X + P(I' \cap I)X - P(I' \cap J)P(I' \cap I)X, w \rangle = 0$ for all $w \in L(I' \cap J) \cap L(I' \cap I) = L(I')$

and since we now know $P(I' \cap J)P(I' \cap I)X$ to be in $L(I')$ by the Characterisation of Projection again this tells us that $P(I' \cap J)P(I' \cap I)X = P(I')X$.

(c) We show that if $x_0 \in M$ with M locally represented as $ZN(I;J)$ and $d > 0$ then there exists a neighbourhood U_1 of x_0 in R^n such that for any $K' \subset I \cup J$ we have $\langle P(T_x Z(K'))X_e(x), \Sigma_{i \in I} \mathrm{grad} f_i(x) \rangle < d/2$ for all $x \in U_1 \cap Z(K')$. X_e is our C^r extension of X on $ZN(I;J)$ to a neighbourhood of $ZN(I;J)$ in R^n near x_0, and this result is saying that for any $K' \subset I \cup J$ (in Figure 1.15 there are 4 such K') the projection of $X_e(x)$ onto $T_x Z(K')$ has arbitrarily small component parallel to $\Sigma_{i \in I} \mathrm{grad} f_i(x)$ if x is arbitrarily close to x_0 in R^n.

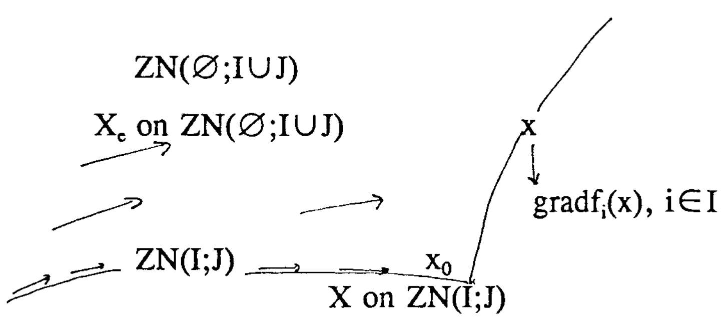


Figure 1.15

The map $x \to \langle P(T_x Z(K'))X_e(x), \Sigma_{i \in I} \mathrm{grad} f_i(x) \rangle$ is clearly continuous in x for fixed K', so since there are only finitely many K' it suffices to show that for each $K' \subset I \cup J$ $\langle P(T_{x_0} Z(K'))X(x_0), \Sigma_{i \in I} \mathrm{grad} f_i(x_0) \rangle = 0$ (recall $X_e(x) = X(x)$ for all $x \in ZN(I;J)$).

Set $X(x_0) = X$, $\mathrm{grad} f_i(x_0) = e_i$. By the construction of the representation of M

$\langle \text{grad} f_i(x_0), \text{grad} f_j(x_0)\rangle = 0$ for all $i \in I$, $j \in J$, so we may apply (b) to conclude that for any $K' \subset I \cup J$ and any $i \in I$ $P(K')e_i = P(K' \cap I)P(K' \cap J)e_i$. Morover $P(I' \cap J)e_i = e_i$ since $e_i \in L(J) \subset L(I' \cap J)$, so $P(K')e_i = P(K' \cap I)e_i$. Then since $X(x_0) \in T_{x_0}Z(I)$, ie in our notation $X = P(I)X$, and $P(K')$ is self-adjoint, we obtain for any $i \in I$
$\langle P(K')X, e_i\rangle = \langle X, P(K' \cap I)e_i\rangle = \langle P(I)X, P(K' \cap I)e_i\rangle$
$= \langle P(K' \cap I)P(I)X, e_i\rangle = \langle P(I)X, e_i\rangle = 0$ since $i \in I$, as required.

(d) $x \in A \backslash M$ implies $x \in Z(I' \cup J')$ with $I' \not\supset I$, hence $P(T_xA)Y(x) = P(T_xZ(K'))Y(x)$ some $I' \cup J' \supset K' \not\supset I$, and applying (a) and (c) we get for $x \in U_0 \cap U_1$ but x not in M that $\langle P(T_xA)(X_e(x) - \Sigma_{i \in I} \text{grad} f_i(x)), \Sigma_{i \in I} \text{grad} f_i(x)\rangle =$
$\langle P(T_xZ(K'))(X_e(x) - \Sigma_{i \in I} \text{grad} f_i(x)), \Sigma_{i \in I} \text{grad} f_i(x)\rangle =$
$\langle P(T_xZ(K'))X_e(x), \Sigma_{i \in I} \text{grad} f_i(x)\rangle + \langle P(T_xZ(K'))(-\Sigma_{i \in I} \text{grad} f_i(x)), \Sigma_{i \in I} \text{grad} f_i(x)\rangle < 0$, which gives (2) with V set to $U_0 \cap U_1$. ∎

Proof of Theorem 1.1

Chikin has established [10] the following: if A is an admissable subset (see above) of R^n and $f: R \times R^n \to R$ is a continuous function satisfying for all $x, y \in R^n$
$|\langle f(t,x) - f(t,y), x-y\rangle| < l(t)|x-y|^2$, for some $l(t)$ summable on finite intervals, then the problem: find absolutely continuous $x:[0,\infty) \to R^n$ satisfying $D_t x(t) = P(T_xA)f(t,x)$ for a.a. $t \in [0,\infty)$, $x(0) = x_0 \in A$, $x(t) \in A$ for all $t \in [0,\infty)$, has a unique solution.
From [19, Section 5] we may furthermore infer that the solution is continuous in initial conditions.

For $x_0 \in M$ extend M locally represented as ZN(I;J) locally to A as above, and choose U as in Lemma 1.3, and consider the vector field $Y = X_e - \Sigma_{i \in I} \text{grad} f_i$ on $A \cap \bar{B}_r(x_0)$ ($B_r(x_0)$ being the open ball with centre x_0 and radius r), where r is chosen so small that $\bar{B}_r(x_0) \subset V$, $\partial \bar{B}_r(x_0)$ is transverse to all the strata of A as a submanifold with corners (and hence $A \cap \bar{B}_r(x_0)$ is admissable), and so that for some $c > 0$ $\langle Y(x) - Y(y), x-y\rangle < c|x-y|^2$ for all $x, y \in \bar{B}_r(x_0)$ (this being possible by smoothness of Y).

Applying Chikin's Theorem to admissable $A \cap \bar{B}_r(x_0)$ we obtain absolutely continuous $x:[0,\infty) \to V$ satisfying $x(0) = x_0$, $x(t) \in A \cap \bar{B}_r(x_0)$ for all $t \in [0,\infty)$, and $D_t x(t) = P(T_{x(t)}(A \cap \bar{B}_r(x_0)))Y(x(t))$ for almost all $t \in [0,\infty)$, and hence since for some $T > 0$ $x(t) \in B_r(x_0)$ for $0 \le t < T$ that $D_t x(t) = P(T_{x(t)}A)Y(x(t))$ all $t \in [0,T)$. We have from Lemma 1.3 and almost everywhere differentiability of x that $\Sigma_{i \in I}(f_i x(t) - f_i x(0)) =$

$\Sigma_{i\in I}\int_0^t f_i'(x(s))\dot{x}(s)ds = \Sigma_{i\in I}\int_0^t\langle \text{grad}f_i(x(s)),P(T_xA)Y(x(s))\rangle ds \leq 0$.

If $x\in A$ then $x\in M$ iff $f_i(x)=0$ for all $i\in I$. For all $x\in A$, $f_i(x)\geq 0$ for all $i\in I$. Hence if $x(0)\in M$ then $x(t)\in M$ for all $t\in[0,T)$, and hence by Lemma 1.3 part 1 $P(T_{x(t)}A)Y(x(t)) = P(T_{x(t)}M)X(x(t))$ for all $t\in[0,T)$, and hence $D_tx(t)=P(T_{x(t)}M)X(x(t))$ for a.a. $t\in[0,T)$, which is the trajectory we seek, is unique by [10] and continuous in x(0) by [19, Section 5].

In the usual way if M is compact and t_x is maximal such that there exists a solution on $[0,t_x)$, then if $t_x<\infty$ $\lim_{t\uparrow t_x} x(t)\in M$ (by compactness) and repeating the construction at the limit point we may extend x(t) past $t=t_x$ contrary to the maximality of t_x, hence we must have $t_x=\infty$. ∎

Remarks

(1) Evidently the map $\phi(M,X)$ need not be differentiable in x or t - consider Examples 0.1, 0.2 or 1.1 above - but we shall show that the points of $[0,t_x)$ where $\phi(M,X)(x)$ is not differentiable are countable and rare (Proposition 5.2) and that it has one sided derivatives at all points (Theorem 3.1 and Proposition 5.1).

(2) Smoothness of data is not essential for Theorem 1.1: the lower bound on differentiability is determined by Chikin's Theorem (X C^1, the f_i's C^2)

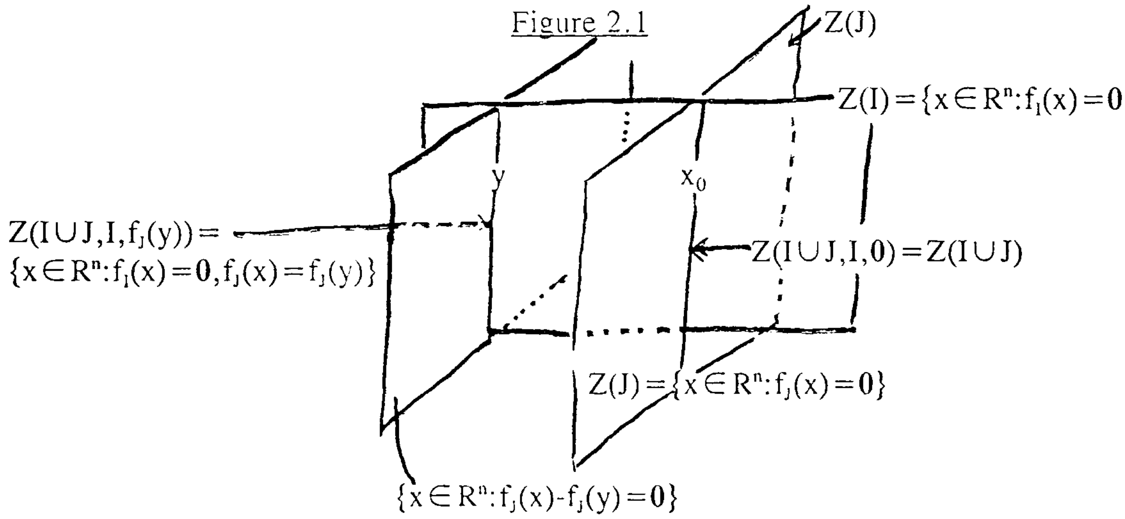
(3) A rather obvious generalisation of the context we have adopted, where M has been a submanifold with corners of Euclidean Space R^n, is to have M a submanifold with corners of a Riemannian manifold N. This means that there exists a C^r map Φ on N such that for each $x\in N$ $\Phi(x):T_xN\times T_xN\rightarrow R$ is bilinear, symmetric and positive definite and so for finite dimensional N makes T_xN a Hilbert Space. Thus setting for any $u\in T_xN$ $\| u \| =\Phi(x)(u,u)^{1/2}$ by [48,4.10] we know that for any $X(x)\in T_xN$ there exists a unique element X(M)(x) in T_xM satisfying
$\| X(x)-X(M)(x) \| =\min\{ \| X(x)-v \| :v\in T_xM\}$ and we may proceed as above. In fact, since the construction of X(M)(x) is invariant under isometries we could use Nash's result on the isometric embedding of Riemannian manifolds in Euclidean space (See [7]) to generalise Theorem 1.1 as follows: if M is a submanifold with corners of Riemannian N, X a smooth vector field on M (or more probably on N), then at each point $x\in M$ there exists $t_x>0$ and a unique trajectory $\phi(M,X)(x):[0,t_x)\rightarrow M$ with $\phi(M,X)(x)(t)=X(M)\phi(M,X)(x)(t)$ for a.a. $t\in[0,t_x)$, and if M is compact we may take $t_x=\infty$.

Chapter Two
The Iteration

We showed in Chapter One that if M is a submanifold with corners and X is a smooth vector field on M then for the system (M,X) there exists at each $x \in M$ a unique trajectory $\phi(M)(x):[0,t_x)\to M$ in the sense described there. We also recall that on some neighbourhood of any $x \in M$ we may establish a local representation of M of the form ZN(I;J). It is fairly clear, and is shown in Lemma 2.1 below, that if M (meaning the f_i's) is C^r and X is C^r then for each K with $I \subset K \subset I \cup J$ the projection of X onto Z(K), or onto the stratum ZP(K;J\K) which is an open subset of Z(K), is C^r, as is its integral flow. In this chapter we shall develop an algorithm to determine if there exists a stratum ZP(K;J\K) of ZN(I;J) such that $\phi(M)(x)(0,\delta)=\{\phi(M)(x,t):0<t<\delta\}$ is the integral curve of the projection of X onto Z(K) for some $0<\delta<t_x$, and if there is to determine what it is.

We begin by making some general constructions involving sets of independent functions and establishing a few elementary facts about them. Suppose $f_1,..,f_{k+m}$ are functions independent at and hence on a neighbourhood of a common zero x_0, with $(1,..,k)=I$ and $(k+1,..,k+m)=J$. Denoting $(f_{k+1}(y),..,f_{k+m}(y))$ by $f_J(y)$ we may foliate Z(I) near x_0 by the manifolds $\{y \in Z(I): f_J=\text{constant}\}$ and will denote each leaf $\{y \in Z(I): f_J(y)=a\}$ by $Z(I \cup J,I,a)$ (Figure 2.1). We see $Z(I \cup J,I,a)=Z(f_1,..,f_k;f_{k+1}-a_{k+1},..,f_{k+m}-a_{k+m})$ where $a=(a_{k+1},..,a_{k+m})$. Plainly $y \in Z(I \cup J,I,f_J(y))$.

Figure 2.1



We can form vector fields $X(I)$ and $X(I\cup JeI)$ on $Z(I)$ (the e is for "extension"; $X(I\cup JeI)$ is a C^r extension of the vector field $X(I\cup J)$ on $Z(I\cup J)$ to a neighbourhood of $Z(I\cup J)$ in $Z(I)$) by projecting at each $x\in Z(I)$ the vector $X(x)$ in T_xR^n onto respectively the subspaces $T_xZ(I)$ and $T_x(Z(I\cup J,I,f_J(x))$, ie $X(I)(x)=P(T_xZ(I))X(x)$, $X(I\cup JeI)(x)=P(T_xZ(I\cup J,J,f_J(x)))X(x)$. We shall show in Lemma 2.1 that these vector fields and their integral flows are C^r, but first some preliminary remarks.

Remarks 2.1

(1) We shall show that $X(I)(x)-X(I\cup J)(x)\in \mathrm{span}\{P(I)\mathrm{grad}f_i(x):i\in J\}$. Working in T_xR^n and writing $X(I)(x)$ as $X(I)$, $T_xZ(I)=L(I)$, $\mathrm{grad}f_i(x)=e_i$ etc it suffices to show that if $I=(1,..,k)$, $J=(k+1,..,k+m)$, then for any $Y\in R^n$

$P(I\cup J)Y-P(I)Y\in \mathrm{span}\{P(I)e_{k+1},P(I\cup\{k+1\})e_{k+2},..,P(I\cup J\backslash\{k+m\})e_{k+m}\}$ because then re-applying this result with $I'=I$, $J'=J\backslash\{k+m\}$, $Y'=e_{k+m}$ we get

$P(I\cup J\backslash\{k+m\})e_{k+m}-P(I)e_{k+m}\in \mathrm{span}\{P(I)e_{k+1},..,P(I\cup J\backslash\{k+m,k+m-1\})e_{k+m-1}\}$, ie

$P(I\cup J\backslash\{k+m\})e_{k+m}\in P(I)e_{k+m}\cup \mathrm{span}\{P(I)e_{k+1},..,P(I\cup J\backslash\{k+m,k+m-1\})e_{k+m-1}\}$ and inserting this into the previous result and repeating we eventually obtain

$P(I\cup J)Y-P(I)Y\in \mathrm{span}\{P(I)e_{k+1},P(I)e_{k+2},..,P(I)e_{k+m}\}$ as required.

Decomposing

$P(I\cup J)-P(I)Y=P(I\cup J)Y-P(I\cup J\backslash\{k+m\})Y+P(I\cup J\backslash\{k+m\})Y-..-P(I)Y$ it suffices to show that

$P(I\cup\{k+1,..,k+p+1\})Y-P(I\cup\{k+1,..,k+p\})Y\in \mathrm{span}P(I\cup\{k+1,..,k+p\})e_{k+p+1}$.

We have by definitions that the vector

$P(I\cup\{k+1,..,k+p+1\})Y-P(I\cup\{k+1,..,k+p\})Y\in L(I\cup\{k+1,..,k+p\})$ and by the Characterisation of Projection for linear subspaces that

$\langle P(I\cup\{k+1,..,k+p+1\})Y-P(I\cup\{k+1,..,k+p\})Y,w\rangle=0$ for all

$w\in L(I\cup\{k+1,..,k+p+1\})$, and these two conditions determine

$P(I\cup\{k+1,..,k+p+1\})Y-P(I\cup\{k+1,..,k+p\})Y$ up to a non-zero scalar. Thus it suffices to check that $P(I\cup\{k+1,..,k+p\})e_{k+p+1}$ satisfies these two conditions.

$P(I\cup\{k+1,..,k+p\})e_{k+p+1}\in L(I\cup\{k+1,..,k+p\})$, and

$\langle P(I\cup\{k+1,..,k+p\})e_{k+p+1},w\rangle=\langle e_{k+p+1},P(I\cup\{k+1,..,k+p\})w\rangle$

$=\langle e_{k+p+1},w\rangle$ if $w\in L(I\cup\{k+1,..,k+p\})$

$=0$ if $w\in L(I\cup\{k+1,..,k+p+1\})\subset L(k+p+1)$, and so the result follows.

(2) It follows that if $\{\mathrm{grad}f_i(x):i\in I\cup J\}$ are independent vectors in R^n then $\{P(I)\mathrm{grad}f_i(x):i\in J\}$ are independent vectors in $T_xZ(I)$. For if $\{P(I)e_i\}_{i\in J}$ were not

independent then there would exist $\{\lambda_i\}_{i\in J}$ with λ_i not all zero and $\Sigma_{i\in J}\lambda_i P(I)e_i=\mathbf{0}$. By (1) above for any $v\in R^n$ $P(I)v-v\in \text{span}\{e_j:j\in I\}$, hence $P(I)e_i-e_i=\Sigma_{j\in I}\mu_{ij}e_j$, some μ_{ij}, hence $\Sigma_{i\in J}\lambda_i(\Sigma_{j\in I}\mu_{ij}e_j+e_i)=\mathbf{0}$. But the λ_i are not all zero so this implies linear dependence of $\{e_i:i\in I\cup J\}$, contrary to the assumption.

(3) Normal Spaces. If S_1 is a submanifold of S_2, with S_1 and S_2 smooth boundaryless submanifolds of R^n, define at $x\in S_1$ the <u>normal space to S_1 in S_2</u> (a topologist might have preferred "perpendicular space") to be $\{y\in T_xS_2:\langle y,z\rangle=0 \ \forall\ z\in T_xS_1\}$ written $N_x(S_1 \text{ in } S_2)$; if S_1, S_2 are $Z(I)$, $Z(I\cup J)$, zero sets of functions independent near $x\in Z(I\cup J)$, $N_x(Z(I\cup J) \text{ in } Z(I))$ may be written $N_x(I\cup J \text{ in } I)$. Thus $y\in N_x(I\cup J \text{ in } I)$ iff $y\in T_xZ(I)$ and $\langle y-0,0-z\rangle=0$ for all $z\in T_xZ(I\cup J)$ and hence from the Charactisation of Projection $N_x(I\cup J \text{ in } I) = \{y\in T_xZ(I):P(I\cup J)y=\mathbf{0}\}$. From the Subspace Projection Theorem ([5, pp 4 and 8]) we get $T_xZ(I)=T_xZ(I\cup J)\oplus N_x(I\cup J \text{ in } I)$, so $\dim N_x(I\cup J \text{ in } I) = |J| = m$, and since by (2) above $\{P(I)\text{grad}f_i(x)\}_{i\in J}$ are independent vectors in $T_xZ(I)$ and $\langle P(I)\text{grad}f_i(x),z\rangle=0$ for all $z\in T_xZ(I\cup J)$, any $j\in J$, we must have for any $x\in Z(I\cup J)$ that $N_x(I\cup J \text{ in } I) = \text{span}\{P(I)\text{grad}f_i(x)\}_{i\in J}$. By the same argument for any $x\in Z(I)$, $N_x(Z(I\cup J,I,f_J(x)) \text{ in } Z(I))=\text{span}\{P(I)\text{grad}f_i(x)\}_{i\in J}$.

Thus $X(I\cup J)(x)-X(I)(x)\in \text{span}\{P(I)\text{grad}f_i(x)\}_{i\in J}$, ie for any $x\in Z(I\cup J)$ $X(I\cup J)(x)-X(I)(x)\in N_x(I\cup J \text{ in } I)$. Furthermore since for $x\in Z(I)$ $X(I\cup J e I)(x)$ is just $X(I\cup J)(x)$ with different but still independent functions we have at $x\in Z(I)$ $X(I\cup JeI)(x)-X(I)(x)\in N_x(Z(I\cup J,I,f_J(x)) \text{ in } Z(I))$.

<u>Lemma 2.1</u> If X is a C^r vector field on R^n and $\{f_i\}$ are C^r functions with $x_0\in Z(I\cup J)$ and such that $\{\text{grad}f_i(x_0)\}_{i\in I\cup J}$ is an independent set of vectors , there exists a neighbourhood U of x_0 in Z(I) such that X(I) and $X(I\cup JeI)$ are C^r vector fields on U and their integral flows, denoted $\phi(I)$ and $\phi(IUJeI)$ respectively, are also C^r.

<u>Proof</u> Take U_1 open in R^n a neighbourhood of x_0 so small that $\{\text{grad}f_i(x)\}_{i\in IUJ}$ is an independent set for all $x\in U_1$. Then for each $i\in I=(1..k)$, $x\in U_1$ and $\lambda\in R^k$ define a C^r map $g_i:R^{n+k}\to R$ by $g_i(x,\lambda)=\langle X(x)-\Sigma_{j\in I}\lambda_j \text{grad}f_j(x),\text{grad}f_i(x)\rangle$ and $g:R^{n+k}\to R^k$ by $g(x,\lambda)= ((g_1(x,\lambda),..,g_k(x,\lambda))$. The matrix A with coefficients $A_{ij}=\langle\text{grad}f_i(x_0),\text{grad}f_j(x_0)\rangle$ $i,j\in I$ is invertible since $\{\text{grad}f_i(x_0)\}_{i\in I}$ are independent, and setting $\lambda(x_0)=A^{-1}b$ where $b_i=\langle X(x_0),\text{grad}f_i(x_0)\rangle$ we have $g(x_0,\lambda(x_0))=\mathbf{0}$ and by the C^∞ or C^ω Implicit Function Theorem ([14, Chapter 10]) there exists an open

neighbourhood U_2 of x_0 contained in U_1 and a unique C^r map $\lambda:U_2 \to R^k$ such that for all $x \in U_2$ $g(x,\lambda(x))=0$. Since this is saying $\lambda(x)$ is the *unique* $\lambda \in R^k$ such that $X(x)-\Sigma_{j\in I}\lambda_j \text{grad}f_j \in T_x Z(I,\varnothing,f_I(x))$ and since by Remark 2.1(1) above we know $X(Ie\varnothing)(x)=X(x)-v$ with $v\in \text{span}\{\text{grad}f_i(x):i\in I\}$ it follows $v=\Sigma_{j\in I}\lambda_j \text{grad}f_j(x)$ and (1) $X(Ie\varnothing)(x)=X(x)-\Sigma_{j\in I}\lambda_j \text{grad}f_j(x)$ for all $x\in U_2$ and is therefore C^r, and since if $x\in U_2 \cap Z(I)$ then $X(Ie\varnothing)(x)=X(I)(x)$, (2) that for all $x\in U_2\cap Z(I)$ $X(I)(x)=X(x)-\Sigma_{j\in I}\lambda_j \text{grad}f_j(x)$ and is C^r. Furthermore if β, L(I) are as in the definition of submanifold with corners $\beta^* X(I)$ is C^r in L(I) and hence X(I) is a C^r vector field on $Z(I)\cap U_2$ in the sense of Chapter One. If we apply (1) with I replaced by IUJ we get an open neighbourhood U_3 of x_0 in R^n with $X(IUJe\varnothing)$ a C^r vector field on U_3. If we now apply (2) with X replaced by $X(IUJe\varnothing)$ (which we may do because we now know it to be C^r) we get a neighbourhood U_4 contained in U_3 of x_0 with $X(I\cup Je\varnothing)(I)$, the projection of $X(I\cup Je\varnothing)$ onto Z(I), a C^r vector field on $U_4\cap Z(I)$. We may check from definitions that for all $x\in Z(I)$ $X(I\cup JeI)(x) =X(I\cup Je\varnothing)(I)(x)$ and we conclude that $X(I\cup JeI)$ is a C^r vector field on $U_4\cap Z(I)$.

We may use classical theory (eg [1, Chapter 4]) to infer that the flows $\phi(I)$ and $\phi(IUJeI)$ are also C^r. ∎

<u>Remark 2.2</u> Using that $\langle X(I)(x),\text{grad}f_j(x)\rangle=0$ for all $j\in I$ and writing $\lambda(x)$ for the column vector with components $\lambda_i(x)$ we find that in the equation in (2) of Lemma 2.1 that $\lambda(x)=M(x)^{-1}N(x)^T X(x)$ where M(x) is the $k\times k$ symmetric matrix with elements $M(x)_{ij} =\langle \text{grad}f_i(x),\text{grad}f_j(x)\rangle$ (which is invertible because $\{\text{grad}f_i(x)\}$ are independent) and N(x) the $n\times k$ matrix with ith column $\text{grad}f_i(x)$; hence in this notation $X(I)(x)=X(x)-N(x)M(x)^{-1}N(x)^T X(x)$.

We have established in Lemma 1.2 the relation between P(LC(I;J))X and P(L(K))X for $I\subset K\subset I\cup J$ and much of Chapters 2 to 5 is concerned with the relation between $\phi(M)$ = the unique semiflow of X(M) provided by Theorem 1.1 and $\phi(K)$ =unique C^r integral flow of X(K) (if M is locally LC(I;J) this K must lie in the range $I\subset K\subset I\cup J$) as provided above by Lemma 2.1.

We firstly show that for some systems there exist points where no $\delta>0$ can be found satisfying the condition that for some K there exists $\delta>0$ such that for all $t\in(0,\delta)$ $\phi(M)(x,t)=\phi(K)(x,t)$.

Example 2.1

Take $X(x,y)=(1,f(x))$ with

$$f(x)=\begin{cases} (1/x^2)\exp(-1/x)(\sin(1/x)-\cos(1/x)) & \text{if } x>0 \\ 0 & \text{if } x\leq 0 \end{cases}$$

on $M=\{(x,y)\in R^2: y\geq 0\}$.

If $1/x_m=m\pi+\pi/4$, then $f(x_m)=0$,

$$f'(x_m)=\exp(-1/x_m)/x_m^4 \begin{cases} -\sqrt{2} & \text{if m is even} \\ +\sqrt{2} & \text{if m is odd} \end{cases}$$

So if m is odd $f(x_m)=0$, $f'(x_m)>0$.

For $F'(x)=f(x)$ we have the integral $F(x)-F(y)=[\exp(-1/t)\sin(1/t)]_y^x$

and since $F(x_{m-2})-F(x_m)=-(1/\sqrt{2})(\exp(2\pi)-1)\exp(-\pi(m+1/4))<0$

the integral curve at $(x_m,0)$ with m odd has the form shown in Figure 2.2.

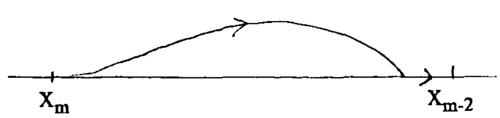


Figure 2.2

By Theorem 1.1 there exists an absolutely continuous trajectory $\phi(M)(0)$ based at the origin, which is clearly not C^1 on any deleted neighbourhood of 0, and in fact in any such neighbourhood there are countably infinitely many points where $\phi(M)(0)$ is not differentiable.

Figure 2.3

Let us consider now a simple situation where such a K does exist

Example 2.2

Suppose $M=\{x\in R^3: x_1\geq 0, x_2\geq 0\}$ and $X(x_1,x_2,x_3)=(x_3^3-x_2^2,-1,1)$. We seek K such that $\phi(M)(0,(0,\delta))\subset ZP(K;(1,2)\backslash K)$ and $\phi(M)(0,t)=\phi(K)(0,t)$ for $t\in[0,\delta)$, some $\delta>0$.

Since X(0)=(0,-1,1), by continuity we must have that X(M)(y)=X(2)(y) for all y near the origin in ZP(2;1). Hence if a single stratum does contain ϕ(M)(0,t) for small t>0 it must be either Z(1,2) or ZP(2;1). We decide which by considering $X(2)(x_1,x_3)=(x_3^2,1)$: the first nonvanishing time derivative of $\langle\phi(2)(x=0,t=0),n_1\rangle$ is the third, which is >0, and hence $\phi(M)(0,t)\subset ZP(2;1)$ with $\phi(M)(0,t)=\phi(2)(0,t)$ for all small t>0 (Figure 2.4).

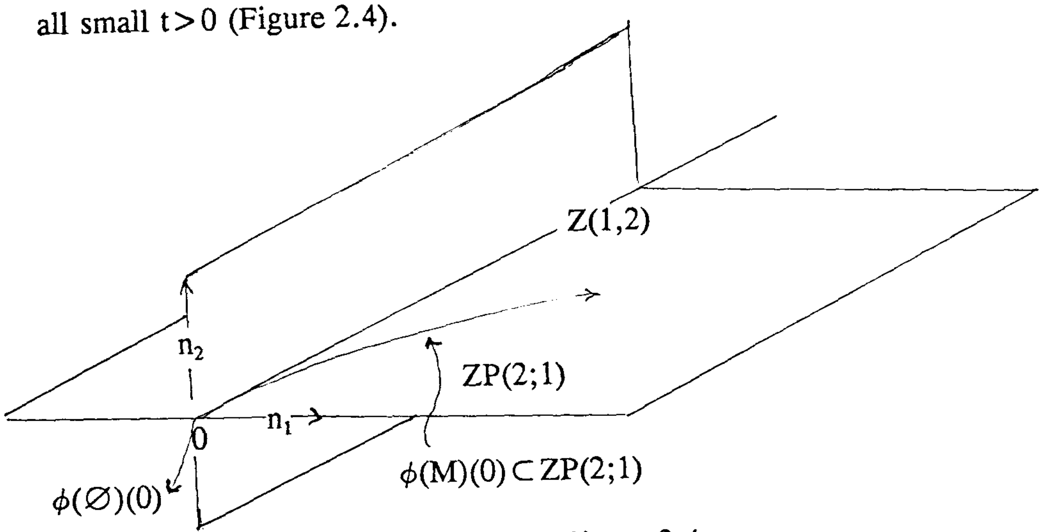


Figure 2.4

We could not have inferred this by for example considering time derivatives of the unconstrained equation: we see in fact the unconstrained trajectory $\phi(\varnothing)(0)$ heads into $\{x:x_1<0,x_2<0\}$.

Our general algorithm will provide us at the ith stage with a subcorner $ZPN(K_1;K_2;I\cup J\backslash(K_1\cup K_2))$ with the property that if there exists a K such that $\phi(M)(x,t)=\phi(K)(x,t)$ for all $t\in(0,\delta)$ then it is a stratum of the subcorner; then by suitable comparison of (i+1)th time derivatives (in fact of $\phi(K_1)(x)$ and $\phi(K_1\cup K_2)(x)$ at t=0) it will provide us with a yet smaller subcorner with this property. The result underlying the iterative step in this shrinking process is Lemma 1.2.

With M as usual locally represented as ZN(I;J) we begin by taking the differences between the first time derivatives of $\phi(I)(x)$ and $\phi(I\cup J)(x)$ at t=0, ie $D_t\phi(I)(x,t=0)-D_t\phi(I\cup J)(x,t=0)$, and use Lemma 1.2 to find the subcorner of $T_xM=T_xZN(I;J)=LC(I;J)$ such that projecting this quantity onto T_xM gives the same result as projecting onto the affine spans of the strata in this subcorner. (We use the notation $D_tf(t=0)$ to denote the time derivative of f evaluated at t=0). If this subcorner is $LCO(K_1;K_2;I\cup J\backslash(K_1\cup K_2))$ at the second stage we work with the closed corner $LC(K_1;K_2)$, seeking the subcorner of this such that the projection of

$D_t^2\phi(I)(x,t=0)-D_t^2\phi(I\cup J)(x,t=0)$ onto $LC(K_1;K_2)$ gives the same result as projecting onto the affine spans of the strata in this subcorner. We see that in this way we obtain a contracting sequence of strata $I=S^0_1(x)\subset S^0_2(x)\subset\ldots\subset S_2(x)\subset S_1(x)=I\cup J$ (where in the notation of this paragraph, $S^0_2(x)=K_1, S_2(x)=K_1\cup K_2$) and in view of this contracting property this sequence must converge. (Whether to a single set of indices or not being a matter of some import). We see that the sets of indices $S_\infty(x)=\cap_{i\geq 1}S_i(x)$ and $S^0_\infty(x)=\cup_{i\geq 1}S^0_i(x)$ have the property that for all sufficiently large i $S_i(x)=S_\infty(x)$ and $S^0_i(x)=S^0_\infty(x)$. The strata $ZP(K;J\backslash K)$ with $S^0_\infty(x)\subset K\subset S_\infty(x)$ will be the candidates for the stratum we seek.

In Lemma 2.4 we show that $X(M)(x)=X(K)(x)$ is equivalent to a pair of conditions, that $X(K)(x)$ points into $T_xZN(K;J\backslash K)$ and that $\langle X(x),P(K\backslash j)\text{grad}f_j(x)\rangle\leq 0$ for all $j\in K\backslash I$. With $S_\infty(x)$ and $S^0_\infty(x)$ as above we show in Lemma 2.5 that for all t in some $(0,\delta_1)$ $\phi(S_\infty(x))(x,t)\in ZP(S_\infty(x);J\backslash S_\infty(x))$, and in Lemma 2.6 that for all t in some $(0,\delta_2)$ $\langle X(S^0_\infty(x)\backslash j)\phi(S^0_\infty(x))(x,t),\text{grad}f_j(\phi(S^0_\infty(x))(x,t))\rangle<0$ for all $j\in S^0_\infty(x)\backslash S^0_1(x)$, which together imply after some manipulation that if $S_\infty(x)=S^0_\infty(x)$ then the conditions for Lemma 2.4 apply on $t\in(0,\min(\delta_1,\delta_2))$ with $K=S_\infty(x)=S^0_\infty(x)$, ie that $X(M)\phi(K)(x,t)=X(K)\phi(K)(x,t)$ on $(0,\min(\delta_1,\delta_2))$. If $S^0_\infty(x)\neq S_\infty(x)$ but the data (ie X and the f_i's) are analytic it is still the case that $\phi(M)(x,t)=\phi(K)(x,t)$ any $S^0_\infty(x)\subset K\subset S_\infty(x)$ for $t>0$ sufficiently small.

Formally, the <u>iteration</u> at $x\in M$ is defined as follows: if M is locally ZN(I;J) (and we recall that by convention we suppose x itself is in $Z(I\cup J)$) we set $S^0_1(x)=I$, $S_1(x)=I\cup J$, and $S^0_i(x)$, $S_i(x)$ are defined iteratively using Lemma 1.2: this tells us that for given $S^0_{i-1}(x), S_{i-1}(x)$, there exist unique $S^0_i(x)$, $S_i(x)$ with $S^0_{i-1}(x)\subset S^0_i(x)\subset S_i(x)\subset S_{i-1}(x)$ such that

$$P(T_xZN(S^0_{i-1}(x),S_{i-1}(x)\backslash S^0_{i-1}(x)))(D_t^{i-1}\phi(S^0_{i-1}(x))(x,t=0)-D_t^{i-1}\phi(S_{i-1}(x))(x,t=0))$$
$$=P(T_xZ(K))(D_t^{i-1}\phi(S^0_{i-1}(x))(x,t=0)-D_t^{i-1}\phi(S_{i-1}(x))(x,t=0))$$

iff $S^0_i(x)\subset K\subset S_i(x)$. We observe that the sets of indices $S^0_j(x)$, $S_j(x)$ obtained depend on M, X and on the point $x\in M$ of evaluation, so written out in full are $S^0_j(x,M,X)$ and $S_j(x,M,X)$, but these will usually be abbreviated to $S^0_j(x),S_j(x)$.

Working through Example 2.2 above for example we find $S^0_1(0)=(\varnothing), S^0_i(0)=(2)$ for all $i\geq 2$, $S_i(0)=(1,2)$ $i=1,2,3$, and $S_i(0)=(2)$ for all $i\geq 4$, so $S_\infty(0)=S^0_\infty(0)=(2)$, and Theorem 2.1 below tells us (as we reasoned above directly) that $\phi(M)(0,t)=\phi(2)(0,t)$ for all $t\in[0,t_0)$, some $t_0>0$, and that $\phi(M)(0,t)\in ZP(2;1)$ on $(0,t_0)$. A more complete understanding of the iteration will follow from Chapter Three (where we relate the iterates $S^0_i(x)$, $S_i(x)$ to the right hand derivatives $D_t^{+i}\phi(M)$ at x) and Chapter Four (where we relate the iteration at each $x \in M$ to the decomposition of M into generalisations of the classical tangency sets).

Remark 2.3 The reader will have observed that the iteration is an operator, and we can formalize this as follows.

Define the operator ITN acting on triples of the form $((I,J),\{n_i:i \in I \cup J\},\{\psi(K):I \subset K \subset I \cup J\})$ where the first argument (I,J) is a pair of sets of indices, the second argument $\{n_i\}$ is an independent set of vectors in R^n, and the third argument is a collection of smooth functions $\psi(K):U \to R^n$ for $I \subset K \subset I \cup J$, where U is some interval of the real line containing the origin. By Lemma 1.2 there exists a unique pair of sets of indices I',J' with $I \subset I' \subset I' \cup J' \subset I \cup J$ such that $P(LC(I;J))(D_t\psi(I)(t=0)-D_t\psi(I \cup J)(t=0))=P(K)(D_t\psi(I)(t=0)-D_t\psi(I \cup J)(t=0))$ for all $I' \subset K \subset I' \cup J'$, and we then set $ITN((I,J),\{n_i:i \in I \cup J\},\{\psi(K):I \subset K \subset I \cup J\})=((I',J'),\{n_i:i \in I' \cup J'\},\{\psi(K):I' \subset K \subset I' \cup J'\})$. We see this yields $ITN^j((S^0_1(x),S_1(x)\backslash S^0_1(x)),\{gradf_i(x):i \in S_1(x)\},\{\phi(K)(x):S^0_1(x) \subset K \subset S_1(x)\})=((S^0_{j+1}(x),S_{j+1}(x)\backslash S^0_{j+1}(x)),\{gradf_i(x):i \in S_{j+1}(x)\},\{D_t^j\phi(K)(x):S^0_{j+1}(x) \subset K \subset S_{j+1}(x)\})$ for any $j \geq 0$.

Theorem 2.1 (1) If X is a smooth vector field on a smooth submanifold with corners M of R^n, with M near x locally represented as ZN(I;J), then if $S^0_\infty(x)=S_\infty(x)$ there exists $t_0>0$ such that the trajectory $\phi(M,X)(x,t)= \phi(S_\infty(x))(x,t)$ for all $t \in [0,t_0)$, with $\phi(S_\infty(x))(x,t) \in ZP(S_\infty(x);S_1(x)\backslash S_\infty(x))$ for all $t \in (0,t_0)$
(2) If the data (ie, X and the f_i's) are analytic there exists $t_0>0$ such that $\phi(M,X)(x,t)=\phi(K)(x,t)$ on $t \in [0,t_0)$, any $S^0_\infty(x) \subset K \subset S_\infty(x)$, and for $t \in (0,t_0)$ $\phi(M,X)(x,t) \in ZP(S_\infty(x);S_1(x)\backslash S_\infty(x))$.
In either case $X(M)\phi(M,X)(x,t)=X(K)\phi(M,X)(x,t)$ on $t \in [0,t_0)$, any $S^0_\infty(x) \subset K \subset S_\infty(x)$.

This is proved after Lemma 2.6. We recall and shall use without further mention basic facts about P(K): that it is self-adjoint, that $P(K)^2=P(K)$, that if $k \in K$ then $P(K)n_k=\mathbf{0}$, etc.

In Lemmas 2.2 and 2.3 we shall write $D_t^i\phi(I)$ for $D_t^i\phi(I)(x,t=0)$.

Lemma 2.2 If $D_t^i(\phi(I \cup j)-\phi(I))=0$ $i=0,..,k-1$, then
$D_t^k(\phi(I)- \phi(I \cup j)) =D_t^k(f_j\phi(I))P(I)gradf_j(x)/ \mid P(I)gradf_j(x) \mid ^2$

Proof We showed in Remarks 2.1 that we have $X(I \cup j)(x)-X(I)(x) \in \text{span}(P(I)\text{grad}f_j(x))$. Since $\langle P(I)\text{grad}f_j(x),X(I\cup j)(x)\rangle = \langle P(I)\text{grad}f_j(x),P(I\cup j)X(x)\rangle = \langle P(I\cup j)\text{grad}f_j(x),X(x)\rangle = 0$ (because $P(I\cup j)\text{grad}f_j(x)=P(I\cup j)P(j)\text{grad}f_j(x)$ and $P(j)\text{grad}f_j(x)=\mathbf{0}$) we must have $\langle P(I)\text{grad}f_j(x),X(I)(x)-X(I\cup j)(x)\rangle = \langle P(I)\text{grad}f_j(x),X(I)(x)\rangle$ and therefore $X(I)(x)-X(I\cup j)(x) = \langle P(I)\text{grad}f_j(x),X(I)(x)\rangle P(I)\text{grad}f_j(x)/ \mid P(I)\text{grad}f_j(x) \mid ^2$.
Then since (by definition of grad) $D_t(f_j\phi(I)) = f_j'(\phi(I))D_t\phi(I) = \langle \text{grad}f_j(x),X(I)(x)\rangle = \langle P(I)\text{grad}f_j(x),X(I)(x)\rangle$ the Lemma is true for k=1. Suppose it is true for k-1, and that $D_t^i(\phi(I\cup j)- \phi(I))=0$ $i=0,..,k-1$. Since it is true for k-1 we know that $D_t^{k-1}(f_j\phi(I))=0$. Using $X(I\cup j)(x)=X(I)(x)-\langle X(I)(x),\text{grad}f_j(x)\rangle P(I)\text{grad}f_j(x)/ \mid P(I)\text{grad}f(x) \mid ^2$ we have that $D_t^k\phi(I\cup j) = D_t^{k-1}(X(I)\phi(I\cup j)-D_t(f_j\phi(I))P(I)\text{grad}f_j(\phi(I\cup j))/ \mid P(I)\text{grad}f_j(\phi(I\cup j)) \mid ^2)$. $D_t^{k-1}(X(I)\phi(I\cup j))$ involves terms in $D_t^i\phi(I\cup j)$ up to $i=k-1$, so since these all equal $D_t^i\phi(I)$ we must have $D_t^{k-1}X(I)\phi(I\cup j)=D_t^k\phi(I)$. Furthermore since $D_t^if_j\phi(I)=0$ for $i\leq k-1$ we have $D_t^{k-1}(D_t(f_j\phi(I))P(I)\text{grad}f_j(\phi(I\cup j))/ \mid P(I)\text{grad}f_j(\phi(I\cup j)) \mid ^2) = D_t^k(f_j\phi(I))P(I)\text{grad}f_j/ \mid P(I)\text{grad}f_j \mid ^2$, and so the Lemma is true for k.

∎

Lemma 2.3 If for some $i\geq 1$ K is such that $S^0_i(x)\subset K\subset S_i(x)$ then $P(K)D_t^i(\phi(S^0_i(x))-\phi(S_i(x)))=D_t^i(\phi(K)-\phi(S_i(x)))$.

Proof If $i=1$; $P(K)D_t(\phi(S^0_1(x))- \phi(S_1(x)))=P(K)(X(S^0_1(x))-X(S_1(x)))=X(K)-X(S_1(x))$ as required.

Suppose the result is true for i-1.
We have $P(K)D_t^i(\phi(S^0_i(x))-\phi(S_i(x))=P(K)D_t^i(\phi(S^0_i(x))- \phi(S^0_i(x)\cup j(1))+ \phi(S^0_i(x)\cup j(1))-..- \phi(K)+ \phi(K)-.. \phi(S_i(x)))$
where $K=S^0_i(x)\cup j(1)\cup..\cup j(k)$ and $S_i(x)=S^0_i(x)\cup j(1)\cup..\cup j(k)\cup..\cup j(m)$ for $m\geq k\geq 0$. We know $S^0_{i-1}(x)\subset S^0_i(x)\subset K\subset S_i(x)\subset S_{i-1}(x)$ and by the inductive hypothesis we also have
$P(S^0_i(x))D_t^{i-1}(\phi(S^0_{i-1}(x))-\phi(S_{i-1}(x)))=D_t^{i-1}(\phi(S^0_i(x))-\phi(S_{i-1}(x)))$ and
$P(S^0_i(x)\cup j(1))D_t^{i-1}(\phi(S^0_{i-1}(x))-\phi(S_{i-1}(x)))=D_t^{i-1}(\phi(S^0_i(x)\cup j(1))-\phi(S_{i-1}(x))$.
Since $S^0_i(x)\subset S^0_i(x)\cup j(1)\subset S_i(x)$ the left hand sides are equal; subtracting we get therefore $D_t^{i-1}(\phi(S^0_i(x))-\phi(S^0_i(x)\cup j(1)))=0$ to which we may apply Lemma 2.2 to obtain $D_t^i(\phi(S^0_i(x))-\phi(S^0_i(x)\cup j(1)))$
$=D_t^i(f_{j(1)}\phi(S^0_i(x)))P(S^0_i(x))\text{grad}f_{j(1)}(x)/ \mid P(S^0_i(x))\text{grad}f_{j(1)}(x) \mid ^2$,
and similarly for all other terms which are of the form $D_t^i(\phi(S^0_i(x)\cup j(1)\cup..\cup j(r))-$

$\phi(S^0_i(x)\cup j(1)\cup..\cup j(r+1)))=$
$D_t^i f_{j(r+1)}\phi(S^0_i(x)\cup j(1)U..\cup j(r)))P(S^0_i(x)\cup j(1)\cup..\cup j(r))gradf_{j(r+1)}(x)/\ |P(S^0_i(x)\cup j(1)\cup..\cup j(r))gradf_{j(r+1)}(x)|^2.$ (*)

Then if $r+1\leq k$, $P(K)P(S^0_i(x)\cup j(1)\cup..\cup j(r))=P(K)$ and $P(K)gradf_{j(r+1)}(x) = P(K)P(j(r+1))gradf_{j(r+1)}(x)=\mathbf{0}$. Hence from (*) it follows $P(K)D_t^i(\phi(S^0_i(x)-\phi(K))=\mathbf{0}$, and so $P(K)D_t^i(\phi(S^0_i(x))-\phi(S_i(x)))=P(K)D_t^i(\phi(K)-\phi(S_i(x)))$. (**)

If $r+1>k$, ie if $r\geq k$, $P(K)P(S^0_i(x)\cup j(1)U..\cup j(r))=P(S^0_i(x)\cup j(1)\cup..\cup j(r))$ and so using (*)

$P(K)D_t^i(\phi(S^0_i(x)\cup j(1)\cup..\cup j(r))-\phi(S^0_i(x)\cup j(1)\cup..\cup j(r+1)))=$
$D_t^i\phi(S^0_i(x)\cup j(1)\cup..\cup j(r))-\phi(S^0_i(x)\cup j(1)\cup..\cup j(r+1)))$, and hence
$P(K)D_t^i(\phi(K)-\phi(S_i(x)))=P(K)D_t^i(\phi(K)-\phi(K\cup j(k+1))+\phi(K\cup j(k+1))-...-\phi(S_i(x)))=$
$D_t^i(\phi(K)-\phi(K\cup j(k+1))+\phi(K\cup j(k+1))-...-\phi(S_i(x)))=D_t^i(\phi(K)-\phi(S_i(x)))$.

Combining with (**) $P(K)D_t^i(\phi(S^0_i(x))-\phi(S_i(x)))=D_t^i(\phi(K)-\phi(S_i(x)))$ as required. ∎

Corollary 2.1 If $S^0_{i+1}(x)\subset K_1,K_2\subset S_{i+1}(x)$ then $D_t^j\phi(K_1)(x,t=0)=D_t^j\phi(K_2)(x,t=0)$ for all $j\leq i$

Corollary 2.2 $D_t^i(\phi(S^0_i(x))(x,t=0)-\phi(S_i(x))(x,t=0))\in N_x(S_i(x)$ in $S^0_i(x))$

Proof By Corollary 2.1 $D_t^{i-1}(\phi(K)(x,t=0)-\phi(K\cup j)(x,t=0)) =0$ for all $S^0_i(x)\subset K\subset K\cup j\subset S_i(x)$; therefore by Lemma 2.2 if $S_i(x)=S^0_i(x)\cup(1,..,k)$ then $D_t^i(\phi(S^0_i(x))(x,t=0)-\phi(S_i(x))(x,t=0))=\alpha_1P(S^0_i(x))gradf_1(x)+\alpha_2P(S^0_i(x)\cup\{1\})gradf_2(x)+...+\alpha_kP(S_i(x)\backslash\{k\})gradf_k(x)$, some scalars $\alpha_1,\alpha_2,..,a_k$, which by the reasoning used to prove Remark 2.1 parts 1 and 2 is contained in $span\{P(S^0_i(x))gradf_j(x):j=1,..,k\}=span\{P(S^0_i(x))gradf_j(x):j\in S_i(x)\backslash S^0_i(x)\}$, which equals $N_x(S_i(x)$ in $S^0_i(x))$ by Remark 2.1(3). ∎

Remark 2.3 In the case that the submanifold with corners is (at least locally) an intersection of linear corners (ie LC(I;J) rather than ZN(I;J)) we may dispense with the $S_i(x)$ term in the definition and construction of the iteration and obtain an equivalent iteration with the $S^0_i(x)$ term only (the $S_i(x)$ term is subtacted ultimately to take care of the bending of the strata away from the tangent cone). Also we can use Corollary 2.2 to show that an equivalent iteration may be obtained by replacing $T_xZN(S^0_i(x);S_i(x)\backslash S^0_i(x))$ with $T_xZN(S^0_i(x);S_i(x)\backslash S^0_i(x))\cap N_x(S_i(x)$ in $S^0_i(x))$; we shall not need either fact.

Lemma 2.4 Suppose X is a vector in R^n and K satisfies $I \subset K \subset I \cup J$; the following are equivalent

1. $P(LC(I;J))X = P(K)X$
2.(a) $P(LC(I;K \backslash I))X \in L(K)$ and (b) $P(K)X \in LC(K;J \backslash K)$
3.(a) $\langle X, P(K \backslash j)n_j \rangle \leq 0$ for all $j \in K \backslash I$ and (b) $P(K)X \in LC(K;J \backslash K)$

Eg if $I = \varnothing$, $J = (1,2,3)$, and $K = (1)$:

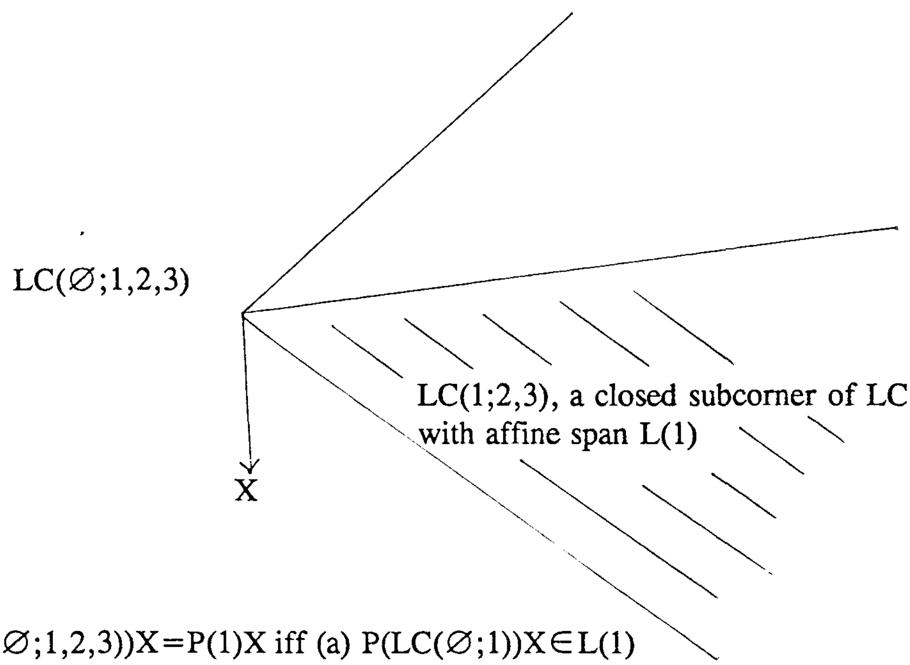


Figure 2.5

Proof

1→2 We can check from definitions that if $I \subset K \subset I \cup J$ then $P(LC(I;J))X = P(K)X$ iff $P(LC(I;J))P(I)X = P(K)P(I)X$, and that $P(LC(I;K \backslash I))X \in L(K)$, $P(K)X \in LC(K;J \backslash K)$ iff $P(LC(I;K \backslash I))P(I)X \in L(K)$, $P(K)P(I)X \in LC(K;J \backslash K)$, and so it suffices to prove the result with X replaced by $P(I)X \in L(I)$, which is equivalent to proving the result with I set to $\varnothing$. We first show 1→2a; we show that if for $\varnothing \subset K \subset J$ $P(LC(\varnothing;J))X = P(K)X$, then $\langle X - P(K)X, P(K)X - y \rangle \geq 0$ for all $y \in LC(\varnothing;K)$, which by the Characterisation of Projection implies $P(K)X = P(LC(\varnothing;K))X$. Consider $(y + L(K)) \cap L(J \backslash K)$ (see Figure 2.6); L(K) and $L(J \backslash K)$ are transverse and hence $y + L(K)$ and $L(J \backslash K)$ are transverse for all y. Hence (on dimensional grounds) $(y + L(K)) \cap L(J \backslash K) \neq \varnothing$. Furthermore if $z \in (y + L(K)) \cap L(J \backslash K)$ then $z - y \in L(K)$ so $\langle z - y, n_i \rangle = 0$ for all $i \in K$. $z \in L(J \backslash K)$ means $\langle z, n_i \rangle = 0$ for all $i \in J \backslash K$. Since $\langle y, n_i \rangle \geq 0$ for all $i \in K$ we have therefore that $z \in LC(J \backslash K;K)$ which is contained in $LC(\varnothing;J)$.

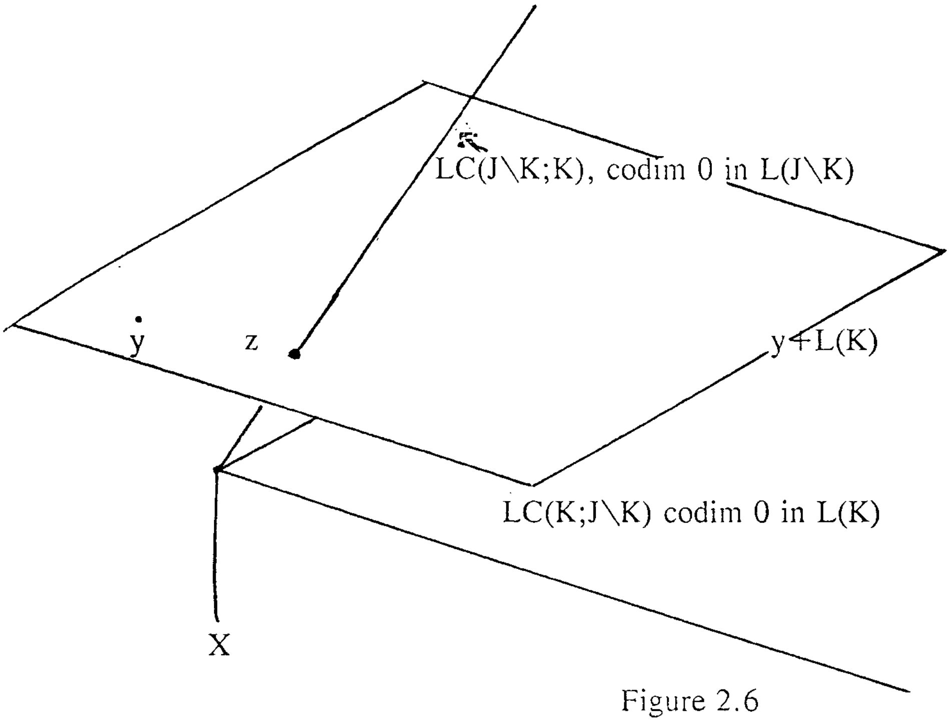


Figure 2.6

Since $y-z \in L(K)$ $\langle X-P(K)X, P(K)X-y\rangle = \langle X-P(K)X, P(K)X-(y-z+z)\rangle =$ $\langle X-P(K)X, P(K)X-z\rangle$, 1. tells us that $P(K)X = P(LC(\varnothing;J))X$ and since $z \in LC(\varnothing;J)$ by the Charactisation of Projection again we know $\langle X-P(LC(\varnothing;J))X, P(LC(\varnothing;J))X-z\rangle \geq 0$ and so $\langle X-P(K)X, P(K)X-z\rangle \geq 0$. This gives $P(LC(I;K\backslash I))X = P(K)X$ (ie 1→2a). 1. also tells us that $P(K)X \in LC(I;J) \cap L(K) = LC(K;J\backslash K)$ (ie 1→2b)

<u>2→1</u> 2a. is equivalent to $P(LC(I;J))X = P(K)X$ hence under conditions 2a. and 2b. $P(K)X \in LC(I;K\backslash I) \cap LC(K;J\backslash K) = LC(K;J)$. So $P(LC(I;K\backslash I))X =$ $P(K)X \in LC(K;J) \subset LC(I;J)$ (since $I \subset K \subset I \cup J$) and hence since $LC(I;K\backslash I) \supset LC(I;J)$, by Lemma 1.1 $P(LC(I;K\backslash I))X = P(LC(I;J))X$ and hence $P(K)X = P(LC(I;J))X$ as was to be shown.

<u>2a⇔3a</u> <u>Fact 1</u> $\text{Span}\{P(I)e_i\}_{i\in K\backslash I} = \text{span}\{P(K\backslash i)e_i\}_{i\in K\backslash I}$ any $I \subset K$.

Clearly the right hand side $= \text{span}\{P(K\backslash i)P(I)e_i\}_{i\in K\backslash I}$, so since for $i \in K\backslash J$ (see Remarks 2.1) $P(K\backslash i)P(I)e_i - P(I)e_i \in \text{span}\{P(I)e_j\}_{j\in K\backslash I\backslash i}$ we have $\text{span}\{P(K\backslash i)P(I)e_i\}_{i\in K\backslash I} \subset \text{span}\{P(I)e_i\}_{i\in K\backslash I}$. Suppose $K\backslash I = (1,..,m)$. For $i,j \in (1,..,m)$

$$\langle P(I)e_i, P(K\backslash j)P(I)e_j\rangle = \langle P(K\backslash j)e_i, e_j\rangle \begin{cases} =0 & \text{if } i \neq j \\ \neq 0 & \text{if } i=j \end{cases}$$

- this is so because $i \in K\backslash j$ if $i \neq j$, while if $i=j$ $\langle P(K\backslash j)e_i, e_j\rangle = | P(K\backslash j)e_j |^2$, so equals 0 iff $P(K\backslash j)e_j = \mathbf{0}$, which (by Remarks 2.1(1)) is so iff $e_j \in \text{span}\{e_i : i \in K\backslash j\}$ which would contradict the linear independence of $\{e_j : j \in K\}$. Thus the matrix A with elements $A_{ij} = \langle P(I)e_i, P(K\backslash j)P(I)e_j\rangle$ for $i,j \in (1,..,m)$ is invertible from which it follows that $(P(K\backslash 1)P(I)e_1,..,P(K\backslash m)P(I)e_m)$ has rank m, which proves the result.

Fact 2 For $I \subset K$ any $w \in LC(I;K\backslash I)$ may be expressed as $w=P(K)w+\Sigma_{i\in K\backslash I}a_iP(K\backslash i)e_i$ for some sequence of reals $\{a_i\}$ with each $a_i \geq 0$.

Write $w=P(K)w+w-P(K)w$, then if $w \in L(I)$,

$w-P(K)w=P(I)w-P(K)P(I)w \in span\{P(I)e_i\}_{i\in K}$ (by Remark 2.1(2))

$=span\{P(I)e_i\}_{i\in K\backslash I}$ (since $P(I)e_i=0$ if $i \in I$)

$=span\{P(K\backslash i)e_i\}_{i\in K\backslash I}$ by Fact 1. Hence $w=P(K)w+\Sigma_{i\in K\backslash I}a_iP(K\backslash i)e_i$ for some sequence of reals $\{a_i\}$. $w \in LC(I;K\backslash I)$ iff $w \in L(I)$ and $\langle w,e_j\rangle \geq 0$ for all $j \in K\backslash I$; hence $w \in LC(I;K\backslash I)$ iff $\Sigma_{i\in K\backslash I}\langle a_iP(K\backslash i)e_i,e_j\rangle \geq 0$ for all $j \in K\backslash I$ iff $\langle a_iP(K\backslash i)e_i,e_i\rangle \geq 0$ for all $j \in K\backslash I$ (all other terms $=0$) and since as in Fact 1 $\langle P(K\backslash i)e_i,e_i\rangle = |P(K\backslash i)e_i|^2 \neq 0$ $w \in LC(I;K\backslash I)$ iff $a_i \geq 0$ for all $i \in K\backslash I$.

We now establish that 2a⇔3a. $P(LC(I;K\backslash I))X \in L(K)$ iff $P(LC(I;K\backslash I))X=P(K)X$ (via Lemma 1.1 since L(K) is a closed convex subset of $LC(I;K\backslash I)$)
iff for all $w \in LC(I;K\backslash I)$ $\langle X-P(K)X,P(K)X-w\rangle \geq 0$, ie $\langle X-P(K)X,w\rangle \leq 0$, which using Fact 2 is equivalent to saying iff for all sequences $\{a_i\}$ with each $a_i \geq 0$
$\langle X-P(K)X,P(K)w+\Sigma_{i\in K\backslash I}a_iP(K\backslash i)e_i\rangle \leq 0$, ie iff $\Sigma_{i\in K\backslash I}a_i\langle X-P(K)X,P(K\backslash i)e_i\rangle \leq 0$. Then since $\langle P(K)X,P(K\backslash i)e_i\rangle=\langle X,P(K)e_i\rangle=0$ for all $i \in K\backslash I$ this is so iff for all $i \in K\backslash I$
$\langle X,P(K\backslash i)e_i\rangle \leq 0$, as was claimed. ∎

We have observed that because of the property $S^0_1(x) \subset S^0_i(x) \subset S_i(x) \subset S_1(x)$ $\cap_{i\geq 1}S_i(x)$ exists, is contained in every $S_j(x)$ for $j \in Z^+$ and equals $\lim_{i\to\infty}S_i(x)$; we call this $S_\infty(x)$. Similarly $\cup_{i\geq 1}S^0_i(x)$ exists, contains every $S^0_j(x)$ for $j \in Z^+$ and is contained in every $S_j(x)$ and hence in $S_\infty(x)$, and equals $\lim_{i\to\infty}S^0_i(x)$. We have for all i
$S^0_1(x) \subset S^0_i(x) \subset S^0_\infty(x) \subset S_\infty(x) \subset S_i(x) \subset S_1(x)$. In Lemmas 2.5 and 2.6 we consider the flows $\phi(S^0_\infty(x))$ and $\phi(S_\infty(x))$. We shall show that on some $(0,T)$ $\phi(S_\infty(x))(x,t)$ lies in $ZP(S_\infty(x);S_1(x)\backslash S_\infty(x))$ (Figure 2.7a) and that for all $j \in S^0_\infty(x)\backslash S^0_1(x)$ and at every point y of $\phi(S^0_\infty(x))(x,t)$ that $\langle X(y),P(S^0_\infty(x)\backslash j)\mathrm{grad}f_j(y)\rangle \leq 0$ (Figure 2.7b).

If then $S^0_\infty(x)=S_\infty(x)$ we may combine these results and use Lemma 2.4 to infer that at every point $y=\phi(S_\infty(x))(x,t)$ with $t \in (0,T)$ $P(T_yM)X(y)=P(K)X(y)$ and hence this is the K alluded to in the overview above - $\phi(M)(x,t)= \phi(K)(x,t)$ on $(0,T)$. In the analytic case we get a result even if we do not have $S^0_\infty(x) = S_\infty(x)$.

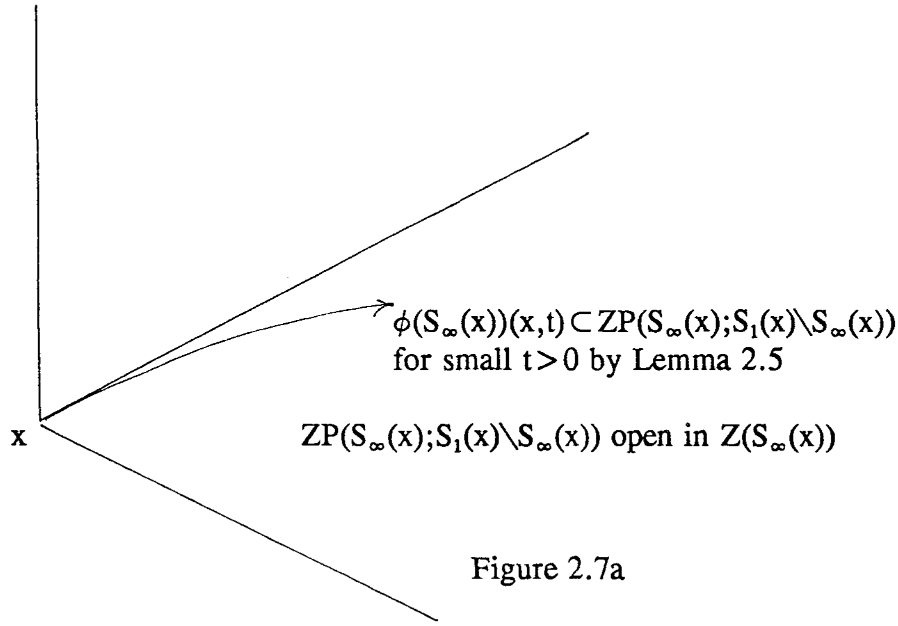


Figure 2.7a

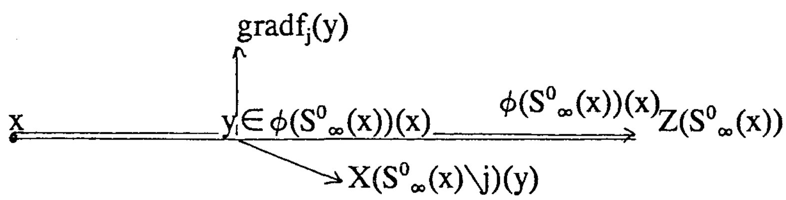


Figure 2.7b. $\langle X(\phi(S^0_\infty(x))(x,t), P(S^0_\infty(x)\backslash j)\text{gradf}_j(\phi(S^0_\infty(x)\backslash j)(x,t))\rangle \leq 0$ for small $t \geq 0$, each $j \in S^0_\infty(x)\backslash S^0_1(x)$, by Lemma 2.6

<u>Lemma 2.5</u> If $j \in S_r(x)\backslash S_{r+1}(x)$ then

$$D_t^i f_j \phi(S_\infty(x))(x,t=0) \begin{cases} =0 \text{ if } i<r \\ >0 \text{ if } i=r \end{cases}$$

<u>Proof</u> We have $S^0_r(x) \subset S_\infty(x) \subset S_\infty(x) \cup j \subset S_r(x)$ but $S_\infty(x) \cup j \not\subset S_{r+1}(x)$. By Corollary 2.1 $D_t^i\phi(S_\infty(x))(x,t=0) = D_t^i\phi(S_\infty(x)\cup j)(x,t=0)$ for all $i<r$. By Lemma 2.2 $D_t^i f_j\phi(S_\infty(x))(x,t=0)=0$ for all $i<r$. Since $S^0_r(x) \subset S_\infty(x) \cup j \subset S_r(x)$ but $S_\infty(x) \cup j \not\subset S_{r+1}(x)$ we know $P(S_\infty(x)\cup j)D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0)) \neq P(S_\infty(x))D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0))$ (by definition of the iteration). We shall write $T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x))$ as C_r and $D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0))$ as X_r, so by the construction of the iteration again $P(S_\infty(x))D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0))=P(C_r)X_r$. We have therefore $\langle P(S_\infty(x))X_r, \text{gradf}_j(x)\rangle = \langle P(C_r)X_r, \text{gradf}_j(x)\rangle \geq 0$ for all $j \in S_r(x)$ and by the beginning of Lemma 2.2 we know that $\langle P(S_\infty(x))X_r, \text{gradf}_j(x)\rangle = 0$ iff $P(S_\infty(x))X_r = P(S_\infty(x)\cup j)X_r$.

Furthermore $\langle P(S_\infty(x)\cup j)X_r, \text{grad}f_j(x)\rangle = \langle P(S_\infty(x)\cup j)X_r, P(S_\infty(x))\text{grad}f_j(x)\rangle = 0$ for all j by self-adjointness of P and since $P(S_\infty(x)\ \cup j) = P(S_\infty(x))P(S_\infty(x)\cup j)$. By Lemma 2.3 we know that $P(S_\infty(x)\cup j)X_r = D_t^r(\phi(S_\infty(x)\cup j)(x,t=0) - \phi(S_r(x))(x,t=0))$ and $P(S_\infty(x))X_r = D_t^r(\phi(S_\infty(x))(x,t=0) - \phi(S_r(x))(x,t=0))$, hence $\langle \text{grad}f_j(x), D_t^r(\phi(S_\infty(x))(x,t=0) - \phi(S_r(x))(x,t=0))\rangle > 0$ (≥ 0 because $\langle \text{grad}f_j(x), P(S_\infty(x))X_r\rangle = \langle \text{grad}f_j(x), P(T_xZN(S^0_r(x); S_r(x)\backslash S^0_r(x))X_r\rangle \geq 0$ as $j \in S_r(x)\backslash S_{r+1}(x)$ and hence $j \in S_r(x)\backslash S^0_r(x)$; $\neq 0$ because we noted above that $\langle \text{grad}f_j(x), P(S_\infty(x))X_r\rangle = 0$ iff $P(S_\infty(x))X_r = P(S_\infty(x))X_r = P(S_\infty(x)\cup j)X_r$, and we also saw that the construction of the iteration implied that this was not the case if $j \in S_r(x)\backslash S_{r+1}(x)$). We have also that $\langle \text{grad}f_j(x), D_t^r(\phi(S_\infty(x)\cup j)(x,t=0) - \phi(S_r(x))(x,t=0))\rangle = 0$ (since $D_t^r(\phi(S_\infty(x)\cup j)(x,t=0) - \phi(S_r(x))(x,t=0)) = P(S_\infty(x)\cup j)X_r$), thus $\langle \text{grad}f_j(x), D_t^r(\phi(S_\infty(x))(x,t=0) - \phi(S_\infty(x)\cup j)(x,t=0))\rangle > 0$ and hence by Lemma 2.2 $D_t^r(f_j\phi(S_\infty(x))(x,t=0)) > 0$. ∎

<u>Lemma 2.6</u> Suppose $S^0_r(x) \subset S^0_r(x)\cup j \subset S^0_{r+1}(x) \subset K \subset S_{r+1}(x)$. If we set $g_j(K)(t) = \langle X(K\backslash j)\phi(K)(x,t), P(K\backslash j)\text{grad}f_j\phi(K)(x,t)\rangle$, then

$$D_t^i g_j(K)(t=0) \begin{cases} =0 & \text{if } i<r-1 \\ <0 & \text{if } i=r-1 \end{cases}$$

<u>Proof</u> Since $j \in K$, for all $y \in Z(K)$ $\langle X(K)(y), P(K\backslash j)\text{grad}f_j(y)\rangle = \langle X(K)(y), \text{grad}f_j(y)\rangle = 0$; thus $g_j(K)(t) = \langle X(K\ \backslash j)\phi(K)(x,t) - X(K)\phi(K)(x,t), P(K\backslash j)\text{grad}f_j\phi(K)(x,t)\rangle$. By Corollary 2.1 we know $D_t^i\phi(K\backslash j)(x,t=0) = D_t^i\phi(K)(x,t=0)$ $i=0,..,r-1$, hence $D_t^i g_j(K)(t=0) = 0$ $i=0,..,r-2$, and $D_t^{r-1}g_j(K)(t=0) =$ $\langle D_t^r\phi(K\backslash j)(x,t=0) - D_t^r\phi(K)(x,t=0), P(K\backslash j)\text{grad}f_j\phi(K)(x,t=0)\rangle$. By Lemma 2.3 we know $P(K)X_r = D_t^r(\phi(K)(x,t=0) - \phi(S_r(x))(x,t=0))$ and $P(K\backslash j)X_r = D_t^r(\phi(K\backslash j)(x,t=0) - \phi(S_r(x))(x,t=0))$ (X_r as defined in Lemma 2.5). By definition of the iteration $P(T_xZN(S^0_r(x), S_r(x)\backslash S^0_r(x)))X_r\ =\ P(K)X_r$ iff $S^0_{r+1}(x) \subset K \subset S_{r+1}(x)$ (X_r as defined in Lemma 2.5) and hence by Lemma 2.4 for any $S^0_{r+1}(x) \subset K \subset S_{r+1}(x)$ $\langle X_r, P(K\backslash j)\text{grad}f_j(x)\rangle \leq 0$ for all $j \in K\backslash S^0_r(x)$. Since $K\backslash j$ is not between $S^0_{r+1}(x)$ and $S_{r+1}(x)$ we must have $P(K\backslash j)X_r \neq P(K)X_r$. We know from Remarks 2.1 that $P(K\backslash j)X_r - P(K)X_r =$ $\langle P(K\backslash j)\text{grad}f_j(x), X_r\rangle P(K\backslash j)\text{grad}f_j(x) / \mid P(K\backslash j)\text{grad}f_j(x) \mid^2$

so $\langle \text{grad}f_j(x), P(K\backslash j)X_r\rangle \neq 0$, so by the above <0. By the above

$$D_t^r(\phi(K\backslash j)(x,t=0)-\phi(K)(x,t=0))=P(K\backslash j)X_r-P(K)X_r. \qquad (*)$$

Acting on both sides of (*) with $P(K\backslash j)$, which leaves the right hand side unchanged, we get $\langle P(K\backslash j)D_t^r(\phi(K\backslash j)(x,t=0)-\phi(K)(x,t=0)), \text{grad}f_j(x)\rangle <0$, and hence $D_t^{r-1}g_j(K)(t=0)<0$ as required. ∎

Proof of Theorem 2.1

(1) By Lemma 2.5 we know that on some $(0,t_0)$ $f_j\phi(S_\infty(x))(x,t)>0$ for all $j\in S_1(x)\backslash S_\infty(x)$, ie that $\phi(S_\infty(x))(x,t)\in ZP(S_\infty(x),S_1(x)\backslash S_\infty(x))$ on $(0,t_0)$.

For any $j\in S^0_\infty(x)\backslash S_1(x)$ there exists r such that $j\in S^0_{r+1}(x)\backslash S^0_r(x)$, and Lemma 2.6 tells us that with this j then for any $S^0_{r+1}(x)\subset K\subset S_{r+1}(x)$ there exists $t(j,K)>0$ such that $\langle X\phi(K)(x,t), P(K\backslash j)\text{grad}f_j\phi(K)(x,t)\rangle<0$ on $(0,t(j,K))$. $S^0_\infty(x)$ satisfies $S^0_{r+1}(x)\subset S^0_\infty(x)\subset S_{r+1}(x)$ for all r and hence there exists $t_0>0$ such that for all $t\in(0,t_0)$ and for all $j\in S^0_\infty(x)\backslash S_1(x)$

$\langle X\phi(S^0_\infty(x))(x,t), P(S^0_\infty(x)\backslash j)\text{grad}f_j(\phi(S^0_\infty(x))(x,t))\rangle<0$. If then $S^0_\infty(x)=S_\infty(x)$ setting $S_\infty(x)=S^0_\infty(x)=K$, then for $y=\phi(K)(x,t)$ with $t\in(0,t_0)$ we have $y\in ZP(K;S_1(x)\backslash K)$ so $T_yM=T_yZN(S^0_1(x);K\backslash S^0_1(x))$ and Lemma 2.4 tells us that $P(T_yZN(S^0_1(x);K\backslash S^0_1(x)))X(y)=P(K)X(y)$ iff

(a) $\langle X(y), P(K\backslash j)\text{grad}f_j(y)\rangle\leq 0$ for all $j\in K\backslash S^0_1(x)$, and

(b) $P(K)X(y)\in T_yZN(K;K\backslash K)$

so (b) is satisfied vacuously, and (a) is satisfied by the above.

We know $P(T_xM)X(x)=P(K)X(x)$ because by definition $X(M)(x)=X(K)(x)$ any $S^0_2(x)\subset K\subset S_2(x)$ and we know $S^0_2(x)\subset S^0_\infty(x)=K=S_\infty(x)\subset S_2(x)$. Thus for all $t\in[0,t_0)$ $P(T_{\phi(K)(x,t)}M)X(\phi(K)(x,t))=P(K)X(\phi(K)(x,t))$. The left hand side is by definition $X(M)\phi(K)(x,t)$, the right hand side is $D_t\phi(K)(x,t)$ so we have $X(M)(\phi(K)(x,t))=D_t\phi(K)(x,t)$ for all $t\in[0,t_0)$, and so by uniqueness (Theorem 1.1) of the solution to the equation $X(M)\phi(M)(x,t)=D_t\phi(M)(x,t)$ for a.a. $t\in[0,t_0)$ we must have $\phi(M)(x,t)=\phi(K)(x,t)$ on $[0,t_0)$.

(2) By Corollary 2.1 we have if $S^0_\infty(x)\subset K_1,K_2\subset S_\infty(x)$ $D_t^i\phi(K_1)(x,t=0)=D_t^i\phi(K_2)(x,t=0)$ for all i and so if the data, and hence the $\phi(K_i)$'s, are analytic we have $D_t^i\phi(K_1)(x,t)=D_t^i\phi(K_2)(x,t)$ for all t.

We shall show that $\phi(S^0_\infty(x))(x,t)=\phi(M)(x,t)$ for all sufficiently small $t\geq 0$, and hence by the above $\phi(K)(x,t)=\phi(S^0_\infty(x))(x,t)$ for all $S^0_\infty(x)\subset K\subset S_\infty(x)$. We showed in (1) that on some $(0,t_0)$ (i) $\langle X\phi(S^0_\infty(x))(x,t), P(S^0_\infty(x)\backslash j)\text{grad}f_j\phi(S^0_\infty(x)(x,t))\rangle<0$ for all

$j \in S^0_\infty(x) \setminus S^0_1(x)$, and (ii) $\phi(S_\infty(x))(x,t) \in ZP(S_\infty(x);S_1(x)\setminus S_\infty(x))$. Since we have shown $\phi(K_1)(x,t)=\phi(K_2)(x,t)$ for all $S^0_\infty(x) \subset K_i \subset S_\infty(x)$ we can rework this second condition as (ii') $\phi(S^0_\infty(x))(x,t) \in ZP(S_\infty(x);S_1(x)\setminus S_\infty(x))$. For $t \in (0,t_0)$ writing $y=\phi(S^0_\infty(x))(x,t)$ $(=\phi(S_\infty(x))(x,t)$) we have therefore $T_yM=T_yZN(S^0_1(x);S_\infty(x)\setminus S^0_1(x))$. If we show $P(T_yM)X(y)=X(S^0_\infty(x))(y)$ it will follow by the same reasoning as in the last part of (1) that $\phi(M)(x,t)=\phi(S^0_\infty(x))(x,t)$ for all $t \in [0,t_0)$.

Lemma 2.4 tells us that $P(T_yZN(S^0_1(x);S_\infty(x)\setminus S^0_1(x)))X(y)=P(S^0_\infty(x))X(y)$ iff

(a) $\langle X(y),P(S^0_\infty(x)\setminus j)\mathrm{grad}f_j(y)\rangle \leq 0$ for all $j \in S^0_\infty(x)\setminus S^0_1(x)$, and

(b) $P(S^0_\infty(x))X(y) \in T_yZN(S^0_\infty(x);S_\infty(x)\setminus S^0_\infty(x))$.

(a) holds by (i) above. We know $y=\phi(S^0_\infty(x))(x,t)=\phi(S_\infty(x))(x,t)$ and that $D_t\phi(S^0_\infty(x))(x,t)=D_t\phi(S_\infty(x))(x,t)$ and so that $X(S^0_\infty(x))(y)=X(S_\infty(x))(y)$. Then since $X(S_\infty(x))(y) \in T_yZ(S_\infty(x)) \subset T_yZN(S^0_\infty(x);S_\infty(x)\setminus S^0_\infty(x))$ (b) also follows, and hence the result. ∎

Example 2.3 For the biological models [60] which originally inspired the thesis we have a situation of the following form: $M=\{x \in R^n: \langle x,n_i\rangle \geq p_i \;\; i=1,..,n\}$, where $\{n_i\}$ are an orthonormal set, and $X(x)=Ax$ where $A \in L(R^n,R^n)$. Suppose x is a point such that $\langle x,n_i\rangle=p_i \;\; \forall\, i=1,..,k$. We seek the stratum of M containing $\phi(M)(x,t)$ for all $t \in (0,\delta)$, some $\delta>0$.

We see we have $S_1(x)=(1..k), S^0_1(x)=\varnothing$, and using the definition of iteration and Remark 2.5(1) below we get

$S_2(x)=\{i \in S_1(x)\setminus S^0_1(x):\langle Ax,n_i\rangle \leq 0\}$

$S^0_2(x)=\{i \in S_1(x)\setminus S^0_1(x):\langle Ax,n_i\rangle < 0\}$,

and generally

$S_m(x)=\{i \in S_{m-1}(x)\setminus S^0_{m-1}(x):\langle (P(S^0_{m-1}(x))A)^{m-1}x,n_i\rangle \leq 0\} \cup S^0_{m-1}(x)$

$S^0_{m-1}(x)=\{i \in S_{m-1}(x)\setminus S^0_{m-1}(x):\langle (P(S^0_{m-1}(x))A)^{m-1}x,n_i\rangle < 0\} \cup S^0_{m-1}(x)$, where

$P(K)A=A-\Sigma_{i \in K}n_in_i^TA$.

If $S_m(x)=S^0_m(x)$ we know by Theorem 2.1(1) that

$\{x \in R^n:\langle x,n_i\rangle=p_i \; \forall i \in S_m(x), \langle x,n_i\rangle>p_i \; \forall i \in (1..k)\setminus S_m(x)\}$ is the stratum we seek.

Alternatively if we arrive at a state where it is evident that $S^0_r(x)=S^0_m(x)$, $S_r(x)=S_m(x) \;\forall\, r \geq m$, then since the system is analytic we may apply Theorem 2.1(2) with the same result.

Remark 2.5

A submanifold with orthogonal corners is a submanifold with corners such that for some neighbourhood of each point there exists *some* local representation $ZN(I;J)=ZN(f_1,..,f_k;f_{k+1},..,f_{k+m})$ with $\langle gradf_i(x),gradf_j(x)\rangle=0$ for all $x\in ZN(I;J)$ and for all $i,j\in I\cup J$ with $i\neq j$. This category includes orthants and sets formed out of orthants and balls, such as $\{x\in R^n:x_i\geq 0$ for all $i\in K_1$, $a^2\leq\Sigma_{i\in K_2}x_i^2\leq b^2\}$ for $K_1,K_2\subset(1,..,n)$ and not necessarily distinct reals a,b (Figure 2.8), as well as of course submanifolds with smooth boundary. While this notion of the corners of a submanifold with corners being orthogonal will suffice for our purposes it is obviously very crude: it has in particular the drawback that the defining property will not hold for all local representations even if it holds for one - eg, $M=\{x\in R^n:x_1\geq 0$, $\frac{1}{2}\leq x_1^2+x_2^2\leq 1\}$ (Figure 2.8) is a submanifold with orthogonal corners of R^n, and near a point with $x_1=0$, $x_2^2+x_1^2=1$ we can choose for the functions in our representation $f_1(x)=x_1/x_2, f_2(x)=1-x_2^2-x_1^2$ whose gradients are othogonal near this point, but if we choose instead $f_1(x)=x_1, f_2(x)=1-x_2^2-x_1^2$, which is also a local representation of M, the gradients are only orthogonal if $x_1=0$.

This situation arises because the "intrinsic" property of a submanifold with orthogonal corners is that $gradf_i(y),gradf_j(y)$ are perpendicular at every point y where $f_i(y)$ and $f_j(y)$ are both zero, not everywhere on M.

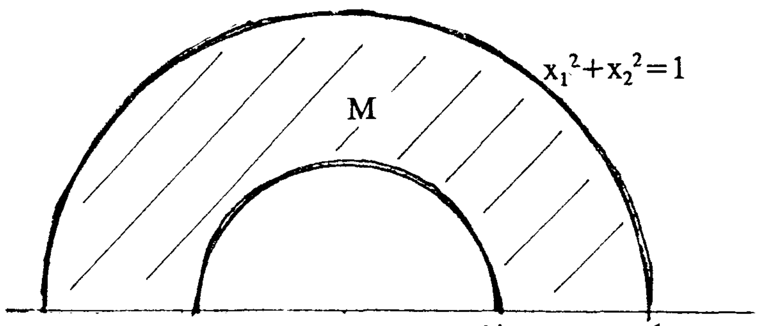


Figure 2.8. A submanifold with orthogonal corners

(1) We show that if M is a submanifold with orthogonal corners (with local orthogonal representation ZN(I;J)) then for each y in the domain of this representation $X(M)(y)=X(S_1(y))(y)+\Sigma_{i\in S_1(y)\cup J}(gradf_i(y)/|gradf_i(y)|^2)\max(\langle gradf_i(y),X(y)\rangle,0)$.

From Chapter One X(M)(y) is the unique vector in T_yM such that

$|X(M)(y)-X(y)| = \min\{|X(y)-Y| : Y \in T_yM\}$. Near y M is locally represented as $ZN(I;S_1(y)\backslash I)$, with $T_yM = T_yZN(I;S_1(y)\backslash I)$. By Remark 2.1 we have $Y-P(T_yZ(S_1(y))Y \in \text{span}\{P(I)\text{gradf}_i(y):i \in S_1(y)\}$ and by orthogonality of $\{\text{gradf}_i(y)\}$ we find this gives $Y=P(T_yZ(S_1(y)))Y+\Sigma_{i\in S_1(y)}\langle Y,\text{gradf}_i(y)\rangle\text{gradf}_i(y)/|\text{gradf}_i(y)|^2$ and similarly $X(y)=X(S_1(y))+\Sigma_{i\in S_1(y)}\langle X(y),\text{gradf}_i(y)\rangle\text{gradf}_i(y)/|\text{gradf}_i(y)|^2$.

Thus $|X(y)-Y|$ is minimized over $Y \in T_yZN(I;S_1(y)\backslash I)$ by choosing

$P(T_yZ(S_1(y)))Y=X(S_1(y))$

$\langle Y,\text{gradf}_i(y)\rangle=0$ if $i \in I$

$\langle Y,\text{gradf}_i(y)\rangle=\max\{\langle X(y),\text{gradf}_i(y)\rangle,0\}$ if $i \in S_1(y)\backslash I$

which is the required result.

(2) We show why in Example 0.2 we needed the angle between F_1 and F_2 to be greater than a right-angle if the transition at y_3 was to occur. In fact we show in general that if M is a submanifold with orthogonal corners then for any $x \in M$ and $t>0$ if $\phi(M)(x,t) \in$ a stratum σ then $X(M)\phi(M)(x,t)=X(\sigma)\phi(M)(x,t)$ (such clearly was not the case at y_3 in Figure 0.2).

If $\phi(M)(x,t)=y \in \sigma$ (see Figure 2.9) then working with the usual local representation of M near y (ie $ZN(S^0_1(y);S_1(y)\backslash S^0_1(y))$) $X(\sigma)y=X(S_1(y))(y)$: from (1) above therefore if $X(M)(y) \neq X(\sigma)(y)$ then $\langle\text{gradf}_i(y),X(y)\rangle>0$ some $i \in S_1(y)\backslash I$. By continuity (of $y \to \langle\text{gradf}_i(y),X(y)\rangle$) there exists $h>0$ with $t-h>0$ such that $\langle\text{gradf}_i\phi(M)(x,s),X(\phi(M)(x,s))\rangle>0$ for all $s \in (t-h,t)$, which implies by (1) again that $\langle X(M)\phi(M)(x,s),\text{gradf}_i\phi(M)(x,s)\rangle=\langle X\phi(M)(x,s),\text{gradf}_i\phi(M)(x,s)\rangle$ for all $s \in (t-h,t)$.

Hence $f_i\phi(M)(x,t)-f_i\phi(M)(x,t-h)= \int_{t-h}^{t}\langle\text{gradf}_i(\phi(M)(x,s)),X(M)\phi(M)(x,s)\rangle ds=$
$\int_{t-h}^{t}\langle\text{gradf}_i(\phi(M)(x,s)),X(\phi(M)(x,s))\rangle ds>0$.

We know $f_i\phi(M)(x,t-h)\geq 0$ since $\phi(M)(x,t-h)\in M$, hence $f_i\phi(M)(x,t)>0$ which is a contradiction to $i \in S_1(y)$. ∎

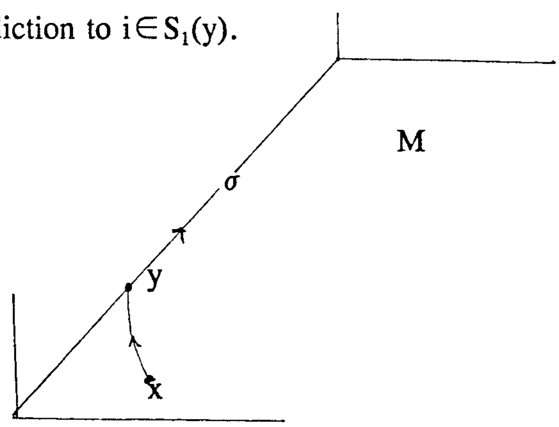


Figure 2.9

Chapter Three
The Iteration In Relation To Right-hand Derivatives

We have seen in Chapter Two that if the iteration at x converges to a single stratum (or to be precise a single set of indices representing a single stratum) then this stratum contains $\phi(M)(x,t)$ for all sufficiently small positive t. This chapter and part of Chapter 5 will be concerned with what the pair $(S^0_i(x),S_i(x))$ is telling us at each stage, even if the data is only smooth and we have no a priori reason to suppose that the iteration will ever converge to a single stratum. If Theorem 2.1 applies at x then for some $t_0>0$ $\phi(M)(x,t)=\phi(S_\infty(x))(x,t)$ for all $t\in[0,t_0)$, and so $\phi(M)(x,t)$ has right-hand time derivatives at t=0, denoted $D_t^{+i}\phi(M)(x,t=0)$, of all orders and equal to the two-sided time derivatives of $\phi(S_\infty(x))(x,t)$ at t=0. Since (by Corollary 2.1) $D_t^i\phi(K_1)(x,t=0)=D_t^i\phi(K_2)(x,t=0)$ for all $S^0_{i+1}(x)\subset K_1,K_2\subset S_{i+1}(x)$ it follows that in these cases $D_t^{+i}\phi(M)(x,t=0)=D_t^i\phi(K)(x,t=0)$ for all $S^0_{i+1}(x)\subset K\subset S_{i+1}(x)$, and we show in this chapter that that this remains so if the (i+1)th stage of the iteration is the last we reach and where even had we continued ad infinitum it may still not have provided us with a single stratum containing $\phi(M)(x,(0,\delta))$. We shall in the process obtain an alternative definition of trajectory (=solution): $\phi(M)(x)$ is a trajectory iff $D_t^+\phi(M)(x,t)=X(M)\phi(M)(x,t)$ for all $t\in[0,t_x)$ (this has been established by Henry in [31] for the case M is an orthant).

Definition (One sided derivatives) If ϕ is a map $\phi:[0,T)\to R^n$ some $T>0$, such that $\lim_{h\downarrow 0}(\phi(h)-\phi(0))/h$ exists, say the limit is $D^+\phi(0)$ and inductively if $D^{+i}\phi(h)$ exists for all $h\in[0,T)$ some $T>0$ and $\lim_{h\downarrow 0}(D^{+i}\phi(h)-D^{+i}\phi(0))/h$ exists denote the limit by $D^{+(i+1)}\phi(0)$. Similarly for left-hand derivatives: if $D^{-i}\phi(h)$ exists on (-T,0] some $T>0$ and $\lim_{h\downarrow 0}(D^{-i}\phi(0)-D^{-i}\phi(-h))/h$ exists denote the limit by $D^{-(i+1)}\phi(0)$. If right (left) hand derivatives of all orders exist at 0 say ϕ is $C^{+\infty}$ ($C^{-\infty}$) at 0. ϕ is C^∞ on U open in R iff at every point $t\in U$ ϕ is $C^{+\infty}$,$C^{-\infty}$ and $D^{+i}\phi(t)=D^{-i}\phi(t)$ for all $i\in Z^+$.

Theorem 3.1 If M,X are smooth (with M near x locally represented in the usual way as $ZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$), then for every $x\in M$ $\phi(M)(x,t)$ is $C^{+\infty}$ for all $t\in[0,t_x)$ (where t_x is as in Theorem 1.1) and $D_t^{+i}\phi(M)(x,t=0)=D_t^i\phi(K)(x,t=0)$ for all $S^0_{i+1}(x)\subset K\subset S_{i+1}(x)$, and $D_t^+\phi(M)(x,t=0)= X(M)\phi(M)(x,t=0)$.

This will be proved after Lemma 3.2. The last part (that $D_t^+\phi(M)(x,t=0)=$

$X(M)\phi(M)(x,t=0))$ has been proved in the case M is an orthant by Henry [31]). We can see Theorem 2.1 is plausible by reconsidering Example 2.1:

Example 3.1 In Example 2.1 we had $X(x,y)=(1,f(x))$ where

$$f(x)=\begin{cases} (1/x^2)\exp(-1/x)(\sin(1/x)-\cos(1/x)) & \text{if } x>0 \\ 0 & \text{if } x\leq 0 \end{cases}$$

on $M=\{(x,y)\in R^2: y\geq 0\}$ and established that the trajectory based at the origin looked like

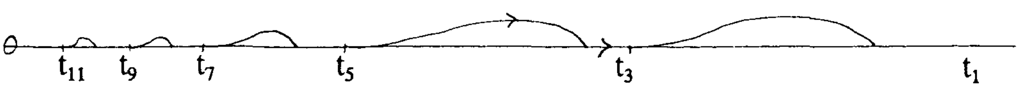


Figure 3.1

where $1/t_m=m\pi+\pi/4$. We can check that $S^0_i(0)=\varnothing$ and $S_i(0)=1$ for all i. Evidently the conclusion of Theorem 1.1 holds away from the origin. We can readily show by ad hoc means that it is $C^{+\infty}$ at 0: on (t_m,t_{m-2}) for m odd $\phi(M)(0,t)=(t,\max(0,\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})$, and thus on (t_m,t_{m-2}) for m odd $D_t^{+i}\phi(M)(0,t)=(D_t^i(t),D_t^{+i}g(t))$ where $D_t^{+i}g(t)=0$ or $D_t^i(\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})$. For large m $t_{m-2}<2t_m$ and hence for all $t\in(t_m,t_{m-2})$ $1/t_m<1/(t-t_m)$ and hence $\sup_{t\in(t_m,t_{m-2})}(1/t)D_t^i(\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})<\sup_{t\in(t_m,t_{m-2})}(1/t_m)D_t^i(\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})<$ $\sup_{t\in(t_m,t_{m-2})}(1/(t-t_m))D_t^i(\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})$. $D_t^i(\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})$ is a sum of terms of the form $(t-t_m)^{-k}\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1}$ or $(t-t_m)^{-k}\exp(-(t-t_m)^{-1})\cos(t-t_m)^{-1}$ $(k\geq 1)$ and since for $t\in(t_m,t_{m-2})$ $1/(t-t_m)>1/(t_{m-2}-t_m)=(2\pi/3)(m+1/4)(m-5/4)$ we can see by substituting $u_m=1/(t_{m-2}-t_m)$ that $\sup_{t\in(t_m,t_{m-2})}(1/t)D_t^i(\exp(-(t-t_m)^{-1})\sin(t-t_m)^{-1})\to 0$ as $m\to\infty$ for all i.

