UNIVERSITY OF SOUTHAMPTON

THE PROCESSING OF DATA FROM MULTI-HYDROPHONE TOWED ARRAYS OF UNCERTAIN SHAPE

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ABSTRACT

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THE PROCESSING OF DATA FROM MULTI-HYDROPHONE TOWED ARRAYS OF UNCERTAIN SHAPE

by Geoffrey William Sweet

An array of omni-directional hydrophones in tow is used to locate distant sources of acoustic radiation.

Where it is impossible, either actually or virtually, either to rotate the antenna or to change its shape, it is expedient to maximize the parallactic angle of point source and antenna through lengthening the antenna, and, since the antenna in question is implemented in the form of a string of discrete elements, to maximize the noise-rejective potential of the antenna by maximizing the number of elements in the string.

Although *a priori* the placing of hydrophones in an array is influenced by an uncertainty in knowledge of array disposition, an uncertainty which increases with distance from the towing vessel, for convenience an actual array with hydrophones spaced equidistantly is assumed for most of the thesis, although a modicum of flexibility of the antenna is allowed. In practice, the appropriateness or otherwise of a particular disposition of hydrophones is a function of the actual location and spectral character of a source.

In virtue of the uncertainty of sensor location as well as of modest relative motion of source and array, phase-differences of signal, reflected by measured pressures compared between hydrophones, are surmised in terms of bands of tolerance. It is shown that three such phase 'bins' per wavelength is optimal in a novel method presented in the thesis for comparing and contrasting the contents of bins such that a maximum may be associated uniquely with the location of a source.

The thesis is submitted with the conviction that a practical solution to a contemporary given problem of 'fuzzy' instrumentation has been found, a solution elaborated upon a theoretical basis with which, taking account of modern facilities for practical implementation, advances in accuracy and speed of processing beyond existing limits may be achieved. The thesis is submitted in the hope that, by varying inductive and deductive patterns of reasoning, a contribution will have been made to the theoretical basis for eliciting unique solutions to fuzzy problems, for which a calculus as well as appropriate modes of algebraic and statistical logic may be requiring to be developed.

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1 Introduction

1.1 The Given Problem and the General History of the Present Solution

The present work started out as an essay on how best to process acoustic pressures measured by means of an uncertainly-shaped towed array of many hydrophones. The research was called for by the Ministry of Defence Procurement Executive in its local manifestation of the Admiralty Research Establishment. However, the administration of the sponsorship was taken over by the Defence Research Agency half-way through the research.

Originally and fundamentally, the object was to estimate the location of a single point source in the presence of random noise by means of a long multi-hydrophone, string-like array, towed in the wake of the ship or submarine, which was likely at any time to be perturbed in a minor and random way from a given shape. 'Long' might mean as much as a kilometre. A great deal of the specification of the issues to be considered was left to the present author. However, it soon became conventional to think in terms if not always of a straight array, then at least of one of a smooth curvature, where the hydrophones normally maintained a mutual uniformity of relative, consecutive interspacing, whatever the operational conditions might be. In other words, a towed array of hydrophones could be distinguished from an arbitrary disposing of hydrophones by an at least piecewise equidistance between them. It seemed that the tautness of the cable under tow allowed that equidistance could be assumed. Concerning the degree of perturbation that might be expected, convention developed only latterly.

At the very beginning, the kind of towed array to be taken into account was adumbrated thus. Its function was circumscribed as follows. The towed array should be long enough such that, if it were fully stretched out flat in a straight line, a target (to be modelled as a monochromatic, spherical, frequency point-source), having reached a certain proximity of interest, could no longer, in virtue of that proximity, be treated with techniques that relied upon the assumption that wave-fronts were planar in incidence upon the array.

While some licence was allowed the imagination of the author for identifying problems, it was emphasized that the prevailing mathematical culture in which the present research should begin contained a bias in favour of plane waves rather than isotropic, spherically-spreading ones, a bias which needed some correction.

It is hoped that the narrative of the present thesis will reflect not only how solutions were sought to problems, but how the problems themselves emerged in the course of the study.

As ideas developed, the scope of the thesis began to broaden, eventually to take in the detecting and isolating of more than one point source too. By the conclusion of the present work, sources were considered which, present at the same time, were either different or similar in spectral character and power, including ones identical in all respects save for their locations, although this latter problem had been deemed to lie on the periphery of the author's remit. Towards the completion of the work, the author's attention was turning to dispositions of hydrophones where a smooth curvilinearity of array was assumed to be only a weak characteristic, whereas hitherto algorithms based upon straight arrays, with and without regular interspacing of hydrophones, had been considered.

The earlier stages of the work were based on the assumption that, during the time in which the acoustic field was being sampled, the source and sensor-system were stationary with respect to each other. But in the later stages the consequences of abandoning such an assumption were considered, and an algorithm was developed for locating point sources with a single snapshot. Without having delved profoundly into the mathematical theory of where time ended and space began in the spatio-temporal *continuum*, the thesis concludes on a note of preference for spatial rather than temporal analysis, although it has been attempted to contribute to both approaches. While, earlier, the basis of the methodology was first to determine the location of a a source uniquely in the absence of noise and then to estimate its location in the presence of noise, later work with the single snapshot approach led to the acknowledgement of a spatio-temporal uncertainty that cast doubt upon the feasibility of such a sequence of activities, and the theoretical possibility of a unique estimate being obtained in a 'fuzzy' context is examined.

In the beginning, the task was to find the best way of using a long towed array of uncertain but essentially curvilinear shape. For convenience, the question of using more than one array has been neglected here, but it is suggested that it may prove to be but a short analytical step from treating a snaking single array to treating a multiplicity of them. For convenience, the study was limited to two dimensions. For the future, it is hoped to tackle the problem in all its three-dimensional complexity and, it is suggested, the possibility exists with the novel phase-binning algorithm (which lies at the heart of the thesis) that taking account of three dimensions, rather than of just two, may actually speed up the process of convergence upon the estimate of the location of the source. However, for the present work it was assumed that the hydrophones and the source defined a plane.

Sources were assumed to be coherent in space and time. 'Monochromatic sources' rather than 'noise sources' were to be the subject of analysis. While 'frequency sources' implied characteristics of the signal which might be elicited from a noisy environment through the analysis of repeated observations, the term 'noise source' was taken by the author to imply an absence of repeatability in the signal in respect of both space and time.[1] For convenience, 'finite-amplitude' effects have been neglected in the present work.

It was assumed, for convenience, that the medium of source and hydrophones was infinite.

Much work had been done with arrays, the hydrophones of which were in exactly known locations. It was commonly agreed, however, that the

[1] It is acknowledged, however, that 'noisy' is sometimes used in the sense of 'unwanted' in the literature, without any connotation of spatial or temporal incoherence. Compare Knudsen *et al.*, who wrote: 'The term *underwater noise* is used to describe unwanted underwater sounds which tend to impair the operation of acoustically operated devices.' V. O. Knudsen, R. S. Alford and J. W. Emling, 'Underwater ambient noise', *Journal of Marine Research*, vol. 7, no. 3, 1948, p. 411. Ross isolates five different types of noise, 'definitions ... consistent with those adopted by the American Standards Association': radiated noise, ambient noise, platform noise, *SONAR* self-noise and *SONAR* background noise. For convenience, 'frequency sources' are distinguished from (random) noise only in the present thesis, although it is acknowledged that there is a case for distinguishing the statistical treatment of noise associated with hydrophone location uncertainty from that required by random noise. Our 'frequency sources' are like what Ross refers to as 'radiated noise': '... noise radiated into the water that can be used by a passive listening *SONAR* to detect the presence of a vehicle at a considerable distance.' D. Ross, *Mechanics of Underwater Noise*, new York, 1976, p. 3f. Specifically towed arrays suffer a penalty of their own. Thus Ketchman: 'Reducing transverse cable vibrations is important because of the associated longitudinal cable vibrations and longitudinal array vibrations. This is especially true since second-order longitudinal vibrations in the array are directly capable of producing hydrophone noise.' J. Ketchman, 'Vibration induced in towed linear underwater array cables', *I. E. E. E. Journal of Oceanic Engineering*, vol. OE-6, no. 3, July, 1981, p. 86. There would seem to be a specific problem too of wake noise. For convenience, wake noise has been left out of the present thesis. However, it is planned to treat the subject in future work.

efficacy of many methods with that basis suffered if the hydrophones were displaced, however slightly, from their assumed positions. For a time, the author thought he might be able to show that the theoretical bases of many published algorithms might have been inappropriate because they had been conceived with a perhaps ill-defined sense of dimensionality. Working in two dimensions, it had been hoped that a small random displacement of a hydrophone would not greatly undermine the robustness of an algorithm. The author felt, however, that it might be possible to demonstrate that the penalty would be less if the problem were to be looked at as one in which the presence of noise damaged a whole dimension of the possible resolution, to the extent indeed in some cases that the dimension might need to be neglected altogether. That is, instead of trying to isolate a point straightaway, it might perhaps be better to examine more closely an interim stage of eliciting a line or curve from the data. In short, rather than try to distil a point from data obtained ostensibly in two dimensions, the mutual consistency of which might be weak if the locations of the hydrophones were uncertain, it might be better to try to draw a line or curve on the basis of the data available and hope to approach an estimate of the location of the point gradually along it. In fact, as will be claimed in the Literature Review, much work with towed arrays has been concentrated upon direction-finding rather than upon locating with Cartesian co-ordinates. The present author calls direction-finding a one-dimensional activity, and he regards the locating of a point (of zero dimensions) as a two-dimensional activity. While it is obvious that a source a long way away is harder to locate, and therefore may be thought of effectively as having only a direction and not a distance as well (this being a familiar reason for concentrating upon direction-finding rather than locating), the author has tried to relate this effect to a theoretical basis to do with dimensionality, and indeed he raises the question whether certain algorithms are not bound to deal with direction-finding, rather than with locating, for the latter rather than former reason.

These sentiments formed the basis of an attempt to apply the theory of the caustic curve to the problem, about which more will be said below in Chapter 5. However, it is acknowledged that a theory of perturbed dimensionality, although it will be much mulled over in what follows, has been little more than adumbrated in the present thesis, it having been the primary purpose of the study to discover practical solutions to problems.

The author's sponsors have recently acquired the means to estimate the gross curvature of the array from the history of the speed and direction of the towing vessel alone, without the use of sources placed deliberately or present fortuitously. A study of the methods used to obtain that estimate lies beyond the scope of the present thesis. They are extrinsic to the signal processing. It may be that the employment of calibrating sources, whether placed deliberately or arising through opportunity, will have been superseded by the new 'historical' techniques, but we shall consider their use briefly below in the Literature Review in especial relation to the work of Rockah and Schultheiss. Roughly speaking, the author's sponsors are able to say that, in practice, over a kilometre or so, the estimated and the true disposition of the array differ by about one part in a thousand. If the shortest distance between two neighbouring hydrophones at any instant was one metre, there would be no possibility of one hydrophone being displaced to such an extent that it might occupy the supposed location of another. If there was an operational requirement that the array be one kilometre long, then the limit of interspacing would be one metre. We shall show below, however, that such a discipline need not be regarded as a considerable constraint on the sensitivity of the antenna.

An error of one part in a thousand with a notional long array of one kilometre seemed to the author to be relatively small. However, since the intelligence of the towed array curvature 'historical' facility only reached him latterly, a small, early part of the thesis is concerned with the case where there is equal uncertainty about the whereabouts of hydrophones, no matter how far away they were supposed to be from the towing vessel. In other words, the problem of the error of location between source and hydrophone was examined briefly as well as the problem of error of location between towing vessel and hydrophone. As has been suggested already, the author tried to tackle that problem in terms of the number of dimensions involved in the problem. In the presence of noise, particularly noise that could not be rejected by any method employing repeated observations, perhaps only a curve could be elicited from the data, a point of no dimensions on the other hand being theoretically beyond reach. Should it not be known exactly where any hydrophones were, in a given exercise in correlation, and if the analysis were carried out in two dimensions straightoff, then the penalty could be very great. Simply put, if one tried to find a pencil of more than two lines with a local error of this order, considerable global error could be incurred. Instead of a cluster of points, a middle location among which might be obvious enough for us to think of as the best estimate of the location of the source, we might be confronted by a confusion of points with no obvious centre. As we shall see in our discussion of the theory of the caustic curve, it may be possible to arrange the data in such a way that we do not get a haphazard confusion of points, but rather a line or curve with an obvious trend.

In fact, there was a case for saying that, in some circumstances, the more hydrophones were used, and the more data were available, the more error could arise. An analogy of the attenuation arising through the serial employment of thick lenses when transmitting light came to mind, and is referred to again below.

Strictly speaking, only a few hydrophones nearest the towing vessel may be said to be in 'known' locations. Beyond some distance, the location of a particular hydrophone becomes 'uncertain'. In the present thesis we shall regard 'uncertainty' not merely in absolute terms but also as relative to the task which the array is being required to perform in a particular context. For instance, should the uncertainty surrounding a hydrophone be greater than the tolerance required to corroborate a particular wavelength, then that wavelength must be neglected in the analysis. So local is the short row of hydrophones nearest the towing vessel, however, that one is scarcely justified in drawing extended $x-$ and $y-$ axes on the basis of it (the hydrophone nearest to the towing vessel being taken as the origin of the system of axes). For an array as a whole, the choice of orientation $x-$ and $y-$ axes is, in the last analysis, arbitrary. Yet because of the availability of information about the upper bound of uncertainty (*i. e.* one part in a thousand in the sense referred to above) there may in practice be an obvious choice of angular orientation of the system of axes. The usual practice seems to be to regard the general trend of an 'almost straight' array as the $x-$ axis. But, strictly speaking, the axes themselves must be said to be uncertain in orientation. Indeed, if the use of calibrating sources is ruled out, and the history of the speed and direction of the array only is employed, there will always be an orientational, or rotational, uncertainty.

We shall see below that this rotational uncertainty is an important outcome of the work of Rockah and Schultheiss, who, however, reach their conclusion by a character of argument fundamentally different from our own. We differ in that we hold out the possibility of removing uncertainty of rotational accuracy in the limit of the analysis of repeated observations with stationary calibrating sources. Such a thing, we argue, is statistically possible with stationary calibrating sources. On the other hand, the statistical treatment of the uncertainty of hydrophone location with the 'historical' approach must be different. Indeed, it must be said that the 'historical' (extrinsic) estimates of array shape can never 'improve' in accuracy with repeated comparisons. In other words, there can be no convergence in the limit as there can be with stationary calibrating sources. This is one of the reasons why, later on, we shall argue forcefully in favour of the single snapshot approach, rather than the time-series approach, if we have only the past history of the speed and direction of the towing vessel to help us decide where hydrophones are roughly located.

However, before learning about the 'historical' facility, and after considering the case where the hydrophones' locations were known precisely, the author set about finding a process of convergence in order to establish Cartesian axes. Perhaps some invariance, some property could be found, which might manifest itself in the light of so many hydrophones and a small error between the estimated and the actual disposition of the array. For, again, even though there was an absence of identifiable points through which to draw a line, one might conversely be able to elicit a line, consisting of points of unknown location. Instead of assuming axes, it was thought that the axes themselves perhaps ought to be thought of as lying in the limit of a process.

It lay then at the heart of that methodological initiative in the mid-term of the thesis that axes about a known origin should be assigned an orientation arising from a convergent process (if one could be found). Not until that was found could one begin to locate a point. It was fundamental to the approach at that stage of the research that the general course of lines and curves should be plotted before going on to attempt to locate points. Furthermore, a point should be regarded *ab initio* as a function of a relation between two lines (or curves), but not of more than two. For more than two lines at once would imply a degree of knowledge of the consistency of the two-dimensional axes than seemed unwarranted in view of the amount of uncertainty about the shape of the array.

In that regard, it seemed that it had turned out to be a weakness of some approaches to the subject that too much information might be applied at once to achieve the solution. It seemed to the present author that the reduction of data obtained in the context of some two dimensions to an obvious landmark without dimensions (*i. e.* a point) needed to be undertaken sequentially, *i. e.* by using two lines to fix a point, which point itself when joined to other such points would help to establish the trend of a line or a curve, and then that line or curve might be found to be disposed in a particular, and informative, way in relation to the convergent line taken to be an axis. In other words, the solution needed to be approached step-wise constructively, with points arising from two lines on each level of the process. On the other hand, much signal processing seemed to try to take in all the stages with a single leap, assuming that the synchronic manipulation of the data could cause the twin requirements of direction and distance to fall out simultaneously. An analogy suggested itself here concerning the transmission of a beam of light through lenses. If the beam were directed through a number of lenses in series, then no matter how small the error might seem to appear to be from one lens to the next, the overall error with many lenses might be very great. Broadly speaking, in the present work there will in the end be greater sympathy for simultaneity than with seriality.

Again, there seemed to be a need to establish axes themselves by a limiting process. Other work, however, had been done on the assumption of axes. Of that work, some was dedicated to the assumption that the hydrophones might be displaced randomly, either in one or two dimensions, which dimensions, again, had been fixed arbitrarily in advance. But it did not seem right to the present author that the hydrophones should be thought of as subject to random displacements in the sense of random noise upon a signal. In that genre of argument he thought he could hear the overtones of the convergences and limits of statistics. He felt, somehow, that the difference between the assumed and true array dispositions was not one that could be accounted for in terms of some limit of probabilities.

Indeed, in the event, the difference between the assumed and the true shapes of the array had to do with an experiment which was not probabilistic but historical, and, in practice, there was not more than one experiment that could be held to help estimate the difference. It turned out that a shape was derived by scrutinizing the history of the speed and the direction of the towing vessel. Thus the 'noise' on the shape of the array was fundamentally different from the noise on the signal. The noise on the signal certainly was something which must be tackled by repeated experiments, or sampling. But the statistical treatment of 'randomness' in hydrophone displacement must be different from that of randomness of noise about a stationary source.

It certainly seemed reasonable to assume that, over short periods of time, the hydrophones were not subject to the same kind or degree of random fluctuation as that random noise which might contaminate the signal. Originally, the author was invited to treat noise-to-signal ratios of one hundred or more. As we have said, the history to be taken into account in estimating the shape of an array was not a 'time history' in the customary sense. It was rather a history of circumstances. Accordingly, it seemed to the author that it might be more appropriate methodologically if the hydrophones were regarded as largely stationary, while, again, the axes should be treated as uncertain. And, again, if one was not sure how consistent a region one's axes implied, it might be wiser to dispense with the notion of two-dimensionality altogether and aim to extract information from the data with one-dimensional implications only.

We have dwelled at some length on the question of dimensionality for another reason too, one to do with the question of using priors in the analysis. We shall treat the use of priors again towards the end of the thesis in a discussion of the concepts of efficiency and of redundancy, but we point out here that the postulation of a specific orientation of axes may effectively be construed as a prior. In suggesting that a line, rather than a point, be elicited from data obtained from a perturbed array, we consider that we are attenuating the possible violence that the use of a 'Procrustean' prior fixing of two axes might do to the efficiency of an algorithm.

Much literature has been concerned with the estimation of direction of arrival of a signal (*DOA*), and many examples of it are based upon the assumption that the source is sufficiently far away from the array that the wavefront will be, for all practical purposes, planar upon arrival at the array. On the other hand, writing on the problem of locating point sources is less common. A conclusion of the present thesis is that we should properly be concerned with spherical waves in general and with the palpably spherical characteristics in particular. It is emphasized, however, that the algorithms presented below are not disabled in the event of effectively planar incidence upon the sensors, but will, beyond certain distances, be unable to convey any more information than that a source lies in a particular direction.

Landau and Lifshitz characterize the spherical wave as follows:

> Unlike a plane wave, whose amplitude remains constant, a spherical wave has an amplitude which decreases inversely as the square of the distance from the centre. The intensity of the wave is given by the square of the amplitude, and falls off inversely as the square of the distance, as it should, since the total energy flux in the wave

is distributed over a surface whose area increases as r^2.[2]

Although we are concerned mainly with spherical waves in the present thesis, nevertheless the theoretical and practical bases for distinguishing plane and spherical waves are examined in a separate chapter, and are referred to with elaboration as the thesis develops.

There has been much discussion in the literature devoted to enumerating and classifying (by relative angles of arrival) the signals incident upon a towed array. Commonly, a single frequency is isolated, whether by means of a Fourier transform or some other method, and the number of directions of arrival that can be elicited for that frequency is estimated. There is a relationship of dependency between the number of hydrophones used and the number of directions of arrival present, for the number of equations must match the number of unknowns.[3] The problem of multiple plane-wave signals of the same frequency is addressed in the present thesis insofar as isotropically radiating point sources are considered which are located a long way out from the array. However, partly because the guidelines conveyed to the present author implied that there was a comparatively scant chance that more than one source could be present that had a frequency of interest, and partly because the use of eigenvalue analysis seems to be already a well-cultivated field of research (and one where plane waves seem to be main concern), not to mention 'beamformers' (where again plane waves seem to be the main concern), the author decided to concentrate his efforts upon estimating locations of sources, whether different or similar, which radiated spherically, and at proximities where sphericity was a palpable characteristic of the radiation in the vicinity of the array. For convenience, the 'multi-path'

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[2] Landau, L. D. and Lifshitz, E. M., *Fluid Mechanics* (translated by J. B. Sykes and W. H. Reid), Oxford, 1987 (first published 1959), p. 269

[3] This would seem to be the view taken, with perhaps some hesitation, in 1983 by Su and Morf:

> It is surprising, at first, to realize that the source location problem is ambiguous when the number of sources is greater than or equal to the number of sensors ... Counterexamples spring readily to mind, for example, the location of two white noise sources by cross correlation of the output of two sensors.

G. Su and M. Morf, 'The signal subspace approach for multiple wide-band emitter location', *I. E. E. E. Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-31, no. 6, December 1983, p. 1507.

problem of radiation from the same source reaching the sensors by routes of different lengths has been neglected in the present thesis.

Concerning, however, the general approach taken by those who are interested in estimating directions of arrival from a string of many hydrophones, we say again that, no matter how local hydrophone displacements may seem to be, the penalty incurred over several hydrophones could, with certain kinds of approach (as will be instanced from time to time below), be immense, because more data might mean exacerbation or compounding of error. Again, we argue that displacements of hydrophones should better be regarded as giving rise to errors of the two-dimensional axes supposed, and, in particular, the further away from the origin, the greater the error is going to be. The analogy with the optical thick lenses may be valid here again.

To recapitulate, then, we argue that small displacements of hydrophones from their nominal locations may seem innocuous insofar as they take place in the immediate vicinity of the nominal locations, yet the cost at long range could, with some approaches, be enormous. To that extent, we shall argue that it is the dimensionality that is disturbed and made inconsistent. To draw conclusions from data in such circumstances about two dimensions may be illegitimate. Indeed, it is perhaps ironically appropriate that most of the literature on the processing of perturbed arrays is concerned precisely with sources radiating in the 'far field', for that fact might seem to be a tacit acknowledgement of the difficulty of estimating the coordinates of a source in the sense we have given and a limiting of analysis to direction-finding only.

Thus far we have been considering processes converging in time. It has been assumed that the towed array has been stationary relative to the target for some time, time enough perhaps to enable us to say that the location of the target was invariant for the purposes of processing, in particular the construction of a time series. But surely it cannot be entirely sound to assume that the target might be stationary relative to the hydrophone array for any length of time. For the target might be moving, and the hydrophones might be moving relative to the target as well as to each other.[4] The target might

[4]Therefore we cannot compare our concerns, in this regard at least, with those interested in Synthetic Aperture image degradation which occurs despite an ability to go over the same ground more than once, *e. g.*

> Unknown ground-height variation ... leaves the image focus unchanged but yields a distorted image when compared with a ground map. It should be noted that repeating the *SAR* imaging over the same scene yields an

turn out to be a vessel, the means of propulsion of which might be altered or cut out completely, in which case the amplitude of it and its spectral character might change from one instant to the next. In such cases, we do not have the justification to rely upon a time series on a line, as it were, but we must see what can be achieved at one point on that line. We must, in other words, abandon a notion of the primacy of the temporal over the spatial. We should see how much we can learn from a single 'snapshot'. Among other things, we cannot then make any use of time-delay techniques. In this sense, we shall abandon, towards the end of the thesis, the notion of a 'virtual' as well as actual rotation or change of shape of the antenna. Nevertheless, the practice of virtual rotation and change of shape of the antenna is considered in the mid-term of the thesis, with particular emphasis upon the obtaining of an alias-free manipulation of the antenna in the absence of noise.

In the mid-term analysis presented in the thesis, it is central to the discussion that a property of invariance be guaranteed by the method. A 'principle of invariance' is canvassed in order to ensure that the method achieves a unique solution in the absence of noise, the better to obtain an estimate of the location of the source in the presence of noise. Graphical evidence is provided of our N-process algorithm, from which, it is trusted, the convergence of the method can readily be appreciated.

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image with the same distortions, so that, even though direct *SAR* to map comparison is not allowed, *SAR* images themselves may be compared without relative distortion.

C. J. Oliver, 'Synthetic-aperture radar imaging', review article, *Journal of Physics* D: Applied Physics 22, 1989, p. 877. Oliver goes on to refer to research in which it is shown that perturbations of the atmosphere mediating between sensor and target may be classified between those which are tantamount to a distortion of range and those which have an effect analogous to a loss of focus. We shall see that this area of the problem is much like the problem of 'binning' data arising from a signal whose period is of the same order as the perturbations of sensor positions.

1.2 The Theoretical Plot of the Narrative

In the background to the narrative of events and the rehearsal of strategies attempted to solve the given problem is a meta-plot, as it were, which serves to unify the theoretical initiatives in the thesis. We believe we may characterize it as follows.

We distinguish three different approaches overall. First, there are the methods and algorithms, the theoretical bases of which are such that a unique determination of the location of a source in the absence of noise is not envisaged or not possible. Whether the 'izes' that may be associated with such methods and algorithms give rise to 'localization' or 'optimization', they have in common a susceptibility to aliases of the true solution. If, before noise, they cannot be relied upon to give up a unique solution, we believe that they cannot be relied upon to produce one in the presence of noise either. The problem of multiple solutions is only partly solved, we believe, by distinguishing between 'optimal' solutions and true solutions, or (with very much indecision) between 'optimal', 'sub-optimal' solutions and true ones. We believe that many such methods and algorithms are obliged to have recourse to what are in effect equivocations because of the uncertainty of their theoretical bases.

Secondly, we introduce a method, the essence of which is that it yields a unique determination of source location in the absence of noise. It is the more effective when noise is added because of that.

Finally, we address a class of problem which cannot be put in the terms of the first two approaches. It is to elicit a unique solution from 'fuzzy' data. A unique determination of location of source in the absence of noise is not possible because noisy hydrophone location is the essence of the problem. On the other hand, we require a theoretical basis upon which an estimate can be obtained, an estimate with which uniquely a source location may be associated.

The first class of methods and algorithms constitutes the subject of the *Literature Review.* The second class is treated in the middle chapters on an N-process algorithm with a stationary target and with a discussion of the possible use of the theory of the caustic. The third class of problem is attended to in the final chapters, where a novel method employing 'phase-binning' is introduced in order to solve what can be characterized as an inverse problem with fuzzy instrumentation.

2 Literature Review

2.1 Discussion of and Contrast with the Approach of Rockah and Schultheiss

Rockah and Schultheiss published two papers on uncertainty of hydrophone location in the *IEEE Transactions* 1987. Their approach to the subject is fundamentally different from the one taken in the present thesis; indeed, in many respects. The titles of their papers alone convey the different emphasis from our own on estimating the hydrophone locations by means of sources (whether targets of opportunity or deliberately placed ones), while we wish conversely to discover sources by means of hydrophones. Despite this and other basic differences of aim, however, it is hoped that ideas of Rockah and Schultheiss may serve as a catalyst to highlight, by contrast, assumptions and attitudes behind the present thesis.

In the first place, Rockah and Schultheiss write in terms of point-sources 'at infinity', whereas we are concerned mainly with sources at finite distances. In fact, we believe the notion of a source at a finite distance to be an indispensible requirement for determining or estimating location. However, it is conceded that detecting or estimating the general direction of arrival of acoustic radiation may not entail the notion of finite distance quite so compellingly. We shall be examining, and indeed questioning below, the theoretical basis and justification for making a distinction between 'bearing estimation' and 'bearing and range estimation', and our conclusion will be that the theoretical basis is uncertain. Much literature on hydrophone arrays treats only or mainly point sources 'at an infinite distance'. In particular, for Rockah and Schultheiss, an infinite distance intervening between the source and the sensor amounts to the presence of a source in the 'far-field':

> The far-field assumption is equivalent to the assertion that the range is known to be infinite ...[5]

However, one of the objects of the present thesis is to examine what is called the 'near-field' problem. Although the matter will be discussed in greater

[5] Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part ii: near-field sources and estimator implementation', *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-35, no. 6, June 1987, p. 725

detail below, it may be remarked in passing here that it would seem as though the line or zone of demarcation between 'near-field' and 'far-field' is sometimes presented in the literature as a kind of factotum, without a defined remit. The distinction may frequently be seen to arise as a sort of reflex of the argument or algorithm being advertized. In other words, the existence of a distinction between the two is not put into the method at the outset, but grows out of the evident limits or limitations of performance of the method. Thus Rockah and Schultheiss in their *Concluding Remarks* to their first paper:

> 3) It is conceptually straightforward (but analytically non-trivial) to extend the far-field bounds discussed here to the near-field case. By introducing source ranges as additional unknowns, one increases the dimension of the estimated parameter vector. As a result, three auxiliary sources may no longer be sufficient for calibration purposes (one finds that the required number of sources depends on the number of sensors in the array). Discussion of these issues or description of efficient calibration procedures would go beyond the scope of the present paper.[6]

On the other hand, it will be one of the tasks of the present thesis to examine and, indeed, to question the theoretical basis of a distinction between 'near' and 'far' fields.

Rockah and Schultheiss reached the following conclusion:

> While array *shape* error tends to zero with increasing source strength, there remains a residual error in array orientation.[7]

The implication of these words is that increasing source strength reinforces the plausibility of a number of orientations that the array could have. In other words, increased source strength does not help to distinguish the true bearing from its aliases, but it can perhaps help to distinguish the aliases from noise. On the other hand, it is one of the theoretical tenets explored in the present thesis to devise a method for determining the location of a

[6] Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part i: far-field sources', *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-35, no. 3, March 1987, p. 299

[7] Y. Rockah and P. M. Schultheiss, 'Array shape calibration ... part ii', *op. cit.* p. 724

source uniquely before noise, trusting thus to avoid aliases at the outset. It also seems possible that Rockah and Schultheiss are able to distinguish 'array shape error' from 'residual error in array orientation' at all because they consider mostly sources at 'infinite' distances. On the other hand, if spherical spreading is assumed, there seems to be no theoretical basis for separating array shape from array orientation, each being mutually inclusive. In any case, with stationary calibrating sources present, the possibility exists that array shape and orientation can be resolved in the limit of analysis of repeated observations. This is not possible where our knowledge of approximate hydrophone locations is based solely upon the history of the speed and direction of the towing vessel. While Rockah and Schultheiss investigate 'intrinsic' methods for estimating hydrophone location, we shall end by presenting a method based upon estimates of hydrophone locations obtained by 'extrinsic' methods. Although Rockah and Schultheiss need certain priors in order to carry out their analysis, and although we shall question the need for priors in the statistical treatment of the problem, we acknowlege that the employment of information of array shape obtained extrinsically amounts in effect to the use of a prior too. However, we might argue that such a prior is founded upon better authority than may be the case with priors generally.

In the *Abstract* of the second paper of Rockah and Schultheiss it is stated:

> Uncertainty concerning sensor locations can seriously degrade the ability of an array to estimate the location of radiating sources. Array calibration then becomes an important issue.[8]

We argue that this and such sentiments suggest the outlook that the uncertainty of the hydrophone locations imposes an extra amount of 'noise' upon an algorithm, an algorithm perhaps not primarily designed to cope with uncertainty. In the present thesis, however, we shall try both to see how an algorithm designed with known hydrophone locations in mind degrades in performance with uncertainty about hydrophone locations and to devise an algorithm based upon uncertainty right from the start. Our discussion of a novel method for using a multi-hydrophone array of hydrophones, a method with a built-in accommodation of uncertainty, can be found at the end of the present thesis.

In a later chapter on the possible implications and applications of the theory of the caustic we shall consider how we might go about establishing

[8] *Ibid.*, p. 724

axes at the end of a limiting process. We shall suggest there that the very choice of orientation of Cartesian coordinates is itself limited by a convergent process. Indeed, in the later stages of the thesis, we shall discuss the general problem of the circumstances in which there is no alternative but to have recourse to a prior, all other possibilities of differentiation and statistical convergence having been exhausted. Our general theme will be that, in a certain paradoxical way, the prior is the last resort, rather than the first. There is no suggestion of a limiting process with Rockah and Schultheiss, who appear to assume an *a priori* fixing of axes:

> While calibration can establish array shape with great accuracy, it cannot resolve a rotational uncertainty in array orientation. This uncertainty translates into a residual error in source bearings, but not in source ranges.[9]

The words quoted appear to suggest that 'range' is largely, if not wholly, a function of the intrinsic amplitude of the signal, as far as Rockah and Schultheiss are concerned. On the other hand, we shall be striving in the present thesis for the determination and estimation of location independently of absolute or relative prior knowledge of the intrinsic amplitude of the signal. That does not mean, however, that we should not be able to determine or estimate the intrinsic amplitude of the source. On the contrary, we shall be presenting algorithms for eliciting such information from the data without the use of priors.

In our discussion of the caustic we shall consider how the rotational uncertainty referred to by Rockah and Schultheiss above might be removed in the limit, particularly if the sources are assumed to be stationary. It may not be fanciful to say that Rockah and Schultheiss have obtained for a result of their efforts the very uncertainty which prompted them to initiate them. In other words, they posited axes arbitrarily from the beginning; but the arbitrariness of the axes (evident in the manifestation of aliases of them) is an important outcome of their investigations.

In their first paper of 1987 in the *IEEE Transactions* it is clear that Rockah and Schultheiss were not primarily concerned to devise an algorithm especially suited to the array of uncertain curvilinearity *ab initio*, and there are indeed many papers in which algorithms designed for hydrophones of

[9] *Ibid.*, p. 724

known locations are 'adapted' to treat systems of hydrophones of uncertain disposition.[10] A common comment on the outcome of such exercises is that uncertain curvature causes a 'degradation' of the performance of the algorithm.[11] Yet we shall argue in favour of a purpose-designed algorithm which does not degrade, but which, in the limit as the array is filled in with hydrophones, converges upon a unique result which can plausibly be taken to be the location of a source. More technically, as we shall argue below, to converge upon a line seems to be a more feasible undertaking than to try to centre upon a point straightaway. We shall argue that aliases arise in results partly because of such a 'jumping' of dimensions in an attempt to master a noisy problem.

Rockah and Schultheiss begin their exposition with the assumption that the exact locations are known. They will then consider the penalty of uncertainty on their algorithm. Again, the present thesis will end by offering an algorithm which proceeds straight-off from the uncertainty. Rockah and Schultheiss set about determining the number of hydrophones required in order to fix the locations of hydrophones, which are otherwise in unknown locations. But, more than this, they inquire about the need to know where the calibrating sources are.

Rockah and Schultheiss declare in their first paper of 1987 in the *IEEE Transactions* as follows:

> The present paper therefore examines the possibilities of achiev-

[10]An example is found in a paper by Hinich, who declares an interest in bearing estimation alone (and therefore not in location, which requires range to be estimated as well). Linearity in the array is a prerequisite for him (whereas we shall argue rather that gross perturbations are desirable), and he examines a time-domain limiting process based upon linearity in part as well as in whole:

> The method which is used to reduce the bending perturbation deflection of the bearing is to group the sensors into adjacent subarrays, process these arrays over short time slices, average the subarray bearings for each time period, and then to average the average over time.

M. J. Hinich, 'Bearing estimation using a perturbed linear array', *Journal of the Acoustical Society of America*, vol. 61, 1977, p. 1540

[11]See Bouvet, who in a literature review writes: 'Most of these works made the assumption of small displacement, or more precisely, of a linear shape slightly perturbed. Unfortunately, this assumption is not true in real situations.' M. Bouvet, 'Beamforming of a distorted line array in the presence of uncertainties in the sensor positions', *Journal of the Acoustical Society of America*, vol. 81, 1987, p. 1833

> ing calibration with auxiliary sources whose locations need not be known initially.[12]

With neither the locations of the calibrating sources nor those of the hydrophones known *a priori*, the task would seem to be theoretically impossible. Technically, there must be an infinite number of unknowns to discover from a finite number of equations, and Rockah and Schultheiss write indeed:

> This paper deals with source localization using a two-dimensional array of sensors whose locations are not known precisely. If only a single source is observed, uncertainties in sensor location increase errors in source bearing and range by an amount which is independent of signal-to-noise ratio and which can easily dominate overall localization accuracy.[13]

We do not believe that even more than one source of unknown location will make it any easier to locate hydrophones. In any case, if the aim of the exercise is to make use of 'sources of opportunity', the circumstances under which such sources might make themselves available would seem to escape rational prediction. But more sources may make for more aliases. Rockah and Schultheiss continue:

> The problem to be examined can now be described as follows. An array of sensors has an arbitrary but known nominal geometry ... The ith sensor experiences random but time-invariant displacements $(\Delta x_i,\ \Delta y_i)$ from its nominal location $(x_i,\ y_i)$. The Δx_i and Δy_i are independent Gaussian random variables, all with zero mean and standard deviation σ. Thus, the initial uncertainty regions of sensor location have circular symmetry.[14]

It is evident here that, with Rockah and Schultheiss, errors from the nominal are experienced by each hydrophone in both the $x-$ and the $y-$ directions. Clearly, the penalty likely to be incurred through errors in two dimensions is likely to be very much greater than if an error in one dimension only were

[12] Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part i: far-field sources', *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-35, no. 3, March 1987, p. 286

[13] *Ibid.*, p.286

[14] *Ibid.*, p. 286

envisaged. The term 'circular symmetry' does not in practice seem to do any more work than would the observation that the hydrophone could be anywhere but not necessarily at its nominal location. On the other hand, we require as a necessary, and indeed sufficient, condition that a hydrophone cannot be displaced to the extent that its location might be confused with that of another hydrophone.

