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    COMPUTATION OF MAGNETIC FIELDS IN SATURATED
IRON STRUCTURES WITH SPECIAL REFERENCE TO THE
COMPUTATION OF SHORT CIRCUIT PERFORMANCE OF
    INDUCTION MACHINES WITH WOUND ROTORS
    A thesis presented for the degree of
        DOCTOR OF PHILOSOPHY
        of the
        UNIVERSITY OF SOUTHAMPTON
        in the
FACULTY OF ENGINEERING AND APPLIED SCIENCE
DEPARTNENT OF ELECTRICAL ENGINEERING
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ELECTRICAL ENGINEERING
Doctor of Philosophy
COMPUTATION OF MAGNETIC FIELDS IN SATURATED IRON STRUCTURES WITH SPECIAL REFERENCE TO THE COMPUTATION OF SHORT CIRCUIT PERFORMANCE OF INDUCTION MACHINES WITH WOUND ROTORS by Ivan Mandić

The thesis is principally concerned with the solution of nonlinear field problems with particular reference to the computation of the magnetostatic field in magnetically saturated electrical machines. The field is divided into a core region for which a twodimensional solution is obtained and an end region, the analysis of which takes account of the three-dimensional geometry using the method of images.

The influence of saturation in the core region is explored by solving the non-linear partial differential equation in terms of véctor potential. The magnetic field in the core region can therefore be described by a two-dimensional mildly non-linear elliptic partial differential equation. This equation can be solved approximately by different discretization techniques in which the problem is transformed into one of solving a set of non-linear equations. Different possibilities for discretization have been compared and it has been found that the discretization mesh consisting of triangles and having free topology has advantages over some other types of discretization. The necessary number of mesh nodes for given accuracy has been found by numerical experimentation. Several methods for the solution of the set of non-linear algebraic equations arising from discretization are compared. Numerical
experimentation shows that two-step line iteration is superior to some other methods.

The computation of the inductance associated with the end region allows for the practical shape of end windings and the influence of the steel core.

The possibilities for the practical application of the proposed method to the computation of short circuit performance of induction machines with wound rotors is illustrated with the detailed computation of an example.

## ACKNOWLEDGEMENT

The work described in this thesis was carried out while $I$ was on leave from the Electrotechnical Institute 'Rade Koncar', Zagreb, Yugoslavia. I am grateful to this Institute for the permission to work at Southampton, and for their financial support during my stay in Southampton. During the three years of my research I held the research studentship of the University of Southampton, to whom I am also very grateful.

The number of people who helped me in my work is large and I cannot mention them all. I should like, however, to express my gratitude to the following people: Professor P. Hammond and Mr. K.J. Binns, my supervisors, for their guidance and encouragement; Dr. B. Frančić of 'Rade Koncar' who helped to organise my stay in Southampton and Miss S. Parker who carefully typed the manuscript. Finally, I wish to express my gratitude to my wife Margareta, who accepted personal sacrifices without questioning and always found words of encouragement when things did not go so well.

## LIST OF PRINCIPAL SYMBOLS

Unless otherwise stated in the text, the symbols used in this work have the following meaning:

A $z$-component of the magnetic vector potential
A magnetic vector potential
B magnetic flux density
B magnetic flux density vector
D electric flux density vector
d differential operator
E electric field strength vector
H magnetic field strength
H magnetic field strength vector
$\underline{i}$ unity vectors in the direction of $x$ and $y$ coordinate axes, respectively

J electric current or current density
J electric current or current density vector
L inductance
M total number of triangles in the mesh
n total number of nodes in the mesh
R resistance
s slip
S area
$t$ time
v volume
X reactance

Cortesian coordinates
$\varepsilon \quad$ permittivity
$\mu \quad$ permeability
$\mu_{0} \quad$ permeability of free space
$\mu_{r} \quad$ relative permeability
$v$ reluctivity
$\pi \quad 3.14 \quad 159 \ldots$
$\rho \quad$ electric charge density
a partial differential operator
$\nabla \quad$ Hamilton's operator

## CHAPTER 1

INTRODUCTION

The induction motor is today one of the most common electrical devices. It is used in homes and in industry, and there is hardly a branch of technology that does not use the induction motor as a source of mechanical power. It is built in sizes ranging from a fraction of a Watt to several Megawatts, and the total number of induction motors is constantly increasing.

A constant desire of the designer of the induction machine to produce a lighter, cheaper and better motor has caused a dramatic increase in power rating per volume (Ref. 1.1). A part of this increase is due to the development of new and better materials, but a good deal of it is due to better and more efficient use of these materials, i.e. to the design of the machine. The trend towards higher power ratings for the same size of the machine has pushed the utilisation of materials to the limits and this has produced new problems.

One of these problems is the non-linear characteristic of the induction machine due to the non-linear magnetic properties of the iron. The behaviour of the machine depends on the distribution of magnetic field in the machine, which in turn depends on the properties of the magnetic steel laminations used in the machine. As the magnetic flux density increases, the non-linearities become more pronounced, the designing process more complicated and less accurate.

In this thesis we shall try to predict the behaviour of the machine more accurately by more accurate computation of the
magnetic field distribution in the machine. We shall restrict our analysis to non-skewed induction machines with wound rotors and to the locked rotor condition only. Non-linear magnetic properties of iron will be fully considered.

In section 1.1 we give the usual equivalent circuit and explain the common method of evaluating the magnitude of the magnetic flux in the machine.

Section 1.2 deals with the behaviour of the induction motor with locked rotor. It is explained that increased current at short circuit may cause saturation of some parts of the magnetic circuit at a much lower voltage than at full load conditions or at synchronism.

In section 1.3 we present a brief survey of some earlier attempts for the computation of short circuit current.

### 1.1 The Equivalent Circuit of the Induction Machine

In this section we shall briefly explain the usual equivalent circuit of the induction motor with wound rotor.

The common approach in the analysis of the induction machines is through so-called equivalent circuits. A simplified equivalent circuit for one phase of the machine is represented in Fig. 1.1. $R_{1}$ and $X_{1}$ represent stator winding resistance and stator leakage reactance, $R_{2}^{\prime}$ and $X_{2}^{\prime}$ rotor winding resistance and rotor leakage reactance referred to the stator. The value of $\mathrm{R}_{2}^{\prime}(1-s) / s$ represents the load, $R_{0}$ losses in the iron core and $X_{o}$ the magnetising reactance.

In the analysis of the induction machine one of the fundamental problems is the computation of inductances associated with

$=$
Fig. 1.1 The Equivalent Circuit
$X_{1}, X_{2}^{\prime}$ and $X_{0}$. These inductances depend on the distribution of the magnetic field in the machine and in order to determine them it is necessary to know the magnetic field distribution. In the design process it is usually assumed that the total flux in the machine can be split into several parts and the magnitudes of these partial fluxes are calculated by relatively simple formulae. Different authors define different partial fluxes, but usūally the following are considered (Ref. 1.2):
main flux
rotor and stator end-winding flux
rotor and stator slot leakage flux
leakage due to skew
differential leakage flux.

These partial fluxes are themselves often calculated as sums, and stator slot leakage, for example, would consist of flux crossing the stator slot plus tooth-tip leakage flux. The differential leakage flux represents in principle an infinite sum of all higher harmonics, but usually only first harmonics caused by phase-belts and slots are considered.

The equivalent circuit of Fig. 1.1 represents the voltage relations in the machine very crudely. The representation can be improved by the addition of elements representing the influence of higher harmonics. To a certain extent the influence of saturation can also be dealt with by the addition of branches representing this influence (see, for example, Ref. 1.3). However, the accuracy that can be achieved by this simplified analysis is limited by the accuracy of the elements of the equivalent circuit, and if this accuracy is not improved, the accuracy of the method cannot be improved beyond a certain limit.

### 1.2 The Induction Machine on Short Circuit

The equivalent circuit diagram of Fig. 1.1 is valid in principle for any working condition, However, if some parts of the magnetic circuit are saturated, the effects of non-linearities introduced in this way may be much stronger at short circuit, and the accuracy of computation lower.

The usual values of stator and rotor resistances $R_{1}$ and $R_{2}{ }^{\prime}$ are of the order of 0.01 to 0.05 per unit, and the magnetizing reactance $X_{0}$ is of the order of 2 to 4 per unit (Ref. 1.4). With the rotor at standstill, the load resistance $R_{2}{ }^{\prime}(1-s) / s$ is zero, and the starting current will be largely determined by the leakage reactances $X_{1}$ and $X_{2}$ '. As these have to be kept small if a high power factor is sought at nominal working conditions, the short circuit current is large, usually about six times the nominal current (Ref. 1.4). Hence the leakage fluxes will be high too. If the flux densities in some parts of the equivalent circuit are high enough to saturate the iron, the leakage reactances will be reduced, resulting in even higher starting current. This effect may occur at much lower voltage than the saturation effects at the nominal working point.

As virtually all modern machines are built with flux densities in the knee of the $\mathrm{B}-\mathrm{H}$ curve, the influence of saturation on starting current may be very high, on average increasing the magnitude of starting current by about $30 \%$ (Ref. 1.5). High starting currents represent difficulties regarding the supply, and the designer is presented with the problem of limiting starting currents to an acceptable level, and of computing these currents accurately. The simplest approach is to adjust the results by some
empirical coefficients, but this may fail to produce acceptable results (Ref. 1.7), and more sophisticated methods are necessary. In the following section we shall briefly describe some earlier solutions.

### 1.3 Note on the Work Carried Out by Previous Researchers

The decrease in reactance of induction motors at short circuit due to the saturation of magnetic paths in the machine was recognised in the late 1920 (Ref. 1.8). Consequently, numerous attempts have been made to develop methods which would enable predictions to be made of the drop in reactance and an accurate evaluation of the short circuit current. In principle, this problem can be solved by three different means: by experiment, by mathematical modelling and by statistical analysis.

The use of statistics is probably the simplest way of achieving the goal, because it does not require a complete understanding of the physical phenomena. However, the degree of saturation is a function of many parameters and difficulties may be encountered in attempting the solution of a problem by purely statistical means. In addition, such methods have only a limited value if applied to new designs.

Experimental investigations of the effect of saturation require a physical model of the machine. This is a major drawback of such methods, as the only accurate model of the machine is the machine itself. Hence, this method is not suitable for prediction of short circuit currents in new designs, although measurements of flux distribution on real machines or models (Refs. 1.5, 1.9) can help in understanding of the physical phenomenon.

The most suitable method for the prediction of short circuit current of induction machines is mathematical modelling. The majority of published reports on the subject belong to this group (Refs. $1.4,1.6,1.10-1.16$ ). In some cases, however, the mathematical modelling and statistical analysis are combined, the prediction based on a mathematical model being corrected by statistical means (Ref. 1.17). A short survey of most of the early findings can be found in Ref. 1.14 , and we shall restrict our consideration to a few more recent papers.

We have mentioned in section 1.2 that the leakage reactances of rotor and stator are determined as a sum of different parts corresponding to slot-leakage flux, tooth-tip leakage flux, etc. The partial fluxes are determined by relatively simple expressions, which are derived on the basis that the flux pattern of partial fluxes is known. This basic procedure is unchanged in all papers dealing with saturated values of short circuit current (Refs. 1.4, 1.6, 1.10-1.17). The difference is in the introduction of 'saturation factors' which are applied to different parts of leakage reactances. Different authors derive these factors in different ways.

The paper by Chang and Lloyd (Ref. 1.11) presents a semiempirical method for the computation of saturated values of short circuit leakage reactances. The authors assumed that only zig-zag leakage flux and tooth-tip leakage flux are affected by saturation. In their method minimum iron areas are found in magnetic paths for zig-zag and tooth-tip fluxes. The values of tooth-tip and zig-zag flux are determined as if there were no saturation. Then the saturation factors for corresponding parts of leakage reactances are determined from the universal saturation curve.

Agarwal and Alger also consider only the influence of toothtip and zig-zag leakage fluxes in their paper (Ref. 1.4). In their method tooth-tip and zig-zag leakage fluxes are combined in order to determine the iron area which will saturate. A step-function approximation for $B-H$ curve is used to determine the maximum value of flux through the saturated area. The saturation factor obtained as a ratio of current at which saturation occurs and the actual current is applied to either both zig-zag and tooth-tip reactance, or to tooth-tip reactance only, depending on the position of the saturated area.

Angst in his paper (Ref. 1.6) used basically the method of Agarwal and Alger and extended it by considering skew leakage flux also.

A series of papers by Cigánek are all related (Refs. 1.121.15). His method consists of an iteration procedure. In the first step the magnetic flux densities in teeth and tooth-tips of the machine are determined with the assumption of constant permeability. The fictitious value of flux density in the teeth obtained in such a way is used to determine the saturated value by a graphical method using the actual $B-H$ curve. The decrease in reactance is computed, new value of current determined and the whole procedure repeated. According to Ref. 1.13, two iteration steps give adequate accuracy.

In a later paper by the same author (Ref. 1.14) a similar method was applied for the determination of the influence of saturation in teeth. The influence of saturation of tooth-tips is taken into account as the effective increase of slot openings. This increase is computed from geometrical dimensions of tooth-tips
and maximum value of current in the slot. A separate correction factor obtained in this way is applied to tooth-tip reactance only.

In Ref. 1.15 the saturation in the tooth body was neglected. It was shown in one example that the flux in tooth-tips does not depend on the overall flux distribution, and that adequate saturation factors can be obtained by considering tooth-tip flux only. The $B-H$ curve was represented by an exponential function, and graphical integration along the flux paths in the tooth-tip areas of stator and rotor was used in order to determine saturation factors for tooth-tip flux.

Of all published reports on the evaluation of short circuit current of induction machines, only Chalmers and Dodgson presented a method primarily intended for use on the computer (Ref. 1.16). Consequently, their method could employ more complex computation as compared with methods intended for manual computation. Their method is related to methods by Agarwal and Alger (Ref. 1.4) and Angst (Ref. 1.6). Like these authors, Chalmers and Dodgson also use the step-function approximation for the $B-H$ curve. In their method they distinguish between several magnetic paths for different partial fluxes corresponding to tooth-tip leakage, zigzag leakage, etc. Different flux paths coincide partially with each other. Partial fluxes are computed as if there were no saturation in per unit values and superimposed to each other. The total flux could be said to pass through a certain number of critical areas where the flux densities will have the highest value. The area with the highest flux density is found, and all partial fluxes passing through that area fixed to the value determined by the maximum flux density from the step function $B-H$ curve.

If the magnetomotive forces are not in balance with the flux, the area which will saturate next is found, etc. The process is repeated until the balance between magnetomotive forces and flux is reached. Then the total reactance of the machine is found by summing the contributions of all partial fluxes. This method used as a part of an iteration procedure will give a starting current at a specified voltage.

Correlation of the results of this method and measurements on actual machines depends on the choice of the maximum flux density in steel laminations. No other method than statistical analysis can provide the best value of this flux density, and this method must also be regarded as semi-empirical to a certain extent.

Although the methods described by these different authors differ from each other, they also have several common points. They are all based on an approximate magnetic field solution of the linear problem. This approximate solution is obtained by superposition of a number of partial fluxes, which are computed assuming that the reluctance of different parts of magnetic circuit (air-gap, tooth body, etc.) is constant and can be computed from geometrical dimensions only. The different parts of the reactance obtained in this way are later adjusted by factors depending on the saturation level in the machine. The accuracy that can be reached by any of these methods is limited by the accuracy of the basic field solution, and it may be expected that the accuracy is lower as the flux density increases, because the flux pattern will differ more from the flux pattern obtained by the usual computation if the relucitvity of steel is higher.

Hence, the key to the more accurate starting current is more accurate field distribution. In the following chapters we shall try to find a method which will enable us to solve the magnetic field in the machine more accurately, and which can be used to determine the short circuit current more accurately than is possible by means of the methods previously described by other authors.

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This chapter is concerned mainly with the computation in the end-region of the machine.

In section 3.1 we point out the basic characteristics of different regions of the machine and suggest different approaches for the computation in the end-region and in the core region.

Necessary simplifications of the geometry of the end-region are introduced and explained in section 2.2 .

Section 2.3 is concerned with the field distribution in the end-region, and it is pointed out that the method of images, which is dealt with in some detail, can form a basis for the computation of end-winding inductance.

The formulae for the computation of the inductance, based on the method of images, are derived in section 2.4 , while in section 2.5 we give a brief description of the computer program developed on the basis of these formulae.

In the final section of this chapter we give a short discussion of some recently published papers dealing with the same problem, and we also point out some limitations and possibilities of our method. Some possibilities for further development are also mentioned.

### 2.1 Splitting of the Machine into Regions

An induction machine is a complex three-dimensional structure. Its behaviour under working conditions depends on the distribution of the electromagnetic field in it. This field is a
a single entity; all active parts of the machine will have an influence on the distribution of the electromagnetic field in different parts of the machine. Strictly, an attempt to solve the field in one part of the machine only, cannot give the correct answer. Yet, from the practical point of view it is desirable to analyse the field in different regions of the machine separately. This simplifies the problem significantly and allows for the use of different computational techniques in different regions.

In our analysis we shall distinguish two different regions in the machine: the end-winding region and the core region. The endwinding region is characterised by the complex three-dimensional geometry of the coils which are situated in the air. The core region is characterised by the much simpler geometry of the windings which are embedded in the steel lamination. In the core region the pattern of the magnetic field distribution is basically two-dimensional and is greatly influenced by the magnetic properties of the steel, which are generally non-linear. In the endwinding region the field pattern is three-dimensional, but the influence of the surrounding steel is not so dominant as in the core region.

These basic differences suggest different approaches to the problem of field solution: in the core region it will be a twodimensional analysis which will take non-linear properties of steel into account while in the end-winding region it will be a three-dimensional analysis, which will take only the basic influence of the surrounding steel into account.

### 2.2 Simplification of the Geometry of the Region ... Assumptions

Fig. 2.1 represents a sketch of a wound rotor induction machine end-winding region. In the analytical approach it is usual to represent the end-coils by some simple shape and ignore the three-dimensional character of the region. The influence of the magnetic core is taken into account either by assuming zero or infinite permeability (Ref. 2.1-2.3) and using the method of images.

If the field distribution is to be solved by a numerical method, some of the simplifications may be abandoned. Some assumptions on the magnetic properties of the core and geometry of the region are still necessary. In this work we have made the following assumptions:
a) The permeability of iron core is constant.
b) The influence of slots and air gap on the field distribution in the air is negligible.
c) The influence of the shaft on the field distribution is negligible.
d) The iron core surface is of infinite extent, i.e. there is no influence of the edge of the core on the magnetic field distribution in the air.
e) Except in their immediate vicinity the coils are represented by single, infinitely thin conductors.

### 2.3 Magnetic Field Distribution in the Air

Under the assumptions of the previous section the magnetic field distribution in the end-winding region can be solved by the method of images.


4

Fig. 2.1 The End-Region

The magnetic field in the air due to current $J$ in a closed loop near a permeable surface (Fig. 2.2a) can be calculated by replacing the surface by the image of the loop carrying current (Fig. 2.2b):

$$
\begin{equation*}
J^{\prime}=\frac{\mu_{r}^{-1}}{\mu_{\mathrm{r}}+1} \mathrm{~J} \tag{2.1}
\end{equation*}
$$

This result has been given as early as 1898 (Ref. 2.4). More recently image systems of circuits partially embedded in iron have been derived (Ref. 2.2, 2.5). Carpenter (Ref. 2.2) gives a particularly clear and simple approach using superposition.

The circuit on Fig. 2.3a can be constructed from a pair of infinitely long straight conductors (Fig. 2.3b) and a semiinfinite circuit (Fig. 2.3c). The field in the air due to current in infinitely long conductors is two-dimensional and it is unaffected by the presence of iron (has no image), while the field due to current in circuit on Fig. 2.3c can be evaluated from that circuit and its image (Fig. 2.3d). The circuits on Figs. 2.3b and 2.3d are now superimposed giving the final image system (Fig. 2.3e) for the calculation of the field in the air.

The influence of the air gap and slots can also be taken into account (Ref. 2.2). As the air gap in induction machines is short, its influence on the field distribution in the air is small. The influence of slots and air gap has been ignored in our calculations.

### 2.4 Inductance Calculation <br> A convenient method for the calculation of inductance is in terms of vector potential. The mutual inductance between two



Fig. 2.2 A Current Loop Near a Permeable Surface

a



Fig. 2.3 The Image System of an End-Coil
current carrying loops, $\ell_{1}$ and $\ell_{2}$, (Fig. 2.4) is given by (Ref. 2.6):

$$
\begin{equation*}
L=\frac{1}{J_{1}} \int_{\ell_{2}} \cdot A_{1} \cdot d \ell_{2} \tag{2.2}
\end{equation*}
$$

where $A_{1}$ is magnetic vector potential due to current in loop $\ell_{1}$. In our case integration will take place along the contour of the coil. It is therefore convenient to divide the integration path into three parts; one corresponding to the core region and two corresponding to two end-coils. It follows that for the calculation of end-winding inductance the integration path has to include only the contour of the end-coil, i.e. the loop should not be closed but include path ABCDE on Fig. 2.3a only.

In free space, the vector potential $A$ due to current filament dJ at distance $r$ from the filament will be (Ref. 2.7):

$$
\begin{equation*}
\underline{A}=\frac{\mu_{0}}{4 \pi} \cdot \frac{d J}{r} \tag{2.3}
\end{equation*}
$$

Consider now two end-coils represented on Fig. 2.5. The influence of the iron has been replaced by the image of the coil; therefore we can write for their mutual inductance:

$$
\begin{align*}
& L_{12}=\frac{\mu_{0}}{4 \pi J} \cdot \int_{A B C D E}\left[J \int_{G H I J K L M} \frac{\frac{d \ell_{1}}{} \cdot \frac{d \ell_{2}}{}}{r}+J^{\prime} \int_{G P M} \frac{d l_{1} \cdot d \ell_{2}}{r}+\left(J+J^{\prime}\right) \cdot\right. \\
& \left.\cdot\left(\int_{F G} \int_{F \rightarrow \infty} \frac{d \ell \cdot d \ell}{-1}+\int_{\substack{\mathbb{N} \\
N \rightarrow \infty}} \frac{\frac{d \ell}{-} \cdot \frac{d \ell}{2}}{r}\right)\right] \tag{2.4}
\end{align*}
$$

An analogue expression would have been obtained for the calculation of self-inductance. A certain caution is necessary in


Fig. 2.4 Two Current Carrying Loops in Arbitrary Position


Fig. 2.5 On the Mutual Inductance of Two End Coils
that case, as some parts of inner and outer integration paths would coincide yielding an infinite result. We shall return to this point later in this section.

For practical purposes it is convenient to replace the endcoil contours and their images by a finite number of straight lines. Any shape of coil can be represented to any desired accuracy in this way. Then the integrals in Eq. (2.4) may be obtained as a sum of the integrals of type (Fig. 2.6):

$$
\begin{equation*}
N=\cos \phi \int_{A}^{B} d l_{1} \int_{a}^{b} \frac{d l_{2}}{r} \tag{2.5}
\end{equation*}
$$

The value of this integral for two lines in a general position is (Ref. 2.8):
$N=\cos \phi \cdot\left[\overline{\mathrm{CB}} \cdot \ln \frac{|\overline{\mathrm{ab}}|+|\overline{\mathrm{bB}}|+|\overline{\mathrm{ab}}|}{|\overline{\mathrm{aB}}|+|\overline{\mathrm{b} B}|-|\overline{\mathrm{ab}}|}-\overline{\mathrm{CA}} \cdot \ln \frac{|\overline{\mathrm{aA}}|+|\overline{\mathrm{bA}}|+|\overline{\mathrm{ab}}|}{|\overline{\mathrm{aA}}|+|\overline{\mathrm{bA}}|-|\overline{\mathrm{ab}}|}+\right.$
$\left.+\overline{\mathrm{cb}} \cdot \ln \frac{|\overline{\mathrm{bA}}|+|\overline{\mathrm{bB}}|+|\overline{\mathrm{AB}}|}{|\overline{\mathrm{BA}}|+|\overline{\mathrm{bB}}|-|\overline{\mathrm{AB}}|}-\overline{\mathrm{ca} \cdot \ln } \frac{|\overline{\mathrm{aA}}|+|\overline{\mathrm{aB}}|+|\overline{\mathrm{AB}}|}{|\overline{\mathrm{aA}}|+|\overline{\mathrm{aB}}|-|\overline{\mathrm{AB}}|}\right]-$
$-|\overline{\mathrm{Cc}}| \cdot \operatorname{ctg} \phi \cdot\left[\operatorname{arctg}\left(\frac{|\overline{\mathrm{Cc}}|}{|\overline{\mathrm{Bb}}|} \cdot \operatorname{ctg} \phi+\frac{\overline{\mathrm{cb}} \cdot \overline{\mathrm{CB}}}{|\overline{\mathrm{Cc}}| \cdot|\overline{\mathrm{Bb}}|} \cdot \sin \phi\right)-\right.$
$-\operatorname{arctg}\left(\frac{|\overline{\mathrm{Cc}}|}{|\overline{\mathrm{BA}}|} \cdot \operatorname{ctg} \phi+\frac{\overline{\mathrm{cb}} \cdot \overline{\mathrm{CA}}}{|\overline{\mathrm{Cc}}| \cdot|\overline{\mathrm{BA}}|} \cdot \sin \phi\right)-\operatorname{arctg}\left(\frac{|\overline{\mathrm{Cc}}|}{|\overline{\mathrm{aB}}|} \cdot \operatorname{ctg} \phi+\right.$ $\left.\left.+\frac{\overline{C a} \cdot \overline{C B}}{|\overline{C C}| \cdot|\overline{a B}|} \cdot \sin \phi\right)+\operatorname{arctg}\left(\frac{|\overline{C C}|}{|\overline{a A}|} \cdot \operatorname{ctg} \phi+\frac{\overline{c a} \cdot \overline{C A}}{|\overline{C c}| \cdot|\overline{a A}|} \cdot \sin \phi\right)\right]$

Overlined letters represent the geometrical distance between corresponding points on Fig. 2.6. Distance $\overline{C c}$ is a common perpendicular to both lines $\ell_{1}$ and $\ell_{2}$. If $\overline{C c}$ is zero, Eq. (2.6) does not


Fig. 2.6 On the Mutual Neumann Integral Between Two Straight Line
give a definite result; the expression is finite, however, and may be readily evaluated, except in the case where $\overline{\mathrm{AB}}$ and $\overline{\mathrm{ab}}$ overlap each other.

If either $\overline{\mathrm{AB}}$ or $\overline{\mathrm{ab}}$ is infinitely long (as for example lines $\overline{\mathrm{FG}}$ or $\overline{\mathrm{MN}}$ on Fig. 2.5), Eq. (2.6) will not give a finite result. In our case lines $\overline{\mathrm{FG}}$ and $\overline{\mathrm{MN}}$ will always be parallel and carry opposite currents. The value of (2.5) for that case is (see Appendix):

$+\overline{C_{1} B} \cdot \ln \frac{\left|\overline{b_{2} B}\right|-\overline{C_{2} B} \cos \phi-\overline{c_{2} b_{2}}}{\overline{a_{1} B}-\overline{C_{1} B} \cos \phi-\overline{C_{1} a_{1}}}+\overline{C_{1} A} \cdot \ln \frac{\left|\overline{a_{1} A}\right|-\overline{C_{1} A} \cos \phi+\overline{c_{1} a_{1}}}{\left|\overline{b_{2} \mathrm{~A}}\right|-\overline{C_{2} \mathrm{~A}} \cos \phi-\overline{\mathrm{C}_{2} \mathrm{~b}_{2}}}+$
$\left.+\overline{C_{2} C_{1}} \cdot \ln \frac{\left|\overline{b_{2} B}\right|-\overline{C_{2} B} \cdot \cos \phi-\overline{C_{2} b_{2}}}{\left|\overline{b_{2} A}\right|-\overline{C_{2} A} \cdot \cos \phi-\overline{C_{2} b_{2}}}\right)-\left|\overline{C_{1} C_{1}}\right| \cdot \operatorname{ctg} \phi \cdot\left[\operatorname{arctg}\left(\frac{\overline{C_{1} B}}{\left|\overline{C_{1} c_{1}}\right|} \cdot \sin \phi\right)-\right.$
$-\operatorname{arctg}\left(\frac{\overline{C_{1} A}}{\left|\overline{C_{1} C_{1}}\right|} \cdot \sin \phi\right)-\operatorname{arctg}\left(\frac{\left|\overline{C_{1} C_{1}}\right|}{\left|\overline{a_{1} B}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} a_{1}} \cdot \overline{C_{1} B}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{a_{1} B}\right|} \cdot \sin \phi\right)+$
$\left.+\operatorname{arctg}\left(\frac{\left|\overline{C_{1} c_{1}}\right|}{\left|\overline{a_{1} A}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} a_{1}} \cdot \overline{C_{1} A}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{a_{1} A}\right|} \cdot \sin \phi\right)\right]+\left|C_{2} c_{2}\right| \cdot \operatorname{ctg} \phi$.
$\cdot\left[\operatorname{arctg}\left(\frac{\overline{C_{2} B}}{\left|\overline{C_{2} C_{2}}\right|} \cdot \sin \phi\right)-\operatorname{arctg}\left(\frac{\overline{C_{2} A}}{\left|\overline{C_{2} C_{2}}\right|} \cdot \sin \phi\right)-\operatorname{arctg}\left(\frac{\left|\overline{C_{2} C_{2}}\right|}{\left|\overline{b_{2} B}\right|}\right.\right.$.
$\left.\left.\cdot \operatorname{ctg} \phi+\frac{\overline{c_{2} b_{2}} \cdot \overline{C_{2} B}}{\left|\overline{C_{2} C_{2}}\right| \cdot\left|\overline{b_{2} B}\right|} \cdot \sin \phi\right)+\operatorname{arctg}\left(\frac{\left|\overline{C_{2} C_{2}}\right|}{\left|\overline{b_{2} A}\right|} \cdot \operatorname{ctg} \phi-\frac{\overline{C_{2} b_{2}} \cdot \overline{C_{2} A}}{\left|\overline{C_{2} c_{2}}\right| \cdot\left|\overline{b_{2} A}\right|} \sin \phi\right)\right]$

Again, overlined letters represent distances between corresponding points on Fig. 2.7. Distances on a certain straight line are always measured in the direction from a to $b$ or $A$ to $B$.


Fig. 2.7 On the Mutual Neumann Integral Between One Finite and Two Semi-Infinite Straight Lines

The most convenient way of calculating the mutual inductance between end-coils 1 and 2 on Fig. 2.5 seem to be to obtain it as a sum of three parts which correspond to mutual inductance between end-coil 1 (line $A B C D E$ ) and: line GHIJKLM ( $L_{1}$ ); line GPM ( $L_{2}$ ); lines $F G$ and $M N\left(L_{3}\right)$. The total mutual inductance is thens

$$
\begin{equation*}
L=L_{1}-\frac{1-\mu_{r}}{1+\mu_{r}} L_{2}+\frac{2 \mu_{r}}{1+\mu_{r}} L_{3} \tag{2.8}
\end{equation*}
$$

The self-inductance of the coil is obtained in a similar way. For the parts of the contours which would overlap, Eq. (2.6) is used with lines of integration parallel but separated by the radius of the coil thickness (corresponding to external inductance). Internal inductance is evaluated by the formula (Ref. 2.9):

$$
\begin{equation*}
S=\frac{\mu_{0}}{8 \pi} \ell \tag{2.9}
\end{equation*}
$$

where $\ell$ is the length of the part of end-coil in consideration, and added to $L_{1}$ in (2.8).