We have set $\phi(M)(0,t)=(t,g(t))$ and the above tells us that $(1/t)D_t^{+i}g(t)\to 0$ as $t\to 0$ for all $i\geq 0$. Thus $g(t)/t\to 0$ so $D_t^+g(t=0)=0$, $(D_t^+g(t)-D_t^+g(0))/t=D_t^+g(t)/t\to 0$ as $t\to 0$ so $D_t^{+2}g(t=0)=0$, and inductively $D_t^{+i}g(t=0)=0$ for all i. Thus at the only point which might have presented a problem (ie the origin) we are saved because there is at this point an infinite order tangency between $\phi(\varnothing)$ and $\phi(1)$ (that is to say, $D_t^i\phi(\varnothing)(0,t=0)=D_t^i\phi(1)(0,t=0)$ for all i).

To recap: to prove Theorem 3.1 we are necessarily interested in infinite order tangencies, because if there aren't any then by Corollary 2.1 $S^0_\infty(x)=S_\infty(x)$, so we could apply Theorem 2.1, and the conclusion of Theorem 3.1 follows immediately.

The above example suggests that if there is an infinite order tangency at x between $\phi(K_1)$ and $\phi(K_2)$ for all $I\subset K_1,K_2\subset I\cup J$ we could stitch together an inductive proof that $\phi(M)(x)$ is right hand smooth at x. We must though consider how to deal with the situation where $S^0_\infty(x) \neq S_\infty(x)$, and hence the flows $\phi(K)$ with K between these bounds are infinitely tangent at x, but $S^0_\infty(x) \neq S^0_1(x)$ or $S_\infty(x) \neq S_1(x)$ and hence other strata are present locally (Figure 3.2); we would like to show that on some $(0,\delta)$ $\phi(M)(x,t)$ is disjoint from these, and hence that we could apply the above idea on a subcorner of ZN(I;J).

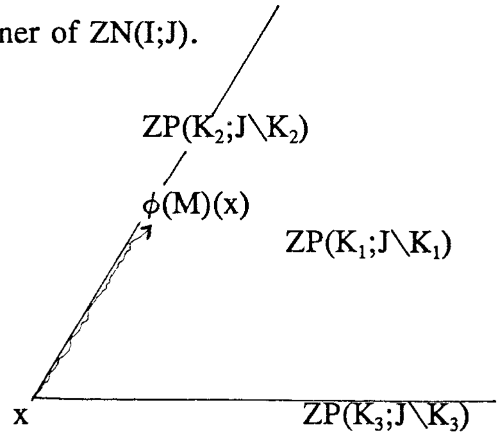


Figure 3.2

For example, in the context of Figure 3.2, if $S^0_\infty(x) \subset K_1, K_2 \subset S_\infty(x)$ and so $\phi(K_1)$ and $\phi(K_2)$ have an infinite order tangency at x, we wish to show that on some $(0,\delta)$ $\phi(M)(x)$ does not intersect any $ZP(K_3;J\backslash K_3)$ where K_3 does not lie between $S^0_\infty(x)$ and $S_\infty(x)$.

None of the results so far will tell us this: to apply Theorem 2.1 we needed $S_\infty(x) = S^0_\infty(x)$. What we do is to use Lemmas 2.5 and 2.6 to show that there exists a finely tapered set in $ZNP(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x);S_1(x)\backslash S_\infty(x))$ which is mapped into itself by the flow and contains x in its boundary, which is exactly the result needed.

Definition We define the canonical r-funnel $F_c(q,r) = \{(t,x) \in R^1 \times R^{q-1} : t \geq 0, |x| \leq t^r\}$. If X is a non-vanishing C^r vector field on a C^r q-dimensional submanifold without corners S, with corresponding flow ϕ, we have by the straightening-out Theorem ([1, Chapter 4], [37, Chapter 5] etc) that any point x in S has a neighbourhood U for which there exists a C^r diffeomorphism $f: U \to R^q$ such that $f.X$ = unit field $\check{e}_1$ on R^q (ie for all $y \in R^q$ $\check{e}_1(y) = e_1 = (1,0) \in R^1 \times R^{q-1}$, with flow $\psi(y,t) = y + te_1$, $y \in R^q$), and so that $f\phi(x',t) = \psi(fx',t)$, for all $x' \in U$. We say $f^{-1}F_c(q,r)$ is an r-funnel about the trajectory $\phi(x)$ in S (Figure 3.3).

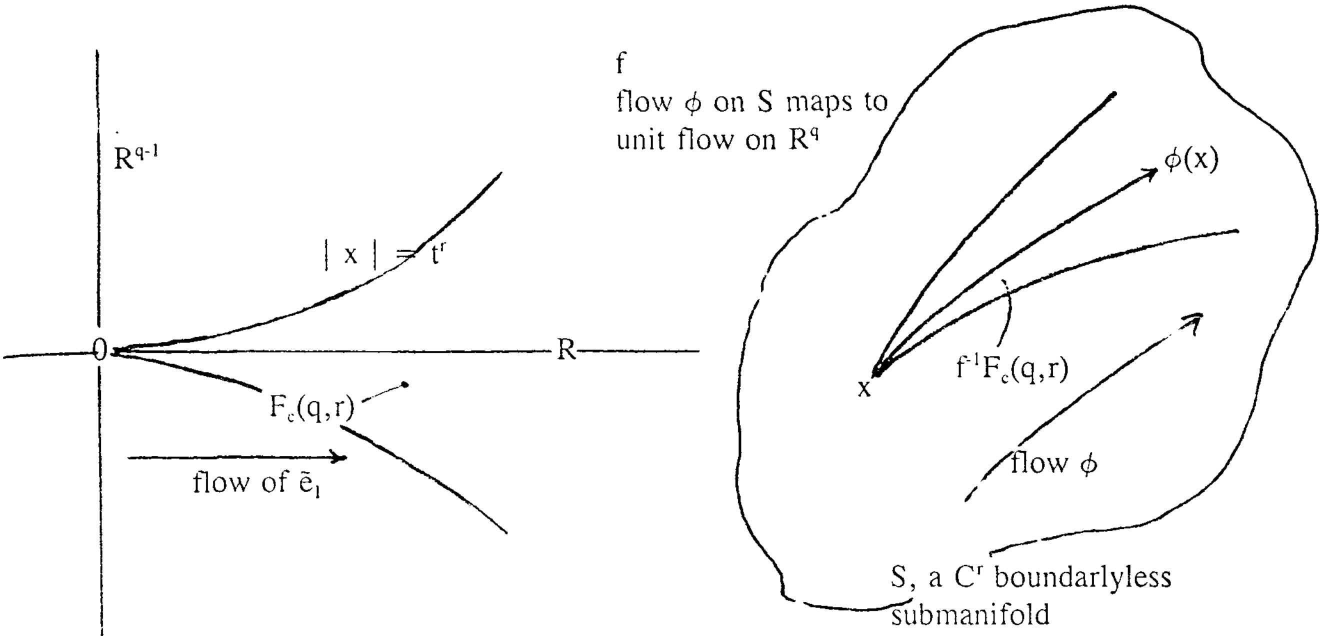


Figure 3.3

Lemma 3.1 If X is any smooth vector field on R^q with corresponding flow ϕ agreeing with the flow of the unit vector field $\tilde{e}_1$ above to infinite order at the origin (ie $D_t\phi(0,t=0)=e_1=(1,0)\in R\times R^{q-1}$, $D_t^i\phi(0,t=0)=(0,0)$ for all $i>1$) then for any $r\in Z^+$ there exists a neighbourhood U of the origin such that for all $x\in\partial F_c(q,r)\cap U$ X(x) points into $F_c(q,r)$ (Figure 3.4).

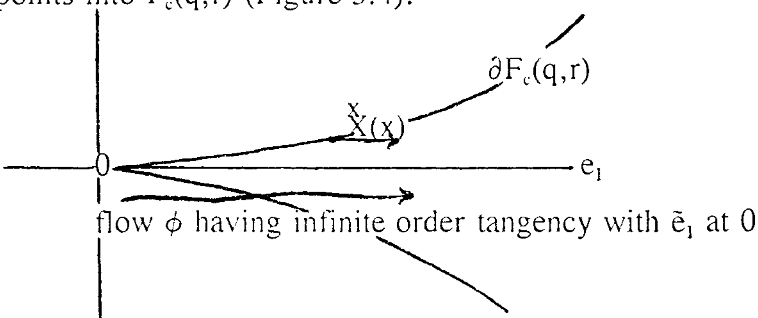


Figure 3.4

Proof We shall denote the set of unit vectors in R^{q-1} by S^{q-2}. Setting $\psi_\theta(t) = t^r\theta + te_1$ where $\theta\in S^{q-2}$ we have $\partial F_c(q,r)=\cup\{\psi_\theta(t):\theta\in S^{q-2},t\geq 0\}$ (see Figure 3.5).

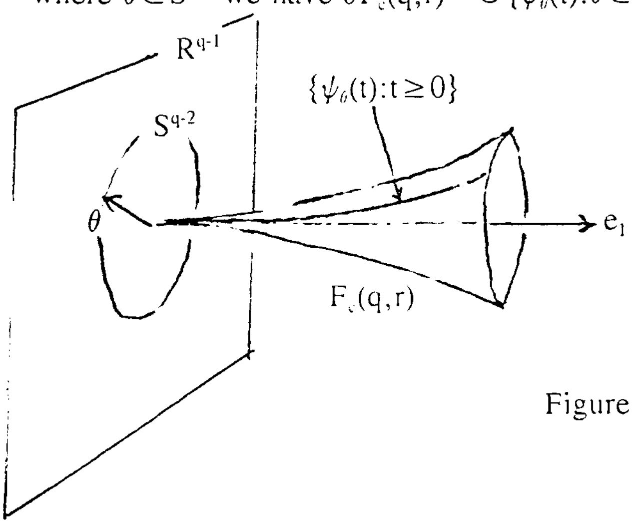


Figure 3.5

The tangent space to $\partial F_c(q,r)$at $\psi_\theta(t)$ has inward pointing normal $n(\psi_\theta(t))=t^{r-1}re_1-\theta$. Setting $g_\theta(t)=\langle n(\psi_\theta(t)),X(\psi_\theta(t))\rangle$ we seek $D_t^i g_\theta(t=0)$. From the definition of ψ_θ we have that $D_t^i\psi_\theta(t=0)=D_t^i\phi(0,t=0)$ if $i\leq r-1$, hence $D_t^iX\psi_\theta(t=0)=D_t^{i+1}\phi(0,0)$ if $i\leq r-1$, so

$$D_t^iX\psi_\theta(t=0)=\begin{cases} e_1 \text{ if } i=0 \\ 0 \text{ if } 1\leq i<r \end{cases}$$

$$\text{Thus}\quad D_t^i g_\theta(t=0)=\begin{cases} r!\,\langle e_1,e_1\rangle \text{ if } i=r-1 \\ 0 \text{ if } 0\leq i<r-1 \end{cases}$$

and hence $g_\theta(t)=t^{r-1}(r+(t/r!)D_t^r g_\theta(\mu t))$ some $\mu\in(0,1)$ by Taylor's Theorem, and if $M=\sup\{\ |\ D_t^r g_\theta(t))/r!r\ |\ :t\in[0,1],\theta\in S^{q-2}\}$ (which is finite for fixed $r>0$ by compactness of S^{q-2} and smoothness of the data) we have $g_\theta(t)>0$ for all θ if $0<t<\min(1,1/M)$, and setting $U=(0,\min(1,1/M))\times R^{q-1}$ (that is, $(0,\min(1,1/M))$ as an open subset of R) we get $\langle X(x),n(x)\rangle>0$ for all $x\in\partial F_c(q,r)\cap U\backslash\{0\}$, while at the origin $X(0)=e_1$. ∎

We will show (Lemma 3.2) that for large enough r the intersection of an r-funnel in $Z(S^0_\infty(x))$ about $\phi(S^0_\infty(x))(x)$ with M is , if x itself is deleted, disjoint from all strata $ZP(K;J\backslash K)$ such that $\phi(S^0_\infty(x))(x)$ is not infinitely tangent to $\phi(K)(x)$ at x, and furthermore is mapped into itself by the flow $\phi(M)$ near x and hence contains $\phi(M)(x)$ (see Figure 3.6).

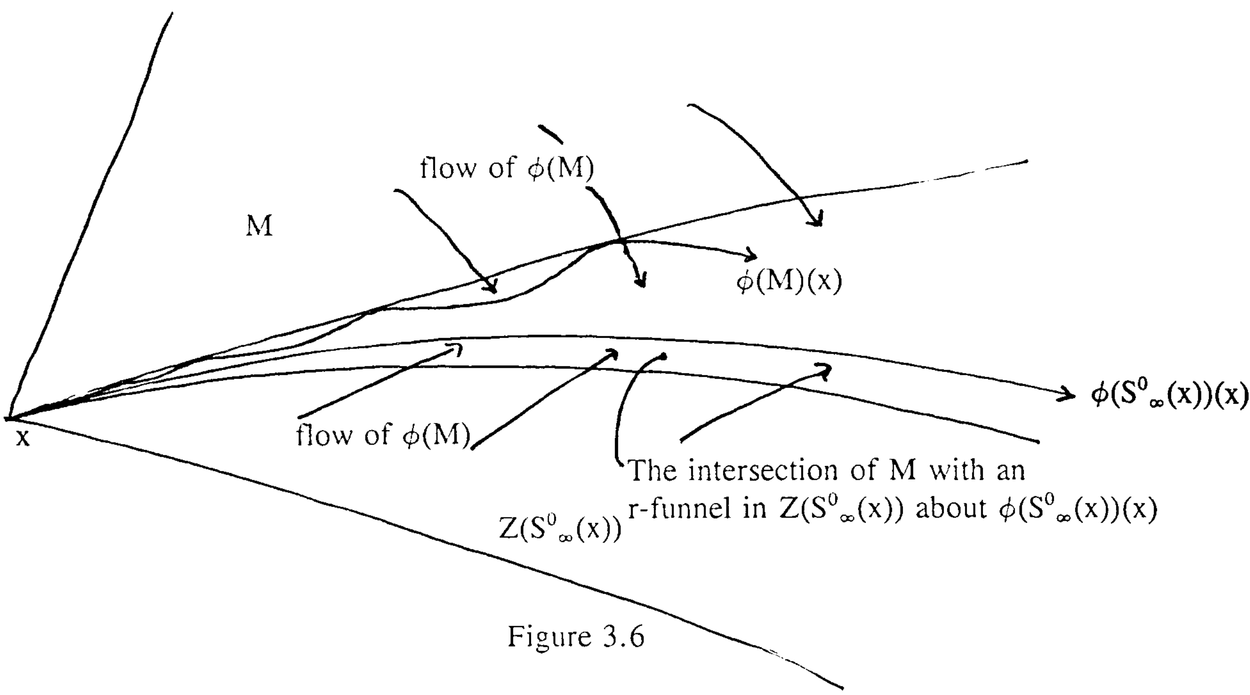


Figure 3.6

These funnels depend on x,M r,X, the straightening-out map f and $S^0_\infty(x,M,X)$, and will usually be denoted by $F_x(r,f)$, with a funnel satisfying the conclusions of Lemma 3.2 usually denoted by F_x (this amounts to choosing r large enough and a particular choice of f), and its intersection with $ZN(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x))$ by F_x'.

The reader may like to think of funnels in the following way. Classically, if X was a smooth vector-field on a smooth manifold the trajectory through a point x, $\phi(x)$, was smooth. Now if M is a smooth vector-field on a smooth submanifold with corners M $\phi(M)(x)$ is only guaranteed to be smooth on a right neighbourhood of t=0 if the conditions to apply Theorem 2.1 apply (viz that the iteration converges to a single set of indices). Otherwise we may have a situation such as illustrated in Example 2.1 where $\phi(M)(0)$ is not smooth on a right neighbourhood of t=0. Thus in general the best we can do (if smoothness is what we're after) is to replace "smooth $\phi(x)$" with "smooth F_x'", which contains the non-smooth $\phi(M)(x)$ and has various additional properties mentioned above and proved in Lemma 3.2 below. These properties of the funnels are the basis for the proof of Theorem 3.1.

Lemma 3.2 If M is a smooth submanifold with corners and X is a smooth vector field on M, with M near x locally represented as $ZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$, then there exists $r_0(x)\in Z^+$ such that for each $r\geq r_0$ there exists a neighbourhood U of x in R^n and a funnel $F_x=F(x,ZN(I;J),r,X,f,S^0_\infty(x))$ in $Z(S^0_\infty(x))$ of $X(S^0_\infty(x))$ about $\phi(S^0_\infty(x))(x)$ and corresponding closed subsets $F_x'=F_x\cap ZN(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x))$ such that

1. $F_x'\cap U\backslash\{x\}\subset ZNP(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x);S_1(x)\backslash S_\infty(x))$ (= a subcorner of $ZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$, so $F_x'\subset M$)
2. $X(M)(y)=X(K)(y)$ some not necessarily constant K with $S^0_\infty(x)\subset K\subset S_\infty(x)$ for all $y\in F_x'\cap U$
3. $X(K)(y)$ points into $\mathrm{int}F_x$ for all $y\in\partial F_x\cap U$ for all $S^0_\infty(x)\subset K\subset S_\infty(x)$
4. $\phi(M)(x)\cap U\subset F_x'\cap U$

Eg take $M=\{x:x_i\geq 0,i=1,2,3\}\subset R^3$, $X(x)=(1,f(x_1),-1)$ with f as in Example 2.1. We have then that $S^0_1(0)=\varnothing$ and $S_1(0)=(1,2,3)$, $S^0_i(0)=(3)$ and $S_i(0)=(1,3)$ for all $i\geq 2$. Then our funnels F_x are of the form $\{x\in R^3:x_3=0, |x_2|\leq x_1^r,x_1\geq 0\}$, and F_x' is of the form $F_x\cap\{x:x_2\geq 0\}$ (Figure 3.7).

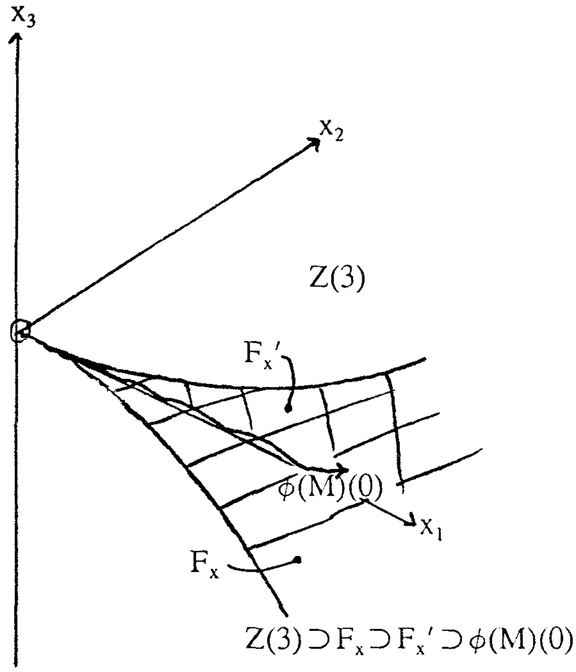


Figure 3.7

Alternatively take M as above and set $X(x)=(1,x_1,f(x_1))$; then $S^0_1(0)=\varnothing$, $S_1(0)=(1,2,3)$, $S^0_2(0)=\varnothing$, $S_2(0)=(1,3)$, $S^0_i(0)=\varnothing$, $S_i(0)=(3)$ for all $i \geq 3$. The funnels F_x are of the form $\{x \in R^3: |\sqrt{(x_2^2+x_3^2)}-\frac{1}{2}x_1^2| \leq x_1^r, x_1 \geq 0\}$, and F_x' is of the form $F_x \cap \{x: x_3 \geq 0\}$. $\phi(M)(0)$ is as in Figure 3.1, with the x-axis there mapped to the curve $\{x_3=0, x_2=\frac{1}{2}x_1^2\}$ and the y-axis pointing in the x_3-direction (Figure 3.8).

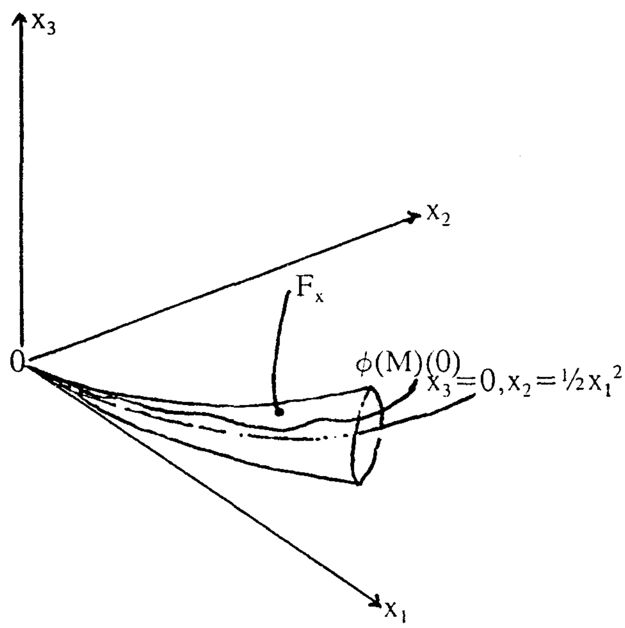


Figure 3.8

Proof The convention in this chapter (and in fact in Chapter Seven) will be that $\dim(Z(S^0_\infty(x)))=q$.

(a) We show that there exists $r_0 \in Z^+$ such that for some neighbourhood U of x a r_0-funnel $F_x(r_0,f)$ about $\phi(S^0_\infty(x))(x)$ in $Z(S^0_\infty(x))$ (and therefore $F_x(r,f)$ for any $r \geq r_0$) has the property that $F_x(r_0,f) \cap U\backslash\{x\} \subset ZP(S^0_\infty(x);S_1(x)\backslash S_\infty(x))$. We have defined $F_x'=F_x \cap ZN(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x))$ so if we show this then $F_x' \cap U\backslash\{x\} \subset ZNP(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x);S_1(x)\backslash S_\infty(x))$ which is part (1) of Lemma 3.2.

For each $j \in S_1(x)\backslash S_\infty(x)$ there exists $r(j) \in Z^+$ such that $j \in S_{r(j)}(x)\backslash S_{r(j)+1}(x)$ and by Lemma 2.5 we know that

$$D_t^i f_j \phi(S_\infty)(x,t=0) = \begin{cases} 0 \text{ for all } i<r(j) \\ k_j>0 \text{ if } i=r(j) \end{cases} .$$

We take $r_0=1+\max\{r(j):j \in S_1(x)\backslash S_\infty(x)\}$. Suppose we set $\psi_\theta'=f^{-1}\psi_\theta$ where f is the straightening out map used in the definition of funnel and ψ_θ is a curve in $\partial F_c(q,r_0)$, as in Lemma 3.1. Since by Corollary 2.1 $D_t^i\phi(S_\infty(x))(x,t=0) = D_t^i\phi(S^0_\infty(x))(x,t=0)$ for all i and by definitions $D_t^i\psi_\theta'(t=0)=D_t^i\phi(S^0_\infty(x))(x,t=0)$ for all $i<r_0$, it follows that

$$D_t^i f_j \psi_\theta'(t=0) = \begin{cases} 0 \text{ for all } i<r(j) \\ k_j>0 \text{ if } i=r(j) \end{cases} .$$

Hence $f_j\psi_\theta'(t)=t^{r(j)}((r(j)+1)k_j+tD_t^{r(j)+1}f_j\psi_\theta'(\mu t))/(r(j)+1)!)$ some $\mu \in (0,1)$, which is positive for $t \in (0,\delta(\theta)]$ say, and by evident continuity of $f_j\psi_\theta'$ in θ, $f_j\psi_{\theta'}'(t)>0$ if $t \in (0,\delta(\theta)]$ and $\theta' \in$ some neighbourhood of θ, U_θ. We get a covering of S^{q-2} by such U_θ's, and by compactness of S^{q-2} there exists a finite subcover $S^{q-2}=\cup\{U_\theta: \theta \in \Theta\}$ where Θ is a finite set in S^{q-2}. So $f_j\psi_\theta'(t)>0$ for all θ if $t \in (0,\min_{\theta\in\Theta}\delta(\theta)]$. Repeating for each $j \in S_1(x)\backslash S_\infty(x)$ we obtain a neighbourhood U of x in $Z(S^0_\infty(x))$ such that for all $y \in \partial F_x(r_0,f) \cap U\backslash\{x\}$ and for all $j \in S_1(x)\backslash S_\infty(x)$ $f_j(y)>0$, and since by definition $F_x(r_0,f) \subset Z(S^0_\infty(x))$ we have $F_x(r_0,f) \cap U\backslash\{x\} \subset ZP(S^0_\infty(X);S_1(x)\backslash S_\infty(x))$, and hence $F_x' \cap U\backslash\{x\} \subset ZNP(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x);S_1(x)\backslash S_\infty(x))$ as required for (1).

(b) We show that if X is a vector in R^n such that for some K with $I \subset K \subset I \cup J$ $\langle X,P(K\backslash j)n_j\rangle<0$ for all $j \in K\backslash I$ then there exists $\epsilon>0$ such that if $|\langle P(K)X,n_i\rangle| / |P(K\backslash j)X,n_j\rangle| <\epsilon$ for all $i \in J\backslash K$ and for all $j \in K\backslash I$ then $P(LC(I;J))X=X(H)$ some $K \subset H \subset I \cup J$.

(i) For any $X \in L(I)$ we have by Remark 2.1(1) that for some $\{\lambda_i,\epsilon_i\}$,

$X\text{-}X(I\cup J)=\Sigma_{i\in K\backslash I}\lambda_i P(I)n_i+\Sigma_{i\in J\backslash K}\epsilon_i P(I)n_i$. Then for any $j\in I\cup J$

$\langle P(K)(X\text{-}P(I\cup J)X),n_j\rangle=\Sigma_{i\in J\backslash K}\langle\epsilon_i P(K)n_i,n_j\rangle$, and since $I\subset K\subset I\cup J$ and $j\in I\cup J$ the left hand side is $\langle P(K)X,n_j\rangle\text{-}\langle P(I\cup J)X,n_j\rangle=\langle P(K)X,n_j\rangle\text{-}0$.

Hence if we set $N_{ij}=\langle P(K)n_i,P(K)n_j\rangle$, for $i,j\in J\backslash K$, N^{-1}=inverse of N in span$\{P(K)n_i;i\in J\backslash K\}$ (this inverse exists by Remark 2.1(2)), and $p_j=\langle P(K)X,n_j\rangle$, then $N\epsilon=p$ or $\epsilon=N^{-1}p$.

If $j\in K\backslash I$ then $\langle P(K\backslash j)X,n_j\rangle=\langle P(K\backslash j)X\text{-}P(I\cup J)X,n_j\rangle=\langle P(K\backslash j)(X\text{-}P(I\cup J)X),n_j\rangle$
$=\langle P(K\backslash j)(\Sigma_{i\in K\backslash I}\lambda_i n_i+\Sigma_{i\in J\backslash K}\epsilon_i n_i),n_j\rangle=\lambda_j\,|\,P(K\backslash j)n_j\,|\,^2+\Sigma_{i\in J\backslash K}\langle\epsilon_i P(K\backslash j)n_i,n_j\rangle$.

It is then straightforward to check that given any $\delta>0$ there exists $1/\epsilon$ so large (how large depends on the invertible matrix N) that if $\langle P(K\backslash j)X,n_j\rangle<0$ for all $j\in K\backslash I$ and $\langle P(K)X,n_i\rangle/\langle P(K\backslash j)X,n_j\rangle<\epsilon$ for all $j\in K\backslash I$ and for all $i\in J\backslash K$, then $\lambda_j<0$ and $|\,\epsilon_i\,|\,/\,|\,\lambda_j\,|\,<\delta$ for all such i,j and ϵ_i,λ_j as above.

(ii) We know eg by Lemma 1.2 that $P(LC(I;J))X=P(H)X$ some $I\subset H\subset I\cup J$. We must have $P(H)X\in LC(H;J\backslash H)$ so $\langle P(H)X,n_j\rangle\geq 0$ for all $j\in J\backslash H$, so

$\langle P(H)X,(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i\rangle\geq 0$ (where K is as above). But since $\langle P(H)P(I\cup J)X,n_i\rangle=\langle P(I\cup J)X,n_i\rangle=0$ if $i\in I\cup J$, and each $\lambda_i<0$, we have

$\langle P(H)X,\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i\rangle=\langle P(H)(X\text{-}P(I\cup J)X),\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i\rangle=$
$\langle P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i+\Sigma_{i\in J\backslash K}\epsilon_i n_i),\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i\rangle\ =$
$-\,|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,^2+\langle\Sigma_{i\in J\backslash K}\epsilon_i n_i,P(H)(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i)\rangle$.

We have $|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,=0$ iff $K\backslash H=\emptyset$, ie iff $K\subset H$. If $K\not\subset H$ then
$-\,|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,^2+\langle\Sigma_{i\in J\backslash K}\epsilon_i n_i,P(H)(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i)\rangle=$
$-\,|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,^2(1+\langle(\Sigma_{i\in J\backslash K}\epsilon_i n_i),P(H)(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i)\rangle/\,|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,^2)$.
$\langle\Sigma_{i\in J\backslash K}\epsilon_i n_i,P(H)(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i)\rangle/\,|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,^2$ is small by (i), and hence
$-\,|\,P(H)(\Sigma_{i\in K\backslash H}\lambda_i n_i)\,|\,^2+\langle\Sigma_{i\in J\backslash K}\epsilon_i n_i,P(H)(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i)\rangle<0$, contrary to $\langle P(H)X,(\Sigma_{i\in K\backslash H}\,|\,\lambda_i\,|\,n_i)\rangle\geq 0$. Hence we must have $H\supset K$ as claimed.

(c) We showed in (a) that there exists an integer r_0 such that for any $r\geq r_0$ there exists an r-funnel $F_x=F_x(r,f)$ about $\phi(S^0_\infty(x))(x)$ satisfying for some neighbourhood V of x $U\cap F_x(r,f)\backslash\{x\}\subset ZNP(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x);S_1(x)\backslash S_\infty(x))$. We now show (2), ie that for all $y\in F_x'\cap U$ $X(M)(y)=X(K)(y)$ some $S^0_\infty(x)\subset K\subset S_\infty(x)$. $F_x'\cap U\backslash\{x\}$ intersects strata $ZP(K;S_1(x)\backslash K)$ for all $S^0_\infty(x)\subset K\subset S_\infty(x)$, and taking $y\in ZP(K;S_1(x)\backslash K)$ we have $T_yM=T_yZN(S^0_1(x);K\backslash S^0_1(x))$ and we must show

$P(T_yZN(S^0_1(x);K\backslash S^0_1(x)))X(y)=X(K')(y)$ some $S^0_\infty(x)\subset K'\subset S_\infty(x)$. By (b) above it suffices to establish two conditions, that $\langle P(S^0_\infty(x)\backslash j)X(y),\text{grad}f_j(y)\rangle<0$ for all $j\in S^0_\infty(x)\backslash S^0_1(x)$, and that $\langle P(S^0_\infty(x))X(y),\text{grad}f_i(y)\rangle/\langle P(S^0_\infty(x)\backslash j)X(y),\text{grad}f_j(y)\rangle$ is arbitrarily small for all $i \in K' \setminus S^0_\infty(x)$ and for all $j \in S^0_\infty(x) \setminus S^0_1(x)$.

If $y \in F_x(r,f) \setminus \{x\}$ then we may write $y = \psi_b(t)$ where $b \in$ the closed ball $\bar{B}^{m-1}$ with $m = n - \dim Z(S^0_\infty(x))$, b being a parameter in the cross-section to $F_x(r,f)$ (with $b \mapsto \psi_b(t)$ continuous for each t) and $D_t^i \psi_b(t=0) = D_t^i \phi(S^0_\infty(x))(x,t=0)$ for all $i < r$ (see Figure 3.9).

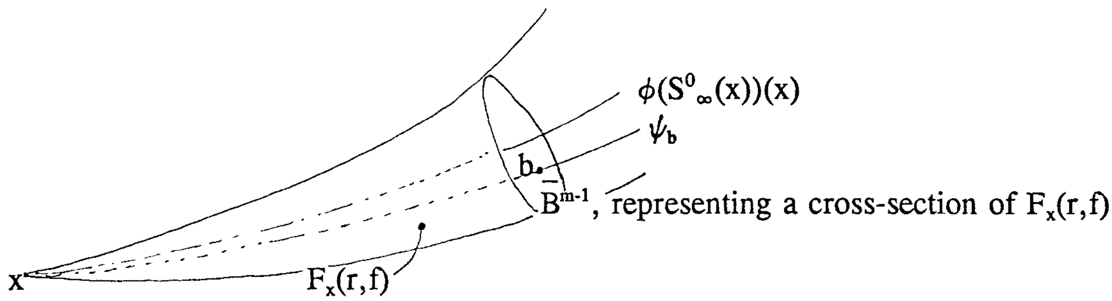


Figure 3.9

We know by Lemma 2.6 that for all K satisfying $S^0_\infty(x) \subset K \subset S_\infty(x)$ and $j \in S^0_\infty(x) \setminus S^0_1(x)$ that there exists $r(j) \in Z^+$ such that $j \in S^0_{r(j)+1}(x) \setminus S^0_{r(j)}(x)$, and setting $g_j(K)(t) = \langle X(K \setminus j)\phi(K)(x,t), \mathrm{grad} f_j \phi(K)(x,t) \rangle$ then

$$D_t^i g_j(K)(t=0) = \begin{cases} 0 \text{ for all } 0 \le i < r(j)-1 \\ k_j' < 0 \text{ if } i = r(j)-1 \end{cases}$$

If we now take $r \ge \max\{\{r_0 \text{ of part (a)}\}, \{r(j)-1 \text{ for } j \in S^0_\infty(x) \setminus S_\infty(x)\}\}$ (where r(j) is as in Lemma 2.6) then setting $h_j(b)(t) = \langle X(S^0_\infty(x) \setminus j)\psi_b(t), \mathrm{grad} f_j \psi_b(t) \rangle$ we have since $D_t^i \psi_b(t=0) = D_t^i \phi(S^0_\infty(x))(x,t=0)$ for all $i < r$ that

$$D_t^i h_j(b)(t=0) \begin{cases} =0 \text{ for all } 0 \le i < r(j)-1 \\ <0 \text{ if } i = r(j)-1 \end{cases}$$

and hence by continuity of $b \mapsto \psi_b$ that $h_j(b')(t) < 0$ for all $t \in (0, T(b)]$ and for all b' in some neighbourhood U_b of b in $\bar{B}^{m-1}$. By the same argument as in (a), we may use compactness of $\bar{B}^{m-1}$ to infer that there exists $T > 0$ such that $h_j(b)(t) < 0$ for all $t \in (0,T]$ and for all $b \in \bar{B}^{m-1}$, ie that there exists a neighbourhood U of x such that for all $y \in F_x(r,f) \cap U \setminus \{x\}$ and for all $j \in S^0_\infty(x) \setminus S^0_1(x)$ $\langle P(S^0_\infty(x) \setminus j)X(y), \mathrm{grad} f_j(y) \rangle < 0$.

Thus we have shown the first of the two conditions. For the second, we have ("theorem of indeterminate forms") that if $g,h: R \to R$ are smooth with $D_t^i g(t=0) = D_t^i h(t=0) = 0$ for all $i < r(j)-1$, and $D_t^{r(j)-1} g(t=0) = 0$, $D_t^{r(j)-1} h(t=0) < 0$, then $\lim_{t \to 0} g(t)/h(t) = 0$.

Thus if we set $g_j(b)(t)=\langle P(S^0_\infty(x))X\psi_b(t),\mathrm{grad}f_j\psi_b(t)\rangle$ for $j\in S_\infty(x)\backslash S^0_\infty(x)$, and $h_j(b)(t)=\langle P(S^0_\infty(x)\backslash j)X\psi_b(t),\mathrm{grad}f_j\psi_b(t)\rangle$ for $j\in S^0_\infty(x)\backslash S^0_1(x)$ (ie, $h_j(b)$ is exactly as above), then the second condition will follow if we can show that $D_t^i g_j(b)(t=0)=0$ for all $i<r(j)$, $D_t^i h_j(b)(t=0)=0$ for all $i<r(j)-1$, and $D_t^{r(j)-1}h_j(b)(t=0)<0$. Dealing first with the $g_j(b)$ term, we have by construction of ψ_b that for all $i<r(j)$
$D_t^i\langle P(S^0_\infty(x))X\psi_b(t=0),\mathrm{grad}f_j\psi_b(t=0)\rangle=$
$D_t^i\langle P(S^0_\infty(x))X\phi(S^0_\infty(x))(x,t=0),\mathrm{grad}f_j\phi(S^0_\infty(x))(x,t=0)\rangle$
$=D_t^{i+1}f_j\phi(S^0_\infty(x))(x,t=0)$, which$=0$ for all $i\geq 0$ and for all $j\in S_\infty(x)\backslash S^0_\infty(x)$ by Lemma 2.2 and Corollary 2.1. $h_j(b)(t)$ we have already dealt with above; we saw that setting $h_j(b)(t)=D_t^i\langle P(S^0_\infty(x)\backslash j)X\psi_b,\mathrm{grad}f_i\psi_b(t)\rangle$ then

$$D_t^i h_j(b)(t=0)\begin{cases}=0 & \text{for all } 0\leq i<r(j)-1\\ <0 & \text{if } i=r(j)-1\end{cases}$$

which is exactly the result we need.

Hence for all $j\in S_\infty(x)\backslash S^0_\infty(x)$ and for all $i\in S^0_\infty(x)\backslash S^0_1(x)$, for all sufficiently small $t>0$ $\langle P(S^0_\infty(x))X(\psi_b(t)),\mathrm{grad}f_i(\psi_b(t))\rangle/\langle P(S^0_\infty(x)\backslash j)X(\psi_b(t)),\mathrm{grad}f_j(\psi_b)(t))\rangle<0$, and hence as above for some neighbourhood U of x this quantity is <0 on $F_x(r,f)\cap U\backslash\{x\}$, which completes the proof of (2).

(d) For all $S^0_\infty(x)\subset K\subset S_\infty(x)$, $f_*X(KeS^0_\infty(x))$ is a smooth vector field on R^q (where f is our straightening-out map and $X(KeS^0_\infty(x))$ is as defined at the beginning of Chapter Two) with integral flow $f_*\phi(Ke\varnothing)$ (the integral flow of the push forward is the push forward of the integral flow - eg [1, Section 4.2]) and furtheremore by Corollary 2.1 $D_t^i f_*\phi(K_1e\varnothing)(x,t=0)=D_t^i f_*\phi(k_2e\varnothing)(x,t=0)$ for all $i\in Z^+$ and for any $S^0_\infty(x)\subset K_1,K_2\subset S_\infty(x)$, so by Lemma 3.1 there exists some neighbourhood U of the origin on which $f_*X(KeS^0_\infty(x))(x)$ points into $F_c(q,r)$ for all $x\in\partial F_c(q,r)$, and hence there exists a neighbourhood V of x in $Z(S^0_\infty(x))$ such that $X(K)(y)$ points into $F_x(r,f)$ for all $y\in\partial F_x(r,f)\cap V$, which is (3).

(e) (2) and (3) imply $F_x'\cap U$ is mapped into itself and we can then use continuous dependence on initial conditions (Theorem 1.1 part 3) to obtain (4). ∎

Proof of Theorem 3.1

(1) By absolute continuity of $\phi(M)(x)$ we have

$\phi(M)(x,h)-\phi(M)(x,0)=\int_0^h X(M)\phi(M)(x,s)ds$ and hence

$|\phi(M)(x,h)-\phi(M)(x,0)-hX(M)\phi(M)(x,0)| \leq$

$\int_0^h |X(M)\phi(M)(x,s)-X(M)\phi(M)(x,0)|\,ds$ (by [48, Chapter 1])

$\leq |h| \sup_{s\in[0,h]} |X(M)\phi(M)(x,s)-X(M)\phi(M)(x,0)|$ (also by [48, Chapter 1]).

By Lemma 3.2 part 4 $\phi(M)(x,s) \in F_x'$ for all sufficiently small $s \geq 0$, and by Lemma 3.2 part 2 for all such s $X(M)\phi(M)(x,s)=X(K)\phi(M)(x,s)$ some $S^0_\infty(x) \subset K \subset S_\infty(x)$. Hence since $X(M)(x)=X(K)(x)$ for all $S^0_\infty(x) \subset K \subset S_\infty(x)$ we must have $\sup\{|X(M)\phi(M)(x,s)-X(M)\phi(M)(x,0)| : s\in[0,h]\} \to 0$ as $h \downarrow 0$ hence $\lim_{h\downarrow 0} |\phi(M)(x,h)-\phi(M)(x,0)-hX(M)\phi(M)(x,0)| / |h| = 0$.

Hence $D_t^+\phi(M)(x,t=0)=X(M)\phi(M)(x,t=0)=X(K)\phi(M)(x,0)$ for any $S^0_\infty(x) \subset K \subset S_\infty(x)$, so by Corollary 2.1 $D_t^+\phi(M)(x,t=0)=X(K)\phi(M)(x,0)$ any $S^0_2(x) \subset K \subset S_2(x)$.

(2) Since for small $t>0$ $\phi(M)(x,t) \in F_x \cap U\backslash\{x\} \subset ZP(S^0_\infty(x),S_1(x)\backslash S_\infty(x))$ (see part (a) of the proof of Lemma 3.2) we have $S_1(\phi(M)(x,t)) \subset S_\infty(x)$ for small $t>0$. Returning to part (c) of the proof of Lemma 3.2 we see that at each point y of $F_x' \cap U$, and hence of $\phi(M)(x,t)$ for t small and >0, $\langle X(K\backslash j)(y), \mathrm{grad} f_j(y)\rangle < 0$ for all $j \in S^0_\infty(x)\backslash S^0_1(x)$ any $S^0_\infty(x) \subset K \subset S_\infty(x)$. Thus by part (c) of the proof of Lemma 3.2 this means (since M near y is represented as $ZN(S^0_1(x);S_\infty(x)\backslash S^0_1(x))$) that we cannot have $X(M)(y)=X(H)(y)$ any $H \subset S^0_\infty(x)$, hence $S^0_2(\phi(M)(x,t)) \supset S^0_\infty(x)$ for t small and positive. Thus for such t we have $S^0_\infty(x) \subset S^0_2(\phi(M)(x,t)) \subset S_1(\phi(M)(x,t)) \subset S_\infty(x)$ so by the construction of the iteration $S^0_1(x) \subset S^0_\infty(x) \subset S^0_2(\phi(M)(x,t)) \subset S^0_\infty(\phi(M)(x,t)) \subset$ $\subset S_\infty(\phi(M)(x,t)) \subset S_1(\phi(M)(x,t)) \subset S_\infty(x) \subset S_1(x)$, for all sufficiently small $t>0$ and for all $x \in M$.

(3) We take as inductive hypothesis that for all $i<k$ and for all $x \in M$ $D_t^{+i}\phi(M)(x,t=0)$ exists, and that $D_t^{+i}\phi(M)(x,t=0)=D_t^i\phi(I)(x,t=0)$ any $S^0_\infty(x) \subset I \subset S_\infty(x)$ (and hence by Corollary 2.1 for any I such that $S^0_{i+1}(x) \subset I \subset S_{i+1}(x)$).

The inductive hypothesis is true if $k=2$ by (1). We have by definition

$$D_t^{+k}\phi(M)(x,t=0)=\lim_{h\downarrow 0}(D_t^{+(k-1)}\phi(M)(x,h)-D_t^{+(k-1)}\phi(M)(x,0))/h \qquad (**)$$

if the right hand side exists, and by the inductive hypothesis $D_t^{+(k-1)}\phi(M)(x,h)=D_t^{k-1}\phi(I)(\phi(M)(x,h),t=0)$ any $S^0_\infty(\phi(M)(x,h)) \subset I \subset S_\infty(\phi(M)(x,h))$, and so by (2) $D_t^{+(k-1)}\phi(M)(x,h)=D_t^{k-1}\phi(I)(\phi(M)(x,h),t=0)$ some $S^0_\infty(x) \subset I \subset S_\infty(x)$. Thus for each small $h>0$ we may select a set of indices $I(h)$ where $S^0_\infty(x) \subset I(h) \subset S_\infty(x)$ such that $D_t^{+(k-1)}\phi(M)(x,h)=D_t^{k-1}\phi(I(h))(\phi(M)(x,h),t=0)$.

$$\text{Setting } \delta_i(h)=\begin{cases} 1 \text{ if } I(h)=I_i \\ 0 \text{ otherwise}\end{cases}$$

where each I_i is one of the $2^{|S_\infty(x)\setminus S^0_\infty(x)|}$ set of indices lying in the range $S^0_\infty(x)\subset I_i\subset S_\infty(x)$, then since $D_t^{+(k-1)}\phi(M)(x,t=0)=D_t^{k-1}\phi(I)(x,t=0)$ for all $S^0_\infty(x)\subset I\subset S_\infty(x)$ the right hand side of (**) is

$\lim_{h\downarrow 0}\Sigma_i(\delta_i(h)(D_t^{k-1}\phi(I_i e\varnothing)(\phi(M)(x,h),t=0)-D_t^{k-1}\phi(I_i)(x,t=0))/h$. (*)

(In this formula we need $\phi(I_i e\varnothing)$ rather than $\phi(I_i)$ because $\phi(I_i)(\phi(M)(x,h))$ is only defined if $\phi(M)(x,h)\in Z(I_i)$. Of course, if $\phi(M)(x,h)\notin Z(I_i)$ then $\delta_i(h)=0$, so the need for $\phi(I_i e\varnothing)$ is purely formal).

(4) If Y is any smooth vector field, ϕ any smooth flow, set $Y^i(x)=D_t^{i-1}Y\phi(x,t=0)$. We show $\lim_{h\downarrow 0}(Y^i(\phi(x,h))-Y^i(\phi(x,0)))/h = Y^{i+1}(x)$. $Y^i(\phi(x,h))=D_t^{i-1}Y\phi(\phi(x,h),t=0)$, but $\phi(\phi(x,h),t)=\phi(x,t+h)$ so $Y^i(\phi(x,h))=D_t^{i-1}Y\phi(x,t+h)|_{t=0}$. Hence $\lim_{h\downarrow 0}(Y^i(\phi(x,h))-Y^i(\phi(x,0)))/h =$
$\lim_{h\downarrow 0}(D_t^{i-1}Y(\phi(x,t+h))|_{t=0}-D_t^{i-1}Y(\phi(x,t))|_{t=0})/h = D_hD_t^{i-1}Y\phi(x,t+h)|_{t=h=0} =$
$D_t^iY\phi(x,t=0)=Y^{i+1}(x)$.

(5) We show that if Y is any C^1 vector field, ϕ a C^0 right differentiable function $\phi:[0,T)\to R^n$ with $|D_t^+\phi|$ bounded on compact intervals, then $\lim_{h\downarrow 0}(Y\phi(h)-Y\phi(0))/h = Y'(\phi(0))D_t^+\phi(0)$.

Proof- by C^1-ness of Y $|Y(\phi(h))-Y(\phi(0))-Y'(x)(\phi(h)-\phi(0))|=k|\phi(h)-\phi(0)|$ where $k\to 0$ as $|\phi(h)-\phi(0)|\to 0$. Taking $\sup_{t\in[0,h]}|D_t^+\phi(t)|=M$ we have by the right sided Mean Value Theorem ([14, Chapter 8.5, problem 2]) that $|\phi(h)-\phi(0)|\leq Mh$ and hence by continuity of ϕ that $|Y\phi(h)-Y(\phi(0))-Y'(\phi(0))(\phi(h)-\phi(0))|\leq khM$ where $k\to 0$ as $h\to 0$, hence result.

(6) Set $D_t^i\phi(Ie\varnothing)(y,t=0)=X_I^i(y)=D_t^{i-1}X(Ie\varnothing)\phi(Ie\varnothing)(y,t=0)$ giving us fields X_I^i on a neighbourhood of x in R^n. Consider

$\lim_{h\downarrow 0}(D_t^{k-1}\phi(I_i e\varnothing)(\phi(M)(x,h),t=0)-D_t^{k-1}\phi(I_i)(x,t=0))/h$
$=\lim_{h\downarrow 0}(X_{I_i}^{k-1}(\phi(M)(x,h))-X_{I_i}^{k-1}(x))/h = X_{I_i}^{k-1\prime}(x)D_t^+\phi(M)(x,t=0)$ by (5). But since by (1) $D_t^+\phi(M)(x,t=0)=D_t\phi(I_i)(x,t=0)$ the above $= X_{I_i}^{k-1\prime}(\phi(I_i)(x,0))D_t\phi(I_i)(x,t=0) =$
$\lim_{h\downarrow 0}(X_{I_i}^{k-1}(\phi(I_i e\varnothing)(x,h))-X_{I_i}^{k-1}(\phi(I_i)(x,0)))/h$ (by (5) backwards) $=X_{I_i}^k(x)$ by (4).

(7) If $\delta_i(h)$ is as defined in (3) and $f_i(h)\to f(h)$ independent of i as $h\downarrow 0$, then $(\Sigma\delta_i(h)f_i(h))-f(h)=\Sigma\delta_i(h)(f_i(h)-f(h))\to 0$ as $h\downarrow 0$. Using this with $f_i(h)=(D_t^{k-1}\phi(I_i e\varnothing)(\phi(M)(x,h),0)-D_t^{k-1}\phi(I_i e\varnothing)(\phi(M)(x,t),0))/h$ (where I_i is as defined in

(3)) which by (6) tends to $D_t^k\phi(I_i)(x,t=0)$ as $h\downarrow 0$, and using that by definition of I_i and Corollary 2.1 $D_t^k\phi(I_i)(x,t=0)=D_t^k(K)(x,t=0)$ for all $S^0_\infty(x)\subset K\subset S_\infty(x)$, we see that $(*)= D_t^k\phi(K)(x,t=0)$ any $S^0_\infty(x)\subset K\subset S_\infty(x)$, and so by induction Theorem 3.1 is true for all k. ∎

Remarks 3.1

(1) Theorem 3.1 part 1 is true so long as X is C^1. Since an absolutely continuous function is a.e. differentiable ([48, Chapter 7] or [4, Section 0]) the following are equivalent definitions (this remark gives (b)→(a), (a)→(b) by Theorem 3.1) of $\phi(M)(x)$ if X is C^1:

(a) $\phi(M)(x):[0,t_x)\to M$ is an absolutely continuous function such that $D_t\phi(M)(x,t)=X(M)\phi(M)(x,t)$ a.e. on $[0,t_x)$

(b) $\phi(M)(x):[0,t_x)\to M$ is an absolutely continuous function such that $D_t^+\phi(M)(x,t)=X(M)\phi(M)(x,t)$ everywhere on $[0,t_x)$.

This second definition defines a trajectory by a property holding at every point of $[0,t_x)$ and given the one-sided "semi" nature of everything in the subject has much to commend it.

(2) If $\phi(M)(x,t)\in ZP(K;J\setminus K)$ for $t\in(0,h)$ we have $D_t^+f_k\phi(M)(x,t=0)=0$ for all $k\in K$ for all $t\in(0,h)$, ie $\langle \text{grad} f_k\phi(M)(x,t),X(M)\phi(M)(x,t)\rangle=0$ for all $k\in K$ and for all $t\in(0,h)$, and since $\phi(M)(x,t)\in ZP(K;J\setminus K)$, so $X(M)\phi(M)(x,t)=X(K')\phi(M)(x,t)$ some $K'\subset K$, we have $X(M)\phi(M)(x,t)=X(K)\phi(M)(x,t)$ for all $t\in(0,h)$. Furthermore by Theorem 3.1 $X(M)(x)=\lim_{t\downarrow 0}X(M)\phi(M)(x,t)$ so we have $X(M)\phi(M)(x,t)=X(K)\phi(M)(x,t)$ for all $t\in[0,h)$. Thus by uniqueness of integral curves (Theorem 1.1(1)) $\phi(M)(x,t)=\phi(K)(x,t)$ for all $t\in[0,h)$.

(3) If we set $\Im_i(x)=\{K:S^0_i(x)\subset K\subset S_i(x)\}$ we have from the construction of the iteration that $\Im_i(x)\supset\Im_j(x)$ for all $j\geq i$. At present $S^0_1(x)$ is merely the set of indices defining the manifold on which X is defined (see the preamble on the iteration in Chapter 2) and might as well be written S^0_1 since it is independent of x. As far as the local trajectory $\phi(M)(x,t)$ is concerned the vector field might though as well have been defined only on $\lim_{h\downarrow 0}S^0_2(\phi(M)(x,t))$ since it is (by Lemma 3.2) contained entirely within this stratum; if we replace the old S^0_1 by $S^0_1(x)=\lim_{h\downarrow 0}S^0_2(\phi(M)(x,t))$ (which certainly does depend on x) we get from part 2 of the proof of Theorem 3.1 $\Im_1(\phi(M)(x,t))\subset\Im_\infty(x)$ if $t>0$ is sufficiently small, which combined with the iteration property (ie that $S^0_j(x)\subset S^0_{j+1}(x)\subset S_{j+1}(x)\subset S_j(x)$ for all $j\geq 1$) gives for all $x\in M$

$\Im_i(x) \supset \Im_j(x) \supset \Im_k(\phi(M)(x,t))$

for all $j \geq i \in Z^+$ for all $k \in Z^+$ and for all sufficiently small $t>0$.

Unless otherwise stated it is though convenient to remain with the original definition of $S^0_1(x)$, $S_1(x)$ - ie, if M near x is locally ZN(I;J), $S^0_1(x)=I$, $S_1(x)=I \cup J$ - in which case the first of the two relations ($\Im_i(x) \supset \Im_j(x)$ for all $j \geq i$) is unaffected, the second is $S^0_i(x) \subset S^0_2(\phi(M)(x,t)) \subset S_1(\phi(M)(x,t)) \subset S_j(x)$ for all $x \in M$, for all $i,j \in Z^+$, for all sufficiently small $t>0$.

(4) We might at this point mention two publications which we have been unable to obtain:

(i) A paper by M.G. Chikin subsequent to [10]: *On The Existence of a Right Derivative in the Solutions of a class of Discontinuous Systems*, 1988 (in Russian), (ii) The thesis of B. Cornet: *Contributions à la theorie des mécanisms dynamiques d'allocation des resources*, Université Paris IX Dauphine, 1981.

It seems likely that in (i) Chikin has proved a version of the part of Theorem 3.1 which says that $D^+\phi(M)(x,t=0)=X(M)\phi(M)(x,t=0)$, and (ii) may contain variants on the existence-uniqueness result in [12].

Remark 3.2 This chapter provides us with a perspective on the iteration as a "selecting" process. At the outset we have $x \in M$, M locally represented as $ZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$, so we know that for small $t>0$ $\phi(M)(x,t) \in ZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$. Lemma 3.2(4) tells us that for small $t>0$ $\phi(M)(x,t) \in ZNP(S^0_\infty(x);S_\infty(x)\backslash S^0_\infty(x);S_1(x)\backslash S_\infty(x))$. We recall that $ZNP(S^0_i(x);S_i(x)\backslash S^0_i(x);S_1(x)\backslash S_i(x)) = \cup_{S^0_i(x) \subset K \subset S_i(x)} ZP(K;S_1(x)\backslash K)$ and so $ZNP(S^0_{i'}(x);S_{i'}(x)\backslash S^0_{i'}(x);S_1(x)\backslash S_{i'}(x)) \subset ZNP(S^0_i(x);S_i(x)\backslash S^0_i(x);S_1(x)\backslash S_i(x))$ for all $i' \geq i$. We know therefore that $\phi(M)(x,t) \subset ZNP(S^0_i(x);S_i(x)\backslash S^0_i(x);S_1(x)\backslash S_i(x))$ for all i. at the (i+1)th stage of the iteration we have therefore a better knowledge of where $\phi(M)(x)$ lies than at the ith.

The ordering we obtained in Remark 3.1(3) tells us that for any $j \geq 0$ that for all sufficiently small $t>0$ $D_t^{+j}\phi(M)(x,t)=D_s^j\phi(K)(\phi(M)(x,t),s=0)$ some $S^0_\infty(x) \subset K \subset S_\infty(x)$ (so since $S^0_i(x) \subset S^0_\infty(x) \subset S_\infty(x) \subset S_i(x)$ we know at the ith stage that the K in this expression lies in the range $S^0_i(x) \subset K \subset S_i(x)$), so the "selecting" is for all the right hand derivatives of $\phi(M)$ on a right neighbourhood of $t=0$.

If $S^0_\infty(x)=S_\infty(x)$ Lemma 3.2(4) tells us $\phi(M)(x,t) \subset ZNP(S^0_\infty(x);\varnothing;S_1(x)\backslash S_\infty(x))$ from which (eg using Remark 3.1(2)) we could recover Theorem 2.1(1).

Chapter Four

Tangencies, The Iteration, and a Refined Iteration

In this chapter we conclude (excepting Corollary 5.2) our study of the iteration. We shall establish the relationship between the iteration and suitably generalised versions of the classical tangency sets, establish the essential properties of the latter, and consider also a generalisation (in fact a refinement) of the iteration which is better suited to local questions.

Tangency Sets

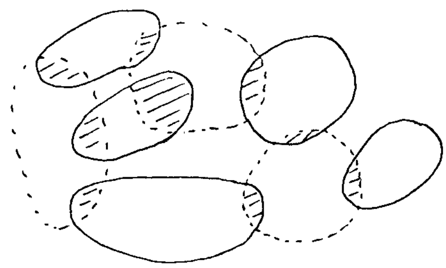
If V_1 is a C^r submanifold of V_2 which is a C^r submanifold of R^n, and if X is a C^r vector field on R^n we know by Lemma 2.1 that we can project X onto V_i to form C^r vector fields $X(V_i)$ and so integrate these vector fields to obtain C^r flows $\phi(V_i)$. We shall set $\Gamma_k^X(V_1 \text{ relative to } V_2)=\{x\in V_1: D_t^i\phi(V_1)(x,t=0)=D_t^i\phi(V_2)(x,t=0) \text{ for all } i<k\}$. We have seen how to represent any submanifold with corners M near any $x\in M$ locally as ZN(I;J); there are $3^{|J|}$ pairs of sets of indices K_1,K_2 with $I\subset K_2\subset K_1\subset I\cup J$ and for each we set $\Gamma_k(K_1 \text{ r } K_2)=\Gamma_k(Z(K_1) \text{ relative to } Z(K_2))=\{x\in Z(K_1): D_t^i\phi(K_1)(x,t=0)=D_t^i\phi(K_2)(x,t=0) \text{ for all } i<k\}$. For example, if $x\in Z(K_1)\subset Z(K_2)$ then $x\in\Gamma_2(K_1 \text{ r } K_2)$ iff $X(K_2)(x)=X(K_1)(x)$ iff $X(K_2)(x)\in T_xZ(K_1)$. We observe that while classically tangency sets were defined with generic restrictions on X our definitions are for any smooth vector field.

It will be evident that these sets are intimately bound up with the detailed behaviour of the semiflow $\phi(M)$ in relation to the strata $ZP(K;J\backslash K)$. In the first place it will seem likely, in view of the construction of the iteration, that they relate to the iteration in a significant way; we obtain in Proposition 4.4 a formula which expresses the subsets ("iteration sets") of M where the iteration achieves a particular value (ie, a particular contracting sequence of sets of indices) in terms of intersections of these tangency sets. Because of this formula and for other reasons we are interested in the intersection of these sets. It is straightforward to check that, for example, if Z(1), Z(2) are hypersurfaces in R^n then $\Gamma_2((1,2) \text{ r } \emptyset)=\Gamma_2((1) \text{ r } \emptyset)\cap\Gamma_2((1,2) \text{ r } (1))$ and that also $\Gamma_2((1,2) \text{ r } \emptyset)=\Gamma_2((1,2) \text{ r } (1))\cap\Gamma_2((1,2) \text{ r } (2))$. In general expressions involving Γ-sets cannot be simplified to a single term but can be simplified to some extent: we seek a canonical "simplification" which will also tell us when they are equal. This is achieved in Proposition 4.1. Proposition 4.2 is concerned with the intersection properties of these tangency sets when X satisfies certain generic conditions.

The iteration at a point does not in itself determine the local behaviour of the semiflow (eg Figure 4.5) and we consider a refinement of the iteration which comes closer to doing so. We show in Proposition 4.3 that this refinement of the iteration (and hence a fortiori the iteration itself, and hence the iteration sets) is, unlike the tangency sets (Example 4.2) preserved by a semiflow preserving diffeomorphism.

The reason for the following discussion will become evident when we state Proposition 4.1 below. We recall from Chapter One that for $I\subset K\subset K\cup L\subset I\cup J$ we have defined a subcorner of ZN(I;J) to be a set of the form $ZNP(K;L;I\cup J\backslash(K\cup L))=\{x\in R^n:\langle x,n_i\rangle=0\ \forall i\in K,\langle x,n_i\rangle\geq 0\ \forall i\in L,\langle x,n_i\rangle>0\ \forall i\in I\cup J\backslash(K\cup L)\}$ which is evidently contained in ZN(I;J) and decomposes as $\cup_{\varnothing\subset M\subset L}\{x\in R^n:\langle x,n_i\rangle=0\ \forall i\in K\cup M,\langle x,n_i\rangle>0\ \forall i\in I\cup J\cup(L\backslash M)\backslash(K\cup L)\}=$ $\cup_{K\subset H\subset K\cup L}ZP(H;I\cup J\backslash H)$. It follows from this decomposition that if the subcorner $ZNP(K;L;I\cup J\backslash(K\cup L))$ is denoted s.c.(K;K∪L) the intersection of two subcorners $s.c.(K_i;K_i\cup L_i)$, $i=1,2$, is the subcorner $s.c.(K_1\cup K_2;(K_1\cup L_1)\cap(K_2\cup L_2))$.

If $\{ZP(I_i;I\cup J\backslash I_i)\}$ is a set of strata of ZN(I;J) the convex hull of the set is defined as the intersection of all subcorners each containing all the strata. Since for any such subcorner $ZNP(K;L;I\cup J\backslash(K\cup L))=\cup_{K\subset H\subset K\cup L}ZP(H;J\backslash H)$ we must have $K\subset I_i,K\cup L\supset J\backslash I_i$ for all i, it equals $ZNP(\cap I_i;\cup I_i\backslash\cap I_i;I\cup J\backslash\cup I_i)$. If $\{c_i\}$ is a set of non-intersecting subcorners of ZN(I;J) and $\{c_i^j\}_{i=1..I(j),j=1..k}$ is a collection of such sets of non-intersecting subcorners we may define the interior intersection of the collection by $\{c_{i(1)}{}^1\cap..\cap c_{i(k)}{}^k:i(s)\in(1..I(s))\text{ each }s=1,..,k\}$. If $(c_{i(1)}{}^1\cap..\cap c_{i(k)}{}^k\cap c_{i(1)'}{}^1\cap..\cap c_{i(k)'}{}^k)\neq\varnothing$ then $i(j)=i(j)'\ \forall j\in(1..k)$ because $c_{i(j)}{}^j\cap c_{i(j)'}{}^j=\varnothing$ if $i(j)\neq i(j)'$; since (as noted above) the intersection of finitely many subcorners is a subcorner we therefore have that the interior intersection of the collection is itself a set of non-intersecting subcorners of ZN(I;J). If $\{\sigma_i^j\}$ is a collection of sets of strata of ZN(I;J) there exists at least one set of non-intersecting subcorners of ZN(I;J) with the property that each set of strata in the collection is contained in a single subcorner, namely the set with one element, ZN(I;J). We define the subcorner decomposition of a collection of sets of strata to be the interior intersection of the collection of sets of non-intersecting subcorners such that each set of non-intersecting subcorners in the collection has this property.



Schematic representation of the interior intersection of a collection of sets of non-intersecting subcorners. We have here k=2; the set j=1 consists of 3 subcorners (regions bounded by dotted curves), the set j=2 of 5 subcorners (regions bounded by full curves). The interior intersection of the collection is shaded.

Diagramatically we may represent a closed corner ZN(I;J) where $J=(j_1,..,j_k)$ by the following table

row 0 $\quad I$

$I\cup j_1 \; . \quad . \quad . \quad . \quad . \; I\cup j_k$

$I\cup j_1\cup j_2 \quad I\cup j_1\cup j_3. \quad . \quad I\cup j_{k-1}\cup j_k$

$. \quad . \quad . \quad .$

row $|J|$ $\quad I\cup J$

where the mth row consists of $\binom{|J|}{m}$ sets of indices, the set of indices K for any $I\subset K\subset I\cup J$ representing the stratum ZP(K;$I\cup J\setminus K$) (Figure 4.1).

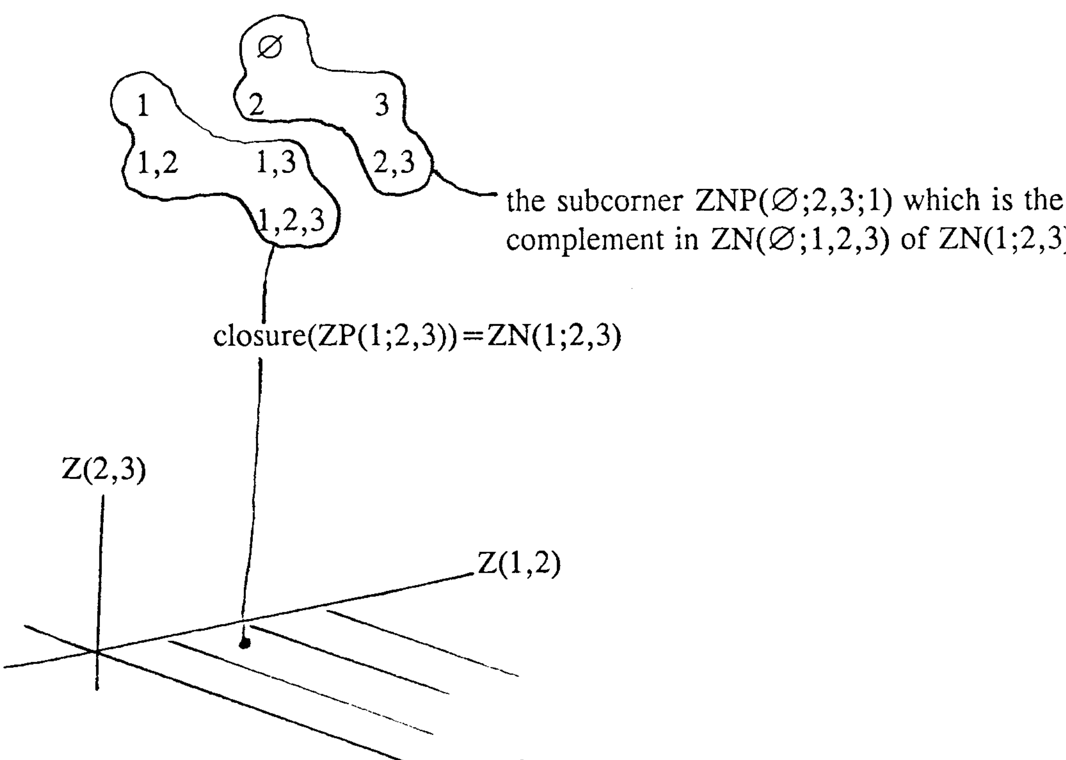


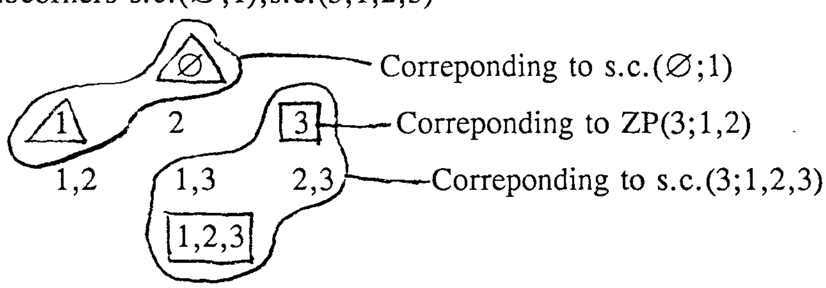
Figure 4.1. Representing a corner ZN(I;J) by a diagram (here I=∅,J=(1,2,3)). ZN(1;2,3) corresponds to the 4 sets of indices indicated because ZN(1;2,3)= ZP(1,2,3;∅)∪ZP(1,2;3)∪ZP(1,3;2)∪ZP(1;2,3) and each ZP(K;(1,2,3)\K) is represented by K; similarly ZNP(∅;2,3;1) is represented by the 4 sets of indices indicated because ZNP(∅;2,3;1)=ZP(∅;1,2,3)∪ZP(2;1,3)∪ZP(1;2,3)∪ZP(3;1,2).

The stratum closure ZN(K;I∪J\K) = $\cup_{K\subset H\subset I\cup J}$ZP(H;I∪J\H) is represented by the set of sets of indices in the diagram containing K; a subcorner ZNP(K;L;I∪J\(K∪L)) by those sets of indices which contain K and are contained in K∪L - see Figure 4.1 above for ZN(∅;1,2,3).

Intuitively the convex hull of a set of strata is the smallest subcorner containing all of them and the subcorner decomposition of a collection of sets of strata is the smallest non-intersecting set of smallest subcorners satisfying the property that each set of strata is contained in a single subcorner.

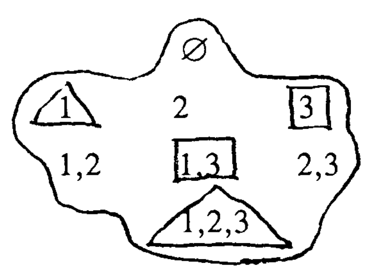
Examples 4.1 Consider the corner ZN(∅;1,2,3). The convex hull of {ZP(∅;1,2,3),ZP(1,2;3)} is the subcorner s.c.(∅;1,2) and of {ZP(1;2,3),ZP(2,3;1)} is the whole corner corresponding to the set of all eight sets of indices;

(a) The subcorner decomposition of {{ZP(∅;1,2,3),ZP(1;2,3)},{ZP(3;1,2),ZP(1,2,3;∅)}} is the non-intersecting pair of subcorners s.c.(∅;1),s.c.(3;1,2,3) -



The strata of one set are triangled and those of the other are squared; the subcorners of the subcorner decomposition for these sets are the two sets of strata within smooth curves.

(b) The subcorner decomposition of {{ZP(1;2,3),ZP(1,2,3;∅)},{ZP(1,3;2),ZP(3;1,2)}} is the whole corner of 8 sets of indices.

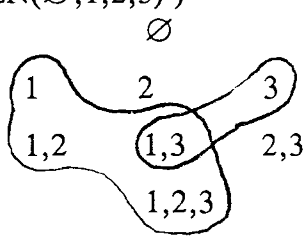


We remind ourselves of our definition ot tangency set, which extends the classical notion of tangency set (see eg [44,45,51,58]) which dealt with a single vector field and was always accompanied by generic restrictions:

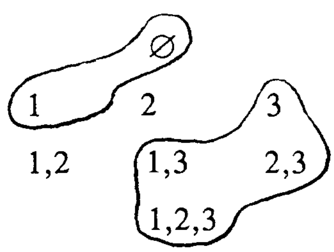
Definition If functions $(f_1,..,f_k;f_{k+1},..,f_{k+m})$ are independent and X is a smooth vector field on R^n, then setting $I=(1,..,k)$ and $J=(k+1,..,k+m)$ we define $\Gamma_k^X(I\cup J \text{ r } J)=\{x\in Z(I\cup J):D_t^i\phi(I)(x,t=0)=D_t^i\phi(I\cup J)(x,t=0)\ \forall i<k\}$ where $\phi(I)$ is the integral flow of X(I) etc; we may abbreviate $\Gamma_k^X(I\cup J \text{ r } J)$ to $\Gamma_k(I\cup J \text{ r } J)$.
The reason for making the above constructions (the subcorner decompositions etc) is the following result, which is the canonical "simplification" of multiple intersections of (generalised) tangency sets alluded to above:

Proposition 4.1 If $I\subset I_i\subset I_i\cup J_i\subset I\cup J$, $i=1,..,m$, and $\Gamma_k(I_i\cup J_i \text{ r } I_i)$ is as defined above, then if $\{s.c.(K^0_i;K_i):i=1,..,r\}$ = the subcorner decomposition of $\{s.c.(I_i;I_i\cup J_i):i=1,..,m\}$ we have $\cap_{i=1,..,m}\Gamma_k(I_i\cup J_i \text{ r } I_i)= \cap_{i=1,..,r}\Gamma_k(K_i \text{ r } K^0_i)$.
We observe that by definition of subcorner decomposition the subcorners $\{s.c.(K^0_i;K_i):i=1,..,r\}$ are disjoint.
Proof after lemma 4.3

Examples
Since the subcorner decomposition of {s.c.(1;1,2,3),s.c.(3;1,3)} is {s.c.(∅;1,2,3)} (=ZN(∅;1,2,3)) -



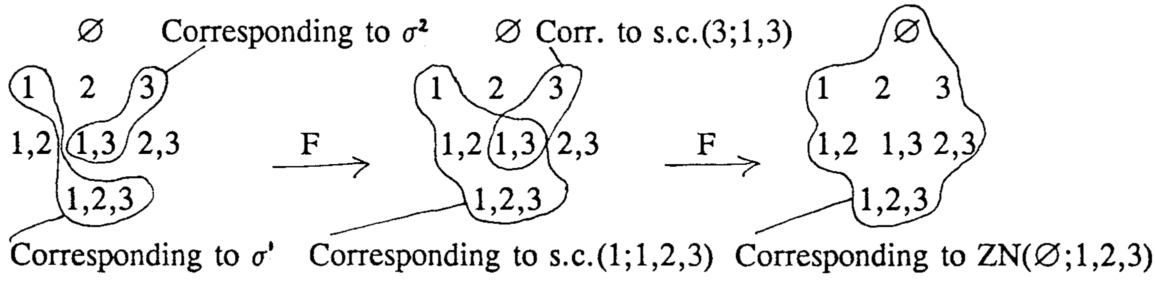
we have for all k that $\Gamma_k((1,2,3) \text{ r } (1))\cap\Gamma_k((1,3) \text{ r } (3))=\Gamma_k((1,2,3) \text{ r } \varnothing)$.
Since the subcorner decomposition of {s.c.(∅;1),s.c.(3;1,2,3)} is itself-



$\Gamma_k(1 \text{ r } \varnothing)\cap\Gamma_k((1,2,3) \text{ r } 3)$ does not simplify.
Corollary 4.1 As special cases of Proposition 4.1 we have
(1) $\cap_{j\in J}\Gamma_k(I\cup j \text{ r } I)=\Gamma_k(I\cup J \text{ r } I)$
(2) $\cap_{j\in J}\Gamma_k(I\cup J \text{ r } I\cup J\backslash j)=\Gamma_k(I\cup J \text{ r } I)$
(3) If $I_1\subset..\subset I_{r+1}$ then $\cap_{i=1}^r\Gamma_k(I_{i+1} \text{ r } I_i)=\Gamma_k(I_{r+1} \text{ r } I_1)$ ∎

Remarks 4.1 (i) We could extend the definition of $\Gamma_k(I_i \cup J_i \text{ r } J_i)$ to $\Gamma_k(G_i \text{ r } H_i)$ where each G_i and H_i are any pair of sets of indices with $I \subset G_i, H_i \subset I \cup J$, and in this case Proposition 4.1 still holds (with exactly the same proof) replacing each s.c.$(I_i;I_i \cup J_i)$ with the pair $\{ZP(G_i;J\backslash G_i),ZP(H_i;J\backslash H_i)\}$.
(ii) By Lemma 2.2 and Corollary 4.1 Parts (i) and (ii) $\Gamma_k(I \cup J \text{ r } J)=$
$\{x \in Z(J):D_t^i f_j\ \phi(J)(x,t=0)=0$ for all $j \in I$ and for all $0 \leq i < k\}$ $=$
$\{x \in Z(I \cup J):D_t^i f_j \phi(I \cup J\backslash j)(x,t=0)=0$ for all $j \in I$ and for all $0 \leq i < k\}$.

If $\{\sigma_i^j\}_{i=1,..I(j)}$ is (for fixed j) a set of strata, denoted by σ^j say, and $\{\sigma^j\}$ is a collection of such sets, say σ^p, σ^q are linked in $(\sigma^1,..,\sigma^k)$ if there exists a sequence of integers $p=s(1)$, $s(2),..,s(m)=q$ with each $s(i) \in (1,..,k)$ and such that $\sigma^{s(i)},\sigma^{s(i+1)}$ have a stratum in common,each $i=1,..,m-1$. If we say p,q are equivalent in (1,..,k) if σ^p,σ^q are linked in $(\sigma^1,..,\sigma^k)$ this yields an equivalence relation on (1..k) and we shall denote the equivalence classes $J_1,..,J_r$, so $(1,..,k)=J_1 \cup .. \cup J_r$ is a disjoint union. If $\{\sigma^j\}$ is a collection of sets of strata we now define a map F mapping one collection of sets of strata to a new collection of sets of strata by $F(\sigma^1,..,\sigma^k)$
$=(\text{conv}(\cup_{i \in J_1}\sigma^i),..,\text{conv}(\cup_{i \in J_r}\sigma^i))$ where $\text{conv}(\cup_{i \in J}\sigma^i)$ denotes the convex hull of all the strata in $\cup_{i \in J}\sigma^i$. If for example we had $\sigma^1=\{ZP(1;2,3),ZP(1,2,3;\varnothing)\}$ and $\sigma^2=\{ZP(3;1,2),ZP(1,3;2)\}$ (which is in fact exactly the data of Example 4.1(b)) then $F(\sigma^1,\sigma^2)=\{$s.c.(1;1,2,3),s.c.(3;1,3)$\}$ and F{s.c.(1;1,2,3),s.c.(3;1,3)}$=ZN(\varnothing;1,2,3)$



Thus denoting the s.c.d. of $\{\sigma^1,..,\sigma^k\}$ by s.c.d.$\{\sigma^1,..,\sigma^k\}$ we therefore have in this example $F^2(\sigma^1,\sigma^2)=$s.c.d.$\{\sigma^1,\sigma^2\}$, and in general -

Lemma 4.1 $F^j(\sigma^1,..,\sigma^k)$ $j=1,2,..$ converges in a finite number of steps to s.c.d.$(\sigma^1,..,\sigma^k)$.

Proof What the process F involves at the jth stage is taking the collection of sets of strata provided by the (j-1)th stage, $F^{j-1}(\sigma^1,..,\sigma^k)=(\hat{\sigma}^1,..,\hat{\sigma}^m)$ say, subdividing $\hat{\sigma}^1,..,\hat{\sigma}^m$ into linked subsets and then taking the convex hull of each of these subsets. We claim that the sets of strata we obtain at each stage are entirely contained within a subcorner of the subcorner decomposition; this is by definition true initially, and if true up to the jth stage where we have sets of strata $\hat{\sigma}^1,..,\hat{\sigma}^m$, if $\hat{\sigma}^r$ and $\hat{\sigma}^s$ have a stratum in common they cannot be in different subcorners of the subcorner decomposition because these are disjoint; hence the sets of strata in any linked subset of $(\hat{\sigma}^1,..,\hat{\sigma}^m)$, $\{\hat{\sigma}^r\}_{r\in J'}$ say, must be in the same subcorner; we have defined the convex hull of $\{\hat{\sigma}^r\}_{r\in J'}$ as the intersection of all the subcorners containing all strata in $\{\hat{\sigma}^r\}_{r\in J'}$ and hence the subcorner of the subcorner decomposition containing $\{\hat{\sigma}^r\}_{r\in J'}$ contains the convex hull $\{\hat{\sigma}^r\}_{r\in J'}$. Hence it is true at the (j+1)th stage that the sets of strata $\tilde{\sigma}^i$ in $F^j(\sigma^1,..,\sigma^k)=(\tilde{\sigma}^1,..,\tilde{\sigma}^p)$ are each contained entirely within a single subcorner of the subcorner decomposition and our claim is true by induction.

It follows straight from the definition of F that the number of sets of strata in $F^j(\sigma^1,..,\sigma^k)$ is no more than that in $F^{j-1}(\sigma^1,..,\sigma^k)$ - with the above notation, $p\leq m\leq k$- and that each set in $F^{j-1}(\sigma^1,..,\sigma^k)$ is contained in a set of $F^j(\sigma^1,..,\sigma^k)$ - ie for any i $\sigma^i\subset\hat{\sigma}^{i'}\subset\tilde{\sigma}^{i''}$, some i′, i″, - so since there are only finitely many strata involved we must reach in finitely many steps a stage where $F^s(\sigma^1,..,\sigma^k)=F^{s+1}(\sigma^1,..,\sigma^k)$. If any two sets in $F^s(\sigma^1,..,\sigma^k)$ intersected or any single set was not equal to its convex hull we would have $F^s(\sigma^1,..,\sigma^k)\neq F^{s+1}(\sigma^1,..,\sigma^k)$, hence at the sth stage we have a disjoint set of subcorners which by the above satisfies the property that each of the sets of strata we began with is contained in a single subcorner; hence since each of these subcorners is contained in a subcorner of the subcorner decomposition, we must by the definition of the subcorner decomposition have $\text{s.c.d.}(\sigma^1,..,\sigma^k)=F^s(\sigma^1,..,\sigma^k)$. ∎

(NB. We have of course that $F^t(\sigma^1,..,\sigma^k)=F^s(\sigma^1,..,\sigma^k)$ for all $t\geq s$, unlike the case with iteration where we may have $S_{i+1}=S_i$, $S^0_{i+1}=S^0_i$ before convergence.)

<u>Lemma 4.2</u> Suppose a set $(\sigma_1,..,\sigma_k)$ of strata of ZN(I;J) is represented by a set $S=(I_1,..,I_k)$ of sets of indices (ie $\sigma_i=ZP(I_i;J\setminus I_i)$, $I\subset I_i\subset I\cup J$) satisfying

(I) If $I_1\in S$ and $I_1\subset I_2\in S$, then $K\in S\ \forall\ I_1\subset K\subset I_2$

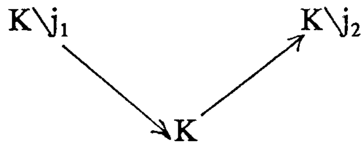
(II) If $I, I\cup i, I\cup j\in S$ then $I\cup i\cup j\in S$

(III) If $I, I\setminus i, I\setminus j\in S$, then $I\setminus i\setminus j\in S$

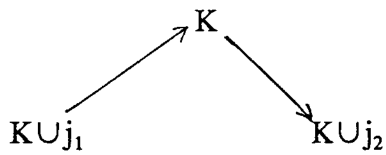
then if S has the property that for each pair $I,I'\in S\ \exists\ \{I_{i_j}\}_{j=1..k}$ with $I_{i_{j+1}}\supset I_{i_j}$ or $I_{i_{j+1}}\subset I_{i_j}$ for each $i=1,..,k'-1$, where $I_{i_1}=I$, $I_{i_{k'}}=I'$, and for all j $I_{i_j}\in S$, (*)

then $\text{conv}(\sigma_1\cup..\cup\sigma_k)=\sigma_1\cup..\cup\sigma_k$.

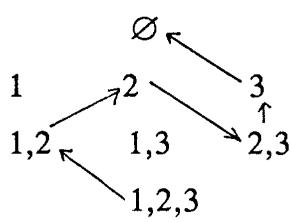
<u>Proof</u> If (*) is satified and (I) is satisfied, then for any $I_1,I_2 \in S$ $\exists$ $I_1=I^1,I^2,..,I^{k'}=I_2$ with $I^1,..,I^{k'} \in S$ and each I^i containing one index more or one index less than I^{i+1} - call such a sequence a <u>path</u> from I_1 to I_2 (so (*) and (I) say S is path connected). A path may contain type (a) sets of indices K satisfying $I^{i-1}=K\backslash j_1, I^i=K, I^{i+1}=K\backslash j_2$,



type (b) sets of indices K such that $I^{i-1}=K\cup j_1, I^i=K, I^{i+1}=K\cup j_2$



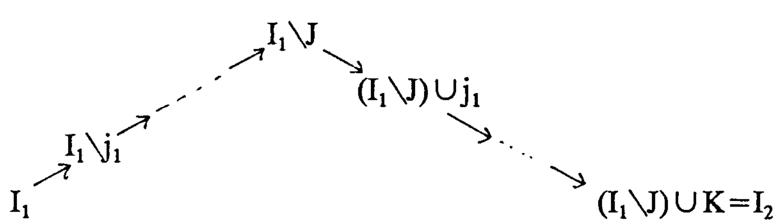
as well as other types.



A path from (1,2,3) to $\varnothing$

(A) If there exists a path $I_1 \rightarrow I_2$ without any type (a) sets of indices it must be of the form

$I_1 \rightarrow I_1\backslash i_1 \rightarrow .. \rightarrow I_1\backslash J \rightarrow I_1\backslash J \cup j_i \rightarrow .. \rightarrow I_1\backslash(J \cup K)=I_2$ (possibly with $I_1\backslash J=I_2$),

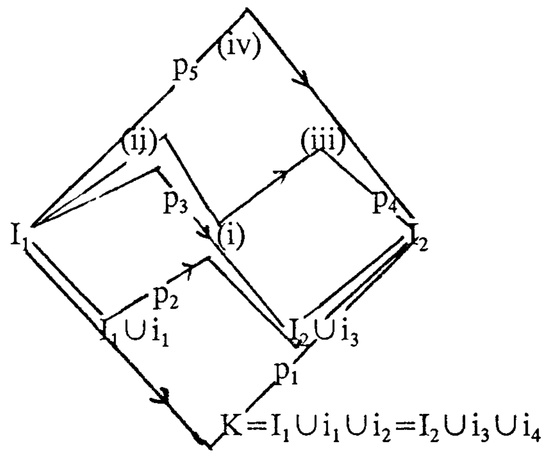


so we have $I_1 \cap I_2=(I_1\backslash J)\cup(I_1 \cap K) \supset I_1\backslash J$, so we have $I_1 \supset I_1 \cap I_2 \supset I_1\backslash J$ with I_1 and $I_1\backslash J \in S$,

and hence by (I) that $I_1 \cap I_2 \in S$

(B) If there exists a path $I_1 \rightarrow I_2$ without any type (b) sets of indices it must be of the form $I_1 \rightarrow I_1 \cup i_1 \rightarrow .. \rightarrow I_1 \cup J \rightarrow I_1 \cup J\backslash j_1 \rightarrow .. \rightarrow I_1 \cup J\backslash K=I_2$ (possibly with $I_1 \cup J=I_2$) so we have $I_1 \subset I_1 \cup I_2 \subset I_1 \cup J$ with I_1 and $I_1 \cup J$ $\in S$ and hence by (I) that $I_1 \cup I_2 \in S$.

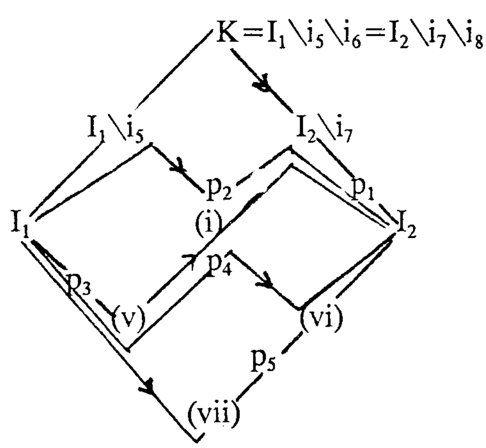
If we begin with an initial path p_1 between I_1 and I_2, and p_1 has a type (a) set of indices, then by repeated application of (III) we may obtain a path P_m with each set of indices in p_m in S and such that p_m contains no type (a) sets of indices; by (A) above $I_1 \cap I_2 \in S$.



(1),(ii),(iii),(iv) are sets of indices (in fact $(i)=(I_1 \cup i_1) \cap (I_2 \cup i_3)$ etc)

Eg for a path p_1 from I_1 to I_2 as shown above K is a type (a) set of indices. Since $K, K \setminus I_2 = I_1 \cup i_1$, and $K \setminus I_4 = I_2 \cup i_3 \in S$ we may use III to infer (i)$\in$S, and similarly infer (ii),(iii), and finally (iv)$\in$S: then $I_1 \to (ii) \to (iv) \to (iii) \to I_2$ is without type (a) sets of indices and as shown in (A) $I_1 \cap I_2$ (=iv)$\in$S.

If an initial path has a type (b) set of indices we may by repeated application of (II) obtain a path $p_{m'}$ with each set of indices in S and such that $p_{m'}$ contains no type (b) sets of indices: by (B) above $I_1 \cup I_2 \in S$.



Eg for a path p_1 $I_1 \to I_2$ as shown above K is a type (b) set of indices and using II and a similar argument to the above we may infer that (i),(v),(vi),(vii)$\in$S.

Hence if $I_1, I_2 \in S$, $I_1 \cap I_2$ and $I_1 \cup I_2 \in S$; hence if $I_1, .., I_k \in S$ $\cap_{i=1,,k} I_i$ and $\cup_{i=1,..,k} I_i \in S$, and so by (I) and definition of convex hull, $\sigma_1 \cup .. \cup \sigma_k = \mathrm{conv}(\sigma_1 \cup .. \cup \sigma_k)$. ∎

In Lemma 4.3 below expressions of the form $D_t^i\phi(K)(x,t=0)$ are abbreviated to $D_t^i\phi(K)$ (and, to be consistent, $\mathrm{grad}f_i(x)$ is abbreviated to $\mathrm{grad}f_i$).

Lemma 4.3

(1) If $x \in Z(I \cup J)$ and $D_t^i\phi(I \cup J) = D_t^i\phi(I)\ \forall i \leq k$ then $D_t^i\phi(K) = D_t^i(I)\ \forall I \subset K \subset I \cup J$ $\forall i \leq k$.

(2) If $x \in Z(I \cup (m',m))$ and $D_t^i\phi(I \cup m') = D_t^i\phi(I \cup m) = D_t^i\phi(I)\ \forall i \leq k$ then $D_t^i(I \cup (m',m)) = D_t^i\phi(I)\ \forall i \leq k$

(3) If $m',m \in I$, $x \in Z(I)$, and $D_t^i\phi(I\backslash m') = D_t^i\phi(I\backslash m) = D_t^i(I) \forall i \leq k$ then $D_t^i\phi(I\backslash(m',m)) = D_t^i\phi(I)\ \forall i \leq k$

Proof

(a) Suppose $I \cup J = I \cup (1,..,j)$. We claim $\text{span}\{P(I)\text{grad}f_1,..,P(I \cup J\backslash j)\text{grad}f_j\} = \text{span}\{P(I)\text{grad}f_1,..,P(I)\text{grad}f_j\}$ (and hence by Remark 2.1(2) vectors $P(I)\text{grad}f_1,..,P(I \cup J\backslash j)\text{grad}f_j$ are independent). Inductively suppose that for some j' with $1 \leq j' \leq j$ $\text{span}(P(I)\text{grad}f_1,..,P(I \cup (1,..,j'-1))\text{grad}f_{j'}) = \text{span}(P(I)\text{grad}f_1,..,P(I)\text{grad}f_{j'})$. By Remarks 2.1 $P(I \cup (1,..,j'))\text{grad}f_{j'+1} - P(I)\text{grad}f_{j'+1} \in \text{span}\{P(I)\text{grad}f_i\}_{i \in 1,..,j'}$, say $= \Sigma_{i=1}^{j'}\lambda_i P(I)\text{grad}f_i$. By Remark 2.1(2) $P(I)\text{grad}f_{j'+1} + \Sigma_{i=1}^{j'}\lambda_i P(I)\text{grad}f_i \neq \mathbf{0}$, so $\mathbf{0} \neq P(I \cup (1,..,j'))\text{grad}f_{j'+1} \in \text{span}\{P(I)\text{grad}f_i\}_{i=1..j'+1}$ but $\langle P(I)\text{grad}f_i, P(I \cup (1,..,j'))\text{grad}f_{j'+1}\rangle = 0$ for all $i = 1,..,j'$, hence since $P(I \cup (1,..,j'))\text{grad}f_{j'+1} \neq \mathbf{0}$ we must have $\text{span}\{P(I)\text{grad}f_i : i = 1..j'+1\} = \text{span}\{P(I)\text{grad}f_i : i = 1..j'\} \oplus P(I \cup (1,..,j'))\text{grad}f_{j'+1}$ and using the inductive assumption the result follows for $j'+1$.

(b) Proofs of (1)-(3)

(1) True from definitions if $k=0$. Suppose true for k-1.

For any K such that $I \subset K \subset I \cup J$ there exists a sequence $I, I \cup (1), I \cup (1,2),..,K,..,I \cup J$ so consider $\mathbf{0} = D_t^k\phi(I) - D_t^k\phi(I \cup J) = D_t^k\phi(I) - D_t^k\phi(I \cup (1)) + D_t^k\phi(I \cup (1)) - .. - D_t^k\phi(I \cup J)$ which by the inductive assumption and Lemma 2.2 is

$\mathbf{0} = D_t^k(f_1\phi(I))P(I)\text{grad}f_1 / |P(I)\text{grad}f_1|^2 +$
$D_t^k(f_2\phi(I \cup 1))P(I \cup (1))\text{grad}f_2 / |P(I \cup 1)\text{grad}f_2|^2 + ..$. By (a) the vectors $P(I)\text{grad}f_1$, $P(I \cup (1))\text{grad}f_2$,.. are independent and hence for each i the premultiplier $D_t^k(f_i\phi(I \cup (1,..,i-1)) = 0$, and hence $D_t^k\phi(I) = D_t^k\phi(K)$.

(2) We have $D_t^i\phi(I) = D_t^i\phi(I \cup (m)) = D_t^i\phi(I \cup m')$ for all $i \leq k$ and claim $D_t^i\phi(I) = D_t^i\phi(I \cup (m,m'))$ for all $i \leq k$.

This is true from definitions if $k=0$, suppose true for k-1.

Writing $D_t^k\phi(I \cup (m,m')) - D_t^k\phi(I) = D_t^k\phi(I \cup (m,m')) - D_t^k\phi(I \cup m') + D_t^k\phi(I \cup m') - D_t^k\phi(I)$ we know $D_t^k\phi(I \cup m') - D_t^k\phi(I) = 0$ by supposition. By the inductive assumption and

Lemma 2.2 $D_t^k\phi(I\cup(m,m'))-D_t^k\phi(I\cup m')=$
$D_t^k(f_m\phi(I\cup m'))P(I\cup m')\mathrm{grad}f_m/\,|\,P(I\cup m')\mathrm{grad}f_m\,|^2$, however $D_t^i\phi(I\cup m')=D_t^i\phi(I)$ for all $i\leq k$ and since $D_t^k(f_m\phi(I))=0$ (because using the supposition and Lemma 2.2 again $0=D_t^k\phi(I\cup m)-D_t^k\phi(I)=D_t^k(f_m\phi(I))P(I)\mathrm{grad}f_m/\,|\,P(I)\mathrm{grad}f_m\,|^2$) the result follows.

(3) By relabelling the sets of indices the assertion is equivalent to saying that if we have $D_t^i\phi(I\cup(m',m))=D_t^i\phi(I\cup m')=D_t^i(I\cup m)$ for all $i\leq k$ then $D_t^i\phi(I\cup(m',m))=D_t^i\phi(I)$ for all $i\leq k$. This is true from definitions if $k=0$, suppose it is true for k-1, and suppose
$D_t^i\phi(I\cup(m',m))=D_t^i\phi(I\cup m')=D_t^i\phi(I\cup m)$ for all $i\leq k$. Then

$$D_t^k\phi(I\cup(m',m))-D_t^k(I)=D_t^k\phi(I\cup(m',m))-D_t^k\phi(I\cup(m))+D_t^k\phi(I\cup(m))-D_t^k\phi(I)$$
$$=0+D_t^k\phi(I\cup m)-D_t^k\phi(I)$$
$$=D_t^k\phi(I\cup(m',m))-D_t^k\phi(I\cup(m'))+D_t^k\phi(I\cup(m'))-D_t^k\phi(I)$$
$$=0+D_t^k\phi(I\cup m')-D_t^k\phi(I)$$
$$=D_t^k(f_m\phi(I))P(I)\mathrm{grad}f_m/\,|\,P(I)\mathrm{grad}f_m\,|^2=D_t^k(f_{m'}\phi(I))P(I)\mathrm{grad}f_{m'}/\,|\,P(I)\mathrm{grad}f_{m'}\,|^2$$

and since (eg by Remark 2.1(2)) these vectors are independent the premultipliers must both be zero. ∎

<u>Proof of Proposition 4.1</u>

We recall that we abbreviate the subcorner $ZNP(I_i;J_i;I\cup J\backslash(I_i\cup J_i))$ of $ZN(I;J)$ to $s.c.(I_i;I_i\cup J_i)$. We have to show that if $s.c.d.\{s.c.(I_i;I_i\cup J_i)\}_{i=1..m}=\{s.c.(K^0_i;K_i)\}_{i=1..r}$ then $\cap_{i=1}^{m}\Gamma_k(I_i\cup J_i\ r\ I_i)=\cap_{i=1}^{r}\Gamma_k(K_i\ r\ K^0_i)$. We shall denote $s.c.(I_i;I_i\cup J_i)$ (we recall this is a union of strata) by σ^i. If $\{\sigma_i\}_{i\in A}$ are strata of $ZN(I;J)$ with $\sigma_i=ZP(K_i;J\backslash K_i)$ some $I\subset K_i\subset I\cup J$, where $x\in Z(I\cup J)$, then we shall say the flows on $\{\sigma_i\}$ are (k-1)th order tangent at x if $D_t^j\phi(K_{i_1})(x,t=0)=D_t^j\phi(K_{i_2})(x,t=0)$ for all $i_1,i_2\in A$ and for all $j<k$.

