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Weak singularities in general relativity

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ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

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WEAK SINGULARITIES IN GENERAL RELATIVITY

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This thesis is concerned with certain types of weak singularity in general relativity for which some geometrical concepts remain well defined at the singularity.

We review the use of holonomy to analyse quasi-regular singularities. We introduce a class of curvature singularities which we call idealised cosmic strings which may provide more general models for cosmic strings than quasi-regular singularities. We analyse these singularities using methods of holonomy and examine the curvature and geometry in their neighbourhoods.

In order to do this we prove a number of results about the behaviour and divergence of tensors in parallelly propagated frames and in pairs of frames related by bounded transformations. Making use of path-ordered exponentials of curvature we give conditions under which we prove that certain elements of holonomy exist even for a curvature singularity. We then present a $2+2$ formalism suited to analysing idealised cosmic strings and show how the geometry of the full connection is related to the geometry of a connection which we call the projected connection. We also apply these results to prove the existence of certain intrinsic and extrinsic holonomy groups which we define.

In addition we prove a number of results about conformal singularities and in particular that the 4-cone is not conformally regular and we examine the effect of conformal transformations on extrinsic curvature.

Finally we prove that coordinates may be found in which a metric has block diagonal form.

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Contents

Introduction

In this thesis we discuss certain "weak" singularities in general relativity from a geometrical viewpoint.

General relativity is described in terms of the differential geometry of a pair (M, g) where M is a manifold and g is a Lorentzian metric. (M, g) are assumed to be smooth, or at least C^2, and g is assumed to be non-singular everywhere on M. This makes it difficult to talk about a space-time being "singular" or "having a singularity".

The most familiar example of a singularity occurs in the Schwarzschild black hole solution. Here the curvature experienced along any timelike curve which crosses the event horizon grows unbounded within a finite proper time. We would like to say that the endpoint of the curve has hit a "singularity", a place perhaps of "infinite curvature", but of course this endpoint cannot be part of the manifold and it is more profitable to think of a singularity as a boundary or "edge" to space-time.

It used to be thought that singularities were an artefact of spherical symmetry and unlikely to occur in physically realistic situations, but theorems due to Penrose and Hawking [HE] show that singularities are in fact a generic feature of space-time. However these theorems do not give any information about the nature of these singularities and it is thus important to investigate singularities more carefully.

The Schwarzschild singularity is an example of a "strong curvature singularity" but there also exist certain types of "weak" singularity. For example a quasi-regular singularity is one where all the components of the Riemann tensor have well defined limits in any frame parallelly propagated along any C^1 curve ending at the singularity. Thus an observer would not notice anything unusual as they approached it. An intermediate singularity, on the other hand, is one for which there exists a non-parallelly propagated frame along a curve ending at the singularity in which the components of the Riemann tensor have well defined limits, despite the fact that in a parallelly propagated frame the curvature is badly behaved. Thus in particular, curvature scalars will be well behaved along such a curve.

So the behaviour of singularities can be subtle. Many examples of singularities can be constructed with unusual behaviours and there is no entirely satisfactory definition of a singularity. Maybe we would like to say that these weak singularities are somehow unphysical, but we are still interested in a mathematical way of handling them, maybe to "factor them out" of our investigations. However we shall see how weak singularities, and quasi-regular ones in particular, can be used as models for cosmic strings, thus here weak singularities have a direct physical interpretation.

Cosmic strings are thin tubes of "false vacuum" which arise from attempts to apply Grand Unified Theories to the early universe. They are usually modelled as 2-dimensional timelike worldsheets using weak field theory on a fixed background. These worldsheets turn out to be minimal and can bend on small length scales and form closed loops. Because of this, they have been proposed as a mechanism for the formation of structure and galaxies in the early universe.

However if we wish to take into account gravitational effects, it is necessary to solve the full coupled equations for a cosmic string. In the axisymmetric case this results in a space-time which outside the string (very nearly) has metric

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\theta^2 + dz^2 \qquad 0 \leq \theta < 2\pi \qquad A \neq 1.$$

This metric is called the *4-cone* and is the simplest example of a quasi-regular singularity.

We start in chapter 1 by reviewing and consolidating the necessary background material on singularities and quasi-regular singularities. We review how the 4-cone, and more generally how *elementary* quasi-regular singularities may be considered to be *totally geodesic*, despite not having a well defined normal metric. We review how *holonomy* may be used to study singularities. A frame parallelly propagated round a closed loop and back to its starting point undergoes a Lorentz transformation: if the loop is homotopic to a point, then this transformation will tend to the identity as the loop shrinks to a point, however if the loop encircles a singularity, then the transformation will not in general tend to the identity, rather it will tend to an element of a so-called singular holonomy group, and these groups will tell us about the structure of the singularity.

We review how quasi-regular singularities give rise to well defined singular holonomy groups. In the case of 2-dimensional timelike singularities (of which the 4-cone is an example), these groups will consist of rotations through multiples of a fixed angle $2\pi(1 - A)$ with the singularity as axis and, subject to mild conditions, be conserved along the singularity. This leads to the result that 2-dimensional quasi-regular singularities may be considered to be totally geodesic.

It turns out that 2-dimensional timelike quasi-regular singularities may provide suitable models for cosmic strings. In particular the conservation of holonomy leads to these singularities having the light bending properties we would expect of a cosmic string. The fact that they are totally geodesic means that they are minimal, which is consistent with the weak field approach, however requiring them to be totally geodesic is a stronger condition and suggests that cosmic strings are really quite inflexible objects. It is of interest to see whether they can nonetheless bend or form closed loops on small length scales. However a relationship can be derived between the size of loops which can form and the curvature of the ambient space-time [CEV]. In particular a cosmic string cannot bend on length scales smaller than the cosmological length scale. This remains true even if we model cosmic strings, not with quasi-regular singularities, but with a regular space-time where a region of high curvature is confined to a narrow tube.

Now a construction of a circular cosmic string of arbitrarily small radius is given in [FIU]. The construction is complicated and it is not obvious that, in a range of cases, it gives rise to a curvature singularity. This is shown in [UHIM], which describes a class of curvature singularities such as this one which are proposed as models for cosmic strings. The claim is made that singularities in this class are nonetheless totally geodesic. However the definition of this class is not particularly rigorous and a number of unnatural restrictive assumptions are made. We analyse this discussion in section 4.1, where we correct a number of claims, make the discussion more rigorous, and point out some results missing in the original discussion.

We then go on in section 4.2 to define a new class of "weak" curvature singularities using a more natural set of assumptions, which we call *idealised cosmic strings*. These are somehow worse than quasi-regular ones but remain weak enough so that certain concepts of differential geometry remain well defined at these singularities. The idea is that a space-time with such a singularity admits a preferred foliation of spacelike 2-surfaces normal to the singularity, each of which has a quasi-regular singularity in the induced 2-metric, while the singularity itself has a perfectly regular Lorentzian 2-metric. Thus the singularity has a well defined dimension and intrinsic geometry. Despite this, the singularity will in general be a curvature singularity. Because of this, they may be able to bend on smaller length scales, though they may have unexpected light bending properties. Some idealised cosmic strings may however be considered to be totally geodesic.

In chapter 5 we describe a $2+2$ formalism which is naturally suited to analysing idealised cosmic strings. In particular, we introduce a new connection called the *projected connection* which contains some, but not all, of the geometrical information of the full space-time connection, and discuss the properties of the projected connection and its curvature, and show how they relate to the properties of the full space-time connection and curvature. We then apply methods of holonomy to study idealised cosmic strings.

However the full holonomy groups will not in general exist for a curvature singularity. We therefore define the *intrinsic holonomy groups*, generated by parallelly propagating frames along loops restricted to lie in the preferred spacelike 2-surfaces with respect to the projected connection, and the *extrinsic holonomy groups*, generated by parallelly propagating frames along loops restricted to lie in the preferred spacelike 2-surfaces with respect to the full connection. We exhibit conditions under which the intrinsic holonomy groups exist and are conjugate to the extrinsic holonomy groups. Thus we use the projected connection as a means of proving results about the full connection. We then exhibit conditions under which the intrinsic and extrinsic holonomy groups are conserved along the singularity.

Chapter 2 is concerned with developing the tools necessary for this. In section 2.1, we show how the result of parallelly propagating a basis along a curve may be expressed as a *path-ordered exponential* of an integral of the connection. In section 2.2 we define an *ω-frame* to be a basis parallelly propagated along a curve with respect to a connection ω and prove a number of results about the behaviour of tensors in such frames. In particular we show how the rate of divergence of a tensor may be well defined, so that statements like "the curvature diverges as $1/r$" make sense. The point of this is that in order to examine the behaviour of the curvature and other tensors along a curve terminating at a singularity, it is not meaningful to examine the components of these tensors in a coordinate basis, since these are not covariant. Rather it is necessary to examine the components of these tensors in a basis parallelly propagated along the curve. We shall also need to express holonomy in terms of path-ordered exponentials of integrals of the connection.

Now we wish to relate the behaviours of the projected and full connections. Therefore in section 2.3, we define an equivalence relation on connections $\omega \sim \overline{\omega}$ if the Lorentz transformation relating any ω-frame with any $\overline{\omega}$-frame has a well defined limit along a curve terminating at a singularity. We prove that if $\omega \sim \overline{\omega}$, then the components of a tensor will be bounded in an ω-frame if and only if they are bounded in a $\overline{\omega}$-frame, and that if they diverge, then they will diverge at the same rate in both frames. We also prove that a curve has finite *b-length* with respect to ω if and only if it has finite b-length with respect to $\overline{\omega}$. Now $\omega, \overline{\omega}$ are not tensors, but the *connection difference* $\sigma = \overline{\omega} - \omega$ is a tensor and we demonstrate conditions on σ which yield $\omega \sim \overline{\omega}$.

In section 2.4, we use the first and second Cartan equations to examine the difference in torsions and curvatures of two connections for which $\omega \sim \overline{\omega}$.

In section 2.5, we show how holonomy may also be expressed as a path-ordered exponential of an integral of the curvature and prove conditions under which certain elements of holonomy exist even for a curvature singularity. We make use of this in chapter 5 to prove the existence of the intrinsic holonomy groups.

In chapter 6, after introducing a form of the Gauss-Codazzi-Ricci equations which relate the curvature of the projected connection with the curvature of the full connection, we examine the various components of the curvature of an idealised cosmic string and show which components converge and which diverge. We also give conditions under which the string can be said to be totally geodesic and discuss some consequences of this. In section 6.3 we give some examples of idealised cosmic strings and examine their behaviours. In section 6.4 we give a proof that a metric may be *block diagonalised*, that is, coordinates may be chosen such that the metric has form

$$g = \begin{pmatrix} g^{\perp} & 0 \\ 0 & g^{\|} \end{pmatrix}$$

where $g^{\perp}$, $g^{\|}$ are 2×2 metrics on orthogonal families of 2-surfaces. Such a coordinate system would be a very natural one in which to discuss idealised cosmic strings.

Chapter 3 is concerned with a different kind of "weak" singularity. A *conformal singularity* is one which may be removed by applying a conformal transformation $g \mapsto \Omega^2 g$ to a space-time (M, g). In other words, if (M, g) is singular but $(M, \Omega^2 g)$ is regular, (M, g) contains a conformal singularity, and this singularity can in some sense be said to be mathematically tractable. However we wish to avoid mapping the singularity away to infinity and we use the results of chapter 2 to examine conditions under which curves b-incomplete with respect to g are mapped to curves b-incomplete with respect to $\overline{g}$ and vice versa. We briefly discuss the consequences of this for the Weyl Curvature Hypothesis.

In section 3.3 we describe the conformal Cartan connection, which is a connection, not on a space-time, but on the bundle of *conformal connections* on a space-time (M, g), which includes all the connections of all metrics conformally related to g. We use this connection to prove that any vacuum space-time with non-trivial singular holonomy cannot be conformally regular, and in particular that the 4-cone cannot be conformally regular. We give a simpler proof of this in section 3.4.

In section 3.5 we examine the effect of conformal transformations on the extrinsic curvatures of a submanifold. We prove that a conformal transformation of the metric can always make the trace of the extrinsic curvature, known as the mean curvature, zero, but that the trace free part of the extrinsic curvature, known as the umbilical curvature, is a conformal invariant.

Chapter 1

Singularities and quasi-regular singularities

1.1 Singular incompleteness

Let (M, g) be a space-time. Since, as we discussed in the introduction, singularities are not part of the space-time, we have to describe their properties in terms of the non-singular (M, g). We wish to examine whether (M, g) is "complete" or whether it admits some kind of "singular boundary" ([ES], [HE] and [TCE]).

Suppose g is positive definite. Then if M is (path) connected, any two distinct points $a, b \in M$ can be connected by a C^1 curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = a$ and $\gamma(1) = b$, and any such curve will have a strictly positive and finite length. Define $d(a, b)$ to be the infimum of the lengths of all such curves from a to b. Then (M, d) is a metric space. A sequence (x_n) in M is a Cauchy sequence if given $\varepsilon > 0 \quad \exists N_\varepsilon$ such that

$$d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N_\varepsilon .$$

A metric space is *complete* if all Cauchy sequences converge to points of the space. If all Cauchy sequences in (M, g) converge to points of M, then (M, g) is said to be *Cauchy complete* or *m-complete*.

m-incompleteness corresponds to the idea that there are points missing from M, or that there exists $\varepsilon > 0$ and a C^1 parametrised curve $\gamma : [0, \varepsilon) \to M$ such that any infinite sequence of points of $Im\, \gamma$ with parameter values accumulating at ε is Cauchy incomplete. In other words, γ cannot be extended any further (in the direction of increasing parameter value) despite having finite length. We think of γ as having "reached the edge of the manifold".

This leads to the idea of *geodesic completeness* or *g-completeness*. A manifold is g-complete if all geodesics extend to infinite parameter value in both directions.

For a positive definite metric, g- and m-completeness are equivalent [KN], and no curve of finite length can ever leave the manifold. For a metric of Lorentzian signature, m-completeness does not make sense: there is no obvious, natural positive definite metric on (M, g). g-completeness does however make sense. Recall that an affinely parametrised geodesic is a C^2 curve $\gamma : s \mapsto \gamma(s) \in M$ with $u^i = d\gamma^i(s)/ds$ and $u^i \nabla_i u^j = 0$. u^i remains one of timelike, null or spacelike for all parameter values. If a geodesic cannot be extended beyond a finite parameter value in a given direction then it is incomplete, and if it is incomplete in one affine parameter, then it is incomplete in all affine parameters. For a timelike geodesic, proper time is an affine parameter. Assuming our manifold to be time orientable, an incomplete timelike geodesic can be future incomplete, past incomplete, or both. A physical object travelling along such a geodesic will leave the manifold in a finite proper time, or will have entered the manifold a finite proper time ago, or both. Proper distance along spacelike geodesics is an affine parameter, but the physical significance of spacelike geodesic incompleteness is less clear. Null geodesic incompleteness is probably important given that light is assumed to travel along null geodesics, however in this case the meaning of an affine parameter is not clear. It is important though to distinguish between the three types of geodesic incompleteness since examples can be constructed which exhibit any one of the three, but not the other two.

g-completeness is not however enough for a Lorentzian manifold. Geroch [G] gives an example of a space-time which is geodesically complete despite the existence of inextendible curves of bounded acceleration on which only a finite proper time elapses to the future of any point. In other words, a rocket-ship with only a finite amount of fuel could traverse such a curve in finite proper time. Going further we can call a space-time (M, g) *timelike incomplete* if there exists an inextendible C^1 timelike curve of bounded acceleration—not necessarily a geodesic—which is future or past incomplete, i.e. which continues to the future or the past for a finite proper time.

We can also consider b-completeness. Given a C^1 curve $\gamma : s \mapsto \gamma(s)$ through $x \in M$, let (e_i) be a basis for the tangent space T_x at x. Now parallelly propagate (e_i) along γ to give a basis for $T_{\gamma(s)}$ for each s. We can express the tangent vector V of γ in terms of this basis

$$V = V^i(s)e_i.$$

Then define $u = \int_x (\sum_i V^i V^i)^{1/2}\, dt$. u is called a *generalised affine parameter* (g.a.p.) along γ and depends on the point x and the frame (e_i) at x. It can be shown that if a curve cannot be extended beyond finite parameter value in one g.a.p., then this holds for all g.a.p. and we say the curve is incomplete with respect to g.a.p. If there are no such curves in M, then (M, g) is *b-complete*. We note that if γ above is a geodesic then u is an affine parameter, thus b-completeness implies g-completeness, but the converse is not true unless g is positive definite.

b-completeness may be too strong a requirement and we could say that a space-time is singularity free if it is non-spacelike b-complete. In particular timelike b-completeness is equivalent to timelike completeness as defined above.

1.2 Singular boundaries

Corresponding to a space-time (M, g) which is incomplete with respect to some definition, we have a class $\mathcal{C}$ of inextendible incomplete parametrised curves. We will assume each curve starts at a point of M and so is inextendibly incomplete in one direction only. We would like to know whether two curves $\gamma_1, \gamma_2 \in \mathcal{C}$ have the same, or distinct, "singular endpoints". So we want an equivalence relation $\sim$ on $\mathcal{C}$ such that $\gamma_1 \sim \gamma_2$ means γ_1 and γ_2 end at the same "singular points" according to some suitable definition. This allows us to form the quotient space $\mathcal{C}/\sim$, the elements of which are the distinct equivalence classes of $\sim$, so each point of $\mathcal{C}/\sim$ represents a different singular point. Then given a topology on $\mathcal{C}$, we obtain a topology for the singularity.

More generally we would like a map

$$\theta : M \rightarrow \overline{M}$$

where $\overline{M}$ is a manifold and $\theta : M \rightarrow Im\,\theta$ is a diffeomorphism such that the closure of $Im\,\theta$ in $\overline{M}$ is $\overline{M}$, and such that $\partial M = \overline{M} - Im\,\theta$ somehow represents the singularity. This makes clear the notion of a singular boundary. We emphasize though, that given a topology on ∂M, the differential structure of $\overline{M}$ is highly non-unique.

One construction is the *b-boundary*. Here we define a positive definite metric e, not on (M, g), but on LM, the bundle of orthonormal frames on M. Above each point $x \in M$, there is a fibre $\pi^{-1}(x)$ consisting of all orthonormal frames at x and diffeomorphic to the Lorentz group L where

$$\pi : LM \rightarrow M.$$

It turns out that (LM, e) is m-complete if and only if (M, g) is b-complete ([S71]). We can extend π to a map $\overline{\pi}$ on $\overline{LM}$, the Cauchy completion of LM. Then we can form the quotient $\overline{M}$ of $\overline{LM}$ by $\overline{\pi}$: $\overline{M}$ is a topological space with

$$\overline{M} = \hat{M} \cup \partial M$$

where $\hat{M}$ is homeomorphic to M and ∂M is our singular boundary. However in general $\overline{M}$ will not be Hausdorff and will not have the structure of a manifold. In fact, if $x \in \partial M$, $\overline{\pi}^{-1}(x)$ will be homeomorphic to the manifold L/G where G is some subgroup of L defined up to conjugacy, and $\overline{\pi}^{-1}(x)$ may turn out to be just one point. Moreover in the $k = 1$ (closed) Robinson-Walker space-time, points of the past singularity are identified with points of the future singularity. It is also hard to compute the b-boundary.

Another construction is the *c-boundary*. Here we take $\mathcal{C}$ to be all b-incomplete timelike (or possibly non-spacelike) C^1 curves. Then

$$\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$$

where $\mathcal{C}^+$ is the class of future incomplete curves and $\mathcal{C}^-$ the class of past incomplete curves. If $\gamma_1, \gamma_2 \in \mathcal{C}^+$ then $\gamma_1 \sim \gamma_2 \iff I^-(\gamma_1) = I^-(\gamma_2)$ and if $\gamma_1, \gamma_2 \in \mathcal{C}^-$ then $\gamma_1 \sim \gamma_2 \iff I^+(\gamma_1) = I^+(\gamma_2)$ where $I^-(\gamma)$ (or $I^+(\gamma)$) is the set of points in M connected to any point of $Im\,\gamma$ by a future (or past) oriented C^1 timelike curve.

This expresses the concept that for two curves to terminate at the same point they must remain in causal contact until they reach their endpoints: the causal structure is a very natural property of the space-time. As defined, the c-boundary is equivalent to the construction of TIPs and TIFs given by [GKP], except that their boundary also includes points at infinity.

There are problems, however, in how to identify points of $\mathcal{C}^+/\sim$ with points of $\mathcal{C}^-/\sim$, and there may be cases where we would like to say $\gamma_1 \sim \gamma_2$, but where there are obstructions preventing γ_1 and γ_2 from being in causal contact.