The direction of forces on end-coils will depend on the direction of the current; i.e. on the shape of the end-coil. For small low voltage induction machines these forces will contribute little to the torque of the machine. This may not be so with large high voltage machines where there is a considerable current flowing in the direction of the shaft in the end-winding region. If necessary, the contribution to the torque of the forces on endcoils may be evaluated by differentation of the magnetic energy by the angle. For the contribution of two end-coils, one on rotor and one on 8 tator, we have:
$T=J_{1} \frac{\partial}{\partial \phi}\left[J_{2} L_{1}-\frac{1-\mu}{1+\mu} J_{2}^{\prime} L_{2}+\frac{2 \mu}{1+\mu} \quad\left(J_{2}^{\prime}+J_{2}\right) L_{3}\right]$
and the overall contribution to the torque is obtained as a double sum, of (2.10) over all stator and rotor end-coils. The differentiation is most easily carried out numerically.

### 2.5 Computer Program and Results

In the previous section we have derived the formulae for the computation of end-winding inductance under the assumptions listed in section 2.2. The computation of the mutual inductance between two end-coils involves numerous evaluation of expressions given by Eqns. (2.6) and (2.7). These two expressions are too complicated for paper and pencil computation and the automatic computation is the only practical solution.

We have developed a computer program for the computation of end-winding inductance by the described method. A simplified self-explanatory flow chart is represented in Fig. 2.8. The program has been written in FORTRAN (extended version for ICL 1900 series computers). The novel feature of this program, as compared with other currently used methods, is the use of Eqn. (2.7), which enables us to compute the inductance for any relative permeability of steel core.

Although the program has been written for computation of stator end-winding inductance, it can be easily adapted for the computation of mutual inductance between rotor and stator endwindings.

Results of the computation by this program for two typical examples are presented on Figs. 2.9 and 2.10. The diagrams represent the mutual inductance between two end-coils for a sixpole and a two-pole machine for relative permeabilities $\mu_{r}=0$ and $\mu_{r}=10000$.


Fig. 2.8 The flow-chart of the program for computation of the end-winding inductance
Inductance in $\mathrm{H} \times 10^{-8}$
Fig. 2.9
$\begin{array}{ll}\text { length of the coil } & \ell=230 \mathrm{~mm} \\ \text { number of turns } & n=1 \\ \text { stator bore diameter } & D=200 \mathrm{~mm}\end{array}$


For the purpose of comparison we have plotted the results obtained by the method of Carpenter (Ref. 2.2) on Fig. 2.9 (dotted line). He also used the method of images, but assumed that all end-coils lie in the same plane and the end coils were represented by two mutually perpendicular straight line segments. The relative permeability was assumed to be zero, thus his curve should be compared with our curve for $\mu_{r}=0$. Our curve shows the same basic pattern, although the discrepancies are quite large, particularly, our curve decreases much faster with the increasing mutual distance between the two 'coils. The discrepancies are mainly due' to the fact that the end-coils do not actually lie in the same plane. The influence of the better representation of the coil shape on these discrepancies is probably much smaller, because our representation was fairly simple, the end-coils being represented by 5 straight line segments (Fig. 2.1).

Another example of the use of this program may be found in Chapter 7, where it is used to compute the end-winding inductances? of a wound rotor induction machine used as our sample problem.

The computation time depends, of course, on the type of computer and compiler used. As a rough guide it can be estimated from $T \simeq 10^{4} s \mathrm{n} \mathrm{m}^{2}$, where $s$ is computer access time, $n$ is number of coils and $m$ is a number of straight line segments used for representation of the end-coil. For our case (s $=2.4 \mu \mathrm{sec}$, $n=28, m=5$, this formular gives computation time of about 17 seconds, which corresponds well with the observed value.

### 2.6 Discussion

The most commonly used methods for the computation of endwinding inductance are based either on some drastic simplifications of the geometry of the end-region (Ref. 2.1) or make extensive use of empirical coefficients (Ref. 2.10, 2.11) The solution of the problem has been an object of many researchers in the past. The most complete list of references can be found in Ref. 2.15, with the exception of some later papers. Until recently, virtually all work on this problem has been based on an analytical solution of the field distribution in the end-region of the machine, although current filaments have been used for the computation of forces on end-coils of large turbogenerators. The analytical approach is complex and often requires ignoring the three-dimensional character of the region.

In 1959 Honsinger (Ref. 2.3) has published a paper on the subject which took the three-dimensional geometry into account. He assumed that infinitely permeable boundaries surround the endwinding consisting of elliptic end-coils. His results are not suitable for computer use because of extensive use of graphs.

Carpenter (Ref. 2.2) developed the method of images and used it in the same paper for the calculation of induction machine endwinding inductances. He simplified the geometry by assuraing that all end-coils have a simple V -shape and all lie in the plane. Further assuming zero permeability be obtained a very simple expression for the overall end-winding inductance.

When the method described here had been developed, Lawrenson published a paper (Ref. 2.12) on the use of a numerical method for the computation of end-winding inductances similar to our method.

Similarly as in our method the three-dimensional geometry was fully taken into account, but the methods differ in several other points. Lawrenson assumed that the vector potential due to current filament $\overline{a b}$ (Fig. 2.6) is constant along the integration line $\overline{A B}$, and the resulting formula in his method is much simpler than Eqn. (2.6). Unlike us, he took into account the effect of other conducting surfaces in the vicinity (shaft and casing) but assumed the permeability of all surrounding metal surfaces to be zero. If his method is simplified so that only the influence of the steel core is taken into account, both his method and our method (with $\mu_{r}=0$ ) will probably give similar results. Computation times are also likely to be similar, because, although his formulae are simpler than ours, his method will generally require more integration steps than ours in order to achieve the same accuracy.

Recently, attempts have been made to solve the end field by direct numerical solution of the three-dimensional vector potential equation (Ref. 2.13-2.14). This method will allow the nonlinear magnetic properties of iron to be taken into account, although it cannot deal with the effect of eddy currents. The possibilities of the method have not yet been fully exploited. It is likely to make very high demands on computer store and come putation times, which can, at present, be hardly justified for the use with induction machines and fields.

Our method for the computation of end-winding inductances takes into account the influence of permeable iron core and the complex three-dimensional geometry of the end region. The accuracy of the method can be improved by considering other
permeable and/or conducting surfaces in the vicinity, representing the end-coil more accurately by splitting it into single conductors and by considering the effect of slots and the air gap.

The limitations of our method lie in the uncertainty of the permeability of the steel core. As the core is both permeable and conducting, the alternating magnetic field will cause eddy currents which will reduce its effective permeability. The influence of this effect on the inductance will depend on the applied frequencies. Our method does not allow for this effect. However, it can be used to determine upper and lower bounds for the end-winding inductance, by computing its magnitude for the two extreme cases, i.e. zero permeability (ideally conducting surface) and the value of permeability bbtained for low flux densities and DC excitation. The influence of different permeability of the core on the mutual inductance of two end-coils can be quite large, as it can be readily seen from Figs. 2.9 and 2.10, and for some relative positions of the coils different permeability can even result in the opposite sign of the mutual inductance.

The magnitude of the end-winding inductance is influenced by many factors, some of which are not easily considered. We believe that our method of computation presents an improvement over the most methods in current use, particularly as it can supply upper and lower bounds for the value of inductance. We also think that any further research in this field should include a considerable amount of measurements on models and real machines. Experimental work in this field would be extremely valuable, as it could provide such information as the influence of finite core diameter, etc., and it could probably also indicate the value of the
'effective' permeability of the core. Unfortunately, due to the lack of time, we were unable to do any experiments in this field.

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## CHAPTER 3

## THE CORE REGION

In this chapter the problem of the field solution in the core region is discussed.

In the first section the necessary assumptions and simplifications are made' and discussed. On the basis of these assumptions the problem is reduced to the two-dimensional one, and the partial differential equation of the magnetic vector potential is derived.

In section 3.2 a short account of different possibilities for the solution of linear and non-linear field problems is given. The methods are divided into four groups: analytical, anologue, graphical and numerical. The basic advantages, disadvantages and limitations are pointed out.

In section 3.3 different possibilities for the discretization of the region are presented in order to obtain a mesh for use with finite difference or finite element methods.

Section 3.4 discusses the difference between the finite difference and finite element methods. Some definitions of these methods, if a partial differential equation is considered as the source equation, are explained. It is also pointed out that it is justified to consider Maxwell's equations as the source equations for the problem in hand, in which case both methods follow the same process. Finally, a historical note on the use of these methods is given, with the emphasis on the use of triangular mesh, which is often associated with the finite element method.

Section 3.5 briefly discusses two possibilities for the field approximation by piecewise polynomials in the triangulated continuum.

In section 3.6 requirements for the method of solution are listed and the choice of the method is made.

In section 3.7 two possibilities for the derivation of equations for the approximate solution are presented and it is shown that these yield identical results. Both approaches are based on a concept of the elimination of a suitably defined error of the approximate solution. The definition of error is different in the two approaches. Some other possibilities for the derivation of equations are also mentioned.

Finally, sections 3.8 and 3.9 briefly discuss boundary conditions and derivation of torque and induced voltage from the field distribution.

### 3.1 Magnetic Field Considerations in the Core Region Assumptions

The magnetic field in the machine is governed by Maxwell's equations:

$$
\begin{align*}
\nabla \times \underline{E} & =-\frac{\partial \underline{B}}{\partial t}  \tag{3.1}\\
\nabla \times \underline{H} & =\underline{J}+\frac{\partial \underline{D}}{\partial t}  \tag{3.2}\\
\nabla \underline{B} & =0  \tag{3.3}\\
\nabla \underline{D} & =\rho \tag{3.4}
\end{align*}
$$

and the subsidiary relations:

$$
\begin{gather*}
\underline{B}-\mu \underline{H}=\frac{1}{V} \underline{H}  \tag{3.5}\\
\underline{D}=\varepsilon \underline{E} \tag{3.6}
\end{gather*}
$$

It would be extremely difficult to solve the field distribution in the core region exactly, even under the assumption made in section 2.1 that the end region has no influence on the field distribution in the core. It is therefore necessary to introduce some additional simplifications:
a) The magnetization curve of the core material is a single valued function; i.e. there is no hysteresis effect.
b) The permeability $\mu$ outside the machine is zero.
c) The permeability $\mu$ of the shaft material is zero.
d) Eddy current effect in the core lamination is ignored.
e) The effect of changes of electric charge distribution is ignored.

Assumption a) simplifies the problem significantly. Strictly, it would not be possible to take hysteresis effect into account completely; for this purpose we would have to know not only the properties of the material, but also its magnetic history. Even if the effect of previous magnetizations is ignored (which is justified for most practical purposes), the problem would remain very complex due to complexity of the magnetic field in the machine. In some parts of the magnetic circuit the field is mainly oscillating in magnitude; in others it is rotating as well. For these reasons it is necessary to ignore the hysteresis effect. The relative influence of the hysteresis on the field distribution decreases as flux density increases, because the saturation effect. becomes predominant.

Assumption b) simplifies the boundary conditions and restricts our calculations to the interior of the machine only. The flux distribution at short circuit is mainly influenced by the
air-gap area, and the magnetic flux outside the machine has little influence on the behaviour of the machine.

Assumption c) may be easily removed. It is believed, however, that this represents the reality reasonably well, because the penetration of the flux into the shaft will be damped by eddy currents.

Assumption d) is also made in order to simplify the problem. The eddy current phenomena in saturable material is a very complex problem itself, even in the much simpler cases. Modern mild magnetic materials have high electric resistivity and laminations are thin, so that the eddy currents are limited by high resistance, and the magnetic field due to eddy currents can be neglected for frequencies of up to a few hundred Hz (Ref. 3.1).

Assumption e) is constantly made for the computation of electric machines based on magnetic principles, although not always explicitly. It is fully justified, because the effect of electric charges on the behaviour of the machine is completely negligible in comparison with other effects.

With the definition of magnetic vector potential $A$ :

$$
\begin{equation*}
\underline{B}=\nabla \times \underline{A} \tag{3.7}
\end{equation*}
$$

we get from Equations (3.2) and (3.5) (second term of the righthand side of Eq. (3.2) is zero according to assumption e)):

$$
\begin{equation*}
\nabla \times[U(\nabla \times \underline{A})]=\underline{J} \tag{3.8}
\end{equation*}
$$

Fig. 3.1 represents a sketch of a part of a cross-section of a machine. The z-axis is directed out of the paper, so the current density $J$ has only a $z$-component. Hence the vector on the


Fig. 3.1 A Cross-Section of an Induction Machine
left-hand side of Eq. (3.8) may have only a z-component as well, which is:

$$
\begin{equation*}
\frac{\partial}{\partial x}\left[v\left(\frac{\partial A_{z}}{\partial x}-\frac{\partial A_{x}}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[v\left(\frac{\partial A_{z}}{\partial y}-\frac{\partial A_{z}}{\partial z}\right)\right]=J_{z} \tag{3.9}
\end{equation*}
$$

The assumption that there is no end effect is equivalent to the statement that all partial derivations in z-direction are zero, so that finally we have:

$$
\begin{equation*}
\frac{\partial}{\partial x}\left(v \frac{\partial A_{z}}{\partial x}\right)+\frac{\partial}{\partial y}\left(v \frac{\partial A_{z}}{\partial y}\right)=J_{z} \tag{3,10}
\end{equation*}
$$

The problem is therefore reduced to a scalar one, and is equivalent to the solution of a second order mildly non-linear (or quasi-linear) partial differential equation of elliptic type.

As vectors $\underline{A}$ and $\underline{J}$ have only $z$-component, the subscript $z$ will be omitted in future.

Reluctivity $v$ is constant for air regions; for iron regions it can be expressed as a function of the absolute value of flux density $|\underline{B}|$ which is (from Eqn. (3.7)):

$$
\begin{equation*}
|\underline{B}|=|B|=\sqrt{B_{x}^{2}+B_{y}^{2}}=\sqrt{\left(-\frac{\partial A}{\partial y}\right)^{2}+\left(\frac{\partial A}{\partial x}\right)^{2}} \tag{3.11}
\end{equation*}
$$

Equation (3.10) describes the magnetic field in the core region completely and it can be considered as the source equation for our computation. It should be emphasized, however, that this equation is derived from Maxwell's equations. It is therefore immaterial whether we derive our algorithms for the field solution from Eq. (3.10) or from some other equation obtained from Maxwell's equations, or from the Maxwell's equations directly,
either in their differential (Eqns. (3.1)-(3.4)) or some other form. In other words we can define as 'the source equation' any other suitable expression derived from Maxwell's equations.

### 3.2 Short Account on Different Methods for Field Problems <br> All presently available methods for the electromagnetic field problems may be classified in several groups:

## 1. Analytical methods

These field computational methods give the direct solution of the field equations in a suitable algebraic form. The most often used method from this group is probably the method of conformal mapping. It was originated by Christoffel (Ref. 3.2) and Schwartz (Ref. 3.3) more than a century ago and used for the first time for electromagnetic field problems by Carter (Refs. 3.4 and 3.5). The problems that could be solved by this method were limited to simple geometry. More recently, the class of problems for which this method is applicable has been widened to more complex geometry by the use of numerical integration techniques (Refs. 3.6-3.8). The major disadvantage of the method is that it cannot be used for nonhomogeneous or non-linear media.

Another well-known analytical method for the Poisson differential equation is method of separation of variables. This method is also restricted to simple geometries and homogeneous linear media. The solution is often represented in a form of infinite series, which may converge slowly (Ref. 3.9).

In principle, non-linear partial differential equations can be solved analytically by different transformation methods. The best known of these are the Kirchoff transformation (Ref. 3.10),
and the hodograph transformation (Ref. 3.11). The hodograph transformation has been successfully applied to the problems of compressible flow. It has been also applied to simple cases of magnetic flux distribution in non-linear media (Ref. 3.12), but like the Kirchoff transformation it is not suitable for nonhomogeneous regions of complex geometry.

Integration methods also belong to the group of analytical methods. Although very powerful for linear media (use of an integration method has been made in Chapter 2), their use for nonlinear regions is not very practical (though possible). For non-homogeneous regions of complex geometry numerical integration is essential (Refs. 3.12 and 3.13).

## 2. Analogue methods

Are based on the fact that many different natural phenomena may be described by the Poisson type differential equation. Some of these phenomena are more easily modelled, or more easily measured, than the magnetic field distribution and they can be used to determine magnetic flux by measurement of the analogue quantity. The best known analogue model is the electrolytic tank, based on the analogy between the magnetic field and the current density field. The conducting sheet analogue is based on the same principle. Both models can treat complicated geometry and inhomogenity, and both models have been used successfully for the solution of linear magnetic field problems (Ref. 3.14-3.16). Unfortunately, the methods cannot be used for non-linear problems.

Another analogue method that has been used for linear magnetic field problems is based on the similarity of the magnetic
field and the laminar flow of incompressible fluids (Ref. 3.17). It can be used also for non-homogeneous, but not non-linear problems.

Elastic membrane analogy can be used also for magnetic field problems (Ref. 3.18). It is possible to extend its use to nonlinear media (Ref. 3.19) if the magnetization curve is simplified. Its use is restricted to simple geometry, particularly for nonlinear problems.

Network analogues have been also used for the determination of the magnetic field problems. Here the magnetic field problem is replaced by a resistor network, and a flux distribution is obtained by measurements of current and voltage distributions on the model. These models have been used widely. Very high accuracy has been claimed (Ref. 3.20), and several ways of dealing with non-isotropic and non-linear media have been presented (Refs. 3.21 and 3.22). The main disadvantages of the method arethe high cost of the model, particularly for the non-linear type models, and very elaborate modelling preparation, when large problems are considered.

This list of analogue methods is not complete. There are several other analogues (heat-conduction analogue, mechanical stress analogue, sand-heap analogue), that might be, at least in principle, used for the determination of magnetic field distribution. However, their use seems to be more complex than the analogues described, and they probably are not suitable for nonlinear non-homogeneous problems.
3. Graphical methods (Method of curvilinear squares)

These are methods in which use is made of the property of orthogonality of flux lines and constant scalar potential lines. The method was originated by Richardson (Ref. 3.23) and Lèhman (Ref. 3.24) at the beginning of the century. The use of graphical field computation methods has been extended to non-linear cases (Refs. 3.25 and 3.26), but its general inaccuracy and computational effort limits its use to simple problems.
4. Numerical methods

Is a term usually used for a wide group of methods, which could be, perhaps, better described as 'approximative algebraic methods', since they are essentially not more numerical than, say, conformal mapping. Basically, they consist of three steps:
a) approximating the exact solution by (a set of) suitable functions;
b) substituting this approximate solution into the source equations; and
c) minimising (or eliminating) the error by adjusting the coefficients associated with the approximating functions.

The most useful approximating functions are polynomials, and the last step leads to a solution of (a set of) algebraic (possibly non-linear) equations. Some of the best known of these methods are: Finite Differences, Finite Elements, Raleigh-Ritz, Galerkin, Method of Moments, Collocation, Point Matching, Weighted Residuals, Reaction Method, Least Squares ... etc. All these methods can be closely related on a more formal basis using the geometrical interpretation in function space. (Ref. 3.27). The
last step, the solution of algebraic equations, is most easily performed by a digital computer, and at this point these methods become numerical. It is possible, however, to solve these equations by analogue computers. In fact, the well-known network analogues mentioned earlier are methods that belong to this group, because unlike the other analogues, they do not represent the field directly, but its approximation by a set of piecewise linear functions. Hybrid computing methods have been also used for the solution of a final set of equations (Ref. 3.28).

A complete survey of the literature on these methods would be a formidable task. The same, or closely related methods have been used for the solution of other technical problems in civil engineering, structural mechanics, mechanics of fluids, etc., and a number of papers and books published on the theoretical aspects of the methods, on the related numerical methods for the solution of equations and on the application to different technical problems is enormous. (For example, Rosenbloom, (Ref. 3.29) quotes more than 700 references on linear partial differential equations only). We shall not attempt to make such a survey, but shall concentrate on the two methods which appear to be most suitable for two-dimensional field problems: finite difference and finite element methods, although a complete survey is virtually impossible even for these. For both methods it is necessary to divide a region into a number of sub-regions; the methods could both therefore be called 'discretization methods'.

### 3.3 Discretization of the Continuum

In this section we shall describe some of the possibilities for the discretization of the region of our particular problem. The resulting division into sub-regions we shall call a mesh. Most of the meshes described have been used for practical problems, and some references are given.

Regular square mesh (Fig. 3.2) has the advantage of geometrical regularity and simplicity of resulting approximating formulae. It cannot represent the complicated geometry faithfully, unless either a large number of mesh nodes is used, or, it is decided for material interfaces not to be mesh lines, which complicates the approximating formulae. This mesh has been used for electromagnetic field problems, usually for relatively simple geometries (Refs, 3.30 and 3.31). Its use becomes complex when periodicity boundary conditions on domains of circular sector shapes (e.g. a pole pitch of a rotational machine) have to be satisfied, because generally the mesh patterns along two boundaries with periodicity condition do not coincide.

Regular square mesh with variable mesh density (Fig. 3.3) may be used for problems where more detailed information about the flux distribution is needed in certain areas of the total flux plot. Special interpolating formulae have to be developed for nodes on and near the border lines where fine and coarse meshes meet (Refs. $3.30 \& 3.32$ ). The mesh is also not convenient for interpreting periodicity boundary conditions.

Irregular rectangular mesh
(Fig. 3.4) is probably the
most extensively used type of mesh for electromagnetic field


Fig. 3.2 Regular Square Mesh


Fig. 3.3 Regular Square Mesh with Variable Density


Fig. 3.4 Irregular Rectangular Mesh
problems in electric machines (Ref. 3.33). Its advantage is that it is more general than the two previous types, thus allowing more flexibility in representing complicated geometry and inhomogenity. Similar difficulties with periodicity boundary conditions would have been encountered as with the previous two types. In works published so far this has been avoided by replacing the rotational geometry by rectangular geometry, distorting the true geometrical picture.

Topologically irregular rectangular mesh (Fig. 3.5) has been introduced in order to deal with difficulties encountered when fitting rotor and stator meshes for doubly slotted machines. The number and dimensions of rotor and stator teeth are not generally the same. As the change of material is allowed on mesh lines only, rotor and stator mesh lines necessary for the representation of teeth geometry will not be the same. In a topologically regular mesh, mesh lines stretch from one boundary to another, almost doubling the number of nodes. This is avoided by constructing the rotor and stator mesh separately and introducing topological irregularities, covered by special interpplating formulae for the 'joint' (Ref. 3.34).

Polar co-ordinate mesh (Fig. 3.6) was introduced in order to avoid distortion of geometry when representing a part of the machine by rectangular mesh. Regular polar mesh, with equal $\Delta \phi$ and $\Delta r$ throughout the mesh is the simplest case, analogous to a rectangular mesh. More flexible is the mesh where $\Delta \dot{\phi}$ and $\Delta r$ are allowed to vary throughout the mesh (Ref. 3.35). Topologically irregular polar mesh may also be used.


Fig. 3.5 Topologically Irregular Rectangular Mesh


Fig. 3.6 Polar Co-Ordinate Mesh

Combined meshes Polar and rectangular meshes can be combined. Different types of mesh may be used to represent faithfully different local geometrical features. This approach is flexible, but complex. However, it has been used for the solution of field problems in electrical machines (Ref. 3.36).

Hexagonal mesh (Fig. 3.7) This type of mesh has been used very rarely (Ref. 3.37). It offers few advantages over the regular square mesh, although it may be found useful for certain types of geometries.

Regular triangular mesh (Fig. 3.8) is an alternative to regular square mesh. It has not been used so widely as the regular square mesh, because the final equations are more complicated (Ref. 3.37).

Irregular triangular mesh with fixed topology (Fig. 3.9) is very flexible. It allows faithful representation of complex geometry, because triangles can vary in size and shape. The topological regularity is an advantage for computer programming purposes, but it is a disadvantage regarding number of mesh nodes. This type of mesh has been used for computing the field of particle accelerator magnets (Ref. 3.38).

Free topology irregular triangle mesh (Fig. 3.10) allows changes in shape and size of triangles more freely than fixed topology triangle mesh. This is an advantage, as careful use may result in significant reduction of the number of nodes. This type of mesh has been in use for structural mechanics problems for some time (Ref. 3.39) and, although suggested for electromagnetic field problems as early as 1949 (Ref. 3.40), it has not been used for


Fig. 3.7 Hexagonal Mesh


Fig. 3.8 Regular Triangular Mesh


Fig. 3.9 Irregular Triangular Mesh with Fixed Topology


Fig. 3. 10
electric machine problems until relatively recently (Ref. 3.41). The mesh is very flexible and allows faithful representation of different geometrical features; different boundary conditions represent no principal difficulties.

Curvilinear meshes Any type of mech described so far can be refined by using 'curvilinear polygons'. Curvilinear triangles and curvilinear quadrilaterals have been described in the literature (Refs. 3.42 and 3.43). Meshes constructed from curvilinear polygonal cells can describe complex geometry with fewer mesh nodes than meshes constructed of normal polygons. The main disadvantage of such meshes is their complexity, which results in complexity of algebraic expressions in the final set of equations and high requirements of computer core storage per node.

The number of different types of meshes that could be constructed is virtually unlimited. It would be possible to generalise by defining a 'general curvilinear polygonal' mesh. Such generalisation would have few practical advantages, because derivation of formulae that would cover such a general case would not have been easy, and hardly very useful.

### 3.4 Finite Difference and Finite Element Methods

These two methods are the most commonly used in the recent technical literature. The difference between them depends on the definition of either of them; these are not consistent in literature. The definition of finite difference method given by J. Walsh (Ref. 3.44) is that 'in the finite difference method the algebraic equations are obtained by direct approximation of the differential equation at mesh points ...' and .... if we require
the solution at all points of the region, it is obtained by interpolating between the mesh values'.

In the same paper the definition of the finite element method reads: 'In the finite element method the region is divided into sub-regions, and the approximate solution is represented by a polynomial over each sub-region, with matching conditions on the boundary between them. Each polynomial is defined by a number of coefficients, or equivalently by values of the function and its derivatives at certain points. The algebraic equation for determining these values are obtained either from a variational principle (Ritz's method), or by the method of Galerkin.'

This definition of the finite difference method does not seem to cover a number of methods described as 'finite differeñe methods' where algebraic equations are obtained by integration (Galerkin's method), (Refs. 3.45-3.46, 3.12). Also, for nonlinear magnetic field problems the non-linear reluctance is usually defined over the mesh cells surrounding a particular mesh node; therefore the values of the vector potential $A$ between the mesh nodes appear in the final set of algebraic equations implicitly.

However, these definitions seem justified if a partial differential equation is considered as the source equation. Following them, the finite difference approximation for our problem would be obtained by substituting the differential operator of Eq. (3.10) by a difference operator. The solution is then obtained by a point matching procedure applied to the approximating difference equations, where these will be satisfied exactly at a certain number of points. The difference operators are
usually obtained by developing the function into a Taylor series and neglecting the members of the series of higher order. Although this will normally lead to a rectangular (or polar) mesh, it is possible to apply this basic procedure to a general triangular mesh (Ref. 3.47).

The finite element method would consist of transforming Eq. (3.10) into some other form (for example this can be done by applying the Euler theorem of the calculus of variations to Eq. (3.10)), defining the approximate solution as a piecewise polynomial over the cells of a chosen mesh, and applying some error minimisation procedure to this approximation, employing the transformed equation.

For our problem it seems more logical to consider Eqns. (3.1)(3.6) as the source equations. In that case both finite element method and finite difference method follow the same basic pattern:
a) transformation of the source equations;
b) definition of the error; and
c) elimination (or minimisation) of the error.

Indeed, if the variables are the same (i.e. vector-potential $A$ ), the mesh chosen is the same, and the order of approximation is the same, both methods may yield identical sets of equations in some cases. (For Laplace's equation even the mesh does not have to be the same, see for example Ref. 3.48).

Often, different names may be found in the literature for very closely related (if not the same) methods; names being chosen to describe a different approach, or a different problem adequately. (For example, while the 'finite element method' seems adequate for structural engineering problems, 'discretization
method' might be a better choice for magnetic field problems; on the other hand, if the method is based on a variational principle, it could also be named 'functional approximation method').

Historically, finite difference methods can be traced back to Gauss (Ref. 3.49). The work of Courant (Ref. 3.39) is usually considered as the first on the finite element techniques, although the term 'finite element' has started being used much later, first in structural engineering. Irregular triangular mesh, which is often associated with finite element method was also used for the first time by Courant, although the regular triangular mesh was considered before (Ref. 3.37). Southwell also uses triangular mesh cells as means of charging mesh density in otherwise regular orthogonal mesh (Ref. 3.50). Prager and Synge use triangular mesh for approximate solution of boundary value problems (Ref. 3.51). Their method is generally known as 'the hypercicle method' (Ref. 3.52) and is based on a geometric representation in function space. In 1949 irregular triangular mesh was suggested for electromagnetic field problems. The equations were derived from the conducting sheet analogy, and the possibilities for the use for non-linear magnetic field problems were pointed out (Ref. 3.40).

Hand computation limited the use of triangular meshes to simple cases. In 1953, McNeal published a paper on the use of irregular, topologically free triangular mesh for electromagnetic field problems. He used integration as a means of derivation of the equations and analogue solution of equations (Ref. 3.45). The same type of mesh has been used for the neutron diffusion equation (Ref. 3.53).

The use of fixed topology triangular mesh for non-linear magnetostatic problems was reported in 1965 (Ref. 3.38), when Zienkiewicz and Cheung also published their paper on the use of triangular finite elements for the solution of field problems (Ref. 3.48). They considered linear cases only. Since 1965 the methods employing discretization by triangles, or more complex polygons have been more widely used, and a number of papers have appeared in the technical literature. Most of them have been in connection with structural mechanics problems. Recently, work on the use of triangular meshes for electrical machines problems has also been reported (Refs. 3.41, 3.54).

### 3.5 Representation of the Approximate Solution in the Discretized Continuum

In section 3.3 we have described several possibilities for the discretization of the region in consideration. Different mesh types will yield 'different' methods. The vast number of methods that result is further widened by the possibility of defining different approximating functions and/or different error criteria that are to be minimised or eliminated. Again, the choice is virtually unlimited. However, although special approximating functions have been considered in order to deal with singularities (Refs. 3.55 and 3.56 ), the most useful approximating functions are, for practical purposes, polynomials, particularly low order polynomials. We shall briefly examine two different possibilities for fitting a low order polynomial surface on a triangular mesh cell (similar approach may be used for other polygons):
a) Use of additional nodes. This type of approximation has been discussed by Silvester (Ref. 3.57). It consists of dividing a triangle into a number of smaller triangles, in a regular pattern (Fig. 3.11), so that the number of nodes in such a triangle corresponds to the degrees of freedom of a polynomial. The coefficients of a polynomial expression are then obtained from the value of the function at all the nodes of the triangle. This approach is general and it is basically the same for a polynomial of any order.
b) Alternatively, coefficients of the approximating polynomial may be determined from the value of the function and its derivatives at triangle vertices. This approach is discussed in Ref. 3.58. It may be necessary to add some nodes inside the triangle in order to obtain the complete polynomial (e.g. for a complete cubic it is necessary to add one node in the middle of the triangle in order to obtain ten coefficients). The complete polynomial is necessary in order to keep the approximation independent of the relative position of the triangle.

The approach in a) has the disadvantage that it will not give a 'smooth' approximation on the intertriangle interfaces. Generally, it will yield a continuous derivative in the direction of the interface only. (In our case it would correspond to continuous normal and discontinuous tangential flux density).

The approach in b) will yield a smooth surface, but it is generally more complex to use, particularly in the non-homogeneous region where smoothness will not always be desirable. For the


Fig. 3.11 Subdivision of a Triangle
simplest polynomials, i.e. linear functions, both approximations reduce to the same, non-smooth surface.

In structural engineering problems these two approximations are often referred to as non-conforming and conforming solutions (Ref. 3.59). The whole problem is also closely related to approximation techniques and the recently developed theory of spline functions (Ref. 3.60), which will probably provide the answers about general accuracy and applicability of the method.