(a) Lemmas 4.2 and 4.3 together tell us that if the flows on strata $\{\sigma_i\}_{i\in A}$ are (k-1)th order tangent at x and $\{K_i\}_{i\in A}$ is path connected (in the sense of page 65) then $\mathrm{conv}\{\sigma_i\}_{i\in A}$ consists of strata the flows on which are (k-1)th order tangent at x.

(b) By Lemma 4.3(i) if $x\in\Gamma_k(I_i\cup J_i\ r\ I_i)$ the flows on all strata in σ^i are (k-1)th order tangent at x and it follows from (a) that if $\{\sigma^i\}_{i\in H}$ is linked the flows on the strata in $\mathrm{conv}(\cup_{i\in H}\sigma^i)$ are (k-1)th order tangent at x.

(c) Since $F(\sigma^1,..,\sigma^m)=(\mathrm{conv}\cup_{i\in H(1)}\sigma^i,..,\mathrm{conv}\cup_{i\in H(r)}\sigma^i)$, where the decomposition of (1,..,m) into $H(1)\cup..H(r)$ is as given on p.63 with each $(\cup_{i\in H(j)}\sigma^i)$ is linked, $F(\sigma^1,..,\sigma^m)$ is a collection of sets (in fact subcorners) of strata with the property that their flows at x are (k-1)th order tangent, inductively we see the subcorner of strata in

$F^q(\sigma^1..\sigma^m)$ have this property for all $q \geq 0$ and hence by Lemma 4.1 s.c.d.$(\sigma^1..\sigma^m)$ has this property.

This tells us that if for each $i=1,..,m$ the flows $\phi(I_i),\phi(I_i \cup J_i)$ are (k-1)th order tangent at x then for each $i=1,..,r$ each pair of flows $\phi(K^0_i),\phi(K_i)$ are (k-1)th order tangent at x, ie $\cap^m_{i=1}\Gamma_k(I_i \cup J_i \text{ r } I_i) \supset \cap^r_{i=1}\Gamma_k(K_i \text{ r } K^0_i)$.

The opposite set inclusion follows from definitions - if $x \in \cap^r_{i=1}\Gamma_k(K_i \text{ r } K^0_i)$ then for each $i=1,..,r$ the flows $\phi(K_i)$, $\phi(K^0_i)$ are (k-1)th order tangent at x, so by Lemma 4.3(1) for each $i=1,..,r$ $K^0_i \subset K^1, K^2 \subset K_i$ the flows $\phi(K^1),\phi(K^2)$ are (k-1)th order tangent at x, which is equivalent to saying that the flows on any two strata in the same subcorner of $\{$s.c.$(K^0_i;K_i):i=1,..,r\}$ = the subcorner decomposition of $\{$s.c.$(I_i;I_i \cup J_i):i=1,..,m\}$ are (k-1)th order tangent at x, and by definition of subcorner decomposition this implies that for each $i=1,..,m$ the flows $\phi(I_i),\phi(I_i \cup J_i)$ are (k-1)th order tangent at x. ∎

Tangency Sets in the Generic Case

Before proceeding to establish the relation between the iteration and tangency sets we establish generic properties of the latter.

Definition

(1) For our purposes a polyhedron is a connected (not necessarily compact) subset of R^n of the form $H=\{x \in R^n:\langle x,n_i\rangle = p_i \ \forall i \in I, \langle x,n_i\rangle \geq p_i \ \forall i \in J\}$ for a finite set of vectors $\{n_i, i \in I \cup J\}$ satisfying the property that if at a point $x \in H$ $\langle x,n_i\rangle = p_i \ \forall i \in I \cup J'$ the set $\{n_i:i \in I \cup J'\}$ is linearly independent (a special kind of submanifold with corners of course). Thus we regard closed corners LC(I;J), n-dimensional cubes, simplices etc as polyhedra.

(2) A r-polynomial vector field on R^n is a vector field $X:R^n \to R^n$ such that in the usual co-ordinates on R^n $X_i(x)=a^0_i+\Sigma_{j=1,..,n}a^1_{ij}x_j+..+\Sigma_{j_1,..,j_r=1,..,n}a^r_{ij_1..j_r}x_{j_1}..x_{j_r}$. An r-polynomial vector field on a polyhedron H (as above) is an r-polynomial vector field on $\{x \in R^n:\langle x,n_i\rangle = p_i \text{ for all } i \in I\} \cong R^{n-|I|}$.

(3) If M,N are respectively smooth, analytic submanifolds with corners and H is a polyhedron, $\Xi_\infty(M),\Xi_\omega(N),\Xi_{\omega,r}(H)$ are the spaces of smooth, analytic, r-polynomial vector fields on respectively M,N,H.

If M is a compact submanifold with corners or a polyhedron , M has a globally finite stratifications into C^r submanifolds. By definition of submanifold with corners there exists a neighbourhood of each $x \in M$ of the form $\beta(U \cap LC(I;J))$ where β,U are as defined in Chapter One: a stratum $\sigma = ZP(K;J\backslash K)$ is locally $\beta(U \cap LO(K;J\backslash K))$ and we may extend σ locally in a C^r way to $\breve{\sigma} = \beta(U \cap L(K))$ (=Z(K) in our local representation) which contains $\bar{\sigma}_i$ (corresponding to ZN(K;J\K)) in its (relative) interior. Thus if σ_i is a stratum of M then at each $x \in \bar{\sigma}_i$ $X(\breve{\sigma}_i)(x)$= the projection of X onto the tangent space (rather than the tangent cone) to $\bar{\sigma}_i$ at x. In the above we have defined $\Gamma_k(Z(I \cup J) \text{ r } Z(I))$ and $\phi(I)$, but we can avoid the need to work via local representations; if σ_1, σ_2 are strata of M with $\sigma_1 \subset \partial\bar{\sigma}_2$ the objects of interest will be $\Gamma_k(\breve{\sigma}_1 \text{ r } \breve{\sigma}_2)$ and $\phi(\breve{\sigma}_2)$ (=the integral flow of $X(\breve{\sigma}_2)$ on $\breve{\sigma}_2$).

We now generalise a classical theorem ([45,44,58] - see Remark 4.2 below) to show that for generic X both the tangency sets themselves and certain intersections of them are submanifolds of readily calculable dimension. This result is crucial for part of Proposition 4.4 below and for Chapter Seven. The two cases - smooth and polynomial - are treated in completely different ways.

Proposition 4.2 If M is respectively a smooth submanifold with corners, a polyhedron, a compact polyhedron, then there exist subsets $\Xi_\infty'(M)$,$\Xi_{\omega,1}'(M)$ and if $r \geq n$ $\Xi_{\omega,r}'(M)$, open dense in $\Xi_\infty(M)$,$\Xi_{\omega,1}(M)$,$\Xi_{\omega,r}(M)$ such that if $\{\sigma_i\}$ are the strata of M as a submanifold with corners, then

(a)If $\sigma_1 \subset \partial\bar{\sigma}_2$ then $\Gamma_k(\breve{\sigma}_1 \text{ r } \breve{\sigma}_2)$ is a C^∞, linear,C^ω submanifold of $\breve{\sigma}_1$ of codimension (k-1)(dimσ_2-dimσ_1)

(b) For any sequence of strata $\sigma_1,..,\sigma_r$ with $\sigma_i \subset \partial\bar{\sigma}_{i+1}$, $\cap_{i=1..r-1}\Gamma_{k(i)}(\breve{\sigma}_i \text{ r } \breve{\sigma}_{i+1})$ is a submanifold of σ_1 of codimension $\Sigma_{i=1..r-1}(K(i)-1)(\dim\sigma_{i+1}-\dim\sigma_i)$

Remark 4.2 (The relation between our tangency sets and classical ones ([45]) for generic X). In [45] Pugh shows that for any smooth compact manifold M and smooth submanifold V there exists an open dense subset $\Xi(V,M)$ of the space of smooth vector fields $\Xi(M)$ on M such that for any X in this subset the sets defined inductively by $\Gamma^0(V,X)=V$,$\Gamma^1(V,X)=\{x \in V: X(x) \in T_xV\}$,
$\Gamma^i(V,X)=\{x \in \Gamma^{i-1}(V,X): X(x) \in T_x\Gamma^{i-1}(V,X)\}$ are all submanifolds of M of codimension in M of i times that of V in M. For M=Z(I) we have setting $V=Z(I \cup J)$ (and working locally, so compactness is not an issue) that $\Gamma^0(V,X)$= our $\Gamma_1(I \cup J \text{ r } I)$, and in fact we now show that for X in Pugh's open-dense set we have

$\Gamma^i(V,X)=\Gamma_{i+1}(I \cup J \text{ r } I)$ $\forall i=0,1,..$ (*)

We have $\Gamma_k(I\cup J \text{ r } I)=\cap_{j\in J}\Gamma_k(I\cup j \text{ r } I)$ (by Proposition 4.1) and using L_X to denote Lie derivatives (see [1, Chapter 4]) we have by Lemma 2.2 and using (from [1, Section 4.2]) that if f is a differentiable function on Z(I) $L_{X(I)}f=D_t f\phi(I)(\ ,t=0)$ and that if ϕ is the flow of X and ϕ^t the time t map of ϕ (ie, $\phi^t(x)=\phi(x,t)$) then $\phi^{s*}L_X\phi^{t*}L_Xf\mid_{s=t=0}=D_sD_t\phi^{s+t*}f\mid_{s=t=0}$ where $\phi^{t*}f=f\phi^t$, we obtain $L_{X(I)}{}^kf=D_t^kf\phi(I)(\ ,t=0)$, and since by part (ii) of Remark 4.1 we know $\cap_{j\in J}\Gamma_k(I\cup j \text{ r } I)=\cap_{j\in J}\{x\in Z(I):D_t^if_j\phi(I)(x,t=0)$ for all $i<k\}$ it follows $\cap_{j\in J}\Gamma_k(I\cup j \text{ r } I)=\cap_{j\in J}\{x\in Z(I):L_{X(I)}{}^if_j(x)=0\ \forall i<k\}$. We have shown above that (*) is true for k=0. Suppose now (*) is true up to k-1. We have then (with $V=Z(I\cup J)$, $M=Z(I)$)

$$\Gamma^k(V,X)=\{x\in\Gamma^{k-1}(V,X):X(I)(x)\in T_x\Gamma^{k-1}(V,X)\} \quad (1)$$

$$=\{x\in\Gamma_k(I\cup J \text{ r } I):X(I)(x)\in T_x\Gamma_k(I\cup J \text{ r } I)\} \quad (2)$$

$$=\{x\in\Gamma_k(I\cup J \text{ r } I):X(I)(x)\in T_x\{x\in Z(I):L_{X(I)}{}^if_j(x)=0\ \forall i<k,\ \forall j\in J\}\} \quad (3)$$

$$=\{x\in\Gamma_k(I\cup J \text{ r } I):d(L_{X(I)}{}^if_j)X(I)(x)=0\ \forall i<k,\ \forall j\in J\} \quad (4)$$

$$=\{x\in\Gamma_k(I\cup J \text{ r } I):L_{X(I)}{}^if_j(x)=0\ \forall i\le k,\ \forall j\in J\} \quad (5)$$

$$=\Gamma_{k+1}(I\cup J \text{ r } J) \quad (6)$$

where (1) holds by definition, (2) is true by the inductive hypothesis, (3) is true by the above, (4) follows from [1,Section 3.5], (5) from [1,Section 4.2], and (6) by the above again, and so the result is true for k. ∎

Suppose now for U a neighbourhood of a relatively compact boundaryless submanifold S in R^n we define (where Ξ refers to any of the spaces of smooth, analytic or polynomial vector fields) $P_S:\Xi(U)\to\Xi(S)$ by, for $X\in\Xi(U)$ and $x\in S$ $(P_SX)(x)=P(T_xS)X(x)$. We topologize $\Xi(U),\Xi(S)$ in the usual way, ie two vector fields are close if their derivatives of all orders are close at every point.

<u>Lemma 4.4</u> P_S is linear, onto and open

<u>Proof</u> Linearity is immediate from the way P_S is defined. For ontoness, we recall from Chapter One the idea of X_e: if X is a C^r vector field on S and V is a neighbourhood in R^n of a point in S, with $V\cap S$ the zero set of C^r independent functions $f_1,\ldots,f_k$, then eg by [35, Section 4.5] each $y\in V$ is uniquely $y=\psi_k^{t(k)}..\psi_1^{t(1)}x$ some unique $x\in S$ where $\psi_i^{t(i)}(x)=\psi_i(t(i),x)$ and each $\psi_i:(-\epsilon,\epsilon)\times V\to R^n$ is the solution to $D_t\psi_i(t,x)=\text{grad}f_i\psi_i(t,x)$. As in Chapter One we set $X_e(y)=\psi_k^{t(k)}\ldots\psi_1^{t(1)}.X(x)$ which has the property that $P_S(X_e)=X$. Hence P_S is onto; openness follows by the Banach-Schauder Open Mapping Theorem [1] (it is to apply this that we need S relatively compact). ∎

Lemma 4.5 Suppose X is the vector field $\dot{x}_1=x_2, \dot{x}_2=1$ on R^n and that $L(1)=\{x\in R^n:\langle x,n_1\rangle=0\}$. Then $\Gamma_2(L(1) \text{ r } R^n)=\{x\in L(1):x_2=0\}$. We choose coordinates $(x_2,x)\in R\times\Gamma_2^X(L(1) \text{ r } R^n)$ on L(1), and we suppose Y is a smooth vector field on R^n transverse to L(1). Then there exists $\epsilon_0>0$, a neighbourhood U of 0 in $\Gamma_2(L(1) \text{ r } R^n)$ and a unique C^∞ map $x_2:(-\epsilon_0,\epsilon_0)\times U\rightarrow R$ such that for y near 0 in L(1) $\langle X(y)+\epsilon Y(y),n_1\rangle=0$ iff $y=(x_2(\epsilon,x),x)$, and the map $(\epsilon,x)\rightarrow(x_2(\epsilon,x),x)$ is a diffeomorphism on $(-\epsilon_0,\epsilon_0)\times U$. In fact $\{(x_2(\epsilon,x),x):x\in U\}=\Gamma_2^{X+\epsilon Y}(L(1) \text{ r } R^n)$ near 0, and $\{\Gamma_2^{X+\epsilon Y}(L(1) \text{ r } R^n):-\epsilon_0<\epsilon<\epsilon_0\}$ foliates a neighbourhood of 0 in L(1) (see Figure 4.2).

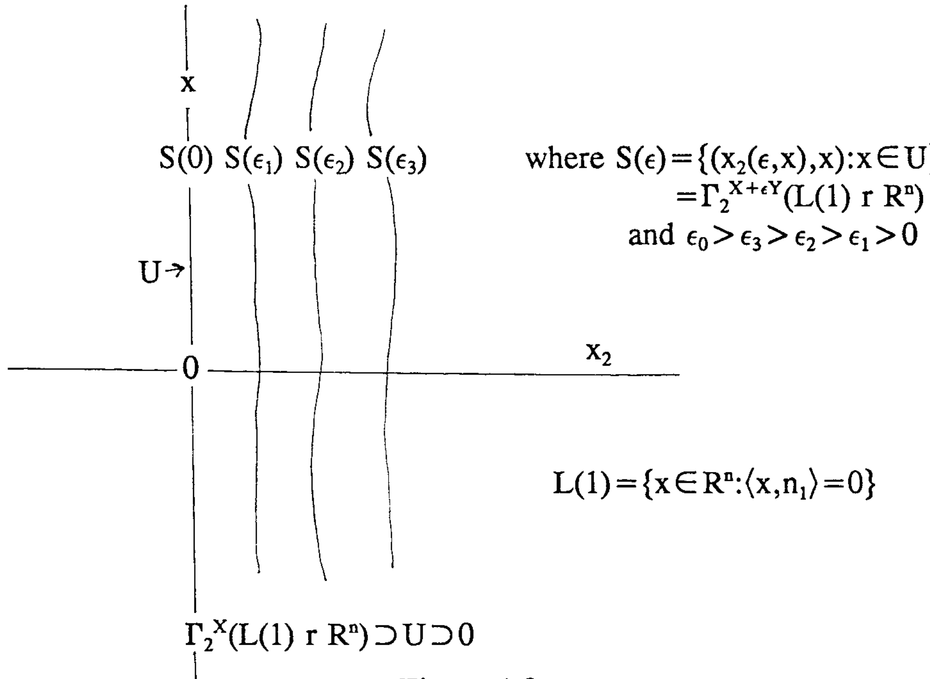


Figure 4.2

The way Lemma 4.5 is used is as follows. (i) By Remark 4.2 and Lemma 4.4 there exists an open-dense subset of $\Xi_\infty(ZN(I;J))$ such that if X is in this subset then for each $j'\in Z^+$, $I\subset K\subset K\cup j\subset I\cup J$ $\Gamma_{j'}^X(j\cup K \text{ r } K)$ is a codimension 1 submanifold of $\Gamma_{j'-1}^X(j\cup K \text{ r } K)$, and we can check furthermore that $\Gamma_k^X(K\cup j \text{ r } K)=\Gamma_2^{X(\Gamma_{k-2}^X(K\cup j \text{ r } K))}(\Gamma_{k-1}^X(K\cup j \text{ r } K)$ relative to $\Gamma_{k-2}^X(K\cup j \text{ r } K))$ where $X(\Gamma_{k-2}^X(K\cup j \text{ r } K))$ is the smooth vector field obtained by projecting X onto $\Gamma_{k-2}^X(K\cup j \text{ r } K)$. (ii) It is straightforward to check (and this is used by Pugh in [45]) that if Y is a vector field tangent to a submanifold S of R^n then $\Gamma_2^{X+Y}(S \text{ r } R^n)=\Gamma_2^X(S \text{ r } R^n)$ and in general that if Y is tangent to $\Gamma_{k-1}^X(S \text{ r } R^n)$ then $\Gamma_k^{X+Y}(S \text{ r } R^n)=\Gamma_k^X(S \text{ r } R^n)$.

Suppose now $M=ZN(I;J)$, $J=(1,..,m)$ and we wish to perturb X so that $\{\Gamma_{k(i)}^X(I\cup i \text{ r } I)\}_{i=1,\dots,m}$ is in general position. We begin by perturbing X into the open-dense subset of $\Xi_\infty(M)$ so that (i) holds. Inductively suppose that $\{\Gamma_{j_i'}^X(I\cup i \text{ r } I)\}_{i=1,\dots,m}$ are in general position for all $j_i'\le j_i$ where each $j_i\le k(i)$,

$i=1,..,m$. If $j_i=k(i)$ for all i then result is shown; otherwise we may assume $j_1<k(1)$, and we show how to perturb X so that $\{\Gamma_{j_i'}{}^X(I\cup i\ r\ I)\}_{i=1,..,m}$ are in general position for all $j_1'\leq j_1+1$, $j_i'\leq j_i$ $i=2,..,m$.

Because $\{\Gamma_{j_i'}{}^X(I\cup i\ r\ I)\}_{i=1,..,m}$ are in general position for all $j_i'\leq j_i$ we can find a vector field $\tilde{Y}$ on R^n tangent to each one of them near their common intersection, and furthermore we may chose it so that $Y_1=\tilde{Y}\mid\Gamma_{j_1-1}{}^X(I\cup 1\ r\ I)$ is transverse to $\Gamma_{j_1}{}^X(I\cup 1\ r\ I)$. We then apply Lemma 4.5 with R^n, L, X, Y of Lemma 4.5 mapped by a diffeomorphism to $\Gamma_{j_1-1}{}^X(I\cup 1\ r\ I)$, $\Gamma_{j_1}{}^X(I\cup 1\ r)$, $X(\Gamma_{j_1-1}{}^X(I\cup 1\ r\ I))$, Y_1 (as is possible by classical canonical form theorems, eg [44,58]). By (i) $\Gamma_{j_i'}{}^{X+\epsilon\tilde{Y}}(I\cup i\ r\ I)=\Gamma_{j_i'}{}^X(I\cup i\ r\ I)$ for all $j_i'\leq j_i$ and all $i=1,..,m$. By Lemma 4.5 and the basic transversality theorem (eg [22,Lemma 4.6]) we know that for almost all and hence arbitrarily small ϵ $\Gamma_2{}^{X(\Gamma_{j_1-1}{}^X(I\cup i\ r\ I))+\epsilon\tilde{Y}_1}(\Gamma_{j_1-1}{}^X(I\cup 1\ r\ I)$ relative to $\Gamma_{j_1}{}^X(I\cup 1\ r\ I))$ is transverse (in $\Gamma_{j_1-1}{}^X(I\cup 1\ r\ I)$) to every intersection $\cap_{i=1,..,m}\Gamma_{j_i'}{}^X(I\cup i\ r\ I)$, $j_i'\leq j_i$, (each such intersection is a submanifold by the inductive assumption that they are in general position). By (i) and (ii) we know $\Gamma_2{}^{X(\Gamma_{j_1-1}{}^X(I\cup i\ r\ I))+\epsilon\tilde{Y}_1}(\Gamma_{j_1-1}{}^X(I\cup 1\ r\ I)$ relative to $\Gamma_{j_1}{}^X(I\cup 1\ r\ I))=\Gamma_{j_1+1}{}^{X+\epsilon\tilde{Y}}(I\cup 1\ r\ I)$ and so the result follows.

This method was used in a draft version of this thesis to prove a primitive version of Corollary 4.1 and we use it below to prove Proposition 4.2(b) in the smooth case.

<u>Proof of Lemma 4.5</u> For points $(x_2,x)\in L(1)$ $X\mid L(1):L(1)\to R^n$ has the form $X(x_1=0,x_2,x)=(x_2,1,0)$ and hence $\langle X(0,x_2,x)+\epsilon Y(0,x_2,x),n_1\rangle=x_2+\langle\epsilon Y(0,x_2,x),n_1\rangle=F(x,x_2,\epsilon)$ say, where $x=(x_3,..,x_n)$. $\partial F/\partial x_2=1$ at $\epsilon=0$ hence $\neq 0$ for all sufficiently small ϵ, and by the Implicit Function Theorem there exists a unique smooth $x_2:(-\epsilon_0,\epsilon_0)\times U\to$ a neighbourhood of 0 in L(1) such that $F(x,x_2(\epsilon,x),\epsilon)=0$ with $x_2(x,0)=0$.

It remains to show that $(\epsilon,x)\to(x_2(\epsilon,x),x)$ is a diffeomorphism on $(-\epsilon_0,\epsilon_0)\times U$. By [14,Chapter 10] we have for $(\epsilon,x)\in(-\epsilon_0,\epsilon_0)\times U$

$$\frac{\partial x_2(\epsilon,x)}{\partial(\epsilon,x)}=(\partial F/\partial x_2)^{-1}\begin{bmatrix}\partial F/\partial\epsilon\\ \partial F/\partial x\end{bmatrix}$$

(where $x=(x_3,..,x_n)$, and both sides of this equation are column vectors of (n-2) elements) so $\partial x_2/\partial\epsilon=(\partial F/\partial x_2)^{-1}\partial F/\partial\epsilon$, and since $\partial F/\partial\epsilon=\langle Y(0,x_2,x),n_1\rangle\neq 0$ (since Y is transverse to L(1)) the derivative of the map $(\epsilon,x)\to(x_2(\epsilon,x),x)$

$$\begin{bmatrix} \partial x_2/\partial\epsilon & \partial x/\partial\epsilon \\ \partial x_2/\partial x & \partial x/\partial x \end{bmatrix} \quad \text{is invertible.}$$

∎

Proof of Proposition 4.2 (smooth case)

By Remark 4.2 and Lemma 4.4 we can find an open-dense subset of $\Xi_\infty(M)$ such that (a) is satisfied, so we must prove part (b). If (a) is satisfied we know that since there are only finitely many strata in M as a submanifold with corners and for any distinct strata σ,σ' $\Gamma_j(\breve{\sigma}\ r\ \breve{\sigma}')$ is empty if $j>n$, there can exist only finitely many "chains" of strata $\sigma_1,..,\sigma_r$ with $\sigma_{i+1}\subset\overline{\sigma}_i$ and only finitely many non-empty tangency sets $\Gamma_j(\breve{\sigma}_{i+1}\ r\ \breve{\sigma}_i)$. The strategy for proving Proposition 4.2(b) is to select a chain $\sigma_1,..,\sigma_r$ as above and a sequence of integers $k_1,..,k_r$ (to guarantee stability of the intersections the order in which the perturbations are made, ie the order in which the chains and sequences are chosen, is critical, see pages 75-6) and use Lemma 4.5 to find X′ arbitrarily close to X such that $\{\Gamma_{k_i}^{X'}(\breve{\sigma}_{i+1}\ r\ \breve{\sigma}_i)\cap\breve{\sigma}_r\ \}_{i=1,...,r-1}$ are in general position; the conclusion of Proposition 4.2(b) then holds for this chain and sequence, and since being in general position involves their satisfying a finite set of transversality conditions the conclusion will still hold for any X″ sufficiently near X′. Thus if we select any other chain and sequence, since we may make the perturbations (to force the tangency sets for this second chain into general position) as small as we wish we can make them small enough to leave the result for the first chain and sequence unaffected, and by making perturbations of diminishing size treat all of the finitely many chains and sequences in this way.

Thus the result follows if we can show that applying Lemma 4.5 to any particular chain and sequence we can perturb $\Gamma_{k_r\text{-}1}(\breve{\sigma}_r\ r\ \breve{\sigma}_{r-1})\cap..\cap\Gamma_{k_1}(\breve{\sigma}_2\ r\ \breve{\sigma}_1)$ into general position in $\breve{\sigma}_r$ with an arbitrarily small perturbation.

If σ_i,σ_{i+1} is any pair of strata of M with $\sigma_{i+1}\subset\partial\overline{\sigma}_i$ then there exist $\breve{\sigma}_i^1$, $\breve{\sigma}_i^2,..,\breve{\sigma}_i^m$ such that $\breve{\sigma}_{i+1}=\breve{\sigma}_i^1\subset\breve{\sigma}_i^2\subset..\subset\breve{\sigma}_i^m=\breve{\sigma}_i$ and dim σ_i^{j+1}-dim$\sigma_i^j=1$ for all $j=1,..,m-1$, and by Proposition 4.1 we may express each $\Gamma_{k(i)}(\breve{\sigma}_{i+1}\ r\ \breve{\sigma}_i)$ as $\cap\{\Gamma_{k(i)}(\breve{\sigma}_i^j\ r\ \breve{\sigma}_i^{j+1}):j=1,..,\dim\sigma_i\text{-}\dim\sigma_{i+1}\}$. Thus without loss of generality we may assume that we begin with tangency sets of the form $\Gamma_{k(1)}(\breve{\sigma}_2\ r\ \breve{\sigma}_1)$, $\Gamma_{k(2)}(\breve{\sigma}_3\ r\ \breve{\sigma}_2),..,\Gamma_{k(r)}(\breve{\sigma}_r\ r\ \breve{\sigma}_{r-1})$, where $\dim\sigma_i\text{-}\dim\sigma_{i+1}=1$. So we wish to show that there exists X′ arbitrarily close to any smooth vector field X such that $\Gamma_{k(1)}{}^{X'}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1) \cap .. \cap$ $\Gamma_{k(r-1)}{}^{X'}(\breve{\sigma}_r \text{ r } \breve{\sigma}_{r-1})$ is a submanifold of σ_r of codimension $\Sigma_{i=1}^{r-1}(k(i)-1)$.

As in our discussion during Lemma 4.5 of how to apply it, we construct the perturbation X′ from X by perturbing X in stages, so that if we begin a stage with $\cap_{1 \leq i \leq r-1} \Gamma_{j_i}(\breve{\sigma}_{i+1} \text{ r } \breve{\sigma}_i)$ in general position in $\breve{\sigma}_r$ we end it having "pushed" $\Gamma_{j_k+1}(\breve{\sigma}_{k+1} \text{ r } \breve{\sigma}_k)$ $\breve{\sigma}_k)$ in $\Gamma_{j_k}(\breve{\sigma}_{k+1} \text{ r } \breve{\sigma}_k)$ (some $1 \leq k \leq r-1$, $j_k < k(j)$) so that $\Gamma_{j_k+1}(\breve{\sigma}_{k+1} \text{ r } \breve{\sigma}_k) \cap_{1 \leq i \leq r-1, i \neq k} \Gamma_{j_i}(\breve{\sigma}_{i+1}$ r $\breve{\sigma}_i)$ are in general position in $\breve{\sigma}_r$.

Example We can see how the idea works with a simple example.

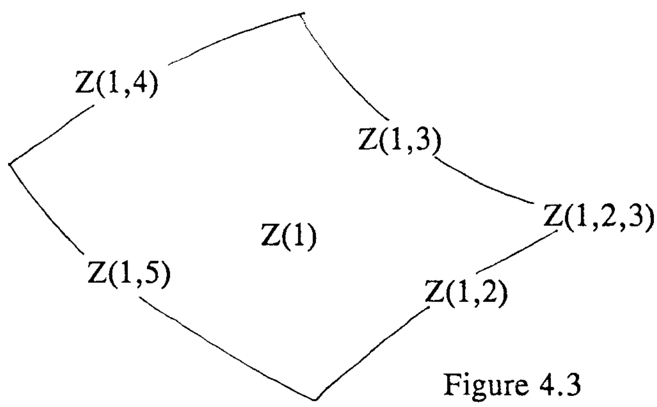


Figure 4.3

Consider $M \subset R^n$ as illustrated in Figure 4.3. As we have observed above, by [44], Lemma 4.4 and Remark 4.2 there exists an open-dense subset of $\Xi_\infty(M)$ such that for X in this subset part (a) of Proposition 4.2 is satisfied. We begin by perturbing X into this subset, so $\Gamma_k(1 \text{ r } \varnothing)$ is codimension k-1 in Z(1), $\Gamma_k((1,j) \text{ r } 1)$ is codimension k-1 in Z(1,j) etc.

Consider a particular chain and a particular sequence, such as $\Gamma_4(1 \text{ r } \varnothing) \cap \Gamma_2(1,2 \text{ r } 1) \cap \Gamma_3(1,2,3 \text{ r } 1,2)$. We would "push" these tangency sets into general position in Z(1,2,3) as follows (the j,i at the beginning of each line are explained below):

j=1,i=2 (1) With Y on R^n transverse to Z(1) push $\Gamma_2(1 \text{ r } \varnothing)$ transverse to Z(1,2) and to Z(1,2,3)

j=1,i=3 (2) With Y tangent to Z(1) and transverse to $\Gamma_2(1 \text{ r } \varnothing)$ push $\Gamma_3(1 \text{ r } \varnothing)$ transverse to Z(1,2) and to Z(1,2,3)

j=1,i=4 (3) With Y tangent to $\Gamma_2(1 \text{ r } \varnothing)$ and transverse to $\Gamma_3(1 \text{ r } \varnothing)$ push $\Gamma_4(1 \text{ r } \varnothing)$ transverse to Z(1,2) and Z(1,2,3)

j=2,i=2 (4) With Y tangent to $\Gamma_3(1 \text{ r } \varnothing)$ and transverse to Z(1,2) push $\Gamma_2(1,2 \text{ r } 1)$ transverse to $\Gamma_4(1 \text{ r } \varnothing) \cap Z(1,2)$ and to $\Gamma_4(1 \text{ r } \varnothing) \cap Z(1,2,3)$

j=3,i=2 (5) With Y tangent to $\Gamma_3(1 \text{ r } \varnothing)\cap Z(1,2)$ and transverse to $Z(1,2,3)$ push $\Gamma_2(1,2,3 \text{ r } 1,2)$ transverse to $\Gamma_4(1 \text{ r } \varnothing)\cap\Gamma_2(1,2 \text{ r } 1)\cap Z(1,2,3)$

j=3,i=3 (6) With Y tangent to $\Gamma_3(1 \text{ r } \varnothing)\cap Z(1,2,3)$ and transverse to $\Gamma_2(1,2,3 \text{ r } 1,2)$ push $\Gamma_3(1,2,3 \text{ r } 1,2)$ transverse to $\Gamma_4(1 \text{ r } \varnothing)\cap\Gamma_2(1,2 \text{ r } 1)\cap Z(1,2,3)$

(End of Example).

Continuing with our discussion before the example we see that in general if we are to perturb $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(r)}(\breve{\sigma}_r \text{ r } \breve{\sigma}_{r-1})$ into general position in $\breve{\sigma}_r$ there are $\Sigma_{j=1}^{r-1}(k(j)-1)$ pushing stages and these are ordered

j=1,i=2,..,k(1)

j=2,i=2,..,k(2)

etc down to

j=r-1 i=2,..,k(r-1).

At the (j,i)th stage the inductive assumption is that we have pushed $\Gamma_{k(j-1)}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})$ transverse to $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-2)}(\breve{\sigma}_{j-1} \text{ r } \breve{\sigma}_{j-2})\cap\sigma_{j'}$ $\forall j\leq j'\leq r$, where each $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-2)}(\breve{\sigma}_{j-1} \text{ r } \breve{\sigma}_{j-2})\cap\sigma_{j'}$ is a submanifold, and also $\Gamma_{i-1}(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$ transverse to $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})\cap\sigma_{j'}\forall j+1\leq j'\leq r$ (these $\sigma_{j'}, j+2\leq j'\leq r$ are the lower dimensional strata), where the first part of the inductive assumption implies that $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})\cap\sigma_{j'}$ are all submanifolds.

Then at the (j,i)th stage

(a) If $2<i$ proceed as follows: we know $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})\cap\sigma_{j'}$ is a submanifold for each $j+1\leq j'\leq r$; these and $\Gamma_m(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$ for $m\leq i-1$ are left fixed by Y tangent to $\Gamma_{k(1)-1}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)-1}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})\cap\Gamma_{i-2}(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$ and we can also choose this Y transverse to $\Gamma_{i-1}(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$, and hence by Lemma 4.5 we may with an arbitrarily small Y push $\Gamma_i(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$ transverse to $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})\cap\sigma_{j'}$ $\forall j+1\leq j'\leq r$.

(b) If $i=2$ proceed as follows; we know that $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)}(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)\cap\sigma_{j'}$ each $j+1\leq j'\leq r$ is a submanifold and is left fixed by Y tangent to $\Gamma_{k(1)-1}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)-1}(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$. We choose Y transverse to $\breve{\sigma}_{j+1}$ and by Lemma 4.5 push $\Gamma_2(\breve{\sigma}_{j+1} \text{ r } \breve{\sigma}_j)$ transverse to $\Gamma_{k(1)}(\breve{\sigma}_2 \text{ r } \breve{\sigma}_1)\cap..\cap\Gamma_{k(j-1)}(\breve{\sigma}_j \text{ r } \breve{\sigma}_{j-1})\cap\sigma_{j'}$ $\forall j+1\leq j'\leq r$.

■

<u>Proof of Proposition 4.2, Polynomial case</u>

The methods used to prove Proposition 4.2 in the smooth case cannot readily be applied in the polynomial case. Pugh's paper [45] and Remarks 4.2 were for the smooth case and, more seriously, to use Lemma 4.5 we needed to be able to select Y at each stage tangent to certain tangency sets, which if Y is only allowed to be polynomial will not generally be possible. Rather that refine the methods above to circumvent this we adopt a quite different strategy. Our submanifold with corners M is (by assumption) a polyhedron, which without loss of generality we can take to be of codimension 0 in R^n, and by definition possesses a finite stratification into submanifolds each of which is an open subset of some $L(K)=\{x\in R^n:\langle x,n_i\rangle=p_i$ for all $i\in K\}$.

For M codimension 0 in R^n $\Xi_{\omega,r}(M)=\Xi_{\omega,r}(R^n)$, which is for the rest of this proof denoted $\Xi_{\omega,r}$. We observe that part (a) of Proposition 4.2 is a special case of part (b) (with $s=1$); we shall therefore prove part (b).

Corresponding to each element of $\Xi_{\omega,r}$ is a sequence $(a^0{}_1,a^0{}_2,..,a^0{}_{11},a^1{}_{11},a^1{}_{12},..,a^1{}_{nn},..,a^r{}_{1..1},..,a^r{}_{n..n})$ which are the coefficients of $X\in\Xi_{\omega,r}$, ie $X_i(x)=a^0{}_i+\Sigma_{j=1,..,n}a^1{}_{ij}x_j+..+\Sigma_{j_1,..,j_r=1,..,n}a^r{}_{ij_1..j_r}x_{j_1}..x_{j_r}$.

Suppose now we have a chain of tangency sets $\Gamma_{k(1)}(L(I_2)\ r\ L(I_1))$, $\Gamma_{k(2)}(L(I_3)\ r\ L(I_2)),..,\Gamma_{k(s)}(L(I_{s+1})\ r\ L(I_s))$ where $L(I_{i+1})\subset L(I_i)$ for all $1\leq i\leq s$ and $\{k(i)\}$ are positive integers. After a translation we may suppose $0\in L(I_{s+1})$. Then the condition that $0\in\Gamma_{k(1)}{}^X(L(I_2)\ r\ L(I_1))\cap..\cap\Gamma_{k(s)}{}^X(L(I_{s+1})\ r\ I(I_s))$ or that $X(I_{s+1})(0)=\mathbf{0}$ may be viewed as conditions on X. We shall show in (1) below that if $r\geq n$ then defining $T^{k(1),..,k(r)}(I_1,..,I_{s+1})=$ $\{X\in\Xi_{\omega,r}:0\in\Gamma_{k(1)}{}^X(L(I_2)\ r\ L(I_1))\cap..\cap\Gamma_{k(s)}{}^X(L(I_{s+1})\ r\ L(I_s))$ and $X(I_{s+1})(0)\neq\mathbf{0}\}$ is an analytic submanifold of $\Xi_{\omega,r}$ of codimension $k(1)\mid I_2\backslash I_1\mid+..+k(s)\mid I_{s+1}\backslash I_s\mid$. For given X we can map each point in $L(I_{s+1})$ to $\Xi_{\omega,r}$ by $x\to G_X(x)\in\Xi_{\omega,r}$ where $G_X(x)$ is the vector field given by $G_X(x)(y)=X(x+y)$. We see that $x\in\Gamma_{k(1)}{}^X(L(I_2)\ r\ L(I_1))\cap..\cap\Gamma_{k(s)}{}^X(L(I_{s+1})\ r\ L(I_s))\backslash\{x\in L(I_{s+1}):X(I_{s+1})(x)=\mathbf{0}\}$ iff $G_X(x)\in T^{k(1)..k(s)}(I_1,..,I_{s+1})$. We shall show in (2) that we can find X′ arbitrarily close to X such that $G_{X'}\pitchfork T^{k(1)..k(s)}(I_1,..,I_{s+1})$ and so that $\{x\in L(I_{s+1}):X'(I_{s+1})(x)=\mathbf{0}\}$ is disjoint from all the tangency sets $\Gamma_{k(i)}{}^{X'}(L(I_{i+1})\ r\ L(I_i))$.

This has two consequences: firstly that $G_{X'}{}^{-1}T^{k(1)..k(s)}(I_1,..,I_{s+1})=$ $\Gamma_{k(1)}{}^{X'}(L(I_2)\ r\ L(I_1))\cap..\cap\Gamma_{k(s)}{}^{X'}(L(I_{s+1})\ r\ L(I_s))$ is a submanifold of $L(I_{s+1})$ of the required codimension, and secondly that so is $\Gamma_{k(1)}{}^{X''}(L(I_2)\ r\ L(I_1))\cap..\cap\Gamma_{k(s)}{}^{X''}(L(I_{s+1})\ r\ L(I_s))$ for all X″ sufficiently near X′.

We can then conclude the proof of Proposition 4.2 by a similar argument to that used in the smooth case: since there are only finitely many chains

$L(I_1) \supset L(I_2) \supset .. \supset L(I_{s+1})$ and finitely many sequences $\{k(i)\}$ with each $k(i) \leq n+1$ by making perturbations to the vector field of diminishing size we may adjust X for each chain and sequence in turn, preserving at each stage the result obtained at all previous stages. The codimension result implies in particular that $\Gamma_{n+1}(L(I) \text{ r } L(I'))$ is empty for all $I \neq I'$, so since by definition $\Gamma_k(L(I) \text{ r } L(I')) \subset L_{n+1}(L(I) \text{ r } L(I'))$ for all $k \geq n+1$ Proposition 4.2 holds for all sequences $\{k(i)\}$.

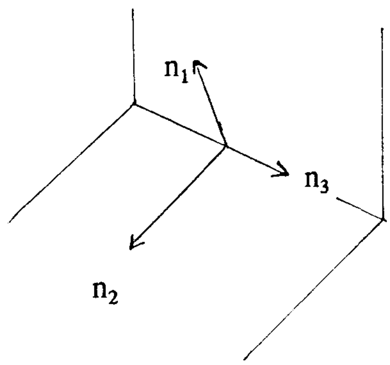
(1)(i) By Proposition 4.1 $\Gamma_{k(i)}{}^X(L(I_{i+1}) \text{ r } L(I_i)) =$
$\cap_{j=1,..,p}\Gamma_{k(i)}(L(I_i^{j+1}) \text{ r } L(I_i^j))$ where $I_{i+1}=I_i^{p+1} \supset I_i^p \supset .. \supset I_i^1 = I_i$
and $| I_{i+1}{}^{m+1} \backslash I_i^m | = 1$, so without loss of generality we may assume we begin with $| I_{i+1} \backslash I_i | = 1$.

(ii) We may choose orthogonal unit vectors $\{n_i\}$ such that
$L(I_2) = \{x \in L(I_1) : \langle x, n_1 \rangle = 0\}$

.

.

$L(I_{s+1}) = \{x \in L(I_s) : \langle x, n_s \rangle = 0\}$.



(iii) With the conventions of (i) and (ii)
$0 \in \Gamma_{k(1)}{}^X(L(I_2) \text{ r } L(I_1)) \cap .. \cap \Gamma_{k(s)}{}^X(L(I_{s+1}) \text{ r } L(I_s))$
iff $D_t^i\langle \phi(I_1)(0,t=0), n_1 \rangle = 0$ for all $i < k(1), \ldots, D_t^i\langle \phi(I_s)(0,t=0), n_s \rangle = 0$ for all $i < k(s)$

(iv) If we express our vector field relative to co-ordinates $x_i = \langle x, n_i \rangle$, supplemented if necessary so that i runs from 1 to n, then since

$$(X(K)(x))_i = \begin{cases} (X(K)(x))_i \text{ if } i \in (1,..,n)\backslash K \\ 0 \text{ if } i \in K \end{cases}$$

we get

$$(D_t\phi(K)(0,t=0))_i = \begin{cases} a^0{}_i \text{ if } i \in (1,..,n)\backslash K \\ 0 \text{ if } i \in K \end{cases}$$

$$(D_t^2\phi(K)(0,t=0))_i = \begin{cases} \sum_{j \in (1,..,n)\backslash K} a^1{}_{ij} a^0{}_j \text{ if } i \in (1,..,n)\backslash K \\ 0 \text{ if } i \in K \end{cases}$$

$$(D_t^3\phi(K)(0,t=0))_i = \begin{cases} \displaystyle\sum_{j,k\in(1,..,n)\setminus K} a^1_{ij}a^1_{jk}+2a^2_{ijk}a^0_j a^0_k & \text{if } i\in(1,..,n)\setminus K \\ 0 & \text{if } i\in K \end{cases}$$

and in general if $q\leq r+1$ then if $i\in(1,..,n)\setminus K$ then

$(D_t^q\phi(K)(0,t=0))_i=\Sigma_{j_1,..,j_{q-1}\in(1,..,n)\setminus K}(\ldots. + \text{ a multiple of } a^{q-1}_{ij_1..j_{q-1}}a^0_{j_1}..a^0_{j_{q-1}}\ )$.

(v) To show that $T^{k(1)..k(s)}(I_1,..,I_{s+1})$ is a submanifold we shall show that the set of vectors $\{grad_X g_{ij}(X),\ j=1,..,s,\ i=1,..,k(j)\}$ is independent at every point $X\in\Xi_{\omega,r}$ such that $X(I_{s+1})(0)\neq\mathbf{0}$, where $g_{ij}(X)=D_t^i\langle\phi(I_j)(x=0,t=0),n_j\rangle$.

This last condition means that if $I_2=I_1\cup(1),..,I_{s+1}=I_s\cup(s)=I_1\cup(1,..,s)\subset(1,..,n)$ then there exists some $j\in(1,..,n)\setminus(I_1\cup(1,..,s))$ such that $a^0_j\neq 0$. Each $X\in\Xi_{\omega,r}$ corresponds to a sequence $a=(a^0_1,a^0_2,..)$ and translating X's into a's so

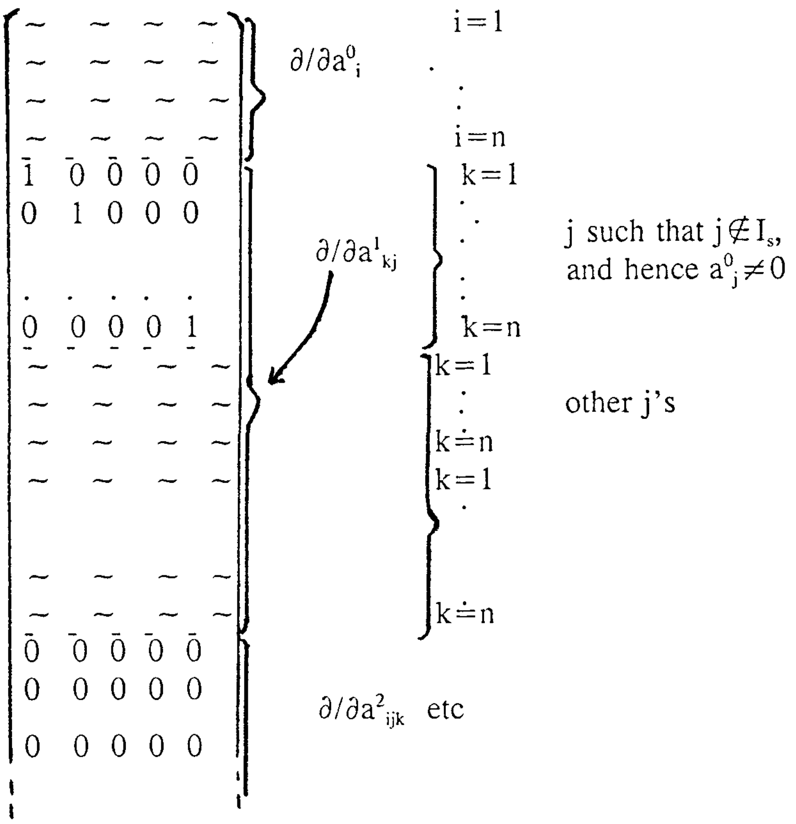
$grad_a=(\partial/\partial a^0_1,\partial/\partial a^0_2,..,\partial/\partial a^0_n,\partial/\partial a^1_{11},..,\partial/\partial a^r_{n..n})$ we show that $\{grad_a g_{ij}(a):j=1,..,s,$ $i=1,..,k(j)\}$ is an independent set for all a such that $a^0_j\neq 0$ some j.

We have $grad_a(g_{11}, g_{12},..,g_{1s})=grad_a(a^0_1,..,a^0_s)=$

$$\begin{array}{r} \partial/\partial a^0_i \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \partial/\partial a^1_{ij} \\ \text{etc} \end{array} \begin{pmatrix} 1 & 0 & . & . & 0 \\ 0 & 1 & . & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & 0 & 1 \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & \bar{0} \\ . & . & . & . & . \\ . & . & . & . & . \\ \vdots & & & & \vdots \end{pmatrix}$$

and so $grad_a g_{11},..,grad_a g_{1s}$ are independent.

We have $grad_a(g_{21},..,g_{2s})=grad_a(\Sigma_{j\notin I_1}a^1_{1j}a^0_j,\Sigma_{j\notin I_2}a^1_{2j}a^0_j,..,\Sigma_{j\notin I_s}a^1_{sj}a^0_j)$ and we know that there exists $j\notin I_s$ such that $a^0_j\neq 0$, so we obtain $grad_a(g_{21},..,g_{2s})=$



hence $\{grad_a g_{11},..,grad_a g_{1s}\}$ are independent and continuing in this way we see that the vectors $\{grad_a g_{ij}(a)\}_{j=1,..,s,i=1,..,n+1}$ are independent, and hence for any $\{k(i)\}_{i=1,..,s}$ with each $k(i) \leq n+1$ $T^{k(1)...k(s)}(I_1,..,I_{s+1})$ is a submanifold of the required codimension.

(2)

(i) For any particular chain $L(I_1) \supset L(I_2) \supset .. \supset L(I_{s+1})$ and sequence $\{k(i)\}$ consider for fixed X the map $M \cap L(I_{s+1}) \times \Xi_{\omega,r} \to \Xi_{\omega,r}$ defined by $(x,Y) \to G_{X+Y}(x)$, which is differentiable (we recall that by definition $G_{X+Y}(x)(y)=(X+Y)(x+y)$). Fixing x,X the map $Y \to G_{X+Y}(x)$ is the composition $Y \to Y+X \to G_{X+Y}(x)$ which has inverse $Z \to Z' \in \Xi_{\omega,r}$, where Z' is the vector field $Z'(y)=Z(y-x)$, followed by $Z' \to Z'-X$. Hence $Y \to G_{X+Y}(x)$ is a diffeomorphism for fixed x,X and so our original map $(x,Y) \to G_{X+Y}(x)$ is a submersion for fixed X.

Thus ([35]) for a.a. and hence arbitrarily small fixed Y and for fixed X the map $x \to G_{X+Y}(x)$ is transverse to our submanifold $T^{k(1)..k(s)}(I_1,..,I_{s+1})$ as well as to $\{X \in \Xi_{\omega,r}: X(I_{s+1})0=0\}$ which we denote $\zeta(I_{s+1})$. We see $\zeta(I_{s+1})=$ those polynomial vector fields with $a^0_i=0$ for all $i \in (1,..,n) \setminus I_{s+1}$ so is certainly a submanifold of $\Xi_{\omega,r}$. Thus we have found $X'=X+Y$ arbitrarily close to X with $G_{X'}^{-1}T^{k(1)..k(s)}(I_1,..,I_{s+1})$ and $G_{X'}^{-1}\zeta(I_{s+1})$ submanifolds of $L(I_{s+1}) \cap M$, of respectively codimension >0 and dimension equal to 0; we can check from definitions that these submanifolds are respectively $\Gamma_{k(1)}^{X'}(I_2 \text{ r } I_1) \cap .. \cap \Gamma_{k(s)}^{X'}(I_{s+1} \text{ r } I_s) \cap M$ and $\{x \in M \cap L(I_{s+1}): X'(I_{s+1})(x)=0\}$.

(ii) If we now add to X' an arbitrarily small constant vector field Y' (ie, $Y' \in \Xi_{\omega,0}$) which is tangent to $L(I_i)$ for all $1 \leq i \leq s+1$, then as in the discussion of Lemma 4.5, $\Gamma_2^{X'+Y'}(I_{i+1}\ r\ I_i) = \Gamma_2^{X'}(I_{i+1}\ r\ I_i)$, and we can therefore perturb any zeros of $X'(I_{s+1})$ off $\Gamma_2^{X'}(I_{i+1}\ r\ I_i)$: in fact we replace X' by X'+Y' such that the zeros of $(X'+Y')(I_{s+1})$ are disjoint from $\Gamma_2^{X'+Y'}(I_{i+1}\ r\ I_i)$ for all i, and since $\Gamma_k(I_{i+1}\ r\ I_i) \subset \Gamma_2(I_{i+1}\ r\ I_i)$ for all $k \geq 2$ these zeros are disjoint from all tangency sets. $X'' = X' + Y'$ may be such that $G_{X''}$ is no longer transverse to $T^{k(1)..k(s)}(I_1,..,I_{s+1})$ (because this submanifold is not closed), but applying (i) again we obtain by an arbitrarily small perturbation $G_{X'''}$ which is transverse to $T^{k(1)..k(s)}(I_1,..,I_{s+1})$ and with the zeros of $X'''(I_{s+1})$ now disjoint from $\Gamma_2^{X'''}(I_{i+1}\ r\ I_i)$ for all i. This means $G_{X'''}(M \cap L(I_{s+1})) \cap \zeta(I_{s+1})$ is disjoint from $\mathrm{clos}(G_{X'''}(M \cap L(I_{s+1})) \cap T^{k(1)..k(s)}(I_1,..,I_{s+1}))$ so we may remove a small open neighbourhood U of $\zeta(I_{s+1})$ from $\Xi_{\omega,r}$ which is disjoint from $G_{X'''}(M \cap L(I_{s+1})) \cap T^{k(1)..k(s)}(I_1,..,I_{s+1})$. $T^{k(1)..k(s)}(I_1,..,I_{s+1}) \backslash U$ is then a closed submanifold of $\Xi_{\omega,r}$ and hence by [35] $G_{\tilde{X}}$ remains transverse to $T^{k(1)..k(s)}(I_1,..,I_{s+1}) \backslash U$ for all $\tilde{X}$ sufficiently near X'''. ▬

The linear case of Proposition 4.2 follows by straightforward linear algebra.

The Iteration Sets

Definition Suppose submanifolds with corners M,M' of R^n carry semiflows respectively ϕ_M and $\phi_{M'}$: we shall say these semiflows are differentiably equivalent if there exists a diffeomorphism $f: M \rightarrow M'$ such that $f\phi_M(x,t) = \phi_{M'}(fx,t)$ for all $x \in M$ and for all $t \in [0,t_x)$.

Example 4.2 In this example we show that there exists a submanifold with corners M of R^3 and vector fields X,X' on R^3, and a diffeomorphism $f: M \rightarrow M$ such that $f\phi(M,X)(x,t) = \phi(M,X')(fx,t)$ for all $x \in M$, and for all $t \geq 0$, but $\Gamma_k^X(1,2\ r\ 2) \neq \varnothing$ any $k \geq 0$ while $\Gamma_k^{X'}(1,2\ r\ 2) = \varnothing$ for all k - ie a differentiable equivalence need not preserve tangency sets.

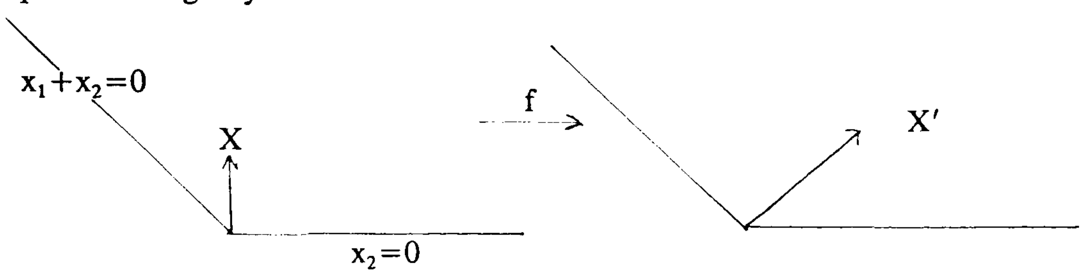


Figure 4.4

Take for our vector field the constant vector field $X=(0,1,0)$ on $M=M'=\{x\in R^3: x_2\geq 0, x_1+x_2\geq 0\}$ and consider the invertible linear map $f:M\rightarrow M$ with matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which clearly maps M to M and sends X to $X'=(1,1,0)$ (see Figure 4.4): then evidently the origin $\in\Gamma_k^X(1,2\ r\ 2)$ for all k but $\notin\Gamma_k^{X'}(1,2\ r\ 2)$ any $k>1$.

In this example the tangency sets are not generic (and do not satisfy Proposition 4.2) but by making the perturbation sufficiently large we can in a similar way construct an example of a generic tangency set not preserved by differentiable equivalence.

We have in Remarks 3.1(3) defined $\Im_i(x)=\{K:S^0_i(x)\subset K\subset S_i(x)\}$ where M locally is $ZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$, and we now set $\Im(x)=(\Im_1(x),\Im_2(x),..)$. We may regard $\Im(x)$ as a "contracting sequence" and for any given contracting sequence $c=(\Im_1,\Im_2,..)$ define the iteration set $\Im^{-1}(c)=\{x\in ZN(I;J):\Im(x)=c\}$.

Proposition 4.3 Differentiable equivalence preserves the iteration and hence the iteration sets.

This will be proved after Lemma 4.8. It is completely straightforward to prove that the $\{S_i(x)\}$ terms are preserved, and all the interest is in the $\{S^0_i(x)\}$ half of the iteration (the reason for this asymmetry is discussed below). This is in fact an opportune moment to introduce a refinement of the $\{S^0_i(x)\}$ half of the iteration which will be called the algorithm sequence $A^1_0,A^1_1,..,A^1_{k_1}=A^2_0,A^2_1,..,A^2_{k_2}=A^3_0,..$, where each A^r_j is a set of strata of $T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x))$ and is determined ultimately by x,M,X. Each subsequence $A^r_0,..,A^r_{k_r}$ is an algorithm for determining $S^0_{r+1}(x)$ given $S^0_r(x),S_r(x)$, and hence this subsequence $A^r_0,A^r_1,..,A^r_{k_r}$ may be viewed as a refinement of the consecutive pair of iterates $S^0_r(x),S^0_{r+1}(x)$ of the iteration.

The reason we introduce the algorithm sequence at this point is that we shall in fact prove a strong form of Proposition 4.3, that the sequence $\{S_i(x)\}$ and the whole algorithm sequence (and hence a fortiori the sequence $\{S^0_i(x)\}$) are preserved by differentiable equivalence. We have good reason to wish to refine the iteration in this way. We saw in Theorem 2.1 covering most cases (and in Remark 3.2 for all cases) that, crudely speaking, the iteration determines into which stratum or strata a trajectory is heading. Thus in a situation such as in Figure 4.5 below on ZN(∅;1,2) where X(M)(x)=**0**, whether X(x) points in the direction (i),(ii), or (iii) makes no difference to the iteration which is the same in all three cases (with $S^0_1(x)=\varnothing$, $S_1(x)=S_2(x)=S^0_2(x)=(1,2)$) despite the fact that the local flows in the three cases are distinguished by differentiable equivalence (as well in fact by the so-called spfp equivalence defined in Chapter Six).

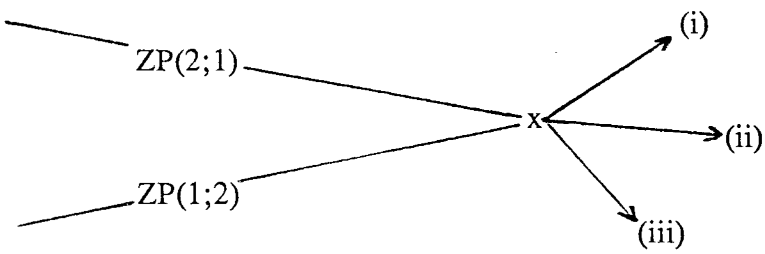


Figure 4.5

Thus the iteration at a point tells us about the trajectory through that point and not much about the local geometry of the semiflow. Our algorithm sequence however will distinguish between such cases as illustrated in Figure 4.5 (see below).

Definition (1) A constant vector field X on LC(I;J) is one where $X\in\Xi_{\omega,0}(LC(I;J))$
(2) A stratum LO(K;J\K) in LC(I;J) is strictly active for a constant vector field X if on LO(K;J\K) X(M)=X(K) (of course for a constant vector field if a property such as X(LC(I;J))(y)=X(K)(y) holds for some y∈LO(K;J\K) then it holds for all of y∈LO(K;J\K)) and X(M) | LO(K;J\K)≠X(K') | LO(K;J\K) any I⊂K'strictly contained in K (see Figure 4.6); by Lemma 2.4 this pair of conditions is equivalent to $\langle X(0),P(K\backslash j)n_j\rangle<0$ for all j∈K\I.

The term "strictly active" is used because in other contexts (eg Chapter 6) one may think of LO(1;2) in Figure 4.6 as active but not strictly active.

We shall show that the following algorithm provides us with the strictly active strata, and has certain other properties.

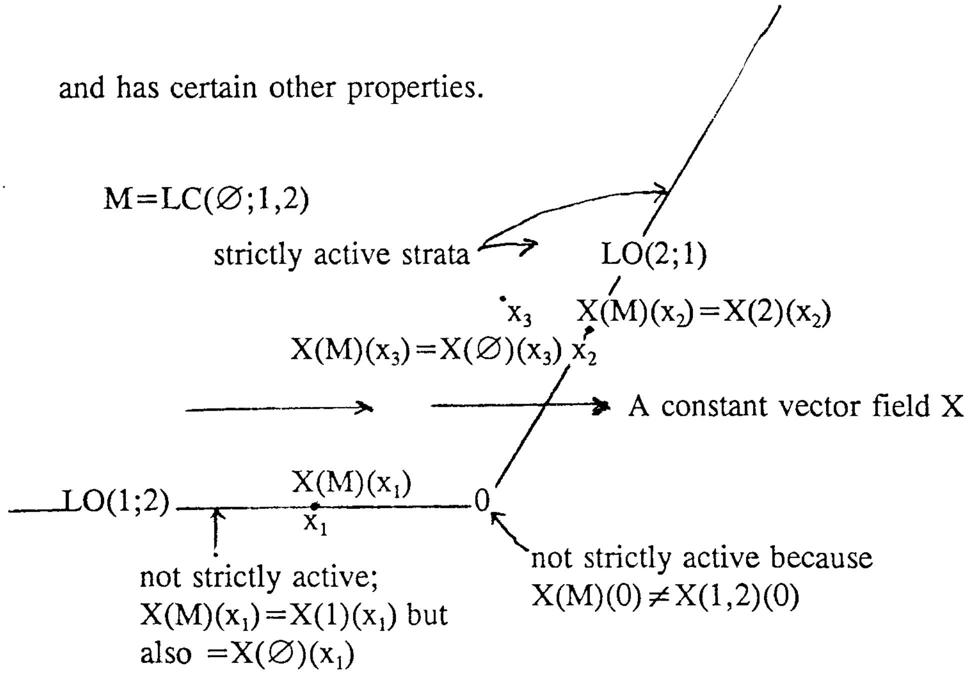


Figure 4.6

Algorithm Set $M=LC(I;J)$ and $A_0=LO(I;J)$, X is a constant vector field. If $X(0)\in LC(I;J)$ stop; otherwise set $A_1=$ those codimension 1 strata σ of $LC(I;J)$ (which can individually be denoted $A_{1,1}$, $A_{1,2}$, etc) such that $X(M) \mid \sigma \neq X$. If $\sigma=LO(K;J\backslash K)$ this means $X(I;K\backslash I)\neq X$. Setting $\breve{A}_{1,i}=$ affine span of $A_{1,i}$, $\bar{A}_{1,i}=$closure of $A_{1,i}$ (ie if $A_{1,i}=LO(K;J\backslash K)$ $\breve{A}_{1,i}=L(K), \bar{A}_{1,i}=LC(K;J\backslash K)$) then if for some i $X(\breve{A}_{1,i})\in \bar{A}_{1,i}$ stop, otherwise set $A_2=$ those codimension 2 strata σ of $LC(I;J)$ such that $X(M) \mid \sigma \neq X(\breve{A}_{j,i})$, j=0 or 1, any i. Inductively if for some i $X(\breve{A}_{j,i})\in \bar{A}_{j,i}$ stop, otherwise set $A_{j+1}$to be those codimension (j+1) strata of $LC(I;J)$ such that $X(M) \mid \sigma\neq X(\breve{A}_{j',i})$, any $j'\leq j$, any i.

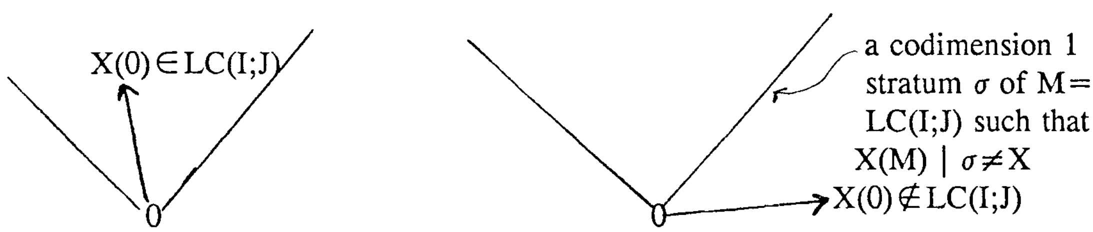
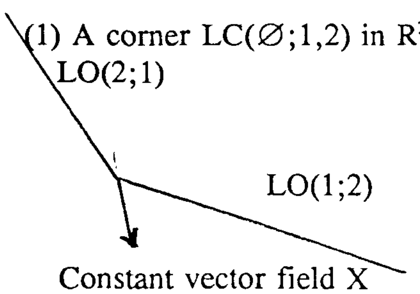


Figure 4.7

Examples 4.3 (Examples of the algorithm).

(1) A corner $LC(\varnothing;1,2)$ in R^2:



$A_0=LO(\varnothing;1,2)$
$A_{1,1}=LO(1;2), A_{1,2}=LO(2;1)$, where we stop, since $X(\breve{A}_{1,1})=X(1)\in \bar{A}_{1,1}=LC(1;2)$

(2) Suppose the corner is LC(∅;1,2,3) (Figure 4.8(a)) and in cross-section the constant vector field X impinges on the strata as shown in Figure 4.8(b), and suppose $\langle X(0),P(1,2)n_3\rangle > 0$.

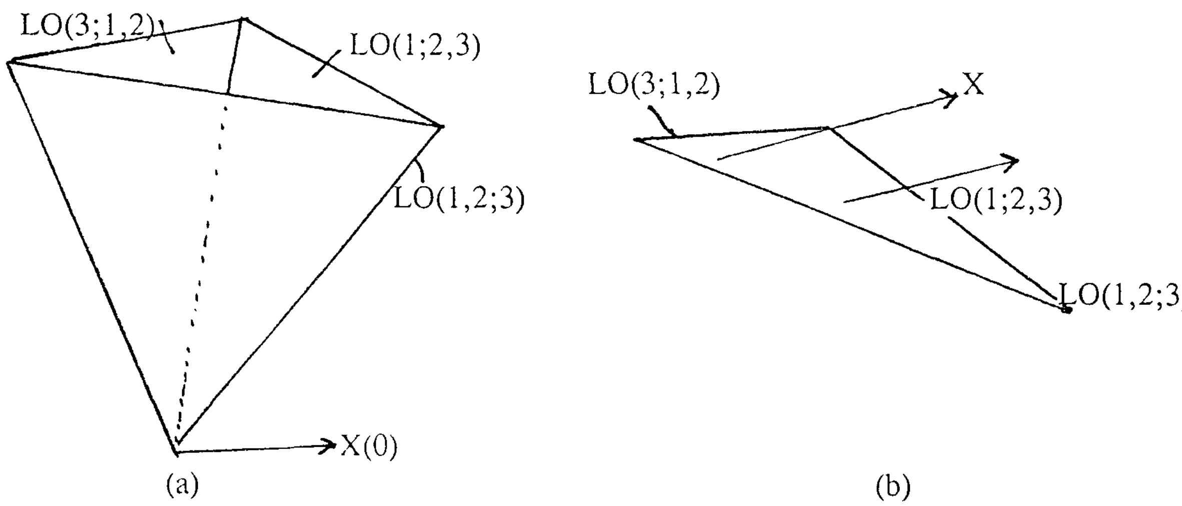


Figure 4.8

Then the algorithm gives A_0=LO(∅;1,2,3),A_1={LO(3;1,2), LO(1;2,3)},A_2={LO(1,2;3)} and then we stop since X(1,2)∈LC(1,2;3).

To prove Lemma 4.6 we shall need the following remark (which follows from definitions)

<u>Remark 4.3</u> If σ=LO(H;J\H) is a stratum of M=LC(I;J), so X(M) | σ=X(LC(I;H))X, then if X(M) | σ=X(K) where necessarily I⊂K⊂H then X(M) | LO(H′;J\H′)=X(K) for all K⊂H′⊂H, ie if X(M) on LO(H;J\H)=X(K) some K satisfying I⊂K⊂H then X(M) on LO(H′;J\H′)=X(K) for all H′ satisfying K⊂H′⊂H (see Figure 4.9).

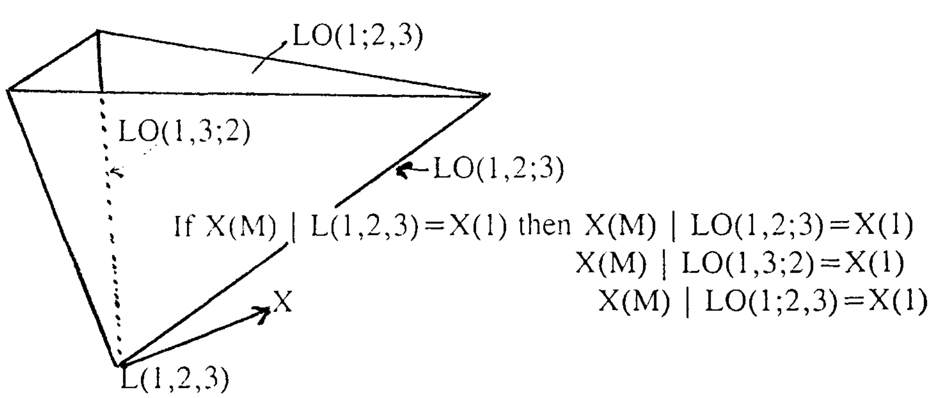


Figure 4.9. The corner is M=LC(∅;1,2,3)⊂R^n

As a prelude to Lemma 4.6 below we may verify that in Example 4.3(2) above

(i) The strictly active strata are LO(∅;1,2,3),LO(3;1,2),LO(1,2;3) and LO(1;2,3)

(ii) That the iteration (which will always have $\mathfrak{I}_j(x)=\mathfrak{I}_2(x)$ for all $j\geq 2$ if the vector field is constant) at 0 is $S^0_1(0)=\varnothing$,$S_1(0)=(1,2,3)$,$S^0_2(0)=(1,2)$ and $S_2(0)=(1,2)$

(iii) That if A_j is known A_{j+1} is determined by $\{\text{sign}\langle X(0),P(K_{j',i'})n_k\rangle : j'\leq j,\ A_{j',i'}$ is strictly active, $k\in J\backslash K_{j',i'}\}$ where $L(K_{j,i})=\breve{A}_{j,i}$.

Eg having established that $A_1=\{LO(3;1,2),LO(1;2,3)\}$ it follows

LO(3,2;1) is not in A_2 because $\langle X(0),P(3)n_1\rangle>0$

LO(3,1;2) is not in A_2 despite $\langle X(0),P(3)n_2\rangle<0$ because $\langle X(0),P(1)n_3\rangle>0$

but LO(1,2;3) is in A_2 because $\langle X(0),P(1)n_2\rangle<0$.

<u>Lemma 4.6</u> If X is a constant vector field on LC(I;J) with the algorithm as defined above then

(1) If the algorithm stops at the j_0th stage then for all $j\leq j_0$ A_j =set of codimension j (in L(I)) strictly active strata

(2) The algorithm stops no later than the $|J|$th stage, if this is the j_0th stage there exists a unique $A_{j_0,i}$ such that $X(\breve{A}_{j_0,i})\in\bar{A}_{j_0,i}$ and this $A_{j_0,i}=$ $LO(S^0_2(\text{origin},LC(I;J),X),J\backslash S^0_2(\text{origin},LC(I;J),X))$

(3) If A_j is known A_{j+1} is determined by $\{\text{sign}\langle X(0),P(K_{j',i'})n_k\rangle : j'\leq j,\ A_{j',i'}$ is strictly active, $k\in J\backslash K_{j',i'}\}$ where $L(K_{j,i})=\breve{A}_{j,i}$.

<u>Proof</u>

(1) Since by definition the vector field X we begin with is on LC(I;J) this is true for j=0. Suppose (1) is true up to j-1. We must show that if LO(K;J\K) is a codimension j stratum of LC(I;J) then LO(K;J\K) is strictly active iff it is in A_j ie iff $X(M)\,|\,LO(K;J\backslash K)\neq X(M)LO(K';J\backslash K')$ any strictly active LO(K';J\K') with $|K'\backslash I|<j$. Since we must have $X(M)\,|\,LO(K;J\backslash K)=X(H)$ some $I\subset H\subset K$ and since by Remark 4.3 we then have LO(H;J\H) strictly active, either H=K in which case LO(K;J\K) is strictly active and $X(K)=X(M)\,|\,LO(K;J\backslash K)\neq X(M)\,|\,LO(K';J\backslash K')$ any K' such that $|K'\backslash I|<j$, or H is strictly smaller than K in which case the stratum LO(H;J\H) is strictly active and $X(M)\,|\,LO(K;J\backslash K)=X(H)=X(M)\,|\,LO(H;J\backslash H)$.

(2) By (1) above we know that each $A_{j,i}$ is strictly active, and by the way the algorithm is constructed any $A_{j,i}$ where the algorithm stops satisfies $X(\breve{A}_{j,i}) \in \bar{A}_{j,i}$. By definition of the iteration $P(LC(I;J))X=P(K)X$ iff $S^0_2(\text{origin},LC(I;J),X) \subset K \subset S_2(\text{origin},LC(I;J),X)$. By Lemma 2.4 we know $P(LC(I;J))X=P(K)X$ iff

(a) $\langle X,P(K\backslash j)n_j\rangle \leq 0$ for all $j \in K\backslash I$, and

(b) $P(K)X \in LC(K;J\backslash K)$ ie $\langle P(K)X,n_j\rangle \geq 0$ for all $j \in J\backslash K$.

If $\langle X,P(K\backslash j)n_j\rangle=0$ then eg by Lemma 2.2 $X(K)=X(K\backslash j)$, hence $S^0_2(\text{origin},LC(I;J),X)$ is characterised by

(a) $\langle X,P(S^0_2\backslash j)n_j\rangle < 0$ for all $j \in S^0_2\backslash I$

(b) $P(S^0_2)X \in LC(S^0_2;J\backslash S^0_2)$ ie $\langle P(S^0_2)X,n_j\rangle \geq 0$ for all $j \in J\backslash S^0_2$.

(a) is just the condition that $LO(S^0_2;J\backslash S^0_2)$ is strictly active, so comparing this characterisation of S^0_2 with that above for the $A_{j,i}$ where the algorithm stops we see that we must have any such $A_{j,i}=LO(S^0_2;J\backslash S^0_2)$ (and so is unique), where $S^0_2=S^0_2(\text{origin},LC(I;J),X)$.

(3) This follows from the construction of the algorithm and (1). ∎

The algorithm we have constructed, A_0, A_1, A_2,.. is determined by a constant vector field X and corner $LC(n_1,..,n_k;n_{k+1},..,n_{k+m})$, so written out in full A_j is an abbreviated form of $A_j(X,LC(n_1,..,n_k;n_{k+1},..,n_{k+m}))=A_j(X,LC(I;J))$, where X is a constant vector field and $I=(1,..,k)$, $J=(k+1,..,k+m)$.

From definitions we know that $S^0_{r+1}(x,M,X)=S^0_2(0,T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x)),X_r)$, where in this expression X_r is interpreted as the constant vector field on the linear corner $T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x))$ which takes the value $D_t^r\phi(S^0_r(x))(x,t=0)-D_t^r\phi(S_r(x))(x,t=0)$ at every point. If we then set $A^r_j(x,M,X)=A_j(X_r,T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x)))$, $j=0,1,..$, $r=1,2,..$, (where we see that each $A^r_j(x,M,X)$ will be a set of strata of the linear corner $T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x))$, which is a subcorner of $T_xZN(S^0_1(x);S_1(x)\backslash S^0_1(x))$) and $A^r(x,M,X)=\{A^r_0(x,M,X),A^r_1(x,M,X),\ldots\}$ it follows from this fact and Lemma 4.6 that

$A^r(x,M,X)=\{A^r_0=T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x))=LC(\text{grad}f_i(x),i \in S^0_r(x);\text{grad}f_i(x),i \in S_r(x)\backslash S^0_r(x))=LC(S^0_r(x);S_r(x)\backslash S^0_r(x))$, $A^r_1,..,A^r_{j_0}=LO(S^0_{r+1}(x);S_r(x)\backslash S^0_{r+1}(x))\}$ and so can be viewed as an algorithm for (in effect) determining $S^0_{r+1}(x)$ given $S^0_r(x),S_r(x)$. We shall call $A(x,M,X)=A^1(x,M,X),A^2(x,M,X),..$ the <u>algorithm sequence</u> and from the foregoing see that (at least if the $\{S_i(x)\}$ are known A(x,M,X) constites a refinement of the sequence $S^0_1(x),S^0_2(x),\ldots$ Important for us is that the individual stages of each $A^r(x,M,X)$ may be used to distinguish between local flows not distinguished by the iteration. We saw in Figure 4.5 three fields for which the iteration at a given point was the same, but our algorithm sequence (in this case we need go no further than A^1) distinguishes between them -

(i) $A^1_0=LO(\varnothing;1,2),A^1_1=LO(2;1),A^1_2=LO(1,2;\varnothing)$

(ii) $A^1_0=LO(\varnothing;1,2),A^1_1=\{LO(1;2),LO(2;1)\},A^1_2=LO(1,2;\varnothing)$

(iii) $A^1_0=LO(\varnothing;1,2),A^1_1=LO(1;2),A^1_2=LO(1,2;\varnothing)$

(by construction of the algorithm and Lemma 4.6 the initial and final sets of strata in each A^r are functions of the iteration, and since we specifically chose (i)-(iii) to have the same iteration $A^1_0=LO(S^0_1(x);S_1(x)\backslash S^0_1(x))$, $A^1_2=LO(S^0_2(x);S_1(x)\backslash S^0_2(x))$ are the same in every case).

We proceed to show that the iteration and the algorithm sequence are preserved by differentiable equivalence, ie if f is a differentiable equivalence between $\phi(M,X)$ near x and $\phi(M',X')$ near x', then up to suitable identification (see below) the iteration and algorithm sequence for (x,M,X) are the same as those for (x',M',X'). In doing so we shall be reversing the emphasis of Theorem 2.1, Remark 3.2 and Corollary 5.2 where we are interested in establishing as much as possible about $\phi(M)(x)$ given the iteration $\{S^0_i(x),S_i(x)\}_{i\in Z^+}$; now we shall be determining the iteration from the semiflow.

We can readily show that the upper bound of the iteration, the S_i, are preserved by a differentiable equivalence, and all the work goes into treating the S^0_i case. The reason for this asymmetry arises from the fact that differentiable equivalence is a condition on $\phi(M)$, not immediately (if M is locally represented as ZN(I;J)) on the $\phi(K)$'s for $I\subset K\subset I\cup J$, and the link between the $S_i(x)$'s and $\phi(M)$ is much simpler than that between the $S^0_i(x)$'s and $\phi(M)$. It is straightforward to show that $\{D_t^{+i}f_j\phi(M)(x,t=0)\}_{i\in Z^+}$ determines $\{S_i(x)\}_{i\in Z^+}$, and since a differentiable equivalence preserves $\{D_t^{+i}f_j\phi(M)(x,t=0)\}_{i\in Z^+}$ (part(a) of the proof of Proposition 4.3) it preserves $\{S_i(x)\}_{i\in Z^+}$. We see in fact that the only part of the semiflow which is used is the single trajectory through $\phi(M)(x)$. $\phi(M)(x)$ is not though in itself enough to determine $\{S^0_i(x)\}_{i\in Z^+}$. Intuitively speaking $S_i(x)$ is the largest set of indices K lying between $S^0_{i-1}(x)$ and $S_{i-1}(x)$ for which $D_t^{+(i-1)}\phi(M)(x,t=0)= D_t^{(i-1)}\phi(K)(x,t=0)$ and $S^0_i(x)$ is the smallest such set; if we take the three fields on R^2 with $M=\{(x_1,x_2): x_2\geq 0\}$ (a) $\dot{x}_1=1,\dot{x}_2=0$, (b) $\dot{x}_1=1,\dot{x}_2=-x_1$, (c) $\dot{x}_1=1,\dot{x}_2=-1$

(a) (b) (c)

then to decide which is the smallest set of indices such that , for example, $D_t^+\phi(M)(0,t=0)=D_t\phi(K)(0,t=0)$ or $X(M)(0)=X(K)(0)$ we cannot just consider $\phi(M)(0)$ but must look locally: we see that in (a) and (b) it is $\varnothing$, in (c) it is (1). To infer from the existence of a differentiable equivalence that the $\{S^0{}_i\}$ are preserved we shall see that we need to find points $x_k \to x$ such that $D_t^{i'}\phi(M)(x_k,t=0)=D_t^{i'}\phi(S^0{}_i(x)\backslash j)(x_k,t=0)$ for all $i' \leq i$, ie we must show that a sequence of points $\{x_k\}$ with $x_k \to x$ exist which satisfy this condition. We shall use the algorithm sequence to do this, and will in fact in the process show the stronger result, that the algorithm sequence is preserved.

We have seen above that $A^r(x,M,X)$=the set of strictly active strata for the algorithm with data $(X_r,T_xZN(S^0{}_r(x),S_r(x)\backslash S^0{}_r(x)))$. We shall make the identifications $T_x(ZN(S^0{}_r(x),S_r(x)\backslash S^0{}_r(x)))$ with $LC(S^0{}_r(x),S_r(x)\backslash S^0{}_r(x))$ etc in the obvious way. We recall from Chapter Three that a funnel $F_x(r,f)$ at x for a flow ϕ_X on a submanifold Z is a set $f^{-1}F_c(n,r)$ where n is the dimension of Z and f is a "straightening-out" map $f:Z \to R^n$ such that $f.X$=unit vector field $\tilde{e}_1$ on R^n,and $F_c(n,r)=\{(t,x)\in R\times R^{n-1}:t\geq 0$ and $|x| \leq t^r\}$.

<u>Lemma 4.7</u> If M near x is locally represented as ZN(I;J) and if $LO(K;S_r(x)\backslash K)\in A^r(x,M,X)$ then there exists a funnel $F_x(r,f)$ for the flow $\phi(S_r(x)eK)$ in Z(K) about $\phi(S_r(x))$ and a neighbourhood U of x in $ZP(K;J\backslash K)$ such that for all $y\in F_x(r,f)\cap U$ $X(M)(y)=X(K)(y)$.

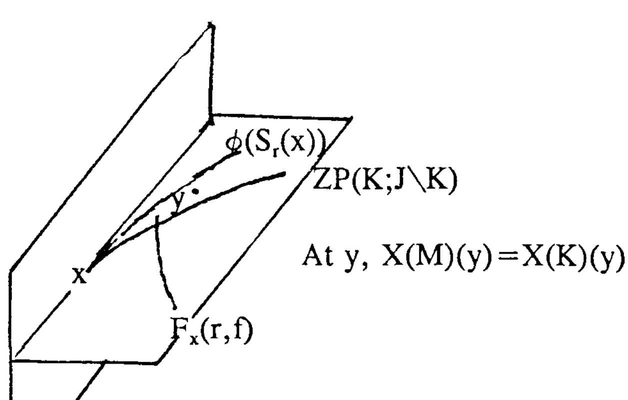


Figure 4.10

Proof The result will follow if we can show that for smooth $\psi:[0,T)\to Z(K)$ (T small and positive) with $\psi(0)=x$ and $D_t^i\psi(t=0)=D_t^i\phi(S_r(x))(x,t=0)$ for all $i<r-1$ that $\langle X(K\backslash j)\psi(t),\mathrm{grad}f_j\psi(t)\rangle<0$ for all small $t>0$ and for all $j\in K\backslash S^0_1(x)$, because if $y\in ZP(K;J\backslash K)$ then $T_yZN(I;J)=T_yZN(I;K\backslash I)$ and so by Lemma 2.4 $X(M)(y)=P(T_yZN(I;J))X(y)=X(K)(y)$ iff $\langle X(K\backslash j)(y),\mathrm{grad}f_j(y)\rangle\leq 0$ for all $j\in K\backslash I$. We consider two cases

(a) $j\in S^0_r(x)\backslash S^0_1(x)$. Then $j\in S^0_q(x)\backslash S^0_{q-1}(x)$ some q such that $1<q\leq r$ and so by Lemma 2.6

$D_t^i\langle P(K\backslash j)X\phi(K)(x,t=0),\mathrm{grad}f_j\phi(K)(x,t=0)\rangle=0$ for all $i\leq q-3$, <0 at $i=q-2$. The supposition that $LO(K;S_r(x)\backslash K)\in A^r(x,M,X)$ implies $S^0_r(x)\subset K\subset S_r(x)$ so by Corollary 2.1 $D_t^i\phi(K)(x,t=0)=D_t^i\phi(S_r(x))(x,t=0)$ for all $i<r$, and hence we may replace $\phi(K)$ by $\phi(S_r(x))$ in the above.

(b) $j\in K\backslash S^0_r(x)$. Then $LO(K;S_r(x)\backslash K)$ strictly active for A^r means $\langle P(K\backslash j)X_r,n_j\rangle<0$ for all $j\in K\backslash S^0_r(x)$ where $X_r=D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0))$

Consider $D_t^i\langle P(K\backslash j)X\phi(S_r(x))(x,t=0),\mathrm{grad}f_j\phi(S_r(x))(x,t=0)\rangle=$

$D_t^i\langle(P(K\backslash j)X\phi(S_r(x))(x,t=0)-P(K)X\phi(S_r(x))(x,t=0)),\mathrm{grad}f_j\phi(S_r(x))(x,t=0)\rangle.$ (*)

From the definition of the iteration and the fact that $S^0_r(x)\subset K\backslash j\subset K\subset S_r(x)$ we have as above that if $i\leq r-2$ then $D_t^iP(K\backslash j)X\phi(S_r(x))(x,t=0)=D_t^iP(K)X\phi(S_r(x))(x,t=0)$ and so (*)=0 if $i\leq r-2$.

By Lemma 2.3 $P(K\backslash j)X_r=D_t^r(\phi(K\backslash j)(x,t=0)-\phi(S_r(x))(x,t=0))$ and $P(K)X_r=D_t^r(\phi(K)(x,t=0)-\phi(S_r(x))(x,t=0))$, so if $i=r-1$

$(*)=\langle(P(K\backslash j)X_r-P(K)X_r),\mathrm{grad}f_j(x)\rangle=\langle P(K\backslash j)X_r,\mathrm{grad}f_j(x)\rangle<0$ by above.

Hence in either case (a) or (b) choosing $\psi:[0,T)\to Z(K)$ such that $D_t^i\psi(t=0)=D_t^i\phi(S_r(x))(x,t=0)$ for all $i<r-1$ we have for all sufficiently small $t>0$ and for all $j\in K\backslash S^0_1(x)$ that $\langle X(K\backslash j)\psi(t),\mathrm{grad}f_j\psi(t)\rangle<0$ as required. ∎

Lemma 4.8 If $j\in S_i(x)\backslash S^0_i(x)$ then

(1) $j\in S^0_{i+1}(x)\backslash S^0_i(x)$ iff $D_t^if_j\phi(S^0_{i+1}(x)\backslash j)(x,t=0)<0$

(2) $j\in S_i(x)\backslash S_{i+1}(x)$ iff $D_t^if_j\phi(S_{i+1}(x))(x,t=0)>0$

(3) $j\in S_{i+1}(x)\backslash S^0_{i+1}(x)$ iff $D_t^if_j\phi(K)(x,t=0)=0\ \forall S^0_{i+1}(x)\subset K\subset S_{i+1}(x)$

Proof (We use without further comment: if $j\in A$ $D_t^if_j\phi(A)=0\ \forall i$, P is self adjoint and idempotent, if $I_1\supset I_2$ $P(I_1)=P(I_1)P(I_2)$, that $S^0_i(x)\subset S^0_{i+1}(x)\subset S_{i+1}(x)\subset S_i(x)$). For this proof quantities of the form $D_t^i\phi(K)(x,t=0)$ will be written $D_t^i\phi(K)$.

(1) We show $j \in S^0_{i+1}(x)\backslash S^0_i(x)$ implies $D_t^i f_j \phi(S^0_{i+1}(x)\backslash j) < 0$: Consider $S^0_{i+1}(x)\backslash j$, any $j \in S^0_{i+1}(x)\backslash S^0_i(x)$. If $P(S^0_{i+1}(x)\backslash j)X_i = P(S^0_{i+1}(x))X_i$, where as in the proof of Lemma 4.7 $X_i = D_t^i(\phi(S_i(x)) - \phi(S^0_i(x)))$, then by construction of the iteration $S^0_{i+1}(x) \subset S^0_{i+1}(x)\backslash j \subset S_{i+1}(x)$ which is a contradiction; hence $P(S^0_{i+1}(x)\backslash j)X_i \neq P(S^0_{i+1}(x))X_i$. $\langle X_i, P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x)\rangle \leq 0$ $\forall j \in S^0_{i+1}(x)$ by Lemma 2.4, $P(S^0_{i+1}(x)\backslash j)X_i - P(S^0_{i+1}(x))X_i =$ $\langle P(S^0_{i+1}(x)\backslash j)\text{grad} f_j, X_i\rangle P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x) / |P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x)|^2$ by eg Lemma 2.2; the left hand side is $\neq 0$ and so $\langle P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x), X_i\rangle \neq 0$, we know it is ≤ 0, hence $\langle P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x), X_i\rangle < 0$. $\langle D_t^i(\phi(S^0_{i+1}(x)\backslash j) - \phi(S^0_{i+1}(x))), \text{grad} f_j(x)\rangle =$ $D_t^i f_j \phi(S^0_{i+1}(x)\backslash j)\langle P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x), \text{grad} f_j(x)\rangle / |P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x)|^2$ by Lemma 2.2. The right hand side $= D_t^i f_j \phi(S^0_{i+1}(x)\backslash j)$,
but $0 > \langle P(S^0_{i+1}(x)\backslash j)\text{grad} f_j(x), X_i\rangle = \langle P(S^0_{i+1}(x)\backslash j)X_i, \text{grad} f_j(x)\rangle =$ $\langle P(S^0_{i+1}(x)\backslash j)X_i - P(S^0_{i+1}(x))X_i, \text{grad} f_j(x)\rangle$ (since $j \in S^0_{i+1}(x) \subset S_{i+1}(x)$).
By Lemma 2.3 $P(S^0_{i+1}(x)\backslash j)X_i = D_t^i(\phi(S^0_{i+1}(x)\backslash j) - \phi(S_i(x)))$

$$P(S^0_{i+1}(x))X_i = D_t^i(\phi(S^0_{i+1}(x)) - \phi(S_i(x)))$$

hence $D_t^i f_j \phi(S^0_{i+1}(x)\backslash j)$ equals $\langle D_t^i\phi(S_{i+1}(x)\backslash j) - D_t^i\phi(S^0_{i+1}(x)), \text{grad} f_j(x)\rangle$ and by the foregoing this last quantity is positive.

(2) We show that $j \in S_i(x)\backslash S_{i+1}(x)$ implies $D_t^i f_j \phi(S_{i+1}(x)) > 0$: By Corollary 2.1 $D_t^i\phi(I) = D_t^i\phi(J)$ for all $S^0_{i+1}(x) \subset I, J \subset S_{i+1}(x)$; hence $D_t^i\phi(S_{i+1}(x)) = D_t^i\phi(S_\infty(x))$ and Lemma 2.5 gives the result.

(3) The fact that $j \in S_{i+1}(x)\backslash S^0_{i+1}(x)$ implies $D_t^i f_j \phi(K) = 0$ $\forall S^0_{i+1}(x) \subset K \subset S_{i+1}(x)$ follows from Corollary 2.1 and Lemma 2.2.

<u>Reverse Implications</u>:

(1) We need to show that if $j \in S_i(x)\backslash S^0_i(x)$ and $j \notin S^0_{i+1}(x)\backslash S^0_i(x)$ then $D_t^i f_j \phi(S^0_{i+1}(x)\backslash j) \nless 0$. If $j \in S_i(x)\backslash S^0_i(x)$ then $j \notin S^0_{i+1}(x)\backslash S^0_i(x)$ iff $j \in S_i(x)\backslash S^0_{i+1}(x)$. If $j \in S_i(x)\backslash S^0_{i+1}(x)$ then $S^0_{i+1}(x)\backslash j = S^0_{i+1}(x)$, therefore we must show that if $j \in S_i(x)\backslash S^0_{i+1}(x)$ then $D_t^i f_j \phi(S^0_{i+1}(x)) \nless 0$: in fact we have by (2) that if $j \in S_i(x)\backslash S_{i+1}(x)$ then $D_t^i f_j \phi(S^0_{i+1}(x)) > 0$ and by (3) that if $j \in S_{i+1}(x)\backslash S^0_{i+1}(x)$ then $D_t^i f_j \phi(S^0_{i+1}(x)) = 0$, so the result follows.

(2) We need to show that if $j \in S_i(x)\backslash S^0_i(x)$ and $j \notin S_i(x)\backslash S_{i+1}(x)$ then $D_t^i f_j \phi(S_{i+1}(x)) \ngtr 0$. Here the condition on j is equivalent to $j \in S_{i+1}(x)\backslash S^0_i(x)$, and hence $D_t^i f_j \phi(S_{i+1}(x)) = 0$ by (3).

(3) Follows from the construction of the iteration. ▬

Proof of Proposition 4.3

In Proposition 4.3 we are claiming that if we have a pair of systems (M,X) and (M′,X′) and a diffeomorphism $f:M\rightarrow M'$ such that $f\phi(M,X)(x,t)=\phi(M',X')(fx,t)$ for all $t\in[0,t_x)$ then the iteration is preserved. If M near x is locally represented as $ZN(I;J)=ZN(f_1,..f_k;f_{k+1},..,f_{k+m})$ then since $f:M\rightarrow M'$ is a diffeomorphism M′ locally is $ZN(f_1f^{-1},..,f_kf^{-1};f_{k+1}f^{-1},..,f_{k+m}f^{-1})$ and the sets of indices defining strata on M,M′ can be identified - near f(x) M′ locally is $ZN(f_1',..,f_k',f_{k+1}',..,f_{k+m}')$ where $f_i'=f_if^{-1}$. By saying that the iteration is preserved we mean that for all i the pairs of sets of indices $(S^0_i(x,M,X),S_i(x,M,X))$, $(S^0_i(f(x),M',X'),S_i(f(x),M',X'))$ can be identified in this way; similarly the algorithm is preserved if under the isomorphism $T_xM\rightarrow T_{f(x)}M'$ the strata in $A^r(x,M,X)$ can be identified with those in $A^r(fx,M',X')$ for each r; we shall denote these identifications with $\cong$. We show (a) that $S_i(x,M,X)\cong S_i(fx,M',X')$ for all i and (b) that $A^r(x,M,X)\cong A^r(fx,M',X')$ for all r (and hence that $S^0_i(x,M,X)\cong S^0_i(fx,M',X')$ for all i).

(a) We know by Theorem 3.1 that $D_t^{+i}\phi(M)(x,t=0)$ exists for all $i\geq 0$ and equals $D_t^i\phi(K)(x,t=0)$ any $S^0_{i+1}(x)\subset K\subset S_{i+1}(x)$. By lemma 4.8(2) we know that if $j\in S_i(x)\backslash S^0_i(x)$ then $j\in S_i(x)\backslash S_{i+1}(x)$ iff $D_t^if_j\phi(S_{i+1}(x))(x,t=0)>0$. In fact this result is true merely requiring that $j\in S_i(x)$, since if $j\in S^0_i(x)$ then $j\in S_{i+1}(x)$ and hence $D_t^if_j\phi(S_{i+1}(x))(x,t=0)=0$. We know (since M′ is diffeomorphic to M) that $S_1(x,M,X)\cong S_1(fx,M',X')$. Suppose inductively that we know $S_i(x,M,X)\cong S_i(fx,M',X')$. Then by the above if $j\in S_i(x,M,X)$ then $j\in S_i(x,M,X)\backslash S_{i+1}(x,M,X)$ iff $0<D_t^if_j\phi(S_{i+1}(x,M,X))(x,t=0)=D_t^{+i}f_j\phi(M)(x,t=0)$

$=D_t^{+i}f_jf^{-1}\phi(M')(fx,t=0)$ (by definition of f as differentiable equivalence)

$=D_t^{+i}f_j'\phi(M')(fx,t=0)=D_t^if_j'\phi(S_{i+1}(f(x),M',X')(fx,t=0)$

which if $j\in S_i(f(x),M',X')$ is >0 iff $j\in S_i(fx,M',X')\backslash S_{i+1}(fx,M',X')$, and hence $S_{i+1}(x,M,X)\}\cong S_{i+1}(fx,M',X')$.