On the other hand, we raise the notion that an error in the location of a hydrophone can be assumed to be one-dimensional. What we are proposing is that our estimation of the gross shape and orientation of the array is correct, except that, piecewise, the estimated and true shapes may vary one with the other. In particular, it seems justified by experience to regard the two arrays, the one true, the other estimated, as running piecewise parallel with each other. If that is accepted, then displacements may be regarded as not point-related, but line-related. In other words, we might regard shorter lengths of the array as being displaced from the nominal corresponding lengths, but assume that the individual hydrophones nevertheless adhere to the line.[15] We shall assume that hydrophones do not move about greatly relative to each other. Again, such a view of the circumstances seems warranted by experience. Perhaps because of the tautness of the cable in tow, hydrophones may justifiably be regarded as equidistant. That is to say, the distance between pairs of neighbouring hydrophones may be assumed to be constant. We shall expand upon this thought in our chapter on the possible use of the theory of the caustic curve.

It can be demonstrated, again by virtue of experience, that the rotational error of axis mentioned by Rockah and Schultheiss is present with a scheme of things such as we have just outlined, except that this error will only be local, and not global over the array as a whole. In our analysis we specify the discipline that no hydrophone can be displaced to the extent that it can occupy a location possibly occupied by another. This amount of error may be fairly small compared with that treated by Rockah and Schultheiss. Nevertheless, we consider devising a convergent process. In particular, we need to look at the circumstances under which such a convergence may take place. But, in practice, we seek somehow to turn to our advantage that disadvantage of uncertainty which besets Rockah and Schultheiss. Instead of

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[15] Thus following Hinich in part. See M. J. Hinich, 'Bearing estimation using a perturbed linear array', *op. cit.*, p. 1540

the displacement of points, we shall speak of the displacement and translation of lines and line-segments. By doing so, we hope to break down ambiguities about the dimensionality, to which we have alluded earlier.

Rockah and Schultheiss continue:

> Several sources (S_1, S_2, ..., S_p) radiate zero mean Gaussian random processes which arrive at the sensors unchanged except for geometrically determined delays.[16]

'Geometrically determined' may refer here to the 'infinite' nature of the extent of the medium only. Later, we shall argue that such delays are rather functions of time, that the geometrical arrangement involved is itself a continuous function of time. We shall have more to say about this below. For the moment let it suffice to say that it would appear that Rockah and Schultheiss take it that any trio of hydrophones may receive a signal such that its amplitudes, when received at each, are all equal. In passing, we note the following observation of Rockah and Schultheiss:

> ... we deal only with the far-field case; the near-field case differs only in algebraic complexity.[17]

In practice, Rockah and Schultheiss do not address the 'near-field case'.

In their first paper of 1987 in the *IEEE Transactions* Rockah and Schultheiss continue:

> If the auxiliary sources are intentionally deployed, one would almost certainly have them radiate signals which do not interfere with each other or with the target signal. We assume that this has been accomplished by confining them to disjoint frequency bands (operation in disjoint time intervals would clearly have a similar effect). In this setting, the distinction between 'target' and 'auxiliary source' becomes quite arbitrary. They all contribute in equivalent fashion to array calibration. When we are concerned with source location, we shall focus on one source - calling it the

[16]Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part i: far-field sources', *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-35, no. 3, March 1987, p. 286

[17]Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part ii: near-field sources', *op. cit.*, p. 733

target - and study the effect of additional sources on our ability to estimate its location.[18]

We believe that it is a problem with their argument that, because of the 'irremovable', if 'residual', errors which they encounter in array orientation, in practice it seems difficult to distinguish between a 'target' and an 'auxiliary source', not to mention an alias of either, or of both. Rockah and Schultheiss say as much, and, in particular, find that it does not help matters (as it shall help us later) either if the signal is palpably spherical in the vicinity of the array. The *Introduction* to their second paper on 'near-field sources' contains the conclusion:

> While array *shape* error tends to zero with increasing source strength, there remains a residual error in array orientation. This translates into a residual error in bearing for any target observed by the array. Only by providing a separate directional reference can one eliminate this error.
>
> The present paper generalizes these results to accommodate near-field calibration sources. It shows that array shape calibration of any desired accuracy can still be achieved with sources of sufficient strength, but that the number of required sources may exceed three (depending on the number of elements in the array). It further shows that target bearing estimates are subject to an irremovable uncertainty in rotation (as in the far-field case), but that there is no such residual error in the range estimate.[19]

One of our aspirations with our novel single snapshot algorithm presented below is to create a method which is independent of the interference mentioned by Rockah and Schultheiss.

In practice, it would seem that 'passive' *SONAR* has the potential to locate targets further away than is possible with *SONAR* proper.[20] Why

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[18]Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part i: far-field sources', *op. cit.*, p. 286f.

[19]Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part ii: near-field sources', *op. cit.*, p. 724

[20]See J. T. Patzewitsch, M. D. Srinath and C. I. Black, 'Near-field performance of passive coherence processing sonars', *I. E. E. E. Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-27, no. 6, December 1979, p. 573ff. They wrote (*Ibid.*, p.

should this be true? In the first place, the outward- and homeward-bound progress of a pulse amounts to a doubling of the distance involved between the initiator and the target. In a noisy medium this can hardly be entirely satisfactory. Secondly, and more importantly, the aim of *SONAR* proper (*i. e.* 'active' *SONAR*) can scarcely be to discover, in the first instance, a vibratory characteristic of the target. With *SONAR* proper, apart from the danger of giving away one's position, we can only hope to recapture on the rebound a characteristic of that which we sent out in the first place and knew already. In regard to the time it takes the 'ping' to complete its round-trip, we might be able to estimate the distance that it has travelled. But we can never be certain how straightforward the reflection off the target may have been.

Thus with 'passive' *SONAR* we hope that the target will manifest itself to the array in all its vibratory parts. Clearly, with passive *SONAR*, a continuity of compressions and rarefactions between source and sensors covers only half of the distance that a 'ping' must make. In a way, the 'ping' of 'active' *SONAR* acts, in effect, like a deliberately-placed source. Unlike Rockah and Schultheiss, we shall make no concessions in favour of calibrating sources, whether deliberately placed or whether arising as opportunity.

Rockah and Schultheiss signal that they are not primarily concerned with a unique determination before noise by referring to the 'Cramér-Rao lower bound' (commonly abbreviated *CLRB*). The 'Cramér-Rao inequality ... bounds the error covariance matrix of all unbiased estimates.'[21] The error covariance matrix, they continue, 'can be approached at sufficiently large signal-to-noise ratios or observation times by using a maximum likelihood estimator.'[22] Thus, it is fair to say, Rockah and Schultheiss require some statistical properties of their system to be stated *a priori*. The concept 'likelihood' must be logically related to the *a priori* assumptions. Indeed, it would seem as though the two must be mutually inclusive. In the concluding sections of the present thesis we shall consider the circumstances under which

573): 'In recent years the use of passive sonars in detection and localization has become attractive. Passive sonars can operate at longer ranges than active sonars and provide more information about the nature of the source.' However, these authors do not go on to explain how.

[21]Y. Rockah and P. M. Schultheiss, 'Array shape calibration using sources in unknown locations - part i: far-field sources', *op. cit.*, p. 287

[22]*Ibid.*, p. 287

priors must be used in an algorithm.

In particular, Rockah and Schultheiss imply that the random displacements of the hydrophones in both $x-$ and $y-$ directions are subject to the same statistical conditions as the signals themselves. But, the two kinds of 'noise' are surely dimensionally different. For the one arises in two dimensions, and the other in one. If we knew exactly where the hydrophones were, there would be little excuse for using a biased estimator of the signal. But with uncertainty of sensor locations as well, there is clearly going to be great difficulty with obtaining an unbiased estimator. This seems to be the point being made by Rockah and Schultheiss when they speak of the use of a 'maximum *a posteriori* estimator'.[23]

In a sense, it can perhaps be argued that the need for a prior in this kind of analysis reflects the ambiguity of dimensionality of which we spoke above. For we argue that with unknown displacements in respect of both axes we cannot hope to recover a direction of arrival, still less a source location, unless an axis *a priori* is postulated. Such a postulation amounts to an assumption upon the orientation of one of the axes. Thus the prior amounts to fixing one of the axes in advance. Again, below, we shall be considering the orientation of axes themselves in terms of the limit of a statistical process.

Rockah and Schultheiss give the parameter vector θ of their problem as follows:

$$\theta = (\alpha, r, \Delta x_2, \ ..., \Delta x_M, \Delta y_2, \ ..., \Delta y_M, \Delta x_1, \Delta y_1)^T$$

If this were regarded as one of a system of equations, then the number of unknowns must match the number of equations in order to solve the system. With frequency sources, there may be an infinity of solutions, for with discrete sensors we have no means of knowing how often the source may be oscillating between the sensors. For this reason, as we shall argue below, in the absence of a continuous sensing strip (let us say), an effectively exponential 'spatial'[24] sampling is what is to be striven for. Indeed, with a perturbed array, as opposed to a dead-straight linear one, the array can be said to function, in the limit as the array may be filled in with hydrophones,

[23]'When the unknown parameters are random variables with known prior distribution, an equivalent formula bounds the average error correlation matrix of estimators which need not be unbiased. This bound can be approached (again under mild restrictions) by the MAP (maximum *a posteriori*) estimator.' *Ibid.*, p. 287

[24]Again, we acknowledge here a naive delineation of the space-time *continuum*.

like an exponential filter. We regard this 'exponential sampling' as methodologically to be preferred to the comparatively insensitive spatial sampling possible with a straight antenna with elements placed at equal intervals.

With the present problem there is a need to distinguish, however naively, between effects in space and those in time. But for each m hydrophone there is an uncertainty Δx_m as well as an uncertainty Δy_m. This means that, as far as space is concerned, there can never be enough equations to match the number of unknowns, since for each hydrophone two variables are requiring to be elicited. Without a calibrating source of known location, the system cannot be solved. Without a calibrating source, a prior must be introduced in order to provide an axis for each of the Δx's and Δy's, and Rockah and Schultheiss attempt to do so by laying down in advance the degree to which the postulation might be at variance with reality. However the exercise is undertaken, the technique employed must essentially be one of least squares.

We note, in passing, that 'covariance' yields a two dimensional extensivity. That is to say, continuously two amplitudes are multiplied with each other and the resulting dimensions are two. On the other hand, the sum of amplitudes yields a point on the real line, so a sum can convey information of only a one dimensional kind. As we shall argue later, it is the use of the ratio of signals that yields the non-dimensional point location with which this thesis is concerned. Thus, in this sense, 'covariance' gives extensivity, while ratio gives intensivity; the one defines an area, but the other a point. With uncertainty in our knowledge of the locations of hydrophones, the notion of eliciting the 'covariance' of measured acoustic pressures compared between one hydrophone and another may require some qualification. We shall argue later on that, in such circumstances of uncertainty, we cannot think in terms of a covariance of point with point. Rather, we shall consider the 'one-dimensional' covariance of line with line, and then indeed a 'two-dimensional' one of region with region.

2.2 The 'Work' Done by a Customary Distinction Between 'Near Field' and 'Far Field'

In the previous chapter we touched upon the distinction between the 'near field' and the 'far field' used explicitly and implicitly by Rockah and Schultheiss. We think of their definition as a technical rather than a general one, to the extent, that is, that a definition is articulated consciously in their text at all. Elsewhere, others' notions of a distinction between the two emerge as adumbrations rather than clear demarcations. Here, for example, is a general statement without further technical elaboration:

> NEAR AND FAR FIELDS
>
> For the more general input it is possible to distinguish the near from the far field, the latter being defined as that region where the pressure decreases inversely with distance. The boundary of the far field can be expressed in terms of the variation with time of the velocity driving pulse. For example, let the input velocity function be a pair of bipolar pulses separated by time T. On the axis the pressure impulse response is a pair of impulses separated by time $\tau = t_2 - t_0$, where these quantities are defined ... The near field in this instance can be defined as that region where τ is greater than T, i.e., the response from one of the inputs at t_2 can interfere with that of the other at t_0. The far field comprises the region where there is only overlapping of adjacent positions of the replicas. [25]

In the present chapter we shall consider how practicable an advancing upon a basis of a distinction between 'near field' and 'far field' is. In the next chapter we shall fix our own data model and evaluate the distinction between 'near field' and 'far field' in the light of it.

In a paper by Hu we read of a 'Fresnel region'.[26] Having elucidated an integral arising from angle of arrival, distance of travel and phase angle, Hu writes that it is very difficult to perform the integration analytically, and

[25] D. E. Robinson, S. Lees and L. Bess, 'Near field transient radiation patterns for circular pistons', *I. E. E. E. Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-22, no. 6, December 1974, p. 395ff.

[26] M.-K. Hu, 'Fresnel region field distributions of circular aperture antennas', *I. R. E. Transactions on Antennas and Propagation*, vol. AP-8, no. 3, May 1960, p. 344ff.

that therefore, in practice, an approximation may be constructed. Depending upon the character of the approximation, ' ... the result may be classified as the far field approximation, the Fresnel region approximation, etc.'[27] Hu continues:

> The most widely used approach to such approximations is to expand ... into a power series ... The truncated series, obtained by neglecting all terms of order higher than one or two, is generally considered as the far-field approximation or the Fresnel region approximation. [28]

However, Hu will proceed differently:

> ... A rather different approach to the Fresnel region approximation is used in this communication. Instead of using the truncated power series expansion, the well-known Newton's iteration formula for finding the square root of a given number is employed.[29]

It seems as though, here, terms like 'near field', 'far field' and 'Fresnel region' arise through circumstances which may have little to do with a real *continuum* of source and medium. In many papers these terms seem to be called upon in order to describe the limits or limitations of competence or efficacy of a particular method or algorithm. In the example just quoted, however, it should be noted that Hu's subject is electromagnetic radiation and reception, and he is not at all directly concerned with underwater acoustics. In passing, we suspect that it might be possible to show that some misconceptions in our subject may have arisen as techniques and methods of radar and other techniques not strictly pertinent to the under-sea were applied to underwater acoustics.

Kay, writing of the 'Fresnel zone', gave the following approximation:

$$\frac{e^{ikR}}{R} \approx \frac{e^{ikR_0}}{R_0} e^{ik[r^2+r'^2-2rr'\cos(\theta-\theta')]/2R_0} \quad ,$$

where (r, θ) and (r', θ') are polar coordinates in the receiving and transmitting apertures respectively. The equation amounts to a second order approximation in the phase and a first order approximation in the amplitude of $\frac{e^{ikR}}{R}$.

[27] *Ibid.*, p. 345
[28] *Ibid.*, p. 345
[29] *Ibid.*, p. 345

The Fresnel zone, wrote Kay, is obtained from the approximate equation.[30] Kay continues:

> The region of validity of this approximation may be estimated by requiring that the relative difference between the exact and approximate values of $\frac{e^{ikR}}{R}$ never exceeds a certain amount. The usual criterion is that no phase error greater than $\frac{\pi}{8}$ be permitted. For gain, this is equivalent to a fractional amplitude error of 0.077 ...
>
> We shall assume that the $\frac{\pi}{8}$ phase criterion and the equally significant amplitude criterion define the limits of the Fresnel zone. We shall now find these limits.
>
> For the present discussion we shall assume that the apertures A and A' can be inscribed in circles of minimum radii r_0 and r'_0 and that the origins of the polar coordinates in these apertures are taken at the centers of these circles. We further assume that the line joining two centers is perpendicular to both aperture planes. In this case the phase error is maximised if $\theta = \theta' + \pi$, $r = r_0$, and $r' = r'_0$. The phase error is then
>
> $$\Delta = R_0(\sqrt{1+b^2} - 1 - \frac{b^2}{2}) \text{ , (16)}$$
>
> where
>
> $$b = \frac{r_0 + r'_0}{R_0}. \text{ (17)}$$
>
> We require
>
> $$\Delta < \frac{\lambda}{16} \text{ (phase criterion). (18)}$$
>
> Similarly the amplitude criterion ... implies that
>
> $$R_0 \geq 0.923\sqrt{R_0^2 + (r+r')^2} \text{ (amplitude criterion). (19)}$$
>
> [31]

[30] A. F. Kay, 'Near-field gain of aperture antennas', *I. R. E. transactions on Antennas and Propagation*, vol. AP-8, no. 6, November 1960, p. 589

[31] *Ibid.*, p. 589

In short, the 'far-field' is roughly conceived as that distance beyond which, from one antenna element to any other antenna element, there is a 'phase curvature' of no more than one sixteenth of a wavelength.

Generally speaking, this view is shared and has been accepted as a rule of thumb by those who work in undersea signal processing as well. For example, MacDonald and Schultheiss specify, characteristically:

> (2) The signal comes from a source sufficiently remote so that its wavefront may be regarded as planar over the dimensions of the receiving array.[32]

We note, in passing, that another of the assumptions at the head of the article just cited is that 'the receiving array is linear',[33] an assumption which we shall try increasingly to do without as the present thesis progresses.

Another context from which a distinction between 'near field' and 'far field' arises is that of the presence of a large object with complex vibratory characteristics close to the array of hydrophones. It may be possible to justify saying that, the further such an 'extended source' is away from the hydrophone array, the more its radiation may seem to be coming from 'something like a point source'. According to this distillation of a distinction, a source is in the 'near field' if it is an extended source, but it is in the 'far field' if it can be construed more as a point source. We shall return to this matter in the next chapter, specifically in a discussion of the signal received by hydrophones from a line radiator .

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[32] V. H. MacDonald and P. M. Schultheiss, 'Optimum passive bearing estimation in a spatially incoherent noise environment', *The Journal of the Acoustical Society of America*, vol. 46, no. 1, 1969, p. 37f

[33] *Ibid.*, p. 37

2.3 The Locating of Point Sources

In general, there would seem to be comparatively infrequent analysis in the literature of the 'locating' or the 'location', as the activity of interest in the present thesis might be termed, of point sources. Instead, perhaps, one encounters a word 'localization' sometimes, which seems somehow to be less explicit in meaning than 'location'. Sometimes the term 'angular location' is used, but this turns out to be an exercise in estimating the direction of arrival only. Whereas much has been published on the sensing of the presence and the estimating of the number of signals requiring to be detected by an extensive towed hydrophone array as well as on the direction of their arrival, little has emerged on the locating of sources. We shall argue, below, that a term such as 'localization' may be appropriate where quantities, properly speaking 'non-rational' (in the technical sense), are elicited by algorithms employing 'rational' numbers. Furthermore, although the term 'fuzzy' may not strictly be thought to apply to the business of analysing non-rational quantities in terms of rational ones, 'localization' may turn out to be an appropriate gloss for an attempt to obtain unique solutions to 'fuzzy' problems. But this is a matter which we shall treat later on in greater detail.

An absence of the required literature was signalled long ago. Gilbrech and Binder wrote about it in 1958:

> There is considerable literature available dealing with the problem of locating noise sources at relatively large distances such as are encountered in ship detection. However, there seems to be very little published information concerned with locating noises at short ranges in which the transmitting medium has little or no effect.[34]

A similar complaint could be heard from Jacobson in 1959:

> Some earlier analyses of two-receiver correlation systems have been concerned, at least in part, with the use of such systems for determining the directions of point sources in a common plane with and at a large distance from the receivers. The distance is

[34] D. A. Gilbrech and R. C. Binder, 'Portable instrument of locating noise sources in mechanical equipment', *The Journal of the Acoustical Society of America*, vol. 30, no. 9, September 1958, p. 842

large to the extent that the signal wave at the receivers may be assumed to be a plane wave. For smaller distances this assumption is no longer valid and a spherical wave must be substituted for the plane wave. If the system is intended for use with plane waves or 'infinite' distance sources, then the predicted source direction for a finite distance source will not converge in general to the actual source direction.[35]

There is perhaps the merest suggestion here that it may be a proper approach to the problem to assume plane waves from the outset, but regard the incapacity of an algorithm based upon that assumption to cope when the source is close to the antenna as a function of 'error'. Indeed, Jacobson goes on to say:

> In the following sections, the predicted source direction and the resulting error in the source direction will be studied as a function of the position of the signal source.[36]

This is an example of a distinction between 'near field' and 'far field' arising out of the evident limits, or limitations, of a method rather than one set in terms of a theoretical basis.

In the end, Jacobson declares that two receivers (of the kind he is using) alone cannot locate a point source:

> It will also be noted that with the two-receiver system being considered here, there is no direct way of distinguishing between a finite distance and an infinite distance source.[37]

If we supposed the case of the source lying on a line, all the points of which were equidistant from either receiver, to be a singular one, Jacobson's conclusion might seem surprising. We need to point out, however, that the rise of the digital computer and a concomitant facility to record and to store data in a structurally efficient manner have made it possible to compare data from different auditory channels both synchronically and diachronically, *i. e.* to measure a system of data from one channel with one from the other. With

[35]M. J. Jacobson, 'Correlation of a finite distance point source', *The Journal of the Acoustical Society of America*, vol. 31, no. 4, April 1959, p. 448

[36]*Ibid.*, p. 448

[37]*Ibid.*, p. 450

such 'modern' facilities it is possible to use two sensors to locate a point source, if, that is, the point source is close enough to locate at all, by comparing, contrasting and taking ratios of data. With the equipment used by Jacobson, that was evidently not possible.

However, the conclusion that Jacobson reached seems to have been shared and elaborated in a paper more than twenty years later by Schultheiss and Weinstein:

> When there are fewer than three sensors, one cannot locate a stationary source at all, at least not in the absence of special and fortuitous multipath links between source and receiver.[38]

This judgement has been questioned implicitly more recently by Segal, Weinstein and Musicus, who assert:

> Time delays between signals radiated from a common point source and observed at two or more spatially separated receivers can be used to determine source location.[39]

We agree. But we shall differ in ambition from Segal, Weinstein and Musicus in that we shall demonstrate a locating algorithm for a single snapshot rather than for multiple snapshots.

In another paper Messer, Rockah and Schultheiss associate themselves with the view that, whatever interpretation we may make of the difference between 'near field' and 'far field', the two differ only in their algebraic structure:

> Extension of the analytical results to the case of near-field sources and interferences is formally trivial ...[40]

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[38]P. M. Schultheiss and E. Weinstein, 'Lower bounds on the localization errors of a moving source observed by a passive array', *I. E. E. E. Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-29, no. 3, June 1981, p. 600

[39]M. Segal, E. Weinstein and B. R. Musicus, 'Estimate-maximize algorithms for multichannel time delay and signal estimation', *I. E. E. E. Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-39, no. 1, January 1991, p. 1

[40]H. Messer, Y. Rockah and P. M. Schultheiss, 'Localization in the presence of coherent interference', *I. E. E. E. Transactions on Acoustics, Speech, and Signal Processing*, vol. ASSP-38, no. 12, December 1990, p. 2029

Again, it is one of the tasks of the present dissertation to supply this 'near field' analysis, insofar, that is, as the distinction between 'near field' and 'far field' is useful at all.

In passing, we might note that the term 'localization' perhaps belongs to the world of 'optimization', while 'location' may be reserved for the activity of 'unique determination' before noise. We shall have more to say about that below. As far as possible, we shall endeavour to apply a 'principle of invariance'. We believe that, as much as possible, our method must be to identify the invariants of our problem before noise is added. For only upon such invariants can a statistical treatment be based. But if no such invariants can be identified in the absence of noise, we must perforce regard the exercise as belonging to the activity of 'optimization', where solutions called 'optimal' or 'sub-optimal' are obtained, and where a decision must be made as to which of them are the true ones or aliases. Often a 'Bayesian' method is used to distinguish the two, a method employing an element of prior expectation or 'inverse probability'.[41] In the present thesis we shall try to balance the merits of the two approaches.

The author was surprised to find that so many of the signal processing techniques customarily employed are not designed to be able to detect the presence of a source within a certain range, let alone locate it. It may be a strange paradox that, the closer is the source of signal, the more difficult it seems to be thought to be to detect and locate. Perhaps the problem is that

[41]We question the logic of a notion of '*a priori* probability' *tout pur*, and we note the words of Sir Ronald Fisher:

> Bayes' introduction of an expression representing probability *a priori* thus contained an arbitrary element, and it was doubtless some consciousness of this that led to his hesitation in putting his work forward. ...
>
> A more important question, however, is whether in scientific research, and especially in the interpretation of experiments, there is cogent reason for inserting a corresponding expression representing probabilities *a priori*. This practical question cannot be answered peremptorily, or in general, for certainly cases can be found, or constructed, in which valid probabilities *a priori* exist, and can be deduced from the data. More frequently, however, and especially when the probabilities of contrasted scientific theories are in question, a candid examination of the data at the disposal of the scientist shows that nothing of the kind can be claimed.

R. A. Fisher, *Statistical Methods and Scientific Inference*, Second Edition, Edinburgh and London, 1959, p. 17

much theory of the radio antenna has been applied to the problem without exploiting enough the discrete individuality of hydrophones in an array.

Only two electrical leads come out of an ordinary television aerial, even though there are usually several elements evident along it. The currents induced in the elements are all added together, without however our being able to tell how much current each element contributes to the sum. But with our towed array, on the other hand we can take measurements from each hydrophone separately. What comes out through the leads of a television aerial is the sum of currents induced in the various quasi-point detectors by a signal. But it is not usually the case that each detector is consulted separately. Nevertheless, that with our towed array each detector can be consulted separately has perhaps not qualified what has proved to be an enduring adherence to the notion 'array gain'. It is a question of exploiting the discrete individuality of the sensors to the full, and that may not be achieved simply by adding up readings taken from them. True, strategies for causing the antenna to discriminate in favour of some particular Direction of Arrival involve some greater differentiation between contributions to the sum, but they commonly involve linear combinations of the contributions, with a simple sum again as the tool of validation. Such discriminatory linear combining of data lies at the heart of 'windowing', 'shading' or 'weighting'. On the other hand, we argue below that it is by taking ratios that the non-dimensionality, with which the location of a point is irreducibly involved, is achieved.

Experiment, indeed some kind of theory, showed that inside a certain range the radio antenna could decreasingly provide evidence of the presence of a signal source. The reason seemed to be that the outputs (although not available individually to be polled) must be tending to cancel each other out as they were summed, a cancelling-out that was bound to occur even before noise was added. Technically speaking, however the amplitude of the signal might vary at an instant from one element to another, if there was a 'phase curvature' of more than one sixteenth of the wavelength of the signal, then cancelling-out would begin, increasingly, to occur, resulting in a much-reduced array gain.[42]

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[42] '... We can ... justify the commonly used relation $D = \frac{2a^2}{\lambda}$ for the minimum permissible distance of the field source from an experimental antenna test site. This distance produces an effective phase curvature of $\frac{\lambda}{16}$.' H. T. Friis and W. D. Lewis, 'Radar anten-

One of the early decisions taken in the present research was to question the standard notion of 'array gain'. The customary notion of 'array gain' did not seem to be appropriate to our task, partly because the practice did not seem to take account of the differential amplitudes of the signal from one sensor to another, but relied instead upon the uniformity of amplitude that evidently belongs to the 'far field' and its characteristic plane-wave processing techniques. A concomitant aspect of the inefficiency of the concept of it, as well as of its practice, was that it relied upon the mechanism of the sum alone, without any recourse to comparison, whether in the activity of mutual multiplication ('convolution') or in that of the ratio. In general, it shall be argued below, we may associate the activity of summing with a 'linear'[43] result, which may run the risk of aliases, and ratio with the locating of a point.

Another notion bound up with 'array gain' is that either the shape of the array, or the extent to which it is filled in with sensing elements, or its length, or a combination of all of these contribute to the noise-rejection capabilities of the instrument. Later, we shall discuss the circumstances in which the length, curvature and density of 'filling-in' can be optimized in order to maximize this quality of 'array gain'.

Consider Figure 1.

We are interested in the mean distance r of D and C, *i. e.*

$$r = \frac{(D+C)}{2}, \tag{1}$$

such that

$$(D-C) \leq \frac{\lambda}{16}, \tag{2}$$

where λ is the wavelength of the signal in question.

Now, by the theorem of Pythagoras,

$$D^2 = C^2 + (\frac{a}{2})^2 \tag{3}$$

and therefore

$$D^2 - C^2 = (\frac{a}{2})^2, \tag{4}$$

nas', *The Bell System Technical Journal*, April, 1947, No. 2, p. 244.

[43] *I. e.* in a technical sense to be described in a later chapter

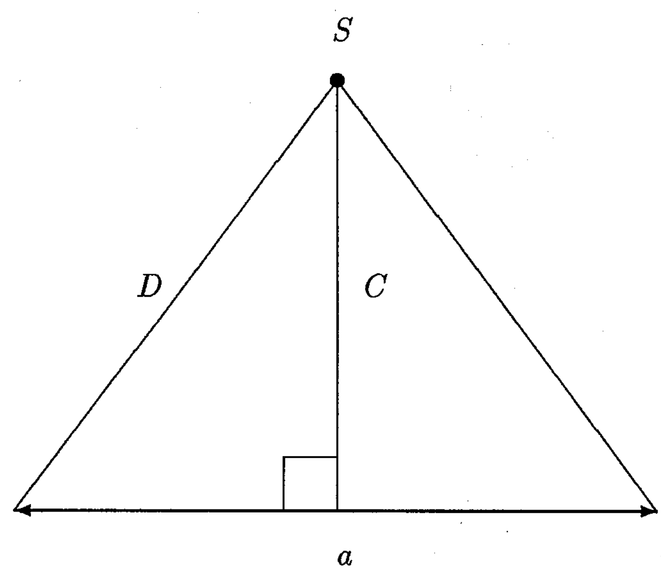


Figure 1: A point source S and an antenna of length a

i. e.

$$(D+C)(D-C) = (\frac{1}{2}a)^2. \tag{5}$$

But, from Equation 2,

$$(\frac{1}{2}a)^2 \leq \frac{\lambda}{16}(D+C), \tag{6}$$

i. e.

$$r = (\frac{D+C}{2}) \geq \frac{2a^2}{\lambda}. \tag{7}$$

In short, at broadside distances of less than twice the square of the length of the array all over the wavelength of the signal, the array gain (*i .e.* as conventionally conceived in terms of a weighted or unweighted sum of induced responses) tends to diminish, and therefore provide less information about the presence of a source. Inside that range, the 'array gain' may amount more or less to nothing. But this analysis has taken no account of the amplitude of the signal. We shall discuss that later on.

2.4 Brief excursus into Beamforming

We might be unusually fortunate if a single source happened to be on a line exactly perpendicular to the line of the antenna, quite apart from it being stationary with respect to the antenna. In practice, small antennae could be rotated to isolate the orientation at which the gain were greatest.[44] Of course, it may not be possible bodily to rotate longer and larger antennas. For them procedures involving time delays is required in order to attempt to perform what is a virtual rotation of the antenna.

A look at the genesis of the term might confirm that 'beamforming' can really only be done with plane waves.[45] Apart from any other consideration, the component 'forming' surely referred originally to the process of creating and transmitting a signal, not just receiving it. More technically, in the absence of prior knowledge of the signal, there must be a test available, the result of which should be the degree to which the antenna must be rotated, whether actually or virtually, in order to obtain the best reception. An expression commonly found in literature on passive *SONAR* such as 'find the direction in which the signal is at its strongest' betrays perhaps an origin in signal-creation, rather than in signal reception, since the orientation of the antenna can scarcely be thought to influence the character of a remote signal source. In adding up the readings taken at the sensor elements the greatest sum is found when the antenna has been rotated around such that the source is to the broadside of the (usually) linear antenna. The 'beam' is associated with this largest sum, while a sinusoidal pattern on either side of it varies with angle of rotation, minor beams being known as sidelobes. [46] We

[44]See C. L. Dolph, 'A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level', *Proceedings of the Institute of Radio Engineers and Waves and Electrons*, vol. 34, June 1946, p. 335ff.

[45]Thus Schelkunoff: '... if the distance of two elements to a distant point differ by $\frac{\lambda}{2}$, the phases of their fields differ by π and the amplitudes are substantially the same.' S. A. Schelkunoff, *Antennas: Theory and Practice*, New York, 1952, p. 30

[46]It is acknowledged that what we have just called 'beam' here is sometimes given a different name, *e. g.* 'mainlobe'. Compare Quartly and Pace:

> Passive sensing using a towed array of hydrophones has long been an important tool in the detection of submarines. The signals received by the hydrophones are delayed relative to one another due to the different transmission times from source to receivers. All methods of detecting and locating sources from these signal records rely on appropriate delays being applied to

consider this phenomenon more technically at the end of the next chapter. Sometimes, of course, it is difficult to distinguish sidelobes from beams, but this is a technical matter that, being much considered in the literature on such themes as 'optimal', 'suboptimal', 'maximum likelihood' or 'Bayesian' beamforming, must lie beyond the scope of the present thesis.

In the literature 'beamforming' is applied in the context of plane waves. But, in its characteristic sense, 'beamforming' could be associated too with the judicious time-delaying of spherically-spreading signals from adjacent transmitting sources, such that the direction of the beam could be altered even if the physical orientation of the array itself could not. The time delays could be such that interfering wave fronts gave rise to a smoothly-sweeping ridge upon a line or a curve. The result is analogous, as will be seen below, to the 'caustic'. But whether a line or a curve, the 'beam' created by such a virtually flexible antenna cannot without prior information be interpreted at the receiving end as evidence of radiation from a point. Without the

---

> these simultaneous records to compensate for those due to the expected bearing of the source. The simplest of these, Conventional Beamforming (CBF), just adds these suitably delayed signals together and squares the result to give a positive value corresponding to the power perceived from a direction.
>
> ... Model errors may be failure to allow for ... near-field sources ...
>
> ... The beam pattern ... has a multi-lobed structure with the main peak corresponding to the bearing of the source, but other (lower) peaks being an artefact of the process rather than indicating extra sources. ... the mainlobe (that in the direction of the source) ... the subsidiary sidelobes...

G. D. Quartly and N. G. Pace, 'Overcoming hydrophone position inaccuracy in Conventional Beamforming', *Proc. I. O. A.*, vol. 13, part 9, 1991, p. 286. By contrast, we shall be presenting, with our phase-binning algorithm, a novel method which does not 'rely on appropriate delays being applied', but which aims to yield an estimate of location with a single snapshot. However, Quartly and Pace envisage only thirty-two or so sensor elements, while we shall require of the order of ten times that number in order to process a single snapshot. In general, as the present thesis proceeds, a preference will emerge in favour of spatial rather than temporal analysis, although we acknowledge that the space-time *continuum* cannot be delineated simply.

In the paper of Quartly and Pace quoted, a single source is modelled. However, we shall be considering how, in the presence of other sources of the same frequency, an 'artefact of the process' can be distinguished from 'extra sources'. That will be one of the objects of our introducing later a method, based upon a principle of invariance, by which a unique determination of the location of a source is made in the absence of noise. We shall argue that it is methodologically appropriate to preclude the possibility of aliases before added noise is treated.

necessary prior information, only a line or a curve may be recovered from the received data. For this reason, it seems reasonable to disclaim any point-locating properties of the beamformer.[47]

In the writing on the signal processing of passive *SONAR*, 'beamforming', on the whole, entails the application to data of a rotational vector in order to determine the angle of maximum strength of signal. The antenna is caused to undergo a virtual rotation.

Insofar as we are concerned here with the locating of point sources, the term 'beamforming' might seem alien to our purpose. 'Beamforming' seems unable to contribute to the locating of point sources, being conceived typically to point in the direction of arrival of a plane wave.

The merits of compasses attached to the array undersea lies beyond the scope of the present thesis. We assume, for present purposes, that our understanding of the deformation of the array must be based solely upon the extrinsic information obtained through a study of the history of the speed and direction of the towing vessel.

For the novel algorithms presented in the present thesis we assume spherical spreading from a point source. In particular, where it seems appropriate to assume that we can make a unique determination of the location before noise, we argue that we must abandon altogether the distinction between 'near field' and 'far field' as burdensome to the conceptual stability of the problem.

With the notion of 'array gain' with radar, the activity used was summing the readings from the individual antenna elements. It will be seen, however, that a sum of amplitudes yields information of a one dimensional kind only. However, if one were to multiply the outputs together, one would have a quantity of two-dimensional implications. A convolution of readings might for example reveal a property of orthogonality, by means of which the rightness of a guess of the frequency of a source could be established. Later, we shall present an algorithm in which it is through ratios of outputs as well as with cross-correlation that the zero-dimensional focussing upon a point comes to pass.

One of the important implications of that line of attack will be that a

[47]But I. Roebuck associates 'beamforming' with 'localization', at any rate: 'beamforming (target is localised)', I. Roebuck, 'The fractal structure of acoustic data', *Defence Oceanography*, 1992, p. 1. It is not clear whether localization of a point or of a direction is meant.

model assuming plane-wave signals without regard to the property of amplitude varying through distance is methodologically flawed, and cannot yield any useful information at all about the coordinates of the source. For covariance of such signals will result in a quantity of ambiguous dimensionality.