### 3.6 Choice of a Mathematic Model

In previous sections we have briefly discussed several methods for the solution of the magnetic field problem in an induction machine. We may summarise in saying that none of the methods, except the numerical ones, have been successfully used for large and complex non-linear problems of which induction machine is a typical example. Without further discussion we shall discard all the methods except the numerical ones as unsuitable for our problem.

In order to make a proper choice of the wide variety of numerical methods, we must summarise the necessities that the method must fulfil:
a) It must represent the geometry accurately, particularly in the air-gap region.
b) It must deal with the periodicity conditions easily.
c) It must provide means for checking the accuracy of the final results.
d) It must provide results from relatively little data (i.e. B-H curve, geometry and current distribution).
e) It must be efficient.

In addition, it is necessary to take into account limitations imposed by the finite store and accuracy of the computer. Generally, the demands and limitations will oppose each other. It is necessary to compromise. Our choice is triangular discretization with free topology and linear approximation of the vector potential A.

This choice is somewhat arbitrary. It has been made on the basis of limited experience of the use of different methods accumulated so far in the technical literature and because it is possible to develop a corresponding computer program to fulfil most of the demands a) - e) satisfactorily. We note that this choice corresponds to most of the work published so far on the use of finite element method.

### 3.7 Derivation of the Algebraic Equations Representing the Field Solution

In the previous section we have chosen the type of our approximate solution. We may represent it graphically (in terms of the vector potential A) as a surface consisting of a number of triangle shaped parts of planes which match together at the mutual boundaries (Fig. 3.12). To obtain the solution we seek the values of vector potential $A$ at the mesh nodes; the approximate solution at all points of the region is obtained from the approximating functions for corresponding triangles. In order to obtain $n$ values of vector potential $\dot{A}$ (for the problem with $n$ mesh nodes) we need $n$ equations. As mentioned in section 3.2 , these are to be obtained by substituting our approximate solution into the source equations,


Fig. 3.12 The Approximate Solution in Discretized Continuum
and by minimising (or eliminating) the error. The error criterion has yet to be defined; before that we consider again the source equations (3.1)-(3.6).

As we have reduced our problem to the magnetostatic one, our system (Eqs. (3.1)-(3.6)) is reduced to three equations: (3.2), (3.3) and (3.5). We write Eq. (3.2) (without the time dependent member) and Eq. (3.3) again in their integral form:

$$
\begin{gather*}
\oint_{\ell} \underline{H} \cdot \underline{d \ell}=\int_{S} \underline{J} \cdot \underline{d S}  \tag{3.12}\\
\oiint_{S} \underline{B} \cdot d S=0 \tag{3.13}
\end{gather*}
$$

Due to the nature of our approximate solution, Eq. (3.13), which is reduced to a two-dimensional line integral, is already satisfied for any integration area chosen; it is therefore clear that we must derive our error criteria either from Eq. (3.12) (or Eq. (3.2)), or some equation that includes Eq. (3.12) implicitly. Our approximation is differentiable only piecewise, inside the triangle areas, and not on the intertriangle boundaries. Furthermore, its second derivative is identically equal to zero over the entire area in consideration (except on the triangle boundaries where it is not defined). The choice of equation for the derivation of our error criterion is therefore further limited to integral type equations (in terms of vector potential A).

Consideration of the field energy provides one such equation. The magnetic field energy in linear media is given by (Ref. 3.61):

$$
\begin{equation*}
U-\frac{1}{2} \int_{V}-\underline{B} d V \tag{3.14}
\end{equation*}
$$

which can be transformed into (in this transformation we make use of Eq. (3.2)):

$$
\begin{equation*}
\mathrm{U}=\frac{1}{2} \int_{V} \underline{A} \cdot \underline{J} \mathrm{dV} \tag{3.15}
\end{equation*}
$$

Eq. (3.15) can also be obtained through the assembly work of current carrying circuits. Eqns. (3.14) and (3.15) yield the same amount of energy. Furthermore, if, due to some imaginary current distribution $J^{*} d V$ the value of the vector potential changes by $d A^{*}$, then the additional work dW corresponds to the change of energy dU . The additional work is:

$$
\begin{equation*}
d W=\frac{\partial}{\partial A}\left(\int_{V} \underline{A} \cdot \underline{J} d V\right) d A^{*} \tag{3.16}
\end{equation*}
$$

and the change of energy from Eq. (3.14):

$$
\begin{equation*}
d U=\frac{\partial}{\partial A}\left(\frac{1}{2} \int_{V} \underline{H} \cdot \underline{B} d V\right) d A^{*} \tag{3.17}
\end{equation*}
$$

which, combined gives:

$$
\begin{equation*}
d U-d W=\frac{\partial}{\partial A}\left(\frac{1}{2} \int_{V} \underline{H} \cdot \underline{B} d V-\int_{V} \underline{A} \cdot \underline{J} d V\right) d A^{*}=0 \tag{3.18}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial}{\partial A}{ }^{\prime}\left(\frac{1}{2} \int_{V} \underline{H} \cdot \underline{B} d V-\int_{V} \underline{A} \cdot \underline{J} d V\right)=0 \tag{3.19}
\end{equation*}
$$

This equation is often regarded as the $s$ tatement that the total potential energy of the system must be stationary.

Different distributions of the vector potential $A$ will. not satisfy Eq. (3.19); we may define our error as:

$$
\begin{equation*}
\varepsilon=\frac{\partial}{\partial A}\left(\frac{1}{2} \int_{V} \underline{H} \cdot \underline{B} d V-\int_{V} \underline{A} \cdot \underline{J} d V\right) \tag{3.20}
\end{equation*}
$$

and seek the distribution of A which.minimises, for example, the mean square error over the region, or which eliminates error defined by Eq. (3.20) completely. (Indeed these two approaches would, in our case, give two equivalent sets of equations).

From geometrical considerations the vector potential $A^{(e)}$ inside the triangle ' $e$ ' is (Fig. 3.13):

$$
\begin{align*}
A^{(e)} & =\frac{1}{2 S^{(e)}}\left\{A_{i}^{(e)}\left[x\left(y_{j}^{(e)}-y_{k}^{(e)}\right)+y\left(x_{k}^{(e)}-x_{j}^{(e)}\right)+x_{j}^{(e)} y_{k}^{(e)}-x_{k}^{(e)} y_{j}^{(e)}\right]+\right. \\
& +A_{j}^{(e)}\left[x\left(y_{k}^{(e)}-y_{i}^{(e)}\right)+y\left(x_{i}^{(e)}-x_{k}^{(e)}\right)+x_{k}^{(e)} y_{i}^{(e)}-x_{i}^{(e)} y_{k}^{(e)}\right]+ \\
& \left.+A_{k}^{(e)}\left[x\left(y_{i}^{(e)}-y_{j}^{(e)}\right)+y\left(x_{j}^{(e)}-x_{i}^{(e)}\right)+x_{i}^{(e)} y_{j}^{(e)}-x_{j}^{(e)} y_{i}^{(e)}\right]\right\}(3.21) \tag{3.21}
\end{align*}
$$

The flux density $\underline{B}^{(e)}$ is obtained from (3.7):

$$
\begin{align*}
& \underline{B}^{(e)}=\frac{1}{2 S}\{(e) \\
&-\underset{i}{ }\left[A_{i}^{(e)}\left(x_{k}^{(e)}-x_{j}^{(e)}\right)+A_{j}^{(e)}\left(x_{i}^{(e)}-x_{k}^{(e)}\right)+A_{k}^{(e)}\left(y_{j}^{(e)}-x_{i}^{(e)}\right)\right]-  \tag{3.22}\\
& k
\end{align*}
$$

The value of $\left.\underline{B}^{(\mathrm{e})}\right)^{2}=\left|\mathrm{B}^{(\mathrm{e})}\right|^{2}$ can be obtained readily from Eq. (3.22). $\mathrm{S}^{(\mathrm{e})}$ denotes the area of the triangle.

As the number of triangles is finite, we may substitute the integrals in Eq. (3.20) by finite sums of the contributions to Eq. (3.20) of all triangles $m$ :


Fig. 3.13 A Typical Triangle of the Discretized Continuum

$$
\begin{align*}
& \varepsilon=\frac{\partial}{\partial A}\left(\frac{1}{2} \int_{V} \underline{H} \cdot \underline{B} d V-\int_{V} \underline{A} \cdot \underline{I} d V\right)=\sum_{e=1}^{m}\left(\frac{\partial}{\partial A^{(e)}} \left\lvert\, \frac{1}{2} \int_{S}^{(e)} \underline{H}^{(e)} \cdot B^{(e)} d S(e)-\right.\right. \\
& \left.-\int_{S}^{(e)} \underline{A}^{(e)} \cdot \underline{J}^{(e)} d S^{(e)} \mid\right) \text {. } \tag{3.23}
\end{align*}
$$

As vector potential $A^{(e)}$ inside the triangle is a function of the nodal values of vector potential $A_{l}$, this expression represents an $n$-dimensional vector function of $A_{l}$, the nodal values of vector potential. By substituting $\left.\underline{H}^{(e)} \cdot \underline{B}^{(e)}=v^{(e)} \underline{B}^{(e)}\right)^{2}$ and Eqn. (3.21) for $A^{(e)}$ into Eqn. (3.23), integrating over the triangle area, and differentiating $n$ times we obtain $n$ components of $\varepsilon$ which we all set equal to zero; we have the set of $n$ equations of which the typical one is:

$$
\begin{gather*}
+\left(y_{j}^{(e)}-y_{k}^{(e)}\right)\left(y_{k}^{(e)}-y_{i}^{(e)}\right)+A_{k}^{(e)}\left(x_{k}^{(e)}-x_{j}^{(e)}\right)\left(x_{j}^{(e)}-x_{i}^{(e)}\right)+\left(y_{j}^{(e)}-y_{k}^{(e)}\right)\left(y_{i}^{(e)}-y_{j}^{(e)} \mid-\right. \\
-\frac{J^{(e)}{ }_{S}^{(e)}}{3}=0 \tag{3.24}
\end{gather*}
$$

where summation is taken over all the triangles adjacent to a certain node. For non-linear media, Eqns. (3.14) and (3.15) do not represent the field energy, but the relation (3.19) should still hold. The only difference is that Eq. (3.24) becomes non-linear, due to non-linear reluctivity $v$, which may be now expressed as a function of flux density $B$, or, more convenient for computation purposes as a function of $\mathrm{B}^{2}$. As in our approximation $\mathrm{B}^{2}$ is constant over the triangle area, the most logical choice is a constant $v=f\left(B^{2}\right)$ inside the triangle.

The approach described is known in technical literature as the 'Ritz's method', 'Rayleigh-Ritz method', 'variational formulation', 'energy approach' or simply 'the finite element method'. Eq. (3.19) is usually derived by applying the Euler theorem of the calculus of variations to Eq. (3.10), which justifies naming the approach 'variational formulation', if Eq. (3.10) is taken as the source equation. If Eq. (3.19) is taken as the source equation, the approach could be also described as 'orthogonality method', or, more particular 'Galerkin's method'.

Another possibility for the derivation of our equations is the direct use of Eq. (3.12). In the current carrying region this equation is not satisfied for any integration loop which does not include at least one mesh node (because our approximation is curl free inside the triangle area), and in the current free area it is satisfied regardless of the distribution of $A$. This is not so if the integration loop includes a mesh node. We may therefore choose a number of integration loops, for which we can apply Eq. (3.12), and define our error vector as:

$$
\begin{equation*}
\{\varepsilon\}=\left\{\int_{\ell} \underline{H} \cdot \underline{d \ell}-\int_{S} \underline{J} \cdot \underline{d S}\right\} \tag{3.25}
\end{equation*}
$$

The number of components of this vector is equal to the number of integration loops chosen and, generally, we can minimise it in the least square sense by adjusting the nodal values of $A$. However, if the number of loops is equal to the number of nodes, we may eliminate $\varepsilon$ defined Eq. (3.25) completely by solving the corresponding equations.

An integration loop may be set in an infinite number of ways. A logical way is to choose the integration line around a mesh node so that it is symmetrical and that the line includes one third of the triangle area (Fig. 3.14). In that case integration around several, or just one mesh node will yield an equivalent system of equations.

Since the field is curl free inside the triangle the value of $\int \underline{H}$. $\mathrm{d} \ell$ will depend on the position of starting and finishing point only. $B y$ substituting $\underline{H}=\underline{\mathcal{B}}$ and using Eq. (3.22) (Fig. 3.14):

$$
\begin{align*}
& \int_{a}^{b} \underline{H} \cdot d \ell=\frac{v^{(e)}}{2 S^{(e)}}\left\{\left[A_{i}^{(e)}\left(x_{k}^{(e)}-x_{j}^{(e)}\right)+A_{j}^{(e)}\left(x_{i}^{(e)}-x_{k}^{(e)}\right)+A_{k}^{(e)}\left(x_{j}^{(e)}-x_{i}^{(e)}\right)\right] \frac{x_{k}^{(e)}-x_{j}^{(e)}}{2}+\right. \\
& \left.+\left[A_{i}^{(e)}\left(y_{j}^{(e)}-y_{k}^{(e)}\right)+A_{j}^{(e)}\left(y_{k}^{(e)}-y_{i}^{(e)}\right)+A_{k}^{(e)}\left(y_{j}^{(e)}-y_{i}^{(e)}\right)\right] \frac{y_{j}^{(e)}-y_{k}^{(e)}}{2}\right\} \tag{3.26}
\end{align*}
$$

The integration of JdS over the corresponding third of the triangle area gives:

$$
\begin{equation*}
\int J d S=\frac{J S}{3} \tag{3.27}
\end{equation*}
$$

If the integration is carried out around one mesh node only we have:

$$
\begin{align*}
& \sum_{e^{=1}}^{p}\left\{\frac { v ^ { ( e ) } } { 4 S ^ { ( e ) } } \left[A_{i}^{(e)}\left\{\left(x_{k}^{(e)}-x_{j}^{(e)}\right)^{2}+\left(y_{j}^{(e)}-y_{k}^{(e)}\right)^{2}\right\}+A_{j}^{(e)}\left\{\left(x_{i}^{(e)}-x_{k}^{(e)}\right)\left(x_{k}^{(e)}-x_{j}^{(e)}\right)+\right.\right.\right. \\
& \left.+\left(y_{k}^{(e)}-y_{i}^{(e)}\right)\left(y_{j}^{(e)}-y_{k}^{(e)}\right)\right\}+A_{k}^{(e)}\left\{\left(x_{j}^{(e)}-x_{i}^{(e)}\right)\left(x_{k}^{(e)}-x_{j}^{(e)}\right)+\left(y_{i}^{(e)}-y_{j}^{(e)}\right)\right. \\
& \left.\left.\left.\left(y_{j}^{(e)}-y_{k}^{(e)}\right)\right\}\right]-\frac{J^{(e)} . S^{(e)}}{3}\right\}=0 \tag{3.24a}
\end{align*}
$$

where summation is taken over all the triangles surrounding the node


Fig. 3.14 Integration Lines Inside a Triangle
in consideration and which is exactly the same as Eq. (3.24). If we choose $n$ integration loops around $n$ mesh nodes we shall get the set of $n$ equations, identical to the set obtained by variational approach applied to the same approximation.

This approach is known in the literature as 'integration method', 'orthogonality method' (or Galerkin's method, as a subgroup of orthogonality methods) or 'small field approximation'.

Although these two approaches are not the only possibility of obtaining the final set of algebraic equations (in Ref. 3.40 an approach that derives the equations from the analogy with a current carrying sheet has been presented, and in Ref. 3.47 a possibility for the use of Taylor expansion, i.e. finite difference method for arbitrary mesh has been presented), they are probably the most convenient ones. The integration method has the advantage of being simple, but it is not quite clear how it can be extended to three-dimensional vector field problems, unlike the energy method, which is, in this sense, more general.

As it is clearly seen from the integration method, our approximation will satisfy Eq. (3.12) only for specially chosen integration loops. The continuity conditions on the interelement boundaries will generally be violated for the tangential component of the magnetic field strength. This is most easily seen for an integration of $\int \underline{H} . d \underline{\ell}$ along the loop which includes only the intertriangle boundary (Fig. 3.15) between two triangles in which the flux lines are not parallel. This violation is due to our approximation. The question therefore arises, whether it is possible to derive an approximation which will not violate the continuity conditions, without increasing the order of


Fig. 3.15 Integration of $\int$ H. dl Along the Intertriangle Boundary
approximation. The answer is clearly negative if the definition of the vector potential as a continuous function is to be kept. If vector potentials are allowed to be discontinuous at triangle boundaries, the continuity of tangential component of magnetic field strength may be preserved (but normal component of flux density will be discontinuous). This approach will give a slightly different set of equations, and a slightly different resulting field. Indeed, there are several possibilities for slightly different approximations and error criterion definitions. These are known as 'minimum complementary energy model', 'Reissner's variational principle model', 'hybrid model', etc. Some of these models will give overestimation (e.g. minimum complementary energy model), and some underestimation (e.g. potential energy model, which we used) of the function $A$ - a useful feature for the practical assessment of accuracy (Ref. 3.62). Unfortunately, all these models are more complex, and more difficult to use than the minimum potential energy model.

### 3.8 Boundary Conditions

The use of Eq. (3.24) for the computation of vector potential is valid only for the internal nodes. For nodes on the boundary another formulae are necessary, depending on the type of the boundary condition.

In our problem we have combined boundary conditions that could be described as Dirichlet plus periodicity boundary conditions. Due to assumptions b) and c) made in section 3.1 there is no magnetic flux outside the machine and in the shaft. This is equivalent to the statement that these two boundaries are flux
lines (or equipotential lines), i.e. we have Dirichlet boundary conditions. An arbitrary constant value of vector potential A may be prescribed to the mesh nodes on these two boundaries, instead of using Eq. (3.24). The simplest choice is $A=0$.

Due to the symmetric electromagnetic structure of the machine, the magnetic field in it will be periodical with the period of two pole pitches and symmetric in two adjacent poles. The computation can be therefore limited to only one pole pitch. The values of vector potential on two sides of the pole will have the same magnitude, but opposite sign. If the distribution of mesh nodes on these two boundaries are the same, Eq. (3.24) may still be used for the nodes on the boundaries. The summation will be partially carried out on one, and partially on the other side of the pole pitch, with the negative sign for vector potential $A$. Although this complicates computer program, it can be solved satisfactorily.

### 3.9 Torque and Induced Voltage Computations

The desirable result of our computation would be torque and impedance of the machine for a given current. As the medium is non-linear it is not safe to talk about impedance (or reactance or inductance), before the definition of these values. The situation is much clearer if we limit ourselves to instantaneous torque, linked flux and induced voltage; these can be also readily evaluated from the field.

The simplest way of torque computation is by the surface integral method (Ref. 3.63). The tangential and normal surface component are given respectively by:

$$
\begin{gather*}
F_{t}=\mu_{0} H_{n} H_{t}  \tag{3.28}\\
F_{n}=\frac{1}{2} \mu_{0}\left(H_{n}^{2}-H_{t}^{2}\right) \tag{3.29}
\end{gather*}
$$

Integration of the tangential component along the air gap and multiplication by rotor radius will give the instantaneous torque per unit axial length of the machine. This integration is easily carried out for our finite element approximation, as the value of $F_{n}$ and $F_{t}$ are constant inside the given triangle and are readily obtainable from the vector potential. Any integration line inside the air gap should yield the same result.

The flux linked by the coil may be evaluated from the vector potential. If we define the value of vector potential as zero at infinity, the value of vector potential $A$ at any point represents the total flux $\psi$ as it would be obtained by integration of $\underline{B}$.dS over the surface extending from infinity to the point in consideration (in our case the surface integral would degenerate to line integral and the result would have been obtained for unit axial length). It follows that the total flux linked by the coil is obtained by simple subtraction of the value of vector potential at the incoming and outcoming side of the coil, multiplied by the number of turns.

From the known value of the instantaneously linked flux $\psi$ the induced voltage $u$ in the coil is obtained from:

$$
\begin{equation*}
u=-\frac{\partial \psi}{\partial t} \tag{3.30}
\end{equation*}
$$

For the approximate solution, the differential operator may be replaced by the difference operator, thus:

$$
\begin{equation*}
u=-\frac{\Delta \psi}{\Delta t} \tag{3.31}
\end{equation*}
$$

It is therefore necessary to evaluate a number of field distributions in order to obtain the induced voltage for a given current.

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## CHAPTER 4

NUMERICAL SOLUTION OF LARGE SPARSE SYSTEMS
OF NON-L INEAR EQUATIONS

This chapter is concerned with some possibilities for the numerical solution of equations arising from discretization of non-linear magnetic field problems.

In section 4.1 we discuss several possibilities for the construction of the triangle mesh and give basic guide-lines on the construction of this mesh. Section 4.2 deals with the approximation of the magnetisation curve. Several possibilities are discussed and the method of piecewise linear approximation is selected.

Section 4.3 discusses the structure of the resulting system of equations. The basic structure of the system is pointed out, which in turn suggests several possibilities for the solution of this system. In section 4.4 we discuss several possibilities for the solution of such systems in the linear case. All available methods are divided in two groups, direct and indirect (iterative) methods and a brief account is given of each of them.

In section 4.5 several methods for the solution of non-linear equations having a single variable are described, and in the next section it is shown how some of these methods can be generalised to apply to systems of equations. It is also shown how these methods can be combined with direct or indirect methods for linear systems, and several known methods of this type are described. Some modifications of these methods are also suggested. Section 4.6 contains a review of the literature of numerical solutions of
non-linear systems related to our problem. The conclusion of this section is that although numerous results have been published, no indirect method of solution has ever been used for non-linear systems arising from free topology triangular meshes, and although these methods seem to be very attractive, some numerical experimentation is thought necessary.

Finally, section 4.8 describes two methods for the acceleration of the convergence of iterative methods for linear and nonlinear problems.

### 4.1 Construction of the Triangle Mesh

In the previous chapter we have discussed several possibilities for discretization of the cross-section of an induction machine. We have chosen a free topology triangular mesh for our problem. This mesh is very flexible, but complex. In order to avoid some of the practical difficulties which would have been encountered in a completely topologically free mesh (organisation of data in the computer core, block iteration techniques), a mild restriction was imposed regarding the topology of the mesh; i.e. it was decided that all mesh nodes must lie on a certain number of lines that stretch from one boundary to another (boundaries with periodicity condition imposed on them, Fig. 4.1). These lines must not intersect each other, but their shape, number and number of nodes on each line is left free. Lines are numbered in the radial direction and nodes are numbered from left to right (or, rather, in the clockwise direction) on each line. Triangles are formed by linking the nodes on the two neighbouring lines.


Fig. 4.1 The Mesh Lines

A typical mesh node for this type of mesh is represented on Fig. 4.2. The number of triangles in the mesh is equal to $p=2 n-m-2$, where $n$ is the total number of nodes and $m$ is the number of nodes on the periphery. Regarding the organisation of data in the computer core it is convenient to associate two triangles with every node ' $i$ ' i.e. triangles ' $a$ ' and ' $b$ ' of Fig. 4.2 and also two nodes ( $A$ and $B$ ) by which the corresponding triangles are formed. When corresponding equations are formed, necessary data include also current densities and permeabilities (reluctivities) for different regions as well as geometry. Seven data for each node are necessary in order to describe the mesh to the computer:
$\left.\begin{array}{l}X \\
v^{(a)} \\
v^{(b)} \\
N_{A} \\
N_{B}\end{array}\right\} \quad$ co-ordinates

$J$$\quad$| reluctivities of triangles a and $b$. |
| :--- |

The accuracy of the field solution depends not only on the mesh, but on the field distribution itself. If, for example, flux density is constant in a certain part of the machine, the vector potential $A$ will change linearly and an accurate representation may be achieved by only few triangles. As the complexity of the field distribution increases, number of nodes must increase as well if the same accuracy of the solution is expected. This demand is more pronounced in non-linear cases, where it is necessary to represent the varying permeability more accurately. In


Fig. 4.2 A Typical Mesh Node
practical terms this means finer division into triangles of air gap and tooth-tip areas than that of teeth and yokes, and also finer division of iron parts than that of air (or copper).

This conclusion is supported by two more considerations:
a) the field distribution in the air gap and tooth tip area has more influence on the performance of the machine; and.
b) the complex geometry of the tooth tips can be better represented by a finer mesh.

The shape of triangles should be kept as near to equilaterals as possible. Equilateral triangles have the best properties regarding the field and reluctivity approximation, and the resulting equations can usually be solved with less difficulties than if triangles with small angles are used.

The basic guide lines for the construction of the triangle mesh are therefore as follows:

The mesh may be relatively coarse in the yokes, but its density should increase towards the air gap. The mesh should be finer in iron (teeth) than in non-magnetic areas (slots). Triangles with small angles should be avoided.

Even for relatively small practical problems the number of nodes will often be as high as several hundred. As seven data for each node are necessary for the complete description of the mesh, the total number of data will usually be about several thousands. Preparing these data by hand would be cumbersome and liable to errors. Some degree of automation in the construction of mesh and preparation of data is imperative if anything but very simple and very few problems are to be solved.

Several works on the automatic generation of triangle meshes have been reported and several different ideas have been exploited.

Winslow (Ref. 4.1) solved the problem by mapping the geometrical picture approximately on to a regular equilateral triangle array, and then found the real node co-ordinates by solving Laplace's equation for the region, where mesh lines are considered as equipotentials. Cardew (Ref. 4.2) used the same idea in his program. Akyuz (Ref. 4.3) introduced a concept of natural coordinate system and used it for an automatic mesh generation program. Reid (Ref. 4.4) constructed a triangle mesh by covering the region with a regular triangle mesh and then adjusted the shape of triangles on the boundary to represent the boundary more accurately. Jensen (Ref. 4.5) defined only the node co-ordinates, and the mesh was then generated automatically by linking the nearest nodes.

These are just a few of the publications in which automatic mesh generation is discussed. The property of all these methods (except the method of Jensen) is that they produce mildly irregular meshes, i.e. the mesh cell size changes slowly throughout the mesh, and triangles are almost equilateral. Unfortunately, these methods are really suitable only for problems with simple internal geometry, and could hardly be applied to our problem because of the complex geometry of slots and teeth.

For this reason a computer program has been developed which generates the triangle mesh for our particular problem of doubly slotted machines. The mesh generation is based on guide lines given earlier in this section. The program is strictly useroriented and is not intended for general application. It is
general, however, in the sense that different relative positions of rotor to stator, as well as different geometrical dimensions and number of slots on rotor and stator can be dealt with.

For control purposes a plotting routine has been written which enables graphical display of a generated mesh. Some of the automatically generated meshes for a small model problem of three slots on stator and two slots on rotor are represented on Figs. 4.3-4.5.

### 4.2 Representation of the Magnetisation Curve

As the reluctivity $v$ of iron is dependent on the local flux density $B$ in a non-trivial way, $v=f(B)$, it is necessary to represent this dependence for use in the computer. A graphical representation which is often used for hand computation cannot be used, and the magnetisation curve must be presented in some form of algebraic or transcendental function.

Fisher and Moser (Ref. 4.6) discussed several possibilities for the representation of magnetisation curves by simple formulae. Although some of their formulae are quite simple and represent the actual magnetisation curve fairly well, their use is not quite adequate for our purpose. The reason for this is that a very simple formula cannot represent the magnetisation curve accurately in the whole range of flux densities necessary for our computation ( $0-2.5 \mathrm{~T}$ ), and it becomes necessary to use some testing device in order to use different formulae for different parts of the curve.

Once a decision has been made to represent the curve by sections, any number of sections can be used without increase in


Fig. 4.3


Fig. 4.4 Computer Generated Mesh


Fig. 4.5 Computer Generated Mesh
computation time. The curve can therefore be represented to any accuracy by the use of very simple expressions, such as piecewise linear approximation. This type of approximation was used by Trutt, Erdelyi and Hopkins (Ref. 4.7).

If non-linear iteration is to be used for the solution of the final equations (see section 4.6 ) representation of the first derivative, $\partial v / \partial B=f(B)$ may also be necessary. If $v=f(B)$ is represented as a piecewise linear function, its derivative is clearly a step function (Fig. 4.6), a point also discussed by Reppe (Ref. 4.8). Obviously, this is a very crude approximation to the actual $\partial V / \partial B=f(B)$, which is a smooth, continuous function. However, it can be used in our computations, as the values of $\partial v / \partial B$ do not have influence on the final result and are used in the iteration process only.

In the actual computer program the functions that have been used were $v=f\left(10 B^{2}\right)$ and $\partial V / \partial\left(10 B^{2}\right)=f\left(10 B^{2}\right)$. This is more convenient as it saves several arithmetic operations for every mesh cell in every iteration. The curves were represented by 80 linear sections and the values of $10 B^{2}$ were used as indices as to which section of the curve was to be used. The curve $v=f\left(10 B^{2}\right)$ as derived from the $B-H$ curve supplied by the manufacturer is represented on Fig. 4.7.