(b) We wish to show that $A^r(x,M,X)\cong A^r(f(x),M',X')$ for all r. The data needed for $A^r(x,M,X)$ are $X_r=D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0))$ and the corner $T_xZN(S^0_r(x);S_r(x)\backslash S^0_r(x))$. We can infer $S^0_r(x)$ from the algorithm $A^{r-1}(x,M,X)$ (which terminates in $LO(S^0_r(x),S_{r-1}(x)\backslash S^0_r(x))$) so if we know that $S_i(x,M,X)\cong S_i(f(x),M',X')$ for all i (which we do by (a)) and that $S^0_1(x,M,X)\cong S^0_1(f(x),M',X')$ (which we do because there exists a diffeomorphism $M\rightarrow M'$ with $x\rightarrow f(x)$) it suffices to show that if (for any fixed $i\geq 1$) $S^0_i(x,M,X)\cong S^0_i(f(x),M',X')$ then $A^i(x,M,X)\cong A^i(f(x),M',X')$, because then the result will follow by induction. To show $A^r(x,M,X) \cong A^r(f(x),M',X')$ it suffices by Lemma 4.6(3) to show that if $LO(K;S_r(x)\backslash K)$ is a strictly active stratum of $LC(S^0_r(x);S_r(x)\backslash S^0_r(x))$ with respect to X_r ie, $LO(K;S_r(x)\backslash K) \in A^r(x,M,X)$, then $sign\langle gradf_j(x),P(K)X_r\rangle = sign\langle gradf_j'(f(x)),P(K)X_r'\rangle$ for all $j \in S_r(x)\backslash K$.

We have if $S^0_r(x) \subset K \subset S_r(x)$

$\langle gradf_j(x),P(K)X_r\rangle = \langle gradf_j(x),P(K)D_t^r(\phi(S^0_r(x))(x,t=0)-\phi(S_r(x))(x,t=0))\rangle$ (by definition of X_r)

$=\langle gradf_j(x),D_t^r(\phi(K)(x,t=0)-\phi(S_r(x))(x,t=0))\rangle$ by Lemma 2.3

$=\langle gradf_j(x),D_t^r(\phi(K)(x,t=0)-\phi(K\cup j)(x,t=0)+\phi(K\cup j)(x,t=0)-\phi(K\cup j\cup j_1)(x,t=0)-..$

$..-\phi(S_r(x))(x,t=0))\rangle$

$=\langle gradf_j(x),D_t^r(f_j\phi(K)(x,t=0))P(K)gradf_j(x)/\mid P(K)gradf_j(x)\mid^2+$

$D_t^r(f_{j_1}\phi(K\cup j)(x,t=0))P(K\cup j)gradf_{j_1}(x)/\mid P(K\cup j)gradf_{j_1}(x)\mid^2+..\rangle$ by Lemma 2.2

$=D_t^r(f_j\phi_X(K)(x,t=0))$

and of course similarly that $\langle gradf_j'(x'),P(K)X_r'\rangle = D_t^r(f_j'\phi_{X'}(K)(x',t=0))$, the suffices X,X' on ϕ in these formulae to remind us that $\phi(K)$ is in these cases the integral flow of respectively $X(K)$ and of $X'(K)$.

We have shown in Lemma 4.7 that for any K such that $S^0_r(x) \subset K \subset S_r(x)$ and such that $LO(K;S_r(x)\backslash K) \in A^r(x,M,X)$ there exists a funnel about $\phi(S_r(x))(x)$ in $Z(K)$ and a neighbourhood U of x in $ZP(K;J\backslash K)$ such that for all y in this funnel and in U (see Figure 4.11) $X(M)(y)=X(K)(y)$.

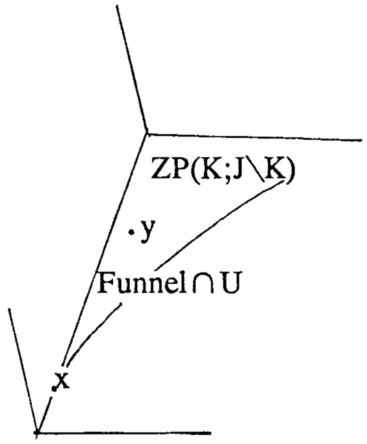


Figure 4.11

Thus if y is such a point then for all small $s>0$ $\phi(M)(y,s)=\phi(K)(y,s)$ and so $D_t^{+i}\phi(M)(y,t=0)=D_t^i\phi(K)(y,t=0)$ for all $i \geq 0$. Hence for each K such that $S^0_r(x) \subset K \subset S_r(x)$ with $LO(K;S_r(x)\backslash K) \in A^r(x,M,X)$ there exists a sequence $\{x_k\}$ with $x_k \rightarrow x$ such that for each x_k $D_t^{+i}\phi(M)(x_k,t=0)=D_t^i\phi(K)(x_k,t=0)$ for all i.

We know by the fact that $\phi(M)$ and $\phi(M')$ are differentiably equivalent that (as in (a)) $D_t^{+i}f_j\phi(M)(x_k,t=0)=D_t^{+i}f_j'\phi(M')(f(x_k),t=0)$ so we have $D_t^if_j'\phi_{X'}(K)(f(x),t=0)=$ $\lim_{x_k \rightarrow x}D_t^if_j'\phi_{X'}(K)(f(x_k),t=0)=$

$\lim_{x_k \to x} D_t^i f_j \phi_X(K)(x_k, t=0) =$
$D_t^i f_j \phi_X(K)(x, t=0)$ for all K as above, and the result follows. ∎

The Relation Between Iteration Sets and (generalised) Tangency Sets

We recall from earlier in this chapter the definition of $\Gamma_k(I \cup J \text{ r } J)$ - if functions $(f_1,..,f_{k+m})$ are independent and setting $I=(1..k)$, $J=(k+1..k+m)$
$\Gamma_k(I \cup J \text{ r } J)=\{x \in Z(I \cup J): D_t^i\phi(I \cup J)(x,t=0)=D_t^i\phi(J)(x,t=0) \text{ for all } i<k\}$. In the same context we the following open subsets of $\Gamma_k(I \cup J \text{ r } J)$:
$\Gamma_k^+(I \cup J \text{ r } J)=\{x \in \Gamma_k(I \cup J \text{ r } J): D_t^k f_j \phi(J)(x,t=0)>0 \ \forall j \in I\}$
$\Gamma_k^-(I \cup J \text{ r } J)=\{x \in \Gamma_k(I \cup J \text{ r } J): D_t^k f_j \phi(I \cup J \backslash j)(x,t=0)<0 \ \forall j \in I\}$.
Using Remark 4.1 (ii) we see
$\Gamma_k^+(I \cup J \text{ r } J)=\{x \in Z(J): D_t^m f_s \phi(J)(x,t=0)=0 \ \forall s \in I, \forall m<k, D_t^k f_s \phi(J)(x,t=0)>0 \ \forall s \in I\}$,
and $\Gamma_k^-(I \cup J \text{ r } J)=$
$\{x \in Z(I \cup J): D_t^m f_s \phi(J)(x,t=0)=0 \ \forall s \in I, \forall m<k, \ D_t^k f_s \phi(I \cup J \backslash s)(x,t=0)<0 \ \forall s \in I\}$.
Sets of this type with $|I|=1$ are used by Pugh [45].

As an example if $(\dot{x}_1,\dot{x}_2,\dot{x}_3)=(x_2,x_3,1)$ and $Z(J)=R^n$, $Z(I \cup J)=\{x \in R^n: x_1=0\}=L(1)$, then these sets stratify L(1) (and hence R^n) - $\Gamma_1^+(1 \text{ r } \varnothing)=\{x \in L(1): x_2>0\}$,
$\Gamma_1^-(1 \text{ r } \varnothing)=\{x \in L(1): x_2<0\},..,\Gamma_3^+(1 \text{ r } \varnothing)=L(1,2,3)$ $(=\Gamma_3(1 \text{ r } \varnothing))$,
$\Gamma_3^-(1 \text{ r } \varnothing)=\Gamma_4(1 \text{ r } \varnothing)=..=\varnothing$ (Figure 4.12).

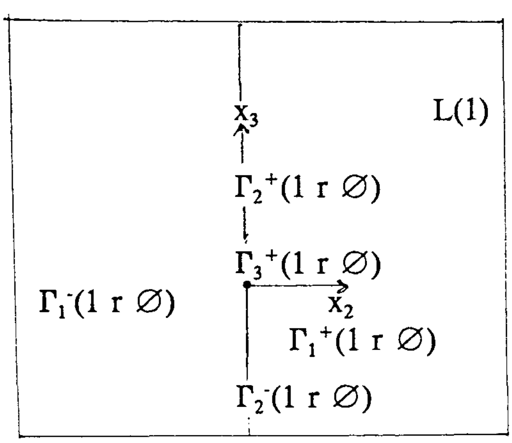


Figure 4.12

We can check that Proposition 4.2 applies to these sets $\Gamma_k^\pm(I \cup J \text{ r } J)$ as well as to $\Gamma_k(I \cup J \text{ r } J)$, ie that if $X \in \Xi'(M)$ (a) the sets $\Gamma_k^\pm(I \cup J \text{ r } J)$ and (b) intersections of them of the type considered in Proposition 4.2(b), are submanifolds (of the same codimension as if the superscripts + or - were absent).

We arrive at the result promised linking the iteration sets to the tangency sets. At the same time we show that for $X \in \Xi'(M)$ the iterationn sets are C^r submanifolds. We recall that for given $ZN(f_1,..,f_k;f_{k+1},..,f_{k+m})$ and vector field X on ZN(I;J), if $c=((S^0{}_1,S_1),(S^0{}_2,S_2),(S^0{}_3,S_3),..,S^0{}_r,S_r))$ is a contracting sequence, ie $I \subset S^0{}_1 \subset S^0{}_2 \subset .. \subset S^0{}_r \subset S_r \subset S_{r-1} \subset .. \subset S_1 \subset I \cup J$, then the iteration set $\mathfrak{I}^{-1}(c)=\{x \in ZN(I;J):S^0{}_i(x)=S^0{}_i,\ S_i(x)=S_i,\ i=1,..,r\}$. We see $\mathfrak{I}^{-1}(c) \supset \mathfrak{I}^{-1}(c')$ if $c' \supset c$. To guarantee that the iteration sets are submanifolds we shall need $X \in \Xi'(ZN(I;J))$, but the formula expressing an iteration set in terms of tangency sets holds for all X.

<u>Proposition 4.4</u> If M locally is ZN(I;J) and $c=((S^0{}_1,S_1),(S^0{}_2,S_2),..(S^0{}_r,S_r))$ is a contracting sequence, then locally $\mathfrak{I}^{-1}((S^0{}_1,S_1),(S^0{}_2,S_2),..,(S^0{}_r,S_r))$
$=ZP(S_1;J\backslash S_1) \cap \Gamma_1{}^+(S_1 \text{ r } S_2) \cap \Gamma_1{}^-(S^0{}_2 \text{ r } S^0{}_1) \cap \Gamma_2{}^+(S_2 \text{ r } S_3) \cap \Gamma_2{}^-(S^0{}_3 \text{ r } S^0{}_2) \cap ..$
$.. \cap \Gamma_{r-1}{}^+(S_{r-1} \text{ r } S_r) \cap \Gamma_{r-1}{}^-(S^0{}_r \text{ r } S^0{}_{r-1}) \cap \Gamma_r(S_r \text{ r } S^0{}_r)$ for any $r \geq 1$, and if furthermore $X \in \Xi'(ZN(I;J))$ it is a C^r boundaryless submanifold of $ZP(S_1;J\backslash S_1)$ of codimension
$|S_1\backslash S_2| + |S^0{}_2\backslash S^0{}_1| + 2|S_2\backslash S_3| + 2|S^0{}_3\backslash S^0{}_2| + ..$
$.. + (r-1)|S_{r-1}\backslash S_r| + (r-1)|S^0{}_r\backslash S^0{}_{r-1}| + r|S_r\backslash S^0{}_r|$ in $ZP(S_1;J\backslash S_1)$.

Eg: ZN(I;J)=ZN(∅;1,2) & c = ((∅,1),(∅,1),(∅,∅))

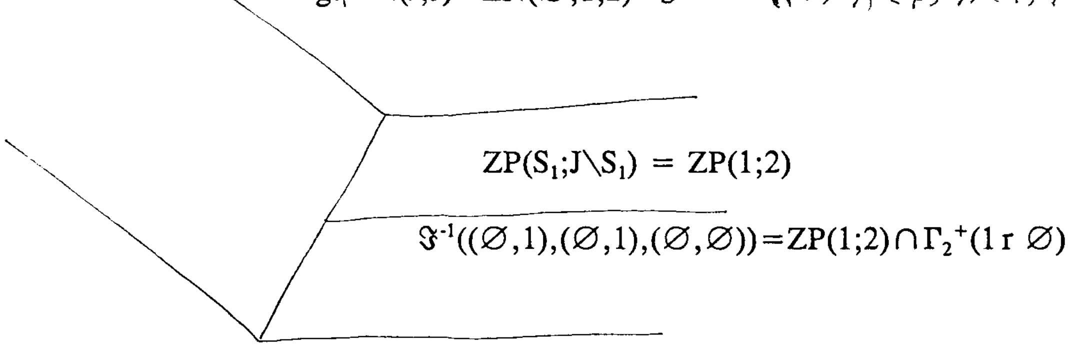


Figure 4.13

One implication of Proposition 4.4 is drawn in Corollary 4.2 - that if $X \in \Xi'(M)$ then the condition to apply Theorem 2.1, ie that $S^0{}_\infty(x)=S_\infty(x)$, applies at every point of M.

<u>Minor Remark</u> In applications the expression for iteration sets in terms of tangency sets would be used in the following form: if the contracting sequence $(s_1,s_1,...s_1,s_2,..,s_2,s_3,....,s_r)=(s_1{}^{i(1)},s_2{}^{i(2)},..s_r{}^{i(r)})$ where $s_i=(S^0{}_i,S_i)$, then $\mathfrak{I}^{-1}(s_1{}^{i(1)},s_2{}^{i(2)},..s_r{}^{i(r)})=$
$\Gamma_{i(1)}{}^+(S_1 \text{ r } S_2) \cap \Gamma_{i(1)}{}^-(S^0{}_2 \text{ r } S^0{}_1) \cap \Gamma_{i(1)+i(2)}{}^+(S_2 \text{ r } S_3) \cap \Gamma_{i(1)+i(2)}{}^-(S^0{}_3 \text{ r } S^0{}_2) \cap ..$
$.. \cap \Gamma_{\Sigma(i(j),j=1..r-1)}{}^+(S_{r-1} \text{ r } S_r) \cap \Gamma_{\Sigma(i(j),j=1..r-1)}{}^-(S^0{}_r \text{ r } S^0{}_{r-1}) \cap \Gamma_{\Sigma(i(j),j=1..r)}(S_r \text{ r } S^0{}_r)$.

Proof of Proposition 4.4 By definition $\Im^{-1}((S^0_1,S_1))=\{x\in ZN(I;J):S_1(x)=S_1\}=ZP(S_1;J\backslash S_1)$, so we suppose the result is true up to r and show that then it is true for r+1. From Lemma 4.8 if $(S^0_r(x),S_r(x))=(S^0_r,S_r)$ then $(S^0_{r+1}(x),S_{r+1}(x))=(S^0_{r+1},S_{r+1})$ iff $\forall\ j\in S^0_{r+1}\backslash S^0_r$ $D_t^r f_j\phi(S^0_{r+1}\backslash j)(x,t=0)<0$ etc. Hence if $(S^0_r(x),S_r(x))=(S^0_r,S_r)$ then using (in (i),(ii), (iii) below respectively) the facts that $S^0_r\subset S^0_{r+1}\backslash j\subset S_r$, $S^r_0\subset S_{r+1}\cup j\subset S_r$, and $S^0_r\subset S^0_{r+1}\subset S_{r+1}\subset S_r$ it follows that $(S^0_{r+1}(x),S_{r+1}(x))=(S^0_{r+1},S_{r+1})$ iff (i) $x\in\Gamma_r^-(S^0_{r+1}\ r\ S^0_{r+1}\backslash j)\ \forall\ j\in S^0_{r+1}\backslash S^0_r$, (ii) $x\in\Gamma_r^+(j\cup S_{r+1}\ r\ S_{r+1})\ \forall\ j\in S_r\backslash S_{r+1}$ and (iii) $x\in\Gamma_{r+1}(S_{r+1}\ r\ S^0_{r+1})$, and by the way $\Gamma_r^\pm(I\cup J\ r\ J)$ have been defined this is so iff $x\in\Gamma_r^+(S_r\ r\ S_{r+1})\cap\Gamma_r^-(S^0_{r+1}\ r\ S^0_r)\cap\Gamma_{r+1}(S_{r+1}\ r\ S^0_{r+1})$. This is the required inductive step: we have verified that the formula is true from definitions if r=1, our inductive hypothesis is that we have for r the form given in the statement of the proposition, and we have now shown that if additionally $(S^0_{r+1}(x),S_{r+1}(x))=(S^0_{r+1},S_{r+1})$ then $x\in\Im^{-1}((S^0_1,S_1),(S^0_2,S_2),..(S^0_r,S_r))\cap\Gamma_r^+(S_r\ r\ S_{r+1})\cap\Gamma_r^-(S^0_{r+1}\ r\ S^0_r)\cap\Gamma_{r+1}(S_{r+1}\ r\ S^0_{r+1})$. Using then that $\Gamma_r(I\cup J\ r\ J)\supset\Gamma_r^\pm(I\cup J\ r\ J)$ and that $\Gamma_r(I\cup J\ r\ J)\supset\Gamma_{r+1}(I\cup J\ r\ J)$ (by definitions) we get $\Gamma_r^+(S_r\ r\ S_{r+1})\cap\Gamma_r^-(S^0_{r+1}\ r\ S^0_r)\cap\Gamma_{r+1}(S_{r+1}\ r\ S^0_{r+1})$ $\subset\Gamma_r(S_r\ r\ S_{r+1})\cap\Gamma_r(S^0_{r+1}\ r\ S^0_r)\cap\Gamma_r(S_{r+1}\ r\ S^0_{r+1})=\Gamma_r(S_r\ r\ S^0_r)$ (the last equality by Corollary 4.1(3)) and hence in the formula we have obtained :

$\Im^{-1}((S^0_1,S_1),(S^0_2,S_2),..(S^0_{r+1},S_{r+1}))=$
$ZP(S_1;J\backslash S_1)\cap\Gamma_1^+(S_1\ r\ S_2)\cap\Gamma_1^-(S^0_2\ r\ S^0_1)\cap\Gamma_2^+(S_2\ r\ S_3)\cap\Gamma_2^-(S^0_3\ r\ S^0_2)\cap..$
$..\cap\Gamma_{r-1}^+(S_{r-1}\ r\ S_r)\cap\Gamma_{r-1}^-(S^0_r\ r\ S^0_{r-1})\cap\Gamma_r(S_r\ r\ S^0_r)\cap\Gamma_r^+(S_r\ r\ S_{r+1})\cap$
$\Gamma_r^-(S^0_{r+1}\ r\ S^0_r)\cap\Gamma_{r+1}(S_{r+1}\ r\ S^0_{r+1})$ the $\Gamma_r(S_r\ r\ S^0_r)$ term (which was the final term in $\Im^{-1}((S^0_1,S_1),..,(S^0_r,S_r))$ and now appears 4th from the end in $\Im^{-1}((S^0_1,S_1),..,(S^0_{r+1},S_{r+1}))$ is redundant and we obtain the desired expression for r+1. The codimension result follows if $X\in\Xi'(ZN(I;J))$ from Proposition 4.2(2). ∎

Corollary 4.2 For $X\in\Xi'(M)$ we may decompose M into submanifolds each contained in strata of M as a submanifold with corners, consisting of iteration sets $\Im^{-1}(c)$ for contracting sequences c, with $\Im^{-1}(c)=\varnothing$ if
$|S_1\backslash S_2|+|S^0_2\backslash S^0_1|+2|S_2\backslash S_3|+2|S^0_3\backslash S^0_2|+..$
$..+(r-1)|S_{r-1}\backslash S_r|+(r-1)|S^0_r\backslash S^0_{r-1}|+r|S_r\backslash S^0_r|>$dimension of M and hence for all but finitely many contracting sequences. It follows that if $X\in\Xi'(M)$ then $S^0_\infty(x)=S_\infty(x)$ for all
$x\in M$ and we may apply theorem 2.1 part 1 at every point of M.

Chapter Five

A Theorem about Recurring Strata and some Implications

Almost every problem in the theory of trajectories is concerned mainly with dealing with the situation where $\phi(M)(x,t)$ makes infinitely many stratum jumps in an arbitrarily small time interval. If instead for $0<t<\delta$ $\phi(M)(x,t)$ lies in a single stratum σ of M as a submanifold with corners, some $\delta>0$, then we saw in Remark 3.1(2) that $\phi(M)(x,t)=\phi(\bar{\sigma})(x,t)$ for all $t\in[0,\delta)$, and in particular $\phi(M)(x)$ is C^r on $(0,\delta)$. Turning from individual trajectories to the local semiflow a crucial result we shall need to prove a local stability theorem (Chapter Seven) is that if there are no points of infinite order tangency between flows on strata of M then the number of stratum jumps made by trajectories in any compact set is bounded uniformly on that compact set. In this chapter we shall establish a theorem (Theorem 5.1) covering both situations and derive some implications.

Throughout this chapter M is a smooth submanifold with corners of R^n and X is a smooth vector field on M.

Definition Suppose $x\in M$ with $\{\sigma_j\}$ the strata of M as a submanifold with corners. A recurring set of strata at x is a set of distinct strata $(\sigma_0,..,\sigma_{r-1})$ such that there exists a sequence of points $\{x_i\}\subset M$ with $x_i\to x$, where $x\in\bar{\sigma}_j$ for all $0\le j\le r-1$, and for each $i\in Z^+$ there exist $0=t_i^0<t_i^1<..<t_i^i\le h_i$ where $h_i\downarrow 0$, $\phi(M)(x_i,t_i^j)\in\sigma_{j,\mathrm{mod}\, r}$ for all $0\le j\le i$ and the trajectory segments $\phi(M)(x_i,[0,h_i))=\{\phi(M)(x_i,t):0\le t<h_i\}\subset\mathrm{conv}(\sigma_0,..,\sigma_{r-1})$, where we recall from Chapter Four that $\mathrm{conv}(\sigma_0,..,\sigma_{r-1})=$(the intersection of all subcorners of M containing $\cup_{j=0}^{r-1}\sigma_j$) = (the smallest subcorner containing $\cup_{j=0}^{r-1}\sigma_j$).

Examples 5.1 (Examples of recurring strata).

(1) In Example 2.1 the strata $ZP(\varnothing;1)$ and $ZP(1;\varnothing)$ are recurring at the origin (take x_i=origin $\forall i$, h_i any sequence $\downarrow 0$)

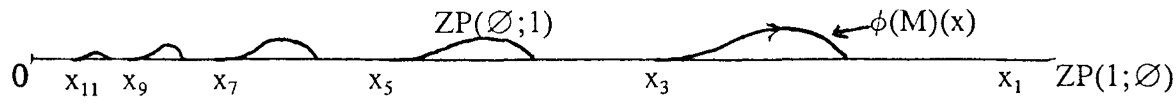


Figure 5.1

(2) We can modify Example 2.1 to obtain an infinite number of hits on the left

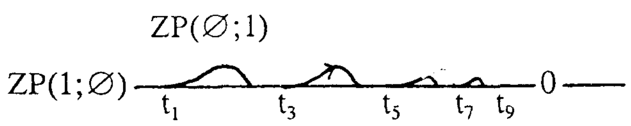


Figure 5.2

and again ZP(1;∅) and ZP(∅;1) are recurring at the origin - take $x_i=t_{2i+1}$, $h_i=|t_{2i+1}|$

(3) We may find a field X such that X(1),X(2) have respectively the properties of Examples (1) and (2) above (see Figure 5.3).

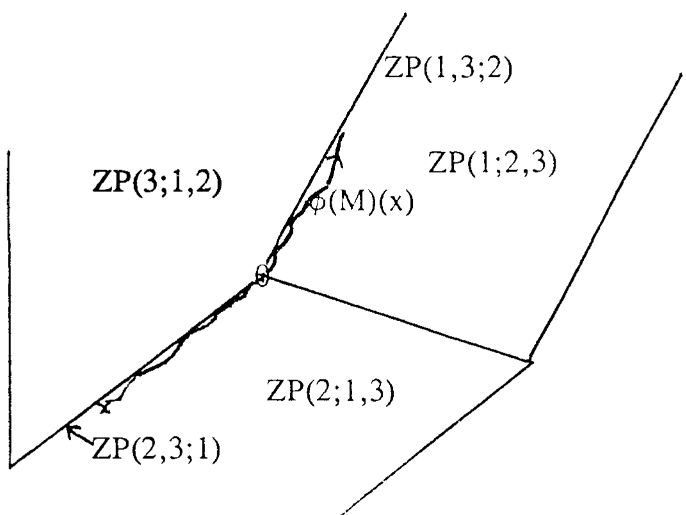


Figure 5.3

For this system we have on ZNP(2;3;1)=ZP(2,3;1)∪ZP(2;1,3) X(M)=X(2) or X(2,3), on ZNP(1;3;2)=ZP(1,3;2)∪ZP(1;2,3) X(M)=X(1) or X(1,3) and the integral curve through a point on ZP(2,3;1) has the form shown. Then the pair (ZP(1;2,3),ZP(1,3;2)) is recurring at 0 and (ZP(2;1,3),ZP(2,3;1)) is recurring at 0, but no subset of size three or more such as

(ZP(1;2,3),ZP(1,3;2),ZP(2;1,3),ZP(2,3;1)) is recurring at 0.

(4) Set

$$X(x_1,x_2,x_3)=\begin{cases}(1,\exp(-1/x_3)\sin(x_1/x_3),0) \text{ if } x_3>0\\ (1,0,0) \text{ if } x_3\leq 0\end{cases}$$

on R^3, with $M=\{x\in R^3:x_2\geq 0\}$.

Looking down onto the plane $x_2=0$ we have trajectories $\phi(M)(x)$ parallel to the x_1-axis (Figure 5.4a).

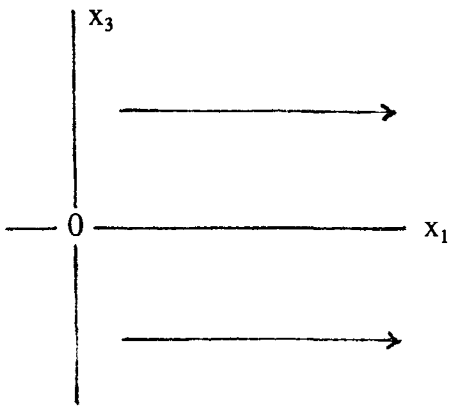


Figure 5.4a

Each trajectory $\phi(M)(0,0,x_3)$ with $x_3>0$ looks like

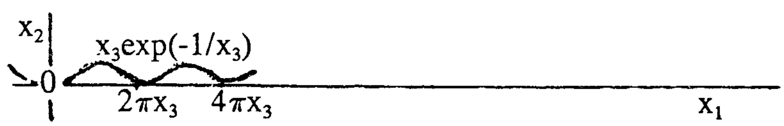


Figure 5.4b

with $x_2(t)=x_3\exp(-1/x_3)(1-\cos(t/x_3))$ and on every compact neighbourhood of 0 any single trajectory makes only finitely many transitions between $\text{int}(M)=ZP(\varnothing;2)$ and $\partial M=ZP(2;\varnothing)$, but $(ZP(2;\varnothing),ZP(\varnothing;2))$ are still recurring at 0 - take $x_i=(0,0,1/i^2)$, $h_i=i(2\pi/i^2)$ and let $i\to\infty$.

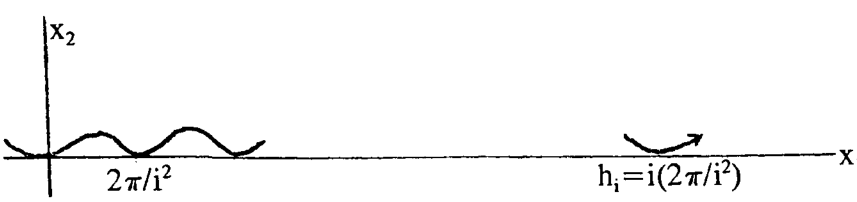


Figure 5.4c. Trajectory through $(0,0,1/i^2)$

(End of Examples 5.1).

We can check that in each of the above examples if a set of strata $(\sigma_0,..,\sigma_{r-1})$ is recurring at x then for any $j,k\in(0,..,r\text{-}1)$ $D_t^i\phi(\breve{\sigma}_j)(x,t=0)=D_t^i\phi(\breve{\sigma}_k)(x,t=0)$ for all i, $\breve{\sigma}_j$ being (as in Chapter Four) the C^r extension of σ_j. Intuitively we think of this as meaning that the flows on the strata (or really the strata extensions since the strata themselves are disjoint) are infinitely tangent at x.

Theorem 5.1 where we establish this fact in general is a key result in this thesis, the implications of which occupy the second half of this chapter and Chapter Seven. It is simpler to prove if M has only orthogonal corners (defined in Remark 2.5) and since this covers the applications we shall restrict ourselves to this context.

Theorem 5.1 If M is a submanifold with orthogonal corners locally represented as ZN(I;J) and if strata $\{ZP(I_j,J\backslash I_j),j=0,..,r\text{-}1\}$ are recurring at $x\in ZN(I;J)$ then $D_t^i\phi(\cap_{j=0}^{r-1}I_j)(x,t=0)=D_t^i\phi(\cup_{j=0}^{r-1}I_j)(x,t=0)$ for all $i\geq 0$; equivalently if strata $\{\sigma_j\}_{j=0,..,r-1}$ are recurring at $x\in M$ then the flows on all the strata in $\text{conv}\{\sigma_j\}_{j=0,..,r-1}$ are infinitely tangent at x.

We re-iterate that the trajectory segments $\phi(M)(x_j,[0,h_j))$ in the definition of recurring strata can all be on the same trajectory or overlap (as in the case of Example 5.1(2) where we can take $x_i=t_{2i+1}$, $h_i=|t_{2i+1}|$, or Example 5.1(1), where we may take $x_i=0$ for all i, h_i any sequence $\downarrow 0$).

To prove Theorem 5.1 we shall have to deal with the fact that the trajectory segments arising in the definition of recurring strata are by their nature highly non-smooth. Let us consider how we would prove it in a particular case.

Example 5.2

$M=\{y\in R^n:y_1,y_2\geq 0\}$. We shall set $\sigma_0=\text{int}(M)$, $\sigma_1=\{y\in R^n:y_1=0,y_2>0\}$, $\sigma_2=\{y\in R^n:y_2=0,y_1>0\}$, $\sigma_3=\{y\in R^n:y_2=0,y_1=0\}$. We suppose there exist $\{x_j\}_{j\in Z^+}\subset M$ and $\{h_j\}_{j\in Z^+}\subset R^+$ where $x_j\rightarrow x\in\sigma_3$, $h_j\downarrow 0$ and so that for each j the trajectory segment $\phi(M)(x_j,[0,h_j))$ hits each of σ_1,σ_2 at least j times (see Figure 5.5). We are claiming that

$D_t^i\phi(1)(x,t=0)=D_t^i\phi(1,2)(x,t=0)=D_t^i\phi(2)(x,t=0)=D_t^i\phi(\emptyset)(x,t=0)$ for all $i\geq 0$.

If we were constructing an analytic (in the sense of concrete) example the following symmetry relation would probably hold (this type of relation does in Examples 5.1(1,2 and 4)): if for points $x,y\in M$ $P(1,2)x=P(1,2)y$ then $X(x)=X(y)$, ie the vector field is independent of y_1 and y_2. This implies $X\phi(M)(x,t)=X\phi(1,2)(P(1,2)x,t)$ for all t, and hence if $\phi(M)(x)$ re-enters σ_0 from σ_1 and σ_2 j times on [0,T], ie $\langle X\phi(M)(x,t),n_m\rangle$ has j zeros on [0,T], m=1,2, then the smooth $\langle X\phi(1,2)(P(1,2)x,t),n_m\rangle$ also has j zeros on [0,T], m=1,2, and the result may be inferred directly. This special type of case provides in fact no hint as to how one proceeds in general.

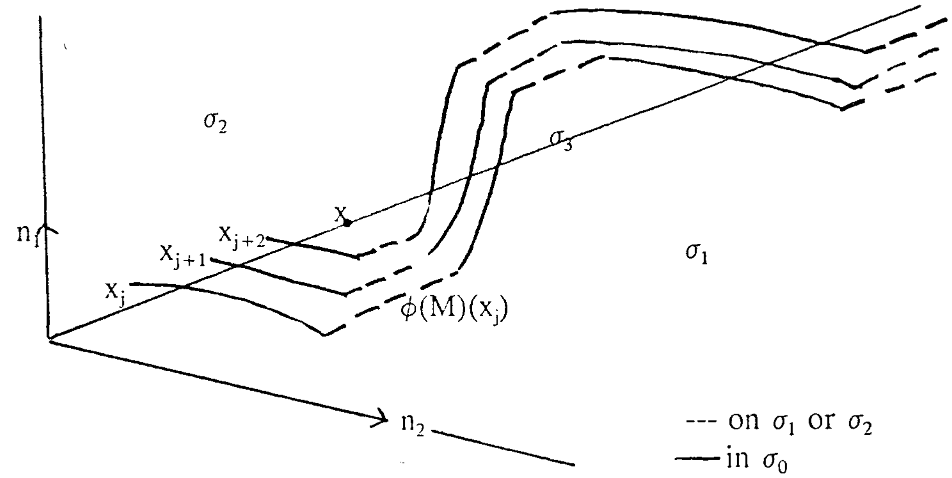


Figure 5.5

We make for fixed k the inductive hypothesis

(1) that there exists a subsequence $\{x^k_j\}$ of $\{x_j\}$ with $x^k_j \to x$ as $j\to\infty$, for each $j \in Z^+$ reals $0=T^k_{j,0}<T^k_{j,1}<..<T^k_{j,j}\downarrow 0$ with points $x^k_{j,j'}=\phi(M)(x^k_j,T^k_{j,j'})$ on $\phi(M)(x^k_j,[0,T^k_{j,j'}))$ such that if j' is odd $X(M)(x^k_{j,j'})=X(1)(x^k_{j,j'})$ and if j' is even $X(M)(x^k_{j,j'})=X(2)(x^k_{j,j'})$, and

(2) that $(1/(T^k_{j,j'+1}-T^k_{j,j'}))^i D_t^{k-i}\langle X(\varnothing)\phi(\varnothing)(x^k_{j,j'},t=0),n_m\rangle$ (*)

is uniformly bounded for all integers j, for all $0\leq i\leq k$ and for all $0\leq j'\leq j$, m=1,2. Since $T^k_{j,j'+1}-T^k_{j,j'}\to 0$ and $x^k_{j,j'}\to x$ as $j\to\infty$ for any $\{(j,j')\}$ with $j'\leq j$ the inductive hypothesis implies in particular that $D_t^{k-1}\langle X(\varnothing)\phi(\varnothing)(x,t=0),n_m\rangle=0$ for m=1,2 which using Lemma 2.2 tells us that $D_t^k\phi(1,2)(x,t=0)=D_t^k\phi(\varnothing)(x,t=0)$ etc, ie, if we can show the inductive hypothesis holds for all k then the result follows.

Evidently this inductive hypothesis is satisfied for k=0, taking $x^0_j=x_j$, $T^0_{j,j}=h_j$, where x_j, h_j are given to us by supposition, and $x^0_{j,j'}$ are points along $\phi(M)(x_j)$ where the curve re-enters σ_0 from σ_1 (if j' odd) and from σ_2 (if j' even) (see Figure 5.5) and we can fix $T^0_{j,j'}$ by $\phi(M)(x_j,T^0_{j,j'})=x^0_{j,j'}$ (if k=0 condition (2) is satisfied so long as the points remain in a bounded set). Thus we must show that if the hypothesis holds for k then it holds for k+1.

Using Lemmas 5.2 and 5.3 we can, if the inductive hypothesis is satisfied for k, for each j find smooth curves $\psi^k_j:[0,T^k_{j,j})\rightarrow M$ which are tangent to $\phi(M)(x^k_j)$ at $x^k_{j,j'}$ for each $j'\leq j$ (and hence to σ_1 for odd j' and to σ_2 for even j') and which furthermore themselves satisfy certain uniform bound conditions of a form similar to (*) (see Figure 5.6). For any $r\geq 0$ such that $j\geq r+2k$ there are therefore between the pair of points $x^k_{j,r}$, $x^k_{j,r+2k}$ along ψ^k_j k points of tangency between ψ^k_j and σ_m, m=1,2, ie k points where $\langle X(\varnothing)\psi^k_j(t),n_m\rangle=0$, and hence on $[T^k_{j,r},T^k_{j,r+2k}]$ there are k-i points where $D_t^i\langle X(\varnothing)\psi^k_j(t),n_m\rangle=0$. This, (*) in the inductive hypothesis and Lemma 5.4 then tell us that $\mathrm{Sup}_{t\in[T^k_{j,r},T^k_{j,r+2k}]}(1/(T^k_{j,r}-T^k_{j,r+2k}))^i(D_t^{k-i}\langle X(\varnothing)\psi^k_j(t),n_m\rangle)$ remain uniformly bounded as $j\rightarrow\infty$, m=1,2 for all $0\leq i\leq k$, which combined with Lemma 5.5 tells us that $(1/(T^k_{j,r}-T^k_{j,r+2k}))^i(D_t^{k-i}\langle X(\varnothing)\phi(\varnothing)(x^k_{j,r},t=0),n_m\rangle)$ is uniformly bounded as $j\rightarrow\infty$ for all $0\leq i\leq k$, m=1,2.

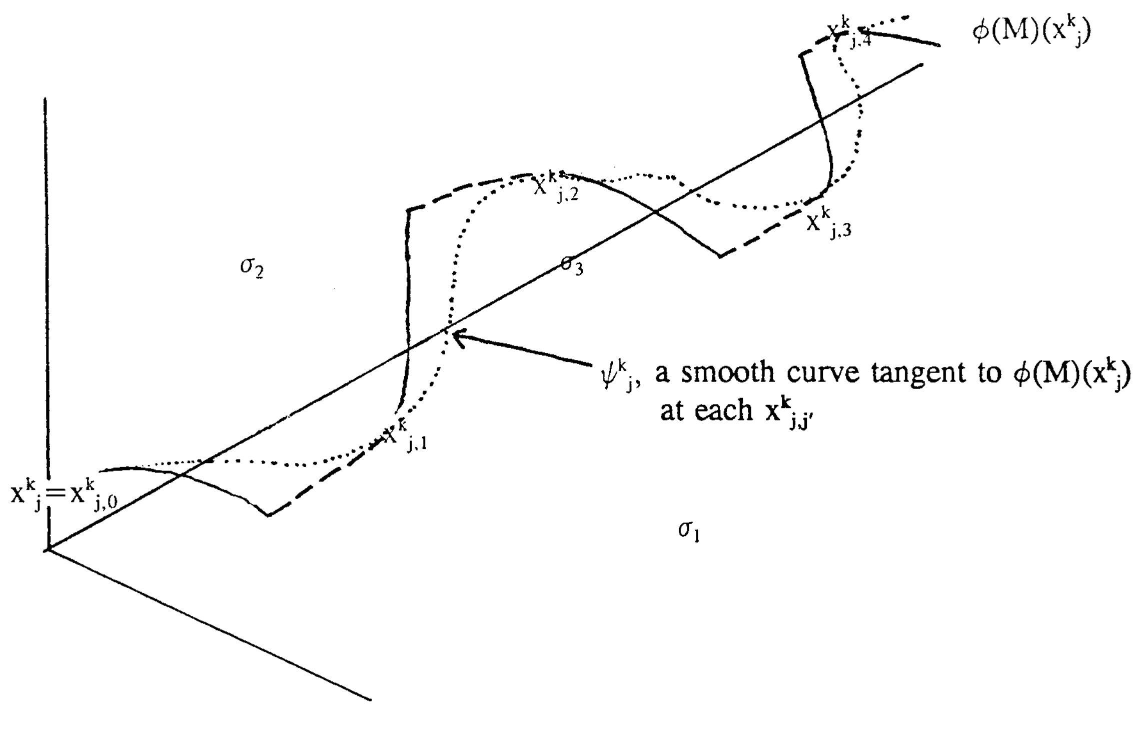


Figure 5.6

If we now set $x^{k+1}_j=x^k_{2jk}$ and $T^{k+1}_{j,j'}=T^k_{2kj,2kj'}$ (so $x^{k+1}_{j,j'}=x^k_{2kj,2kj'}$) so $\{x^{k+1}_j\}$ is a subsequence of $\{x^k_j\}$ and $\{x^{k+1}_{j,j'}\}$ is a subsequence of $\{x^k_{2jk,j'}\}$ (see Figure 5.7), we see that this proves the inductive hypothesis for k+1.

The only extra ingredient in the general case is that more work is involved in selecting the points $x^0_{j,j'}$ (Lemma 5.1). Note incidentally that what the segments

$\phi(M)(x_j,[0,h_j))$ do in *addition* to intersecting $\sigma_0,\sigma_1,..\sigma_{r-1}$ cyclically in turn is irrelevant, so long as they remain in conv$(\sigma_0,..,\sigma_{r-1})$: for example if in Example 5.2 in addition to entering σ_0 from σ_1,σ_2 at $x_{j,j'}$ these curves intersect σ_3 or have tangencies with σ_3 it makes no difference to the result or proof.

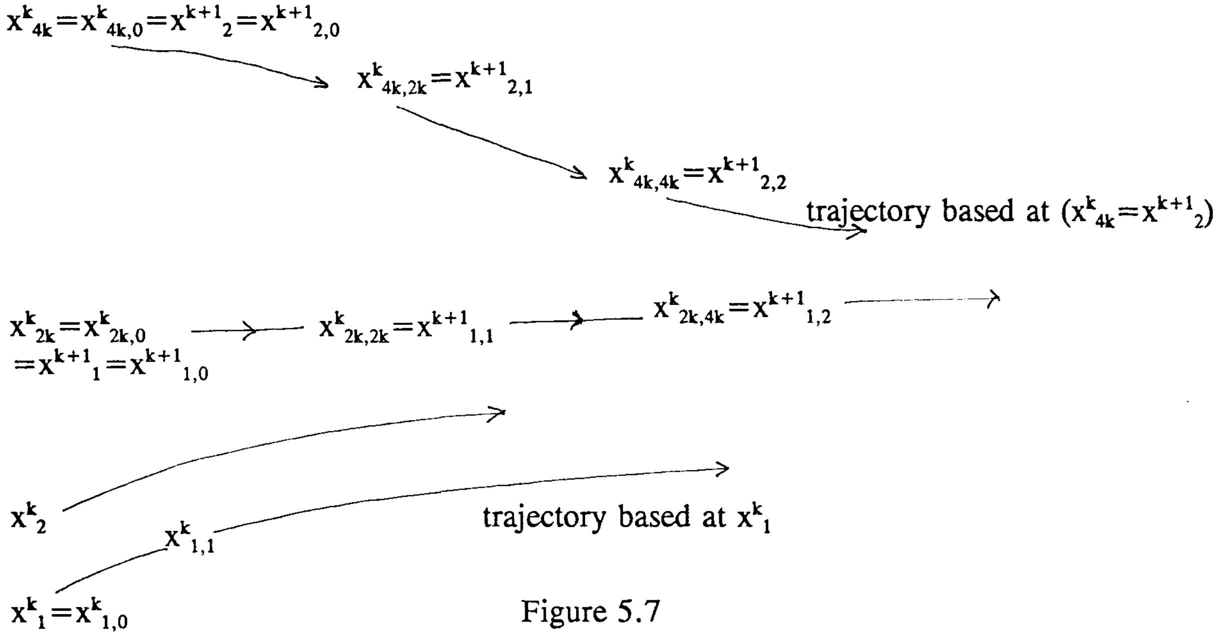


Figure 5.7

(End of Example 5.2)

Lemma 5.1 generalises the fact that if a trajectory bounces several times between strata then we can find a succession of points along the trajectory where certain vector fields $X(K)$ (in the case of Example 5.2, $X(\varnothing)$ and $X(1)$ or $X(\varnothing)$ and $X(2)$) or their smooth extensions coincide.

<u>Lemma 5.1</u> If M is a submanifold with orthogonal corners with $x\in M$, $0=t_0<t_1<..<t_j$ with $\phi(M)(x,[0,t_j))\subset ZN(I;J)=$ a local orthogonal representation of M (with x in $ZN(I;J)$ but not necessarily in $Z(I\cup J)$), and if for $I\subset I_0,..,I_{r-1}\subset I\cup J$ $\phi(M)(x,t_i)\in ZP(I_{i,\mathrm{mod}\ r},J\backslash I_{i,\mathrm{mod}\ r})$, for all $0\le i\le j$, then for each $k\in \cup_{i=0}^{r-1}I_i\backslash\cap_{i=0}^{r-1}I_i$ and for each $m\ge 0$ with $(m+1)r\le j$ there exists a point $t\in(t_{mr},t_{(m+1)r})$ such that $\langle X(\cap_{i=0}^{r-1}I_i)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle=0$.

Eg If in Figure 5.8 below $\phi(M)(x,t)\in ZP(1,2;3)$ for even t and $\phi(M)(x,t)\in ZP(3;2,1)$ for odd t, so $\cap I_i=\varnothing$, $\cup I_i=(1,2,3)$, then on each interval (t_i,t_{i+2}) there exists for each $j=1,2,3$ a point t where $\langle X(\phi(M)(x,t)),\mathrm{grad}f_j(\phi(M)(x,t))\rangle=0$

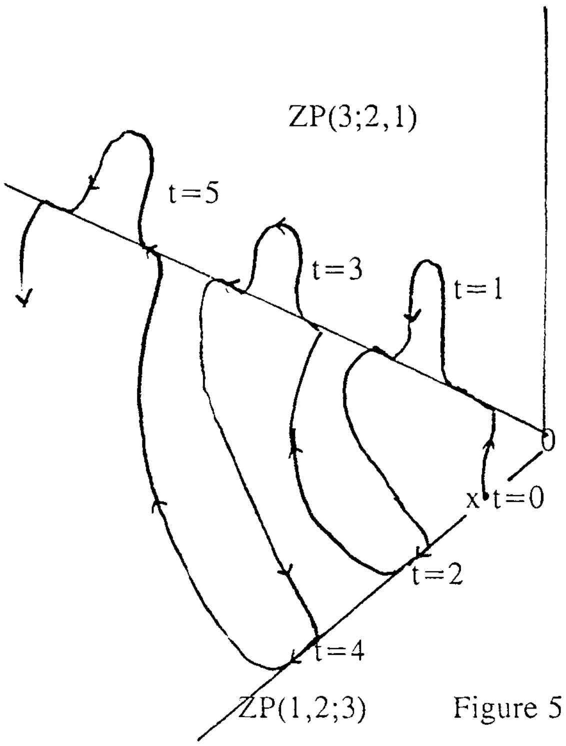


Figure 5.8

Proof

(1) We show that if on the domain of the local orthogonal representation ZN(I;J) $f_i(y)>0$ some $i \in J$ then $\langle X(M)(y), \mathrm{grad}f_i(y)\rangle = \langle X(K)(y), \mathrm{grad}f_i(y)\rangle$ any $K \supset I$ such that $i \in J \setminus K$, from which it follows that $t \to \langle X(M)\phi(M)(x,t), \mathrm{grad}f_i(\phi(M)(x,t))\rangle$ is continuous for as long as $f_i\phi(M)(x,t)>0$ (this is not the case if the corners are not orthogonal). We recall that because X is defined on M locally represented as ZN(I;J) the vector field X we begin with is identical to X(I). From Remark 2.5(1) we know that if $y \in ZN(I;J)$

$X(M)(y) = X(S_1(y))(y) + \Sigma_{i \in S_1(y)\setminus I} \mathrm{grad}f_i(y) \max(\langle \mathrm{grad}f_i(y), X(\emptyset)(y)\rangle, 0) / |\mathrm{grad}f_i(y)|^2$

where of course $I \subset S_1(y) \subset I \cup J$. By Remark 2.1 we know that

$X(S_1(y)) = X(I)(y) - \Sigma_{i \in S_1(y)\setminus I} \lambda_i \mathrm{grad}f_i(y)$ some reals $\{\lambda_i\}$ and

$X(I)(y) - X(K)(y) = \Sigma_{j \in K\setminus I} P(I)\mathrm{grad}f_j(y)$ where the right hand side $= \Sigma_{j \in K \setminus I} \mathrm{grad}f_j(y)$ if as here $\{\mathrm{grad}f_i(y)\}_{i \in I \cup J}$ are orthogonal. Then if $f_i(y)>0$ - ie $i \notin S_1(y)$ - then since for our submanifold with orthogonal corners $\langle \mathrm{grad}f_i(y), \mathrm{grad}f_j(y)\rangle = 0$ for all $i \neq j$ we have for all $i \in J$ $\langle X(M)(y), \mathrm{grad}f_i(y)\rangle = \langle X(I)(y), \mathrm{grad}f_i(y)\rangle = \langle X(K)(y), \mathrm{grad}f_i(y)\rangle$ if $i \in J \setminus K$. Continuity of $t \to \langle X(M)\phi(M)(x,t), \mathrm{grad}f_i(\phi(M)(x,t))\rangle$ for as long as $f_i(\phi(M)(x,t))>0$ follows from this and continuity of $t \to \phi(M)(x,t)$ and of $y \to X(I)(y)$.

(2) If $k \in \cup_{i=0}^{r-1} I_i \setminus \cap_{i=0}^{r-1} I_i$ then there exists $k(0), k(+) \in (0,..,r\text{-}1)$ such that $k \in I_{k(0)}$ and $k \in J \setminus I_{k(+)}$. Hence for every m $f_k\phi(M)(x,t_{mr+k(0)})=0$ and $f_k\phi(M)(x,t_{mr+k(+)})>0$, so $f_k\phi(M)(x,t)$ is both zero and positive on any interval $[t_{mr}, t_{(m+1)r})$ (it can never be negative because $k \in I \cup J$ and $\phi(M)(x,t) \in ZN(I;J)$). If $k \in I_0$ $f_k\phi(M)(x,t_{mr})=0$ for all m, so since by (1) $D_t^+(f_k\phi(M)(x,t)) = \langle X(M)\phi(M)(x,t), \mathrm{grad}f_k\phi(M)(x,t)\rangle$ is continuous

where $f_k\phi(M)(x,t)>0$ (see Figure 5.9) there must be some $t\in(t_{mr},t_{(m+1)r})$ where $f_k\phi(M)(x,t)>0$ and $\langle X(M)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle = D_t^+(f_k\phi(M)(x,t))=0$.

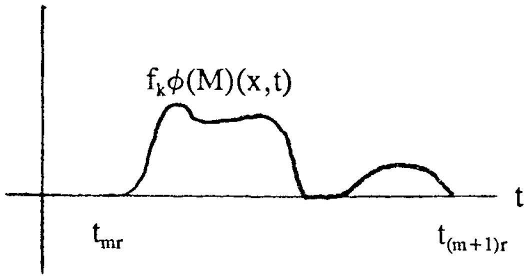


Figure 5.9

By (1) again at this t $\langle X(K)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle=0$ for any $K\supset I$ such that $k\in J\setminus K$, and $K=\cap_{i=0}^{r-1}I_i$ satisfies this condition since for each $I_i\supset I$ $k\in\cup_{i=0}^{r-1}I_i\setminus\cap_{i=0}^{r-1}I_i\subset J\setminus\cap_{i=0}^{r-1}I_i$.

Alternatively if $k\notin I_0$ then $k\in J\setminus I_0$ and $f_k\phi(M)(x,t_{mr})>0$ for all m. We know that there exists $k(0)\in(1,..,r\text{-}1)$ ($k(0)\neq0$ because $k\notin I_0$) with $f_k\phi(M)(x,t_{mr+k(0)})=0$. Set $T=\sup\{t\in(t_{mr},t_{(m+1)r}):f_k\phi(M)(x,t)=0\}$ (necessarily $<t_{(m+1)r}$). Thus $f_k\phi(M)(x,T)=0$, or equivalently $k\in S_1(\phi(M)(x,T))$. By Remark 2.5(2)
$X(M)\ \phi(M)(x,T)=X(S_1(\phi(M)(x,T)))\phi(M)(x,T)$ so
$\langle X(M)\phi(M)(x,T),\mathrm{grad}f_k(\phi(M)(x,T))\rangle=0$. Since $f_k\phi(M)(x,t)>0$ on $(T,t_{(m+1)r})$, by (1) again $\langle X(\cap_{i=0}^{r-1}I_i)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle=\langle X(M)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle$ on $(T,t_{(m+1)r})$, hence

$\langle X(\cap_{i=0}^{r-1}I_i)\phi(M)(x,T),\mathrm{grad}f_k\phi(M)(x,T)\rangle$
$=\lim_{t\downarrow T}\langle X(\cap_{i=0}^{r-1}I_i)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle$ (by continuity of $X(\cap_{i=0}^{r-1}I_i)$ and $\phi(M)$)
$=\lim_{t\downarrow T}\langle X(M)\phi(M)(x,t),\mathrm{grad}f_k\phi(M)(x,t)\rangle$ (by foregoing)
$=\langle X(M)\phi(M)(x,T),\mathrm{grad}f_k\phi(M)(x,T)\rangle$ (by Theorem 3.1))
$=0$ (by above). ∎

We recall from our discussion of Example 5.2 that the kth inductive stage of the proof of Theorem 5.1 involves finding smooth curves $\{\psi^k_j\}_{j=1,2,..}$ where $\psi^k_j(0)=x^k_j$ for all j, each passing through the points $\phi(M)(x^k_j,T_{j,j'})$ for all $j'\leq j$, with $\dot\psi^k_j=X(M)$ at these points, and satifying certain extra conditions; in Lemma 5.2 and 5.3 below we are concerned with constructing these curves.

<u>Lemma 5.2</u> Suppose X is a vector field on Z(I), $\{x_j\}$ is a sequence of points in the submanifold with corners M =ZN(I;J) such that $x_j \to x$ as $j \to \infty$, and suppose $\{T_j\}$ is a sequence of positive reals such that $T_j \downarrow 0$ as $j \to \infty$, and that $(1/T_j)^{k-i} D_t^i \langle X\phi(I)(x_j,t=0), \mathrm{gradf}_m \phi(I)(x_j,t=0)\rangle$ is uniformly bounded for all $i \leq k$, for all $j \in Z^+$ and for all $m \in J$ (ie, there exists A independent of j such that $|(1/T_j^{k-i}) D_t^i \langle X\phi(I)(x_j,t=0), \mathrm{gradf}_m \phi(I)(x_j,t=0)\rangle| < A$ for all $j \geq 0$, for all $i \leq k$ and for all $m \in J$).

Then $\mathrm{Sup}\{ |(1/T_j)^k (X(M)\phi(M)(x_j,t) - X(I)\phi(I)(x_j,t))| : t \in [0,T_j]\}$ and $\mathrm{Sup}\{ |(1/T_j)^{k+1} (\phi(M)(x_j,t) - \phi(I)(x_j,t))| : t \in [0,T_j]\}$ are bounded uniformly over $j \in Z^+$.

<u>Proof</u>

(a) X(y), X(M)(y) are uniformly bounded in any compact region so $\mathrm{Sup}\{ |X(M)\phi(M)(x_j,t) - X(I)\phi(I)(x_j,t)| : t \in [0,T_j]\}$ is uniformly bounded for all j - say this quantity is bounded by A - then since for $0 < t \leq T_j$

$|(1/T_j)(\phi(M)(x_j,t) - \phi(I)(x_j,t))| \leq |(1/t) \int_{s=0}^{t} (X(M)\phi(M)(x_j,s) - X(I)\phi(I)(x_j,s))ds| \leq A$

the result for k=0 follows.

(b) Suppose the result holds up to k-1.

Then we have $X(M)\phi(M)(x_j,t) - X(I)\phi(I)(x_j,t) = X(M)\phi(M)(x_j,t) - X(M)\phi(I)(x_j,t) + X(M)\phi(I)(x_j,t) - X(I)\phi(I)(x_j,t)$. By assumption the trajectory segments $\phi(M)(x_j,[0,T_j))$ are all contained in ZN(I;J) hence $X(M)\phi(M)(x_j,t) = X(K)\phi(M)(x_j,t)$ some $I \subset K \subset I \cup J$ for all j for all $t \in [0,T_j)$, and since there are only finitely many such K the fact that $\sup_{t \in [0,T_j]} (1/T_j)^k (X(M)\phi(M)(x_j,t) - X(M)\phi(I)(x_j,t))$ is uniformly bounded over $j \in Z^+$ will follow if we show that $\sup_{t \in [0,T_j]} (1/T_j)^k (X(K)\phi(M)(x_j,t) - X(K)\phi(I)(x_j,t))$ is uniformly bounded over $j \in Z^+$ for each $I \subset K \subset I \cup J$.

By the Mean Value Theorem $|X(K)(y) - X(K)(z)| \leq |y-z| \sup_{w \in B} |X(K)'(w)|$ where we take B to be a compact convex region large enough to contain all the points $\phi(M)(x_j,t)$ and $\phi(I)(x_j,t)$ for $0 \leq t \leq T_j$ (which can be taken to be compact since $x_j \to x$ and $T_j \downarrow 0$) and hence there exists a constant $A < \infty$ with $\sup_{w \in B} |X(K)'w| = A$. Thus $|(1/T_j)^k (X(K)\phi(M)(x_j,t) - X(K)\phi(I)(x_j,t))| \leq A |(1/T_j)^k (\phi(M)(x_j,t) - \phi(I)(x_j,t))|$ for all K such that $I \subset K \subset I \cup J$, for all j and for all $t \in [0,T_j]$, and the right hand side is uniformly bounded by assumed result for k-1.

We now show that $\sup_{t \in [0,T_j]} (1/T_j)^k (X(M)\phi(I)(x_j,t) - X(I)\phi(I)(x_j,t))$ is uniformly bounded over $j \in Z^+$. As above it suffices to show that for all $I \subset K \subset I \cup J$

$\sup_{t\in[0,T_j]}(1/T_j)^k(X(K)\phi(I)(x_j,t)-X(I)\phi(I)(x_j,t))$ is uniformly bounded over $j\in Z^+$.

By Remark 2.2 we know that for $K\supset I$ $X(K)(y)-X(I)(y)=-NM^TN^TX(I)(y)$ where $(N^TX(I))_k=\langle \mathrm{gradf}_k(y),X(I)(y)\rangle$ and $N(y)M(y)^T$ is a matrix depending smoothly on y. Hence (since $\{\phi(I)(x_j,t):j\in Z^+,\ t\in[0,T_j]\}$ is bounded) $|X(K)\phi(I)(x_j,t)-X(I)\phi(I)(x_j,t)|$ is bounded over $j\in Z^+$, $t\in[0,T_j]$, by some positive constant multiplied by $\sup_{k\in K\setminus I}|\langle \mathrm{gradf}_j\phi(I)(x_j,t),X(I)\phi(I)(x_j,t)\rangle|$. The supposition of the lemma is that $(1/T_j)^{k-i}D_t^i\langle X\phi(I)(x_j,t=0),\mathrm{gradf}_m\phi(I)(x_j,t=0)\rangle$ is uniformly bounded for all $i\le k$, for all $j\in Z^+$ and for all $m\in J$. If we expand out $\langle \mathrm{gradf}_j\phi(I)(x_j,t),X(I)\phi(I)(x_j,t)\rangle$ as a Taylor series we get $\langle \mathrm{gradf}_j\phi(I)(x_j,t),X(I)\phi(I)(x_j,t)\rangle=$
$\langle \mathrm{gradf}_j\phi(I)(x_j,t=0),X(I)\phi(I)(x_j,t=0)\rangle+tD_t\langle \mathrm{gradf}_j\phi(I)(x_j,t=0),X(I)\phi(I)(x_j,t=0)\rangle+$
$\frac{1}{2}t^2D_t\langle \mathrm{gradf}_j\phi(I)(x_j,t=0),X(I)\phi(I)(x_j,t=0)\rangle+\ldots+$
$(1/k!)t^kD_t^k\langle \mathrm{gradf}_j\phi(I)(x_j,\theta t),X(I)\phi(I)(x_j,\theta t)\rangle$ (some $\theta\in(0,1)$) where all but the last term are of the form $t^i\times$(a quantity $\le AT_j^{k-i}$) some constant A where $i=0,..,k-1$; since $t\in[0,T_j]$ these terms are therefore uniformly bounded by T_j^k. The last term is $t^k\times$multiplier where the multiplier is uniformly bounded by suposition at $t=0$ and hence by continuity uniformly bounded on compact sets. Thus
$|(1/T_j)^k(X(M)\phi(I)(x_j,t)-X(I)\phi(I)(x_j,t))|$ is bounded uniformly over $j\in Z^+$, $t\in[0,T_j]$, which combined with the first line yields that
$|(1/T_j)^k(X(M)\phi(M)(x_j,t)-X(I)\phi(I)(x_j,t))|$ is bounded uniformly over $j\in Z^+$, $t\in[0,T_j]$. This is the first half of the result; for the second half, if $0<t\le T_j$ then
$|(1/T_j)^{k+1}(\phi(M)(x_j,t)-\phi(I)(x_j,t))|\le$
$|(1/t)\int_{s=0}^{t}(1/T_j)^k(X(M)\phi(M)(x_j,t)-X(I)\phi(I)(x_j,t))ds|$, the integrand is less than or equal to some constant A by the above and hence
$\sup_{t\in[0,T_j]}|(1/T_j)^{k+1}(\phi(M)(x_j,t)-\phi(I)(x_j,t))|\le A$ ∎

Remark With the same assumptions we can also show that $\mathrm{Sup}\{|(1/T_j)^{k+1-i}(D_t^{+i}\phi(M)(x_j,t)-D_t^{+i}\phi(I)(x_j,t))|:t\in[0,T_j]\}$ is uniformly bounded as $j\to\infty$ for all $0\le i\le k+1$, but this strengthening is not needed.

Lemma 5.3 Suppose a submanifold with corners M is locally represented near x as ZN(I;J) with $\{x_j\}$ a sequence in M such that $x_j\to x$ as $j\to\infty$, and suppose $\{T_{j,j}\}$ is a sequence of non-negative reals with $T_{j,j}\downarrow 0$ as $j\to\infty$,
$[0,T_{j,j}]=[0=T_{j,0},T_{j,1})\cup[T_{j,1},T_{j,2})\cup..\cup[T_{j,j-1},T_{j,j}]$. If
$\sup\{|(1/(T_{j,j'+1}-T_{j,j'}))^{k-1}(X(M)\phi(M)(x_j,t)-X(I)\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'}))|:t\in[T_{j,j'},T_{j,j'+1}]\}$ and

$\sup\{ \mid (1/(T_{j,j'+1}-T_{j,j'}))^k(\phi(M)(x_j,t)-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})) \mid : t\in[T_{j,j'},T_{j,j'+1}]\}$ are uniformly bounded as $j\to\infty$ for all $j'=0,..,j-1$ (ie, there exists a constant $A>0$ such that for all $j\in Z^+$ and for all $j'\leq j$ both these quantities are less than A) then there exist smooth curves $\psi_j:[0,T_j]\to Z(I)$ satisfying

(1) $\psi_j(0)=x_j$ and $\psi_j(T_{j,j'})=\phi(M)(x_j,T_{j,j'})$ for all $0\leq j'\leq j$

(2) $\dot{\psi}_j(T_{j,j'})=X(M)\phi(M)(x_j,T_{j,j'})$ for all $0\leq j'\leq j$

(3) $\sup\{ \mid D_t^k\psi_j(t) \mid : t\in[0,T_{j,j}]\}<A$ for all j, some A independent of j

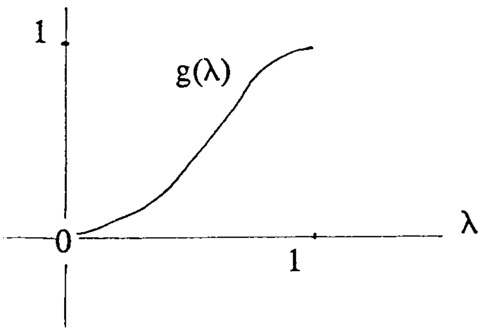
(4) $\sup\{(1/(T_{j,j'+1}-T_{j,j'}))^{k-i} \mid D_t^i(\psi_j(t)-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})) \mid : t\in[T_{j,j'},T_{j,j'+1}]\}<A$ for all $0\leq j'<j$, for all $j\in Z^+$, for all $0\leq i\leq k$, some constant A.

Proof (a) The suppositions imply that if we set

$G_1(t,j,j')=\phi(M)(x_j,T_{j,j'})+(t-T_{j,j'})X(M)\phi(M)(x_j,T_{j,j'})+\frac{1}{2}(t-T_{j,j'})^2D_t^2\phi(I)(\phi(M)(x_j,T_{j,j'}),t=0)+.....$
$+(1/(k+1)!)(t-T_{j,j'})^{k+1}D_t^{k+1}\phi(I)(\phi(M)(x_j,T_{j,j'}),t=0)$, and

$G_2(t,j,j')=\phi(M)(x_j,T_{j,j'+1})-(T_{j,j'+1}-t)X(M)\phi(M)(x_j,T_{j,j'+1})-$
$\frac{1}{2}(T_{j,j'+1}-t)^2D_t^2\phi(I)(\phi(M)(x_j,T_{j,j'+1}),t=0)-..$
$..-(1/(k+1)!)(T_{j,j'}-t)^{k+1}D_t^{k+1}\phi(I)(\phi(M)(x_j,T_{j,j'+1}),0)$ then
$(1/(T_{j,j'+1}-T_{j,j'}))^iD_t^{k-i}(G_1(t,j,j')-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'}))$ and
$(1/(T_{j,j'+1}-T_{j,j'}))^iD_t^{k-i}(G_2(t,j,j')-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'}))$ are uniformly bounded as $j\to\infty$ for all $0\leq j'\leq j-1$, $j\in Z^+, t\in[T_{j,j'},T_{j,j'+1}]$.

(b) We set $g:R\to[0,1]$ to be a C^∞ function infinitely tangent at 0 and 1 to the maps $R\to 0$, $R\to 1$ respectively :



and observe that $\sup_{t\in[0,AT]}D_t^ig(t/A)=(1/A^i)\sup_{t\in[0,T]}D_t^ig(t)$

(c) If we now for any $j\in Z^+$, $0\leq j'\leq j-1$ set on $t\in[T_{j,j'},T_{j,j'+1}]$
$\psi_j(t)=(1-g((t-T_{j,j'})/(T_{j,j'+1}-T_{j,j'})))G_1(t,j,j')+g((t-T_{j,j'})/(T_{j,j'+1}-T_{j,j'}))G_2(t,j,j')$ then this ψ_j is smooth and satisfies (1) and (2) (Figure 5.11).

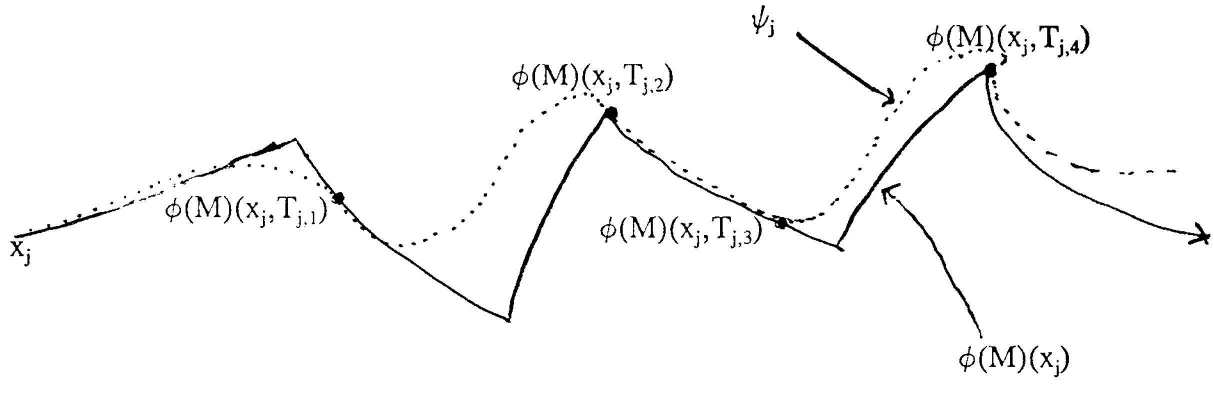


Figure 5.11. ψ_j is a smooth curve coinciding with $\phi(M)(x_j,t)$ at $t=T_{j,j'}$ for $j'=0,..,j$ and at these points the time derivative $\dot{\psi}_j(T_{j,j'})=X(M)\phi(M)(x_j,T_{j,j'})$.

Furthermore, for $t\in[T_{j,j'},T_{j,j'+1}]$ $D_t^m(\psi_j(t)-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'}))=$
$\Sigma^m_{s=0}\binom{m}{s}(D_t^s(1-g((t-T_{j,j'})/(T_{j,j'+1}-T_{j,j'}))))D_t^{m-s}(G_1(t,j,j')-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})+$
$D_t^s g((t-T_{j,j'})/(T_{j,j'+1}-T_{j,j'}))D_t^{m-s}(G_2(t,j,j')-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})))$ and hence by (a) and (b) for $t\in[T_{j,j'},T_{j,j'+1}]$

$$\sup\{\ |\ (1/(T_{j,j'+1}-T_{j,j'}))^{k-m}D_t^m(\psi_j(t)-\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'}))\ |\ :t\in[T_{j,j'},T_{j,j'+1}]\} \qquad (*)$$

is uniformly bounded over $j'\leq j$, $j\in Z^+$. This gives (4) of the list of conditions we claim ψ_j satisfies (see Figure 5.12).

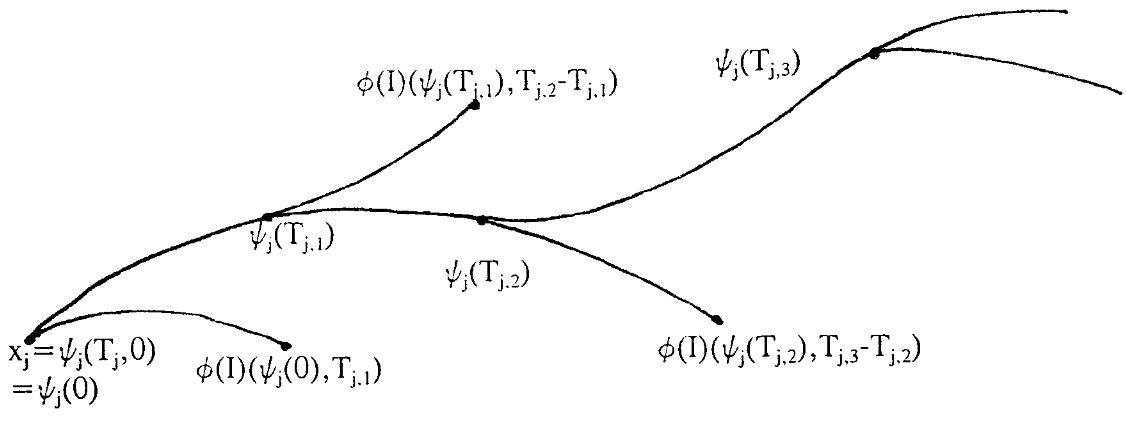


Figure 5.12

As for (3), under the assumptions of this lemma $\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})$ is contained in a compact set for all $t\in[T_{j,j'},T_{j,j'+1}]$, $j'\leq j$ and $j\in Z^+$, hence by smoothness of $\phi(I)$ there exists a constant $A'>0$ such that $|\ D_t^k\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})\ |\ \leq A'$ for all $t\in[T_{j,j'},T_{j,j'+1}]$, $j'\leq j$, $j\in Z^+$. We know that

$$|\ D_t^k\psi_j(t)\ |\ -\ |\ D_t^k\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})\ |\ \leq\ |\ D_t^k\psi_j(t)-D_t^k\phi(I)(\phi(M)(x_j,T_{j,j'}),t-T_{j,j'})\ |$$

$\leq A$ for all $t \in [T_{j,j'}, T_{j,j'+1}]$, $j' \leq j$, $j \in Z^+$, some constant A, by (*) with m set to k, and it follows that for all $j \in Z^+$ and for all $t \in [0, T_{j,j}]$ $|D_t^k \psi_j(t)| \leq A + A'$. ▬

Lemma 5.4 below is a quite general result saying that if $\{\psi_j\}$ is a sequence of smooth curves, $\psi_j:[0,T_j) \to B$ (= a ball in R^n) where $T_j \downarrow 0$, and g is a smooth function $g:R^n \to R$ such that $g\psi_j$ has k zeros on $[0,T_j)$ for each j, then so long as the kth time derivative of $\psi_j(t)$ is uniformly bounded over $t \in [0,T_j)$ and $j \in Z^+$, then for all $0 \leq i \leq k$ $(1/T_j)^i(D_t^{k-i} g\psi_j(t))$ is bounded uniformly in $t \in [0,T_j)$ and $j \in Z^+$.

Lemma 5.4 If $\{\psi_j\}$ is a sequence of smooth functions $\psi_j:[0,T_j) \to B$ (= a ball in R^n) where $T_j \downarrow 0$, and f is a smoooth real valued function with non-vanishing gradient, such that

(i) $\sup\{ |D_t^k \psi_j(t)| : t \in [0,T_j]\} < A$ for all j, some $A > 0$

(ii) $\langle X(\psi_j(t)), \mathrm{grad} f(\psi_j(t))\rangle$ has k zeros on $[0,T_j]$ for all j

then $\sup\{ |(1/T_j)^i D_t^{k-i}\langle X\psi_j(t), \mathrm{grad} f\psi_j(t)\rangle| : t \in [0,T_j]\} < A'$ for all j, $0 \leq i \leq k$, some constant $A' > 0$.

Proof Between any pair of zeros of $\langle X\psi_j(t), \mathrm{grad} f\psi_j(t)\rangle$ there exists a zero of $D_t\langle X\psi_j(t), \mathrm{grad} f\psi_j(t)\rangle$, between any pair of which there exists a zero of $D_t^2\langle X\psi_j(t), \mathrm{grad} f\psi_j(t)\rangle$ etc. Hence there exist $t_1,..,t_{k-1} \in [0,T_j]$ with $D_t^i\langle X\psi_j(t), \mathrm{grad} f\psi_j(t)\rangle = 0$ at $t = t_i$. Writing $Xf\psi_j(t)$ for $\langle X\psi_j(t), \mathrm{grad} f\psi_j(t)\rangle$ then if $t \in [0,T_j]$ $(1/T_j)(D_t^{k-1}Xf\psi_j(t)) = (1/T_j)(D_t^{k-1}Xf\psi_j(t) - D_t^{k-1}Xf\psi_j(t_{k-1})) = 1/T_j \int_{s=t_{k-1}}^{t} D_s^k Xf\psi_j(s)ds$, where the integrand is bounded uniformly over $j \in Z^+$ since X,f are smooth and by supposition the quantities $\sup\{ |D_t^k\psi_j(t)| : t \in [0,T_j]\}$, and hence by the Mean Value Theorem $\sup\{ |D_t^i\psi_j(t)| : t \in [0,T_j]\}$ for all $i \leq k$, are bounded uniformly over $j \in Z^+$. Hence $\sup\{ |(1/T_j)D_t^{k-1}Xf\psi_j(t)| : t \in [0,T_j]\}$ is bounded uniformly over $j \in Z^+$. Similarly by induction, if $\sup\{ |(1/T_j)^i D_t^{k-i}Xf\psi_j(t)| : t \in [0,T_j]\}$ is bounded uniformly over $j \in Z^+$ so is $\sup\{ |(1/T_j)^{i+1} D_t^{k-i-1}Xf\psi_j(t)| : t \in [0,T_j]\}$ (using that for $t \in [0,T_j]$, $(1/T_j)^{i+1}(D_t^{k-i-1}Xf\psi_j(t)) = (1/T_j)^{i+1}(D_t^{k-i-1}Xf\psi_j(t) - D_t^{k-i-1}Xf\psi_j(t_{k-i-1}))$ $= 1/T_j \int_{s=t_{k-i-1}}^{t} (1/T_j^i) D_t^{k-i}Xf\psi_j(s)ds$). ▬

Lemma 5.5 is another quite general result we shall need to prove Theorem 5.1.

Lemma 5.5 If $f:R^n \to R^n$ is smooth, $x \in \bar{B} \subset R^n$ where $\bar{B}$ is a closed n-ball, $\sup_{y\in\bar{B},i\le k} | D_y^i f(y) | = A < \infty$, $\{g_j\}$, $\{h_j\}$ are sequences of smooth functions $g_j,h_j:R\to R^n$ with $g_j(0)=h_j(0)=x_j$, $x_j\to x$ as $j\to\infty$, $\{T_j\}$ a sequence with $T_j \downarrow 0$ and $g_j(0,T_j),h_j(0,T_j) \subset \bar{B}$, then if $(1/T_j)^i D_t^{k-i}(g_j(t=0)-h_j(t=0))$ is bounded uniformly over $j\in Z^+$, for all $i\le k$, then $(1/T_j)^i (D_t^{k-i}(fg_j(t=0)-fh_j(t=0))$ is bounded uniformly over $j\in Z^+$, for all $i\le k$.

Proof If $h:R\to R^n$ is a smooth function with $h(0)=x$, and writing $D_t^i h(t=0)$ as $h^{(i)}$ and $D_x^j f(x)$ as $f^{(j)}(x)$, the chain rule for differentiating compositions of functions gives $D_t^i fh(t=0)=$ a finite sum of terms each of the form $f^{(j)}(x)(h^{(k(1))},..,h^{(k(j))})$ where $\Sigma_{s=1}^j k(s)=i$. Thus with x_j,g_j,h_j,f as above $(1/T_j)^i D_t^{k-i}(fg_j(t=0)-fh_j(t=0))$ is a sum of terms $(1/T_j)^i f^{(m)}(x_j)(g_j^{(k(1))}-h_j^{(k(1))},..,g_j^{(k(m))}-h_j^{(k(m))})$ where $\Sigma_{s=1}^m k(s)=k-i$, which since the $\{x_j\}$ are in a compact set and all functions involved are smooth this typical term has magnitude $\le (1/T_j)^i A \,| g_j^{(k(1))}-h_j^{(k(1))} | .. | g_j^{(k(m))}-h_j^{(k(m))} |$. We know by supposition that $| D_t^{k-i}(g_j(t=0)-h_j(t=0)) | \le CT_j^i$ for all $0\le i\le k$ some C independent of j, hence each typical term has magnitude $\le (1/T_j)^i A(C(T_j)^{k-k(1)})(C(T_j)^{k-k(2)})..(C(T_j)^{k-k(m)})$ and so $| (1/T_j)^i D_t^{k-i}(fg_j(t=0)-fh_j(t=0)) | \le (1/T_j)^i A'(T_j)^{\Sigma(k-k(s):s=1..m)} C^m$ some A' independent of j. We know $\Sigma_{s=1}^m k(s)=k-i$ thus $mk-\Sigma_{s=1}^m k(s)=(m-1)k+i$. Hence $\Sigma_{s=1}^m (k-k(s))=(m-1)k+i$ which is $\ge i$ if $m\ge 1$, and hence $| (1/T_j)^i D_t^{k-i}(fg_j(t=0)-fh_j(t=0)) |$ is uniformly bounded over $j\in Z^+$ for each $0\le i\le k$.

∎

Proof of Theorem 5.1

We are claiming that if M is a submanifold with orthogonal corners with local orthogonal representation ZN(I;J), then if there exists $\{x_j\}_{j=1,..,\infty} \subset ZN(I;J)$ with $x_j \to x \in Z(I\cup J)$, reals $h_j \downarrow 0$ and a subset $\{ZP(I_i;J\backslash I_i):i=0,..,r-1\}$ of strata in ZN(I;J) such that for each j there exist $0=T_{j,0}<T_{j,1}<..<T_{j,j}\le h_j$ with $\phi(M)(x_j,[0,h_j]) \subset conv\{ZP(I_i;J\backslash I_i),\ i=0,..,r-1\}$ for each $j\in Z^+$ and $\phi(M)(x_j,T_{j,i}) \in ZP(I_{i,mod\ r};J\backslash I_{i,mod\ r})$ for each $i\le j$, then $D_t^k \phi(\cap_{i=0}^{r-1} I_i)(x,t=0) = D_t^k \phi(\cup_{i=0}^{r-1} I_i)(x,t=0)$ for all $k\ge 0$.

For example in the case of Example 5.2 $ZN(I;J)=\{y\in R^n: y_1\ge 0, y_2\ge 0\}=ZN(\varnothing;1,2)$, $r=2$, $ZP(I_0;J\backslash I_0)=ZP(\varnothing;1,2)$, $ZP(I_1;J\backslash I_1)=ZP(1;2)$, $ZP(I_2;J\backslash I_2)=ZP(2;1)$, and there existed $\{x_j\} \subset ZN(\varnothing;1,2)$ such that $x_j\to x\in Z(1,2)$ as $j\to\infty$ and for each x_j we could find $0<T_{j,1}<..<T_{j,j}$ where $T_{j,j}\downarrow 0$ as $j\to\infty$ such that $\phi(M)(x_j,T_{j,1})\in ZP(1;2)$, $\phi(M)(x_j,T_{j,2})\in ZP(2,1)$, $\phi(M)(x_j,T_{j,3})\in ZP(\emptyset;1,2)$, $\phi(M)(x_j,T_{j,4})\in ZP(1;2)$,..,up to $\phi(M)(x_j,T_{j,j})$; Theorem 5.1 tells us $D_t^k\phi(\emptyset)(x,t=0)=D_t^k\phi(1,2)(x,t=0)$ for all $k\geq 0$.

We can suppose throughout that $\cap_{i=0}^{r-1}I_i=I$ and $\cup_{i=0}^{r-1}I_i=I\cup J$ (I,J,I_i as in the beginning of this proof). This is because we are assuming that $\phi(ZN(I;J))(x_j,[0,T_j))\subset ZNP(\cap_{i=0}^{r-1}I_i;\cup_{i=0}^{r-1}I_i\backslash\cap_{i=0}^{r-1}I_i;J\backslash\cup_{i=0}^{r-1}I_i)$ and we can check in general that at every point $y\in ZNP(K_1;K_2;J\backslash(K_1\cup K_2))$ (a submanifold with corners and a subcorner of ZN(I;J)), that $T_yZNP(K_1;K_2;J\backslash(K_1\cup K_2))=T_yZN(K_1;K_2)\subset T_yZN(I;J)$ and hence by Theorem 1.1 that $X(ZN((I;J))(y)=X(ZNP(K_1;K_2;J\backslash(K_1\cup K_2)))(y)$ for all $y\in ZNP(K_1;K_2;J\backslash(K_1\cup K_2))$. Hence if $\phi(ZN(I;J))(x,[0,T))\subset ZNP(K_1;K_2;J\backslash(K_1\cup K_2))$ then $D_t\phi(ZN(I;J))(x,t)=X(ZN(I;J))\phi(ZN(I;J))(x,t)=X(ZN(K_1;K_2))(\phi(ZN(I;J))(x,t)$ for all $t\in[0,T)$, so by uniqueness of trajectories (Theorem 1.1) we have $\phi(ZN(I;J))(x,t)=\phi(ZN(K_1;K_2))(x,t)$ for all $t\in[0,T)$.

Additionally $X(ZN(K_1;K_2))(y)=P(T_y(ZN(K_1;K_2)))X(K_1)(y)$, and setting $K_1=\cap_{i=0}^{r-1}I_i,K_1\cup K_2=\cup_{i=0}^{r-1}I_i$ we see that our trajectory segments are those of the vector field $X(\cap_{i=0}^{r-1}I_i)$ projected onto $ZN(\cap_{i=0}^{r-1}I_i;\cup_{i=0}^{r-1}I_i\backslash\cap_{i=0}^{r-1}I_i)$.

(1) The inductive hypothesis (at the kth stage, so k is considered fixed in the following) is that

(i) There exists a subsequence $\{x^k_j\}$ of $\{x_j\}$ (so $x^k_j\to x$ as $j\to\infty$), and for each j reals $\{T^k_{j,j'}\}_{0\leq j'\leq j,j=1,2,..}$ such that $0=T^k_{j,0}<T^k_{j,1}<..<T^k_{j,j}\downarrow 0$ with points $x^k_{j,j'}$ on $\phi(M)(x^k_j,[0,T^k_j))$ given by $x^k_{j,j'}=\phi(M)(x^k_j,T^k_{j,j'})$ satisfying $\phi(M)(x^k_j,[0,T_{j,j}])\subset ZNP(\cap_{i=0}^{r-1}I_i,\cup_{i=0}^{r-1}I_i\backslash\cap_{i=0}^{r-1}I_i,I\cup J\backslash\cup_{i=0}^{r-1}I_i)$ and $\langle X(\cap_{i=0}^{r-1}I_i)(x^k_{j,j'}),\mathrm{grad}f_{j'\bmod m}(x^k_{j,j'})\rangle=0$ for all $j'=0,..,j$ where we have set $\cup_{i=0}^{r-1}I_i\backslash\cap_{i=0}^{r-1}I_i=(0,..,m\text{-}1)$, and furthermore

(ii) that $(1/(T^k_{j,j'+1}-T^k_{j,j'}))^iD_t^{k-i}\langle X(\cap_{i=0}^{r-1}I_i)\phi(\cap_{i=0}^{r-1}I_i)(x^k_{j,j'},t=0),\mathrm{grad}f_s\phi(\cap_{i=0}^{r-1}I_i)(x^k_{j,j'},t=0)\rangle$ is uniformly bounded over $j\geq 0$ for all $0\leq i\leq k$, for all $0\leq j'\leq j$, for all $s\in(0,..,m\text{-}1)$.

(2) We show that the inductive hypothesis is true if $k=0$.

Lemma 5.1 tells us that under the suppositions (restated above) of Theorem 5.1 that if $s\in\cup_{i=0}^{r-1}I_i\backslash\cap_{i=0}^{r-1}I_i$ then for any interval $(T_{j,qr},T_{j,(q+1)r})$ with $(q+1)r\leq j$, there exists a point $T_{j,q,s}$ such that $\langle X(\cap_{i=0}^{r-1}I_i)\phi(M)(x_j,T_{j,q,s}),\mathrm{grad}f_s\phi(M)(x_j,T_{j,q,s})\rangle=0$, so taking $x^0_j=\phi(M)(x_{(j+1)r},T_{(j+1)r,0,0})$ for $j=1,2,..$ and $T^0_{j,0}=T_{(j+1)r,0,0}\text{-}T_{(j+1)r,0,0}=0$,

$T^0_{j,1}=T_{(j+1)r,1,1}-T_{(j+1)r,0,0},$

.

.

$T^0_{j,m}=T_{(j+1)r,m,0}-T_{(j+1)r,0,0},$

$T^0_{j,m+1}=T_{(j+1)r,m+1,1}-T_{(j+1)r,0,0},$

.

.

$T^0_{j,j}=T_{(j+1)r,j,j \bmod m}-T_{(j+1)r,0,0}$

where $T_{(j+1)r,i,i \bmod m}\in(T_{(j+1)r,ir},T_{(j+1)r,(i+1)r})$ and hence all the right hand sides exist. This gives (i). (ii) follows by the boundedness of $X(\cap_{i=0}^{r-1}I_i)$ on compact subsets.

(3) We show that if the inductive result holds for k then it holds for k+1.

(a). Using Lemma 5.2 with k,j,T_j,x_j in Lemma 5.2 set to respectively $k+1,(j,j'),T^k_{j,j'+1}-T^k_{j,j'},x^k_{j,j'}$ (by setting j in Lemma 5.2 to (j,j') here we have in mind something like setting 1,2,3,4,5,.. to (1,0),(1,1),(2,0),(2,1),(2,2),..) we see that the inductive hypothesis for k implies that there exists some constant A such that for all $0\le j'\le j,\ j\in Z^+,\ t\in[0,T^k_{j,j'+1}-T^k_{j,j'}]$

$|(1/(T^k_{j,j'+1}-T^k_{j,j'}))^{k+1}(X(M)\phi(M)(x^k_{j,j'},t)-X(I)\phi(I)(x^k_{j,j'},t)|<A$, and

$|(1/(T^k_{j,j'+1}-T^k_{j,j'}))^{k+2}(\phi(M)(x^k_{j,j'},t)-\phi(I)(x^k_{j,j'},t)|<A.$

If we then apply Lemma 5.3 with $k,\psi_j,x_j,T_{j,j'}$ in Lemma 5.3 set to $k+1,\psi^k_j,x^k_j,T^k_{j,j'}$ it follows there exist smooth curves ψ^k_j for each $j\in Z^+$ such that

(i) $\psi^k_j(0)=x^k_{j,0}$ and $\psi^k_j(T^k_{j,j'})=x^k_{j,j'}$

(ii) $\dot\psi^k_j(T^k_{j,j'})=X(M)\phi(M)(x^k_j,T^k_{j,j'})$

(iii) $\sup\{|D_t^{k+1}\psi^k_j(t)|:t\in[0,T^k_{j,j}]\}<A$ for all j, some $A>0$ independent of j

(iv) $(1/(T_{j,j'+1}-T_{j,j'}))^{k+1-i}|D_t^i(\psi^k_j(t)-\phi(I)(x^k_j,t-T^k_{j,j'}))|<A'$ for all $j'+1\le j, j\in Z^+, 0\le i\le k+1, t\in[T^k_{j,j'},T^k_{j,j'+1}]$ some $A'>0$.

(b) By the inductive hypothesis for each $s\in(0,..,m-1)$ there are in every m(k+1) points $x^k_{j,j'+1},..,x^k_{j,j+m(k+1)}$ k+1 points $x^k_{j,j''}$ where $\langle X(I)x^k_{j,j''}, \mathrm{grad}f_s(x^k_{j,j''})\rangle=0$ - in fact those j″ such that j″ mod m =s. Inserting this and (3)(a)(iv) into Lemma 5.4, with $X,\psi_j(t),T_j,\mathrm{grad}f$ in Lemma 5.4 set to $X(I),\psi^k_j(T^k_{j,j'}+t),T^k_{j,j'+q}-T^k_{j,j'},\mathrm{grad}f_s$, where $q\ge m(k+1)$, we infer $(1/(T^k_{j,j'+q}-T^k_{j,j'}))^i|D_t^{k+1-i}\langle X(I)\psi^k_j(t),\mathrm{grad}f_s\psi^k_j(t)\rangle|<A$ for all $0\le i\le k+1, t\in[0,T^k_{j,j'+q}-T^k_{j,j'}], s\in(0,..,m-1), j\in Z^+$, some $A>0$.

(c) We obtain subsequences $\{x^{k+1}_j\}\subset\{x^k_j\},\{x^{k+1}_{j,j'}\}\subset\{x^k_{j,j'}\}, \{T^{k+1}_{j,j'}\}\subset\{T^k_{j,j'}\}$, as follows.

By part (i) of the inductive hypothesis at k we know

$\langle X(\cap_{i=0}^{r-1}I_i)x^k_{j,j'},\mathrm{grad}f_{j',\bmod m}(x^k_{j,j'})\rangle=0$ for all $j'=0,..,j$ where $\cup_{i=0}^{r-1}I_i\setminus\cap_{i=0}^{r-1}I_i=(0,..,m\text{-}1)$, $x^k_{j,0}=x^k_j$, and if we now take $x^{k+1}_j=x^k_{jm(k+1)+j}$, $x^{k+1}_{j,0}=x^{k+1}_j, x^{k+1}_{j,1}=x^k_{jm(k+1)+j,m(k+1)+1},..,x^{k+1}_{j,j'}=x^k_{jm(k+1)+j,j'(m(k+1)+1)},..$ up to $x^{k+1}_{j,j}$, and similarly $T^{k+1}_{j,j'}=T^k_{jm(k+1),j'(m(k+1)+1)}$, we see these are all defined for $j'\leq j$.

Thus as $j'(m(k+1)+1)$,mod m $=j'$, mod m, it follows that part (i) of the inductive hypothesis holds for $k+1$ with this renumbering. We observe that (3)(a)(iv) will continue to hold replacing $j'+1$ by $j'+q$, any q which is both ≥ 1 and such that $j'+q\leq j$, ie $(1/(T^k_{j,j'+q}\text{-}T^k_{j,j'}))^{k+1-i}\mid D_t^i(\psi_j(t)\text{-}\phi(I)(x^k_j,t\text{-}T^k_{j,j'}))\mid<A$ for all $j'+q\leq j$, $j\in Z^+$, $0\leq i\leq k+1$, $t\in[T^k_{j,j'},T^k_{j,j'+q}]$, some $A>0$.

(d) If we take $q=m(k+1)+1$ then (c) implies that $(1/(T^k_{j,j'+m(k+1)+1}\text{-}T^k_{j,j'}))^{k+1-i}\mid D_t^i(\psi^k_j(t)\text{-}\phi(I)(x^k_j,t\text{-}T^k_{j,j'}))\mid<A$ for all $0\leq i\leq k+1$, $j'+m(k+1)+1\leq j$, $j\in Z^+$, $s\in(0,..,m\text{-}1),t\in[T^k_{j,j'},T^k_{j,j'+m(k+1)+1}]$, some $A>0$, and if we apply Lemma 5.5 with $k,f,g_j(t),h_j(t),T_j$ in Lemma 5.5 set to $k+1$, $X(I)f_s$ (=the Lie derivative of f_s with respect to $X(I)$), $\psi^k_j(t+T^{k+1}_{j,j'}),\phi(I)(x^{k+1}_{j,j'},t)$, $T^{k+1}_{j,j'+1}\text{-}T^{k+1}_{j,j'}$ respectively, where the quantities superfixed by $k+1$ are as given in (c), it follows that $(1/(T^{k+1}_{j,j'+1}\text{-}T^{k+1}_{j,j'}))^i\mid D_t^{k+1-i}(X(I)f_s\psi^k_j(t)\text{-}X(I)\phi(I)(x^k_j,t\text{-}T^k_{j,j'}))\mid$ is bounded uniformly over $0\leq i\leq k+1$, $j'<j$, $j\in Z^+$, $s\in(0,..,m\text{-}1)$. If we then combine this expression with (b), we see that $(1/(T^{k+1}_{j,j'+1}\text{-}T^{k+1}_{j,j'}))^i\mid D_t^{k+1-i}\langle X(I)\phi(I)(x^{k+1}_{j,j'},t=0),\mathrm{grad}f_s\phi(I)(x^{k+1}_{j,j'},t=0)\rangle\mid<A$ for all $0\leq i\leq k+1$, $j'\leq j$, $j\in Z^+$,$s\in(0,..,m\text{-}1)$, some constant $A>0$, which establishes part (ii) of the inductive hypothesis for $k+1$.

(4) The inductive hypothesis which we now know holds for all k tells us that $D_t^k\langle X(\cap_{i=0}^{r-1}I_i)\phi(\cap_{i=0}^{r-1}I_i)(x^k_j,t=0),\mathrm{grad}f_s\phi(\cap_{i=0}^{r-1}I_i)(x^k_j,t=0)\rangle\to 0$ as $j\to\infty$ for all $s\in(0..m\text{-}1)$ and hence since $\{x^k_j\}$ is a subsequence of $\{x_j\}$ where $x_j\to x$ that $D_t^k\langle X(\cap_{i=0}^{r-1}I_i)\phi(\cap_{i=0}^{r-1}I_i)(x,t=0),\mathrm{grad}f_s\phi(\cap_{i=0}^{r-1}I_i)(x,t=0)\rangle=0$ for all k and for all $s\in(0..m\text{-}1)$. It then follows from Lemma 2.2 that $D_t^k\phi(\cap_{i=0}^{r-1}I_i)(x,t=0)=D_t^k\phi(\cap_{i=0}^{r-1}I_i\cup j)(x,t=0)$ for all $j\in(0,..,m\text{-}1)$ for all k, and hence by Proposition 4.1 (in fact Corollary 4.1(1)) and the fact that we have set $(0,..,m\text{-}1)=\cup_{i=0}^{r-1}I_i\setminus\cap_{i=0}^{r-1}I_i$ that $D_t^k\phi(\cap_{i=0}^{r-1}I_i)(x,t=0)=D_t^k\phi(\cup_{i=0}^{r-1}I_i)(x,t=0)$ for all k.