1.3 Classification of singularities

We now consider what goes wrong at a singularity. An incomplete curve $\gamma \in \mathcal{C}$ could arise simply because our manifold is not "big" enough, that is, it does not contain a whole space-time. Suppose that the space-time (M, g) is C^p and that there exists a C^q isometry of (M, g) into a C^q space-time (M', g') for some $q \leq p$ such that there exists an extension of γ into the interior of $M' - M$. If the Riemann tensor of (M', g') is C^r (where $r \leq q$—for example, if (M', g') is C^2 then the Riemann tensor will be C^0) then (M', g') is a C^r extension of (M, g) and γ is said to terminate at a C^r *regular boundary point*.

Alternatively, suppose we pick $x \in Im\,\gamma$ and a frame (e_i) at x and parallelly propagate this frame to the endpoint of γ. Let $\gamma : s \mapsto \gamma(s)$ and $s < s_0$. We can then examine the behaviour of the Riemann tensor in this frame as $s \to s_0$. γ is said to terminate at a C^r (or C^{r-}) *curvature singularity* if there is a component of curvature $R_{ijk}{}^{l}{}_{;u_1 \dots u_r}$ which does not have a C^0 (or C^{0-}) limit as $s \to s_0$. Here ; denotes covariant differentiation and $u_1, \dots, u_r$ are vector fields defined along γ. This expresses the idea that the curvature "blows up" at a curvature singularity, and if this happens it follows that γ cannot terminate at a C^r regular boundary point—otherwise the curvature would have a well behaved limit along γ.

We can also look at the behaviour of curvature scalars along γ. A curvature scalar is a polynomial scalar field constructed from g_{ij} and $R_{ijk}{}^{l}{}_{;u_1 \dots u_r}$. If such a field does not have a C^0 (or C^{0-}) limit as $s \to s_0$ then γ is said to terminate at a C^r (or C^{r-}) *scalar singularity*. These are easier to test for, as scalars are the same in all coordinate systems and we need not worry about parallelly propagating a frame along γ. A C^r scalar singularity will be a C^r curvature singularity, but the converse need not be true.

However it is possible for the curvature along γ to be perfectly well behaved as $s \to s_0$ without γ terminating at a regular boundary point. γ is said to terminate at a C^r (or C^{r-}) *quasi-regular singularity* if all components $R_{ijk}{}^{l}{}_{;u_1 \dots u_r}$ have C^0 (or C^{0-}) limits as $s \to s_0$. The idea is that locally nothing goes wrong with the curvature. The singularity has a global nature and is somehow a topological "defect" of the space-time.

The prototype quasi-regular singularity is the conical singularity

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\theta^2 + dz^2 \quad 0 < \theta < 2\pi.$$

If we set $\tilde{\theta} = A\theta$ we obtain

$$ds^2 = -dt^2 + dr^2 + r^2 d\tilde{\theta}^2 + dz^2 \quad 0 < \theta < 2\pi A$$

which is locally isometric to Minkowski space, thus this metric is locally flat and its Riemann tensor vanishes. However for $A \neq 1$ there is a quasi-regular singularity which we can think of as the $r = 0$ 2-plane. In fact for $A < 1$ the metric can be obtained by taking Minkowski space, removing the wedge $2\pi A < \tilde{\theta} < 2\pi$, and identifying the edges of this wedge. This metric is named by analogy with the 2-metric

$$ds^2 = dr^2 + A^2 r^2 d\theta^2 \quad 0 < \theta < 2\pi$$

which is the metric of a cone. Again the metric has a quasi-regular singularity at $r = 0$, which corresponds to the vertex of the cone. It has an angular deficit of $2\pi(1 - A)$. The singularity causes geodesics to focus for $A < 1$, and to diverge for $A > 1$.

The 4-cone is an example of a "primeval" quasi-regular singularity, in that it has existed for all values of t. It is also, in ways to be made more precise later, an example of a 2-dimensional timelike quasi-regular singularity. However, examples can be constructed of timelike and spacelike quasi-regular singularities with dimensions 0 to 3.

The fact that locally nothing goes wrong with a quasi-regular singularity can be made precise with Clarke's local extension theorem [C73]:

Theorem 1.3.1. *Let γ be a C^1 curve ending at a C^0 (or C^{0-}) quasi-regular singularity. Then there exists an open $U \supset Im\,\gamma$ such that $(U, g_{|U})$ has a C^0 extension (U', g') in which γ has a regular endpoint.*

The point is that we cannot extend the whole of (M, g), only an open neighbourhood of γ.

For example, in the case of the 4-cone suppose $A < 1$ and consider the C^1 curve $\gamma(u) = (t_0, u, \theta_0, z_0)$ on which $r \to 0$ as $u \to 0$. Here $M = \mathbb{R}^4 - \{r = 0\}$ and the coordinate θ can be chosen so that $0 < \theta_0 < 2\pi$. Then $U = \{x \in M : 0 < \theta < 2\pi\}$ is an open neighbourhood of $Im\,\gamma$ which is isometric to a portion of Minkowski space $\{x \in M : 0 < \tilde{\theta} < 2\pi A\}$, though M itself does not admit a regular (i.e. at least C^0) extension.

For future reference we note that the 4-cone can be expressed in Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$ as

$$ds^2 = -dt^2 + \frac{(x^2 + A^2y^2)}{x^2 + y^2}dx^2 + \frac{2xy(1 - A^2)}{x^2 + y^2}dxdy + \frac{(y^2 + A^2x^2)}{x^2 + y^2}dy^2 + dz^2.$$

1.4 Elementary quasi-regular singularities

Given a space-time (M, g) which has a non-trivial group of isometries, there is an easy method due to Ellis and Schmidt [ES] (see also [V87]) which generates a space-time (M', g') with quasi-regular singularities. An *elementary* quasi-regular singularity is a singularity generated in this way. Not all quasi-regular singularities are elementary.

We proceed as follows. Take a non-trivial discrete subgroup G of the group of isometries of (M, g) and form the fixed point set $F = \{x \in M : gx = x \quad \forall g \in G\}$. Delete F from M to give $(M - F, g)$, on which G acts with no fixed points. Now identify points of $(M - F, g)$ related by G, thus forming the quotient space $(\tilde{M}, \tilde{g})$. Finally we may need to delete some points of $\tilde{M}$ to give (M', g') in order to make the resulting space Hausdorff: in particular this may be necessary if a sequence of points (x_n) of $M - F$ related by G have an accumulation point in the fixed point set F. For example, consider Minkowski space in Cartesian coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

and the group of isometries

$$G = \{\begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} \cosh(n\lambda) & \sinh(n\lambda) \\ \sinh(n\lambda) & \cosh(n\lambda) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} \mid n \in \mathbb{Z}\}$$

for some constant $\lambda \neq 0$. The fixed point set is the 2-plane

$$F = \{t = 0, x = 0\}.$$

Now let S^+, S^- be the null 3-surfaces

$$S^+ = \{t = x\} \qquad S^- = \{t = -x\}.$$

Given $x \in S^+$ (or S^-), the set of points Gx lies entirely in S^+ (or S^-) and has an accumulation point in F. The quotient space will not be Hausdorff. In order to obtain a Hausdorff space we need to delete the points in the quotient space which correspond to S^+, S^-.

We will be interested in cases where F is 2-dimensional and timelike in (M, g) and G has one generator, an isometry $f : M \to M$ such that $f^*g = g$. It is helpful to insert an additional step in the above procedure. Given $(M - F, g)$ we pass to the universal covering space $(\hat{M}, \hat{g})$, lift f to $\hat{f} : \hat{M} \to \hat{M}$, and then form the quotient space $(\tilde{M}, \tilde{g})$ by identifying points of $(\hat{M}, \hat{g})$ related by $\hat{f}$. We may still need to delete points of $\tilde{M}$ to give (M', g'). However the lift of f to $\hat{f}$ is not unique in that there will exist a family $\{f_i\}$ of isometries of $(\hat{M}, \hat{g})$, each one of which projects down to f, and we can use any one of the $\{f_i\}$ to give $(\tilde{M}, \tilde{g})$, each one in general giving a different quotient space.

For example, in the case of the 4-cone, we start with Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \quad 0 \leq \theta \leq 2\pi.$$

We then remove the $r = 0$ 2-plane and unwrap the resulting space-time to get its universal covering space: this has the same metric but now $-\infty < \theta < \infty$. Any isometry

$$\hat{f} : \theta \mapsto \theta + \alpha$$

acts on this space with no fixed points and will project down to an isometry of (M, g) with fixed point set $\{r = 0\}$. The space (M', g') will have a quasi-regular singularity unless $\alpha = \pm 2\pi$. In particular $\alpha = 2k\pi$ for $k \in \mathbb{Z} - \{-1, 1\}$ will result in a so-called "covering space singularity", although the corresponding f is the identity on (M, g).

An important property of the fixed point set F of an isometry f of a space-time (M, g) is that it is a *totally geodesic* submanifold of M. Given $x \in M$ and $u \in T_xM$, construct the unique affinely parametrised geodesic $\gamma_u(s)$ in M such that $\gamma_u(0) = x$ and γ_u has tangent u at x. This will exist for $s \in [0, \varepsilon)$ for some $\varepsilon > 0$ where ε depends on x and u. Then a submanifold S of M is totally geodesic if given $x \in S$ and $u \in T_xS$, $\exists \varepsilon > 0$ such that $\gamma_u(s) \in S \quad \forall s \in [0, \varepsilon)$. In other words, a geodesic of M initially tangent to S will remain in S.

If S is non-null (as defined in section 3.5), given $x \in S$ and $u \in T_xS$ we can also construct $\tilde{\gamma}_u(s)$ as above, except that we choose $\tilde{\gamma}_u(s)$ to be a geodesic of S (using the metric induced on S) rather than of M. $\gamma_u(s)$ and $\tilde{\gamma}_u(s)$ will always coincide if S is totally geodesic.

Given the isometry $f : M \to M$, we define its derivation at a point $x \in M$ by

$$D_xf : T_xM \to T_{f(x)}M : u \mapsto \left.\frac{d}{ds}f(\gamma_u(s))\right|_{s=0}.$$

In other words, given $x \in M$ and $u \in T_xM$, then f will map a C^1 curve through x with tangent u at x to a C^1 curve through $f(x)$ with tangent $D_xf(u)$ at $f(x)$.

Now let $x \in F$ so $f(x) = x$ and define $DF = \{u \in T_xM : D_xf(u) = u\}$, the fixed point set of the derived map.

From [V87] we have (though we present a slightly simplified version of the proof)

Theorem 1.4.1. *$DF = T_xF$ and F is totally geodesic.*

Proof. Suppose $u \in T_xF$ and let $\kappa_u(s)$ be some curve in F through x with tangent u at x. This lies in F so $f(\kappa_u(s)) = \kappa_u(s)$. The tangent of $\kappa_u(s)$ at x is mapped to itself. Hence $D_xf(u) = u$ and $u \in DF$.

Conversely suppose $u \in DF$ and consider $\gamma_u(s)$. $f(\gamma_u(s)) = \gamma_{D_xf(u)}(s) = \gamma_u(s)$ since f is an isometry and maps geodesics to geodesics. Hence $\gamma_u(s)$ lies in F and $u \in T_xF$. Hence $DF = T_xF$.

Now if $x \in F$ and $u \in T_xF$, by the above $\gamma_u(s)$ lies in F and hence F is totally geodesic. □

For $x \in F$, the tangent space T_xM becomes degenerate when we identify points under f. Tangent directions of $T_xM - DF$ become identified, which could not happen if x were a regular point of the new space-time. Somehow there are "not enough directions" at x. However T_xF remains well defined and thus we can regard F as totally geodesic even after making the identifications. In this sense, elementary quasi-regular singularities are totally geodesic.

We shall say a space-time $(M, \hat{g})$ has a *locally elementary* quasi-regular singularity if it can be expressed as $\hat{g} = g + \varepsilon$ where (M, g) has an elementary quasi-regular singularity and, for every C^1 curve γ of finite b-length terminating at the elementary quasi-regular singularity, given an open $U \supset Im\,\gamma$ such that $(U, g_{|U})$ has an extension (U', g') in which γ has a regular endpoint $x \in U'$, and given a coordinate patch $V \ni x$, then in V

$$\varepsilon_{ij} \to 0, \quad \varepsilon_{ij,k} \to 0, \quad \varepsilon_{ij,kl} \to 0 \qquad \text{as} \quad u \to u_0$$

where u is b-length measured along γ with supremum u_0. In particular if g is the 4-cone we shall call $(M, \hat{g})$ *locally conical.*

It can be shown that a locally elementary quasi-regular singularity is still quasi-regular, though it may not be elementary. Not all quasi-regular singularities are necessarily locally elementary.

1.5 Cosmic strings

Cosmic strings are objects which arise in a natural way in attempts to apply Grand Unified Theories to the early universe ([V92] and references therein, [I]). Such attempts are tentative but they suggest that the spontaneous symmetry breaking of a gauge field may result in topologically trapped regions of false vacuum. Specifically, as the universe cools below a critical temperature T_0, the potential $V_T(\phi)$ associated with a gauge field ϕ may develop more than one minimum. ϕ will generally assume one of these minima at each point, however causally disjoint regions of space may settle into different minima. Boundaries between these regions may be unable to assume a minimum value and will form narrow regions of false vacuum, or "topological defects" of the gauge field. 2-dimensional defects are called cosmic strings. They would be very thin objects, with almost all the field confined to a tube about 10^3 Planck units across. Though there is no observational evidence for their existence, they should have observational consequences such as gravitational lensing and they may also provide a mechanism for the formation of galaxies and other large scale structure in the early universe.

If we wish to study cosmic strings in general relativity, one approach is to look at weak field theory on a fixed background (M, g) and to represent a thin cosmic string by a 2-dimensional worldsheet S. Given $x_0 \in S$, we can find a coordinate patch U of S such that $x_0 \in U$ and a coordinate patch V of M such that $U \subset V$. Let U have coordinates (u, v) and let the corresponding coordinates in V be $(x^i(u, v))$. It turns out that the behaviour of U is given by the Nambu action

$$I = -2\pi\mu \int\int_U (-\frac{1}{2}F^{ij}F_{ij})^{1/2}\, du dv$$

where μ is the linear density of the string and

$$F^{ij} = \frac{dx^i}{du}\frac{dx^j}{dv} - \frac{dx^j}{du}\frac{dx^i}{dv}.$$

If $X^i = dx^i/du$, $Y^j = dx^j/dv$ then (X^i, Y^j) is a basis of T_xS for each $x \in U$ and

$$\begin{aligned}\frac{1}{2}F^{ij}F_{ij} &= \frac{1}{2}g_{ik}g_{jl}(X^iY^j - X^jY^i)(X^kY^l - X^lY^k)\\ &= g(X,X)g(Y,Y) - g(X,Y)g(X,Y)\end{aligned}$$

which is the determinant of the metric induced on S by restricting g to S. Thus

$$A = \int\int_U (-\frac{1}{2}F^{ij}F_{ij})^{1/2}\, dudv$$

measures the surface area of U. Requiring I to be extremal for all $x_0 \in S$ gives the condition that S must be a minimal surface in the space-time [Ch]. Also S can be given energy-momentum tensor densities $\mathcal{T}_i^j = T_i^j\sqrt{-g}$

$$\mathcal{T}_0{}^0 = \mathcal{T}_3{}^3 = 2\pi\mu\delta_2$$

where ∂_0 and ∂_3 are tangent to S and δ_2 is a 2-dimensional delta function with support on S. We require $\nabla_i T^{ij} = 0$ but this merely gives the condition that S should be minimal again.

However this approach ignores the gravitational effects of the string. We cannot, though, necessarily expect to solve Einstein's equations for a delta function valued energy-momentum tensor. Solving the full coupled equations for the metric and gauge field in the axisymmetric case results in a space-time which is (very nearly) locally flat and conical outside the string. Thus the 4-cone

$$ds^2 = -dt^2 + dr^2 + A^2r^2d\theta^2 + dz^2 \qquad \text{where} \quad A = 1/(1+4\mu)$$

can be used as an idealised model of a thin cosmic string. For a real string $1 - A \approx 10^{-6}$, where we recall that the 4-cone has angular deficit $2\pi(1-A)$.

The 4-cone can be regarded as being the limit of a sequence of space-times with a source consisting of stressed filaments lying along a tube parallel to the z-axis. In the limit that the diameter of the tube tends to zero, the energy-momentum tensor becomes $\mathcal{T}_0{}^0 = \mathcal{T}_3{}^3 = 2\pi\mu\delta_2$ as above. However there appear to be problems with trying to model singularities with distributional valued curvature and in taking a singular space-time to be the limit of non-singular ones ([GT]).

We will show later that 2-dimensional timelike quasi-regular singularities may provide suitable idealised models of cosmic strings in a curved space-time. In particular, they can be given the same energy-momentum tensor as the 4-cone. We will also show that they are totally geodesic. This implies that they are minimal which is consistent with the Nambu action. Requiring them to be totally geodesic is a stronger condition and suggests that cosmic strings are really quite inflexible objects. It is of interest to see whether they can nonetheless bend or form closed loops on small length scales. However a relationship can be derived between the size of loops which can form and the curvature of the ambient space-time [CEV]. In particular a cosmic string cannot bend on length scales smaller than the cosmological length scale. This remains true even if we model cosmic strings, not with quasi-regular singularities, but with a regular space-time where a region of high curvature is confined to a narrow tube.

One way out of this would be to try to find a class of "weak" curvature singularities, somehow worse than quasi-regular ones but which remain weak enough to have nice properties, to model cosmic strings. We propose to describe such a class. These singularities may still be totally geodesic, though they may appear to bend on small length scales. In fact it could be said that cosmic strings never bend: they only bend the space around them. In particular we note a construction of a circular cosmic string given in [FIU] which turns out, in a range of cases, to be a totally geodesic curvature singularity, as shown—though perhaps not entirely satisfactorily—in [UHIM].

We note that there exist metrics describing two cosmic strings moving relative to each other, as well as metrics describing an arbitrary number of non-parallel cosmic strings moving relative to each other [LG]. A metric describing a spinning cosmic string

$$ds^2 = -(dt + 4Jd\theta)^2 + dr^2 + A^2r^2d\theta^2 + dz^2,$$

where J is a constant, is given in [M]. Like the conical metric, this metric describes a vacuum space-time with a quasi-regular singularity at $r = 0$. An interesting feature for $J \neq 0$ is the presence of closed timelike curves. There also exist metrics describing global cosmic strings, which arise from the breaking of a global gauge symmetry, and superconducting cosmic strings [R].

1.6 Holonomy of quasi-regular singularities

Let (M, g) be a space-time and let $x \in M$. A *frame* at x is a basis (e_i) of T_xM. It is *pseudo-orthonormal* if $g(e_i, e_j) = \eta_{ij}$ where $\eta_{ij} = \text{diag}(-1, 1, 1, 1)$. In the following we will take all frames to be oriented, time-oriented pseudo-orthonormal frames.

Let γ be a C^1 curve

$$\gamma : [0, 1] \to M \qquad \gamma(0) = x$$

and let (e_i) be a frame at x. Define $e_i(s)$ by parallelly propagating each e_i along γ to $\gamma(s)$. Then $(e_i(s))$ will still be pseudo-orthonormal because parallel propagation preserves the inner product. In particular, if $\gamma(0) = \gamma(1)$ then γ is a closed loop and $(e_i(0))$ and $(e_i(1))$ will be defined at the same point $x \in M$ but will in general be different. There will be a Lorentz transformation L_i^j such that

$$e_i(1) = L_i^j e_j(0).$$

L_i^j is called an *element of holonomy* generated by the closed loop γ ([V92] and [V90], see also [N]).

More generally we consider the frame bundle (LM, π) where

$$\pi : LM \to M$$

and each fibre $\pi^{-1}(x)$ consists of all (oriented time-oriented pseudo-orthonormal) frames at x. The oriented, time-oriented Lorentz group $L_+^\uparrow$ acts on each fibre transitively and freely (the identity is the only element of $L_+^\uparrow$ which fixes any point of $\pi^{-1}(x)$) and in fact each fibre is homeomorphic to $L_+^\uparrow$. This makes LM a principal fibre bundle.

Given the closed loop γ and a frame (e_i) at $\gamma(0)$ we can lift γ to a curve $\overline{\gamma}$ in LM such that

$$\overline{\gamma} : [0,1] \to LM \qquad \overline{\gamma} : s \mapsto (\gamma(s), (e_i(s)))$$

where $e_i(s)$ is obtained by parallelly propagating e_i round γ to $\gamma(s)$. $\overline{\gamma}$ is called a *horizontal lift* of γ. Thus $\overline{\gamma}(0)$ and $\overline{\gamma}(1)$ are points on the same fibre related by some $g \in L_+^\uparrow$. If γ is homotopic to x, then as γ shrinks to x, g will tend to the identity.

Holonomy is a useful tool in the study of singularities. Loosely speaking, as a loop encircling a singularity shrinks to a point on the singularity, the holonomy generated will not in general tend to the identity, rather it will tend to an element of a so-called singular holonomy group, and these groups will tell us about the structure of the singularity.