### 4.3 Some Properties of the Resulting System of Equations

As we have shown in Chapter 3, our problem has been transformed into a problem of solution of $n$ simultaneous non-linear equations, of which the representative one is Eqn. (3.24). For the purpose of programming it is convenient to introduce the notation


Fig. 4.6 Piecewise Linear Approximation of Reluctivity and its Differentiation as a Function of Flux Density

(see Fig. 4. 8 and 4.9):

$$
\begin{array}{lll}
b_{i}=y_{j}-y_{k} & b_{j}=y_{k}-y_{i} & b_{k}=y_{i}-y_{j} \\
c_{i}=x_{k}-x_{j} & c_{j}=x_{i}-x_{k} & c_{k}=x_{j}-x_{i} \tag{4.1}
\end{array}
$$

$$
\begin{equation*}
S^{(e)}=\text { area of the triangle } \tag{4.2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{e=1}^{p} \frac{J^{(e)} S^{(e)}}{3}=J_{i} \tag{4.3}
\end{equation*}
$$

With this notation Eqn. (3.24) can now be written as:

$$
\begin{align*}
& f_{i}(A)=A_{i}\left\{\frac{b_{i}^{(1)} b_{i}^{(1)}+c_{i}^{(1)} c_{i}^{(1)}}{4 S^{(1)}} \cdot v^{(1)}+\frac{b_{i}^{(2)} b_{i}^{(2)}+c_{i}^{(2)} c_{i}^{(2)}}{4 S^{(2)}} \cdot v^{(2)}+\ldots\right. \\
& \left.+\frac{b_{i}^{(m)} b_{i}^{(m)}+c_{i}^{(m)} c_{i}^{(m)}}{4 S^{(m)}} \cdot v^{(m)}+\ldots+\frac{b_{i}^{(p)} b_{i}^{(p)}+c_{i}^{(p)} c_{i}^{(p)}}{4 S^{(p)}}\right\}+ \\
& +A_{i, 1}\left\{\frac{b_{i}^{(1)} b_{j}^{(1)}+c_{i}^{(1)} c_{j}^{(1)}}{4 S^{(1)}} \cdot v^{(1)}+\frac{b_{i}^{(p)} b_{k}^{(p)}+c_{i}^{(p)} c_{k}^{(p)}}{4 S^{(p)}} \cdot v^{(p)}\right\}+ \\
& +A_{i, 2}\left\{\frac{b_{i}^{(2)} b_{j}^{(2)}+c_{i}^{(2)} c_{j}^{(2)}}{4 S^{(2)}} \cdot v^{(2)}+\frac{b_{i}^{(1)} b_{k}^{(1)}+c_{i}^{(1)} c_{k}^{(1)}}{4 S^{(1)}} \cdot v^{(1)}\right\}+ \\
& \stackrel{\vdots}{\vdots} A_{i, m}\left\{\frac{b_{i}^{(m)} b_{j}^{(m)}+c_{i}^{(m)} c_{j}^{(m)}}{4 S^{(m)}} \cdot v^{(m)}+\frac{b_{i}^{(m-1)} b_{k}^{(m-1)}+c_{i}^{(m-1)} c_{k}^{(m-1)}}{4 S^{(m-1)}} \cdot v^{(m-1)}\right\}+ \\
& \stackrel{+}{\vdots} A_{i, p}\left\{\frac{b_{i}^{(p)} b_{j}^{(p)}+c_{i}^{(p)} c_{j}^{(p)}}{4 S^{(p)}} \cdot v^{(p)}+\frac{b_{i}^{(p-1)} b_{k}^{(p-1)}+c_{i}^{(p-1)} \cdot c_{k}^{(p-1)}}{4 S^{(p-1)}} \cdot v^{(p-1)}\right\}- \\
& -J_{i}=0 \tag{4.4}
\end{align*}
$$



Fig. 4.8 A Typical Triangle


Fig. 4.9 A Mesh Node with the Adjacent Triangles

$$
\begin{equation*}
A_{i} C_{0}+A_{i, 1} C_{1}+\ldots+A_{i, m} C_{m}+\ldots+A_{i, p} C_{p}-J_{i}=0 \tag{4.5}
\end{equation*}
$$

Following Winslow (Ref. 4.1) we call coefficients $C_{m}$ coupling coefficients. Formally, we can write Eqns. (4.5) in a matrix form:

$$
\begin{equation*}
[C]\{A\}=\{J\} \tag{4.6}
\end{equation*}
$$

where the elements $C_{k, \ell}$ of matrix [ $C$ ] are the coupling coefficients. In a non-linear case, these coefficients are functions of the vector $\{A\}$, because of the non-linear reluctivity $v$. As can be easily seen from Eqn. (4.4) and Fig. 4.11, coupling coefficient for two mesh nodes ( $k$ and $\ell$ ) depends only on the properties of the two triangles for which the line $k-\ell$ is a common side. They are also symmetrical, i.e. $C_{k, \ell^{2}} C_{\ell, k}$. Coupling coefficient for the node itself depends on the properties of all surrounding triangles. The matrix [C] is therefore symmetrical. It is also sparse, as in average less than seven entries in every row will be different from zero. As the numbering of nodes is done in an ordered manner, line after line in our mesh, matrix [C] is also tri block-diagonal, i.e. it has a form:

$$
[c]=\left[\begin{array}{llll}
{\left[\begin{array}{l}
\left.c_{1,1}\right]
\end{array}\right.} & {\left[c_{1,2}\right]} & &  \tag{4.7}\\
{\left[\begin{array}{c}
\left.c_{2,1}\right]
\end{array}\right.} & {\left[\begin{array}{c}
\left.c_{2,2}\right]
\end{array}\right.} & {\left[c_{2,3}\right]} & \\
& {\left[c_{3,2}\right]} & {\left[c_{3,3}\right]} & {\left[c_{3,4}\right]}
\end{array}\right] \begin{aligned}
& \\
& 0
\end{aligned}
$$

where $n+2$ is the number of 1 ines in our mesh. All diagonal matrices in (4.7) are square and have $m-1$ rows where $m$ is a number of nodes on a corresponding mesh line..

As the matrix elements are functions rather than constants, we cannot discuss the numerical properties of matrix (4.7). If, however, a linear case is considered, i.e. if the reluctivity $v$ is fixed (although it may vary from one triangle to another), then it can be shown that the matrix $(4.7)$ is diagonally dominant (Refs. 4.2, 4.9). In the next section we shall briefly examine some possibilities for the solution of (4.6) for the linear case.

### 4.4 On the Solution of Large Sparse Sets of Linear Equations <br> In this section we shall briefly examine some possibilities for the solution of Eqn. (4.6) for the linear case, i.e. for problems with constant permeability. We seek:

$$
\begin{equation*}
\{A\}=[C]^{-1}\{J\} \tag{4.8}
\end{equation*}
$$

where [C] is constant, large, sparse, symmetric, diagonally dominant, tri block-diagonal matrix. The product $[C]^{-1}\{J\}$ may be found in a number of different ways. Basically, we shall differentiate two groups of methods for the solution of linear systems: direct and indirect methods. In direct methods the exact solution is found (theoretically) after a finite number of arithmetic operations. In indirect methods the exact solution would be found after an infinite number of arithmetic operations. In indirect methods one starts from a suitable approximate solution and improves the values of unknowns step by step. In practice neither direct nor indirect methods will give the exact solution, since to achieve
this it would be necessary to compute to an infinite number of decimal places.

We shall consider some direct methods first. One standard direct method is, for example, Cramer's rule. It is however completely impractical for anything but very small systems. Gaussian elimination is another well known direct method, and it can be used for large systems very successfully.

As pointed out in section 4.3 , matrix [C] is symmetric, sparse and tri-block-diagonal. Simple application of the Gaussian elimination procedure for general matrices in our problem would not be very suitable, as it would result in waste of computer time. A number of algorithms, based on Gaussian elimination; have been developed, which are more suitable for either sparse, or symmetric, or tri-block-diagonal, or band structured matrices. A detailed analysis of all these algorithms would take us too far, and we mention only two of them, which seem to be most suitable for our problem.

A compact storage scheme originated by Jennings (Ref. 4.10) is very effective for general band-structured matrices. A Cholesky reduction sequence may be used in a similar manner (Ref. 4.11), which may help in the solution of ill-conditioned systems.

Another method, which is known as group or block elimination and which makes use of the tri-block-diagonal structure of matrix [C] has been described, for example, by Zienkiewicz and Cheung (Ref. 4.12). It is in principle also Gaussian elimination, but sub-matrices play the role of elements of standard Gaussian elimination.

The efficiency of block elimination and compact storage scheme cannot be compared for systems arising from general triangulation, but simple analysis shows that block elimination is more effective for matrices derived for the model problem of Fig. 4. 10 for large $n(n>20$, i.e. order of [C] larger than 400), and would probably be faster for our problem. More detailed analysis shows that the necessary computation time for block elimination can be further reduced (Ref. 4.13).

Another possibility for the solution of Eqn. (4.8) is an indirect method. The situation with indirect methods is much more complicated than with direct ones, as there are numerous indirect techniques and the comparison of the efficiency of these is no simple matter. We shall differentiate two groups of indirect methods, namely relaxation and iteration. The term relaxation is due to Southwell (Ref. 4.14), and by it we mean a routine which improves the values of unknowns in Eqn. (4.8) in a non-systematic way. Relaxation is suitable for hand computation, and skill gained in the use of it may improve the efficiency of the procedure significantly. It cannot easily be programmed for an automatic digital computer, and is used in automatic computation only rarely. We shall not consider this technique in any detail. The term 'relaxation' is used often nowadays for a different group of techniques which we shall call 'iteration'. In an iteration procedure the unknowns are improved in an orderly, systematic manner. All the methods we shall discuss are iteration methods, and we shall use the term relaxation only to keep to the established términology.


Fig. 4.10 A Regular Triangulation of a Square

An iteration for Eqn. (4.8) can often be described by a sequence:

$$
\begin{align*}
&\{A\}_{k}= {[N]^{-1}\left([P]\{A\}_{k-1}+\{J\}\right) \cdot=[M]_{k-1}+[N]^{-1}\{J\} } \\
& k=1,2,3 \ldots \tag{4.9}
\end{align*}
$$

where \{A\} is a suitable starting approximate solution, and matrices $[N]$ and $[P]$ are formed by a suitable splitting of matrix [C]:

$$
\begin{equation*}
[\mathrm{C}]=[\mathrm{N}]+[\mathrm{P}] \tag{4.10}
\end{equation*}
$$

Matrix [C] can be split in an infinite number of ways, and in this sense the number of different iteration techniques is infinite. In order to carry out iteration effectively, Eqn. (4.9) must have certain properties. First, the iteration must be convergent, i.e. consecutive estimates of $\{A\}_{k}$ must approach the solution of Eqn. (4.8) as $k$ increases. Additionally, operations on the right-hand side of Eqn. (4.9) must be carried out easily. Whether an iteration is convergent depends, among other things, on the properties of the matrix [C].

A number of convergent iteration procedures has been developed for the solution of systems of equations arising from discretization techniques applied to linear partial differential equations. Some of the best known of these iteration techniques are:

Jacoby iteration.
Gauss or Gauss-Seidel (Liebmann) iteration.
Successive over-relaxation or extrapolated Gauss-Seidel iteration (SOR).

Successive line over-relaxation (SLOR).
Alternating direction implicit method (ADI).

Many of these methods have several different versions (for example 2 line successive over-relaxation (S2LOR), or so-called semi-iterations with the use of Chebysheev polynomials, etc.), and there are also a number of iteration techniques which we have not listed (e.g. method of conjugate gradients). A full comprehensive analysis of these methods is virtually impossible, as new methods (or variations of the old ones) are introduced constantly. The task would be even more difficult if comparisons for the actual computer time had to be made, as it would be necessary to take into account the size and type of computer, etc.

It is therefore essential to select only some of these methods and to consider them in detail. Our system is non-linear, and none of these methods can be used directly. However, they can form a basis for some non-linear iteration methods. We have chosen $S O R$ as a representative of point iteration techniques and SLOR as a representative of block iteration techniques. This choice is based on the fact that SOR is superior to both Jacoby and Gauss-Seidel iteration, and SLOR is superior to line Jacoby and line Gauss-Seidel. The programming for SOR, Jacoby and GaussSeidel iteration are similar, and the same is true when their line versions are compared. The ADI method can be used only for meshes with regular topology, and it is not suitable for our problem. Use of some more sophisticated or recently developed methods like S2LOR, the method of conjugate gradients (Refs. 4.15, 4.16), dynamic programming solutions (Ref. 4.17) or the peripheral block relaxation method (Ref. 4.18 ) have not been considered in much
detail, but it is likely that they would require complex programs for a free topology of triangular meshes. Limited practical experience with these methods does not indicate that significant computer time saving would result from the use of them.

As almost all these iterative methods have been specially developed for systems arising from rectangular meshes, a question arises whether these methods will converge for our equations. Diagonal dominance of matrix [ $C$ ] ensures convergence for SOR.

SLOR is normally faster than SOR for rectangular meshes, and the use of it is very attractive. As our search in literature did not produce a single reference where this method is used for free topology meshes, and as yet there is no theoretical proof that this method will converge for semi-free topology triangular meshes (indeed, recent results by Cardew, Ref. 4.2, suggest that there never will be such a proof, as he had experienced divergence for some cases of the use of SLOR for fixed topology triangular meshes), it was decided that the suitability of these two methods for our particular problem should be checked on examples (see Chapter 5).

An important item is a comparison of the efficiency of direct and indirect methods. This comparison is difficult to make because of the many parameters involved (for example: is the use of backing store on the computer necessary for the problem in hand or not, accuracy of the result, etc.). In the literature usually only one group of methods is discussed in detail, while the other group is often avoided completely. An interesting fact can be observed: In papers dealing with the finite element method, i.e. problems connected closely with structural mechanics, the solution
of the final set of equations is rarely dealt with in detail. Presumably a direct solution is assumed. On the other hand, papers dealing with field problems often discuss an iterative method of solution. This seems to indicate that the experience gained from the use of a particular method might play an important role in the decision whether a direct or indirect method is to be used.

Different methods for the solution of systems of linear equations described in this section can form a basis for the solution of non-linear systems. We shall describe several methods for non-linear systems in section 4.6. In the next section, however, we shall turn our attention to non-linear equations having a single variable.

### 4.5 An Account on the Solution of Non-Linear Equations

In this section we shall briefly describe some methods for the solution of the non-linear equation having a single variable:

$$
\begin{equation*}
f(x)=0 \tag{4.11}
\end{equation*}
$$

Generally, this equation cannot be solved directly. Its numerical solution is possible by the use of different iteration techniques. An iteration procedure for Eqn. (4.11) can be usually described as a sequence:

$$
\begin{equation*}
x_{k+1}=g\left(x_{k}, x_{k-1}, \ldots, x_{0}\right) \quad k=0,1,2 \ldots \tag{4.12}
\end{equation*}
$$

using a suitable function $g$ and an initial estimate $x_{0}$. Similarly to the iterative solution of linear systems the iteration is said to converge when the consecutive estimates of $x_{k}$ approach the root
of the equation as $k$ increases. An analogue procedure can be described for non-linear systems by the use of matrices. Some of the better known iteration methods for Eqn. (4.11) are the chord method, Newton's method, the method of false position, Muller's method, Aitken's $\Delta^{2}$ method, Steffensen's method, etc. We shall not discuss the theoretical basis of these methods nor their convergence, but a brief description of some of them follows.
Chord Method (Whittaker's method) (Ref. 4.19, 4.20)
consists of a sequence:

$$
\begin{equation*}
x_{k+1}=x_{k}-m f\left(x_{k}\right) \quad k=0,1,2 \ldots \tag{4.13}
\end{equation*}
$$

where $m$ is a constant.

Newton's Method (Newton-Raphson method) (Ref. 4.19, 4.21) is obtained when the constant $m$ of the chord method is replaced by $\frac{1}{m_{k}}=\frac{\partial f\left(x_{k}\right)}{\partial x_{k}}$, thus:

$$
\begin{equation*}
x_{k+1}=x_{k}-\frac{f\left(x_{k}\right)}{\frac{\partial f\left(x_{k}\right)}{\partial x_{k}}} \quad k=0,1,2 \ldots \tag{4.14}
\end{equation*}
$$

The method of false position (Regula falsi) (Ref. 4.20, 4.22) is obtained by approximating the derivative in Eqn. (4.14) by a difference quotient:

$$
\begin{equation*}
\frac{f\left(x_{k}\right)}{x_{k}}=\frac{f\left(x_{k}\right)-f\left(x_{k-1}\right)}{x_{k}-x_{k-1}} \tag{4.15}
\end{equation*}
$$

$$
\begin{equation*}
x_{k+1}=x_{k}-f\left(x_{k}\right) \frac{x_{k}-x_{k-1}}{f\left(x_{k}\right)-f\left(x_{k-1}\right)} \quad k=1,2,3 \ldots \tag{4.16}
\end{equation*}
$$

In this method two initial estimates, $x_{0}$ and $x_{1}$ have to be made.

Muller's method (Ref. 4.23, 4.24) The method of false position is equivalent to approximating the function by a straight line in the neighbourhood of the root. In Muller's method the function is approximated by a low order (quadratic) polynomial. As quadratic polynomial has normally two zeros, the working algorithm is somewhat more complicated than algorithms already described. It also needs three initial estimates, $x_{0}, x_{1}$ and $x_{2}$.

$$
\text { Aitken's } \Delta^{2} \text { method (Ref. } 4.25,4.26 \text { ) was originally }
$$

proposed to accelerate convergence of any sequence of numbers. It therefore can be used to accelerate the convergence of any of the methods already described. If the sequence of numbers is a sequence formed by, for example, chord iteration (Eqn. 4.13), then Aitken's $\Delta^{\mathbf{2}}$ method consist of forming a new sequence:

$$
\begin{equation*}
x_{k}^{*}=x_{k}-\frac{\left(x_{k+1}^{\left.-x_{k}\right)^{2}}\right.}{x_{k+2^{-2 x_{k+1}}}^{-x_{k}}} \quad k=1,2 \ldots \tag{4.17}
\end{equation*}
$$

which will normally converge faster than the original sequence $X_{k}$. The new estimate $x_{k}^{*}$ may now be used as a starting value for two more steps of chord iteration, after which Eqn. (4.12) is applied again. This method is known as diagonal Aitken's $\Delta^{2}$ procedure, or also Steffensen's method.

Basically, all these methods can be generalised for the solution of non-linear systems. A disadvantage of simple generalisation
is that every iteration step requires a solution of a linear system of the same order. Such methods have been used, however, for the solution of non-linear systems arising from general triangulation of non-linear magnetic field problems.

We shall discuss some of these methods and also look at some other possibilities for the solution of non-linear systems of equations in the next section.

### 4.6 On the Solution of Simultaneous Non-Linear Equations <br> In the previous two sections we have described some of the possibilities for the solution of large linear systems and also some methods for non-linear equations having a single variable. In this section we shall discuss some methods for non-linear systems.

Our system of equations may be symbolically written:

$$
\begin{equation*}
\underline{F}(A)=0 \tag{4.18}
\end{equation*}
$$

where $F$ is an n-dimensional column vector of which a typical element is given by Eqn. (4.4). Analogously to single non-linear equation, Eqn. (4.18) cannot be solved directly for a general case. An iteration procedure for system (4.18) can be described by an equation which is analogous to Eqn. (4.12):

$$
\begin{equation*}
A_{k+1}=G\left(A_{-k}, A_{k-1}, \cdots, A_{0}\right) \quad k=1,2, \ldots \tag{4,19}
\end{equation*}
$$

An infinite number of iteration procedures can be defined by a choice of function $G$. We cannot discuss these methods in the general case, but must restrict ourselves to methods which are either in common use, or are obvious extensions of some other
well-known methods. Even so, we can hardly do more than briefly describe the basic procedure for different methods, as detailed analysis would be too complex and too lengthy.

We shall differentiate two main groups of methods for systems of equations, direct and indirect methods. In direct methods one iteration step will normally require solution of a linear system of order $n$ in each iteration. In indirect methods the iteration is performed by the solution of smaller linear or non-linear systems. Analogously to a single equation, we can define a chord method for systems, which can form a basis for several direct and indirect methods:

Method A (Chord method for systems, Ref. 4.27) is defined by a sequence:

$$
\begin{equation*}
\{A\}_{k+1}=\{A\}_{k}-[B] \cdot\{F(A)\}_{k} \quad k=1,2, \ldots \tag{4.20}
\end{equation*}
$$

where $[B]$ is a constant square matrix of order $n$.

Method B (Linearised iteration). In this iteration matrix $[B]$ is replaced by matrix $\left[C\{A\}_{k}\right]^{-1}$ of Eqn. (4.6) in evezy iteration:

$$
\begin{equation*}
\{A\}_{k+1}=\{A\}_{k}-\left[C\{A\}_{k}\right]^{-1} \cdot\{F(A)\}_{k} \quad k=1,2, \ldots \tag{4.21}
\end{equation*}
$$

Method C (Newton's method, Newton-Raphson method) is defined similarly, but matrix $[B]$ of Eqn. $(4.20)$ is replaced by the inverse of the Jacobian matrix: •

$$
\begin{equation*}
[J(A)]_{k}=\left[\frac{\partial f_{i}(A)_{k}}{\partial A_{j}}\right] \tag{4.22}
\end{equation*}
$$

and we have:

$$
\{A\}_{k+1}=\{A\}_{k}-[J(A)]_{k}^{-1} \cdot\{F(A)\}_{k} \quad k=1,2, \ldots \quad \text { (4.23) }
$$

Convergence theorems for these methods can usually be proved only locally, i.e. for the initial vector ${\underset{\sim}{0}}^{0}$ in the neighbourhood of the root. In this case method $C$ shows the property of quadratic convergence, similar to Newton's method for a single equation. Methods B and C both require a solution of a linear system of order n in each iteration. This solution may be carried out either by a direct method (e.g. Gaussian elimination) or by an indirect method (e.g. SOR). In the latter case we may talk about indirect (two-step, nested, two-level) iteration as every step of Eqn. (4.21) or (4.23) consist of an iteration process itself.

Two-step iterative methods of this kind offer an attractive possibility. Namely as one iteration step in Eqn. (4.21) or (4.23) gives only an approximation of the root of Eqn. (4.18), the solution of a linear system need not be accurate. This means that the inner iteration may consist of only a few iteration steps. However, the total number of outer iterations may be significantly larger than in the case where Eqns. (4.21) or (4.23) are solved accurately.

Method B forms a basis for several two-step'iterative methods of this kind. The outer iteration in these methods consists of evaluating the elements of matrix [C\{A\}]by the formula:

$$
\begin{equation*}
\left(C_{i, j}\{A\}\right)_{k}=\left[C_{i, j}\{A\}^{*}-\left(C_{i, j}\{A\}\right)_{k-1}\right] \cdot B+\left(C_{i, j}\{A\}\right)_{k-1} \tag{4.24}
\end{equation*}
$$

where \{A\}* is an estimate of vector $A$ at the end of inner iteration
cycle, and $\beta$ is an acceleration factor. Depending on the choice of inner iteration we have:

Method B1 Two step linearised point Jacobi iteration.
Method B2 Two step linearised point Gauss-Seidel iteration.
Method B3 Two step linearised point SOR (alternating iteration, alternating relaxation, small field approximation).

Method B4 Two step linearised line Jacobi iteration.
Method B5 Two step linearised line Gauss-Seidel iteration.
Method B6 Two step linearised line SOR (alternating iteration, alternating relaxation).

Method B2 can be considered as a special case of Method B3, and B5 as a special case of B6. Methods B3 and B6 have been used successfully.

Analogously to methods Bl-B6 we could define another group of methods where Method $C$ would serve as a basis. We are not aware of any attempts of considering these methods either theoretically or in practice. Instead of these we shall consider another group of two step iterations for which method $C$ also serves as a basis.

We have mentioned that Method C converges quadratically, but its use requires a solution of a large linear system in each iteration. Convergence of Method A for a given system (4.18) will depend on the matrix $[B]$ in Eqn. (4.20). By analogy between Eqn. (4.20) and (4.23) we can expect fast convergence with Method $A$, if

$$
\begin{equation*}
[B]=[J(A)]^{-1} \tag{4.25}
\end{equation*}
$$

One possibility for approximation of the Jacobian matrix is to neglect some or all off-diagonal elements of this'matrix. If all
off-diagonal elements are neglected, then the inversion of a matrix is replaced by simple division, in fact, a point iteration technique results. Similarly, if a Jacobian matrix is reduced to a tri-diagonal strip, then a line iteration technique results. These techniques can be put on a firm theoretical basis (Ref. 4.28) .

Analogously to methods B1-B6 two-step iteration techniques can be defined in this case as well. We shall call these techniques two-step non-linear iteration. The outer iteration consists of recalculating the elements of the Jacobian matrix which have not been neglected (diagonal elements for point iteration and elements of the tri-diagonal strip for line iteration) by the formula:

$$
\begin{equation*}
\left(\frac{\partial f_{i}(A)}{\partial A_{j}}\right)_{k}=\left(\frac{\partial f_{i}(A)}{\partial A_{j}}\right)_{k-1}-\left[\left(\frac{\partial f_{i}(A)}{\partial A_{j}}\right)_{k-1}-\frac{\partial f_{i}(A)_{k}}{\partial A_{j}}\right] \cdot B \quad k=1,2, \ldots \tag{4.26}
\end{equation*}
$$

where $\beta$ is an acceleration factor. Depending on the choice of inner iteration we have:

Method C1 Two-step non-linear point Jacobi iteration. The outer iteration is defined by (4.26) and the inner iteration by a sequence:
$A_{i, \ell}=A_{i, \ell-1}-\frac{f_{i}\left(A_{1, \ell-1}, A_{2, \ell-1}, \ldots, A_{n, \ell-1}\right)}{\left(\frac{\partial f_{i}(A)}{\partial A_{i}}\right)_{k}} \quad i=1 \ldots n, \ell=1, \ldots K$
$K$ is a number of inner iterations per outer iteration.

Method C2 Two-step non-linear point Gauss-Seidel iteration. The outer iteration defined by Eqn. (4.26) and the inner by:

$$
\begin{gather*}
A_{i, \ell}=A_{i, \ell-1}-\frac{f_{i}\left(A_{1, \ell}, A_{2, \ell}, \ldots, A_{i-1, \ell}, A_{i, \ell-1}, A_{i+1, \ell-1}, \ldots, A_{n, \ell-1}\right)}{\left(\frac{\partial f_{i}(A)}{\partial A_{i}}\right)_{k}} \\
i=1 \ldots n, \ell=1 \ldots K \tag{4.28}
\end{gather*}
$$

Method C3 Two-step non-linear point SOR. The outer iteration is defined by Eqn. (4.26) and the infier by:

$$
\begin{gather*}
A_{i, \ell}=A_{i, \ell-1}-\frac{f_{i}\left(A_{1, \ell}, A_{2, \ell}, \ldots, A_{i-1, \ell}, A_{i, \ell-1}, A_{i, \ell-1}, \ldots, A_{n, \ell-1}\right)}{\left(\frac{\partial f_{i}(A)}{\partial A_{i}}\right)_{k}} \omega \\
i=1 \ldots n, \ell=1 \ldots K \tag{4.29}
\end{gather*}
$$

where $\omega$ is the over-relaxation factor.

Two-step line iteration methods of this kind are also obtained by using Eqn. (4.26) for outer iteration, but line iteration for inner iterations. We quote the following methods without explicitly writing the exact procedure:

Method C4 Two-step non-linear line Jacobi iteration.
Method C5 Two-step non-linear line Gauss-Seidel iteration.
Method C6 Two-step non-linear line SOR.

Basically, every outer iteration of methods B1-B6 and C1-C6 contains several inner iterations. An extreme in the use of these schemes is only one inner iteration per outer iteration. It is possible, of course, to construct iteration schemes in which the
outer iteration as defined for methods B1-B6 and C1-C6 contains less than one iteration. Non-linear point iteration of this type will consist of the use of any of Eqns. (4.27)-(4.29), but the.. coefficient $\partial f_{i}(A) / \partial A_{i}$ is recalculated as soon as any new estimate of $A_{i}$ is known. As the procedure reduces to a single-step iteration, we shall call these methods one-step iteration. Linearised iteration of this type can be defined, but we consider only nonlinear iteration of this type.

Method D1 One-step non-linear point Jacobi iteration (Jacobi-Newton Process, $J-N-P$ ) is defined by a sequence:

$$
\begin{align*}
& A_{i, k}=A_{i, k-1}-\frac{f_{i}\left(A_{k-1}\right.}{\partial f_{i}\left(A_{1, k}, A_{2, k}, \cdots, A_{i-1, k}, A_{i, k-1}, A_{i+1, k-1}, \cdots, A_{n, k-1}\right)}{ }_{\partial A_{i}} \\
& i=1 \ldots n, k=1,2 \ldots  \tag{4.30}\\
& \text { Method D2 One-step non-linear point Gauss-Seidel iteration } \\
& \text { (Gauss-Seidel-Newton-Process, G-S-N-P) is defined by: }
\end{align*}
$$

$$
\begin{gather*}
A_{i, k}=A_{i, k-1}-\frac{f_{i}\left(A_{i, k}, A_{2, k}, \ldots, A_{i-1, k}, A_{i, k-1}, A_{i+1, k-1}, \ldots, A_{n, k-1}\right)}{\frac{\partial f_{i}\left(A_{1, k}, A_{2, k}, \ldots, A_{i-1, k}, A_{i, k-1}, A_{i+1, k-1}, \ldots, A_{n, k-1}\right)}{\partial A_{i}}} \\
 \tag{4.31}\\
i=1 \ldots n, k=1,2 \ldots
\end{gather*}
$$

Method D3 One-step non-linear point SOR (Extrapolated-Gauss-Seidel-Newton-Process, E-G-S-N-P, Generalised Newton's method, Non-linear Over-relaxation, Non-linear Successive Overrelaxation) is defined by:

$$
\begin{gather*}
A_{i, k}=A_{i, k-1}-\frac{f_{i}\left(A_{1, k}, A_{2, k}, \ldots, A_{i-1, k}, A_{i, k-1}, A_{i+1, k-1}, \ldots, A_{n, k-1}\right)}{\partial f_{i}\left(A_{1, k}, A_{2, k}, \ldots, A_{i-1, k}, A_{i, k-1}, A_{i+1, k-1}, \ldots, A_{n, k-1}\right)} \\
\partial A_{i} \tag{4.32}
\end{gather*}
$$

Analogously to methods D1-D3, we have corresponding one-step non-linear line iteration techniques:

| Method D4 | One-step non-linear line Jacobi iteration. |
| :--- | :--- |
| Method D5 | One-step non-linear line Gauss-Seidel iteration. |
| Method D6 | One-step non-linear line SOR. |

These methods are analogous to methods D1-D3, but line iteration techniques are used instead.

In the indirect iteration methods listed so far the basic procedure is to use an algorithm for one equation (or group of equations). In doing so a very inaccurate solution of a nonlinear equation corresponding to a certain node is usually found. Another group of methods can be defined in which an accurate solution of (for point methods) a non-linear equation in a single variable $A_{i}$ is found assuming the remaining values $A_{1} \ldots A_{i-1}$, $A_{i+1} \ldots A_{n}$ constant. Any method for non-linear equation in a single variable can be used, and normally several steps of, for example, Newton's method will be necessary until another equation is dealt with. These methods can also be called two-step iteration, but to differentiate these methods from two-step methods defined earlier, we shall call them non-linear iteration. This group of methods is wide, and only as an example we define:

$$
\begin{aligned}
& \text { Method E2 Non-linear Newton point Gauss-Seidel iteration } \\
& \text { (non-linear Gauss-Seidel (Liebmann) Process, G-S-P, extended } \\
& \text { Liebmann iteration, Liebmann method). } \\
& A_{i, k, \ell}=A_{i, k, \ell-1}- \\
& \left.-\frac{f_{i}\left(A_{1, k, K}, A_{2, k, K}, \cdots, A_{i-1, k, K}, A_{i, k, \ell-1}, A_{i+1, k-1, K}, \cdots, A_{n, k-1, K}\right)}{\partial f_{i}\left(A_{1, k, K}, A_{2, k, K}, \cdots, A_{i-1, k, K}, A_{i, k, \ell-1}, A_{i+1, k-1, K}, \ldots, A_{n, k-1, K}\right)}\right) \\
& \ell=1 \ldots \mathrm{~K}, \mathrm{i}=1 \ldots \mathrm{n}, \mathrm{k}=1,2, \ldots \quad \text { (4.33) }
\end{aligned}
$$

where n is a number of nodes, K is a number of steps of non-linear iteration to be carried out for every node. $K$ need not be fixed but may vary from node to node and from iteration to iteration, and the inner iteration can be stopped when a certain accuracy is: reached.

In this section we have briefly described a number of differ:ent iteration techniques for non-linear systems. Some possibilities have only been mentioned, and it is also clear that numerous additional related iteration techniques can be defined. This vast number of available methods makes the choice for the method for our system very difficult. Although some methods are closely related it is by no means true that their rate of convergence will be similar, nor is it simple to say which method will be faster. The situation is even more complex if actual computation times are considered. As a comprehensive analysis of all these methods is virtually impossible, either theoretically, or by practical comparison of the efficiency of different methods, we shall make a
brief survey of published results on the use of different iteration techniques for the solution of non-linear systems obtained by discretization techniques applied tọ quasilinear elliptic partial differential equations.

### 4.7 Review of Literature on the Numerical Solutions of Quasi-linear Elliptic Partial Differential Equations

In principle, the methods described in section 4.6 can be used to solve any system of non-linear equations. Our equations have some special properties, and if an efficient method is sought we must restrict our consideration to the use of published methods applied to systems arising from discretization.

The paper by Bers (Ref. 4.29 ) has become a classic in this field. He discussed a general quasilinear partial differential equation of the elliptic type and showed that a normal finite difference approximation (rectangular mesh) exists, is unique and converges to the true solution as the mesh size decreases. He also showed that non-linear point Jacobi and non-linear point Gauss-Seidel methods converge for this case (method E2 is an example of this iteration, where Newton iteration is used as inner iteration). Douglas (Ref. 4.30) considered a somewhat simpler case, namely the equation:

$$
\begin{equation*}
\frac{\partial^{2} A}{\partial x^{2}}+\frac{\partial^{2} A}{\partial y^{2}}=F(x, y, A) \tag{4.34}
\end{equation*}
$$

and the application of the ADI iteration in a form of a two-step linearised iteration. However, the outer iteration was set up differently from that one described by Eqn. (4.24). The possibility
of the use of SOR instead of ADI for inner iteration was also pointed out, with the remark that the total computation time is likely to be significantly larger, but no detailed analysis was given.

