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AN INVESTIGATION INTO MODULAR FORMS, Q-SERIES, PARTITIONS AND RELATED APPLICATIONS.

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Abstract

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An Investigation into Modular Forms, q-series, Partitions, and Related Applications.

This thesis presents some new identities between Ramanujan's arithmetical function $\tau(n)$ and the divisor functions $\sigma_k(n) = \sum_{d|n} d^k$. Known congruence properties of $\tau(n)$ are used to derive an upper bound M for which it can be shown that $\tau(n) \neq 0$ for all $n \leq M$.

Jacobi's Triple Product Identity and the Quintuple Product Identity along with the Chebyshev Polynomials are used to derive many summation theorems in real α and β, where $\alpha\beta = \pm 1$. It is shown how these can be applied to sums of reciprocals of Fibonacci and Lucas numbers, to produce many new and interesting identities. In fact these results are applicable to any sequence of numbers defined by a second order linear recurrence relation of the form $U_{n+1} = \lambda U_n + U_{n-1}$. It is shown how the well known modular transformations of the standard theta functions θ_2, θ_3 and θ_4 can be used to produce similar results.

Many new and beautiful results of the previous character are arrived at using an elementary idea, without the help of the more advanced theory of elliptic functions used in the derivation of earlier results. Again, these theorems are applicable to any sequence defined by the above second order linear recurrence relation.

Some new polynomial indentities involving Fibonacci and Lucas numbers are derived using Chebyshev-like polynomials. These include a generalisation of the well-known identity $F_{3n} = F_n\{5F_n{}^2 + 3(-1)^n\}$.

Finally some previous results are applied to the theory of highly restricted partitions, identities involving sums of binomial coefficients and representations of the unrestricted partition function.

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Preface

This thesis is an investigation into some strongly interrelated areas of mathematics. Chapter 1 has been inspired by Ramanujan's remarkable paper [25] **"On Certain Arithmetical Functions"**. Chapters 2 and 3 are based on the idea of using Chebyshev and Chebyshev-like polynomials to transform various identities. Since some of these identities involve modular forms, it is not surprising that in chapter 2 we find a number of its results relating back to those of chapter 1. Chapter 4 contains relevant material which has surfaced throughout the course of investigations *into this thesis.*

The new results are fairly evenly distributed throughout this thesis, with perhaps the exception of chapter 1. Here the nature of the subject dictates that a reasonable amount of introductory material is necessary.

Reading Guide

An effort has been made to create an interesting and readable thesis. Therefore it is hoped that the reader will find a reasonable balance has been achieved between too much unnecessary clutter, which would result from unconditionally including the proofs of all the theorems, and the unreadability which results from insufficient explanation. Therefore when several results are very similar, such as lemmas 3.4.1–3.4.6, only the proof of one may be given, the proofs of the remainder being clarified by the given example.

The introductory and concluding sections always refer to the next higher level section. For example §**3.1 Introduction** introduces chapter 3 and §**2.3.4 Closing Remarks** comments on §**2.3 The Chebychev Polynomial Transformations.**

In chapters 1 and 4 equation numbers, theorem numbers, etc ... are all of the form $1.n$ and $4.n$ respectively. In chapters 2 and 3 they are of the form $2.m.n$ and $3.m.n$, where m is the subsection and n is the number within that subsection. For example section 2.2 contains equations 2.2.1–2.2.22.

Introduction

This thesis investigates some intimately interwoven areas of mathematics. Chapter 1 presents some identities between Ramanujan's arithmetical function $\tau(n)$ [25] and the divisor functions $\sigma_k(n) = \sum_{d|n} d^k$. Some of these identities have appeared in the literature before, in various forms, others are new. By explaining why they should exist, these identities are placed in their modern setting. Chapter 1 also highlights the fact that these identities lead to some well known congruence properties for $\tau(n)$. These, plus other congruence properties of $\tau(n)$, are important because they can be used to derive an upper bound M for which it can be shown that $\tau(n) \neq 0$ for all $n \leq M$ [18, 19]. In fact the most famous unsolved problem concerning $\tau(n)$ is a suggestion of D.H. Lemher's, who conjectured that $\tau(n) \neq 0$ for all n. Using congruence properties of $\tau(n)$, readily available in the literature, it is shown how to establish such a bound M slightly larger than any previously published figure.

For general interest, and because of an enthusiasm for numbers, table i of [25] has been strengthened by the addition of the next 16 entries. It is also pointed out how Ramanujan's results, developed in [25], relate to the results of chapter 2. In fact the tables given in chapter 1 can be used to extend some of the results in chapter 2. Chapter 2 presents a transformation of Jacobi's triple product identity (**JTP**) and B. Gordon's quintuple product identity. This transformation uses essentially what are Chebyshev polynomials. Some new and some well known theta function identities are corollaries of this transformation. The transformation is also used to derive many summation theorems in real α and β, where $\alpha\beta = \pm 1$. From these theorems chapter 2 goes on to show how they can be used to derive many identities between sums of reciprocals of Fibonacci and Lucas numbers. Also, the well known modular transformations of the standard theta functions θ_2, θ_3 and θ_4, are used to further enhance these results. Finally, in chapter 2 the transformation is modified to produce summation theorems of a slightly different type.

Chapter 3 presents a much more elementary approach to some of the previously derived theorems. It shows how this new approach enables us to derive some remarkable, given the elementary nature of the proofs, generalisations of previous theorems between sums of reciprocals of Fibonacci and Lucas numbers. This new approach naturally produces some new results of its own.

The transformation of the **JTP** leads directly to some results in the theory of highly restricted partitions. These results are extended by the application of an idea of L.J. Rogers [31]. Using essentially what is the same transformation idea, it is shown how to produce some new polynomial identities between the Fibonacci and the Lucas numbers [16]. These polynomial identities have as corollaries some familiar properties of the Fibonacci and Lucas numbers.

The Chebyshev polynomial transformation also yields some combinatorial results involving sums of binomial coefficients. Some more elementary combinatorial identities are derived in §**3.5** using the lemmas of §**3.4**, plus one additional lemma.

Finally, chapter 4 presents some representations of the unrestricted partition function $p(n)$, where $\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1}$ for $|q| < 1$, which have surfaced throughout the course of the investigations into this thesis.

Chapter 1

Modular forms and Ramanujan's tau function.

§1.1 Introduction

Ramanujan's arithmetical function $\tau(n)$ is defined by the equation

$$\sum_{n=1}^{\infty} \tau(n)x^n = x \prod_{n=1}^{\infty} (1 - x^n)^{24} \qquad \text{for } |x| < 1. \tag{1.1}$$

Ramanujan was the first person to investigate the divisibility properties of $\tau(n)$ in his remarkable 1916 paper **"On Certain Arithmetical Functions"** [25]. He also made some important conjectures on the order of magnitude of $\tau(n)$. The function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)x^n \qquad \text{where } x = e^{2\pi i z} \quad \text{for } Im(z) > 0 \tag{1.2}$$

was one of the earliest known modular forms. So before saying any more, here is the definition of an entire modular form of weight k, where k denotes an integer (positive, negative or zero).

Definition: A function $f(z)$ is said to be an entire modular form of weight k if it satisfies the following conditions:

(i) $f(z)$ is analytic in the upper half plane $H = \{z \mid Im(z) > 0\}$.
(ii) $f(Az) = (cz + d)^k f(z)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the modular group (described below).
(iii) $f(z)$ has a Fourier expansion of the form $f(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi i n z}$.

The inhomogeneous modular group $PSL(2, Z)$, often denoted by Γ, is the set of Möbius transformations

$$T : z \mapsto \frac{az + b}{cz + d} \qquad \text{where } a, b, c \text{ and } d \in Z \quad \text{and} \quad ad - bc = 1.$$

$PSL(2, Z)$ is a discrete subgroup of $PSL(2, R)$. That is to say it is a Fuchsian group [17].

For our discussions we also need the definition of a cusp form.

Definition: A function $f(z)$ is said to be a cusp form of weight k if it satisfies the conditions for an entire modular form of weight k and $c(0) = 0$, where $f(z) = \sum_{n=0}^{\infty} c(n)e^{2\pi inz}$.

The function $\Delta(z)$ is a cusp form of weight 12. To show this one only needs to prove the following two equations.

(i) $\Delta(z+1) = \Delta(z)$.
(ii) $\Delta(-1/z) = z^{12}\Delta(z)$.

This is because $z \to z+1$ corresponds to the transformation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $z \to -1/z$ corresponds to the transformation $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Together these two transformations generate the modular group Γ, and so modularity follows. The proof of (i) is trivial, since $z \to z+1$ leaves $x = e^{2\pi iz}$ fixed. For a proof of (ii) see [2].

§1.2 Some Background

Before we proceed with the main results the following background information is certainly worth mentioning in any discussion of Ramanujan's $\tau(n)$ function. The main conjectures concerning $\tau(n)$ in Ramanujan's 1916 paper [25] were as follows:

(i) $\tau(mn) = \tau(m)\tau(n)$ whenever $(m,n) = 1$. ie. $\tau(n)$ is a multiplicative function. Here (m,n) stands for the greatest common divisor of m and n.

(ii) The Dirichlet series
$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}.$$
The product is taken over all primes. It is known as an Euler product expansion. This conjecture is equivalent to
$$\tau(p)\tau(p^r) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1}) \qquad \text{for prime } p \text{ and integer } r \geq 1.$$

(iii) $|\,\tau(n)\,| \geq n^{11/2}$ for an infinity of n. ie. $\limsup_{n\to\infty} |\,\tau(n)n^{-11/2}\,| > 0$.

(iv) $|\,\tau(p)\,| \leq 2p^{11/2}$ for all primes p.

(v) $|\,\tau(n)\,| \leq d(n)n^{11/2}$ for all n. Here $d(n)$ is the number of divisors of n.

The first two conjectures were proved by Louis J. Mordell in 1917 [21]. His paper marked the beginnings of the theory of Hecke operators. Conjecture (iii) was proved by G. H. Hardy in 1917, but was not published until 1927 [11]. The last two conjectures on the order of magnitude of $\tau(n)$ remained open until as recently as 1974, when they were finally proved by Paul Deligne [6]. In 1937 E. Hecke published the full exposition of his work on Euler products and associated modular forms [14]. He determined all entire modular forms of weight k whose Fourier coefficients satisfy the following multiplicative property.

$$c(m)c(n) = \sum_{d|(m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right) \tag{1.3}$$

The sum in (1.3) is taken over all the divisors of the greatest common divisor of m and n. Ramanujan's $\tau(n)$ function satisfies such a relation. In fact the modular discriminant $\Delta(z)$ is a cusp form of weight 12, so $\tau(n)$ has the following multiplicative property.

$$\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11} \tau\left(\frac{mn}{d^2}\right) \tag{1.4}$$

Notice that when $(m,n) = 1$ equation (1.4) reduces to $\tau(m)\tau(n) = \tau(mn)$. The set of entire modular forms of weight k forms a linear space M_k over the complex numbers. In his characterisation of the modular forms whose Fourier coefficients satisfy equation (1.3) he defined on M_k the map T_n, now known as the Hecke operator.

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{j=0}^{d-1} f\left(\frac{nz+jd}{d^2}\right) \qquad \text{where } f \in M_k \tag{1.5}$$

Hecke showed that $T_n : M_k \to M_k$. A non-zero function f satisfying a relation of the form

$$T_n f = c(n)f,$$

for some complex scalar $c(n)$, is called an eigenfunction of T_n. If f is an eigenfunction for every Hecke operator T_n, $n \geq 1$, then f is called a simultaneous eigenfunction. It is said to be normalised if $c(1) = 1$.

What Hecke showed was that if $f \in M_{2k,0}$, the linear space of cusp forms of weight $2k$ ($M_{2k,0}$ is a subspace of M_{2k} of dimension one less than M_{2k}), then f is a normalised simultaneous eigenfunction if and only if its Fourier coefficients satisfy equation (1.3). Hecke also wanted to show that $M_{2k,0}$ has a basis which consists entirely of normalised simultaneous eigenfunctions. However, it was Hans Petersson [23] who first proved this in 1939.

§1.3 A conjecture of D. H. Lehmer's

The most outstanding unsolved problem concerning $\tau(n)$ is a suggestion of D. H. Lehmer's who conjectured that $\tau(n) \neq 0$ for all n. Using congruence properties of $\tau(n)$ this conjecture was verified by D. H. Lehmer for all $n \leq 113\,740\,236\,287\,998$ [18]. The previous number is the largest M published in the literature for which it has been proved that $\tau(n) \neq 0$ for all $n \leq M$. However, using some readily available results, we note that it is a trivial matter to slightly improve upon this bound.

It can be shown fairly easily [18] that if n is the smallest integer for which $\tau(n) = 0$, then n is a prime number. From [33] we know that $\tau(n)$ satisfies the following congruences, where $\sigma_k(n) = \sum_{d|n} d^k$.

$$\begin{aligned} \tau(n) &\equiv \sigma_{11}(n) \bmod 2^{11} && n \equiv 1 \bmod 8 \\ \tau(n) &\equiv 1217\sigma_{11}(n) \bmod 2^{13} && n \equiv 3 \bmod 8 \\ \tau(n) &\equiv 1537\sigma_{11}(n) \bmod 2^{12} && n \equiv 5 \bmod 8 \\ \tau(n) &\equiv 705\sigma_{11}(n) \bmod 2^{14} && n \equiv 7 \bmod 8 \end{aligned}$$

So if p_0 is the smallest integer for which $\tau(p_0) = 0$ we know that p_0 is prime, and from the above congruences, since $\sigma_{11}(p_0) = p_0^{11} + 1$, we have

$$p_0{}^{11} \equiv -1 \bmod 2^{11}.$$

But this implies $p_0 \equiv 7 \bmod 8$, and so by the last of the above congruences we know that

$$p_0{}^{11} \equiv -1 \bmod 2^{14}.$$

Now by using the Euclidean algorithm it is fairly easy to prove the following result.

Theorem: *If $p^m \equiv -1 \bmod q^n$, for positive integers m and n, where q is a prime number and $(p,q) = 1$, then we have $p^d \equiv (-1)^d \bmod q^n$, where $d = (m, q-1)$.*

Since we must have $p_0{}^{11} \equiv -1 \bmod 2^{14}$, if $\tau(p_0) = 0$ and $\tau(n) \neq 0$ for all $n < p_0$, then by the above theorem we have

$$p_0 \equiv -1 \bmod 2^{14}.$$

Similarly we can use the following congruences for $\tau(n)$ to show that we must have

$$\begin{aligned} p_0 &\equiv -1 \bmod 3^7 \\ p_0 &\equiv -1 \bmod 5^3 \\ p_0 &\equiv -1 \bmod 691 \end{aligned}$$

plus some other congruence properties modulo 7 and 23.

$$n^{610}\tau(n) \equiv \sigma_{1231}(n) \bmod 3^6 \qquad n \equiv 1 \bmod 3$$
$$n^{610}\tau(n) \equiv \sigma_{1231}(n) \bmod 3^7 \qquad n \equiv 2 \bmod 3$$
$$n^{30}\tau(n) \equiv \sigma_{71}(n) \bmod 5^3$$

$$\tau(n) \equiv n\sigma_9(n) \bmod 7 \qquad \left(\tfrac{n}{7}\right) = 1$$
$$\tau(n) \equiv n\sigma_9(n) \bmod 7^2 \qquad \left(\tfrac{n}{7}\right) = -1$$

where $\left(\frac{n}{7}\right)$ is the Legendre symbol.

$$\tau(n) \equiv 0 \bmod 23 \qquad \left(\tfrac{n}{23}\right) = -1$$
$$\tau(n) \equiv \sigma_{11}(n) \bmod 691$$

Using the above congruences we have

$$p_0 \equiv -1 \bmod 2^{14}3^7 5^3 691.$$

Since 113 740 236 288 000 $\equiv 0 \bmod 2^{12}$ and is $\not\equiv 0 \bmod 2^{13}$, we can slightly improve the previous bound of D. H. Lehmer's. Combining the information contained in the above congruences we can conclude that the first possible value of p_0 is the first prime > 113740236288000 of the form $ny-1$, where $y = 2^{14}3^7 5^3 691$. Hence we must have $n \geq 37$. Since

$$y \equiv 3 \bmod 7$$

and

$$y \equiv -1 \bmod 23$$

the above congruences for $\tau(n)$ modulo 7 and 23 show that n must be one of the forms

$$\{7^2 m + k \mid k = 0,\ 30 \text{ or } 48 \quad m = 0, 1, 2 \ldots\}$$

AND one of the forms

$$\{23m + k \mid k = 0, 1, 2, 3, 5, 7, 8, 11, 12, 15 \text{ or } 17 \quad m = 0, 1, 2 \ldots\}.$$

Thus the first possibility for n is 48. But $y \equiv 3 \bmod 11$, so $48y - 1 \equiv 0 \bmod 11$. Hence this p_0 is divisible by 11. The next possibilities are $n = 49, 97, 146 \ldots$ until we encounter the first $ny - 1$ which is prime. This turns out to be $n = 392$ and hence we have

Theorem: $\tau(n) \neq 0$ for all $n \leq 1\,213\,229\,187\,071\,998$.

D. H. Lehmer's conjecture is related to the vanishing of the Poincaré series $G_{12}(z, m)$ [28], where $G_k(z, m)$ is defined by

$$G_k(z, m) = \frac{1}{2} \sum_T (cz + d)^{-k} e^{2\pi i m T(z)},$$

where m is any positive integer, $z \in H$ (the upper half-plane) and $T(z) = \frac{az+b}{cz+d}$. The summation is extended over all matrices $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with different second rows in the homogeneous modular group $\{T \mid a, b, c, d \in Z,\ ad - bc = 1\}$. In fact

$$G_{12}(z, m) = c_{12} m^{11} \tau(m) \Delta(z)$$

where $\Delta(z)$ is the modular discriminant and c_{12} is a constant such that

$$c_{12} = \frac{4\pi^8}{21.691\phi(8)\phi(11)}$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \qquad \text{for} \qquad Re(s) > 13/2.$$

Hence the Poincaré series vanishes identically if and only if $\tau(m)$ vanishes. The problem of the non-vanishing of $\tau(n)$ is a difficult one. A Russian mathematician, N. V. Kuznetsov, claimed in the early eighties that he had proved D. H. Lehmer's conjecture. He set out to furnish a proof in two papers, the first of which is readable, but the second of which has never been published. Mathematicians who have seen the manuscript of the second paper have been unable to follow his very complicated arguments. So the current status of the conjecture is that it is still open.

§1.4 Some new identities for Ramanujan's $\tau(n)$ function

Why they exist: If $M_{2k,0}$ is the linear space of cusp forms of weight $2k$, where $2k \geq 12$, then the dimension of $M_{2k,0}$ [2] is given by

$$dim\, M_{2k,0} = \left[\frac{2k}{12}\right] - 1 \qquad \text{for } 2k \equiv 2 \bmod 12$$

$$dim\, M_{2k,0} = \left[\frac{2k}{12}\right] \qquad \text{for } 2k \not\equiv 2 \bmod 12. \tag{1.6}$$

Now since the modular discriminant $\Delta(z) \in M_{12,0}$, and by equations (1.6) $dim\, M_{12,0} = 1$. We have that every cusp form of weight 12 is a constant multiple of $\Delta(z)$. In fact $\Delta(z)$ is a normalised eigenfunction for each T_n (defined by equation 1.5) with corresponding eigenvalue $\tau(n)$.

The Eisenstein series E_{2k} defined by

$$E_{2k}(z) = \frac{1}{2\zeta(2k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^{2k}} \qquad \text{for } Im(z) > 0, \tag{1.7}$$

where the summation extends over all integral m and n not both equal to 0, are entire modular forms of weight $2k$ (for $k \geq 2$) for the full modular group [2]. Here $\zeta(z)$ is the Riemann zeta function, given by

$$\zeta(z) = \prod_p (1 - p^{-s})^{-1} \qquad \text{for } Re(s) > 1.$$

The product being taken over all primes.