∎

Theorem 5.1 is the basis for Chapter 7 in the form of Corollary 5.1 below, which says that in the absence of infinite order tangencies the number of stratum jumps made by the trajectories on any compact set is bounded by a constant for that compact set (cf example 5.1(4) where no such constant exists).

Let us formalize "number of stratum jumps". We recall from the definitions of Chapter 2 that if M locally is represented as ZN(I;J) then $S_1(x)=\{i\in I\cup J:f_i(x)=0\}$, ie $x\in ZP(S_1(x);J\setminus S_1(x))$.

Definition $|\phi(M)(x,[0,h))|=\sup\{k\geq 1$: there exists $0\leq t_1<t_2<..<t_k<h$ with $S_1\phi(M)(x,t_i)\neq S_1\phi(M)(x,t_{i+1})$ each $i=1..k-1\}$ if the quantity between the braces is finite, and infinity otherwise.

There is evidently a link between this definition and that of recurrence:

Lemma 5.6 If M is a submanifold with corners, if there exists a sequence of points $\{x_i\}\subset M$ with $x_i\to x$, reals $h_i\downarrow 0$ with $|\phi(M)(x_i,[0,h_i))|\to\infty$, then for some $r>1$ there exists a subset $(\sigma_0,..,\sigma_{r-1})$ of strata which recur at x.

Proof We must remember that merely showing that there exist $\{x_j,t_{j,j'}\}_{j'\leq j,j\in Z^+}$ with $\phi(M)(x_j,t_{j,j'})\in\sigma_{j',\mathrm{mod}\,m}$ (where m is as in the proof of Theorem 5.1) is not sufficient: we must also show $\phi(M)(x_j,[0,t_{j,j}))\subset\mathrm{conv}(\sigma_0,..,\sigma_{r-1})$.

Suppose $(\sigma_0,..,\sigma_{s-1})$ is the set of all strata σ_i with $x\in\bar{\sigma}_i$. We are given that there exists for each j $0\leq t^j_1<t^j_2<..<t^j_{k(j)}\leq h_j$ with $S_1\phi(M)(x_j,t_{j,i})\neq S_1\phi(M)(x_j,t_{j,i+1})$ for all $1\leq i<k(j)$, with $k(j)\to\infty$ as $j\to\infty$.

For each j set $t_{j,0}=0$ and inductively define for $j'=1,2,..$ $t_{j,j'}=\inf\{t>t_{j,j'-1}:t<h_j$ and $\phi(M)(x,t)\in\sigma_{j',\mathrm{mod}\,s}\}$ if the right hand side exists, so we have $t_{j,0}=t_{j,1}=..=t_{j,i_0}<t_{j,i_0+1}=..=t_{j,i_0+i_1}<t_{j,i_0+i_1+i_2}=..$ some $i_0,i_1,i_2,..\geq 0$. Set $N(j)=\max\{k:t_{j,j'}$exists for all $0\leq j'\leq k\}$ if finite, and ∞ otherwise (it exists because $t_{j,0}=0$ by definition).

(1) We show that if $N(j)\to\infty$ as $j\to\infty$ then $(\sigma_0,..,\sigma_{s-1})$ recur.

(i) We first show that if $t_{j,j'}=t_{j,j'+1}=...=t_{j,j'+k}$ some $j'\leq j'+k\leq N(j)$ (eg, if $\phi(M)(x_j,t_{j,j'}+1/i)\in\sigma_{j'+1,\mathrm{mod}\,s}$ for all $i\geq 1$, then $t_{j,j'+1}=t_{j,j'}$), then there exist $t_{j,j'}=t_{j,j'}'<t_{j,j'+1}'<...<t_{j,j'+k}'$ $(<t_{j,j'+1+k}$ if $N(j)\geq j'+1+k)$ such that $\phi(M)(x_j,t_{j,j''})\in\sigma_{j'',\mathrm{mod}\,s}$ for all $j'\leq j''\leq j'+k$. By definition of $t_{j,j'}$ above we have for each $j''=j',..,j'+k$ a sequence $\tau^i_{j''}\downarrow t_{j,j'}$ (with possibly $\tau^i_{j'}=t_{j,j'}$ for all i) such that for all i $\phi(M)(x_j,\tau^i_{j''})\in\sigma_{j'',\mathrm{mod}\,s}$. We can therefore find $t_{j,j'+k-1}'<t_{j,j'+k}'$ and $t_{j,j'+k-2}'<t_{j,j'+k-1}'$ etc with each $t_{j,j''}$ for $j'<j''\leq j'+k$ satisfying $t_{j,j'}=t_{j,j'}'<t_{j,j''}'<t_{j,j''+1}'$ and $\phi(M)(x_j,t_{j,j''}')\in\sigma_{j'',\mathrm{mod}\,s}$.

(ii) If $N(j) \to \infty$ as $j \to \infty$ we can take a subsequence of $\{x_j\}$ and renumber the j's so that $N(j) \geq j$ for each j. By definition $t_{j,1} \leq t_{j,2} \leq .. \leq t_{j,j}$ and so by using (i) we can find $t_{j,1}' < t_{j,2}' < .. < t_{j,j}'$ with each $t_{j,j'}'$ satisfying $\phi(M)(x_j, t_{j,j'}') \in \sigma_{j',\text{mods}}$ and hence (because $(\sigma_0, .., \sigma_{s-1})$ = all the strata, so $\phi(M)(x_j, [0, t_{j,j}))$ is guaranteed to be in $\text{conv}(\sigma_0, .., \sigma_{s-1})$) $(\sigma_0, ., \sigma_{s-1})$ recur.

(2) If N(j) remains bounded as $j \to \infty$ we show there exists $x_j' \to x$, $h_j' \downarrow 0$ and a subset $(\sigma_0', .., \sigma_{s'-1}')$ of $(\sigma_0, .., \sigma_{s-1})$ such that $\phi(M)(x_j, [0, h_j')) \subset (\sigma_0' \cup .. \cup \sigma_{s'-1}')$, some $s' < s$, and such that $| \phi(M)(x_j', [0, h_j')) | \to \infty$ as $j \to \infty$ (ie, we show that in this case the suppositions of Lemma 5.6 hold with $M = (\sigma_0 \cup .. \cup \sigma_{s-1})$ replaced by $\sigma_0' \cup .. \cup \sigma_{s'-1}'$, ie reducing the number of strata by at least one). Setting $t_{j,n(j)+1} = h_j$ and decomposing $[0, h_j) = [0, t_{j,1}) \cup .. \cup [t_{j,N(j)}, t_{j,N(j)+1})$ we have $| \phi(M)(x_j, [0, h_j)) | = | \phi(M)(x_j, [0, t_{j,1})) | + .. + | \phi(M)(x_j, [t_{j,N(j)}, t_{j,N(j)+1})) |$ and since $| \phi(M)(x_j, [0, h_j)) | \to \infty$ while N(j) remains bounded there exists some bounded sequence $\{i(j), j = 1, 2, \ldots \infty\}$ with $| \phi(M)(x_j, [t_{j,i(j)-1}, t_{j,i(j)})) | \to \infty$. Since there are only finitely many strata in $(\sigma_0, .., \sigma_{s-1})$ there exists some stratum $\sigma_{s'}' \in (\sigma_0, .., \sigma_{s-1})$ and a subsequence of this sequence, which we shall also denote $\{i(j), j = 1, 2, .., \infty\}$, such that $s' = i(j) \bmod s$ for all j. By definition of $t_{i(j)-1}$, $t_{i(j)}$ we must have $\phi(M)(x_j, t) \notin \sigma_{s'}'$ for all $t \in [t_{j,i(j)-1}, t_{j,i(j)})$ and hence the sequence $x_j' = \phi(M)(x_j, t_{j,i(j)-1}), h_j' = t_{j,i(j)} - t_{j,i(j)-1}$ satisfies the required properties.

(3) If N(j) as defined above $\to \infty$ then $(\sigma_0, .., \sigma_{s-1})$ recur: if N(j) remains bounded as $j \to \infty$ we infer from (2) that there exists some strict subset $(\sigma_0', .., \sigma_{s'-1}')$ of $(\sigma_0, .., \sigma_{s-1})$ such that the suppositions of Lemma 5.6 are satisfied with additionally $\phi(M)(x_i, [0, h_i)) \subset (\sigma_0' \cup .. \cup \sigma_{s'-1}')$ for all i. Hence replacing s by s' in the definition of the sequence $\{t_{j,j'}\}$ (and hence indirectly in the definition of N(j)) we can repeat the process; either our new $N(j) \to \infty$ and, arguing as in (1) above, $(\sigma_0', .., \sigma_{s'-1}')$ recur, or we can find a strict subset of $(\sigma_0', .., \sigma_{s'-1}')$ such that the suppositions of Lemma 5.6 are satisfied and additionally $\phi(M)(x_i, [0, h_i)) \subset$ the union of these strata. Continuing in this way we at each stage either obtain a recurring set of strata or a strictly smaller subset with the suppositions of Lemma 5.6 holding for this subset; if no subset of size > 2 recurs we get eventually some pair σ_0'', σ_1'' with $| \phi(M)(x_j'', [0, h_j'')) | \to \infty$ as $j \to \infty$ and $\phi(M)(x_j'', [0, h_j'')) \subset \sigma_0'' \cup \sigma_1''$, and hence σ_0'', σ_1'' must recur. ∎

Theorem 5.1 and Lemma 5.6 provide us in the first place with Corollary 5.1 below, which is the result upon which Chapter Seven hinges. We remark that since Theorem 5.1 was only proved for a submanifold with orthogonal corners (mainly because

Lemma 5.1 was only proved for a submanifold with orthogonal corners - Lemmas 5.2 and 5.3 are true for any submanifold with corners) for all results which use Theorem 5.1 the corners must be orthogonal too.

Definition If V is a neighbourhood of y in M set $T(V,y)=\sup\{t\geq 0:\phi(M)(y,\tilde{t})\in V\ \forall\ 0\leq\tilde{t}\leq t\}$, if finite, and infinity otherwise
(ie $T(V,y)$=time it takes a trajectory starting at $y\in V$ to reach ∂V).

We recall that there is an infinite order tangency between flows on σ_1,σ_2 at x if $x\in\bar{\sigma}_1\cap\bar{\sigma}_2$ and $D_t^i\phi(\breve{\sigma}_1)(x,t=0)=D_t^i\phi(\breve{\sigma}_2)(x,t=0)$ for all i.

Corollary 5.1 If M is a submanifold with orthogonal corners, X is a smooth vector field on M and $x\in M$ is such that $X(M)(x)\neq\mathbf{0}$ and if there are no infinite order tangencies between flows on strata at x, there exists a neighbourhood V of x in M and $N>0$ such that for all $y\in V$ $|\phi(M)(y,[0,T(V,y)))|\leq N$.

Proof

(1) We show that if $X(M)(x)\neq\mathbf{0}$ then if $V_i=B_{r(i)}(x)\cap M$, where $B_r(x)$=the open ball in R^n of radius r and centre x, then $\sup\{T(V_i,y):y\in V_i\}\to 0$ as $r(i)\downarrow 0$.
If not there exists $y_i\to x$ and $r(i)\downarrow 0$ and constant $\delta>0$ such that $|\phi(M)(y_i,t)-x|<r(i)$ for all i and for all $t\in[0,\delta]$. By continuous dependence on initial conditions (Theorem 1.1(3)) for fixed $\epsilon>0$ $|\phi(M)(x,\epsilon)-\phi(M)(y_i,\epsilon)|\to 0$ as $y_i\to x$. For all i $|\phi(M)(x,\epsilon)-\phi(M)(y_i,\epsilon)|\geq|\phi(M)(x,\epsilon)-x|-|\phi(M)(y_i,\epsilon)-x|$ and if $0\leq\epsilon<\delta$ it follows (since we know that if $\epsilon<\delta$ that $|\phi(M)(y_i,\epsilon)-x|<r(i)\downarrow 0$) by taking the limit $i\to\infty$ that $|\phi(M)(x,\epsilon)-x|=0$. By Theorem 3.1 $\phi(M)(x,t)\neq 0$ for all sufficiently small $t>0$ (because if it was zero for $t_i\downarrow 0$ then we would have $\lim_{t_i\downarrow 0}(1/t_i)(\phi(M)(x,t_i)-x)=0$, contrary to $X(M)(x)\neq\mathbf{0}$), hence result.
(2) If there was no V, N as claimed in Corollary 5.1 we could find $V_i=B_{r(i)}(x)\cap M$ with $r(i)\downarrow 0$ and $y_i\in V_i$ such that $|\phi(M)(y_i,[0,T(V_i,y_i)))|\to\infty$, by (1) $T(V_i,y_i)\to 0$ and hence by Lemma 5.6 there exists a subset of strata which recur at x, and hence by Theorem 5.1 the flows on these strata are infinitely tangent at x. ∎

We shall now derive some other implications of Theorem 5.1. As part of Theorem 3.1 we showed the trajectories $\phi(M)(x)$ to be $C^{+\infty}$ and we now show them to be $c^{-\infty}$. The idea is to show that if $\phi(M)(x,t)$ makes infinitely many stratum jumps in any left neighbourhood of some $t_0 \in (0,t_x)$ (and so we could not infer the result by a simple argument) then the strata intersected infinitely often are recurring and so by Theorem 5.1 all the flows projected onto them are infinitely tangent at $\phi(M)(x,t_0)$. We can then put together an inductive argument somewhat similar to Theorem 3.1 to show that left hand time derivatives of all orders exist, and in fact equal the two-sided time derivatives at $\phi(M)(x,t_0)$ of the flows on any of the recurring strata.

Definition If $x \in M$, M locally represented near $\phi(M)(x,t_0)$ as ZN(I;J), $t_0 \in (0,t_x)$, set

$I^-(t_0) = \cup\{K: \exists \{h_j(K)\}_{j=1,\ldots,\infty}$ with $h_j(K) \downarrow 0$ as $j \to \infty$ with $K = S_1(\phi(M)(x,t_0 - h_j(K)))\ \forall j\}$

$I_-(t_0) = \cap\{K: \exists \{h_j(K)\}_{j=1,\ldots,\infty}$ with $h_j(K) \downarrow 0$ as $j \to \infty$ with $K = S_1(\phi(M)(x,t_0 - h_j(K)))\ \forall j\}$

$I^+(t_0) = \cup\{K: \exists \{h_j(K)\}_{j=1,..,\infty}$ with $h_j(K) \downarrow 0$ as $j \to \infty$ with $K = S_1(\phi(M)(x,t_0 + h_j(K)))\ \forall j\}$

$I_+(t_0) = \cap\{K: \exists \{h_j(K)\}_{j=1,..,\infty}$ with $h_j(K) \downarrow 0$ as $j \to \infty$ with $K = S_1(\phi(M)(x,t_0 + h_j(K)))\ \forall j\}$

Eg. If M=ZN(∅;1,3) is as illustrated in Figure 5.13 with vector field $\dot{y}_3 = f(y_1 + y_2)$, $-\dot{y}_1 = \dot{y}_2 = 1$, where

$$f(y) = \begin{cases} (1/y^2)\exp(-1/|y|)(\sin(1/y) - \cos(1/y)) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

then if x=(1,-1,0) and $t_0 = 1$, so $x_0 = \phi(M)(x,t_0) = 0$, then $\phi(M)(x)$ intersects ZP(3;1) and ZP(∅;1,3) infinitely often in any left neighbourhood of t_0, ZP(1,3;∅) and ZP(1;3) infinitely often in any right neighbourhood of t_0, and $I_-(t_0) = \varnothing$, $I^-(t_0) = (3)$, $I_+(t_0) = (1)$, and $I^+(t_0) = (1,3)$.

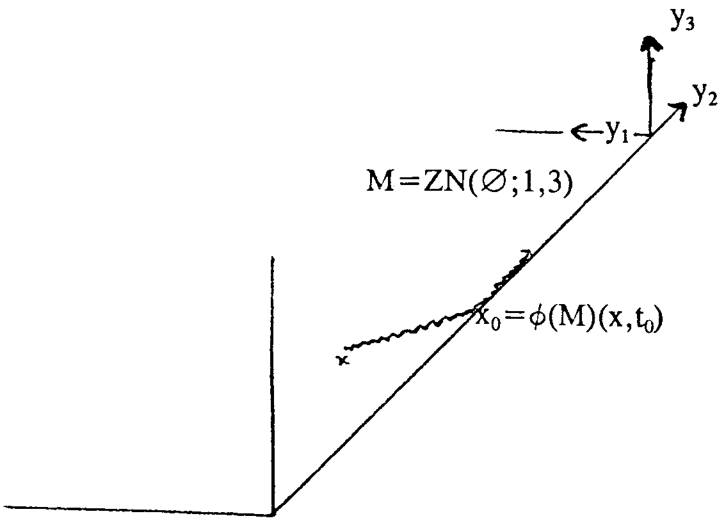


Figure 5.13

We shall show in part (c) of the proof of Corollary 5.2 that $S^0_\infty(x) \subset I_+(0) \subset I^+(0) \subset S_\infty(x)$. The quantities $I^\pm(t_0), I_\pm(t_0)$ satisfy the following lemma (it is straightforward to check that its conclusions are satisfied with the data of the example above). For the same reason as Corollary 5.1 Theorem 5.1 restricts us to submanifolds with orthogonal corners:

Lemma 5.7 Setting $\phi(M)(x,t_0)=x_0$ with M a submanifold with orthogonal corners locally represented near x_0 as $ZN(I;J)=ZN(S^0_1(x_0);S_1(x_0)\backslash S^0_1(x_0))$ as usual, we have

(1) $I \subset I_-(t_0) \subset I^-(t_0) \subset I \cup J$ and

$I \subset I_+(t_0) \subset I^+(t_0) \subset I \cup J$

(2) $D_t^i\phi(K)(x_0,t=0)=D_t^i\phi(K')(x_0,t=0)$ for all $I_-(t_0) \subset K,K' \subset I^-(t_0)$, $I_+(t_0) \subset K,K' \subset I^+(t_0)$ and for all i

(3) For given $x \in M$, $t_0 \in (0,t_x)$ there exists $h>0$ such that

(i) $I^-(t_0)=\cup\{S_1(\phi(M)(x,t)):t_0-h<t<t_0\}$

(ii) $I_-(t_0)=\cap\{S_1(\phi(M)(x,t)):t_0-h<t<t_0\}$

(iii) $I^+(t_0)=\cup\{S_1(\phi(M)(x,t)):t_0<t<t_0+h\}$

(iv) $I_+(t_0)=\cap\{S_1(\phi(M)(x,t)):t_0<t<t_0+h\}$

and such that

(v) for all $t \in (t_0-h,t_0)$ $I_\pm(t)=I_-(t_0), I^\pm(t) \subset I^-(t_0)$ and

(vi) for all $t \in (t_0,t_0+h)$ $I_\pm(t)=I_+(t_0), I^\pm(t) \subset I^+(t_0)$.

Proof

(1) Follows from definitions (use that for any $y \in ZN(I;J)$, $I \subset S_1(y) \subset I \cup J$)

(2) Since $\{ZP(K;J\backslash K)$:there exist $\{h_j(K)\}_{j\in Z^+}$ with $h_j(K)\downarrow 0$ as $j\to\infty$ and $K=S_1(\phi(M)(x,t_0-h_j(K)))$ for all j$\}$ and $\{ZP(K;J\backslash K)$:there exist $\{h_j(K)\}_{j\in Z^+}$ with $h_j(K)\downarrow 0$ as $j\to\infty$ and $K=S_1(\phi(M)(x,t_0+h_j(K)))$ for all j$\}$ are plainly each recurring at x_0, (2) follows by Theorem 5.1.

(3) We do the first two as specimens:

(i) By definition $I^-(t_0) \subset \cup\{S_1(\phi(M)(x,t)):t_0-h<t<t_0\}$ any $h>0$. If there does not exist $\{h_j(K)\}_{j\in Z^+}$ with $h_j(K)\downarrow 0$ such that $K=S_1(\phi(M)(x,t_0-h_j(K)))$ $\forall j$ then there exists $h(K)>0$ such that $K\neq S_1(\phi(M)(x,t))$ for all $t_0-h(K)<t<t_0$. Taking $h=\min_{K\notin I^-(t_0)}h(K)$ we see that $k' \in \{S_1(\phi(M)(x,t)):t_0-h<t<t_0\}$ iff

$k' \in \{K$:there exist $h_j(K)\downarrow 0$ with $K=S_1(\phi(M)(x,t_0-h_j(K)))\}$, hence

$\cup\{S_1\phi(M)(x,t):t_0-h<t<t_0\}=\cup\{K$:there exist $\{h_j(K)\}_{j\in Z^+}$ with $h_j(K)\downarrow 0$ and $K=S_1\phi(M)(x,t_0-h_j(K))$ $\forall j\}$ as required.

(ii) We have defined $I_-(t_0)=\cap\{K$:there exist $\{h_j(K)\}_{j\in Z^+}$ with $h_j(K)\downarrow 0$ and

$K=S_1(\phi(M)(x,t_0-h_j(K)))\ \forall j\}$ and we now show that $I_-(t_0)=\{i\in I\cup J: f_i\phi(M)(x,t_0-h)=0$ for all arbitrarily small $h>0\}$. If $i\notin\{i\in I\cup J: f_i\phi(M)(x,t_0-h)=0$ for all arbitrarily small $h>0\}$ then there exists a sequence $\{h_j(i)\}_{j\in Z^+}$ such that $h_j(i)\downarrow 0$ as $j\to\infty$ with $f_i\phi(M)(x,t_0-h_j(i))\neq 0$, and since there are only finitely many values which $S_1(\phi(M)(x,t_0-h_j(i)))$ may take on there exists a subsequence with $S_1(\phi(M)(x,t_0-h_j(i)))$ equal to some constant set of indices not including i, and hence $i\notin I_-(t_0)$. Conversely, if $i\in\{I\cup J: f_i\phi(M)(x,t_0-h)=0$ for all arbitrarily small $h>0\}$ then $i\in$ any K for which there exists $\{h_j(K)\}_{j\in Z^+}$ with $h_j(K)\downarrow 0$ and $K=S_1\phi(M)(x,t_0-h_j(i)))\ \forall j$, so $i\in I_-(t_0)$. Hence $I_-(t_0)=\{i$:there exists $h>0$ such that $f_i\phi(M)(x,t)=0$ for all $t_0-h<t<t_0\}$. Then since $S_1\phi(M)(x,t)=\{i: f_i\phi(M)(x,t)=0\}$, it follows $\cap\{S_1\phi(M)(x,t): t_0-h<t<t_0\}=$ $\{i: f_i\phi(M)(x,t)=0$ for all $t_0-h<t<t_0\}=I_-(t_0)$ which gives the required result. ▬

The following result may superficially appear to be a left hand version of Theorem 3.1: an important difference is that while in Theorem 3.1 the right hand derivatives were expresed in terms of the iterates $S^0_i(x)$, $S_i(x)$, which were calculated by a simple algorithm determined by X and the f_i's defining M near x, the quantities $I_-(t)$, $I^-(t)$ are functions of the trajectories (see definition above) and are not directly calculable

Proposition 5.1 If M is a submanifold with orthogonal corners and X is a smooth vector field on M, then for each $x\in M$ at each $t\in(0,t_x)$ $\phi(M)(x)$ is $C^{-\infty}$, and if M is represented near $\phi(M)(x,t)$ as ZN(I;J) then for all $j\geq 0$ $D_t^{-j}\phi(M)(x,t)=D_s^j\phi(K)(\phi(M)(x,t),s=0)$, any $I_-(t)\subset K\subset I^-(t)$.

Proof We shall prove the result at $t_0\in(0,t_x)$.
(a) We show there exists $h>0$ such that if $t\in(t_0-h,t_0)$ then $I_-(t_0)\subset S_j\phi(M)(x,t)\subset I^-(t_0)$ for all $j\geq 1$.
From Lemma 5.7 we know there exists $h>0$ such that $f_i\phi(M)(x,t)=0$ for all $i\in I_-(t_0)$ for all $t\in(t_0-h,t_0)$ and hence $D_t^{+j}f_i\phi(M)(x,t)=0$ for all $i\in I_-(t_0)$, $j\geq 0$, $t\in(t_0-h,t_0)$. By Theorem 3.1 $D_t^j\phi(S_\infty(y))(y,t=0)=D_t^{+j}\phi(M)(y,t=0)$ for all i and for all j; hence $D_s^jf_i\phi(S_\infty(\phi(M)(x,t)))(\phi(M)(x,t),s=0)=0$ for all $i\in I_-(t_0)$. However by Lemma 5.7(3,ii) we have $I_-(t_0)\subset S_1(\phi(M)(x,t))$ any $t\in(t_0-h,t_0)$, so if $I_-(t_0)$ was not contained in $S_\infty(\phi(M)(x,t))$ then for each $i\in I_-(t_0)\backslash S_\infty(\phi(M)(x,t))$ there exists $j\in Z^+$ such that $i\in S_j(\phi(M)(x,t))\backslash S_{j+1}(\phi(M)(x,t))$ and by Lemma 2.5 we would have $D_s^jf_i\phi(S_\infty(\phi(M)(x,t))(\phi(M)(x,t),s=0)>0$, a contadiction. By Lemma 5.7 again we know $S_1\phi(M)(x,t)\subset I^-(t_0)$ for all $t\in(t_0-h,t_0)$ so by the iteration property (specifically, $S_1(y)\supset S_j(y)$

for all $j \geq 1$)we have $I_-(t_0) \subset S_j(\phi(M)(x,t)) \subset S_1(\phi(M)(x,t)) \subset I^-(t_0)$ for all $j \geq 1, t \in (t_0-h,t_0)$, as required.

(b) We show $D_t^-\phi(M)(x,t_0)$ exists $=D_t\phi(K)(x_0)$ any $I_-(t_0) \subset K \subset I^-(t_0)$ where again we are setting $\phi(M)(x,t)=x_0$.

We have for any $h>0$

$$\left| \phi(M)(x,t_0)-\phi(M)(x,t_0-h)-hD_t\phi(K)(x_0,t=0)\right| = \left|\int_{t_0-h}^{t_0}(X(M)\phi(M)(x,s)-X(K)(x_0))ds\right|$$

$\leq |h| \operatorname{Sup}\{ |X(M)\phi(M)(x,s)-X(K)(x_0)| : s \in (t_0,t_0-h)\}$. By (a) for $h>0$ sufficiently small $X(M)\phi(M)(x,s)=X(K')\phi(M)(x,s)$ some $I_-(t_0) \subset K' \subset I^-(t_0)$ and in fact if we set

$$\delta(K,s) = \begin{cases} 1 \text{ if } K=S_2(\phi(M)(x,s)) \\ 0 \text{ otherwise} \end{cases}$$

we have for all $s \in (t_0-h,t_0)$ that $X(M)\phi(M)(x,s)=\Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\delta(K,s)X(K)\phi(M)(x,s)$ (we are using that $X(M)(y)=X(S_2(y))(y)$). Thus with $X(K'e\varnothing)$ the C^r extension of $X(K')$ to R^n of Chapter Two (we use $X(K'e\varnothing)$ rather than $X(K')$ because $X(K')\phi(M)(x,s)$ is only defined if $\phi(M)(x,s) \in Z(K')$. Of course if $\phi(M)(x,s) \notin Z(K')$ then $\delta(K',s)=0$, so this change makes no essential difference. A similar situation occurred in the proof of Theorem 3.1) we have

$|X(M)\phi(M)(x,s)-X(K)(x_0)| = |\Sigma_{I_-(t_0)\subset K'\subset I^-(t_0)}\delta(K',s)(X(K'e\varnothing)\phi(M)(x,s)-X(K)(x_0))|$
$<\Sigma_{I_-(t_0)\subset K'\subset I^-(t_0)} |X(K'e\varnothing)\phi(M)(x,s)-X(K)(x_0)| \to 0$ as $s \uparrow t_0$ by Lemma 5.7(2), and the result follows.

(c) Suppose the proposition is true up to $j=k-1$. Then by the (k-1)th result we know that for $0<t_0-h<t_0<t_x$ $D_t^{-(k-1)}\phi(M)(x,t_0-h)=D_t^{k-1}\phi(K)(\phi(M)(x,t_0-h),t=0)$ any $I_-(t_0-h) \subset K \subset I^-(t_0-h)$. By Lemma 5.7(3) we know that for sufficiently small $h \geq 0$ $I_-(t_0-h)=I_-(t_0)$, and that $I^-(t_0-h) \subset I^-(t_0)$. Thus for each sufficiently small $h \geq 0$ we can choose a single K with $I_-(t_0) \subset K \subset I^-(t_0)$ such that $D_t^{-(k-1)}\phi(M)(x,t_0-h)=D_t^{k-1}\phi(K)(\phi(M)(x,t_0-h),t=0)$: we set for each $I_-(t_0) \subset K \subset I^-(t_0)$ $\tilde{\delta}(K,h)=1$ if K has been chosen at h, and $\tilde{\delta}(K,h)=0$ otherwise. Thus for each fixed h $\Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\tilde{\delta}(K,h)=1$. Setting as usual $x_0=\phi(M)(x,t_0)$ we therefore have for any $I_-(t_0) \subset K \subset I^-(t_0)$

$D_t^{-(k-1)}\phi(M)(x,t_0)-D_t^{-(k-1)}\phi(M)(x,t_0-h)-hD_t^k\phi(K)(x_0,t=0)=$
$\Sigma_{I_-(t_0)\subset K'\subset I^-(t_0)}\tilde{\delta}(K',h)[D_t^{(k-1)}\phi(K')(x_0,t=0)-D_t^{(k-1)}\phi(K'e\varnothing)(\phi(M)(x,t_0-h),t=0)+$
$D_t^{(k-1)}\phi(K')(x_0,-h)-D_t^{(k-1)}\phi(K')(x_0,-h)-hD_t^k\phi(K)(x_0,t=0)]$ (in the second term $\phi(K'e\varnothing)$ is used rather than $\phi(K')$ for the same reason as in (b), to guarantee that the term is defined) and the result (ie, that $D_t^{-k}\phi(M)(x,t_0)=D_t^k\phi(K)(x_0,t=0)$ any $I_-(t_0) \subset K \subset I^-(t_0)$)

will follow if we can show that for every K,K' such that $I_-(t_0) \subset K, K' \subset I^-(t_0)$ that

$(1/h) \mid D_t^{(k-1)}\phi(K')(x_0,t=0) - D_t^{(k-1)}\phi(K'e\varnothing)(\phi(M)(x,t_0-h),t=0) +$
$D_t^{(k-1)}\phi(K')(x_0,-h) - D_t^{(k-1)}\phi(K')(x_0,-h) - hD_t^k\phi(K)(x_0,t=0) \mid$ (*)
$\to 0$ as $h \downarrow 0$.

By Lemma 5.7(2) we know $D_t^k\phi(K')(x_0,t=0) = D_t^k\phi(K)(x_0,t=0)$ any $I_-(t_0) \subset K, K' \subset I^-(t_0)$ so,

$(1/h)(D_t^{(k-1)}\phi(K')(x_0,t=0) - D_t^{(k-1)}\phi(K')(x_0,-h) - hD_t^k\phi(K)(x_0,t=0)) =$
$(1/h)(D_t^{(k-1)}\phi(K')(x_0,t=0) - D_t^{(k-1)}\phi(K')(x_0,-h) - hD_t^k\phi(K')(x_0,t=0))$ which $\to 0$ as $h \downarrow 0$ by smoothness of $\phi(K')$. This takes care of the first, fourth and fifth terms of (*). We know $D_t^-\phi(M)(x_0,t=0) - D_t\phi(K')(x_0,t=0)$ any $I_-(t_0) \subset K' \subset I^-(t_0)$
$= \lim_{h\downarrow 0}(1/h)(\phi(M)(x,t_0) - \phi(M)(x,t_0-h) - \phi(K')(x_0,0) + \phi(K')(x_0,-h))$
$= \lim_{h\downarrow 0}(1/h)(\phi(K')(x_0,-h) - \phi(M)(x,t_0-h))$, hence setting $Y(x) = D_t^{k-1}\phi(K'e\varnothing)(x,t=0)$, which is smooth, and supposing B is a convex compact set containing $\phi(M)(x,t_0-h')$ and $\phi(K')(x_0,-h')$ $\forall\ 0 \le h' \le t_0$, by the Mean Value Theorem
$\mid D_s^{k-1}\phi(K'e\varnothing)(\phi(M)(x,t_0-h),s=0) - D_s^{k-1}\phi(K'e\varnothing)(\phi(K')(x_0,-h),s=0) \mid \le$
$\mid \phi(M)(x,t_0-h) - \phi(K)(x_0,-h) \mid \sup_{x\in B} \mid D_xY(x) \mid$, hence using that
$D_s^{k-1}\phi(K'e\varnothing)(\phi(K')(x_0,-h),s=0) = D_s^{k-1}\phi(K')(\phi(K')(x_0,-h),s=0) = D_t^{k-1}\phi(K')(x_0,-h)$ we have $(1/h)(D_s^{k-1}\phi(K'e\varnothing)(\phi(M)(x,t_0-h),s=0) - D_t^k\phi(K')(x_0,-h)) \to 0$ as $h \downarrow 0$, which deals with the second and third terms of (*), and it follows that $D_t^{-k}\phi(M)(x,t_0)$ exists and equals $D_t^k\phi(K)(x_0,t=0)$ any $I_-(t_0) \subset K \subset I^-(t_0)$. ▬

We can now establish the following "approximation" result which has several implications (see Corollaries 5.2 and 5.3 and Proposition 5.2 below):

<u>Lemma 5.8</u> If M is a submanifold with orthogonal corners, $x \in M$, $x_0 = \phi(M)(x,t_0)$, M locally represented as ZN(I;J) near x_0, then for any real $\epsilon > 0$ and integer $i > 0$ there exists $h_i(\epsilon) > 0$ such that for any $0 < h \le h_i(\epsilon)$ (the condition $h > 0$ is essential) and for all $0 \le j \le i$

(1) $\mid D_t^{\pm j}\phi(M)(x,t_0-h) - D_t^j\phi(K)(x_0,-h) \mid \le \epsilon h^{i-j}$ any $I_-(t_0) \subset K \subset I^-(t_0)$

(2) $\mid D_t^{\pm j}\phi(M)(x,t_0+h) - D_t^j\phi(K)(x_0,h) \mid \le \epsilon h^{i-j}$ any $I_+(t_0) \subset K \subset I^+(t_0)$

<u>Proof</u>

We observe we are making here four assertions. We shall do (1), (2) being similar. Consider the following assertion, which will be called assertion (*):

For any given $\epsilon > 0$ and non-negative integers i,j, there exists $h(i,j,\epsilon) > 0$ such that for all $0 < h \le h(i,j,\epsilon)$ $\mid D_t^{\pm j}\phi(M)(x,t_0-h) - D_t^j\phi(K)(x_0,-h) \mid \le \epsilon h^{i-j}$ for all $I_-(t_0) \subset K \subset I^-(t_0)$.

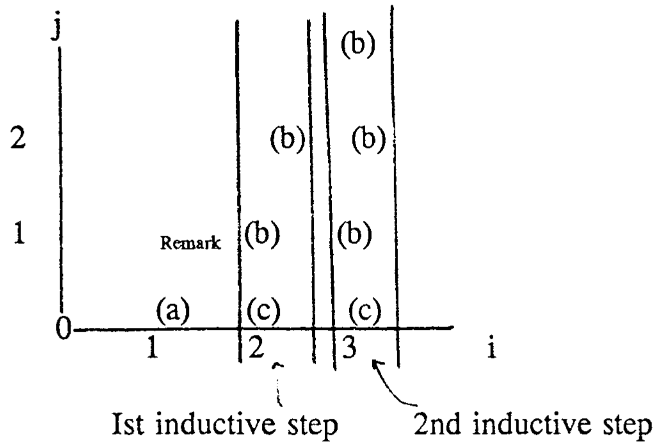
Then Lemma 5.8 will follow if we show that for any given $\epsilon>0$

(a) (*) holds with $(i=1,j=0)$

(b) That if (*) holds for $(i\text{-}1,j=0)$ with $i\text{-}1\geq 1$ then it holds for (i,j) for all $1\leq j\leq i$

(c) That if (*) holds for $(i,j=1)$ some $i\geq 0$ then it holds for $(i,0)$

since we can then take $h_0(\epsilon)=h_1(\epsilon)=h_2(\epsilon)=\min\{h(2,0,\epsilon),h(2,1,\epsilon),h(2,2,\epsilon)\}$, and for all $i\geq 2$ $h_i(\epsilon)=\min\{h(i,j,\epsilon):0\leq j\leq i\}$. The figure below shows which part of the proof verifies (*) for each (i,j). After showing (*) to hold (for any given $\epsilon>0$) for $(i=1,j=0)$ the first inductive step is to infer that it holds for $(i=1,j=1,2)$ (by (b) below) and for $(i=1,j=0)$ (by (c) below), the second inductive step is to infer that therefore it holds for $(i=2,1\leq j\leq 3)$ (by (b)) and for $(i=2,j=0)$ (by (c)) etc. The remark is: that if (*) holds for given ϵ for (i,j) then (by inspection) it holds a fortiori for all (i',j) with $i'\leq i$.



(a) We saw in part (c) of the proof of Proposition 5.1 that $\lim_{h\downarrow 0}(1/h)(\phi(M)(x,t_0\text{-}h)\text{-}\phi(K)(x_0,\text{-}h))=0$ for all $I_-(t_0)\subset K\subset I^-(t_0)$ (where as usual $x_0=\phi(M)(x,t_0)$), and hence for any $\epsilon>0$ there exists $h(1,0)>0$ with $|\ \phi(M)(x,t_0\text{-}h)\text{-}\phi(K)(x_0,\text{-}h)\ | <\epsilon h$ for all $0\leq h\leq h(1,0)$

(b)

(i) We know that for each sufficiently small $h>0$ and any $j\geq 0$ there exists K with $I_-(t_0)\subset K\subset I^-(t_0)$ such that $D_t^{\pm j}\phi(M)(x,t_0\text{-}h)=D_t^j\phi(K)(\phi(M)(x,t_0\text{-}h),t=0)$. This is so in the -j case because by Proposition 5.1 $D_t^{-j}\phi(M)(x,t_0\text{-}h)=D_t^j\phi(K)(\phi(M)(x,t_0\text{-}h),t=0)$ some $I_-(t_0\text{-}h)\subset K\subset I^-(t_0\text{-}h)$ and by Lemma 5.7 for $h>0$ sufficiently small $I_-(t_0\text{-}h)=I_-(t_0),I^-(t_0\text{-}h)\subset I^-(t_0)$, and in the +j case by part (a) of the proof of Proposition 5.1, where we showed $I_-(t_0)\subset S_j(\phi(M)(x,t))\subset I^-(t_0)$ for all $t\in(t_0\text{-}h,t_0)$ some $h>0$; then by definition $D_t^{+j}\phi(M)(x,t)=D_s^j\phi(S_{j+1}(\phi(M)(x,t)))(\phi(M)(x,t),s=0)$.

(ii) We can therefore find $h_0>0$ and choose for each $t\in(t_0\text{-}h_0,t_0)$ and $I_-(t_0)\subset K\subset I^-(t_0)$ $\delta^+(K,t)$, $\delta^-(K,t)=0$ or 1 such that

$D_t^{\pm j}\phi(M)(x,t)=\Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\delta^{\pm}(K,t)D_s^j\phi(Ke\varnothing)(\phi(M)(x,t),s=0)$ each $t\in(t_0-h_0,t_0)$ (again we have replaced $\phi(K)$ by $\phi(Ke\varnothing)$ to guarantee this term is defined even when $\phi(M)(x,t)\notin Z(K)$) where for each fixed t $\Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\delta^{+}(K,t)=\ \Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\delta^{-}(K,t)=1$. Hence for $0<h<h_0$ $|\ D_t^{\pm j}\phi(M)(x,t_0-h)-D_t^j\phi(I_-(t_0))(x_0,-h)\ |$ =

$|\ \Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\delta^{\pm}(K,t_0-h)(D_s^j\phi(Ke\varnothing)(\phi(M)(x,t_0-h),s=0)-D_s^j\phi(K)(\phi(K)(x_0,-h),s=0)+$
$D_s^j\phi(K)(\phi(K)(x_0,-h),s=0)-D_t^j\phi(I_-(t_0))(x_0,-h))\ |\ \le$
$\Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\ |\ D_s^j\phi(Ke\varnothing)(\phi(M)(x,t_0-h),s=0)-D_s^j\phi(K)(\phi(K)(x_0,-h),s=0)\ |\ +$
$\Sigma_{I_-(t_0)\subset K\subset I^-(t_0)}\ |\ D_s^j\phi(K)(\phi(K)(x_0,-h),s=0)-D_t^j\phi(I_-(t_0))(x_0,-h)\ |$, and the result will follow if we can show that for any given $\epsilon>0$, for each $1\le j\le i$ and for all $I_-(t_0)\subset K\subset I^-(t_0)$ each of the two quantities between | | signs is $\le\epsilon h^{i-j}$ for all $0<h\le$ some $h(i,j)>0$.

(iii) We first deal with the

$|\ D_s^j\phi(Ke\varnothing)(\phi(M)(x,t_0-h),s=0)-D_s^j\phi(K)(\phi(K)(x_0,-h),s=0)\ |$ term. Setting $Y(x)=D_t^j\phi(Ke\varnothing)(x,t=0)$ and $\bar{B}$ a compact convex set containing for all $0\le h\le t_0$ $\phi(M)(x,t_0-h)$, $\phi(K)(x_0,-h)$ we have by the Mean Value Theorem

$|\ D_s^j\phi(Ke\varnothing)(\phi(M)(x,t_0-h,s=0)-D_s^j\phi(Ke\varnothing)(\phi(I_-(t_0)(x_0,-h),s=0)\ |\ \le$
$|\ \phi(M)(x_0,t_0-h)-\phi(K)(x_0,-h)\ |\ \mathrm{Sup}_{x\in B}\ |\ D_xY(x)\ |$. However by the fact that (*) is true for (i-1,j=0) we know there exists $h(i-1,0)>0$ such that for all $0<h\le h(i-1,0)$
$|\ \phi(M)(x,t_0-h)-\phi(K)(x_0,-h)\ |\ <\epsilon h^{i-1}$ any $I_-(t_0)\subset K\subset I^-(t_0)$ which gives
$|\ D_s^j\phi(Ke\varnothing)(\phi(M)(x,t_0-h),s=0)-D_s^j\phi(Ke\varnothing)(\phi(K)(x_0,-h),s=0)\ |\ \le\epsilon h^{i-j}$ any $1\le j\le i$ for all sufficiently small $h>0$.

(iv) We deal with the second term in the formula of (ii)
$|\ D_s^j\phi(K)(\phi(K)(x_0,-h),s=0)-D_t^j\phi(I_-(t_0))(x_0,-h)\ |$. We set $f(h)=D_t^j\phi(K)(x_0,-h)-D_t^j\phi(I_-(t_0))(x_0,-h)$ and we want to show that for any $\epsilon>0$ and $I_-(t_0)\subset K\subset I^-(t_0)$ that $|\ f(h)\ |\ \le\epsilon h^{i-j}$ for all $0<h<$ some $h(i,j)>0$. We know by Lemma 5.7(2) that $D_h^i f(h=0)=0$ for all i, hence
$f(h)=f(0)+hD_hf(h=0)+..+(h^{i-j}/(i-j)!)D_h^{i-j}f(\theta h)=(h^{i-j}/(i-j)!)D_h^{i-j}f(\theta h)$ some $\theta\in(0,1)$, where since $D_h^{i-j}f(0)=0$ $\sup_{\theta\in[0,1]}D_h^{i-j}f(\theta h)\to 0$ as $h\to 0$. Since there are only finitely many possible K the result follows.

(c) We show that if (*) holds for $j=1$, some fixed $i\ge j$, then it holds for $j=0$, same i (ie that then $|\ \phi(M)(x,t_0-h)-\phi(K)(x_0,-h)\ |\ \le\epsilon h^i$ for all $I_-(t_0)\subset K\subset I^-(t_0)$ and for all sufficiently small $h\ge 0$):this follows because we have

$$|\ \phi(M)(x,t_0-h)-\phi(K)(x_0,-h)| = \left|\int_{t_0-h}^{t_0}(X(M)\phi(M)(x,t)-X(K)\phi(K)(\phi(M)(x,t_0),t-t_0))dt\right|$$

and if $I_-(t_0)\subset K\subset I^-(t_0)$ the integrand is $<\epsilon h^{i-1}$ by the result with $j=1$. ∎

We showed in Remark 3.2 that the iteration can be viewed as a selecting process, and now we show that it is also providing us with an increasingly accurate approximation to the location and (one-sided) derivatives of $\phi(M)(x,t)$ for t small and positive (Corollary 5.2 below). If given some data we do the calculation to find $S^0_{m+1}(x),S_{m+1}(x)$ for some $m\geq 0$ at a point $x\in M$ with M locally repesented as ZN(I;J) we know from Remark 3.2 that for t sufficiently small and positive $\phi(M)(x,t)\in \cup_{S^0_{m+1}(x)\subset K\subset S_{m+1}(x)}ZP(K;J\backslash K)$; Corollary 5.2 is now telling us that in addition for any $\epsilon>0$ there exists $h>0$ such that for all $0<t<h$ $|\phi(M)(x,t)-\phi(K)(x,t)|<\epsilon t^m$, and generally that $|D_t^{\pm i}\phi(M)(x,t)-D_t^i\phi(K)(x,t)|<\epsilon t^{m-i}$, any $0\leq i\leq m$, any $S^0_{m+1}(x)\subset K\subset S_{m+1}(x)$. In the case of Example 5.1 (=Example 2.1) for instance where $S^0_i(0)=\varnothing$ for all i, $S_i(0)=(1)$ for all i, Corollary 5.2 below tells us (as we could in this case verify directly) that for any $\epsilon>0$ and any $m\in Z^+$ there exists $h>0$ such that on $0<t<h$ $|\phi(M)(x,t)-(t,0)|<\epsilon t^m$. From the point of view of applications the usefulness of this result (and Remark 3.2) rests on only having to calculate finitely many iterates $(S^0_i(x),S_i(x))$; we do not need to know the whole series.

<u>Corollary 5.2</u> For each $x\in M$, M locally represented as ZN(I;J), $i>0$, $\epsilon>0$ there exists $h_i>0$ such that for all $0\leq j\leq i$ and for all $S^0_{i+1}(x)\subset I\subset S_{i+1}(x)$ $|D_t^{\pm j}\phi(M)(x,t)-D_t^{\pm j}\phi(I)(x,t)|\leq\epsilon t^{i-j}$ any $0<t\leq h_i$.

<u>Proof</u>

(a) We show that for all K such that $I_+(0)\subset K\subset I^+(0)$ $D_t^i\phi(S_\infty(x))(x,t=0)=D_t^i\phi(K)(x,t=0)$ for all i. By Theorem 3.1 $\lim_{h\downarrow 0}D_t^{+i}\phi(M)(x,t=h)=D_t^{+i}\phi(M)(x,t=0)$ hence $D_t^{+i}\phi(M)(x,t=0)-D_t^i\phi(K)(x,t=0)$ $=\lim_{h\downarrow 0}(D_t^{+i}\phi(M)(x,t=h)-D_t^i\phi(K)(x,t=h))$ which by Lemma 5.8(2) $=0$ if $I_+(0)\subset K\subset I^+(0)$. Since also by Theorem 3.1 $D_t^{+i}\phi(M)(x,t=0)=D_t^i\phi(S_\infty(x))(x,t=0)$ for all i we have $D_t^if_j\phi(S_\infty(x))(x,t=0)=D_t^if_j\phi(K)(x,t=0)=0$ for all $j\in K$ and $D_t^if_j\phi(K)(x,t=0)=D_t^if_j\phi(S_\infty(x))(x,t=0)=0$ for all $j\in S_\infty(x)$, which by Lemma 2.2 imply respectively $D_t^i\phi(S_\infty(x))(x,t=0)=D_t^i\phi(S_\infty(x)\cup K)(x,t=0)$ and $D_t^i\phi(K)(x,t=0)=D_t^i\phi(K\cup S_\infty(x))(x,t=0)$, for any set of indices K with $I_+(t_0)\subset K\subset I^+(t_0)$, for all i, which gives the result.

(b) We show that for any $\epsilon>0$ there exists $h_i>0$ such that $|D_t^j\phi(K_1)(x,h)-D_t^j\phi(K_2)(x,h)|\leq\epsilon h^{i-j}$ any $0\leq h\leq h_i$ any $S^0_{i+1}(x)\subset K_1,K_2\subset S_{i+1}(x)$. $D_t^j\phi(K_1)(x,h)-D_t^j\phi(K_2)(x,h)=D_t^j\phi(K_1)(x,0)-D_t^j\phi(K_2)(x,0)+$

$h(D_t^{j+1}\phi(K_1)(x,0)-D_t^{j+1}\phi(K_2)(x,0))+..+h^{i+1-j}/(i+1-j)!(D_t^{i+1}\phi(K_1)(\theta h)-D_t^{i+1}\phi(K_2)(\theta h))$ some $\theta\in(0,1)$ with all terms but the last $=0$ by Corollary 2.1, while clearly $h/(i+1-j)!(D_t^{i+1}\phi(K_1)(\theta h)-D_t^{i+1}\phi(K_2)(\theta h))\to 0$ as $h\to 0$.

(c) From Lemma 2.3 and the construction of the iteration (or by other methods, eg, if K does not lie between $S^0_\infty(x)$ and $S_\infty(x)$ there exists $j\in K\backslash S_\infty(x)\cup S^0_\infty(x)\backslash K$ and the result follows from Lemma 2.5 and 2.6) it follows that if $D_t^i\phi(K)(x,t=0)=D_t^i\phi(S_\infty(x))(x,t=0)$ for all i then $S^0_\infty(x)\subset K\subset S_\infty(x)$. Hence (a) tells us that $S^0_\infty(x)\subset I_+(0)\subset I^+(0)\subset S_\infty(x)$ (these inequalities may be strict - see Remark 5.1 below). Then using the triangle inequality to combine Lemma 5.8(2) with part (b) above the result follows. ∎

Remark 5.1 We showed in the above proof that $S^0_\infty(x)\subset I_+(0)\subset I^+(0)\subset S_\infty(x)$. We note that the inclusions may be strict: in Figure 5.14(i) $S^0_\infty(0)$ is strictly contained in $I_+(0)$, in Figure 5.14(ii) $I^+(0)$ is strictly contained in $S_\infty(0)$ -

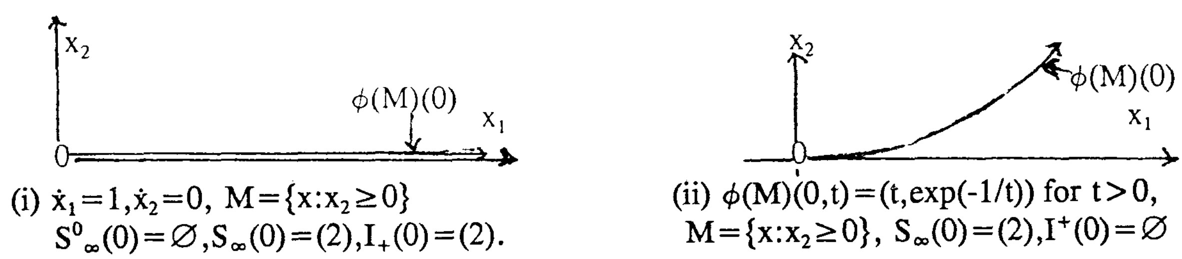
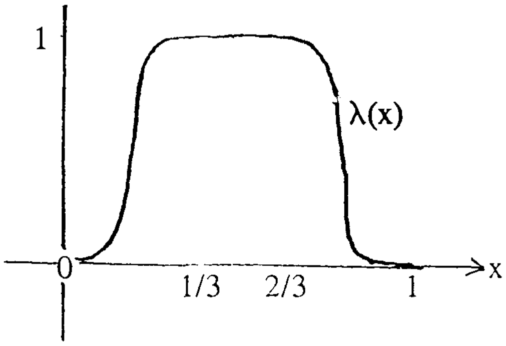


Figure 5.14

We may also use Lemma 5.8 to show that the number of points t in $(0,t_x)$ where $D_t^{+i}\phi(M)(x,t)\neq D_t^{-i}\phi(M)(x,t)$, some i, is countable (Proposition 5.2). An example will put this result in perspective.

Example 5.3

Construct a middle 1/q Cantor set for $q\geq 3$ in the usual way, ie remove from [0,1] A_1=centrally placed open interval of length 1/q, from the two closed intervals $[0,1]\backslash A_1$ remove centrally placed open intervals A_2^1, A_2^2 of length $1/q^2$ etc. We see measure$(\cup_{i,j}A_i^j)=1/(q-2)$. We now set $g(x)=6(3x-1)-9(3x-1)^2$, and $\lambda:R\to[0,1]$ a smooth bump function with graph



and set $h(x)=\lambda(x)g(x)$. $g(x)$ has zeros at $x=1/3$, $5/9$ so if $(\dot{x},\dot{y})=(1,h(x))$ and $M=\{(x,y)\in R^2:y\geq 0\}$ we obtain the integral curve shown in Figure 5.15.

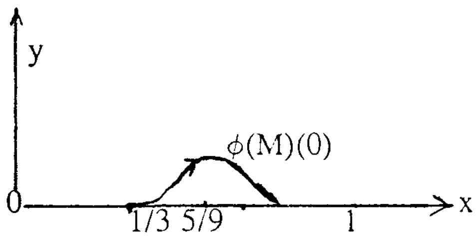


Figure 5.15

If the left edge point of A_i^j is x_i^j define f_i on $[0,1]$ by $f_i=0$ on $[0,1]\setminus\cup_j A_i^j$, and on each A_i^j set $f_i(x)=h(q^i(x-x_i^j))$ ie, fit h into each segment A_i^j . **Each f_i is smooth on** $[0,1]$, and setting $M_i=\sup_{j\leq i,x\in[0,1]}|D_t^j f_i(x)|$ it follows by the usual uniform convergence argument used in such situations (eg as in Proposition 4.8 of [22]) that $f:[0,1]\to R$ defined by $f(x)=\Sigma_{i=1}^{\infty} f_i(x)/(2^i M_i)$ is smooth. We set $Y(x,y)=(1,f(x))$: if $M=\{(x,y)\in R^2:y\geq 0\}$ we see $\phi(M)(0)$ has the form shown in Figure 5.16 below.

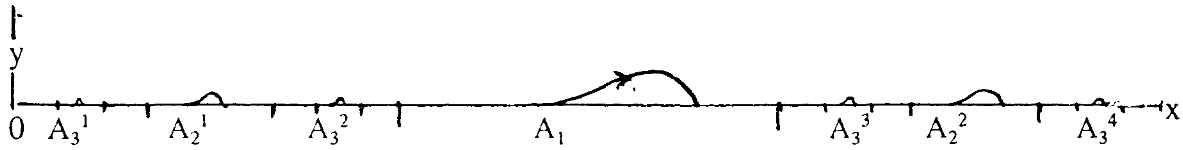


Figure 5.16

(i) There are uncountably many points $t\in[0,1]$ such that for any $h>0$ $|\phi(M)(x,[t-h,t))|=\infty$ and $|\phi(M)(x,[t,t+h))|=\infty$, (ii) setting $D=\{t\in[0,1]:D_t^{-i}\phi(M)(0,t)\neq D_t^{+i}\phi(M)(0,t)$ some $i\}$ $\bar{D}$ contains every point in $(\cup_{i,j}A_i^j)^c$ (since every point in $[0,1]\setminus\cup_{i,j}A_i^j$ is a limit of points x_0 in $\cup_{i,j}A_i^j$ with $D_t^-\phi(M)(x_0,0)\neq D_t^+\phi(M)(x_0,0)$), so is uncountably infinite, (iii) if $q=3$ measure$(\bar{D})=0$, if $q>3$ measure$(\bar{D})>0$.

However by the way Y was constructed we see D is countable and this is always the case.

Proposition 5.2 If M is a submanifold with orthogonal corners and X is a smooth vector field on M then for each $x \in M$ $\phi(M)(x)$ is smooth on an open-dense subset of $(0,t_x)$ and the set $\{t \in (0,t_x): D_t^{+i}\phi(M)(x,t) \neq D_t^{-i}\phi(M)(x,t)\}$ is countable (including finite or zero).

Proof The fact that $\phi(M)(x)$ is smooth on an open-dense subset of $(0,t_x)$ is immediate - for each $t \in (0,t_x)$ and $i \in S_1(\phi(M)(x,t))$ either $f_i\phi(M)(x,t)=0$ on $(t-\epsilon,t+\epsilon)$ or arbitrarily close to t there exists t′ with $f_i\phi(M)(x,t') \neq 0$: then by continuity of $\phi(M)(x)$ in t and of f_i we have $f_i\phi(M)(x,t) \neq 0$ on $(t'-\epsilon',t'+\epsilon')$, some $\epsilon'>0$. Repeating for all $i \in S_1(\phi(M)(x,t))$ we obtain t_0 arbitrarily close to t such that for some $\delta>0$ and for each $i \in S_1(\phi(M)(x,t))$ either $f_i\phi(M)(x,t'')$ is zero on $(t_0-\delta,t_0+\delta)$ or non-zero on $(t_0-\delta,t_0+\delta)$, ie $\phi(M)(x,t'') \in$ single stratum in this t-range, and hence by remark 3.1(2) is C^∞ there. Hence $\phi(M)(x)$ is smooth on an open-dense subset of $(0,t_x)$.
As regards the countability assertion, by Lemma 5.8 we obtain for each $t_0 \in (0,t_x)$, positive integer $i \geq j$ and $\epsilon>0$ a $\delta>0$ such that on $(t_0-\delta,t_0) \cup (t_0,t_0+\delta)$
$| D_t^{+j}\phi(M)(x,t)-D_t^{-j}\phi(M)(x,t) | \leq \epsilon | t-t_0 |^{i-j}$, and hence by [14, Section 3.9] the set $\{t \in (0,t_x): | D_t^{+i}\phi(M)(x,t)-D_t^{-i}\phi(M)(x,t) | > 1/n\}$ is countable for fixed i,n, and hence $\{t \in (0,t_x): | D_t^{+i}\phi(M)(x,t)-D_t^{-i}\phi(M)(x,t) | > 0 \text{ some } i \geq 0\}$ is countable. ∎

A second application of the ideas of this chapter is given in Corollary 5.3 below, for which we shall need the following lemma, which is true for any submanifold with corners (not necessarily with orthogonal corners), and is used again in Chapter Six:

Lemma 5.9 If M is a submanifold with corners then for each $x \in M$ and any $\epsilon>0$ there exists a neighbourhood U of x in M such that for all $y \in U$
$| X(M)x |^2-\epsilon < | \langle X(M)y,X(M)x\rangle | < | X(M)y |^2+\epsilon$.

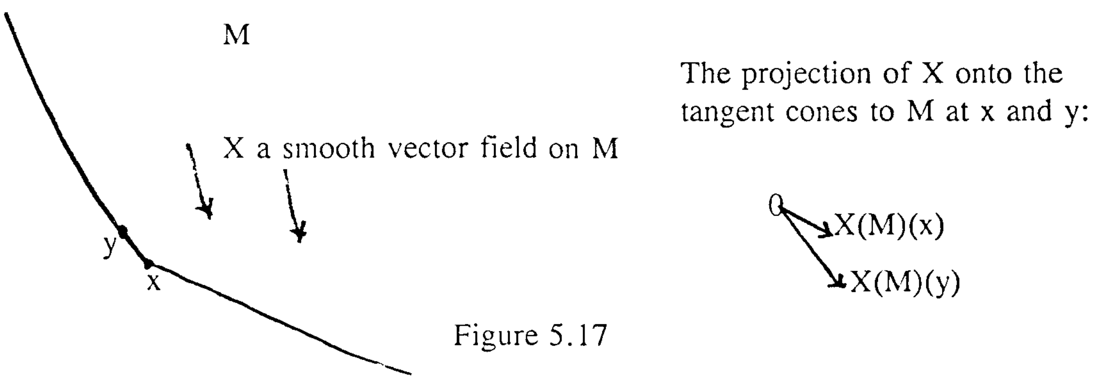


Figure 5.17

Proof

(1) We show that if C_1,C_2 are closed linear corners of R^n with $C_1 \supset C_2$ and X a vector in R^n then setting $P(C_1)X=v_1$ and $P(C_2)X=v_2$ $|v_1|^2 \geq \langle v_1,v_2\rangle \geq |v_2|^2$. For a closed linear corner C $P(C)X=P(L)X$ some linear subspace L (eg Lemma 1.2), so since P(L) is self-adjoint and idempotent $\langle X,P(C)X\rangle = |P(C)X|^2$. By the Characterisation of Projection we have $\langle X-v_1,v_1-z\rangle \geq 0$ for all $z \in C_1$, hence since $C_2 \subset C_1, \langle X-v_1,v_1-v_2\rangle \geq 0$, hence $\langle v_1,v_2\rangle - \langle X,v_2\rangle \geq 0$ (use $\langle X,v_1\rangle = \langle v_1,v_1\rangle$). Since $\langle X,v_2\rangle = \langle v_2,v_2\rangle$ this gives $\langle v_1,v_2\rangle - |v_2|^2 \geq 0$, and since we also have $\langle v_1-v_2,v_1-v_2\rangle \geq 0$, adding this to $\langle v_1,v_2\rangle - \langle v_2,v_2\rangle \geq 0$ we obtain $|v_1|^2 - \langle v_1,v_2\rangle \geq 0$, which are the two inequalities required.

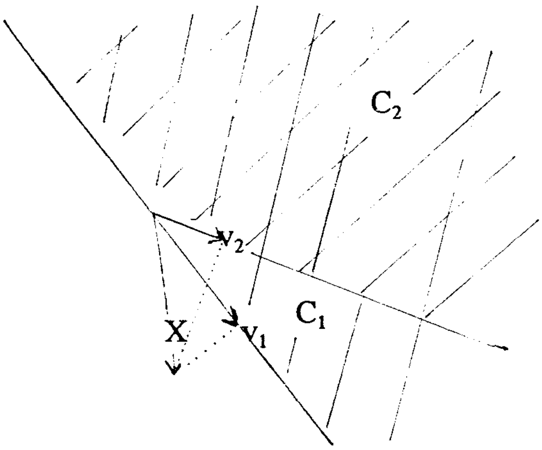


Figure 5.18

(2) Suppose near x M is represented as ZN(I;J). If y is a point in M near x then for some K $y \in ZP(K;J\setminus K)$ (where $I \subset K \subset I \cup J$) and $X(M)(y) = P(T_yZN(I;K\setminus I))X(y)$. By [13] the map $y \to T_yZN(I;K\setminus I)$ is continuous in y for as long as $y \in ZP(K;J\setminus K)$, hence for each $I \subset K \subset I \cup J$ and for any $\epsilon' > 0$ there exists a neighbourhood of x in R^n whose intersection U(K) with $ZP(K;J\setminus K)$ is such that if $y \in U(K)$ then $|PT_xZN(I;K\setminus I)X(x) - PT_yZN(I;K\setminus I)X(y)| < \epsilon'$. $T_xZN(I;K\setminus I)$ is a closed linear corner containing $T_xZN(I;J) = T_xM$ and hence setting $PT_xZN(I;K\setminus I)X(x) = v(K)$, so $v(I \cup J) = X(M)(x)$, we have by (1) $|v(K)|^2 \geq \langle v(K),v(I \cup J)\rangle \geq |v(I \cup J)|^2$ for all $I \subset K \subset I \cup J$. We have from the above that for any $\epsilon' > 0$ there exists $U(K) \subset ZP(K;J\setminus K)$ such that $|v(K) - X(M)(y)| < \epsilon'$ if $y \in U(K)$, so using $|v(K)| - |X(M)(y)| \leq |v(K) - X(M)(y)|$ we have
$|X(M)(x)|^2 \leq \langle X(M)(y) + v(K) - X(M)(y), X(M)(x)\rangle \leq |X(M)(y)|^2 + \epsilon'^2 + 2\epsilon' |X(M)(y)|$; but

$\langle X(M)(y)+v(K)-X(M)(y),X(M)(x)\rangle=\langle X(M)(y),X(M)(x)\rangle+\langle v(K)-X(M)(y),X(M)(x)\rangle$ and $\langle v(K)-X(M)(y),X(M)(x)\rangle\leq\epsilon' \mid X(M)(x) \mid$, so for any $\epsilon(K)>0$ we may find U(K) such that for all $y\in U(K)$

$\mid X(M)(x) \mid^2-\epsilon(K)< \mid \langle X(M)(y),X(M)(x)\rangle \mid < \mid X(M)(y) \mid^2+\epsilon(K)$, and if we now apply this result with $\epsilon(K)=\epsilon$ for all of the finitely many K with $I\subset K\subset I\cup J$ and take $U=\cap_{I\subset K\subset I\cup J}U(K)$ the result follows. ∎

We showed (in Theorem 3.1) that $\phi(M)(x)$ is $C^{+\infty}$ for all $t\in[0,t_x)$ and (in Proposition 5.1) that $\phi(M)(x)$ is $C^{-\infty}$ for all $t\in(0,t_x)$, so $\lim\{D_t^{+i}\phi(M)(x,t),t\downarrow t_0\} = D_t^{+i}\phi(M)(x,t_0)$ and $\lim\{D_t^{-i}\phi(M)(x,t),t\uparrow t_0\} = D_t^{-i}\phi(M)(x,t_0)$: we can now deal with the two remaining cases, $\lim\{D_t^{+i}\phi(M)(x,t):t\uparrow t_0\}$ and $\lim\{D_t^{-i}\phi(M)(x,t):t\downarrow t_0\}$. We shall also use Lemmas 5.8 and 5.9 to derive inequalites relating the magnitudes and scalar products of the right and left first order time derivatives of $\phi(M)(x)$.

Corollary 5.3 If M is a submanifold with orthogonal corners then at any $t_0\in(0,t_x)$

(1)(i) $\lim\{D_t^{+i}\phi(M)(x,t),t\uparrow t_0\}$ exists and $= D_t^{-i}\phi(M)(x,t_0)$

(ii) $\lim\{D_t^{-i}\phi(M)(x,t),t\downarrow t_0\}$ exists and $= D_t^{+i}\phi(M)(x,t_0)$

(2) $\mid D_t^-\phi(M)(x,t_0) \mid^2\geq\langle D_t^-\phi(M)(x,t_0),D_t^+\phi(M)(x,t_0)\rangle \geq \mid D_t^+\phi(M)(x,t_0) \mid^2$ with $\mid D_t^-\phi(M)(x,t_0) \mid = \mid D_t^+\phi(M)(x,t_0) \mid$ iff $X(M)\phi(M)(x)$ is continuous at t_0.

Proof

(1)(i) By Lemma 5.8(1) $\lim_{t\uparrow t_0}D_t^{+i}\phi(M)(x,t)=D_t^i\phi(K)(x_0,t)$ any $I_-(t_0)\subset K\subset I^-(t_0)$, which by Proposition 5.1 $=D_t^{-i}\phi(M)(x,t_0)$

(ii) By Lemma 5.8(2) $\lim_{t\downarrow t_0}D_t^{-i}\phi(M)(x,t)=D_t^i\phi(K)(x_0,t=0)$ any $I_+(t_0)\subset K\subset I^+(t_0)$ which by part (a) of the proof of Corollary 5.2 $=D_t^{+i}\phi(M)(x,t_0)$.

(2) By Lemma 5.9 we may find $\epsilon_i\downarrow 0$, $\tilde{t}_i\uparrow t_0$, such that for all i

$\mid X(M)\phi(M)(x,t_0) \mid^2-\epsilon_i< \mid \langle X(M)\phi(M)(x,\tilde{t}_i),X(M)\phi(M)(x,t_0)\rangle \mid <$ $\mid X(M)\phi(M)(x,\tilde{t}_i) \mid^2+\epsilon_i$, by Theorem 3.1 $X(M)\phi(M)(x,\tilde{t}_i)=D_t^+\phi(M)(x,\tilde{t}_i)$ while by (1)(i) $\lim_{\tilde{t}_i\uparrow t_0}X(M)\phi(M)(x,\tilde{t}_i)=D_t^-\phi(M)(x,t_0)$, and taking limits the inequality part of the result follows.

Finally, $X(M)\phi(M)(x)$ is continuous at t_0 iff $\lim_{t\uparrow t_0}X(M)\phi(M)(x,t)=\lim_{t\downarrow t_0}X(M)\phi(M)(x,t)$, which by (1) means iff $D_t^-\phi(M)(x,t_0)=D_t^+\phi(M)(x,t_0)$. It only remains to show that $\mid D_t^-\phi(M)(x,t_0) \mid = \mid D_t^+\phi(M)(x,t_0) \mid$ implies $D_t^-\phi(M)(x,t_0)=D_t^+\phi(M)(x,t_0)$: by the inequality we have just proved if they were equal we would have $\langle D_t^-\phi(M)(x,t_0)-D_t^+\phi(M)(x,t_0),D_t^+\phi(M)(x,t_0)\rangle =$ $\langle D_t^-\phi(M)(x,t_0),D_t^-\phi(M)(x,t_0)-D_t^+\phi(M)(x,t_0)\rangle =0$ and subtracting these gives the required result. ∎

*

Chapters Two to Five represent the major part of the contribution made by this thesis to understanding the class of system under investigation. It was however an initial aim to study the local and global geometry of these semidynamical systems in the spirit of the way that this was done for smooth unconstrained systems in [37,42], and it is to these matters (including in Chapter Eight specific consideration of the systems of this type occurring in [60]) that we turn in the remainder of this thesis.

Chapter Six
Local Geometric Theory

In this chapter we shall investigate how far the local geometric results of classical dynamical systems (see eg [42, Chapter 2] or [37, Chapters 4-6]) have analogues for these systems. For classical systems we know that there exists a dense subset of systems (i) which are differentiably stable at all points, (ii) where the flow near each zero is homeomorphic to that of the linearization, and (iii) where the zeros have C^r stable manifolds; furthermore we could always "straighten-out" away from zeros. We shall show that naive generalisations of these all fail. In place of (i) we shall consider stability with respect to a form of equivalence between semiflows (stratum preserving flow preserving, or spfp, equivalence) which is weaker than the existence of a diffeomorphism $f:M\to M'$ conjugating the semiflows, but which is still a homeomorphism of $M\to M'$ which preserves strata (as a diffeomorphism would) and semiflows, and we establish a necessary condition for two semiflows to be equivalent in this sense. We find that even with this weaker equivalence straightening out (which in the context of these systems we interpret as meaning that the semiflows $\phi(ZN(I;J))$ on $ZN(I;J)$ near x and $\phi(LC(I;J))$ on $LC(I;J)=T_xZN(I;J)$ near the origin are spfp equivalent) is not usually possible but that there is still a useful relation between the two. We shall generalise the definition of hyperbolic zero to regular zero and show that regular zeros have most of the properties which hyperbolic zeros have on boundaryless manifolds, and furthermore that in the case of submanifolds with orthogonal corners (which are the only submanifolds with corners occurring in applications) have C^1 but not generally C^2 stable manifolds.

Stratum Preserving Flow Preserving Homeomorphisms and Stability

To simplify matters we suppose M is a compact submanifold with corners. Thus M has a globally finite stratification into C^r submanifolds which we denote $(\sigma_1,\sigma_2,..)$. If M and M′ are diffeomorphic the diffeomorphism f relating them preserves strata, ie $f\sigma_i=\sigma_i'$ where $(\sigma_1',\sigma_2',..)$ is the corresponding stratification of M′. In Chapter Four we defined semiflows $\phi(M,X)$ and $\phi(M',X')$ on M,M′ obtained by integrating $X(M),X'(M')$ to be differentiably equivalent if there existed a diffeomorphism $f:M\to M'$ satisfying $f\phi(M,X)(x,t)=\phi(M',X')(fx,t)$ for all $t\geq 0$. We may say that the semiflows $\phi(M,X)$ and $\phi(M',X')$ are <u>differentiably equivalent at x,x'</u> if there exists a neighbourhood U of x in M and a diffeomorphism $f:U\to U'$= a neighbourhood of $f(x)=x'$ in M' such that for each $y\in U$ $f\phi(M)(y,t)=\phi(M')(fy,t)$ for all $t\geq 0$ with $\phi(M)(y,t)\in U$.

<u>Definition</u> A semiflow $\phi(X,M)$ is <u>differentiably stable at $x\in M$</u> if for any X′ sufficiently near X there exists x′ near x such that $\phi(M,X)$ and $\phi(M,X')$ are differentiably equivalent at x,x′. $\phi(X,M)$ is <u>locally differentiably stable</u> if it is differentiably stable at every $x\in M$.

<u>Examples 6.1</u> We show that by contrast with the classical unconstrained case locally differentiably stable systems are not dense in $\Xi_\infty(M)$ or $\Xi_{\omega,r}(M)$, any $r\geq 0$. Take for M the closed corner $\{x\in R^3: x_1\geq 0, x_2\geq 0, x_3-x_1-x_2\geq 0\}$ and suppose X(0) is chosen so that 0 has a preimage by $\phi(M)$ in int(M) and in each 2-dimensional stratum of M, with furthermore $X_i(0)\neq 0$, $i=1,2,3$ (Figure 6.1).

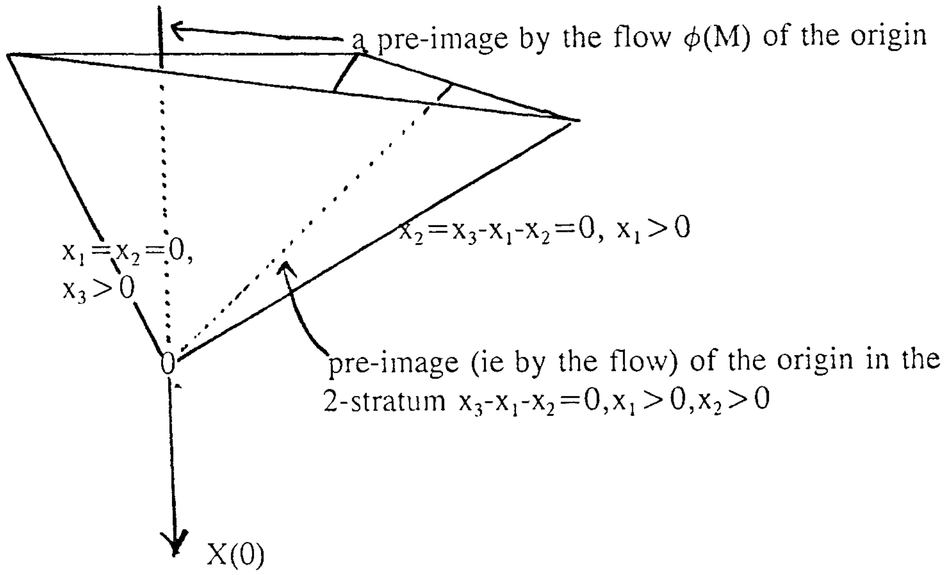


Figure 6.1

Suppose X is perturbed to X_ϵ such that $X_\epsilon(0)=X(0)+(\epsilon,0,0)$. Any differentiable equivalence f between $\phi(M,X)$ and $\phi(M,X_\epsilon)$ is a diffeomorphism $f:M\to M$ which must preserve the strata ($\{x\in R^3: x_1=x_2=0, x_3>0\}$ etc) of M, and in particular map the origin to the origin and hence (since it is flow preserving) preserve the preimages by $\phi(M)$ of the origin in each stratum; furthermore as a diffeomorphism its derivative map will be an invertible linear map at each point.

Setting Df(0) =A, the fact that the 1-dimensional strata of M as a submanifold with corners are preserved implies

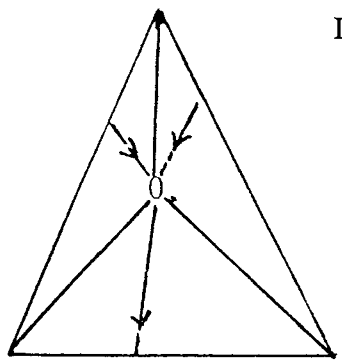
$$A\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix} = \begin{bmatrix}\lambda&0&0\\0&\mu&0\\\lambda&\mu&\nu\end{bmatrix} \quad \text{some } \lambda,\mu,\nu\neq 0, \qquad \text{hence } A=\begin{bmatrix}\lambda&0&0\\0&\mu&0\\\lambda-\nu&\mu-\nu&\nu\end{bmatrix}.$$

The tangent spaces to the pre-images by the flow of the origin in int(M), $\{x:x_1=0,x_2>0,x_3\text{-}x_1\text{-}x_2>0\}$ and $\{x:x_2=0,x_1>0,x_3\text{-}x_1\text{-}x_2>0\}$ are respectively X(0),P(1)X(0) and P(2)X(0) which must be mapped by A to $(X_\epsilon(0),P(1)X_\epsilon(0),P(2)X_\epsilon(0))$, ie if X(0)=(a,b,c)

$$A\begin{bmatrix}0&a&a\\b&0&b\\c&c&c\end{bmatrix} = \alpha\begin{bmatrix}0&a+\epsilon&a+\epsilon\\b&0&b\\c&c&c\end{bmatrix}$$

and these are incompatible unless $\epsilon=0$: hence for $\epsilon\neq 0$ there is no differentiable equivalence between $\phi(M,X)$ and $\phi(M,X_\epsilon)$ at the origin.

In this example the origin is a sink but we could re-work the example with the non-right angle, which was acute in the above (we need a non right angle because if the corner is orthogonal we *can* get a C^1 differentiable equivalence at 0) replaced by one which is obtuse, eg $M=\{x\in R^3: x_1\geq 0,\ x_2\geq 0,\ x_3+x_1+x_2\geq 0\}$, and X chosen so that the origin has pre-images in two of the 2-dimensional strata but is mapped by the flow into the third (Figure 6.2). Then exactly as above we can show that for no X′ near X is there a differentiable equivalence at the origin between $\phi(X)$ and $\phi(X')$.

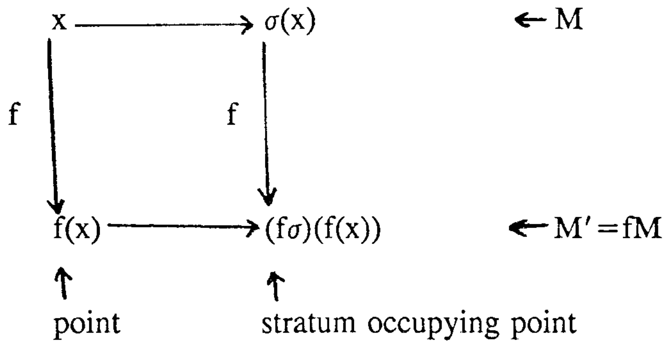


Looking down into M

Figure 6.2

Differentiable equivalence is therefore too strong for most purposes, but it seems reasonable to ask that any equivalence between two systems should at least preserve strata and if in the notion of equivalence we replace "diffeomorphism" with "homeomorphism" with no further conditions all points in ∂M look the same. Denoting for the moment the stratum occupied by x as $\sigma(x)$ we therefore make the following definitions:

<u>Definitions</u> If M and M′ are diffeomorphic submanifolds with corners of R^n (so with the above convention, for each $x \in M$ $f(\sigma(x)) = (f\sigma)(f(x))$ or the diagram



commutes)

we shall say

(1) A homeomorphism $h: M \to M'$ is <u>stratum preserving</u> if for all $x \in M$ $(f\sigma)(h(x)) = f(\sigma(x))$, ie for each k dimensional stratum m of M (as a submanifold with corners) h(m) is a k-dimensional stratum of M′, and vice versa.

(2) A homeomorphism $h: M \to M'$ is <u>flow-preserving</u> if it preserves trajectories, ie $h\phi(M)(x) = \phi(M')(hx)$ for all $x \in M$, ie for each $x \in M$ there exists a continuous strictly increasing $\tau: [0, \infty) \to [0, \infty)$ such that $h\phi(M,X)(x,t) = \phi(M',X')(hx, \tau(t))$ for all $x \in M$, for all $t \geq 0$

(3) $\phi(M,X), \phi(M',X')$ are <u>stratum preserving flow preserving (spfp) equivalent at x,x′</u> if there exists a neighbourhood U of $x \in M$ and a stratum preserving homeomorphism $h: U \to U'$ = neighbourhood of $h(x) = x'$ in M′, and for each $y \in U$ there exists a continuous strictly increasing $\tau: [0, T(U,y)) \to [0, T(U', h(y)))$ (τ will of course depend on y) such that $h\phi(M,X)(y,t) = \phi(M',X')(h(y), \tau(t))$ for all $t \geq 0$ such that $\phi(M)(y,t) \in U$.

(4) $\phi(M,X), \phi(M',X')$ are <u>stratum preserving flow preserving (spfp) equivalent</u> if there exists a stratum preserving flow preserving homeomorphism $h: M \to M'$.

Definition A semiflow ϕ(X,M) is (spfp) stable at x∈M if for any X′ sufficiently near X there exists x′ near x such that ϕ(M,X) and ϕ(M,X′) are spfp equivalent at x,x′. It is locally spfp stable if it is spfp stable at every x∈M. It is (spfp) stable if for any X′ sufficiently near X ϕ(M,X) and ϕ(M,X′) are spfp equivalent.

We show in Chapter Seven that for X linear and M an orthant or for r-polynomial X with r≥n and M a cube there exist open-dense subsets of $\Xi_{\omega,1}$(M) and $\Xi_{\omega,r}$(M) respectively consisting of fields which are locally (spfp) stable.

Remarks (1) In the definition of stability what is being tweaked is X, not (directly) X(M): in the unconstrained case we tweak X and must preserve ϕ(X); here we tweak X giving rise to a tweak of X(M) and we must preserve ϕ(M,X).

(2) We could strengthen the definition of stability by tweaking the manifold at the same time, but in applications the manifold is fixed (in fact it seems in the cases considered in this thesis that we would get the same result with this strengthened definition).

We obtain a necessary condition for two semiflows to be spfp equivalent which incorporates the intuitive requirement that intersections with the strata of M made by backward as well as forward trajectories must be preserved.

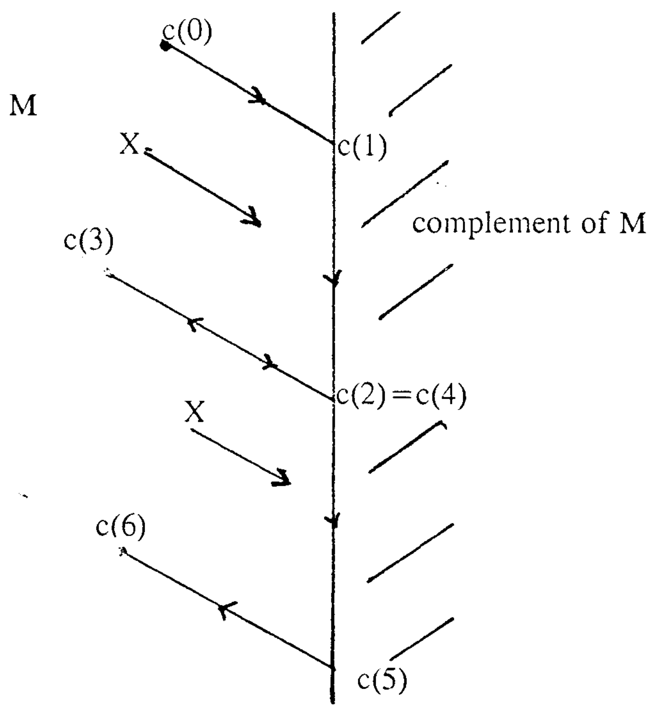
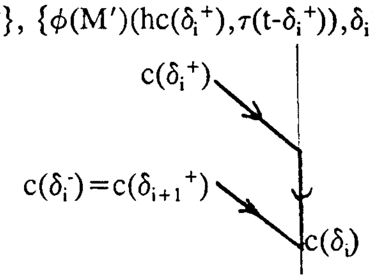


Figure 6.3

An <u>invariant curve</u> for (M,X) is (for our purposes) an absolutely continuous map $c:[0,\delta)=\Delta_1^+ \cup \Delta_1^- \cup \Delta_2^+ \cup .. \to M$ where each Δ_i^+ is an interval $[\delta_i^+,\delta_i)$, each Δ_i^- is an interval $[\delta_i,\delta_i^-)$ and setting $\cup\Delta_i^-=\Delta^-$, $\cup\Delta_i^+=\Delta^+$ satisfies $D_tc(t)=X(M)c(t)$ for almost all $t\in\Delta^+$, $D_tc(t)=-X(M)c(t)$ for almost all $t\in\Delta^-$ (Eg Figure 6.3).
If M,M′ are diffeomorphic submanifolds with corners then invariant curves $c:[0,\delta)\to M$, $c':[0,\delta')\to M'$ for (M,X),(M′,X′) are <u>equivalent</u> if there exists a continuous strictly increasing $\tau:[0,\delta)\to[0,\delta')$ such that $\tau(\Delta^+)=\Delta'^+$, $\tau(\Delta^-)=\Delta'^-$ and such that for all $t\in[0,\delta)$ $f(\sigma(c(t)))=(f\sigma)(c'(\tau(t)))$, where f is the diffeomorphism $f:M\to M'$. Then a necessary condition for a homeomorphism $h:M\to M'$ to be spfp is:

<u>Lemma 6.1</u> If h is a spfp homeomorphism between ϕ(M,X) and ϕ(M′,X′) then for each invariant curve c of (M,X) there exists an equivalent invariant curve c′ of (M′,X′).

<u>Proof</u> (1) We know the result is true if the invariant curve is a trajectory segment $\phi(M)(x,[0,\delta))$ mapped by h to $\phi(M')(hx,[0,\tau(\delta)))$, since if h is a spfp homeomorphism we have by definition that for each $x\in M$ there exists a continuous strictly increasing $\tau:[0,\delta)\to[0,\delta')$ such that
$h\phi(M)(x,t)=\phi(M')(hx,\tau(t))$ for $0\le t\le\delta$, and $(f\sigma)(h(y))=f(\sigma(y))$ for all $y\in M$,
which together give $f(\sigma(\phi(M)(x,t)))=(f\sigma)(\phi(M')(hx,\tau(t)))$ for each $x\in M$, for all $0\le t<\delta$, which means $\phi(M')(hx,\tau[0,\delta))$ is equivalent to $\phi(M)(x,[0,\delta))$ (with $[0,\delta)=\Delta^+,[0,\delta')=\Delta'^+$, $\Delta^-=\Delta'^-=\varnothing$).
(2) We have for each i $D_tc(t)=X(M)c(t)$ a.a. $t\in\Delta_i^+$,
$D_tc(t)=-X(M)c(t)$ a.a. $t\in\Delta_i^-$. On each Δ_i^+ c(t) satisfies the condition to be a trajectory of X(M), hence we have for $t\in\Delta_i^+$ $c(t)=\phi(M)(c(\delta_i^+),t-\delta_i^+)$. Similarly we have on $t\in\Delta_i^-$, setting $s=\delta_i^--t$, $D_sc(\delta_i^--s)=-(-X(M)c(\delta_i^--s))$, hence $c(\delta_i^--s)$ satisfies the condition to be a traδectory of X(M) and for $t\in\Delta_i^-$ $c(t)=\phi(M)(c(\delta_i^-),\delta_i^--t)$.
(3) By (1) each trajectory segment $\{\phi(M)(c(\delta_i^-),\delta_i^--t):\delta_i\le t\le\delta_i^-\}$ and $\{\phi(M)(c(\delta_i^+),t-\delta_i^+):\delta_i^+\le t\le\delta_i\}$ of c is mapped by a spfp homeomorphism to respectively $\{\phi(M')(hc(\delta_i^-),\tau(\delta_i^--t)),\delta_i\le t\le\delta_i^-\}$, $\{\phi(M')(hc(\delta_i^+),\tau(t-\delta_i^+)),\delta_i^+\le t\le\delta_i\}$



their ends meet at $c(\delta_i)$, and so we may construct piecewise an invariant curve c′ and a continuous function τ which (inductively in i) has the required properties. ∎

For example in the case of Figure 6.3 above we would set
$\Delta_1^+=[0,2), \Delta_1^-=[2,3), \Delta_2^+=[3,5), \Delta_2^-=[5,6)$, $\delta_1^+=0$, $\delta_1^-=3, \delta_2^+=3$, $\delta_2^-=6$, so
$c(t)=\phi(M)(c(0),t)$ on $[0,2)$, $=\phi(M)(c(3),3-t)$ on $[2,3)$ etc, and
$c'(t)=hc(t)=h\phi(M)(c(0),t)$ on $[0,2)$, $c'(t)=h\phi(M)(c(3),3-t)$ on $[2,3)$ etc.

If we say two points $x,y\in M$ are equivalent if for any invariant curve based at x there exists an invariant curve equivalent to it based at y, we may partition M into equivalence classes all of which must (by Lemma 6.1) be preserved by a spfp homeomorphism. A tame example is illustrated in Figure 6.4, where there are 17 equivalence classes (6 points, 8 1-manifolds, 3 2-manifolds).

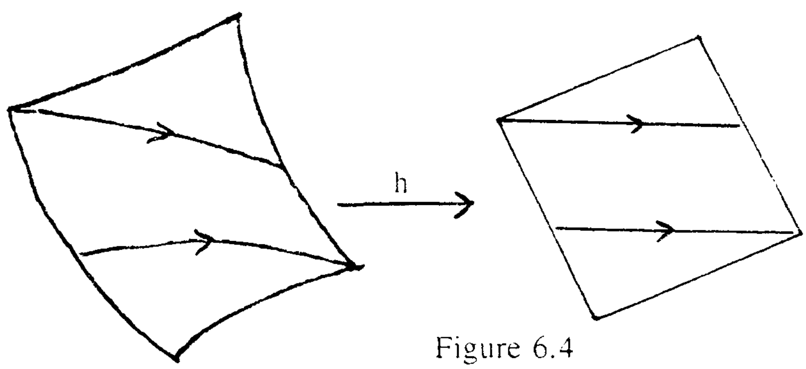


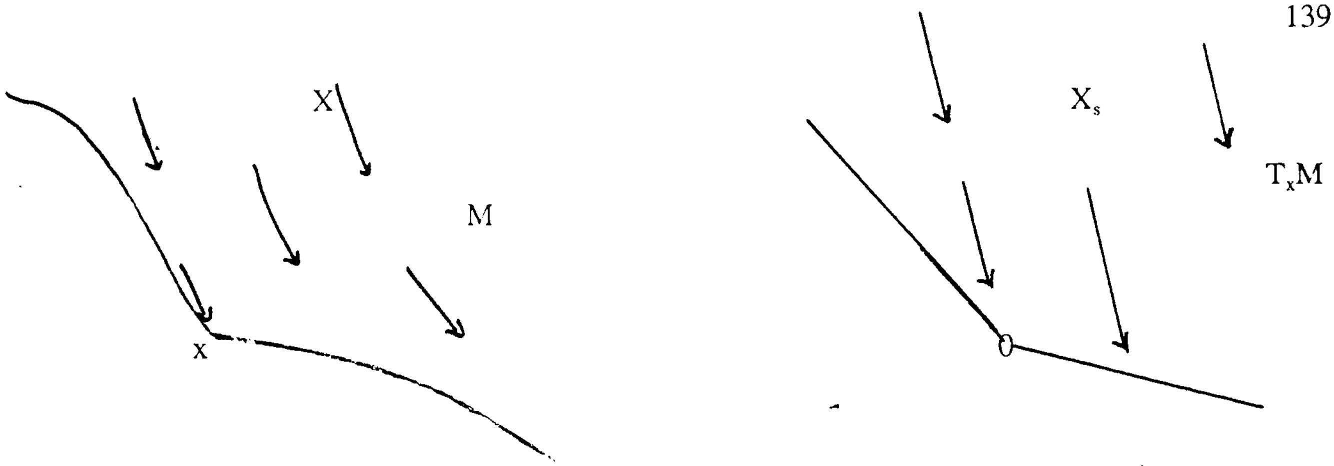
Figure 6.4

In fact we show in Example 6.6(1) below that in some circumstances at least a spfp homeomorphism will preserve the tangency sets $\Gamma_k(I\cup J \text{ r } I)$ too. These sets may then be added to the strata of M in the foregoing to provide a stronger necessary condition for a homeomorphism to be spfp.

We can see that the existence, even locally of a spfp homeomorphism between two semiflows places exacting requirements upon them (and hence the requirements upon a semiflow to be spfp stable are highly exacting too). We shall though show in Chapter Seven that for M a polyhedron there exists an open-dense subset of polynomial vector fields X with $\phi(M,X)$ locally spfp stable.

Constant Systems and Straightening-Out

A system (M_0,X) is termed constant if M_0 is a (linear) corner LC(I;J) and the vector field $X\in\Xi_{\omega,0}(M_0)$ (so for all $x,y\in M_0$ $X(x)=X(y)$). We shall investigate the relation between the semiflow of a system (M,X) near $x\in M$ and that of its straightening-out at x, the constant system (T_xM,X_s), where X_s is the constant vector field on T_xM given by $X_s(y)=X(x)$ for all $y\in T_xM$ (Figure 6.5).



A vector field on M near x Figure 6.5 The straightening-out at x

For (M_0,X) a constant system we are interested in the different values which $X(M_0)$ may take on - eg, in Figure 6.6 below $X(M_0)=X(\varnothing)$ on $LO(\varnothing;1,2)\cup LO(1;2)$ and $X(M_0)=X(2)$ on $LO(2;1)\cup LO(1,2;\varnothing)$. We see that this set of possible values of $X(M_0)$ may be strictly smaller than $\{K:X(M_0)(x)=X(K)(x) \text{ some } x\in M_0\}$ because , as on LO(1;2) in Figure 6.6, more than one K might satisfy the condition $X(M_0)(x)=X(K)(x)$.

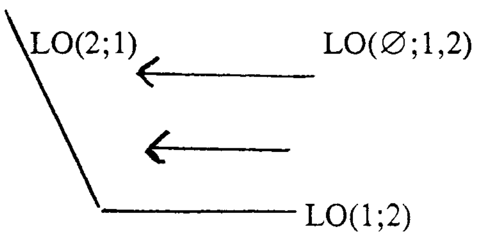


Figure 6.6. If X is as shown on $M_0=LC(\varnothing;1,2)$, then $S^0_2(LC(\varnothing;1,2),X)=\{\varnothing,(1)\}$ and the regions on which $S^0_2(x)$ is constant are:
$LCO(\varnothing;1;2)=LO(\varnothing;1,2)\cup LO(1;2)$ (corresponding to $K=\varnothing$) and
$LC(2;1)=LO(1,2;\varnothing)\cup LO(2;1)$ (corresponding to $K=(2)$)

To overcome this we recall that from the construction of the iteration we know that $S^0_2(x)=\cap\{K:S^0_1(x)\subset K\subset S_1(x) \text{ and } X(M)(x)=X(K)(x)\}$ and $S_2(x)=\cup\{K:S^0_1(x)\subset K\subset S_1(x) \text{ and } X(M)(x)=X(K)(x)\}$. Thus $\{K:K=S^0_2(x) \text{ some } x\in M_0\}$ provides us with a set of sets of indices such that at each point $x\in M_0$ $X(M_0)(x)=X(K)(x)$ for one and only one K in this set, and in fact it follows from Lemma 6.2 below that the set of distinct values of $\{S^0_2(x):x\in M_0\}$ corresponds exactly to the set of distinct values of $X(M_0)$.

If $(LC(I;J),X)$ is a constant system we call the set of distinct values of $\{S^0_2(x):x\in LC(I;J)\}$ (these are necessarily in the range $I\subset K\subset I\cup J$) $S^0_2(LC(I;J),X)$. So in Figure 6.6 $S^0_2(LC(\varnothing;1,2),X)=\{\varnothing,(2)\}$. We see that for $K=\varnothing$ or (2) in Figure 6.6 $\{x\in LC(\varnothing;1,2):S^0_2(x)=K\}$ is a subcorner satisfying $LO(K;J\backslash K)\subset\{x\in LC(I;J):S^0_2(x)=K\}\subset LC(K;J\backslash K)$, and this is always the case:

Lemma 6.2 If (LC(I;J),X) is a constant system then for each $K \in S^0_2(LC(I;J),X)$ $\{x \in LC(I;J): S^0_2(x)=K\}$ is a subcorner of LC(I;J) satisfying $LO(K;J\backslash K) \subset \{x \in LC(I;J): S^0_2(x)=K\} \subset LC(K;J\backslash K)$.