Greenspan and Yohe (Ref. 4.31) considered the threedimensional case of Eqn. (4.34) and the application of one-step linearised SOR (called by them non-linear over-relaxation, or Method D). A comparison with method E2 of section 4.6 was also given, and it was stated that method E2 was inferior for that particular problem.

Greenspan and Parter (Ref. 4.32) considered theoretical aspects of several direct iterative methods and their indirect (point type) counterparts for Eqn. (4.34). The iterations were set up differently, but are related to our linearised iteration and Newton-Raphson iteration. Two examples were given, and it was shown that different problems may favour direct or indirect iteration techniques, respectively. Although SOR was used for the solution of the linear system in their direct method, it is likely that this conclusion will stand even if a direct method of solution is chosen - unless the number of equations is very large.

Ortega and Rockoff (Ref. 4.33) compared the rate of convergence of methods D1, D2, D3, E2 and non-linear Newton point SOR (which is analogous to E2 except that an over-relaxation factor is used). Method D3 showed superior performance for the problems they examined (usual difference equations for Eqn. 4.34).

Greenspan, in another paper (Ref. 4.34), considered Method D3 and also two modifications of the same method where the basic nonlinear iteration is applied to an enlarged system of equations.

This enlarged system is introduced in order to simplify equations of the original system. Several one- and two-dimensional examples are given, and in most cases Method D3 gives the shortest computation time.

Meis and Tornig (Ref. 4.35) gave the convergence theorems for Method D3 and also linearised ADI (introduced by Douglas). No comparison of these methods was given.

These are only some of the publications dealing with nonlinear systems arising from discretization. They have been chosen almost randomly, and the list of quoted references is by no means complete. A number of other works are quoted in, for example, Ref. 4.35. Most of the papers quoted here discuss a solution of Eqn. (4.34), and the results cannot be directly applied to our problem. This is the main reason why we did not include more papers into this review. None of these papers give 'the best method" for a general case, rather, they suggest that different problems may favour different methods, even where only the mesh size is changed (Ref. 4.32). Thus, if we want to chose the method which is most suitable for our particular problem, we must restrict our attention to works which deal with non-linear magnetostatic problems.

As far as we are aware, the first published results on the numerical solution of non-linear magnetic fields appeared in 1963, (Refs. 4.36-4.38). This was followed by a series of publications by several authors, mainly in collaboration with Erdelyi, in which the basic technique introduced in Refs. 4.36-4.38 was refined or applied to different problems (Refs. 4.39-4.61). In all these papers, the authors use regular or semiregular meshes and with
few exceptions consider problems arising in rotating electrical machines.

Another group of reports has been published in connection with several programs developed mainly for the purpose of computation of electromagnets for particle accelerators. Several laboratories have been developing these programs: Lawrence Radiation Laboratory, University of California; Aragone National Laboratories, Midwestern Universities Research Association; Brookhaven National Laboratory; Stanford Linear Accelerator Center, Stanford University and also European CERN. In the literature these programs are often referred to by their names, and we quote names of some of these programs for future reference: LINDA, TRIM, POISSON, NUTCRACKER, MARE, GRACY, SYBIL. The literature on these programs is extensive and consists mainly of internal reports. However, several papers have been published in periodicals (Ref. 4.1, 4.62-4.65). Further references can be found in quoted literature. Some of these programs differ considerably from each other. Problems for which they were developed have relatively simple geometry, but the accuracy required is high. The mesh used is regular or mildly irregular, rectangular or triangular.

Recently, several reports have been published on the use of irregular triangular meshes for non-linear magnetic problems. Although the problems considered are closely related to problems in Refs. 4.36-4.61, (i.e. mainly rotating electrical machines), methods of computation differ from those in Refs. 4.36-4.61 to a considerable degree (Refs. 4.66-4.69).

Direct comparison of the efficiency of different methods used in these reports is virtually impossible for several reasons. The problems considered by different authors are different, the discretization mesh is different, and different computers have been used. Data given in these reports about the efficiency of a particular method used differ from paper to paper. While some authors use mathematical terms like rate of convergence, others simply quote actual computation time, sometimes without specifying computer or compiler. A number of papers, moreover, do not give any details about the efficiency of the numerical method used for computation. We shall therefore consider different groups of reports separately, because otherwise the conclusions could be quite misleading.

A common feature in Refs. $4.36-4.61$ is the use of mildly irregular meshes of rectangular or polar type, resulting in a large number of equations (up to several thousand). In their first paper, Trutt, Erdelyi and Jackson (Ref. 4.36) introduce Method B3 of section 4.6. Other methods are not considered but no details about the efficiency of this method are given. In Ref. 4.45, Ahamed and Erdelyi describe a block-acceleration technique (see section 4.8 ), but continue to use two-step point SOR as a basic numerical procedure. Reichert (Ref. 4.51) compared the performance of Method B3, and linearised two-step ADI for a simple problem with square mesh and 255 mesh nodes. He also used a block-acceleration technique, and the results do not bear a direct relationship to the efficiency of the same methods without acceleration. However, the results indicate very slightly better performance of the two-step linearised point SOR (Method B3). In

- another paper, (Ref. 4.52), the same author advocates the use of two-step linearised SLOR (method B6) for meshes with more than about 500 nodes and two-step linearised SOR (method B3) for smaller meshes. In Ref. 4.58, Erdelyi and Fuchs describe Method B6 in more detail. An example is given in Ref. 4.59. The total average computer time is quoted, and although there is no comparison with other methods, the results suggest that B 6 converges faster than Method B3 for large meshes. Von Zweygbergk and Hultin state in their paper (Ref. 4.55) that Method B2 did not converge for any example they tried, and they developed another two-step linearised procedure which is closely related to direct methods, but iteration is used as means for the solution of the linear system. No discussion of the rate of convergence is given, except the statement that the method is fast and reliable. Trying to summarise these results we can say that for this type of mesh methods B3 and B6 can be considered as reliable. Comparisons, as well as the recent trend towards the use of Method B6 indicate Method B6 is faster for meshes with more than about 500 mesh nodes.

Methods used in another group of reports (Refs. 4.1, 4.624.65) differ from one another, and some of these programs use rather special techniques. So programs MARE, SYBIL and LINDA evaluate the field separately in air and iron regions and then combine these solutions in an iteration scheme. A rectangular discretization mesh was used. Programs GRACY and NUTCRACKER use rectangular or polar meshes and Method B3 for the solution of the equations. In this respect they do not differ from methods of the first group. Of special interest to us is program TRIM. The
discretization mesh used is a topologically regular triangular mesh. Although the mesh in our program is more general, this is the only program that uses a triangular mesh and indirect methods of solution. It is believed that basically the same methods can be used successfully for both type of meshes. The basic iteration procedure used in program TRIM is Method B3. However, tests have been carried out with non-linear Newton point $S O R$ and non-linear Aitken's $\Delta^{2}$ point $S O R$ (both called non-linear over-relaxation) which both showed inferior performance as compared with two-step linearised iteration (Ref. 4.1). All these programs are intended for use with large meshes (e.g. a version of NUTCRACKER program can solve problems of up to 22500 mesh nodes (Ref. 4.64)).

A third group of reports are Refs. 4.66-4.69. They are characterised by the use of free-topology irregular meshes. The number of nodes is relatively small (a few hundred) and the resulting systems were solved by direct methods. The procedure consists of a few steps of chord iteration (Method A), which generate the initial values of the vector potentials, followed by Newton-Raphson iteration (Method C). Comparison of Method A and $C$ is given, and Method $C$ is about 3 to 4 times faster. Come parison with indirect methods was not attempted.

Finally, we consider two reports in which comparison of the efficiency of several different methods for a particular problem (or problems) have been presented. In his paper, Concus (Ref. 4.70) compared the performance of Methods B3, D3 and two versions of Method $C$, where point $S O R$ and line $S O R$ was used for the solution of the resulting linear system. A rectangular mesh was used and results for two problems with 90 and 870 nodes were compared.

For the small problem Method B3 was marginally superior to the other three methods, but for the larger problem Method D3 gave the shortest computation time. Another report due to Reichert was published as his discussion to Ref. 4.67. He compared Method B6, D3, D6 and C (direct solution of the linear system in Method C was used). A rectangular mesh with 224 mesh nodes was used, and his results show superior performance of Method D6.

On the basis of this review no definite conclusion may be drawn as to which method will be most suitable for our problem. The only practical computations that use a non-uniform freetopology mesh use direct iterative methods, but the number of mesh nodes is rather small. The indications are that for large meshes indirect iterative methods might be superior.

Of indirect iterative methods two-step linearised point SOR has shown better performance when compared with non-linear Newton point SOR for a mildly non-uniform triangular mesh. This is also true for other types of regular meshes.

For mildly irregular rectangular meshes different two-step linearised iteration methods and one-step non-linear itesation methods have been compared. In most cases non-linear type iterations have been shown to be superior to two-step linearised iteration. Block (line) type iterations have always shown better performance than their point-type counterparts. Finally, a number of methods that are obvious modifications of some well-known methods have never been tested on systems arising from discretization.

We have therefore decided to compare some of these methods, namely Methods B3, B6, D3, C3, C6. Results of this comparison
will be given in Chapter 5. Before that, in the next section we consider some block-acceleration techniques that have been successfully used for various problems.

### 4.8 On Block-Methods for Acceleration of Convergence

Unlike the over-relaxation, in which the acceleration of convergence is achieved by extrapolating the new estimates of vector potential, but the basic iteration procedure is unchanged, block methods for acceleration of convergence consist of altering the basic iteration procedure by adding one or more equations which are iterated alongside with the basic system. This type of acceleration is suitable for indirect iterative methods and so far it has been used only for linearised two-step iteration techniques.

The basic idea in block acceleration techniques is due to Southwell (see for example Ref. 4.71) and in principle it consists of altering the values of vector potential at more than one mesh node by some simple procedure (e.g. increasing the values of a group of unknowns by the same amount, a method which Southwell called block relaxation, but this term is used nowadays in another sense). If an area on our mesh, containing more than one mesh node is encircled by a closed contour C (Fig. 4.11) and if all equations (4.4) corresponding to the enclosed nodes are added, the coefficients of $A_{i}$ in Eqn. (4.4) will partially cancel out. The resulting equation will be satisfied if all Eqns. (4.4) are satisfied, i.e. when the solution has converged. The converse is not valid. However, if this resulting equation is not satisfied, the values of all vector-potentials inside the contour can be changed in such a way that this equation becomes satisfied. The_idea can


Fig. 4.11 Several Mesh Nodes Enclosed by a Closed Contour C
also be given a physical meaning, namely the equation obtained by adding several basic equations (4.4) is equivalent to the equation which would be obtained by integrating $\oint \mathrm{H} d \ell$ along the contour $C$. On these grounds the method is sometimes called 'acceleration based on physical grounds'. Another common name for this process is 'non-stationary methods'.

The first modern use of this method as far as we are aware of is due to Ahamed (Ref. 4.72). The method consists of choosing the contour $C$ so that it encircles the entire mesh except the nodes on the boundaries. The ratio:

$$
\begin{equation*}
K=\frac{\iint_{S} \underline{J} \cdot d \underline{S}}{\oint_{C} H \cdot d \ell} \tag{4.35}
\end{equation*}
$$

is found using appropriate difference approximation after every iteration and the entire array of vector potentials is multiplied. by this ratio. The performance of this method is remarkable for both linear and non-linear problems and concentrated excitation. Results published by Ahamed show that the total computation time can be reduced ten times. However, the method does not behave so well with distributed excitation (indeed in cases where $\iint \underline{J} \cdot d \underline{S}=0$ it cannot be used) and modifications of the method were necessary, which unfortunately complicate the basic method a good deal (Ref. 4.49). The method can be used only when at least some of the boundary potentials are zero, and although modifications can be made to include other types of boundaries, or cases when the integration contour encloses only part of the entire area, (Ref. 4.53), another method, due to de La Vallée Poussin and Lion is then more suitable.

De La Vallée Poussin and Lion have found (Ref. 4.73) that one of the causes for poor convergence of linear magnetic field problems is the existence of 'windows' or 'quasiwindows' of low permeability in highly permeable area (slots of the machine). The rate of convergence can be improved if such windows are enclosed by an integration contour and the value:

$$
\begin{equation*}
\Delta A=\frac{\iint_{S} \underline{J} \cdot d \underline{S}+\oint_{C} \underline{H} \cdot d \underline{Z}}{\oint_{C} \frac{\partial \underline{H}}{\partial(\Delta \mathrm{~A})} \cdot d \underline{\ell}} \tag{4.36}
\end{equation*}
$$

is added to the vector potentials internal to the contour after every iteration. Similarly to the method by Ahamed the finite difference approximation to Eqn. (4.36) is used. This method corrects the value of $\oint \underline{H} \cdot d \underline{l}$ by adding $\Delta A$ to the nodes interior to the integration contour, while Ahamed's method does the same by multiplication. These methods are sometimes also called 'additive' or'multiplicative' acceleration, respectively.

Although these methods have been used only with rectangular meshes, corresponding algorithms for triangular meshes can readily be developed. Both these methods can be said to be linear. For non-linear problems, if these are solved by some non-linear indirect technique, modifications which convert these two methods into non-linear ones suggest themselves. However, practical application of these non-linear versions might become lengthy and increase the total computation time.

These block-acceleration methods are also related to variational methods of Wachspress, (Ref. 4.74), which were developed for the neutron diffusion equation.

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| CHAPTER 5 |
| :---: |
| COMPARISON OF DIFFERENT ITERATION METHODS |

In Chapter 4 we have defined several iteration schemes for the solution of our non-linear system of equations. In this chapter we shall describe the computer program in more detail and also present some results from the use of different iteration schemes in one example.

In section 5.1 we give the basic flow charts for different iteration methods. Section 5.2 describes the overall computer program. The program consists of three main parts corresponding to mesh generation, solution of equations and output.

In section 5.3 different iteration methods are compared. The section consists of four sub-sections. In sub-section 5.3.1 we describe the model problem for which comparison is made and also made some general remarks on the usefulness of such comparison. In sub-section 5.3 .2 criteria for convergence are defined, while in 5.3 .3 we discuss the problem of the estimation of iteration parameters. A trial and error method was adopted. In the final sub-section the number of iterations for different iteration schemes as well as the computation times are compared for different schemes.

Section 5.4 is concerned with convergence as a function of iteration parameters.
5.1 Algorithms for Different Iteration Schemes

In Chapter 4 we have defined several iteration schemes. In this section we shall describe the algorithms and flow charts for
several different methods in more detail. On the basis of these algorithms the actual computer program was written.

Method B3 (see section 4.6) . The basic flow chart for this method is represented on Fig. 5.1. The computation is started with prescribed constant values of reluctivities $v$ for iron parts. With these values the coupling coefficients of Eqn. 4.5 are computed using corresponding expressions of Eqn. 4.4 for all nodes $n$ (box 2). Since the coupling coefficients are symmetrical, it is necessary to compute only four coefficients for every node: coefficient $C_{i, i}$, which corresponds to the node itself and also three coefficients that correspond to three linking lines with nodes $A, B$ and $i+1$ (Fig. 5.2), $C_{i, A}, C_{i, B}$ and $C_{i, i+1}$. Some of these coefficients are not defined for boundary nodes and the computation of these is bypassed.

Now the system of equations is completely defined and box 3 of the program is entered. This box solves approximately the system of equations by a standard point $S O R$ technique. The prom cedure can be easily described as follows: Assuming all values of vector potential constant, except for node ' $i$ ', the linear equation 4.5 is solved for $A_{i}$; the value obtained we call $A^{*}$. The new value of $A_{i, k}$ is then extrapolated (overrelaxed) to:

$$
\begin{equation*}
A_{i, k}=A_{i, k}+\left(A^{*}-A_{i, k}\right) \cdot \omega \tag{5,1}
\end{equation*}
$$

The sub-script $k$ denotes the iteration number, and $\omega$ is the acceleration factor. This procedure is repeated systematically for all nodes $K$ times. A periodicity boundary condition is maintained by setting $A_{\ell, k}=-A_{f, k}$ where $A_{\ell}$ is the vector potential


Fig. 5.1 Basic Flow Chart for Linearised Iteration


Fig. 5.2 Coupling Coefficients for Node 'i'
of the last node on a certain mesh line and $A_{f}$ is the vector potential of the first node on the same line. This procedure is carried out in every iteration for every mesh line as soon as all vector potentials on that line are estimated.

Every cycle of inner iterations needs an estimate for initial values of $A$. The first cycle is usually started with all $A=0$, although in the program provision is made to start computation from any other starting vector $\{A\}$, generated, for example, by previous iteration with a different current distribution. The consequent iteration cycles are started with $\{A\}$ as at the end of the previous cycle.

After $K$ iteration steps box 4 of the program is entered in which new values of reluctivities for iron parts are computed on the basis of new values of vector potential. In terms of vector potential the flux density is:

$$
\begin{equation*}
|B|=\sqrt{\left(\frac{\partial A}{\partial x}\right)^{2}+\left(\frac{\partial A}{\partial y}\right)^{2}} \tag{5.2}
\end{equation*}
$$

and for a triangle ' $e$ ' with nodes $i, j, k$ (Fig. 5.3) the value $10\left(B^{(e)}\right)^{2}$ is readily evaluated:

$$
\begin{align*}
10\left(B^{(e)}\right)^{2}= & \frac{10}{4\left(S^{(e)}\right)^{2}}\left[\left(b_{i}^{(e)} A_{i}^{(e)}+b_{j}^{(e)} A_{j}^{(e)}+b_{k}^{(e)} A_{k}^{(e)}\right)^{2}+\right. \\
& \left.\left(c_{i}^{(e)} A_{i}^{(e)}+c_{j}^{(e)} A_{j}^{(e)}+c_{k}^{(e)} A_{k}^{(e)}\right)^{2}\right] \tag{5,3}
\end{align*}
$$

where $S^{(e)}$ is the triangle area and $b^{(e)}$ and $c^{(e)}$ are constants defined in Eqn. (4.1). This value of $10\left(B^{(e)}\right)^{2}$ is used to evaluate the new value of reluctivity $v^{(e)}$ from $v=f\left(10 B^{2}\right)$ as


Fig. 5.3 A Mesh Triangle
explained in section 4.2. This value we call $\nu^{*}$ and the reluctivity $v^{(e)}$ is then set to

$$
\begin{equation*}
v^{(e)}=v_{p}^{(e)}+\left(v^{*}-v_{p}^{(e)}\right) \cdot B \tag{5.4}
\end{equation*}
$$

where $v_{p}^{(e)}$ is the previous estimate for $v$ for that triangle. $\beta$ is an acceleration factor. This computation is repeated for all triangles in iron. Computation in the air gap and slots is bypassed.

After that convergence of the process is checked (box 5). If the convergence criteria are not satisfied the control is transferred again to box 2. Otherwise the iteration is stopped and output follows.

Method B6 The flow chart for this method is the same as for method B3, but block iteration is used in box 3. The block of nodes corresponds to one mesh line. In this technique all vector potentials are assumed constant except those on a particular mesh line. The system of equations is reduced to a form


This system is symmetric, but not strictly tri-diagonal, due to periodicity boundary conditions. The usual Thomas algorithm cannot therefore be used for the solution of Eqn. (5.5). Solution
can be achieved in two ways: either the system (5.5) is reduced to strictly tri-diagonal form by transferring the products $C_{1, n}, A_{1}$ and $C_{1, n} \cdot A_{n}$ on the right-hand side by the use of previous estimates of $\{A\}$ or, alternatively, the algorithm is adapted to suit Eqn. (5.5). The first approach is described in detail in Ref. 5.1. We have chosen the second approach, in which Eqn. (5.5) is solved exactly for each mesh line. We do not give the precise details of the algorithm, but it consists of simple recursion formulae, and was derived from Jenning's compact data storage solution (Ref. 5.2). Full advantage was taken of a special form of the matrix [C] in Eqn. (5.5).

Method C3 (see section 4.6) The flow chart for this method is represented in Fig. 5.4. Similarly as in B3 and B6 the computation is started with prescribed reluctivities $v$ for iron parts and these values are used in the computation of the coupling coefficients (box 2). In addition to the coupling coefficients, values of $\partial f\left(A_{i}\right) / \partial A_{i}$ are also computed for all nodes. The expression for $\partial f\left(A_{i}\right) / \partial A_{i}$ is easily evaluated from Eqn. (4.4):

$$
\begin{aligned}
& \frac{\partial f\left(A_{i}\right)}{\partial A_{i}}=\frac{b_{i}^{(1)} b_{i}^{(1)}+c_{i}^{(1)} c_{i}^{(1)}}{4 S_{i}^{(1)}} \cdot v_{i}^{(1)}+\frac{b_{i}^{(2)} b_{i}^{(2)}+c_{i}^{(2)} c_{i}^{(2)}}{4 S_{i}^{(2)}} \cdot v_{i}^{(2)}+\ldots+ \\
& +\frac{b_{i}^{(m)} b_{i}^{(m)}+c_{i}^{(m)} c_{i}^{(m)}}{4 S_{i}^{(m)}} \cdot v_{i}^{(m)}+\ldots+\frac{b_{i}^{(p)} b_{i}^{(p)}+c_{i}^{(p)} c_{i}^{(p)}}{4 S_{i}^{(n)}} \cdot v_{i}^{(p)}+ \\
& +A_{i, i}\left(\frac{b_{i}^{(1)} b_{i}^{(1)}+c_{i}^{(1)} c_{i}^{(1)}}{4 S_{i}^{(1)}} \cdot \frac{\partial v_{i}^{(1)}}{\partial A_{i, i}}+\frac{b_{i}^{(2)} b_{i}^{(2)}+c_{i}^{(2)} c_{i}^{(2)}}{4 S_{i}^{(2)}} \cdot \frac{\partial v_{i}^{(2)}}{\partial A_{i, i}}+\ldots+\right. \\
& \left.+\frac{b_{i}^{(m)} b_{i}^{(m)}+c_{i}^{(m)} c_{i}^{(m)}}{4 S_{i}^{(m)}} \cdot \frac{\partial v_{i}^{(m)}}{\partial A_{i, i}}+\ldots+\frac{b_{i}^{(p)} b_{i}^{(p)}+c_{i}^{(p)} c_{i}^{(p)}}{4 S_{i}^{(n)}} \cdot \frac{\partial v_{i}^{(p)}}{\partial A_{i, i}}\right)+
\end{aligned}
$$



Fig. 5.4 Plow Chart for Two Step Nonlinear Point SOR

$$
\begin{align*}
& \left.+A_{i, 1} \frac{b_{i}^{(1)_{b}^{(1)}+c_{i}^{(1)} c_{j}^{(1)}}}{4 S_{i}^{(1)}} \cdot \frac{\partial \nu_{i}^{(1)}}{\partial A_{i, i}}+\frac{b_{i}^{(p)} b_{k}^{(p)}+c_{i}^{(p)} c_{k}^{(p)}}{4 S_{i}^{(p)}} \cdot \frac{\partial v_{i}^{(p)^{-}}}{\partial A_{i, i}}\right)+ \\
& +A_{i, m}\left(\frac{b_{i}^{(m)} b_{j}^{(m)}+c_{i}^{(m)} c_{j}^{(m)}}{4 S_{i}^{(m)}} \frac{\partial v_{i}^{(m)}}{\partial A_{i, i}}+\frac{b_{i}^{(m-1)} b_{k}^{(m-1)}+c_{i}^{(m-1)} c_{k}^{(m-1)}}{4 S_{i}^{(m-1)}} \frac{\partial v_{i}^{(m-1)}}{\partial A} A_{i, i}^{(m)}\right)+ \\
& +A_{i, p}\left(\frac{b_{i}^{(p)} b_{j}^{(p)}+c_{i}^{(p)} c_{j}^{(p)}}{4 S_{i}^{(p)}} \frac{\partial v_{i}^{(p)}}{\partial A_{i, p}}+\frac{b_{i}^{(p-1)} \cdot b_{k}^{(p-1)}+c_{i}^{(p-1)} \cdot c_{k}^{(p-1)}}{4 S_{i}^{(p-1)}} \frac{\partial v_{i}^{(p-1)}}{\partial A_{i, i}}\right) \tag{5.6}
\end{align*}
$$

In this expression values of $\partial \nu^{(e)} / \partial A_{i}$ are necessary. These values are all set to zero before the first iteration cycle. For consecutive iterations values of $\partial \nu / \partial A$ are computed in box 4 .

After the computation of the coupling coefficients and values $\partial f\left(A_{i}\right) / \partial A_{i}$ for all nodes, control is transferred to box 3 . The process in box 3 is described by Eqn. (4.29), i.e. with the values of coupling coefficients and $\partial f\left(A_{i}\right) / \partial A_{i}$ fixed, new values of vector potential are estimated by the formula:

$$
\begin{equation*}
A_{i}=A_{i}^{*}-\frac{f\left(A_{i}\right)}{\partial f\left(A_{i}\right) / \partial A_{i}} \cdot \omega \tag{5.7}
\end{equation*}
$$

where $A_{i}^{*}$ is the previous estimate of the vector potential of node ' $i$ ' and $\omega$ is the acceleration factor. In the evaluation of $f\left(A_{i}\right)$ the latest known estimates of \{A\} are always used. Eqn. (5.7) is applied systematically for all nodes. The periodicity boundary condition is preserved in a similar manner as in $B 3$, i.e. the last node on a mesh line is assigned the negative value of the vector potential of the first node on that line after the computation of all vector potentials on the line.

After Eqn. (5.7) has been applied for all nodes $K$ times box 4 of the program is entered. Similarly to the two methods already described, new values of reluctivity $v$ are computed in this box. The procedure is exactly the same as described in B3. However, in addition to the reluctivity values for $\partial \nu / \partial A$ are also computed. For non-magnetic regions these values are zero. For a triangle ' $e^{\prime}$ in the iron region simple differentiation gives (Fig. 5.3):

$$
\begin{align*}
& \frac{\partial v^{(e)}}{\partial A_{i}^{(e)}}=\frac{\partial v^{(e)}}{\partial\left(10\left(B^{(e)}\right)^{2}\right)} \cdot \frac{\partial\left(10{\left.\left(B^{(e)}\right)^{2}\right)}_{\partial A_{i}^{(e)}}^{\partial(e)}=\beta .2 b\left(10\left(B^{(e)}\right)^{2}\right) \cdot \frac{10}{4\left(S^{(e)}\right)^{2}}\right.}{} \\
& \quad\left(b_{i}^{(e)}{ }_{B}^{(e)}+c_{i}^{(e)} \cdot B_{x}^{(e)}\right) \tag{5.8}
\end{align*}
$$

where $\beta$ is the acceleration factor used for reluctivities and $b\left(10 B^{2}\right)$ is a slope of a corresponding section of $V=f\left(10 B^{2}\right)$ approximation. As it can be easily seen three values of $\partial v / \partial A$ are necessary for every triangle.

After box 4 convergence is checked and if necessary control returned to box 2 .

Method C6 (see section 4.6) (version A) The flow chart for this method is given on Fig. 5.5. The basic structure is the same as for method C3 and consists of three main blocks.

In box 2 coupling coefficients as well as values $\partial f\left(A_{i}\right) / \partial A_{i, i}$ and $\partial f(A i) / \partial A_{i, i+1}$ for all nodes are computed. Formula (5.6) is used for $\partial f\left(A_{i}\right) / \partial A_{i, i}$. The values of $\partial f\left(A_{i}\right) / \partial A_{i, i+1}$ are computed from (we set $A_{i, i+1}=A_{i, 1}$ ):


Fig. 5.5 Flow Chart for Two-Step Nonlinear Line SOR

$$
\begin{align*}
& \frac{\partial f\left(A_{i}\right)}{\partial A_{i, 1}}=A_{i, i} \frac{b_{i}^{(1)} b_{i}^{(1)}+c_{i}^{(1)} c_{i}^{(1)}}{4 S_{i}^{(1)}} \cdot \frac{\partial v_{i}^{(1)}}{\partial A_{i, 1}}+\frac{b_{i}^{(p)} b_{i}^{(p)}+c_{i}^{(p)} c_{i}^{(p)}}{4 S_{i}^{(n)}} \cdot \frac{\partial v_{i}^{(p)}}{\partial A_{i, 1}} \\
& + \\
& +\frac{b_{i}^{(1)} b_{j}^{(1)}+c_{i}^{(1)} c_{j}^{(1)}}{4 S_{i}^{(1)}}+v_{i}^{(1)}+\frac{b_{i}^{(p)} b_{k}^{(p)}+c_{i}^{(p)} c_{k}^{(p)}}{4 S_{i}^{(p)}} \cdot v_{i}^{(p)}+  \tag{5.9}\\
& +A_{i, 2} \cdot \frac{b_{i}^{(1)} b_{k}^{(1)}+c_{i}^{(1)} c_{k}^{(1)}}{4 S_{i}^{(1)}} \cdot \frac{\partial v_{i}^{(1)}}{\partial A_{i, 1}}+A_{i, p} \frac{b_{i}^{(p) b_{j}^{(p)}+c_{i}^{(p)} c_{j}^{(p)}}}{4 S_{i}^{(p)}} \cdot \frac{\partial v_{i}^{(p)}}{\partial A_{i, 1}}
\end{align*}
$$

This expression is obtained from Eqn. (4.4) by direct computation. It is easily shown by direct computation that these coefficients are symmetrical, i.e. $\partial f\left(A_{i}\right) / \partial A_{i+1}=\partial f\left(A_{i+1}\right) / \partial A_{i}$.

After the computation of the coupling coefficients and the quantities given by Eqns. (5.6) and (5.9) for all nodes box 3 of the program is entered. In this box new estimates of \{A\} are made. By the definition in section 4.6 , method C 6 is obtained from Eqn. (4.23) by approximating the Jacobian maxtrix by its tridiagonal strip. If periodicity conditions are to be maintained, it is more convenient to include into the reduced Jacobian matrix also the off-diagonal coefficients that are due to periodicity condition, similarly as in method B6. Thus, the diagonal submatrices of the Jacobian matrix have the same form as matrix [c] in Eqn. (5.5), and the reduced Jacobian matrix is also symmetrical.

By the definition one inner iteration step can be described by

$$
\begin{equation*}
\{A\}_{k+1}=\{A\}_{k}-[J]^{-1} \cdot\left\{F\left(A^{*}\right)\right\} \cdot \omega \tag{5.10}
\end{equation*}
$$

where $A^{*}$ is a latest known estimate of $\{A\}$. Due to the block structure of $[J],[J]^{-1} \cdot\left\{F\left(A^{*}\right)\right\}$ can be evaluated block by block
(blocks correspond to mesh lines). The elements of the vector $\left\{F\left(A^{*}\right)\right\}$ are evaluated as they are required, i.e. line by line. The latest known estimate of the vector potential is always used, and vector $\left\{A^{*}\right\}$ consists partially of values of $A$ from ( $k-1$ ) th and $k$-th iteration. The procedure is repeated $K$ times for all lines.

In box 4 the newly estimated values of vector potential are used to recalculate reluctivities and values $\partial \nu / \partial A$ for all triangles in iron. Formulae 5.4 and 5.8 are used. After that convergence is checked and control transferred again to box 2 if necessary.

Method D3 (see section 4.6 ) The flow chart for this method is shown on Fig. 5.6. The computation is started from some approximate solution of $A$ and distribution of $v$. These approximate values are used to evaluate new values of reluctivities $v$ of the surrounding triangles of node ' $i$ ' by the Eqn. (5.4) and $\partial v / \partial A$ by Eqn. (5.8)., These values are then used to compute $f\left(A_{i}\right)$ and $\partial f\left(A_{i}\right) / \partial A_{i}$ and the new approximation of $A_{i}$ is obtained from:

$$
\begin{equation*}
A_{i, k+1}=A_{i, k}-\frac{f\left(A_{i}\right)}{\frac{\partial f\left(A_{i}\right)}{\partial A_{i}}} \cdot \omega \tag{5.11}
\end{equation*}
$$

$\omega$ is the acceleration factor. The new values of $A$ are always used in computing $f\left(A_{i}\right)$ and $\partial f\left(A_{i}\right) / \partial A_{i}$. The procedure is applied systematically to all nodes, which is followed by a convergence check. This process differs from the one described by Eqn. (4.32) in the use of the acceleration factor $\beta$ for reluctivities. Introduction of this factor was necessary in order to achieve convergence.