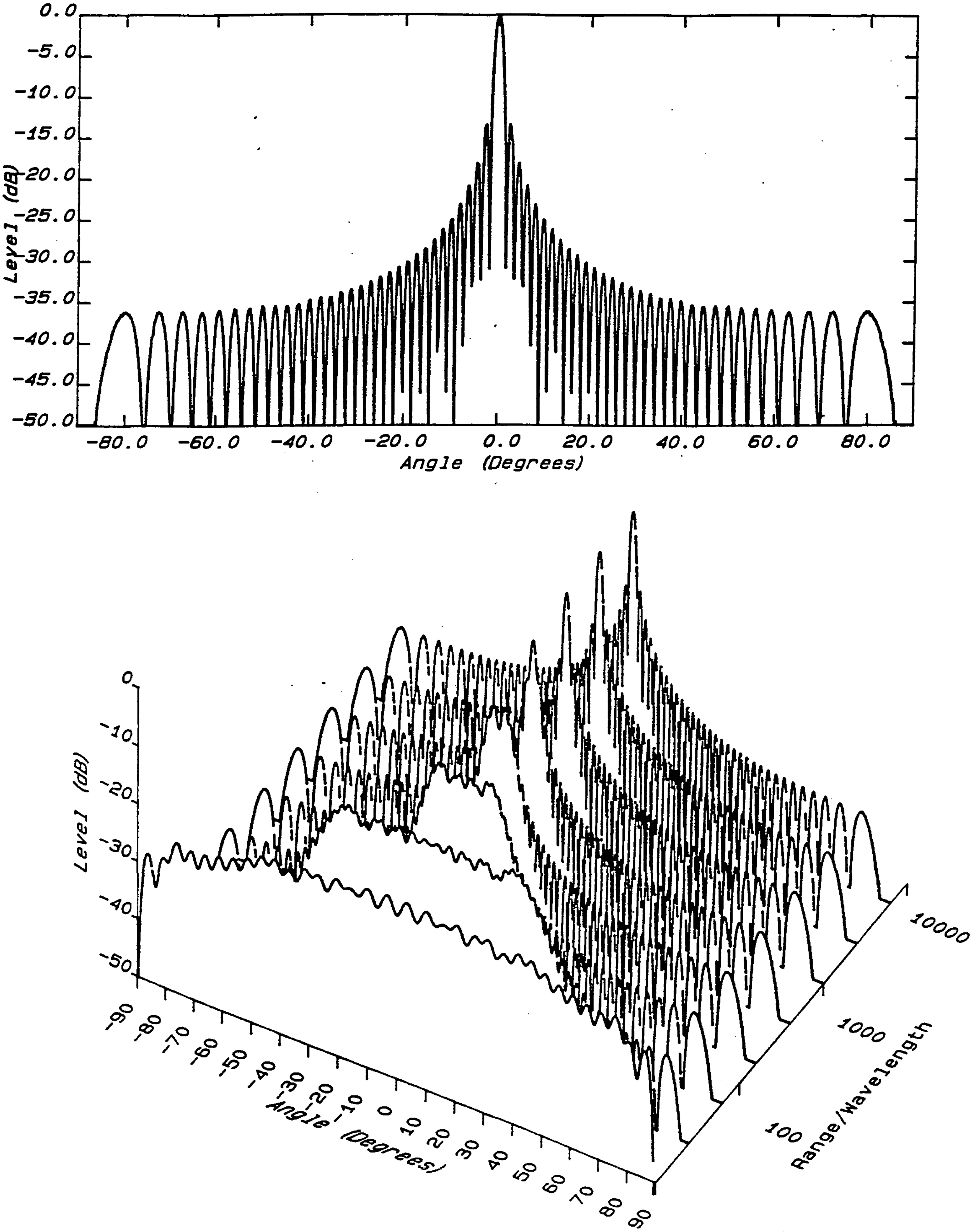
In anticipation of the introduction of our novel 'phase-binning' algorithm below, where we believe that we are able to prove that the minimal, optimal number of bins to be employed is three, we may characterize 'beamforming' as a 'binning' of data, whether straightforward or in linear combination, in one bin only. It is because of that, we may argue, that 'beamforming' is not suited to locating point sources at finite distances and, concomitantly, cannot avoid aliasing (which, in practice, it does not).

2.5 The Non-applicability of the 'Conventional Beamformer' to the Problem

We include a graphical illustration of the non-applicability of the 'Conventional Beamformer' to our problem.[48] In Figure 2 below sixty-four hydrophones have been used in a straight antenna of length $L = 32\lambda$, with λ the wavelength of the source. Plainly, the algorithm employed cannot detect a source within a certain distance of the array.

[48] We should like to thank the Admiralty Research Establishment, Portland, for volunteering this evidence.

Ex. L=32λ (N=64)



Showing The Near Field Effect

Figure 2: The non-applicability of the 'Conventional Beamformer' to the problem

3 The Theoretical Basis of a Distinction Between Near-Field and Far-Field

3.1 The Data Model Assumed in the Thesis

In order to describe the propagation of acoustic disturbances in a fluid medium, we adopt a common approach and assume, *inter alia* and fundamentally, that physical quantities in fluid mechanics may be expressed as sums of state-steady values plus acoustic ones.[49] In the plane $z = 0$, the pressure $p(x, y, t)$ comprises the hydrostatic pressure $p_0(x, y)$ obtaining in the absence of disturbance and the instantaneous value $p'(x, y, t)$ arising from disturbance. Thus

$$p(x, y, t) = p_0(x, y) + p'(x, y, t). \tag{8}$$

In order to derive an expression for p', it is convenient to introduce the velocity potential ϕ, whence the fluid velocity $\mathbf{v}$ due to the disturbance is given by

$$\mathbf{v} = \mathbf{grad}\phi. \tag{9}$$ [50]

Now, by the equation of motion, we have

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\mathbf{grad}p' = 0, \tag{10}$$

where ρ_0 is the density of the undisturbed fluid assumed uniform. Hence

$$\frac{\partial \mathbf{grad}\phi}{\partial t} + \frac{1}{\rho_0}\mathbf{grad}p' = 0. \tag{11}$$

Hence

$$\frac{\partial \phi}{\partial t} + \frac{p'}{\rho_0} = 0. \tag{12}$$

Thus

$$p' = -\rho_0\frac{\partial \phi}{\partial t}. \tag{13}$$

[49]See D. Ross, Mechanics of Underwater Noise, New York, 1976, p. 23

[50]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (transl. J. B. Sykes and W. H. Reid), 1st edition, 1959, p. 245f. *et passim*

Now, ϕ satisfies the wave equation

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}, \quad ^{51} \tag{14}$$

where c is the velocity of sound.

Take the origin at the centre of a small pulsating spherical surface radiating sound in all directions. Then the wave equation can be written as

$$\nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi}{\partial r}), \tag{15}$$

where r is the distance from the source. Therefore

$$\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi}{\partial r}) = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \tag{16}$$

Write

$$\phi = \frac{f(r,t)}{r}. \tag{17}$$

Then

$$\frac{\partial^2 f}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2}. \tag{18}$$

The general solution of Equation 18 is

$$f = f_1(ct - r) + f_2(ct + r), \tag{19}$$

where f_1 and f_2 are arbitrary functions. Thus the general solution of Equation 15 is of the form

$$\phi = \frac{f_1(ct - r)}{r} + \frac{f_2(ct + r)}{r}. \tag{20}$$

The first term on the RHS of Equation 20 is an out-going wave, while the second is a wave converging on the origin. For an oscillating sphere in an unbounded medium there is no incoming wave, *i. e.*

$$\phi = \frac{f_1(ct - r)}{r}. \tag{21}$$

[51] *Ibid.*, p. 246

But, for a fluctuating sphere oscillating at frequency ν

$$\phi \propto \cos(2\pi\nu t + \epsilon), \tag{22}$$

at a given point, where ϵ is a constant.

Therefore, from Equation 21,

$$f_1(ct - r) = A\cos\left(2\pi\nu(t - \frac{r}{c}) + \epsilon\right), \tag{23}$$

where $(A, \epsilon) =$ const. Therefore from Equation 21

$$\phi(r,t) = \frac{A}{r}\cos\left(2\pi\nu(t - \frac{r}{c}) + \epsilon\right). \tag{24}$$

To determine A and ϵ, consider the radial velocity $v(r,t)$. Now

$$\begin{aligned} v(r,t) &= \frac{\partial\phi}{\partial r} \\ &= -\frac{A}{r}\left(\frac{1}{r}\cos(2\pi\nu(t - \frac{r}{c}) + \epsilon) + \frac{2\pi\nu}{c}\sin(2\pi\nu(t - \frac{r}{c}) + \epsilon)\right). \end{aligned} \tag{25}$$

Let $a(t)$ be the radius of the sphere at time t. Then

$$a(t) = a_0 + \alpha\cos(2\pi\nu t + \epsilon_0), \tag{26}$$

where a_0 is the mean radius and α the amplitude, and where α, ν and ϵ_0 are prescribed. But

$$v(a_0, t) = \dot{a}(t) = -2\pi\nu\alpha\sin(2\pi\nu t + \epsilon_0). \tag{27}$$

Hence, putting $r = a_0$ in Equation 25, we have

$$\begin{aligned} -2\pi\nu\alpha\sin(2\pi\nu t + \epsilon_0) &= -\frac{A}{a_0}(\frac{1}{a_0}\cos(2\pi\nu(t - \frac{a_0}{c}) + \epsilon) \\ &\quad + \frac{2\pi\nu}{c}\sin(2\pi\nu(t - \frac{a_0}{c}) + \epsilon)). \end{aligned} \tag{28}$$

Equation 28 determines A and ϵ in terms of the prescribed α and ϵ_0. Thus Equation 24 uniquely determines ϕ. Therefore, from Equations 13 and 24 we have

$$p'(r,t) = \frac{2\pi\nu\rho_0 A}{r}\sin(2\pi\nu(t - \frac{r}{c}) + \epsilon). \tag{29}$$

3.2 Spherical Spreading and Plane Waves

It will be convenient to introduce complex numbers $\underline{p}'$, $\underline{\phi}$, for which p' and ϕ in the previous section are the real parts. Thus, from Equation 24 above, we have

$$\underline{\phi}(r,t) = \frac{A}{r} e^{i(2\pi\nu(t-\frac{r}{c})+\epsilon)}, \tag{30}$$

and from Equation 13

$$\underline{p}'(r,t) = -\rho_0 \frac{\partial \underline{\phi}}{\partial t} = -\frac{2\pi i \nu \rho_0 A}{r} e^{i(2\pi\nu(t-\frac{r}{c})+\epsilon)}. \tag{31}$$

We may now consider the pressure field, measurable by a single hydrophone h, created by a continuous line radiator of length L in radial pulsation lying on the ξ-axis, as in Figure 3.

Since we have an extended source, we replace the factor $Ae^{i\xi}$ in Equation 31 by $\underline{A}(\xi)\frac{d\xi}{L}$, the strength of an element $(d\xi)$ of the line at q. Let $d\underline{\phi}$ be the contribution of that element to the potential at h at time t. Then

$$d\underline{\phi} = \frac{\underline{A}(\xi)}{L}\, d\xi\, \frac{e^{i(2\pi\nu t - kr')}}{r'}, \tag{32}$$

where

$$k = \frac{2\pi\nu}{c}. \tag{33}$$

Therefore

$$\underline{\phi}(r,t) = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underline{A}(\xi)\, \frac{e^{i(2\pi\nu t - kr')}}{r'}\, d\xi. \tag{34}$$

For large distances $r \gg L$, the triangle of the origin, h and q, cosine rule, gives

$$r' = r - \xi \sin\theta + O(\frac{L^2}{r}). \tag{35}$$

Hence

$$\underline{\phi}(r,t) = \frac{1}{L}\frac{e^{i(2\pi\nu t - kr)}}{r}\int_{-\frac{L}{2}}^{\frac{L}{2}} \underline{A}(\xi)\, e^{ik\xi\sin\theta}\, d\xi \tag{36}$$

correct to the zero order in $\frac{L}{r}$. Equation 36 refers to the far-field. But the source is uniformly distributed by hypothesis, therefore

$$\underline{A}(\xi) = \underline{A} \text{ (constant)}. \tag{37}$$

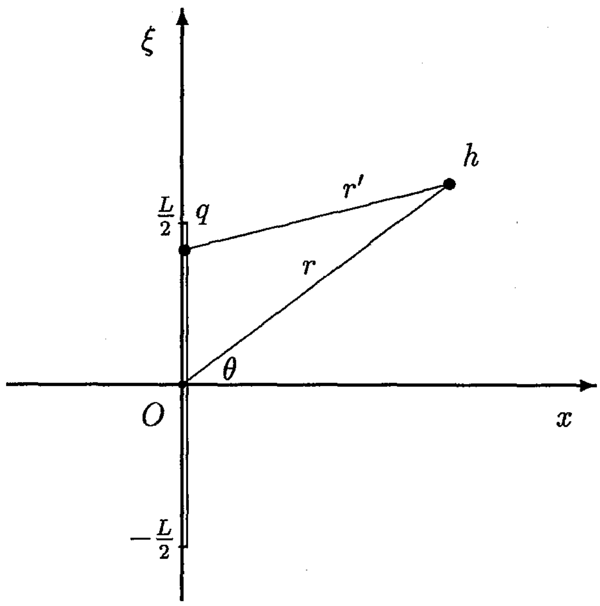


Figure 3: A continuous line radiator in radial pulsation and a hydrophone h

Hence, to order zero,

$$\underline{\phi}(r,t) = \frac{\underline{A}}{L}\frac{e^{i(2\pi\nu t-kr)}}{r}\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ik\xi\sin\theta}\,d\xi \tag{38}$$

$$= \frac{\underline{A}}{r}e^{i(2\pi\nu t-kr)}\frac{\sin(\frac{1}{2}\,kL\sin\theta)}{\frac{1}{2}\,kL\sin\theta}, \tag{39}$$

therefore, from Equation 31 above,

$$\underline{p}'(r,t) = -\frac{2\pi i\nu\rho_0\underline{A}}{r}e^{i(2\pi\nu t-kr)}\frac{\sin(\frac{1}{2}\,kL\sin\theta)}{\frac{1}{2}\,kL\sin\theta} \quad \text{(far-field)}. \tag{40}$$

We consider now the measuring by two hydrophones of the pressure field of a signal radiating from a single monopole. Take the z-plane to be the plane through the source and the antenna. Refer to Figure 4.

In Figure 4 source S is to be taken as a small pulsating spherical surface radiating sound in all directions and h_1 and h_2 are the two hydrophones lying on an x-axis; d is the distance between the two hydrophones; the origin O is at the mid-point between them; r_1 and r_2 are the distances between the source and the respective hydrophones; r is the distance from S to the origin, and θ the angle between the direction of the source, as seen from the origin, and the y-axis.

We require to know the pressure at field points h_1 and h_2. The quantities r_1^2 and r_2^2 are made up as follows:

$$r_1^2 = r^2 + (\frac{d}{2})^2 - rd\sin\theta; \tag{41}$$

$$r_2^2 = r^2 + (\frac{d}{2})^2 + rd\sin\theta. \tag{42}$$

They contribute to the obtaining of $\overline{r}$, the root-mean-square distance, as follows:

$$\begin{aligned}\overline{r} &= \sqrt{\overline{r^2}}\\ &= \sqrt{\frac{r_1^2 + r_2^2}{2}}\\ &= \sqrt{r^2 + (\frac{d}{2})^2}.\end{aligned} \tag{43}$$

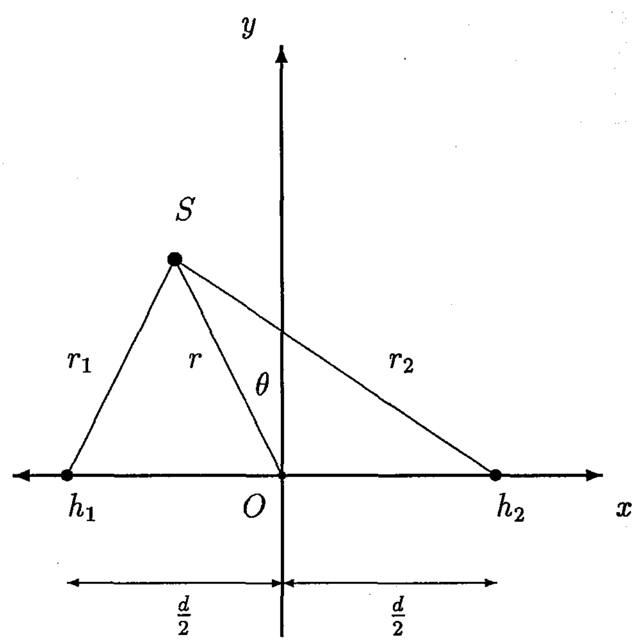


Figure 4: A single monopole radiating upon two hydrophones

We define a dimensionless parameter β, as follows:

$$\begin{aligned}\beta &= \frac{rd}{r_1^2 + r_2^2}\sin\theta \\ &= \frac{rd}{\overline{2r^2}}\sin\theta \\ &= \frac{rd}{2r^2 + \frac{d^2}{2}}\sin\theta \\ &= \frac{1}{2\frac{r}{d} + \frac{d}{2r}}\sin\theta. \end{aligned} \tag{44}$$

Then, from Equations 41 and 42,

$$\begin{aligned} r_1 &= \overline{r}\sqrt{1 - \frac{rd}{\overline{r^2}}\sin\theta} \\ &= \overline{r}\sqrt{1 - 2\beta}; \end{aligned} \tag{45}$$

and

$$\begin{aligned} r_2 &= \overline{r}\sqrt{1 + \frac{rd}{\overline{r^2}}\sin\theta} \\ &= \overline{r}\sqrt{1 + 2\beta}. \end{aligned} \tag{46}$$

From Equation 31 the instantaneous pressures $\underline{p}'_{h_1}$ and $\underline{p}'_{h_2}$, at hydrophones h_1 and h_2, are as follows:

$$\underline{p}'_{h_1} = \frac{2\pi\nu\rho_0 A}{r_1}\, e^{i(2\pi\nu t - kr_1 + \psi)}; \tag{47}$$

$$\underline{p}'_{h_2} = \frac{2\pi\nu\rho_0 A}{r_2}\, e^{i(2\pi\nu t - kr_2 + \psi)}, \tag{48}$$

where $\psi = \epsilon - \frac{\pi}{2}$ is the phase angle.

Consider the quantity $\underline{p}'_{av} = \frac{1}{2}(\underline{p}'_{h_1} + \underline{p}'_{h_2})$ as compared with the pressure $\underline{p}'_0$ at distance $\overline{r}$ from the source on the line from S to the origin O in Figure 4 above. Now

$$\underline{p}'_0 = \frac{2\pi\nu\rho_0 A}{\overline{r}}\, e^{i(2\pi\nu t - k\overline{r} + \psi)}. \tag{49}$$

Therefore

$$\begin{aligned}
\underline{p}'_{av} &= \frac{1}{2}(\underline{p}'_{h_1} + \underline{p}'_{h_2}) \\
&= \frac{1}{2}\frac{2\pi\nu\rho_0 A}{\overline{r}} e^{i(2\pi\nu t - k\overline{r} + \psi)} \left(\frac{e^{ik\overline{r}(1-\sqrt{1-2\beta})}}{\sqrt{1-2\beta}} + \frac{e^{ik\overline{r}(1-\sqrt{1+2\beta})}}{\sqrt{1+2\beta}}\right) \\
&= \frac{1}{2}\underline{p}'_0 \left(\frac{e^{ik\overline{r}(1-\sqrt{1-2\beta})}}{\sqrt{1-2\beta}} + \frac{e^{ik\overline{r}(1-\sqrt{1+2\beta})}}{\sqrt{1+2\beta}}\right). \qquad (50)
\end{aligned}$$

Now

$$\beta = \frac{rd}{2\overline{r^2}} \sin\theta \qquad (51)$$

and

$$\overline{r^2} = r^2 + (\frac{d}{2})^2, \qquad (52)$$

from Equations 43 and 44. If r is very much greater than d, then we can write

$$\overline{r} \approx r \qquad (53)$$

in Equation 51, so that

$$\beta \approx \frac{(\frac{d}{2})\sin\theta}{r}. \qquad (54)$$

Write

$$|E| = |\frac{2\underline{p}'_{av}}{\underline{p}'_0}|. \qquad (55)$$

When $r \gg L$, β very small compared with unity, then, combining Equations 50, 53, 54 and 55, we have

$$\begin{aligned}
|E| &\approx |(1+\beta)\, e^{ikr(1-(1-\beta))} + (1-\beta)\, e^{ikr(1-(1+\beta))}| \text{ (to } O(\beta)) \qquad (56) \\
&= |e^{ikr(\frac{(\frac{d}{2})}{r}\sin\theta)} + e^{-ikr(\frac{(\frac{d}{2})}{r}\sin\theta)}| \qquad (57) \\
&= |e^{i\frac{kd}{2}\sin\theta} + e^{-i\frac{kd}{2}\sin\theta}|. \qquad (58)
\end{aligned}$$

It is to be noted that the terms of $O(\beta)$ have now disappeared, and therefore Equation 58 is exactly the same as for a plane wave. Therefore we must work with $O(\beta^2)$ with distant point sources.

We are now concerned with plane waves. Consider the four hydrophone array, as in Figure 5.

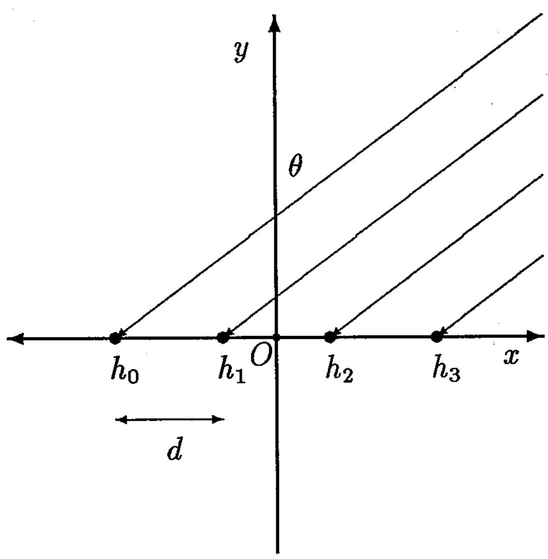


Figure 5: Plane waves incident upon a rigid array of four equidistantly-spaced sensors

Define

$$E_{2N}(\theta) = \sum_{k=0}^{N-1} w_k \cos{(2k+1)\,u}, \tag{59}$$

where the w_k are weighting factors, and where

$$u = \frac{d\pi}{\lambda}\sin\theta. \tag{60}$$

Then, for a four hydrophone array,

$$\begin{aligned} E_4(\theta) &= w_0 \cos u + w_1 \cos 3u & (61)\\ &= w_0 \cos u + w_1(4\cos^3 u - 3\cos u). & (62)\end{aligned}$$

Write

$$x = \cos u, \tag{63}$$

then

$$-1 \leq x \leq 1. \tag{64}$$

Define

$$\begin{aligned} G_3(x) &= w_0 x + w_1(4x^3 - 3x) & (65)\\ &= x(4w_1x^2 + (w_0 - 3w_1)). & (66)\end{aligned}$$

It can be seen that $G_3(x)$ is equal to zero at $x = 0$ and at values of x satisfying

$$4w_1x^2 + w_0 - 3w_1 = 0, \tag{67}$$

whence

$$x = \pm\sqrt{\frac{3 - \frac{w_0}{w_1}}{4}}. \tag{68}$$

We can find turning points of $G_3(x)$ as follows:

$$\frac{d(G_3(x))}{dx} = 0 \tag{69}$$

$\Longrightarrow$

$$x = \pm\sqrt{\frac{3 - \frac{w_0}{w_1}}{12}}. \tag{70}$$

However, for the main lobe we must satisfy

$$\frac{d(G_3(x))}{du} = 0. \tag{71}$$

Now

$$\begin{aligned}
\frac{d(G_3(x))}{du} &= \frac{d(G_3(x))}{dx} \times \frac{dx}{du} &(72)\\
&= (12w_1x^2 + (w_0 - 3w_1)) \times (-\sin u) &(73)\\
&= -(12w_1x^2 + (w_0 - 3w_1)) \sin u &(74)\\
&= -(12w_1x^2 + (w_0 - 3w_1)) \sqrt{1-\cos^2 u}. &(75)
\end{aligned}$$

But $\cos u = x$ from Equation 63, hence Equation 75 $\Longrightarrow$

$$\frac{d(G_3(x))}{du} = -(12w_1x^2 + (w_0 - 3w_1))\sqrt{1-x^2}. \tag{76}$$

Returning to the pressure field measurable by two hydrophones of a signal radiating from a single monopole in the 'far field', far enough away, that is, for plane waves to be assumed, define E_2 as follows:

$$\begin{aligned}
E_2 &= \frac{\underline{p}_2'}{\underline{p}_0'} &(77)\\
&= \frac{1}{2}(e^{i\frac{kd}{2}\sin\theta} + e^{-i\frac{kd}{2}\sin\theta}), &(78)
\end{aligned}$$

as earlier. Using Equations 60 and 78,

$$\begin{aligned}
E_2(\theta) &= \frac{1}{2}(e^{iu} + e^{-iu}) &(79)\\
&= \cos u. &(80)
\end{aligned}$$

The 'angle of incidence' θ may be as follows:

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}. \tag{81}$$

Now

$$\cos u = \frac{\sin 2u}{2\,\sin u}. \tag{82}$$

We now generalize to N hydrophones. Define $E_N(\theta)$ as follows:

$$E_N(\theta) = \frac{1}{N}\sum_{n=1}^{N}\frac{\underline{p}'_n}{\underline{p}'_0}, \tag{83}$$

with

$$n = 1,\, 2,\, \dots N, \tag{84}$$

where $\underline{p}'_n(t)$ is the complex pressure at the nth hydrophone at time t, and where $\underline{p}'_0(t)$ is the complex pressure at the mid-point of the array. After a little reduction we have

$$E_N(\theta) = \frac{\sin Nu}{N\,\sin u}. \tag{85}$$

We hope that we have put what is essentially a customary distinction between 'near field' and 'far field' into theoretical perspective. Even if the criterion of 'wavefront curvature' less than one sixteenth of a wavelength is allowed, we see no compelling reason to employ it unless we wished, for convenience perhaps, to neglect the problem of the amplitude of the signal. So rather than devise an algorithm on the premise of a finite distinction between 'near field' and 'far field', we shall proceed only in terms of spherical spreading, and assume that 'range' and 'bearing' are mutually complementary properties in that light. Methodologically speaking, we believe it to be better that an algorithm should be developed on that basis. For the sake of good method, we should start with locating sources arbitrarily close to the hydrophone array, and then work outwards. We consider this approach to be preferable to working from afar inwards. Naively, we believe that a 'good' algorithm is one which can locate a point source where it is near enough to the array to be locatable at all. Of course, it will gradually cease to be able to locate the source as it is moved into the distance. We think that such an approach, from the hydrophone array outwards, where we strive for resolution at ever greater distances, is better than one which, as a prior, proceeds from an assumption of a distinction between 'near' and 'far' fields, a distinction which may be arbitrary.

4 A 'Principle of Invariance' in the Determination of the Location of a Stationary Point Source in the Absence of Noise

In this chapter we follow the procedure of creating a method with which we can locate a point source uniquely in the absence of noise, the better to treat noise when it has been added. The procedure follows from a principle of invariance, in accordance with which the statistical treatment of a noisy problem may be administered only where a unique solution is guaranteed before noise. A corollary of the Principle is that in problems where it can be proved that unpredictable aliases of the solution will compete for attention, a statistical treatment cannot distinguish the solution from its aliases, and that therefore there is no theoretical basis for the statistical treatment.

We present an exposition of an algorithm for determining the location of a single point source uniquely in the absence of noise. Then we shall present an N-process for treating noise on the signal.

4.1 The Location of a Point Source Under Water Arbitrarily Close to a Linear Array of Hydrophones

A method for locating a point source is presented, based on the fact that the locus of points in a plane, the distances of which from two fixed points are in a prescribed ratio, is a circle. A third fixed point allows a second circle to be drawn, the intersection of which with the first circle uniquely locates a source, known, for convenience, *a priori* to lie on one side of the array.

An N-process is presented for estimating the frequency and amplitude of a signal in the presence of noise. It is to be contrasted favourably, in terms of speed, with the discrete Fourier transform, implemented in the form of the Fast Fourier Transform.

4.2 The Geometrical Construction for the Location of a Point Source with a Three-Hydrophone Array

Consider an array of three hydrophones at points A, B and C and a point source S, as shown in Figure 6 below.

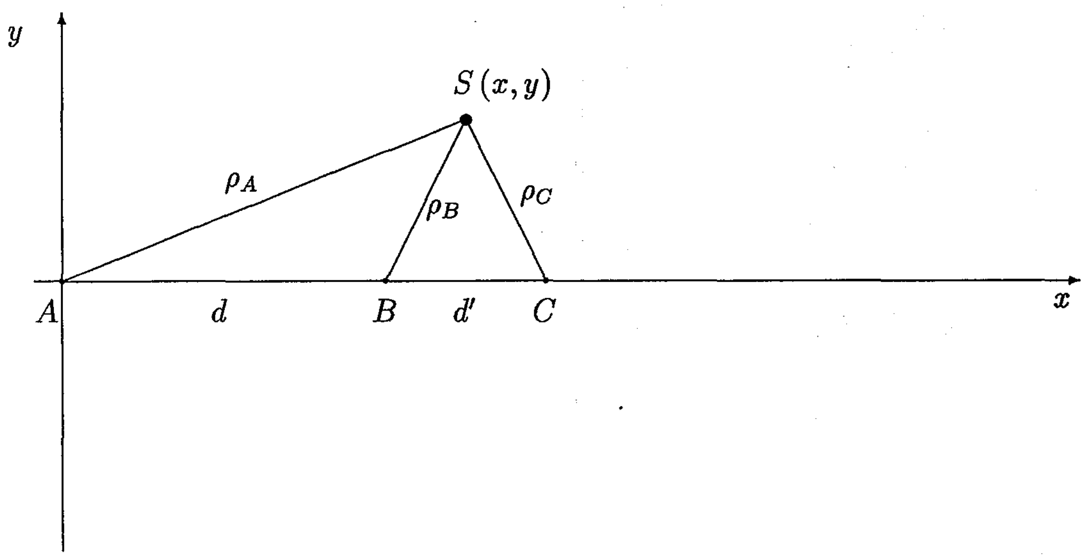


Figure 6: A linear array of three hydrophones and a point source

Let

$$\frac{\rho_B}{\rho_A} = \beta \tag{86}$$

and

$$\frac{\rho_C}{\rho_A} = \gamma\,. \tag{87}$$

We note that the ratios β and γ are obtainable from measurements of hydrostatic pressure at the three hydrophones.

Thus, in virtue of the ratio β, the source must lie on the circle C_β and, in virtue of the ratio γ, it must also lie on the circle C_γ, as shown in Figure 7 below.

The circle C_β has its centre at the point $(\frac{d}{1-\beta^2}, 0)$ and has radius $\frac{\beta d}{1-\beta^2}$, and the circle C_γ has its centre at the point $(\frac{d+d'}{1-\gamma^2}, 0)$ and has radius $\frac{\gamma(d+d')}{1-\gamma^2}$. The construction is easily generalised to the case where the three hydrophones are not collinear.

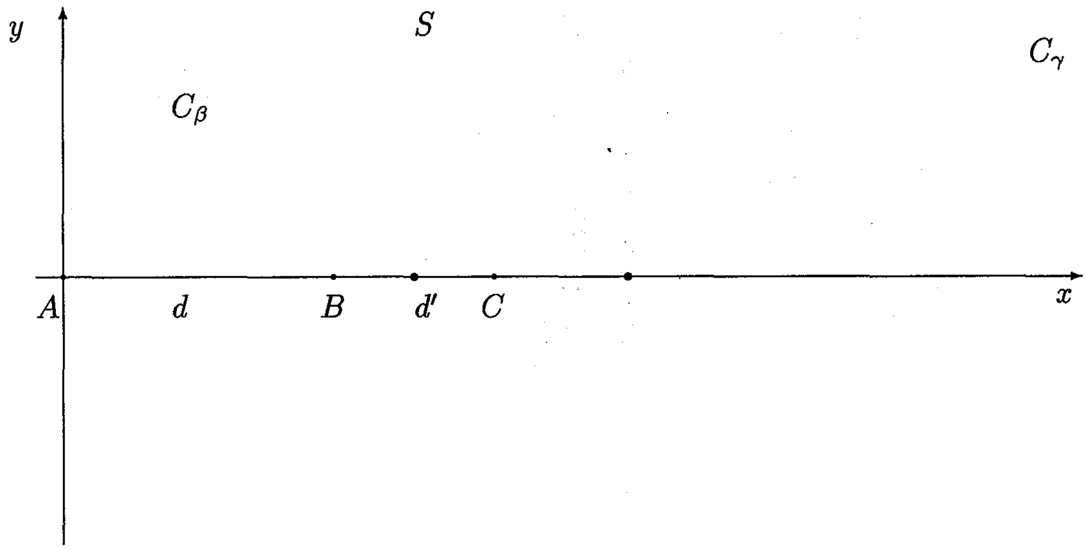


Figure 7: Geometrical construction for locating a point source

4.3 The Determination of the Frequency of the Source in the Absence of Noise

The true acoustic pressure $p^{(0)}(t)$ due to a monochromatic, spherical source at a distance ρ from a hydrophone at time t is given as follows:

$$p^{(0)}(t) = \frac{a_0}{\rho} \cos\left(2\pi\nu(t - \frac{\rho}{c}) + \phi\right), \tag{88}$$

where a_0 is the (small) amplitude of the disturbance at unit distance from the source, ν the frequency of the disturbance, c the speed of the acoustic wave in the medium and ϕ the phase angle.

Let τ be the sampling interval and let $\varpi = 2\pi\nu\tau$. Then the true pressures at A at times $t = -\tau, 0, \tau$ are as follows:

$$P_{A_1}^{(0)} = \frac{a_0}{\rho_A} \cos\left(-\varpi - 2\pi\nu\frac{\rho_A}{c} + \phi\right); \tag{89}$$

$$P_{A_2}^{(0)} = \frac{a_0}{\rho_A} \cos\left(-2\pi\nu\frac{\rho_A}{c} + \phi\right); \tag{90}$$

$$P_{A_3}^{(0)} = \frac{a_0}{\rho_A}\cos\left(\varpi - 2\pi\nu\frac{\rho_A}{c} + \phi\right). \tag{91}$$

On eliminating a_0 and ϕ from Equations 89, 90 and 91 we have

$$\cos\varpi = \frac{P_{A_1}^{(0)} + P_{A_3}^{(0)}}{2P_{A_2}^{(0)}}. \tag{92}$$

Equation 92 is valid only if $\sin\varpi \neq 0$ and $P_{A_2}^{(0)} \neq 0$. Also, it does not in general define ϖ uniquely. These limitations are all overcome by taking τ less than one half of the period of the source (Nyquist condition). This ensures that $\sin\varpi \neq 0$. Further, Equation 92 holds for any three consecutive observations at equal intervals τ. Thus, in the event that $P_{A_2}^{(0)}$ vanished, we would apply Equation 88 at a fourth time $t = 2\tau$. The condition $\varpi < \pi$ ensures that $p^{(0)}(t)$ cannot vanish at two consecutive times, hence $P_{A_3}^{(0)} \neq 0$, and a valid Equation 92 results. In any case, no divisions by $p(t)$ are involved in the algorithm as developed later in the present chapter.

4.3.1 The Determination of the Amplitude $a = \frac{a_0}{\rho}$ in the Absence of Noise

From Equations 89, 90 and 91 we have

$$a_A^2 = \left(\frac{a_0}{\rho_A}\right)^2 = \frac{P_{A_2}^{(0)^2} - P_{A_1}^{(0)}P_{A_3}^{(0)}}{\sin^2\varpi}, \tag{93}$$

and similarly for a_B and a_C.

4.3.2 The Determination of x and y from the Amplitudes a_A, a_B and a_C in the Absence of Noise

If $d = d'$, then

$$\frac{x}{d} = \frac{1}{2}\frac{1/a_C^2 - 4/a_B^2 + 3/a_A^2}{1/a_C^2 - 2/a_B^2 + 1/a_A^2}. \tag{94}$$

We shall have estimates for $a^2\sin^2\varpi$ rather than a^2, directly from the measurements. Therefore it will be appropriate to re-express Equation 94 in the form

$$\frac{x}{d} = \frac{1}{2}\frac{1/(a^2\sin^2\varpi)_C - 4/(a^2\sin^2\varpi)_B + 3/(a^2\sin^2\varpi)_A}{1/(a^2\sin^2\varpi)_C - 2/(a^2\sin^2\varpi)_B + 1/(a^2\sin^2\varpi)_A}. \tag{95}$$

y^2 is obtained uniquely from

$$\frac{(x-d)^2+y^2}{x^2+y^2}=\beta^2. \tag{96}$$

4.4 N-Process for the Estimation of the Frequency of a Single Sinusoidal Signal in the Presence of Noise

Let the function, representing a noise-free signal, at a hydrophone be

$$p^{(0)}(t) = a\cos(2\pi\nu t + \phi). \tag{97}$$

Let $p_1^{(0)}$, $p_2^{(0)}$, ..., $p_N^{(0)}$ be the values of the function at times $t = \tau, 2\tau, ..., N\tau$. Thus

$$p_n^{(0)} = a\cos(n\overline{\omega} + \phi), \tag{98}$$

where $\overline{\omega} = 2\pi\nu\tau$, as before.

Define

$$q_n^{(0)} = \frac{1}{\sqrt{2}}(p_{n-1}^{(0)} + p_{n+1}^{(0)}) \ \ (n = 2, 3, ..., N-1). \tag{99}$$

Then, denoting

$$a\cos(n\overline{\omega} + \phi) = \alpha_n, \tag{100}$$

we have

$$p_n^{(0)} = \alpha_n, \tag{101}$$

$$q_n^{(0)}/\sqrt{2} = \alpha_n \cos\overline{\omega}. \tag{102}$$

On eliminating α_n between Equations 101 and 102, we have

$$-p_n^{(0)}\sqrt{2}\cos\overline{\omega} + q_n^{(0)} = 0 \ \ (n = 2, 3, ..., N-1).^{52} \tag{103}$$

Denoting

$$-\sqrt{2}\cos\overline{\omega} = X, \tag{104}$$

Equation 103 can be written

$$p_n^{(0)}X + q_n^{(0)} = 0. \tag{105}$$

It is noted that, when noise is present, none of the $p_n^{(0)}$ is known. In preparation for the statistical treatment, it is convenient to take the moment of

[52] Equation 103 is a rearrangement of Equation 92 in a form suitable for subsequent generalisation to a signal comprising the sum of sinusoidal functions of more than one frequency, a problem which we intend to address in future work.

Equation 105 by $p_n^{(0)}$ and sum over n, likewise taking the moment by $q_n^{(0)}$. This produces the two moment equations

$$c_{11}^{(0)}X + c_{12}^{(0)} = 0, \tag{106}$$

$$c_{21}^{(0)}X + c_{22}^{(0)} = 0, \tag{107}$$

in which

$$\left.\begin{array}{ll} c_{11}^{(0)} = \frac{1}{N-2}\sum_{n=2}^{N-1} p_n^{(0)^2} & c_{12}^{(0)} = \frac{1}{N-2}\sum_{n=2}^{N-1} p_n^{(0)} q_n^{(0)} \\ c_{21}^{(0)} = \frac{1}{N-2}\sum_{n=2}^{N-1} q_n^{(0)} p_n^{(0)} & c_{22}^{(0)} = \frac{1}{N-2}\sum_{n=2}^{N-1} q_n^{(0)^2} \end{array}\right\} \tag{108}$$

Thus the $c_{ij}^{(0)}$, $(i,j) = 1,\ 2$, are cross-correlation coefficients among the true values $p_n^{(0)}$ $(n = 2, 3, ..., N-1)$. Again, it is to be remembered that the $c_{ij}^{(0)}$ are unknowns when noise is present. It will be important also to note that, because only one unknown X appears in Equations 106 and 107, the matrix $\left[c_{ij}^{(0)}\right]$ is certainly not of rank 2, but is in general of rank 1.