Let $\kappa : (0,1] \to M$ be a C^1 curve of finite b-length terminating at a quasi-regular singularity. Pick a frame (e_i) at $\kappa(1)$ and lift κ to give $\overline{\kappa} : (0,1] \to LM$ by parallelly propagating (e_i) along κ. Define the *loop space* Ω_κ of κ to be the set of all C^1 maps of the form (see diagram 1.6.1) $\gamma : [0,1] \times (0,1] \to M : (s,u) \mapsto \gamma(s,u) = \gamma_u(s)$ such that

(a) $\gamma(0,u) = \gamma(1,u) = \kappa(u)$

(b) the b-length of γ_u, measured in the frame $\overline{\kappa}(u)$ parallelly propagated round γ_u, $\to 0$ as $u \to 0$.

We call the elements of Ω_κ *lassos*. If we lift γ_u to $\overline{\gamma}_u$ by parallelly propagating $\overline{\kappa}(u)$ round γ_u, (b) is equivalent to

(b)′ the b-length of $\overline{\gamma}_u$, $l(\overline{\gamma}_u) \to 0$ as $u \to 0$

which is independent of the initial choice of frame (e_i). $\overline{\gamma}_u(0)$ and $\overline{\gamma}_u(1)$ are frames defined at the same point of M but will in general be different so for some $L(\overline{\gamma}_u) \in L^{\uparrow}_{+}$

$$\overline{\gamma}_u(1) = L(\overline{\gamma}_u)\overline{\gamma}_u(0).$$

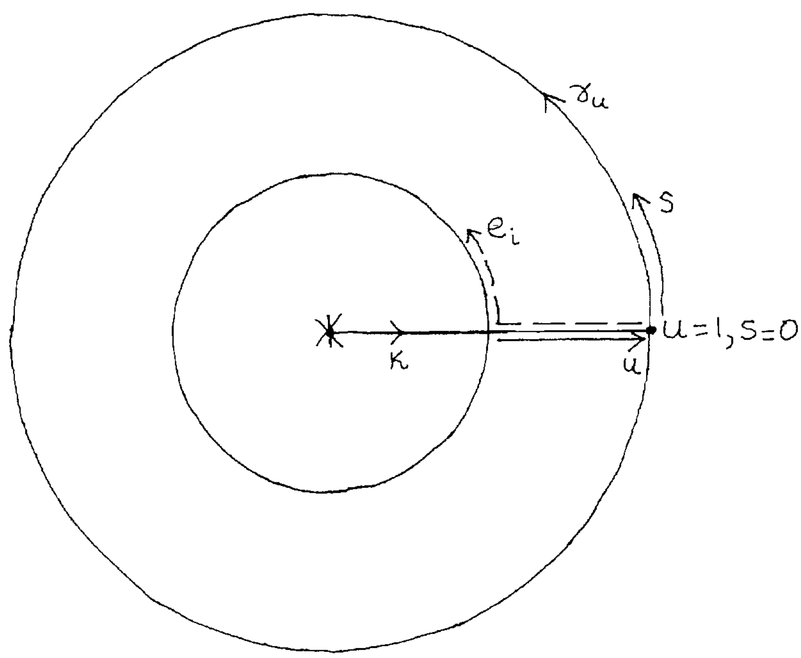

Diagram 1.6.1

We will say that $\overline{\gamma}$ satisfies the *area condition* if the area of $\overline{\gamma}([0,1],[u_0,u_1]) \to 0$ as $u_0, u_1 \to 0$ where we measure area with respect to the positive definite metric on LM used to construct the b-boundary.

The following theorem is quoted in [V90] and [V92] without the area condition.

Theorem 1.6.1. *If $\overline{\gamma}$ satisfies the area condition then $L(\overline{\gamma}_u)$ tends to some well defined limit $L(\overline{\gamma})$ as $u \longrightarrow 0$.*

Suppose that $\overline{\gamma}$ satisfies the area condition. Note that the b-distance between $\overline{\gamma}_u(0)$ and $\overline{\gamma}_u(1)$ tends to zero and so $\overline{\gamma}_u(0)$ and $\overline{\gamma}_u(1)$ are in fact tending to the same point p on the b-boundary $\overline{LM} - LM$, though $L(\overline{\gamma})$ need not be the identity. The point is, $L_+^\uparrow$ does not in general act freely on $\overline{\pi}^{-1}(x)$, where $x = \overline{\pi}(p)$, and p will be a fixed point of $L(\overline{\gamma})$. The subgroup of elements of $L_+^\uparrow$ which fix p is called the *isotropy subgroup* G_p of p. In fact $\overline{\pi}^{-1}(x)$ is homeomorphic to the manifold $L_+^\uparrow/G_p$ where G_p is defined up to conjugacy. Note that G_p need not be a normal subgroup of $L_+^\uparrow$.

If the area condition is satisfied for a particular lift of γ obtained from a horizontal lift of κ then it will be satisfied for all lifts of γ obtained from horizontal lifts of κ. We therefore let

$$\Omega_\kappa^A = \{\gamma \in \Omega_\kappa \mid \text{lifts of } \gamma \text{ obtained from horizontal lifts of } \kappa$$
$$\text{satsify the area condition}\}.$$

There is a natural group structure on Ω_κ. Given $\gamma, \delta \in \Omega_\kappa$ let

$$(\gamma * \delta)_u(s) = \begin{cases} \delta_u(2s) & 0 \le s < \frac{1}{2} \\ \gamma_u(2s-1) & \frac{1}{2} \le s < 1. \end{cases}$$

We claim that Ω_κ^A is non-empty and a subgroup of Ω_κ. Let $\gamma, \delta \in \Omega_\kappa^A$. Given a frame (e_i) at $\kappa(0)$ we see that $L(\overline{\gamma * \delta}) = L(\overline{\gamma})L(\overline{\delta})$. Thus the set of Lorentz transformations generated by Ω_κ^A and a frame (e_i) at $\kappa(0)$ form a group $H_{\overline{\kappa}}$, called the *singular* or *s-holonomy* group. If we start with a different frame $(e'_i) = (L_i^j e_j)$ then $H_{\overline{\kappa}'} = L^{-1} H_{\overline{\kappa}} L$.

From [V90] we have (see also [C78])

Theorem 1.6.2. *Let κ be as above and $\overline{\kappa}$ a horizontal lift of κ. Suppose $\overline{\kappa}$ terminates at $p \in \overline{LM} - LM$. Then $H_{\overline{\kappa}} = G_p$.*

Note that $\overline{\kappa}$ has finite b-length if and only if κ has finite b-length, and in this case it follows that $\overline{\kappa}$ must have a well defined limit point p. So $H_{\overline{\kappa}}$ depends only on p and encodes information about the singularity.

With κ as above we know that there exists an open $U \supset Im\,\kappa$ such that $(U, g_{|U})$ has an extension (U', g') in which γ has a regular endpoint $x \in U'$. We say κ terminates at a *good* quasi-regular singularity if U can be chosen to be "wedge" shaped rather than "cusp" shaped, or in other words, if U can be chosen so that the tangent directions at x which point to the interior of U form an open set. Such a wedge shaped U can be found for most quasi-regular singularities of interest, however there exist quasi-regular singularities which are accumulation points of a sequence of quasi-regular singularities for which it is not clear if this can be done.

Now consider the above homeomorphism $\overline{\pi}^{-1}(x) \simeq L_+^\uparrow/G_p$. If $p_1, p_2 \in L_+^\uparrow$ are in the same equivalence class then the two frames p_1 and p_2 have become identified at the b-boundary. In other words the tangent space at a point of a quasi-regular singularity is degenerate. It turns out however that it is degenerate only in directions which are, in some limiting sense, not tangent to the singularity and that therefore vectors tangent to a quasi-regular singularity are well defined. This in turn means that such a singularity has a well defined dimension and induced metric. Specifically, it can be shown that the elements of $H_{\overline{\kappa}}$ leave the components of vectors tangent to a quasi-regular singularity unchanged.

From [V90] we have

Theorem 1.6.3. *Let $f : [0,1] \times (0,1] \to M$ be a C^1 map such that $\kappa_s : u \mapsto f(s,u)$ terminates at a good quasi-regular singularity, thus $f(s,0)$ will be a curve along the singularity. Let $\kappa = \kappa_0$ (though f need not be in Ω_κ) and $\overline{\kappa}$ a horizontal lift of κ. Let*

$X^i(u)$ be the components of $X(u) = f_u{}'(0)$ in the frame $\overline{\kappa}(u)$ where $f_u(s) = f(s, u)$. Let $X^i = \lim_{u \to 0} X^i(u)$. Then $\forall L_i^j \in H_{\overline{\kappa}}$, $L_i^j X^i = X^j$.

In the case that the quasi-regular singularity is 2-dimensional and timelike, which we hope would make it a suitable model for a cosmic string, it follows that the elements of $H_{\overline{\kappa}}$, which preserve vectors tangent to the singularity, must be rotations with axis tangent to the singularity and it can be shown that $H_{\overline{\kappa}}$ is generated by the rotation $L(\overline{\gamma})$. A point of such a singularity will have the same tangent space as a point of the conical singularity

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\theta^2 + dz^2$$

where A is determined by $L(\overline{\gamma})$. A could in principle vary over the singularity. However, the following result, known as the conservation of holonomy, shows that it does not.

From [V85] we have (see also [V90] and [V92]) (see diagram 1.6.2)

Theorem 1.6.4. *Let κ_0 and κ_1 be C^1 curves terminating at points of a quasi-regular singularity connected by a curve $c : [0,1] \longrightarrow \partial M$ where ∂M is the b-boundary of M. Let $\gamma_0 \in \Omega^A_{\kappa_0}$ and $\gamma_1 \in \Omega^A_{\kappa_1}$. Suppose there exists a C^1 homotopy*

$$h : [0,1] \times (0,1] \times [0,1] \longrightarrow M : (s,u,v) \longmapsto h_u(s,v)$$

such that

(a) $h_u(s,0) = \gamma_0(s,u) \qquad h_u(s,1) = \gamma_1(s,u)$

(b) $h_u(0,v) = h_u(1,v)$

(c) $\lim_{u \to 0} h_u(s,v) = c(v)$.

Now let $\overline{\kappa}_0$ be a horizontal lift of κ_0 in LM. Lift h to $\overline{h}$ by letting $\overline{h}_u(0,0) = \overline{\kappa}_0(u)$ and parallelly propagating along $s = 0$, $u =$ constant and then round the loops $u =$ constant, $v =$ constant. Also, let $\overline{\kappa}_1$ be a horizontal lift of κ_1. In general

$\overline{\kappa}_1(u) \neq \overline{h}_u(0,1)$ *but we can choose* $\overline{\kappa}_1$ *such that* $\lim_{u\to 0}\overline{\kappa}_1(u) = \lim_{u\to 0}\overline{h}_u(0,1)$ *where the right hand limit can be shown to exist. Suppose further that*

(d) area $\overline{h}_u([0,1],[0,1]) \to 0$ *as* $u \to 0$

where the area is defined with the positive definite metric on LM *used to construct the b-boundary. In other words, the area of the* $u =$ *constant tubes tends to zero.*

Then $L(\overline{\gamma}_0) = L(\overline{\gamma}_1)$.

Given κ_0, κ_1, γ_0, γ_1 and c, most quasi-regular singularities of interest will admit a homotopy satisfying (a)–(d). However again there exist quasi-regular singularities for which this is not clear. We will call a 2-dimensional quasi-regular singularity *good* if in addition to to being good according to our previous definition this homotopy exists. An example of a good 2-dimensional timelike quasi-regular singularity is the 4-cone

$$ds^2 = -dt^2 + dr^2 + A^2r^2d\theta^2 + dz^2.$$

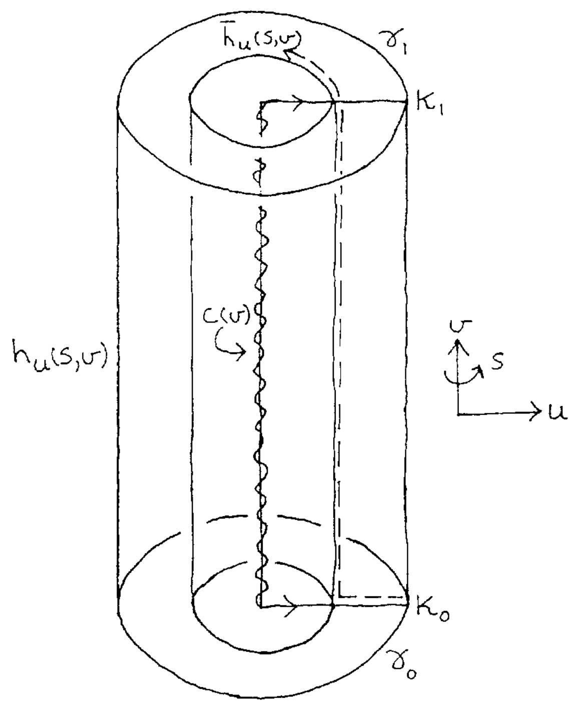

Diagram 1.6.2

From [V90] and [V92] we have

Corollary 1.6.5. *The singular holonomy groups which arise from a (path connected) good 2-dimensional timelike quasi-regular singularity are all generated by a rotation through the same angle β.*

Thus these singularities have the same light bending properties as a straight string.

As we demonstrated in Theorem 1.4.1, the fixed point set of an isometry of a space-time is totally geodesic and in this way, the elementary quasi-regular singularity which it gives rise to may be considered to be totally geodesic. We now show in the following corollary of Theorem 1.6.4 that, in a sense made clear by the proof, a good 2-dimensional quasi-regular singularity may also be considered to be totally geodesic ([V90] and [V92]). We note that a (non-null) submanifold S is totally geodesic if and only if vectors initially tangent to S remain tangent to S under parallel propagation (Proposition 3.5.5).

Corollary 1.6.6. *A good 2-dimensional quasi-regular singularity is totally geodesic.*

Proof. Let x, y be points on such a singularity connected by a curve c with curves κ_0, κ_1 terminating at x, y respectively as in Theorem 1.6.4. Let $\overline{\kappa}_0$ be a horizontal lift of κ_0 and let $\overline{\kappa}_1$ be the horizontal lift of κ_1 defined in Theorem 1.6.4 (for choices of γ_0, γ_1 and h). This gives us equivalence classes of frames $p = \lim_{u \to 0} \overline{\kappa}_0(u)$, $q = \lim_{u \to 0} \overline{\kappa}_1(u)$ on the b-boundary of LM. Provided q can be shown to depend only on p and c, q can be defined to be the parallel transport of p along c. Let $X(0)$ be tangent to the singularity at x. Pick frames $p_0 \in p$, $q_0 \in q$. Let $X(1)$ be a vector with the same components in q_0 as $X(0)$ has in p_0. By Theorem 1.6.3 $X(0)$ is fixed by $H_{\overline{\kappa}_0}$ and by Theorem 1.6.4 $X(1)$ is fixed by $H_{\overline{\kappa}_1}$. Because $H_{\overline{\kappa}_0} = G_p$, $H_{\overline{\kappa}_1} = G_q$ it follows that $X(0)$, $X(1)$ have the same components in any frame $p_0 \in p$, $q_0 \in q$

and that $X(1)$ is the well defined parallel transport of $X(0)$ along c. By Theorem 1.6.3, $X(1)$ must be tangent to the singularity. Hence the result. $\square$

We note that a p-dimensional submanifold S of an n-dimensional manifold (M, g) has $n-p$ extrinsic curvatures K^{μ}_{ij} at each point of S and is totally geodesic if $K^{\mu}_{ij} = 0$. This is equivalent to being minimal $K^{\mu} = g^{ij}K^{\mu}_{ij} = 0$ and totally umbilic $K^{\mu}_{ij} = K^{\mu}g^{\|}_{ij}$ where $g^{\|}_{ij}$ is the metric induced on S by g.

Chapter 2

Path-ordered exponentials and holonomy

2.1 Path-ordered exponentials of the connection

Let M be a (4-dimensional connected Lorentzian) manifold and $GL(M)$ the principal fibre bundle with structure group $GL_4(\mathbb{R})$ such that each fibre $\pi^{-1}(x)$ consists of all bases at x where $GL(M)$ has projection $\pi : GL(M) \longrightarrow M$. Let ω be a (not necessarily metric) connection on $GL(M)$.

Given a tensor, or matrix, or tensor valued matrix $U^{j_1 \dots j_q}_{i_1 \dots i_p}$ it will be useful to define its (basis dependent) Euclidean norm

$$\|U^{j_1 \dots j_q}_{i_1 \dots i_p}\| = (\sum_{i_i} \dots \sum_{i_p} \sum_{j_1} \dots \sum_{j_q} (U^{j_1 \dots j_q}_{i_1 \dots i_p})^2)^{1/2}.$$

Thus $\| \quad \|$ is a continuous map onto the non-negative reals. Given a tensor, or matrix, or tensor valued matrix $V^{j'_1 \dots j'_{q'}}_{i'_1 \dots i'_{p'}}$, it can be shown that

$$\|c(U^{j_1 \dots j_q}_{i_1 \dots i_p} \otimes V^{j'_1 \dots j'_{q'}}_{i'_1 \dots i'_{p'}})\| \leq \|U^{j_1 \dots j_q}_{i_1 \dots i_p} \otimes V^{j'_1 \dots j'_{q'}}_{i'_1 \dots i'_{p'}}\| = \|U^{j_1 \dots j_q}_{i_1 \dots i_p}\| \|V^{j'_1 \dots j'_{q'}}_{i'_1 \dots i'_{p'}}\|$$

where $c(U^{j_1 \dots j_q}_{i_1 \dots i_p} \otimes V^{j'_1 \dots j'_{q'}}_{i'_1 \dots i'_{p'}})$ is any contraction of $U^{j_1 \dots j_q}_{i_1 \dots i_p} \otimes V^{j'_1 \dots j'_{q'}}_{i'_1 \dots i'_{p'}}$. We note that the b-length of $\gamma_{[\alpha,s]}$ with respect to the basis $(\tilde{e}_i)$ is

$$l(\gamma_{[\alpha,s]}) = \int_\alpha^s (\sum_i u^i(s_0) u^i(s_0))^{1/2} \, ds_0 = \int_\alpha^s \|u^i(s_0)\| \, ds_0$$

Let $x_0 \in M$ and let $\gamma : s \mapsto \gamma(s)$ be a C^1 curve in M with $\gamma(\alpha) = x_0$ and tangent u. Pick a basis (e_i) of T_{x_0} and define $(e_i(s))$ by parallelly propagating (e_i) along γ to $\gamma(s)$. Then, expressing components with respect to a reference basis $(\tilde{e}_i)$,

$$u^i \nabla_i e_j = 0.$$

If $e_i(s) = L_i^j(s)\tilde{e}_j(s)$ then $L_i^j(s) \in GL_4(\mathbb{R})$ and

$$u^k(s)\partial_k L_i^j(s) + u^k(s)\omega_{kl}^j(s)L_i^l(s) = 0$$

where $\nabla_i \tilde{e}_j = \omega_{ij}^k \tilde{e}_k$. Thus

$$\frac{dL_i^j}{ds}(s) = -u^k(s)\omega_{kl}^j(s)L_i^l(s)$$

and

$$L_i^j(s_0) = L_i^j(\alpha) + \int_\alpha^{s_0} A_k^j(s_1)L_i^k(s_1)\, ds_1$$

where $A_i^j = -u^k\omega_{ki}^j$. By repeated iteration

$$L_i^j(s_0) = L_i^j(\alpha) + \sum_{m=1}^N \int_\alpha^{s_0} \dots \int_\alpha^{s_{m-1}} A_{k_1}^j(s_1)\dots A_l^{k_{m-1}}(s_m)L_i^l(\alpha)\, ds_m \dots ds_1 + R_i^j(s_0)$$

where the remainder term

$$R_i^j(s_0) = \int_\alpha^{s_0} \dots \int_\alpha^{s_N} A_{k_1}^j(s_1)\dots A_l^{k_N}(s_{N+1})L_i^l(s_{N+1})\, ds_{N+1}\dots ds_1.$$

Proposition 2.1.1. *$\|R_i^j(s_0)\| \to 0$ as $N \to \infty$.*

Proof.

$$\|R_i^j(s_0)\| \le \int_\alpha^{s_0} \dots \int_\alpha^{s_N} \|A_{k_1}^j(s_1)\| \dots \|A_l^{k_N}(s_{N+1})\|\|L_i^l(s_{N+1})\|\, ds_{N+1}\dots ds_1.$$

By continuity $\exists M, M' > 0$ such that $\|A_i^j(s)\| \le M$, $\|L_i^j(s)\| \le M'$ for $s \in [\alpha, s_0]$. Hence

$$\begin{aligned}\|R_i^j(s_0)\| &\le \int_\alpha^{s_0} \dots \int_\alpha^{s_N} M^{N+1}M'\, ds_{N+1}\dots ds_1 \\ &= \frac{1}{(N+1)!}\int_\alpha^{s_0} \dots \int_\alpha^{s_0} M^{N+1}M'\, ds_{N+1}\dots ds_1\end{aligned}$$

since the region of integration $\{\alpha \le s_i \le s_0 \mid i = 1, \dots, N+1\}$ may be split into $(N+1)!$ regions in each one of which the ordering of $(s_1, \dots, s_{N+1})$ into descending order of value is different. Hence

$$\|R_i^j(s_0)\| \le \frac{M'}{(N+1)!}\left(\int_\alpha^{s_0} M\, ds\right)^{N+1} = M'\frac{\lambda^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty$$

where $\lambda = \int_\alpha^{s_0} M\, ds$. $\square$

Thus

$$L_i^j(s_0) = L_i^j(\alpha) + \sum_{m=1}^{\infty} \int_\alpha^{s_0} \cdots \int_\alpha^{s_{m-1}} A_{k_1}^j(s_1) \ldots A_l^{k_{m-1}}(s_m)\, ds_m \ldots ds_1 L_i^l(\alpha).$$

Suppressing indices and using matrix notation we have

$$L(s_0) = L(\alpha) + \sum_{m=1}^{\infty} \frac{1}{m!} \int_\alpha^{s_0} \cdots \int_\alpha^{s_0} : A(s_1) \ldots A(s_m) : \, ds_m \ldots ds_1 L(\alpha)$$

where : : indicates that the expression should be ordered so that terms with a larger s value precede those with a smaller s value. Without the ordering this would be the expansion of an exponential function. Instead, expressing components with respect to the reference basis $(\tilde{e}_i)$, we write

$$L_i^j(s) = P \exp \int_\alpha^s -u^k(s_0)\omega_{kl}^j(s_0)\, ds_0\, L_i^l(\alpha)$$

which we call the *path-ordered exponential* of the connection ω along γ.