Fig. 5.6 Flow Chart for One-Step Nonlinear SOR

Method C6 (version B) The similarity between two-step and one-step non-linear methods suggests that it might be possible to define convergent methods which do not belong. to either of these groups, but lie between them. In these methods elements of the reduced Jacobian matrix are recalculated in every step on the basis of new estimates of $A$, but reluctivities are not computed in every step. Acceleration factors for both vector potentials and reluctivities can be used.

A program for line iteration of this type was written. It is basically a two-step procedure and the flow chart on Fig. 5.5 was followed, except that the values of $\partial f\left(A_{i}\right) / \partial A_{i}$ and $\partial f\left(A_{i}\right) / \partial A_{i+1}$ were computed in box 3, alongside with the values of $f\left(A_{i}\right)$. The latest known estimates of $A$ were always used.

In the next section we shall briefly describe the computer program which was written on the basis of the different iteration methods described here.

### 5.2 The Computer Program

The block diagram of the computer program is shown on Fig. 5.7. The program consists of three main blocks. The first block contains subroutines for data preparation, i.e. subroutines for reading in and checking data, mesh generation and generation of initial values of vector potential.

The second block of the program contains six subroutines for computation of vector potentials. These six subroutines correspond to the six different methods described in the previous section.

The last block contains output subroutines.


Fig. 5.7 Block Diagram of the Computer Program

The entire program has been written in FORTRAN (extended version for ICL 1900 series computers). The source program contains about 7,000 statements. While the program was being written, emphasis was put on the efficiency of the code, particularly of the iteration subroutines. Preliminary tests showed that both available compilers (XFAT and XFEW) produce object codes which are rather slow in dealing with integers. This imposed severe limitations on the use of multi-dimensional arrays, and these were used only rarely. The result was a program much more complex than originally expected which was, however, reasonably efficient. We wish to emphasize, however, that really fast computation can be achieved only by writing a program in machine code.

In the next section we shall give some results on the use of our program. Although the total computation time can be further reduced by writing the program in machine language, we believe that the comparison of the efficiency of different methods has general relevance.

### 5.3 Performance of Different Iteration Methods

5.3.1 The model problem

In Chapter 4 we have pointed out that the convergence of a particular numerical method for non-linear problems depends on the problem itself. Practical investigations into the convergence of numerical methods are usually carried out on so called 'model problems' which often have very simple geometry (e.g. a unit square with a hole in the middle). Unfortunately, results obtained from such simple problems cannot always be easily extrapolated to more complex problems. An illustration of this
difficulty could be the multiplicative acceleration introduced by Ahamed (see section 4.8). This method shows excellent performance for a simple geometry and concentrated excitation, but results for distributed excitation are rather poor. It is therefore necessary for a model problem to be representative of a class of problems for which a particular computer program is intended. This cannot be achieved easily with programs intended for general use, where a class of problems may not even be known.

We are in a somewhat better situation as our program is intended for computation of magnetic field distribution of doubly slotted electrical machines. Thus; we can choose our model problem to represent a pole pitch of a doubly slotted machine, with combined periodicity and Dirichlet boundary conditions and with a current distribution corresponding to real problems. The magnetization curve for the model problem can be the same as for the actual problem. The size of problem regarding both number of slots (i.e. geometry) and number of nodes (i.e. mesh density) can also be chosen, in principle, to match real problems. Hence good correlation can be expected between the performance of a particular method for a model problem and for real problems. Unfortunately, the size of such a model problem would impose severe limitations on numerical experimentation with different iteration schemes, different parameters, etc., because computation would tend to be lengthy and costly. Thus it becomes necessary to restrict the size of the model problem, in order to carry out numerical experiments, with the danger that the results of such experiments may not always indicate the best method for large, real problems.

The model problem we have chosen for our experiments is a doubly slotted structure with two slots on the rotor and three slots on the stator. The boundary conditions are periodicity plus Dirichlet boundary conditions, which correspond to the natural boundary conditions in electrical machines. The current distribution in slots has been chosen to correspond roughly to short circuit conditions. The mesh for this problem is represented on Fig. 4.3. The mesh has 87 nodes and 145 triangles of which 84 cover iron parts and the rest the slots and the air gap. As some of the nodes lie on the boundaries, the number of equations generated from this mesh is 68. This is a rather small system and the number of equations for practical problems can be expected to be about ten times higher. It was believed, however, that some basic features of several numerical methods that were to be tested could be established even on this small model. Such features include the question whether the method is convergent or not, the dependence of the convergence on the choice of iteration parameters, and to a certain extent also the relative efficiency of different. iteration methods.

### 5.3.2 Initial conditions and convergence criteria

In order to compare the convergence of different iteration schemes, as well as the influence of different parameters, it is necessary to determine initial values of vector potential and reluctivity of the iron parts. In all cases the starting values of reluctivity were chosen to be $v_{\text {iron }}=1000$ ( $\mu_{\text {rel }} \simeq 795$ ) over the entire iron region. The initial values of vector potential were obtained by approximate solution of this linear problem. Point SOR was used in order to obtain this approximation.

50 iterations were performed and these values of vector potential were used as the initial vector $\{\mathrm{A}\}$ 。"

One important problem in iterative solutions is when to stop the iteration process. Ideally for this purpose we should examine the error vector

$$
\begin{equation*}
\{\varepsilon\}_{k}=\{A\}_{k}-\{A\} \tag{5.12}
\end{equation*}
$$

where $\{A\}_{k}$ is the approximate solution after $k$ iterations and $\{A\}$ is the exact solution. Unfortunately, vector $\{A\}$ is generally not known, and theoretical investigations of $\{\varepsilon\}$ are possible only in special cases. Thus, some other quantities have to be used. Two sets of quantities are easily monitored during computation: the displacement vector $\{d\}$ :

$$
\begin{equation*}
\{d\}_{k}=\{A\}_{k}-\{A\}_{k-1} \tag{5.13}
\end{equation*}
$$

and the change in reluctivity

$$
\begin{equation*}
\mu=\nu_{m}-\nu_{m-1} \tag{5.14}
\end{equation*}
$$

We decided to monitor the displacement vector's first power norm $\|\mathrm{d}\|$, maximum norm $\|\mathrm{d}\|^{\infty}$ and also the maximum absolute ( $\mu_{\max }$ ) and relative ( $\mu_{r e l}$ ) changes in reluctivity at any one place in the iron region. For our tests the prescribed values were

$$
\begin{aligned}
& \|\mathrm{d}\|=1.0 \times 10^{-6}|\mathrm{~Wb}| \\
& \|\mathrm{d}\|^{\infty}=1.0 \times 10^{-7}|\mathrm{~Wb}| \\
& \mu_{\max }=10.0 \quad|\mathrm{~m} / \mathrm{H}| \\
& \mu_{\mathrm{re} ~}=1.0 \times 10^{-2}
\end{aligned}
$$

The iteration was carried on until all the corresponding values in our iteration process had fallen below these prescribed values. The iteration was then stopped. We shall return to the question of convergence criteria in section 6.3 when we discuss errors associated with our iteration process. In the next sub-section we shall turn our attention to the problem of the estimation of iteration parameters.

### 5.3.3 Estimation of iteration parameters

In all the iteration schemes described in section 5.2 it is necessary to determine the acceleration factors $\omega$ for inner iteration and $\beta$ for outer iteration, and for two-step methods also $K$, the number of inner iterations per outer iteration. This is not a simple problem if optimum parameters are sought. An adequate choice of iteration parameters is necessary even if iteration time is not critical, because the wrong choice may cause prolongation of iteration time by more than one order of magnitude, or even divergence.

This problem has not been completely solved even for much simpler linear cases where only one iteration parameter, $\omega$, is necessary, and only in the simplest cases is it possible to determine an optimum $\omega$ in advance. For the majority of practical problems $\omega$ cannot easily be determined in advance. However, the underlying theory is well developed and several techniques have been developed in which iteration is started with a value of $\omega$ expected to be in the neighbourhood of the optimum value and this value is then adjusted during iteration.

Estimation of iteration parameters for non-linear problems is more complicated, because not only are there more than one
parameter to be determined, but there is also hardly any theoretical background. In practical applications a'trial and error' method is often adopted, i.e. several short runs are performed with different iteration parameters, and then parameters which appear to give the fastest convergence are used for actual computation. Another possibility that has also been exploited (Ref. 5.3-5.5) is to keep some of the parameters constant during iteration, and change others. Usually $\beta$ and $K$ are fixed, and $\omega$ is changed on the basis of linear theory. Although linear theory is not strictly valid, both Anderson (Ref. 5.3) and Winslow (Ref. 5.4) have reported good performance by means of this method.

Such methods are very valuable if numerous different problems have to be solved. It was felt, however, that for our problem a 'trial and error' method would be more suitable. The reason for this is that it makes possible the choice of optimum values of all necessary parameters, unlike the methods described by Anderson or Winslow. Also, solution of our problem requires several magnetic field solutions with slightly different current distributions, other physical parameters being unchanged. Thus, the optimum iteration parameters for one field distribution could be expected to give also good convergence for the others. In section 5.4 we shall give some results which show that the choice of iteration parameters is not very critical, which supports this assumption. Before giving these results we shall, in the next sub-section, compare the performance of different iteration schemes for our model problem.

### 5.3.4 Comparison of computation times for different methods

In sub-section 5.3 .1 we have described a model problem for which different iteration schemes have been tested, and in 5.3 .2 we have given the convergence criteria. As explained in 5.3.3, a 'trial and error' method was adopted for the estimation of iteration parameters. Such a method can never give the optimum values with $100 \%$ security. However, the number of trials was large, and it is believed that these parameters are fairly near to the optimum values.

Optimum iteration parameters as well as total number of iterations and computation times for different iteration methods are given in Table 5.1. Methods correspond to those described in section 5.1 .

TABLE 5.1 Comparison of Different Iteration Methods

| Column 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Method | $\omega$ | $\beta$ | R | iterations to converge | time (sec) | Convergence to three significant digits $\qquad$ |  |
|  |  |  |  |  |  | iterations | sec |
| B3 | $1 \cdot 63$ | $0 \cdot 18$ | 4 | 267 | $48 \cdot 0$ | 150 | 27.0 |
| B6 | 1.76 | $0 \cdot 29$ | 4 | 113 | 24.4 | 52 | 11.0 |
| C3 | 1.92 | 0.205 | 4 | 157 | 34.0 | 76 | 16.0 |
| C6 (A) | 1.83 | 0.25 | 1 | 61 | $30 \cdot 0$ | 39 | 19.0 |
| C6 (B) | not convergent |  |  |  |  |  |  |
| D3 | 1.75 | 0.04 | - | 171 | $105 \cdot 6$ | 91 | 56.0 |
| $\begin{aligned} & \text { B3 with } \\ & \text { block } \quad \text { * } \\ & \text { acceleration } \end{aligned}$ | 1-70 | $0 \cdot 15$ | 4 | 280 | $69 \cdot 6$ | 117 | 29.0 |
| Newton <br> Raphson iteration | - | - | - | 8 | $20 \cdot 0$ | 5 | $12 \cdot 5$ |

[^0]optimum values, but are fairly near to the optimum. The number of iterations without block acceleration with these parameters was 282 with running time of 50.4 sec .
**
The values given are estimated, not measured. The computation time was estimated on the assumption of use of block elimination scheme (see section 4.4).
$\dagger$ The values represent the number of iterations and the computation time if other convergence criteria are applied. These values are more realistic than the values in columns 5 and 6 if compatation is carried out with the purpose of short circuit current estimation (see section 6.3 ).

Column 6 in Table 5.1 represents time measured by the internal computer clock, and it includes also some printing time. These times were obtained by the use of XFAT compiler on ICL 1907 computer. Use of optimising compiler XFEW reduces the computation time by about 10\%. Thus, the indicated times must not be considered as absolute values, but should be used only for comparison with each other.

The times given in Table 5.1 indicate the better efficiency of line iterative methods compared with point interation methods (methods $B 6$ versus $B 3$ and $C 6(A)$ versus $C 3$ ). This follows a pattern recognised in the literature for other types of meshes (see section 4.7). Two-step non-linear point iteration (method C3) was significantly.faster than two-step linearised point iteration (method B3). However, two-step non-linear line iteration (method C6(A)) was slower than two-step linearised line iteration (method B6) which showed the best performance of all tested methods.

Method D3 (one-step non-linear point iteration) showed very inferior performance and so did method C6(B) (a version of twostep non-linear line iteration, see section 5.1), which did not converge at all for any combination of iteration parameters tried. We do not know the reason for this, and among other things a possible error in the program cannot be positively excluded, although the program was tested most carefully. It was originally planned to write two more subroutines, one for method D6 (one-step nonlinear line iteration) and also $B$ version of method C3, which would correspond to a point version of method $C 6(B)$. However, after the very discouraging results shown by methods C6(B) and D3, it was decided not to do so, as it is believed that the computation speed of these methods would be inferior to that of method B6.

The block acceleration method was tested only with two-step linearised point iteration (method B3). The result shown in Table 5.1 was typical for this method. In some cases (i.e. for some combinations of iteration parameters) the total number of iterations was slightly increased, in other cases it decreased by several iterations. In all cases the total computation time was longer than computation with the same iteration parameters but without acceleration. Use of Ahamed's type (multiplicative) acceleration showed similar results. The increase in computation time is due to the considerable amount of computation which is necessary in order to carry out the procedure. The amount of computation for triangular meshes is much higher than for rectangular meshes to which this type of acceleration has been applied successfully by others.

The last row in table 5.1 represents an estimation. The number of the iterations was based on results published by Silvester and Chari (Ref. 5.6), and computation time was obtained by assuming that a block elimination scheme was used for the solution of a linear system. The time necessary for one arithmetic operation was assumed to be about $50 \mu \mathrm{sec}$, which corresponds to measured values if all variables are stored in one-dimensional arrays.

The conclusions that can be drawn from these results are as follows:

A11 tested methods (except method $C 6(B)$ ) have converged for our model problem and have reached the solution to the required accuracy in a reasonable computation time. We believe that all these methods can be used with some confidence for the computation of other magnetostatic problems in connection with the use of irregular triangular meshes. Two-step linearised line iteration has shown the best performance of all tested methods. Although it cannot be said with complete confidence without any further tests, we believe that this method will be even more superior to point methods (B3, C3 and D3) for larger meshes as normally used in linear problems and regular meshes. As regarding the two-step nonlinear line iteration (method $C 6(A)$ ), we know of no reason why the relative efficiency of this method would increase over than one of method B6 for larger meshes. We can therefore expect method B6 to be superior to all other tested methods for larger meshes as well. Use of block acceleration techniques is not expected to improve the performance of any of the tested methods. Use of direct iterative methods, like Newton Raphson method will probably be inferior
to method $B 6$ for larger meshes because the necessary computation time is approximately proportional to $\mathrm{n}^{3}$ for direct iterative methods where n is the number of nodes, while for indirect iterative methods the computation time is increasing more slowly than $n^{2}$ (computation time for one iteration is proportional to $n$, while the necessary number of iteration increases more slowly than $n$. Even systems of several thousand unknowns require usually only several hundred iterations, see for example Ref. 5.3).

A further insight into the behaviour of different iteration schemes can be obtained by the analysis of the displacement vector as a function of the number of iterations, or for the purpose of comparison as a function of computation time. The curves in Fig. 5.8 represent the first power norm for different iteration methods plotted against computation time. (Since with some methods these curves show quite large irregular oscillations, the diagrams were obtained by computing the average norm for 4 consecutive iterations, and by plotting these values).

As it can be readily seen from the diagram, almost all tested methods show acceleration of convergence as iteration progesses. The degree of acceleration varies for different methods, but they all fall between virtually linearly convergent method B 6 and virtually quadratically convergent method C6(A). With this information we can say that method $B 6$ would be even faster compared with other methods if the accuracy required were lower. As we shall see in section 6.3, the accuracy for practical computation can indeed be much lower which puts method B6 even further ahead.

Before arriving at a final conclusion as to which method is to be used for the computation of our real problem, we shall


Fig. 5.8 First Power Norm of the Displacement Vector for Different Iteration Schemes

# $=$ <br> briefly examine in the next section the sensitivity of different iteration schemes to the choice of iteration parameters. 

### 5.4 Dependence of Convergence on the Choice of Iteration Parameters

As already mentioned in section 5.3.3, the wrong choice of iteration parameters can result in severe penalties in terms of computation time. In this section we shall examine the dependence of the convergence of different iteration schemes on the choice of iteration parameters for our model problem. This is a very important question if a 'trial and error' method is used for the estimation of the parameters. Namely, if a method converges fast with optimum parameters, but is very sensitive to the choice of iteration parameters, then the trial and error method will normally require many trials before a combination of parameters can be found which gives fast convergence.

It was therefore decided to carry out tests which will give some insight on the computation speed as a function of the iteration parameters. It was soon realised that the best choice for a number of inner iterations per outer iteration $K$ lies between 3 and 4 for methods B3, B6 and C3 for virtually any choice of $\omega$ and $B$, while the fastest convergence for method $C 6(A)$ was achieved with $K=1$. This fact simplified the tests, as for all two-step methods $K$ could be fixed and only $\beta$ and $\omega$ changed, in the same way as in tests for method D3. Figs. 5.9-5.13 represent the results of these tests in a form of curves in the $\omega-\beta$ co-ordinate system. The position of the best combination of these parameters is indicated on all figures. If the combination of parameters is chosen


Fig. 5.9 Dependence of Convergence on Iteration Parameters


$\omega$
$1.9+$


Two-step nonlinear point
iteration (Method C3)


Fig. 5.12 Dependence of Convergence on Iteration Parameters

to be on the inner curve, the computation time is increased by $25 \%$, while the outer curve gives an increase of $50 \%$.

All tests for diagrams on Figs: 5.9-5.13 were carried out for our model problem described in section 5.3.1. The diagrams are approximate, because, due to long computation time it was not possible to carry out extensive tests. However, the value of more accurate diagrams is doubtful, due to another phenomenon which we observed. Namely, the computation time as a function of the iteration parameters $\omega$ and $\beta$ is not a smooth function in the neighbourhood of the optimum values of $\omega$ and $\beta$. Several local minima can usually be found, and as an example we give the diagram on Fig. 5.14. This diagram represents a number of iterations as a function of $\omega$ for a two-step linearised line iteration. $\beta$ and $K$ were fixed $(\beta=0.24, K=4)$. The oscillations appear to be quite irregular, with an amplitude of as much as $25 \%$ of the total computation time. We observed these oscillations for all iteration methods we tested, except for method D3 (one-step non-linear point iteration).

These results indicate that it may be extremely difficult to find the overall optimum values of iteration parameters, either with an automatic routine, or with a trial and error method. On the other hand, it seems reasonably easy to find the combination of parameters which gives the computation time within about $125 \%$ of the optimum value as the diagrams 5:9-5.13 show. On these grounds there seems to be no reason why we should not chose twostep linearised line iteration (method B6) as a method for the computation of fields for our problem. This method is the fastest of all tested methods (see table 5.1), and the diagram on Fig. 5.10


Fig. 5.14 Number of Iterations for Two-Step Linearised Line Iteration Versus Overrelaxation Factor
does not indicate that there will be any great difficulties in finding the combination of iteration parameters which gives the computation time within the range of $125 \%$ of the optimum value, although, of course these parameters will generally be different from those obtained for our model problem.

With this choice we conclude this chapter. In the next chapter we shall turn our attention to different errors that are present in our analysis.

## References to Chapter 5

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In this chapter we shall examine the errors which are present in our analysis.

Our analysis of the machine can be split into two steps: 1. definition of a perfect machine (neglection of eddy currents, hysteresis, etc); and
2. definition and solution of a mathematical model. Both steps involve the introduction of several types of errors.

In section 6.1 different types of errors are listed according to their origin. In section 6.2 the errors of the first group are examined, while in section 6.3 the errors of the second group are analysed. The analysis of different types of errors is not detailed, but an attempt was made to determine the orders of magnitude of different types of errors, although in some cases it is merely a guess.

The conclusion of this chapter is that the overall accuracy in the computation of voltage with prescribed currents by our method lies in the range of $10 \%$ - $15 \%$.

### 6.1 Types of Errors

Our analysis of the machine can be considered to consist of several 'steps'. Every step introduces some uncertainties, which cause discrepancies between computed results and the actual physical phenomenon. These discrepancies we shall call the 'error'. Hence, the error will be defined as:

$$
\begin{equation*}
\varepsilon=A_{a}-A \tag{6.1}
\end{equation*}
$$

where $A_{a}$ is our approximate solution and $A$ represents the numerical value of the actual physical phenomenon. The magnitude of the error may be quite different for different quantities (e.g. local flux densities and induced voltage), and we shall always state in what terms the error is expressed. In most cases we shall examine the error in the effective value of total flux for prescribed values of currents.

We shall differentiate two basic steps in our analysis of the machine:

1. definition of the ideal machine; and
2. definition and solution of the mathematical model.

The major causes of error in the first step are:

1. Mechanical inaccuracy.
2. Inhomogeneous magnetic properties of steel laminations.
3. Approximation of the $B-H$ curve by single valued function.
4. Neglect of influences of non-active parts of the machine (casing, bearings, etc.).

By neglecting all these influences we have defined the 'perfect' machine. Our mathematical model is based on this perfect machine. However, it does not represent the perfect machine exactly, because several simplifications were introduced in order to carry out the computation effectively. These simplifications introduced further errors. Also, the numerical solution of our mathematical model causes errors. The major causes of errors of this step are:
6. Approximation of the $\mathrm{B}-\mathrm{H}$ curve by piecewise linear functions.
7. Neglect of end effects. .
8. Discretization.
9. Iterative solution of equations.
10. Truncation of numbers in the computer core.

The border line between these two groups of causes of errors is not sharp. The common feature of the first group is that the causes are not known exactly and some additional data or measurements would be necessary in order to determine the errors exactly, even in principle (e.g. in order to determine the effect of eddy currents the resistivity of the laminations must be known).

The causes of the second group are known exactly (e.g. we know that the numbers in the computer are represented only to a certain number of decimal places).

There is also a third group of errors caused by measurements. The computed results can only be compared with the results of measurements, not with the machine itself. Although these errors do not belong to either of the groups mentioned above, their magnitude must be considered if the validity of a certain method of computation is assessed by comparison of computed and measured results, hence we have:
11. Measurement errors.

An analysis of all these errors in detail would be very complicated. Fortunately, the computation of the short circuit current need not be particularly accurate (the accuracy of $10 \%$ can be considered as fairly good; for example, various national
standards allow tolerances in short circuit current of $15 \%-20 \%$ of their nominal value), so that the error analysis need not be very accurate either, and only major causes of errors must be considered in more detail. If the computation is used in order to determine for example the influence of different designs of tooth tips, etc., only errors $8-10$ have to be considered, and these are dealt with in more detail.

In the next section we shall first examine the errors of the first group.

### 6.2 Errors Due to Idealisation of the Machine

The exact estimate of errors of this gorup is not possible even in principle, because this would require some additional data that can be obtained only by measurements, which are themselves liable to errors. Due to the lack of such measurements our analysis of error will be only qualitative and can, at the very best, indicate only the orders of magnitude of the errors involved. We shall now examine briefly different influences listed in section 6.1.

Mechanical inaccuracy of the machine can cause serious discrepancies between computed results and measurements on the actual machine. The main cause of these discrepancies is a non-uniform air-gap. The largest portion of this non-uniformity is caused by the eccentric position of rotor in the stator bore. The air-gap in induction machines is short and tolerances in the positioning of bearings are often as high as $10 \%$ of the air-gap length. This eccentricity causes unbalanced magnetic pull. If the winding is connected in series, and the tolerances of $\pm 10 \%$ of the air-gap
length are allowed, then the $f l u x$ density may be expected to vary also $\pm 10 \%$ (assuming relative permeability of iron $\mu_{r e l}=\infty$ and the machine without slots). The forces on the magnetised iron parts are proportional to $B^{2}$ where $B$ is the flux density, hence the forces on the rotor may be expected to vary approximately $\pm$ $20 \%$ along the rotor periphery, and a large unbalanced radial force may result.

The influence on the total flux will probably be much smaller, due to the fact that the total flux will depend on the average airgap length. The average air-gap length is not influenced by the eccentric position of the rotor, but only on the actual dimensions of stator bore and rotor, which are manufactured with lower tolerances.

However, different levels of flux density will have some influence on total flux because of saturation, which will be uneven as the flux rotates in the machine. Simple analysis of the influence of non-uniform air-gap was carried out under the following assumptions:
a) Air-gap length has a tolerance of $\pm 10 \%$.
b) The stator and rotor core backs of the machine are infinitely permeable (this is allowed for short circuit computation as the main flux is low).
c) The flux density is constant along the pole pitch.
d) The ratio of flux path length through iron and air is $r=200$ (this is realistic since the paths through the rotor and stator cores have been neglected).
c) The leakage fluxes represent about $50 \%$ of the total flux, and they are not affected by the non-uniform air-gap.

The maximum error may be expected to occur with four-pole machines and with working point in the knee of the $B-H$ curve. The flux density was chosen to be 1.5 T , and even in this most unfavourable case our analysis showed that the relative error in computation of total flux lies in the range $0.01>\varepsilon>0$. The leakage flux is hardly affected at all by the non-uniform air-gap, so that the error in computation of total flux will probably lie well under $1 \%$, unless the eccentricity is higher than $\pm 10 \%$ of the airgap length. If the winding has parallel branches, these will probably further reduce the influence of the non-uniform air-gap.

Errors listed in 2,3 and 4 in section 6.1 can together be called errors due to imperfect steel laminations. Of these, errors due to non-homogeneous material will probably be quite small because the steel sheets with different magnetic properties will be randomly distributed in the machine and the influence of different permeability of different sheets on the machine characteristics will be small. We expect these errors to be about $\pm 1 \%$ in terms of the total flux, although they may cause larger variations in local flux densities. It must also be emphasized that different batches of magnetic steel with the same nominal $\mathrm{B}-\mathrm{H}$ curve may differ from each other to some extent, and that it is important that the analysis is carried out with the actual $B-H$ curve for the material used in the machine.

The problem of hysteresis is a complicated one and to account for it fully it would be necessary to know the magnetic history of the material. Modern magnetic materials intended for use in electrical machines have relatively narrow hysteresis loops. The hysteresis data are often not given by the manufacturers, but the
order of magnitude of its influence can be established by simple analysis. The losses in steel laminations are always given and for usual laminations of 0.5 mm they vary between $3.0-8.0 \mathrm{~W} / \mathrm{kg}$ at 50 Hz and $f 1 u x$ density $1 \cdot 5 \mathrm{~T}$, depending on the type of material. Approximately one half of these losses is due to hysteresis, the other half to eddy currents (Ref. 6.1). This corresponds to the area of hysteresis loop of about $0.5 \times 10^{3} \mathrm{Ws} / \mathrm{m}^{3}$. Assuming furthermore a rectangular hysteresis loop (Fig. 6.1) and flux density $\pm 1.5 T$, it gives the width of hysteresis loop of $166 \mathrm{~A} / \mathrm{m}$ $(\simeq \pm 80 \mathrm{~A} / \mathrm{m})$. This means that the necessary field strength for given flux density lies in the range $\pm 80 \mathrm{~A} / \mathrm{m}$, as computed from B-H curve as single valued function, depending on the branch of the loop. The maximum values of flux densities will not greatly be affected by the hysteresis, we can therefore say that in the first approximation the effect of hysteresis will be a time lag between excitation current and flux. This time lag can easily be determined, and from our data its value is about $2.5^{\circ}$ (el). The induced voltage will show similar time lag. If the errors of the instantaneous values of total flux are considered, then these may be presented as the percentage values of the maximum value of flux. In this case the maximum error is about $\pm 4 \%$. The influence of hysteresis on the effective value of total flux will be lower, we expect it to be about $\pm 2 \%$. Eddy currents will cause a similar effect as the hysteresis. Study of eddy currents in non-linear media is not simple. One of the basic concepts in the study of eddy currents is the skin depth:

$$
\begin{equation*}
\delta=\sqrt{\frac{\rho}{\pi f \mu}} \tag{6.2}
\end{equation*}
$$



Fig. 6.1 A Rectangular Hysteresis Loop
where $\rho$ is the resistivity of the material, $f$ the applied frequency and $\mu$ the permeability of the material. In order to simplify the matter we have assumed $\mu$ to be constant with a value of $\mu=\mu_{r} \cdot \mu_{0}=600 \times 4 \pi \times 10^{-7} \mathrm{H} / \mathrm{m}$, which corresponds approximately to the flux density of 1.5 T . With $\rho=0.4 \times 10^{-6} \Omega \mathrm{~m}$ (usual material, see for example Ref. 6.1), and $f=50 \mathrm{~Hz}$ the skin depth is $\delta=1.8 \times 10^{-3} \mathrm{~m}$. As the sheets are only $5 \times 10^{-4} \mathrm{~m}$ thick, it can be assumed that the eddy current density and the phase shift of eddy currents change linearly across the sheet, without making any significant error. The maximum eddy current density occurs at the surface of the sheet and for our data its value is about $1 \times 10^{6} \mathrm{~A} / \mathrm{m}^{2}$. The maximum field strength due to eddy currents occurs in the middle of the sheet and its value is about $250 \mathrm{~A} / \mathrm{m}$, which is about $12 \%$ of the excitation current for 1.5 T . The average flux density is reduced by about $10 \%$. The maximum phase shift of eddy currents occurs in the middle of the sheet and it corresponds to about $60^{\circ}$ (el). The average phase shift of eddy currents is about $10^{\circ}(\mathrm{el})$, which will cause a time lag between excitation current and flux of about $1^{\circ}$ (el).

Although the eddy currents damp the flux significantly, this will not cause a significant error, because usually $B-H$ curves supplied by the manufacturers give the maximum flux density versus the effective value of alternating excitation current, thus the damping is already allowed for. However, the influence of the phase shift remains and if expressed as the percentage of the maximum value of total flux for given current its value is $\varepsilon \cong \pm$ $2 \%$. The influence of eddy currents on the effective value of total flux will be lower. We expect it to be about $\pm 1 \%$.

Influence of non-active parts of the machine will depend on the construction of the machine. Influence of other non-active parts except the casing and the shaft on the flux distribution will probably be quite negligible. We expect this error to be about one order of magnitude smaller than the errors caused by, for example, eddy currents.

The influence of the casing and shaft must be examined separately for the core region and the end region. In our computation of end-winding inductances in Chapter 2 we have assumed that the core extends to infinity and that the shaft and the casing have no influence on the flux distribution. The influence of the casing on the end-winding inductance can amount to about $25 \%$ of the total end-winding inductance (Ref. 6.2) for ideally permeable (or ideally non-permeable) material. The sign of the error will depend on the permeability, and for ferromagnetic materials the flux will be underestimated, while for non-magnetic conducting material the flux will be overestimated. Most medium power machines have casings made of cast iron, for which material the relative permeability is about $\mu_{r}=200$, but due to eddy currents it is effectively much lower. However, it is still well above $\mu_{r}=1$, and in this case the error of about $\varepsilon=-20 \%$ can be expected in terms of the end-winding flux. The influence on the total flux will be approximately by one order of magnitude lower because the end flux represents only about $10 \%$ of total flux.