Let p_n be the measured values, thus

$$p_n = p_n^{(0)} + \epsilon_n, \tag{109}$$

where ϵ_n are the errors for a given realisation. From the assumptions made about the noise we have

$$E\left\{\epsilon_n\right\} = 0 \tag{110}$$

and

$$E\left\{\epsilon_m \epsilon_n\right\} = \sigma^2 \delta_{m,n} \tag{111}$$

over all realisations, where σ^2 is the (unknown) common variance.

From the measured p_n we construct

$$q_n = \frac{1}{\sqrt{2}}(p_{n-1} + p_{n+1}). \tag{112}$$

Thence

$$\left.\begin{array}{ll} c_{11} = \frac{1}{N-2}\sum_{n=2}^{N-1} p_n^2 & c_{12} = \frac{1}{N-2}\sum_{n=2}^{N-1} p_n q_n \\ c_{21} = \frac{1}{N-2}\sum_{n=2}^{N-1} q_n p_n & c_{22} = \frac{1}{N-2}\sum_{n=2}^{N-1} q_n^2 \end{array}\right\} \tag{113}$$

The c_{ij} are the cross-correlations among the measurements in a given realisation, and are known quantities.

On substituting for the p_n from Equation 109 into Equations 113 (using Equation 112) and taking expectation values using Equations 110 and 111, we have

$$E\{c_{ij}\} = c_{ij}^{(0)} + \sigma^2 \delta_{ij} \quad ((i,j) = 1,\, 2)\,. \tag{114}$$

On substituting for the $c_{ij}^{(0)}$ from Equation 114, Equations 106 and 107 can be written

$$\left(E\{c_{11}\} - \sigma^2\right) X + E\{c_{12}\} = 0 \tag{115}$$

and

$$E\{c_{21}\} X + E\{c_{22}\} - \sigma^2 = 0. \tag{116}$$

In Equations 115 and 116 the $E\{c_{ij}\}$ are also unknown, since we have only one realisation. But this time we have

$$E\{c_{ij}\} - c_{ij} \to 0, \tag{117}$$

as $N \to \infty$ (see 'The Statistical Treatment' below).

Accordingly, in Equations 115 and 116 we replace the unknown $E\{c_{ij}\}$ by the known c_{ij} and the true values (X, σ^2) by new quantities (X_N, σ_N^2) to be determined; thus

$$(c_{11} - \sigma_N^2) X_N + c_{12} = 0 \tag{118}$$

and

$$c_{21} X_N + c_{22} - \sigma_N^2 = 0. \tag{119}$$

Thence, in virtue of Equation 117 we have

$$\left.\begin{array}{l} X_N \to X \\ \sigma_N^2 \to \sigma^2 \end{array}\right\}$$

as $N \to \infty$.

$$\tag{120}$$

(X_N, σ_N^2) are therefore estimates of (X, σ^2) which converge to these latter as $N \to \infty$. Equations 118 and 119 are solved in the usual way by first eliminating X_N, giving

$$\Delta_N(\sigma_N^2) = \left|[c_{ij}] - \sigma_N^2 I\right| = 0. \tag{121}$$

Now $[c_{ij}]$ is symmetric, hence the eigenvalues σ_N^2 are real. Moreover, bearing in mind the definitions of the c_{ij} in Equation 113, Schwarz's inequality shows that both eigenvalues are positive.

Further, following arguments reported by Courant and Hilbert,[53] the smaller of the two eigenvalues gives the valid estimate of σ^2. Therefore σ_N^2, the lesser root of Equation 121 is given by the equation

$$\sigma_N^2 = \frac{1}{2}\left[c_{11} + c_{22} - \sqrt{(c_{22} - c_{11})^2 + 4c_{12}^2}\right]. \tag{122}$$

Thence the estimate of $\overline{\omega}_N$ is defined by

$$\cos\overline{\omega}_N = -\frac{X_N}{\sqrt{2}} = \frac{\sqrt{2}c_{12}}{\left[\sqrt{(c_{22} - c_{11})^2 + 4c_{12}^2} - (c_{22} - c_{11})\right]}. \tag{123}$$

4.4.1 The Estimation of the Amplitude a in the Presence of Noise

Introduce

$$P_n^{(0)} = p_n^{(0)^2} - p_{n-1}^{(0)} p_{n+1}^{(0)} \quad (n = 2, 3, ..., N-1). \tag{124}$$

Then

$$P_n^{(0)} = a^2 \sin^2 \overline{\omega}. \tag{125}$$

Thus $P_n^{(0)}$ is the same for all n, and is therefore a *quadratic invariant* of the true pressures.

In preparation for obtaining an estimate of the amplitude in the presence of noise, 124 is summed over the valid values of n, giving

$$a^2 = \frac{C^{(0)}}{\sin^2 \overline{\omega}}, \tag{126}$$

where

$$C^{(0)} = \frac{1}{N-2} \sum_{n=2}^{N-1} P_n^{(0)}. \tag{127}$$

Thus $C^{(0)}$ is a normalised cross-correlation coefficient defined for any $N \geq 3$.

[53]Namely, the Gram determinant, Courant R., and Hilbert D., ***Methods of Mathematical Physics***, New York, 1953 (first published 1937), p. 35f.

As previously, it is noted that $C^{(0)}$ is unknown when noise is present, and we therefore construct

$$C = \frac{1}{N-2} \sum_{n=2}^{N-1} (p_n^2 - p_{n-1} p_{n+1}) \tag{128}$$

from the measurements.

On taking expectation values, using the statistical conditions in Equations 110 and 111, we have:

$$E\{C\} = C^{(0)} + \sigma^2. \tag{129}$$

Thence, following arguments given earlier, namely, substituting for $C^{(0)}$ in terms of $E\{C\}$ from Equation 129 into Equation 126 (*i. e.* $a^2 \sin^2 \varpi = E\{C\} - \sigma^2$), and recognizing that $E\{C\} \to C$ as $N \to \infty$, an estimate of $a^2 \sin^2 \varpi$ may be defined by

$$(a^2 \sin^2 \varpi)_N = C - \sigma_N^2. \tag{130}$$

An estimate of a^2 itself, if required, would be given by

$$a_N^2 = \frac{(a^2 \sin^2 \varpi)_N}{\sin^2 \varpi_N}. \tag{131}$$

Following the arguments set out in 'The Statistical Treatment' below, we shall expect

$$E\left\{(a_N - a)^2\right\} = O(N^{-1}) \tag{132}$$

for large N, where the coefficient for the RHS may be obtained explicitly for a given N in terms of the probability distribution of the ϵ_n and the elements a, ν and ϕ of the true signal. This coefficient depends on the joint probability over all N points, the expression for which is not derived in the present thesis.

Finally, define the estimate of x using Equation 94, thus

$$\frac{x_N}{d} = \frac{1}{2} \frac{1/(a^2 \sin^2 \varpi)_{N_C} - 4/(a^2 \sin^2 \varpi)_{N_B} + 3/(a^2 \sin^2 \varpi)_{N_A}}{1/(a^2 \sin^2 \varpi)_{N_C} - 2/(a^2 \sin^2 \varpi)_{N_B} + 1/(a^2 \sin^2 \varpi)_{N_A}}. \tag{133}$$

A similar definition of the estimate of y follows straightforwardly. Thus

$$\frac{(x_N - d)^2 + y_N^2}{x_N^2 + y_N^2} = \beta_N^2 = \frac{(a \sin^2 \varpi)_{N_A}}{(a \sin^2 \varpi)_{N_B}} = \frac{C_B - \sigma_{N_A}^2}{C_A - \sigma_{N_B}^2}. \tag{134}$$

4.5 The Statistical Treatment

4.5.1 The Joint Probability Distribution for the Errors at Two Different Data Points

Let (ϵ_m, ϵ_n) be the errors at two different data points (m, n). Define $P_{mn}(x, y)$ $dxdy$ = proportion of realisations in which (ϵ_m, ϵ_n) lie in the ranges $(x, x + dx), (y, y + dy)$ respectively. Then

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_{mn}(x, y) dxdy = 1 \tag{135}$$

by definition.

Denote $P_m(x)dx$ = proportion of realisations in which ϵ_m lies in $(x, x+dx)$ irrespective of the values of ϵ_n. Then

$$P_m(x) = \int_{-\infty}^{+\infty} P_{mn}(x, y) dy. \tag{136}$$

(Similarly, $P_n(y) = \int_{-\infty}^{+\infty} P_{mn}(x, y)dx$.)

Let $f(\epsilon_m, \epsilon_n)$ be any function of (ϵ_m, ϵ_n), and denote

$$E\{f(\epsilon_m, \epsilon_n)\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y) P_{mn}(x, y) dxdy.^{54} \tag{137}$$

4.5.2 Inter Data-Point Correlation

The two sets (ϵ_m, ϵ_n) are uncorrelated if $P_{mn}(x, y)$ takes the form

$$P_{mn}(x, y) = Q_m(x) Q_n(y), \tag{138}$$

where $Q_m(x)$, $Q_n(y)$ are any two functions.[55]

Applying Equation 136 to Equation 138 we have $Q_m(x) = P_m(x)$, $Q_n(y) = P_n(y)$, hence

$$P_{mn}(x, y) = P_m(x) P_n(y) \tag{139}$$

[54]Following Kolmogorov, A., *Grundbegriffe der Wahrscheinlichkeitsrechnung*, New York, 1946 (first published 1933), p. 35ff. and Cramér, H., *Mathematical Methods of Statistics*, Princeton, N. J., 1946 (first published 1945), p. 170ff.

[55]See Whittaker E. and Robinson G., *The Calculus of Observations*, 4th Edn., London, 1925, p. 317ff.

(if ϵ_m,ϵ_n are uncorrelated).

It is assumed that the noise at the different data points is uncorrelated about zero mean at each point, and that the probability distribution of ϵ_n is the same at each data point. In this case we have

$$P_{mn}(x, y) = P(x)P(y) \;\; ((m, n) = 1, 2, ..., N) \tag{140}$$

and

$$\left.\begin{array}{l} \mu_n = 0 \\ \sigma_n^2 = \sigma^2 \end{array}\right\} \tag{141}$$

Then, from these arguments,

$$\left.\begin{array}{l} E\{\epsilon_n\} = 0 \\ E\{\epsilon_n^2\} = \sigma^2 \end{array}\right\} (n = 1, 2, ..., N). \tag{142}$$

Define the correlation coefficients of ϵ_m, ϵ_n:

$$c_{mn} = E\{\epsilon_m \epsilon_n\} = 0, \;\; m \neq n. \tag{143}$$

Equations 142 and 143 can be written as a single equation:

$$E\{\epsilon_m \epsilon_n\} = \sigma^2 \delta_{mn}. \tag{144}$$

4.5.3 The Estimates for Uniform and Gaussian Distributions of ϵ_n

From Equations 113 and 140 above we have

$$c_{11} - E\{c_{11}\} = \frac{1}{N-2} \sum_{n=2}^{N-1} p_n^2 - \frac{1}{N-2} \sum_{n=2}^{N-1} p_n^{(0)^2} - \sigma^2. \tag{145}$$

On substituting $p_n = p_n^{(0)} + \epsilon_n$, Equation 145 can be written

$$c_{11} - E\{c_{11}\} = \frac{2}{N-2} \sum_{n=2}^{N-1} p_n^{(0)} \epsilon_n + \frac{1}{N-2} \sum_{n=2}^{N-1} \epsilon_n^2 - \sigma^2. \tag{146}$$

The magnitude of $c_{11} - E\{c_{11}\}$ is given by considering $(c_{11} - E\{c_{11}\})^2$. From Equation 146

$$
\begin{aligned}
(c_{11} - E\{c_{11}\})^2 &= \frac{4}{(N-2)^2}\left(\sum_{n=2}^{N-1} p_n^{(0)}\epsilon_n\right)^2 + \frac{1}{(N-2)^2}\left(\sum_{n=2}^{N-1}\epsilon_n^2\right)^2 + \sigma^4 \\
&+\frac{4}{(N-2)^2}\left(\sum_{n=2}^{N-1} p_n^{(0)}\epsilon_n\right)\left(\sum_{n=2}^{N-1}\epsilon_n^2\right) - \frac{4\sigma^2}{N-2}\sum_{n=2}^{N-1} p_n^{(0)}\epsilon_n \\
&-\frac{2\sigma^2}{N-2}\sum_{n=2}^{N-1}\epsilon_n^2. \qquad (147)
\end{aligned}
$$

Hence

$$
\begin{aligned}
E\left\{(c_{11} - E\{c_{11}\})^2\right\} &= \frac{4}{(N-2)^2}E\left\{\left(\sum_{n=2}^{N-1} p_n^{(0)}\epsilon_n\right)^2\right\} \\
&+\frac{1}{(N-2)^2}E\left\{\left(\sum_{n=2}^{N-1}\epsilon_n\right)^2\right\} \\
&\sigma^4 - \frac{2\sigma^2}{N-2}E\left\{\sum_{n=2}^{N-1}\epsilon_n^2\right\}. \qquad (148)
\end{aligned}
$$

The fourth and fifth terms in Equation 146 have vanished since they involved products of three ϵ_n's and single ϵ_n's respectively. Now

$$
\left(\sum_{n=2}^{N-1} p_n^{(0)}\epsilon_n\right)^2 = \sum_{n=2}^{N-1} p_m^{(0)}p_n^{(0)}\epsilon_m\epsilon_n, \qquad (149)
$$

therefore

$$
\begin{aligned}
E\left\{\left(\sum_{n=2}^{N-1} p_n^{(0)}\epsilon_n\right)^2\right\} &= \sum_{m=2}^{N-1}\sum_{n=2}^{N-1} p_m^{(0)}p_n^{(0)}E\{\epsilon_m\epsilon_n\} \\
&= \sigma^2\sum_{n=2}^{N-1} p_n^{(0)^2}. \qquad (150)
\end{aligned}
$$

Similarly

$$
\left(\sum_{n=2}^{N-1}\epsilon_n^2\right)^2 = \sum_{m=2}^{N-1}\sum_{n=2}^{N-1}\epsilon_m^2\epsilon_n^2, \qquad (151)
$$

therefore

$$E\left\{\left(\sum_{n=2}^{N-1} \epsilon_n^2\right)^2\right\} = \sum_{m=2}^{N-1}\sum_{n=2}^{N-1} E\left\{\epsilon_m^2 \epsilon_n^2\right\}. \tag{152}$$

Now

$$E\left\{\epsilon_m^2 \epsilon_n^2\right\} = \begin{cases} \sigma^4 & , m \neq n \\ k\sigma^4 & , m = n \end{cases}, \tag{153}$$

where

$$k = \begin{cases} 9/5 & \text{(uniform distribution of } \epsilon_n) \\ 3 & \text{(Gaussian distribution of } \epsilon_n) \end{cases}, \tag{154}$$

hence

$$\begin{aligned} E\left\{\left(\sum_{n=2}^{N-1} \epsilon_n^2\right)^2\right\} &= (N-2)(N-3)\sigma^4 \\ &\quad +(N-2)k\sigma^4 \\ &= (N-2)(N-3+k)\sigma^4. \end{aligned} \tag{155}$$

Again,

$$\begin{aligned} E\left\{\sum_{n=2}^{N-1} \epsilon_n^2\right\} &= \sum_{n=2}^{N-1} E\left\{\epsilon_n\right\}^2 \\ &= (N-2)\sigma^2. \end{aligned} \tag{156}$$

On substituting these expressions into Equation 148 we have

$$\begin{aligned} E\left\{(c_{11} - E\left\{c_{11}\right\})^2\right\} &= \frac{4\sigma^2}{N-2}.\frac{1}{N-2}\sum_{n=2}^{N-1} p_n^{(0)^2} \\ &\quad + \frac{1}{N-2}(N-3+k)\sigma^4 \\ &\quad + \sigma^4 - 2\sigma^4 \\ &= \frac{\sigma^2}{N-2}\left[\frac{4}{N-2}\sum_{n=2}^{N-1} p_n^{(0)^2} + (k-1)\sigma^2\right]. \end{aligned} \tag{157}$$

The term $\frac{1}{N-2}\sum_{n=2}^{N-1} p_n^{(0)^2}$ is given in terms of the parameters of the true signal by the following lemma.

4.5.4 Lemma

Provided $\sin\varpi \neq 0$, then

$$\frac{1}{N-2}\sum_{n=2}^{N-1} p_n^{(0)^2} = \frac{1}{2}a^2\left[1+\frac{1}{N-2}\cos\left((N-1)\varpi+2\phi\right)\frac{\sin(N-2)\varpi}{\sin\varpi}\right]. \tag{158}$$

4.5.5 Proof

Taking $t_n = (n-1)\tau$ we have, from 98,

$$p_n^{(0)} = a\cos\left((n-1)\varpi+\phi\right). \tag{159}$$

The phase angle ϕ is thus the phase of the wave at the detector, at the instant of the first observation. Thence

$$p_n^{(0)^2} = \frac{1}{2}a^2\left(1+\cos 2(n\varpi+\phi)\right), \tag{160}$$

giving

$$\frac{1}{N-2}\sum_{n=2}^{N-1} p_n^{(0)^2} = \frac{1}{2}a^2\left[1+\frac{1}{N-2}\cos\left((N-1)\varpi+2\phi\right)\frac{\sin(N-2)\varpi}{\sin\varpi}\right], \tag{161}$$

quod erat demonstrandum.

Hence Equation 157 becomes

$$E\left\{(c_{11}-E\left\{c_{11}\right\})^2\right\} =$$

$$\frac{\sigma^2}{N-2}\left[2a^2\left(1+\frac{1}{N-2}\cos\left((N-1)\varpi+2\phi\right)\frac{\sin(N-2)\varpi}{\sin\varpi}\right)+(k-1)\sigma^2\right]. \tag{162}$$

Thus $c_{11}-E\left\{c_{11}\right\} = O(N^{-\frac{1}{2}})$, as expected. Similar expressions may be derived for $E\left\{(c_{12}-E\left\{c_{12}\right\})^2\right\}$ and $E\left\{(c_{22}-E\left\{c_{22}\right\})^2\right\}$, from which $E\left\{(\cos\varpi_N-\cos\varpi)^2\right\}$ may be derived to give $\cos\varpi_N-\cos\varpi = O(N^{-\frac{1}{2}})$.

Note that, from Equation 123 above, $\cos\varpi_N$ is a non-linear function of the c_{ij}, hence $\cos\varpi_N$ will have a bias; but this bias will be of $O(N^{-\frac{1}{2}})$.

4.6 Graphical Evidence of the 'Amplitude Algorithm'

Graphical evidence of the performance of our Amplitude Algorithm follows in the form of three Figures.[56] The estimates cluster round the target with successive realisations of the noise. The trial source was located at $(1, 1)$, and the antenna has hydrophones at the origin, at $(1, 0)$ and at $(2, 0)$. There is no uncertainty about the locations of the hydrophones. It may be noticed that early estimates of the location of the point-source suggest a 'pull' towards the centre of the antenna, and the visual effect is not unlike a coma. We conjecture that the 'coma' is born of the bias in convergence referred to above. Ten thousand data samples were made available for each of the three elements of the antenna, and the noise-to-signal ratio (being the mean square of the noise upon the mean square of the signal) was ten.

[56]We thank the Admiralty Research Establishment for the graph-plots.

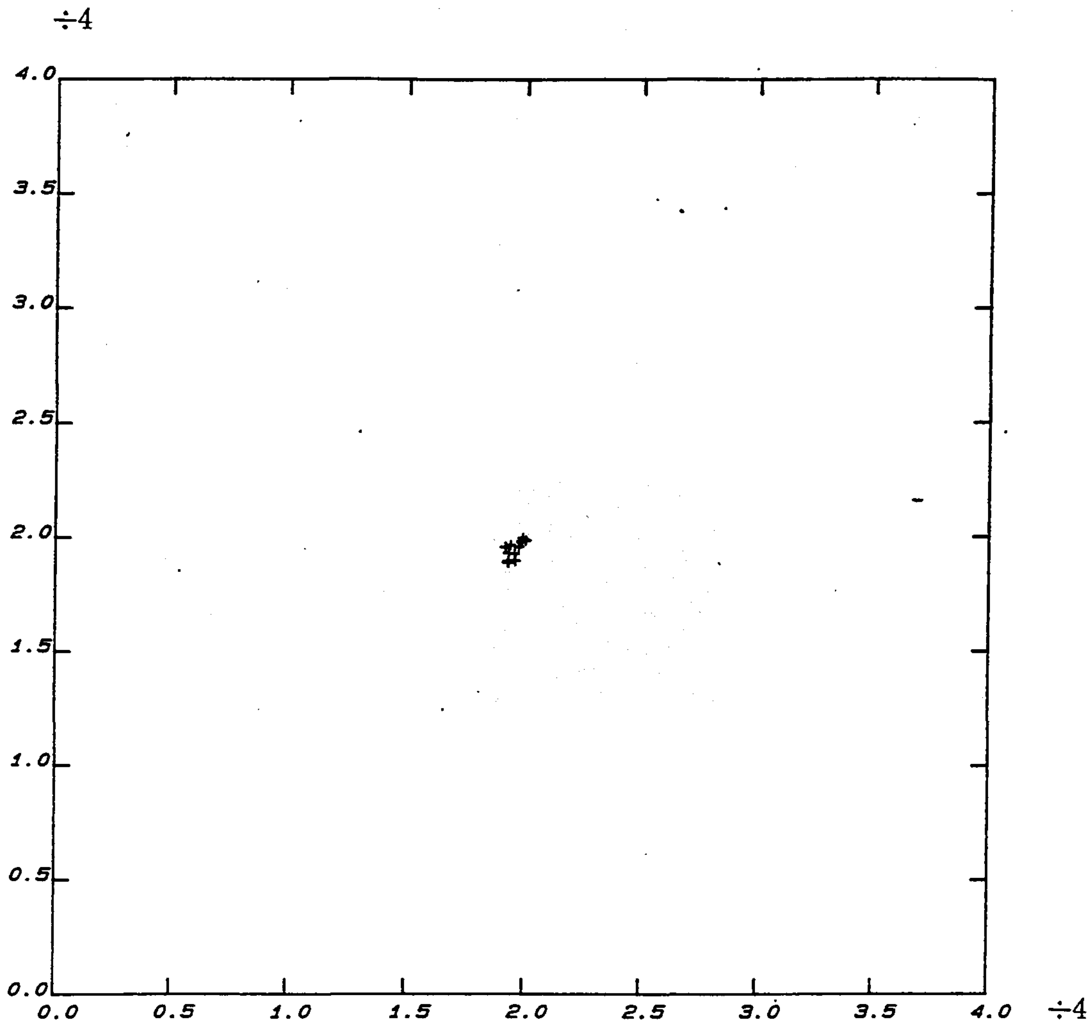


Figure 8: The 'coma' on the convergence upon the target with the Amplitude Algorithm: Graph 1

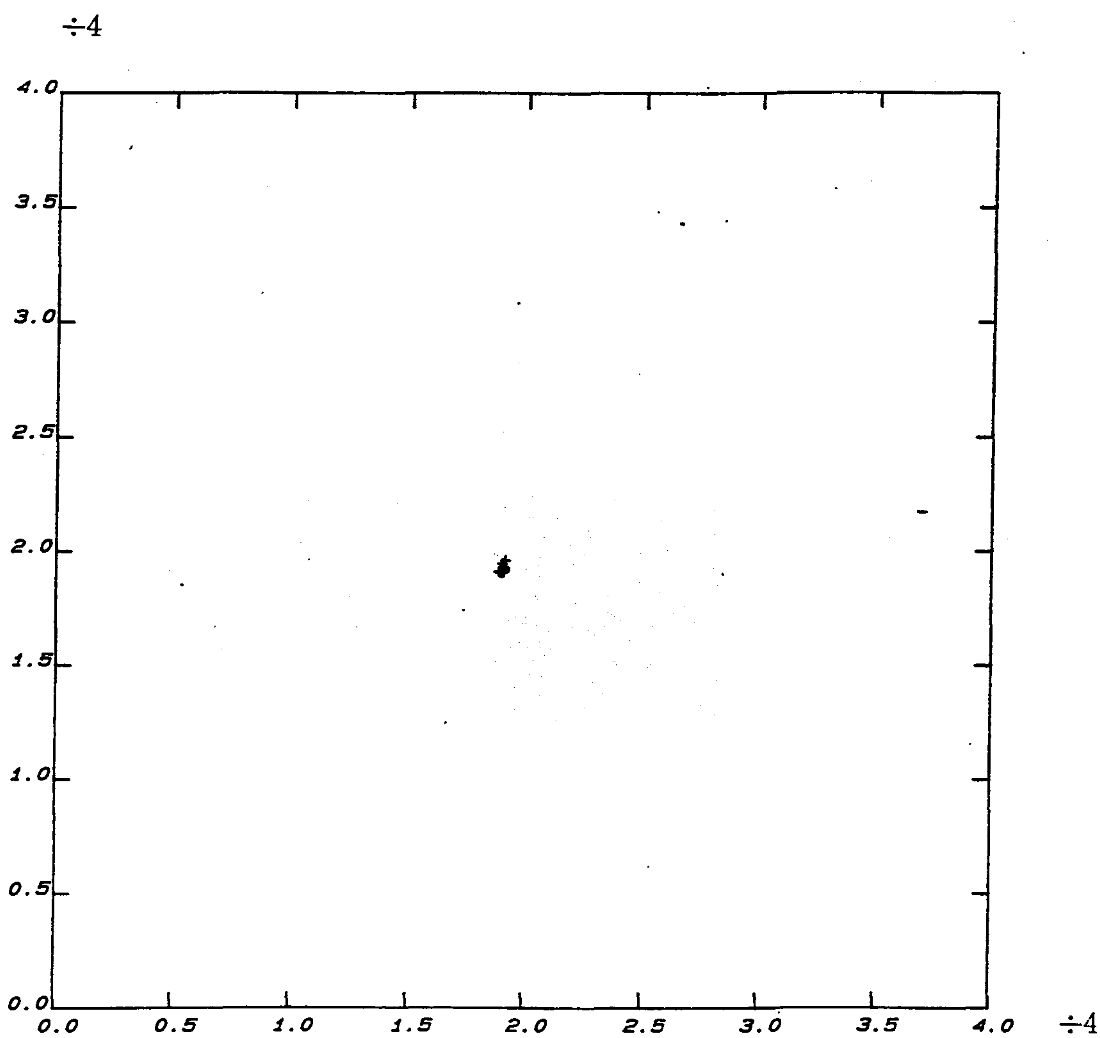


Figure 9: The 'coma' on the convergence upon the target with the Amplitude Algorithm: Graph 2

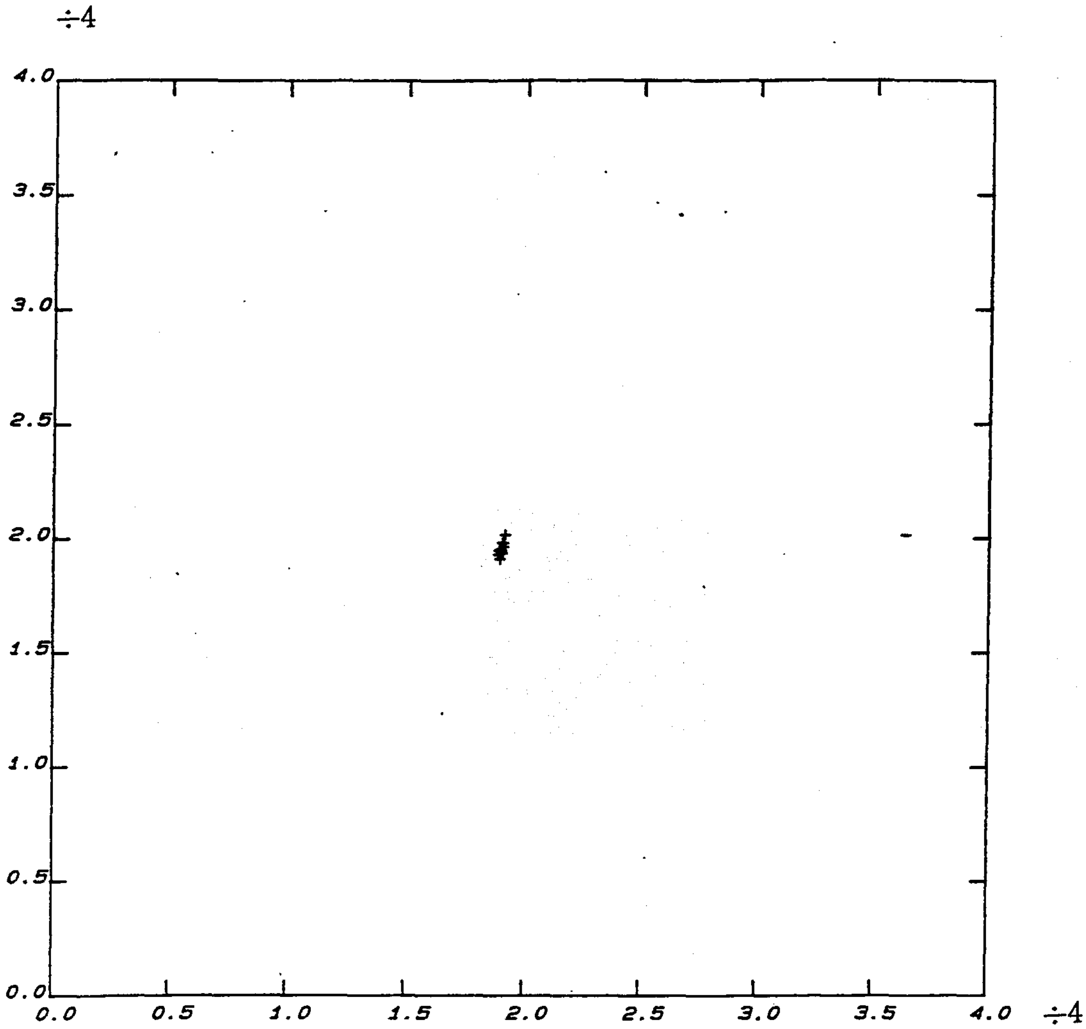


Figure 10: The 'coma' on the convergence upon the target with the Amplitude Algorithm: Graph 3

5 The Implications of Noise for the Dimensionality of the Problem

5.1 The Caustic Curve

In this chapter the dimensionality of a system of source and hydrophones contaminated by noise is considered. The hypothesis is ventured that the presence of noise has implications for the dimensionality of the resolution of the problem. In particular, it is suggested that, with noise present, the conceptual feasibility of locating a point at all is damaged in the technical sense too that whereas a point has no dimensions, and whereas the data were obtained within a nominal framework of two dimensions, the actual framework itself is obscured in uncertainty. Given such a perturbation of the dimensionality of the means by which the data are acquired, we argue that, in the presence of random noise, the lowest order of dimension of resolution possible is actually one and not zero, which latter we might expect to be the case where we knew exactly where the hydrophones were located. In other words, instead of conceiving of a convergence centring upon a point, we might aspire instead to a convergence settling down about a line or curve.

The activity of array shape calibration by means of frequency sources of opportunity or deliberately-placed frequency sources may be thought of as an effort to stabilize the two-dimensional framework within which the data are acquired. But it may be possible to justify a particular estimate of the shape of an array in the limit of some convergent process without recourse to sources of opportunity or deliberately-placed sources. We considered above, for example, the limits that Rockah and Schultheiss reached in their particular approach. They found that the positions of hydrophones relative to each other could be estimated well, but that a residual error of overall array orientation remained. There would be such an error regardless of the number of data samples taken. Given such an independence of the sheer number of data samples taken, we raise the question whether there is, in fact, a need to take more than one snapshot at all. Or can there be an optimal number of snapshots that should be taken? The answer to the latter question may lie beyond the scope of the present thesis, but we will conjecture here in passing that the answer may lie in the conceptual difficulties involved in describing non-rational quantities in terms of rational ones, and thus may have something to do with the theory of rational approximation.

Instead, we shall end our thesis by concentrating upon the possibilities of the single snapshot. However, the present chapter should be regarded as an interim stage in arriving at that goal. If our reasoning here is valid, it can be used as an argument in favour of filling in an array with as many hydrophones as possible. To that extent the trend of the present thesis may be seen to be moving away somewhat from analysis based upon time-series in favour of greater spatial filtering. Indeed, in the later stages of the thesis we shall argue in favour of a strong methodological bias in favour of the spatial at the expense of the temporal.

However, for the present analysis we use the phenomenon of the caustic curve as catalyst, and we find that it provides a conceptual support for some thoughts about the integrability of a system, thoughts which we shall develop later in the thesis.

Picture a multi-hydrophone array bent in a semi-circle with a single point source far away from it on its concave side. If we knew exactly where all the hydrophones were, and if the Nyquist condition were satisfied, *i .e.*

$$\frac{d}{\lambda} < \frac{1}{4}, \tag{163}$$

with d the hydrophone interspacing and λ the wavelength of the signal, and if there was no noise on the signal, we should have little difficulty in locating the point source. That is to say, we could calculate the direction of arrival of the cone of rays observed at pairs of neighbouring hydrophones round the array, and we could draw the hyperbolas from neighbouring points on the array that would all meet at the point source. More precisely, we could draw the hyperbolas corresponding to the phase differences observed at pairs of hydrophones. The hyperbolas meet in a point, and we have a 'pencil'. A pencil is a concurrence of lines (or, more precisely here, of hyperbolas).

Suppose, however, we did not know exactly where all the hydrophones were located. The direction of arrival lines we drew might not meet at a point; they might not be concurrent; they might not make a pencil. Instead, the lines might criss-cross haphazardly. It might be impossible to tell where the point source was in the resulting confusion.

One abstract explanation as to why we do not get a pencil when there is uncertainty about the locations of the hydrophones might be as follows. When we know for certain where the hydrophones are, it is easy to refine the two-dimensional information into the zero-dimensional point of the source. But if there is uncertainty about the locations of the hydrophones, then in an (admittedly naive) sense we are working with three dimensions rather than with two, the uncertainty being enshrined, as it were, in the third dimension.

Evidently we cannot whittle down the three-dimensional information to a zero-dimensional point so easily. But it might be possible to construe the data as containing information of a three-dimensional kind, and to distil from it a one-dimensional line rather than a zero-dimensional point.

Once again, the uncertainty of location of hydrophones is a different kind of 'noise' from the noise on the signal. Our estimation of the gross curvature of the array arises from a study of the past history of the speed and direction of the ship. But we cannot make repeated estimations of the shape of the array with any hope that, in the long run, and in the limit, our estimations will converge to the true shape. In any case, as we shall argue later on below, there is every theoretical justification for regarding the array shape as varying continuously with time. In the spatio-temporal *continuum*, the estimation of the gross shape of the array is neutral with respect to time. Whatever we might assume about the motion or otherwise of the source, there is ultimately no foundation for regarding the array as a stationary system. For the present, however, we assume that the array is stationary.

5.2 Noise Rejection Through Simple Integration

The amount of uncertainty of array shape we have to deal with is, however, a calculable one and is small compared with the kind of noise levels that are likely to contaminate the signal. With such a comparatively meagre quantity of uncertainty about the array shape we have a noise-to-signal ratio which is small enough for us to be able to envisage a simple integrating-out of the noise.

The question now arises how to construe the integral. We shall have more to say later about a 'philosophy of integration', but in the present chapter it may be sufficient to think of integrating in terms of a simple covering of a familiar or customary shape of area on a flat surface by a number of strips. Later on, of course, we shall have to recognize shapes other than strips and fill spaces with less familiar perimeters. In what follows in the present chapter it is suggested that an integral is taken over an appropriate area in the plane of the source and the array.

A caustic is a curve that joins the points of intersection of rays (*i. e.* the envelope of those rays) reflected off neighbouring points round a curved reflector. For convenience, let the angle of reflection of the rays be equal to the angle of incidence. That is not a necessary condition, but the point is that the angle of reflection should be chosen so that points of intersection of neighbouring reflected rays are guaranteed. For example, points of intersection are guaranteed if the array is semi-circular and the point source is out on its concave side. Thus we assume for convenience that our array is semi-circular. See Figure 11. In the Figure let point P be the location of a point source of light. Let the curve connecting points A and B be regarded as the half section of a semi-spherical reflecting surface with centre C.

The caustic has two branches which meet in a cusp, *i. e.* the two branches gradually bend round until they touch at a (double) point. In Figure 11 only one such branch is indicated. In terms of the Figure, the cusp is at point Q. In terms of a semi-circular multi-hydrophone array, the two branches of the caustic touch at a point, that is, if there is no uncertainty about the hydrophone locations. If there is no uncertainty about the hydrophone locations, then the position of the cusp will give uniquely the distance and direction of the source. But what happens when hydrophone locations are uncertain is that we find a line on which the cusp must lie, but we do know exactly where it is on the line. However, the line points in the direction

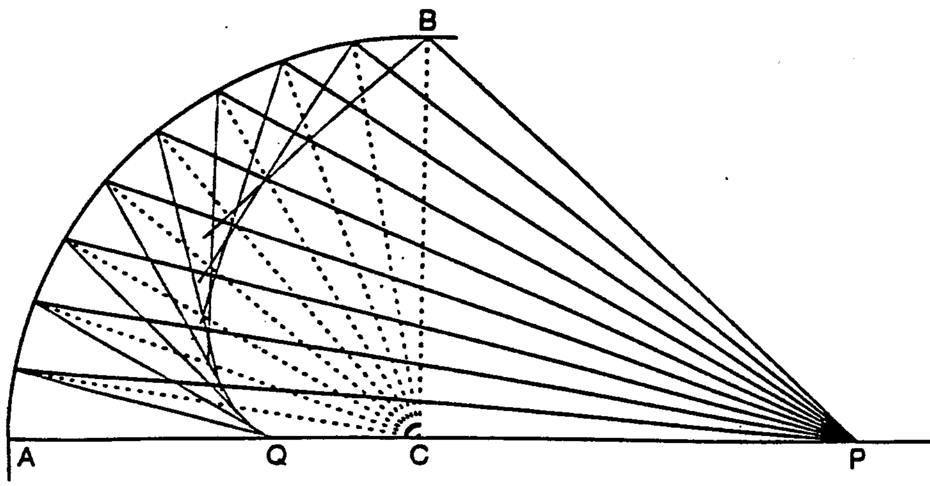


Figure 11: The caustic

of the source and therefore we can estimate the direction of the source as seen from the cusp. That line is the line that joins the source to the centre of the semicircle of the array. Furthermore, the line is perpendicular to the tangent to the array at the point of intersection of the line and the array. The cusp lies on that line through the source which intersects the array at right angles. It can readily be shown that the two caustic branches will converge more and more to that line. In short, when there is no uncertainty, we obtain a point which conveys direction and distance of the source; but when there is uncertainty, we only get a line on which the source must lie. In the former case a point is determined uniquely, while where there is uncertainty, a line only may be extrapolated from the data (*i. e.* interpolated among points).