Proof (1) We show that if x, x_0 are points in LC(I;J) then $x \in LO(S^0_2(x_0), J\backslash S^0_2(x_0))$ implies $S^0_2(x)=S^0_2(x_0)$. By Lemma 2.4 we know $S^0_2(x)$ is characterised as the unique set of indices $S^0_1(x) \subset S^0_2(x) \subset S_1(x)$ such that

(i) $\langle X(S^0_2(x)\backslash j), n_j \rangle < 0$ for all $j \in S^0_2(x)\backslash S^0_1(x)$

(ii) $\langle X(S^0_2(x)), n_j \rangle \geq 0$ for all $j \in S_1(x)\backslash S^0_2(x)$.

If then $x \in LO(S^0_2(x_0); J\backslash S^0_2(x_0))$ we have $S_1(x)=S^0_2(x_0)$, so $S^0_2(x_0)$ is a candidate for $S^0_2(x)$ and since it satisfies (ii) vacuously (since $S_1(x)=S^0_2(x)$) and (i) (because $S^0_1(x_0)=S^0_1(x)=I$ for all $x, x_0 \in LC(I;J)$, and (i) is satisfied by x_0) we must therefore have that $S^0_2(x)=S^0_2(x_0)$.

(2). We show $S^0_2(x)=S^0_2(x_0)$ implies $x \in LC(S^0_2(x_0); J\backslash S^0_2(x_0))$. Since $S^0_2(x) \subset S_1(x)$ we must have $LO(S_1(x); J\backslash S_1(x)) \subset \cup_{S^0_2(x) \subset K \subset I \cup J} LO(K;J\backslash K)$ and so $x \in LO(S_1(x); J\backslash S_1(x)) \subset \cup_{S^0_2(x) \subset K \subset I \cup J} LO(K;J\backslash K) = LC(S^0_2(x); J\backslash S^0_2(x))$, so if then $S^0_2(x)=S^0_2(x_0)$ the claim follows.

(1) and (2) together prove the inclusions

$LO(K;J\backslash K) \subset \{x \in LC(I;J): S^0_2(x)=K\} \subset LC(K;J\backslash K)$ if $K=S^0_2(x_0)$ some x_0.

(3) We show $\{x \in LC(I;J): S^0_2(x)=S^0_2(x_0)\}$ any fixed $x_0 \in LC(I;J)$ is a subcorner. First we show that $S^0_2(x)$ is a constant on strata. It follows from definitions (as we observed above) that if $x \in LO(K;J\backslash K)$ then $S^0_2(x) = \cap\{K' : I \subset K' \subset K$ and $X(LC(I;J))(x) = X(K')(x)\}$. If $x \in LO(K;J\backslash K)$ then

$X(LC(I;J))(x) = P(T_x LC(I;J))X(x)$ (by definition)

$= P(LC(I;K\backslash I))X(x)$ (since $T_x LC(I;J) = T_x LC(I;K\backslash I)$)

$= P(LC(I;K\backslash I))X(0)$ (the vector field is constant, so $X(x)=X(0)$)

so is independent of $x \in LO(K;J\backslash K)$. Returning to our characterisation above of $S^0_2(x)$, since $X(K')(x) = P(T_x L(K'))X(x) = P(K')X(x) = P(K')X(0)$ independent of x the constancy of $S^0_2(x)$ on strata follows. Returning to the claim that $\{x \in LC(I;J): S^0_2(x)=S^0_2(x_0)\}$ is a subcorner, suppose we show that if $P(LC(I;K_1\backslash I))X = P(LC(I;K_2\backslash I))X = P(S^0_2(x_0))X$ then

(a) $P(LC(I;K_1 \cup K_2\backslash I))X = P(S^0_2(x_0))X$ and

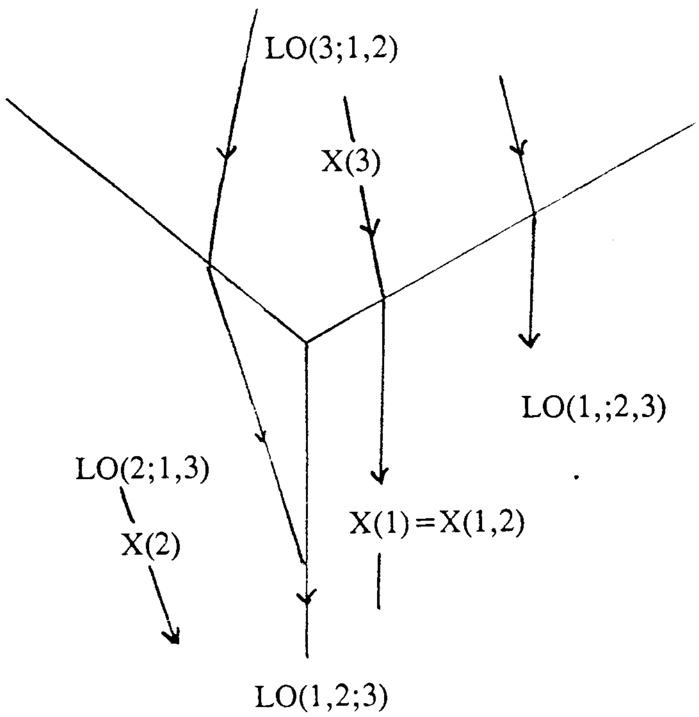
(b) $P(LC(I;K_1 \cap K_2\backslash I))X = P(S^0_2(x_0))X$. Then since we have observed that if $x \in LO(K;J\backslash K)$ then $S^0_2(x) = \cap\{K' : I \subset K' \subset K : P(LC(I;K\backslash I))X = P(K')X\}$ it follows that if there exists $x_i \in LO(K_i;J\backslash K_i)$, $i=1,2$, with $S^0_2(x_i)=K_0$ some K_0, then for any

$x\in LO(K_1\cup K_2;J\backslash(K_1\cup K_2))$ or $x\in LO(K_1\cap K_2;J\backslash(K_1\cap K_2))$ $S^0_2(x)=K_0$. We saw in Chapter One that any subcorner of LC(I;J) is a union of strata $\cup_{K_i\in E}LO(K_i;J\backslash K_i)$ characterised by the fact that $K_1,K_2\in E$ implies $K_1\cap K_2$ and $K_1\cup K_2\in E$, and so the claim that $\{x\in LC(I;J):S^0_2(x)=S^0_2(x_0)\}$ is a subcorner follows.

Setting $C_i=LC(I;K_i\backslash I)$ for i=1,2, if $P(C_1)X=P(C_2)X$ then by Lemma 1.1(1) $P(C_i)X=P(C_1\cap C_2)X$ which gives (a). By Lemma 1.1(2) $P(C_i)X=P(conv(C_1\cup C_2))X$, and since by a similar argument to that in Lemma 1.2 we obtain $P(conv(LC(I;K_1\backslash I)\cup LC(I;K_2\backslash I))X=P(LC(I;K_1\cap K_2\backslash I))X$, (b) follows. ▬

For a constant system (LC(I;J),X) we saw in part (3) of the proof of Lemma 6.2 that $S^0_2(x)$ depends only on which stratum x occupies; thus for any stratum LO(K;J\K) we may define $S^0_2(LO(K;J\backslash K))$ by $S^0_2(LO(K;J\backslash K))=S^0_2(x)$, for any $x\in LO(K;J\backslash K)$ and $K\in S^0_2(LC(I;J),X)$. Then for each $K\in S^0_2(LC(I;J),X)$ set $E(K)=\{K':S^0_2(LO(K';J\backslash K'))=S^0_2(LO(K;J\backslash K))\}$. We know then by Lemma 6.2 that $\cup\{LO(K';J\backslash K'):K'\in E(K)\}=\{x\in LC(I;J):S^0_2(x)=K\}$ is a subcorner, and that $K\subset E(K)\subset\cup\{K':K\subset K'\}$.

Example 6.2



Looking down into LC(∅;1,2,3)

Figure 6.7

On LC($\varnothing$;1,2,3), with suitable vector field X (see Figure 6.7), S^0_2(LC($\varnothing$;1,2,3),X)={$\varnothing$,(3),(2),(1)}, ie $S^0_2(x)$ may take on one of 4 possible values, $K_1=\varnothing$, K_2=(3), K_3=(2), or K_4=(1), ie there are 4 distinct values which $\check{X}$(LC($\varnothing$;1,2,3))(x) may take on. Then by Lemma 6.2 {x$\in$LC($\varnothing$;1,2,3):$S^0_2(x)=K_i$} is a subcorner containing LO(K_i;(1,2,3)\$K_i$) and contained in LC($K_i$;(1,2,3)\$K_i$). We have in fact E($K_1$)={$\varnothing$}, E($K_2$)={(3)}, E($K_3$)={(2),(2,3)}, E($K_4$)={(1),(1,3),(1,2),(1,2,3)}.

If M is a submanifold with corners and x$\in$M the <u>straightening-out</u> at x is the constant system (T_xM,X_s) where X_s is the constant vector field on T_xM given by $X_s(y)=X(x)$ for all y$\in T_xM$. If M is locally represented as ZN(I;J) (with x$\in$Z(I$\cup$J)) then $T_xM\cong$LC(I;J). By the above we may partition the strata of T_xM into subsets {LO(K;J\K):K$\in$E(K_i)} for i=1,..,r, where $\{K_i\}_{i=1,\ldots,r}=S^0_2(T_xM\cong LC(I;J),X_s)$ (this will in fact be a notational convention throughout the remainder of this section). Then for each K_i, i=1,..,r, we define M(x,K_i) = $\cup$\{ZP(K;J\K):K$\in$E(K_i)}. By Lemma 6.2 each M(x,K_i) is a subcorner of ZN(I;J), and we see {M(x,K_i):i=1..r} is a partition of M near x.

For instance, if a submanifold with corners M locally represented as ZN($\varnothing$;1,2) and vector field X straighten out at x$\in$M to form a constant system as in Figure 6.6 above, then r=2 with M(x,K_1)=ZP($\varnothing$;1,2)$\cup$ZP(1;2) and M(x,K_2)=ZN(2;1). Thus {M(x,K_i)} is a partition of M near x into unions of strata, where which strata go into which union is determined by the straightening out at x.

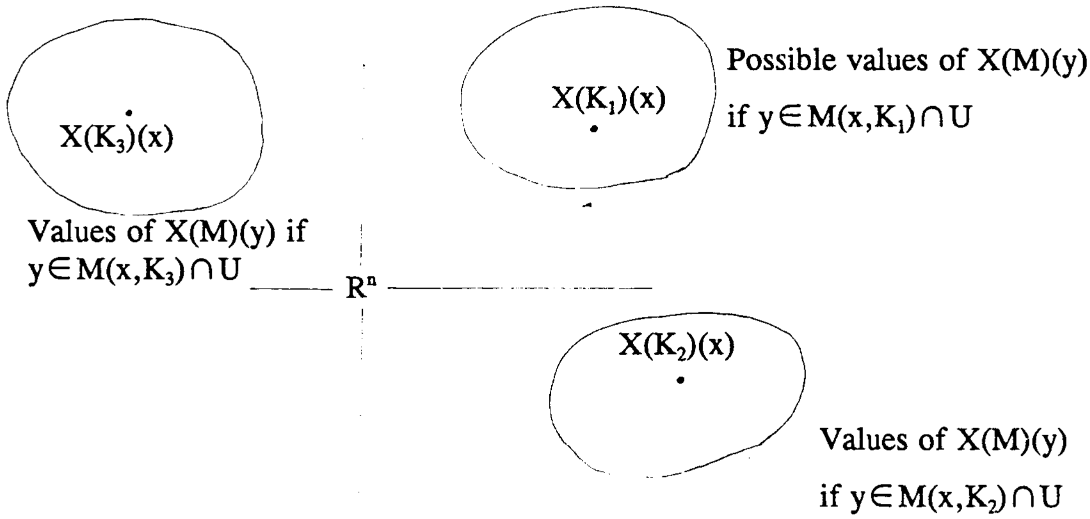
The idea now (Lemma 6.3 and Proposition 6.1) is to infer as much as possible about the original non-constant system *near* x from the straightening out *at* x (very much of course in the spirit of classical geometric theory). As a constant system the straightening out is very easy to analyse, and for example determining the subdivision of the sets of indices K in I$\subset$K$\subset$I$\cup$J into the {E(K_i)} consists of finitely many operations involving only a finite set of vectors.

<u>Lemma 6.3</u> With $K_1,\ldots,K_r=S^0_2(T_xM\cong LC(I;J),X_s)$ and M(x,K_i) as defined above there exists a neighbourhood U of x in M such that the relation $\geq$ defined on {M(x,K_i), i=1..r} by M(x,K_i)$\geq$M(x,K_j) if there exists a trajectory from U$\cap$M(x,K_i) to U$\cap$M(x,K_j) is a partial order (ie, for as long as a trajectory remains in some neighbourhood of x once it has vacated M(x,K_i) it cannot return to it).

We observe this means that no trajectory can make more than r-1 transitions between the sets $M(x,K_i)$ for as long as it remains in U.

Proof (1) We show if as $x_i \to x$ $|X(M)(x_i)| - |X(M)(x)| \to 0$ then $|X(M)(x_i)-X(M)(x)| \to 0$. From Lemma 5.9 we know given $\epsilon>0$ there exists a neighbourhood U of x such that for all $y \in U$ $|X(M)(x)|^2-\epsilon < \langle X(M)(y),X(M)(x)\rangle < |X(M)(y)|^2+\epsilon$. Thus if $|X(M)(x_i)| - |X(M)(x)| \to 0$ we must have $|X(M)(x)|^2-\langle X(M)(x_i),X(M)(x)\rangle \to 0$ and $|X(M)(x_i)|^2-\langle X(M)(x_i),X(M)(x)\rangle \to 0$ and hence $|X(M)(x)-X(M)(x_i)|^2 = |X(M)(x)|^2 + |X(M)(x_i)|^2 - 2\langle X(M)(x_i),X(M)(x)\rangle \to 0$.

(2) We show that given any $\epsilon>0$ there exists a neighbourhood U of x in M so small that $\mathrm{Sup}\{|X(M)(y)-X(K_i)(x)| : y \in M(x,K_i) \cap U,\ i=1..r\} < \epsilon$.
Beginning with the straightening out of (M,X) at x, since if $K \in E(K_i)$ $S^0_2(y',T_xM \cong LC(I;J),X_s)=K_i$ for all $y' \in LO(K;J\backslash K)$, and if $y' \in LO(K;J\backslash K)$ $P(T_{y'}LC(I;J))X_s = P(LC(I;K\backslash I))X_s$, and by definition $P(T_{y'}LC(I;J))X_s = P(S^0_2(y',LC(I;J),X_s))X_s = P(K_i)X_s$, we have for all $K \in E(K_i)$ $P(LC(I;K\backslash I))X_s = P(K_i)X_s$.
We have defined $M(x,K_i) = \cup\{ZP(K;J\backslash K):K \in E(K_i)\}$ where we recall $E(K_i)=\{K:S^0_2(LO(K;J\backslash K))=S^0_2(LO(K_i;J\backslash K_i))\}$. If $y \in M(x,K_i)$ $y \in ZP(K;J\backslash K)$ some $K \in E(K_i)$, so $X(M)(y)=P(T_yM)X(y)=P(T_yZN(I;K\backslash I))X(y)$. By [13] $y \to P(T_yM)X(y)$ is continuous as long as $y \in$ a single stratum, hence as $y \to x$ $X(M)(y) \to P(T_xZN(I;K\backslash I))X(x)$. Then since the constant vector field X_s takes the value X(x) at all points, the result follows.
(3) Since by definition all the values $P(K_i)X_s$ are distinct $\delta_1 = \inf_{i \neq j} |P(K_i)X_s - P(K_j)X_s|$ is positive. The quantity $\delta_2 = \min\{\,||P(K_i)X_s| - |P(K_j)X_s|| : |P(K_i)X_s| \neq |P(K_j)X_s|\,\}$ is indisputably positive, and we shall set $\delta = \min\{\delta_1,\delta_2\}$ (the need for the δ_2 term will arise in (5) below). By (2) we may choose our neighbourhood U of x in M so small that $\sup\{|X(M)(y)-X(K_i)(x)| : y \in M(x,K_i) \cap U, i=1,..,r\} < \delta/3$.



etc..

Suppose $y \in M(x,K_1) \cap U$. Set $t_1 = \inf\{t>0: \phi(M)(y,t) \notin M(x,K_1)\}$. $\phi(M)(y,t)$ is in $M(x,K_1)$ for small $t>0$ because $\phi(M)(y,t) \in$ some $M(x,K_i)$, and by the foregoing if $\phi(M)(y,t) \in M(x,K_i)$ then $|X(M)\phi(M)(y,t)-X(K_i)(x)| < \delta/3$, while $|X(K_i)(x)-X(K_j)(x)| > \delta$ if $i \neq j$. Since (by Theorem 3.1) $\lim_{t\downarrow 0} X(M)\phi(M)(y,t) = X(M)(y)$ (and so $X(M)\phi(M)(y,t)$ is close to $X(M)(y)$ for t small and positive) we must by the above have that $\phi(M)(y,t) \in M(x,K_1)$ for small $t>0$, and hence $t_1>0$.

(4) We claim there exists a neighbourhood U of x such that for any $y \in U$ and $0 \leq t \leq T(U,y)$, if on any left neighbourhood of t there exist points s such that $\phi(M)(x,s) \in M(x,K_i)$, and on any right neighbourhood of t there exist points s such that $\phi(M)(x,s) \in M(x,K_j)$ (by (3) this means $\phi(M)(x,t) \in M(x,K_j)$) then

$|P(K_j)X_s| < |P(K_i)X_s|$.

Suppose there exist sequences $\{x_k\},\{t_k\},\{s_k^m\}$ with $x_k \to x$, $t_k \downarrow 0$, $0<s_k^m<t_k$ and $s_k^m \uparrow t_k$ for each k as $m\to\infty$, such that $\phi(M)(x_k,s_k^m) \in M(x,K_i)$, $\phi(M)(x_k,t_k) \in M(x,K_j)$ (see Figure 6.8), and $|P(K_i)X_s| \geq |P(K_j)X_s|$. Since $x_k \to x$ and $t_k \downarrow 0$ we have by (2) $X(M)\phi(M)(x_k,t_k) \to X(K_j)(x)$ (as $k\to\infty$) and $X(M)\phi(M)(x_k,s_k^m) \to X(K_i)(x)$ (as $k\to\infty$).

By Lemma 5.9 $\lim_{m\to\infty} |X(M)\phi(M)(x_k,s_k^m)| \geq |X(M)\phi(M)(x_k,t_k)|$ for all k, hence we have $|X(K_i)(x)| \geq |X(K_j)(x)|$. If we had equality then

$|X(M)\phi(M)(x_k,t_k)| - |X(M)\phi(M)(x_k,s_k^m)| \to 0$ (as $k\to\infty$) and so by (1)

$|X(M)\phi(M)(x_k,t_k)-X(M)\phi(M)(x_k,s_k^m)| \to 0$ (as $k\to\infty$) and hence

$|X(K_j)(x)-X(K_i)(x)| = 0$, which is only possible if $i=j$.

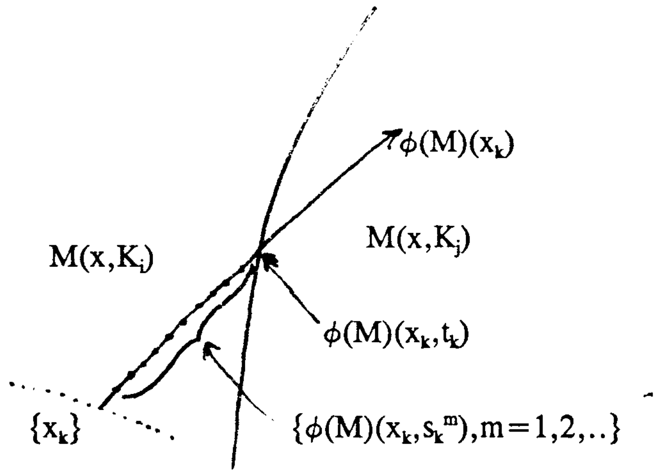


Figure 6.8

(5) Since by (2) there exists a neighbourhood U of x such that if $y \in U \cap M(x,K_j)$ then $|X(M)(y)-X(K_j)(x)| < \delta/3$ and by (3) each maximally connected interval of $[0,T(U,y))$ on which $\phi(M)(y,t)$ is contained in a single $M(x,K_j)$ is of the form $[T,T')$ with $T'>T$, if we choose a pair of adjacent such intervals $[T_{i-1},T_i)$ and $[T_i,T_{i+1})$ and relabel the K_j's so that $\phi(M)(y,t) \in U \cap M(x,K_j)$ on $[T_j,T_{j+1})$ $j=i-1,i$, then for all $t \in [T_i,T_{i+1})$ $|X(K_i)(x)| - |X(M)\phi(M)(y,t)| < \delta/3$ while $|X(K_{i-1})(x)| - |X(M)\phi(M)(y,t)| > \delta/3$ (by definition of $[T_i,T_{i+1})$ and (4) $|X(K_{i-1})(x)| - |X(K_i)(x)| > 0$, therefore by definition of δ in (3) $|X(K_{i-1})(x)| - |X(K_i)(x)| \geq \delta$, and add this to $|X(K_i)(x)| - |X(M)\phi(M)(y,t)| > -\delta/3$). Hence inductively once a trajectory has vacated a region $M(x,K_i)$ it cannot return to it. ∎

Remark If (M,X) is a constant system (so equals its own straightening-out at the origin) then we may take U=M in Lemma 6.3, ie if we partition LC(I;J) into the subcorners $\{\cup LO(K;J\backslash K):K \in E(K_i)\}_{i=1,\ldots,r}$ then for any $y \in LC(I;J)$ once $\phi(LC(I;J))(y)$ has left any such region it can never return to it (re-work the proof above or use that for a constant system (M_0,X), $\phi(M_0)(x,t)=(1/\epsilon)\phi(M_0)(\epsilon x,\epsilon t)$, so the global result follows from the local one near 0).

Examples 6.3

(1) If (M,X) straightens out at x to yield a constant system with the data as in Example 6.2 above we have (denoting our partial order by $\geq$)
$M(x,K_1) \geq M(x,K_2) \geq M(x,K_3) \geq M(x,K_4)$ where
$M(x,K_1)=ZP(\varnothing;1,2,3), M(x,K_2)=ZP(3;1,2)$,

$M(x,K_3)=ZNP(2;3;1)=ZP(2;1,3)\cup ZP(2,3;1), M(x,K_4)=ZN(1;2,3)$

(2) The partial order need not be a total order. If (M,X) straightens out at x (see Figure 6.9) to yield a slightly different constant system to that in Example 6.2 (but on the same corner $LC(\varnothing;1,2,3)$), with $S^0_2(LC(\varnothing;1,2,3),X_s)=\{\varnothing,(3),(2),(1,2)\}$, ie $S^0_2(y)$ now takes on 5 values as y varies over $LC(\varnothing;1,2,3)$: $K_1=\varnothing, K_2=(3)$, $K_3=(2)$, $K_4=(1)$, $k_5=(1,2)$, and we have $E(K_1)$, $E(K_2)$, $E(K_3)$ as in Example 6.2, $E(K_4)=\{(1),(1,3)\}$ and $E(K_5)=\{(1,2),(1,2,3)\}$. $M(x,K_1), M(x,K_2), M(x,K_3)$, are as in (1) above, with $M(x,K_4)=ZP(1;2,3)\cup ZP(1,3;2)=ZNP(1;3;2)$ and $M(x,K_5)=ZP(1,2;3)\cup ZP(1,2,3;\varnothing)=ZN(1,2;3)$, and

$$M(x,K_1)\geq M(x,K_2)\geq \begin{Bmatrix} M(x,K_3) \\ M(x,K_4) \end{Bmatrix} \geq M(x,K_5)$$

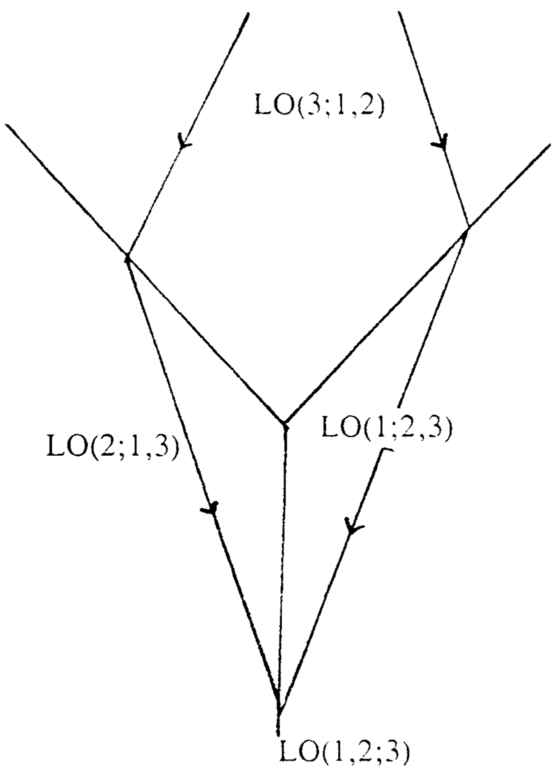


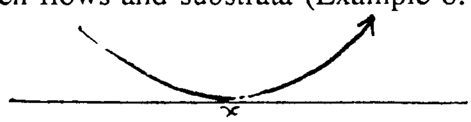
Figure 6.9. The straightening out. Looking down into $LC(\varnothing;1,2,3)$

<u>Remark</u> Given this partial order and the fact that $\{M(x,K_i):i=1,..,r\}$ is a finite partition of M near x it follows that there must exist one or more $M(x,K_i)$ such that

there is no $M(x,K_j)$ with $M(x,K_j) \geq M(x,K_j)$ and $j \neq i$ (geometrically a union of strata acting as a "sink" for the local flow - for example in Example 6.3(1) this is $M(x,K_4)$ and in Example 6.3(2) it is $M(x,K_5)$). Using Lemmas 2.4, 4.6 and 6.2 we can show that there is in fact exactly one such set: it is $ZN(S^0_2(0,LC(I;J),X_s),J \setminus S^0_2(0,LC(I;J),X_s)$, but we shall not use this fact.

We next show (Example 6.4) that even away from tangencies we cannot in general

establish a spfp homeomorphism between $\phi(M,X)$ near $x \in M$ and $\phi(T_xM,X_s)$ near the origin , or for that matter necessarily be able to find *any* constant system for which we can establish a spfp homeomorphism between $\phi(M,X)$ near $x \in M$ and $\phi(T_xM,X_s)$ near the origin. Clearly no constant system can be locally (spfp) equivalent to a neighbourhood of a point x where $\Im_\infty(x) \neq \Im_2(x)$ (crudely speaking a point where there is a tangency between the semiflow and a lower dimensional stratum) (Figure 6.10), but it may not be possible to establish an equivalence even when there are no tangencies between flows and substrata (Example 6.4).



$\Im_\infty(x)=(1) \neq \Im_2(x)=(1) \cup (\varnothing)$, where $M=\{x \in R^n: x_1 \geq 0\}$

Figure 6.10

Example 6.4 Consider the orthogonal 3-dimensional corner $\{x \in R^3: x_i \geq 0, i=1,2,3\}$ and non-constant vector field X with constant part X_0 close to (-1,1,1). Consider the subsets of LO(1;2,3) $V_1=\{x \in LO(1;2,3): x=\phi(M)(y,t)$ some $t>0$, some $y \in LO(2,3;1)\}$ and $V_2=\{x \in LO(1;2,3): x=\phi(1)(0,t)$ some $t>0\}$ (see figure below). V_1 then is the intersection of LO(1;2,3) with the surface obtained by acting on LO(2,3;1) (= span$\{n_1\}$) with the unconstrained flow, and V_2 is the image by the flow of the origin in LO(1;2,3). In the straightening out at 0 (which in this case is obtained by replacing X by X_s everywhere equal to $\dot{X}_0$, since we already have $T_0LC(\varnothing;1,2,3) \cong LC(\varnothing;1,2,3)$) we see that the subsets in the straightening out corresponding to V_1,V_2, which we shall denote $\tilde{V}_1,\tilde{V}_2$, are $\tilde{V}_1=LO(1;2,3) \cap \text{span}\{n_1,X(0)\}$ which we see will coincide exactly with $\tilde{V}_2=\{\lambda(X(0)-\langle X(0),n_1\rangle n_1):\lambda>0\}$: in the original V_1 and V_2 will be tangent at 0 but not in general coincident (Figure 6.11).

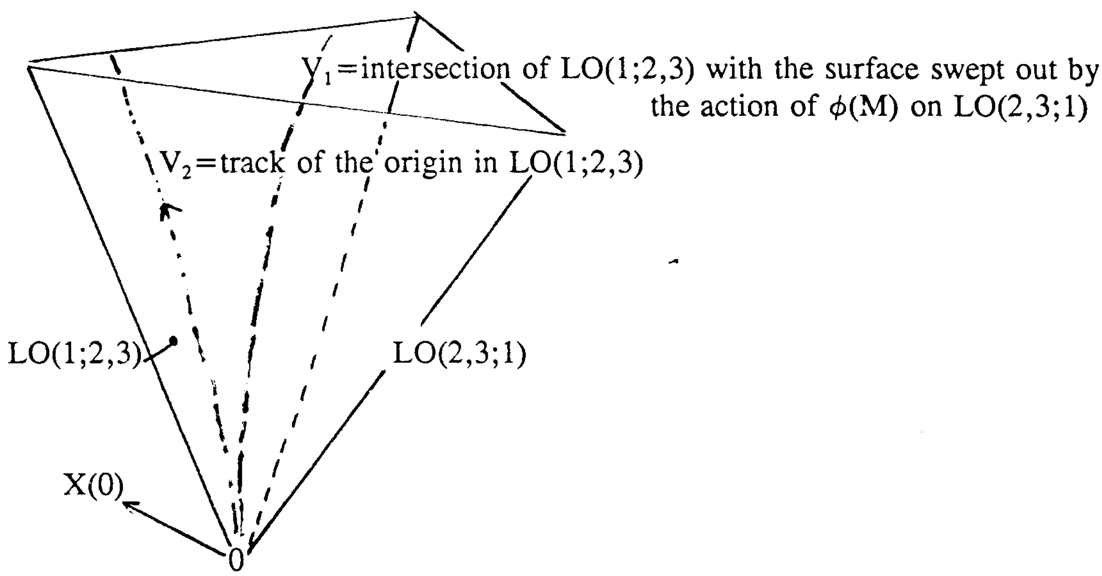
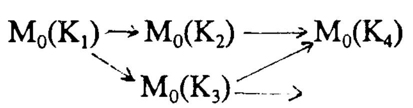


Figure 6.11

By Lemma 6.1 there therefore cannot be a spfp homeomorphism between (M,X) and (T_xM=M,X_0) near the origin. We note also that this phenomenon cannot be perturbed away. Within the class of constant systems the semiflow is spfp stable, and by suitable choice of X we can make it spfp stable within for example the class of linear vector fields, but the two classes are distinguished by spfp homeomorphism.

We have partitioned M near x∈M locally represented as ZN(I;J) into regions {M(x,K_i),i=1,..,r} with each M(x,K_i) a subcorner of ZN(I;J) such that at every point of the corresponding subcorner in the straightening-out (T_xM,X_s), X_s(T_xM) is a constant, and saw in Lemma 6.3 that once a trajectory has left M(x,K_i) it cannot return to it. In Proposition 6.1 we improve upon this. Suppose we denote the subcorner in the straightening-out corresponding to M(x,K_i) by M_0(K_i) (ie it equals ∪{LO(K;J\K):K∈E(K_i)}, cf M(x,K_i)=∪{ZP(K;J\K):K∈E(K_i)}), we shall say M_0(x,K_i)→M_0(x,K_j) if there exists a trajectory ϕ(M_0,X_s) of X_s on M_0=LC(I;J) passing from M_0(K_i) to M_0(K_j). Since there are only finitely many K_i we can partition M_0 into M_0(K_i), and form a finite diagram (hereafter called the diagram of the straightening out) of the form



We remarked after Lemma 6.3 that if the system is constant Lemma 6.3 applies globally, that is, having left a set $M_0(K_i)$ a trajectory cannot return to it, which means that there are no loops in the diagram obtainable by following arrows. Some examples of diagrams are given in Examples 6.5 below.

In Proposition 6.1 we establish the relation between the sequence of sets $M(x,K_i)$ a trajectory $\phi(M,X)(y)$ may occupy near x, and the diagram of the straightening out at x:

<u>Proposition 6.1</u> If $S^0_2(T_xM \cong LC(I;J),X_s)=K_1,..,K_r$ and $M(x,K_i)$, $M_0(K_i)$ are as defined above, then there exists a neighbourhood U of $x \in M$ such that for any $y \in M \cap U$ we can partition $[0,T(U,y))$ into $[0,T_1) \cup .. \cup [T_{s-1},T_s)$ some $s \leq r$, such that for each $i=1,..,s$ $\phi(M)(y,[T_{i-1},T_i))$ is contained in a single set $M(x,K_{j(i)})$ and for each $i=1..s-1$ $M_0(K_{j(i-1)}) \rightarrow M_0(K_{j(i)})$, ie the sequence of sets $\phi(M)(y)$ occupies is drawn from the diagram of the straightening-out.

By the no-loops remark above Proposition 6.1 implies Lemma 6.3.

<u>Examples 6.5</u>

(1) Suppose the straightening-out at x is as illustrated in Figure 6.12a below, with $X_s(LC(\varnothing;1,2,3))$ taking on 3 distinct values, $X_s(\varnothing)=X_s(3)$, $X_s(2)=X_s(3,2)$, $X_s(1)=X_s(3,1)$ with $E(\varnothing)=\{(\varnothing),(3)\}$, $E((2))=\{(2),(2,3)\}$ and $E((1))=\{(1),(1,2),(1,3),(1,2,3)\}$ and the diagram of the corresponding regions as follows:

$M_0(\varnothing)=LCO(\varnothing;3;1,2) \rightarrow M_0(2)=LCO(2;3;1)$

$\downarrow \qquad\qquad \swarrow$

$M_0(1)=LC(1;2,3)$.

Proposition 6.1 (and in fact in this case also Lemma 6.3) tells us that the only transitions between subcorners $M(x,K_i)$ a trajectory of the original can make near x are $ZNP(\varnothing;3;1,2) \rightarrow ZNP(2;3;1)$ (ie, $ZP(\varnothing;1,2,3) \cup ZP(3;1,2) \rightarrow ZP(2;3,1) \cup ZP(2,3;1)$) or $ZNP(\varnothing;3;1,2) \rightarrow ZN(1;2,3)$, or $ZNP(2;3;1) \rightarrow ZN(1;2,3)$, for example this permits a situation such as that in Example 5.1(3) (Figure 6.12b)

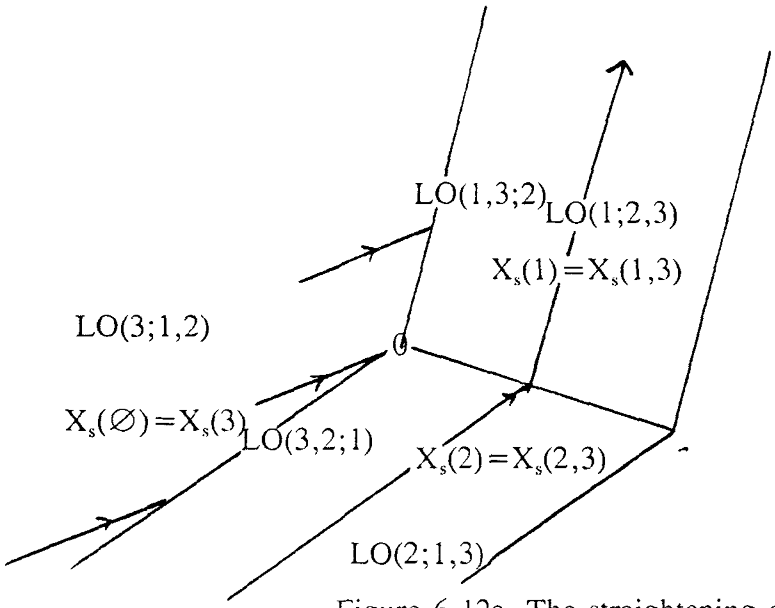


Figure 6.12a. The straightening out at x

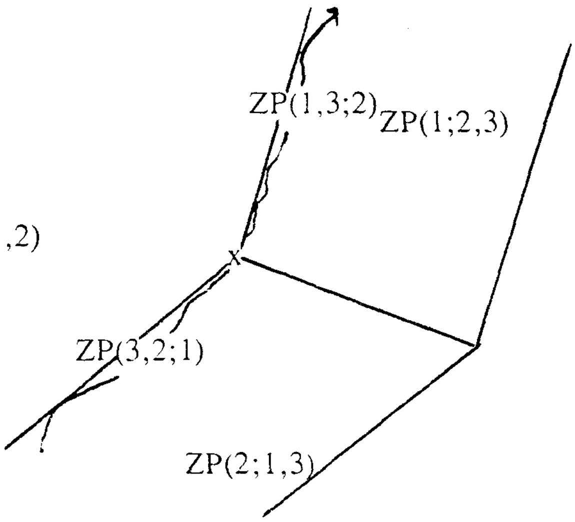


Figure 6.12b

(2) If the straightening-out at x is as illustrated in Figure 6.13

with diagram LO(2,3;1) ← LCO(3;1,2) ← LCO(∅;2;1,3)

↘ ↙ ↙

LC(1,2;3) ← LO(1;2,3)

then for some neighbourhood U of x in M the transitions made by $\phi(M)(y)$ on U are drawn from this diagram (replacing L,C,O by respectively Z,N,P).

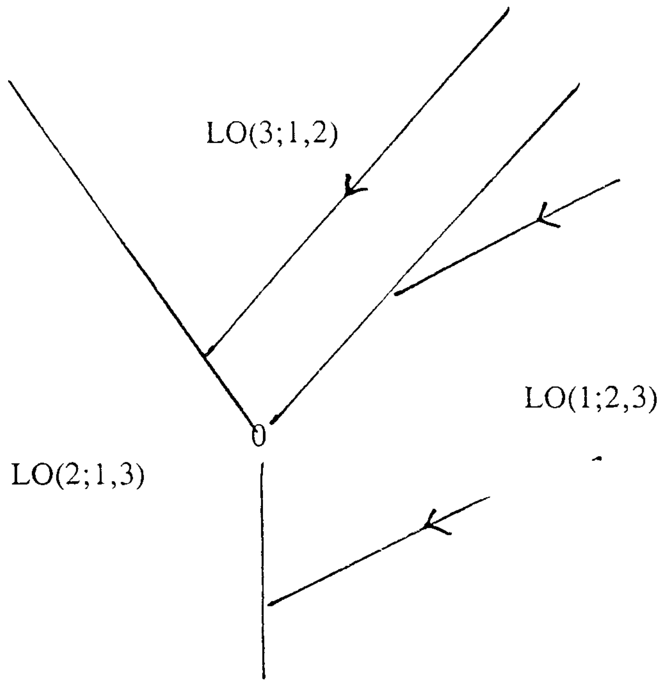


Figure 6.13. Looking down into LC($\varnothing$;1,2,3)

Proof of Proposition 6.1

In Lemma 6.3 we saw there exists a neighbourhood U of x in M such that for any $y \in U$ we can partition $[0,T(U,y))$ into $[0,T_1) \cup .. \cup [T_{s-1},T_s)$ (s bounded by some constant r on U) and reorder the $K_1,..,K_r$ such that for each $i=1,..,s-1$ $\phi(M)(y,[T_{i-1},T_i)) \subset M(x,K_i)$. To prove Proposition 6.1 we must show that in addition that if arbitrarily close to x there exists y making the transition from $M(x,K_i)$ to $M(x,K_j)$ then $M_0(K_i) \to M_0(K_j)$.

(1) We show that if there exists a sequence $\{x_i\} \subset ZP(K;J\backslash K)$ with $x_i \to x$ and sequences $\{t_i\},\{t_i'\}$ with $t_i,t_i' \downarrow 0$ such that $x_i=\phi(M)(y_i,t_i)$, $z_i=\phi(M)(x_i,t_i')$ with $\phi(M)(y_i,[0,t_i)) \subset M(x,K_1)$ and $\phi(M)(x_i(0,t_i')) \subset M(x,K_2)$ (see Figure 6.14), then $-X(K_1)(x)$ points into $T_xZN(K_1;K\backslash K_1)$ and $X(K_2)(x)$ points into $T_xZN(K_2;K\backslash K_2)$.

$-X(K_1)(x)$ points into $T_x(ZN(K_1;K\backslash K_1))$ iff $\langle X(K_1)x,\mathrm{grad}f_i(x)\rangle \leq 0$ for all $i \in K\backslash K_1$. Suppose in fact $\langle X(K_1)(x),\mathrm{grad}f_i(x)\rangle > 0$ some $i \in K\backslash K_1$. We have $f_i(x_j)-f_i(y_j) = \int_0^{t_j}\langle \mathrm{grad}f_i\phi(M)(y_j,t),X(M)\phi(M)(y_j,t)\rangle dt$. We also have by Lemma 6.2 $\sup_{t\in[0,t_j]} \mid X(M)(y_j,t)-X(K_1)(y_j,t) \mid \to 0$ as $j\to\infty$. Because $\langle X(K_1)(x),\mathrm{grad}f_i(x)\rangle > 0$ we must have $\lim_{j\to\infty}\inf_{t\in[0,t_j]}\langle X(K_1)\phi(M)(y_j,t),\mathrm{grad}f_i\phi(M)(y_j,t)\rangle > 0$, and hence $\lim_{j\to\infty}\inf_{t\in[0,t_j]}\langle X(M)\phi(M)(y_j,t),\mathrm{grad}f_i\phi(M)(y_j,t)\rangle > 0$, and hence there exists $\epsilon > 0$ and $j_0 \in Z^+$ such that for all $j \geq j_0$ $\lim_{j\to\infty}\inf_{t\in[0,t_j]}f_i\phi(M)(x_j,t)-f_i\phi(M)(y_j,t) > 0$ $f_i(x_j)-f_i(x_j) > \epsilon t_j$ which is a contradiction because $f_i(x_j)=0$ for all $i \in K$ by construction, while $f_i(y_j) \geq 0$ for all $y_j \in ZN(I;J)$ for all $i \in I \cup J$.

Similarly we get a contadiction if we suppose $\langle X(K_2)x, \mathrm{grad} f_i(x)\rangle < 0$ some $i \in K\backslash K_2$, ie if $X(K_2)(x)$ does not point into $T_x ZN(K_2;K\backslash K_2)$.

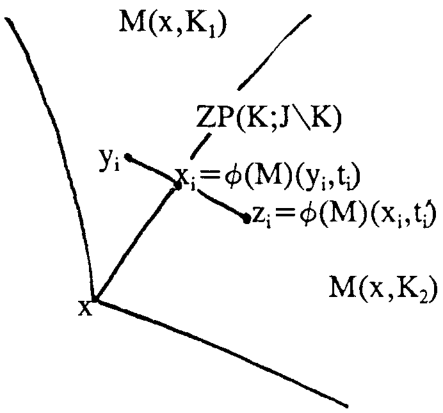


Figure 6.14

(2) We show that if the conclusion of (1) holds then there exists a trajectory $\phi(LC(I;J),X_s)$ of the straightening-out at x passing from $LC(K_1;J\backslash K_1)$ through $LO(K;J\backslash K)$ to $LC(K_2;J\backslash K_2)$. By (1) we know that if there are points on $ZP(K;J\backslash K)$ arbitrarily close to x through which a trajectory makes the transition from $M(x,K_1)$ to $M(x,K_2)$ then $-X_s(K_1)$ points into $LC(K_1;K\backslash K_1)$ and $X_s(K_2)$ points into $LC(K_2;K\backslash K_2)$. Near any point $y \in LO(K;J\backslash K)$ $LC(I;J)$ is locally $LC(I;K\backslash I)$, so (1) tells us we have $-X_s(K_1)(y)$ pointing into $T_y(LC(I;J))$ and $X_s(K_2)(y)$ pointing into $T_y(LC(I;J))$. By Lemma 6.2, for all $y \in LO(K_i;J\backslash K_i)$ $S^0_2(y) = K_i$, so for all $y \in LO(K_i;J\backslash K_i)$ $X_s(LC(I;J))(y) = X_s(K_i)(y)$, and hence we see $\{y - tX_s(K_1), y + tX_s(K_2) : 0 \leq t < \delta\}$ some $\delta \geq 0$ is a trajectory of the straightening-out passing through y. ▬

Linearization

A system (M_1,X) is <u>linear</u> if M_1 is a (linear) corner $LC(I;J)$ and $X \in \Xi_{\omega,1}(M_1)$, ie for each $x \in L(I)$ $X(x) = a + Ax$ some $a \in L(I)$ and linear map $A: L(I) \rightarrow L(I)$ (in fact it is the vector field which is constant or linear if the system is described as such. In both cases the f_i's forming M are linear, so our terminology is not ideal). The biological model which inspired the thesis is of this form and we consider these further in Chapter Eight. For the moment we merely show that linear systems have no advantage over constant systems as far as representativeness of systems in general is concerned.

We shall say that (M,X) can be <u>linearized</u> on a neighbourhood of $x \in M$ if there exists a neighbourhood U of x and a spfp homeomorphism h between the semiflows of $\phi(M,X)$ on U and $\phi(T_xM,X_L)$ on a neighbourhood of the origin in T_xM, where $X_L(y)=X(x)+DX(x)y$.

<u>Example 6.6</u> We show that there is no dense subset of $\Xi_{\omega,r}(M)$ with $r>1$ and M a half-space of R^n consisting of fields each of which can be linearized on a neighbourhood of each point in M.

(1) We show that if X,X′ are vector fields on $M=ZN(I;i)$, $M'=ZN(I';i')$ (where $|I| = |I'|$) with $X\in\Xi'(M)$, $X'\in\Xi'(M')$ then any spfp homeomorphism $h:M\to M'$ preserves $\Gamma_k^X(I\cup i \text{ r } I)$, any k, ie $h\Gamma_k^X(I\cup i \text{ r } I)=\Gamma_k^{X'}(I'\cup i' \text{ r } I')$ any k.
Working with (M,X), set $Z_+=\{x\in Z(I\cup i)$:there exists $\delta>0$ such that $f_i\phi(M)(x,t)>0$ for all $t\in(0,\delta)\}$ and $Z_-=\{x\in Z(I\cup i)$:there exists $\delta>0$ such that $f_i\phi(M)(x,t)=0$ for all $t\in[0,\delta)\}$. Plainly Z_+ and Z_-, and hence $\bar{Z}_+$, $\bar{Z}_-$, are preserved by any spfp homeomorphism. But for $X\in\Xi_\infty'(M)$ we have from definitions that $Z_+=\cup_{k\geq1}\Gamma_k^+(I\cup i \text{ r } I)$ and
$\bar{Z}_+=\Gamma_1^+(I\cup i \text{ r } I)\cup\Gamma_2(I\cup i \text{ r } I)$, and that $Z_-=\cup_{k\geq1}\Gamma_k^-(I\cup i \text{ r } I)$ and
$\bar{Z}_-=\Gamma_1^-(I\cup i \text{ r } I)\cup\Gamma_2(I\cup i \text{ r } I)$ (see Figure 6.15), so
$\Gamma_2(I\cup i \text{ r } I)=\bar{Z}_+\cap\bar{Z}_-$, and so must be preserved by h. Similarly since for $X\in\Xi_\infty'(M)$ and any integer k $\Gamma_k(I\cup i \text{ r } I) = \text{closure}(Z_+\cap\Gamma_{k-1}(I\cup i \text{ r } I))\cap\text{closure}(Z_-\cap\Gamma_{k-1}(I\cup i \text{ r } I))$ (see Figure 6.15) it follows by induction that each $\Gamma_k(I\cup i \text{ r } I)$ is preserved by a spfp homeomorphism h.

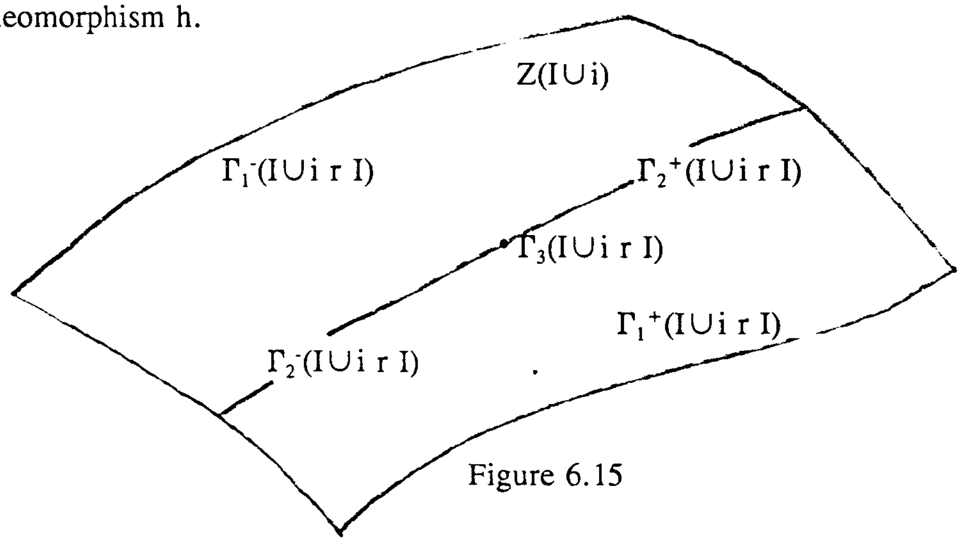


Figure 6.15

(2) If M is locally represented as ZN(I;J) with $x\in Z(I\cup J)$ then $T_xM\cong LC(I;J)$; we show by example that in general, taking $M=ZN(I;i)$, if $X\in\Xi'(M)$ and $x\in\Gamma_k^X(Z(I\cup i) \text{ r } Z(I))$ then $0\notin\Gamma_k^{X_L}(L(I\cup i) \text{ r } L(I))$ where $(T_xM\cong LC(I;J),X_L)$ is the linearization of X,M at x, and

hence by (1) above linearization is not generally possible.

Consider $M=\{x\in R^n: \langle x,n_1\rangle \geq 0\}$, $n\geq 3$, and suppose near 0 $X\in \Xi_\infty'(M)$ has the form $(\dot{x}_1,\dot{x}_2,\dot{x}_3)=(x_2+f_1(x),x_3+f_2(x),1+f_3(x))$ each $f_i:R^n\to R$ is such that $f_i(0)=0$, $Df_i(0)=\mathbf{0}$, $i=1,2,3$. For $x=0$ we have then $T_xM\cong M$, $X_L(x_1,x_2,x_3)=(\dot{x}_1,\dot{x}_2,\dot{x}_3)=(x_2,x_3,1)$, so $X_L\in \Xi_{\omega,1}'(T_xM)$. Then $0\in \Gamma_3^{X_L}(1 \text{ r } \varnothing)$ but for (M,X) we have $\dot{x}_1(0)=x_2+f_1(x)\mid_{x=0}=0, \ddot{x}_1(0)=x_3+f_2(x)+L_Xf_1(x)\mid_{x=0}=0$ (written out in co-ordinates we have $L_Xf_1(0)=\Sigma X_i(0)\partial f_1(x)/\partial x_i$ and hence $=0$) but in general $x_1^{(3)}(0)\neq 0$ and hence $0\in \Gamma_2^X(1 \text{ r } \varnothing)$ but $0\notin \Gamma_3^X(1 \text{ r } \varnothing)$. This phenomenon is stable under perturbations in X. If we perturb X we will perturb the location of points x in ∂M such that in the linearization at x $0\in \Gamma_3^{X_L}(1 \text{ r } \varnothing)$ but such a point x will still not generally be in $\Gamma_3^X(1 \text{ r } \varnothing)$. Thus for a non-empty open subset of vector fields with M a half space in R^n, $n\geq 3$, there exists points where (M,X) cannot be linearized.

Example 6.6 leaves open the possibilities (1) that by perturbing M as well as X we could everywhere linearize, and (2) that even if linearization in the given sense is not always possible, we could find (for generic X,M) for each $x\in M$ *some* linear system locally spfp equivalent to (M,X) near x. In fact neither of these two possibilities holds:

Example 6.7 We show there exist M and $X\in \Xi_\infty(M)$ such that for any Y sufficiently near X and N near M there exists $x\in N$ such that no linear system (M_1,X_1) exists for which we can find a spfp homeomorphism $h:N\to M_1$ between the semiflows $\phi(N,Y)$ on a neighbourhood of x in N and the semiflow $\phi(M_1,X_1)$ near any point $x_L\in M_1$.

(1) Plainly in a linear system the set $\Gamma_k(K_1 \text{ r } K_2)$ (where $K_2\subset K_1$) is affine, ie a translate of a linear subspace. If $\Gamma_k(K_1 \text{ r } K_2)=\{x\in L(I):\langle x,\tilde{n}_i\rangle =p_i, i=1,..,r\}$ where $K_1\supset K_2\supset I$, some independent set $\{\tilde{n}_i\}\subset L(I)$, then the normal space to $\Gamma_k(K_1 \text{ r } K_2)$ in $L(I)$, denoted $N(\Gamma_k(K_1 \text{ r } K_2) \text{ in } L(I))$, equals $\text{span}\{P(I)\tilde{n}_i:i=1,..,r\}$. We show (still in the context of linear systems) that if $N(\Gamma_k(K_1 \text{ r } K_2) \text{ in } L(K_2))\subset L(K)$ some $K\supset K_2$ then $\Gamma_{k+1}(K_1\cup K \text{ r } K)=\Gamma_{k+1}(K_1 \text{ r } K_2)\cap L(K)$

(i) First we show that if L_1,L_2 are linear subspaces of L with $X\in L$ then $N(L_1 \text{ in } L)\subset L_2$ implies $P(L_2)X\in L_1\cap L_2$ iff $X\in L_1$. Since if L_a,L_b are subspaces of L then $L_a\subset L_b$ iff $N(L_b \text{ in } L)\subset N(L_a \text{ in } L)$, and since $N(N(L_a \text{ in } L) \text{ in } L) = L_a$, we see that the supposition $N(L_1 \text{ in } L)\subset L_2$ is equivalent to $N(L_2 \text{ in } L)\subset L_1$.

By Remark 2.1 $P(L_2)X-X\in N(L_2 \text{ in } L)$ so $P(L_2)X-X\in L_1$, therefore

$P(L_1)(P(L_2)X\text{-}X)=P(L_2)X\text{-}X$, and since if $X \in L_1$ $P(L_1)(P(L_2)X\text{-}X)=P(L_1)P(L_2)X\text{-}X$ if $X \in L_1$ $P(L_1)P(L_2)X=P(L_2)X$ and hence $P(L_2)X \in L_1 \cap L_2$. Conversely, if $P(L_1)X \in L_1 \cap L_2$ since $P(L_1)X\text{-}X \in N(L_2 \text{ in } L) \subset L_1$ we must have $X \in L_1$.

(ii) The result is true by definitions if $k=0$. Suppose it is true for $k\text{-}1$. $\Gamma_{k-1}(K_1 \text{ r } K_2) \supset \Gamma_k(K_1 \text{ r } K_2)$, if $N(\Gamma_k(K_1 \text{ r } K_2) \text{ in } L(K_2)) \subset L(K)$ then $N(\Gamma_{k-1}(K_1 \text{ r } K_2) \text{ in } L(K_2)) \subset L(K)$ and so by the (k-1)th result $\Gamma_k(K_1 \cup K \text{ r } K)=\Gamma_k(K_1 \text{ r } K_2) \cap L(K)$ and hence $T_x\Gamma_k(K_1 \cup K \text{ r } K)=T_x\Gamma_k(K_1 \text{ r } K_2) \cap L(K)$. Then using (i) with L set to $L(K_2)$, L_2 set to $L(K)$ and L_1 set to $T_x\Gamma_k(K_1 \text{ r } K_2)$ and using that $K \supset K_2$ we obtain $X(K)(x)= P(L(K))X(K_2)(x) \in T_x\Gamma_k(K_1 \text{ r } K_2) \cap L(K)$ iff $X(K_2)(x) \in T_x\Gamma_k(K_1 \text{ r } K_2)$. Since $\Gamma_{k+1}(K_1 \cup K \text{ r } K)=\{x \in \Gamma_k(K_1 \cup K \text{ r } K): X(K_2 \cup K)(x) \in T_x\Gamma_k(K_1 \cup K \text{ r } K)\}$ by the above $\Gamma_{k+1}(K_1 \cup K \text{ r } K)=\{x \in \Gamma_k(K_1 \text{ r } K_2) \cap L(K): X(K_2)(x) \in T_x\Gamma_k(K_1 \text{ r } K_2)\}$, and we obtain $\Gamma_{k+1}(K_1 \cup K \text{ r } K)=\Gamma_{k+1}(K_1 \text{ r } K_2) \cap L(K)$ as required.

(2) We exhibit a vector field X and a submanifold with corners M of R^n $(n \geq 4)$ such that for any X′ near X and any M′ near M there is a point x of M′ for which there is no linear system (M_1,X_1) such that the semiflow $\phi(M',X')$ near x is equivalent to $\phi(M_1,X_1)$ near some point x_L of M_1.

If $M = ZN(\varnothing;1,2)$ and $x \in Z(1,2)$ then we see that $x \in \Gamma_2(1 \text{ r } \varnothing) \cap \Gamma_2(1,2 \text{ r } 2)$ iff $\langle X(x),\text{grad}f_1(x)\rangle=0$ and $\langle X(x),\text{grad}f_1(x)\rangle\text{-}\langle X(x),\text{grad}f_2(x)\rangle\langle \text{grad}f_1(x),\text{grad}f_2(x)\rangle=0$ iff $x \in \Gamma_2(1 \text{ r } \varnothing)$ (ie $\langle X(x),\text{grad}f_1(x)\rangle=0$ and either (i) $x \in \Gamma_2(2 \text{ r } \varnothing)$ (ie $\langle X(x),\text{grad}f_2(x)\rangle=0$) (so by Proposition 4.1 $x \in \Gamma_2(1,2 \text{ r } \varnothing)$) or (ii) $\langle \text{grad}f_1(x),\text{grad}f_2(x)\rangle=0$ (Figure 6.16).

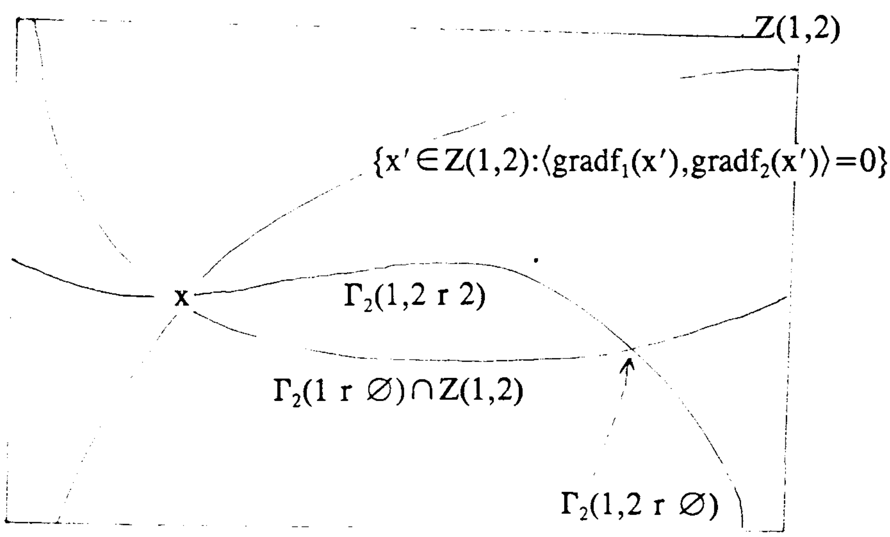


Figure 6.16

We now show that there is no linear system with a semiflow anywhere spfp equivalent to that near x in the Figure 6.16 above. By an argument similar to Example 6.6(1) we can show that $\Gamma_2(1,2\ r\ \emptyset)$ and one or more branches ending in x of each of the curves $\Gamma_2(1\ r\ \emptyset)\cap Z(1,2)$ and $\Gamma_2(1,2\ r\ 2)$ must be preserved by a spfp homeomorphism.

If we now try to construct a linear system (M_1,X_1) with semiflow spfp equivalent near some $x_L \in M_1$ to $\phi(M,X)$ near x we must have $M_1=LC(\emptyset;1,2)$ with $x_L \in L(1,2)$. If L(i) has normal n_i then either $\langle n_1,n_2\rangle \neq 0$ or $\langle n_1,n_2\rangle=0$. If $x_L \in \Gamma_2^{X_1}(1,2\ r\ 2)\cap\Gamma_2^{X_1}(1\ r\ \emptyset)$ and $\langle n_1,n_2\rangle\neq 0$ then by the above we must have case (i), ie $x_L \in \Gamma_2^{X_1}(1,2\ r\ \emptyset)$:

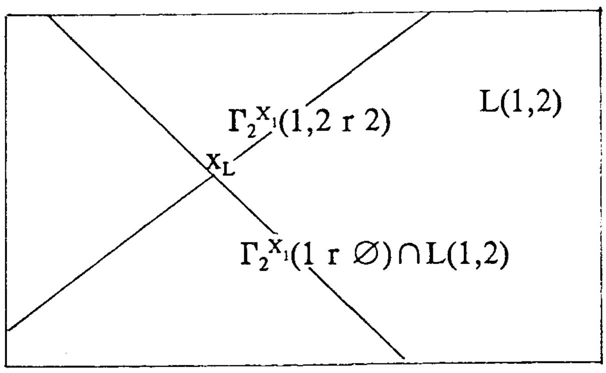


Figure 6.17a. $x_L\in\Gamma_2^{X_1}(1,2\ r\ 2)\cap\Gamma_2^{X_1}(1\ r\ \emptyset)$ and $\langle n_1,n_2\rangle\neq 0$ implies $x_L\in\Gamma_2(1,2\ r\ \emptyset)$

Alternatively, if $\langle n_i,n_2\rangle=0$ then by (1) we have $\Gamma_2^{X_1}(1,2\ r\ 2)=\Gamma_2^{X_1}(1\ r\ \emptyset)\cap L(1)=\Gamma_2^{X_1}(1\ r\ \emptyset)\cap L(1,2)$:

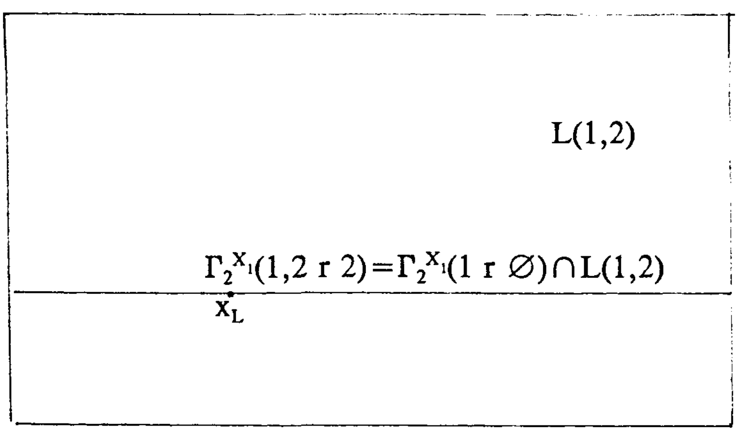


Fig. 6.17b. $x_L\in\Gamma_2^{X_1}(1,2\ r\ 2)\cap\Gamma_2^{X_1}(1\ r\ \emptyset)$ and $\langle n_1,n_2\rangle=0$ implies $\Gamma_2^{X_1}(1,2\ r\ 2)=\Gamma_2^{X_1}(1\ r\ \emptyset)\cap L(1,2)$

The two possible candidates for a linear system locally spfp equivalent to (M,X) near x in Figure 6.16 are as in Figures 6.17a and 6.17b near x_L. But in a. $x_L\in\Gamma_2^{X_1}(1\ r\ \emptyset)$ and in b. $\Gamma_2^{X_1}(1,2\ r\ 2)=\Gamma_2^{X_1}(1\ r\ \emptyset)\cap L(1,2)$, neither relation holding at x in Figure 6.16. This argument does not rest upon any special choice of X or M (special in the sense that any perturbation would destroy it) and the assertion follows.

Remark In view of the limitations of differentiable equivalence (Examples 6.1) it would be worthwhile combining the idea of Example 6.6 part 1 (that if $X \in \Xi'(ZN(I;i))$, $X' \in \Xi'(ZN(I';i'))$ then an spfp homeomorphism $h:ZN(I;i) \to ZN(I';i')$ preserves $\Gamma_k(I \cup i \text{ r } I)$) with that of Proposition 4.3 (that differentiable equivalence preserves the algorithm etc) to show that, under generic restrictions on X, spfp homeomorphisms preserve the algorithm sequence and the iteration.

Regular Zeros and their Stable Manifolds

We recall that a zero of a smooth vector field X (ie, a point x such that $X(x)=0$) is hyperbolic if $DX(x)$ has no pure imaginary eigenvalue. The Stable (unstable) manifold of a zero x is the set of points y such that $|\phi(y,t)-x| \to 0$ as $t \to \infty$ $(t \to -\infty)$ where ϕ is the flow of X; if X is C^r they are C^r injectively immersed submanifolds tangent at x to the stable and unstable manifolds of the linearization $\dot{\eta}=DX(x)\eta$, and if the vector field X is a C^r function of $\lambda \in R^p$ (so by [42] the map $\lambda \to$zero of $X(\lambda)$ is C^r) the graph of $\lambda \to$stable manifold of zero of $X(\lambda)$ is C^r ([42] for $r=\infty$, [49] for $r=\omega$).
We make a straightforward generalisation of hyperbolicity as follows:

Definition If a submanifold with corners M is locally represented as $ZN(I;J)$ a zero x_0 of $X(M)$ in $Z(I \cup J)$ is regular if
(1) $X(I \cup J)$ has a hyperbolic zero at x_0
(2) For all $K \subset J$ $S^0_2(x_0,ZN(I;K\backslash I),X)=S_2(x_0,ZN(I;K\backslash I),X)$.
Via Lemmas 2.4 and 4.6 we see (2) is equivalent to
(2′) If we straighten (M,X) out at x_0 to give the constant system $(T_xM \cong LC(I;J),X_s)$ then for all $y \in T_xM$ $S^0_2(y,T_xM,X_s)=S_2(y,T_xM,X_s)$, and to
(2″) $X(x_0) \in \cap\{U(K):K \subset J\}$ where $U(K)=\{X:\langle X(K\backslash j),n_j\rangle \leq 0$ for all $j \in K\backslash I$ implies $\langle X(K\backslash j),n_j\rangle < 0$ for all $j \in K\backslash I$ and $\langle X(K),n_j\rangle \neq 0$ for all $j \in J\backslash K\}$. We can interpret (2) geometrically as follows. For any constant system $(LC(I;J),X)$ if $X(LC(I;J))(y)=X(K)(y)$ for some $y \in$ a stratum $LO(K';J\backslash K')$ then (eg by Part (3) of the proof of Lemma 6.2) $X(LC(I;J))(y)=X(K)(y)$ for all $y \in$ that stratum; we can think of these $X(K)$'s as the "active" vector fields of the constant system, and condition (2) in the definition of regularity says that if $X_s(K_1)$, $X_s(K_2)$ are active for the straightening out then $X_s(K_1) \neq X_s(K_2)$.

Figure 6.18 illustrates two zeros of X(M) where condition (2) does not hold. We observe that in the first, a trajectory could alternate infinitely often between ZP(1;2) and ZP(∅;1,2) on any left neighbourhood of x (as in Example 5.1(2) for example); in the second, an arbitrarily small perturbation could destroy the zero entirely. Neither phenomenon can occur if x is a regular zero (Remark 6.1 and Proposition 6.1(1) respectively).

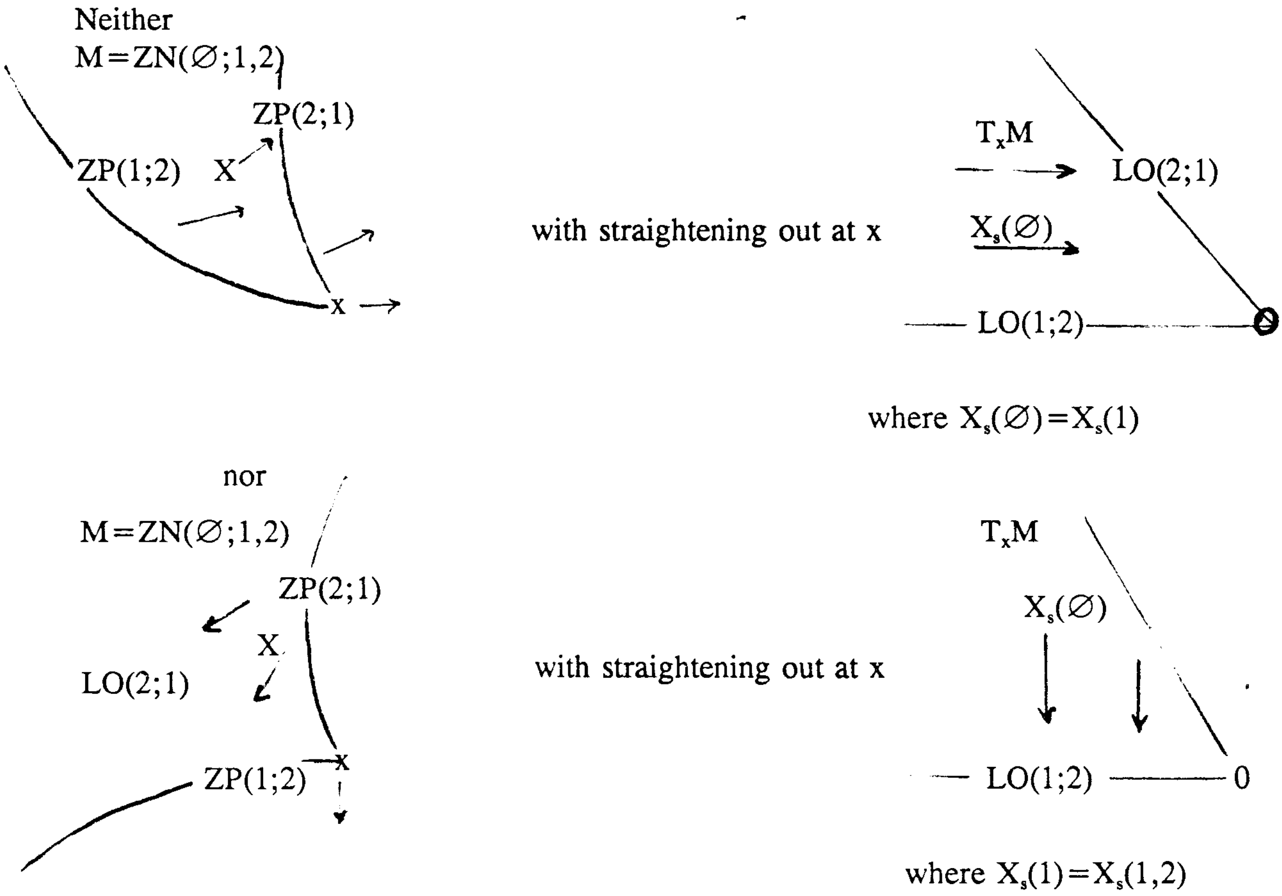


Figure 6.18

is regular, but a system on a three dimensional corner with straightening out

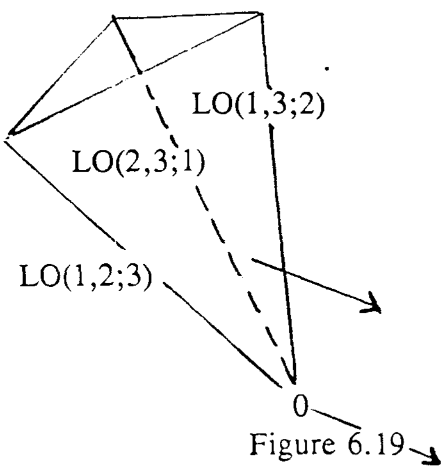


Figure 6.19

is regular despite eg $X_s(2,3)=X_s(2)$ if neither is "active" .

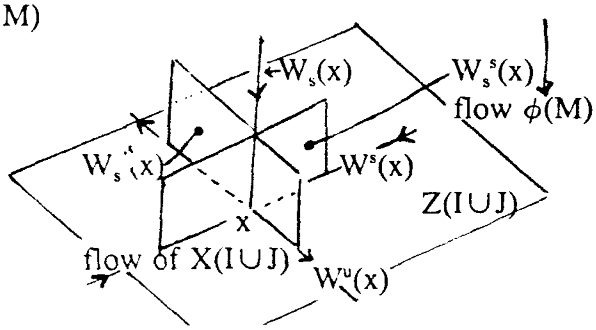
Remark 6.1 We can straightaway derive one useful property about the local flow near a regular zero from Lemma 6.3. There we saw that given any $x \in M$ there exists a neighbourhood U of x such that if $y \in U$ then we could get a finite partition of $[0,T(U,y))$ into $[0,T_1) \cup [T_1,T_2) \cup ..$ and reorder the $K_1,..,K_r \in S^0_2(LC(I;J) \cong T_xM,X_s)$ such that on $[T_{i-1},T_i)$ $\phi(M)(y,t) \in M(x,K_i)$ with $\phi(M)(y,t) \notin M(x,K_i)$ for all $t > T_i$. Each $M(x,K_i)$ was a certain union of strata and in general on an interval $[T_{i-1},T_i)$ the trajectory could move about between these strata - for example, in Example 6.5(1) (which was derived from Example 5.1(3)) a trajectory would make infinitely many stratum jumps on the interval $[T_{i-1},T_i)$. However we can now show that in the case x is a regular zero $\phi(M)(y,t) \subset ZP(K_i,J\backslash K_i)$ on (T_{i-1},T_i) and so intersects at most two strata on $[T_{i-1},T_i)$.

By definition $M(x,K_i) = \cup\{ZP(K;J\backslash K):K \in E(K_i)\}$ where $E(K_i)$ was defined using data from the straightening out at x, $E(K_i) = \{K:I \subset K \subset I \cup J:S^0_2(y) = K_i$ for all $y \in LO(K;J\backslash K)\}$. We saw in part (2) of the proof of Lemma 6.3 that as the neighbourhood U shrinks to x so $\sup\{\ |\ X(M)(y)-X(K_i)(y)\ |\ :y \in U \cap M(x,K_i)\} \downarrow 0$.

We saw in Lemma 6.2 that if $K_i \in S^0_2(LC(I;J),X)$ for X a constant vector field then for all $y \in LO(K_i;J\backslash K_i)$ $X(LC(I;J))(y) = X(K_i)(y)$, so if $K_i \in S^0_2(T_xM,X_s)$ $X_s(K_i)$ is active, so by condition (2) of regularity $\langle X(K_i)(x),\text{grad}f_j(x)\rangle \neq 0$ for all $j \in J\backslash K_i$. Hence on a small enough neighbourhood U of x $\langle X(K_i)(y),\text{grad}f_j(y)\rangle \neq 0$ for all $y \in U \cap M(x,K_i), j \in J\backslash K_i$, hence $\langle X(M)(y),\text{grad}f_j(y)\rangle \neq 0$ for all $y \in U \cap M(x,K_i), j \in J\backslash K_i$. Hence for as long as $\phi(M)(y,t) \subset M(x,K_i) \cap U$ $f_j\phi(M)(y,t)$ is strictly monotone, and since $f_j(y) \geq 0$ for all $j \in J$ and $y \in M(x,K_i) \cap U$, if at $t > T_{i-1}$ $f_j\phi(M)(y,t) = 0$ then $\phi(M)(y)$ leaves $M(x,K_i) \cap U$ at t, so since $ZP(K_i;J\backslash K_i) \cap U \subset M \cap U \subset ZN(K_i;J\backslash K_i)$ for all $t \in (T_{i-1},T_i)$ $\phi(M)(y,t) \in ZP(K_i;J\backslash K_i)$ as claimed.

In summary: if x is a regular zero of (M,X) there exists a neighbourhood U of x in M such that if $y \in U$ then $[0,T(U,y)) = [0,T_1) \cup .. \cup [T_{s-1},T_s)$ some finite s with each $T_i > T_{i-1}$ and such that for all $t \in (T_{i-1},T_i)$ $\phi(M)(y,t) \in ZP(K_i;J\backslash K_i)$. This fact is used in proving our stable manifold theorem (see eg Figure 6.24).

If x is a regular zero of X(M) with M locally represented near x as ZN(I;J) we define local invariant manifolds (for U a neighbourhood of x in M)



$W_s(x)=\{y \in U:\phi(M)(y,t)=x \text{ some } t \geq 0\}$

$W_s^s(x)=\{y \in U:\phi(M)(y,t) \in W^s(x) \text{ some } t \geq 0\}$

$W_s^u(x)=\{y \in U:\phi(M)(y,t) \in W^u(x) \text{ some } t \geq 0\}$

where $W^s(x)$ and $W^u(x)$ are the stable and unstable manifolds of $X(I \cup J)$ at x. We see $W_s^s(x) = \{y \in U:\phi(M)(y,t) \rightarrow x \text{ as } t \rightarrow \infty\}$.

Proposition 6.2

(1) There exists an open-dense subset of $\Xi_\infty(M)$ where M is a compact submanifold with corners such that if X is in this subset all zeros of X(M) are regular. Regular zeros are isolated (and hence finite in number on a compact submanifold with corners). Regular zeros survive as regular zeros on perturbing X (in fact X→each zero of X is C^r for each zero).

(2) If M is a submanifold with orthogonal corners the local invariant manifolds of a regular zero z_0 are C^1 not necessarily C^2 piece-wise C^r submanifolds with corners, and if M is locally represented near z_0 as $ZN(I;J)$

$\text{Codim}(W_s^{s,u}(z_0) \text{ in } M)=\text{codim}(W^{s,u}(z_0) \text{ in } Z(I \cup J))$

$T_{z_0}W_s(z_0)=N_{z_0}(I \cup J \text{ in } I) \cap T_{z_0}M$

$T_{z_0}W_s^s(z_0)=(N_{z_0}(I \cup J \text{ in } I) \times E^s) \cap T_{z_0}M$

$T_{z_0}W_s^u(z_0)=(N_{z_0}(I \cup J \text{ in } I) \times E^u) \cap T_{z_0}M$ where E^s, E^u are the stable and unstable manifolds of the linearization $\dot{\eta}=DX(I \cup J)(z_0)\eta$ and we recall $T_{z_0}M$ is the tangent cone to M at z_0.

Remarks 6.2 While in Chapter Five the requirement that the submanifold had orthogonal corners was merely to simplify the proof and exposition, here the orthogonality condition is essential, as we now show.

We saw in Remark 2.5 that if M had only orthogonal corners the transitions possible between strata were much restricted, and it turns out that whether the invariant manifolds are C^1 or not depends on the type of transitions which occur between strata. Evidently our invariant manifolds are the preimage by the semiflow of an invariant manifold of $\phi(I \cup J)$ in $Z(I \cup J)$. We can see in a crude way how orthogonality affects whether pre-images are C^1 or not by leaving aside for the moment both the fact that $Z(I \cup J)$ is locally the deepest stratum, and the invariance with respect to $\phi(I \cup J)$ of the submanifold we begin with on $Z(I \cup J)$, and considering a point x_0 near a non-orthogonal intersection Z(1,2) of surfaces Z(1), Z(2) as shown in Figure 6.20. If we consider the pre-images by the semiflow of the point x_0, we get a trajectory γ

running down ZP(1;2), across Z(1,2) and into ZP(2;1) (we saw in Remark 2.5 that this would not happen if Z(1) and Z(2) intersected at right angles) and each of the three C^r components $\gamma \cap ZP(1;2)$, $\gamma \cap Z(1,2)$ and $\gamma \cap ZP(2;1)$ has a C^r pre-image V_1, V_{12}, V_2. If then we take the sequences $\{x_i^1\} \subset \gamma \cap ZP(1;2)$ with $x_i^1 \to x$ and $\{x_i^2\} \subset \gamma \cap ZP(2;1)$ with $x_i^2 \to x$ then $T_{x_i^1}V_1 = \mathrm{span}\{X(x_i^1), \mathrm{grad}f_1(x_i^1)\}$, $T_{x_i^2}V_2 = \mathrm{span}\{X(x_i^2), \mathrm{grad}f_2(x_i^2)\}$. Thus in the limit as $i \to \infty$ and $x_i^1, x_i^2 \to x$ $\lim_{i\to\infty} T_{x_i^1}V_1 = \lim_{i\to\infty} T_{x_i^2}V_2$ iff $X(x) \in \mathrm{span}\{\mathrm{grad}f_1(x), \mathrm{grad}f_2(x)\}$; in general we get a crease along V_{12} where V_1 meets V_2 (Figure 6.20).

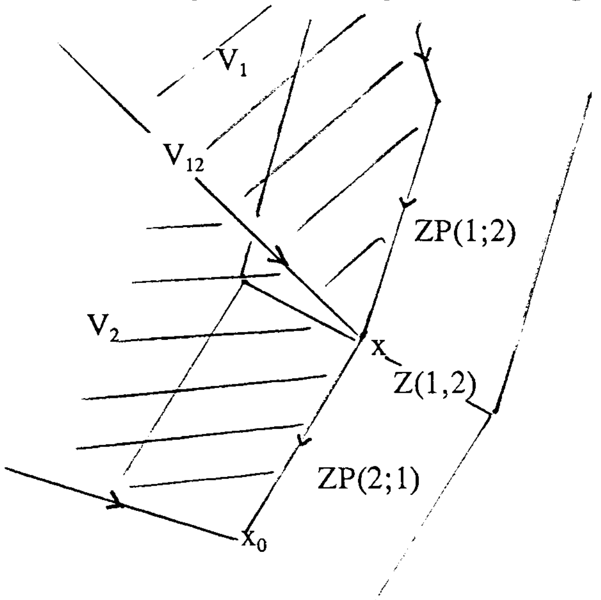


Figure 6.20

We can construct an example in four or more dimensions where this phenomenon prevents even a local invariant manifold such as we are considering being C^1 on any neighbourhood of the zero. The example will be on $M = LC(\varnothing;1,2,3)$ in R^n, $n \geq 4$. We shall use co-ordinates $x = (PL(1,2,3)x, x - PL(1,2,3)x) = (x_{123}, x_{123}')$ and will consider a vector field X on M which is independent of x_{123}. Hence for all $x \in M$ $X(M)(x)$ is independent of x_{123}, so $P(\mathrm{span}\{n_1,n_2,n_3\})\phi(M)(x,t) = \phi(M \cap \mathrm{span}\{n_1,n_2,n_3\})(P(\mathrm{span}\{n_1,n_2,n_3\})(x),t)$ for all $x \in M$, and we can represent the system by the projection of X onto a cross-section $M \cap \mathrm{span}\{n_1,n_2,n_3\}$, thus exhibiting it in R^3. This is analogous to the way that if the vector field X in R^3 is independent of x_{12} with $M = LC(\varnothing;1,2) \subset R^3$ (see Figure 6.21 below),

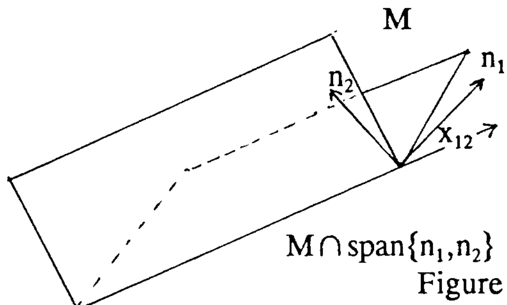


Figure 6.21

then $P(\mathrm{span}\{n_1,n_2\})\phi(M)(x,t) =$
$\phi(M \cap \mathrm{span}\{n_1,n_2\})(P(\mathrm{span}\{n_1,n_2\})(x),t)$ for all $x \in M$, enabling us to represent the system in R^2.

Suppose the projection of our system onto $\mathrm{span}\{n_1,n_2,n_3\}$ has straightening out

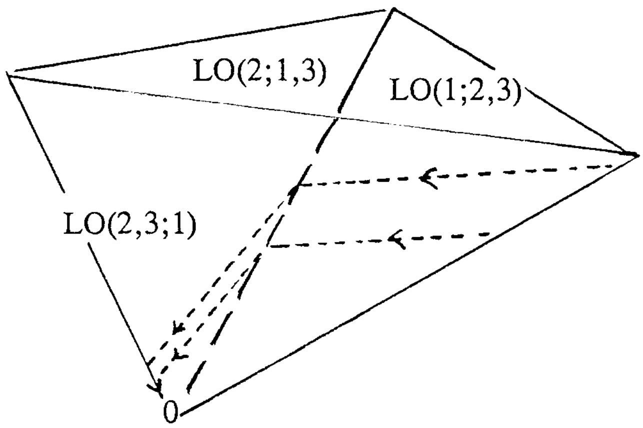


Figure 6.22

With diagram

$$\begin{array}{l} LO(\varnothing;1,2,3) \\ \downarrow \quad\searrow \\ LO(1,3;2)\cup LO(1;2,3) \longrightarrow LO(1,2;3)\cup LO(2;1,3) \longrightarrow LO(2,3;1)\rightarrow L(1,2,3). \end{array}$$

Suppose X(1,2,3) has a hyperbolic repelling zero at x_0: if the straightening-out is as given this zero is then regular for $X(LC(\varnothing;1,2,3))$. The stable manifold $W_s^s(x_0)=W_s(x_0)$ is the pre-image by $\phi(M)$ of x_0 and is three-dimensional: by Proposition 6.1 (or specifically Remark 6.1) we know that near enough to x_0 the sequence of strata the trajectories occupy on the way to x_0 must be those which the trajectories in the straightening out at x_0 follow, ie working backwards along the diagram of the straightening out (above) we see that $W_s(x_0)$ is formed by taking

(i) The pre-image by $\phi(2,3)$ in LC(2,3;1) of x_0 (ie, $\{x \in LC(2,3;1):\phi(2,3)(x)\cap x_0 \neq \varnothing\}=\{\phi(2,3)(x_0,-t):t\geq 0\}$): call this V(2,3) (it is a 1-dimensional manifold with corners)

(ii) The pre-image by $\phi(2)$ of V(2,3) in LC(2;1,3): call this V(2) (it is a 2-dimensional manifold with corners) and the intersection of V(2) with L(1,2), call this V(1,2) (which is a 1-dimensional manifold with corners).

(iii) The pre-image by $\phi(1)$ of V(1,2) in LC(1;2,3): call this V(1) (it is a 2-dimensional manifold with corners)

(iv) The pre-images by $\phi(\varnothing)$ of V(2), V(1,2), V(1) in LC($\varnothing$;1,2,3): call these V($\varnothing$;2), V($\varnothing$;1,2) and V($\varnothing$;1) (respectively 3,2 and 3 dimensional manifolds with corners).

We recall that if V is a submanifold with corners we may denote the tangent *space* to V at any $x \in V$ by $T_x\check{V}$. We claim that in general if $x \in V(\varnothing;1,2)\backslash\{x_0\}$ then $T_x\check{V}(\varnothing;2) \neq T_x\check{V}(\varnothing;1)$. At x_0 itself the tangent spaces do coincide: because x_0 is a zero we must by Remark 2.1 or the Characterisation of Projection have $X(x_0) \in \mathrm{span}\{n_1,n_2,n_3\}$ so since $T_{x_0}\check{V}(1,2)=P(2,3)X(x_0)+\lambda P(2)X(x_0)$ some λ we have $T_{x_0}\check{V}(1,2) \in \mathrm{span}\{n_1,n_2,n_3\}$. Then since $T_{x_0}\check{V}(\varnothing;1) = \mathrm{span}\{T_{x_0}\check{V}(1,2),X(1)(x_0)\}$ and $T_{x_0}\check{V}(\varnothing;2) = \mathrm{span}\{T_{x_0}\check{V}(1,2),X(2)(x_0)\}$ we have $T_{x_0}\check{V}(\varnothing;1)=T_{x_0}\check{V}(\varnothing;2) = \mathrm{span}\{n_1,n_2,n_3\}$. However on $V(\varnothing;1,2)\backslash\{x_0\}$, while by continuity the tangent spaces $T_x\check{V}(\varnothing;1)$, $T_x\check{V}(\varnothing;2) \rightarrow T_{x_0}\check{V}(\varnothing;1), T_{x_0}\check{V}(\varnothing;2)$ as $x \rightarrow x_0$, for general X (general, that is, subject to the straightening out having the form illustrated) no relation binds them to be equal and V($\varnothing$;1,2) represents a two-dimensional "crease" between the three dimensional V($\varnothing$;1) and V($\varnothing$;2).

<u>Remark 6.2(2)</u> Even if M has orthogonal corners the local invariant manifolds need not be C^2. This should be evident from the way they are proved to be C^1 (Proposition 6.1(2), see below) but for a concrete example consider $\dot{x}_2=-1$, $\dot{x}_3=-1$, $\dot{x}_1=x_3-x_1-x_2$ on the orthogonal corner $\{x \in R^3 : x_2 \geq 0, x_3 \geq 0\}$ which has a regular zero at the origin (see figure 6.23).

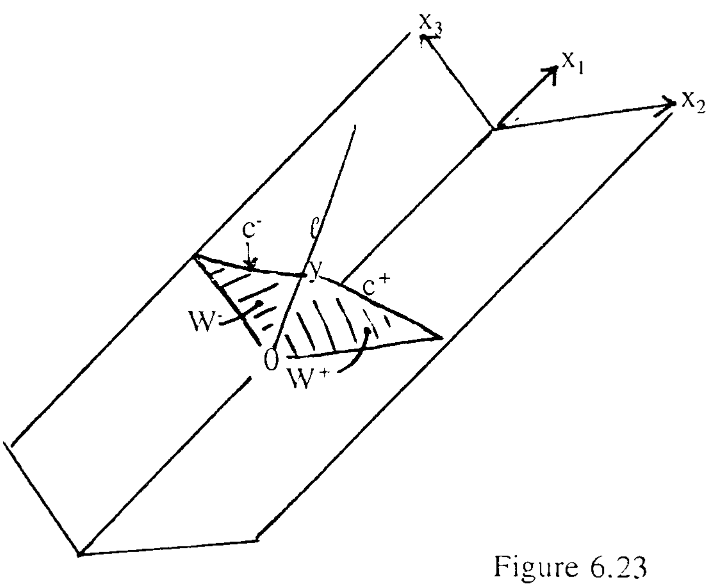


Figure 6.23

Its stable manifold contains $\ell=\{x:x_1\geq 0,x_2=x_3=0\}$ which separates the stable manifold $W_s^s(0)$ into two parts, $W^+=W_s^s(0)\cap\{x:x_2>x_3\}$, $W^-=W_s^s(0)\cap\{x:x_2<x_3\}$. Suppose $y\in\ell\setminus\{0\}$ and c is the intersection of $W_s^s(0)$ with the plane $\{x\in R^3:x_2+x_3=y_2+y_3\}$, so $c=c^-\cup y\cup c^+$ where $c^+=W^+\cap\{x_2+x_3=y_2+y_3,\ x_2>x_3\}$, $c^-=W^-\cap\{x:x_2+x_3=y_2+y_3,\ x_2<x_3\}$.

Then if λ is a parameter along c with $\lambda(0)=y$ (such as $\lambda=x_2-x_3$) then we can check by a straightforward computation that $dc(\lambda)/d\lambda\mid_{left}=dc(\lambda)/d\lambda\mid_{right}$ (as we know from Proposition 6.2(2)) but $d^2c(\lambda)/d\lambda^2\mid_{left}\neq dc^2(\lambda)/d\lambda^2\mid_{right}$, ie $W_s^s(0)$ is not C^2.

Proof of Proposition 6.2

(1)(i) Condition (2) in the definition of regular zero is a condition on x,M,X (without necessarily requiring that $X(M)(x)=0$) and we show that for each stratum σ of M the set of $x\in\sigma$, smooth f_i's defining M and $X\in\Xi_\infty(M)$ such that x,M,X satisfies condition (2), is open.

Suppose locally $\sigma=Z(I\cup J)$ with M locally ZN(I;J), then (using (2″)) (2) is not satisfied iff for some $K\subset J$ $\langle X(K\setminus j)(x),\text{grad}f_j(x)\rangle\leq 0$ for all $j\in K\setminus I$ and either $\langle X(K\setminus j)(x),\text{grad}f_j(x)\rangle=0$ some $j\in K\setminus I$ or $\langle X(K)(x),\text{grad}f_j(x)\rangle=0$ some $j\in J\setminus K$. Hence by continuity of $x\rightarrow X(x)$, $x\rightarrow T_xM$ for $x\in\sigma$ (by [13] again), the set of points in σ,$X\in\Xi_\infty(M)$ and smooth real valued functions f_i, $i\in I\cup J$, on R^n such that (2) is not satisfied is closed.

(ii) We can see that if x is a regular zero of (M,X) with $x\in\sigma$ then (with M,σ as in (i)) $X(I\cup J)(x)=\mathbf{0}$ but $X(K)(x)\neq\mathbf{0}$ for all $I\subset K$ strictly contained in $I\cup J$. For since $X(M)(x)=\mathbf{0}$ we must have $X(K)(x)=\mathbf{0}$ some $I\subset K\subset I\cup J$; by Remark 2.1 if $X(K)(x)=\mathbf{0}$ some K strictly contained in $I\cup J$ then $X(K')(x)=\mathbf{0}$ for all $K\subset K'\subset I\cup J$. By definition of $S^0_2(0,T_xM,X_s)$, $S_2(0,T_xM,X_s)$ we would have $S^0_2(0,T_xM,X_s)\subset K'\subset S_2(0,T_xM,X_s)$ for all $K\subset K'\subset I\cup J$ so if $K\neq I\cup J$ $S^0_2(0,T_xM,X_s)\neq S_2(0,T_xM,X_s)$ which is a contradiction to condition (2) of regularity.

(iii) We can now show that regular zeros are isolated, and hence finite in number on our compact M, and that if all the zeros of X(M) are regular and if X is perturbed to

X' then for each regular zero x of X(M) in σ there exists a regular zero x' of $X'(M)$ in σ near x.

Consider the hyperbolic zeros of $X(\breve{\sigma})$ for each stratum σ of M. We know from [42] that these are isolated and hence finite in number on $\bar{\sigma}$. The regular zeros x of X(M) on $\bar{\sigma}$ are the subset of these satisfying (2), hence by (ii) are disjoint from $\partial\bar{\sigma}$ (because $X(\breve{\sigma})(x)=0$), by [42] again remain hyperbolic and move only slightly under perturbations in X, and so by (i) remain regular under perturbations in X.

(iv) To complete the proof of Proposition 6.2(1) it remains to show that there exists an open-dense subset of $\Xi_\infty(M)$ with all zeros regular. By [42] we know that for each stratum σ of M there exists an open dense subset of $\Xi_\infty(\breve{\sigma})$ with all zeros of $X(\breve{\sigma})$ hyperbolic, and hence by Lemma 4.4 an open dense subset of $\Xi_\infty(M)$ with all zeros of $X(\breve{\sigma})$ hyperbolic for every σ in M. For X in this subset each $X(\breve{\sigma})$ has only finitely many zeros, and we can perturb the vector field such that each satisfies (2) (possibly removing some entirely) as follows. Order the strata according to increasing codimension in M (and strata of the same codimension arbitrarily) and taking each in order we shall for $\breve{\sigma}$ locally represented as Z(K) add to X an arbitrarily small vector field $\Sigma_{i\in K\setminus I}\lambda_i \mathrm{grad} f_i$ which by Remarks 2.1 leaves X(K) and hence the location of the zeros of $X(\breve{\sigma})$ unaffected. Using condition (2) in the form of (2″) the result follows if we show that we may perturb X in this way so that at each zero x of X(K) and for all $I\subset K'\subset K$ $\langle X(K'\setminus j)(x),\mathrm{grad}f_j(x)\rangle\neq 0$ for all $j\in K'\setminus I$, and $\langle X(K')(x),\mathrm{grad}f_j(x)\rangle\neq 0$ for all $j\in K\setminus K'$, ie if for all $I\subset K'\subset K$ and for all $j\in K\setminus K'$ $\langle X(K')(x),\mathrm{grad}f_j(x)\rangle\neq 0$. If $\langle X(K')(x),\mathrm{grad}f_j(x)\rangle=0$ some $j\in K\setminus K'$
$\langle P(K')(X(x)+\Sigma_{i\in K\setminus I}\lambda_i(x)\mathrm{grad}f_i(x)),\mathrm{grad}f_j(x)\rangle=\langle\Sigma_{i\in K\setminus I}\lambda_i P(K')\mathrm{grad}f_i(x),\mathrm{grad}f_j(x)\rangle$ and choosing $\lambda_j\neq 0$ $\lambda_{j'}=0$ for all $j'\neq j$, the above$=\lambda_j\,|\,P(K')\mathrm{grad}f_j(x)\,|\,^2\neq 0$, and hence for arbitrarily small λ_j we can perturb X to $X'=X+\lambda_j\mathrm{grad}f_j$ so that $\langle X'(K')(x),\mathrm{grad}f_j(x)\rangle\neq 0$.