Proposition 2.1.2. *If A_i^j is a matrix valued function defined along γ then*

(a) $\|P \exp \int_\alpha^{s_0} A_i^j(s)\, ds - \delta_i^j\| \leq \exp \int_\alpha^{s_0} \|A_i^j(s)\|\, ds - 1$

(b) $\|P \exp \int_\alpha^{s_0} A_i^j(s)\, ds\| \leq \exp \int_\alpha^{s_0} \|A_i^j(s)\|\, ds + 1.$

Proof. (a)

$$\begin{aligned}
&\|P \exp \int_\alpha^{s_0} A_i^j(s)\, ds - \delta_i^j\| \\
&= \|\delta_i^j + \sum_{m=1}^{N} \int_\alpha^{s_0} \cdots \int_\alpha^{s_{m-1}} A_{k_1}^j(s_1) \ldots A_l^{k_{m-1}}(s_m)\, ds_m \ldots ds_1 + R_i^j(s_0) - \delta_i^j\| \\
&\leq \sum_{m=1}^{N} \int_\alpha^{s_0} \cdots \int_\alpha^{s_{m-1}} \|A_{k_1}^j(s_1)\| \ldots \|A_l^{k_{m-1}}(s_m)\|\, ds_m \ldots ds_1 + \|R_i^j(s_0)\| \\
&= \sum_{m=1}^{N} \frac{1}{m!} \int_\alpha^{s_0} \cdots \int_\alpha^{s_0} \|A_{k_1}^j(s_1)\| \ldots \|A_l^{k_{m-1}}(s_m)\|\, ds_m \ldots ds_1 + \|R_i^j(s_0)\|
\end{aligned}$$

since the ordering within the integral is unimportant. Hence

$$\begin{aligned}
\|P \exp \int_\alpha^{s_0} A_i^j(s)\, ds - \delta_i^j\| &\leq \sum_{m=1}^{N} \frac{1}{m!} \left(\int_\alpha^{s_0} \|A_i^j(s)\|\, ds \right)^m + \|R_i^j(s_0)\| \\
&\to \exp \int_\alpha^{s_0} \|A_i^j(s)\|\, ds \text{ as } N \to \infty
\end{aligned}$$

and hence

$$\|P\exp\int_{\alpha}^{s_0} A_i^j(s)\,ds - \delta_i^j\| \le \exp\int_{\alpha}^{s_0}\|A_i^j(s)\|\,ds - 1.$$

(b) This is proved in a similar fashion where we note that $\|\delta_i^j\| = 2$.

2.2 ω-frames

Let γ be a C^1 inextendible curve

$$\gamma : (0,\alpha] \to M : s \mapsto \gamma(s) \qquad \alpha > 0.$$

Define an *ω-frame* to be a basis parallelly propagated along γ with respect to ω.

Suppose a tensor U is defined along γ. U is *C^0-ω-quasi-regular* if its components in an ω-frame have C^0 limits as $s \to 0$. For example, if ω is a Levi-Civita connection and γ terminates at a C^0 quasi-regular singularity then the curvature tensor $R_{ijk}{}^l$ will be C^0-ω-quasi-regular.

Now suppose U is defined in a neighbourhood of γ. Using ∇ to denote the covariant derivative with respect to ω, U is *C^r-ω-quasi-regular* for $r \ge 1$ if the components of $\nabla_{i_1}\ldots\nabla_{i_r}U$ in an ω-frame have C^0 limits as $s \to 0$ and in addition U is C^{r-1}-ω-quasi-regular. This is a recursive definition.

If $r \ge 0$ and U is C^r-ω-quasi-regular with respect to one ω-frame, then it will be with respect to all ω-frames since if $(e_i(s))$, $(\tilde{e}_i(s))$ are two ω-frames

$$e_i(s) = a_i^j\tilde{e}_j(s)$$

for some constant $a_i^j \in GL_4(\mathbb{R})$. We now prove the following propositions.

Proposition 2.2.1. *If $r \ge 0$ and U, V are C^r-ω-quasi-regular then so are $U + V$ and λU where $\lambda \in \mathbb{R}$ is constant along γ.*

Proof. ∇ is linear. $\square$

Proposition 2.2.2. *If $r \geq 1$ and U is C^r-ω-quasi-regular then ∇U is C^{r-1}-ω-quasi-regular.*

Proof. $(\nabla_{i_1} \dots \nabla_{i_r})U = (\nabla_{i_1} \dots \nabla_{i_{r-1}})(\nabla_{i_r} U)$. □

Proposition 2.2.3. *If $r \geq 0$ and U, V are C^r-ω-quasi-regular then so is $U \otimes V$.*

Proof. U, V are C^0-ω-quasi-regular hence so is $U \otimes V$. If $r \geq 1$ then for $1 \leq k \leq r$

$$\begin{aligned}\nabla_{i_1} \dots \nabla_{i_k}(U \otimes V) = U \otimes \nabla_{i_1} \dots \nabla_{i_k} V + \sum_{l=1}^{k-1} \binom{k}{l} \nabla_{i_1} \dots \nabla_{i_l} U \otimes \nabla_{i_{l+1}} \dots \nabla_{i_k} V \\ + \nabla_{i_1} \dots \nabla_{i_k} U \otimes V\end{aligned}$$

each term of which is by definition C^0-ω-quasi-regular. Hence $\nabla_{i_1} \dots \nabla_{i_k}(U \otimes V)$ is C^0-ω-quasi-regular. □

Proposition 2.2.4. *If $r \geq 0$ and U, V are C^r-ω-quasi-regular then so is any contraction of $U \otimes V$.*

Proof. Contractions of $U \otimes V$ are formed by taking the tensor product of $U \otimes V$ and Kronecker tensors. Now

$$\nabla_i \delta^k_j = \omega^k_{il} \delta^l_j - \omega^l_{ij} \delta^k_l = \omega^k_{ij} - \omega^k_{ij} = 0$$

so Kronecker tensors are C^r-ω-quasi-regular. □

Suppose U is an m-form then for vectors $X_1, \dots, X_{m+1}$ recall

$$dU(X_1, \dots, X_{m+1}) = (m+1) \underbrace{(X_1 U(X_2, \dots, X_{m+1}) - mU([X_1, X_2], X_3, \dots, X_{m+1}))}_{\text{antisymmetrised over } X_1 \dots, X_{m+1}}$$

(where the antisymmetrisation includes a factor of $1/n!$). In a basis (e_i)

$$dU_{i_1 \dots i_{m+1}} := dU(e_{i_1}, \dots, e_{i_{m+1}}) = (m+1)(\partial_{[i_1} U_{i_2 \dots i_{m+1}]} - mU_{k[i_3 \dots i_{m+1}} c^k_{i_1 i_2]})$$

where $\partial_i f := e_i(f)$ for a scalar $f : M \to \mathbb{R}$ and $[e_i, e_j] = c^k_{ij} e_k$. Thus

$$\begin{aligned}
dU_{i_1\dots i_{m+1}} &= (m+1)(\nabla_{[i_1} U_{i_2\dots i_{m+1}]} + \omega^k_{[i_1 i_2} U_{|k|i_3\dots i_{m+1}]} + \dots + \omega^k_{[i_1 i_{m+1}} U_{i_2\dots i_m]k} \\
&\quad - mU_{k[i_3\dots i_{m+1}} c^k_{i_1 i_2]}) \\
&= (m+1)(\nabla_{[i_1} U_{i_2\dots i_{m+1}]} + m\omega^k_{[i_1 i_2} U_{|k|i_3\dots i_{m+1}]} - mU_{k[i_3\dots i_{m+1}} c^k_{i_1 i_2]}) \\
&= (m+1)(\nabla_{[i_1} U_{i_2\dots i_{m+1}]} + mT^k_{[i_1 i_2} U_{|k|i_3\dots i_{m+1}]})
\end{aligned}$$

where ω has torsion

$$T^k_{ij} = \omega^k_{ij} - \omega^k_{ji} - c^k_{ij}.$$

Thus dU is clearly a tensor.

Now if a tensor W obeys

$$W^{k_1\dots k_q}_{i_1\dots i_m j_1\dots j_p} = W^{k_1\dots k_q}_{[i_1\dots i_m]j_1\dots j_p}$$

then it can also be treated as an m-form valued tensor $W^{k_1\dots k_q}_{j_1\dots j_p}$ of valence $\binom{q}{p}$. Thus when we say that the m-form valued tensor $W^{k_1\dots k_q}_{j_1\dots j_p}$ is C^r-ω-quasi-regular we shall mean that the tensor $W^{k_1\dots k_q}_{i_1\dots i_m j_1\dots j_p}$ is C^r-ω-quasi-regular.

Let $U^{k_1\dots k_q}_{j_1\dots j_p}$ be an m-form valued tensor. Since $U(e_{j_1},\dots,e_{j_p},e^{k_1}\dots,e^{k_q})$ is an m-form we define

$$dU^{k_1\dots k_q}_{i_1\dots i_{m+1}j_1\dots j_p} = d(U(e_{j_1},\dots,e_{j_p},e^{k_1}\dots,e^{k_q}))_{i_1\dots i_{m+1}}$$

however this expression is basis dependent and not a tensor.

Suppressing form indices we now define the *exterior covariant derivative* of U

$$\begin{aligned}
DU^{k_1\dots k_q}_{j_1\dots j_p} =& dU^{k_1\dots k_q}_{j_1\dots j_p} + \omega^{k_1}_k \wedge U^{kk_2\dots k_q}_{j_1\dots j_p} + \dots + \omega^{k_q}_k \wedge U^{k_1\dots k_{q-1}k}_{j_1\dots j_p} \\
&+ U^{k_1\dots k_q}_{kj_2\dots j_p} \wedge \omega^k_{j_1} + \dots + U^{k_1\dots k_q}_{j_1\dots j_{p-1}k} \wedge \omega^k_{j_p}
\end{aligned}$$

where ω^k_j is the 1-form defined by $\omega^k_j(e_i) = \omega^k_{ij}$ and $\wedge$ acts on the form indices. As above we can show

$$DU^{k_1\dots k_q}_{i_1\dots i_{m+1}j_1\dots j_p} = \nabla_{[i_1} U^{k_1\dots k_q}_{i_2\dots i_{m+1}]j_1\dots j_p} + mT^k_{[i_1 i_2} U^{k_1\dots k_q}_{|k|i_3\dots i_{m+1}]j_1\dots j_p}$$

thus DU is both an $(m+1)$-form valued tensor and a tensor of valence $\binom{q}{m+p+1}$. We note that if U is just an m-form then $dU = DU$. Thus we have proved

Proposition 2.2.5. *If $r \geq 1$ and U is C^r-ω-quasi-regular then DU is C^{r-1}-ω-quasi-regular, provided that the torsion T of ω is C^{r-1}-ω-quasi-regular (for example zero).*

Now let U be a tensor defined along $\gamma : (0, \alpha] \to M : s \mapsto \gamma(s)$. Given an ω-frame and $r > 0$ we write

$$U = o(s^{-r}) \iff \text{the } \omega\text{-frame components of } s^r U(s) \to 0 \text{ as } s \to 0$$

$$U = O(s^{-r}) \iff \text{the } \omega\text{-frame components of } s^r U(s) \text{ are bounded as } s \to 0.$$

Whether or not $U = o(s^{-r})$ or $U = O(s^{-r})$ does not depend on the choice of ω-frame but does in general depend on the parameter s.

Theorem 2.2.6. *If $\gamma : (0, \alpha] \to M : s \mapsto \gamma(s)$ has finite b-length and s_1, s_2 are two parametrisations of γ which both measure b-length and U is a tensor defined along γ then*

$$U = o({s_1}^{-r}) \iff U = o({s_2}^{-r})$$

and

$$U = O({s_1}^{-r}) \iff U = O({s_2}^{-r}).$$

First we prove

Lemma 2.2.7. *If $a_i^j \in GL_4(\mathbb{R})$ then $\exists m, M > 0$ such that $\forall u^i \in \mathbb{R}^4$*

$$m(\sum_i u^i u^i)^{1/2} \leq (\sum_i \tilde{u}^i \tilde{u}^i)^{1/2} \leq M(\sum_i u^i u^i)^{1/2}$$

where $\tilde{u}^j = a_i^j u^i$.

Proof. For $a, b \in \mathbb{R}$

$$0 \leq (a-b)^2 \Rightarrow (a+b)^2 \leq (a-b)^2 + (a+b)^2 = 2(a^2 + b^2)$$

thus

$$\begin{aligned}\sum_i a^i_j u^j a^i_k u^k &= \sum_i (a^i_0 u^0 + \ldots + a^i_3 u^3)^2 \leq 4 \sum_i (a^i_0 u^0)^2 + \ldots + (a^i_3 u^3)^2 \\ &= 4((a^0_0)^2 + \ldots + (a^3_0)^2)(u^0)^2 + \ldots + 4((a^0_3)^2 + \ldots + (a^3_3)^2)(u^3)^2 \\ &\leq M^2(u^0 u^0 + \ldots + u^3 u^3) = M^2 \sum_i u^i u^i\end{aligned}$$

where $M = 4\max_i((a^0_i)^2 \ldots (a^3_i)^2)^{1/2} > 0$.

Similarly $\frac{1}{m} = 4\max_i((b^0_i)^2 \ldots (b^3_i)^2)^{1/2} > 0$ where $b^j_i = (a^{-1})^j_i$. $\square$

Now we proceed with the proof of Theorem 2.2.6.

Proof. Let (e_i), $(\tilde{e}_i)$ be the ω-frames along γ with respect to which s_1, s_2 are measured. Thus $e_i = a^j_i \tilde{e}_j$ where $a^j_i \in GL_4(\mathbb{R})$ is constant along γ. Let $\gamma : s \mapsto \gamma(s)$ have tangent $u = u^i e_i = \tilde{u}^i \tilde{e}_i$. Then

$$s_1(s) = \int_0^s (\sum_i u^i(s_0) u^i(s_0))^{1/2}\, ds_0$$

$$s_2(s) = \int_0^s (\sum_i \tilde{u}^i(s_0) \tilde{u}^i(s_0))^{1/2}\, ds_0$$

where in general $s_1(s) \neq s_2(s)$. Also $u^i e_i = u^i a^j_i \tilde{e}_j$ so $\tilde{u}^j = a^j_i u^i$.

By Lemma 2.2.7 $\exists m, M > 0$ such that

$$m s_1(s) \leq s_2(s) \leq M s_1(s).$$

Let U have components $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ in an ω-frame. Now suppose $U = o({s_1}^{-r})$. Then

$${s_1}^r U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s(s_1)) \to 0 \text{ as } s_1 \to 0$$

but

$$|{s_2}^r U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s(s_2))| \leq M^r |{s_1}^r U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s(s_1))|$$

since $s(s_2) = s(s_1)$. Hence

$${s_2}^r U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s(s_2)) \to 0 \text{ as } s_2 \to 0$$

and $U = o({s_2}^{-r})$.

Similarly $U = o({s_2}^{-r}) \Rightarrow U = o({s_1}^{-r})$ and $U = O({s_1}^{-r}) \iff U = O({s_2}^{-r})$. $\square$

Thus b-length is a natural parameter along a b-incomplete curve. We will therefore say a tensor U defined along γ is *C^{-r}-ω-quasi-regular* for $r > 0$ if γ has finite b-length and $U = o(s^{-r})$ when s is a parametrisation of γ which measures b-length with respect to an ω-frame.

Proposition 2.2.8. *If $r \geq 1$ and U, V are C^{-r}-ω-quasi-regular then so are $U + V$ and λU where $\lambda \in \mathbb{R}$ is constant along γ.*

Proof. $s(U + V) = sU + sV$ and $s(\lambda U) = \lambda(sU)$. $\square$

The following lemma will prove to be useful.

Lemma 2.2.9. *Let $\gamma : (0, \alpha] \to M \quad \alpha > 0$ be a C^1 curve of finite b-length, let s be a parametrisation of γ which measures b-length, let (e_i) be the ω-frame along γ with respect to which s is measured, and let $\gamma : s \mapsto \gamma(s)$ have tangent $u = u^i e_i$. Then $\|u^i(s)\| = 1$.*

Proof. The b-length of $\gamma_{(0,s]}$ is

$$s = \int_0^s \|u^i(s_0)\| \, ds_0$$

$$\Rightarrow 1 = \|u^i(s)\|.$$

$\square$

Finally we prove

Theorem 2.2.10. *Let $\gamma : s \mapsto \gamma(s)$ be a C^1 curve with tangent u. If u is everywhere non-zero then an ω-frame (e_i) can be extended to a neighbourhood of γ so that $\omega^k_{ij} = 0$ on γ where ω has components ω^k_{ij} in (e_i).*

Proof. For each $x = \gamma(s)$ make a C^1 choice $V_x < T_x M$ (where $<$ denotes vector subspace) such that $T_x M = T_x \gamma \oplus V_x$ (possible since $u(s)$ is C^0 and non-zero). Thus V_x is 3-dimensional and nowhere tangent to γ. For each $v \in V_x$ and each $x = \gamma(s)$, parallelly propagate (e_i) along the unique geodesic through x in the direction of v. This will extend (e_i) to a neighbourhood of γ (since V_x is C^1).

Now for each $v \in V_x$

$$v^i \nabla_i e_j = 0$$

since the geodesic through x in the direction of v can be parametrised to have tangent v at x. In addition

$$u^i \nabla_i e_j = 0$$

and hence $\nabla_i e_j = \omega^k_{ij} e_k = 0$ at x. Thus $\omega^k_{ij} = 0$ on γ. □

2.3 Connection difference

Let ω, $\overline{\omega}$ be (not necessarily metric) connections on $GL(M)$. ω, $\overline{\omega}$ are not tensors on M, however the connection difference

$$\sigma = \overline{\omega} - \omega$$

is a tensor on M.