Thus we have a convergence for direction. What now about distance? Here the integral applies. It can easily be shown that for a given direction from the centre of the semi-circle there is a one-to-one relationship between the length of a caustic branch and the distance of the source from the centre. For example, if there is no uncertainty about the array shape, then a point-source at an 'infinite' distance will produce a cusp exactly halfway along the radius of the array, and the caustic will be at its longest. On the other hand, if the point source is right in the centre of semicircle of the array, then the cusp will be in the same place and the caustic will have no length, but be just a point. As the point source is moved away from the array, the caustic curve gets longer and more curved.

Of course, if there is no uncertainty about the array shape, then there is no need to measure the length of the caustic to determine the distance of the point source, for both direction and distance are contained in the location of the cusp. But if there is uncertainty about the array shape, then we must take the length and curvature of the caustic into account in order to estimate the position of the source.

Clearly it will not be feasible to measure the length of the caustic straight off. Instead, it must be done by degrees, indeed, by a limiting process. A considerable problem arises, however, if we try to characterize the curvature of the caustic. A solution offered here is to calculate the size of the area between the caustic curve and the array. The more hydrophones there are available, the more strips can be drawn, and therefore the more 'comprehensive' will be the integration. The size of the area has a direct relationship with the distance of the point-source along the cusp-line. It is hoped that the comparatively minor uncertainties of array shape will be ironed out in the integration. Thus the 'noise' is 'integrated out'.

The longer the length of curve of the array, the further out we may be able to measure. The more hydrophones filling up the array, the more precisely we can measure the distance, and the better the convergence to the direction of the source.

It may be noted that the caustic curve is 'strictly concave', *i. e.* it has a single minimum, which is global. Likewise, the directional convergence is 'strictly convex', *i. e.* has a single maximum, which is global. These properties will be important in the binning algorithm, which will be the centre-piece of the present thesis later.

For convenience, we took the angle of reflection to be equal to the angle of incidence. That does not have to be the case generally. Our underlying concern is to minimize the penalty of the uncertainties by ensuring that, first, we can form a caustic at all, and, second, one which is as close to the array as possible. But if we were to project the direction-of-arrival hyperbolas back to the source (*i. e.* with zero angle of reflection), then the result would be difficult to interpret. With an angle of reflection of zero, it would be particularly difficult if the point source were very far away, because any uncertainties of array shape, even quite small ones, would be greatly exacerbated. However, if we can keep the caustic close in to the array, then the uncertainties will not contribute anything like so much to the error. Put another way, if one tries to project lines into the far distance, the distortion is much magnified, but if one keeps them close by, one gets less distortion. By the same token, perhaps, a contact lens is much better than a spectacle lens because it is so much closer to the cornea.

We need to choose a suitable curvature for the array, and then we need to choose appropriate angles of reflection. But suppose the array is not actually shaped so conveniently? A solution may be to map the array on to a virtual array of the desired shape. Of course uncertainties will be exacerbated if this is attempted, but it may work if the actual array is not so very much different in shape from the virtual one. And again, the distortions introduced by doing this are as nothing when compared with the penalties incurred by trying to project the reflected rays too far away from the array. It is all a matter of degree. Then there is the possibility of flexing the array successively into different shapes and comparing the results. However, since we have assumed a stationary source for this analysis, we believe that any further elaboration of the uses of the theory of the caustic should be the subject of further work beyond the bounds of the present thesis. For example, if the tangent to the array at the point opposite to the source were to be moved slightly, owing to a change in position of the source, then it will no longer be at right-angles to the line joining it to that point. The point on the array at which the tangent is now perpendicular to the line joining it to the source is therefore shifted.

Our consideration of the caustic curve may, for example, have implications for the tracking of moving targets, but that cannot be anything more than a peripheral object for the present thesis.

Apart from that, it seemed that the properties of the caustic curve could be utilized only with a somewhat narrow choice of array shapes. For this reason too the particular methodological initiative was abandoned with the hope of reviving it in a future examination.

However, the idea of employing the theory of the caustic did lead on to the notion of integration, and of 'integrating out' noise in particular.

As an exercise in integration the simple sum of strips envisaged here may not convey all the facets of a signal. We need an integral which takes account of many aspects of the problem, *i. e.* it needs to be based upon a more differentiated dissection of the problem. However, the need to have as many hydrophones as possible as an aid to integration, in order to have as many strips as possible, is an outcome of the present argument. The relationship of integration and differentiation and the implications of it for the number of hydrophones required for a given problem constitute the theme of the remainder of the thesis.

The analysis in this chapter has been based upon an assumption that the system array-plus-source is stationary long enough to carry out the processing. From now on, however, we shall concentrate our attention upon single snapshot processing and, without involving ourselves too deeply with the related field of target tracking, venture a general concept of a 'quasi-stationary' system of sources and hydrophone array. In particular, motion between sources and hydrophones must be sub-sonic.

6 Single Snapshot Frequency-Source Location with a Multi-Hydrophone Array of Uncertain Shape

6.1 Recapitulation and Introduction

Thus far in our research we have demonstrated what we believe to be a novel method, using three hydrophones in known locations, for determining uniquely the location of a single point source in the absence of random noise and an N-process for estimating its location in the presence of noise. In the previous chapter we also considered some properties of a statistically convergent process for locating point sources by means of an array, the global curvi-linearity of which was given, but of which the local disposition of hydrophones was subject to unknown random perturbation. All novel approaches ventured thus far are founded upon time-series and are largely conditional upon the source being stationary with respect to the array whilst the samples are taken. The methods proceeded from an ambition to be able to locate a point source arbitrarily close to the antenna, but the distant range attainable by them has turned out to be little more than three or four antenna-lengths. We should like to extend the range. As a requirement, a rule-of-thumb distance of about ten times the length of the antenna away developed latterly in convention between the author and his Sponsor.

Because the array shape with which the present thesis is properly concerned is given in terms which reflect the history of the speed and direction of the towing vessel rather than the limit of a convergent process of estimation based upon many repeated observations, we believe that, theoretically, little is to be gained from a using time series with a few hydrophones that cannot be supplied as well by using a single snapshot of many hydrophones.

Furthermore, if repeated observations cannot form the basis for rejecting noise on hydrophone locations, we believe that we must think not in terms of measuring the phase-angle of a source at a hydrophone, but of supposing a measured pressure to lie somewhere in a band of phase angles. Since the gross array shape is given with an unalterable amount of uncertainty, there is no basis for supposing that repeated observations over time can improve upon a guess as to which phase-angle an acoustic pressure measured at a particular hydrophone might be associated with.

6.2 Phase Binning

In a final-year undergraduate project the present author had developed a method of 'optimal binning'.[57] The problem was to elicit the periodicities of *Sigma Scorpii*, a close binary star, from noisy data. In particular, the measurements of the radial velocity of the star-system had been taken over a long period of time, of the order of decades indeed, and at wildly irregular known intervals.

It was found that the irregularity of the sampling intervals was responsible for the presence of aliases of the true periods on the periodogram obtained from applying the Fourier Transform to the data. The present author conducted a simple experiment with artificial data. First, he applied the Fourier Transform to samples taken at regular intervals. The periods of the artificial data appeared as expected with only their expected integer multiples as aliases. Next, he applied the Fourier Transform to the artificial data at the same irregular intervals as those at which the radial velocities of *Sigma Scorpii* had been measured. This time, however, not only did the true periods and their integer-multiple aliases appear, but also a number of aliases associated with other, spurious periods occurring on the periodogram.

Thus the present author had had some experience of dealing with irregular sampling intervals before embarking upon the present research. However, the difference was that, with *Sigma Scorpii* the irregularity was known, whereas with the towed array it was not.

In his dissertation upon *Sigma Scorpii* the present author developed a rapid method for estimating periods where the sampling was irregular. For an arbitrary period, the entire sample-series was 'folded', for which process the analytical expression follows.

6.2.1 Analytical Expression for the Folding Integral

Let $f(t)$ be an arbitrary signal defined in $0 \leq t \leq T$. Define

$$F_K(\tau, P) = \frac{1}{K} \sum_{r=1}^{K} f(\tau + (r-1)P) \tag{164}$$

[57]See G. W. Sweet, *The Periodicities of* Sigma Scorpii, Dissertation, Oxford Polytechnic, 1990

defined in $0 \leq \tau \leq P$, where

$$K = \left[\frac{T}{P}\right]. \tag{165}$$

Construct

$$FI(P) = \frac{1}{P}\int_0^P F_K^2(\tau, P)\, d\tau. \tag{166}$$

Equation 166 is the definition of the folding integral.

Each sample is assigned a phase 'bin' rather than a phase point. A bin is a band of phase points. When every 'folded' sample had been assigned its appropriate bin, the contents of each bin were averaged and squared then added together. If a period of the radial velocity had been found, the sum was large. But if the test period was not a period of the source, the contents of the bins tended to cancel each other out, and the sum was negligibly small.

The author found that the optimal number of bins was three, and he will justify this assertion with a theorem and proof below.

The phase-binning method is quicker than a discrete Fourier Transform because it does not involve convolution with sines or cosines. Instead, it employs only the 'mod' operation.

The algorithm, when applied to the problem of the point source and the towed array, proceeds as follows. The wavelength of the source is divided into equal parts called 'phase bins'. The choice of bin in which a pressure, measured by means of a hydrophone, is placed is determined by the supposed frequency and location of the source of interest. Naively, if the binning is right, the measured pressures will reinforce each other. But if measured pressures are assigned to the wrong bins, they will tend to cancel each other out.

Because of the uncertainty of hydrophone location, a measured pressure should be thought of as containing potential phase angle information about a range between points on the real line rather than about a unique point. In other words, we regard data coming from hydrophones as being capable of yielding 'fuzzy' information at best.

For a single snapshot, in the absence of noise, there must be at least three bins per wavelength. Three is the minimum number of bins required to corroborate a wavelength without regard to the phase angle of the signal. If d is the distance between hydrophones reached consecutively by a wavefront, the condition

$$d < \frac{\lambda}{2} \tag{167}$$

applies.

In general, while the period or location of the source of a signal may be conceived of in terms of non-rational numbers, they cannot in practice be measured or characterized except in terms of rational ones. To this extent, all solutions to such problems are optimal at best, and any analysis based upon the convolution of a measured signal with a notional signal is limited. The method of phase binning requires the employment of notional periods and locations but not the convolution with a notional signal. Indeed, the convolution with a notional signal may in any case be limited by the uncertainty of hydrophone location, especially if the convolution was of such a kind that phase angle was requiring to be matched with phase angle, *i. e.* point with point. Instead, we propose a sort of 'convolution' of a band with band of points. Indeed, if the parallactic angle of source and array is sufficiently wide to allow a good sense of the distance of the source, we believe it possible to think of our 'convolution' in terms of region with region. In other words, we are trying to stretch the notion of convolution from point to line and then to area, and thus expand upon the dimensional potential of convolving.

In presenting now the algebraic elements of our method, we hope not to do violence to our policy of specializing in the location of point sources near enough to be locatable in any event if we treat plane waves in what follows. This is done for passing convenience. We shall go on to consider the general case of palpably spherical spreading subsequently, for which a more differentiated approach will be required.

6.3 The Ratio P/a^2

In the absence of noise and in a medium of infinite extent let p be measured pressures with

$$p = a\sin\theta \ . \tag{168}$$

Let B_n $(n = 1, 2, 3)$ denote phase bins. Then

$$B_1 = (\overline{p})_1 = \frac{1}{2\pi/3}\int_0^{\frac{2\pi}{3}} a\sin\theta d\theta = \frac{9a}{4\pi} \tag{169}$$

$$B_2 = (\overline{p})_2 = \frac{1}{2\pi/3}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} a\sin\theta d\theta = 0 \tag{170}$$

$$B_3 = (\overline{p})_3 = \frac{1}{2\pi/3}\int_{\frac{4\pi}{3}}^{2\pi} a\sin\theta d\theta = -\frac{9a}{4\pi} \ . \tag{171}$$

Construct the power integral P as follows:

$$P = (B_1)^2 + (B_2)^2 + (B_3)^2 = \frac{81a^2}{8\pi^2} = 1.0259a^2. \tag{172}$$

It will be shown below that, in general, with N bins

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}) \tag{173}$$

in the limit either as $\lambda \to \infty$ with length of array l (assuming, for convenience, a straight antenna of known orientation) and the number of hydrophones M constant or, with a given λ and l, as $M \to \infty$.

To treat noise as effectively as possible, we must make sure that $\frac{P}{a^2}$ is kept as small as possible. This is achieved when $N = 3$, which is the smallest possible value for N.

6.4 The Binning Procedure in Practice

The measured pressures p are folded in the manner of Equations 164, 165 and 166 above. In the case of the single snapshot, the pressures p_m measured at the M hydrophones are folded in respect of the number of whole test wavelengths λ_0 intervening between the hydrophones and the supposed location of the source, the remainder giving the phase angle of the signal. The aim is to determine which of the three phase bins of λ_0 is appropriate for each measured pressure.

If the reference wavelength λ_0 is the same as the true wavelength λ, then the folding integral will be maximal. If it is not, then pressures p will tend to cancel each other out.

The discretized form of the Bs in Equations 169, 170 and 171 above is as follows

$$B \approx \frac{1}{\mu} \sum_{\rho=1}^{\mu} (p_\rho + \epsilon_\rho), \tag{174}$$

where μ is the number of pressure measurements assigned to the particular bin, the p_ρ the true pressures and the ϵ_ρ the errors due to noise of the kind treated in Section 4.5 above. In the model employed in the Appendix (Section 10 below), the true pressures are as in Equation 88 above, while the errors are given as follows:

$$\epsilon_\rho = R\frac{a_0}{\rho}, \tag{175}$$

where R is the noise level times a random number between $+1$ and -1.

The general form of $P(\lambda_0, x_0, y_0)$ differentiable in terms of reference wavelength λ_0 and test-source coordinates (x_0, y_0) is as follows:

$$P(\lambda_0, x_0, y_0) = (B_{\mu_1}(\lambda_0, x_0, y_0))^2 + (B_{\mu_2}(\lambda_0, x_0, y_0))^2 + (B_{\mu_3}(\lambda_0, x_0, y_0))^2, \tag{176}$$

with $M = \mu_1 + \mu_2 + \mu_3$. P will be maximal for the correct wavelength and source coordinates.

6.5 The Merits of an 'Exponential' Spacing of Hydrophones

The two Figures following have been included to point up what we believe to be an advantage of spacing hydrophones non-equidistantly along a straight array. Suppose the antenna were straight, and that a far distant point source were radiating upon the antenna at 'end-fire', *i. e.* along the line of the antenna. Suppose, further, that the wavelength of the source was $2d$. Then all items in each bin would have the same value, and no integration could be undertaken over the period of the signal in the sense of the Method outlined above.

We believe that singularities arising for that and such reasons with an array of equidistantly-spaced hydrophones may be treated by spacing the hydrophones non-equidistantly. The graph in Figure 12 shows the plot of a curve which is not smooth. But it has become much smoother in Figure 13, where the coordinates $(x_m, 0)$ $(m = 1, 2, ...)$ of the hydrophones h_m are given as follows:

$$x_m = 1000(e^{\frac{m-1}{1000}} - 1) \ . \qquad (177)$$

The thus 'exponential' spacing would seem to allow a better spread of values across a bin. As we shall argue shortly, a more or less good spread of values across bins becomes possible with greater gross flexing of the array.

With Figures 12 and 13 there is no noise on the signal, but there is an uncertainty of one part in a thousand in the coordinates of the hydrophones.

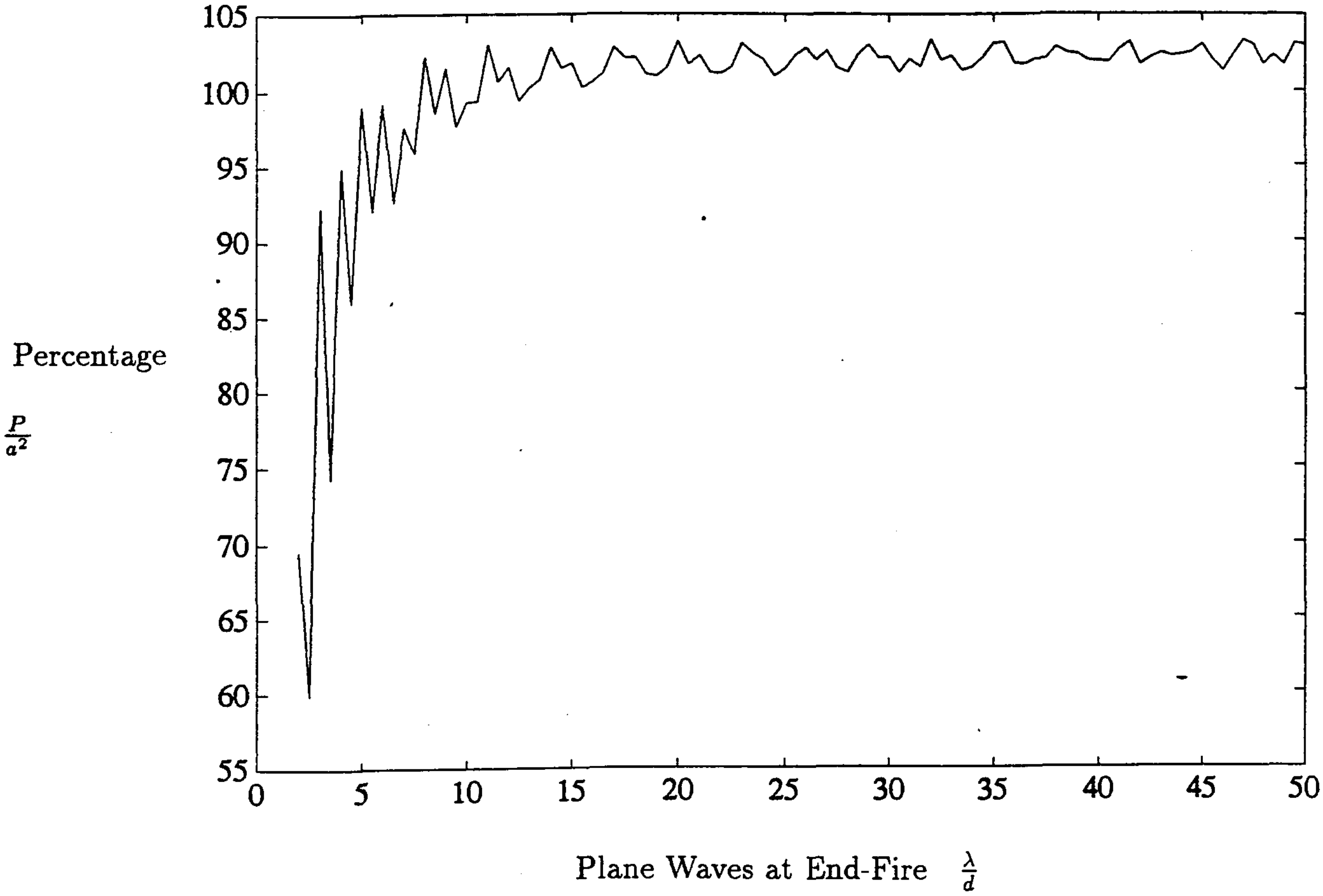


Figure 12: Singularities with even spacing of hydrophones

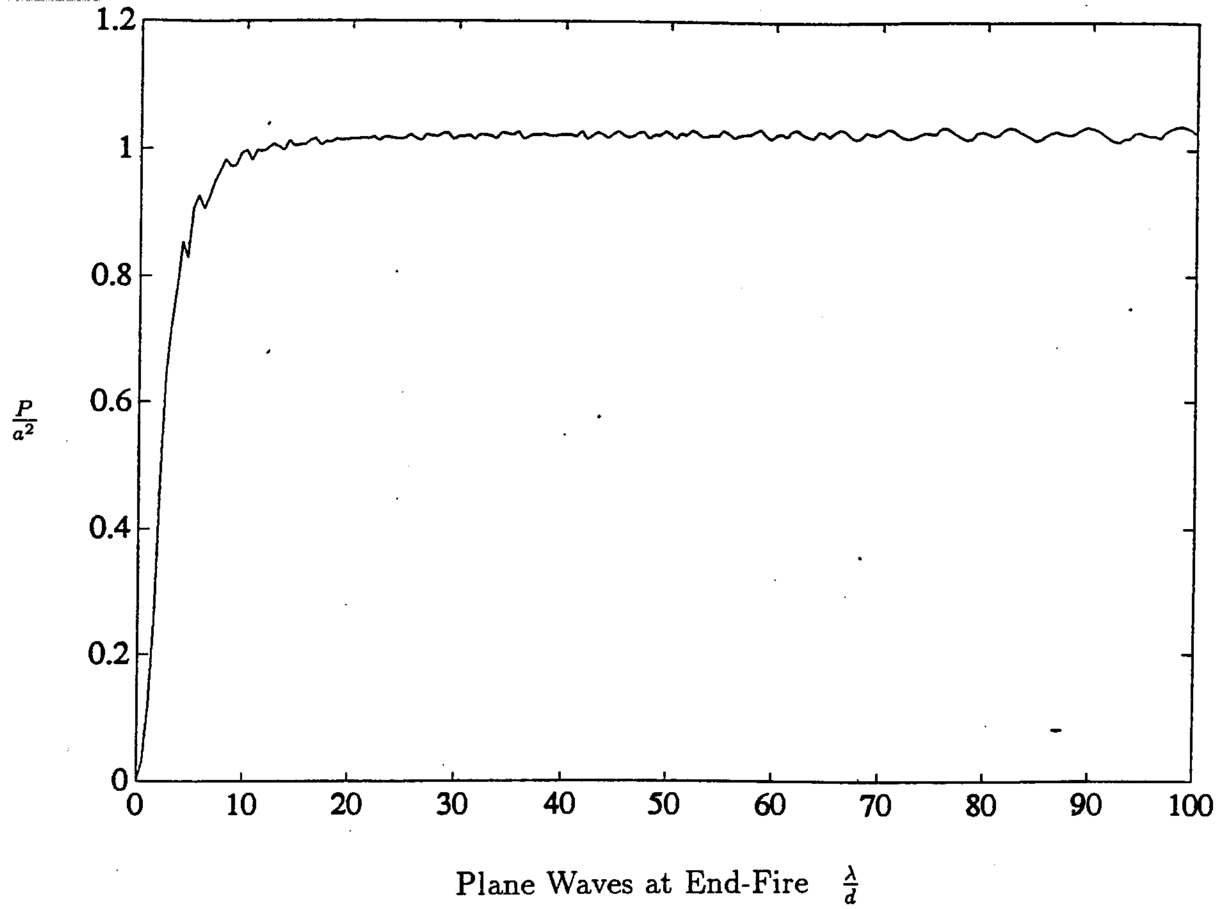


Figure 13: Treatment of singularities with 'exponential' spacing of hydrophones

6.6 Determination of Amplitude of Plane Wave

For simplicity, let the array be assumed to be straight, the hydrophones spaced equidistantly, and the source sufficiently far away for the wave front to be planar upon arrival at the array. Let L be the length of the array, M the number of hydrophones, λ_0 the wavelength of the plane wave, a the amplitude of the wave, θ the angle of incidence of the wave upon the antenna, N the number of bins and P the power in the binning periodogram at $\lambda = \lambda_0 \sec\theta$.

6.6.1 Theorem

In the limit as $M \to \infty$, and for values of λ_0 for which $L\cos\theta/\lambda_0$ is an integer K,

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}) \ . \tag{178}$$

E. g.

$$\begin{aligned} \frac{P}{a^2} &= 1.0259, \quad N=3 \\ &= 1.6211, \quad N=4 \\ &= 2.1879, \quad N=5 \\ &\sim \frac{1}{2}N, \quad \text{large } N. \end{aligned} \tag{179}$$

Thus $\frac{P}{a^2}$ is least when $N = 3$. It can be shown that this makes the estimate of a^2 least sensitive to noise on the signal. Hence the term 'optimal binning' when $N = 3$.

6.6.2 Proof

$$p(\underline{r}, t) = a\cos(2\pi\nu_0 t - 2\pi\underline{r}.\underline{l}/\lambda_0 + \phi_0) \ , \tag{180}$$

where $\underline{l} = (\cos\theta, -\sin\theta)$.

Now $\underline{r} = (x, 0)$ at point x on the array, therefore at time $t = t_0$

$$p(x) = a\cos(\frac{2\pi\nu}{\lambda} + \phi) = a\cos(\chi + \phi) =, \quad \lambda = \lambda_0\sec\theta, \tag{181}$$

where

$$\begin{aligned} \chi &= \frac{2\pi x}{\lambda}, \\ \phi &= -2\pi\nu_0 t_0 - \phi_0. \end{aligned} \tag{182}$$

In binning procedure (in limit $M \to \infty$)

$$\begin{aligned} B_1 = (\overline{p})_1 = \frac{N}{2\pi}\int_0^{\frac{2\pi}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{2\pi}\left[\sin(\chi+\phi)\right]_0^{\frac{2\pi}{N}} \\ &= \frac{Na}{2\pi}\left(\sin(\frac{2\pi}{N}+\phi)-\sin\phi\right). \end{aligned}$$

Therefore

$$B_1 = \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos(\phi+\frac{\pi}{N}). \tag{183}$$

Similarly

$$\begin{aligned} B_2 = (\overline{p})_2 = \frac{N}{2\pi}\int_{\frac{2\pi}{N}}^{\frac{4\pi}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos(\phi+\frac{3\pi}{N}) \\ &\vdots \\ B_n = (\overline{p})_n = \frac{N}{2\pi}\int_{\frac{2\pi(n-1)}{N}}^{\frac{2\pi n}{N}} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos\left(\phi+\frac{(2n-1)\pi}{N}\right) \\ &\vdots \\ B_N = (\overline{p})_N = \frac{N}{2\pi}\int_{\frac{2\pi(N-1)}{N}}^{2\pi} a\cos(\chi+\phi)\,d\chi &= \frac{Na}{\pi}\sin(\frac{\pi}{N})\cos\left(\phi+\frac{(2N-1)\pi}{N}\right) \end{aligned}$$

$\Rightarrow$

$$P = B_1^2 + B_2^2 + ... + B_N^2 = a^2(\frac{N}{\pi})^2\sin^2(\frac{\pi}{N})\sum_{n=1}^{N}\cos^2\left(\phi+\frac{(2n-1)\pi}{N}\right) = \frac{1}{2}N \tag{184}$$

from Lemma below (all integer $N > 2$), therefore

$$\frac{P}{a^2} = \frac{N^3}{2\pi^2}\sin^2(\frac{\pi}{N}), \quad \text{QED.} \tag{185}$$

6.6.3 Lemma

$$\sum_{n=1}^{N} \cos^2\left(\phi + \frac{(2n-1)\pi}{N}\right) = \frac{1}{2}N, \quad \text{all integer } N > 2. \tag{186}$$

6.6.4 Proof

$$\begin{aligned} I_N = \sum_{n=1}^{N} \cos^2\left(\phi + \frac{(2n-1)\pi}{N}\right) &= \frac{1}{2}\sum_{n=0}^{N-1}\left(1 + \cos\left(2\phi + \frac{2\pi(2n+1)}{N}\right)\right) \\ &= \frac{1}{2}N - \frac{1}{2}\Re \sum_{n=0}^{N-1} e^{2i\phi + \frac{2\pi i}{N}} . e^{\frac{4\pi i n}{N}} \\ &= \frac{1}{2}N - \frac{1}{2}\Re e^{2i\phi+\frac{2\pi i}{N}} \underbrace{\sum_{n=0}^{N-1} e^{\frac{4\pi i n}{N}}}_{= \frac{e^{\frac{4\pi i N}{N}} - 1}{e^{\frac{4\pi i}{N}} - 1} = 0, \text{ all integer } N > 2} \end{aligned}$$

$\Rightarrow$

$$I_N = \frac{1}{2}N, \text{ integer } N > 2, \text{ QED.} \tag{187}$$

N. B. When $L\cos\theta/\lambda_0 \neq$ integer, then $B_n \neq (\overline{p})_n$ exactly in the binning procedure; for, after taking the mean of the sums over the $K = [L/\lambda]$ wavelengths λ (where $\lambda = \lambda_0 \sec\theta$) along the array, there is a quantity of order K left over at the far end.

6.7 Limits

In practice, it is found that the general case of a non-straight array as well as uncertainty in hydrophone location contribute to remove any bias in averaging that might arise if many pressure measurements reflected a common phase angle.

Such a bias can be seen most clearly with sources at greater distances from the array. At distances where the amplitude of the signal is much the same whichever hydrophone is involved, P is sensitive to phase angle, and a periodicity can be seen in the P as successive distances are tried out in the binning, but no P is found that is maximal. At lesser distances, on the other hand, the amplitude of the signal is different according to which hydrophone is in question, and Equation 88 above (rather than Equation 180) applies, in which a_0 is the intrinsic amplitude of the signal at the instant of interest. The average of the measured pressures in a bin turns out to be much closer to an integral B, and thus the maximal P is much the more conspicuously so. Because there are as many different amplitudes as there are hydrophones, any tendency for many hydrophone readings to reflect the same values is impeded, and a better spread of values is allowed in the bin.

6.8 Possible Application of Theorem to 'Beamforming' and Convolution

It may be useful in future work to investigate the extent to which Conventional Beamforming and convolution may be interpreted in the light of our Theorem above. We might, for instance, liken the (in terms of the present thesis)[58] 'undifferentiated' summing of measured acoustic pressures along the array as a one-bin exercise, *i. e.* as a square of sums with $N = 1$. On the other hand, in order to capture the periodicity, we require a minimum of three bins, else phase-angle must be taken into account. Aliases do not arise with our present binning method, although we shall not be able to prove that within the scope of the present thesis, but will regard such a proof as an exercise for further work. On the other hand, the 'beamforming' methods do not escape aliases, as can readily be proved.

We saw that, as N tended to infinity, so the ratio P/a^2 would settle to $\frac{1}{2}N$.

[58]See particularly Chapter 7 below.

We might liken the use of N bins to a 'weighting', 'shading' or 'windowing' convolution. Such a convolution might be thought of as an assigning of a special bin to each data-point. The problem here, as we indicated above, may be that the a^2 may become increasingly difficult to elicit from a very noisy P as the ratio P/a^2 rises. The a^2 might, as it were, become increasingly buried in noise, a noise created by the method used itself. We might be able to show that with noise, and particularly the irreducible noise on sensor location, it is better not to convolve point with point, but rather band with band.[59] We believe that our present method is the more effective and efficient for doing that.

[59]Indeed, it might be argued that where both 'range' and 'bearing' are requiring to be discovered, area with area need to be convolved. This will be the sense of our emphasis upon the radiality of a source in arguments below.

6.9 Graphical Evidence of the Binning Method

We show now graphical evidence of our binning method. Because of our concern that the activity of 'locating' a point source should include finding its distance as well as its direction, the graphical evidence following has been selected to demonstrate the effectiveness of the Method in this regard in particular. In general, it may well be true that the many published algorithms available specialize in direction-finding, and we have referred to such a tendency from time to time in our thesis. For that reason, we assume, for the first graph, for example, that we have the direction of the source already. We shall see in later graphs the effectiveness of the Method for getting the right direction too.

On the first graph is a plot of the binning procedures applied to the data along an interval on a line, an interval within which, for present convenience, it is known that the source lies. Notice that we have chosen an incremental step of $37d/40$, where d is the unit of distance measurement of the system. We have no particular reason for choosing this number, but we wish to 'get away' somewhat from uniformity and 'rationality' in our use of numbers in our analysis. We said earlier, for example, that we should in general prefer not to have uniform sampling of the signal, and that we regarded an 'exponential' filtering of the data as the *limes* to strive for. The incremental step $37d/40$ applies for all fourteen graphs following. An example of an incremental step of unity may be found in the Appendix at the end of the thesis, where a listing of the computer program that generated it is printed too. For reasons which might be better explained in further work a smoother curvature of profile is obtained with $37d/40$ than with d. Naively, we wish to try out as many distances as possible. But we cannot try out them all. Since a distinct pattern, indeed a quasi-periodicity, can be seen with the Method, we wish to 'sample' the pattern as well as possible, and the best way of doing it is to sample at a 'non-rational' interval.

6.9.1 A Principle of Falsification and the Use of a 'Non-rational' Sampling Interval

It is evident that where it is given that a phenomenon has a periodicity, the period is bound to yield eventually to a double sampling, such that the ratio of one sampling interval over the other is an irrational number. We qualify Nyquist in that we regard the satisfying of the Nyquist condition as an attempt to verify the supposition of a particular period, while we believe that an approach designed to reach the truth through falsification has merit too. Although in practice it is impossible to achieve a ratio of sampling intervals which is an irrational number, nevertheless we argue that we might try to show that a period which we had elicited from the data was the correct one by proving, say, that half of it was not a period of them. Strictly speaking, such a falsification could be achieved only with an irrational ratio of sampling intervals. To the extent that such a strategy cannot be carried out in practice, it may be that the 'verification' approach is to be preferred. It is a matter to which we intend to return in future work.

In any case, the nature of the 'periodicity' in our present results must remain obscure in the limits of competence of the present thesis. But to the extent that further work may go on to establish a repeatability, we should prefer a sampling strategy that may be more efficient than one based exlusively upon the unit of measurement of the system.

Falsification applies to the whole use of the Method for locating a point source. We must look at all the points at which the source might reasonably lie. However, given the presence of a source, along a given line a maximum should be apparent, and, if there is merit in our distinction between 'syntactical' and 'paratactical' sums (to be introduced in the next chapter), a maximum exists if a source is present, and that maximum may be associated uniquely with the location of the source.

For all the fourteen graphs following, as well for the graph preceding the computer-program listing in the Appendix, an array of sinusoidal shape was chosen for convenience. Variously, ten d to fifty d is its amplitude, and that is recorded on the figures themselves. On the other hand, the example shown in the Appendix arose from an array amplitude of a hundred metres, which had evident implications for the sensitivity of the analysis. On all graphs, hydrophones are located with a random uncertainty of one part in a thousand in respect of their x- and y-coordinates. For convenience, the nominal spacing of the x-coordinates is the uniform unit distance d. By this means, we have hoped to give some consistency to a degree of 'randomness' not only in respect of the irreducible uncertainty, but also in respect of the gross curvature and disposition of the array before noise. Again, we do not believe that there is any compelling reason for uniformity of sampling.

Each array used in the graphical evidence has 693 hydrophones. Thus the baseline presented to a source could be $693/d$ at most.

For the first graph it had been decided to place the point source at a distance of $5221/d$ and at angle of $\pi/3.1$ degrees to the baseline, the origin falling together with the hydrophone nearest the towing vessel. The wavelength of the source was chosen as $856/d$. The amplitude of the array was determined to be $50/d$. For this graph only there was no noise on the signal. The profile of the graph is not smooth, but there is a unique 'global' maximum where we should hope it to be.

The second graph was based upon slightly different ingredients, with the noise-to-signal ratio now unity. A certain amount of 'furring' is beginning to smooth the profile in a useful way.

The third graph and the five following were based upon a true source location of $5184/d$ distance with baseline angle of $\pi/3.2$. The first of this run of six reflects a noise-to-signal ratio of $(15)^2$, while the remaining five reflect noise-to-signal ratios of $(30)^2$. 'Noise-to-signal ratio' applies to the noise on the signal, of course, not to that on the hydrophone locations.

The effects of not applying the true angle of the point source are shown in the last four of the run of six. We believe that these graphs show that the maximum binning sum of squares of sums (or 'integral' in the sense to be developed below) may be associated with the true location of the source.

The first eight graphs arise from a comparatively generous width of baseline 'seen' by the source. The ninth graph, on the other hand, shows a nascent difficulty when the baseline becomes narrow. In general, the narrower is the baseline, the more difficulty there is in resolving distance. Such difficulty is reflected in the increasing uniformity of the plot. With a wider baseline, on the other hand, it can be shown that the periodicity is 'stretched' and distorted in a hyperboloid fashion. It has proved to be very difficult to obtain an analytical account of this hyperboloid effect, particularly since our general approach with the Method was to let it perform with an arbitrary array shape, rather than begin with a calculable or uniform one, and then perturb it gradually. We propose to pursue the point, however, in further work.

The tenth graph shows the benefit of widening the baseline at 'end-fire' by increasing the amplitude of the sinusoidal array to $67/d$.

The following three graphs show the decreasing power of the Method to resolve distance, even with a comparatively favourable width of baseline.

The final graph has been included to show that the Method is effective for frequency of 1.5kHz, which is twice the rule-of-thumb frequency to aim for which emerged latterly in the author's discussion with the Sponsor.

For convenience, we have shown graphical evidence for a single point source only. The distinguishing of more than one point source of the same frequency will be a function of obtaining a better understanding of the hyperboloid structure of the ambient acoustic field mapped by the Binning Method than has been possible within the scope of the present thesis. However, early work by the author would appear to show that iso-contours can readily be associated with the maximum in their middle, that there is a reasonably small difficulty of confusing one such system with another, either in whole or in part, provided there is not an unreasonable surfeit of them.

It can easily be shown that the presence of signals with other frequencies and amplitudes does not interfere with the Binning Method. Indeed, such signals get 'chopped' up and mixed about by the discriminatory binning to such an extent they can be said to contribute noise rather than signal to the construction of the binned sums.