Since

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}$$

where B_{2k} are the Bernoulli numbers, defined by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \qquad \text{for } |z| < 2\pi,$$

and we can show by differentiation of the partial fraction decomposition formula for the cotangent

$$\pi \cot \pi z = \frac{1}{z} + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \left\{ \frac{1}{z+m} - \frac{1}{m} \right\} \qquad \text{for } z \neq 0$$

that

$$\sum_{(m,n)\neq(0,0)} \frac{1}{(m+nz)^{2k}} = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)e^{2\pi i n z} \qquad \text{for } Im(z) > 0,$$

where $\sigma_k(n) = \sum_{d|n} d^k$. We have

$$E_{2k}(z) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)e^{2\pi i n z} \qquad \text{for } Im(z) > 0. \tag{1.8}$$

Now every modular form for the full modular group is a polynomial in E_4 and E_6 [27]. So if we use the Eisenstein series to construct cusp forms of weight 12, because $dim\, M_{12,0} = 1$ these will necessarily be constant multiples of $\Delta(z)$. Therefore we can expect identities between the divisor functions $\sigma_{2k-1}(n)$ and $\tau(n)$ to exist. Ramanujan [25] showed how to express the series

$$\begin{aligned}\phi_{r,s}(x) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn} \\ &= \sum_{n=1}^{\infty} n^r \sigma_{s-r}(n)x^n \qquad \text{for integer } r \text{ and } s \geq 0, \text{ and } |x| < 1,\end{aligned} \tag{1.9}$$

as a polynomial in P, Q and R, where

$$\begin{aligned} P &= 1 - 24\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} \\ Q &= 1 + 240\sum_{n=1}^{\infty} \frac{n^3x^n}{1-x^n} \\ R &= 1 - 504\sum_{n=1}^{\infty} \frac{n^5x^n}{1-x^n}. \end{aligned} \tag{1.10}$$

Note that P, Q and R are the familiar Eisenstein series E_2, E_4 and E_6. $P = E_2$ is not a modular form, but is the logarithmic derivative of the discriminant function $\Delta(z)$. The normalised modular discriminant is given by

$$\begin{aligned} 1728\Delta(z) &= {E_4}^3 - {E_6}^2 \\ &= Q^3 - R^2. \end{aligned} \tag{1.11}$$

The procedure employed is to express $Q^3 - R^2$ as a polynomial in the $\phi_{r,s}(x)$, defined by equation 1.9, using the results in Ramanujan's tables (i)-(iii) of [25], plus an additional entry which is calculated later on. This procedure is effectively that used by D. Niebur [22] to produce the identity

$$\tau(n) = n^4\sigma(n) - 24\sum_{k=1}^{n-1}(35k^4 - 52k^3 n + 18k^2n^2)\sigma(k)\sigma(n-k), \tag{1.12}$$

where $\sigma(n)$ stands for the sum of the divisors of n. However, after an extensive search of the literature most of the identities appear to be new, with the exceptions of theorem 1.7, which is D. H. Lehmer's equation (10) of [19], and theorems 1.1 and 1.2 which are equations (52) and (53) respectively (in a slightly different form) of [34]. Of course all the formulae are essentially variants of each other, differing only in the known expressions for $\sigma_k(n)$. So they can be transformed into D. Niebur's equation (1.12) above or some of the formulae in B. Van der Pol's paper [34]. However, I do not think this detracts from their beauty. Moreover, different congruence properties satisfied by $\tau(n)$ are immediate from the different representations.

To prove theorems 1.1–1.7 we need the following tables from [25].

TABLE i

1. $1 - 24\phi_{0,1}(x) = P$
2. $1 + 240\phi_{0,3}(x) = Q$
3. $1 - 504\phi_{0,5}(x) = R$
4. $1 + 480\phi_{0,7}(x) = Q^2$
5. $1 - 264\phi_{0,9}(x) = QR$
6. $691 + 65520\phi_{0,11}(x) = 441Q^3 + 250R^2$
7. $1 - 24\phi_{0,13}(x) = Q^2R$
8. $3617 + 16320\phi_{0,15}(x) = 1617Q^4 + 2000QR^2$
9. $43867 - 28728\phi_{0,17}(x) = 38367Q^3R + 5500R^3$
10. $174611 + 13200\phi_{0,19}(x) = 53361Q^5 + 121250Q^2R^2$
11. $77683 - 552\phi_{0,21}(x) = 57183Q^4R + 20500QR^3$
12. $236364091 + 131040\phi_{0,23}(x) = 49679091Q^6 + 176400000Q^3R^2 + 10285000R^4$
13. $657931 - 24\phi_{0,25}(x) = 392931Q^5R + 265000Q^2R^3$
14. $3392780147 + 6960\phi_{0,27}(x) = 489693897Q^7 + 2507636250Q^4R^2 + 395450000QR^4$
15. $1723168255201 - 171864\phi_{0,29}(x) = 815806500201Q^6R + 88134070500Q^3R^3 + 26021050000R^5$
16. $7709321041217 + 32640\phi_{0,31}(x)$
    $= 764412173217Q^8 + 5323905468000Q^5R^2 + 1621003400000Q^2R^4$

TABLE ii

1. $288\phi_{1,2}(x) = Q - P^2$
2. $720\phi_{1,4}(x) = PQ - R$
3. $1008\phi_{1,6}(x) = Q^2 - PR$
4. $720\phi_{1,8}(x) = Q(PQ - R)$
5. $1584\phi_{1,10}(x) = 3Q^3 + 2R^2 - 5PQR$
6. $65520\phi_{1,12}(x) = P(441Q^3 + 250R^2) - 691Q^2R$
7. $144\phi_{1,14}(x) = Q(3Q^3 + 4R^2 - 7PQR)$

TABLE iii

1. $1728\phi_{2,3}(x) = 3PQ - 2R - P^3$
2. $1728\phi_{2,5}(x) = P^2Q - 2PR + Q^2$
3. $1728\phi_{2,7}(x) = 2PQ^2 - P^2R - QR$
4. $8640\phi_{2,9}(x) = 9P^2Q^2 - 18PQR + 5Q^3 + 4R^2$
5. $1728\phi_{2,11}(x) = 6PQ^3 - 5P^2QR + 4PR^2 - 5Q^2R$
6. $6912\phi_{3,4}(x) = 6P^2Q - 8PR + 3Q^2 - P^4$
7. $3456\phi_{3,6}(x) = P^3Q - 3P^2R + 3PQ^2 - QR$
8. $5184\phi_{3,8}(x) = 6P^2Q^2 - 2P^3R - 6PQR + Q^3 + R^2$
9. $20736\phi_{4,5}(x) = 15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5$
10. $41472\phi_{4,7}(x) = 7(P^4Q - 4P^3R + 6P^2Q^2 - 4PQR) + 3Q^3 + 4R^2$

To produce theorem 1.7 we need an additional formula for $\phi_{5,6}(x)$, which is not given in Ramanujan's tables. So as an illustration of how we can simply derive the formulae in tables (ii) and (iii) from those in table (i) the calculation of $\phi_{5,6}(x)$ is presented below.

From table (iii), entry 9, we have

$$20736\phi_{4,5}(x) = 15PQ^2 - 20P^2R + 10P^3Q - 4QR - P^5.$$

Now if we differentiate the above formula with respect to x and multiply through by x. Then use the following formulae

$$x\frac{dP}{dx} = \frac{P^2 - Q}{12}$$
$$x\frac{dQ}{dx} = \frac{PQ - R}{3}$$
$$x\frac{dR}{dx} = \frac{PR - Q^2}{2}$$

we obtain

$$\begin{aligned}20736\phi_{5,6}(x) = {} & 15Q^2\frac{(P^2-Q)}{12} + 30PQ\frac{(PQ-R)}{3} \\ & - 40PR\frac{P^2-Q}{12} - 20P^2\frac{(PR-Q^2)}{2} \\ & + 30P^2Q\frac{(P^2-Q)}{12} + 10P^3\frac{(PQ-R)}{3} \\ & - 4R\frac{(PQ-R)}{3} - 4Q\frac{(PR-Q^2)}{2} - 5P^4\frac{(P^2-Q)}{12}.\end{aligned}$$

This simplifies to

$$248832\phi_{5,6}(x) = 5(45P^2Q^2 - 24PQR - 40P^3R + 15P^4Q - P^6) + 9(Q^3 - R^2) + 25R^2. \qquad (1.13)$$

As an example of how to derive theorems 1.1–1.7, below is presented the proof of theorem 1.7.

Proof:

$$\begin{aligned}35(3PQ - 2R - P^3)^2 = {} & -35(45P^2Q^2 - 24PQR - 40P^3R + 15P^4Q - P^6) \\ & + 315(6P^2Q^2 - 4PQR - 4P^3R + P^4Q) + 140R^2\end{aligned} \qquad (1.14)$$

Now from table (iii), entry 1, we have

$$35(3PQ - 2R - P^3)^2 = 35.1728^2\phi_{2,3}{}^2(x) \qquad (1.15)$$

and from table (iii), entry 9, we have

$$315(6P^2Q^2 - 4PQR - 4P^3R + P^4Q) = 45.41472\phi_{4,7}(x) - 45(3Q^3 + 4R^2). \qquad (1.16)$$

Now if we use equations (1.13), (1.15) and (1.16) to replace $-35(45P^2Q^2 - 24PQR - 40P^3R + 15P^4Q - P^6)$, $35(3PQ - 2R - P^3)^2$ and $315(6P^2Q^2 - 4PQR - 4P^3R + P^4Q)$ in equation (1.14) we obtain

$$72(Q^3 - R^2) = 1866240\phi_{4,7}(x) - 1741824\phi_{5,6}(x) - 104509440{\phi_{2,3}}^2(x). \tag{1.17}$$

Using equation (1.11) to substitute for $Q^3 - R^2$ in equation (1.17) we obtain

$$\sum_{n=1}^{\infty} \tau(n)x^n = 15\phi_{4,7}(x) - 14\phi_{5,6}(x) - 840{\phi_{2,3}}^2(x). \tag{1.18}$$

Equating coefficients of x in equation (1.18) gives

$$\tau(n) = 15n^4\sigma_3(n) - 14n^5\sigma(n) - 840\sum_{k=1}^{n-1} k^2(n-k)^2\sigma(k)\sigma(n-k),$$

where $\sigma(n)$ stands for the sum of the divisors of n. □

The identities: All the following theorems were derived in a similar way to that given in the proof of theorem 1.7. We take $\sigma_k(0) = \frac{1}{2}\zeta(-k) = -\frac{1}{2}B_{k+1}/(k+1)$.

Theorem 1.1: $\tau(n) = 70\sum_{k=0}^{n}(2n-5k)\sigma_3(k)\sigma_5(n-k)$

Theorem 1.2: $\tau(n) = 60\sum_{k=0}^{n}(n-3k)(2n-3k)\sigma_3(k)\sigma_3(n-k)$

Theorem 1.3: $\tau(n) = 30\sum_{k=0}^{n}(18k^2-5n^2)\sigma_3(k)\sigma_3(n-k)$

Theorem 1.4: $\tau(n) = n^2\sigma_7(n) - 540\sum_{k=1}^{n-1}k(n-k)\sigma_3(k)\sigma_3(n-k)$

Theorem 1.5: $\tau(n) = n^5\sigma(n) - 120\sum_{k=1}^{n-1}k^2(n-k)(4n-7k)\sigma(k)\sigma(n-k)$

Theorem 1.6: $\tau(n) = n^4\sigma_3(n) - 168\sum_{k=1}^{n-1}k^2(n-k)(3n-5k)\sigma(k)\sigma(n-k)$

Theorem 1.7: $\tau(n) = 15n^4\sigma_3(n) - 14n^5\sigma(n) - 840\sum_{k=1}^{n-1}k^2(n-k)^2\sigma(k)\sigma(n-k)$

As another example of the derivation of the above theorems, and because it is probably the most appealing of the above formulae, below is presented the proof of theorem 1.1.

Proof: From table (ii), entry 5, we have

$$1584\phi_{1,10}(x) = 3(Q^3-R^2) - 5R(PQ-R) \tag{1.19}$$

and from table (ii), entry 2, and table (i), entry 3, we have respectively

$$720\phi_{1,4}(x) = PQ - R \tag{1.20}$$

and

$$1 - 504\phi_{0,5}(x) = R. \tag{1.21}$$

So by substituting for Q^3-R^2, $PQ-R$ and R in equation (1.19) from equations (1.11), (1.20) and (1.21) we obtain

$$36\sum_{n=1}^{\infty}\tau(n)x^n = 11\phi_{1,10}(x) + 25\phi_{1,4}(x) - 12600\phi_{0,5}(x)\phi_{1,4}(x). \tag{1.22}$$

Equating coefficients of x in equation (1.22) gives us

$$36\tau(n) = 11n\sigma_9(n) + 25n\sigma_3(n) - 12600\sum_{k=1}^{n-1} k\sigma_3(k)\sigma_5(n-k). \tag{1.23}$$

Now from entries 2, 3 and 5 we can derive the following formula

$$\textstyle\sum_{3,5}(n) = 11\sigma_9(n)/5040 \tag{1.24}$$

where

$$\textstyle\sum_{r,s}(n) = \sigma_r(0)\sigma_s(n) + \sigma_r(1)\sigma_s(n-1) + \sigma_r(2)\sigma_s(n-2) + \cdots + \sigma_r(n)\sigma_s(0) \quad \text{for } r \text{ and } s \geq 1.$$

Alternatively equation (1.24) is entry 5 of table (iv) of [25]. We now substitute for $\sigma_9(n)$ in equation (1.23), using equation (1.24), and note that $\sigma_3(0) = \frac{1}{240}$ and $\sigma_5(0) = -\frac{1}{504}$ to give theorem 1.1. □

The following congruence properties of $\tau(n)$ are immediate corollaries of theorems 1.4–1.7.

Corollary 1.4: $\tau(n) \equiv n^2\sigma_7(n) \bmod 540$

Corollary 1.5: $\tau(n) \equiv n^5\sigma(n) \bmod 120$

Corollary 1.6: $\tau(n) \equiv n^4\sigma_3(n) \bmod 168$

Corollary 1.7: $\tau(n) \equiv 15n^4\sigma_3(n) - 14n^5\sigma(n) \bmod 840$

§1.5 Some More Entries for Ramanujan's Table i

Using the methods of Ramanujan what would be the next 16 entries of table (i) of [25] have been calculated. They are presented below.

17.
$$\begin{aligned}151628697551 - 24\phi_{0,33}(x) = {} & 55884495051Q^7R \\ & + 88296652500Q^4R^3 \\ & + 7447550000QR^5\end{aligned}$$

18.
$$\begin{aligned}26315271553053477373 + 138181680\phi_{0,35}(x) = {} & 179233997323066062 3Q^9 \\ & + 16289678778066366750Q^6R^2 \\ & + 8095347742018950000Q^3R^4 \\ & + 137905059737500000R^6\end{aligned}$$

19.
$$\begin{aligned}154210205991661 - 24\phi_{0,37}(x) = {} & 43635626965161Q^8R \\ & + 94782461476500Q^5R^3 \\ & + 15792117550000Q^2R^5\end{aligned}$$

20.
$$\begin{aligned}26108271849644912 2051 + 1082400\phi_{0,39}(x) = {} & 1221524232081296705 1Q^{10} \\ & + 14035566485639853000 0Q^7R^2 \\ & + 1032762120134876250 00Q^4R^4 \\ & + 52355993057500000 00QR^6\end{aligned}$$

21.
$$\begin{aligned}1520097643918070802691 - 151704\phi_{0,41}(x) = {} & 3265679818668716666 91Q^9R \\ & + 933063597397822086000Q^6R^3 \\ & + 257701549269802050000Q^3R^5 \\ & + 2764515383575000000R^7\end{aligned}$$

22.
$$\begin{aligned}2530297234481911294093 + 5520\phi_{0,43}(x) = {} & 813213049056512303 43Q^{11} \\ & + 1152360895010751483750Q^8R^2 \\ & + 1176474112693246080000Q^5R^4 \\ & + 120140921872262500000Q^2R^6\end{aligned}$$

23. $$2593265702582226796 8607 - 1128\phi_{0,45}(x) = 4191478936397766158607Q^{10}R + 152360117156642957 10000Q^7R^3 + 6297623266945481100000Q^4R^5 + 2075431068147250000000QR^7$$

24. $$5609403368997817686249127547 + 4455360\phi_{0,47}(x) = 1238394326019673615113175 47Q^{12} + 212125575430247047456041 0000Q^9R^2 + 286871497998232180192740 0000Q^6R^4 + 49205290771498346100000000 0Q^3R^6 + 3540294396074587250000000R^8$$

25. $$990114410482159296424955 05 - 1320\phi_{0,49}(x) = 11948877704441488092133005Q^{11}R + 53837124192236422014862500Q^8R^3 + 31124299240028012035500000Q^5R^5 + 21011399115100075000000000Q^2R^7$$

26. $$6162813216426845825753269 1681 + 12720\phi_{0,51}(x) = 934610402421629413310453931Q^{13} + 1903437829609499496202433 7750Q^{10}R^2 + 3293228338672630027057290 0000Q^7R^4 + 853336897794279142912500 0000Q^4R^6 + 193491101082742182500000000QR^8$$

27. $$29149963634884862421418123812691 - 86184\phi_{0,53}(x) = 26098339647980415756221226 70191Q^{12}R + 1427329813105840771970258 2042500Q^9R^3 + 1099897738269374153136041 9100000Q^6R^5 + 1261469523817891331905500 000000Q^3R^7 + 638463251678026282750000 0000R^9$$

28.

$$
\begin{aligned}
&3541989899018895362407736770947 47 + 13920\phi_{0,55}(x) \\
&= 3689846390899369128325431357747Q^{14} \\
&+ 8811919413730566821462868601 2000Q^{11}R^2 \\
&+ 1898340217467421244853221597 25000Q^8R^4 \\
&+ 6929068999599642770042490000 0000Q^5R^6 \\
&+ 3265237630945946712072500000 0000Q^2R^8
\end{aligned}
$$

29.

$$
\begin{aligned}
&2913228046513104891794716413587449 - 1416\phi_{0,57}(x) \\
&= 1924389278698819966495885228064 49Q^{13}R \\
&+ 1255715586766721522493597349881000Q^{10}R^3 \\
&+ 1244173044364725746319365790900000Q^7R^5 \\
&+ 2173600551276028370495422500000000Q^4R^7 \\
&+ 354043238417278928262250000000 0QR^9
\end{aligned}
$$

30.

$$
\begin{aligned}
&12152331404837555720403049940798202460414 91 + 6814407600\phi_{0,59}(x) \\
&= 869620814471833296898145965448830881524 1Q^{15} \\
&+ 2406706956202871992626106937720091574762 50Q^{12}R^2 \\
&+ 6317106874861647392785435592755311547500 00Q^9R^4 \\
&+ 3090709285498832742607789808169041250000 00Q^6R^6 \\
&+ 2499225031461919019835085324526250000000 00Q^3R^8 \\
&+ 9237036808283607103944731562500000000 0R^{10}
\end{aligned}
$$

31.

$$
\begin{aligned}
&396793078518930920708162576045270521 - 24\phi_{0,61}(x) \\
&= 192466693873389102033980300149095 21Q^{14}R \\
&+ 147704550023299495208656602247011000Q^{11}R^3 \\
&+ 182972719041223793685248595133350000Q^8R^5 \\
&+ 453157658219000306299335961500000000Q^5R^7 \\
&+ 155337424516869098092575250000000 0Q^2R^9
\end{aligned}
$$

32.

$$
\begin{aligned}
&10678383014786652988638544497914264794201 7 + 65280\phi_{0,63}(x) \\
&= 524910392452989599674733535167186302017Q^{16} \\
&+ 1666463802594818121548947837141819452 0000Q^{13}R^2 \\
&+ 5235483455119738350304375992260966712 0000Q^{11}R^4 \\
&+ 3308439071688447549338519488277760000 00000Q^8R^6 \\
&+ 4105361257321586439281874247170000000 00000Q^5R^8 \\
&+ 496952040619136355104040200000000000000Q^2R^{10}
\end{aligned}
$$

§1.6 Closing Remarks

As has been shown, congruence properties of $\tau(n)$, such as those given in corollaries 1.4–1.7, are important because they can be used to give a bound M for which we can prove $\tau(n) \neq 0$ for all $n \leq M$.