Let $\gamma : s \mapsto \gamma(s)$ be a C^1 curve in M with $\gamma(\alpha) = x_0$ and tangent u. Pick bases (e_i), $(\overline{e}_i)$ of $T_{x_0}M$ and parallelly propagate them along γ to $\gamma(s)$ with respect to ω, $\overline{\omega}$ respectively to give $(e_i(s))$, $(\overline{e}_i(s))$. Set

$$\overline{e}_i(s) = L^j_i(s) e_j(s)$$

where $L^j_i \in GL_4(\mathbb{R})$ though in general $L^j_i(0) \neq \delta^j_i$. If we extend (e_i), $(\overline{e}_i)$ to a neighbourhood of γ and work in the basis (e_i) we may define

$$\nabla_k e_i = \omega^j_{ki} e_j \qquad \overline{\nabla}_k e_i = \overline{\omega}^j_{ki} e_j$$

however since

$$u^k \nabla_k e_i = u^k \omega^j_{ki} e_j \qquad u^k \overline{\nabla}_k e_i = u^k \overline{\omega}^j_{ki} e_j$$

the values of $u^k \omega^j_{ki}$, $u^k \overline{\omega}^j_{ki}$ depend only on the values of (e_i), $(\overline{e}_i)$ along γ and in the following we shall only need these values. Now in the basis (e_i)

$$u^k(s) \overline{\nabla}_k \overline{e}_i(s) = u^k(s) \overline{\nabla}_k (L^j_i(s) e_j(s)) = u^k(s)(\partial_k L^j_i(s) + \overline{\omega}^j_{kl}(s) L^l_i(s)) = 0$$

$$\Rightarrow \frac{dL_i^j}{ds}(s) + u^k(s)\overline{\omega}^j_{kl}(s)L^l_i(s) = 0$$

and in the basis (e_i)

$$u^k(s)\nabla_k e_i(s) = u^k(s)\omega^j_{ki}(s)e_j(s) = 0 \Rightarrow u^k(s)\omega^j_{ki}(s) = 0.$$

Hence

$$\frac{dL_i^j}{ds}(s) + u^k(s)\sigma^j_{kl}(s)L^l_i(s) = 0$$

and as before

$$L^j_i(s) = P\exp\int_\alpha^s -u^k(s_0)\sigma^j_{kl}(s_0)\,ds_0\, L^l_i(\alpha) = P\exp\int_s^\alpha +u^k(s_0)\sigma^j_{kl}(s_0)\,ds_0\, L^l_i(\alpha) \tag{2.3.1}$$

which we can either regard as a matrix equation for L^j_i, in which case everything must be expressed in the basis (e_i), or as a tensor equation, in which case the components L^j_i of $\overline{e}_i$ are now basis dependent.

Now let γ be a C^1 inextendible curve

$$\gamma : (0, \alpha] \longrightarrow M : s \mapsto \gamma(s) \qquad \alpha > 0.$$

We do not in general know how $L^j_i(s)$ behaves as $s \longrightarrow 0$.

Theorem 2.3.1. *Let $(e_i(s))$, $(e'_i(s))$ be ω-frames, let $(\overline{e}_i(s))$, $(\overline{e}'_i(s))$ be $\overline{\omega}$-frames and let*

$$\overline{e}_i(s) = L^j_i(s)e_j(s) \qquad \overline{e}'_i(s) = L'^j_i(s)e'_j(s).$$

Then $\lim_{s\to 0} L^j_i(s)$ exists and is in $GL_4(\mathbb{R})$ $\iff$ $\lim_{s\to 0} L'^j_i(s)$ exists and is in $GL_4(\mathbb{R})$.

Proof. $\exists$ constant $a^j_i, b^j_i \in GL_4(\mathbb{R})$ such that

$$e'_i(s) = a^j_i e_j(s) \qquad \overline{e}'_i(s) = b^j_i \overline{e}_j(s).$$

Thus

$$b^j_i\overline{e}_j(s) = L'^j_i(s)a^k_j e_k(s) \Rightarrow \overline{e}_i(s) = (b^{-1})^j_i L'^k_j(s)a^l_k e_l(s) \Rightarrow L^j_i(s) = (b^{-1})^k_i L'^l_k(s)a^j_l$$

and hence $\lim_{s\to 0} L_i^j(s)$ exists and is in $GL_4(\mathbb{R}) \iff \lim_{s\to 0} L'^j_i(s)$ exists and is in $GL_4(\mathbb{R})$. $\square$

Thus we define

$$\omega \sim \overline{\omega} \iff L_i^j(0) := \lim_{s\to 0} L_i^j(s) \text{ exists and is in } GL_4(\mathbb{R})$$

where $L_i^j(s)$ may be defined with respect to any choice of ω- and $\overline{\omega}$-frame.

Theorem 2.3.2. *$\sim$ is an equivalence relation on the set of connections on $GL(M)$.*

Proof. Let ω be a connection on $GL(M)$ and $(e_i(s))$ an ω-frame. Now

$$e_i(s) = \delta_i^j e_j(s)$$

and $\lim_{s\to 0} \delta_i^j = \delta_i^j$. Thus $\omega \sim \omega$.

Let $\overline{\omega}$ be a connection on $GL(M)$ and $(\overline{e}_i(s))$ an $\overline{\omega}$-frame and suppose $\omega \sim \overline{\omega}$. Thus if we set

$$\overline{e}_i(s) = L_i^j(s) e_j(s)$$

then $\lim_{s\to 0} L_i^j(s)$ exists and is in $GL_4(\mathbb{R})$ but

$$e_i(s) = (L^{-1})_i^j(s)\overline{e}_j(s)$$

and so $\lim_{s\to 0}(L^{-1})_i^j(s)$ exists and is in $GL_4(\mathbb{R})$. Thus $\overline{\omega} \sim \omega$.

Now let $\tilde{\omega}$ be a connection on $GL(M)$ and $(\tilde{e}_i(s))$ an $\tilde{\omega}$-frame. Suppose $\omega \sim \overline{\omega}$ and $\overline{\omega} \sim \tilde{\omega}$. If we set

$$\tilde{e}_i(s) = \overline{L}_i^j(s)\overline{e}_j(s) \qquad \overline{e}_j(s) = L_j^k(s) e_k(s)$$

then

$$\tilde{e}_i(s) = \overline{L}_i^j(s) L_j^k(s) e_k(s).$$

Since $\lim_{s\to 0} \overline{L}_i^j(s)$, $\lim_{s\to 0} L_j^k(s)$ both exist and are in $GL_4(\mathbb{R})$, $\lim_{s\to 0} \tilde{L}_i^k(s)$ exists and is in $GL_4(\mathbb{R})$, where $\tilde{L}_i^k(s) = \overline{L}_i^j(s) L_j^k(s)$. Thus $\omega \sim \tilde{\omega}$. $\square$

Let ω be a connection on $GL(M)$. If γ has finite b-length with respect to a particular ω-frame, then it will have finite b-length with respect to all ω-frames, and we say that γ has *ω-finite b-length*. In this case the b-length l of γ with respect to a particular ω-frame (e_i) is given by

$$l = \int_0^\alpha (\sum_i u^i(s)u^i(s))^{1/2}\, ds$$

where γ has tangent $u = u^i e_i$. Although u in fact depends on the parametrisation s of γ, l does not.

Let $\overline{\omega}$ be a another connection on $GL(M)$. γ need not in general have $\overline{\omega}$-finite b-length, even if it has ω-finite b-length.

Now let (e_i) be an ω-frame and let $(\overline{e}_i)$ be an $\overline{\omega}$-frame, let $\sigma = \overline{\omega} - \omega$, and let γ have tangent $u = u^i e_i = \overline{u}^i \overline{e}_i$. Thus $u^j(s) = L_i^j(s)\overline{u}^i(s)$ where $\overline{e}_i(s) = L_i^j(s)e_j(s)$.

Theorem 2.3.3. *If $\omega \sim \overline{\omega}$ then*

$$\gamma \text{ has } \omega\text{-finite b-length} \iff \gamma \text{ has } \overline{\omega}\text{-finite b-length.}$$

First we prove

Lemma 2.3.4. *If $\omega \sim \overline{\omega}$ then $\exists M > 0$ such that*

$$(\sum_i u^i(s)u^i(s))^{1/2} \leq M(\sum_i \overline{u}^i(s)\overline{u}^i(s))^{1/2}.$$

Proof. $L_i^j(s) \in GL_4(\mathbb{R})$ and $\overline{u}^i(s) \in \mathbb{R}^4$, so by Lemma 2.2.7 $\exists M(s) > 0$ such that

$$(\sum_i u^i(s)u^i(s))^{1/2} \leq M(s)(\sum_i \overline{u}^i(s)\overline{u}^i(s))^{1/2}$$

and

$$M(s) = 4\max_i((L_i^0(s))^2 + \ldots + (L_i^3(s))^2)^{1/2} > 0.$$

Now $\omega \sim \overline{\omega}$ so $\lim_{s\to 0} L_i^j(s)$ exists and is in $GL_4(\mathbb{R})$. Thus $M(s)$ is continuous and strictly positive on $[\alpha, 0]$. Therefore set

$$M = \max_{0\leq s\leq\alpha} M(s)$$

where $M > 0$. $\square$

We now prove Theorem 2.3.3.

Proof. Suppose γ has ϖ-finite b-length. The b-length of $\gamma|_{[\alpha,s)}$ with respect to (e_i) is

$$l(s) = \int_s^\alpha (\sum_i u^i(s_0)u^i(s_0))^{1/2}\,ds_0$$

and with respect to $(\overline{e}_i)$ is

$$\overline{l}(s) = \int_s^\alpha (\sum_i \overline{u}^i(s_0)\overline{u}^i(s_0))^{1/2}\,ds_0.$$

Since $\omega \sim \varpi$, $\exists M > 0$ such that

$$l(s) \leq M\overline{l}(s)$$

but $\lim_{s\to 0}\overline{l}(s) < \infty$ hence $\lim_{s\to 0} l(s) < \infty$ and γ has ω-finite b-length.

Now $\varpi \sim \omega$ and thus if γ has ω-finite b-length, then it has ϖ-finite b-length. □

Proposition 2.3.5. *Let U be a tensor defined along γ. If $\omega \sim \varpi$ then U is C^0-ω-quasi-regular $\Longleftrightarrow$ U is C^0-ϖ-quasi-regular.*

Proof. Let U have components $U^{j_1\ldots j_q}_{i_1\ldots i_p}$ in (e_i) and $\overline{U}^{j_1\ldots j_q}_{i_1\ldots i_p}$ in $(\overline{e}_i)$. Then

$$\overline{U}^{j_1\ldots j_q}_{i_1\ldots i_p}(s) = (L^{-1})^{j_1}_{l_1}(s)\ldots(L^{-1})^{j_q}_{l_q}(s)L^{k_1}_{i_1}(s)\ldots L^{k_p}_{i_p}(s)U^{l_1\ldots l_q}_{k_1\ldots k_p}(s)$$

where we note that $\overline{e}_i(s) = L^j_i(s)e_j(s) \Rightarrow \overline{e}^j(s) = (L^{-1})^j_i(s)e^i(s)$. Since $\omega \sim \varpi$ it follows that $\lim_{s\to 0} L^j_i(s)$ and $\lim_{s\to 0}(L^{-1})^j_i(s)$ both exist and thus

$$\lim_{s\to 0} U^{j_1\ldots j_q}_{i_1\ldots i_p}(s) \text{ exists} \Longleftrightarrow \lim_{s\to 0}\overline{U}^{j_1\ldots j_q}_{i_1\ldots i_p}(s) \text{ exists.}$$ □

Proposition 2.3.6. *Let U be a tensor defined in a neighbourhood of γ. If $\omega \sim \varpi$ and $r \geq 1$, and σ is C^{r-1}-ω-quasi-regular, then U is C^r-ω-quasi-regular $\Longleftrightarrow$ U is C^r-ϖ-quasi-regular.*

Proof. Suppose U is C^r-ω-quasi-regular. Define

$$T^{(n)} = \overline{\nabla}T^{(n-1)} \quad 1 \leq n \leq r \qquad T^{(0)} = U.$$

Working in the basis (e_i)

$$
\begin{aligned}
T^{(n)}{}^{j_1\ldots j_q}_{k_n\ldots k_1 i_1\ldots i_p} &= \overline{\nabla}_{k_n} T^{(n-1)}{}^{j_1\ldots j_q}_{k_{n-1}\ldots k_1 i_1\ldots i_p} \\
&= \nabla_{k_n} T^{(n-1)}{}^{j_1\ldots j_q}_{k_{n-1}\ldots k_1 i_1\ldots i_p} \\
&\quad + \sigma^{j_1}_{k_n l} T^{(n-1)}{}^{l j_2\ldots j_q}_{k_{n-1}\ldots k_1 i_1\ldots i_p} + \ldots + \sigma^{j_q}_{k_n l} T^{(n-1)}{}^{j_1\ldots j_{q-1} l}_{k_{n-1}\ldots k_1 i_1\ldots i_p} \\
&\quad - \sigma^{l}_{k_n k_{n-1}} T^{(n-1)}{}^{j_1\ldots j_q}_{l k_{n-2}\ldots k_1 i_1\ldots i_p} - \ldots - \sigma^{l}_{k_n i_p} T^{(n-1)}{}^{j_1\ldots j_q}_{k_{n-1}\ldots k_1 i_1\ldots i_{p-1} l}
\end{aligned}
$$

where U has valence $\binom{q}{p}$.

$T^{(0)} = U$ is C^r-ω-quasi-regular, so assume inductively that $T^{(n-1)}$ is C^{r-n+1}-ω-quasi-regular for $1 \leq n \leq r$. Thus $\nabla T^{(n-1)}$, $T^{(n-1)}$, and σ are all C^{r-n}-ω-quasi-regular (since σ is C^{r-1}-ω-quasi-regular), and hence so is $T^{(n)}$.

In particular $T^{(n)}$ is C^0-ω-quasi-regular, and thus by Proposition 2.3.5 $T^{(n)}$ is C^0-$\overline{\omega}$-quasi-regular. Therefore U is C^r-$\overline{\omega}$-quasi-regular.

Applying the above to σ we see that σ is C^{r-1}-$\overline{\omega}$-quasi-regular. Since $\overline{\omega} \sim \omega$ we have by symmetry that if U is C^r-$\overline{\omega}$-quasi-regular, then U is C^r-ω-quasi-regular. □

Proposition 2.3.7. *Let U be a tensor defined along $\gamma : s \mapsto \gamma(s)$. If $\omega \sim \overline{\omega}$ and $r > 0$ then U is C^{-r}-ω-quasi-regular $\iff$ U is C^{-r}-$\overline{\omega}$-quasi-regular.*

Proof. Suppose U is C^{-r}-ω-quasi-regular. Thus γ has ω-finite b-length (by definition) and since $\omega \sim \overline{\omega}$, γ has $\overline{\omega}$-finite b-length. Let s_1, s_2 be parametrisations of γ which measure b-length with respect to (e_i), $(\overline{e}_i)$. Then

$$
\begin{aligned}
s_1(s) &= \int_0^s \Big(\sum_i u^i(s_0) u^i(s_0)\Big)^{1/2} ds_0 \\
s_2(s) &= \int_0^s \Big(\sum_i \overline{u}^i(s_0) \overline{u}^i(s_0)\Big)^{1/2} ds_0.
\end{aligned}
$$

By Lemma 2.3.4 $\exists M > 0$ such that

$$s_2(s) \leq M s_1(s).$$

Now let U have components $U^{j_1\ldots j_q}_{i_1\ldots i_p}$ in (e_i). Thus

$$s_1{}^r\, U^{j_1\ldots j_q}_{i_1\ldots i_p}(s(s_1)) \to 0 \text{ as } s_1 \to 0$$

but

$$\left|s_2{}^r\, U^{j_1\ldots j_q}_{i_1\ldots i_p}(s(s_2))\right| \leq M^r \left|s_1{}^r\, U^{j_1\ldots j_q}_{i_1\ldots i_p}(s(s_1))\right|$$

since $s(s_2) = s(s_1)$. Hence

$$s_2{}^r\, U^{j_1\ldots j_q}_{i_1\ldots i_p}(s(s_2)) \dot{\longrightarrow} 0 \text{ as } s_2 \longrightarrow 0.$$

Now let U have components $\overline{U}^{j_1\ldots j_q}_{i_1\ldots i_p}$ in $(\overline{e}_i)$. Since $\omega \sim \overline{\omega}$, $\lim_{s\to 0} L^j_i(s)$ exists and

$$s_2{}^r\, \overline{U}^{j_1\ldots j_q}_{i_1\ldots i_p}(s(s_2)) \longrightarrow 0 \text{ as } s_2 \longrightarrow 0.$$

Thus U is C^{-r}-$\overline{\omega}$-quasi-regular.

Now since $\overline{\omega} \sim \omega$, by symmetry if U is C^{-r}-$\overline{\omega}$-quasi-regular, then U is C^{-r}-ω-quasi-regular. □

Corollary 2.3.8. *If $\omega \sim \overline{\omega}$ and $r < 0$ or $r \in \mathbb{N}$ then σ is C^r-ω-quasi-regular $\Longleftrightarrow$ σ is C^r-$\overline{\omega}$-quasi-regular.*

Now let $L^1(0,\alpha) = \{f : (0,\alpha) \to \mathbb{R} \,|\, f \text{ is integrable}\}$. In the following when we say $A^j_i(s) \in L^1(0,\alpha)$ we shall mean that each component $A^j_i(s) \in L^1(0,\alpha)$.

We note that for a function $f : (0,\alpha) \to \mathbb{R}$, $f(s) = o(s^{-1})$ is neither necessary nor sufficient for $f \in L^1(0,\alpha)$. For example $f(s) = 1/(s \log s) \notin L^1(0,\alpha)$ despite $f(s) = o(s^{-1})$. On the other hand $f(s) = o(s^{-r})$ for some $r < 1$ is sufficient for $f \in L^1(0,\alpha)$, though not necessary. For example $f(s) = 1/(s(\log s)^k) \in L^1(0,\alpha)$ for $k > 1$ despite the fact that $f(s) \neq o(s^{-r})$ for $r < 1$, but $f(s) = o(s^{-1})$.

Let ω, $\overline{\omega}$ be connections on $GL(M)$. Let U be a tensor defined along γ with components $U^{j_1\ldots j_q}_{i_1\ldots i_p}$ in an ω-frame (e_i). Whether or not $U^{j_1\ldots j_q}_{i_1\ldots i_p} \in L^1(0,\alpha)$ does not depend on the choice of ω-frame but does in general depend on the parameter s.

Proposition 2.3.9. *Suppose $\gamma : (0, \alpha] \to M : s \mapsto \gamma(s)$ has finite b-length and let s_1, s_2 be two parametrisations of γ which both measure b-length. Then*

$$U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1) \in L^1(0, \alpha_1) \iff U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2) \in L^1(0, \alpha_2)$$

where $s_1 = \alpha_1$ when $s = \alpha$ and $s_2 = \alpha_2$ when $s = \alpha$.

Proof. Let (e_i), $(\tilde{e}_i)$ be the ω-frames along γ with respect to which s_1, s_2 are measured. Let U have components $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1)$ in (e_i). Suppose $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1) \in L^1(0, \alpha_1)$. Then

$$\int_0^{\alpha_1} U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1)\, ds_1 = \int_0^{\alpha_2} U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2) \frac{ds_1}{ds_2}(s_2)\, ds_2 < \infty$$

and so $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2)(ds_1/ds_2)(s_2) \in L^1(0, \alpha_2)$. Now $e_i = a_i^j \tilde{e}_j$ where $a_i^j \in GL_4(\mathbb{R})$ is constant along γ. Let $\gamma : s \mapsto \gamma(s)$ have tangent $u = u^i e_i = \tilde{u}^i \tilde{e}_i$. Then $u^i e_i = u^i a_i^j \tilde{e}_j$ so $\tilde{u}^j = a_i^j u^i$ and

$$\begin{aligned} s_1(s) &= \int_0^s \|u^i(s_0)\|\, ds_0 \\ s_2(s) &= \int_0^s \|\tilde{u}^i(s_0)\|\, ds_0. \end{aligned}$$

Hence $(ds_1/ds)(s) = \|u^i(s)\|$ and $(ds_2/ds)(s) = \|\tilde{u}^i(s)\|$ and thus $(ds_1/ds_2)(s_2) = \|u^i(s_2)\|/\|\tilde{u}^i(s_2)\|$. By Lemma 2.2.7 $\exists m, M > 0$ such that

$$m\|u^i\| \le \|\tilde{u}^i\| \le M\|u^i\|$$

and thus

$$\frac{\|u^i(s_2)\|}{M\|u^i(s_2)\|} \le \frac{ds_1}{ds_2}(s_2) \le \frac{\|u^i(s_2)\|}{m\|u^i(s_2)\|} \Rightarrow \frac{1}{M} \le \frac{ds_1}{ds_2}(s_2) \le \frac{1}{m}$$

from which it follows that

$$U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2) \in L^1(0, \alpha_2).$$

Similarly

$$U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2) \in L^1(0, \alpha_2) \Rightarrow U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1) \in L^1(0, \alpha_1).$$ □

We shall therefore say U is *ω-integrable* if γ has ω-finite b-length and, given a parametrisation s of $\gamma : (0, \alpha] \to M$ which measures b-length, $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s) \in L^1(0, \alpha)$.

Proposition 2.3.10. *If $\omega \sim \overline{\omega}$ then U is ω-integrable $\iff$ U is $\overline{\omega}$-integrable.*

Proof. Let U have components $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ in an ω-frame (e_i) and components $\overline{U}^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ in a $\overline{\omega}$-frame $(\overline{e}_i)$. Let $\overline{e}_i = L^j_i e_j$.