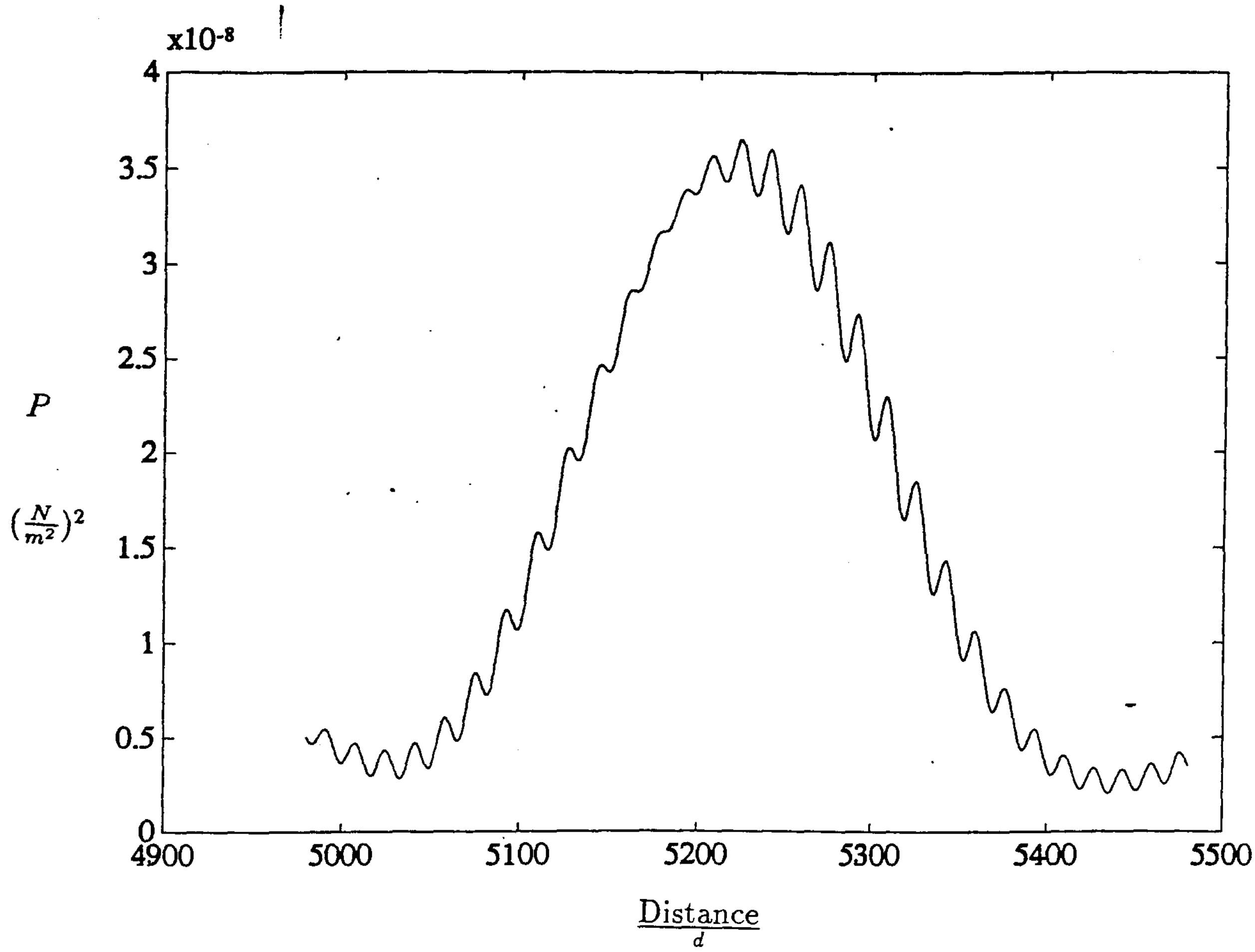


Figure 14: Located by 'binning method': point source at a distance of $\frac{5221}{d}$ and angle of $\frac{\pi}{3}$ degrees with $NSR = 0$

In Figure 14 the point source has frequency 856Hz. There is no noise on the signal. The hydrophone array has 693 elements and is shaped sinusoidally, with an amplitude of $\frac{50}{d}$. The uncertainty in hydrophone location is one part in a thousand. The search has been conducted along the true bearing.

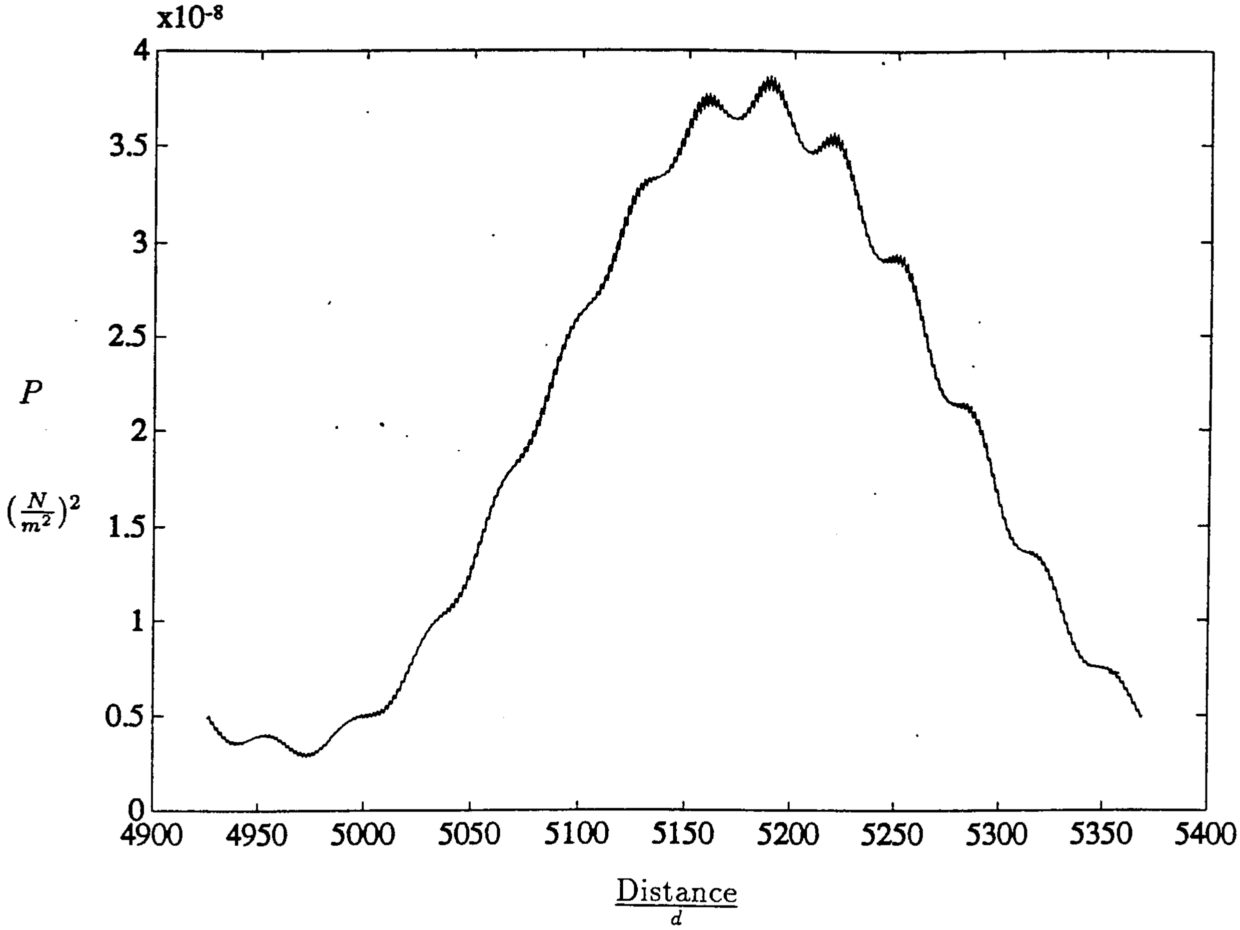


Figure 15: Located by 'binning method': point source at a distance of $\frac{5184}{d}$ and angle of $\frac{\pi}{3.2}$ degrees with $NSR = 1$

In Figure 15 the point source has frequency 791Hz. The noise-to-signal ratio is unity. The hydrophone array has 693 elements and is shaped sinusoidally, with an amplitude of $\frac{47}{d}$. The uncertainty in hydrophone location is one part in a thousand. The search has been conducted along the true bearing.

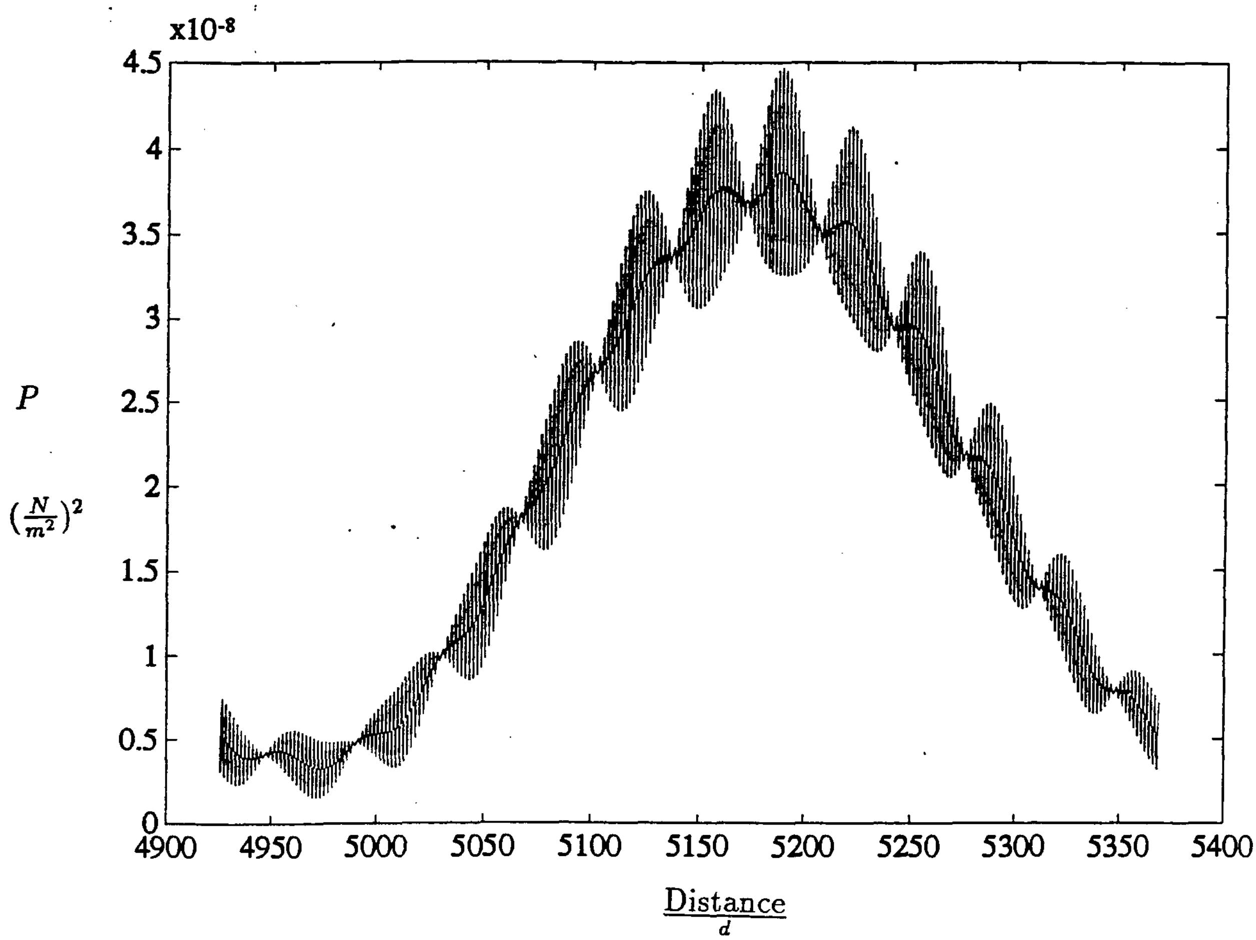


Figure 16: Located by 'binning method': point source at a distance of $\frac{5184}{d}$ and angle of $\frac{\pi}{3.2}$ degrees with $NSR = (15)^2$

In Figure 16 the point source has frequency 791Hz. The noise-to-signal ratio is $(15)^2$. The hydrophone array has 693 elements and is shaped sinusoidally, with an amplitude of $\frac{47}{d}$. The uncertainty in hydrophone location is one part in a thousand. The search has been conducted along the true bearing.

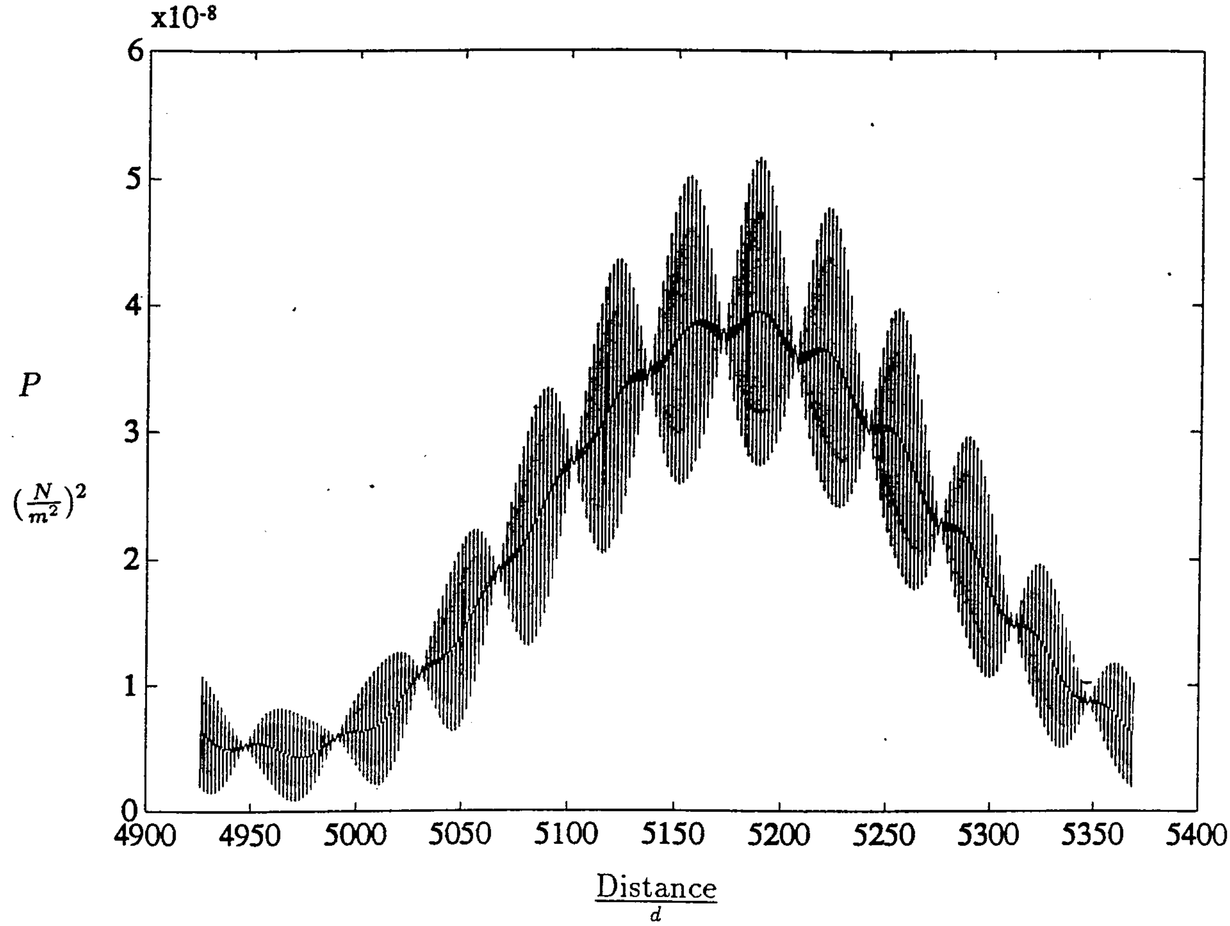


Figure 17: Located by 'binning method': point source at a distance of $\frac{5184}{d}$ and angle of $\frac{\pi}{3.2}$ degrees with $NSR = (30)^2$

In Figure 17 the point source again has 791Hz., but the noise-to-signal ratio has been increased to $(30)^2$. Apart from that, the characteristics of the system are the same as those for Figure 16 above.

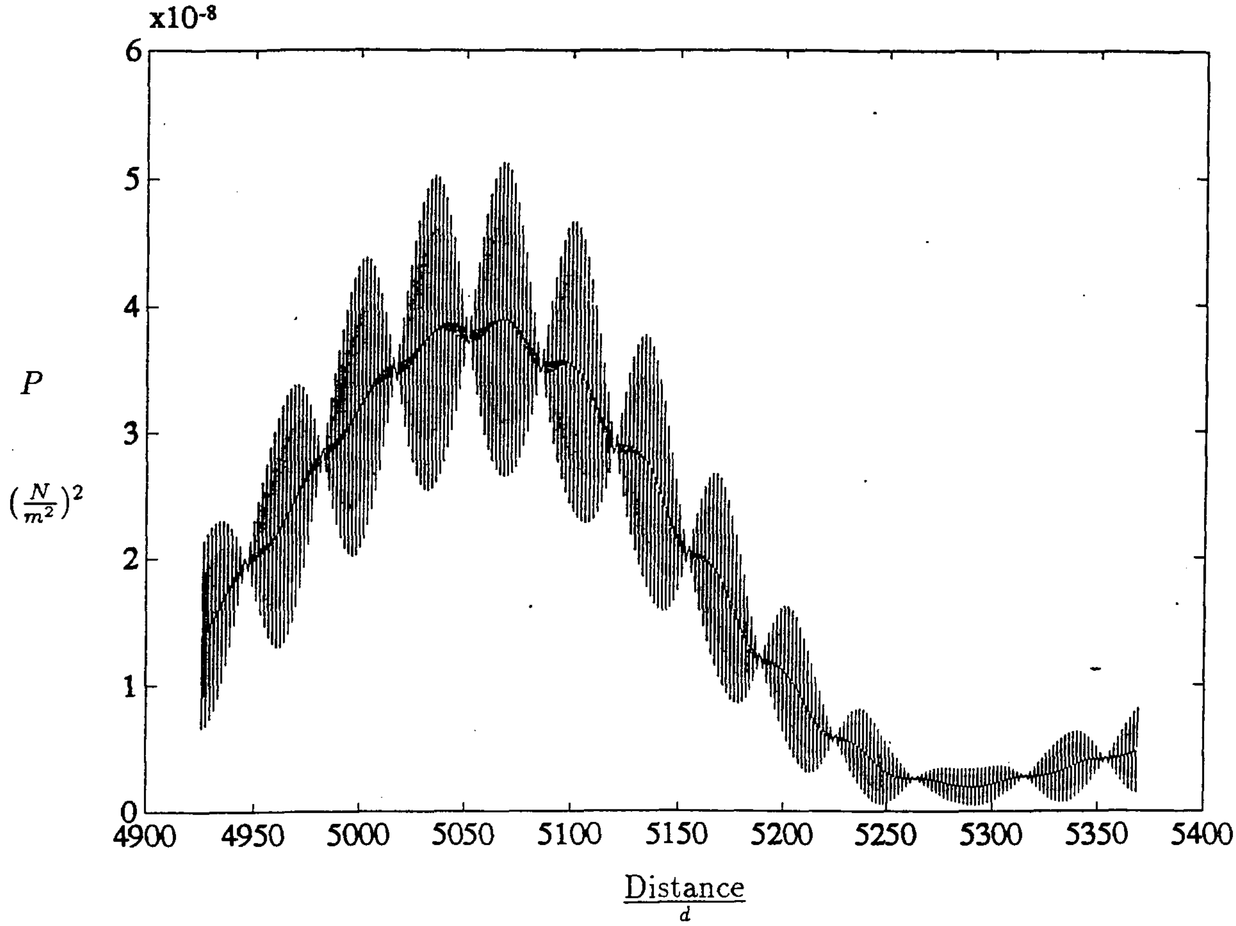


Figure 18: The 'binning method': illustration of reduced maximum owing to search along wrong bearing of $\frac{\pi}{3.205}$ instead of $\frac{\pi}{3.2}$ degrees

In Figure 18 the search has been along a slightly wrong angle. Apart from that, the characteristics of the system are the same as for Figure 17 above.

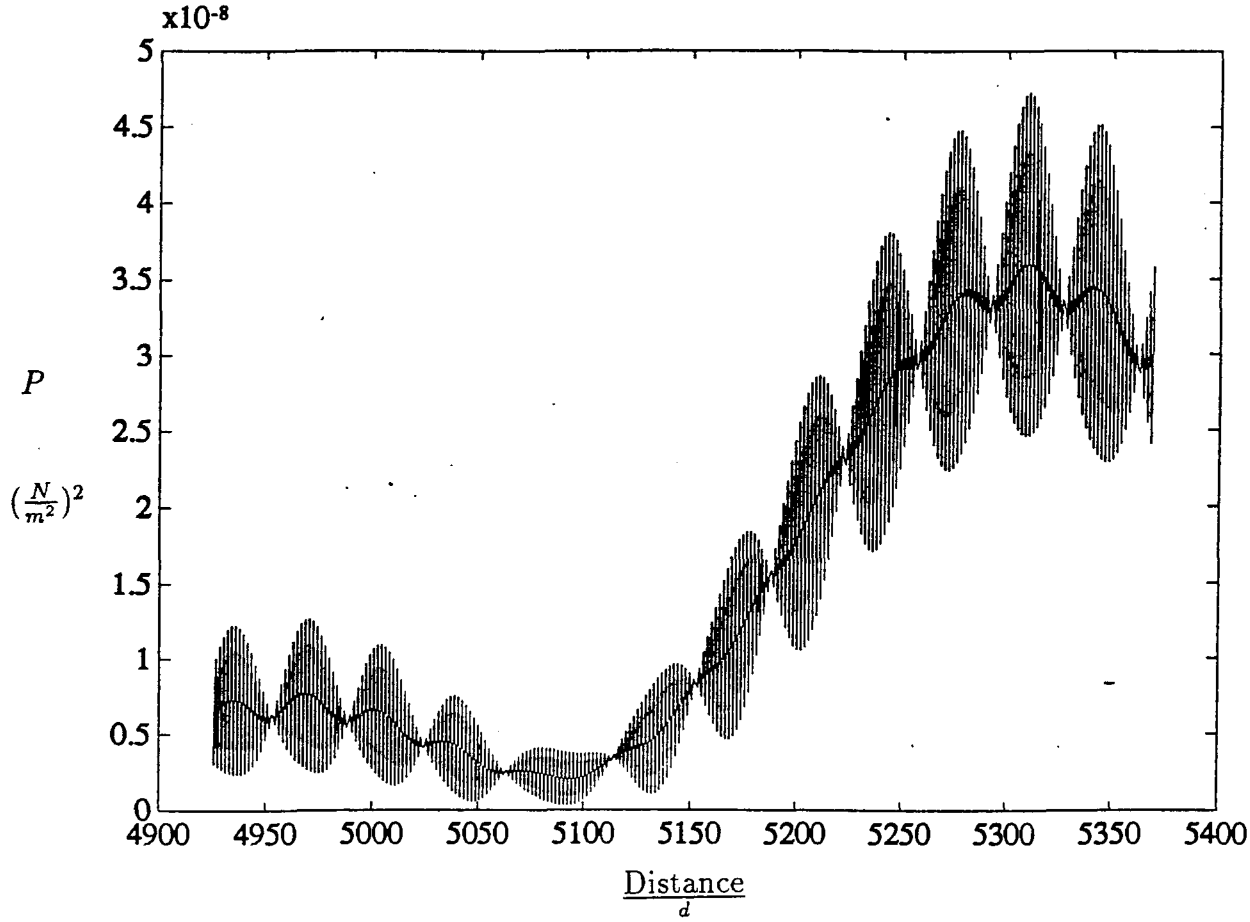


Figure 19: The 'binning method': illustration of reduced maximum owing to search along wrong bearing of $\frac{\pi}{3.195}$ instead of $\frac{\pi}{3.2}$ degrees

Compare remarks for Figure 18 above.

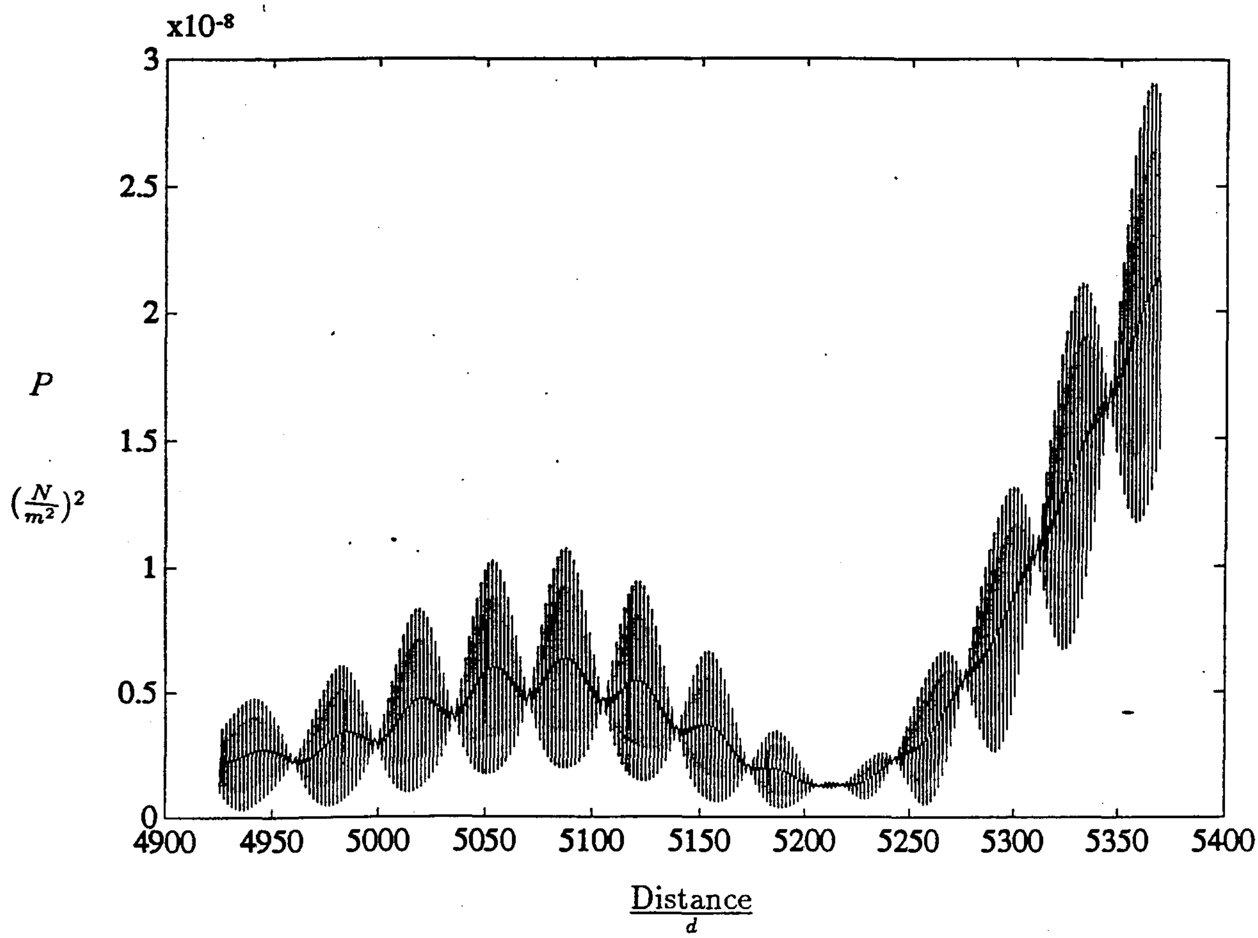


Figure 20: The 'binning method': illustration of reduced maximum owing to search along wrong bearing of $\frac{\pi}{3.19}$ instead of $\frac{\pi}{3.2}$ degrees

Compare remarks for Figure 18 above.

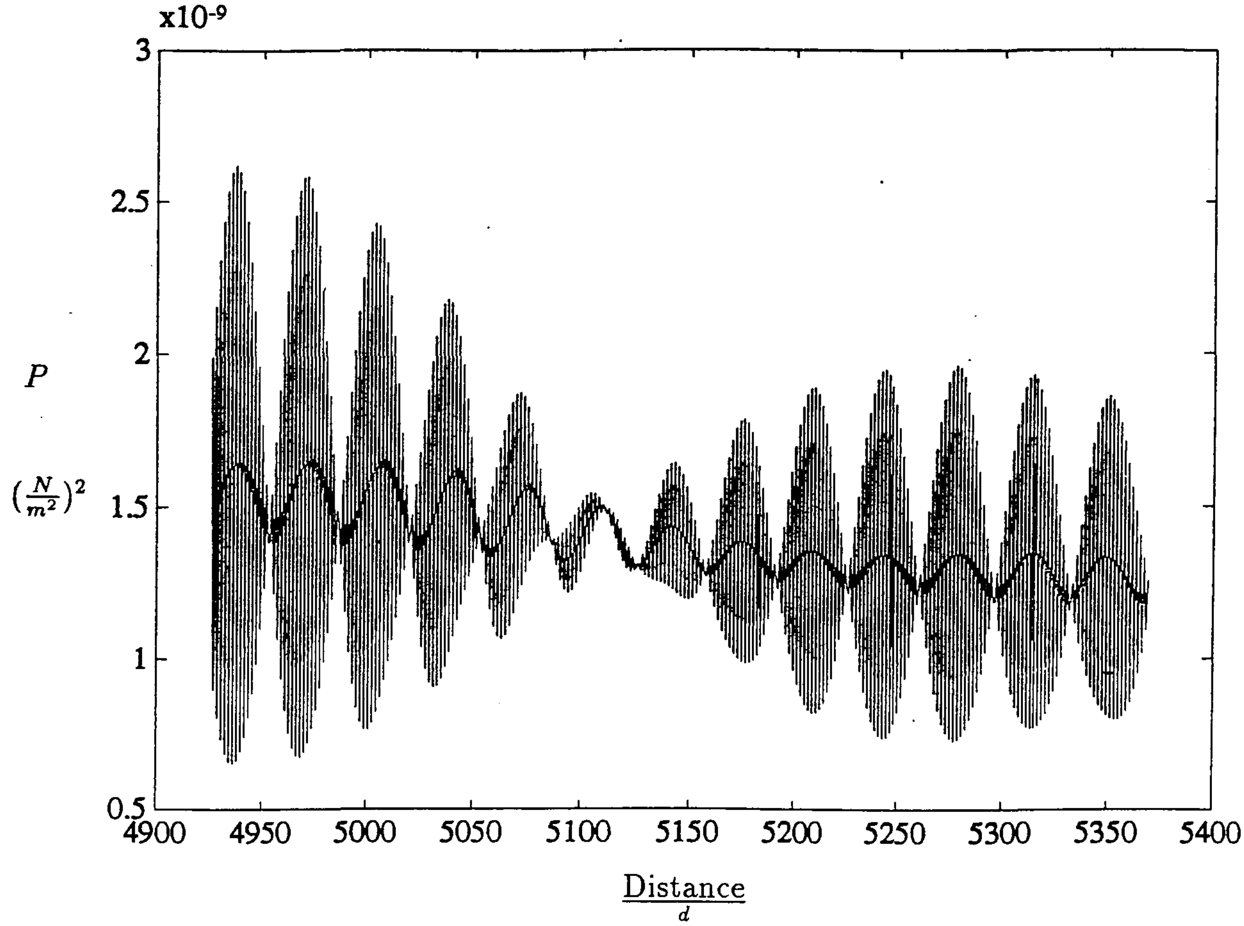


Figure 21: The 'binning method': illustration of reduced maximum owing to search along wrong bearing of $\frac{\pi}{3.5}$ instead of $\frac{\pi}{3.2}$ degrees

Compare remarks for Figure 18 above.

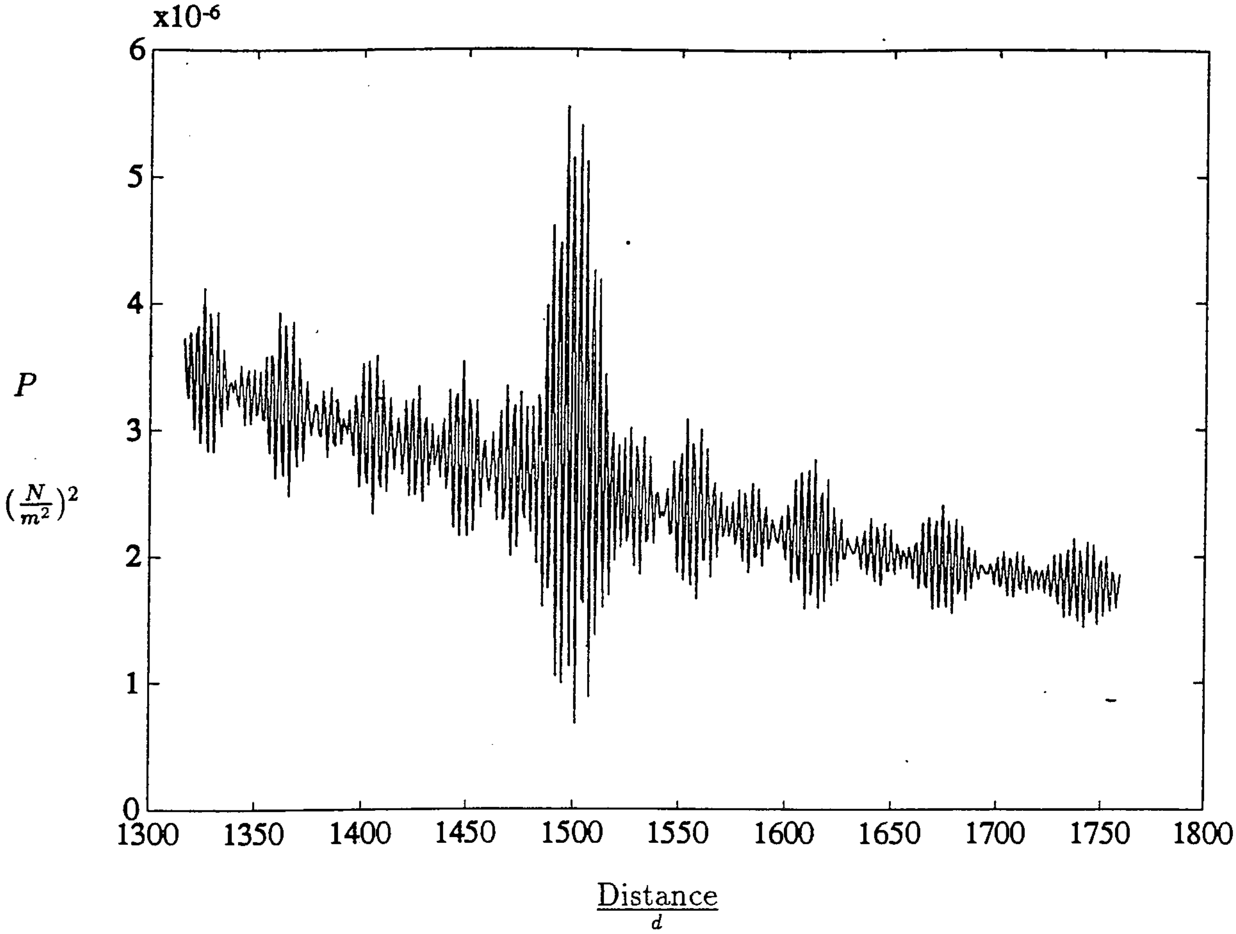


Figure 22: Located by 'binning method': point source at a distance of $\frac{1500}{d}$ and angle of $\frac{\pi}{20}$ degrees with $NSR = (30)^2$

In Figure 22 the point source has frequency 500Hz. The noise-to-signal ratio is $(30)^2$. The hydrophone array has 693 elements and is shaped sinusoidally, with an amplitude of $\frac{50}{d}$. The hydrophone location uncertainty is one part in a thousand. The search has been conducted along the true bearing.

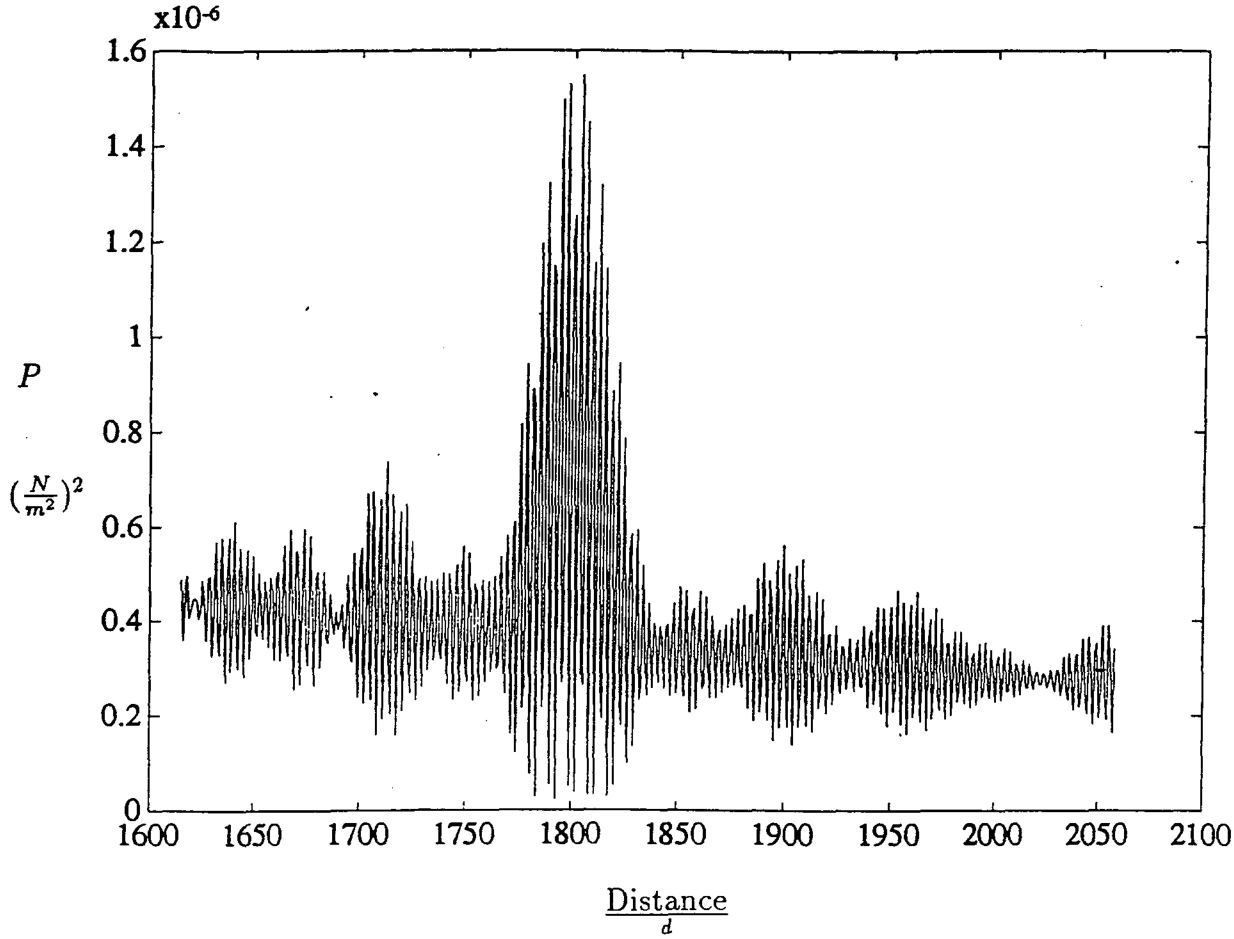


Figure 23: Located by 'binning method': point source at a distance of $\frac{1799}{d}$ and angle of $\frac{\pi}{180}$ degrees with $NSR = (30)^2$

In Figure 23 the point source has frequency 500Hz. The noise-to-signal ratio is $(30)^2$. The hydrophone array has 693 elements and is shaped sinusoidally, with an amplitude of $\frac{67}{d}$. The hydrophone uncertainty is again one part in a thousand. The search has been conducted along the true bearing.