None of the congruences given by corollaries 1.4–1.7 are new. In fact 1.4–1.6 all appear in Ramanujan's lost notebook [26], along with many similar congruences. It is interesting to note that congruences similar to those given in section 1.4, for example

$$\tau(n) \equiv \sigma_3(n) \bmod 32 \qquad \text{for odd } n$$

$$\tau(n) \equiv n\sigma_9(n) \bmod 25 \qquad \text{for all } n$$

of paper [4], appeared in the literature in 1946, but Ramanujan had discovered the stronger congruences

$$\tau(n) \equiv \sigma_3(n) \bmod 256 \qquad \text{for odd } n$$

$$\tau(n) \equiv n\sigma_9(n) \bmod 1050 \qquad \text{for all } n$$

some years previously [26]. It can also be shown [33] that there are no congruences modulo primes other than 2, 3, 5, 7, 23 and 691.

Unfortunately the formulae in theorems 1.1–1.7, nice as they are, cannot be used to shed any more light on the problem of the vanishing of $\tau(n)$. For example, because there is so much cancellation taking place between the terms of the series, they give only trivial upper bounds on the order of magnitude of $\tau(n)$.

Chapter 2

A Transformation using Chebyshev Polynomials.

§2.1 Introduction

The general theme of this chapter is a transformation using essentially what are Chebyshev polynomials. After some explanatory background information, it is shown how this transformation is applied to Jacobi's triple product identity (equation 2.2.5) and B.Gordon's analogous quintuple product identity (equation 2.2.1) [8] to produce some familiar, and other, theta function identities. We are also lead to some interesting summation identities involving real α and β, where $\alpha\beta = \pm 1$ and $|\beta| < 1$. These have as corollaries some identities involving sums of reciprocals of Fibonacci and Lucas numbers. It is also shown how these identities can be enhanced using results from the theory of elliptic functions, and the results of Ramanujan's paper [25] of chapter 1.

Finally, it is shown how to modify the transformation to produce some results of a slightly different nature.

§2.2 Some Background

§2.2.1 The Chebyshev Polynomials

These polynomials were discovered more than a century ago by the Russian mathematician Chebyshev (the spelling has many variations). They are more at home in the field of numerical analysis, which with the advent of the computer has come to prominence over the last few decades. The Chebyshev polynomials are defined on the closed interval $[-1,1]$ by

$$T_n(x) = \cos n\theta \qquad \text{where } \cos\theta = x, \quad -1 \le x \le 1.$$

From the trigonometric identity $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta$, it follows that the Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad \text{with } T_0(x) = 1 \text{ and } T_1(x) = x.$$

Solving this recurrence relation we obtain

$$2T_{2n}(x) = \sum_{j=0}^{n} (-1)^{n+j} \frac{2n}{n+j} \binom{n+j}{2j} 2^{2j} x^{2j} \qquad \text{for } n \ge 0 \tag{2.2.1}$$

and

$$2T_{2n+1}(x) = \sum_{j=0}^{n} (-1)^{n+j} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} 2^{2j+1} x^{2j+1} \qquad \text{for } n \ge 0 \tag{2.2.2}$$

Now we replace x with $\cos\theta$ in equations (2.2.1) and (2.2.2). Then let $z = e^{i\theta}$, so that $z + 1/z = 2\cos\theta$. So we have

$$z^{2n} + \frac{1}{z^{2n}} = \sum_{j=0}^{n} (-1)^{n+j} \frac{2n}{n+j} \binom{n+j}{2j} \left(z + \frac{1}{z}\right)^{2j} \qquad \text{for } n \ge 1 \tag{2.2.3}$$

and

$$z^{2n+1} + \frac{1}{z^{2n+1}} = \sum_{j=0}^{n} (-1)^{n+j} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} \left(z + \frac{1}{z}\right)^{2j+1} \qquad \text{for } n \ge 0 \tag{2.2.4}$$

It is equations (2.2.3) and (2.2.4) that are applied extensively in this chapter.

§2.2.2 Jacobi's Triple Product Identity

Jacobi's triple product identity (**JTP**) states that for complex q and z with $|q| < 1$ and $z \neq 0$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n. \tag{2.2.5}$$

For fixed $z \neq 0$ the series and products represent analytic functions of q in the disk $|q| < 1$. Although Gauss was first to discover equation (2.2.5) this identity is named after Jacobi who discovered it in the course of his work on theta functions, where it arises naturally. Hereafter equation (2.2.5) shall be referred to as Jacobi's triple product identity, or for short **JTP**.

The **JTP** has numerous consequences of interest in number theory and combinatorial analysis. Among these are

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \tag{2.2.6}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} \tag{2.2.7}$$

Equation (2.2.6) is due to Euler and is known as Euler's pentagonal number theorem. Equation (2.2.7) is a famous result of Jacobi's, which can be easily derived from the **JTP**.

There are many proofs of the **JTP** in the literature, including some combinatorial proofs. For completeness an analytic proof is included below. The quintuple product identity can be proved along similar lines.

Proof: For $|q| < 1$ and $z \neq 0$ let

$$\phi(z) = \prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}).$$

Then

$$\phi(zq^2) = \prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n+1})(1+z^{-1}q^{2n-3})$$

$$= \frac{1+z^{-1}q^{-1}}{1+zq}\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})$$

$$= z^{-1}q^{-1}\phi(z). \tag{2.2.8}$$

Now $\phi(z)$ can be expanded as a Laurent series in the deleted neighbourhood of $z = 0$. Hence $\phi(z) = \sum_{n=-\infty}^{\infty} A_n(q)z^n$. So using the functional equation (2.2.8) for $\phi(z)$, we have

$$\begin{aligned}\phi(z) &= \sum_{n=-\infty}^{\infty} A_n(q)z^n \\ &= zq\phi(zq^2) \\ &= \sum_{n=-\infty}^{\infty} A_n(q)z^{n+1}q^{2n+1}.\end{aligned}$$

Hence by equating coefficients of z^n we have

$$A_n(q) = q^{2n-1}A_{n-1}(q).$$

So by the appropriate iteration we have

$$A_n(q) = q^{n^2}A_0(q) \qquad \text{for all } n.$$

Hence

$$\phi(z) = A_0(q)\sum_{n=-\infty}^{\infty} q^{n^2}z^n. \tag{2.2.9}$$

Therefore to complete the proof we need only show that $A_0(q) = 1$. Letting $z = e^{i\pi/2} = i$ in equation (2.2.9) and writing $\phi_q(z)$ for $\phi(z)$, to show the dependence on q, we have

$$\frac{\phi_q(i)}{A_0(q)} = \sum_{n=-\infty}^{\infty} q^{n^2}i^n$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n)^2}, \qquad \text{since } i^n = -i^{-n} \text{ for odd } n.$$

But from equation (2.2.9) we have

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(2n)^2} = \frac{\phi_{q^4}(-1)}{A_0(q^4)}.$$

Therefore

$$\frac{\phi_q(i)}{A_0(q)} = \frac{\phi_{q^4}(-1)}{A_0(q^4)}. \tag{2.2.10}$$

From the definition of $\phi(z)$ we have

$$\begin{aligned}
\phi_q(i) &= \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{4n-2}) \\
&= \prod_{n=1}^{\infty} (1 - q^{4n})(1 - q^{4n-2})(1 + q^{4n-2}) \\
&= \prod_{n=1}^{\infty} (1 - q^{8n})(1 - q^{8n-4})(1 - q^{8n-4}) \\
&= \phi_{q^4}(-1).
\end{aligned}$$

So from equation (2.2.10) we have $A_0(q) = A_0(q^4)$. Replacing q by q^4, q^{4^2}, ... we obtain

$$A_0(q) = A_0(q^{4^k}) \qquad \text{for } k = 1, 2, \ldots$$

But $q^{4^k} \to 0$ as $k \to \infty$ and $A_0(q) \to 1$ as $q \to 0$. So we must have $A_0(q) = 1$ for all q. □

§2.2.3 The Quintuple Product Identity

The quintuple product identity is an elegant fivefold analogue of Jacobi's triple product identity. It was first presented by B. Gordon [8]. However, M. V. Subbarao and M. Vidyasagar [32] have observed that Gordon was anticipated some 32 years earlier by G. N. Watson [35], who stated and proved a fivefold product identity easily seen to be equivalent to equation (2.2.11). The quintuple product identity states that for complex q and z, with $|q| < 1$ and $z \neq 0$, we have

$$\prod_{n=1}^{\infty}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1})(1-z^2q^{2n-1})(1-z^{-2}q^{2n-1}) = \sum_{n=-\infty}^{\infty}(z^{3n}-z^{-3n-1})q^{n(3n+1)/2}. \tag{2.2.11}$$

Like the **JTP** the quintuple product identity also has numerous consequences of interest in number theory and combinatorial analysis. Among these are

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2(1-q^{4n})^2 = \sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n} \tag{2.2.12}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{2n-1})^2 = \sum_{n=-\infty}^{\infty}(6n+1)q^{n(3n+1)/2} \tag{2.2.13}$$

The quintuple product identity can be proved in a similar way to the proof given previously for the **JTP**. Having established the **JTP** we can evaluate the constant term in the proof of the quintuple product identity by setting $z = -1$.

§2.2.4 The Theta Functions

For our discussions it is also necessary to introduce the theta functions. The basic theta functions are defined for complex z and τ, with $Im(\tau) > 0$ and $q = e^{i\pi\tau}$, by

$$\theta_1(z,\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)i\pi z}$$

$$\theta_2(z,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)i\pi z}$$

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2i\pi n z}$$

$$\theta_4(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2i\pi n z}.$$

The theta functions are of great importance in number theory, particularly in the theory of partitions and in the theory of the representation of a number as a sum of squares [24]. Only θ_2, θ_3 and θ_4, with z set equal to zero, will be used in our discussions. So that with $Im(\tau) > 0$ and $q = e^{i\pi\tau}$ we have

$$\theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \tag{2.2.14}$$

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{2.2.15}$$

$$\theta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \tag{2.2.16}$$

where $\theta_2(0) = 0$, $\theta_3(0) = 1$ and $\theta_4(0) = 1$. The functions θ_2, θ_3 and θ_4 are all entire modular forms of weight 1/2. In fact they are modular forms for the groups $\Gamma_U(2)$, $\Gamma_V(2)$ and $\Gamma_W(2)$ [27], where

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and $\Gamma_U(2) = \{S \in \Gamma \mid S \equiv I \text{ or } U \bmod 2\}$. $\Gamma_V(2)$ and $\Gamma_W(2)$ being similarly defined. Here

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \text{ and } a,\, b,\, c,\, d \in Z\}$$

is the homogeneous modular group. It is easy to show using the **JTP** that the theta functions have the following product expansions.

$$\theta_2(q) = 2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2 \tag{2.2.17}$$

$$\theta_3(q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2 \tag{2.2.18}$$

$$\theta_4(q) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2 \tag{2.2.19}$$

Also, as a consequence of their modularity, the theta functions satisfy the following transformation formulae [5, 24, 27]. For $Re(s) > 0$ we have

$$\sqrt{s}\theta_3(e^{-\pi s}) = \theta_3(e^{-\pi/s}) \tag{2.2.20}$$

$$\sqrt{s}\theta_2(e^{-\pi s}) = \theta_4(e^{-\pi/s}) \tag{2.2.21}$$

$$\sqrt{s}\theta_4(e^{-\pi s}) = \theta_2(e^{-\pi/s}). \tag{2.2.22}$$

Here equation (2.2.21) is just (2.2.22) with $s := s^{-1}$.

§2.3 The Chebyshev Polynomial Transformations

§2.3.1 Introduction

In this section the Chebyshev polynomial transformation is applied to Jacobi's triple product identity and the quintuple product identity to derive theorems 2.3.1 and 2.3.2. From these theorems are deduced some theta function identities.

§2.3.1 The Transformation of Jacobi's Triple Product Identity

Theorem 2.3.1 For complex q and z with $|q| < 1$ and $z \neq 0$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})\left\{1+\left(z+\frac{1}{z}\right)q^{2n-1}+q^{4n-2}\right\}$$
$$=\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j}\frac{2n}{n+j}\binom{n+j}{2j}q^{(2n)^2}\left(z+\frac{1}{z}\right)^{2j}$$
$$+\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}(-1)^{n+j}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}q^{(2n+1)^2}\left(z+\frac{1}{z}\right)^{2j+1},$$

where $2n/(n+j)$ is taken to be 1 for $n = j = 0$.

Proof: The **JTP** states that for complex q and z with $|q| < 1$ and $z \neq 0$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1})=\sum_{n=-\infty}^{\infty}q^{n^2}z^n.$$

Therefore

$$\prod_{n=1}^{\infty}(1-q^{2n})\left\{1+\left(z+\frac{1}{z}\right)q^{2n-1}+q^{4n-2}\right\}=1+\sum_{n=1}^{\infty}\left(z^n+\frac{1}{z^n}\right)q^{n^2}.$$

Applying equations (2.2.3) and (2.2.4) to the above the **RHS** is seen to be equal to

$$1+\sum_{n=1}^{\infty}\sum_{j=0}^{n}(-1)^{n+j}\frac{2n}{n+j}\binom{n+j}{2j}\left(z+\frac{1}{z}\right)^{2j}q^{(2n)^2}$$
$$+\sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^{n+j}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}\left(z+\frac{1}{z}\right)^{2j+1}q^{(2n+1)^2}.$$

Which upon interchanging the order of summation of n and j completes the proof of theorem 2.3.1. □

Below are presented some corollaries of theorem 2.3.1. These are derived by equating the coefficients of $(z+1/z)^k$ for $k=0$, 1 and 2 to obtain corollaries 2.3.1, 2.3.2 and 2.3.3 respectively. Note that although it is easy to determine the coefficient of $(z+1/z)^k$ for $k \leq 2$ in the **LHS** of theorem 2.3.1 by inspection, generally we can evaluate the coefficient as follows.

Let $x = z + 1/z$, so that the **LHS** of theorem 2.3.1 is given by

$$\begin{aligned} g(x) &= \prod_{n=1}^{\infty}(1-q^{2n})(1+xq^{2n-1}+q^{4n-2}) \\ &= \sum_{n=0}^{\infty} a_n(q)x^n. \end{aligned} \tag{2.3.1}$$

Then we can determine the $a_n(q)$ by differentiating equation (2.3.1) with respect to x, and setting $x=0$. So that

$$a_n(q) = \frac{1}{n!}g^{(n)}(0),$$

where $g^{(n)}(x)$ is the nth derivative of $g(x)$.

Corollary 2.3.1 For $|q| < 1$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{4n^2}$$
$$= \theta_4(q^4).$$

Corollary 2.3.2 For $|q| < 1$ we have

$$\theta_2(q^2)^2 = 4\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}.$$

Corollary 2.3.3 For $|q| < 1$ we have

$$\frac{1}{2}\theta_4(q^4)\left\{\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\right)^2 - \sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1+q^{4n-2})^2}\right\} = \sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{4n^2}.$$

The last equality in corollary 2.3.1 is straight from the definition of $\theta_4(q)$ (equation 2.2.16). Corollaries 2.3.1 and 2.3.3 come *directly* from equating coefficients in theorem 2.3.1. Corollary 2.3.2 requires a small amount of extra work. So its proof is presented below.

Proof: Equating coefficients of the term $(z + 1/z)$ in theorem 2.3.1 we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}} = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(2n+1)^2}.$$

We now use equation (2.2.7), $\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}$, theorem 357 of [13], to obtain

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}} = q\prod_{n=1}^{\infty}(1-q^{8n})^3.$$

Hence

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}} = q\prod_{n=1}^{\infty}(1-q^{8n})^2(1+q^{4n})^2,$$

which from equation (2.2.17) is $\frac{1}{4}\theta_2(q^2)^2$. □

Corollaries 2.3.1 and 2.3.2 are certainly well known. Corollary 2.3.3 is unlikely to have appeared in the literature before because it is cumbersome. However, it is useful because we require it in the proof of some later theorems.

One further corollary of theorem 2.3.1 is presented because it is important in the proof of theorem 2.4.24, which itself can be used to derive a beautiful identity involving sums of reciprocals of Fibonacci and Lucas numbers – theorem 2.4.25. Equating the coefficients of $(z + 1/z)^3$ in theorem 2.3.1 we obtain

Corollary 2.3.4 For $|q| < 1$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})$$
$$\times\left\{\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\right)^2 - 3\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1+q^{4n-2})^2} + 2\sum_{n=1}^{\infty}\frac{q^{6n-3}}{(1+q^{4n-2})^3}\right\}$$
$$=\sum_{n=1}^{\infty}(-1)^{n+1}n(n+1)(2n+1)q^{(2n+1)^2}.$$

§2.3.3 The Transformation of the Quintuple Product Identity

Theorem 2.3.2 For complex q and z with $|q| < 1$ and $z \neq 0$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})\left\{1+\left(z+\frac{1}{z}\right)q^{2n-1}+q^{4n-2}\right\}\left\{(1+q^{4n-4})^2-\left(z+\frac{1}{z}\right)^2 q^{4n-4}\right\}$$
$$=\sum_{j=0}^{\infty}\sum_{n=[(j+3)/3]}^{\infty}(-1)^{n+j}q^{12n^2-4n}\frac{6n-2}{3n-1+j}\binom{3n-1+j}{2j}\left(z+\frac{1}{z}\right)^{2j}$$
$$+\sum_{j=0}^{\infty}\sum_{n=[(j+2)/3]}^{\infty}(-1)^{n+j}q^{12n^2-4n}\frac{6n}{3n+j}\binom{3n+j}{2j}\left(z+\frac{1}{z}\right)^{2j}$$
$$+\sum_{j=0}^{\infty}\sum_{n=[(j+1)/3]}^{\infty}(-1)^{n+j}q^{12n^2+4n}\frac{6n+2}{3n+1+j}\binom{3n+1+j}{2j}\left(z+\frac{1}{z}\right)^{2j}$$
$$+\sum_{j=0}^{\infty}\sum_{n=[(j+2)/3]}^{\infty}(-1)^{n+j}q^{12n^2+8n+1}\frac{6n+1}{3n+1+j}\binom{3n+1+j}{2j+1}\left(z+\frac{1}{z}\right)^{2j+1}$$
$$+\sum_{j=0}^{\infty}\sum_{n=[(j+1)/3]}^{\infty}(-1)^{n+j}q^{12n^2+8n+1}\frac{6n+3}{3n+2+j}\binom{3n+2+j}{2j+1}\left(z+\frac{1}{z}\right)^{2j+1}$$
$$+\sum_{j=0}^{\infty}\sum_{n=[j/3]}^{\infty}(-1)^{n+j}q^{12n^2+16n+5}\frac{6n+5}{3n+3+j}\binom{3n+3+j}{2j+1}\left(z+\frac{1}{z}\right)^{2j+1}.$$

where $[x]$ stands for the greatest integer less than or equal to x and $6n/(3n+j)$ is taken to be 1 for $n = j = 0$.

Proof: To prove theorem 2.3.2 we use B. Gordon's [8] quintuple product identity in the following form [5].

For complex q and z with $|q| < 1$ and $z \neq 0$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}z^{-1})(1-q^{4n-4}z^2)(1-q^{4n-4}z^{-2})$$
$$=\sum_{n=-\infty}^{\infty}q^{3n^2-2n}\left\{(z^{3n}+z^{-3n})-(z^{3n-2}+z^{-3n+2})\right\}.$$

The proof is essentially the same as that for theorem 2.3.1. That is to say, we use equations (2.2.3) and (2.2.4) to substitute for $(z^{3n}+z^{-3n})$ and $(z^{3n-2}+z^{-3n+2})$ in the **RHS** of the above, the main difference being that we must treat separately the cases when $3n$ and $3n-2$ are odd and even. For example, after letting $z := -z$ in the above, we need to transform

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n}(z^{3n-2}+z^{-3n+2}) \\
&= \sum_{n=-\infty}^{\infty} q^{12n^2-4n}(z^{6n-2}+z^{-6n+2}) - \sum_{n=-\infty}^{\infty} q^{12n^2+8n+1}(z^{6n+1}+z^{-6n-1}).
\end{aligned}$$

The first summation on the **RHS** of the above is

$$\sum_{n=0}^{\infty} q^{12n^2+4n}\left(z^{6n+2}+\frac{1}{z^{6n+2}}\right)+\sum_{n=0}^{\infty} q^{12n^2+20n+8}\left(z^{6n+4}+\frac{1}{z^{6n+4}}\right).$$

We can now use equations (2.2.3) and (2.2.4) to obtain

$$\left(z^{6n+2}+\frac{1}{z^{6n+2}}\right)=\sum_{j=0}^{3n+1}(-1)^{n+j+1}\frac{6n+2}{3n+1+j}\binom{3n+1+j}{2j}\left(z+\frac{1}{z}\right)^{2j}$$

and

$$\left(z^{6n+4}+\frac{1}{z^{6n+4}}\right)=\sum_{j=0}^{3n+2}(-1)^{n+j}\frac{6n+4}{3n+2+j}\binom{3n+2+j}{2j}\left(z+\frac{1}{z}\right)^{2j}.$$

The full details are long and laborious, so that it is hoped that the above is enough to convey the idea to the reader. However, it is worth mentioning that when we interchange the order of summations we use the following rules.