Suppose U is ω-integrable. Then γ has ω-finite b-length. Let s_1 be a parametrisation of $\gamma : (0, \alpha_1] \to M$ which measures b-length. Then $U^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1) \in L^1(0, \alpha_1)$. Since $\omega \sim \overline{\omega}$, $\lim_{s_1 \to 0} L^j_i(s_1)$ exists and $L^j_i(s_1)$ is continuous and bounded on $[0, \alpha_1]$ and

$$\overline{U}^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1) = (L^{-1})^{j_1}_{l_1}(s_1) \ldots (L^{-1})^{j_q}_{l_q}(s_1) L^{k_1}_{i_1}(s_1) \ldots L^{k_p}_{i_p}(s_1) U^{l_1 \ldots l_q}_{k_1 \ldots k_p}(s_1)$$

lies in $L^1(0, \alpha_1)$. Therefore

$$\int_0^{\alpha_1} \overline{U}^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1)\, ds_1 = \int_0^{\alpha_2} \overline{U}^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_1(s_2)) \frac{ds_1}{ds_2}(s_2)\, ds_2 < \infty$$

and $\overline{U}^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2)(ds_1/ds_2)(s_2) \in L^1(0, \alpha_2)$ where $s_2(\alpha_1) = \alpha_2$.

Let s be any parametrisation of $\gamma : (0, alpha] \to M$ and let $\gamma : s \mapsto \gamma(s)$ have tangent $u = u^i e_i = \overline{u}^i \overline{e}_i$. Thus $u^j = (L^{-1})^j_i \overline{u}^i$. As in the proof of the previous proposition

$$(ds_1/ds_2)(s_2) = \|u^i(s_2)\| / \|\overline{u}^i(s_2)\|$$

and from Lemma 2.3.4 $\exists m, M > 0$ such that

$$m\|\overline{u}^i(s_2)\| \leq \|u^i(s_2)\| \leq M\|\overline{u}^i\|$$

and hence

$$m\frac{\|\overline{u}^i(s_2)\|}{\|\overline{u}^i(s_2)\|} \leq \frac{ds_1}{ds_2}(s_2) \leq M\frac{\|\overline{u}^i(s_2)\|}{\|\overline{u}^i(s_2)\|} \Rightarrow m \leq \frac{ds_1}{ds_2}(s_2) \leq M$$

from which it follows that $\overline{U}^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s_2) \in L^1(0, \alpha_2)$. Since $\omega \sim \overline{\omega}$, γ has $\overline{\omega}$-finite b-length and U is $\overline{\omega}$-integrable. Similarly if U is $\overline{\omega}$-integrable then it is ω-integrable.

□

Finally we give conditions on σ under which $\omega \sim \overline{\omega}$ holds.

Lemma 2.3.11. *Let A_i^j be a matrix valued function defined along γ. Then*

$$\|A_i^j(s)\| \in L^1(0,\alpha) \Rightarrow \lim_{s\to 0} P\exp\int_s^\alpha A_i^j(s_0)\,ds_0 \text{ exists and is in } GL_4(\mathbb{R}).$$

Proof. By Proposition 2.1.2

$$\|P\exp\int_{s_0}^{s_1} A_i^j(s)\,ds - \delta_i^j\| \leq \exp\int_{s_0}^{s_1} \|A_i^j(s)\|\,ds - 1.$$

Since $\|A_i^j(s)\| \in L^1(0,\alpha)$ it follows that $P\exp\int_{s_0}^{s_1} A_i^j(s)\,ds \to \delta_i^j$ as $s_0, s_1 \to 0$ and $\lim_{s\to 0} P\exp\int_s^\alpha A_i^j(s_0)\,ds_0$ exists and is in $GL_4(\mathbb{R})$. □

It can be shown that $A_i^j : s \mapsto A_i^j(s)$ satisfies

$$\|A_i^j(s)\| \in L^1(0,\alpha) \iff |A_i^j(s)| \in L^1(0,\alpha)$$

where by $|A_i^j(s)| \in L^1(0,\alpha)$ we mean that each component $A_i^j(s)$ obeys $|A_i^j(s)| \in L^1(0,\alpha)$. We define a relation on the set of connections on $GL(M)$. For connections ω, $\overline{\omega}$ on $GL(M)$ and $\sigma = \overline{\omega} - \omega$

$\omega \simeq \overline{\omega} \iff \exists$ a parameterisation s of γ (not necessarily measuring b-length)

such that in an ω-frame $|u^k(s)\sigma_{ki}^j(s)| \in L^1(0,\alpha)$.

Whether or not $\omega \simeq \overline{\omega}$ does not depend on the choice of ω-frame. Note that s does not have to measure b-length for $\omega \simeq \overline{\omega}$ to hold, however the tangent u of γ depends on the parametrisation s of γ.

Proposition 2.3.12.

(a) If $\omega \simeq \overline{\omega}$ then $|u^k(s)\sigma_{ki}^j(s)| \in L^1(0,\alpha)$ will hold for any parametrisation s of γ.

(b) If $\omega \simeq \overline{\omega}$ then $\omega \sim \overline{\omega}$.

(c) $\simeq$ is an equivalence relation on the set of connections on $GL(M)$.

Proof. (a) Since $\omega \simeq \overline{\omega}$ there exists a parametrisation s of $\gamma : (0,\alpha] \to M$ such that $|u^k(s)\sigma_{ki}^j(s)| \in L^1(0,\alpha)$. Let s' be another parametrisation of $\gamma : (0,\alpha] \to M$. Then

$$\int_0^\alpha |u^k(s)\sigma_{ki}^j(s)|\,ds = \int_0^\alpha |\tilde{u}^k(s)\sigma_{ki}^j(s)|\frac{ds'}{ds}\,ds = \int_0^\alpha |\tilde{u}^k(s')\sigma_{ki}^j(s')|\,ds'$$

where $\gamma : s \mapsto \gamma(s)$ has tangent u and $\gamma : s' \mapsto \gamma(s')$ has tangent $\tilde{u}$. Thus $|\tilde{u}^k(s')\sigma^j_{ki}(s')| \in L^1(0,\alpha)$.

(b) If $\omega \simeq \overline{\omega}$ then $|u^k(s)\sigma^j_{ki}(s)| \in L^1(0,\alpha)$ and hence by Lemma 2.3.11 and equation 2.3.1 $\omega \sim \overline{\omega}$.

(c) Let s be a parametrisation of $\gamma : (0,\alpha] \to M$. Let ω be a connection on $GL(M)$. Then $\omega \simeq \omega$ since $\omega - \omega = 0$ and $0 \in L^1(0,\alpha)$. Let $\overline{\omega}$ be a connection on $GL(M)$ and $\sigma = \overline{\omega} - \omega$. Suppose $\omega \simeq \overline{\omega}$. Then in an ω-frame $|u^k(s)\sigma^j_{ki}(s)| \in L^1(0,\alpha)$, which also holds in a $\overline{\omega}$-frame since by (b) $\omega \sim \overline{\omega}$. Hence in a $\overline{\omega}$-frame $|u^k(s)(-\sigma^j_{ki}(s))| \in L^1(0,\alpha)$ and $\overline{\omega} \sim \omega$. Now let $\tilde{\omega}$ be a connection on $GL(M)$, $\sigma' = \tilde{\omega} - \omega$ and $\sigma'' = \tilde{\omega} - \omega$. Suppose $\omega \simeq \overline{\omega}$ and $\overline{\omega} \simeq \tilde{\omega}$. Then working in a ω-frame

$$\begin{aligned} |u^k(s){\sigma''}^j_{ki}(s)| &= |u^k(s){\sigma'}^j_{ki}(s) + u^k(s)\sigma^j_{ki}(s)| \\ &\leq |u^k(s){\sigma'}^j_{ki}(s)| + |u^k(s)\sigma^j_{ki}(s)| \\ &\in L^1(0,\alpha) \end{aligned}$$

by Proposition 2.3.10, since by (b) $\omega \sim \overline{\omega}$. Hence $\omega \simeq \tilde{\omega}$. □

We now define

$$\mathcal{P} = \{p : TM \to TM \,|\, \text{for each } x = \gamma(s),\ p : T_xM \to T_xM \text{ is a linear } C^0 \text{ map such that } p(v) = v \quad \forall v \in T_x\gamma\}.$$

Thus if $p \in \mathcal{P}$ then p is a tensor of valence $\binom{1}{1}$. In particular $\mathcal{P}$ contains the set of C^0 projection tensors $\mathcal{P}_0 = \{p \in \mathcal{P} \,|\, \text{for each } x = \gamma(s),\ p : T_xM \to T_x\gamma\}$. We note that the identity map $\imath \in \mathcal{P}$ (where in components $\imath^j_i = \delta^j_i$).

Now suppose that $p \in \mathcal{P}$. We define two relations on the set of connections on $GL(M)$ for which γ has ω-finite b-length. For connections ω, $\overline{\omega}$ in this set and $\sigma = \overline{\omega} - \omega$

$$\omega \approx_p \overline{\omega} \iff |p^l_k \sigma^j_{li}| \text{ is } \omega\text{-integrable}$$

$$\omega \cong_p \overline{\omega} \iff p^l_k \sigma^j_{li} \text{ is } C^0\text{-}\omega\text{-quasi-regular.}$$

We note that if $p^j_i = \delta^j_i$ then $\omega \approx_p \overline{\omega} \iff \sigma^j_{ki}$ is ω-integrable and $\omega \cong_p \overline{\omega} \iff \sigma^j_{ki}$ is C^0-ω-quasi-regular.

Theorem 2.3.13. *If $\omega \cong_p \varpi$ then $\omega \approx_p \varpi$, if $\omega \approx_p \varpi$ then $\omega \sim \varpi$.*

Proof. Suppose $\omega \cong_p \varpi$. Then γ has ω-finite b-length and $p^l_k \sigma^j_{li}$ is C^0-ω-quasi-regular. Let s be a parametrisation of γ which measures b-length with respect to ω. Working in the basis (e_i) with respect to which s is measured

$$\lim_{s\to 0} p^l_k(s)\sigma^j_{li}(s) \text{ exists} \Rightarrow |p^l_k(s)\sigma^j_{li}(s)| \in L^1(0,\alpha)$$

and hence $\omega \approx_p \varpi$.

Now suppose $\omega \approx_p \varpi$. As before γ has ω-finite b-length. Let s be a parametrisation of γ which measures b-length with respect to ω. Working in the basis (e_i) with respect to which s is measured, Lemma 2.2.9 implies that $\|u^i\| = 1$ and hence the components of u obey $|u^i(s)| \leq 1$. Thus

$$|u^k(s)\sigma^j_{ki}(s)| = |u^k(s)p^l_k(s)\sigma^j_{li}(s)| \leq |u^k(s)||p^l_k(s)\sigma^j_{li}(s)| \leq |p^l_k(s)\sigma^j_{li}(s)| \in L^1(0,\alpha)$$

and hence $|u^k(s)\sigma^j_{ki}(s)| \in L^1(0,\alpha)$. Therefore $\omega \simeq \varpi$ and $\omega \sim \varpi$. □

Theorem 2.3.14. *$\cong_p$ and $\approx_p$ are equivalence relations on the set of connections on $GL(M)$ for which γ has finite b-length.*

Proof. Let ω be a connection on $GL(M)$ for which γ has finite b-length and let $\sigma = \omega - \omega$. Then $\sigma = 0$ and $p^l_k\sigma^j_{li}$ is C^0-ω-quasi-regular. Thus $\omega \cong_p \omega$.

Let ϖ be a connection on $GL(M)$ for which γ has finite b-length. Suppose $\omega \cong_p \varpi$ and let $\sigma = \varpi - \omega$. Then $p^l_k\sigma^j_{li}$ is C^0-ω-quasi-regular. By Theorem 2.3.13 $\omega \sim \varpi$ and thus $-p^l_k\sigma^j_{li}$ is C^0-ϖ-quasi-regular. Thus $\varpi \cong_p \omega$.

Now let $\tilde{\omega}$ be a connection on $GL(M)$ for which γ has finite b-length. Suppose $\omega \cong_p \varpi$ and $\varpi \cong_p \tilde{\omega}$. Let $\sigma = \varpi - \omega$, $\sigma' = \tilde{\omega} - \varpi$ and $\sigma'' = \tilde{\omega} - \omega$. Then $p^l_k\sigma^j_{li}$ is C^0-ω-quasi-regular and $p^l_k{\sigma'}^j_{li}$ is C^0-ϖ-quasi-regular. By Theorem 2.3.13 $\omega \sim \varpi$ and $p^l_k{\sigma'}^j_{li}$ is C^0-ω-quasi-regular. Now $\sigma'' = \sigma' + \sigma$ and thus $p^l_k{\sigma''}^j_{li}$ is C^0-ω-quasi-regular. By Theorem 2.3.13 $\varpi \sim \tilde{\omega}$ thus $\omega \sim \tilde{\omega}$ and therefore $-p^l_k\sigma^j_{li}$ is C^0-$\tilde{\omega}$-quasi-regular and $\omega \cong_p \tilde{\omega}$.

We can show that $\approx_p$ is an equivalence relation in a manner similar to the proof of 2.3.12(c). $\square$

2.4 First and second Cartan equations

Let $x_0 \in M$ and let U be a neighbourhood of x_0. Let (e_i) be a C^2 section of $GL(M)$ above U (thus for each $x \in U$, $(e_i)|_x$ is a basis of T_xM). Let ω, ϖ be connections on $GL(M)$ and let $\sigma = \varpi - \omega$. ω has torsion T^k where T^k is a 2-form valued tensor given by the first Cartan equation

$$T^k = de^k + \omega^k_j \wedge e^j$$

where $\omega^k_j = \omega^k_{ij} e^i$ and (e^i) are the 1-forms dual to (e_i). Similarly ϖ has torsion $\overline{T}^k$ given by

$$\overline{T}^k = de^k + \varpi^k_j \wedge e^j$$

where $\varpi^k_j = \varpi^k_{ij} e^i$. Thus

$$\overline{T}^k - T^k = \sigma^k_j \wedge e^j$$

where $\sigma^k_j = \sigma^k_{ij} e^i$ and hence in the basis (e_i)

$$\overline{T}^k_{ij} - T^k_{ij} = \sigma^k_{[ij]}.$$

In particular if $\overline{T}^k_{ij} = T^k_{ij}$, for example if ω and ϖ both have zero torsion, then $\sigma^k_{[ij]} = 0$ and $\sigma^k_{ij} = \sigma^k_{ji}$.

ω has curvature Ω^j_i where Ω^j_i is a 2-form valued tensor given by the second Cartan equation

$$\Omega^j_i = d\omega^j_i + \omega^j_k \wedge \omega^k_i.$$

Similarly ϖ has curvature $\overline{\Omega}^j_i$ given by

$$\overline{\Omega}^j_i = d\varpi^j_i + \varpi^j_k \wedge \varpi^k_i$$

and thus

$$\begin{aligned}\overline{\Omega}{}^j_i - \Omega^j_i &= d\sigma^j_i + \overline{\omega}{}^j_k \wedge \overline{\omega}{}^k_i - \omega^j_k \wedge \omega^k_i \\ &= d\sigma^j_i + (\sigma^j_k + \omega^j_k) \wedge (\sigma^k_i + \omega^k_i) - \omega^j_k \wedge \omega^k_i \\ &= d\sigma^j_i + \sigma^j_k \wedge \sigma^k_i + \sigma^j_k \wedge \omega^k_i + \omega^j_k \wedge \sigma^k_i \\ &= D\sigma^j_i + \sigma^j_k \wedge \sigma^k_i\end{aligned}$$

where D is the exterior covariant derivative associated with ω.

Now let γ be a C^1 inextendible curve

$$\gamma : (0, \alpha] \longrightarrow M : s \longmapsto \gamma(s) \qquad \alpha > 0.$$

From the above we have

Theorem 2.4.1. *Let $r \geq 0$. Suppose that along γ, Ω^j_i is C^r-ω-quasi-regular and σ^j_i is C^{r+1}-ω-quasi-regular. If $\omega \sim \overline{\omega}$ then $\overline{\Omega}{}^j_i$ is C^r-$\overline{\omega}$-quasi-regular.*

We note that if γ has ω-finite b-length then by Theorem 2.3.13 it follows that if σ^j_i is C^{r+1}-ω-quasi-regular for $r \geq 0$, then $\omega \sim \overline{\omega}$.

2.5 Path-ordered exponentials of the curvature

Let ω be a connection on $GL(M)$. We have shown how the result of parallelly propagating a basis along a curve with respect to ω may be expressed in terms of the path-ordered exponential of ω. It follows that elements of holonomy may be expressed in this way. We now show how elements of holonomy may also be expressed in terms of path-ordered exponentials of the curvature of ω.

Let γ be a C^1 map

$$\gamma : [0,1] \times [u_0, u_1] \longrightarrow M : (s, u) \longmapsto \gamma(s, u) = \gamma_u(s)$$

where $\gamma_u(0) = \gamma_u(1)$. Thus $s \longmapsto \gamma_u(s)$ is a closed loop for $u_0 \leq u \leq u_1$. Now pick a basis $e_i(0, u_1)$ at $\gamma(0, u_1)$ and parallelly propagate it along $\kappa(u) = \gamma(0, u)$ to give $e_i(0, u)$ and then along $\gamma_u(s)$ to give $e_i(s, u)$. Thus $e_i(s, u)$ is a lift of $\gamma(s, u)$.

Since γ_u is a closed loop, $e_i(0, u)$ and $e_i(1, u)$ are defined at the same point. If we set

$$e_i(1, u) = L_i^j(u) e_j(0, u)$$

then $L_i^j(u) \in GL_4(\mathbb{R})$ is the element of holonomy generated by parallelly propagating $e_i(0, u)$ round γ_u. From [V85]

$$L_i^j(u_1) - L_i^j(u_0) = \int_{u_0}^{u_1} \int_0^1 L_k^j(u) \Omega_i^k(X(s, u), Y(s, u))\, ds du \tag{2.5.1}$$

where ω has 2-form valued curvature Ω_i^j and X, Y are the images of ∂_s, ∂_u induced by γ. (∂_s, ∂_u are tangent to $[0, 1] \times [u_0, u_1]$ whereas X, Y are tangent to M.) It is assumed in [V85] that the connection is metric and torsion free, but the proof given will in fact work for any connection.

By repeated iteration

$$\begin{aligned} L_i^j(u_1) = L_i^j(u_0) + \sum_{m=1}^{N} \int_{u_0}^{u_1} \int_0^1 \cdots \int_{u_0}^{u_m} \int_0^1 L_k^j(u_0) \Omega_{i_2}^k(X(s_{m+1}, u_{m+1}), Y(s_{m+1}, u_{m+1})) \\ \ldots \Omega_i^{i_m}(X(s_2, u_2), Y(s_2, u_2))\, ds_{m+1} du_{m+1} \ldots ds_2 du_2 + R_i^j(u_1) \end{aligned}$$

where the remainder term

$$\begin{aligned} R_i^j(u_1) = \int_{u_0}^{u_1} \int_0^1 \cdots \int_{u_0}^{u_{N+1}} \int_0^1 L_k^j(u_{N+2}) \Omega_{i_2}^k(X(s_{N+2}, u_{N+2}), Y(s_{N+2}, u_{N+2})) \\ \ldots \Omega_i^{i_{N+1}}(X(s_2, u_2), Y(s_2, u_2))\, ds_{N+2} du_{N+2} \ldots ds_2 du_2 \end{aligned}$$

can be shown to obey $\|R_i^j(u_1)\| \to 0$ as $N \to \infty$. As before, suppressing indices and using matrix notation, we have

$$\begin{aligned} L(u_1) &= L(u_0) \\ &+ L(u_0) \sum_{m=1}^{\infty} \frac{1}{m!} \int_{u_0}^{u_1} \int_0^1 \cdots \int_{u_0}^{u_1} \int_0^1 : \Omega(X(s_{m+1}, u_{m+1}), Y(s_{m+1}, u_{m+1})) \\ &\ldots \Omega(X(s_2, u_2), Y(s_2, u_2)) : \, ds_{m+1} du_{m+1} \ldots ds_2 du_2 \end{aligned}$$

where : : indicates that the expression should be ordered so that terms with a *smaller* u value precede those with a *larger* u value. Note that this is the reverse of the ordering in the path-ordered exponential of the connection, and that : : denotes u ordering and not s ordering. We write

$$L_i^j(u_1) = L_k^j(u_0)\, P_u \exp \int_{u_0}^{u_1} \int_0^1 \Omega_i^k(X(s,u), Y(s,u))\, ds du. \qquad (2.5.2)$$

Since the inner integral is an ordinary integral we can set

$$W_i^j(u) = \int_0^1 \Omega_i^j(X(s,u), Y(s,u))\, ds$$

which gives

$$L_i^j(u_1) = L_k^j(u_0) P_u \exp \int_{u_0}^{u_1} W_i^k(u)\, du$$

where $P_u \exp \int_{u_0}^{u_1} \dots du$ is the more usual path-ordered integral, though still ordered so that terms with a smaller u value precede those with a larger u value.