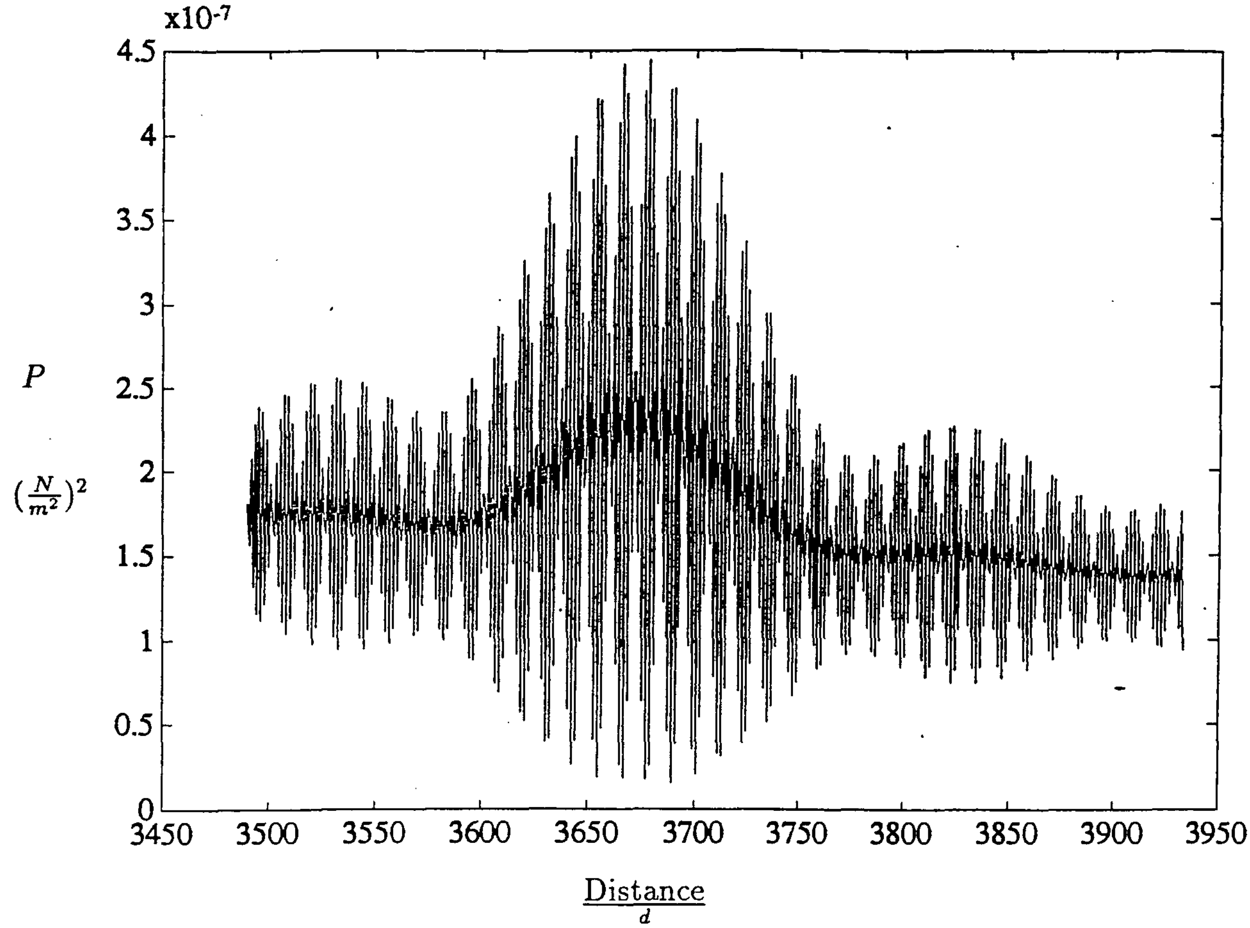


Figure 24: Located by 'binning method': point source at a distance of $\frac{3674}{d}$ and angle of $\frac{\pi}{2}$ degrees with $NSR = (30)^2$

In Figure 24 the point source has frequency 750Hz. The noise-to-signal ratio is $(30)^2$. The hydrophone array has 693 elements and is shaped sinusoidally, with an amplitude of $\frac{10}{d}$. The hydrophone uncertainty is again one part in a thousand. The search has been conducted along the true bearing.

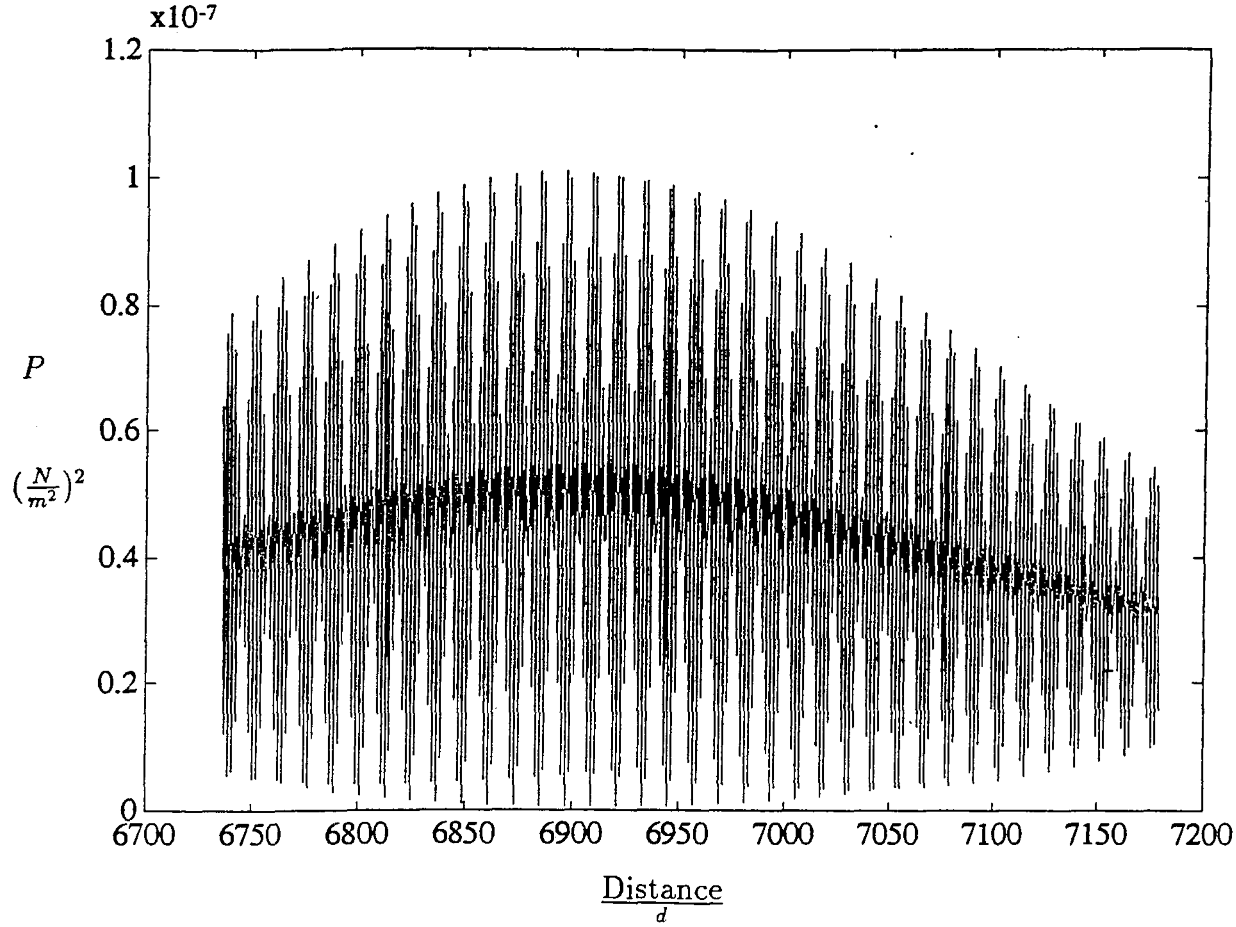


Figure 25: Located by 'binning method': point source at a distance of $\frac{6920}{d}$ and angle of $\frac{\pi}{2}$ degrees with $NSR = (30)^2$

Apart from the distance, the characteristics of the system for Figure 25 are the same as those for Figure 24 above.

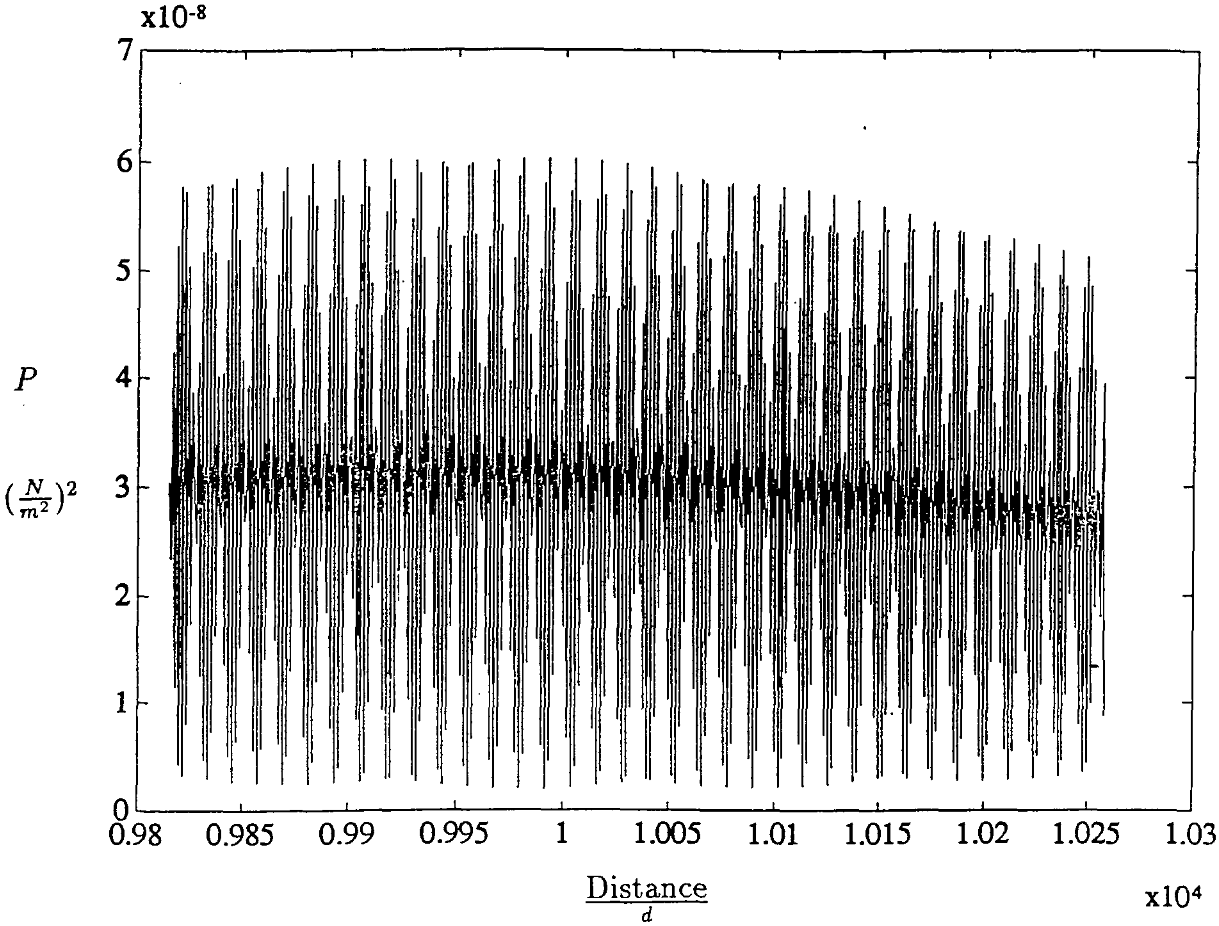


Figure 26: Located by 'binning method': point source at a distance of $\frac{9999}{d}$ and angle of $\frac{\pi}{2}$ degrees with $NSR = (30)^2$

Apart from the distance, the characteristics of the system for Figure 26 are the same as those for Figure 24 above.

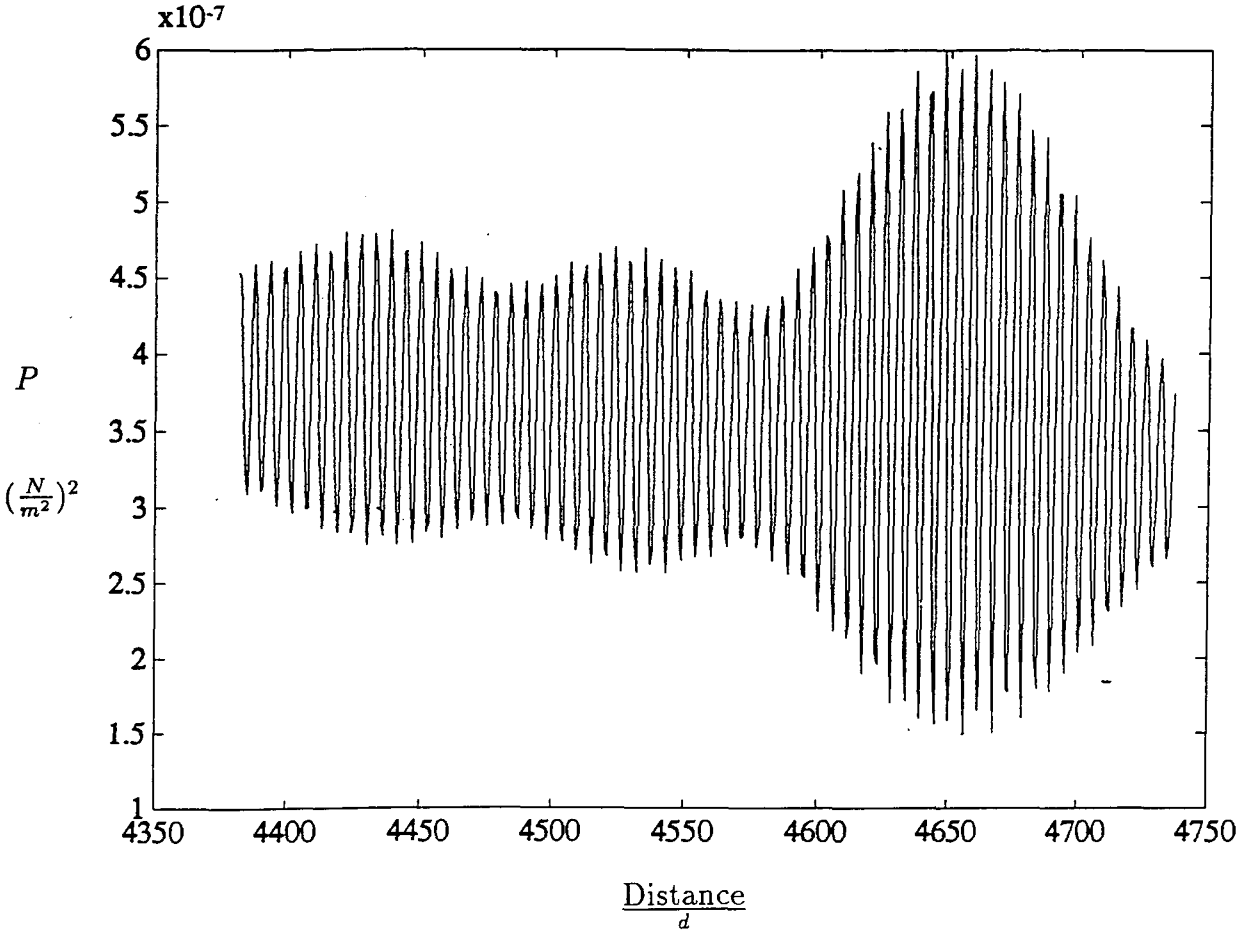


Figure 27: Located by 'binning method': point source at a distance of $\frac{4652}{d}$ and angle of $\frac{\pi}{2.6}$ degrees with $NSR = (30)^2$

In Figure 27 the point source has frequency 1500Hz. The noise-to-signal ratio is $(30)^2$. The hydrophone array has 693 elements, is shaped sinusoidally with amplitude $\frac{24}{d}$. The hydrophone location uncertainty is one part in a thousand. The search was conducted along the true bearing.

7 Theory of Partial Integration with Finite Syntactical Sums

We have applied to the algebra only the plane-wave case, in order to explain why our phase-binning algorithm should be effective. That it is effective, we believe to be evident by its fruits. However, we have not attempted to prove its worth with amplitudes differentiated by distance, believing that to be a specialized mathematical task beyond the scope of the present 'engineering' thesis. However, as a pointer to further work, we venture here a 'philosophy' of our phase-binning algorithm and, by way of summary and recapitulation, an account of its place in the general business of locating frequency point-sources with any disposition of hydrophones.

Given partial knowledge of the recent and local history of a function, it may be possible to recover it whole by making an approximation of the integral of the system available by means of a syntactical sum. A 'paratactical' system of relationships of measured quantities upon a region may be thought of as only a loose alignment, while a syntactical sum can be interpreted as the economical cohesion of them.

Let it be assumed that a typical process of interest is teleological, that is to say, its nature and purpose become evident upon its completion, *i. e.* it is known by its fruits. Similarly, in Mathematics the plausibility of the interpretation of a phenomenon is to do with the quality of the reasoning behind it. In the technical problem of integrating an inarticulate system, the justification of the procedure of approximate integration can be shown only by its completion. In what follows it will be argued that there is a test of integrability that can be applied to a system which will help to decide how plausible an explanation of it is likely to turn out to be. Indeed, the theoretical possibility should be considered of algorithms calling themselves in respect of their suitability for treating different problems.

Naively, the exercise may be likened to the solving of a jigsaw puzzle without prior knowledge of the image. The more that pieces evidently tend to cohere in part, the greater becomes the likelihood of achieving the whole. We will argue below that 'likelihood' itself, in relation to a particular problem can be circumscribed as a function of integrability and differentiability. A jigsaw puzzle usually has a unique solution, which implies the 'syntactical' disposition of the pieces on the board. If it is assumed that only the solution is sought, there is no basis for 'preferring' one paratactical arrangement to another. Thus there is no basis for 'differentiating' between one paratactical configuration and another in that sense, yet any of them may be distinguished freely from the syntactical one.

Likewise, there are properties of the focussed image on the retina that the brain distinguishes straightaway by means which are understood well enough, for example, for a lens to be characterized that will correct a myopic eye. Ultimately, of course, there can be no principles before experience, but such as are used are to be justified in the light of experience. Experience shows, in fact, that principles and experience nourish each other mutually, and, indeed, some would argue that the dialectic is teleological.

7.1 Proposal of a Proof by Syllogism

Consider the syllogism that that which is an integral is differentiable, and that that which is differentiable is integrable, therefore an integral is such if it is differentiable. On the mutual inclusivity of integration and differentiation we may quote Courant, who addressed 'die wechselseitige Beziehung zwischen Differentiation und Integration' as 'Fundamentalsatz der Differential- und Integralrechnung'[60] ('the mutuality of differentiation and integration as fundamental theorem of differential and integral calculus'). In practice, it is argued here that if one arrangement of finite differences is uniquely differentiable from another in terms of qualities, the possession of which by the function is requiring to be shown, then a syntactical sum may be construed, otherwise it is a total of paratactical relationships.

In searching for the integral, the convergence of the parts of the search procedure is bound up with the need to show the respects in which the system may be differentiated uniquely, and to show how the evidence may be arranged to reflect them.

Our inverse problem with 'fuzzy' instrumentation from underwater acoustics may serve as an illustration. Consider, for convenience, a section of the acoustic field created by a monochromatic, spherically-spreading point source. Several hydrophone sensors are placed at a distance from the source, the object being to estimate the wavelength with the location of the point

[60] R. Courant, *Vorlesungen über Differential- und Integralrechnung*, vol. 2, 4th edition, Heidelberg 1972, p. 357

source in the presence of noise. With reference to an arbitrary system of Cartesian axes, there is a small uncertainty of hydrophone location which increases with distance from the origin. The uncertainty is, from the statistical point of view, not removable with time. Without going into the technicalities of how estimates of hydrophone locations are obtained, it should be understood that there is no convergence upon their 'true' locations, no matter how often the estimates are made. The uncertainty is to be understood as fixed and immutable for all time, a limited incalculability which is independent of time. The source may be moving slowly relative to the hydrophones and certainly its speed must be sub-sonic. An instantaneous pressure p measured by means of a hydrophone in the absence of noise is a function of the intrinsic amplitude a of the source, the distance ρ between it and the hydrophone, its frequency ν, the speed of sound in water c and the phase-angle ϕ of the source.

Provided that data are obtained from the hydrophones at a single instant, only small changes df are represented in them of a function f, the local and recent history of which is supposed to be manifest in the measured pressures p. f is a function of a, ρ, ν, c and ϕ, all continuous functions of time. Totals are founded upon approximate values of the variables, *i.e.* upon central values a_0, ρ_0, ν_0, c_0 and ϕ_0 at time $t = t_0$.

If it be assumed that we are dealing with small differences df, rather than with less calculable Δf, it will not be fanciful to work within a 'synchronical' framework. 'Synchronical' shall convey the sense of samples taken at the same time as well as an assumption that little has changed in the ingredients of the source requiring to be detected in the time taken for the signal to manifest itself to the different hydrophones. We understand Δf to convey either the actuality or the potential for larger-scale changes in the ingredients of the signal from one sample to the next. Whether such changes are actual or potential makes little difference to the theoretical basis of the analysis. The possibility of either alone renders any integration inexact in the strict sense of $f = \int df$, and certainly it will be harder to justify the result of a syntactical sum rather than a paratactical total with Δf rather than df. But it is emphasized that we are dealing with matters of degree. For whichever method we choose, as we shall see below, there will always be an element of uncertainty. But we are inclined to place right above certainty and analysis above synthesis. This latter point means that we wish to avoid, as far as possible, undifferentiated, indiscriminate *a priori* or perhaps even 'Bayesian' priors, so long as they have not been justified analytically. We call the more or less bold imposition of expectations before experience 'synthetic'. Many 'optimal' or 'sub-optimal' methods may fail to apply a test of possibility before being applied to a problem, and their results may accordingly be ambiguous and requiring of interest as well as objectivity on the part of the observer to interpret them. It may not be fanciful to suggest that the terms 'optimal' and 'suboptimal' are coined in order to reflect that very state of affairs.

In terms of the present analysis, it is argued that the case for using an *a priori* or 'Bayesian' approach might profitably be considered in the light of whether one is dealing with uncalculable Δfs or relatively reliable dfs. Too great a reliance upon 'prior' or 'Bayesian' approaches in a method may mean missing or glossing over the analytical stage of reasoning and going straight, sometimes indiscriminately, for a synthetic solution, and to that extent they may countenance the triumph of hope over expectation born of experience.

With our scheme of things, we will perforce be working with differences of the order and detail Δf. In order to achieve control over those differences, we concentrate upon the single snapshot, which is at the heart of the present exercise. Only by positing the approximate synchronicity of sampling can we hope to gain control over the otherwise undifferentiable Δfs.

A synchronical system is said to be optimally differentiable with respect to source location and wavelength if there are as many different distances ρ as there are hydrophones. Likewise, it is said to be maximally differentiable with respect to wavelength $\lambda = \frac{c}{\nu}$, supposing a certain location, if both each hydrophone location represents a different phase of the source in its periodic diachronical propagation through the medium and all cover the greatest proportion of the period. The greatest part of the period is requiring to be differentiated, else the period itself can be only partially differentiated. If substantial tracts of the period are left unaccounted for, there is little justification for claiming knowledge of them. It is not merely a matter of failing to meet the 'Nyquist' criterion in such a case, we are quite simply not entitled to claim any more than the merest knowledge of the period. All sorts of effects might be occurring in the tracts which we are not in a position to analyse. A synchronical system is said to be optimally differentiable with respect to distance to the extent that the baseline open to the source is longer.

If these conditions are satisfied, the system is said to be partially integrable with respect to location and wavelength in the absence of noise. But if they cannot be satisfied, *i.e.* if the system contains redundancy in the sense of there being two or more ρs equal or two or more hydrophones located at the same phase of λ, thus reducing the differentiability of the system, a syntactical sum may still be possible. There is perhaps a paradox here, to the extent that redundancy is usually considered to be a factor in noise-rejection in a system of communication. The theory is that a right ratio of signal and redundancy allows for some immunity against noise. In our scheme of things, two or more ρs equal mean duplication, triplication of the same potential piece of information, and so, in that sense, redundancy does no work of the kind implied in the mutual balance of signal and redundancy and noise. Where one has no basis for predicting the quantity and quality of random noise, one might at least be able to reassure oneself that the framework of one's data processing was more or less secure against what may come. Again, it is important to differentiate between one articulation of the framework and some other. The criterion for choosing one or another will be its efficiency.

In any case, the commonly assumed interrelationship of noise, signal and redundancy is perhaps requiring of further comment. There are no absolute standards by which to assay the *quantum* of redundancy (for it is an element greatly subordinate to the semantics of the signal) in a system of communication, where both the message and the channel of communication must be taken into account. But one thing is certain, the redundancy must not be so great that the signal becomes unintelligible because repetitious and therefore non-developing in the sense that the stages of the argument cannot be differentiated one from another. Again, the signal must never be conceived with a particular or specified *quantum* of redundancy in mind beforehand, since the semantics have ever primacy over noise or redundancy considerations. Indeed, the redundancy is a consideration to do with the channel of communication alone. Considerations about how the channel of communication should be set up are quite independent of the message requiring to be communicated. It cannot be the concern, let us say, of the communications engineer what message might be transmitted by means of his system. Likewise, we shall argue below that some aspects of 'prior' or 'Bayesian' smack of isolating or hedging the semantics before they are transmitted. In other words, the doubtful theoretical proposition there is that the semantics of communications can somehow be calculated in advance. We think this an unsound basis upon which to proceed. To use the optical analogy again, we must make sure that the lens we characterize to view any object is appropriate to the task before we begin. If all the ρs on offer are different in our pre-analytical system, and if other differences are taken into account (see below), then we can be sure that our lens will be of the highest quality for the task in hand. The differences just referred to (which we shall go into in more detail below) will specify the task which the system is being called upon to treat.

In our scheme, the less redundancy there is the better. In fact, the desirable condition of all ρs different and no two or more hydrophones representing the same phase of λ (to say nothing of the need to 'differentiate' hydrophone locations *vis-à-vis* radiality) is the best kind of 'convolution' the system of communication could have. Here again, a contrast of method is acknowledged with some customary theories of communication, where convolution or modulation schemes are requiring to be calculable, or periodic, whereas we ask only that ours be unique, as free from periodicity or ambiguity or aliases as possible. Indeed, we shall show later that, for analysing signals characterized by periodicity, the channel of communication, the 'lens' of inspection, should resist such a chracteristic of a signal as much as possible. For to do otherwise would make it difficult to distinguish signal from channel. A case in point here was the present author's work on the binary star *Sigma Scorpii.* Irregular intervals of sampling showed up on the periodogram. However, the author was in fact able to turn the irregularity to his advantage. For irregularity is only a local difficulty. Globally, it offers possibilities for orthogonal discrimination that no periodic or calculable scheme could provide, with their aliases and integer multiplicities. It offers a theoretical basis for arriving at a unique solution to a problem. A catch-all lens is not what we want; we want the right lens for the job. As we have suggested before, an 'exponential' sampling strategy is very greatly to our advantage over and against a regular sampling strategy *à la* Nyquist. What more poignant example could there be of a 'redundant' sampling strategy, where a Procrustean phase-difference is imposed upon the processing right from the very start?

Whether a syntactical sum is actually the result of the data processing depends upon selecting the appropriate location and wavelength of the source. In addition, the system must be optimally differentiable with respect to the radiality of the source.

If the system is integrable in the sense above, then the likelihood of being able to distinguish a syntactical sum from a paratactical total will be the greater. For the more redundancy there is in the system, the fewer the degrees of freedom it will have to do its work. This will be a consideration with noise-rejection too.

These conditions are paramount before going to work. The degree to which they are satisfied will give an idea of the ultimate success of the data-processing. The point is that the system must by examined for its integrability before sums are attempted.

Where there is doubt about the location of hydrophones, it makes sense to have the system as differentiable as possible. If we cannot 'fill out' the period of the source *a priori*, then we are going to get clusters of values about isolated phase-points, which will merely serve to increase the ill-effects of the uncertainty. But, on the other hand, if we have a good spread across the period *a priori*, then there will be a better chance of noise cancelling itself out (*i. e.* in the technical sense of our binning algorithm). Again, if only isolated phase-points are on offer, the system is far from being maximally differentiable, and if only a part of the period is on offer, the system is only partially differentiable.

We inspect first the 'Jacobian' of differences of ρs. For maximal efficiency and integrability of the system the Jacobian must not vanish. We then look at the differences seen in terms of different wavelengths in respect of different phase-points and portions of the period, second and repeated differences in a sort of qualified and iterated Hessian.

7.2 Efficiency of Convolution and the Gram Determinant

We have argued that the convolution or the modulation of the acoustic radiation will be the more efficient for becoming more individual and unmistakeable. If we have only a limited number of hydrophones at our disposal, in given exercise, and if, moreover, we can count on their locations becoming increasingly uncertain as they become remote from the origin, it is important to minimize the effects of such Procrustean shortenings by having as much control over them as possible. We are wasting time in a search procedure if, for example, we find, for a given location, that two or more distances ρ are equal. In fact, we want the distances ρ to be as different as possible. The more variety there is among the different distances ρ, the more individual will be the convolution of modulation of the radiation. The more starkly individual the convolution or modulation is, the more easily we might be able to distinguish the effects of its characteristics upon the signal processing from effects due to noise of various kinds. The more starkly characterized and featured a convolution or modulation system is, the more efficiently we shall be able to distinguish the work it does in the signal processing from the interference caused by other factors, including noise.

It can readily be seen that as a source is removed away from a system of hydrophones, the differences between the distances ρ become more uniform. But uniformity is something we wish to avoid; on the contrary, we wish character and individuality in the mechanism of convolution or of modulation. At great distances from the hydrophones, examining the changes in differences between distances ρ is likely to be comparatively unprofitable. At closer distances, on the other hand, there is much more to be gained by comparing sets of differences of distances ρ.

As an abstracted measure of the variety of differences there may be for a given system of hydrophones and source, we offer the Gram determinant. Let $v_1,\ v_2, \ldots,\ v_m$ be vectors. The Gram determinant Γ is given as follows:

$$\Gamma = \begin{vmatrix} v_1^2 & v_1 v_2 & \ldots & v_1 v_m \\ v_2 v_1 & v_2^2 & \ldots & v_2 v_m \\ \vdots & \vdots & \vdots & \vdots \\ v_m v_1 & v_m v_2 & \ldots & v_m^2 \end{vmatrix}$$

We quote from Courant and Hilbert as follows:

> Notwendig und hinreichend für die lineare Abhängigkeit der Vektoren $v_1, \ldots ,v_m$ ist das Verschwinden der 'Gramschen Determinante' ...[61]

There follows:

> Der Wert der Gramschen Determinante ... stellt ein Maß für die lineare Unabhängigkeit der Vektoren $v_1, \ldots ,v_m$ dar.[62]

In our case, the vectors v are made up as follows. Taking the M different distances ρ from a putative source to a known disposition of hydrophones, the vector element v_m, is the difference between the mth distance ρ and the nth distance ρ, with $m \neq n$. We are interested in getting an idea of how much variety there is of distances ρ. We are after variety more than uniformity.

For example, let the source be thought to be a long way away from a (for convenience) straight antenna with uniform spacing between antenna elements. At longer distances the differences between distances ρ will be the more uniform. In fact, if θ is the angle between a line leading from the origin to the source and d the element inter-spacing, the differences between distances ρ will all be roughly $d \cos \theta$. The further away is the source in such a system, the more likely the Gram determinant will be to vanish. We consider such a system comparatively inefficient, because there is a great deal of redundancy in it. We want to find out what is happening with the radiation between the limits Procrusteanly set by $d \cos \theta$. It may not be fanciful to suggest again that, for far-flung sources, an exponential spacing would be better, for, with radiation characterized by a periodicity, rather fewer aliases would escape through the comparatively finer mesh of the net of the exponential rather than Procrustean uniform spacing.

Of gross perturbations of an antenna, we have argued before, for a given system of antenna and putative source, the sampling of the field of radiation would be at its most efficient where an exponential sampling were available.

So far we have been speculating about the work that a particular system of antenna and source might do. The value of the Gram determinant would be

[61]'A necessary and sufficient condition for the linear dependence of the vectors v_1, ... ,v_m is that the "Gram determinant" should vanish ...'

[62]'The value of the Gram determinant represents a measure of the linear independence of the vectors $v_1, \ldots ,v_m$.' R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, 2nd edition, Berlin, 1931, vol. 1, p. 29f.

very small at far distances, and even if the hydrophones were spread about irregularly, the source would still have to be moved in quite close to the antenna before the value of the Gram determinant began to rise appreciably. Thus there are absolute and relative uses for the Gram determinant in the present problem. We might, for instance, wish to consider changes in the Gram determinant by infinitesimal changes in distance, *i. e.* $\frac{d(Gramdeterminant)}{d(distance)}$, as it were.

One might start with a particular trial location, take the Gram determinant for that with the antenna disposition available. That exercise in itself might not begin to yield information, or do work, until a similar exercise will have been performed with another location. How much work the system will be able to achieve can thus be measured by how much the value of the Gram determinant varies with location. In general, it can readily be seen that the value of the Gram determinant will increase as the source approaches the antenna. Indeed, because the rate of change of value increases as it does so, the resolution of the signal processing in estimating the location of a source of radiation may be fairly said to improve, but we shall have more to say on that in future work. For the time being, we are assessing how sensitive, and therefore how integrable, the system might be before attempting the signal-processing itself. We believe that a distinction between 'infinite' and 'finite' in source distances does not do much work in itself. By the means and method just outlined we hope to differentiate the notion 'finite'. 'Infinite' must lie outside the scope of the present thesis.

Having weighed up the absolute or relative values of the Gram determinant, we proceed now to differentiate the sytem in terms of a putative wavelength of a source. The object of this exercise is to satisfy ourselves that we can sample over the supposed period of a source of radiation sufficiently well to verify the presence of that periodicity, if it is present, and to accommodate aliases, if present. Ultimately, of course, with M hydrophones we want M distinct phase points of the period accounted for, with M sufficiently large that the period is covered more or less comprehensively.

Let us return to the vectors v_m of differences between the different distances ρ. We want to look at the modulus with respect to the wavelength λ of the differences $v_{m,n}$. In the first place, it makes sense to have as much variety as possible here again. Again, uniformity of sampling is of comparatively little use, because much redundancy is involved, and little work is done. Let us develop the example given above of a source a long way away from a straight antenna. If the wavelength λ happened to be exactly $d \cos \theta$, we would have a rather Procrustean system that would do little work, for example, if there was another wavelength $\frac{\lambda}{2}$ present as well. Again, we might venture to suggest that an exponential spacing, if spacing there must be,[63] would be more efficient in such a circumstance.

It is well if the system captures the radiality of the source as broadly as possible. If the parallactic angle is more or less narrow, the central-value $\frac{a_0}{\rho}$ must be sought as the centroid of a density of possible pretenders on a region possibly as long as it is broad. But if the parallactic angle is more or less wide, then there will be a greater homogeneity of the $\frac{a}{\rho}$ and the $\frac{a_0}{\rho}$ may be sought more along a curve than in the middle of a region which may be as long as it is broad. It is plainly easier to conduct a search along the 'one dimension' than in among the 'two dimensions', as it were. With the more or less narrow parallactic angle, two degrees of freedom must be accommodated in order to find the centroid. But with a more or less wide parallactic angle, the statistical centre (*i. e.* the a_0) will be on a line rather than in an area. Therefore it is with the wider parallactic angles that legitimacy for surmising the central value a_0 begins to develop. Naively, it might be easier to distinguish the contribution of the ρs from that of a_0 with a wider parallactic angle than with a narrower one.

7.3 Avoidance of Search for Centroid

Two considerations stand out among those that make the handling of the $\frac{a_0}{\rho}$ more calculable. In the first place, we shall show that, as frequencies are higher which are requiring to be treated, the more 'mixing' of adjacent ρs will take place in a bin. In the second, we shall address the need to achieve the widest parallactic angle possible in order to account best for the radiality of the point source.