Subject to conditions of convergence we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{3n+2} f_x(n,j) = \sum_{j=0}^{\infty}\sum_{n=[j/3]}^{\infty} f_x(n,j)$$
$$\sum_{n=0}^{\infty}\sum_{j=0}^{3n+1} f_x(n,j) = \sum_{j=0}^{\infty}\sum_{n=[(j+1)/3]}^{\infty} f_x(n,j)$$
$$\sum_{n=0}^{\infty}\sum_{j=0}^{3n} f_x(n,j) = \sum_{j=0}^{\infty}\sum_{n=[(j+2)/3]}^{\infty} f_x(n,j),$$

where $f_x(n,j)$ is some function of n and j, dependent on x, and $[y]$ stands for the greatest integer less than or equal to y. □

As with theorem 2.3.1, we can now equate the coefficients of $(z+1/z)^k$ for $k=0$, 1 and 2 in theorem 2.3.2. When we do this we obtain corollaries 2.3.4, 2.3.5 and 2.3.6 respectively. As before, we can evaluate the higher order coefficients of $(z+1/z)$ in the **LHS** of theorem 2.3.2 by letting

$$\begin{aligned} g(x) &= \prod_{n=1}^{\infty}(1-q^{2n})(1+xq^{2n-1}+q^{4n-2})\left\{(1+q^{4n-4})^2 - x^2q^{4n-4}\right\} \\ &= \sum_{n=0}^{\infty} b_n(q)x^n. \end{aligned}$$

So that

$$b_n(q) = \frac{1}{n!}g^{(n)}(0),$$

where $g^{(n)}(x)$ is the nth derivative of $g(x)$.

Corollary 2.3.4 For $|q| < 1$ we have

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}$$

Proof: From the constant term in theorem 2.3.2 we obtain

$$4\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{4n-2})(1+q^{4n-4})^2 = 4\sum_{n=-\infty}^{\infty} (-1)^n q^{4n(3n+1)}.$$

We note that the **LHS** is just $4\prod_{n=1}^{\infty}(1-q^{8n})$. We then let $q^8 := q$. □

Of course corollary 2.3.4 is just Euler's pentagonal number theorem.

Corollary 2.3.5 For $|q| < 1$ we have

$$\prod_{n=1}^{\infty}(1-q^{2n})^3(1+q^n)^2 = \sum_{n=-\infty}^{\infty} (-1)^n (3n+1) q^{3n^2+2n}.$$

Proof: Equating the coefficient of $(z+1/z)$ in theorem 2.3.2 we obtain

$$\begin{aligned}&4\prod_{n=1}^{\infty}(1-q^{8n})\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}\\ &=\sum_{n=0}^{\infty}(-1)^n\left\{(6n+1)q^{12n^2+8n+1}+(6n+3)q^{12n^2+8n+1}(1+q^{8n+4})+(6n+5)q^{12n^2+16n+5}\right\}.\end{aligned}$$

Using corollary 2.3.2 to substitute for $\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1+q^{4n-2}}$ in the **LHS** and simplifying the **RHS** we have

$$4\prod_{n=1}^{\infty}(1-q^{8n})(1+q^{4n})^2 = \sum_{n=0}^{\infty}(-1)^n\left\{(12n+4)q^{12n^2+8n+1}+(12n+8)q^{12n^2+16n+5}\right\}.$$

Which if we divide through by 4 and let $q^4 := q$, completes the proof of corollary 2.3.5. □

Note that corollary 2.3.5 is equivalent to equation (2.2.12).

Corollary 2.3.6 For $|q| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 4\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}} = \sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2}.$$

Proof: Equating the coefficient of $(z+1/z)^2$ in theorem 2.3.2 we obtain

$$4\prod_{n=1}^{\infty}(1-q^{8n})\left\{\frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}\right)^2 - \frac{1}{2}\sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} - \sum_{n=1}^{\infty} \frac{q^{4n-4}}{(1+q^{4n-4})^2}\right\}$$
$$= \sum_{n=-\infty}^{\infty} (-1)^{n+1}\left\{(3n+1)^2 + (3n)^2\right\} q^{12n^2+4n}.$$

Now if we let $q := q^8$ in Euler's pentagonal number theorem (corollary 2.3.4) and logarithmically differentiate, we obtain

$$\frac{\sum_{n=-\infty}^{\infty} (-1)^{n+1}\left\{(3n+1)^2+(3n)^2\right\} q^{12n^2+4n}}{4\prod_{n=1}^{\infty}(1-q^{8n})} = 3\sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}} - \frac{1}{4}.$$

Therefore

$$\frac{1}{2}\left\{\left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}\right)^2 - \sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2}\right\} - \sum_{n=1}^{\infty} \frac{q^{4n}}{(1+q^{4n})^2} = 3\sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}}. \qquad (2.3.2)$$

To complete the proof we need to substitute for the term in braces on the **LHS** of the above equation. Corollary 2.3.3 provides the clue. From equations (2.2.16) and (2.2.19) we have

$$\theta_4(q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2$$
$$= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}.$$

So if we logarithmically differentiate the above, then multiply through by q, we obtain

$$\frac{1}{2}\theta_4(q)\left\{\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{(4n-2)q^{2n-1}}{1-q^{2n-1}}\right\} = \sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{n^2}. \tag{2.3.3}$$

But

$$\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}},$$

so the term in braces in equation (2.3.3) is

$$\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{(4n-2)q^{2n-1}}{1-q^{2n-1}} = 2\left\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right\}.$$

Therefore

$$\theta_4(q)\left\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\right\} = \sum_{n=1}^{\infty}(-1)^{n+1}n^2q^{n^2}. \tag{2.3.4}$$

Now we just let $q := q^4$ in equation (2.3.4) and use corollary 2.3.3 to show that the term in braces in equation (2.3.2) is equal to

$$\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}} - \sum_{n=1}^{\infty}\frac{nq^{8n}}{1-q^{8n}}.$$

Substituting the last expression into equation (2.3.2) and letting $q^4 := q$ completes the proof of corollary 2.3.6. □

Although we were led to corollary 2.3.6 by the transformation of the quintuple product identity it is worth mentioning that there is a much more illuminating proof of this result, which is presented in section 2.4.

§2.3.4 Closing Remarks

By equating coefficients of $(z + 1/z)^k$ for $k = 0, 1, 2, \ldots$ in theorems 2.3.1 and 2.3.2 we obtain a set of identities which involve sums of the form

$$\sum_{n=1}^{\infty} \frac{q^{j(2n-1)}}{(1 + q^{4n-2})^j} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{q^{2jn}}{(1 + q^{4n})^j} \qquad \text{for } j = 1, 2, 3, \ldots$$

As the results in the next section will demonstrate, because these identities generally contain sums of the above form some interesting summation theorems involving real α and β, where $\alpha\beta = \pm 1$, are derivable from them. From these results we can go on to derive some identities between sums of reciprocals of Fibonacci and Lucas numbers. Equating the coefficients of $(z + 1/z)^0$ and $(z + 1/z)^1$ gives known results. Equating the coefficients of higher powers of $(z + 1/z)$ tends to give results which are unlikely to have appeared in the literature before because they are more cumbersome. However, as is shown by the next section, these identities do give us something extra over what we would obtain had we just differentiated the **JTP** and the quintuple product identity, without first applying the Chebyshev polynomial transformation.

§2.4 Some Summation Identities and Applications

§2.4.1 Introduction

In this section the results of section 2.3 are used to derive some summation identities involving real α and β, where $\alpha\beta = \pm 1$. In addition, some preliminary elementary results are included which produce identities of a similar type. From the $\alpha\beta$ identities some results between sums of reciprocals of Fibonacci and Lucas numbers are deduced. Some theorems from the theory of elliptic functions are used to obtain more results of the same type. It is also noted that Ramanujan's functions P, Q and R (from chapter 1) can be utilised to produce similar results.

The material is ordered with the more elementary matter presented first and that which requires a bit more work, later on (such as equating coefficients in theorem 2.3.1 of $(z + 1/z)^k$ for $k \geq 3$).

The preliminary results are presented first as lemmas because most are required later on. Included here as lemma 2.4.1 is corollary 2.3.6 from the previous section. A more natural proof is presented here than that which was given in the previous section. We briefly digress to show how we can extend the proof to produce further identities for

$$\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^k} \qquad \text{where } |q| < 1 \text{ and } k = 3, 4, 5, \ldots$$

§2.4.2 Some Preliminary Results

Lemma 2.4.1 For $|q| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 2^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}.$$

Proof:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} m q^{mn} \\ &= \sum_{n=1}^{\infty} \{\sigma_1^o(n) - \sigma_1^e(n)\}\, q^n, \end{aligned} \tag{2.4.1}$$

where for integer $k \geq 0$ we have

$$\sigma_k^o(n) = \sum_{\substack{d|n \\ d \text{ odd}}} d^k \qquad \text{and} \qquad \sigma_k^e(n) = \sum_{\substack{d|n \\ d \text{ even}}} d^k.$$

Now $\sigma_1^o(n) - \sigma_1^e(n) = \sigma_1(n) - 2^2\sigma_1(n/2)$, where $\sigma_k(n) = \sum_{d|n} d^k$ and $\sigma(x) = 0$ for non-integral x. So from equation (2.4.1) we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^2} &= \sum_{n=1}^{\infty} \{\sigma_1(n) - 2^2\sigma_1(n/2)\}\, q^n \\ &= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 2^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}. \end{aligned}$$

□

The two summations on the **RHS** of lemma 2.4.1 are of a type known as a Lambert series [13].

Lemma 2.4.2 For $|q| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^3} = \frac{1}{2}\left\{\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 2^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}\right\} + \frac{1}{2}\left\{\sum_{n=1}^{\infty} \frac{n^2q^n}{1-q^n} - 2^3 \sum_{n=1}^{\infty} \frac{n^2q^{2n}}{1-q^{2n}}\right\}.$$

Proof:

$$\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^3} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (-1)^{m+1} \frac{m(m+1)}{2} q^{mn}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} \left\{\sigma_1^o(n) - \sigma_1^e(n)\right\} q^n + \frac{1}{2}\sum_{n=1}^{\infty} \left\{\sigma_2^o(n) - \sigma_2^e(n)\right\} q^n.$$

Now we note that $\sigma_1^o(n) - \sigma_1^e(n) = \sigma_1(n) - 2^2\sigma_1(n/2)$ and $\sigma_2^o(n) - \sigma_2^e(n) = \sigma_2(n) - 2^3\sigma_2(n/2)$. Therefore

$$\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^3} = \frac{1}{2}\sum_{n=1}^{\infty} \left\{\sigma_1(n) - 2^2\sigma_1(n/2)\right\} q^n + \frac{1}{2}\sum_{n=1}^{\infty} \left\{\sigma_2(n) - 2^3\sigma_2(n/2)\right\} q^n$$
$$= \frac{1}{2}\left\{\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 2^2\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}\right\} + \frac{1}{2}\left\{\sum_{n=1}^{\infty} \frac{n^2q^n}{1-q^n} - 2^3\sum_{n=1}^{\infty} \frac{n^2q^{2n}}{1-q^{2n}}\right\}.$$

□

The equivalent results for $\sum_{n=1}^{\infty} \frac{q^n}{(1+q^n)^k}$ for $k = 4, 5, 6, \ldots$ proceed as in the proofs of the previous theorems. We simply need to observe that $\sigma_k^o(n) - \sigma_k^e(n) = \sigma_k(n) - 2^{k+1}\sigma_k(n/2)$ for all $k \geq 0$.

Lemma 2.4.3 For $|q| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}}.$$

Proof:

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mq^{m(2n-1)}$$
$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mq^{mn} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} mq^{2mn}$$
$$= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}$$
$$= \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}}.$$

□

With $q := -q$ in lemma 2.4.3 we have

Lemma 2.4.4 For $|q| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{nq^n}{1-q^{2n}}.$$

See section 3.2 for a generalisation of lemmas 2.4.3 and 2.4.4.

Lemma 2.4.5 For $|q| < 1$ we have

$$\sum_{n=1}^{\infty} \frac{nq^n}{1+q^n} = \sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}.$$

Proof: Logarithmically differentiate the following identity of Euler's.

$$\prod_{n=1}^{\infty} (1+q^n)(1-q^{2n-1}) = 1 \qquad \text{for } |q| < 1.$$

□

The above lemma appears on page 70 of [5]. The identity of Euler's used in its proof is trivial, but has an interesting combinatorial interpretation i.e. *for any natural number n, the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.*

§2.4.3 Some Summation Identities in Real α and β, where $\alpha\beta = \pm 1$

Theorem 2.4.1 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} + \beta^{2n-1})^2} + 2\sum_{n=1}^{\infty} \frac{n}{\alpha^{4n} - \beta^{4n}} = \left(\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n-1} + \beta^{2n-1}}\right)^2$$

Proof: From equation (2.3.4) with $q := q^4$ we have

$$\theta_4(q^4)\left\{\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}}\right\} = \sum_{n=1}^{\infty} (-1)^{n+1} n^2 q^{4n^2}.$$

We now use corollary 2.3.3 to substitute for the **RHS** in the above equation, to obtain

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}\right)^2 - \sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} &= 2\left\{\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{4n}} - \sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}}\right\} \\ &= 2\sum_{n=1}^{\infty} \frac{nq^{4n}}{1-q^{8n}}. \end{aligned} \qquad (2.4.2)$$

Now let α and β satisfy the conditions of theorem 2.4.1. Then set $q = \beta$ to complete the proof. □

Theorem 2.4.2 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})^2} + 2\sum_{n=1}^{\infty} \frac{n}{\alpha^{4n} - \beta^{4n}} = \left(\sum_{n=1}^{\infty} \frac{1}{\alpha^{2n-1} - \beta^{2n-1}}\right)^2$$

Proof: Let α and β satisfy the conditions of the theorem. Then set $q = \beta$ in equation (2.4.2) to complete the proof. □

Next we use a theorem originally due to Gauss (also discovered by Legendre, Jacobi and Genocchi [7]) to derive two more results which we can use to modify theorems 2.4.1 and 2.4.2. For $|q| < 1$ we have

$$\left(\sum_{n=0}^{\infty} q^{(2n+1)^2}\right)^4 = \sum_{n=0}^{\infty} \frac{(2n+1)q^{8n+4}}{1-q^{16n+8}}. \tag{2.4.3}$$

From the definition of $\theta_2(q)$, equation (2.2.14), the **LHS** of the above is easily seen to be equal to $\{\theta_2(q^4)/2\}^4$. So using corollary 2.3.2 in the **LHS** of equation (2.4.3) and letting $q^2 := q$ we have

$$\left(\sum_{n=0}^{\infty} \frac{q^{2n+1}}{1+q^{4n+2}}\right)^2 = \sum_{n=0}^{\infty} \frac{(2n+1)q^{4n+2}}{1-q^{8n+4}}. \tag{2.4.4}$$

We now use equation (2.4.4) to derive the following theorems.

Theorem 2.4.3 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\left(\sum_{n=0}^{\infty} \frac{1}{\alpha^{2n+1}+\beta^{2n+1}}\right)^2 = \sum_{n=0}^{\infty} \frac{(2n+1)}{\alpha^{4n+2}-\beta^{4n+2}}.$$

Theorem 2.4.4 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\left(\sum_{n=0}^{\infty} \frac{1}{\alpha^{2n+1}-\beta^{2n+1}}\right)^2 = \sum_{n=0}^{\infty} \frac{(2n+1)}{\alpha^{4n+2}-\beta^{4n+2}}.$$

To prove the above theorems we just let α and β satisfy the conditions of the theorems and use equation (2.4.4) with $q = \beta$. However, since we started with equation (2.4.3) a short proof of this equation is presented below.

Proof: From the expansions of $\theta_3(q)$ and $\theta_4(q)$ (equations 2.2.18 and 2.2.19) we have

$$\frac{\theta_4(q)}{\theta_3(q)} = \prod_{n=1}^{\infty} \left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^2.$$

Logarithmically differentiating the above with respect to q, and multiplying through by q, we obtain

$$\frac{q}{\theta_4}\frac{d\theta_4}{dq} - \frac{q}{\theta_3}\frac{d\theta_3}{dq} = -4\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}. \tag{2.4.5}$$

Now from equations (2.2.20) to (2.2.22) we can show

$$\frac{1}{\theta_4}\frac{d\theta_4}{ds} - \frac{1}{\theta_3}\frac{d\theta_3}{ds} = \frac{\pi}{4}\theta_2^4,$$

where $q = e^{-\pi s}$. But $\frac{dq}{ds} = -\pi q$, so from the above equation we obtain

$$\frac{1}{\theta_4}\frac{d\theta_4}{dq} - \frac{1}{\theta_3}\frac{d\theta_3}{dq} = -\frac{1}{4q}\theta_2^4.$$

We now use the above equation to substitute for the **LHS** of equation (2.4.5) to obtain

$$\theta_2(q)^4 = 16\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}.$$

Finally letting $q := q^4$ we obtain equation (2.4.3). □

Using theorems 2.4.3 and 2.4.4 to substitute for the **RHS** of theorems 2.4.1 and 2.4.2 respectively, we have

Theorem 2.4.5 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty}\frac{1}{(\alpha^{2n-1}+\beta^{2n-1})^2} + 2\sum_{n=1}^{\infty}\frac{n}{\alpha^{4n}-\beta^{4n}} = \sum_{n=1}^{\infty}\frac{(2n-1)}{\alpha^{4n-2}-\beta^{4n-2}}.$$

Theorem 2.4.6 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty}\frac{1}{(\alpha^{2n-1}-\beta^{2n-1})^2} + 2\sum_{n=1}^{\infty}\frac{n}{\alpha^{4n}-\beta^{4n}} = \sum_{n=1}^{\infty}\frac{(2n-1)}{\alpha^{4n-2}-\beta^{4n-2}}.$$

If we now add $\sum_{n=1}^{\infty}\frac{2n}{\alpha^{4n}-\beta^{4n}}$ to both sides of theorems 2.4.5 and 2.4.6, we can restate these more neatly as

Theorem 2.4.7 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} + \beta^{2n-1})^2} + \sum_{n=1}^{\infty} \frac{4n}{\alpha^{4n} - \beta^{4n}} = \sum_{n=1}^{\infty} \frac{n}{\alpha^{2n} - \beta^{2n}}.$$

Theorem 2.4.8 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})^2} + \sum_{n=1}^{\infty} \frac{4n}{\alpha^{4n} - \beta^{4n}} = \sum_{n=1}^{\infty} \frac{n}{\alpha^{2n} - \beta^{2n}}.$$

We need a few more results before we are in a position to present some applications. Using the substitution $q = \beta$ in lemmas 2.4.3 and 2.4.4 we obtain

Theorem 2.4.9 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{n}{\alpha^{2n} - \beta^{2n}}.$$

Theorem 2.4.10 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} + \beta^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{\alpha^{2n} - \beta^{2n}}.$$

Similarly we also have

Theorem 2.4.11 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} + \beta^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{n}{\alpha^{2n} - \beta^{2n}}.$$

Theorem 2.4.12 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1} - \beta^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{\alpha^{2n} - \beta^{2n}}.$$

Notice that we could use theorem 2.4.9 and the equivalent result for $\sum_{n=1}^{\infty} \frac{n}{\alpha^{4n}-\beta^{4n}}$ (obtained by letting $q := q^4$ in equation 2.4.3) to substitute for $\sum_{n=1}^{\infty} \frac{n}{\alpha^{2n}-\beta^{2n}}$ and $\sum_{n=1}^{\infty} \frac{n}{\alpha^{4n}-\beta^{4n}}$ in theorem 2.4.7, to obtain

(A) For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}+\beta^{2n-1})^2} + 4\sum_{n=1}^{\infty} \frac{1}{(\alpha^{4n-2}-\beta^{4n-2})^2} = \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}-\beta^{2n-1})^2}.$$

Similarly we could use theorem 2.4.11 to modify theorem 2.4.8 to obtain

(B) For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}-\beta^{2n-1})^2} + 4\sum_{n=1}^{\infty} \frac{1}{(\alpha^{4n-2}-\beta^{4n-2})^2} = \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}+\beta^{2n-1})^2}.$$

However, results (A) and (B) are quite trivial, since if α and β satisfy the conditions of (A) and (B) we can drop the summation signs to obtain some trivially true identities, from which these follow.