The flux in the core region will be affected in a similar way. Assuming the area of the cross-section of the casing to be about one fifth of the area of the cross-section of the stator core, relative permeability of the casing $\mu_{r}=50$ and relative
permeability of the core $\mu_{r}=500$ (corresponds to flux density of $1 \cdot 5 \mathrm{~T}$ ), the casing will take about $2 \%$ of the flux. However, the influence on the total flux will be about one order of magnitude lower, because the stator core contributes only marginally towards the total magnetic resistance. The influence of the shaft will be similar, and except probably with two-pole machines, it need not be taken into account. We can therefore say that the combined errors due to casing and shaft will be about - $2.5 \%$ in terms of the effective value of total flux.

In this section we have examined different errors due to idealization of the machine. We have determined the orders of magnitude of these errors, and to a certain extent we were also able to determine the direction in which these errors are likely to influence our results. In the next section we shall turn our attention to errors caused by imperfections of our mathematical model.

### 6.3 Errors Due to Imperfections of the Mathematical Model

The errors of this group are listed under $6-10$ in section 6.1 . Some of these errors can be simply reduced by improving our mathematical model or solution methods. So for example the influence of truncation error can be greatly diminished by the use of double precision arithmetic in the computer. Such improvements will generally be costly because they would reduce computation speed. We have therefore to make a compromise; i.e. reduce these errors to the level acceptable from the engineering point of view, but not any further because of the severe penalties in the computation cost.

Approximation of $\mathrm{B}-\mathrm{H}$ curve (or, better, approximation of the function $v=f\left(B^{2}\right)$ ) causes only negligible error. The approximation of $v=f\left(B^{2}\right)$ by 80 straight line segments gives the average error of less than $1 \%$. This error is further reduced by at least one order of magnitude during computation due to the fact that $v$ will be sometimes overestimated and sometimes underestimated.

End effects must be examined in two ways. First, there is an influence of the magnetic core on the flux in the end region. This influence amounts to about $10 \%$ of the end-winding flux (Ref. 6.2). However, this influence has been taken into account, and we can expect the remaining error to be of the order of $\pm 1 \%$ of the end-winding flux, or $\pm 0.1 \%$ of the total flux, because the end-winding flux contributes only about $10 \%$ towards the total flux.

The influence of the finite core length on the core flux will depend largely on the size of the air gap, and can be taken into account by increasing the effective core length by approximately one air-gap length on both sides of the machine (Ref. 6.3). The air-gap of induction machines is of the order of $\delta=0 \cdot 002 \mathrm{D}$ where $D$ is rotor diameter, and if the core length is $\ell \simeq D$, the error would be $\varepsilon=0.002 \mathrm{D} \times 2 / \mathrm{D}=0.004$. If the increase of the effective core length is taken into account, the error will be reduced by approximately one order of magnitude. High level of flux density will increase the error, because of saturation but we believe that it will still remain under $\pm 0.5 \%$ in terms of the total flux.

Discretization may be expected to be one of the major causes of error in our analysis. In linear cases and regular rectangular
discretization the magnitude of error can be determined from (Ref. 6.4):

$$
\begin{equation*}
\varepsilon=\frac{h^{4}}{24}\left(\frac{\partial^{4} A}{\partial x^{4}}+\frac{\partial^{4} A}{\partial y^{4}}\right) \tag{6.1}
\end{equation*}
$$

where ' $h$ ' is the mesh cell size. In non-linear cases and irregular discretization the error is larger. For rectangular meshes de La Vallée Poussin and Lion have verified that the error is (Ref. 6.5):

$$
\begin{equation*}
\varepsilon=o\left(h^{2}\right) \tag{6.2}
\end{equation*}
$$

For irregular triangular meshes the error is also influenced by the degree of irregularity (triangles with small angles may cause large errors). The problem of discretization error is by no means completely solved and we refer the reader to Ref. 6.6 , in which several other works on the subject are quoted.

Being unable to find a reliable practical method for determination of the discretization error for non-linear problems and irregular triangulation we have decided to find this errar by experiments. The mesh density for our model problem was increased in several steps, so that the finest mesh contained 392 nodes and 721 triangles, i.e. the finest mesh contained approximately four times as many nodes as the coarsest mesh. Seven different mesh grades were used in all. The meshes for three different grades (corresponding to 87,234 and 392 mesh nodes) are represented on Figs. 4.3-4.5, and the corresponding flux plots are given on Figs. 6.2-6.4.


Fig. 6.2 Flux Plot for Mesh from Fig. 4.3


Fig. 6.3 Flux Plot for Mesh from Fig. 4.4


Fig. 6.4 Flux Plot for Mesh from Fig. 4.5

We shall start our error analysis from the diagram on Fig. 6.5. This diagram represents the computed values of vector potential at four different points of our model problem (points are marked on Fig. 4.3) as a function of a number of nodes. (Total number of mesh nodes was chosen for the abscissa for this and several subsequent diagrams, rather than the mesh cell size, definition of which would be rather vague for irregular meshes). The first obvious conclusion is that coarse mesh gives underestimated values of vector potential (the theory predicts this for linear problems, see section 3.7 and Ref. 3.62 , but the generalisation of this result to non-linear cases is not quite obvious). The values of vector potential converge well as the number of nodes increases, and for the purpose of an error estimate we shall assume the values obtained from computation in the finest mesh as exact. The largest discrepancies between results for coarser and finer meshes shows curve B on Fig. 6.5. The numerical values of errors for this curve are:

| number of mesh <br> nodes | 87 | 132 | 171 | 235 | 299 | 352 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| error (Wb) | $12.0 \times 10^{-4}$ | $7 \cdot 7 \times 10^{-4}$ | $3 \cdot 7 \times 10^{-4}$ | $1 \cdot 6 \times 10^{-4}$ | $1 \cdot 2 \times 10^{-4}$ | $1 \cdot 0 \times 10^{-4}$ |

These values are larger than $0\left(h^{2}\right)$ if the dimension of the smallest triangle is used as $h$, but smaller than $O\left(h^{2}\right)$ if the largest triangle is used. Thus. Eqn. (6.2) is valid for our model problem if $h$ is defined as the average value of mesh cell size.

The vector potential itself is not our final result. We are interested in the linked flux and the torque. The linked flux is obtained by integration of vector potential over the slot area.
A in $\mathrm{Wb} \times 10^{-2}$
C

In a certain sense it represents the average value of vector potential and it could be expected that the computation of it is only marginally less sensitive to mesh cell size than the vector potential, because the vector potential shows systematic errors. Fig. 6.6 represents the linked flux for three different coils of our model problem. The relative errors for different mesh densities are virtually the same as for curves on Fig. 6.5.

In our program the torque is computed from the flux densities in the air gap, and the flux densities are obtained as the first order differences from the vector potential. Due to the nature of this computation the flux densities are much less accurate than the vector potential. As an illustration we give Figs. 6.7-6.9 which represent the radial flux density for three different meshes with 87,235 and 391 mesh nodes respectively. Position of rotor and stator teeth are also sketched for reference. The discrepancies between these diagrams are large and obvious. So for example the differences in local maxima for meshes with 235 and 391 nodes (Figs. 6.8 and 6.9) are more than $15 \%$ in comparison with about $1 \%$ discrepancies in vector potential. Fig. 6.7 which corresponds to a mesh with 87 mesh nodes hardly represents more than a rough guess of the actual flux density distribution.

The computation of torque is somewhat less critical because this computation involves integration along the air gap. However, computation of the torque is still much less accurate than the computation of linked flux as Fig. 6.10 shows. The diagram represents the computed torque versus number of mesh nodes. The errors are about one order of magnitude larger than the corresponding errors on Fig. 6.6.
Linked flux in $\mathrm{Wb} \times 10^{2}$
mesh nodes
$100 \quad 200$
Linked Flux for Different Coils as a Function of a Number of Nodes
Fig. 6.6

T $|$| B |
| :--- | :--- |

Rotor surface
$--\infty-\infty$ Stator surface

Fig. 6.8 Radial Flux Density in the Air Gap for Mesh from Fig. 4.4


Fig. 6.10 Influence of the Number of Mesh Nodes on the Computed Value of Torque

We can now answer the very important question of the minimum necessary number of nodes. For our model problem it is the mesh with 235 mesh nodes. This mesh produces the linked flux with an accuracy of about $1 \%$ of its maximum value and torque to about $5 \%$. The local flux densities are much less accurate and the errors of up to $25 \%$ can be expected. This is the accuracy which is adequate for most practical purposes. We can conclude that meshes with 4050 mesh nodes per slot pitch (stator plus rotor slots of the sector of the machine considered) for doubly slotted machines will give results which are acceptable from the engineering point of view.

Iterative solution of equations. As an iterative solution of equations requires an infinite number of arithmetic operations to reach the exact solution, our solution will be only approximate. Similarly as with discretization error we can improve the accuracy at the expense of more lengthy and more costly computations. As discretization and other factors are likely to cause errors of up to $5 \%$, there is not much point in continuing the iteration after the vector potentials have reached the accuracy of about $1 \%$.

Without the exact solution we do not know the error at any stage of computation. The quantity which can be easily evaluated is the displacement vector $d$ (Eqn. 5.13). In section 5.3 we have used the magnitude of the first power norm of the displacement vector as one of convergence criteria. However, the displacement vector bears no direct or simple relation to the error vector, and the magnitude of some norm of this vector should not be rased as criterion for terminating the iteration.

Without going into any theoretical details, we state that the rate of decrease of error vector is equal to the rate of decrease
of displacement vector (see for example Ref. 6.7) for linear problems. Thus, in linear cases we know that, for example, the error vector is reduced to $1 \%$ of its initial value at the same time when the displacement vector is reduced to $1 \%$ of its initial value. We have no theoretical proof of this for non-linear problems. However, practical tests on our model problem with different iteration schemes and different mesh grades have shown that this relam tion holds for our problems as well. This enables us to define a very simple practical criterion as to when the iteration process can be terminated. Namely, although we do not know the error vector, we know that it is reduced at the same rate as the displacement vector, and we can stop the iteration when the displacement vector is reduced to $1 \%$ of its initial value, under the assumption that the initial error was $100 \%$. As it is likely that the starting error will be in most cases of the order of magnitude of $10 \%$ (supposing that initial A-vector is obtained by the solution of a linear problem), this criterion should give us a good safety margin.

Any norm of the displacement vector can be used for this purpose. The first power norm is suitable because it does not require a lot of computation and it oscillates less than the maximum norm. For practical application it is useful to take the average value of $||d||$ for several consecutive iterations in order to smooth the oscillations further.

Although this convergence test has no firm theoretical basis for the time being, it has proved valid for all cases we have tried, using different mesh densities and different iteration schemes. When this test was applied the errors were under $1 \%$ of
the final values of vector potentials, the vector potentials being accurate in approximately three decimal digits. The 'final values' were obtained by long runs in which.the first power norm of the displacement vector was reduced to less than 0.0001 of its initial value.

We can now determine the computation time for our model problem from Chapter 5 on the basis of this convergence test. Points where different iteration processes could be terminated are denoted with asterisks on Fig. 5.8 and corresponding numbers of iterations and computation times are listed in Table 5.1 in columns 7 and 8. It is clear that this criterion favours Method B6 even more than the criteria defined in subsection 5.3.

It should be emphasized that even with linear problems this test would be strictly valid only after a large number of iterations. It may therefore be necessary to increase the ratio between initial and final values of displacement vector for some cases, namely if convergence speed is decreased after a certain number of iterations (i.e. if the slope of the curve representing ||d|| plotted against iteration number becomes less steep after a certain number of iterations, as for example the curve for Method B3 with acceleration on Fig. 5.8), but we believe that the ratio $10^{3}: 1$ should suffice for virtually all practical purposes. On the other hand, the iteration can be terminated earlier if it is known that the initial error is small.

This error is of course in terms of the vector potential. As with the discretization errors, local flux densities show larger discrepancies than the vector potentials. However, the influence on the computed torque is considerably smaller than on the flux
densities because errors due to inaccurate solution are much more randomly distributed than errors due to discretization, which are usually systematic. The same is valid for computation of linked flux, which is generally more accurate than the computation of vector potentials.

Truncation error. In the iteration processes we use the truncation error does not accumulate from iteration to iteration. This is probably the reason why the analysis of it is completely avoided in virtually all published work. However, although we do not expect this error to be of any significant amount, it is useful to know its order of magnitude. If the vector potentials are truncated with the error $\Delta$, and the mesh cell size is $d$, then the first order differences which we basically use are computed with the accuracy $2 \Delta /$ d. Computation on the ICL 1907 computer in single precision corresponds to approximately 10 decimal digits, or relative error of $10^{-10}$. The order of magnitude of vector potentials for medium size induction machines is about 0.1 Tm which results in the absolute error of $10^{-11} \mathrm{Tm}$. Minimum d is about 0.1 mm for our mesh and the truncation error is $\varepsilon=2 \times 10^{-11} \mathrm{Tm} /$ $10^{-4} \mathrm{~m}=2 \times 10^{-7} \mathrm{~T}$. This error is below our required accuracy by several orders of magnitude. Its influence can be observed as slowing down of convergence if the iteration is carried on long enough. This happens when some of the components of displacement vector are reduced to the order of magnitude of the truncation error. Again, flux densities are more influenced than the vector potentials.

We have now examined all errors listed in section 6.1 except the errors due to measurements. In the next section we shall briefly examine errors due to measurements and we shall
also make a summary of all the errors involved in order to predict the likely correspondence between our computed results and the results obtained by measurements on, the actual machine.
6.4 Errors Due to Measurements ... Summary

The prediction of measurement errors is important because the computed results can only be compared with the results of measurements. Measurement errors must therefore be carefully determined, and they must not be attributed to the computation. Different quantities can be measured with different accuracy, but unfortunately hardly anything can be said about the accuracy of a particular type of measurement without knowing details on the equipment used. The choice of equipment depends partially on the speed the measurements have to be taken. We must therefore first determine whether the quantities to be measured can be considered as static or time dependent.

We are interested in steady-state and we must ensure that the transient currents are damped enough to be negligible. With the power factor of 0.25 at 50 Hz , which is a realistic value for usual machines at short circuit, the time constant is only about. 0.012 sec and the transient currents will be below $0.1 \%$ of their initial value after about 0.085 sec.

Another factor we have to consider is the increase of resistance due to the increase of the temperature of the winding. Assuming windings to be of copper, and with the current density of $15 \mathrm{~A} / \mathrm{mm}^{2}$, the heat dissipated in the winding is about $4 \times 10^{6} \mathrm{~W} / \mathrm{m}^{3}$. With the specific heat for copper of $7.86 \times 10^{5} \mathrm{~W} \mathrm{sec} / \mathrm{m}^{30} \mathrm{C}$ the temperature rise of copper is about $5^{\circ} \mathrm{C}$ per second. The
corresponding rise in resistivity is about $3.5 \times 10^{-10} \Omega \mathrm{~m}$ per second, or about $2 \%$. Due to the low power factor this increase in resistance will influence the current only by less than $0 \cdot 3 \%$, but the test cannot last longer than a few seconds because of overheating problems. Thus the readings of instruments should be taken preferably during the first second of the test.

This is a fairly short period of time and some sort of recording instrument is essential. Digital recording instruments are very accurate, but they tend to be very expensive. We have already seen that the accuracy of our computation is not very high, and that errors of several percent can be expected. Hence, an analogue recording instrument can be used. The accuracy of such instruments is usually within few percent, but unfortunately no more can be said without details of the particular instrument used.

In the previous two sections we have discussed different errors introduced during our analysis of the machine. The relation between different errors is not simple and in mathematical terms it depends on the type of mathematical operation that link different quantities which are in error. However, unless the errors form a substantial part of the quantities involved, in most circumstances it can be assumed that the errors are not related, i.e. that the magnitude of the error induced by one cause does not influence the magnitude of the error induced by the other cause. In this case the total error is simply obtained by summation of all the errors.

Any cause of error can have very different influence on the accuracy of different quantities (e.g. accuracy of local flux
densities will always be lower than the accuracy of total magnetic flux), and this accuracy will depend on the type of mathematical operation that link different quantities. In most cases systematic errors will be linearly related (i.e, the percentage value of the error will be the same in all related quantities), but randomly distributed errors may be increased by several orders of magnitude if the value of some quantity is obtained by differentiation of another quantity in error, or by subtraction of two quantities in error. On the other hand, integration will tend to reduce randomly distributed errors.

Most of the causes listed in section 6.1 will produce a systematic error which will have similar influence on all computed quantities. However, Eterative solution of equations and truncation will produce randomly distributed errors, and these may cause difficulties. In section 6.3 and 6.4 we have mostly expressed the error in terms of total flux, or linked flux. The linked flux is only an intermediate quantity, which will give induced voltage by differentiation in time. Hence, in order to keep the error of the computed voltage as low as possible, the values of linked flux obtained by several consecutive field computation must be suitably smoothed (for examply by approximating the curve by a polynomial fitted by the least squares method). In that case the errors in computed voltage will be similar to the errors in the computed linked flux.

We have listed different causes of errors and their expected influence on different quantities in Table 6.1. Two values are given for every cause and the top number represents the estimated upper bound and the bottom number represents the estimated lower
Table 6.1 : Errors in Computation with Prescribed Currents

|  | Cause of error | Error in the computed effective value of linked flux (\%) | Error in the computed effective value of induced voltage (\%) | Error in the computed value of torque (\%) |
| :---: | :---: | :---: | :---: | :---: |
| 1 | Mechanical inaccuracy | $\begin{aligned} & +1.0 \\ & -0.0 \end{aligned}$ | $\begin{aligned} & +1.0 \\ & -0.0 \end{aligned}$ | $\begin{aligned} & +2.0 \\ & -0.0 \end{aligned}$ |
| 2 | Inhomogeneous material | $\begin{aligned} & +1.0 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +1 \cdot 0 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +2.0 \\ & -2.0 \end{aligned}$ |
| 3 | Hysteresis | $\begin{aligned} & +2.0 \\ & -2.0 \end{aligned}$ | $\begin{aligned} & +2.0 \\ & -2.0 \end{aligned}$ | $\begin{aligned} & +4 \cdot 0 \\ & -4 \cdot 0 \end{aligned}$ |
| 4 | Eddy currents | $\begin{aligned} & +1.0 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +1.0 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +2.0 \\ & -4.0 \end{aligned}$ |
| 5 | Casing and shaft | $\begin{aligned} & -1.0 \\ & -2.5 \end{aligned}$ | $\begin{aligned} & -1 \cdot 0 \\ & -2 \cdot 5 \end{aligned}$ | $\begin{aligned} & -2.0 \\ & -5.0 \end{aligned}$ |
| 6 | Approximation of $\mathrm{B}-\mathrm{H}$ curve | $\begin{aligned} & +0 \cdot 1 \\ & -0 \cdot 1 \end{aligned}$ | $\begin{aligned} & +0.1 \\ & -0.1 \end{aligned}$ | $\begin{aligned} & +0.2 \\ & -0.2 \end{aligned}$ |
| 7 | End effects | $\begin{aligned} & +0.6 \\ & -0.6 \end{aligned}$ | $\begin{aligned} & +0.6 \\ & -0.6 \end{aligned}$ | $\begin{aligned} & +1 \cdot 2 \\ & -1 \cdot 0 \end{aligned}$ |
| 8 | Discretization* | $\begin{aligned} & -0.5 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & -0.5 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +5.0 \\ & -5.0 \end{aligned}$ |
| 9 | Iteration | $\begin{aligned} & +1.0 \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +2.0 * * \\ & -1.0 \end{aligned}$ | $\begin{aligned} & +4.0 \\ & -4.0 \end{aligned}$ |
| 10 | Truncation*** | < 0.01 | $<0.01$ | $<0.01$ |
| 11 | Total | $\begin{aligned} & +5 \cdot 2 \\ & -9.2 \end{aligned}$ | $\begin{array}{r} +6.2 \\ -9.2 \\ \hline \end{array}$ | $\begin{aligned} & +18.4 \\ & -\quad 24.2 \\ & \hline \end{aligned}$ |
| 12 | Measurements**** |  | $\begin{aligned} & +4.0 \\ & -4.0 \end{aligned}$ | $\begin{aligned} & +10 \cdot 0 \\ & -10 \cdot 0 \end{aligned}$ |
| 13 | Grand total | $\cdots$ | $\begin{aligned} & +10 \cdot 2 \\ & -13.2 \end{aligned}$ | $\begin{aligned} & +28 \cdot 4 \\ & -34 \cdot 2 \end{aligned}$ |

[^1]bound. The bounds for the total error were obtained by summation of all contributions. The given figures represent the maximum expected error. The most probable error is lower and we believe that the computed effective value of induced voltage and the measured value of induced voltage will differ by more than $10 \%$ only rarely. The accuracy of the computed torque is lower and errors of up to $20 \%$ can be expected.

## References to Chapter 6

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In the preceding chapters we have discussed methods for the solution of magnetostatic fields in the end-region (Chapter 2) and core region (Chapters 3-5) of induction motors. The time varying problem can be described as a series of magnetostatic problems and in section 3.9 it was shown how the non-linear field solutions can be used in order to determine the voltage at the terminals if the variations of currents with time in different windings of the machine are known.

In this chapter we shall show by one example how our magnetostatic field solutions can be used to solve the more complicated problem where the voltage at terminals is known and the short circuit currents in the windings are sought.

In section 7.1 we describe the problem and suggest two different methods for the solution of it. The second of these methods is an iterative procedure in which the magnetostatic field solutions form an essential part.

In section 7.2 we describe in detail a modification of this basic iteration in which only a few non-linear field solutions are necessary, and in section 7.3 we give the results of a computation of a sample problem which was chosen to be that of a 15 kW , six pole motor with wound rotor.

### 7.1 The Time-Dependent Problem With Prescribed Voltage

As the magnetic properties of steel in the machine are nonlinear, the inductance between different windings is a function of current in the windings. The terminal voltage can be simply obtained from a series of magnetostatic field solutions, if the currents in windings are known (section 3.9).

The practical problems will be posed in this way only rarely. In the vast majority of practical problems the voltage at the terminals will be known, and the current in different windings will be sought. This is a more complicated problem than the computation of voltage for the prescribed values of current.

This problem can be described by a system consisting of a partial differential equation of the magnetic vector potential (Eqn. (3.10)) and a system of ordinary differential equations describing the current-voltage relation in the windings of the machine. The right-hand side of Eqn. (3.10) will no longer be a function of $x$ and $y$ only, but will be obtained from the solution of the system of ordinary differential equations.

Consider, for example, the problem of a star-connected three phase wound rotor machine represented schematically on Fig. 7.1 (the end-inductances have been omitted for clarity). The problem can be described by a system:

$$
\begin{equation*}
\frac{\partial}{\partial x}\left(v \frac{\partial A}{\partial x}\right)+\frac{\partial}{\partial y}\left(v \frac{\partial A}{\partial y}\right)=f\left(x, y, i_{1}, \ldots, i_{6}\right) \tag{7.1a}
\end{equation*}
$$



Fig. 7.1 Schematic Representation of a Three-Phase, Star-Connected Wound Rotor Motor

$$
\begin{aligned}
& i_{1} R_{1}-i_{2} R_{2}+u_{1}-u_{2}=u_{A B} \\
& i_{2} R_{2}-i_{3} R_{3}+u_{2}-u_{3}=u_{B C} \\
& i_{1}+i_{2}+i_{3}=0 \\
& i_{4} R_{4}-i_{5} R_{5}+u_{4}-u_{5}=0 \\
& i_{5} R_{5}-i_{6} R_{6}+u_{5}-u_{6}=0 \\
& i_{4}+i_{5}+i_{6}=0
\end{aligned}
$$

where $f\left(x, y, i_{1}, \ldots, i_{6}\right)$ is a known function that depends on the distribution of windings in the cross-section of the machine, $R_{1} \ldots R_{6}$ are the resistances of the windings (which may possibly include any outer resistance), $i_{1} \ldots i_{6}$ are currents in the six windings, $u_{A B}$ and $u_{B C}$ are the voltages at the stator terminals and $u_{1} \ldots u_{6}$ are voltages induced in the six windings $W_{1} \ldots W_{6}$. Their values are obtained by the time differentiation of the total flux $\psi$ linked by the corresponding coil:

$$
\begin{equation*}
u_{k}=-\frac{d \Psi_{k}}{d t} \tag{7.1c}
\end{equation*}
$$

where $\psi_{k}$ is computed from the field solution in a manner explained in section 3.9 .

There are several possibilities for an approximate solution of system (7.1a-7.1c). Eqn. (7.1a) is similar to the parabolic equations obtained in the solution of diffusion problems. The difference is in the right-hand side, which in our case is the function of the complete field solution. It can be expected that methods analogous to those in the solution of diffusion equation could be used. The discretization in space can be semi-regular triangulation as used for our magnetostatic field solutions. However, it would be probably more convenient to use a fixed
time-step corresponding to the usual finite difference method. Timewise our problem represents a boundary value problem with periodicity boundary conditions because we are only interested in a steady-state solution. However, by this method our problem could also be treated as an initial value problem, in which case it would also give the transient solution as a result. In either case this approach would require somewhat different techniques from those used for the magnetostatic field solutions, and the results of our investigations in the previous chapters would not be directly applicable.

Another possibility is an iterative procedure consisting of alternate solutions of Eqn. (7.1a) and (7.1b). An approximate solution is chosen for currents $i_{1} \ldots i_{6}$ and Eqn. (7.1a) solved for different instants of time by one of the methods described in Chapters 3-6. Then the values of induced voltage in every winding are computed by the finite difference approximation to Eqn. (7.1c), and the system (7.1b) solved (i.e. its finite-difference approximation). The new values of current at different instants of time are used to compute a new set of field solutions and the procedure is repeated until the difference of two successive current estimates fall below a certain specified limit. The transient solution cannot be obtained by this method, but it has the advantage that it is based on magnetostatic field solutions which we know how to solve. The disadvantage of this method is that it requires numerous non-linear field solutions and therefore it is likely that the computation time will be long (although reduction in computation time can be achieved by the use of higher order differences in the time direction and by extrapolation of vector potentials from
two or more time steps in order to form starting values for the new field solution). Fortunately, a modification of this approach is possible in which it is necessary to compute only one non-linear field in every iteration step. This method will be described in detail in the following section.

### 7.2 A Practical Method for the Solution of a Time-Varying Problem With Prescribed Voltages

The system of equations (7.1b - 7.1c) can be written in the form:

$$
\begin{align*}
& i_{1} R_{1}-i_{2} R_{2}-\frac{d}{d t} \sum_{k=1}^{6}\left(L_{1, k} i_{k}-L_{2, k} i_{k}\right)=u_{A B} \\
& i_{2} R_{2}-i_{3} R_{3}-\frac{d}{d t} \sum_{k=1}^{6}\left(L_{2, k} i_{k}-L_{3, k} i_{k}\right)=u_{B C} \\
& i_{1}+i_{2}+i_{3}=0  \tag{7.2}\\
& i_{4} R_{4}-i_{5} R_{5}-\frac{d}{d t} \sum_{k=1}^{6}\left(L_{4, k} i_{k}-L_{5, k} i_{k}\right)=0 \\
& i_{5} R_{5}-i_{6} R_{6}-\frac{d}{d t} \sum_{k=1}^{6}\left(L_{5, k} i_{k}-L_{6, k} i_{k}\right)=0 \\
& i_{4}+i_{5}+i_{6}=0
\end{align*}
$$

where $L_{m, k}$ represents the inductance between the windings $W_{m}$ and $\mathrm{W}_{\mathrm{k}}$. (The inductance in this context is the so-called 'apparent' inductance. If there is only one winding then the apparent selfinductance is defined as a total flux linked by the winding divided by the current through the winding. See for example Ref. 7.1)

Because of saturation the inductance $L_{m, k}$ will be reduced in comparison with its non-saturated value. Indeed, the inductances are not constants, but functions of current, $L_{m, k}=f\left(i_{1}, \ldots i_{6}\right)$. Alternatively, they can be expressed as functions of time $L_{m, k}=g(t)$. However, if the variations of the values of inductances with time are small compared with the variations of current, an approximate solution of the system (7.2) can be obtained by setting $L_{m, k}=$ constant, this value being smaller than the value for non-saturated machine. Strictly, every current distribution will have corresponding saturated values of inductances. In our approximate computation we shall compute only one set of values of inductances, corresponding to a current distribution at a certain instant of time. The saturated values of inductances $L_{m, k}, k=1,6$ for the given current distribution can be obtained if firstly the non-linear field solution is computed for the given set of currents, and then with the reluctivities fixed to the values obtained in this non-linear field solution, a linear field solution is computed with the unity current in winding $W_{m}$. The inductance $L_{m, k}$ will then have the numerical value of the total flux linkage with the winding $W_{k}$.

The basic iteration procedure as described in section 7.1 remains unchanged, but instead of several non-linear field solutions in every interation step only one non-linear field is needed. This non-linear field solution provides only the saturated values of reluctivities of iron parts. The saturated values of inductances are computed from the linear field solutions. Hence, in addition to one non-linear field solution, it is also necessary to compute several linear fields in every iteration step. In the case
of symmetrical three-phase windings and symmetrical supply, only two linear fields are needed, one for one stator current equal to unity and another for one rotor current equal to unity. Linear field solutions can be computed with much less effort than the non-linear ones, and it is likely that in most cases this procedure would result in considerable reduction of total computation time in comparison with the basic iteration described in section 7.1 .

This approach is based on the assumption that the variations of the values of inductances $L_{m, k}$ with time are small and that the influence of saturation on the inductances can be approximated by simply reducing the values of inductances. In the case of symmetrical windings some indication of the variation of the values of inductances with time can be obtained from the comparison of the mutual inductances of different phases. So for example $L_{1,2}$ will be different from $L_{1,3}$ (both values can be obtained from the same linear field solution), unlike in the linear case where these two inductances have the same value. Variation of the values of inductances will be periodical with the period of $T / 2$ where $T$ is the period of the supply voltage. We can therefore approximate the inductances by the truncated Fourier series*. $L_{m, k}=g(t)$ cannot have any sharp peaks, and it can be expected that the magnitude of higher harmonics in $L_{m, k}=g(t)$ will decrease very rapidly as the order of harmonic increases. We believe that in most cases it will be sufficient to determine the magnitude of the second harmonic

[^2]only, and its influence on the current, without the need for any further computation.

By neglecting the higher harmonics we can represent the value of inductance $L_{m, k}$ by

$$
\begin{equation*}
L_{m, k}=g(t) \simeq C_{1}+C_{2} \sin \left(\frac{4 \pi}{T} \cdot t-C_{3}\right) \tag{7.3}
\end{equation*}
$$

The values of the three constants $C_{1} \ldots C_{3}$ can be computed from the three values of inductance $L_{m, k}$, which correspond to three different instants of time. In the case of symmetrical threephase windings it is not necessary to compute three non-linear field distributions in order to determine three values of $L_{m, k}$. Instead, self and mutual inductances of all three phases can be determined from the corresponding three linear field distributions. The linear field distributions are computed with the values of reluctivities obtained in the non-linear field distribution, and with the unity current in windings $W_{1}, W_{2}$ and $W_{3}$, respectively. The values of inductances obtained in this way will be the same as the values of the inductance of one phase corresponding to three different instants of time separated by $T / 6$, because the flux in the machine is rotating and the winding is symmetrical. The inductances of the rotor winding can be obtained in an analogous way.

This computation of constants $C_{1} \ldots C_{3}$ need not be carried out in every iteration step, but only when the iteration has reached the required accuracy. Consequently, the system (7.2) has to be solved only once with variable values of inductances.

The complete procedure for problem of Fig. 7.1 including initialization is presented on Fig. 7.2 in a form of a flow chart.


Fig. 7.2 The Flow Chart for the Computation of Current With Prescribed Voltage

In the case of unsymmetrical windings it may be necessary to adapt this procedure to suit the problem in hand. Also, the iteration need not be stopped as indicated in Fig. 7.2, instead, the procedure can be continued in an analogous way in order to determine the influence of harmonics of order higher than two, in which case more than three field solutions would be required in every iteration. Whether or not to continue the computation is best decided at the end of the process described in Fig. 7.2, when the influence of the second harmonic will be known. When this decision is being made the overall accuracy of our method must also be considered, as it is possible that the influence of hysteresis, eddy currents, etc., on the short circuit current are higher than the influence of the higher harmonics in $L_{m, k}=g(t)$.