Let us consider the consequences of the higher frequencies first. There is a greater likelihood here that successive hydrophones encountered by the wavefront on its course of propagation will belong in consecutively different bins. It is perhaps readily seen that, the longer is the wavelength, the more consecutive 'suites' of hydrophones will belong to one bin. On the other

[63] *I. e.*, there might be little advantage in having a sensor spacing which was less than the correlation distance of the noise.

hand, what we require is for there to be as near equal a distribution of $\frac{a_0}{\rho}$s in each bin. Otherwise, the distribution will be lop-sided as between one bin and another. If, for example, we had a straight antenna and our analysis were tuned to a longer wavelength, then there would be a tendency for ρs of a certain size to be concentrated in one bin, to the disadvantage of the other two. It might be, for example, that the shorter ρs were all in one bin together, all the medium-length ρs in another distinct bin and all the longer ρs in a group in the third, there would quite clearly be a mal-adjustment of distribution across the bins.

Whatever the wave-length, however, it would be advantageous if all the ρs were distributed about a curve, rather than dotted here and there upon a region. Once again, we are not advocating an array curve which runs exactly along the wavefront, since that would compromise one of our criteria of efficiency. For in such a case all the hydrophones would represent one single phase-point, and no information could be gleaned therefrom about the period of the signal.

But to the extent that such a singular case is avoided, a sinewy straddling of a curve would be advantageous. It may not be fanciful to argue that such a configuration licensed a better-founded assumption of a central value a_0 for a. The problem with the region is that a centroid must be posited, and, whatever the circumstances, a search in two dimensions must be undertaken. Along the line, however, the a_0 might well be thought of as lying somewhere on the curve, possibly even in the middle. But whichever may be its location, it at least lies upon the curve and does not need to be sought, if it could be sought at all in practice (it seems difficult to conceive how it could be), somewhere upon a region.

The matter might be likened to the caustic curve. Here, briefly to recapitulate, a point source is focussed upon a spot as a convergent process approaches a point. Instead of a point being located uniquely, it can be surmised in terms of the convergence of lines. As is well known, the convergence of such lines is 'at infinity'. Whether it is a point 'at infinity' or a pencil that is the result is a matter that cannot be resolved in the present thesis. But it is the use of one-dimensional lines to adumbrate a point, which has no dimensions, which is the essence of the present observation. It is a convergence of a serial disposition of lines which encourages one to isolate a special point. It is a step-wise analysis with a theoretically guaranteed convergence, whereas theory falls short of the mark with the two-dimensional region, where, by definition, the solution could lie anywhere, and therefore, as often as not, a less tautly disciplined ('non-linear') is all that is practically available.

So, with longer wavelengths, we want the ρs to be much the same, for the purposes of justifying an assumption that we can construe a central value a_0 of the as. But, by doing that, particularly with the longer wavelengths, we run the risk of each hydrophone landing in one bin only, and all the hydrophones are not spread satisfactorily across all three bins. As ever, different considerations apply, and must be balanced. But, after all, it it one of the main hopes arising from the present thesis that it may be possible for an algorithm to call itself, and to decide for itself the merits of one particular course of action rather than another.

But with the shorter wavelengths, a curvi-linear rather than two-dimensional distribution of $\frac{a}{\rho}$ is to be wished for. It would seem that this stipulation is cognate with a requirement that the parallactic angle be as great as possible. The hydrophones, in their elongity, should present as broad a front to the target as possible.

It is emphasized that the criterion of efficiency that we have associated here with the greatest variety of differences (*i. e.* a high Gram determinant) is not directly to be associated with the signal processing itself. It is hoped that it is seen in terms of an attempt at an independent, abstract yardstick of efficiency. It should always be born in mind that all ρs equal mean all phase points equal in a period, and thus the system is not differentiable with respect to wavelength.

8 Conclusion and Plans for Further Work

8.1 Optimality, Efficiency; Self-Selection of Algorithm: Data Flow Implementation?

Ours being a system of several interlocking parts, the 'optimality' of it is a function of several variables, of which one is the interested curiosity of the user of the system. The Algorithm must be designed to accommodate it. In this section we take a particular look at the problem of sensor uncertainty.

The uncertainty we have had to come to terms with in the present research is an error in sensor location of no more than one part in a thousand of its x- and y- coordinates. Essentially, this means that, in a given system, no sensor can be displaced enough to occupy the same location as another. If we need as wide a parallactic angle as possible in order to maximize our differentiation with respect to the radiality of the signal, it is clear that the condition that no sensor can be displaced to the extent that it can be confused with another is very important. If widening the parallactic angle meant increasing the noise or redundancy of the signal, it should not be done. The advantages of a flexible antenna are manifest, but if there is confusion possible of one element with another, then the system is bound to degrade in efficiency, whatever other factors might be. Otherwise, mere multiplicity of sensors, without further qualification, would not merely cease to do work, but would, in fact, tend increasingly to cause a degradation of the system. Thus independence of location must be a *sine qua non* whatever the degree of 'fuzziness' might be.

Although there is some uncertainty, provided there is a good spread in the three phase bins, an error of placing in one band is likely to be compensated by an error in another. The problem here, however, is that there are three bins. Statistically, there is perhaps a better case for saying that, the more limited is the uncertainty, the lesser will be the possibility of placing a measured pressure in the wrong one of several bins. With the constraint that width of parallactic angle must be consistent with independence of sensor locations (whatever the uncertainty might be), it seems logical to expect that the chances of getting the wrong bin out of three bins, rather than two, are reduced. If, in the limit of the analysis, there is a danger of confusion between only two bins, then, with the same system, there must be a reduced risk of confusing between three. Perhaps these sentiments might be developed into a rule of thumb, the substance of which would be that the Algorithm may be employed for wavelengths such that, with two bins, it is becoming increasingly difficult to distinguish them. But it would be safe to use three bins with those wavelengths. The decision whether to use the Algorithm or not in such limiting cases could easily be made automatic. Much could be gained for economy and efficiency by allowing the Algorithm to decide upon its own usefulness locally. It could perhaps be easily implemented in a data-flow architecture. A decision to attempt to advance the solution by performing a further iteration, for example, could be made automatically on the basis of examining the degrees of differentiability in respect of direction, wavelength and radiality in turn. Weak differentiability in any respect should dissuade from pursuing the exercise.

Another kind of reasoning suggests itself for preferring three bins to two, and why three should be preferred to more than three. A facility of binaurality for the purposes of locating a source of sound or noised, does comparatively little work unless the 'ear'-system can be orientated to find the direction in which both ears hear the same thing. Thus, although only two ears are required, the system must have a third degree of freedom. If one ear is used in creating a synthetic aperture, for example, an interest must be declared at the outset that the source be stationary with respect to the aperture as the serial (as opposed to parallel) operation is carried out. Human binaurality is a fixed parallel system that can be orientated without the exclusive use of time-delays. To that extent it saves time.

Enough time is saved to be able to discover the general direction of a source. But more time is required in order to work out how far it is away. We have only relative means to assess whether the source is getting louder or softer, so we can only really measure more or less small changes in the signal. But we cannot get any integral from such samples which would entail knowledge (however acquired) of the changes in the distance that were taking place.

With our phase-binning, single snapshot algorithm, small changes in position of the source may not be great enough to jump the dividing line between one bin and another. If the system is coupled loosely enough for avoiding too much crossing between two bins, then this may be a necessary and sufficient condition for it to be quite reasonably unlikely that three might be crossed. Here again, a limiting process for the self-selection of the Algorithm may be something associated with the limit of competence of two bins in the first instance.

We might perhaps want to isolate the point at which it becomes impossible to decide which of two bins to assign to a hydrophone. That could be a yardstick at the sharp end of the analysis, while, at the blunt end, so to speak, we want there to be as many in one bin as there are in either of the other two.

It can be seen that a self-selection of the Algorithm might involve many and various factors. A quantity cannot be assigned here to the complexity, however, because of the infinite variety of interest that might motivate a user of the system. However, methodologically, we are duty-bound to enumerate the factors involved. In other words, we must try to differentiate the system as much as possible. By doing this, we can get closer to the desired integration. Because of the interest of the potential user, which is infinite, no upper bound can be placed upon the efficiency of the system. The lower bound, on the other hand, can be set in terms of the self-selection criteria adumbrated above.

8.2 The Use of Interest in Integration; 'Entropy' Increasing with Repeated Differentiations; Methodological Indissolubility of Source, Medium and Sensors

A. N. Kolmogorov put the following rhetorical questions:

> But what real meaning is there, for example, in asking how much information is contained in *War and Peace*? Is it reasonable to include this novel in the set of 'possible novels', or even to postulate some probability distribution for this set? Or, on the other hand, must we assume that the individual scenes in this book form a random sequence with 'stochastic relations' that damp out quite rapidly over a distance of several pages?[64]

One answer may be that it depends upon the interest and curiosity of the reader. Nevertheless, the channels by which any information might be conveyed are capable of a certain amount of characterization. For instance, Tolstoy and his reader share some comprehension of the changing patterns of aristocratic life in certain parts of Russia. The rites of passage of individuals and such a society are broadly agreed upon between reader and novelist. What the novelist could not legislate for are the reader's interest and curiosity.

Of less interest and curiosity value to a reader might be a telegram of congratulations (to use Kolmogorov's example). To the extent that interest and curiosity play a diminished role in such a case, Kolmogorov grants the validity of an 'abstract' mathematical treatment:

> In practice, for example, it can be assumed that the problem of finding the 'entropy' of a flow of congratulatory telegrams and the channel 'capacity' required for timely and undistorted transmission is validly represented by a probabilistic treatment even with the usual substitution of empirical frequencies for probabilities.[65]

But it would perhaps be disappointing if mathematical analysis were to be reduced to such a routine subordination in the matter. Certainly, entropy

[64] A. N. Kolmogorov, 'Three approaches to the quantitative definition of information', *Problemy Peredachi Informatsii*, vol. 1, no. 1, 1965, p. 3f.

[65] *Ibid.*, p. 3

increases with frequency of occurrence, and entropy may be to probability what justice is to certainty. On the whole, it seems apparent that Kolmogorov might have been more exercised about certainty than about probability. In a much earlier work than the one we have been quoting from here, Kolmogorov set up and sought to justify 'axioms'.[66] His desire to do so there may be evidence of some impatience with the 'semantics' of the real world, and a desire to prove that Mathematics might be a means of getting round them or dispensing with them altogether. But it need not be so, despite Kolmogorov's later remark (which may have autobiographical implications):

> ... the problem lies in the vagueness of our ideas of the relationship between mathematical probability theory and real random events in general.[67]

With our problem of the source and hydrophones there might be an infinite number of point sources requiring to be located, with an infinite number of wavelengths. But we define 'information' in terms of both source and hydrophones' disposition. It is not just a matter of acquiring information about the source in the abstract. For example, it is plainly idle to submit for analysis (with a view to distilling information) data which we cannot acquire in first place or, indeed, in any event. The notion of a 'prior' or 'Bayesian' supply of 'information' may have no theoretical basis. Methodologically, it seems better to see information-gathering in the light of what is possible. Otherwise an infinity of possibilities must be catered for, and an infinity of choice or possibilities must rule out the feasibility of differentiating, and therefore of there being much information conveyable at all. At infinity, one might say, data and information are indistinguishable.

An example of an attempt to establish the probability of principles applying before experience is J. M. Keynes' effort to vindicate and circumscribe the 'principle of indifference'. Of the Principle wrote at the outset of his *A Treatise on Probability*:

> The Principle of Indifference asserts that if there is no *known* reason for predicating of our subject one rather than another of several alternatives, then relatively to such knowledge the assertions of each of

[66]See A. N. Kolmogorov, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, New York, 1946 (first published 1933), *Vorwort*, p. iii.

[67]A. N. Kolmogorov, 'Three approaches to the quantitative definition of information', *Problemy Peredachi Informatsii*, vol. 1, no. 1, 1965, p. 3

> these alternatives have an *equal* probability. Thus *equal* probabilities must be assigned to each of several arguments, if there is any absence of positive ground for assigning *unequal* ones.
>
> This rule, as it stands, may lead to paradoxical and even contradictory conclusions. I propose to criticise it in detail, and then to consider whether any valid modification of it is discoverable.[68]

Later, Keynes alerts the reader's attention to his (*i. e.* Keynes') original contribution, about to unfold, to an evolving discussion:

> The principle states that 'there must be no known reason for preferring one of a set of alternatives to any other'. What does this mean? What are 'reasons', and how are we to know whether they do or not justify us in preferring one alternative to another. I do not know any discussion of Probability in which this question has been so much as asked.[69]

Keynes seems to think it unfair if one alternative has more to be said for it than another. It as as if, perhaps, he might consider it to be more commendable, as it were, to see to it that each alternative is allowed a comparable *quantum* of evidence in its favour. Indeed, it would seem as though a principle of fair play was behind Keynes' approach:

> This distinction enables us to formulate the Principle of Indifference at any rate more precisely. There must be no *relevant* evidence relating to one alternative, unless there is *corresponding* evidence relating to the other; our relevant evidence, that is to say, must be symmetrical with regard to the alternatives, and must be applicable to each in the same manner. This is the rule at which the Principle of Indifference somewhat obscurely aims. We must first determine what parts of our evidence are relevant, on the whole by a series of judgements of relevance, not easily reduced to rule, of the type described above. If this relevant evidence is *of the same form* for both alternatives, then the Principle authorises a judgement of indifference.[70]

In short:

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[68]Keynes, J. M., *A Treatise on Probability*, London, 1929 (first published 1921), p. 42
[69]*Ibid.*, p. 53
[70]*Ibid.*, p.56

> The apparent contradictions arose from paying attention to what we may term the *extraneous* evidence only, to the neglect of such part of the evidence as bore upon the form and meaning of the alternatives.[71]

But we contend that interest and curiosity are essential to the pursuit of the truth. We believe that the truth is discovered by degrees. If a prior is to be applied at the outset of an analysis, then we should be prepared to modify it, as the further course of the analysis might indicate. We have written above of the plausibility of a sum being an approximation of an integral to the extent that it arises in the limit of repeated differentiations with respect to what we know of the locations of the sensor elements. The 'prior', as it were, must keep pace with the iterations; it too must adapt itself in the progress towards convergence. Repeated differentations, guided by interest (as opposed to the apathy of indifference), give rise to an increase in 'entropy' rather than to the vindication of the choice of some prior.

We cannot dispense with interest. However, although an infinite number of point sources with an infinite number of frequencies may be 'out there', the Algorithm is sufficiently continuous and flexible for 'a miss is a good as a mile' to be avoided. Because the semantics of the sources of interest in the present research are patently simple compared with those of *War and Peace*, the resistance to noise of them is rather greater. For example, if we get close to a frequency of the source, we shall soon know it. We shall see a total rise to a sum as we get to the truth of the matter, the true frequency. Likewise, if we are a little off the target, we can work out fairly quickly in what direction we should go to get to it. Plainly, *War and Peace* does not contain that degree of redundancy. There is a radically smaller theoretical basis for guessing what piece of text, no matter how small, has been lost, if such has been lost.

8.3 Isotropy; Binaurality and the Decibel

Consider the consequences of a source not seeming the same whichever direction we perceived it from. The undersea acoustic environment is not one in which human beings are at home. The question of how principle relates to experience in that ambience (as opposed to man's natural habitat) does not arise. The acoustic activity, by its very nature, is one that does not

[71] *Ibid.*, p. 61

employ the whole human sensorium. With our problem of passive *SONAR*, it remains to be seen what progress can be made by using optical sensing.

Man is accustomed to using his two ears to determine the direction of a sound system of interest. He moves his head from side to side, or moves whole-bodily, in order to get the particular direction of interest. The exercise is based upon a series of comparisons, a convergent one indeed. The notion of convergence is important, and it is indissoluble with getting the right answer. It is a case of applying successive approximations. For there to be convergence, there must be a signal of interest. Again, the system consists fundamentally of the mutually-inclusive source and auditory sensors.

A corollary of these considerations is that convergence cannot take place in the abstract. There must be interest for there to be a goal to aspire to. We argue that interest is the phenomenon with which convergence may properly be associated. On the other hand, it may be that an approach predicated upon a more or less arbitrary prior cannot theoretically entail convergence, a prior being regarded here as an abstraction and, to that extent, deprived of the subjectivity that informs interest. Whatever the practice may be of approaches involving 'priors', we suggest that the theoretical basis of using a prior effectively involves guessing the probabilities which Borel called 'discontinuous'.[72] Such discontinuous probabilities can never be the basis for

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[72]Borel was faced with an apparently customary discontinuity and irreconcilability between 'discontinuous probabilities', on the one hand, and 'continuous probabilities' (or 'geometric probabilities'), on the other:

> On distingue généralement, dans les problèmes de probabilités, deux catégories principales, suivant que le nombre des cas possibles est fini ou infini: la première catégorie constitue ce que l'on appelle les *probabilités discontinues*, ou probabilités dans le domaine du discontinu, tandis que la seconde catégorie comprend les *probabilités continues* ou *probabilités géométriques.*

(See E. Borel, 'Les probabilités dénombrables et leurs applications arithmétiques', *Rend. Circ. Matem. Palermo*, t. XXXIII (1^0 sem. 1909), p. 247.) However, Borel continues, the notion and theory of the ensemble may seem to cause this classification to appear incomplete. Borel proposes to interpose 'les probabilités dénombrables':

> Une telle classification apparaît comme incomplète, lorsque l'on se reporte aux résultats acquis dans la théorie des ensembles; entre la puissance des ensembles finis et la puissance du continu se place la puissance des ensembles dénombrables; je me propose de montrer

convergence, since a prior, by its very nature, is posited at the beginning of a process, it being the hope that its correctness will be vindicated by the completion of the process. The problem is then how to decide whether the result one has achieved vindicates the prior or not. Again, the terms 'optimal' or 'suboptimal' may be thought of as conveying this difficulty. Perhaps Borel's 'probabilités dénombrables' may help to provide an appropriate theoretical basis for seeking unique solutions to 'fuzzy' problems.

'Interest' involves curiosity, which requires discrimination and differentiation, rather than the potentially arbitrary decision-making that may be involved with a prior or indeed 'Bayesian' approach. Indeed, there is disagreement about the theoretical basis of the notion of 'inverse probability', with which the so-called Bayesian methods are often associated.[73]

We believe, indeed, that, with 'optimal' or 'suboptimal' results, the choice of one result rather than another is itself going to be a matter of preference arising from interest.

With two ears, a series of comparisons, one side with the other, is set up, with a view to convergence. Convergence is readily feasible, for example, if the source of the signal is relatively stationary. The police-car siren shows the implications of the absence of stationarity, or at least of the case where the hearer has no control over his location with respect to a possible source.

It is so much a comparative exercise, one side, then the other, or two different states on the one side compared with each other (either way, motion, change of position, must be involved, or else redundancy in the sense used above will be excessive), that, in much conventional hydrophone array processing, binary comparative levels are the norm. That is to say, a 'reference' level is compared with an actually discovered acoustic phenomenon. The two are combined in the *decibel*. Clearly, the argument of the logarithmic function must have zero dimension, so numerator and denominator must be homogeneous.

> brièvement l'intérêt qui s'attache aux questions de probabilités dans l'énoncé desquelles interviennent de tels ensembles; je les appellerai, pour abréger, *probabilités dénombrables.*

(*Ibid.*, p. 247.) That notwithstanding, Borel is content, at the end, to agree with Georg Cantor that 'le continu n'est pas dénombrable' (*ibid.*, p. 271), that is to say, in the geometric continuity the existence of elements that cannot be defined cannot be denied.

[73]See again R. A. Fisher, *Statistical Methods and Scientific Inference*, Second Edition, Edinburgh and London, 1959, p. 17.

We argue that much hydrophone array processing is founded on the assumption of binaurality. To avoid the confusion of a 'Doppler effect', the source must be stationary. The Kalman filter seems not to vitiate the reservations we have rehearsed above about the relationship of prior approaches and interest.[74] After all, the actuality of a police car can be determined in the last analysis only by seeing that that is what is making the sound. Again, we may not be able to see the source under water.

However, the property of isotropy is to our advantage, given the limited portion of the sensorium available to us for undersea detection and location estimation. We argue that, so much is isotropy a spatial concept, that it belongs properly to the tactile and visual sphere of competence. The very notion of isotropy in respect to a frequency point-source makes up for the absence of the sense of vision with our problem.

The decibel may turn out to be based at root on the notion and practice of binaurality. Binaurality is competent for pin-pointing remote sources if the listener is able to move around, thus creating a 'synthetic aperture' for his audio-sensual apparatus. He has, by virtue of his ability to move about, the potential to be for his purposes an infinitely long antenna. Few difficulties are involved here if the source is not moving unduly with respect to the human listener. For the quasi-instantaneous location estimation which we have pursued in the present thesis, we require as much stationarity as possible. Of course, we cannot request and require that the particular source of acoustic radiation be stationary, but we can at least see to it that our auditorium is as stationary as possible. Our walking-about, as it were, must be accomplished without having to walk a step. For that to be feasible, we must have a greater choice available than just the interplay between two ears. We need multi-aurality rather than binaurality. For this reason, we have seen fit not to try to convey our information in the present thesis in terms of the decibel.

Binaurality may be seen at work in many information systems. The Shannon system of telephonic communication involved, in essence and at root, the idea of comparing two states, true or false, in a decision-making process. Such a system has been thought to convey information optimally

[74]If '... $S(n)$, the signal, is a zero-mean Gaussian sequence ... $S(1)$ is a zero-mean Gaussian variable ... Its variance is also assumed to be known.' K. S. Shanmugan and A. M. Breipohl, *Random Signals: Detection, Estimation and Data Analysis*, New York, 1988, pp. 419 and 421

with respect to noise. But it may be that its slowness is a price one has to pay for such an optimal treatment of noise. For seriality is given absolute primacy over parallelity, and that cannot in itself make for a plausible case of efficiency. We argue that the multi-hydrophone disposition provides the parallelism that many systems do not. 'Parallel' is perhaps not a good term, because, as we have said before, simultaneously-taken hydrophone readings are functions of time. But we can say of the readings that they are synchronous rather than parallel in that sense, and in the sense of Borel's 'probabilités dénombrables' we might postulate erecting an argument upon central values a_0, ρ_0, ν_0, c_0 and ϕ_0 with, if not infinitesimal, then at least very small differences in the variables from one hydrophone to another at the time the single snapshot is taken.

We raise for future discussion the theory that binaurality has fuelled the notion of the point source 'at infinity', that it has to some extent imposed the one-dimensionality of 'direction of arrival' upon the business of using towed arrays, and perhaps, because of that, has cast a shadow over the search for an algorithm for locating a source in two dimensions, a search we have sought to pursue in the present thesis. Indeed, multi-hydrophone systems have to an extent been treated redundantly as if their potentially multi-facetted information could give up only one-dimensional intelligence.

8.4 Direct and Indirect Evidence of Elapsed Time

The ur-function, as it were, archaeological evidence of which is available in a single snapshot, has been assumed in the present analysis to be a continuous function of time. True, we have isolated five factors of it, namely a, ρ, ν, c and ϕ, but they have been given from a process of induction and deduction. The integral $f = \int df$ cannot be reconstructed without knowing how each of a, ρ, ν, c and ϕ have altered, if at all, with time. We conceive of the integral alone; we have no means of controlling the degree of rate of differentiating.

8.5 The Undercorrecting Lens

A disposition of hydrophones may be thought to relate to the radiation field in the medium in some ways as an optical lens interacts with that which passes through it.

It is in the nature of the present thesis that we are not concerned with a taxonomy of 'expected' or 'wanted' or otherwise 'desirable' arranging of sensors. Indeed, we argue that the usefulness of any arrangement of hydrophones must be tested in relation to the possibility of sources being in various different places, and that, in particular, no 'ideal' hydrophone array can be conceived of that is independent of sources and their locations. No one arrangement of hydrophones is a horse of all work in this regard.

Nevertheless, there is no need here to embrace a counsel of despair. It is possible, for example, to see a distortion in things rather then the mere absence of them. Objects do not necessarily disappear, but they can change shape, as a lens is changed in its constituency. It is not necessarily only the shape that changes. In some cases interest is a more important consideration than the characteristics of the lens. If an object is out of focus, it does not mean that all sense of distance, let us say, is absent too. It is really interest that causes one object to be distinguished from another on what is essentially the flat surface which the retina presents to the world. Motion is surely the key factor in being able to differentiate between one quantity or quality and another. One needs to compare channels of perspective, one with another. A single hydrophone can do little to help us, unless a number of singular circumstances obtain, including no changes in a, ρ, ν, c and ϕ. The more hydrophones there are available to poll, the more lines of sight upon the source are afforded. Only under singular circumstances is there anything to be said for using one particular channel over and over again. In passing, we may refer to the notion of parallel lines all meeting, as it is said sometimes, at a 'point at infinity'. In the terms of our discussion above, assuming there is any merit in the argument at all, such channels can scarcely be said to be doing work which is not being duplicated by more or most of their neighbours. Here again, a perhaps unnecessarily high degree of redundancy through duplication is the consequence of the 'point source at infinity', if our arguments about work, semantics, teleology, noise, signal and redundancy are accepted in the present context.

8.6 Work, Noise, Signal, Semantics and Redundancy

The notion of redundancy had no place in semantics. Redundancy is proper to the mode and means of expression. But the mode and means of expression and communication are not passively dependent upon the meaning (semantics) of a message. On the contrary, they must be brought forward to meet the semantic substance. They must do work in this sense. By all means, the characteristics of the particular channel of communication can be prescribed so that, if there is risk of losing part of the message, the endangered part can be recommitted to the system of communications, whether it be a serial activity or a parallel one. But the provision of suchlike safeguards can create redundancy in other, ancillary parts of the system. Rather than dwell upon the passive resistance to noise of redundancy, we look to see what work the channel or channels of communication contribute to moving the message through the system.

We have tried to establish and justify a few abstract notions of efficiency and work. Flexibility and adaptivity are not in themselves easy qualities to define. Indeed, it is central to the present thesis that the source and hydrophones cannot be distinguished in a system, but that indeed both are continuous with respect to each other and are mutually contiguous elements of the system. The key abstraction of 'flexibility' and 'adaptivity' or 'adaptibility' is enshrined, in the present argument, in the word 'differentation'.

We look at an object along different lines of sight and we assume ('integrate') an image for our own use. Whether we call it a comprehensive or accurate image (that we put together in the brain), depends upon our interest.

As indissoluble systematically as sources and hydrophones are 'integration' and 'differentiation'. 'Integration' is like the principles, not known before experience, but, like the principles (there are no absolute ones), must be justified in the light of experience.

8.7 Three Arguments in Favour of Binning

We may think of binning as an embracing of a phenomenon, the quantity and quality of which defy known instruments of measurement. We avoid applying the strait-jacket to a thing which eludes the rational weighing and measuring. Instead of trying to re-section the unknowable to fit the Procrustes-bed of our rationality, we accommodate it as well as our generosity allows. We do not nail it to the mast of preconception, but let it enjoy the degrees of its freedom.

We are remote from the source of the acoustic disturbance, in both space and time. We may know when the effects occur, which are our evidence of activity elsewhere and in the past. But we cannot be at the places or at the times at which what we have recorded, in spatial succession with our hydrophones, were excited. We can only say to what, within reason, we might have reason to attend. We apply the binning to what we suppose might have happened diachronically, trusting in a synchronicity of our measurement-taking. In what is an otherwise uncertain enterprise, the synchronicity of sample-taking may close down on some of the more disturbing vicissitudes that might take place, were more time available for their completion.

We must be indulgent with an instrument designed to serve us in places where we ourselves could not survive. An uncertainty of some kind is bound to arise with an instrument designed for such a purpose. But uncertainty of hydrophone-location is not the only uncertainty, or even the most intractable uncertainty, of the system with which we must work. Our difficulty arises fundamentally from our remoteness, in space as well as time, from something the relevance of which to us we cannot be certain of.

9 Apology

I have been trying to solve a problem in hydroacoustics involving the locating of a spherically-spreading point-source by means of a finite number of omni-directional hydrophone sensors.

In the past, the problem had been posed in such a way that I could proceed by first proving that the method I used was such that a unique determination of the location of the source was obtained in the absence of noise.

But then I wished to solve the problem of how to prove that an estimate might be associated uniquely with the location of a source. For I now posed the substantive problem in terms which did not permit a unique determination in the absence of noise. That is to say, the 'noise' I introduced was of a kind that it could not be treated in the limit of a process which exploited repeated observations. Nevertheless, I have devised an algorithm which produces an estimate with which evidently the location of the source may be associated uniquely.

I believe that there is a technical term 'fuzzy' which may be appropriate for my new formulation of the problem. I should like to be able to prove that an estimate or approximation of an integral is obtained in the limit of a process involving repeated differentiations with respect to what is known about the locations of the hydrophone sensors.

I believe that I have completed the 'Engineering' phase of my work. I should like now to prosecute my search for the theoretical basis of the practical result in the context of Analysis and of the 'philosophy of integration'.

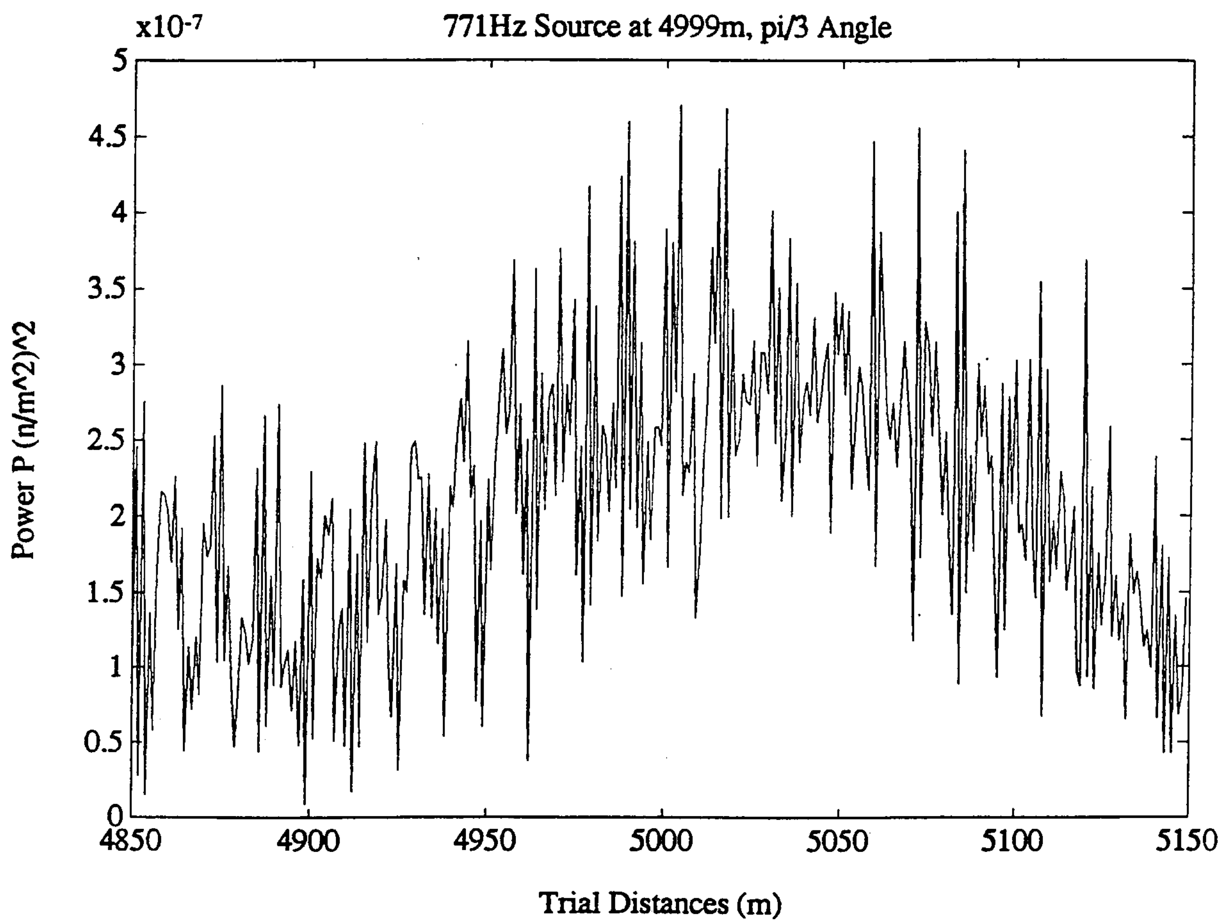
10 Appendix: Example and Computer-Program Listing of the Binning Algorithm

The example of graphical evidence following arises from an attempt to locate a point source at 4999 metres distance and $\pi/3$ degrees to the longest baseline presentable by a sinusoidally-shaped hydrophone array of 693 hydrophones in the presence of noise, with noise-to-signal ratio of 693. The source has a frequency of 771Hz.

It may be noticed that the graph-plot profile here is ragged in comparison with the graphical evidence in Chapter 6 above. This may be laid to the comparatively greater amplitude of the array sinusoid. Here it is 100m, while it was not more than $\frac{67}{d}$ in Chapter 6. It may be noted too that the step of trial distance increment is 1m in the present case, while it was $\frac{37d}{40}$ in Chapter 6.

The suite of six *MATLAB* program-units which generated Figure 28 follows.

Figure 28: Located by 'binning method': point source at a distance of 4999m and angle of $\frac{\pi}{3}$ with $NSR = 693$



```
%program LAUNCH.M;
%supplies variables for
%program ASHAPE.M;
%program PANGL.M;
%program SOURCE.M;

%for ASHAPE.M;
seednumber=673;
numhphones=693;
lateral=100;
q=1/1000;

%for PANGL.M;
initd=4850;
tangle=pi/3;
ntestds=301;
c=1500;
freq=771;
dro=1;

%for SOURCE.M;
sdista=4999;
sangle=tangle;
newsnum=451;
intramp=1;
phi=0;
nsr=693;

%G. W. Sweet, 24/3/93;
```

```
%program ASHAPE.M;
%generates the x- and y- coordinates of
%hydrophones in a sinusoidal array.
%lateral: lateral amplitude of the array;
%numhphones: number of hydrophones;
%seednumber: seed number;
%q: the local hydrophone location uncertainty
%(hx(i),hy(i)): the nominal coordinates;
%(hxtrue(i),hytrue(i)): the actual coordinates;

rand('seed',seednumber)
hx(1)=0;
hy(1)=0;
hxtrue(1)=0;
hytrue(1)=0;
for i=2:numhphones;
   hx(i)=i-1;
   hy(i)=lateral*sin(2*pi*(i-1)/numhphones);
   hxtrue(i)=(1+q*2*(rand(1)-1/2))*hx(i);
   hytrue(i)=(1+q*2*(rand(1)-1/2))*hy(i);
end;

%G. W. Sweet, 15/3/93;16/3;17/3;
```

```
%program SOURCE.M;
%generates a signal.
%intrinsamp: intrinsic amplitude of the signal;
%pressure(i):  the acoustic pressures in the
%absence of noise;
%p(i): the acoustic pressures measured in the presence
%of noise;
%sdista: distance of the source from the origin;
%sangle: angle of the source;
%newsnum: new seed number;
%nsr: noise to signal ratio;
%phi: phase angle;

sx=sdista*cos(sangle);
sy=sdista*sin(sangle);
rand('seed',newsnum);
noise=sqrt(nsr);
for i=1:numhphones;
   sd(i)=sqrt((sx-hxtrue(i))^2+(sy-hytrue(i))^2);
   pressure(i)=(intramp/sd(i))*cos(2*pi*(freq/c)*sd(i)+phi);
   p(i)=pressure(i)+noise*2*(rand(1)-1/2)*(intramp/sd(i));
end;

%G. W. Sweet, 24/3/93;
```

```
%program PANGL.M;
%for locations and frequency of interest,
%calculates the phase angles of signal
%upon arrival at the hydrophones' nominal locations.
%initd: initial trial distance from origin;
%tangle: angle of interest (held constant in this program);
%freq: frequency of interest;
%c: speed of sound in water;
%numhphones: number of hydrophones;
%dro: distance increment in loop;
%ntestds; number of test distances;
%The phangle(i,j) are the phase angles;
%they are input to program BINNING.M.

tdista=initd;
for i=1:ntestds;

    r(1)=tdista;
    tx=tdista*cos(tangle);
    ty=tdista*sin(tangle);

    for j=2:numhphones;
        r(j)=sqrt((tx-hx(j))^2+(ty-hy(j))^2);
        pangle(i,j)=rem(r(j),c/freq);
    end;

    tdista=tdista+dro;
end;


%G. W. Sweet, 24/3/93;
```

```
%program BINNING.M;
%generates binning sets
%for the phase angles ('phangle(i,j)')
%generated by program PANGL.M.
%REM i:index of test distance;
%REM j:index of hydrophone;
%REM ntestds: number of test distances;

for i=1:ntestds;
    for j=1:numhphones;
       if pangle(i,j)==(c/freq)/3
          binno(i,j)=2;
       elseif pangle(i,j)<(c/freq)/3
          binno(i,j)=1;
       elseif pangle(i,j)==2*(c/freq)/3
          binno(i,j)=3;
       elseif pangle(i,j)>2*(c/freq)/3
          binno(i,j)=3;
       else
          binno(i,j)=2;
       end;
    end;
end;

%G. W. Sweet, 24/3/93;
```

```
%program PROC.M;
%matches source
%with the binning sets generated
%by program BINNING.M.
%The 'b1', 'b2' and 'b3' are the measured pressures
%accumulated in the three bins.
%The average in each bin is obtained by dividing
%each by the appropriate 'numinb1', 'numinb2' or
%'numinb3'.

for i=1:ntestds;
   b1=0;
   numinb1=0;
   b2=0;
   numinb2=0;
   b3=0;
   numinb3=0;
   for j=1:numhphones;
      if binno(i,j)==1
         b1=b1+p(j);
         numinb1=numinb1+1;
      elseif binno(i,j)==2
         b2=b2+p(j);
         numinb2=numinb2+1;
      elseif binno(i,j)==3;
         b3=b3+p(j);
         numinb3=numinb3+1;
      end;
   end;
   if numinb1==0
      a=0;
   else
      a=(b1/numinb1)^2;
   end;
   if numinb2==0
      b=0;
   else
      b=(b2/numinb2)^2;
   end;
   if numinb3==0
      cc=0;
   else
      cc=(b3/numinb3)^2;
   end;
   power(i)=a+b+cc;
end;

%G. W. Sweet, 24/3/93;
```

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