Finally, differentiating lemma 2.4.5 and letting $q := q^4$, then $q = \beta$ we obtain

Theorem 2.4.13 For $|\beta| < 1$ and $\alpha\beta = \pm 1$ we have

$$\sum_{n=1}^{\infty} \frac{n^2}{(\alpha^{2n}+\beta^{2n})^2} = \sum_{n=1}^{\infty} \frac{(2n-1)^2}{(\alpha^{4n-2}-\beta^{4n-2})^2}.$$

An obvious alternative proof of theorem 2.4.13 is to notice that if α and β satisfy the conditions of the theorem then we have

$$\frac{4}{(\alpha^{4n}-\beta^{4n})^2} = \frac{1}{(\alpha^{2n}-\beta^{2n})^2} - \frac{1}{(\alpha^{2n}+\beta^{2n})^2}.$$

Multiply through by n^2 and sum from one to infinity to obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2}{(\alpha^{2n}+\beta^{2n})^2} &= \sum_{n=1}^{\infty} \frac{n^2}{(\alpha^{2n}-\beta^{2n})^2} - \sum_{n=1}^{\infty} \frac{(2n)^2}{(\alpha^{4n}-\beta^{4n})^2} \\ &= \sum_{n=1}^{\infty} \frac{(2n-1)^2}{(\alpha^{4n-2}-\beta^{4n-2})^2}.\end{aligned}$$

§2.4.4 Some Applications

The Fibonacci numbers are defined by the recurrence relation $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, where $F_0 = 0$ and $F_1 = 1$. The Lucas numbers are defined by the same recurrence relation, where $L_0 = 2$ and $L_1 = 1$. See section 3.4.1 for more information on the Fibonacci and Lucas numbers. We have

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n,$$

where $\alpha + \beta = 1$, and $\alpha\beta = -1$. So $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Several identities between sums of reciprocals of Fibonacci and Lucas numbers are immediate from the preceeding results. For example, if we let $\beta = (1-\sqrt{5})/2$ in theorem 2.4.2 we obtain

Theorem 2.4.14

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} + 2\sqrt{5}\sum_{n=1}^{\infty} \frac{n}{F_{4n}} = \left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2$$

With $q := q^4$ in lemma 2.4.3, then $q = (1-\sqrt{5})/2$, we obtain

Theorem 2.4.15

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^2} = \sqrt{5}\sum_{n=1}^{\infty} \frac{n}{F_{4n}}$$

Theorems 2.4.14 and 2.4.15 combine to give the more pleasant

Theorem 2.4.16

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} + 2\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^2} = \left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2$$

Similarly setting $\beta = (1-\sqrt{5})/2$ in theorems 2.4.4 and 2.4.8 we obtain respectively

Theorem 2.4.17

$$\left(\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right)^2 = \sqrt{5}\sum_{n=1}^{\infty} \frac{(2n-1)}{F_{4n-2}}$$

Theorem 2.4.18

$$\frac{1}{\sqrt{5}}\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} + \sum_{n=1}^{\infty} \frac{4n}{F_{4n}} = \sum_{n=1}^{\infty} \frac{n}{F_{2n}}$$

Notice that theorem 2.4.18 is equivalent to the following theorem 2.4.19, which is easily obtained by letting $\beta = (1-\sqrt{5})/2$ in theorem 2.4.12. Also theorem 2.4.18 is a simple combination of theorems 2.4.14 and 2.4.17.

Theorem 2.4.19

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} = \sqrt{5} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{F_{2n}}$$

The above result is certainly not new, since it appears on page 98 of [5]. From theorem 2.4.11 we have

Theorem 2.4.20

$$\sqrt{5} \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^2} = \sum_{n=1}^{\infty} \frac{n}{F_{2n}}$$

With $q := q^4$, then $q = (1 - \sqrt{5})/2$, in lemma 2.4.4 we have

Theorem 2.4.21

$$\sqrt{5} \sum_{n=1}^{\infty} \frac{1}{L_{4n-2}^2} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{F_{4n}}$$

Finally, we have from theorem 2.4.13

Theorem 2.4.22

$$5 \sum_{n=1}^{\infty} \frac{n^2}{L_{2n}^2} = \sum_{n=1}^{\infty} \frac{(2n-1)^2}{F_{4n-2}^2}$$

Some of the above results are the first cases of more general theorems presented in chapter 3.

§2.4.5 Further theorems in real α and β, where $\alpha\beta = \pm 1$, plus applications

Theorem 2.4.23 For $0 < \beta < \alpha$ and $\alpha\beta = 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}+\beta^{2n-1})^3} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n-1}+\beta^{2n-1}} \left\{ \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}+\beta^{2n-1})^2} - \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n}+\beta^{2n})^2} \right\}$$

Theorem 2.4.24 For $-1 < \beta < 0$ and $\alpha\beta = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}-\beta^{2n-1})^3} = \sum_{n=1}^{\infty} \frac{1}{\alpha^{2n-1}-\beta^{2n-1}} \left\{ \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n-1}-\beta^{2n-1})^2} - \sum_{n=1}^{\infty} \frac{1}{(\alpha^{2n}+\beta^{2n})^2} \right\}$$

As with the previous theorems, these are a pair. We present their proof below.

Proof: Both are derived from the following identity, with $q = \beta$.

$$\sum_{n=1}^{\infty} \frac{q^{6n-3}}{(1+q^{4n-2})^3} = \sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \left\{ \sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} - \sum_{n=1}^{\infty} \frac{q^{4n}}{(1+q^{4n})^2} \right\} \qquad \text{for } |q| < 1.$$

Equating the coefficients of $(z+1/z)^3$ in theorem 2.3.1 (corollary 2.3.4) we obtain

$$\prod_{n=1}^{\infty} (1-q^{2n})(1+q^{4n-2}) \left\{ \left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \right)^2 - 3\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} + 2\sum_{n=1}^{\infty} \frac{q^{6n-3}}{(1+q^{4n-2})^3} \right\}$$
$$= \sum_{n=1}^{\infty} (-1)^{n+1} n(n+1)(2n+1) q^{(2n+1)^2}. \qquad (2.4.6)$$

We now logarithmically differentiate equation (2.2.7) with $q := q^8$, ie.

$$\prod_{n=1}^{\infty} (1-q^{8n})^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{4n(n+1)},$$

to obtain

$$\sum_{n=1}^{\infty} (-1)^{n+1} n(n+1)(2n+1)q^{(2n+1)^2} = 6q \prod_{n=1}^{\infty} (1-q^{8n})^3 \sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}}. \qquad (2.4.7)$$

Using equation (2.3.2) to substitute for $\sum_{n=1}^{\infty} \frac{nq^{8n}}{1-q^{8n}}$ in the **RHS** of equation (2.4.7) we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} (-1)^{n+1} n(n+1)(2n+1)q^{(2n+1)^2} \\ &= q \prod_{n=1}^{\infty} (1-q^{8n})^3 \left\{ \left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \right)^2 - \sum_{n=1}^{\infty} \frac{q^{4n-2}}{(1+q^{4n-2})^2} - 2\sum_{n=1}^{\infty} \frac{q^{4n}}{(1+q^{4n})^2}. \right\} \qquad (2.4.8)\end{aligned}$$

We now equate equations (2.4.6) and (2.4.8). Then multiply through by $\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}$, and using equation (2.3.5) we can cancel out the term $q\prod_{n=1}^{\infty} (1-q^{8n})^3$. The term in

$$\left(\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}} \right)^3$$

also conveniently cancels out. This completes the proof of the theorem. □

As before with $\beta = (1-\sqrt{5})/2$ in theorem 2.4.24 we obtain an interesting identity for the Fibonacci and Lucas numbers, namely **[16b]**

Theorem 2.4.25

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^3} = \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} \left\{ \sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^2} - 5\sum_{n=1}^{\infty} \frac{1}{L_{2n}^2} \right\}$$

The above result is quite pleasing. However, the identities obtained by equating coefficients of higher powers of $(z+1/z)$ in theorem 2.3.1 are less so. For example, the coefficient of $(z+1/z)^4$ in the **RHS** of theorem 2.3.1 is

$$\sum_{n=2}^{\infty} (-1)^n \frac{2n}{n+2} \binom{n+2}{4} q^{4n^2},$$

which is just

$$q^2 \frac{d^2\theta_4(q)}{dq^2}$$

with $q := q^4$. The equivalent identity for the Fibonacci numbers is

Theorem 2.4.26

$$20\sum_{n=1}^{\infty}\frac{n^2}{F_{4n}^2} - 10\sum_{n=1}^{\infty}\frac{n^2}{F_{2n}^2} = 2\left(\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^2}\right)^2 - 4\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^2}\left(\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}\right)^2$$

$$-5\left(\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}\right)^2 + 5\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^2} + 8\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}}\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^3} - 6\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^4}.$$

§2.5 A modification of the transformation

Thus far we have applied the Chebyshev polynomials to transform the **JTP** and quintuple product identity in the form given in equations (2.2.3) and (2.2.4). However, we can obtain some new results by slightly modifying equation (2.2.3). If we let $n := 2n$ in equation (2.2.3) we have for $z \neq 0$

$$z^{4n} + \frac{1}{z^{4n}} = \sum_{j=0}^{2n} (-1)^j \frac{4n}{2n+j} \binom{2n+j}{2j} \left(z + \frac{1}{z}\right)^{2j} \tag{2.5.1}$$

Now if we let $z := z^4$ in the **JTP**, equation (2.2.5), and use equation (2.5.1) we have for $|q| < 1$ and $z \neq 0$

$$\prod_{n=1}^{\infty} (1 - q^{2n}) \left\{1 + \left(z^4 + \frac{1}{z^4}\right) q^{2n-1} + q^{4n-2}\right\} = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} (-1)^j \frac{4n}{n+j} \binom{2n+j}{2j} \left(z + \frac{1}{z}\right)^{2j} q^{n^2}. \tag{2.5.2}$$

Letting $x = (z + 1/z)^2$, we have $z^4 + 1/z^4 = x^2 - 4x + 2$. So upon interchanging the order of summation and substituting for $z^4 + 1/z^4$ in equation (2.5.2) we obtain

Theorem 2.5.1 For complex q and z with $|q| < 1$ and $z \neq 0$ we have

$$\prod_{n=1}^{\infty} (1 - q^{2n}) \left\{(1 + q^{2n-1})^2 + (x^2 - 4x) q^{2n-1}\right\} = \sum_{j=0}^{\infty} \sum_{n=[(j+1)/2]}^{\infty} (-1)^j \frac{4n}{2n+j} \binom{2n+j}{2j} q^{n^2} x^j,$$

where $x = (z + 1/z)^2$ and $[y]$ stands for the greatest integer less than or equal to y.

Equating coefficients of $(z+1/z)$ in theorem 2.3.1 produced identities involving sums where the summand contained $(1 + q^{4n-2})^k$, for $k = 1, 2, 3, \ldots$, in the denominator. Equating coefficients of x^k in theorem 2.5.1 will produce identities involving sums where the summand contains $(1 \pm q^{2n-1})^{2k}$, for $k = 1, 2, 3, \ldots$, in the denominator. For example, if we equate coefficients in x we can deduce the following corollary (which is just lemma 2.4.3).

Corollary 2.5.1

$$\sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1 - q^{2n-1})^2} = \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^{2n}} \qquad \text{for } |q| < 1.$$

Proof: Equating coefficients of x in theorem 2.5.1 we obtain

$$\theta_3(q)\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}=\sum_{n=1}^{\infty}n^2q^{n^2}.$$

To complete the proof we now let $q := -q$ and use equation (2.3.4). $\square$

As before, more interesting results are possible by equating higher powers of x. For example, on equating the coefficients of x^2 we obtain

$$\theta_3(q)\left\{8\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}\right)^2+\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}-8\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1+q^{2n-1})^4}\right\}$$
$$=\frac{1}{3}\sum_{n=1}^{\infty}n^2(4n^2-1)q^{n^2}. \tag{2.5.3}$$

If we let $q := -q$ in equation (2.5.3) we obtain

$$\theta_4(q)\left\{8\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)^2-\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-8\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{2n-1})^4}\right\}$$
$$=\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n n^2(4n^2-1)q^{n^2}. \tag{2.5.4}$$

Now we can easily show that the **RHS** of equation (2.5.4) is equal to

$$\frac{1}{2}q\dot{\theta}_4+\frac{2}{3}q^2\ddot{\theta}_4,$$

where the dot notation stands for differentiation with respect to q. Also by logarithmically differentiating the product representation for $\theta_4(q)$ we obtain

$$\frac{1}{2}q\dot{\theta}_4+\frac{2}{3}q^2\ddot{\theta}_4$$
$$=\frac{1}{3}\theta_4(q)\left\{8\left(\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\right)^2+\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}-4\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2}+8\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{(1-q^{2n})^2}\right\}. \tag{2.5.5}$$

So we can set the **LHS** of equation (2.5.4) equal to the **RHS** of equation (2.5.5), cancel the $\theta_4(q)$, and use corollary 2.5.1 to substitute for $\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}$ to obtain

Corollary 2.5.2 For $|q| < 1$ we have

$$6\sum_{n=1}^{\infty}\frac{q^{4n-2}}{(1-q^{2n-1})^4}+\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}-4\left(\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}\right)^2=\sum_{n=1}^{\infty}\frac{n^2q^n}{(1-q^n)^2}-2\sum_{n=1}^{\infty}\frac{n^2q^{2n}}{(1-q^{2n})^2}$$

From the above corollary, with $q := q^4$ and then $q = (1-\sqrt{5})/2$, we have

Theorem 2.5.2

$$6\sum_{n=1}^{\infty}\frac{1}{F_{4n-2}^4}+5\sum_{n=1}^{\infty}\frac{1}{F_{4n-2}^2}-4\left(\sum_{n=1}^{\infty}\frac{1}{F_{4n-2}^2}\right)^2=5\sum_{n=1}^{\infty}\frac{n^2}{F_{2n}^2}-10\sum_{n=1}^{\infty}\frac{n^2}{F_{4n}^2}$$

Notice that the **RHS** of theorem 2.5.2 is just minus twice the **LHS** of theorem 2.4.26.

§2.6 Closing Remarks

More results of the type in this section are possible from the theory in Ramanujan's paper [25] **"On certain Arithmetical Functions"**, which was discussed in chapter 1. For example Ramanujan's function $\phi_{1,k}(q)$ can be written

$$\phi_{1,k}(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{(1-q^n)^2} \qquad \text{for } |q| < 1 \text{ and } k = 1, 2, 3, \ldots$$

so that we have

$$\sum_{n=1}^{\infty} \frac{n^k}{F_{2n}} = 5\phi_{1,k}\left(\frac{7-3\sqrt{5}}{2}\right) \qquad \text{for } k = 1, 2, 3, \ldots$$

Moreover, it is fairly easy to show that for the modular form E_4, denoted by Ramanujan as Q, we have [34]

$$\begin{aligned} E_4(\tau) &= \{\theta_2^8(q) + \theta_3^8(q) + \theta_4^8(q)\}/2 \\ &= 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^{2n}, \end{aligned} \tag{2.6.1}$$

where $q = e^{i\pi\tau}$ for $Im(\tau) > 0$. So that using the following modular transformations

$$\begin{aligned} \theta_2^2(q) &= 2\theta_2(q^2)\theta_3(q^2) \\ \theta_3^2(q) &= \theta_3^2(q^2) + \theta_2^2(q^2) \\ \theta_4^2(q) &= \theta_3^2(q^2) - \theta_2^2(q^2), \end{aligned} \tag{2.6.2}$$

we have from equation (2.6.1)

$$2E_4(\tau/2) = \{2\theta_2(q)\theta_3(q)\}^4 + \{\theta_3^2(q) + \theta_2^2(q)\}^4 + \{\theta_3^2(q) - \theta_2^2(q)\}^4.$$

Therefore

$$E_4(\tau/2) = \theta_2^8(q) + 14\theta_2^4(q)\theta_3^4(q) + \theta_3^8(q). \tag{2.6.3}$$

Also, using the modular transformations

$$\begin{aligned}\theta_2^2(q^2) &= \left\{\theta_3^2(q) - \theta_4^2(q)\right\}/2 \\ \theta_3^2(q^2) &= \left\{\theta_3^2(q) + \theta_4^2(q)\right\}/2 \\ \theta_4^2(q^2) &= \theta_3(q)\theta_4(q),\end{aligned} \tag{2.6.4}$$

we have from equation (2.6.1)

$$2E_4(\tau) = \left\{\theta_3^2(q^{1/2}) - \theta_4^2(q^{1/2})\right\}^4/16 + \left\{\theta_3^2(q^{1/2}) + \theta_4^2(q^{1/2})\right\}^4/16 + \theta_3^4(q^{1/2})\theta_4^4(q^{1/2}).$$

Therefore

$$E_4(\tau) = \left[\left\{\theta_3^4(q^{1/2}) - \theta_4^4(q^{1/2})\right\}^2 + 16\theta_3^4(q^{1/2})\theta_4^4(q^{1/2})\right]/16,$$

which on using $\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q)$ is equal to

$$\theta_2^8(q^{1/2})/16 + \theta_3^4(q^{1/2})\theta_4^4(q^{1/2}).$$

Therefore using equation (2.6.2) we have

$$\begin{aligned}E_4(\tau) &= \theta_2^4(q)\theta_3^4(q) + \left\{\theta_3^2(q) + \theta_2^2(q)\right\}^2\left\{\theta_3^2(q) - \theta_2^2(q)\right\}^2 \\ &= \theta_2^8(q) - \theta_2^4(q)\theta_3^4(q) + \theta_3^8(q).\end{aligned} \tag{2.6.5}$$

Hence, subtracting equation (2.6.5) from equation (2.6.3) we obtain

$$\begin{aligned}E_4(\tau/2) - E_4(\tau) &= 15\theta_2^4(q)\theta_3^4(q) \\ &= \frac{15}{16}\theta_2^8(q^{1/2}),\end{aligned} \tag{2.6.6}$$

by the first of equations (2.6.2). But from equation (2.6.1) we have

$$E_4(\tau/2) - E_4(\tau) = 240\left\{\sum_{n=1}^{\infty} \sigma_3(n)q^n - \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}\right\}$$
$$= 240\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^{2n}}.$$

So from equation (2.6.6) and the above, letting $q := q^2$ we obtain

$$\theta_2^8(q) = 256\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{4n}}.$$

Now from corollary 2.3.2 we have

$$\theta_2^8(q^2) = 256\left\{\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}\right\}^4.$$

Hence

$$\sum_{n=1}^{\infty} \frac{n^3 q^{4n}}{1-q^{8n}} = \left\{\sum_{n=1}^{\infty} \frac{q^{2n-1}}{1+q^{4n-2}}\right\}^4, \tag{2.6.7}$$

and using equation (2.4.4) we obtain

$$\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{4n}} = \left\{\sum_{n=1}^{\infty} \frac{(2n-1)q^{2n-1}}{1-q^{4n-2}}\right\}^2. \tag{2.6.8}$$

So from equations (2.6.7) and (2.6.8) respectively we have

Theorem 2.6.1

$$\sum_{n=1}^{\infty} \frac{n^3}{F_{4n}} = \frac{1}{5\sqrt{5}}\left\{\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}\right\}^4$$

Theorem 2.6.2

$$\sum_{n=1}^{\infty} \frac{n^3}{F_{2n}} = \sqrt{5}\left\{\sum_{n=1}^{\infty} \frac{(2n-1)}{L_{2n-1}}\right\}^2$$

As another example, from table (i) of [25] we can obtain

Theorem 2.6.3

$$\sum_{n=1}^{\infty} \frac{n^8}{F_{2n}^2} \sum_{n=1}^{\infty} \frac{(2n-1)}{F_{4n-2}} = \sum_{n=1}^{\infty} \frac{n^4}{F_{2n}^2} \sum_{n=1}^{\infty} \frac{(2n-1)^5}{F_{4n-2}}$$

However, in general the identities for the Fibonacci and Lucas numbers produced in this way tend to be rather messy. So just a few of the more pleasing examples have been highlighted. We could go on to use

$$E_8(\tau) - E_8(2\tau) = \frac{15}{16}\theta_2^8(q)\left\{2\theta_3^8(q^2) + \frac{13}{16}\theta_2^8(q) + 2\theta_2^8(q^2)\right\},$$

where $E_8(\tau) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}$, $q = e^{i\pi\tau}$ for $Im(\tau) > 0$, to produce yet more examples. Fortunately we resist the temptation in favour of some much more elementary identities (though hopefully none the less attractive for this) which are presented in the next chapter.