By (iii), a regular zero of X stays regular for all X' near X and hence repeating the above for all zeros of $X(\breve{\sigma})$ we may with perturbations of diminishing size perturb each to be regular leaving the regularity of those already perturbed unaffected (since here we're primarily interested in the smooth case we could alternatively have used bump functions). By (ii) all zeros of $X(\breve{\sigma})$ are then disjoint from $\partial\bar{\sigma}$, and hence we may treat lower dimensional strata in $\partial\bar{\sigma}$ in a similar way leaving the result for σ unaffected.

(2) The local invariant manifolds of a regular zero on a submanifold with orthogonal corners are C^1 submanifold with corners.

(a) We show that if z_0 is a regular zero of a submanifold with orthogonal corners locally represented as ZN(I;J) with $z_0 \in Z(I \cup J)$ then on some neighbourhood U of z_0

(i) $\langle X(y), P(I)\mathrm{grad}f_j(y)\rangle < 0$ for all $j \in J$ for all $y \in U$, and

(ii) For all $y \in ZP(K;J\backslash K) \cap U$ $X(M)(y) = X(K)(y)$, for all K such that $I \subset K \subset I \cup J$.

(i) implies (ii): At $y \in ZP(K;J\backslash K)$ $T_yM \cong LC(I;K\backslash I)$. Then by Lemma 2.4

$P(T_yM)X(y) = X(K)(y)$ iff $\langle X(K\backslash i)(y), \mathrm{grad}f_i(y)\rangle \leq 0$ for all $i \in K\backslash I$

but since the corner is orthogonal $P(K\backslash i)\mathrm{grad}f_i(y) = \mathrm{grad}f_i(y)$ and hence

$PT_yMX(y) = X(K)y$ iff $\langle \mathrm{grad}f_i(y), X(y)\rangle \leq 0$ for all $i \in K\backslash I$, which follows from (i) (we know $X(y) = X(I)(y)$ because X is on ZN(I;J)).

Proof of (i): By continuity it suffices to show that for all $j \in J$

$\langle X(z_0), P(I)\mathrm{grad}f_j(z_0)\rangle < 0$. By Remark 2.1 we have

$X(I)(z_0) - X(I \cup J)(z_0) \in \mathrm{span}\{P(I)\mathrm{grad}f_j(z_0) : j \in J\}$ and since $X(I \cup J)(z_0) = \mathbf{0}$

$X(I) \in \mathrm{span}\{P(I)\mathrm{grad}f_j(z_0) : j \in J\}$. By the Characterisation of Projection we have

$\langle X(I)(z_0) - P(I \cup J)X(z_0), P(I \cup J)X(z_0) - v\rangle \geq 0$ for all $v \in T_{z_0}M$.

$\langle X(I)(z_0) - P(I \cup J)X(z_0), P(I \cup J)X(z_0)\rangle = 0$ and since the corner is orthogonal

$P(I)\mathrm{grad}f_j(z_0) = \mathrm{grad}f_j(z_0) \in T_{z_0}M$ for all $j \in J$, hence $\langle X(I)z_0, P(I)\mathrm{grad}f_j(z_0)\rangle \leq 0$ for all $j \in J$ and equality for any $j \in J$ contradicts (2″) of the definition of regular zero, hence the result.

(b) A <u>partition</u> $(I_1,..,I_r)$ of $I, I \cup J$ is a sequence of subsets $I \subset I_r \subset I_{r-1} \subset .. \subset I_1 \subset I \cup J$. A <u>positive time sequence</u> on a partition $(I_1,..,I_r)$ is a sequence of r positive reals $(t(I_1),..,t(I_r))$. Call the set of all positive time sequences of all partitions of $I, I \cup J$ T(J). If $z_0 \in Z(I \cup J)$ is a regular zero of X(M) and if S is any subset of $Z(I \cup J) \cap U$ (where U is as in (a) above) and $S^* = \{y \in ZN(I;J) \cap U : \phi(M)(y) \cap S \neq \emptyset\}$ we show there exists a bijection between $T(J) \times S$ and S^*. First we show that for any $y \in S^*$ there exists a unique partition $(I_1,..,I_r)$ and sequence of positive times $t_1,..,t_r$ such that $y = \phi(I_r)..\phi(I_2)(\phi(I_1)(P(y), -t(I_1)), -t(I_2))..), -t(I_r))$ (which we shall abbreviate to $t(I_r)t(I_{r-1})..t(I_1)P(y)$) for some unique $P(y) \in S$. If $x \in ZP(K;J\backslash K)$ then since by (a)(ii) $X(M)(x') = X(K)(x')$ for all $x' \in ZP(K;J\backslash K)$ it follows that if $t_1 = \sup\{t > 0 : \phi(K)(x,t) \in ZP(K;J\backslash K)\}$ then on $[0,t_1)$ $X(K)\phi(K)(x,t) = X(M)\phi(K)(x,t)$ and hence by uniqueness of solutions $\phi(M)(x,t) = \phi(K)(x,t)$ on $[0,t_1)$, and since at t_1 we must have $\phi(K)(x,t) \in ZP(K';J\backslash K')$ with $K' \supset K$ we may repeat the argument and inductively it follows for any $x \in U$ there exists $K'' \supset .. K' \supset K$ with

$\phi(K'')(..\phi(K')(\phi(K)(x,t_1),t_1'),..),t_1'')\in Z(I\cup J)$ and hence that any $x\in S^*$ may be represented as a point in $T(J)\times S$.

Conversely, if $y=t(I_r)tI_{r-1})..t(I_1)P(y)$ some $P(y)\in S$ then as long as each $t(I_i)$ is small enough - say less than ϵ - we claim $y\in S^*$. Since $(0,\epsilon)$ is homeomorphic to $(0,\infty)$ it will follow that there exists a bijection between $T(J)\times S$ and S^*. Suppose $x_0\in S$ and inductively that for $s-1\geq 0$ $x_{s-1}=t(I_{s-1})..t(I_1)P(y)\in S^*$. Then $x_{s-1}\in ZN(I_s;J\backslash I_s)$ and by a(ii) again $X(I_s)\phi(I_s)(x_{s-1},-t)=X(M)\phi(I_s)(x_{s-1},-t)$ for all sufficiently small $t\geq 0$, say for all $0\leq t\leq\epsilon$. Hence setting $x_s=\phi(I_s)(x_{s-1},-t(I_s))$ for some $0<t(I_s)<\epsilon$ we have $\phi(I_s)(x_s,t)=\phi(M)(x_s,t)$ for all $0\leq t\leq t(I_s)$, and our claim follows by induction (illustrated in Figure 6.24 with $s=2$).

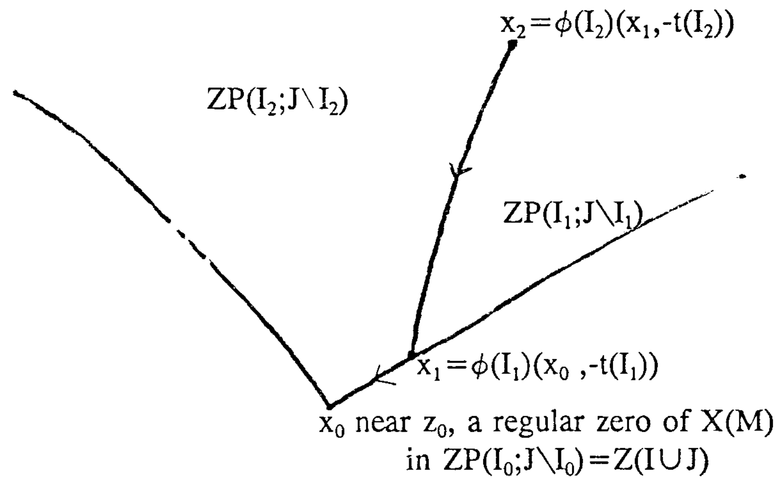


Figure 6.24

With this bijection in mind we shall for any point $x\in S^*$ set $\mathcal{P}(x)=$ the ordered set of sets of indices $(I_1,..,I_r)$, $t(x)=(t(I_1),..,t(I_r))$ and $P(x)=$projection along flow of x onto $S\subset Z(I\cup J)$.

(c) With $I_1\supset I_2\supset..\supset I_r$ as above we say $\mathcal{P}'=(I_1^1,..,I_1^{s_1},..,I_r^1,..,I_r^{s_r})$ <u>subdivides</u> $\mathcal{P}=(I_1,..,I_r)$ if $I\cup J\supset I_1^1\supset..\supset I_1^{s_1}=I_1\supset I_2^1\supset..\supset I_2^{s_2}=I_2\supset..\supset I_r^{s_r}=I_r$. If $\mathcal{P}(y_i)$ subdivides $\mathcal{P}(x)$ for $\{y_i\}$ a sequence of points converging to x we say $t(y_i)\to t(x)$ if as $i\to\infty$ each $t(I_i^{s_i})-t(I_i)\to 0$ and $t(I_i^j)\to 0$ for all $j<s_i$. We can then topologize $T(J)\times S$ by saying $(t(y),P(y))\to(t(x),P(x))$ ($\to$ in the sense of converges to) if $t(y)\to t(x)$ and $P(y)\to P(x)$. We claim that with this topology the bijection of (b) above is a homeomorphism. We shall use the following fact: that there exists $\delta>0$ such that for all y in the submanifold with orthogonal corners ZN(I;J) sufficiently near a regular zero $z_0\in Z(I\cup J)$

$\inf\{\,|\,X(K_1)(y)-X(K_2)(y)\,|\,:I\subset K_1\neq K_2\subset I\cup J\}>\delta$. This is so because

$X(ZN(I;K\backslash I))(z_0)=\lim_{y\to z_0,y\in Z(K)}X(ZN(I;K\backslash I))(y)$ (by [13])

$=\lim_{y\to z_0,y\in ZP(K;J\backslash K)}X(ZN(I;J))(y)$ any $I\subset K\subset I\cup J$, so by (a)(ii)

$X(ZN(I;K\backslash I))(z_0)=X(K)(z_0)$ any $I\subset K\subset I\cup J$. If $X(K_1)(z_0)=X(K_2)(z_0)$ with $K_1\neq K_2$ we would therefore have $X(ZN(I;K_1\backslash I))(z_0)=X(ZN(I;K_2\backslash I))(z_0)$ and both sides therefore equal $X(ZN(I;K_1\cup K_2\backslash I))(z_0)$ contrary to condition (2) of the definition of regular zero. Therefore $X(K_1)(z_0)\neq X(K_2)(z_0)$ for all $I\subset K_1,K_2\subset I\cup J$ and by continuity the result follows.

We have seen $x_0\in S$ (S as in (b) above) is mapped by $t(I_r)..t(I_1)$ to $t(I_r)..t(I_1)x_0=x_r\in ZP(I_r;J\backslash I_r)$ where each $t(I_i)..t(I_1)x_0\in ZP(I_i;J\backslash I_i)$. Suppose we show that for any $1\leq j<r$ and x_{j+1}' near x_{j+1} there exists x_j' near x_j and sequence $I_j\supset I_{j+1}^1\supset I_{j+1}^2\supset .\supset I_{j+1}^{s_{j+1}}=I_{j+1}$ such that $x_{j+1}'=t(I_{j+1}^{s_{j+1}})..t(I_{j+1}^1)x_j'$ (see Figure 6.25). Then inductively for any x_r' near x_r in $ZP(I_r;J\backslash I_r)$ we may find a sequence $\mathcal{P}'$ subdividing $\mathcal{P}(x)$ and x_0' close to x_0 and $t(x_0')$ close to $t(x_0)$ such that x_r' corresponds to $(t(x_0'),x_0')$.

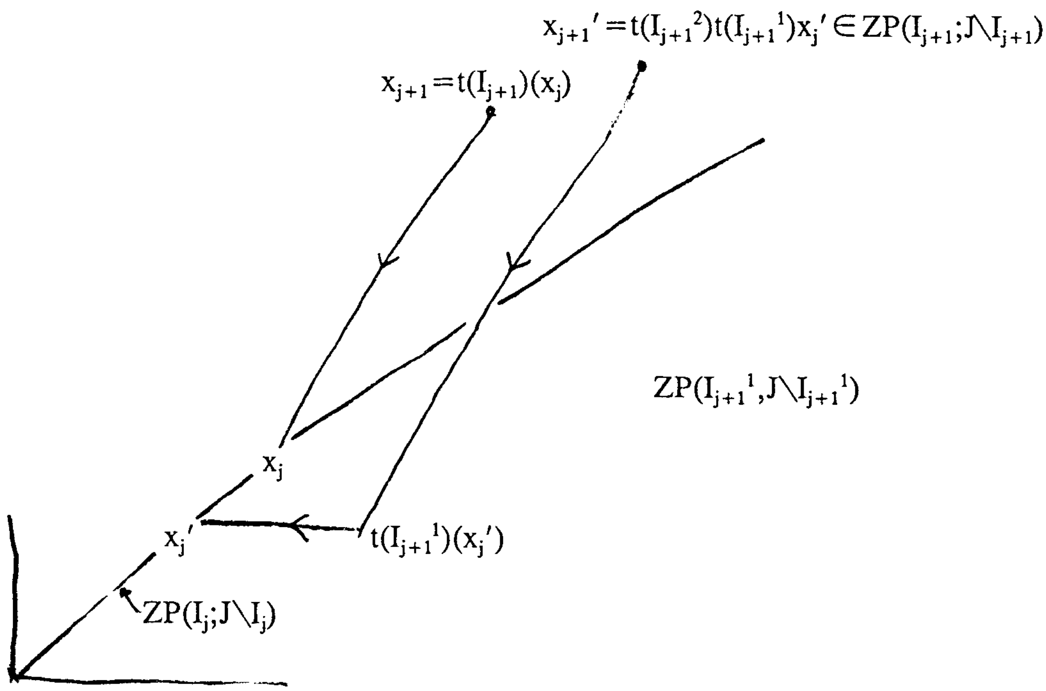


Figure 6.25

By continuous dependence on initial conditions we know that as $x_{j+1}'\rightarrow x_{j+1}$ with $x_{j+1}'\in ZP(I_{j+1};J\backslash I_{j+1})$ we have $\phi(M)(x_{j+1}',t)\rightarrow\phi(M)(x_{j+1},t)$ for all $0\leq t\leq t(I_{j+1})$. The only strata intersecting the region $\{\phi(M)(y,t'):y\in ZP(I_{j+1};J\backslash I_{j+1})$ with y near x_{j+1}, $0\leq t'\leq t\}$ are $ZP(K;J\backslash K)$ with $I_{j+1}\subset K\subset I_j$ and since by the same argument as in (b) the only transitions $\phi(M)(y,t)$ for increasing t can make are those into strata of decreasing dimension we must have $x_{j+1}'=t(I_{j+1}^{s_{j+1}})..t(I_{j+1}^{1})x_j'$ for some $I_j \supset I_{j+1}^1 \supset I_{j+1}^2 \supset .. \supset I_{j+1}^{s_{j+1}}=I_{j+1}$: furthermore since $x_{j+1}=t(I_{j+1})x_j$, if for for any k with $1 \le k < s_{j+1}$ $t(I_{j+1}^k)$ does not tend to 0 as $x_j' \to x_j$ we would have (since by the above $X(I_{j+1}^k)$ is bounded away from $X(I_{j+1})$) that x_{j+1}' does not tend to x_{j+1}, and hence as $x_{j+1}' \to x_{j+1}$ each $t(I_{j+1}^k) \to 0$ $1 \le k \le s_{j+1}$, and so $t(I_{j+1}^{s_{j+1}}) \to t(I_{j+1})$.

(d) It follows from (c) that if z_0^a, z_0^b are regular zeros of submanifolds with orthogonal corners M^a, M^b locally represented as $ZN(I_a;J_a)$, $ZN(I_b;J_b)$ respectively with $|I_a| = |I_b|$, $|J_a| = |J_b|$, then if S_a, S_b are homeomorphic submanifolds of $Z(I_a \cup J_a)$, $Z(I_b \cup J_b)$, then S^*_a, S^*_b are homeomorphic. In particular therefore $W_s^s(z_0)$, $W_s^u(z_0)$ and $W_s(z_0)$ are homeomorphic to
$T_{z_0}W^s(z_0) \times \mathrm{span}\{\mathrm{grad}f_i(z_0):i \in J\}$, $T_{z_0}W^u(z_0) \times \mathrm{span}\{\mathrm{grad}f_i(z_0):i \in J\}$ and
$\mathrm{span}\{\mathrm{grad}f_i(z_0):i \in J\}$ respectively.

Finally we show these local invariant manifolds are C^1.

We have for a point $x_r \in ZP(I_r;J \backslash I_r)$ in S^* a sequence $x_0 \to x_1 \to .. \to x_r$ with each $x_i \in ZP(I_i;J \backslash I_i)$, $I_0 = I \cup J \supset I_1 \supset .. \supset I_r \supset I$, with $x_0 \in S \subset Z(I \cup J)$. We shall denote a neighbourhood of x_i in $S^* \cap ZP(I_i;J \backslash I_i)$ by $S(I_i)$. Since $S(I_i) \subset S^*$ each $S(I_i)$ is invariant by $X(M) \mid ZP(I_i;J \backslash I_i)$, and since $X(M)(x)=X(I_i)(x)$ for all $x \in ZP(I_i;J \backslash I_i) \cap U$ (by a(ii) of course) this means that $X(I_i)(x) \in T_xS(I_i)$ for all $x \in S(I_i)$. We show by induction on i that each $S^*(I_i)$ is a C^1 submanifold if $S \supset S(I_0)$ is.

Suppose this is true up to I_i. Setting $\phi(K)(y,-t(K))=t(K)y$ we have by (b,c) that $S(I_{i+1}) = \bigcup_{I_i \supset I_{i+1}^1 \supset .. \supset I_{i+1}^j \supset I_{i+1}} \bigcup t'(I_{i+1})t(I_{i+1}^j)..t(I_{i+1}^1)S(I_i)$ where the inner union is taken over $t'(I_{i+1})$ near $t(I_{i+1})$ and $t(I_{i+1}^{j'})$, $1 \le j' \le j$, small and positive. We see that the outer union consists of $2^{|I_i \backslash I_{i+1}|}-1$ C^r manifolds (one for each subdivision of $I_i \supset .. \supset I_{i+1}$) and we must show that their tangent spaces where they meet at $x_{i+1} \in t(I_{i+1})S(I_i)$ are all contained in a single space which is $|I_i \backslash I_{i+1}|$ dimensions larger than $T_{x_i}S(I_i)$.

For example, if $I_i=(1,2)$ and $I_{i+1}=\varnothing$ (Figure 6.26)
the outer union consists of $2^2-1=3$ manifolds

$\{t'(\varnothing)t(1)S(1,2):t'(\varnothing)$ is near $t(\varnothing)$ and $t(1)$ is small and positive$\}$

$\{t'(\varnothing)t(2)S(1,2):t'(\varnothing)$ is near $t(\varnothing)$ and $t(2)$ is small and positive$\}$

$\{t'(\varnothing)S(1,2):t'(\varnothing)$ is near $t(\varnothing)$ $\}$

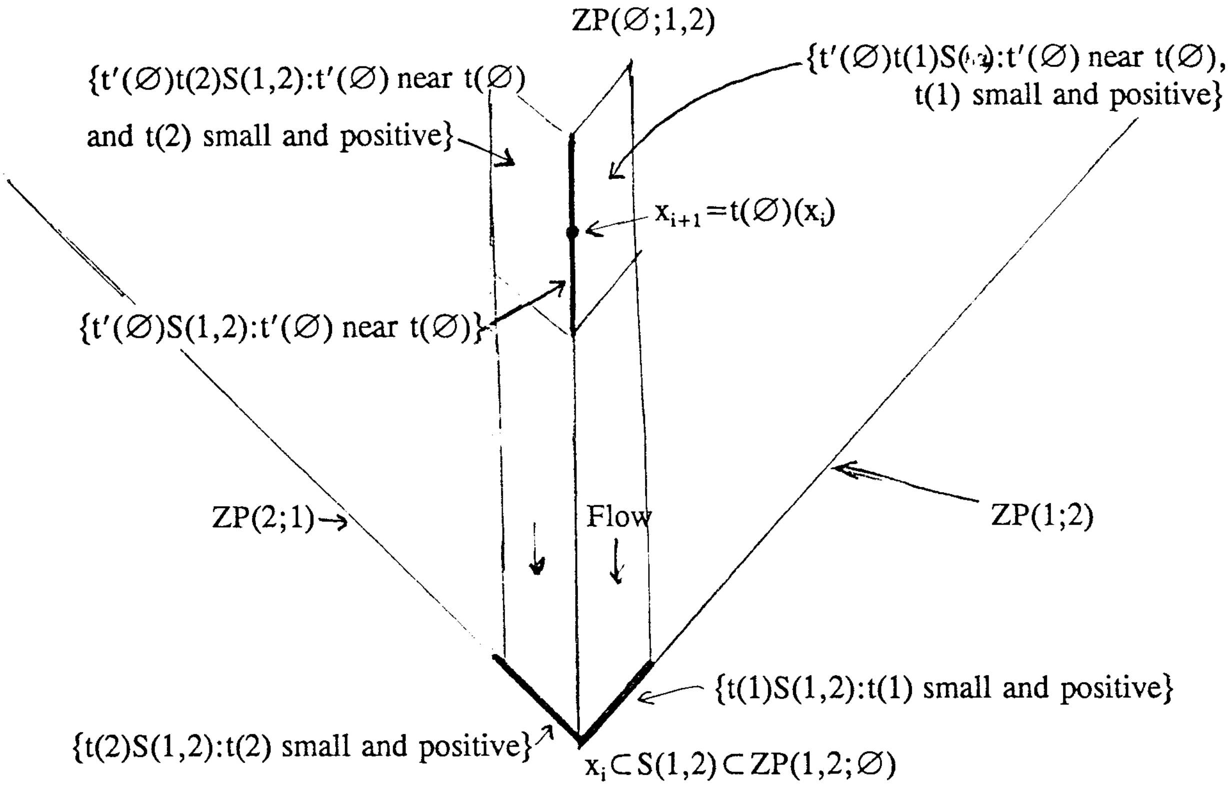


Figure 6.26

we must show that the tangent spaces at x_{i+1} of these manifolds are contained in a single $(\dim S(1,2)+2)$-dimensional space.

(End of example)

Taking a member of the outer union (ie the inner union with the variables varying in the outer union held fixed) $\bigcup_{t'(I_{i+1})\text{ near }t(I_{i+1}),\ t(I_{i+1}^{j'})\text{ small and positive each }1\leq j'\leq j} t'(I_{i+1})t(I_{i+1}^{j})..t(I_{i+1}^{1})S(I_i)$, and letting $x_i' \to x_i$ (so each $t(I_{i+1}^{j'}) \to 0$) we obtain a tangent space at x_{i+1} $\phi(I_{i+1})(-t(I_{i+1})).\mathrm{span}\,\{T_{x_i}S(I_i), X(I_{i+1}^{1})(x_i),..,X(I_{i+1}^{j})(x_i)\}$, (where $\phi(I_{i+1})(-t)$ is defined by $\phi(I_{i+1})(-t)(x)=\phi(I_{i+1})(x,-t)$). Since $S(I_i)$ is invariant we have $X(I_i)(x_i)\in T_{x_i}S(I_i)$ and hence since by Remarks 2.1 $X(I_{i+1}^{j'})(x_i)-X(I_{i+1})(x_i)\in$ $\mathrm{span}\{P(I_{i+1})\mathrm{grad}f_j(x_i):j\in I_{i+1}^{j'}\backslash I_{i+1}\}\subset \mathrm{span}\{P(I_{i+1})\mathrm{grad}f_j(x_i):j\in I_i\backslash I_{i+1}\}$ and $X(I_i)(x_i)-X(I_{i+1})x_i\in \mathrm{span}\{P(I_{i+1})\mathrm{grad}f_j(x_i):j\in I_i\backslash I_{i+1}\}$ we must have $X(I_{i+1}^{j'})(x_i)\in \mathrm{span}\{T_{x_i}S(I_i),P(I_{i+1})\mathrm{grad}f_j(x_i):j\in I_i\backslash I_{i+1}\}$ for all $j'\leq j$. Hence each tangent space is contained in $\phi(I_{i+1})(-t(I_{i+1})).\mathrm{span}\{T_{x_i}S(I_i),P(I_{i+1})\mathrm{grad}f_j(x_i):j\in I_i\backslash I_{i+1}\}$ and the result follows. ∎

Chapter Seven

Polynomial Systems are Generically Locally Stable

We established in Lemma 6.1 and Example 6.6(1) necessary conditions for two semiflows to be spfp equivalent, and used these results in Examples 6.4, 6.6(2) and 6.7 to exhibit semiflows which could not even locally be straightened out (in the sense of establishing a spfp homeomorphism between a local flow near x and the straightening out at x) or linearized, and these examples were not atypical and the phenomena concerned could not be perturbed away. The impression may have begun to form that establishing a spfp homeomorphism between two systems is rarely possible in circumstances of interest. We show that in one important context the opposite is true: we will show that polynomial systems on polyhedra with orthogonal corners are generically locally spfp stable at points x where the projected vector field $X(M)(x) \neq 0$. The original inspiration for working in this context was that the biological systems considered in [60] are of this form.

M is a polyhedron with orthogonal corners means M is a connected subset of R^n locally of the form $\{x \in R^n : \langle x,n_i\rangle = a_i \text{ for all } i \in I, \langle x,n_i\rangle \geq a_i \text{ for all } i \in J\}$, where $\{a_i\}_{i \in I \cup J}$ is a set of reals and $\{n_i\}$ is a set of independent vectors in R^n such that $\langle n_i,n_j\rangle = 0$ if $i \neq j$.

Figure 7.1. A polyhedron with orthogonal corners

Without much loss of generality we may suppose M is codimension 0 in R^n (ie that $I = \emptyset$) and then since any polynomial vector field is globally determined by its value on any open set we have $\Xi_{\omega,r}(M) = \Xi_{\omega,r}(R^n)$. We recall from Chapter 4 that $\Xi_{\omega,r}'(M)$ is that open (by Proposition 4.2) subset of $\Xi_{\omega,r}(M)$ satisfying conditions concerning the relation between the flows of $X(\breve{\sigma})$ and $X(\breve{\sigma}')$ for strata σ,σ' in M with $\overline{\sigma}' \cap \overline{\sigma} \neq \emptyset$, so (unless $M = R^n$) $\Xi_{\omega,r}'(M)$ is a strict subset of $\Xi_{\omega,r}'(R^n) = \Xi_{\omega,r}(R^n)$.

We recall from Chapter 6 that the system (M,X) is spfp stable at $x\in M$ if there exists a neighbourhood U_X of X in $\Xi_{\omega,r}(M)$ and U_x of x in M such that for any $X'\in U_X$ there exists a stratum preserving homeomorphism $h:U_x\to U_{h(x)}'$ = a neighbourhood of h(x) in M such that for each $y\in U_x$ $h\phi(M,X)(y,t)=\phi(M,X')(h(y),\tau(t))$ some continuous strictly increasing $\tau:[0,T(U_x,y))\to[0,T(U_x',h(y)))$, where (as in Chapter Five) $T(U_x,y)$ is the time for which $\phi(M,X)(y,t)$ remains in U_x.

Proposition 7.1 If M is a polyhedron with orthogonal corners in R^n then there exists
(1) a residual subset $\Xi_{\omega,1}''(M)$ of $\Xi_{\omega,1}(M)$, and
(2) if $r\geq n$ and M is compact a residual subset $\Xi_{\omega,r}''(M)$ of $\Xi_{\omega,r}(M)$,
such that the semiflow $\phi(M,X)$ is spfp stable at each $x\in M\backslash\{x\in M:X(M)(x)=\mathbf{0}\}$, and in either case we may take τ=identity.
The proof is after Lemma 7.2.

Throughout this chapter M is taken to be a polyhedron with orthogonal corners of and codim 0 in R^n. If then $\dim(\Xi_{\omega,r}(M))=p$ (a notational convention for this chapter) the space of r-polynomial systems on M, $M\times\Xi_{\omega,r}(M)$, is a finite dimensional polyhedron with orthogonal corners of R^{n+p}, each leaf $M\times\{X\}$ of which represents an individual system. We shall make the following definitions:
$X^p:R^n\times\Xi_{\omega,r}(M)\to R^n\times\Xi_{\omega,r}(M)$ is the r-polynomial vector field on $R^n\times\Xi_{\omega,r}(M)$ (ie, $X^p\in\Xi_{\omega,r}(R^n\times\Xi_{\omega,r}(M)))$ defined by $X^p(x,X)=(X(x),0)$. Since $M\times\Xi_{\omega,r}(M)$ is a submanifold with corners we may also define $X^p(M\times\Xi_{\omega,r}(M))$=projection of X^p onto $M\times\Xi_{\omega,r}(M)$, defined in the same way as for X(M), ie
$X^p(M\times\Xi_{\omega,r}(M))(x,X)=PT_{(x,X)}(M\times\Xi_{\omega,r}(M))X^p(x,X)$, which since $T_xM\times\mathbf{0},\mathbf{0}\times T_X\Xi_{\omega,r}(M)$ are orthogonal in R^{n+p} equals $(P(T_xM)X(x),0)=(X(M)(x),0)$. Thus if we define correspondingly $\phi^p:R^n\times\Xi_{\omega,r}(M)\to R^n\times\Xi_{\omega,r}(M)$ to be the integral flow of X^p, and $\phi^p(M\times\Xi_{\omega,r}(M))$ to be the integral flow of $X^p(M\times\Xi_{\omega,r}(M))$, then by the above we have $\phi^p(M\times\Xi_{\omega,r}(M))((x,X),t)=(\phi(M,X)(x,t),X)$ and furthermore setting $Z^pN^pP^p(I;J;K)$ = $\{(x,X)\in R^{n+p}:x\in ZNP(I;J;K)\}=ZNP(I;J;K)\times\Xi_{\omega,r}(M)$ we obtain
$\phi^p(K)((x,X),t)=\phi^p(Z^p(K))((x,X),t)=(\phi_X(K)(x,t),X)$ (where the suffix X in ϕ_X designates the vector field being projected onto Z(K) to yield $\phi(K)$) etc.
If σ_i are strata of M, $\sigma_i\times\Xi_{\omega,r}(M)$ are strata of the polyhedron $M\times\Xi_{\omega,r}(M)$, and if $\breve{\sigma}_i$ are the affine spans of $\breve{\sigma}_i$, $\breve{\sigma}_i\times\Xi_{\omega,r}(M)$ are the affine spans of $\sigma_i\times\Xi_{\omega,r}(M)$. We have
$\Gamma_k^{X^p}(\breve{\sigma}_1\times\Xi_{\omega,r}(M)\ r\ \breve{\sigma}_2\times\Xi_{\omega,r}(M))=$

$\{(y,Y)\in M\times\Xi_{\omega,r}(M):D_t^i\phi^p(\breve{\sigma}_1\times\Xi_{\omega,r}(M))((y,Y),t=0)=D_t^i\phi^p(\breve{\sigma}_2\times\Xi_{\omega,r}(M))((y,Y),t=0)$ for all $i<k\}$ which by the above equals

$\{(y,Y)\in M\times\Xi_{\omega,r}(M):D_t^i\phi_Y(\breve{\sigma}_1)(y,t=0)=D_t^i\phi_Y(\breve{\sigma}_2)(y,t=0)$ for all $i<k\}$ and so

$\Gamma_k^X(\breve{\sigma}_1 \text{ r } \breve{\sigma}_2) = \Gamma_k^{X^p}(\breve{\sigma}_1\times\Xi_{\omega,r}(M) \text{ r } \breve{\sigma}_2\times\Xi_{\omega,r}(M))\cap\Pi_X$ where

$\Pi_X=\{(x,Y)\in R^n\times\Xi_{\omega,r}(M):Y=X\}$. In particular if $X\in\Xi_{\omega,r}'(M)$, so by Proposition 4.2 for all X' near X $\Gamma_k^{X'}(\breve{\sigma}_1 \text{ r } \breve{\sigma}_2)=\varnothing$ for all but finitely many k,σ_1,σ_2, it follows that if V^p is a neighbourhood of (x,X) in $M\times\Xi_{\omega,r}(M)$ then

$\Gamma_k^{X^p}(\breve{\sigma}_1\times\Xi_{\omega,r}(M) \text{ r } \breve{\sigma}_2\times\Xi_{\omega,r}(M))\cap V^p=\varnothing$ for all but finitely many k,σ_1,σ_2, so there are no infinite order tangencies on V^p and if $X^p(M\times\Xi_{\omega,r}(M))(x,X)\neq\mathbf{0}$ we can apply Corollary 5.1 (with M,X,x,V in Corollary 5.1 set to $M\times\Xi_{\omega,r}(M)$, X^p, (x,X), V^p) to infer that if V^p is small enough there exists N with

$|\ \phi^p(M\times\Xi_{\omega,r}(M))((y,Y),[0,T(V^p,(y,Y))))\ |\ \leq N$ for all $(y,Y)\in V^p$.

We recall from Chapter Three the idea of the funnel at x about the trajectory $\phi(S^0_\infty(x))(x)$.

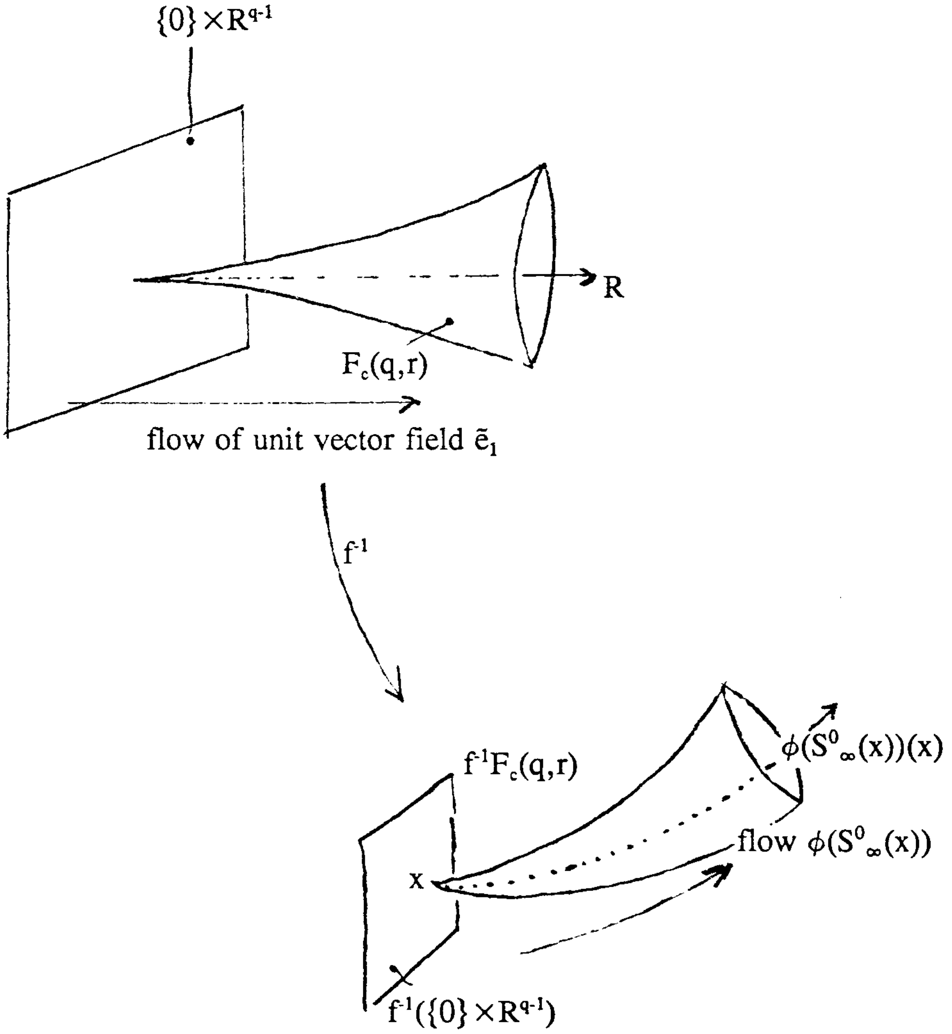


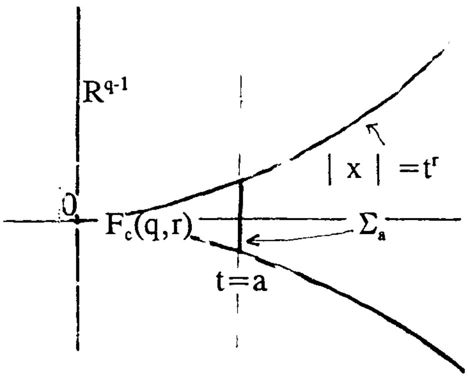
Figure 7.2

In the context of funnels f is a C^r straightening-out map between the flow $\phi(S^0{}_\infty(x))$ on $Z(S^0{}_\infty(x))$ and the unit flow on R^q, where as in Chapter Three q is the dimension of $Z(S^0{}_\infty(x))$, ie there exists a q-1 dimensional section transverse to $X(S^0{}_\infty(x))$ which f maps C^r-diffeomorphically to $\{0\}\times R^{q-1}\subset R^q$, and for each point y near x and t small $\phi(S^0{}_\infty(x))(y,t)$ is mapped by f to $f(y)+te_1$. The canonical (q,r) funnel was a set $F_c(q,r)=\{(t,x)\in R^1\times R^{q-1}:t\geq 0,\ |x|\leq t^r\}$.

A funnel for $\phi(S^0{}_\infty(x))(x)$ was a set $f^{-1}F_c(q,r)$ where r was determined by $\Im_X(x)$. By virtue of Lemma 3.2 and Theorem 1.1(3) (continuous dependence on initial conditions) any y near x enters $f^{-1}F_c$ near x. In summary, for the funnels of Chapter Three, which we denote by $f^{-1}F_c(q,r)$, the diffeomorphism f^{-1} mapped

(i) $0\rightarrow x$
(ii) $R^q\rightarrow Z(S^0{}_\infty(x))$
(iii) The unit vector field $\tilde{e}_1$ on $R^q\rightarrow$the vector field $X(S^0{}_\infty(x))$
(iv) The unit flow ψ where $\psi(y,t)=y+te_1\rightarrow$ the flow $\phi(S^0{}_\infty(x))$
(v) $\{0\}\times R^{q-1}\rightarrow$a section in $Z(S^0{}_\infty(x))$ through x transverse to $X(S^0{}_\infty(x))$
(vi) The canonical funnel $F_c(q,r)\rightarrow f^{-1}F_c(q,r)$
(see Figure 7.2)

It follows from Lemma 3.2 that if we set $\Sigma_a=\{(t,x)\in F_c(q,r):t=a\}$ then for any $a>0$ sufficiently small there exists a neighbourhood of x in M such that if $\phi(M)(x,T)\in f^{-1}\Sigma_a$, where f is the funnel map, then the trajectory based at any point y in the neighbourhood has a unique point of intersection $\phi(M)(y,t)$ with $f^{-1}\Sigma_a$ near $\phi(M)(x,T)$ with t near T. $f^{-1}\Sigma_a$ will be called a funnel cross-section (see Figure 7.3).



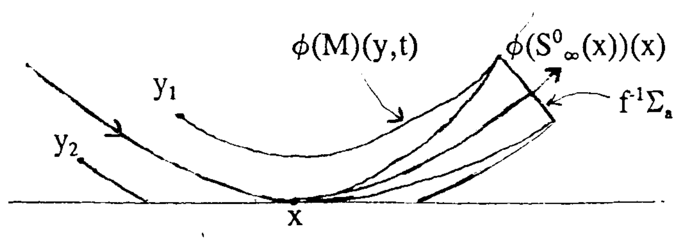


Figure 7.3

The point of making these constructions is that since $\phi^p(M\times\Xi_{\omega,r}(M))((x,X),t)=(\phi(M,X)(x,t),X)$, if we set $F_c{}^p=F_c(q+p,r)$ (q+p is the dimension of $Z^p(S^0{}_\infty(x,X))$), where r is determined by $\Im_{X^p}(x,X)$, $\Sigma_a{}^p=\{(t,x)\in F_c{}^p:t=a\}$ and f^p the straightening-out map between the flow $\phi^p(S^0_\infty(x,X))$ and the unit flow on R^{q+p}, then for any (y,Y) near (x,X) $\phi^p(M\times\Xi_{\omega,r}(M))(y,Y)$ has a unique point of intersection with $(f^p)^{-1}\Sigma_a^p$ (see Figure 7.4).

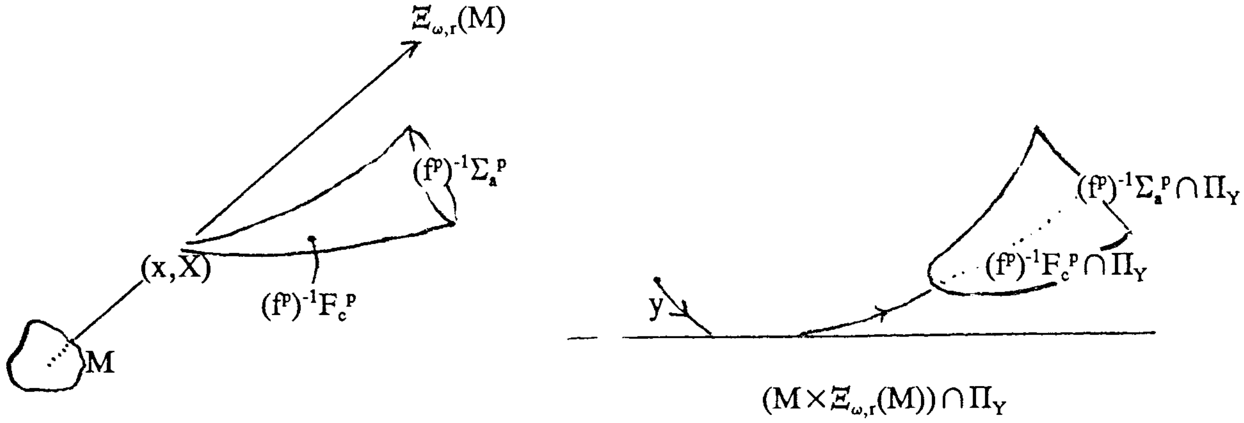


Figure 7.4

Before proceeding further we sketch the strategy for proving Proposition 7.1. Suppose X is an r-polynomial vector field and x∈M. We want to show that if $X(M)(x\,)\neq\mathbf{0}$ then we can find a r-polynomial vector field X′ arbitrarily close to X such that for all X″ near X′ there exists a spfp homeomorphism from a neighbourhood of x to itself conjugating the flows of ϕ(M,X′) and ϕ(M,X″). We have already established a necessary condition for a homeomorphism to be spfp: in Chapter Six we showed how to chop M up into subsets consisting of points which are "equivalent" insofar as the trajectories and invariant curves through them make the same sequence of stratum intersections, and we saw in Lemma 6.1 that any spfp homeomorphism must preserve these subsets.

We shall construct a special stratification (a ϕ(M,X)-compatible stratification, defined below) of a neighbourhood of x in M with the property (this is part of the proof of Proposition 7.1) that if a pair of systems (M,X) and (M′,X′) have compatible stratifications $\mathfrak{S}_1(U,X)$ and $\mathfrak{S}_1(U',X')$ of open subsets U,U′ of M,M′ and if there exists a stratum preserving homeomorphism between $\mathfrak{S}_1(U,X)$ and $\mathfrak{S}_1(U',X')$ then there exists a spfp homeomorphism between ϕ(M,X) on U and ϕ(M′,X′) on U′. In order to obtain a stratum preserving homeomorphism between ϕ(M,X)- and ϕ(M,X′)-compatible stratifications of a neighbourhood of x in M, X′ near X, we consider the total space of all r-polynomial systems on M, $M\times\Xi_{\omega,r}(M)$, of which each system is a leaf, M×{X}. In Lemma 7.1 we show that if $X\in\Xi_{\omega,r}'(M)$ and $X(M)(x)\neq\mathbf{0}$ then there exists a neighbourhood U^p of (x,X) in $M\times\Xi_{\omega,r}(M)$ for which a $\phi^p(M\times\Xi_{\omega,r}(M),X^p)$- compatible stratification $\mathfrak{C}_1(U^p,X^p)$ exists (Corollary 5.1 and the fact that subanalytic sets admit Whitney regular stratifications is used heavily in this) and furthermore that for all X′ near X $\Pi_{X'} \cap \mathfrak{C}_1(U^p,X^p)$ is a $\phi(M,X')$-compatible stratification of $U=\Pi_{X'} \cap U^p$. In Lemma 7.2 we show that if $\Pi_X \pitchfork \mathfrak{C}_1(U^p,X^p)$ then for all X′ near X there exists a stratum preserving homeomorphism between $\mathfrak{C}_1(U^p,X^p) \cap \Pi_X$ and $\mathfrak{C}_1(U^p,X^p) \cap \Pi_{X'}$; it only remains (the other part of the proof of Proposition 7.1) to show that there exists a covering of $M \times \Xi_{\omega,r}(M) \backslash \{(y,Y):Y(M)(y)=0\}$ by such open sets U^p such that for any X in a residual subset of $\Xi_{\omega,r}(M)$, $\Pi_X \pitchfork \mathfrak{C}_1(U^p,X^p)$ for all U^p. A couple of details concerned with quantities yet to be defined have been omitted from this sketch. (The feature of funnels that we use is that by Lemma 3.2, and continuing with the above notation, if the quantity a is small then every point in a sufficiently small neighbourhood of a point $(x,X) \in M \times \Xi_{\omega,r}(M)$ has a unique point of intersection with the funnel cross section $(f^p)^{-1}\Sigma_a{}^p$).

Suppose M is a submanifold with corners of R^n, X a C^r vector field on M.

Definitions

(a) If $\mathfrak{C}$ is a stratification of a subset M of R^n and Y is a vector field (not necessarily continuous) on M we shall say a stratum s of $\mathfrak{C}$ is

(I) Of type I if for all $x \in s$ $Y(x) \in T_x s$, ie Y is tangent to s

(II) Of type II if for all $x \in s$ $Y(x) \notin T_x s$ ie Y nowhere tangent to s.

These of course are extreme possibilities, although it will be a feature of our constuctions that all the strata in our flow compatible stratifications are of one or other type.

We shall denote the strata as m_1, m_2,.. and if we wish to denote the dimension this is done with a superfix (ie $m_1{}^{r_1}$, etc). "r-dimensional stratum" is abbreviated to r-stratum.

(b) If $U \subset M$ a <u>$\phi(M,X)$- compatible stratification</u> $\mathfrak{C}_1(U,X)$ of U is a Whitney-regular (see [59]) stratification of U refining the stratification of M | U as a submanifold with corners satisfying

1. On each stratum m_i of $\mathfrak{C}_1(U,X)$ the map $x \to X(M)(x)$ is C^r
2. Every stratum in $\mathfrak{C}_1(U,X)$ is of type I or type II
3. For each r-stratum $m_i{}^r$ of $\mathfrak{C}_1(U,X)$ there exists a continuous map $m_i:[-1,1]^r=\bar{I}^r \to R^n$ which is an analytic diffeomorphism on $I^r=(-1,1)^r$ with $m_i{}^r=m_i(I^r)$ and $\bar{m}_i{}^r=m_i(\bar{I}^r)$

(Figure 7.5 below)*.

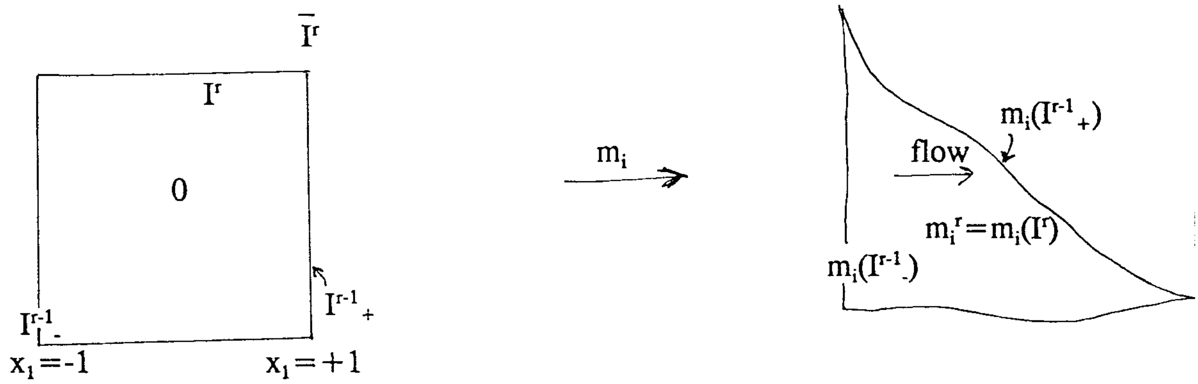


Figure 7.5

Furthermore if m_i^r is of type I (so $\phi(M)=$ the C^r flow $\phi(m_i^r)$ on m_i^r), if we then set $I^{r-1}_\pm=\{x\in R^r: x_1=\pm 1,\ |x_j|<1 \text{ for all } j=2,..,r\}$, then for each $x\in m_i^r$

(i) The quantities $t_\alpha(x)=\inf\{t\in R: \phi(m_i^r)(x,s)\in m_i^r \text{ for all } t<s\leq 0\}$ and $t_\omega(x)=\sup\{t\in R: \phi(m_i^r)(x,s)\in m_i^r \text{ for all } 0\leq s<t\}$ exist, are finite, and are continuous in $x\in m_i^r$, and there exist unique (r-1)-strata, which may be taken to be $m_i(I^{r-1}_+)$, $m_i(I^{r-1}_-)$, such that for all $x\in m_i^r$ the projections along the flow

$\alpha(x)=\lim_{s\to t_\alpha(x)}\phi(m_i^r)(x,s)$ is contained in $m_i(I^{r-1}_-)$

$\omega(x)=\lim_{s\to t_\omega(x)}\phi(m_i^r)(x,s)=\phi(M)(x,t_\omega(x))$ is contained in $m_i(I^{r-1}_+)$

and these maps are continuous in $x\in m_i^r$). All strata in $m_i(\bar{I}^r\backslash\bar{I}^{r-1}_\pm)$ are of type I and in $m_i(\bar{I}^{r-1}_-\backslash\bar{I}^{r-1}_+)$ are of type II, the vector field $X(M)$ is continuous on $m_i(I^r\cup I^{r-1}_-)$ and has a continuous extension to $\bar{m}_i^r$.

(ii) The flow $\phi(M)$ induces a homeomorphism (called H in part (3) of the proof of Proposition 7.1) between the set $m_i(I^r\cup I^{r-1}_-\cup I^{r-1}_+)$ and $m_i(I^{r-1}_+)\times\bar{I}$ (and hence with $I^{r-1}\times\bar{I}$) by $x\to(\omega(x),t_\omega(X)/(t_\omega(x)-t_\alpha(x)))$ (Figure 7.6)

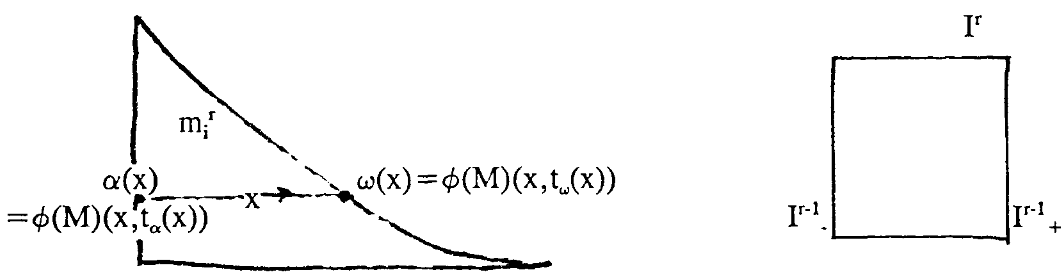


Figure 7.6

* We observe that $I,J,I_i,..$ are sets of indices while $I^r, I^{r-1}, \bar{I}^r, \bar{I}^{r-1}, I^{r-1}_\pm, \bar{I}^{r-1}_\pm, \bar{I}$ are subsets of Euclidean space

4. There exists a C^r submanifold with corners Σ and integer N such that for each point $y \in U$ $\phi(M)(y) \mid U$ has a unique point of intersection with Σ, and $\phi(M)(y) \mid U$ decomposes into $\le N$ C^r segments
$\phi(M)(y,[0,t_1)) \cup \phi(M)(y,[t_1,t_2)) \cup .. \cup \phi(M)(y,[t_s,t_{s+1}))$ where $\phi(M)(y,[t_i,t_{i+1}))$ is contained in a single stratum of $\mathfrak{C}_1(U,X)$.

If $y \in$ stratum of type II (type I) the sequence of strata $\phi(M)(y)$ occupies is (after renumbering the strata) of the form $(m_1^k), m_2^{k+1}, m_3^k, m_4^{k+1}, .., m_{r(1)}^k, m_{r(1)+1}^{k-1}, m_{r(1)+2}^k,$ $m_{r(1)+3}^{k-1}, .., m_{r(2)}^{k-1}, m_{r(2)+1}^{k-2}, m_{r(2)+3}^{k-1}, .., m_{r(j-1)}^{k-(j-1)}, m_{r(j-1)+1}^{k-j}, .., m_{r(j)}^{k-j} \in \Sigma$ where $r(j) \le 2N$, each $m_i^{k-j'}$ with $i < r(j')$ is of type I, each $m_i^{k-j'}$ with $i \ge r(j')$ of type II, and where m_1^k is mapped by the flow $\phi(M) \mid \bar{m}_2^{k+1}$ to m_3^k, by $\phi(M) \mid \bar{m}_4^{k+1}$ to m_5^k etc, until reaching $m_{r(1)}^k$ itself part of a similar sequence of k-1 and k strata.

Eg.

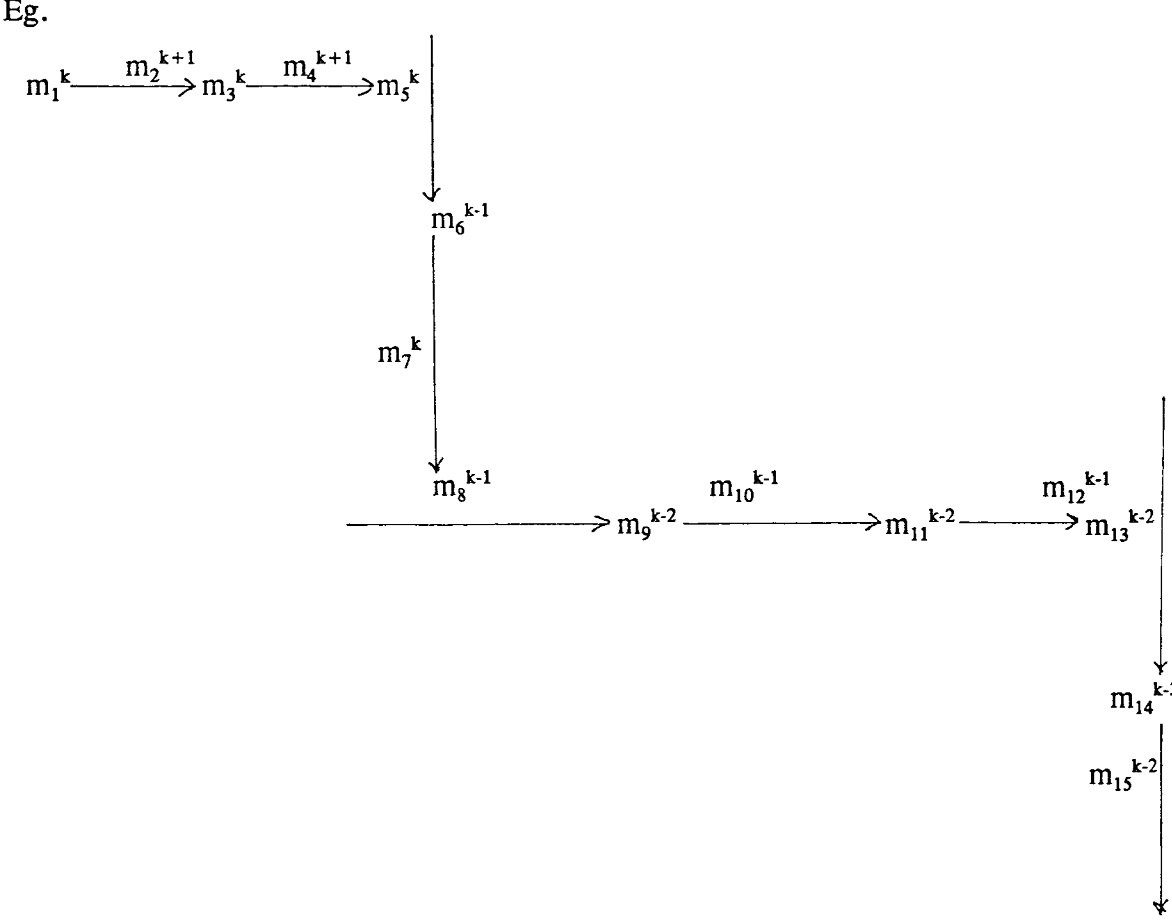


Figure 7.7a. $m_1^k, m_3^k, m_6^{k-1}, m_9^{k-2}, m_{11}^{k-2}$, are m_{14}^{k-3} of type II, the remaining strata are of type I - if k=1 the flow $\phi(M)$ relates to the strata in the first part the sequence as shown in a schematic way in Figure 7.7b

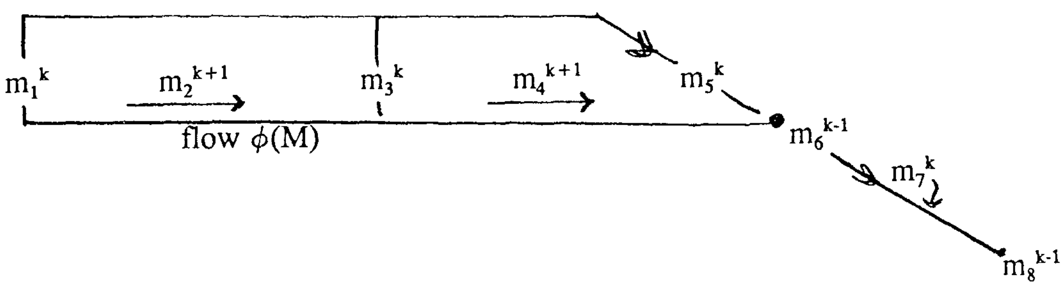


Figure 7.7b

An example of a $\phi(M,X)$-compatible stratification is given in Figure 7.14a.

We shall now (Lemma 7.1) construct a $\phi^p(M\times\Xi_{\omega,r}(M),X^p)$-compatible stratification $\mathfrak{S}_1(U^p,X^p)$ of $U^p\subset M\times\Xi_{\omega,r}(M)$ with special properties (in particular that each slice $\pi_{X'}\cap\mathfrak{S}_1(U^p,X^p)$ for X' near X is a $\phi(M,X')$-compatible stratification).

Lemma 7.1 If $x\in M$, $X\in\Xi_{\omega,r}'(M)$ and $X(M)(x)\neq\mathbf{0}$ there exists a neighbourhood U_X of X in $\Xi_{\omega,r}(M)$ and U^p of (x,X) in $M\times\Xi_{\omega,r}(M)$ for which there exists an analytic $\phi^p(M\times\Xi_{\omega,r}(M),X^p)$-compatible stratification $\mathfrak{S}_1(U^p,X^p)$ of U^p such that for any $X'\in U_X$, $\mathfrak{S}_1(U^p,X^p)\cap\Pi_{X'}$ is an analytic $\phi(M,X')$-compatible stratification of $U=U^p\cap\Pi_{X'}$, and the type (ie, I or II) of each stratum $m\in\mathfrak{S}_1(U^p,X^p)$ is the type of $m\cap\Pi_X\in\mathfrak{S}_1(U^p,X^p)\cap\Pi_X$.

Proof

This result hinges on Corollary 5.1 and the fact that subanalytic sets admit Whitney regular stratifications [28-30,34]. A subset of R^n is semianalytic if it is locally defined by finitely many real analytic equalities and inequalities, and subanalytic if a real analytic image of a semianalytic set. Hardt [28-30] and Hironaka [34] have shown that subanalytic sets admit (locally finite) stratifications into C^ω strata (see Chapter One) and that this stratification may be refined to satisfy the Whitney regularity conditions (see [59],[57b]), and to have certain additional features. In particular it is shown in [29] that the stratification may be refined in such a way that for each r-stratum m there exists a continuous map $f_m:\bar{I}^r\rightarrow\bar{m}$ whose restriction to I^r is an analytic diffeomorphism. [6] is a recent review of the theory of semianalytic and subanalytic sets.

(1). We shall refer at intervals to the constructions and discussion in the preamble earlier in this chapter. Beginning with $x\in M$ such that $X(M)(x)\neq\mathbf{0}$ and $X\in\Xi_{\omega,r}'(M)$

we can choose (see the preamble) a ball $B_{r_2}(X) \subset \Xi_{\omega,r}(M)$ of radius r_2 centred on X so small and a ball $\bar{B}_{r_1}(x) \subset R^n$ so small that if $X(M)(x) \neq \mathbf{0}$ (and so $X^p(M \times \Xi_{\omega,r}(M))(x,X) = (X(M)(x),\mathbf{0}) \neq \mathbf{0}$) that by Corollary 5.1 there is a uniform (over $(y,Y) \in (\bar{B}_{r_1}(x) \cap M) \times B_{r_2}(X)$) bound on the number of stratum intersections $\phi^p(M \times B_r(X))(y,Y)$ makes, and in fact also (see proof of Corollary 5.1) on $T((\bar{B}_{r_1}(x) \cap M) \times B_{r_2}(X),(y,Y))$.

(2) We obtain a stratification $\{\sigma_i'\}$ of $M \times B_{r_2}(X)$ which has the property that $(y,Y) \to X^p(M \times \Xi_{\omega,r}(M))(y,Y)$ is analytic for as long as $(y,Y) \in \sigma_i'$. We saw in the preamble that there exist only finitely many non-empty sets $\Gamma_k^{X^p}(\breve{\sigma}_1 \text{ r } \breve{\sigma}_2)$, where here (and henceforth) the σ_i are strata of $M \times B_{r_2}(X)$ as a submanifold with corners, and by Proposition 4.4 every iteration set is an intersection of some of these finitely many sets. Rather than perturb X to ensure that all possible intersections and differences of these sets form a stratification, since these sets are subanalytic and there are only finitely many of them we can use [28-30] to stratify $M \times B_{r_2}(X)$ in such a way that each set formed by intersections and differences from these sets is a union of strata. We shall call these strata $\{\sigma_i'\}$. By Proposition 4.4 this stratification refines the decomposition of $M \times B_{r_2}(X)$ into iterations sets for X^p, and so in particular $S_2(y,Y)$ is a constant on each σ_i' (from definitions if $(B_{r_1}(x) \cap M) \times B_{r_2}(X)$ is represented near (y,Y) as $Z^pN^p(I;J)$ then $S_2(y,Y)$ has the property that $X^p(S_2(y,Y))(y,Y) = X^p(M \times \Xi_{\omega,r}(M))(y,Y)$). Hence the map $(y,Y) \to X^p(M \times \Xi_{\omega,r}(M))(y,Y))$ is analytic on each σ_i'.

(3) We construct a closed subanalytic neighbourhood U^p of (x,X), with $(x,X) \subset U^p \subset (\bar{B}_{r_1}(x) \cap M) \times B_{r_2}(X)$, as follows. In the preamble we constructed a funnel $(f^p)^{-1}F_c{}^p$ and a funnel cross section $(f^p)^{-1}\Sigma_a{}^p$ with the property that given any sufficiently small $a>0$ the trajectory based at any point (y,Y) in $M \times \Xi_{\omega,r}(M)$ sufficiently near (x,X) crosses $(f^p)^{-1}\Sigma_a{}^p$ in a unique point (see the preamble). By choosing the quantity $a>0$ sufficiently small we can arrange for $(f^p)^{-1}\Sigma_a{}^p$ to be contained in $(B_{r_1}(x) \cap M) \times B_{r_2}(X)$. If the strata of $M \times \Xi_{\omega,r}(M)$ as a submanifold with corners near (x,X) are $\sigma_1,..,\sigma_r$ and (y,Y) is near (x,X) the trajectory $\phi^p(M \times \Xi_{\omega,r}(M))(y,Y)$ is a union of components

$\phi^p(M \times \Xi_{\omega,r}(M))(y,Y) = \phi^p(\breve{\sigma}_{i(1)})((y,Y),[0,t_1)) \cup \phi^p(\breve{\sigma}_{i(2)})((y,Y),[t_1,t_2)) \cup ..$ etc where $\phi^p(M \times \Xi_{\omega,r}(M))((y,Y),t) \in \sigma_{i(j)}$ on (t_{j-1},t_j) and $(y_j,Y_j) = \phi^p(\breve{\sigma}_{i(j-1)})((y_{j-1},Y_{j-1}),t_j - t_{j-1})$. We have arranged (in (1)) that the r_1 and r_2 are so small that that we can apply Corollary 5.1 to infer that there exists N_0 independent of (y,Y) such that the number of analytic components of any trajectory segment in $(\bar{B}_{r_1}(x)\cap M)\times B_{r_2}(X)$ is $\leq N_0$. Since on each stratum σ_i' of our stratification $\{\sigma_i'\}$ of (1) the iteration is constant the region R_i of $(\bar{B}_{r_1}(x)\cap M)\times B_{r_2}(X)$ on which $X^p(M\times\Xi_{\omega,r}(M))=X(\breve{\sigma}_i)$ is for each $i=1,..,r$ a union of strata $\{\sigma_i'\}$, and hence is subanalytic. If for $(y,Y)\in M\times B_{r_2}(X)$ we set $\phi^-(\breve{\sigma}_i)(y,Y)=\{\phi^p(\sigma_i)((y,Y),t):t\leq 0\}\cap R_i$ we then take $U^p=(\bar{B}_{r_1}(x)\cap M)\times B_{r_2}(X))\cap(\cup\phi^-(\sigma_{i(1)})\phi^-(\sigma_{i(2)})..\phi^-(\sigma_{i(s)})((f^p)^{-1}\Sigma_a^p)$ where the union is over all possible sequences $i(1),..,i(s)$, $s\leq N_0$, with each $i(j)$ such that $1\leq i(j)\leq r$. This is evidently subanalytic (because Σ_a^p, R_i are subanalytic and $\phi(\breve{\sigma}_i)$, f^p are analytic) and is closed (because $\bar{B}_{r_1}(x)$ and Σ_a^p are closed and f^p and $\phi(\breve{\sigma}_i)$ are continuous) and by the foregoing is a neighbourhood of (x,X) in $M\times\Xi_{\omega,r}(M)$. A "slice" of U^p (in fact $U^p\cap\Pi_X$) is illustrated in Figure 7.8 below.

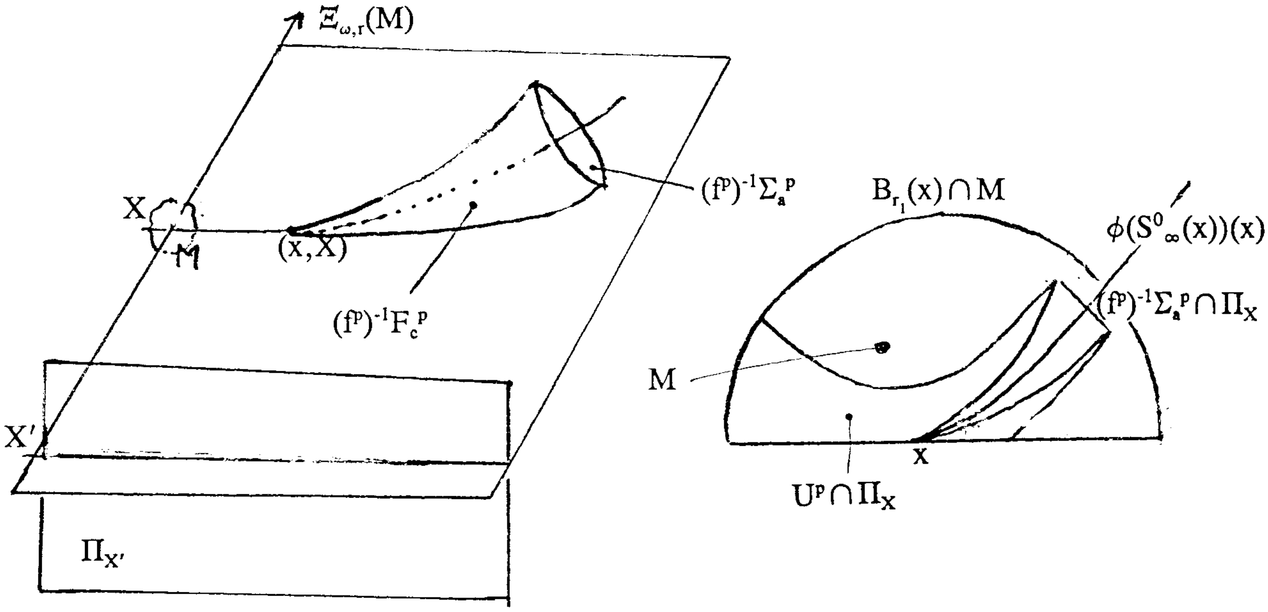
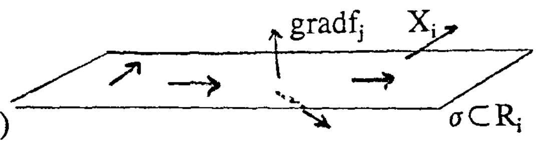


Figure 7.8

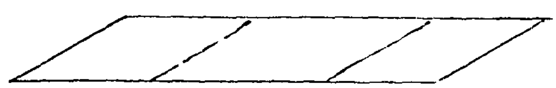
We shall construct a $\phi^p(M\times\Xi_{\omega,r}(M),X^p)$-compatible stratification of U^p with Σ in part 4 of the definition of compatible stratification equal to $(f^p)^{-1}\Sigma_a^p$, and denoting this stratification by $\mathfrak{S}_1(U^p,X^p)$ will show that for any X' near X $\mathfrak{S}_1(U^p,X^p)\cap\Pi_{X'}$ is a $\phi(M,X')$-compatible stratification of $U=U^p\cap\Pi_{X'}$, with Σ in the definition of compatible stratification now equal to $\Pi_{X'}\cap(f^p)^{-1}\Sigma_a^p$.

(4) If U and $\partial\bar{U}$ are subanalytic subsets of R^n with X a non-vanishing analytic vector field on U and X_i analytic vector fields on subanalytic subsets of $\partial\bar{U}$ the following will be our standard procedure for forming an analytic stratification of $\bar{U}$ which has the property that all the strata of U are of type I or II and the (forward) projection ("projection" in this context always means forward projection along the flow) of each stratum onto $\partial\bar{U}$ is a union of strata in $\partial\bar{U}$.

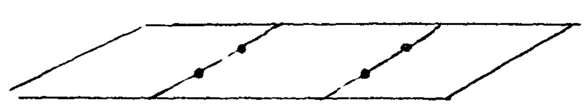
Since U is subanalytic we may by [28-30] obtain an analytic stratification of it into strata each of which is by [29] C^ω diffeomorphic to $(-1,1)^r$, some r, ie if σ is a r-stratum of U then σ is the set of common zeros of r independent C^ω functions $g_j:R^n \to R$. If we now form the subanalytic subsets of σ by taking all equalities and inequalities $\{x \in \sigma: \langle \mathrm{grad} g_j(x), X(x) \rangle = 0, >0, <0\}$ (see Figure 7.9) we may by [28-30] again form an analytic stratification of σ compatible with these sets. We then take each one of them of codimension >0 and repeat, and keep going until we run out of dimensions. We repeat for all strata σ of U. We obtain a stratification of U such that all strata are of type I or II. Finally we project forwards by the flow all these strata onto $\partial\bar{U}$, and (using [28-30] as above) form a stratification of $\partial\bar{U}$ compatible with these images.



(a)



(b) A stratification of σ compatible with $\langle \mathrm{grad} g_j(y), X_i \rangle = 0, >0, <0$



(c) A stratification of the strata of codimension >0 in σ by the same method

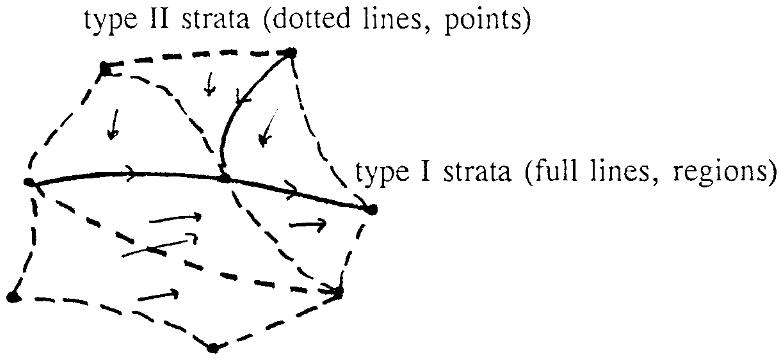
Figure 7.9

(5) In (3) we constructed a subanalytic set U^p in $M\times\Xi_{\omega,r}(M)$ containing (x,X) and subanalytic subsets R_i of $(B_{r_1}(x)\cap M)\times B_{r_2}(X)$ such that $X^p(M\times\Xi_{\omega,r}(M))=X^p(\breve{\sigma}_i)$ on R_i, where each σ_i was a stratum of $M\times\Xi_{\omega,r}(M)$ as a submanifold with corners. We shall now set $U_i=R_i\cap U^p$, ie $U_i=\{(y,Y)\in U^p:X^p(M\times\Xi_{\omega,r}(M))(y,Y)=X^p(\breve{\sigma}_i)(y,Y)\}$ and again is subanalytic. Order the strata of $M\times\Xi_{\omega,r}(M)\mid U^p$ as a submanifold with corners in an arbitrary way, $\sigma_1,..,\sigma_r$. For each $i\in(1,..,r)$ $\sigma_i\cap U_j$ is subanalytic (possibly empty) and for each i,j we stratify by the procedure of (4) the subanalyic set $\bar{U}=\text{clos}(\sigma_i\cap U_j)$ (throughout part (5) of this proof each $\bar{U}$, $\partial\bar{U}$, U is as in part (4)) with the vector field X in (4) set to $X^p(\breve{\sigma}_i)$. We recall that in (4) we finished by stratifying $\partial\bar{U}$ in a way which would be compatible with the projection onto $\partial\bar{U}$ of each of the strata we have formed in U. We then take any one of these strata, say σ', and by the procedure of (4) stratify the subanalytic set $\bar{U}$ where $U=\sigma'\cap U_k$ with X of (4) set to $X^p(\breve{\sigma}_k)$. We carry on repeating until the process terminates (as we know by the construction of U^p it will do in a finite number of stages) with every point of U^p mapped by a sequence of projections onto $(f^p)^{-1}\Sigma_a{}^p$. We have then obtained a stratification of U^p such that

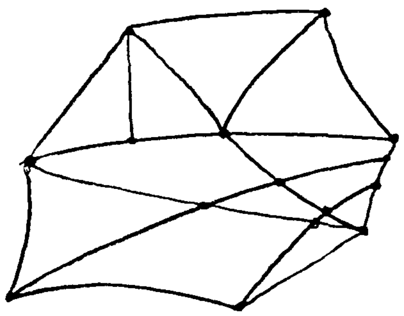
(i) Every stratum is of type I or II

(ii) The forward projection onto $\partial\bar{U}_i$ of every stratum in U_i equals a union of strata in $\partial\bar{U}_i$, and hence the forward projection onto any stratum σ in any $\partial\bar{U}_i$ of any σ' in U^p either contains σ or is disjoint from it.

(iii) The action of the flow $\phi^p(M\times\Xi_{\omega,r}(M))$ on each stratum of $M\times\Xi_{\omega,r}(M)\mid U^p$ as a submanifold with corners is a union of strata.



(a) The strata of (3) with arrows indicating the direction of flow $\phi^p(M\times\Xi_{\omega,r}(M))$



(b) The refined stratification at the end of (5)

Figure 7.10

(6) Taking the stratification of (5) above we perform finally a sequence of projections in the reverse direction to the flow, beginning with the stratification of $(f^p)^{-1}\Sigma_a{}^p$. For each stratum m of (5) there exists a set of strata A(m) with the property that for any $m'\in A(M)$, $\phi^p(M\times\Xi_{\omega,r}(M))(m',t)\cap m\neq\varnothing$ for arbitrarily small $t>0$ (see Figure 7.11).

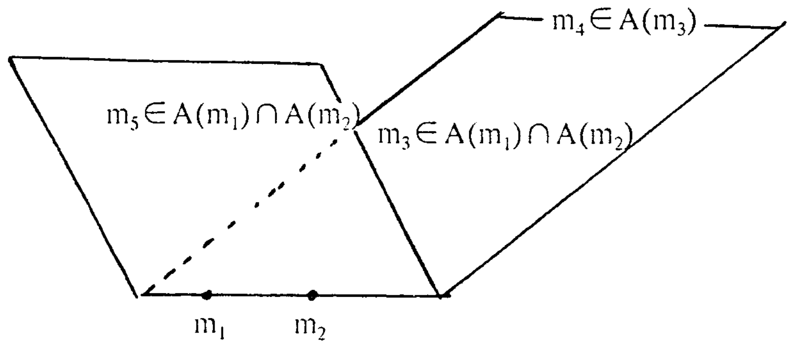


Figure 7.11

We take each stratum m of $(f^p)^{-1}\Sigma_a{}^p$ of (5) and project backwards by the flow through the strata A(m), and continue backwards until reaching ∂U^p (Figure 7.12).

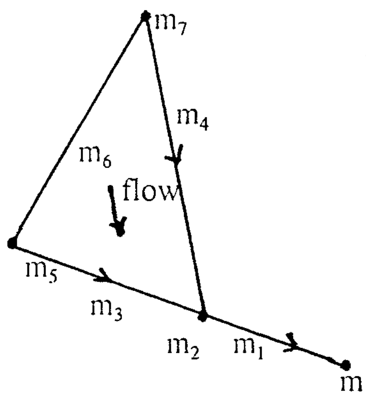
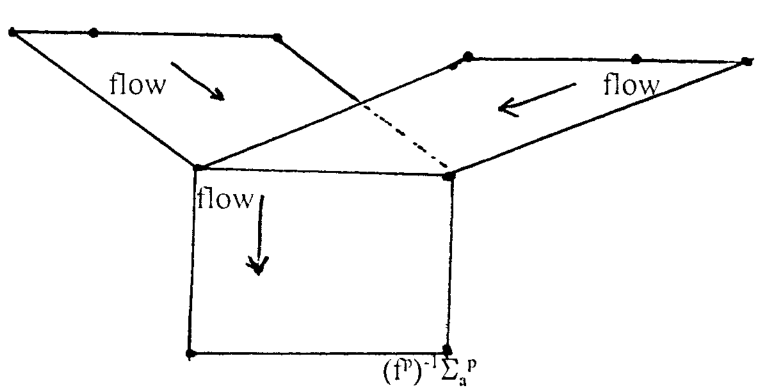


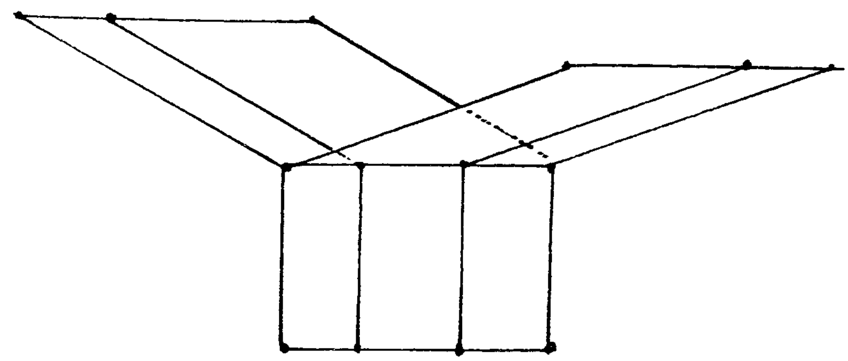
Figure 7.12

By single valuedness of the flow each stratum of U^p is reached by backwards projection in exactly one way. By finiteness of the stratification (since U^p is closed) we reach ∂U^p in finitely many steps. Since the stratification of $\partial\bar{U}_i$ refines the projection onto $\partial\bar{U}_i$ of the stratification of U_i and each stratum of (5) is either invariant by the flow or nowhere tangent to it, we shall by taking pre-images in this way further refine the stratification of (5), yielding a stratification of U^p which we can check satisfies (3) and (4) of the definition of $\phi^p(M \times \Xi_{\omega,r}(M), X^p)$-compatible stratification in addition to (1) and (2), and which is therefore the required stratification $\mathfrak{C}_1(U^p,X^p)$ (Figure 7.13).

Stratification at the beginning of (5)



Stratification at the end of (5)



Stratification at the end of (6)

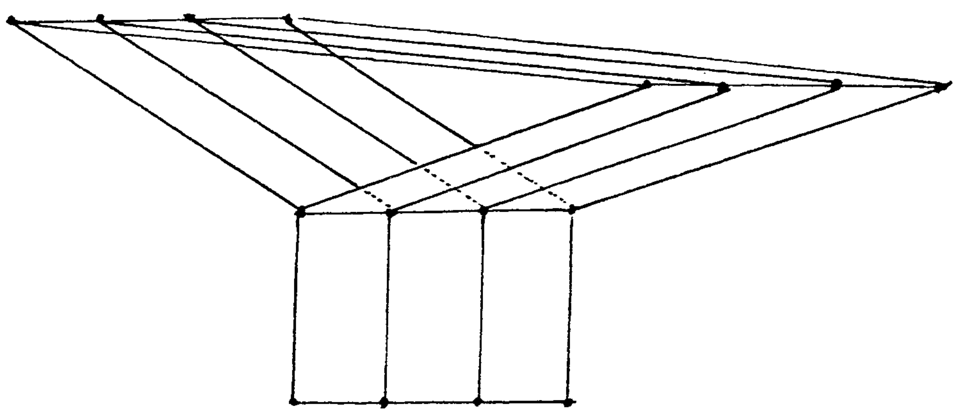


Figure 7.13

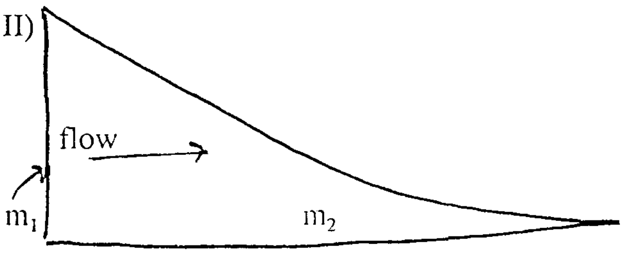
(7) It remains to show that $\mathfrak{C}_1(U^p,X^p)\cap\Pi_{X'}$ is for any X′ near X an analytic $\phi(M,X')$-compatible stratification of $U=U^p\cap\Pi_{X'}$. We saw above that for any $x'\in M$ $\phi^p(M\times\Xi_{\omega,r}(M))((x',X'),t)=(\phi(M,X')(x',t),X')$ and hence $\phi(M,X')(x',t)=\phi^p(M\times\Xi_{\omega,r}(M))((x',X'),t)\cap\Pi_{X'}$. If then we follow through each stage of the construction of the stratification $\mathfrak{C}_1(U,X')$ of M near x in the same way as that for our $\phi^p(M\times\Xi_{\omega,r}(M),X^p)$ compatible stratification $\mathfrak{C}_1(U^p,X^p)$, we will obtain the stratification $\mathfrak{C}_1(U^p,X^p)\cap\Pi_{X'}$ (and for the same reason the type of $m\in\mathfrak{C}_1(U^p,X^p)$ is that of $m\cap\Pi_X\in\mathfrak{C}_1(U^p,X^p)\cap\Pi_X$). ∎

<u>Definitions</u> Suppose $\mathfrak{S}_1(U,X)$ is a $\phi(M,X)$-compatible stratification of $U \subset M$. If m_1, m_2 are strata of $\mathfrak{S}_1(U,X)$ we shall say

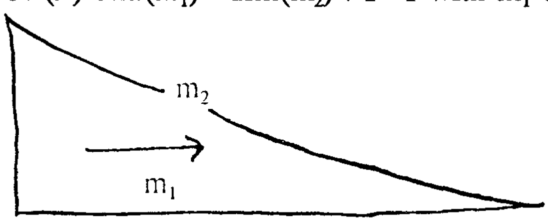
(a) $m_1 \twoheadrightarrow m_2$ if $m_2 \subset \partial \text{clos}(m_1)$ (this is more or less the notation of [57])

(b) Continuing with the notation and conventions of part 3 of the definition of $\phi(M,X)$-compatible stratification, we shall say $m_1 \Rightarrow m_2$ if either

(i) $\dim(m_2) = \dim(m_1) + 1 = r$ with m_2 of type I and $m_1 = m_2(I^{r-1}_{-})$ (and hence m_1 of type II)



or (ii) $\dim(m_1) = \dim(m_2) + 1 = r$ with m_1 of type I and $m_2 = m_1(I^{r-1}_{+})$



<u>Example</u> (of a $\phi(M,X)$-compatible stratification, and of the above defintions)

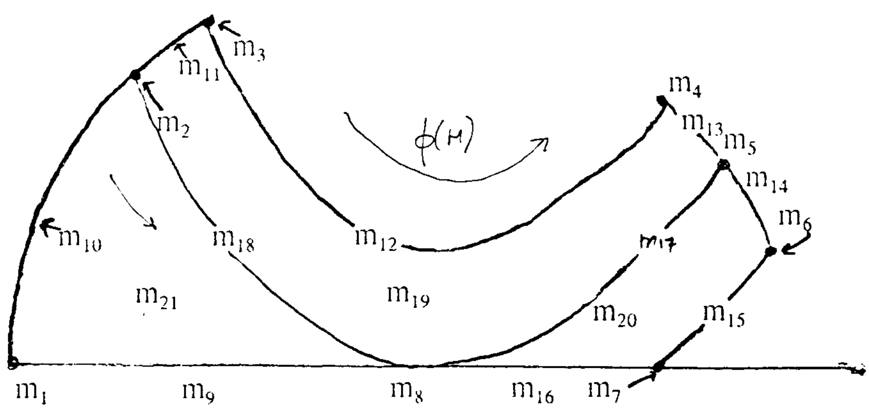


Figure 7.14a

A $\phi(M,X)$-compatible stratification

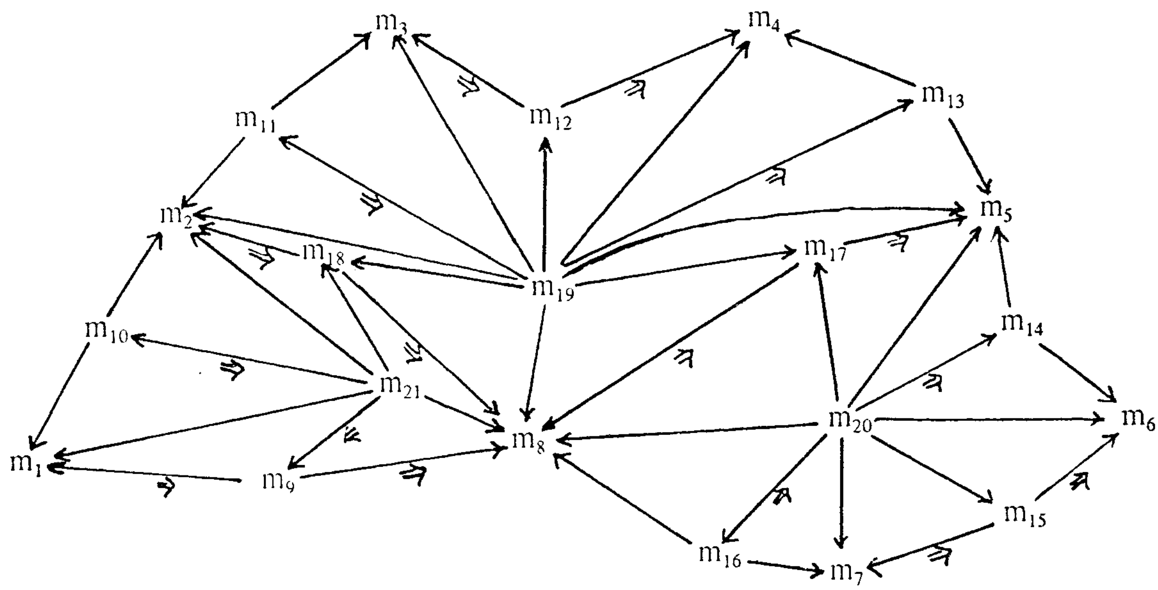


Figure 7.14b

The strata of Figure 7.14a with the relations $\rightarrow, \Rightarrow$ between them

To remind us that $\Rightarrow$ is a feature of the flow we may suffix it by X: $\Rightarrow_X$.

We recall if $\mathfrak{S}_1(U)$ is a stratification of $U \subset R^n \times \Xi_{\omega,r}(R^n)$ that a stratum preserving homeomorphism $h: \mathfrak{S}_1(U) \cap \Pi_X \rightarrow \mathfrak{S}_1(U) \cap \Pi_{X'}$ is a homeomorphism of $U \cap \Pi_X \rightarrow U \cap \Pi_{X'}$ such that if the strata of $\mathfrak{S}_1(U) \cap \Pi_X$ are $(s_1,..,s_m)$ then the strata of $\mathfrak{S}_1(U) \cap \Pi_{X'}$ are $(hs_1,..,hs_m)$ (and consequently $s_1 \rightarrow s_2$ iff $hs_1 \rightarrow hs_2$).

We constructed in Lemma 7.1 a $\phi^p(M \times \Xi_{\omega,r}(M), X^p)$-compatible stratification $\mathfrak{S}_1(U^p, X^p)$ of $U^p \subset M \times \Xi_{\omega,r}(M)$ such that for all X' near X $\mathfrak{S}_1(U^p, X^p) \cap \Pi_{X'}$ is a $\phi(M, X')$- compatible stratification.

Lemma 7.2

If $\mathfrak{S}_1(U^p, X^p)$ is the $\phi^p(M \times \Xi_{\omega,r}(M), X^p)$-compatible stratification of U^p constructed in Lemma 7.1 and if $\Pi_X \pitchfork \mathfrak{S}_1(U^p, X^p)$ then for any X' near X there exists a stratum preserving homeomorphism $h: \mathfrak{S}_1(U^p, X^p) \cap \Pi_X \rightarrow \mathfrak{S}_1(U^p, X^p) \cap \Pi_{X'}$ and furthermore for strata m_i of $\mathfrak{S}_1(U^p, X^p)$ $m_1 \cap \Pi_X \Rightarrow_X m_2 \cap \Pi_X$ iff $m_1 \cap \Pi_{X'} \Rightarrow_{X'} m_2 \cap \Pi_{X'}$.

Proof

We recall we have set $\Pi_X = \{(y,Y) \in R^n \times \Xi_{\omega,r}(R^n) : Y = X\} = R^n \times \{X\}$, with the consequence that $\Pi_X \cap (M \times \Xi_{\omega,r}(R^n))$ corresponds to the system (M,X). By Lemma 7.1

we know that for any $\phi^p(M\times\Xi_{\omega,r}(M),X^p)$-compatible stratificaton $\mathfrak{S}_1(U^p,X^p)$ of $U^p\subset M\times\Xi_{\omega,r}(M)$ and for any X' sufficiently near X that $\mathfrak{S}_1(U^p,X^p)\cap\Pi_{X'}$ is a $\phi(M,X')$-compatible stratification of $U^p\cap\Pi_{X'}$.

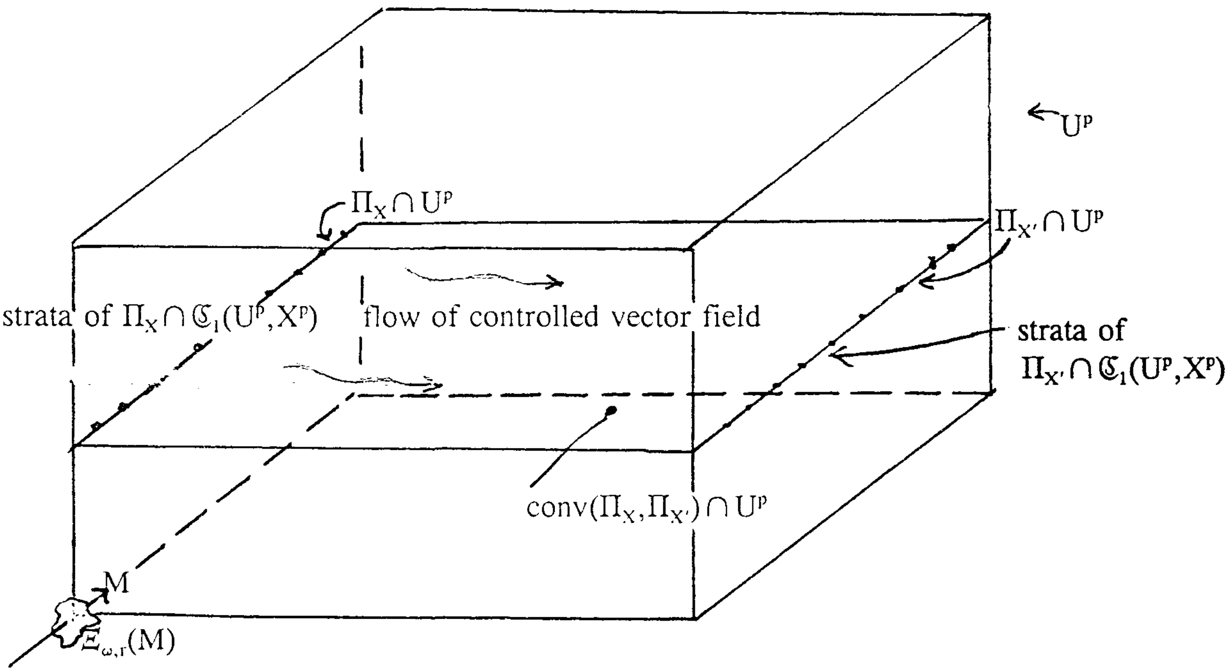


Figure 7.15

If Π_X is transverse to $\mathfrak{S}_1(U^p,X^p)$ (ie, Π_X is transverse to each stratum of $\mathfrak{S}_1(U^p,X^p)$) and so by [59] $\Pi_{X'}$ is transverse to $\mathfrak{S}_1(U^p,X^p)$ for all X' near X, we shall construct a "controlled" (see [59,57b]) vector field on a surface joining $U^p\cap\Pi_X$ to $U^p\cap\Pi_{X'}$ in U^p which will push the strata of $\mathfrak{S}_1(U^p,X^p)\cap\Pi_X$ onto those of $\mathfrak{S}_1(U^p,X^p)\cap\Pi_{X'}$ (see Figure 7.15).

1. Set $\Pi_{XX'}$=affine span of $(\Pi_X,\Pi_{X'})$ in $R^n\times\Xi_{\omega,r}(R^n)$. Since by hypothesis $\Pi_X\pitchfork\mathfrak{S}_1(U^p,X^p)$ it follows $\Pi_{X'}\pitchfork\mathfrak{S}_1(U^p,X^p)$ for all X' sufficiently near X, and that $\Pi_{XX'}\pitchfork\mathfrak{S}_1(U^p,X^p)$ on the convex hull of $(\Pi_X,\Pi_{X'})$. It follows from [59, 57b] that if $\mathfrak{S}_1(U^p,X^p)$ is a Whitney regular stratification and L is a submanifold with boundary of $R^n\times\Xi_{\omega,r}(R^n)$ then if $L,\partial L\pitchfork\mathfrak{S}_1(U^p,X^p)$ then $\mathfrak{S}_1(U^p,X^p)\cap L$ is a Whitney regular stratification of L. Thus on $\mathrm{conv}(\Pi_X,\Pi_{X'})\cap U^p\subset\Pi_{XX'}$ it follows that $\Pi_{XX'}\cap\mathfrak{S}_1(U^p,X^p)$ is a Whitney stratification. We now set $N_{XX'}=(\mathbf{0},X-X')$ (where here we are regarding X,X' as point vectors in $\Xi_{\omega,r}(R^n)$) so $N_{XX'}$ is a constant vector field on $\Pi_{XX'}$, normal in $\Pi_{XX'}$ to Π_X and $\Pi_{X'}$. We shall show that if s is any stratum of the stratification $\mathrm{conv}(\Pi_X,\Pi_{X'})\cap\mathfrak{S}_1(U^p,X^p)$ then $N_{XX'}(s)$ is a smooth non-vanishing vector field on s, where $N_{XX'}(s)$ is defined in the usual way, viz $N_{XX'}(s)(y,Y)=PT_{(y,Y)}sN_{XX'}$, for all $(y,Y)\in s$.