By swapping u_0, u_1 in equation (2.5.1) we also obtain

$$L_i^j(u_0) = L_k^j(u_1)\, P_u \exp \int_{u_1}^{u_0} \int_0^1 \Omega_i^k(X(s,u), Y(s,u))\, ds du. \qquad (2.5.3)$$

Let $U = Im\,\gamma$. If γ is a diffeomorphism then U is a 2-surface with disjoint boundaries γ_{u_0} and γ_{u_1}. If however γ_{u_0} is a point then

$$L_i^j(u_1) = P_u \exp \int_{u_0}^{u_1} W_i^j(u)\, du.$$

Under suitable conditions U will be a 2-surface with boundary γ_{u_1}, however $L_i^j(u_1)$ will depend only on γ_{u_1} and the choice of basis $e_i(0,u_1)$ at $\gamma(0,u_1)$, but not on the spanning surface U or its parametrisation.

Now suppose $\kappa : (0,1] \to M$ is a C^1 curve of finite b-length terminating at a singularity. Let $\gamma \in \Omega_\kappa$ (where Ω_κ is the loop space defined in section 1.6) so

$$\gamma : [0,1] \times (0,1] \to M : (s,u) \mapsto \gamma(s,u) = \gamma_u(s).$$

As above let $e_i(1,u) = L_i^j(u) e_j(0,u)$ where a choice of $e_i(0,1)$ is parallelly propagated along $\kappa(u)$ to give $\overline{\kappa}(u)$ and then along γ_u for each u to give a lift $e_i(s,u)$ of γ. Thus

$$(L^{-1})_k^j(u_0) L_i^k(u_1) = P_u \exp \int_{u_0}^{u_1} W_i^j(u)\, du$$

for $0 < u_0, u_1 \leq 1$. The lift $\overline{\gamma}$ of γ we have chosen will generate a well-defined element of s-holonomy $L^j_i(0) := \lim_{u \to 0} L^j_i(u)$ if and only if

$$(L^{-1})^j_k(u_0)L^k_i(u_1) \longrightarrow \delta^j_i \text{ as } u_0, u_1 \longrightarrow 0$$

which by Lemma 2.3.11 will hold if $\dot{W}^j_i(u) \in L^1(0,1)$. If κ terminates at a quasi-regular singularity then by Theorem 1.6.2, $H_{\overline{\kappa}}$ will exist and by Theorem 1.6.1, $L^j_i(0)$ will exist if $\overline{\gamma}$ satisfies the area condition. However if κ terminates at a curvature singularity we do not in general expect that $L^j_i(0)$ or $H_{\overline{\kappa}}$ will exist.

However we now discuss conditions under which $L^j_i(0)$ will exist even if κ does terminate at a curvature singularity. (We do not impose an area condition on $\overline{\gamma}$.)

We shall say that the curves $\gamma_u : s \longmapsto \gamma_u(s)$ are *parametrised proportional to b-length* if the b-length $l(s,u)$ of $\gamma_u|_{[0,s]}$ measured in the basis $(e_i(s,u))$ obeys

$$l(s,u) = sl(u)$$

where $l(u)$ is the b-length of γ_u, measured in the basis $(e_i(s,u))$. In this case

$$\int_0^s \|X^i(s_0,u)\|\, ds_0 = l(s,u) = sl(u) \Rightarrow \|X^i(s,u)\| = l(u) \longrightarrow 0 \text{ as } u \longrightarrow 0.$$

Given any $\gamma \in \Omega_\kappa$ and lift of γ, the γ_u curves can be reparametrised proportional to b-length. The value of $(L^{-1})^j_k(u_0)L^k_i(u_1)$, and thus of $L^j_i(0) := \lim_{u \to 0} L^j_i(u)$ if it exists, will not be affected by this reparametrisation.

We now consider the curves $\kappa_s(u) = \gamma(s,u)$. These curves do depend on the parametrisation of the curves γ_u and need not in general have finite b-length unless $s = 0$ in which case $\kappa_0(u) = \kappa(u)$.

We note that $(e_i(s,u))$ will not in general be an ω-frame along κ_s since it is obtained by first parallelly propagating $(e_i(0,1))$ along $\kappa(u)$ and then parallelly propagating $(e_i(0,u))$ along $\gamma_u(s) = \gamma(s,u)$. We define another basis $(\tilde{e}_i(s,u))$ as follows. Let $(\tilde{e}_i(0,1)) = (e_i(0,1))$, parallelly propagate $(\tilde{e}_i(0,1))$ along γ_1 to give $(\tilde{e}_i(s,1))$, and parallelly propagate $(\tilde{e}_i(s,1))$ along κ_s to give $(\tilde{e}_i(s,u))$. Thus $(\tilde{e}_i(s,u))$ will be an ω-frame along κ_s (see diagram 2.5.1). We define

$$\tilde{e}_i(s,u) = l_i^j(s,u)e_j(s,u)$$

where $l_i^j(s,u) \in GL_4(\mathbb{R})$. We note that since $\tilde{e}_i(s,1) = e_i(s,1)$, $l_i^j(s,1) = \delta_i^j$.

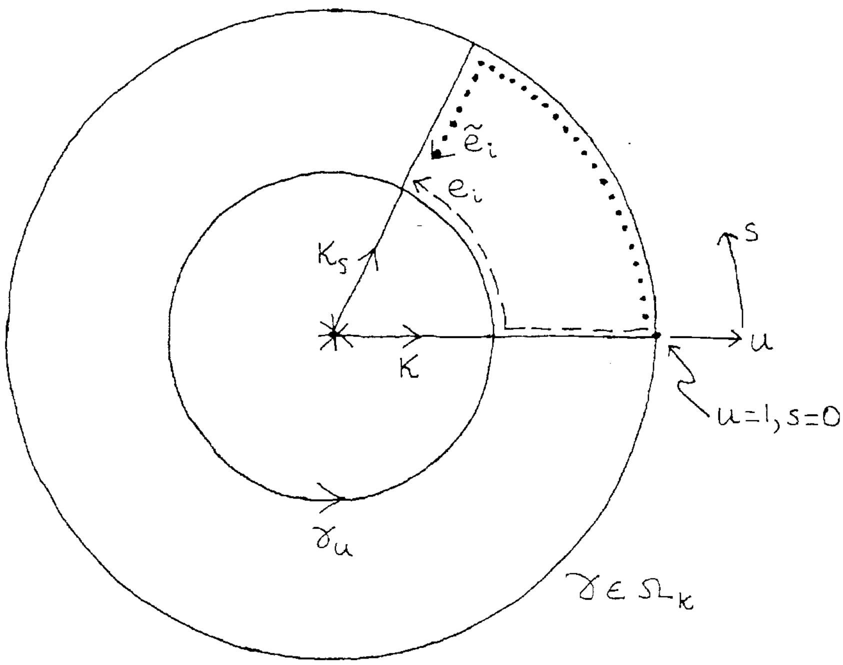

Diagram 2.5.1

We shall say that γ is *sufficiently regular with respect to* Ω_i^j if

(a) the curves γ_u are parametrised proportional to b-length, and for each $s \in [0,1]$, κ_s has finite b-length, and the b-length $\lambda(s)$ of κ_s measured in the basis $(\tilde{e}_i)$ is continuous in s, and $Y(s,u) \neq 0$

(b) $\exists \phi \in L^1(0,1), \psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ such that, in the basis $(\tilde{e}_i)$, where for each $s \in [0,1]$, $\tilde{u}$ measures b-length along κ_s with respect to $(\tilde{e}_i)$ such that $\tilde{u} \to 0$ as $u \to 0$, and $l(s,\tilde{u}) = l(u)$ where we regard u as a function of s and $\tilde{u}$,

$$\|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\| l(s,\tilde{u}) \leq \phi(s)\psi(\tilde{u}).$$

Condition (a) imposes constraints on the geometry of γ while condition (b) imposes constraints on the curvature Ω_i^j. In particular condition (b) is certainly satisfied if there exist $\phi \in L^1(0,1)$ and $\psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ such that, in the basis $(\tilde{e}_i)$,

$$\|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\| \leq \phi(s)\psi(\tilde{u})$$

or even

$$\|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\| \leq \psi(\tilde{u})$$

though these are stronger conditions than condition (b).

We might expect that $l(u) = O(u)$, at least if u measures b-length along κ, but it may happen that the γ_u loops "crinkle up" as $u \longrightarrow 0$ and in fact γ may be chosen so that $l(u) \longrightarrow 0$ arbitrarily slowly. We could, though we shall not, restrict Ω_κ to contain those lassos for which $l(u) = O(u)$ when u measures b-length along κ, though we note that even if u measures b-length along κ, it might not do so along κ_s for $s \neq 0$.

We shall also need the following condition. We shall say that γ is *well bounded with respect to* ω if $\exists\alpha > 0$ such that

$$\|(l^{-1})_i^j(s,u)X^i(s,u)\| \leq \alpha\|X^j(s,u)\|$$

where X^j are the components of X in the basis (e_i).

This condition may seem unduly restrictive but we shall see that it holds in an important case in section 5.4.

We shall prove the following two theorems.

Theorem 2.5.1. *Let γ be sufficiently regular with respect to Ω_i^j and well bounded with respect to ω. Then for each $s \in [0,1]$, $l_i^j(s) := \lim_{u\to 0} l_i^j(s,u)$ exists, and $l_i^j(s)$ is continuous.*

Theorem 2.5.2. *Let γ be sufficiently regular with respect to Ω_i^j and well bounded with respect to ω. Then $L_i^j(0) := \lim_{u\to 0} L_i^j(u)$ exists.*

We note that this theorem tells us conditions under which $\lim_{u\to 0} L_i^j(u)$ exists but does not tell us its value.

The remainder of this section is devoted to proving Theorems 2.5.1 and 2.5.2. We will first need to prove a number of intermediate results.

Suppose now that γ is sufficiently regular with respect to Ω_i^j. For each $s \in [0,1]$, let $\tilde{u}$ to be the parametrisation of κ_s which measures b-length along κ_s with respect to $(\tilde{e}_i)$ such that $\tilde{u} \to 0$ as $u \to 0$ and let $\lambda(s)$ be the b-length of κ_s measured in the basis $(\tilde{e}_i)$. Since γ is sufficiently regular, we make a choice of $\phi \in L^1(0,1), \psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ such that

$$\|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\| l(s,\tilde{u}) \leq \phi(s)\psi(\tilde{u})$$

where $\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}$ are the components of Ω_i^j in the basis $(\tilde{e}_i)$.

Now define the function $h : (0,1] \to \mathbb{R}$

$$h(u_1) = \int_0^{\sup_{s\in[0,1]} \tilde{u}(s,u_1)} \psi(\tilde{u})\, d\tilde{u}$$

where we regard $\tilde{u}$ as a function of s and u.

Lemma 2.5.3.

(a) $\|\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\tilde{l}}(s,\tilde{u})\| = 1$ *where Y has components $Y^{\tilde{l}}$ in the basis $(\tilde{e}_i)$.*

(b) $h(u_1) \to 0$ *as* $u_1 \to 0$.

Proof. (a) Let the curve $\kappa_s : \tilde{u} \mapsto \kappa_s(\tilde{u})$ have tangent $\tilde{Y}$. Now $\partial_{\tilde{u}} = \frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})\partial_u$ so

$$\tilde{Y} = \gamma_*(\partial_{\tilde{u}}) = \gamma_*(\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})\partial_u) = \frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})\gamma_*(\partial_u) = \frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y.$$

By Lemma 2.2.9, $\|\tilde{Y}^{\tilde{l}}\| = 1$ where $\tilde{Y}$ has components $\tilde{Y}^{\tilde{l}}$ in the basis $(\tilde{e}_i)$. Hence

$$\|\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\tilde{l}}(s,\tilde{u})\| = 1.$$

(b)

$$\tilde{u} = \tilde{u}(s,u) = \int_0^u \|Y^{\bar{i}}(s,u_0)\|\, du_0 = \lambda(s) - \int_u^1 \|Y^{\bar{i}}(s,u_0)\|\, du_0.$$

Since γ is sufficiently regular with respect to Ω_i^j, $\lambda(s)$ and $Y^{\bar{i}}(s,u)$ are continuous and $Y(s,u) \neq 0$. It follows that $\tilde{u} = \tilde{u}(s,u)$ is continuous in s, u and for fixed s, strictly monotonic in u. Hence $u = u(s,\tilde{u})$ is also continuous in s, u and for fixed s, strictly monotonic in u.

It follows from this (and the compactness of $[0,1]$) that $\sup_{s\in[0,1]} \tilde{u}(s,u_1) \to 0$ as $u_1 \to 0$. Now $\psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ and hence

$$h(u_1) = \int_0^{\sup_{s\in[0,1]} \tilde{u}(s,u_1)} \psi(\tilde{u})\, d\tilde{u} \to 0 \text{ as } u_1 \to 0. \qquad \square$$

Given $0 \leq s_0 \leq s_1 \leq 1$ and $0 < u_0 \leq u_1 \leq 1$ we now define a map $\rho : [0,1] \times [s_0, s_1] \to M : (\sigma,\tau) \mapsto \rho(\sigma,\tau)$ as follows (see diagram 2.5.2)

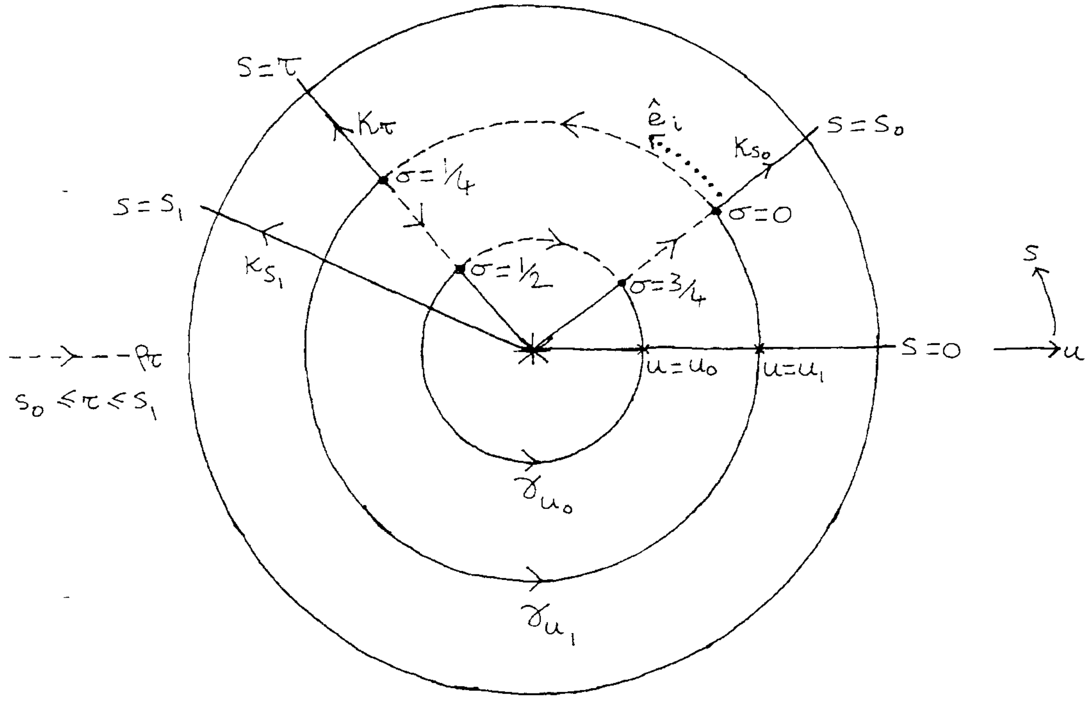

Diagram 2.5.2

$$\rho_\tau(\sigma) = \rho(\sigma, \tau) = \begin{cases} \gamma(s_0 + (\tau - s_0)4\sigma, u_1) & 0 \le \sigma \le \frac{1}{4} \\ \gamma(\tau, u_1 - (u_1 - u_0)4(\sigma - \frac{1}{4})) & \frac{1}{4} \le \sigma \le \frac{1}{2} \\ \gamma(\tau - (\tau - s_0)4(\sigma - \frac{1}{2}), u_0) & \frac{1}{2} \le \sigma \le \frac{3}{4} \\ \gamma(s_0, u_0 + (u_1 - u_0)4(\sigma - \frac{3}{4})) & \frac{3}{4} \le \sigma \le 1. \end{cases}$$

Thus each ρ_τ is a closed loop and $\rho_\tau([0,1/4])$ lies on γ_{u_1}, $\rho_\tau([1/4,1/2])$ lies on κ_τ, $\rho_\tau([1/2,3/4])$ lies on γ_{u_0}, and $\rho_\tau([3/4,1])$ lies on κ_{s_0}. ρ_{s_0} lies entirely on κ_{s_0}. We note that the image of the curve $\tau \mapsto \rho(0,\tau)$ is the fixed point $\gamma(s_0, u_1)$.

We define a basis $(\hat{e}_i(\sigma,\tau))$ at $\rho(\sigma,\tau)$ by setting $\hat{e}_i(0,\tau) = e_i(s_0,u_1)$ and parallelly propagating $\hat{e}_i(0,\tau)$ along ρ_τ to give $\hat{e}_i(\sigma,\tau)$. We set

$$\hat{e}_i(1,\tau) = \hat{l}_i^j(\tau)\hat{e}_j(0,\tau)$$

where $\hat{l}_i^j(\tau) \in GL_4(\mathbb{R})$. Since the image of ρ_{s_0} lies entirely on κ_{s_0}, $(\hat{e}_i(1,s_0))$ is obtained by parallelly propagating $(\hat{e}_i(0,s_0))$ along κ_{s_0} from $\kappa_{s_0}(u_1)$ to $\kappa_{s_0}(u_0)$ and back again to $\kappa_{s_0}(u_1)$, and thus $\hat{l}_i^j(s_0) = \delta_i^j$.

We now suppose that in addition to being sufficiently regular with respect to Ω_i^j, γ is well bounded with respect to ω. We therefore make a choice of $\alpha > 0$ such that

$$\|(l^{-1})_i^j(s,u)X^i(s,u)\| \le \alpha\|X^j(s,u)\|$$

where X^j are the components of X in the basis (e_i).

Lemma 2.5.4.

$$\|\hat{l}_i^j(s_1)\| \le \exp \alpha h(u_1) \int_{s_0}^{s_1} \|l_i^j(s,u_1)\|\|(l^{-1})_i^j(s,u_1)\|\phi(s)\,ds + 1$$

$$\|\hat{l}_i^j(s_1) - \delta_i^j\| \le \exp \alpha h(u_1) \int_{s_0}^{s_1} \|l_i^j(s,u_1)\|\|(l^{-1})_i^j(s,u_1)\|\phi(s)\,ds - 1$$

$$\|(\hat{l}^{-1})_i^j(s_1)\| \le \exp -\alpha h(u_1) \int_{s_0}^{s_1} \|l_i^j(s,u_1)\|\|(l^{-1})_i^j(s,u_1)\|\phi(s)\,ds + 1.$$

Proof. Working in the basis $(\hat{e}_i)$ we have from equation (2.5.2)

$$(\hat{l}^{-1})_k^j(s_0)\hat{l}_i^k(s_1) = P_\tau \exp \int_{s_0}^{s_1}\int_0^1 \Omega_{kli}{}^j(\sigma,\tau)\hat{X}^k(\sigma,\tau)\hat{Y}^l(\sigma,\tau)\,d\sigma d\tau$$

where $\hat{X}$, $\hat{Y}$ are the images of ∂_σ, ∂_τ induced by ρ.