Chapter 3

Further Consequences of the Chebyshev Polynomial Transformation and Related Ideas.

§3.1 Introduction

In this chapter we explore some further consequences of the Chebyshev transformation and Chebyshev-like transformations in the area covered by this thesis. Some of the results concerning sums of reciprocals of Fibonacci and Lucas numbers from chapter 2 are approached in a more elementary way, and extended.

The Chebyshev polynomial transformation is used to derive some results in the theory of restricted partitions. Also, using Chebyshev-like tranformations some new polynomial identities for the Fibonacci and Lucas numbers are derived [16a]. The Chebyshev transformation also enables us to derive some combinatorial results involving sums of binomial coefficients and with the help of some lemmas from chapter 2, plus one additional lemma, further combinatorial identities are derived.

§3.2 Elementary identities between sums of reciprocals of Fibonacci and Lucas numbers

§3.2.1 Introduction

This section presents some elementary identities between sums of reciprocals of Fibonacci and Lucas numbers [**16c**]. These results generalise lemmas 2.4.3 and 2.4.4 from chapter 2. In fact some of the theorems from chapter 2 are merely the first cases of the following results.

It was remarked at the end of section 2.3 that identities obtained by equating coefficients of $(z+1/z)^k$, for $k = 1, 2, 3, \ldots$, in theorem 2.3.1 involve sums of the form

$$\sum_{n=1}^{\infty} \frac{q^{k(2n-1)}}{(1+q^{4n-2})^k} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{q^{2kn}}{(1+q^{4n})^k} \qquad \text{for } k = 1, 2, 3, \ldots \text{ and } |q| < 1.$$

It is sums of the above type which are used to derive the theorems of section 3.2.2. These theorems include the following results as special cases.

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^3} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{L_{2n-1}}$$

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{n(n-1)}{F_{4n-2}}$$

§3.2.2 The main theorems

Our first two theorems are

Theorem 3.2.1

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^{2k+1}} = \sum_{n=1}^{\infty} \binom{n+k-1}{n-k-1} \frac{1}{L_{2n-1}} \qquad \text{for } k = 0, 1, 2, 3, \ldots$$

Proof: Set $q = (1-\sqrt{5})/2$ in theorem 3.2.3.

Theorem 3.2.2

$$\sum_{n=1}^{\infty} \frac{1}{F_{4n-2}^{2k+1}} = 5^k \sum_{n=1}^{\infty} \binom{n+k-1}{n-k-1} \frac{1}{F_{4n-2}} \qquad \text{for } k = 0, 1, 2, 3, \ldots$$

Proof: Let $q := q^2$ in theorem 3.2.3. Then set $q = (1-\sqrt{5})/2$.

Theorem 3.2.3 For $|q| < 1$ and $k = 0, 1, 2, 3, \ldots$ we have

$$\sum_{n=0}^{\infty} \frac{q^{(2k+1)(2n+1)}}{(1-q^{4n+2})^{2k+1}} = \sum_{n=k}^{\infty} \binom{n+k}{n-k} \frac{q^{2n+1}}{1-q^{4n+2}}.$$

Proof: For $|q| < 1$ and $k = 0, 1, 2, 3, \ldots$ we have by the binomial theorem

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{(2k+1)(2n+1)}}{(1-q^{4n+2})^{2k+1}} &= \sum_{n=0}^{\infty} q^{(2k+1)(2n+1)} \sum_{m=0}^{\infty} (-1)^m \binom{-2k-1}{m} q^{2m(2n+1)} \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{-2k-1}{m} \sum_{n=0}^{\infty} q^{(2m+2k+1)(2n+1)} \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{-2k-1}{m} \frac{q^{(2m+2k+1)}}{1-q^{2(2m+2k+1)}}.
\end{aligned}$$

But for the binomial coefficients we have the relation $(-1)^m\binom{-2k-1}{m} = \binom{m+2k}{m}$. Hence

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{(2k+1)(2n+1)}}{(1-q^{4n+2})^{2k+1}} &= \sum_{m=0}^{\infty}\binom{m+2k}{m}\frac{q^{2m+2k+1}}{1-q^{2(2m+2k+1)}}\\ &= \sum_{n=k}^{\infty}\binom{n+k}{n-k}\frac{q^{2n+1}}{1-q^{4n+2}},\end{aligned}$$

on setting $n = m+k$ to obtain the last line. □

Another similar identity, proved in the same way as theorem 3.2.3 is

Theorem 3.2.4 For $|q| < 1$ and $k = 1, 2, 3, \ldots$ we have

$$\sum_{n=0}^{\infty}\frac{q^{k(2n+1)}}{(1-q^{2n+1})^{2k}} = \sum_{n=k}^{\infty}\binom{n+k-1}{n-k}\frac{q^n}{1-q^{2n}}.$$

Proof: For $|q| < 1$ and $k = 1, 2, 3, \ldots$ we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{k(2n+1)}}{(1-q^{2n+1})^{2k}} &= \sum_{n=0}^{\infty}q^{k(2n+1)}\sum_{m=0}^{\infty}(-1)^m\binom{-2k}{m}q^{m(2n+1)}\\ &= \sum_{m=0}^{\infty}(-1)^m\binom{-2k}{m}\sum_{n=0}^{\infty}q^{(m+k)(2n+1)}\\ &= \sum_{m=0}^{\infty}(-1)^m\binom{-2k}{m}\frac{q^{m+k}}{1-q^{2(m+k)}}\\ &= \sum_{m=0}^{\infty}\binom{m+2k-1}{m}\frac{q^{m+k}}{1-q^{2(m+k)}}\\ &= \sum_{n=k}^{\infty}\binom{n+k-1}{n-k}\frac{q^n}{1-q^{2n}},\end{aligned}$$

on setting $n = m+k$ to obtain the last line. □

If we let $q := -q$, then $q := q^2$, and set $q = (1-\sqrt{5})/2$ in theorem 3.2.4, we obtain

Theorem 3.2.5 For $k = 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{1}{{F_{2n-1}}^{2k}} = 5^{k-\frac{1}{2}} \sum_{n=1}^{\infty} (-1)^{n+k} \binom{n+k-1}{n-k} \frac{1}{F_{2n}}$$

If we let $q := q^2$ and set $q = (1-\sqrt{5})/2$ in theorem 3.2.4 we obtain

Theorem 3.2.6 For $k = 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{1}{{L_{2n-1}}^{2k}} = \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \binom{n+k-1}{n-k} \frac{1}{F_{2n}}$$

Notice that theorem 2.4.19 of the previous chapter is just the first case of theorem 3.2.5 and that theorem 2.4.20 is just the first case of theorem 3.2.6. The same idea can be used to produce equivalent theorems where the summand on the **LHS** contains n in the numerator instead of unity, theorems 3.2.9–3.2.11. The latter follow from theorems 3.2.7 and 3.2.8.

Theorem 3.2.7 For $|q| < 1$ and $k = 0, 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{nq^{n(2k+1)}}{(1-q^{2n})^{2k+1}} = \sum_{n=k}^{\infty} \binom{n+k}{n-k} \frac{q^{2n+1}}{(1-q^{2n+1})^2}.$$

Proof: For $|q| < 1$ and $k = 0, 1, 2, 3, \ldots$ we have by the binomial theorem

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{nq^{n(2k+1)}}{(1-q^{2n})^{2k+1}} &= \sum_{n=1}^{\infty} nq^{n(2k+1)} \sum_{m=0}^{\infty} (-1)^m \binom{-2k-1}{m} q^{2mn} \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{-2k-1}{m} \sum_{n=1}^{\infty} nq^{n(2m+2k+1)} \\
&= \sum_{m=0}^{\infty} \binom{m+2k}{m} \frac{q^{2m+2k+1}}{(1-q^{2m+2k+1})^2},
\end{aligned}$$

which on setting $n = m + k$ completes the proof of the theorem. ◻

Theorem 3.2.8 For $|q| < 1$ and $k = 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{nq^{2nk}}{(1-q^{2n})^{2k}} = \sum_{n=k}^{\infty} \binom{n+k-1}{n-k} \frac{q^{2n}}{(1-q^{2n})^2}.$$

Proof: For $|q| < 1$ and $k = 1, 2, 3, \ldots$ we have by the binomial theorem

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{nq^{2nk}}{(1-q^{2n})^{2k}} &= \sum_{n=1}^{\infty} nq^{2nk} \sum_{m=0}^{\infty} (-1)^m \binom{-2k}{m} q^{2mn} \\
&= \sum_{m=0}^{\infty} (-1)^m \binom{-2k}{m} \sum_{n=1}^{\infty} nq^{2n(m+k)} \\
&= \sum_{m=0}^{\infty} \binom{m+2k-1}{m} \frac{q^{2(m+k)}}{(1-q^{2(m+k)})^2},
\end{aligned}$$

which on setting $n = m + k$ completes the proof of the theorem. ◻

If we let $q := q^2$ and set $q = (1-\sqrt{5})/2$ in theorem 3.2.7 we obtain

Theorem 3.2.9 For $k = 0, 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{n}{{F_{2n}}^{2k+1}} = 5^{k+\frac{1}{2}} \sum_{n=1}^{\infty} \binom{n+k-1}{n-k-1} \frac{1}{{L_{2n-1}}^2}.$$

Notice that the first case of theorem 3.2.9 is theorem 2.4.20 of the previous chapter.

If we let $q := -q$ and set $q = (1-\sqrt{5})/2$ in theorem 3.2.7 we obtain

Theorem 3.2.10 For $k = 0, 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{{F_{2n}}^{2k+1}} = 5^{k-\frac{1}{2}} \sum_{n=1}^{\infty} \binom{n+k-1}{n-k-1} \frac{1}{{F_{2n-1}}^2}.$$

The first case of theorem 3.2.10 is theorem 2.4.19 of the previous chapter.

If we let $q := q^2$ and set $q = (1-\sqrt{5})/2$ in theorem 3.2.8 we obtain

Theorem 3.2.11 For $k = 1, 2, 3, \ldots$ we have

$$\sum_{n=1}^{\infty} \frac{n}{{F_{2n}}^{2k}} = 5^{k-1} \sum_{n=1}^{\infty} \binom{n+k-1}{n-k} \frac{1}{{F_{2n}}^2}.$$

§3.2.3 Closing Remarks

The identities of this section are really quite elementary. However, it is hoped that the reader finds them none the less attractive for this. In fact it is felt that part of their appeal lies in the very straightforward nature of their proof. We have already demonstrated how some of the theorems in this section can arise in more complex ways.

§3.3 Some theorems in the theory of restricted partitions

§3.3.1 Introduction

The Chebyshev polynomial transformation has some consequences in the theory of highly restricted partitions [10]. The sets of partitions which we will consider in this section are all subsets of the set of partitions of an integer *into parts where each summand may appear at most twice in the partition.* The partition where each summand is restricted to at most two occurences will be referred to as a *2-repetition* partition (short for up to two times repetition). For example, all the 2-repetition partitions of 7 are

$$7,\ 34,\ 25,\ 2^23,\ 16,\ 13^2,\ 124,\ 1^25,\ 1^223.$$

The notation 2^23 is short for $2+2+3$. The first type of highly restricted partition which we will consider is the 2-repetition partition into odd integers. For example, from the above list of partitions we see that all the 2-repetition partitions of 7 into odd integers are

$$7,\ 13^2,\ 1^25.$$

The first two theorems tell us the number of 2-repetition partitions of an integer into odd parts, which contain a given number of distinct parts (i.e. parts appearing exactly once in a partition).

§3.3.2 The main theorems

Theorem 3.3.1 The number of 2-repetition partitions of an integer m into odd parts, where each partition contains exactly $2j$ distinct summands, is given by

$$\sum_{n\geq 0} (-1)^n p\left\{m/2 - 2(n+j)^2\right\} \frac{2n+2j}{n+2j}\binom{n+2j}{2j},$$

where $(2n+2j)/(n+2j)$ is taken to be 1 when $n = j = 0$.

In theorem 3.3.1 $p(n)$ is the usual unrestricted partition function [13] given by

$$\prod_{n=1}^{\infty} (1-q^n)^{-1} = \sum_{n=0}^{\infty} p(n)q^n \qquad \text{for } |q| < 1.$$

We take $p(0) = 1$, and $p(x) = 0$ for $x < 0$ and non-integral values of x.

Theorem 3.3.2 The number of 2-repetition partitions of an integer m into odd parts, where each partition contains exactly $2j+1$ distinct summands, is given by

$$\sum_{n\geq 0} (-1)^n p\left\{m/2 - (2n+2j+1)^2/2\right\} \frac{2n+2j+1}{n+2j+1}\binom{n+2j+1}{2j+1}.$$

For example, with $m = 7$ and $j = 1$ in theorem 3.3.1, the theorem says that there are no 2-repetition partitions of 7 into odd integers, where the partitions contain exactly 2 distinct summands. This is clear from section 3.3.1. With $m = 7$ and $j = 0$ theorem 3.3.2 gives the number of 2-repetition partitions of 7 into odd integers, where each partition contains exactly 1 distinct summand. From section 3.3.1 this should be 3, corresponding to the partitions 7, 13^2, $1^2 5$, and indeed we have

$$\sum_{n\geq 0} (-1)^n (2n+1) p\left\{7/2 - (2n+1)^2/2\right\} = p(3) = 3.$$

Proof of theorems 3.3.1 and 3.3.2: The coefficient of $x^r q^s$ in

$$\prod_{n=1}^{\infty}(1 + xq^{2n-1} + q^{4n-2})$$

enumerates the number of 2-repetition partitions of s into odd parts, where each partition contains exactly r distinct parts. This is clear because the term q^{2n-1} corresponds to those odd parts which appear only once in the partition and the term q^{4n-2} corresponds to those odd parts appearing twice in the partition. If we write $z + 1/z = x$ in theorem 2.3.1 we obtain

$$\begin{aligned}
&\prod_{n=1}^{\infty}(1 + xq^{2n-1} + q^{4n-2}) \\
&= \prod_{n=1}^{\infty}(1 - q^{2n})^{-1} \sum_{j=0}^{\infty} x^{2j} \sum_{n=0}^{\infty} (-1)^n \frac{2n+2j}{n+2j}\binom{n+2j}{2j} q^{(2n+2j)^2} \\
&+ \prod_{n=1}^{\infty}(1 - q^{2n})^{-1} \sum_{j=0}^{\infty} x^{2j+1} \sum_{n=0}^{\infty} (-1)^n \frac{2n+2j+1}{n+2j+1}\binom{n+2j+1}{2j+1} q^{(2n+2j+1)^2}. \qquad (3.3.1)
\end{aligned}$$

To obtain the coefficient of $x^{2j} q^s$ from the **RHS** of equation (3.3.1) we note that the coefficient of x^{2j} is the product of

$$\sum_{n=0}^{\infty} p(n) q^{2n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} (-1)^n \frac{2n+2j}{n+2j}\binom{n+2j}{2j} q^{(2n+2j)^2}.$$

So we take the coefficient of q^s from the Cauchy product of the above two summations to obtain the coefficient of $x^{2j} q^s$ in equation (3.3.1). This completes the proof of theorem 3.3.1. Similarly, theorem 3.3.2 follows from the last double summation in equation (3.3.1). □

The next two theorems concern 2-repetition partitions where we allow zero to be a valid part of the partition. For example, the 2-repetition partitions of 4, counting zero as a valid part, are given by

4	04	$0^2 4$
2^2	02^2	$0^2 2^2$
13	013	$0^2 13$
$1^2 2$	$01^2 2$	$0^2 1^2 2$.

For a given integer the theorems give us the number of partitions of the above type which contain exactly j distinct members.

Theorem 3.3.3 The number of 2-repetition partitions of an integer m, counting zero as a valid part, where each partition contains exactly $2j$ distinct summands is given by

$$\sum_{n\geq 0}(-1)^n p\left\{m+j-2(n+j)^2\right\}\frac{2n+2j}{n+2j}\binom{n+2j}{2j},$$

where $(2n+2j)/(n+2j)$ is taken to be 1 when $n=j=0$.

Theorem 3.3.4 The number of 2-repetition partitions of an integer m, counting zero as a valid part, where each partition contains exactly $2j+1$ distinct summands is given by

$$\sum_{n\geq 0}(-1)^n p\left\{m-2(n+j)^2-2n-j\right\}\frac{2n+2j+1}{n+2j+1}\binom{n+2j+1}{2j+1}.$$

In theorems 3.3.3 and 3.3.4 $p(n)$ is again the unrestricted partition function. As an example we consider the 2-repetition partitions of 10, counting zero as a valid part, which contain exactly two distinct parts. These are

01^22^24	0^219	03^24
01^223^2	19	0^237
1^226	02^26	37
1^235	024^2	0^246
01^28	0^228	46
12^25	28	$0,10$

Now by theorem 3.3.3 with $m=10$ and $j=1$ we have

$$\sum_{n\geq 0}(-1)^n(n+1)^2 p\left\{11-2(n+1)^2\right\}=p(9)-4p(3)=30-4.3=18.$$

Which corresponds to those partitions above.

As an extension, we now count the number of 2-repetition partitions of a natural number t which have exactly r distinct parts **AND** contain exactly s parts. The number of these partitions is enumerated by the coefficient of $x^r z^s q^t$ in the expansion of

$$\prod_{n=1}^{\infty}(1+xzq^n+z^2q^{2n}).$$

For our next theorems we need the following notation.

$$(q)_n = (1-q)(1-q^2)\dots(1-q^n) \qquad \text{for } n \geq 1.$$
$$(q)_0 = 1.$$

Theorem 3.3.5 The number of 2-repetition partitions of an integer m, with exactly $2j$ distinct parts and exactly $2s$ parts is the coefficient of q^m in the expansion of

$$q^{s(s-1)} \sum_{n=0}^{s-j} (-1)^n \frac{2n+2j}{n+2j} \binom{n+2j}{2j} \frac{q^{(n+j)^2}}{(q)_{s-n-j}(q)_{s+n+j}}.$$

Clearly if we have an even number of members then we can only have an even number of distinct members.

Theorem 3.3.6 The number of 2-repetition partitions of an integer m, with exactly $2j+1$ distinct parts and exactly $2s+1$ parts is the coefficient of q^m in the expansion of

$$q^{s^2} \sum_{n=0}^{s-j} (-1)^n \frac{2n+2j+1}{n+2j+1} \binom{n+2j+1}{2j+1} \frac{q^{(n+j)(n+j+1)}}{(q)_{s-n-j}(q)_{s+n+j+1}}.$$

Clearly if we have an odd number of members then we can only have an odd number of distinct members.