We are choosing X' so near X that $s \pitchfork \Pi_Y$ for all $Y \in \mathrm{conv}(X,X')$ which means that for each $(y,Y) \in s$ the normal space to s in $\Pi_{XX'}$ is independent of the normal space to Π_Y in $\Pi_{XX'}$, ie their only point of intersection is $\mathbf{0} \in T_{(y,Y)}(R^n \times \Xi_{\omega,r}(R^n))$. The normal space to Π_Y in $\Pi_{XX'}$ is span$\{N_{XX'}\}$, so we must have $N_{XX'} \notin$ normal space to s in $\Pi_{XX'}$. However (eg by Characterisation of Projection) $N_{XX'}(s) = \mathbf{0}$ iff $N_{XX'} \in$ normal space to s in $\Pi_{XX'}$.

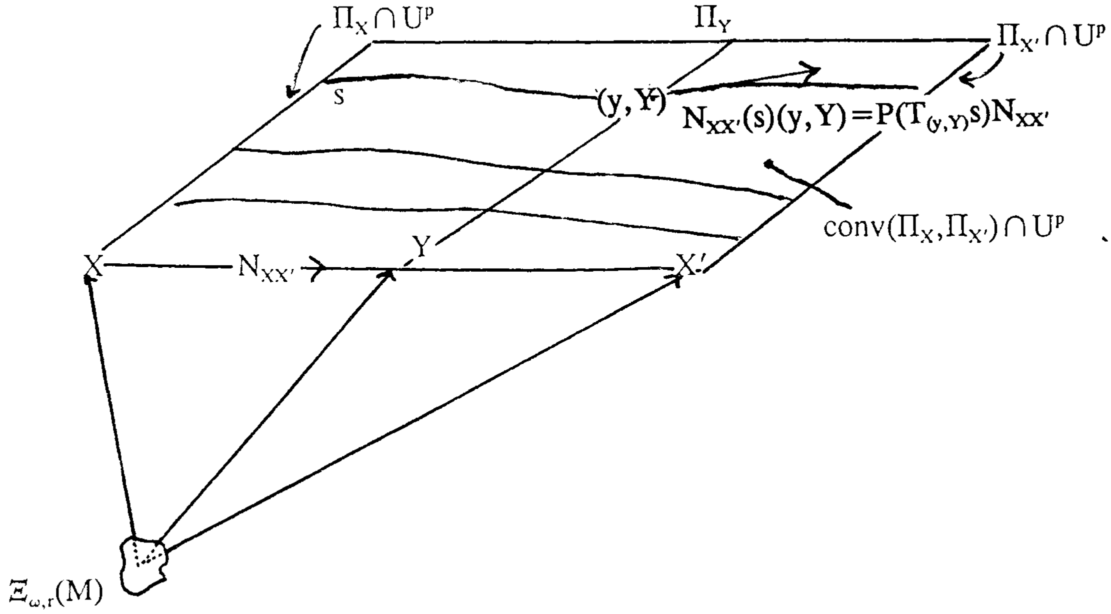


Figure 7.16 - Exactly as in Figure 7.15 above

2. We show we can find a non-vanishing vector field Z on $\mathrm{conv}(\Pi_X,\Pi_{X'}) \cap U^p$ whose integral flow ϕ_Z satisfies

(i) $\phi_Z{}^T$ is a homeomorphism $\mathrm{conv}(\Pi_X,\Pi_{X'}) \cap U^p \to \mathrm{conv}(\Pi_X,\Pi_{X'}) \cap U^p$ and a homeomorphism $\Pi_X \cap U^p \to \Pi_{X'} \cap U^p$, some $T>0$, where $\phi_Z{}^T$ is the time T map of the flow ϕ_Z

(ii) $\phi_Z{}^T$ is a diffeomorphism of each stratum of $\Pi_{XX'} \cap \mathfrak{C}_1(U^p,X^p)$ to itself (see Figure 7.16).

Consider a stratum s of $\mathfrak{C}_1(U^p,X^p) \cap \mathrm{conv}(\Pi_X,\Pi_{X'})$ such that $s=\bar{s}$, ie $\bar{s}$ contains no lower dimensional strata in its boundary. By 1. we can find for each such stratum a non-vanishing vector field $N_{XX'}(s)$ = projection onto s of $N_{XX'}$. Since for all $(y,Y) \in s$ $0 \neq |N_{XX'}(s)(y,Y)|^2 = \langle PT_{(y,Y)}sN_{XX'}, N_{XX'}\rangle$ we have $\langle PT_{(y,Y)}sN_{XX'}, N_{XX'}\rangle > 0$ for all $(y,Y) \in s$ and hence for each $(y,Y) \in s$ there exists $T(s,(y,Y)) > 0$ such that $\phi_{N_{XX'}(s)}((y,Y),T(s,(y,Y))) \in \Pi_{X'}$. Since $\mathfrak{C}_1(U^p,X^p)$ is locally finite the strata s such that

$s=\bar{s}$ have the property that there exists $\delta>0$ such that any pair of such strata have disjoint δ-neighbourhoods. As in the proof of [57b,Lemma 2.3] we can by rescaling $N_{XX'}(s)$ arrange that $T(s,(y,Y))$ is independent of $(y,Y)\in s$. Then the set of vector fields $N_{XX'}(s)$ for such strata s form a controlled vector field (see [57b]) on the union of such strata and by [57b, Lemma 2.4] this may be extended to a controlled vector field Z on $U^p\cap \mathrm{conv}(\Pi_X,\Pi_{X'})$ whose integral flow is a homeomorphism, is a diffeomorphism on each stratum, and again as in the proof of [57b, Lemma 2.3] may be rescaled so that there exists $T>0$ such that for any $(y,Y)\in\Pi_X\cap U^p$ $\phi_Z((y,Y),T)\in\Pi_{X'}$.

This shows that there exists a stratum preserving homeomorphism of $\mathfrak{C}_1(U^p,X^p)\cap\Pi_X\rightarrow\mathfrak{C}_1(U^p,X^p)\cap\Pi_{X'}$ (and in fact that for some neighbourhood U_X of X in $\Xi_{\omega,r}(M)$ that $\mathfrak{C}_1(U^p,X^p)\cong\ \mathfrak{C}_1(U^p,X^p)\cap\Pi_{X'}\times U_X$)

3. We show that under the hypotheses of the Lemma that for X′ sufficiently near X and strata m_1, m_2 of $\mathfrak{C}_1(U^p,X^p)$, $m_1\cap\Pi_X\Rightarrow_X m_2\cap\Pi_X$ iff $m_1\cap\Pi_{X'}\Rightarrow_{X'} m_2\cap\Pi_{X'}$. It suffices to show that for any strata m_i in $\mathfrak{C}_1(U^p,X^p)$ $m_1\Rightarrow_{X^p} m_2$ iff for any X′ near X $m_1\cap\Pi_{X'}\Rightarrow_{X'} m_2\cap\Pi_{X'}$. This follows from the definition of $\Rightarrow$, the fact that $\Pi_{X'}\pitchfork\mathfrak{C}_1(U^p,X^p)$ for all X′ near X, so $\mathrm{codim}(m_i\cap\Pi_{X'})=\mathrm{codim}(m_i)+\mathrm{codim}(\Pi_{X'})$, $i=1,2$, so $\dim(m_1)-\dim(m_2)=\dim(m_1\cap\Pi_{X'})-\dim(m_2\cap\Pi_{X'})$, and from the fact that $\phi^p(M\times\Xi_{\omega,r}(M))((x,X),t)=(\phi(M,X)(x,t),X)$ (for which see the preamble). ∎

<u>Proof of Proposition 7.1</u>

(1). We show that $\{(y,Y)\in M\times\Xi_{\omega,r}(M):Y\in\Xi_{\omega,r}'(M),\ Y(M)y\neq\mathbf{0}\}$ is open in $M\times\Xi_{\omega,r}(M)$.

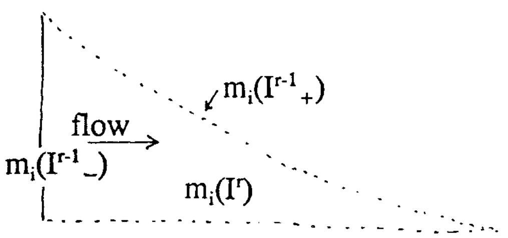
We know that $\Xi_{\omega,r}'(M)$ is open in $\Xi_{\omega,r}(M)$ by Proposition 4.2. We must show that $\{(y,Y)\in M\times\Xi_{\omega,r}(M):Y(M)(y)=\mathbf{0}\}$ is closed. If E_1 and E_2 are subsets of Euclidean space a <u>correspondence</u> F (see [12]) is a map from E_1 to the set of subsets of E_2. A correspondence is closed at $x\in E_1$ if for all sequences $\{x_n\}\subset E_1$, $\{y_n\}\subset E_2$ with $x_n\rightarrow x$ and each $y_n\in F(x_n)$, $\lim_{n\rightarrow\infty}y_n\in F(x)$.

By [12] the map $x\rightarrow(T_xM)^*$ is closed (where $(T_xM)^*$ is the polar cone to T_xM, ie $(T_xM)^*=\{y\in R^n:\langle y,v\rangle\leq 0 \text{ for all } v\in T_xM\}$). From the Characterisation of Projection we know $X(M)(x)=\mathbf{0}$ iff $X(x)\in(T_xM)^*$, hence if $x_i\rightarrow x$, and X_i is a sequence in $\Xi_{\omega,r}(M)$ with $X_i\rightarrow X$ and $X_i(M)(x_i)=\mathbf{0}$ for all i, so $X_i(x_i)\in(T_{x_i}M)^*$ for all i, by closedness of

$y \to (T_yM)^*$ we have therefore $X(x) \in (T_xM)^*$, so $X(M)(x) = \mathbf{0}$, and the result follows.

(2). By Lemma 7.1 for each point (x,X) in the set $M \times \Xi_{\omega,r}'(M) \backslash \{(y,Y): Y(M)(y) = \mathbf{0}\}$ (which we know by (1) to be open) there exists a neighbourhood U^P and a $\phi^P(M \times \Xi_{\omega,r}(M), X^P)$-compatible stratification $\mathfrak{C}_1(U^P,X^P)$ of U^P such that for all X' near X $\Pi_{X'} \cap \mathfrak{C}_1(U^P,X^P)$ is a $\phi(M,X')$-compatible stratification of $U = U^P \cap \Pi_{X'}$. If we take a countable subcover of $M \times \Xi_{\omega,r}'(M) \backslash \{(y,Y): Y(M)(y) = \mathbf{0}\}$ by such neighbourhoods the result is a countable collection of finite collections of analytic strata, so a countable number of analytic submanifolds of $M \times \Xi_{\omega,r}(M)$ (not necessarily forming a stratification). Therefore by [35, Theorem 2.7] we may find a residual subset of $\Xi_{\omega,r}'(M)$ (and hence of $\Xi_{\omega,r}(M)$) such that for all X in the subset Π_X is transverse to every one of these submanifolds. If we choose X in this residual subset and if $X(M)(x) \neq \mathbf{0}$ then $(x,X) \in$ some U^P with $\Pi_X \pitchfork \mathfrak{C}_1(U^P,X^P)$, and so Lemma 7.1 combined with Lemma 7.2 tells us that for all X' sufficiently near X there exists a stratum-preserving type-preserving homeomorphism $\mathfrak{C}_1(U,X) \to \mathfrak{C}_1(U',X')$ (the "type" alluded to is I or II) such that for any strata m_1, m_2 in $\mathfrak{C}_1(U,X)$ $m_1 \Rightarrow_X m_2$ iff $m_1' \Rightarrow_{X'} m_2'$.

(3). To complete the proof of Proposition 7.1 we shall use the following notions.
(i) An r-box of a compatible stratification is a pair of strata $m_i(I^r) \cup m_i(I^{r-1}_-) = m_i(I^r \cup I^{r-1}_-)$, where m_i is a type-I stratum



$m_i(I^r \cup I^{r-1}_-)$ (= a "box" in our terminology)

Figure 7.17

(ii) If $m_i(I^r \cup I^{r-1}_-), m_i'(I^r \cup I^{r-1}_-)$ are r-boxes of stratifications $\mathfrak{C}_1(U,X), \mathfrak{C}_1(U',X')$ and h is a homeomorphism between $m_i(I^{r-1}_+)$ and $m_i'(I^{r-1}_+)$ (see Figure 7.17) the linear extension of h to $m_i(I^r \cup I^{r-1}_-)$ is defined as follows. By property 3(ii) of $\phi(M,X)$-compatible stratification $\mathfrak{C}_1(U,X)$ we know that the flow induces a homeomorphism, say $H: m_i(I^r \cup I^{r-1}_+ \cup I^{r-1}_-) \to m_i(I^{r-1}_+) \times \bar{I}$ by $x \to (\omega(x), t_\omega(x)/(t_\omega(x) - t_\alpha(x)))$ (notation as in the definition of $\phi(M,X)$-compatible stratification) and we may extend the domain of definition of h to $m_i(I^r \cup I^{r-1}_+ \cup I^{r-1}_-)$ by requiring that the diagram

$$\begin{array}{ccc} m_i(I^r \cup I^{r-1}_+ \cup I^{r-1}_-) & \longrightarrow & m_i'(I^r \cup I^{r-1}_+ \cup I^{r-1}_-) \\ H \downarrow & & \downarrow H \\ m_i(I^{r-1}_+) \times \bar{I} & \xrightarrow{h \times Id} & m_i'(I^{r-1}_+) \times \bar{I} \end{array}$$

commutes.

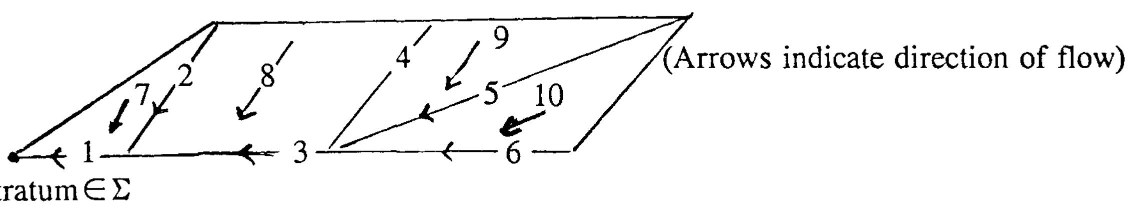
(a) We number the strata of $\mathfrak{C}_1(U,X)$ in a way which depends only on the types of the strata in $\mathfrak{C}_1(U,X)$ and the relations $\rightarrow, \Rightarrow$ between them, and therefore if $\mathfrak{C}_1(U_i,X_i)$, $i=1,2$, is a pair of $\phi(M_i,X_i)$-compatible stratifications where there is a stratum-preserving, type-preserving, $\Rightarrow$-preserving homeomorphism beween them (as is the case for $\mathfrak{C}_1(U,X)$, $\mathfrak{C}_1(U',X')$ of (2) above), then the numbering will also be preserved by the homeomorphism. We observe that the existence of this numbering follows directly from the "abstract" definition of $\phi(M,X)$-compatible stratification.

We know (by property (3) of $\phi(M,X)$-compatible stratification) that every stratum in $\mathfrak{C}_1(U,X)$, other than those in Σ (Σ as defined in the definition of compatible stratification), is part of a box, and so if we number all the boxes we will have numbered all the strata other than those in Σ.

Inductively, suppose we have numbered i boxes, 1,2,..,i. We then select a box satisfying

(i) There does not exist an un-numbered box of lower dimension

(ii) If the box is $m(I^r \cup I^{r-1}_-)$ then $m(I^{r-1}_+)$ is a stratum of a numbered box or is a stratum of Σ, (usually there will be several possibilities) and number the pair of strata in this box $(i+1)$. It follows from Property 4 in the definition of $\phi(M,X)$-compatible stratification that every stratum not in Σ eventually gets numbered, and from single-valuedness of the flow that no stratum receives two numbers (see Figure 7.18 for an example).



The numbers are for the pair of strata in each box -

eg, box number 1 is the pair
number 7 is the pair
etc.

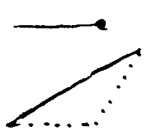


Figure 7.18

(b) We are finally going to show that with the data of (2) above we can establish a spfp homeomorhism between $\phi(M,X) \mid U$ and $\phi(M,X') \mid U'$, which proves the proposition. We have by Lemma 7.2 a stratum preserving homeomorphism between $\Pi_X \cap \mathfrak{C}_1(U^p,X^p)$ and $\Pi_{X'} \cap \mathfrak{C}_1(U^p,X^p)$ and in particular between $\Pi_X \cap (\mathfrak{C}_1(U^p,X^p) \mid (f^p)^{-1}\Sigma_a^p)$ and $\Pi_{X'} \cap (\mathfrak{C}_1(U^p,X^p) \mid (f^p)^{-1}\Sigma_a^p)$, and we extend this by linear extension (as described above) to each stratum in $\Pi_X \cap \mathfrak{C}_1(U^p,X^p) = \mathfrak{C}_1(U,X)$, ie to $U^p \cap \Pi_X = U$, in the order of the numbering of the strata given above. Thus we get a bijection $h: U^p \cap \Pi_X \rightarrow U^p \cap \Pi_{X'}$ which maps the strata of $\mathfrak{C}_1(U^p,X^p) \cap \Pi_X$ onto those of $\mathfrak{C}_1(U^p,X^p) \cap \Pi_{X'}$, and preserves the strata and relations $\rightarrow, \Rightarrow$ (it is a spfp bijection). We must show it is continuous. Since there are only finitely many strata any sequence $\{y_j\} \subset U$ partitions into finitely many subsequences $\{y_j^1\},..,\{y_j^k\}$ with $\{y_j^i\}$ in a single stratum m_i. Thus it suffices to show that if $\{y_j\} \subset m_i$ and $y_j \rightarrow y$ then $hy_j \rightarrow hy$.

Lemma 7.2 has provided us with a homeomorphism of $(f^p)^{-1}\Sigma_a^p \cap \Pi_X \rightarrow (f^p)^{-1}\Sigma_a^p \cap \Pi_{X'}$, so assume inductively that for any sequence $\{y_j\} \subset m_i$ with $i<k$, that $y_j \rightarrow y$ implies $hy_j \rightarrow hy$. We show then that if $y_j \rightarrow y$ for any sequence $\{y_j\} \subset m_k(I^r \cup I^{r-1}_-)$ then $hy_j \rightarrow hy$ (see Figure 7.19).

Because of condition (i) of the algorithm for numbering strata (that if an r-box is numbered, so already have been all s-boxes for $s<r$) all boundary strata of $\bar{m}_k^r$ except $m_k(I^{r-1}_-)$have been numbered with some $i<k$ and so h has already been extended by linear extension to these and by the inductive hypothesis is a homeomorphism on their union.

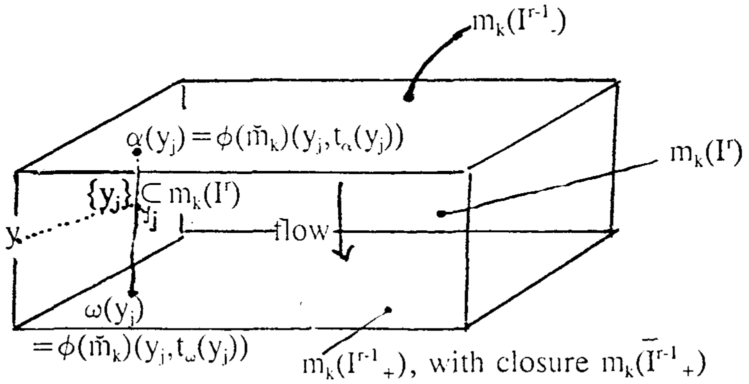


Figure 7.19

By the definition of linear extension to $m_k(I^r \cup I^{r-1}_-)$ we have $hy_j = H^{-1}(h,Id)Hy_j$ (notation as above) where $H(x) = (\omega(x), t_\omega(x)/(t_\omega(x)-t_\alpha(x)))$. We know by the inductive assumption that for any sequence $\{z_j\} \subset m_k(\overline{I}^{r-1}_+)$ that $hz_j \to hz$ as $z_j \to z$. By definition $\{\omega(y_j)\} \subset m_k(\overline{I}^{r-1}_+)$, we have $\omega(y_j) \to \omega(y)$ by continuity of ω, therefore $h\omega(y_j) \to h\omega(y)$. We know by property (3) of $\phi(M,X)$-compatible stratification that $t_\alpha(y_j) \to t_\alpha(y)$, $t_\omega(y_j) \to t_\omega(y)$ and that H^{-1} is a homeomorphism, so putting this together $hy_j = H^{-1}(h\omega(y_j), t_\omega(y_j)/(t_\omega(y_j)-t_\alpha(y_j))) \to H^{-1}(h\omega(y), t_\omega(y)/(t_\omega(y)-t_\alpha(y)))$, and by definition of H the right hand side equals hy, hence the result. ∎

Example Willis Models (see [60] or the Introduction) satisfy the conditions for Proposition 7.1(1)

Remark Using critically that for $X \in \Xi_{\omega,r}(M)$ the map $X \to$(classical) stable or unstable manifold of each regular zero of X is C^ω in X (see [49]) we could by similar methods treat the case $X(M)(x) = \mathbf{0}$.

Chapter Eight
Linear Systems

We recall from Chapter Six that we are calling a system (M,X) linear if $X \in \Xi_{\omega,1}(M)$ and M is a closed linear corner. We saw in Example 6.2 that linear systems are not even locally representative of generic non-linear systems (unlike of course the classical unconstrained case), but the biological models (see [60] or the Introduction) which inspired the thesis are linear and we have therefore made special provision for this case, or cases intermediate between it and general non-linear systems, in Example 2.3, parts of Propositions 4.2 and 4.4, Examples 6.5 and 6.6, and proposition 7.1. All of these results have been local, but in this chapter we establish an important global property of a class of linear system occurring in mathematical biology (Proposition 8.1 below). Before coming to that we make a few general observations about the special properties (local and global) which the semiflow $\phi(M,X)$ has when M and X are linear.

Generalities on Linear Systems

(1) Without much loss of generality we may suppose our closed linear corner is $M=LC(\varnothing;J)$. Evidently each of the fields P(K)X for $\varnothing \subset K \subset J$ is linear, but from the point of view of integrating these systems the situation is much better just than that; if $X(x)=Ax+b$ one readily establishes that if on [S,T) $X(M)(x(t))=P(I_i)X(x(t))$ (some $I_i \subset J$) then $x(T)-x(S)=\exp(-P(I_i)A(T-S))(x(0)-A^{-1}b)+A^{-1}b$, the constant terms outside the exponential are independent of I_i, and hence for $0=t_1<t_2<..<t_r=T$, with $X(M)(x(t))=P(I_i)X(x(t))$ for all $t \in [t_i,t_{i+1})$,
$x(T)=e^{-P(I_{r-1})A(t_r-t_{r-1})}..e^{-P(I_1)A(t_2-t_1)}(x(0)-A^{-1}b)+A^{-1}b$.

(2) Beginning with the linear corner $LC(I;J)=\{x \in R^n:\langle x,n_i\rangle=0 \ i \in I, \ \langle x,n_i\rangle \geq 0 \ i \in J\}$ and linear vector field $X(x)=Ax+b$ we may by a linear change of variables find $\{p_i\}_{i \in I \cup J}$ such that each $L(i)=\{x \in R^n:\langle x,n_i\rangle=p_i\}$ and $X(x)=Ax$, and for these these coordinates we have

$\Gamma_k(I \cup J \text{ r } J) = \{x \in L(I \cup J): \langle (P(J)A)^i x, P(J)n_j \rangle = 0$ for all $i=1,..,k-1, j \in I\}$ and in particular is affine. Furthermore since the map $x \to \phi(I)(x,t)$ is affine in x if X is linear, the manifold swept out by the action of X(I) on $\Gamma_2^{\pm}(I \cup j \text{ r } I)$ is locally convex in L(I), since it is locally convex near t=0 (it has supporting hyperplane $L(I \cup j)$). Hence also the intersection of this codimension 1 submanifold of L(I) with any $L(I \cup K)$ is also locally convex. All of this would be very visible if one graphically portrayed numerically integrated systems - one plot occurs in [60].

(3) We saw in our formula at the end of (1) above how to express x(t) in terms of products of exponentials $e^{tP(I)A}$, and of course we know how to calculate these quantities analytically once the eigenvalues and eigenvectors of P(I)A are known. Because of the form of this formula, because in applications symmetries are likely to exist in A (they do in [60]) making their eigenvalues and eigenvectors readily obtainable and for other reasons connected with the analysis of these systems we are interested in the relation between the eigenvalues and eigenvectors of P(I)A and those of A. Setting K=(1..k) so by Remark 2.2 $P(K)X(x) = X(x) - NM^{-1}N^T X(x)$ where $N = (n_1,..,n_k)$ and $M_{ij} = \langle n_i, n_j \rangle$, and setting P(K)Identity =Id(K), an eigenvalue λ and eigenvector β of P(K)A will satisfy $(P(K)A - \lambda Id(K))\beta = 0$ and since $P(K)Id(K) = P(K)^2 Identity = P(K)Identity = Id(K)$ we may write this as $P(K)(A - \lambda Id(K))\beta = 0$, and hence if $A - \lambda Id(K) \in GL(n)$, we must have $\beta \in (A - \lambda Id(K))^{-1} span(n_1,..,n_k)$. We know also of course that $\beta \in L(K)$. Define $f: R^k \to R^n$ by $f(x) = (A - \lambda Id(K))^{-1}(n_1..n_k)x$ and $g: R^n \to R^k$ by

$$g(x) = \begin{bmatrix} n_1^T \\ \cdot \\ \cdot \\ n_k^T \end{bmatrix} x .$$

Since $(n_1,..n_k)$ are independent $g \pitchfork 0$, ie g is a submersion at 0, and we can use [1, Section 3.5] ("if $g \pitchfork 0$ then $f \pitchfork g^{-1}(0)$ iff $gf \pitchfork 0$") to infer that $\dim(im(f) \cap ker(g)) = 0$ iff $\det(Dgf) \neq 0$. Ker(g) is L(K) so we get a non-trivial β iff $\dim(im(f) \cap ker(g)) \neq 0$ iff $\det(N^T(A - \lambda Id(K))^{-1}N) = 0$, where as in Remark 2.2 $N = (n_1,..,n_k)$, which provides a (n-k)th order equation. We get λ by solving these equations, and then β for each λ is given by the expression $L(K) \cap (A - \lambda Id(K))^{-1}(n_1..n_k)$.

For example, in the biological models described briefly in the Introduction (and in more detail in [60]) the submanifold with corners is an orthant and the vector field is linear where the coefficients in the matrix A in $X(x)=Ax+b$ are $A_{ij}=\alpha_{j-i+n,\ \mathrm{mod}\ n}$, for $\alpha_0,..,\alpha_{n-1} \in R$. A has eigenvalues $\lambda_k=\alpha_0+\omega^k\alpha_1+..+\omega^{(n-1)k}\alpha_{n-1}$, k=0,..,n-1 where $\omega=\exp(2\pi i/n)$, and eigenvectors

$$\begin{bmatrix} 1 \\ \omega^k \\ \cdot \\ \cdot \\ \omega^{(n-1)k} \end{bmatrix} .$$

If $L=\{x\in R^n:x_1=0\}$ P(1)A has eigenvalues $\mu_1,..,\mu_{n-1}$ given by $\Sigma_{i=0}^{n-1}1/(\lambda_i-\mu) =0$ and eigenvectors given by

$$\Omega^* \begin{bmatrix} \dfrac{1}{\lambda_0-\mu_i} \\ \cdot \\ \cdot \\ \dfrac{1}{\lambda_{n-1}-\mu_i} \end{bmatrix} \quad i=1,..,n-1 \quad \text{where} \quad \Omega^*=\overline{\Omega^T} = \Omega^{-1},\ \Omega_{ij}= \omega^{(i-1)j}$$

(up to uninteresting scaling factors) etc.

In summary we see that linear (some people might prefer the the terminology "piece-wise" linear) constrained systems share with their classical counterparts the possessing of several simplifying properties. Where the gulf becomes pronounced is when we consider the global dynamics of the systems, since for constrained systems it is clear that by judicious choice of linear vector field and arrangement of hyperplanes forming the boundary the way that we are gluing together individual linear systems means that in suficiently high dimensions we can conjure up some highly non-linear phenomena. We can imagine a volume increment of flow perhaps beginning in int(M) and hitting part of ∂M, sliding along and intersecting lower dimensional strata (and dropping in dimension when it does so) or lifting off to higher dimensional ones. We can see (an example in three dimensions below will make this clear, see Figure 8.2) that interest centres on the iterated maps formed when a cross-section of the flow maps into itself.

From Theorem 2.1 and Corollary 4.2 we know that if $X \in \Xi_{\omega,1}'(LC(\varnothing;J))$ the flow on LO(K;J\K) enters a higher dimensional stratum LO(K';J\K'), $K' \subset K$, at x iff $S_1(x)=K$ and for sufficiently large m $S_m(x)=S^0_m(x)=K'$. Most of the flow on LO(K;J\K) makes this type of transition along whichever iteration set has lowest codimension, which by Proposition 4.4 is $\{\Gamma_2^+(K \text{ r } K') \mid LO(K;J\backslash K): \mid K\backslash K' \mid =1\}$ (where $\Gamma_2^+(K \text{ r } K') \mid LO(K;J\backslash K)$ means $\Gamma_2^+(K \text{ r } K')$ restricted to LO(K;J\K)). Such sets are codimension 1 in LO(K;J\K) and the flow may induce an iterated map on parts of them $\Gamma_2^+(K_1 \text{ r } K_1') \mid LO(K_1;J\backslash K_1) \rightarrow \Gamma_1^-(K_2 \text{ r } K_1') \mid LO(K_2;J\backslash K_2) \rightarrow \Gamma_2^+(K_2 \text{ r } K_2') \mid LO(K_2;J\backslash K_2) \rightarrow \ldots \ldots \rightarrow \Gamma_2^+(K_1 \text{ r } K_1') \mid LO(K_1;J\backslash K_1)$ where each $K_i \supset K_i'$, $K_i' \subset K_{i+1}$, and $\mid K_i \mid - \mid K_j' \mid =1$ for all i,j (see Figure 8.1).

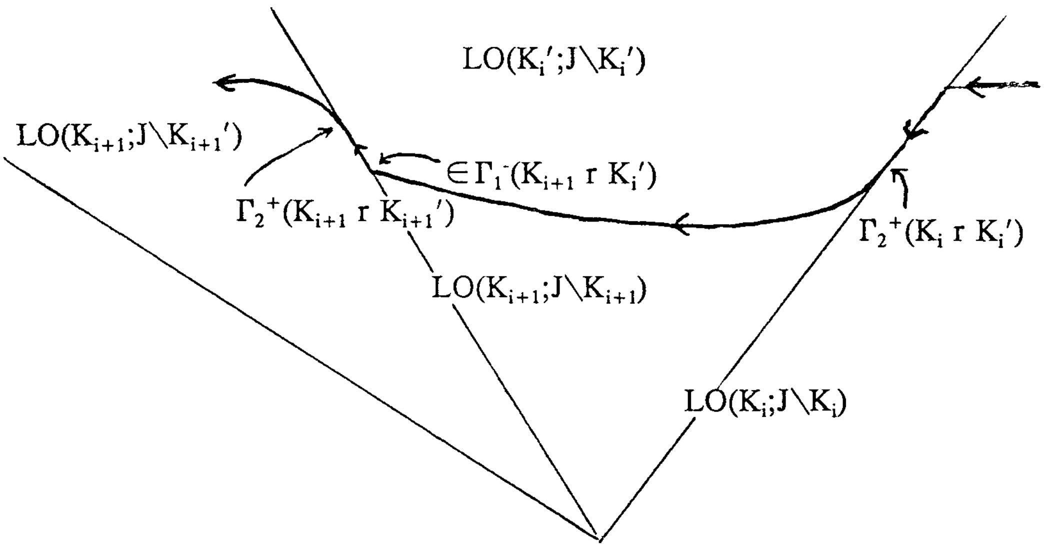


Figure 8.1

For example we can choose a linear vector field on M=LC(∅;1,2,3) in R^3 such that any trajectory initially in int(M)=LO(∅;1,2,3) eventually intersects ∂M and thereafter oscillates between LO(i;j,k) (i,j,k)∈(1,2,3) and LO(∅;1,2,3); the flow on LO(3;2,1) leaves LO(3;1,2) along $\Gamma_2^+(3 \text{ r } \varnothing)$ and subsequently intersects LO(1;2,3) along Q_3 (see figure 8.2 below) flows along LO(1;2,3) until leaving it along $\Gamma_2^+(1 \text{ r } \varnothing)$ and so on in a circuit: we thereby obtain iterated maps on the sets $\Gamma_2^+(i \text{ r } \varnothing)$ (and under some circumstances non-trivial periodic orbits).

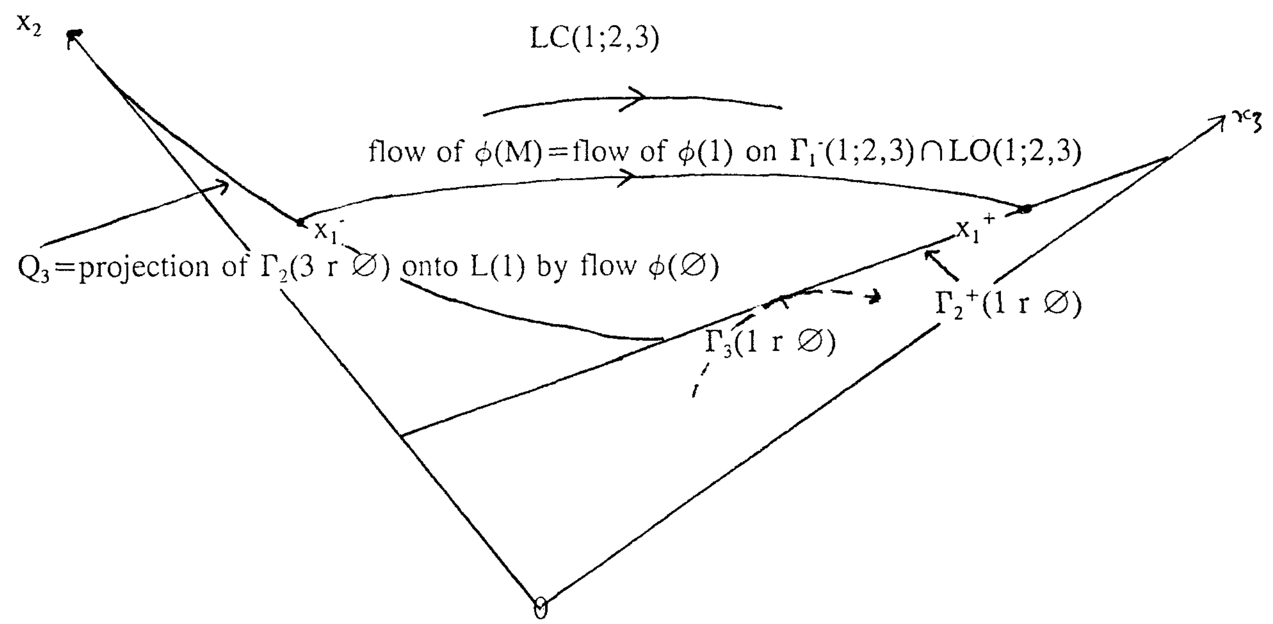


Figure 8.2a. A trajectory of X(M) intersects LO(1;2,3) at x_1^- and leaves LO(1;2,3) at $x_1^+ \in \Gamma_2^+(1\ r\ \varnothing)$

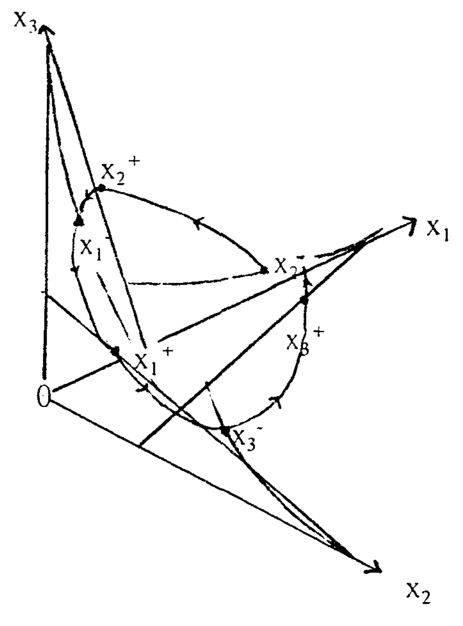
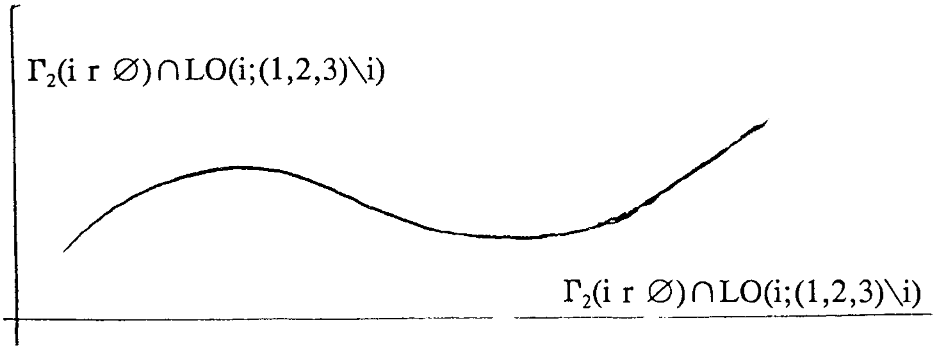


Figure 8.2b. A trajectory beginning at x_2^- makes a complete circuit

We can see that in the general way the iterated maps we have formed on **$\Gamma_2^+(i\ r\ \varnothing) \cap LO(i;(1,2,3)\backslash i)$ will not be invertible -**



but a striking feature of the linear systems which arise in mathematical biology ([60] or the Introduction) is

Proposition 8.1 Suppose M is the orthant $\{x\in R^n: x_i\geq 0,\ i=1,..,n\}$, $X\in \Xi_{\omega,1}'(M)$ (which is open-dense in $\Xi_{\omega,1}(M)$ by Proposition 4.2), where the $(n\times n)$ matrix A in $X(x)=k-Ax$ is non-negative (ie $A_{ij}\geq 0$ for all i,j) and satisfies the following condition:

For each subset I of (1,..,n) with $|I|<n$ and for any pair $j,k\notin I$ there exists $m(j,k)>0$ such that $\langle[(P(I)A)^m n_j],n_k\rangle>0$, where n_i is the unit vector such that $\langle n_i,x\rangle=x_i$.

Then all iterated maps of the form $\Gamma_2^+(K_1\ r\ K_1')\ |\ LO(K_1;J\backslash K_1)\rightarrow \Gamma_1^-(K_2\ r\ K_1')\ |\ LO(K_2;J\backslash K_2)\rightarrow\Gamma_2^+(K_2\ r\ K_2')\ |\ LO(K_2;J\backslash K_2)\rightarrow ..$ $..\rightarrow\Gamma_2^+(K_1\ r\ K_1')\ |\ LO(K_1;J\backslash K_1)$, where each $K_i\supset K_i', K_i'\subset K_{i+1}$ and $|K_i| = |K_j'|+1$ for all i,j (see Figure 8.1) are invertible.

Remark The condition on the matrix A is clearly satisfied by any positive matrix (ie one such that $A_{ij}>0$ for all i,j): in the Willis models certain coefficients may be zero, but it is straightforward to check that in all cases of interest they still satisfy the condition to apply Proposition 8.1.

Proof of Proposition 8.1

Each step in the iterated map desribed in the statement of Proposition 8.1 is of the form $\Gamma_2^+(K_i\ r\ K_i')\ |\ LO(K_i;J\backslash K_i)\rightarrow\Gamma_1^-(K_{i+1}\ r\ K_i')\ |\ LO(K_{i+1};J\backslash K_{i+1})\rightarrow$ $\Gamma_2^+(K_{i+1}\ r\ K_{i+1}')\ |\ LO(K_{i+1};J\backslash K_{i+1})$. Let us suppose $x_i^+\in\Gamma_2(K_i\ r\ K_i')$, $Q_i=R^+.\Gamma_2(K_i\ r\ K_i')\cap\Gamma_1^-(K_{i+1}\ r\ K_i')$=the projection along the flow of $\Gamma_2(K_i\ r\ K_i')$ onto $\Gamma_1^-(K_{i+1}\ r\ K_i')$ (see Figure 8.3) and the flow maps $x_i^+\in\Gamma_2^+(K_i\ r\ K_i')$ to $x_{i+1}^-\in\Gamma_1^-(K_{i+1}\ r\ K_{i+1}')$ to ..to $x_i^-\in\Gamma_1^-(K_i\ r\ K_i')$ to $\Gamma_2^+(K_i\ r\ K_i')$. It follows from

Lemma A.1 that the induced map

$\Gamma_2^+(K_i \text{ r } K_i') \mid LO(K_i;J\backslash K_i) \rightarrow \Gamma_1^-(K_{i+1} \text{ r } K_i') \mid LO(K_{i+1};J\backslash K_{i+1}) \rightarrow$

$\Gamma_2^+(K_{i+1} \text{ r } K_{i+1}') \mid LO(K_{i+1};J\backslash K_{i+1})\ \Gamma_2(K_i \text{ r } K_i')$ is locally a diffeomorphism at

$x_i^+ \in \Gamma_2(K_i \text{ r } K_i')$ iff

(i) $X(K_i')(x_i^+) \notin T_{x_i^+}\Gamma_2(K_i \text{ r } K_i')$

(ii) $X(K_i')(x_{i+1}^-) \notin T_{x_{i+1}^-}L(K_{i+1})$

(iii) $X(K_{i+1})(x_{i+1}^-) \notin T_{x_{i+1}^-}Q_i$

(iv) $X(K_{i+1})(x_{i+1}^+) \notin T_{x_{i+1}^+}\Gamma_2(K_{i+1} \text{ r } K_{i+1}')$.

We shall show that the assumptions of Proposition 8.1 guarantee that these conditions hold, and since the step and $x_i^+ \in \Gamma_2(K_i \text{ r } K_i')$ were chosen arbitrarily we infer that the iterated map from $\Gamma_2^+(K_1 \text{ r } K_1') \mid LO(K_1;J\backslash K_1)$ to itself is a diffeomorphism.

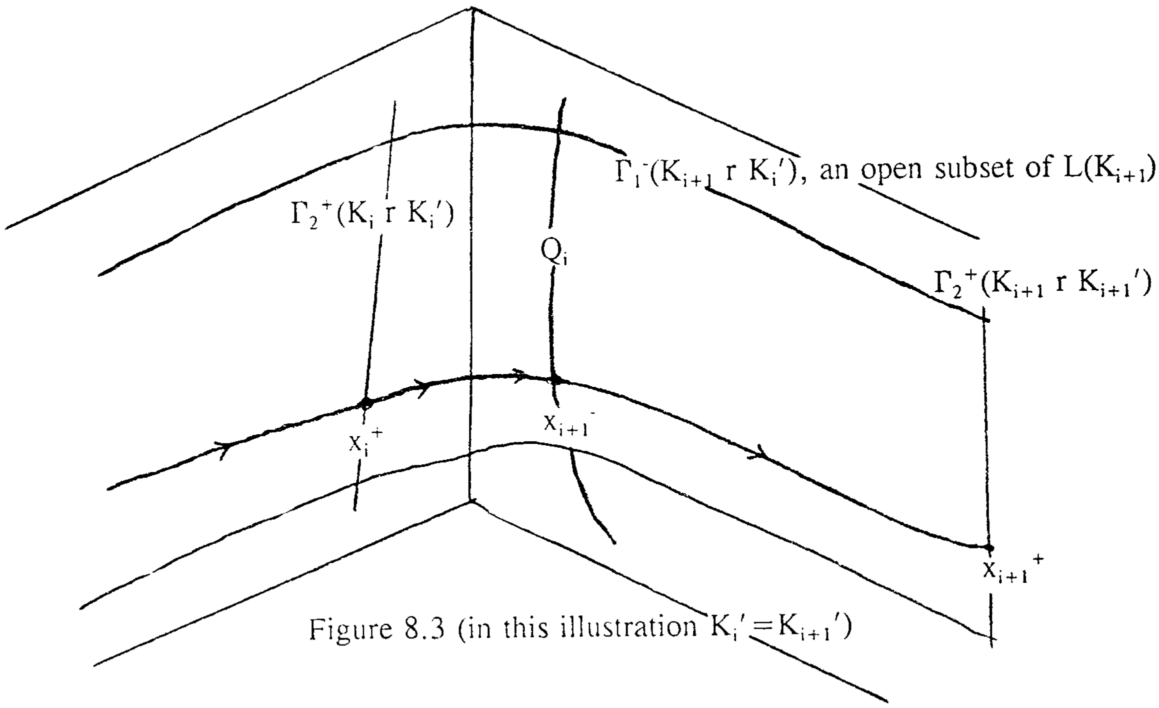


Figure 8.3 (in this illustration $K_i' = K_{i+1}'$)

(i) is equivalent to $x_i^+ \notin \Gamma_3(K_i \text{ r } K_i')$, (ii) is equivalent to $x_{i+1}^- \notin \Gamma_2(K_{i+1} \text{ r } K_i')$, (iv) is equivalent to $x_{i+1}^+ \notin \Gamma_3(K_{i+1} \text{ r } K_{i+1}')$. These conditions are satisfied automatically for the type of trajectory described in the statement of Proposition 8.1, which leaves (iii); we must show that under the assumption of Proposition 8.1 condition (iii) holds for every $x_i^+ \in \Gamma_2^+(K_i \text{ r } K_i')$.

$R^+.\Gamma_2(K_i \text{ r } K_i')$ and $\Gamma_1^-(K_{i+1} \text{ r } K_i')$ are transverse at x_{i+1}^- by condition (ii): hence since

$X(K_{i+1})(x_{i+1}^-) \notin T_{x_{i+1}^-}Q_i$ iff

$\langle X(K_{i+1})x_{i+1}^-$, normal to $T_{x_{i+1}^-}[R^+.\Gamma_2(K_i \text{ r } K_i') \cap \Gamma_1^-(K_{i+1} \text{ r } K_i')]$ in $T_{x_{i+1}^-}L(K_{i+1})\rangle \neq 0$ and we know by Remarks 2.1 that the

normal to $T_{x_{i+1}^-}[R^+.\Gamma_2(K_i \text{ r } K_i') \cap \Gamma_1^-(K_{i+1} \text{ r } K_i')]$ in $T_{x_{i+1}^-}L(K_{i+1}) =$

$P(T_{x_{i+1}^-}\Gamma_1^-(K_{i+1} \text{ r } K_i'))N_{x_{i+1}^-}(R^+.\Gamma_2(K_i \text{ r } K_i')$ in $L(K_i'))$, if condition (ii) applies then

condition (iii) is equivalent to $\langle X(K_{i+1})(x_{i+1}^-), P(T_{x_{i+1}^-}L(K_{i+1}))N_{x_{i+1}^-}(R^+.\Gamma_2(K_i \text{ r } K_i') \text{ in } L(K_i'))\rangle \neq 0$, which is equivalent to $\langle X(K_{i+1})(x_{i+1}^-), N_{x_{i+1}^-}(R^+.\Gamma_2(K_i \text{ r } K_i') \text{ in } L(K_i'))\rangle \neq 0$ (using self-adjointness of P). Since $T_{x_{i+1}^-}R^+.\Gamma_2(K_i \text{ r } K_i') \ni X(K_i')(x_{i+1}^-)$ (and so $\langle X(K_i')(x_{i+1}^-), N_{x_{i+1}^-}(R^+.\Gamma_2(K_i \text{ r } K_i') \text{ in } L(K_i'))\rangle = 0$) and $X(K_{i+1})(x_{i+1}^-) = X(K_i')(x_{i+1}^-) - \langle X(K_i')(x_{i+1}^-), \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i')\rangle \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i')$, where $\hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i')$ is the unit normal to $L(K_{i+1})$ in $L(K_i')$, and using $|K_i| = |K_j'| + 1$ for all i,j (by supposition), this is true iff
$0 - \langle X(K_i')(x_{i+1}^-), \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i')\rangle \langle \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i'), N_{x_{i+1}^-}(R^+.\Gamma_2(K_i \text{ r } K_i') \text{ in } L(K_i'))\rangle \neq 0$
and if condition (ii) is satisfied then $\langle X(K_i')x_{i+1}^-, \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i')\rangle \neq 0$, so this condition holds iff $\langle \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i'), N_{x_{i+1}^-}(R^+.\Gamma_2(K_i \text{ r } K_i') \text{ in } L(K_i'))\rangle \neq 0$.

We have also $T_{x_{i+1}^-}R^+.\Gamma_2(K_i \text{ r } K_i') = \text{span}\{X(K_i')(x_{i+1}^-), \phi(K_i')^t.T_{x_i^+}\Gamma_2(K_i \text{ r } K_i')\}$ $= \phi(K_i')^t.T_{x_i^+}L(K_i)$ (where $\phi(K)^t =$ time t map of $\phi(K)$) and hence condition (iii) holds if condition (ii) holds and $\langle \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i')$,(normal to $\phi(K_i)^t.T_{x_i^+}L(K_i)$ in $T_{x_{i+1}^-}L(K_i'))\rangle \neq 0$. (*)

All of this has been true for any system, but if furthermore the vector field is linear, with $X(x) = k - Ax$, then $X(K_i')(x) = P(K_i')k - P(K_i')Ax$, say $= k' - A'x$, and any vector v in $T_xL(K_i')$ is mapped by the flow $\phi(K_i')$ to $\phi(K_i')^t.v = D_x\phi(K_i')^t v = e^{-tA'}v$. Hence
$N_{x_{i+1}^-}(\phi(K_i')^t.L(K_i) \text{ in } L(K_i')) = \exp(tA'^T)N_{x_i^+}(K_i \text{ in } K_i')$ (**)
since for any $w \in \phi(K_i')^t.L(K_i)$, $w = \phi(K_i')^t.v$ with $v \in T_{x_i^+}L(K_i)$, we must have $w = e^{-tA'}v$, and hence $\langle w, \exp(tA'^T)\hat{N}_{x_i^+}(K_i \text{ in } K_i')\rangle =$
$\langle \exp(-tA')v, \exp(tA'^T)\hat{N}_{x_i^+}(K_i \text{ in } K_i')\rangle = \langle v, \hat{N}_{x_i^+}(K_i \text{ in } K_i')\rangle = 0$ since $(e^{-tA'})^T = \exp(-tA'^T)$ and $v \in T_{x_i^+}L(K_i)$.

Inserting (**) into (*) we see that if condition (ii) holds then condition (iii) holds if $\langle \hat{N}_{x_{i+1}^-}(K_{i+1} \text{ in } K_i'), \exp(tA'^T)\hat{N}_{x_{i+1}^-}(K_i \text{ in } K_i')\rangle \neq 0$.

If the suppositions of Proposition 8.1 are satisfied we have, possibly after renumbering the vectors $n_1,..,n_n$, that $A' = P(I)A$, $\hat{N}_{x_i^+}(K_{i+1} \text{ in } K_i') = n_{i+1}$, $\hat{N}_{x_{i+1}^+}(K_i \text{ in } K_i') = n_i$ where $i, i+1 \notin I \subset (1,..,n)$. The condition above becomes $\langle n_{i+1}, \exp(tP(I)A^T)n_i\rangle \neq 0$ and expanding out the exponential $\exp(tP(I)A^T) = I + tP(I)A^T + ..$ the condition on A guarantees that this above condition is satisfied, and hence under the assumptions of Proposition 8.1 condition (iii) is satisfied. ∎

Remarks One implication of Proposition 8.1 is that the type of complexity arising in non-invertible maps (tent-type maps ([23, Chapter 5] and their higher dimensional analogues) cannot occur in the Willis models in the iterated maps of the type described. Non-negativity of the matrix A has other implications - for example it is easy to see that coupled with the fact that M is an orthant it means that for sufficiently large $|x|$ with $x \in M$ that $\langle X(x),x \rangle \leq 0$ and hence that orbits are bounded - all interesting behaviour occurs in a compact subset of M. Non-negativity also suggests applying the Perron-Frobenius Theorem (see [21]) which implies that on each subsystem $\{x \in R^n : x_i = 0 \ i=1,..,k, \ x_i \geq 0 \ i=k+1,..,n\}$, $k \leq n$, we get for $\dot{x} = P(K)k - P(K)Ax$ a dominant eigenvector with all components positive and with corresponding eigenvalue real and negative and exceeding in magnitude all other eigenvalues. If this implies that for any periodic orbit γ the codimension of the stable manifold of γ is codimension 1 in the deepest stratum through which it passes this would have significant implications for the 4-dimensional Willis model (for example, in conjunction with Proposition 8.1 it would rule out the possibility of chaos in the form of sensitive dependence on initial conditions).

Appendix

Remarks on The Global Properties of the Semiflows

In this appendix we shall sketch some global theory for these systems. We prove Lemma A.1 (which is needed in Chapter 8) but after *** the results are stated without proof. In the case of Conjecture A.1 there are questions outstanding in the local theory we would need to settle first (although there is probably enough in Chapter 5 to establish it on submanifolds with orthogonal corners),in the case of Conjecture A.2 (a two-dimensional result not true in greater than two dimensions) the method of proof we would adopt would provide negligible insight into the global behaviour of these systems in general. In fact insofar as little use is made of the material in Chapters 2-5 this appendix is the least advanced part of the thesis.

We shall suppose we are working with a C^r submanifold with corners M and a C^r vector field X. We observed in Chapter Four that since by definition of submanifold with corners M for each $x \in M$ there exists a neighbourhood of x in M of the form $\beta(U \cap LC(I;J))$ (Chapter One) each stratum m_i of M as a submanifold with corners admits a smooth extension $\check{m}_i$ containing $\bar{m}_i$ in its (relative) interior. Thus for each $x \in \bar{m}_i$, $X(\check{m}_i)(x) =$ the projection of X onto the tangent space (rather than tangent cone) to $\bar{m}_i$ at x.

In constructing a global theory it is convenient to begin with a stratification of M into strata $\{\sigma_i\}$ such that for each σ_i the map $x \to X(M)(x)$ is C^r for as long as x is in σ_i - for example , any stratification $\{\sigma_i\}$ of M which refines the decomposition of M into iteration sets (which itself of course refines the stratification of M as a submanifold with corners) will do. In the case of submanifolds with orthogonal corners a simple stratification with the above property may if $X \in \Xi'(M)$ be constructed by exploiting the fact that in this case if the manifold is locally ZN(I;J) and $I \subset I \cup K_1 \subset I \cup K_1 \cup K_2 \subset I \cup J$ with $K_1 \cap K_2 = \emptyset$ then $\Gamma_2(I \cup K_1 \cup K_2 \text{ r } I \cup K_2) =$ $\Gamma_2(I \cup K_1 \text{ r } I) \cap Z(I \cup K_2)$. This is so because in part (1) of Example 6.7 we saw that if L, L_1, L_2 are linear subspaces of R^n then if $X \in L$ and $N(L_1 \text{ in } L) \subset L_2$ then $P(L_2)X \in L_1 \cap L_2$ iff $X \in L_1$. Since ZN(I;J) is orthogonal we know that for any $x \in Z(I \cup K_1 \cup K_2)$ $N(T_xZ(I \cup K_2) \text{ in } T_xZ(I)) =$ $\text{span}\{\text{grad}f_i(x) : i \in K_2\} \subset \{y : \langle y, \text{grad}f_i(x)\rangle = 0 \text{ for all } i \in I \cup K_1\} = T_xZ(I \cup K_1)$, so applying the above with L, L_1, L_2 set to $T_xZ(I)$, $T_xZ(I \cup K_1)$, $T_xZ(I \cup K_2)$ respectively it follows that $x \in Z(I \cup K_1 \cup K_2)$ and $X(x) \in T_xZ(I \cup K_1)$ iff $x \in Z(I \cup K_1 \cup K_2)$ and

$X(I\cup K_2)(x)\in T_xZ(I\cup K_1\cup K_2)$, ie $x\in Z(I\cup K_1\cup K_2)\cap\Gamma_2(I\cup K_1 \text{ r } I)$ (which by definitions $= Z(I\cup K_2)\cap\Gamma_2(I\cup K_1 \text{ r } I)$) iff $x\in\Gamma_2(I\cup K_1\cup K_2 \text{ r } I\cup K_2)$, and the claim follows.

If $X\in\Xi'(ZN(I;J))$ and we take all possible intersections of sets

$\{x\in Z(I\cup i):\langle \text{grad}f_j(x),X(x)\rangle=0\}$

$\{x\in Z(I\cup i):\langle \text{grad}f_j(x),X(x)\rangle>0\}$

$\{x\in Z(I\cup i):\langle \text{grad}f_j(x),X(x)\rangle<0\}$

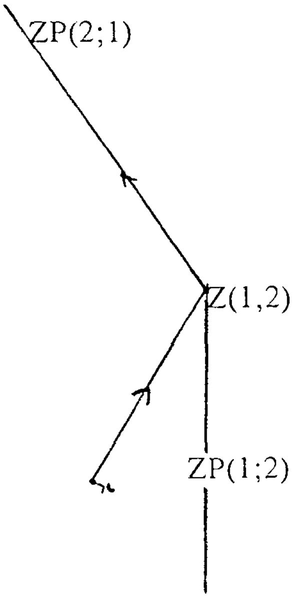
for $i,j\in J$, each intersection is by virtue of the foregoing and Proposition 4.1 an open subset of some $\Gamma_2(K \text{ r } K')$ each of which is by Proposition 4.2 a submanifold of $Z(I)$. If the open subset had infinitely many components this would contradict classical normal form theorems (see [44,45,58]) for classical tangency sets (since using $\Gamma_2(I\cup K_1\cup K_2 \text{ r } I\cup K_2)=\Gamma_2(I\cup K_1 \text{ r } I)\cap Z(I\cup K_2)$ and Proposition 4.1 it follows that there would exist some $\Gamma_2(K \text{ r } I)$ with infinitely many components, which if $X\in\Xi'(M)$ is disallowed by classical normal forms), hence taking all intersections of the sets above yields a locally finite decomposition of $ZN(I;J)$ into submanifolds. Clearly the boundary of the closure of any such set is a union of sets of lower dimension ot the same form, so these sets form in fact a stratification. By its construction and Proposition 4.4 the second iterate $S_2(y)$ is constant for as long as y is in any stratum in this stratification, and hence (since by defintion $X(M)(y)=X(S_2(y))(y)$) for any submanifold with orthogonal corners M we have constructed a stratification refining the stratification of M as a submanifold with corners and such that the map $y\to X(M)(y)$ is C^r on each stratum.

If we are to establish geometric results which will be closely analogous to classical ones [1,37,42] for the behaviour of the semiflow near a trajectory it will be evident that we must exclude from consideration trajectories such as illustrated in Figure 2.3 (Example 2.1) Figures 5.1 - 5.4 (Examples 5.1) and Figure 5.16 (Example 5.3). If $\mathfrak{C}$ is our stratification of M refining that of M as submanifold with corners and such that $x\to X(M)(x)$ is C^r on strata, and if a trajectory segment $\phi(M)(x,[0,T))$ satisfies the regularity condition below with $x\in\sigma\in\mathfrak{C}$, then if $y\in\sigma$ is sufficiently near x there is (Lemma A.1(1)) a T′ near T and a $\mathfrak{C}$-preserving homeomorphism of $M\to M$ mapping $\phi(M)(x,[0,T))$ to $\phi(M)(y,[0,T'))$ (see Figures A.1-A.4). If additionally $\phi(M)(x,[0,T))$ satisfies condition (*) of Lemma A.1 and Σ is a section transverse to $\phi(M)(x)$ in σ then the map $\Sigma\to\phi^T\Sigma$ induced by the semiflow is a diffeomorphism (Lemma A.1(2), see Figures A.5 and A.6).

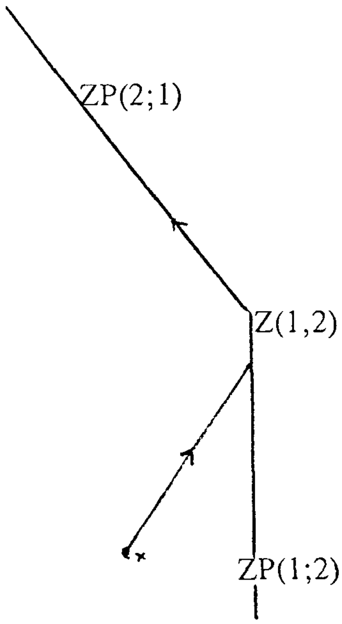
We shall set $\mathfrak{C}_0$=the set of strata of $\mathfrak{C}$ each of which is codimension 0 in the stratum of M as a submanifold with corners which contains it.

<u>Definition</u> A trajectory segment $\phi(M)(x,[0,T))$ is <u>regular</u> for $\mathfrak{C}$ if $[0,T)=\cup_{i=1}^{m}[t_{i-1},t_i)$, where the partition of $[0,T)$ is finite on any bounded subset of $[0,T)$ (so is finite if T is finite) with each $\phi(M)(x,(t_{i-1},t_i))$ contained in a single stratum σ_i of $\mathfrak{C}_0$, and if m_i is the stratum of M as a submanifold with corners containing σ_i (so σ_i is open in m_i by defintion of $\mathfrak{C}_0$, and $\phi(\sigma_i)(x)=\phi(m_i)(x)$ on (t_{i-1},t_i)) then $\phi(\breve{m}_i)(x)$ is transverse in m_i at $\phi(M,X)(x,t_i)$ to the stratum σ_i' of $\mathfrak{C}$ (which may be in σ_{i+1}) occupied by $\phi(M,X)(x,t_i)$ (see Figures A.1 - A.3).

The stratum σ_i' must evidently be in $\bar{\sigma}_i$ and since $\phi(M)(x)$ has dimension 1 must have codimension 1 in $\breve{m}_i$.

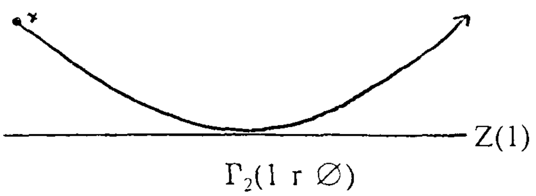


Not regular since Z(1,2) not transverse to $\phi(M)(x)$ in R^2

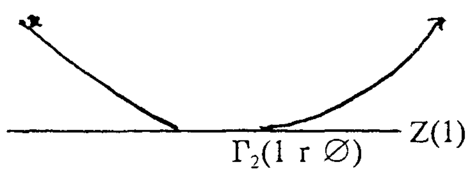


Regular since ZP(1;2) transverse to $\phi(M)(x)$ in R^2 and Z(1,2) transverse to $\phi(M)(x)$ in Z(1)

Figure A.1



Nor regular since $\Gamma_2(1\ r\ \varnothing)$ not transverse to $\phi(M)(x)$ in R^2



Regular since Z(1) transverse to $\phi(M)(x)$ in R^2 and $\Gamma_2(1\ r\ \varnothing)$ transverse to $\phi(M)(x)$ in Z(1)

Figure A.2

Figure A.3 shows a regular trtajectory $\phi(M)(x)$ on $M=ZN(\emptyset;1,2)$ with $\phi(M)(x,t)\in\bar{\sigma}_1,\bar{\sigma}_2,\bar{\sigma}_3,..,\bar{\sigma}_q$ on respectively $t\in[0,t_1),[t_1,t_2),[t_2,t_3),..,[t_{q-1},t_q)$ where in the figure $q=6$ and $\sigma_1=\sigma_3=\sigma_6=ZP(\emptyset;1,2)$, $\sigma_2=\sigma_4=ZP(1;2)$, $\sigma_5=ZP(2;1)$, $\sigma_1'=\sigma_3'=ZP(1;2)$, $\sigma_2'=\Gamma_2^+(1\ r\ \emptyset)$, $\sigma_4'=Z(1,2)$, $\sigma_5'=\Gamma_2^+(2\ r\ \emptyset)$.

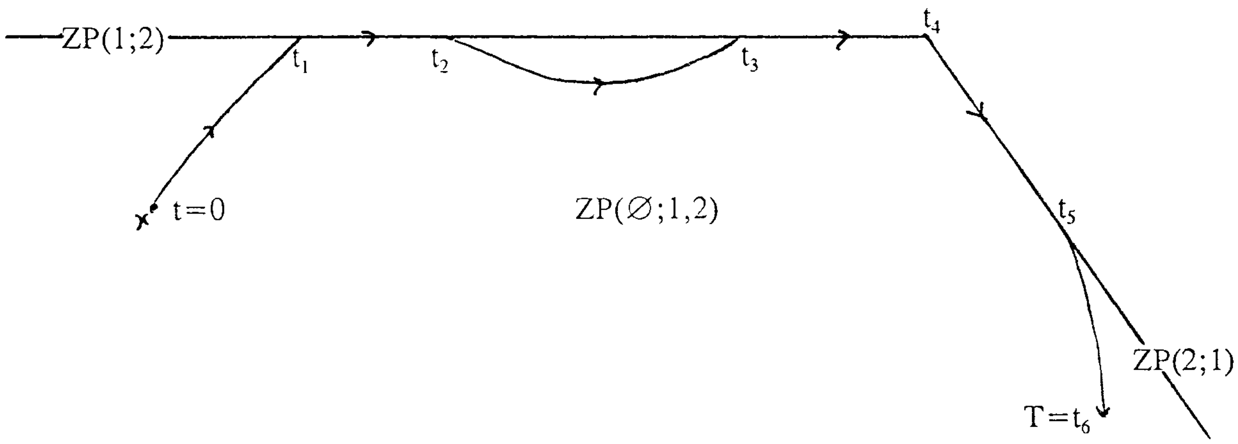


Figure A.3

Remark If M is a submanifold with orthogonal corners and $X\in\Xi'(M)$ with $\mathfrak{C}$ the stratification constructed above then the condition that $\phi(M)(x,[0,T))$ has a finite decomposition into C^r segments $\cup\phi(M)(x,[t_{i-1},t_i))$ with each $\phi(M)(x,(t_{i-1},t_i))$ contained in a single σ_i of $\mathfrak{C}$, where σ_i is open in some stratum of M as a submanifold with corners, holds for any $x\in M$ and $0<T<\infty$.

Proof By Theorem 5.1 if $\phi(M)(x)$ makes infinitely many stratum jumps on a neighbourhood of any point there exists an infinite order tangency between the flows obtained by projecting onto strata at that point, which is not allowed if $X\in\Xi'(M)$. Hence $\phi(M)(x)$ decomposes into finitely many segments, each contained in a single stratum m_i of M as a submanifold with corners; by Remark 3.1(2) $\phi(M)(x)$ is C^r as long as it is contained in a single stratum of M as a submanifold with corners. Finally, it follows from the way we constructed $\mathfrak{C}$ that if $X\in\Xi'(M)$ then no C^r segment $\phi(\check{m}_i)(x,(t_{i-1},t_i))$ can make infinitely many intersections with those strata of $\mathfrak{C}$ which are contained in $\bar{m}_i$, and the result follows. ∎

Lemma A.1 For any submanifold with corners M and stratification $\mathfrak{C}$ of M as above (1) If $\phi(M)(x,[0,T))$ is regular then if $\tilde{x}$ is both near x and in the same stratum of $\mathfrak{C}$ as x, then $\phi(M)(\tilde{x},[0,\tilde{T}))$ is regular some $\tilde{T}$ near T and there exists strictly increasing $\tau:[0,T)\to[0,\tilde{T})$ such that for $0\le t<T$ the stratum of $\mathfrak{C}$ occupied by $\phi(M)(x,t)$ is that occupied by $\phi(M)(\tilde{x},\tau(t))$, and hence if $\phi(M)(x)$ is single valued on [0,T] there exists a $\mathfrak{C}$-preserving homeomorphism of M which maps $\phi(M)(x,[0,T))\to\phi(M)(\tilde{x},[0,\tilde{T}))$ (Fig. A.4).

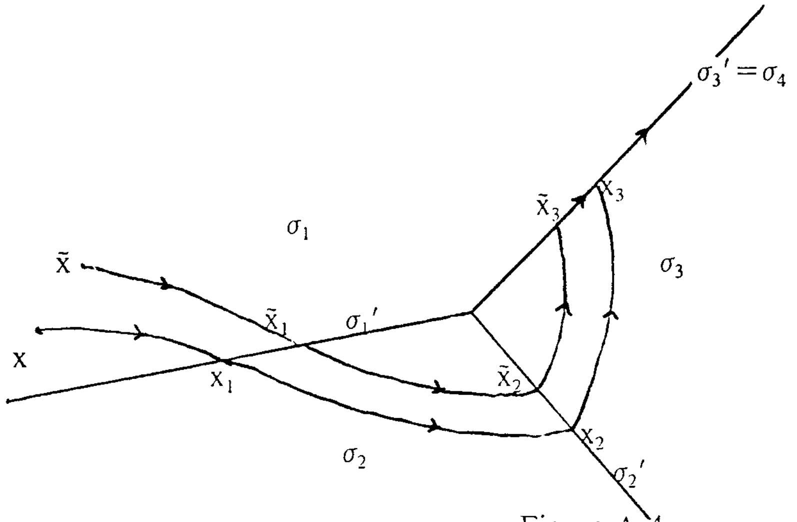


Figure A.4

(2) Suppose $\phi(M)(x,[0,T))$ is regular and suppose with $0=t_0<t_1<..<t_m=T$, with **each $\phi(M)(x,t_{i-1},t_i)) \subset \sigma_i \subset \mathfrak{S}_0$ (where as above σ_i is codimension 0 in m_i). Set** $x_i=\phi(M)(x,t_i)$ and denote the codimension 1 stratum in the boundary of $\bar{\sigma}_i$ which the trajectory intersects at x_i when $t=t_i$ σ_i' (which may be σ_{i+1}). Suppose Σ is a section transverse in σ_1 to $\phi(M)(x)$ at x, ie $T_x\sigma_1=X(\sigma_1)(x)\oplus T_x\Sigma$. Consider a sequence of projections along the flow, of Σ onto the codimension 1 stratum σ_1' of $\bar{\sigma}_1$ at x_1, of this image onto the codimension 1 stratum σ_2' of $\bar{\sigma}_2$ at x_2 etc (see Figure A.5). Call these $\Sigma_1,\Sigma_2,..$ Then if $X(\tilde{m}_{i+1})(x_i) \notin T_{x_i}\Sigma_i$, ie if for each i

$$\dim\mathrm{span}(X(\tilde{m}_{i+1})(x_i),T_{x_i}\Sigma_i)=\dim T_{x_i}\Sigma_i+1 \qquad (*)$$

then there exists a neighbourhood U of x in Σ such that for each $y\in U$ the induced map $y\in\Sigma\rightarrow y_1\in\Sigma_1\subset\sigma_1'\rightarrow y_2\in\Sigma_2\subset\sigma_2'..\rightarrow y_m\in\Sigma_m\subset\sigma_m'$, where y_i is the intersection of $\phi(M)(y)$ with σ_i' at time t near t_i (this is defined and unique for $\phi(M)(x,[0,T))$ regular by Part (1) of the result) is a diffeomorphism.

Remark. Clearly if a regular trajectory segment $\phi(M)(x,[0,T))$ satisfies (*) and $\phi(M)(x,[0,T))$ passes through σ_i', $i=1,..,q$ then $\dim\Sigma\leq\dim\sigma_i'$ for all $i=1,..,q$ - for instance, if Σ was at x in Figure A.4 a trajectory as shown would not satisfy condition (*) because a cross-section in Σ_4 would have dimension less than that of Σ.

To apply Lemma A.1 on the trajectory segment shown in Figure A.5 below there are three regularity conditions R1-R3 and three (*) conditions *1-*3 which must be satisfied, and if they are the conclusion of Lemma A.1 is that the map induced by the flow of $\Sigma\rightarrow\Sigma_3$ is a diffeomorphism. This is close to the form in which the lemma is used in Chapter Eight.

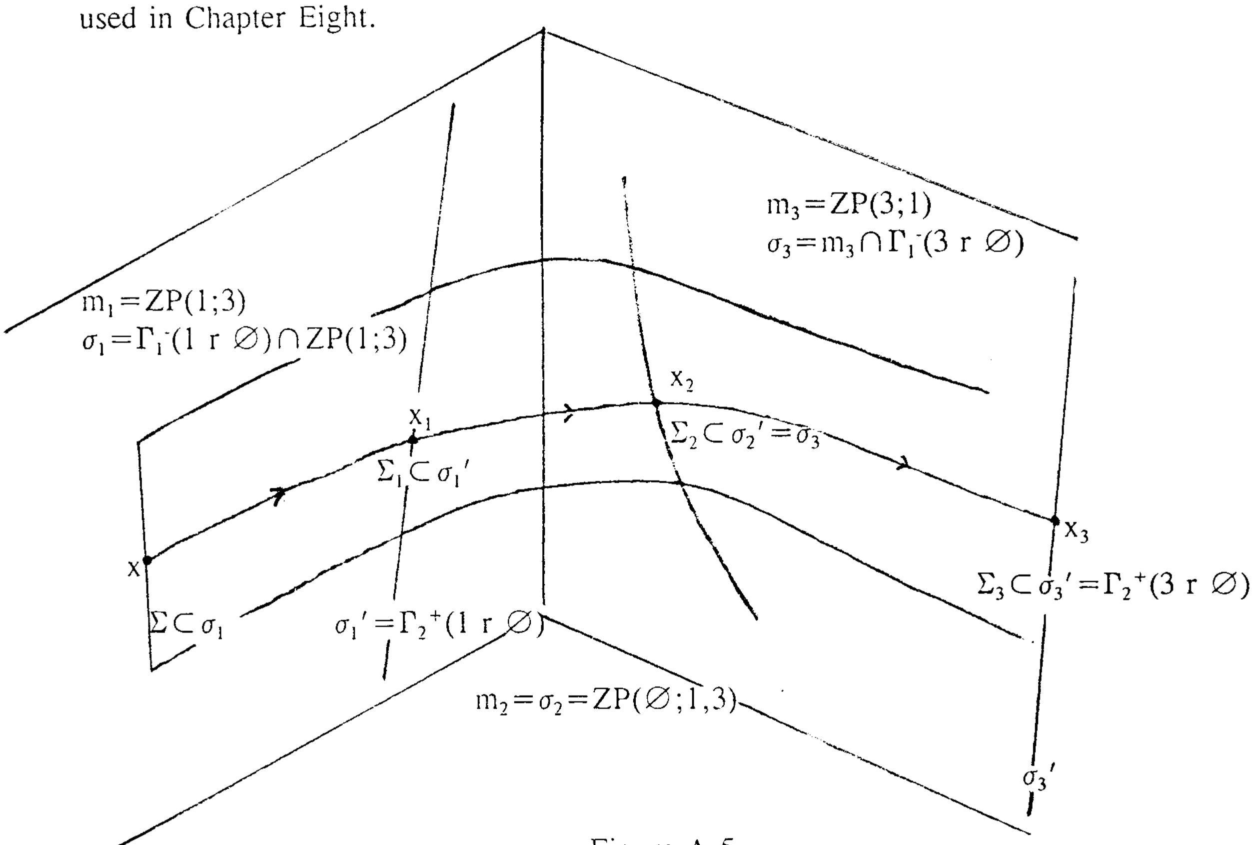


Figure A.5

R1 $X(\breve{m}_1)(x_1) \pitchfork \sigma_1'$ ie $X(1)x_1 = X(x_1) \notin \Gamma_3(1 \text{ r } \emptyset)$

R2 $X(\breve{m}_2)(x_2) \pitchfork \sigma_2'$ ie $X(x_2) \notin \Gamma_2(2 \text{ r } \emptyset)$

R3 $X(\breve{m}_3)(x_3) \pitchfork \sigma_3'$ ie $X(3)x_3 = X(x_3) \notin \Gamma_3(3 \text{ r } \emptyset)$

*1 $X(x_1) \notin T_{x_1}\Sigma_1$ which is equivalent to R1

*2 $X(3)(x_2) \notin T_{x_2}\Sigma_2$

*3 $X(x_3) \notin T_{x_3}\Sigma_3$ which is equivalent to R3

Proof of Lemma A.1

(1) Inductively suppose the result holds up to x_{i-1}, $i \geq 1$. Then $\tilde{x}_{i-1}$ is near x_{i-1} with $\tilde{x}_{i-1} \in \sigma_{i-1}'$ and by smoothness of $X(\breve{m}_i)$ (where we recall $\breve{m}_i$ is the C^r extension of the stratum m_i of M as a submanifold with corners containing σ_i as an open subset) and by openness of transverse intersection if the trajectory $\phi(M)(x_{i-1})$ intersects σ_i' transversely (in $\breve{m}_i$) at $\phi(M)(x_{i-1}, t_i - t_{i-1}) = x_i$ then so will $\phi(M)(\tilde{x}_{i-1})$ at some time close to $t_i - t_{i-1}$ (see Figure A.4).

(2) Suppose this result is true up to the ith stage, ie $\Sigma \rightarrow \Sigma_i$ is a diffeomorphism. We know (by the remark at the end of the definition on page 207) that σ_{i+1}' is locally $\{x \in \breve{m}_{i+1} : f(x) = 0\}$ for some C^r $f: \breve{m}_{i+1} \rightarrow R$, so Σ_{i+1} is locally

$\{\phi(\check{m}_{i+1})(x,t):x\in\Sigma_i,\ f\phi(\check{m}_{i+1})(x,t)=0\}$. By the Implicit Function Theorem we know that if $D_t f\phi(\check{m}_{i+1})(x,t)\neq 0$, ie if $\langle \mathrm{grad} f\phi(\check{m}_{i+1})(x,t), X(\check{m}_{i+1})\phi(\check{m}_{i+1})(x,t)\rangle \neq 0$, which is the case by regularity, then there exists a unique C^r $t:\check{m}_{i+1}\rightarrow R$ with $f\phi(\check{m}_{i+1})(x,t(x))=0$. Hence $\Sigma_i\rightarrow\Sigma_{i+1}$ is C^r. For each $x\in\Sigma_{i+1}$ near x_{i+1} we may map back to Σ_i by $\phi(\check{m}_{i+1})(x,-t(x))$, possibly with more than one value of t(x) for given x if condition (*) does not hold (see Figure A.6).

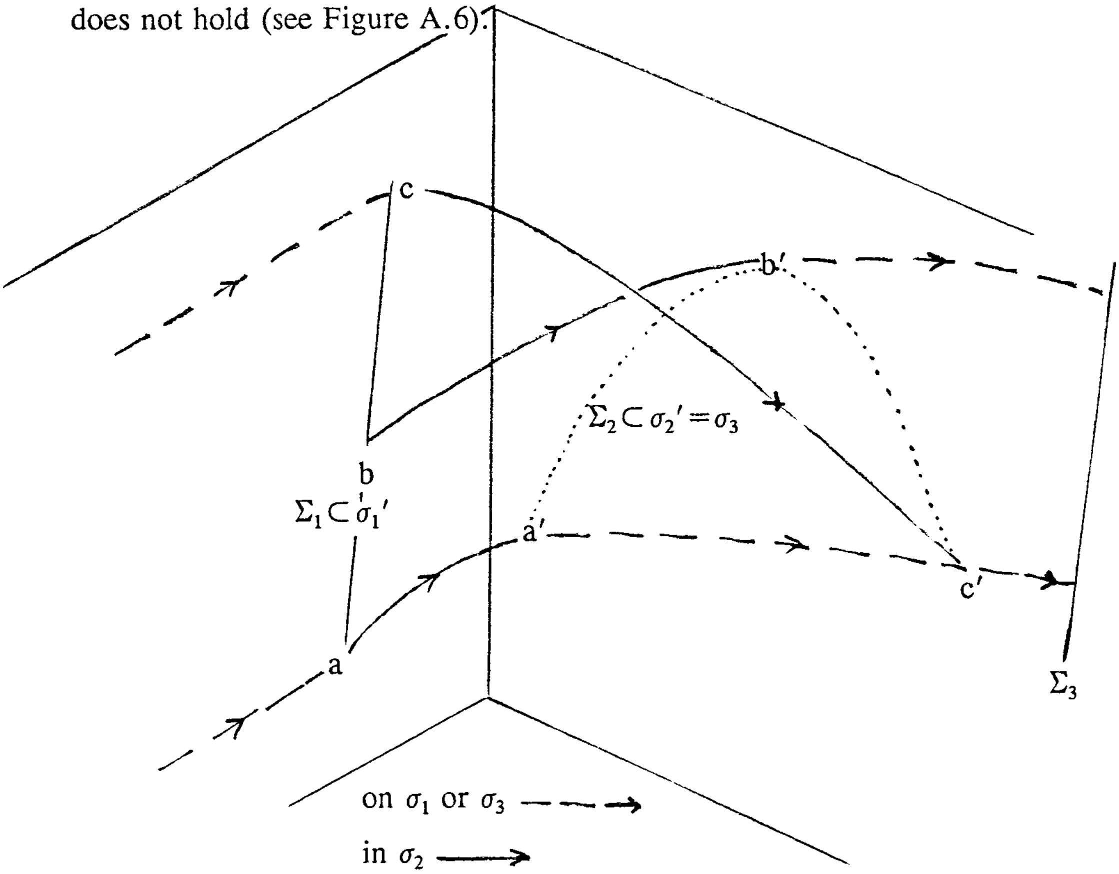


Figure A.6. Manifold and notation exactly as in Figure A.5, but without condition (*) holding at b', where $X(\check{m}_3)(b')\in T_{b'}\Sigma_2$, with the result that $\Sigma_1\rightarrow\Sigma_3$ is C^r but not invertible.

However if condition (*) does hold then $X(\check{m}_{i+1})(y)$ is not tangent to $T_y\Sigma_i$ for all $y\in\Sigma_i$ on a neighbourhood of x_i, and hence we have a C^r submanifold S connecting a neighbourhood of x_i in Σ_i with a neighbourhood of x_{i+1} in Σ_{i+1} formed by acting with the flow $\phi(\check{m}_{i+1})$ on Σ_i, where Σ_i itself is codimension 1 in S (see for instance Figure A.5). Σ_i is locally of the form $\{x\in S:g(x)=0\}$ for some C^r $g:S\rightarrow R$, so near x_i Σ_i is locally the set $\{\phi(\check{m}_{i+1})(x,-t):g\phi(\check{m}_{i+1})(x,t)=0, x\in\Sigma_{i+1}\}$. By condition (*) $D_t g\phi(\check{m}_{i+1})(x,-t)=\langle -\mathrm{grad} g\phi(\check{m}_{i+1})(x,-t), X(\check{m}_{i+1})\phi(\check{m}_{i+1})(x,-t)\rangle\neq 0$ at x_i (and hence on a neighbourhood of x_i) and so by the Implicit Function Theorem again we obtain a unique C^r map from Σ_{i+1} to Σ_i. Hence the induced map $\Sigma_i \to \Sigma_{i+1}$ is a diffeomorphism, which is the required inductive step. ∎

* * *

We end with a discussion of some other global features of these semiflows. Suppose γ is a periodic orbit, ie $\gamma=\phi(M)(x,[0,T))$ for some $x \in M$ where $\phi(M)(x,0)=x=\phi(M)(x,T)$. To obtain analogues of classical results about periodic orbits we will want the conditions to apply Lemma A.1 to hold (so as to get a diffeomorphism on a transverse section) and at its fixed point we will want this diffeomorphism to be hyperbolic.

Defintion With Σ, x, Σ_i and x_i as in Lemma A.1 a periodic orbit is regular if
(1) The segment $\phi(M)(x,[0,T))$ is regular (in the sense defined above)
(2) Condition (*) of Lemma A.1 applies, ie $X(\tilde{m}_{i+1})(x_i) \notin T_{x_i}\Sigma_i$ at each x_i
(3) The diffeomorphism induced on Σ has a hyperbolic fixed point (see [42]) at x,
and we propose in the spirit of Proposition 6.2 part 1:

Conjecture A.1
(1) Under small pertubations in X a regular periodic orbit remains regular and depends continuously on the vector field X
(2) There exists for any compact M a residual subset of $\Xi_\infty(M)$ such that for any X in this subset all zeros and periodic orbits are regular and respectively finite and countable in number.

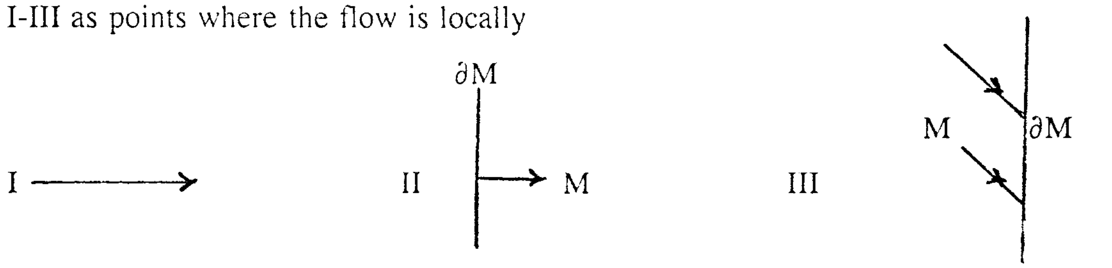
The following is (in view of Chapter Six) the inevitable definition of global structural stability:

Definition A system (M,X) is structurally stable if for any Y sufficiently close to X there exists a spfp homeomorphism conjugating the flows of X(M) and Y(M).

By Lemma 6.1 it follows that if a system is structurally stable the iterated maps obtained when transverse sections are mapped into themselves are stable also, an exacting requirement when $\dim\Sigma \geq 1$ (ie $\dim M \geq 3$) given that this map need not be invertible and there exists all the scope for the complexity of tent-type maps (see [23, Chapter 6]) and their higher dimensional analogues.

These complications do not arise if either X is gradient or $\dim M = 2$. In the gradient case we have as mentioned in Chapter One that on each stratum σ $\mathrm{grad}(f \mid \sigma) = P(T_x\sigma)\mathrm{grad} f(x)$ which implies f is monotone on trajectories , and so here as much as in the unconstrained gradient case ([50]) for generic X or f the non-wandering set is no worse than a finite set of regular zeros.

Finally let us briefly consider two-dimensional systems. To keep matters simple let us suppose M is homeomorphic to $\bar{D}^2$. Let $X \in \Xi_\infty(M)$. We may define Roman points I-III as points where the flow is locally



and Arabic points 1-12 as given in Figure A.7. Let us say a system satisfies condition A if all non-Roman points are Arabic. In the terminology of Chapter Six this means that all the zeros are regular (this includes all the obvious requirements about zeros of X being disjoint from ∂M etc) and that $X \in \Xi_\infty'(M)$. By Proposition 6.2 and Chapter 4 Arabic points are isolated and condition A holds on an open-dense subset $\Xi_A(M) \subset \Xi_\infty(M)$. About each saddle x of X (which will be in int(M) if $X \in \Xi_A(M)$) we may choose a disc $\bar{B}_\epsilon(x)$ so small as to be disjoint from other zeros and ∂M: call the four points on $\partial \bar{B}_\epsilon(x) \cap (W^s(x), W^u(x))$ circle points. Then through any circle point or Arabic point 1-6 there is exactly one trajectory of $X \mid \mathrm{int}(M)$ intersecting it. We say a system satisfying condition A satisfies condition B if no such trajectory intersects another or itself. From the foregoing and classical theory we obtain an open-dense subset $\Xi_{AB}(M)$ of $\Xi_\infty(M)$ of smooth vector fields such that X(M) satisfies A and B (if it does so no periodic orbit of X touches ∂M etc).

A system satisfying A and B satisfies condition C if the non-wandering set of $X \mid int(M)$ consists of a finite number of hyperbolic zeros and hyperbolic periodic orbits (regular =hyperbolic in this case) and again by classical theory the smooth vector fields satisfying A,B and C, $\Xi_{ABC}(M)$ form an open-dense subset of $\Xi_\infty(M)$. This gives Part (1) of the following; the necessity half of Part (2) is immediate.

Conjecture A.2 If M is a smooth 2-dimensional submanifold with corners of R^n
(1) The subset $\Xi_{ABC}(M)$ of vector fields in $\Xi_\infty(M)$ such that X(M) satisfies A,B and C is open-dense in $\Xi_\infty(M)$
(2) The system (M,X) is structurally stable iff $X \in \Xi_{ABC}(M)$.

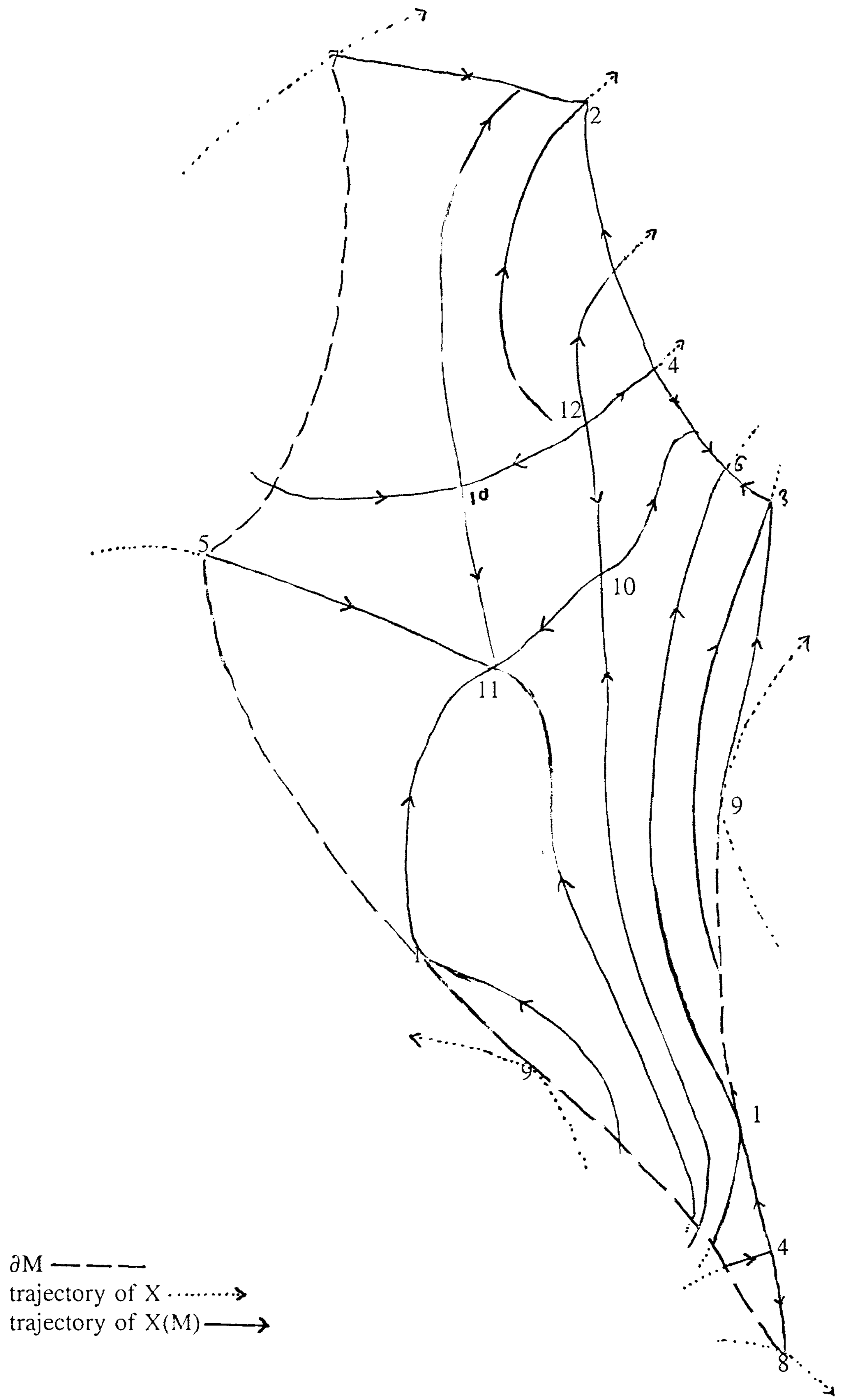


Figure A.7

Notes

Introduction

The type of constraint we consider arises in mechanics as Gauss's "Principle of Least Constraint" ([11,2]): this says that if a body at p of mass m is constrained to lie in a region M but is otherwise freely acted upon by a force-field F then the motion of the body is such as to minimize $G=m(\ddot{p}-F/m)^2$. It follows from the Characterisation of Projection that if M is a submanifold with corners then this implies that the body moves under an effective force-field F_{eff} where $F_{eff}(x)=P(T_xM)F(x)$. The trajectories of Theorem 1.1 are therefore those the body would follow in the limit $m \rightarrow 0$. Modern treatments of mechanical constraints of this type may be found in the book *Rational Mechanics* by C.W Kilmister and J.E. Reeve (London 1966) and in [39].

[60] is the only occurrence known to us of this type of system in mathematical biology, but they occur frequently in mathematical economics - see for instance [3,36,17], with [4, Sections 5.5 and 5.6] containing a good review.

A slightly different type of constraint has been considered by Takens [54-56]; in his version the trajectory moves across a submanifold until encountering one of a set of critical points, where it projects instantaneously across to another part of the submanifold.

Chapter One

Three approaches to proving the results concerning differential equations with discontinuous right hand side upon which Theorem 1.1 is based are due to

(i) Claude Henry, who establishes [31,32] existence of solutions (in our notation) $\phi(M,X)$ for M an orthant. This result is based on the Lasota-Opial Existence Theorem [38].

(ii) M.G. Chikin, who builds on various results of Filippov [19] to establish the result [10] used as the starting point for our Theorem 1.1

(iii) Benard Cornet, in *Existence of Slow Solutions For a Class of Differential Inclusion* [12]. A differential inclusion [4] is of the form $\dot{x}(t) \in F(x)$ where $x \in R^n$ and F is a set valued map. Their connection with differential equations with discontinuous right hand side arises in the following way. If $\dot{x}(t)=f(x)$ with f not necessarily continuous the regularisation of this equation is the differential inclusion $\dot{x}(t) \in F(x)$ defined [4] by taking $F(x) = \cap_{\epsilon>0} conv(f(x+\epsilon B))$: F(x) then has certain desirable properties ($f(x) \in F(x)$ for all x, if f is continuous at x then $F(x)=\{f(x)\}$, the map $x \rightarrow F(x)$ is upper semi-continuous with convex values, [4]). In the case $f(x)=P(T_xM)X(x)$ with M a submanifold with corners we see that $f(x)=(F(x))\bullet$ = by definition the unique element of F(x) with minimal norm, and hence the solution to $\dot{x}(t)=P(T_xM)X(x)$ is the "slow" solution $\dot{x}(t)=[\cap_{\epsilon>0}\{P(T_yM)X(y):y \in x+\epsilon B\}]\bullet$. Cornet's work builds on that of Haddad [24]. For further discussion of differential equations with discontinuous right hand side, multivalued differential equations, and differential inclusions, see in addition to the above [20,4,9,25,33,18].

Chapter Six and The Appendix

The results and discussion of these chapters are related to classical work on the geometric theory of unconstrained flows on manifolds with boundary. The Peixotos characterised structurally stable flows on two-manifolds with boundary in [43]. Sotomayor generalised the Palis-Smale conditions for structural stability on boundarylyess n-manifolds to manifolds with boundary in [51]. Percell characterised structural stability on manifolds with boundary with empty non-wandering set (and so was able to improve conjugating homeomorphism to conjugating diffeomorphism) in [44]. Clark Robinson weakened the defintion of structural stability (his boundary is not fixed) in [47].

Following the work of Newhouse, Palis, Sotomayor and Takens et al on bifurcations of flows on boundarylyess manifolds some study has recently been made of bifurcations of flows on manifolds with boundary, see [52,53]. Structural stability of semiflows in general terms has been considered by Quandt in [46] (see also the references therein).

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Symbols, Notation and Notational Conventions

Greek

Other

C^*, for C a set   Polar cone to C   9

f^*, for f a function   Pull back by f

$\beta_* Y$, for β a function   Push forward of Y by β 14

$X \mid V$, for V a set   X restricted to V   14

0   Zero vector   20

$\langle v_1, v_2 \rangle$ Euclidean inner product   7

$\mid v \mid$ Norm of a vector   7

$\mid J \mid$ Number of elements in the set of indices J   8

$\mid \phi(M)(x,[0,h)) \mid$ 115

$\mathfrak{C}$ A stratification   7

$\mathfrak{C}_1(U,X)$ A $\phi(M,X)$-compatible stratification   176

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$\check{A}$, for A a stratum of a linear corner   see under A

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$m_1 \Rightarrow_X m_2$ abbr. to $m_1 \Rightarrow m_2$, for m_1, m_2 strata of a $\phi(M,X)$-compatible statification   187

$\bar{\sigma}$ Closure of σ

∂A Boundary of A

A^T, for A a matrix   The transpose of A   13

$\pitchfork$ Transverse

$\dot{x}$ time derivative

* C^r if the data is C^r ($r = \infty$ or ω)

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