The procedure suggested and described in this section was tested on one example. The problem and the results are given in the next section.

### 7.3 The Sample Problem

As our sample problem we have chosen a 15 kW , six-pole wound rotor machine, star connected on both stator and rotor with 54 slots on stator and 36 on rotor. The winding was two layer with five turns per coil on the stator and seven turns per coil on the rotor, series connected, $60^{\circ}$ phase belt spread, full pitch on rotor, $7 / 9$ pitch on stator. The active length of the machine was $\ell=200 \mathrm{~mm}$. The cross-section of one pole on the machine is shown on Fig. 7.3, and the magnetising curve of the steel laminations on Fig. 7.4.


Fig. 7.3 The Cross-Section of One Pole of a Machine Chosen as a Sample Problem

Fig. 7.4 The $B-H$ Curve

The end-winding inductances were calculated by the method described in Chapter 2. The end-windings on both stator and rotor were represented by 20 straight line segments, resembling the usual almost elliptical shape of low voltage machines fairly accurately. The complete end-winding inductance matrix is given in Table 7.1. (The matrix is, of course, symmetrical.)

TABLE 7.1
The End-Winding Inductances
(All values in $\mathrm{H} \times 10^{-5}$ )

|  |  | Stator |  |  | Rotor |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | A | B | C | A | B | C |
| Stator | A | 23.19 | 8.75 | 8.75 | 1.90 | 1.52 | 1.10 |
|  | B |  | 23.19 | 8.75 | 1.10 | 1.90 | 1.52 |
|  | C |  |  | 23.19 | 1.52 | 1.10 | 1.90 |
| Rotor | A |  |  |  | 20.76 | 7.08 | 7.08 |
|  | B |  |  |  |  | 20.76 | 7.08 |
|  | C |  |  |  |  |  | 20.76 |

A computer-constructed triangle mesh used for the computation of magnetic field in the core region is shown on Fig. 7.5. The mesh density was chosen according to the rules given in Chapter 6. The complete mesh contained 1505 triangles and 790 nodes resulting in a system of 740 equations.

For the solution of system (7.2) a separate computer program was written which takes the periodicity conditions fully into account so that a solution over a period $T / 6$ only is necessary. First order central differences were employed.


Fig. 7.5 A Computer Generated Mesh

The computation followed the procedure described in section 7.2. The initial values of inductances were obtained by the solution of linear problems with $v_{\text {iron }}=1200 \mathrm{~A} / \mathrm{Tm}$ which corresponds to the flux density of 1.5 T . System (7.2) was solved with a time step of $5.56 \times 10^{-5} \mathrm{sec}(1 \mathrm{el}$. deg.) and $50 \mathrm{~Hz}, 415 \mathrm{volts}$ line to line, which produced an initial estimate of stator current of 188 A . The optimum iteration parameters were found to be $\omega=1 \cdot 80$, $\beta=0.08$ and $K=4$. The linear fields were computed by successive line over-relaxation. The optimum over-relaxation factor (assymptotic) was found by the well-known power method (see for example Ref. 7.2) and its value was about $\omega_{b}=1.956$ (it varies slightly as the reluctivities change from iteration to iteration).

The variations of the effective value of stator and rotor current in the iteration process described by boxes 4-7 of Fig. 7.2 are shown on Fig. 7.6. Convergence appears to be fairly fast, and the difference between the current estimates in the 2nd and 3rd iteration is only about $0.5 \%$. This figure is not directly related to the accuracy of our iteration process, but as the iteration converges rapidly, we have reasons to believe that the error of our current estimate is also of the order of magnitude of $1 \%$. (This accuracy corresponds to the iteration procedure only. The error introduced in this way should be added to the errors introduced by other factors, see Chapter 6).

This accuracy was considered to be adequate for the purpose of short circuit current computation, the iteration was terminated, and the influence of higher harmonics determined (box 8). The variations of the value of inductances with time is illustrated in Fig. 7.7. The complete matrix of core-inductances and the


Fig. 7.6 Effective Value of Stator and Rotor Current Versus Number of Iterations

magnitudes of their 2nd harmonic is given in Table 7.2.

TABLE 7.2
The Core-Region. Inductances
(All values in $\mathrm{H} \times 10^{-2}$ )

|  |  | Stator |  |  | Rotor |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | A | B | C | A | B | C |
| Stator | A | $\begin{aligned} & 1.386 \\ & 0.011 \end{aligned}$ | $\begin{aligned} & 0.641 \\ & 0.016 \end{aligned}$ | $\begin{aligned} & 0.641 \\ & 0.016 \end{aligned}$ | $\begin{aligned} & 1.334 \\ & 0.012 \end{aligned}$ | $\begin{aligned} & 0.618 \\ & 0.015 \end{aligned}$ | $\begin{aligned} & 0.612 \\ & 0.004 \end{aligned}$ |
|  | B |  | $\begin{aligned} & 1.386 \\ & 0.011 \end{aligned}$ | $\begin{aligned} & 0.641 \\ & 0.016 \end{aligned}$ | $\begin{aligned} & 0.612 \\ & 0.004 \end{aligned}$ | $\begin{aligned} & 1.334 \\ & 0.012 \end{aligned}$ | $\begin{aligned} & 0.618 \\ & 0.015 \end{aligned}$ |
|  | C |  |  | $\begin{aligned} & 1.386 \\ & 0.011 \end{aligned}$ | $\begin{aligned} & 0.618 \\ & 0.015 \end{aligned}$ | $\begin{aligned} & 0.612 \\ & 0.004 \end{aligned}$ | $\begin{aligned} & 1.334 \\ & 0.012 \end{aligned}$ |
| Rotor | A |  |  |  | $\begin{aligned} & 1.527 \\ & 0.007 \end{aligned}$ | $\begin{aligned} & 0.574 \\ & 0.016 \end{aligned}$ | $\begin{aligned} & 0.574 \\ & 0.016 \end{aligned}$ |
|  | B |  |  |  |  | $\begin{aligned} & 1.527 \\ & 0.007 \end{aligned}$ | $\begin{aligned} & 0.574 \\ & 0.016 \end{aligned}$ |
|  | C |  |  |  |  |  | $\begin{aligned} & 1.527 \\ & 0.007 \end{aligned}$ |

Top and bottom figures represent the values of $C_{1}$ and $C_{2}$ in Eqn. (7.3), respectively.

On average the magnitude of the second harmonic is about $3 \%$ of the value of inductance. The second harmonic of the inductance will in turn produce some change in the values of current. This value was computed and it amounts to about $3 \%$ of the effective value of stator current. The increased current will, of course, have some influence on the values of inductance, and strictly, we should continue the iteration with this new value of current. The difference of $3 \%$ was considered acceptable, however, because the
overall accuracy of our computation is not better than some $\pm 10 \%$ (see Chapter 6).

As an illustration we have also plotted the field for the last step in our iteration (Fig. 7.8). The flux densities in some parts of the cross-section reach a value of up to 2.5 T . The relative permeability of iron for this high value of flux density is only about $\mu_{r}=5$, but the overall effect of saturation on the value of current is not very large. The stator current is increased by about 9\%. This increase was determined by the computation of nonsaturated values of inductances and the corresponding value of current. The non-saturated values of both stator and rotor current are also plotted on Fig. 7.6. The torque was also computed from the field distribution in the last iteration and its value was $T=139 \mathrm{Nm}$.

All computations were performed on the ICL 1907 computer. In average the non-linear field solutions required about 250 iterations for the accuracy of about $1 \%$, and the linear field solutions required about 150 iterations. A considerable amount of computation time was saved by the use of the estimate of the vector potential distribution from the preceding iteration as the initial values for the new field distribution. By the use of the optimising compiler XFEW the total computation time was about 1 hour. This figure does not include input and output operations, compilation, backing store operations, etc., which can be expected to increase the actual computer time between $10 \%$ and $100 \%$, depending on the amount of output, backing store media, etc. Hence a realistic figure for the total computation time for this type of computer is about two hours. This is about two orders of magnitude


Fig. 7.8 The Flux Distribution in One Pole Pitch
more than the computation time that can be expected from the use of computational methods which are not based on the solution of magnetic field in the machine. So for example the method of Chalmers and Dodgson (Ref. 7.3 ) would require a computation time of the order of one minute.

In this example we have seen that practical computations of short circuit current of induction motors with wound rotors by the use of numerical field solutions is feasible. Our sample problem is relatively simple, but the method described in this chapter is not limited to star-connected windings, nor is it limited to symmetrical three-phase power supply. In fact, any type of induction machine at short circuit can be dealt with in an analogous way, although clearly more complex problems will require more computer time. In the next chapter we shall summarise some of the possibilities of our method and also look at some possibilities for future development.

## References to Chapter 7

7.1 ALGER, P.L. : 'The Nature of Polyphase Induction Machines', p. 27, John Wiley \& Sons, New York, 1951.
7.2 CARRE, B.A. : The Determination of the Optimum Accelerating Factor for Successive Overrelaxation', The Computer, Journal, Vol. 4, pp. 73-78, 1961.
7.3 CHALMERS, B.J., DODGSON, R.: 'Saturated Leakage Reactance of Cage Induction Motors', Proceedings of the IEE, Vol. 116, pp. 1395-1404, 1969.
CONCLUSIONS AND SUGGESTIONS FOR FUTURE WORK

### 8.1 The End-Winding Inductances

In Chapter 2 of this thesis we have described a novel method for the computation of end-winding inductances. Our method is an integration routine based on the method of images.

In the earlier methods described by other authors the permeability of the iron core could have only the values of $\infty$ or 0 (ideally permeable or ideally conductive material). In our method effects of both permeable and conductive properties of the steel laminations can be considered by choosing the 'effective' value of permeability which gives more accurate results.

The accuracy of our method is limited by two major factors. The first one is the value of the effective permeability. This value could be obtained from theoretical considerations or from experiments, and there is room for further research in this direction.

The other factor limiting the accuracy is the influence of other conducting/permeable surfaces in the vicinity of the endwinding (casing, shaft). Their effect can be considered by an approach analogous to the treatment of the effect of the core.

A method of Lawrenson is related to our method (see section 2.6). In Chapter 2 we have suggested that both methods could yield similar accuracy and would probably require comparable computation time. This statement was not based on any detailed analysis, and we believe that in the search for the most accurate
and the most efficient method a detailed comparison of the method of Lawrenson and our method would be a valuable contribution.

### 8.2 Field Solutions

The major part of the thesis is concerned with the problem of . computation of two-dimensional non-linear magnetostatic fields. Several possibilities have been considered and the numerical method based on discretisation techniques appears to be the only feasible way of effective and accurate computation of field problems. Of different possibilities for discretisaion, triangular meshes with free or semi-free topology have several advantages over the other methods, particularly in flexibility and accurate representation of complicated boundaries.

The set of algebraic equations representing the problem in a discretised form was derived in two different ways. Both approaches are based on a principle of the elimination of a suitably defined error of our approximate solution. Of the two approaches the one based on integration is particularly simple and is, in this sense, more suitable than the more common approach based on the calculus of variations.

The mathematical model we have chosen is commonly known as the 'potential energy model'. Several other possibilities for a different type of mathematical model have only been mentioned. This is a field that is in need of more detailed analysis, particularly as some of these mathematical models offer better accuracy without the increase of the order of approximation. 'Hybrid' models are particularly attractive in this sense.

For the approximation of our field solution we have chosen piecewise linear functions. Although this type of approximation gave us a fair accuracy, a further development by the use of piecewise polynomials seems inevitable. Cubics offer the advantage of relative simplicity and greatly improved approximation in comparison with piecewise linear functions. The investigation of the use of piecewise cubic approximation for non-linear problems is recommended as one of the future topics.

We have investigated several methods for the solution of large sets of non-linear algebraic equations arising from discretisation. Numerical experimentation has shown that different iterative methods can be used successfully for such problems, and that some of these techniques are superior to direct methods of solution. Two-step linearised line iteration has shown the best performance of all tested methods, and the use of this method is recommended for future work. However, the performance of this method is not much superior to the performance of some other iteration methods, and it may be that another method will be more suitable for another problem. This may be particularly true if higher order approximation is employed, for which the iteration methods will be less suitable than for the systems arising from low order approximation.

Determination of optimum iteration parameters by trial and error method leaves a lot to be desired. This method can provide the best combination of iteration parameters for a particular problem, but its use is neither very efficient nor elegant. A method which does not require a prior estimate of any parameters and in which the iteration parameters are adjusted during iteration
in order to achieve the best possible convergence would be ideal and further theoretical investigations as well as numerical experimentation in this field are necessary.

### 8.3 Applications

The aim of this research was to find a suitable method for the computation of magnetostatic fields in saturated induction motors with wound rotors, with the purpose of determination of the performance of the machine with locked rotor.

In Chapter 7 we have seen that there are several methods by which the performance of the machine can be determined from the field solutions. Consequently, a method was chosen which appeared to require the least amount of computation.

The method we have selected for the computation of our sample problem gives the accuracy acceptable from the practical point of view and it is reasonably efficient. The use of our method is not. restricted to the computation of short circuit current of machines with wound rotors. We believe that our method can be successfully used for the computation of the performance of non-skewed squirrel cage induction motors in short circuit, and the application of our method to this problem seems to be the logical step. However, the total computation time for the solution of this problem can be expected to be considerably longer, possibly up to five times more than the computation time required for machines with wound rotors. Another problem that is also in the scope of our method is the short-circuit performance of machines with skewed rotors. Two dimensional field solutions can still serve as a basis for computation, because the axial flux can be neglected in most circumstances.

This problem is, nevertheless, much more complicated than the problem of unskewed machines. The total computation time for the solution of this problem may easily be by one order of magnitude higher than the computation time required for the solution of our sample problem, particularly if solutions for different relative positions of rotor to stator are sought.

The behaviour of an electrical machine is determined by the electromagnetic phenomena in the machine. The electromagnetic phenomena can be suitably described by Maxwell's equations. Hence, any design process of the machine incorporates in itself an approximate solution of Maxwell's equations. These equations are too complicated to be solved exactly, and simpler mathematical models have to be used to represent the physical phenomena in the machine in order to carry out the practical computation. In the past these mathematical models have often been very crude, due to the limited possibilities of hand computation. The development of digital computers offers a possibility of more accurate solutions. The final goal is a complete numerical model of the machine, which will enable accurate prediction of the behaviour of the machine at any working conditions. This thesis represents a small contribution towards this goal.

## APPENDIX

A Mutual Neumann Integral Between One Finite and Two Semi-Infinite Anti-Parallel Straight Lines

Referring to Fig. 2.7 and Eqn. (2.5) and (2.6) the mutual Neumann integral between a finite straight line $\overline{\mathrm{AB}}$ and two mutually anti-parallel straight lines is given by $N=\lim _{\substack{b_{1}+\infty \\ a_{2} \rightarrow \infty}}\left[\cos \phi \cdot\left\{\overline{C_{1} B} \cdot \ln \frac{\left|\overline{a_{1} B}\right|+\left|\overline{b_{1} B}\right|+\left|\overline{a_{1} b_{1}}\right|}{\left|\overline{a_{1} B}\right|+\left|\overline{b_{1} B}\right|-\left|\overline{a_{1} b_{1}}\right|}-\overline{C_{1} A} \cdot \ln \frac{\left|\overline{a_{1} A}\right|+\left|\overline{b_{1} A}\right|+\left|\overline{a_{1} b_{1}}\right|}{\left|\overline{a_{1} A}\right|+\left|\overline{b_{1} A}\right|-\left|\overline{a_{1} b_{1}}\right|}+\right.\right.$
$+\overline{c_{1} b_{1}} \cdot \ln \frac{\left|\overline{b_{1} A}\right|+\left|\overline{b_{1} B}\right|+|\overline{A B}|}{\left|\overline{b_{1} A}\right|+\left|\overline{b_{1} B}\right|-|\overline{A B}|}-\overline{c_{1} a_{1}} \cdot \ln \frac{\left|\overline{a_{1} A}\right|+\left|\overline{a_{1} B}\right|+|\overline{A B}|}{\left|\overline{a_{1} A}\right|+\left|\overline{a_{1} B}\right|-|\overline{A B}|}+$
$+\overline{C_{2} B} \cdot \ln \frac{\left|\overline{a_{2} B}\right|+\left|\overline{b_{2} B}\right|+\left|\overline{a_{2} b_{2}}\right|}{\left|\overline{a_{2} B}\right|+\left|\overline{b_{2} B}\right|-\left|\overline{a_{2} b_{2}}\right|}-\overline{C_{2} A} \cdot \ln \frac{\left|\overline{a_{2} A}\right|+\left|\overline{b_{2} A}\right|+\left|\overline{a_{2} b_{2}}\right|}{\left|\overline{a_{2} A}\right|+\left|\overline{b_{2} A}\right|-\left|\overline{a_{2} b_{2}}\right|}+$
$\left.+\overline{c_{2} b_{2}} \cdot \ln \frac{\left|\overline{b_{2} A}\right|+\left|\overline{b_{2} B}\right|+|\overline{A B}|}{\left|\overline{b_{2} A}\right|+\left|\overline{b_{2} B}\right|-|\overline{A B}|}-\overline{c_{2} a_{2}} \cdot \ln \frac{\left|\overline{a_{2} A}\right|+\left|\overline{a_{2} B}\right|+|\overline{A B}|}{\left|\overline{a_{2} A}\right|+\left|\overline{a_{2} B}\right|-|\overline{A B}|}\right\}-$
$-\left|\overline{C_{1} c_{1}}\right| \cdot \operatorname{ctg} \phi .\left\{\operatorname{arctg}\left[\frac{\left|\overline{C_{1} c_{1}}\right|}{\left|\overline{b_{1} B}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} b_{1}} \cdot \overline{C_{1} B}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{b_{1} B}\right|} \cdot \sin \phi\right]-\right.$
$-\operatorname{arctg}\left[\frac{\left|\overline{C_{1} c_{1}}\right|}{\left|\overline{b_{1} A}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} b_{1}} \cdot \overline{C_{1} A}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{b_{1} A}\right|} \cdot \sin \phi\right]-$
$-\operatorname{arctg}\left[\frac{\left|\overline{C_{1} c_{1}}\right|}{\left|\overline{a_{1} B}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} a_{1}} \cdot \overline{C_{1} B}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{a_{1} B}\right|} \cdot \sin \phi\right]+$
$\left.+\operatorname{arctg}\left[\frac{\left|\overline{C_{1} c_{1}}\right|}{\left|\overline{a_{1} A}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} a_{1}} \cdot \overline{C_{1} A}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{a_{1} A}\right|} \cdot \sin \phi\right]\right\}-$
$-\left|\overline{C_{2} C_{2}}\right| \cdot \operatorname{ctg} \phi .\left\{\operatorname{arctg}\left[\frac{\left|C_{2} c_{2}\right|}{\left|\overline{b_{2} B}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{C_{2} b_{2}} \cdot \overline{C_{2} B}}{\left|\overline{C_{2} C_{2}}\right| \cdot\left|\overline{b_{2} B}\right|} \cdot \sin \phi\right]-\right.$
$-\operatorname{arctg}\left[\frac{\left|\overline{C_{2} C_{2}}\right|}{\left|\overline{\mathrm{B}_{2} \mathrm{~A}}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{\mathrm{C}_{2} \mathrm{~b}_{2}} \cdot \overline{\mathrm{C}_{2} \mathrm{~A}}}{\left|\overline{\mathrm{C}_{2} \mathrm{C}_{2}}\right| \cdot\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|} \cdot \sin \phi\right]-$
$-\operatorname{arctg}\left[\frac{\left|\overline{C_{2} C_{2}}\right|}{\left|\overline{a_{2} B}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{2} a_{2}} \cdot \overline{C_{2} B}}{\left|\overline{C_{2} C_{2}}\right| \cdot\left|\overline{a_{2} B}\right|} \cdot \sin \phi\right]+$
$\left.\left.+\operatorname{arctg}\left[\frac{\left|\overline{C_{2} C_{2}}\right|}{\left|\overline{a_{2} A}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{2} a_{2}} \cdot \overline{C_{2} \mathrm{~A}}}{\left|\overline{C_{2} C_{2}}\right| \cdot\left|\overline{a_{2} A}\right|} \cdot \sin \phi\right]\right\}\right]$

After some transformation and application of L'Hospital's rule we get:
$\lim _{b_{1} \rightarrow \infty}\left(\overline{c_{1} b_{1}} \cdot \ln \frac{\left|\overline{b_{1} A}\right|+\left|\overline{b_{1} B}\right|+|\overline{A B}|}{\left|\overline{b_{1} A}\right|+\left|\overline{b_{1} B}\right|-|\overline{A B}|}\right)=|\overline{A B}|$
and analogousiy:
$\lim _{a_{2} \rightarrow \infty}\left(\overline{c_{2} a_{2}} \cdot \ln \frac{\left|\overline{a_{2} A}\right|+\left|\overline{a_{2} B}\right|+|\overline{A B}|}{\left|\overline{a_{2} A}\right|+\left|\overline{a_{2} B}\right|-|\overline{A B}|}\right)=|\overline{A B}|$.

Also, by substituting:

$$
\begin{align*}
& \overline{C_{2} B}=\overline{C_{1} B}+\overline{C_{2} C_{1}}  \tag{A.4}\\
& \overline{C_{2} A}=\overline{C_{1} A}+\overline{C_{2} C_{1}}
\end{align*}
$$

and some manipulation we obtain

$$
\begin{equation*}
\lim _{b_{1} \rightarrow \infty}\left(\frac{\left|\overline{C_{1} c_{1}}\right|}{\left|\overline{B b_{1}}\right|}\right)=0 \quad \lim _{a_{2} \rightarrow \infty} \frac{\left|\overline{C_{2} c_{2}}\right|}{\left|\overline{a_{2} B}\right|}=0 \tag{A.6}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{b_{1}+\infty}\left(\frac{\overline{c_{1} b_{1}} \cdot \overline{c_{1} B}}{\left|\overline{C_{1} c_{1}}\right| \cdot\left|\overline{B b_{1}}\right|}\right)=\frac{\overline{C_{1} B}}{\left|\overline{C_{1} c_{1}}\right|} \quad \lim _{a_{2} \rightarrow \infty} \frac{\overline{c_{2} a_{2}} \cdot c_{2} B}{\left|\overline{C_{2} c_{2}}\right| \cdot\left|\overline{a_{2} B}\right|}=-\frac{\overline{C_{2} B}}{\left|\overline{C_{2} c_{2}}\right|} \tag{A.7}
\end{equation*}
$$

By substituting these expression in Eqn. (A.1) we obtain

$$
\begin{aligned}
N & =\cos \phi \cdot\left\{-\overline{c_{2} b_{2}} \cdot \ln \frac{\left|\overline{b_{2} \mathrm{~A}}\right|+\left|\overline{\mathrm{b}_{2} \mathrm{~B}}\right|+|\overline{\mathrm{AB}}|}{\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|+\left|\overline{\mathrm{b}_{2} \mathrm{~B}}\right|-|\overline{\mathrm{AB}}|}-\mathrm{c}_{1} \mathrm{a}_{1} \cdot \ln \frac{\left|\overline{a_{1} \mathrm{~A}}\right|+\left|\overline{\mathrm{a}_{1} \mathrm{~B}}\right|+|\overline{\mathrm{AB}}|}{\left|\overline{\mathrm{a}_{1} \mathrm{~A}}\right|+\left|\overline{\mathrm{a}_{1} \mathrm{~B}}\right|-|\overline{\mathrm{AB}}|}+\right. \\
& +\overline{\mathrm{C}_{1} \mathrm{~B}} \cdot \ln \frac{\left|\overline{\mathrm{~b}_{2} \mathrm{~B}}\right|-\overline{\mathrm{C}_{2} \mathrm{~B}} \cdot \cos \phi-\overline{\mathrm{C}_{2} \mathrm{~b}_{2}}}{\left|\overline{\mathrm{a}_{1} \mathrm{~B}}\right|-\overline{\mathrm{C}_{1} \mathrm{~B}} \cdot \cos \phi+\overline{\mathrm{c}_{1} \mathrm{a}_{2}}}+\overline{\mathrm{C}_{1} \mathrm{~A}} \cdot \ln \frac{\left|\overline{\mathrm{a}_{1} \mathrm{~A}}\right|-\overline{\mathrm{C}_{1} \mathrm{~A}} \cdot \cos \phi+\overline{\mathrm{c}_{1} \mathrm{a}_{1}}}{\left|\overline{\mathrm{~b}_{2} \mathrm{~A}}\right|-\overline{\mathrm{C}_{2} \mathrm{~A}} \cdot \cos \phi-\overline{\mathrm{C}_{2} \mathrm{~B}_{2}}}
\end{aligned}
$$

$$
\begin{align*}
& \lim _{\substack{b_{1} \rightarrow \infty \\
a_{2} \rightarrow \infty}}\left(\overline{C_{1} B} \cdot \ln \frac{\left|\overline{a_{1} B}\right|+\left|\overline{b_{1} B}\right|+\left|\overline{a_{1} b_{1}}\right|}{\left|\overline{a_{1} B}\right|+\left|\overline{b_{1} B}\right|-\left|\overline{a_{1} b_{1}}\right|}+\overline{C_{2} B} \cdot \ln \frac{\left|\overline{a_{2} B}\right|+\left|\overline{b_{2} B}\right|+\left|\overline{a_{2} b_{2}}\right|}{\left|\overline{a_{2} B}\right|+\left|\overline{b_{2} B}\right|-\left|\overline{a_{2} b_{2}}\right|}-\right. \\
& \left.-C_{2} A \cdot \ln \frac{\left|\overline{a_{2} \mathrm{~A}}\right|+\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|+\left|\overline{a_{2} \mathrm{~b}_{2}}\right|}{\left|\overline{\mathrm{a}_{2} \mathrm{~A}}\right|+\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|-\left|\overline{\mathrm{a}_{2} \mathrm{~B}_{2}}\right|}-\overline{\mathrm{C}_{1} \mathrm{~A}} \cdot \ln \frac{\left|\overline{\mathrm{a}_{1} \mathrm{~A}}\right|+\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|+\left|\overline{\mathrm{a}_{1} \mathrm{~b}_{1}}\right|}{\left|\overline{\mathrm{a}_{1} \mathrm{~A}}\right|+\left|\overline{\mathrm{b}_{1} \mathrm{~A}}\right|-\left|\overline{\mathrm{a}_{1} \mathrm{~B}_{1}}\right|}\right)= \\
& =\overline{C_{1} B} \cdot \ln \frac{\left|\overline{\mathrm{~B}_{2} B}\right|-\overline{C_{2} B} \cdot \cos \phi-\overline{C_{2} b_{2}}}{\left|\overline{a_{1} B}\right|-\overline{C_{1} B} \cdot \cos \phi+\overline{C_{1} a_{1}}}+\overline{C_{1} A} \cdot \ln \frac{\left|\overline{a_{1} A}\right|-\overline{C_{1} A} \cdot \cos \phi+\overline{C_{1} a_{1}}}{\left|\overline{b_{2} A}\right|-\overline{C_{2} A} \cdot \cos \phi-\overline{c_{2} b_{2}}}+ \\
& +\overline{C_{2} C_{1}} \cdot \ln \frac{\left|\overline{b_{2} \mathrm{~B}}\right|-\overline{\mathrm{C}_{2} \mathrm{~B}} \cdot \cos \phi-\overline{\mathrm{c}_{2} \mathrm{~b}_{2}}}{\left|\overline{\mathrm{~b}_{2} \mathrm{~A}}\right|-\mathrm{C}_{2} \mathrm{~A} \cdot \cos \phi-\overline{\mathrm{c}_{2} \mathrm{~b}_{2}}} \tag{A.5}
\end{align*}
$$

$\left.+\overline{C_{2} C_{1}} \cdot \ln \frac{\left|\overline{b_{2} B}\right|-\overline{C_{2} B} \cdot \cos \phi-\overline{c_{2} b_{2}}}{\left|\overline{b_{2} \mathrm{~A}}\right|-\overline{C_{2} A_{2}} \cdot \cos \phi-\overline{\mathrm{C}_{2} \mathrm{~b}_{2}}}\right\}-\left|\overline{\mathrm{C}_{1} \mathrm{c}_{1}}\right| \cdot \operatorname{ctg} \phi \cdot\left\{\operatorname{arctg} \frac{\overline{\mathrm{C}_{1} \mathrm{~B}} \cdot \sin \phi}{\left|\overline{\mathrm{C}_{1} \mathrm{C}_{1}}\right|}-\right.$
$-\operatorname{arctg} \frac{\overline{C_{1} A} \cdot \sin \phi}{\left|\overline{C_{1} C_{1}}\right|}-\operatorname{arctg}\left(\frac{\left|\overline{C_{1} C_{1}}\right|}{\left|\overline{a_{1} B}\right|} \cdot \operatorname{ctg} \phi+\frac{\overline{c_{1} a_{1}} \cdot \overline{C_{1} B} \cdot \sin \phi}{\left|\overline{C_{1} C_{1}}\right| \cdot\left|\overline{a_{1} B}\right|}\right)+$
$\left.+\operatorname{arctg}\left(\frac{\left|\overline{C_{1} C_{1}}\right| \cdot \operatorname{ctg} \phi}{\left|\overline{a_{1} A}\right|}+\frac{\overline{c_{1} a_{1}} \cdot \overline{C_{1} A} \cdot \sin \phi}{\left|\overline{C_{1} C_{1}}\right| \cdot\left|\overline{a_{1} A}\right|}\right)\right\}+\left|\overline{C_{2} C_{2}}\right| \cdot \operatorname{ctg} \phi \cdot\left\{\operatorname{arctg} \frac{\overline{C_{2} B} \cdot \sin \phi}{\left|\overline{C_{2} C_{2}}\right|}-\right.$
$-\operatorname{arctg} \frac{\overline{C_{2} A} \cdot \sin \phi}{\left|\overline{C_{2} C_{2}}\right|}-\operatorname{arctg}\left(\frac{\left|\overline{C_{2} C_{2}}\right|}{\left|\overline{b_{2} B}\right|} \cdot \operatorname{ctg} \phi-\frac{\overline{C_{2} b_{2}} \cdot \overline{C_{2} B} \cdot \sin \phi}{\left|\overline{C_{2} C_{2}}\right| \cdot\left|\overline{b_{2} B}\right|}\right)+$
$\left.+\operatorname{arctg}\left(\frac{\left|\overline{\mathrm{C}_{2} \mathrm{C}_{2}}\right| \cdot \operatorname{ctg} \phi}{\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|}-\frac{\overline{\mathrm{C}_{2} \mathrm{~B}_{2}} \cdot \overline{\mathrm{C}_{2} \mathrm{~A}} \cdot \sin \phi}{\left|\overline{\mathrm{C}_{2} \mathrm{C}_{2}}\right| \cdot\left|\overline{\mathrm{b}_{2} \mathrm{~A}}\right|}\right)\right\}$
which completes the proof.

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[^0]:    * Additive acceleration of Poisson type (see section 4.8 ) was used
    after every 4 iterations. Iteration parameters $\omega, \beta$ and $K$ are not

[^1]:    Top values represent the upper bounds and bottom values the lower bounds of the error.

    * for marks with about 50 nodes per slot (tooth); $* *$ if the values of linked flux are suitably smoothed; $* * *$ for eleven digits arithmetic; $* * * *$ depends largely on the instrunents used.

[^2]:    * Truncated Fourier series is just one possibility for the representation of $L_{m}=g(t)$. Polynomials, or piecewise polynomials (i.e. spline functions) could also be used for this approximation and it may well be that some other approximation would be more suitable than the Fourier series.