Proof of theorems 3.3.5 and 3.3.6: We use an identity from a paper of L. J. Rogers [31] in which he gives his second proof of the Rogers–Ramanujan identities. From the well known

$$\prod_{n=0}^{\infty} (1+zq^n) = \sum_{n=0}^{\infty} \frac{z^n q^{\binom{n}{2}}}{(q)_n} \qquad \text{for } |q| < 1, \tag{3.3.2}$$

with $z := ze^{i\theta}$ and $z := ze^{-i\theta}$ respectively, we can evaluate

$$\prod_{n=0}^{\infty} (1 + 2zq^n \cos\theta + z^2 q^{2n}) = \prod_{n=0}^{\infty} (1 + zq^n e^{i\theta})(1 + zq^n e^{-i\theta}). \tag{3.3.3}$$

If we write

$$\prod_{n=0}^{\infty}(1+2zq^n\cos\theta+z^2q^{2n})=\sum_{n=0}^{\infty}r_n(\theta)z^n,$$

we have by equations (3.3.2) and (3.3.3)

$$r_{2n}(\theta)=\frac{q^{n(n-1)}}{(q)_n^2}+q^{n(n-1)}\sum_{r=1}^{n}\frac{q^{r^2}}{(q)_{n-r}(q)_{n+r}}2\cos 2r\theta \qquad \text{for } n\geq 1 \qquad (3.3.4)$$

and

$$r_{2n+1}(\theta)=q^{n^2}\sum_{r=0}^{n}\frac{q^{r(r+1)}}{(q)_{n-r}(q)_{n+r+1}}2\cos(2r+1)\theta \qquad \text{for } n\geq 0. \qquad (3.3.5)$$

Theorems 3.3.5 and 3.3.6 now follow by setting $x=2\cos\theta$ and using the Chebyshev polynomials, equations (2.2.3) and (2.2.4), in equations (3.3.4) and (3.3.5) respectively. □

§3.3.3 Closing Remarks

As has been shown, the Chebyshev transformation enables us to evaluate the size of various subsets of the set of 2-repetition partitions. Using an idea of L. J. Rogers [31] we can get a handle on the size of an even more highly restricted set of partitions of this type. More generally, the coefficient of $x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} q^s$ in

$$\prod_{n=1}^{\infty} (1 + x_1 q^n + x_2 q^{2n} + \cdots + x_k q^{kn})$$

enumerates the number of partitions of s into parts of which a_1 are distinct, a_2 appear twice, a_3 appear three times, etc ... (so we have $a_1 + 2a_2 + 3a_3 + \cdots + ka_k$ parts). The problem of evaluating the coefficient of $x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} q^s$ is an obvious extension of the work in this section. However, there is clearly no simple closed form (as we have for theorems 3.3.1–3.3.4) for this coefficient, though it seems likely that there are some interesting results to be had here (see section 12, chapter 7 of [10]).

§3.4 Polynomial Identities for the Fibonacci and Lucas Numbers

§3.4.1 Introduction

We have already seen in chapter 2 how we can derive identities involving Fibonacci and Lucas numbers from the transformation of Jacobi's triple product using Chebyshev polynomials. In this section it is shown how to apply a related idea to produce more directly some polynomial identities for the Fibonacci and Lucas numbers [16a].

The Fibonacci numbers are named after Leonardo of Pisa (*c.*1180–1250), better known as Fibonacci (son of Bonaccio), an Italian merchant. They arise in the solution of a famous problem he posed in his Liber Abaci (book of the abacus). Namely, *how many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on?*

The Lucas numbers are named after E. Lucas, who was first to develop the general theory in a seminal paper which appeared in volume one of the *American Journal of Mathematics* in 1878 [20]. They are defined by

$$U_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad V_n(P,Q) = \alpha^n + \beta^n \qquad \text{for } n \geq 0,$$

where P, Q are non-zero integers and α, β are the roots of

$$x^2 - Px + Q = 0.$$

For every $n \geq 2$ we have

$$U_n(P,Q) = PU_{n-1}(P,Q) - QU_{n-2}(P,Q)$$

and

$$V_n(P,Q) = PV_{n-1}(P,Q) - QV_{n-2}(P,Q).$$

The sequence corresponding to $P = 1$, $Q = -1$, $U_0 = 0$ and $U_1 = 1$ defines the Fibonacci numbers, hereafter denoted by F_n. The sequence corresponding to $P = 1$, $Q = -1$, $V_0 = 2$ and $V_1 = 1$ defines the Lucas numbers, hereafter denoted by L_n. So we have

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \qquad \text{for } n \geq 2 \text{ and } F_0 = 0,\ F_1 = 1, \\ L_n &= L_{n-1} + L_{n-2} \qquad \text{for } n \geq 2 \text{ and } L_0 = 2,\ L_1 = 1. \end{aligned} \tag{3.4.1}$$

The Fibonacci and Lucas numbers satisfy many interesting recurrence relations. Among them is the well known identity

$$F_{3n} = F_n \left\{5{F_n}^2 + 3(-1)^n\right\} . \tag{3.4.2}$$

Other identities of the same type as equation (3.4.2) are

$$F_{5n} = F_n \left\{25{F_n}^4 + 25(-1)^n {F_n}^2 + 5\right\} \tag{3.4.3}$$

and

$$F_{7n} = F_n \left\{125{F_n}^6 + 175(-1)^n {F_n}^4 + 70{F_n}^2 + 7(-1)^n\right\} . \tag{3.4.4}$$

Some general theorems encompassing equations (3.4.2)–(3.4.4) are now presented. The theorems show how we can express F_{mn} (for odd m) as a polynomial in F_n and F_{mn}/F_n as a polynomial in L_n. The idea is essentially the same as that used to transform Jacobi's triple product identity in chapter 2. That is, we express the sum of terms of the form $x^n \pm 1/x^n$ as a polynomial in $x \pm 1/x$.

§3.4.2 The main theorems

Theorem 3.4.1 For integers n and $q \geq 0$ we have

$$F_{(2q+1)n} = F_n \sum_{k=0}^{q} (-1)^{n(q+k)} \frac{2q+1}{q+k+1} 5^k \binom{q+k+1}{2k+1} F_n^{\,2k}.$$

From theorem 3.4.1 we can obtain the following well known result as a corollary. If we let $p = 2q+1$, where p is prime, and use Euler's criterion [3] to show

$$5^{(p-1)/2} \equiv \left(\frac{5}{p}\right) \bmod p$$

where $\left(\frac{5}{p}\right)$ is the Legendre symbol, we obtain

Corollary 3.4.1 For $n \geq 0$ and p prime we have

$$F_{pn} \equiv \left(\frac{5}{p}\right) F_n \bmod p.$$

From corollary 3.4.1, with $n = 1$ and $n = q$ (a prime) respectively, we have

$$F_p \equiv \left(\frac{5}{p}\right) \bmod p$$

and

$$F_{pq} \equiv \left(\frac{5}{p}\right) F_q \bmod p.$$

Therefore we obtain

Corollary 3.4.2 For primes p and q we have

$$F_{pq} \equiv F_p F_q \bmod pq.$$

Before proving theorem 3.4.1, some lemmas required for its proof and the proof of two further theorems are presented.

Lemma 3.4.1 For $|x| \neq 0$ and integer $m \geq 0$ we have

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1$$

$$= \sum_{k=0}^{m} \frac{2m+1}{m+k+1} \binom{m+k+1}{2k+1} \left(x - \frac{1}{x}\right)^{2k}$$

Lemma 3.4.2 For $|x| \neq 0$ and integer $m \geq 0$ we have

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1} \left(x^2 + \frac{1}{x^2}\right) + (-1)^m$$

$$= \sum_{k=0}^{m} (-1)^{m+k} \frac{2m+1}{m+k+1} \binom{m+k+1}{2k+1} \left(x + \frac{1}{x}\right)^{2k}$$

Lemma 3.4.3 For $|x| \neq 0$ and integer $m \geq 0$ we have

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1} \left(x^2 + \frac{1}{x^2}\right) + (-1)^m$$

$$= \sum_{k=0}^{m} \binom{m+k}{2k} \left(x - \frac{1}{x}\right)^{2k}$$

Lemma 3.4.4 For $|x| \neq 0$ and integer $m \geq 0$ we have

$$\left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1$$

$$= \sum_{k=0}^{m} (-1)^{m+k} \binom{m+k}{2k} \left(x + \frac{1}{x}\right)^{2k}$$

Lemma 3.4.5 For $|x| \neq 0$ and integer $m \geq 1$ we have

$$\left(x^{2m-1} - \frac{1}{x^{2m-1}}\right) - \left(x^{2m-3} - \frac{1}{x^{2m-3}}\right) + \cdots + (-1)^m \left(x^3 - \frac{1}{x^3}\right) + (-1)^{m+1} \left(x - \frac{1}{x}\right)$$

$$= \sum_{k=1}^{m} \binom{m+k-1}{2k-1} \left(x - \frac{1}{x}\right)^{2k-1}$$

Lemma 3.4.6 For $|x| \neq 0$ and integer $m \geq 1$ we have

$$\left(x^{2m-1} + \frac{1}{x^{2m-1}}\right) + \left(x^{2m-3} + \frac{1}{x^{2m-3}}\right) + \cdots + \left(x^3 + \frac{1}{x^3}\right) + \left(x + \frac{1}{x}\right)$$

$$= \sum_{k=1}^{m} (-1)^{m+k} \binom{m+k-1}{2k-1} \left(x + \frac{1}{x}\right)^{2k-1}$$

Proof of theorem 3.4.1 Solving the recurrence relation given by equation (3.4.1) for F_n we obtain

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{where } \alpha + \beta = 1 \text{ and } \alpha\beta = -1.$$

So for integer n and $p \geq 1$, writing $x = \alpha^n$ and $y = \beta^n = (-1)^n/x$, we have

$$\begin{aligned} \frac{F_{pn}}{F_n} &= \frac{\alpha^{pn} - \beta^{pn}}{\alpha^n - \beta^n} \\ &= \frac{x^p - y^p}{x - y} \\ &= x^{p-1} + x^{p-2}y + x^{p-3}y^2 + \cdots + xy^{p-2} + y^{p-1}. \end{aligned} \tag{3.4.5}$$

Now for $p \equiv 1 \bmod 4$ the **RHS** of equation (3.4.5) is equal to

$$\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n \left(x^{p-3} + \frac{1}{x^{p-3}}\right) + \cdots + (-1)^n \left(x^2 + \frac{1}{x^2}\right) + 1 \tag{3.4.6}$$

and for $p \equiv 3 \bmod 4$ the **RHS** of equation (3.4.5) is equal to

$$\left(x^{p-1} + \frac{1}{x^{p-1}}\right) + (-1)^n \left(x^{p-3} + \frac{1}{x^{p-3}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + (-1)^n \tag{3.4.7}$$

Since $x + 1/x = \alpha^n + 1/\alpha^n = \alpha^n + (-1)^n \beta^n$, we have

$$x + \frac{1}{x} = (\alpha - \beta)F_n \qquad \text{for odd n} \tag{3.4.8}$$

and

$$x - \frac{1}{x} = (\alpha - \beta)F_n \qquad \text{for even n.} \tag{3.4.9}$$

Now $\alpha - \beta = \sqrt{5}$. Hence from equations (3.4.8) and (3.4.9) we obtain

$$\left(x + \frac{1}{x}\right)^2 = 5{F_n}^2 \qquad \text{for odd n} \tag{3.4.10}$$

and

$$\left(x - \frac{1}{x}\right)^2 = 5{F_n}^2 \qquad \text{for even n.} \tag{3.4.11}$$

Therefore if we let $p = 2m + 1$ and assume n is even, from equations (3.4.5) and (3.4.6) we obtain

$$\frac{F_{pn}}{F_n} = \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1. \tag{3.4.12}$$

Now we use lemma 3.4.1 in equation (3.4.12) and apply equation (3.4.11) to obtain theorem 3.4.1 for even n. Similarly, letting $p = 2m + 1$ and assuming n is odd we have from equations (3.4.5) and (3.4.7)

$$\frac{F_{pn}}{F_n} = \left(x^{2m} + \frac{1}{x^{2m}}\right) - \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + (-1)^{m+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^m. \tag{3.4.13}$$

Now we use lemma 3.4.2 in equation (3.4.13) and apply equation (3.4.10) to obtain theorem 3.4.1 for odd n. This completes the proof of theorem 3.4.1. □

Of course we now need to prove the lemmas. They can all be proved by induction. So as an example the proof of lemma 3.4.1 is presented below.

Proof: Let $P_0(x) = 1$ and for integer $m \geq 1$

$$P_m(x) = \left(x^{2m} + \frac{1}{x^{2m}}\right) + \left(x^{2m-2} + \frac{1}{x^{2m-2}}\right) + \cdots + \left(x^2 + \frac{1}{x^2}\right) + 1.$$

Now we note that

$$\left(x^2+\frac{1}{x^2}\right)\left(x^{2m}+\frac{1}{x^{2m}}\right)=\left(x^{2m-2}+\frac{1}{x^{2m-2}}\right)+\left(x^{2m+2}+\frac{1}{x^{2m+2}}\right)$$

to obtain

$$\left(x^2+\frac{1}{x^2}\right)P_m(x)=P_{m+1}(x)+P_{m-1}(x) \qquad \text{for } m\geq 1.$$

Hence

$$P_{m+1}(x)=\left\{\left(x-\frac{1}{x}\right)^2+2\right\}P_m(x)-P_{m-1}(x).$$

By the induction hypothesis we have

$$\begin{aligned} P_{m+1}(x)=&\sum_{k=1}^{m+1}\frac{2m+1}{m+k}\binom{m+k}{2k-1}\left(x-\frac{1}{x}\right)^{2k}\\ &+2\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}\left(x-\frac{1}{x}\right)^{2k}\\ &-\sum_{k=0}^{m-1}\frac{2m-1}{m+k}\binom{m+k}{2k+1}\left(x-\frac{1}{x}\right)^{2k}. \end{aligned} \tag{3.4.14}$$

From equation (3.4.14) the constant term in $P_{m+1}(x)$ is

$$\frac{2(2m+1)}{m+1}\binom{m+1}{1}-\frac{2m-1}{m}\binom{m}{1}=2m+3,$$

and the coefficient of $(x-1/x)^{2k}$ for $1\leq k\leq m-1$ is

$$\frac{2m+1}{m+k}\binom{m+k}{2k-1}+\frac{2(2m+1)}{m+k+1}\binom{m+k+1}{2k+1}-\frac{2m-1}{m+k}\binom{m+k}{2k+1}$$

$$=\frac{(2m+1)(m+k)!}{(m+k)(2k-1)!(m-k+1)!}+\frac{(4m+2)(m+k+1)!}{(m+k+1)(2k+1)!(m-k)!}-\frac{(2m-1)(m+k)!}{(m+k)(2k+1)!(m-k-1)!}$$

$$= \frac{1}{m+k+2}\binom{m+k+2}{2k+1}$$

$$\times\left\{\frac{(2m+1)(2k)(2k+1)+(4m+2)(m+k)(m-k+1)-(2m-1)(m-k)(m-k+1)}{(m+k)(m+k+1)}\right\}.$$

The numerator inside the braces above equals

$$2m^3+5m^2+3m+3k^2+3k+2mk^2+8mk+4m^2k=(2m+3)(m+k)(m+k+1).$$

Hence the coefficient of $(x-1/x)^{2k}$ for $1 \le k \le m-1$ equals

$$\frac{2m+3}{m+k+2}\binom{m+k+2}{2k+1}.$$

From equation (3.4.14) the coefficient of $(x-1/x)^{2m}$ for $1 \le k \le m-1$ equals

$$\frac{2m+1}{2m}\binom{2m}{2m-1}+2=2m+3,$$

and the coefficient of $(x-1/x)^{2m+2}$ equals 1. Since

$$\sum_{k=0}^{m}\frac{2m+1}{m+k+1}\binom{m+k+1}{2k+1}\left(x-\frac{1}{x}\right)^{2k}$$

equals $P_0(x)$ and $P_1(x)$ for $m=0$ and $m=1$ respectively, lemma 3.4.1 is proved by induction. □

It should be noted that lemma 3.4.1 can be derived easily from equation (2.2.4), by letting $z := iz$ in equation (2.2.4) and using the fact that the **LHS** of the lemma is $(x^{2m+1}-1/x^{2m+1})/(x-1/x)$, the sum of a geometrical progression.

It is possible to extract another couple of polynomial identities from equation (3.4.5). These are the following theorems.

Theorem 3.4.2 For integers n and $q \ge 0$ we have

$$F_{(2q+1)n}=F_n\sum_{k=0}^{q}(-1)^{(n+1)(q+k)}\binom{q+k}{2k}L_n{}^{2k}.$$

Theorem 3.4.3 For integers $n \geq 0$ and $q \geq 1$ we have

$$F_{2qn} = F_n \sum_{k=1}^{q} (-1)^{(n+1)(q+k)} \binom{q+k-1}{2k-1} L_n{}^{2k-1}.$$

Proof of theorem 3.4.2: The proof is very similar to that of theorem 3.4.1. We again use equations (3.4.6) and (3.4.7) with $p = 2m + 1$. Then we note that

$$x - \frac{1}{x} = \alpha^n + \beta^n = L_n \qquad \text{for odd } n \tag{3.4.15}$$

and

$$x + \frac{1}{x} = \alpha^n + \beta^n = L_n \qquad \text{for even } n. \tag{3.4.16}$$

and use equations (3.4.15) and (3.4.16) along with lemmas (3.4.3) and (3.4.4) to complete the proof of the theorem. □

Proof of theorem 3.4.3: The proof is again very similar to that of theorems (3.4.1) and (3.4.2). We now use lemmas (3.4.5) and (3.4.6) along with equations (3.4.15) and (3.4.16) to complete the proof of the theorem. □

It is worth noting that if we take $n = 1$ in theorems 3.4.2 and 3.4.3, since $L_1 = 1$, we have two well known results as corollaries.

Corollary 3.4.2 For integer $q \geq 0$ we have

$$F_{2q+1} = \sum_{k=0}^{q} \binom{q+k}{2k}.$$

Corollary 3.4.3 For integer $q \geq 1$ we have

$$F_{2q} = \sum_{k=1}^{q} \binom{q+k-1}{2k-1}.$$

§3.4.3 Closing Remarks

The formulae in theorems 3.4.1–3.4.3 are very pleasing, yet fairly simple. So it is surprising that they have not appeared in the literature before [16a]. They are presented here not only because they are new but also because they come out with such a straight-forward development, exploiting an extension of the idea of the Chebyshev polynomial transformation.

§3.5 Some Combinatorial Identities

§3.5.1 Introduction

This section presents some combinatorial identities, involving binomial coefficients, which fit neatly into the general theme of chapter 2. This is because the main results, theorems 3.5.1 and 3.5.2, arise as a result of transforming the sum $\sum_{n=-\infty}^{\infty} q^{n^2} z^n$ (valid for complex q and z, such that $|q| < 1$ and $z \neq 0$) using the Chebyshev polynomials, equations (2.2.3) and (2.2.4). Theorems 3.5.1 and 3.5.2 do not appear to have been stated explicitly in the literature before (see §3.5.3 for further comment on the following results). The search included J. Riordan's "Combinatorial identities" [30]. Theorems 3.5.3–3.5.6 are included here because they arise naturally as consequences of lemmas contained in section 3.4.

§3.5.2 The main theorems

Theorem 3.5.1 For $0 \leq k < n$ we have

$$\sum_{j=k}^{n} (-1)^j \frac{2n}{n+j} \binom{n+j}{2j} \binom{2j}{j+k} = 0$$

Theorem 3.5.2 For $0 \leq k < n$ we have

$$\sum_{j=k}^{n} (-1)^j \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} \binom{2j+1}{j+k+1} = 0$$

It is worth noting that since each of these combinatorial indentities is valid for $0 \leq k \leq n-1$ they both give rise to a set of combinatorial identities. For example, with $k = 0$ we have

Corollary 3.5.1

$$\sum_{j=0}^{n} (-1)^j \frac{2n}{n+j} \binom{n+j}{2j} \binom{2j}{j} = 0$$

Corollary 3.5.2

$$\sum_{j=0}^{n} (-1)^j \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} \binom{2j+1}{j+1} = 0$$

Proof of theorems 3.5.1 and 3.5.2: Using equations (2.2.3) and (2.2.4), for complex q and z such that $|q| < 1$ and $z \neq 0$, we have

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{n^2} z^n &= 1 + \sum_{n=1}^{\infty} \left(z^n + \frac{1}{z^n}\right) q^{n^2} \\ &= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} (-1)^{n+j} \frac{2n}{n+j} \binom{n+j}{2j} q^{(2n)^2} \left(z + \frac{1}{z}\right)^{2j} \\ &+ \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} (-1)^{n+j} \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1} q^{(2n+1)^2} \left(z + \frac{1}{z}\right)^{2j+1}. \end{aligned} \tag{3.5.1}$$

We now let $z := q^{1/2}$ and $q := q^{1/2}$ in equation (3.5.1) to obtain

$$2\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \sum_{j=0}^{\infty}\sum_{n=j}^{\infty} (-1)^{n+j} \frac{2n}{n+j}\binom{n+j}{2j} q^{2n^2}\left(q^{1/2}+\frac{1}{q^{1/2}}\right)^{2j}$$
$$+\sum_{j=0}^{\infty}\sum_{n=j}^{\infty} (-1)^{n+j} \frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1} q^{2n^2+2n+1/2}\left(q^{1/2}+\frac{1}{q^{1/2}}\right)^{2j+1}. \qquad (3.5.2)$$

By the binomial theorem we have the following two results

$$\left(q^{1/2}+\frac{1}{q^{1/2}}\right)^{2j} = \sum_{k=-j}^{j}\binom{2j}{j+k} q^k \qquad \text{for } j \geq 0, \qquad (3.5.3)$$

and

$$q^{1/2}\left(q^{1/2}+\frac{1}{q^{1/2}}\right)^{2j+1} = \sum_{k=-j}^{j+1}\binom{2j+1}{j+k} q^k \qquad \text{for } j \geq 0. \qquad (3.5.4)$$

We now use equations (3.5.3) and (3.5.4) to substitute for

$$\left(q^{1/2}+\frac{1}{q^{1/2}}\right)^{2j} \qquad \text{and} \qquad q^{1/2}\left(q^{1/2}+\frac{1}{q^{1/2}}\right)^{2j+1}$$

in the **RHS** of equation (3.5.2). Hence

$$2\sum_{n=0}^{\infty} q^{\binom{n+1}{2}} = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=-j}^{j} (-1)^{n+j} \frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j+k} q^{2n^2+k}$$
$$+\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=-j}^{j+1} (-1)^{n+j} \frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}\binom{2j+1}{j+k} q^{2n^2+2n+k} \qquad (3.5.5)$$

Now consider the first triple summation on the **RHS** of equation (3.5.5). By removing the $k=0$ term and noting that $\binom{2j}{j+k}=\binom{2j}{j-k}$ for $0 \leq k \leq j$, the first triple summation equals

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n} (-1)^{n+j} \frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j} q^{2n^2}$$
$$+\sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{k=1}^{j} (-1)^{n+j} \frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j+k}(q^{2n^2-k}+q^{2n^2+k}). \qquad (3.5.6)$$

By interchanging the order of summation of j and k equation (3.5.6) equals

$$\sum_{n=0}^{\infty}(-1)^n q^{2n^2}\sum_{j=0}^{n}(-1)^j\frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j}$$
$$+\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^n(q^{2n^2-k}+q^{2n^2+k})\sum_{j=k}^{n}(-1)^j\frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j+k}. \tag{3.5.7}$$

Considering the second triple summation on the **RHS** of equation (3.5.5). Noting that $\binom{2j+1}{j+k}=\binom{2j+1}{j-k+1}$, for $0\le k\le j+1$, and after letting $k:=k+1$ we find that this equals

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{n+j}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}\binom{2j+1}{j+k+1}(q^{2n^2+2n-k}+q^{2n^2+2n+k+1}). \tag{3.5.8}$$

By interchanging the order of summation of j and k equation (3.5.8) equals

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n(q^{2n^2+2n-k}+q^{2n^2+2n+k+1})\sum_{j=k}^{n}(-1)^j\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}\binom{2j+1}{j+k+1}. \tag{3.5.9}$$

Therefore, from equations (3.5.5), (3.5.7) and (3.5.9) we have

$$\begin{aligned}2\sum_{n=0}^{\infty}q^{\binom{n+1}{2}}=&\sum_{n=0}^{\infty}(-1)^n q^{2n^2}\sum_{j=0}^{n}(-1)^j\frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j}\\&+\sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^n(q^{2n^2-k}+q^{2n^2+k})\sum_{j=k}^{n}(-1)^j\frac{2n}{n+j}\binom{n+j}{2j}\binom{2j}{j+k}\\&+\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^n(q^{2n^2+2n-k}+q^{2n^2+2n+k+1})\sum_{j=k}^{n}(-1)^j\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}\binom{2j+1}{j+k+1}.\end{aligned} \tag{3.5.10}$$

To complete the proof we only have to note that for the ranges of k and given values of n below, the following are all distinct numbers (in fact they are all the non-triangular numbers).

$$\begin{array}{lr}
2n^2 & \text{for } n > 0. \\
2n^2 - k,\ 2n^2 + k & \text{for } 1 \le k < n \text{ and } n > 1. \\
2n^2 + 2n - k,\ 2n^2 + 2n + k + 1 & \text{for } 0 \le k < n \text{ and } n > 0.
\end{array}$$

That is to say $R \cap S = R \cap T = S \cap T = \emptyset$ (the null set) where

$$\begin{aligned}
R &= \left\{2n^2 \mid n \in Z,\ n > 0\right\}, \\
S &= \left\{2n^2 - k,\ 2n^2 + k \mid n \in Z,\ 1 \le k < n,\ n > 1\right\}, \\
T &= \left\{2n^2 + 2n - k,\ 2n^2 + 2n + k + 1 \mid n \in Z,\ 0 \le k < n,\ n > 0\right\}.
\end{aligned}$$

To obtain theorems 3.5.1 and 3.5.2 we equate the coefficients of q^n in equation (3.5.10). □

The next theorems are of a similar nature, that is they are all valid for a given range of values of k. With the exception of theorem 3.5.6, they are consequences of lemmas 3.4.2, 3.4.4 and 3.4.6. For theorem 3.5.6 we require one additional result, given as lemma 3.5.1. The proof of lemma 3.5.1 is similar to the proofs of lemmas 3.4.1–3.4.6.

Theorem 3.5.3 For $0 \le k \le n$ we have

$$\sum_{j=k}^{n} \frac{2j}{j+k} \binom{j+k}{2k} = \frac{2n+1}{n+k+1} \binom{n+k+1}{2k+1}$$

where $2j/(j+k)$ is taken to be 1 for $j = k = 0$.

Proof: From lemma 3.4.2 we have for $x \neq 0$

$$\left(x^{2n} + \frac{1}{x^{2n}}\right) - \left(x^{2n-2} + \frac{1}{x^{2n-2}}\right) + \cdots + (-1)^{n+1}\left(x^2 + \frac{1}{x^2}\right) + (-1)^n$$
$$= \sum_{k=0}^{n} (-1)^{n+k} \frac{2n+1}{n+k+1} \binom{n+k+1}{2k+1} \left(x + \frac{1}{x}\right)^{2k}$$

and from equation (2.2.3) we have

$$(-1)^j \left(x^{2j} + \frac{1}{x^{2j}}\right) = \sum_{k=0}^{j} (-1)^k \frac{2j}{j+k} \binom{j+k}{2k} \left(x + \frac{1}{x}\right)^{2k}.$$

Hence

$$\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\frac{2j}{j+k}\binom{j+k}{2k}\left(x+\frac{1}{x}\right)^{2k}=\sum_{k=0}^{n}(-1)^k\frac{2n+1}{n+k+1}\binom{n+k+1}{2k+1}\left(x+\frac{1}{x}\right)^{2k}.$$

Therefore on interchanging the order of summation we have

$$\sum_{k=0}^{n}(-1)^k\left(x+\frac{1}{x}\right)^{2k}\sum_{j=k}^{n}\frac{2j}{j+k}\binom{j+k}{2k}=\sum_{k=0}^{n}(-1)^k\frac{2n+1}{n+k+1}\binom{n+k+1}{2k+1}\left(x+\frac{1}{x}\right)^{2k}.$$

Equating the coefficients of $(x+1/x)^{2k}$ now completes the proof of the theorem. □

In a similar way one can prove from lemma 3.4.4 the following theorem.

Theorem 3.5.4 For $0 \leq k \leq n$ we have

$$\sum_{j=k}^{n}(-1)^j\frac{2j}{j+k}\binom{j+k}{2k}=(-1)^n\binom{n+k}{2k}$$

where $2j/(j+k)$ is taken to be 1 for $j=k=0$.

Another similar identity is

Theorem 3.5.5 For $0 \leq k \leq n-1$ we have

$$\sum_{j=k}^{n-1}(-1)^j\frac{2j+1}{j+k+1}\binom{j+k+1}{2k+1}=(-1)^{n+1}\binom{n+k}{2k+1}.$$

Proof: Using equation (2.2.4) in lemma 3.4.6 we have for $x \neq 0$

$$\sum_{j=0}^{n-1}\sum_{k=0}^{j}(-1)^{j+k}\frac{2j+1}{j+k+1}\binom{j+k+1}{2k+1}\left(x+\frac{1}{x}\right)^{2k+1}=\sum_{k=0}^{n-1}(-1)^{n+k+1}\binom{n+k}{2k+1}\left(x+\frac{1}{x}\right)^{2k+1}.$$

Therefore, interchanging the order of summation we have

$$\sum_{k=0}^{n-1}(-1)^k\left(x+\frac{1}{x}\right)^{2k+1}\sum_{j=k}^{n-1}(-1)^j\frac{2j+1}{j+k+1}\binom{j+k+1}{2k+1}=\sum_{k=0}^{n-1}(-1)^{n+k+1}\binom{n+k}{2k+1}\left(x+\frac{1}{x}\right)^{2k+1}.$$

Equating coefficients of $(x+1/x)^{2k+1}$ completes the proof of the theorem. □

For the final theorem of this section we need the following lemma.

Lemma 3.5.1 For $|x| \neq 0$ and $n \geq 1$ we have

$$\left(x^{2n-1}+\frac{1}{x^{2n-1}}\right)-\left(x^{2n-3}+\frac{1}{x^{2n-3}}\right)+\cdots+(-1)^n\left(x^3+\frac{1}{x^3}\right)+(-1)^{n-1}$$
$$=\sum_{k=0}^{n-1}(-1)^{n+k-1}\frac{2n}{n+k+1}\binom{n+k+1}{2k+2}\left(x+\frac{1}{x}\right)^{2k+1}.$$

Proof: Similar to the proof given for lemma 3.4.1. □

Theorem 3.5.6 follows from lemma 3.5.1. The proof runs along the same lines as that of theorem 3.5.5.

Theorem 3.5.6 For $0 \leq k \leq n-1$ we have

$$\sum_{j=k}^{n-1}\frac{2j+1}{j+k+1}\binom{j+k+1}{2k+1}=\frac{2n}{n+k+1}\binom{n+k+1}{2k+2}.$$

§3.5.3 Closing Remarks

Theorems 3.5.3–3.5.6 are reasonably elementary identities. They are presented here because they are discernible from the previous work contained in this thesis, from which they appear with little effort. Theorems 3.5.1–3.5.2 are slightly more involved. Actually, all of the results in this section are specializations of well-known hypergeometric series identities. In particular 3.5.1 and 3.5.2 are disguised forms of the *Chu-Vandermonde summation*. However, there are many combinatorial identities in the literature; in fact the supply is inexhaustible. They quite often arise in unpredictable ways, making it difficult to place them in coherent mathematical settings and order of interest. Therefore quite often we find, that in addition to the identity itself, the mathematical interest lies equally in the method of derivation. This is felt to be the case for the theorems presented in this section.

For a general method of proving all identities of a similar nature to those in this section, see the very interesting paper by Wilf and Zeilberger [**36**].

Chapter 4

Some Representations of the Unrestricted Partition Function.

§4.1 Introduction

This chapter presents some representations of the unrestricted partition function which have arisen as a result of work on the previous chapters. A representation of the partition function is an identity where on one side of the equation we have $\sum_{n=0}^{\infty} p(n)q^n$, $p(n)$ being the unrestricted partition function, and on the other side we have some aesthetically pleasing expression. For example from [9], referred to as Gordon's identity in [10], we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q)_n (q)_{n+1}} = \sum_{n=0}^{\infty} p(n)q^n. \tag{4.1}$$

Another example, from [1a] we have a generalisation of equation (4.1)

$$\sum_{n=0}^{\infty} \frac{q^{n(n+a)}}{(q)_n (q)_{n+a}} = \frac{1}{(q)_\infty} \qquad \text{for } a \in Z. \tag{4.2}$$

The $(q)_n$ notation was defined in section 3.3. Here $(q)_n^{-1} = 0$ for $n < 0$ and $(q)_\infty = \prod_{n=1}^{\infty}(1 - q^n)$, so that

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n.$$

§4.2 The main theorems

Theorem 4.1

$$\sum_{n=r}^{\infty} \frac{q^{n^2}}{(q)_{n-r}(q)_{n+r}} = \frac{q^{r^2}}{(q)_\infty} \qquad \text{for } r \geq 0.$$

Theorem 4.2

$$\sum_{n=r}^{\infty} \frac{q^{n(n+1)}}{(q)_{n-r}(q)_{n+r+1}} = \frac{q^{r(r+1)}}{(q)_\infty} \qquad \text{for } r \geq 0.$$

Proof of theorems 4.1 and 4.2: From section 3.3, see also [31], if

$$\prod_{n=0}^{\infty} (1 + 2zq^n \cos\theta + z^2 q^{2n}) = \sum_{n=0}^{\infty} r_n(\theta) z^n. \tag{4.3}$$

Then we have

$$r_{2n}(\theta) = \frac{q^{n(n-1)}}{(q)_n^2} + q^{n(n-1)} \sum_{r=1}^{n} \frac{q^{r^2}}{(q)_{n-r}(q)_{n+r}} 2\cos 2r\theta \qquad \text{for } n \geq 1 \tag{4.4}$$

and

$$r_{2n+1}(\theta) = q^{n^2} \sum_{r=0}^{n} \frac{q^{r(r+1)}}{(q)_{n-r}(q)_{n+r+1}} 2\cos(2r+1)\theta \qquad \text{for } n \geq 0. \tag{4.5}$$

Now if we let $q := q^{1/2}$ and $z = e^{i\theta}$ in Jacobi's triple product identity, equation (2.2.5), we have

$$\prod_{n=1}^{\infty} (1 - q^n)(1 + 2\cos\theta\, q^{n-1/2} + q^{2n-1}) = 1 + \sum_{n=1}^{\infty} 2\cos n\theta\, q^{n^2/2}.$$

Therefore

$$\prod_{n=0}^{\infty} (1 + 2\cos\theta\, q^{n+1/2} + q^{2n+1}) = \frac{1}{(q)_\infty} \left\{ 1 + \sum_{n=1}^{\infty} 2\cos n\theta\, q^{n^2/2} \right\}. \tag{4.6}$$

Now if we let $z := q^{1/2}$ in equation (4.3), using equation (4.6) we have

$$\sum_{n=0}^{\infty} r_n(\theta) q^{n/2} = \frac{1}{(q)_\infty}\left\{1+\sum_{n=1}^{\infty} 2\cos n\theta\, q^{n^2/2}\right\}. \tag{4.7}$$

Therefore we can use equations (4.4), (4.5) and (4.7) to obtain, after interchanging the order of summation of n and r

$$\begin{aligned}\frac{1}{(q)_\infty}\sum_{r=1}^{\infty} q^{r^2/2} 2\cos r\theta &= \sum_{r=1}^{\infty} q^{r^2} 2\cos 2r\theta \sum_{n=r}^{\infty} \frac{q^{n^2}}{(q)_{n-r}(q)_{n+r}} \\ &+ q^{1/4}\sum_{r=0}^{\infty} q^{r(r+1)} 2\cos(2r+1)\theta \sum_{n=r}^{\infty} \frac{q^{(n+1/2)^2}}{(q)_{n-r}(q)_{n+r+1}}.\end{aligned} \tag{4.8}$$

To complete the proof of theorem 4.1 we equate the coefficients of $2\cos 2r\theta$ in equation (4.8) and to complete that of theorem 4.2 we equate the coefficients of $2\cos(2r+1)\theta$ in equation (4.8). □

The next set of representations of the partition function is

Theorem 4.3

$$\frac{1}{1+q^a}\sum_{n=0}^{\infty} \frac{q^{n(n+a-1)}}{(q)_n (q)_{n+a}} = \frac{1}{(q)_\infty} \qquad \text{for } a \in Z.$$

Proof: Jacobi's triple product identity may be written in the form

$$\prod_{n=1}^{\infty} (1+zq^n)(1+z^{-1}q^{n-1}) = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2}. \tag{4.9}$$

Using the well known

$$\prod_{n=0}^{\infty}(1+zq^n) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q)_n} z^n, \qquad \text{for } |q|<1,$$

and equation (4.9) we have

$$\begin{aligned}\frac{(1+z)}{(q)_\infty}\sum_{n=-\infty}^{\infty} z^n q^{n(n+1)/2} &= \prod_{n=0}^{\infty}(1+zq^n)(1+z^{-1}q^n) \\ &= \left\{\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q)_n} z^n\right\}\left\{\sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q)_m} z^{-m}\right\}.\end{aligned} \tag{4.10}$$

Now if we write $n = m + a$, where $n,\ m \in Z^+$, equating the coefficients of the term in z^a (by taking the product of all the terms in z^n and z^{-m}, on the **RHS** of equation 4.10, such that $n = m + a$) we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)/2+(n-a)(n-a-1)/2}}{(q)_n (q)_{n-a}} = \frac{1}{(q)_\infty} \left\{ q^{a(a+1)/2} + q^{a(a-1)/2} \right\}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{q^{n(n-1)-na}}{(q)_n (q)_{n-a}} = \frac{1}{(q)_\infty} \left\{ 1 + q^{-a} \right\}. \tag{4.11}$$

Letting $a := -a$ in equation (4.11) we obtain theorem 4.3. □

§4.3 Closing Remarks

Theorems 4.1 and 4.2 are presented here because they are particularly nice results which can be deduced as a simple consequence of an idea of L. J. Rogers [31]. The title of [9] suggests that the fact that theorems 4.1 and 4.2 are implicit in [31] has been overlooked. However, it should be noted that theorem 4.1 is the case $A = B = r$ in (7.13) of the more recent [1b] and theorem 4.2 is the case $A = r$, $B = r + 1$ in (7.13) of [1b]. It should also be noted that theorem 4.3 is not new, since it is the case $A = a$, $B = a - 1$ in (2.57) of [1b]. However, the theorem is very appealing and the derivation given here is both quick and simple.

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Notation Guide

Symbol	Description
B_n	Bernoulli Numbers $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$ for $\vert x\vert < 2\pi$.
$E_{2k}(z)$	Eisenstein series $E_{2k}(z) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i z}$ for $Im(z) > 0$.
F_n	Fibonacci Numbers $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$.
$G_k(z,m)$	Poincaré series.
H	Upper half-plane $\{z \in C \mid Im(z) > 0\}$
L_n	Lucas Numbers $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$.
M_k	Linear space of modular forms of weight k.
$M_{k,0}$	Linear space of cusp forms of weight k.
P, Q and R	Eisenstein series E_2, E_4 and E_6.
$p(n)$	Unrestricted partition function $\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} (1-q^n)^{-1}$ for $\vert q\vert < 1$.
$(q)_n$	Product $(1-q)(1-q^2)\ldots(1-q^n)$, $(q)_0 = 1$, $(q)_n^{-1} = 0$ for $n < 0$. $\vert q\vert < 1$.
$(q)_\infty$	$\prod_{n=1}^{\infty}(1-q^n)$ for $\vert q\vert < 1$.
T_n	Hecke operator.
$T_n(x)$	Chebyshev Polynomial.
Γ	The modular group.
$\Delta(z)$	The modular discriminant $\Delta(z) = \frac{1}{1728}(E_4^3 - E_6^2)$.
$\zeta(z)$	Riemann's zeta function. $\zeta(s) = \prod_P (1-p^{-s})^{-1}$ for $Re(s) > 1$.
θ_1	$\theta_1(z,\tau) = -i\sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)i\pi z}$ for $Im(z) > 0$ and $q = e^{i\pi\tau}$.
θ_2	$\theta_2(z,\tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)i\pi z}$ for $Im(z) > 0$ and $q = e^{i\pi\tau}$.
θ_3	$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2i\pi n z}$ for $Im(z) > 0$ and $q = e^{i\pi\tau}$.
θ_4	$\theta_4(z,\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2i\pi n z}$ for $Im(z) > 0$ and $q = e^{i\pi\tau}$.
$\theta_i(q)$	$\theta_i(z,\tau)$ with $z = 0$ and $i = 1, 2, 3, 4$ and $\vert q\vert < 1$.
$\sum_{r,s}(n)$	$\sum_{k=0}^{n} \sigma_r(k)\sigma_s(n-k)$ for r, $s \geq 1$.
$\sigma^o(n)$	$\sum_{\substack{d\mid n \\ d\ odd}} d$.
$\sigma^e(n)$	$\sum_{\substack{d\mid n \\ d\ even}} d$.
$\sigma_k(n)$	$\sum_{d\mid n} d^k$.
$\tau(n)$	Ramanujan's tau function $\sum_{n=1}^{\infty} \tau(n) x^n = x\prod_{n=1}^{\infty}(1-x^n)^{24}$ for $\vert x\vert < 1$.
$\phi_{r,s}(x)$	$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^r n^s x^{mn}$ for $\vert x\vert < 1$.