Now $\hat{l}_i^j(s_0) = \delta_i^j$ and

$$\hat{X} = \frac{\partial s}{\partial \sigma} X + \frac{\partial u}{\partial \sigma} Y \qquad \hat{Y} = \frac{\partial s}{\partial \tau} X + \frac{\partial u}{\partial \tau} Y$$

where $\rho(\sigma, \tau) = \gamma(s(\sigma,\tau), u(\sigma,\tau))$. Hence

$$(\hat{X}, \hat{Y}) = \begin{cases} (4(\tau - s_0)X, 4\sigma X) & 0 \le \sigma \le \frac{1}{4} \\ (-4(u_1 - u_0)Y, X) & \frac{1}{4} \le \sigma \le \frac{1}{2} \\ (-4(\tau - s_0)X, 4(\frac{3}{4} - \sigma)X) & \frac{1}{2} \le \sigma \le \frac{3}{4} \\ (4(u_1 - u_0)Y, 0) & \frac{3}{4} \le \sigma \le 1 \end{cases}$$

and

$$\begin{aligned} \hat{l}_i^j(s_1) &= P_\tau \exp \int_{s_0}^{s_1} \int_0^{1/4} \Omega_{kli}{}^j(\sigma,\tau) 4(\tau - s_0) X^k(\sigma,\tau) 4\sigma X^l(\sigma,\tau)\, d\sigma \\ &+ \int_{1/4}^{1/2} \Omega_{kli}{}^j(\sigma,\tau) - 4(u_1 - u_0) Y^k(\sigma,\tau) X^l(\sigma,\tau)\, d\sigma \\ &+ \int_{1/2}^{3/4} \Omega_{kli}{}^j(\sigma,\tau) - 4(\tau - s_0) X^k(\sigma,\tau) 4(\frac{3}{4} - \sigma) X^l(\sigma,\tau)\, d\sigma \\ &+ \int_{3/4}^{1} \Omega_{kli}{}^j(\sigma,\tau) 4(u_1 - u_0) Y^k(\sigma,\tau).0\, d\sigma d\tau \\ &= P_\tau \exp \int_{s_0}^{s_1} \int_{1/4}^{1/2} \Omega_{kli}{}^j(\sigma,\tau) X^k(\sigma,\tau) Y^l(\sigma,\tau) 4(u_1 - u_0)\, d\sigma d\tau \end{aligned}$$

since $\Omega_i^j(X,X) = 0$ and $\Omega_i^j(X,Y) = -\Omega_i^j(Y,X)$. Now for $\frac{1}{4} \le \sigma \le \frac{1}{2}$,

$$u = u_1 - (u_1 - u_0)4(\sigma - \frac{1}{4}) \Rightarrow du = -4(u_1 - u_0)d\sigma \qquad s = \tau \Rightarrow ds = d\tau$$

and hence by Proposition 2.1.2

$$\|\hat{l}_i^j(s_1)\| \le \exp \int_{s_0}^{s_1} \int_{u_0}^{u_1} \|\Omega_{kli}{}^j(s,u) X^k(s,u) Y^l(s,u)\|\, du ds + 1.$$

Since $(\hat{e}_i(s, u_1))$ is obtained by parallelly propagating $(e_i(s_0, u_1))$ along γ_{u_1}, $(\hat{e}_i(s,u_1)) = (e_i(s,u_1))$ and for $\frac{1}{4} \le \sigma \le \frac{1}{2}$,

$$\Omega_{ijk}{}^l(s,u) X^k(s,u) Y^l(s,u) = l_i^{\bar{m}}(s,u_1)(l^{-1})_{\bar{n}}^j(s,u_1)\Omega_{\bar{k}\bar{l}\bar{m}}{}^{\bar{n}}(s,u) X^{\bar{k}}(s,u) Y^{\bar{l}}(s,u)$$

where Ω_i^j, X, Y have components $\Omega_{\bar{k}\bar{j}\bar{m}}{}^{\bar{n}}$, $X^{\bar{k}}$, $Y^{\bar{l}}$ in the basis $(\tilde{e}_i)$. Hence

$$\|\hat{l}_i^j(s_1)\| \le \exp \int_{s_0}^{s_1} \int_{u_0}^{u_1} \|l_i^{\bar{m}}(s,u_1)(l^{-1})_{\bar{n}}^j(s,u_1)\Omega_{\bar{k}\bar{l}\bar{m}}{}^{\bar{n}}(s,u) X^{\bar{k}}(s,u) Y^{\bar{l}}(s,u)\| + 1.$$

Since γ is sufficiently regular with respect to Ω_i^j,

$$\|\Omega_{\bar{k}\bar{l}\bar{i}}{}^{\bar{j}}(s,\tilde{u})\| l(s,\tilde{u}) \leq \phi(s)\psi(\tilde{u})$$

and hence by Lemma 2.5.3(a)

$$\begin{aligned}\|\Omega_{\bar{k}\bar{l}\bar{i}}{}^{\bar{j}}(s,\tilde{u})\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\bar{l}}(s,\tilde{u})\| l(s,\tilde{u}) &\leq \|\Omega_{\bar{k}\bar{l}\bar{i}}{}^{\bar{j}}(s,\tilde{u})\|\|\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\bar{l}}(s,\tilde{u})\| l(s,\tilde{u}) \\ &\leq \phi(s)\psi(\tilde{u}).\end{aligned}$$

Therefore

$$\begin{aligned}\|\hat{l}_i^j(s_1)\| \leq \exp \int_{s_0}^{s_1} &\|l_i^{\bar{m}}(s,u_1)\|\|(l^{-1})_{\bar{n}}^j(s,u_1)\| \\ &\int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \|\Omega_{\bar{k}\bar{l}\bar{m}}{}^{\bar{n}}(s,\tilde{u})Y^{\bar{l}}(s,\tilde{u})\|\|X^{\bar{k}}(s,\tilde{u})\|\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})\,d\tilde{u}ds + 1.\end{aligned}$$

Since γ is sufficiently regular with respect to Ω_i^j,

$$\|X^{\bar{k}}(s,\tilde{u})\| = \|(l^{-1})_p^{\bar{k}}(s,\tilde{u})\overline{X}^p(s,\tilde{u})\| \leq \alpha\|\overline{X}^k(s,\tilde{u})\| = \alpha l(s,\tilde{u})$$

where X has components $\overline{X}^k$ in the basis (e_i). Hence

$$\begin{aligned}\|\hat{l}_i^j(s_1)\| &\leq \exp \int_{s_0}^{s_1} \|l_i^{\bar{m}}(s,u_1)\|\|(l^{-1})_{\bar{n}}^j(s,u_1)\| \\ &\qquad \int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \|\Omega_{\bar{k}\bar{l}\bar{m}}{}^{\bar{n}}(s,\tilde{u})Y^{\bar{l}}(s,\tilde{u})\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})\|\alpha l(s,\tilde{u})\,d\tilde{u}ds + 1 \\ &\leq \exp \alpha \int_{s_0}^{s_1} \|l_i^{\bar{m}}(s,u_1)\|\|(l^{-1})_{\bar{n}}^j(s,u_1)\| \int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \|\phi(s)\psi(\tilde{u})\|\,d\tilde{u}ds + 1 \\ &\leq \exp \alpha \int_{s_0}^{s_1} \|l_i^{\bar{m}}(s,u_1)\|\|(l^{-1})_{\bar{n}}^j(s,u_1)\|\phi(s)\,ds \int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \psi(\tilde{u})\,d\tilde{u} + 1.\end{aligned}$$

Now

$$\int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \psi(\tilde{u})\,d\tilde{u} \leq h(u_1)$$

and hence

$$\|\hat{l}_i^j(s_1)\| \leq \exp \alpha h(u_1) \int_{s_0}^{s_1} \|l_i^j(s,u_1)\|\|(l^{-1})_i^j(s,u_1)\|\phi(s)\,ds + 1.$$

From Proposition 2.1.2 we also have

$$\|\hat{l}_i^j(s_1) - \delta_i^j\| \leq \exp \int_{s_0}^{s_1}\int_{u_0}^{u_1} \|\Omega_{kli}{}^j(s,u)X^k(s,u)Y^l(s,u)\|\,duds - 1$$

and so analogously to the above

$$\|\hat{l}^j_i(s_1) - \delta^j_i\| \leq \exp \alpha h(u_1) \int_{s_0}^{s_1} \|l^j_i(s, u_1)\| \|(l^{-1})^j_i(s, u_1)\| \phi(s)\, ds - 1.$$

Now from equation (2.5.3) we have

$$(\hat{l}^{-1})^j_k(s_1)\hat{l}^k_i(s_0) = P_\tau \exp \int_{s_1}^{s_0} \int_0^1 \Omega_{kli}{}^j(\sigma, \tau)\hat{X}^k(\sigma, \tau)\hat{Y}^l(\sigma, \tau)\, d\sigma d\tau$$

and so analogously to the above by Proposition 2.1.2

$$\|(\hat{l}^{-1})^j_i(s_1)\| \leq \exp \int_{s_1}^{s_0} \int_{u_0}^{u_1} \|\Omega_{kli}{}^j(s, u)X^k(s, u)Y^l(s, u)\|\, duds + 1$$

and as before

$$\begin{aligned} \|(\hat{l}^{-1})^j_i(s_1)\| &\leq \exp \alpha h(u_1) \int_{s_1}^{s_0} \|l^j_i(s, u_1)\| \|(l^{-1})^j_i(s, u_1)\| \phi(s)\, ds + 1 \\ &= \exp -\alpha h(u_1) \int_{s_0}^{s_1} \|l^j_i(s, u_1)\| \|(l^{-1})^j_i(s, u_1)\| \phi(s)\, ds + 1. \end{aligned}$$

□

We now use a similar method to prove the next two results.

For $0 \leq a \leq 1, 0 < b \leq 1$ let $\rho_{a,b} : [0, 1] \longrightarrow M$ be the map (see diagram 2.5.3)

$$\rho_{a,b}(\sigma) = \begin{cases} \gamma(4\sigma a, 1) & 0 \leq \sigma \leq \frac{1}{4} \\ \gamma(a, 1 - 4(\sigma - \frac{1}{4})(1 - b)) & \frac{1}{4} \leq \sigma \leq \frac{1}{2} \\ \gamma(a - 4(\sigma - \frac{1}{2})a, b) & \frac{1}{2} \leq \sigma \leq \frac{3}{4} \\ \gamma(0, b - 4(\sigma - \frac{3}{4})(b - 1)) & \frac{3}{4} \leq \sigma \leq 1. \end{cases}$$

Thus $\rho_{a,b}$ is a closed loop and $\rho_{a,b}([0, 1/4])$ lies on γ_1, $\rho_{a,b}([1/4, 1/2])$ lies on κ_a, $\rho_{a,b}([1/2, 3/4])$ lies on γ_b, and $\rho_{a,b}([3/4, 1])$ lies on κ.

We define a basis $(e^{a,b}_i(\sigma))$ at $\rho_{a,b}(\sigma)$ by setting $e^{a,b}_i(0) = e_i(0, 1)$ and parallelly propagating $e^{a,b}_i(0)$ along $\rho_{a,b}$ to give $e^{a,b}_i(\sigma)$.

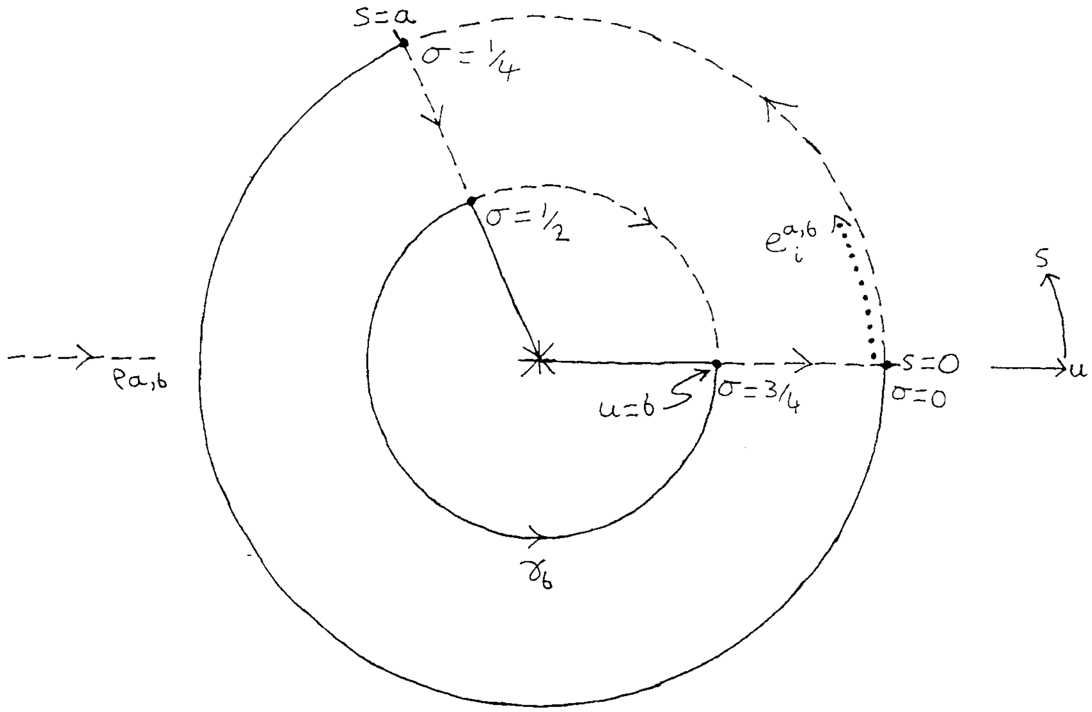

Diagram 2.5.3

Lemma 2.5.5.

$$e^{a,b}_i(1) = l^j_i(a,b)e^{a,b}_j(0).$$

Proof. $\tilde{e}_i(a,b)$ is obtained by parallelly propagating $e_i(0,1)$ along $\rho_{a,b}$ from $\rho_{a,b}(0)$ to $\rho_{a,b}(1/2)$, and $e_i(a,b)$ is obtained by parallelly propagating $e_i(0,1)$ along $\rho_{a,b}$ from $\rho_{a,b}(1)$ to $\rho_{a,b}(1/2)$. Now

$$\tilde{e}_i(a,b) = l^j_i(a,b)e_j(a,b)$$

so if we parallelly propagate $\tilde{e}_i(a,b), e_i(a,b)$ along $\rho_{a,b}$ from $\rho_{a,b}(1/2)$ to $\rho_{a,b}(1)$, $\tilde{e}_i(a,b)$ becomes $e^{a,b}_i(1)$ and $e_i(a,b)$ becomes $e^{a,b}_i(0)$. Thus

$$e^{a,b}_i(1) = l^j_i(a,b)e^{a,b}_j(0). \qquad \square$$

Proposition 2.5.6. *$\exists$ C^0 $M(s), m(s) > 0$ such that*

$$\|l^j_i(s,u)\| \leq M(s) \qquad \|(l^{-1})^j_i(s,u)\| \leq m(s).$$

Proof. Let $s_0 = 0$ and $u_1 = 1$. Then $\rho_{s_1}(\sigma) = \rho_{s_1,u_0}(\sigma)$ and $\hat{e}_i(\sigma, s_1) = e_i^{s_1,u_0}(\sigma)$. Hence by Lemma 2.5.5

$$\hat{e}_i(1, s_1) = l_i^j(s_1, u_0)\hat{e}_j(0, s_1).$$

Thus $\hat{l}_i^j(s_1) = l_i^j(s_1, u_0)$ and by Lemma 2.5.4

$$\|l_i^j(s_1, u_0)\| \leq \exp \alpha h(1) \int_0^{s_1} \|l_i^j(s, 1)\| \|(l^{-1})_i^j(s, 1)\| \phi(s)\, ds + 3$$

$$\|(l^{-1})_i^j(s_1, u_0)\| \leq \exp -\alpha h(1) \int_0^{s_1} \|l_i^j(s, 1)\| \|(l^{-1})_i^j(s, 1)\| \phi(s)\, ds + 3$$

but since $l_i^j(s, 1) = \delta_i^j$ we have

$$\|l_i^j(s_1, u_0)\| \leq \exp \beta \int_0^{s_1} \phi(s)\, ds + 3$$

$$\|(l^{-1})_i^j(s_1, u_0)\| \leq \exp -\beta \int_0^{s_1} \phi(s)\, ds + 3$$

where $\beta = 4\alpha h(1)$ is a constant. Hence $\|l_i^j(s, u)\| \leq M(s) = \exp \beta \int_0^s \phi(s')\, ds' + 3$ and $\|(l^{-1})_i^j(s, u)\| \leq m(s) = \exp -\beta \int_0^s \phi(s')\, ds' + 3$. $M(s)$, $m(s)$ are continuous in s since $\phi \in L^1(0, 1)$. □

Proposition 2.5.7. *Let T be a tensor with components $T^{\tilde{j}_1 \ldots \tilde{j}_q}_{\tilde{i}_1 \ldots \tilde{i}_p}$ in the basis $(\tilde{e}_i)$ and components $T^{j_1 \ldots j_q}_{i_1 \ldots i_p}$ in the basis (e_i). If there exist $\phi' \in L^1(0, 1)$ and $\psi' \in L^1(0, \sup_{s \in [0,1]} \lambda(s))$ such that*

$$\|T^{\tilde{j}_1 \ldots \tilde{j}_q}_{\tilde{i}_1 \ldots \tilde{i}_p}(s, \tilde{u})\| l(s, \tilde{u}) \leq \phi'(s)\psi'(\tilde{u})$$

then there exists $\chi \in L^1(0, 1)$ such that

$$\|T^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s, \tilde{u})\| l(s, \tilde{u}) \leq \chi(s)\psi'(\tilde{u}).$$

Proof.

$$T^{j_1 \ldots j_q}_{i_1 \ldots i_p}(s, \tilde{u}) = (l^{-1})^{\tilde{k}_1}_{i_1}(s, \tilde{u}) \ldots (l^{-1})^{\tilde{k}_p}_{i_p}(s, \tilde{u}) l^{j_1}_{\tilde{l}_1}(s, \tilde{u}) \ldots l^{j_q}_{\tilde{l}_q}(s, \tilde{u}) T^{\tilde{l}_1 \ldots \tilde{l}_q}_{\tilde{k}_1 \ldots \tilde{k}_p}(s, \tilde{u})$$

hence by Proposition 2.5.6 there exist C^0 $M(s), m(s) > 0$ such that

$$\begin{aligned}\|T^{j_1\ldots j_q}_{i_1\ldots i_p}(s,\tilde{u})\|l(s,\tilde{u}) &\leq m(s)^p M(s)^q \|T^{\tilde{l}_1\ldots\tilde{l}_q}_{\tilde{k}_1\ldots\tilde{k}_p}(s,\tilde{u})\|l(s,\tilde{u})\\ &\leq m(s)^p M(s)^q \phi'(s)\psi'(\tilde{u})\\ &\leq \chi(s)\psi'(\tilde{u})\end{aligned}$$

where since $M(s), m(s)$ are continuous, $\chi(s) = m(s)^p M(s)^q \phi'(s) \in L^1(0,1)$. □

We now prove Theorems 2.5.1 and 2.5.2.

Proof of Theorem 2.5.1. Let $s_0 = 0$, $0 \leq s_1 \leq 1$ and $0 < u_0 \leq u_1 \leq 1$. By Lemma 2.5.5

$$e^{s_1,u_0}_i(1) = l^j_i(s_1,u_0)e^{s_1,u_0}_j(0) \qquad e^{s_1,u_1}_i(1) = l^j_i(s_1,u_1)e^{s_1,u_1}_j(0).$$

Thus if we parallelly propagate $e_i(0,1)$ first round ρ_{s_1,u_0} from $\rho_{s_1,u_0}(1)$ to $\rho_{s_1,u_0}(0)$ (i.e. in the reverse sense) and then round ρ_{s_1,u_1} from $\rho_{s_1,u_1}(0)$ to $\rho_{s_1,u_1}(1)$, $e_i(0,1)$ undergoes a transformation

$$e_i(0,1) \longmapsto l^j_i(s_1,u_1)(l^{-1})^k_j(s_1,u_0)e_k(0,1).$$

However parallelly propagating $e_i(0,1)$ in this manner is equivalent to parallelly propagating $e_i(0,1)$ along κ from $\kappa(1)$ to $\kappa(u_1)$, and then round ρ_{s_1} from $\rho_{s_1}(0)$ to $\rho_{s_1}(1)$, and then back along κ from $\kappa(u_1)$ to $\kappa(1)$. Hence

$$l^j_i(s_1,u_1)(l^{-1})^k_j(s_1,u_0)e_k(0,1) = \hat{l}^j_i(s_1)e_j(0,1)$$

and by Lemma 2.5.4 and Proposition 2.5.6 there exist C^0 $M(s), m(s) > 0$ such that

$$\begin{aligned}\|l^j_i(s_1,u_1)(l^{-1})^k_j(s_1,u_0) - \delta^j_i\| &= \|\hat{l}^j_i(s_1) - \delta^j_i\|\\ &\leq \exp \alpha h(u_1)\int_0^{s_1} \|l^j_i(s,u_1)\|\|(l^{-1})^j_i(s,u_1)\|\phi(s)\,ds - 1\\ &\leq \exp \alpha h(u_1)\int_0^{s_1} M(s)m(s)\phi(s)\,ds - 1\\ &\longrightarrow 0 \text{ as } u_0, u_1 \longrightarrow 0\end{aligned}$$

where h, ϕ are as above and since $M(s), m(s)$ are continuous, $M(s)m(s)\phi(s) \in L^1(0,1)$. Hence $\lim_{s\to 0} l^j_i(s)$ exists.

Now let $0 \leq s_0 \leq s_1 \leq 1$, $u_1 = 1$ and $0 < u_0 \leq 1$. Again by Lemma 2.5.5

$$e_i^{s_0,u_0}(1) = l_i^j(s_0,u_0)e_j^{s_0,u_0}(0) \qquad e_i^{s_1,u_0}(1) = l_i^j(s_1,u_0)e_j^{s_1,u_0}(0).$$

Thus if we parallelly propagate $e_i(0,1)$ first round ρ_{s_1,u_0} from $\rho_{s_1,u_0}(0)$ to $\rho_{s_1,u_0}(1)$ and then round ρ_{s_0,u_0} from $\rho_{s_0,u_0}(1)$ to $\rho_{s_0,u_0}(0)$ (i.e. in the reverse sense), $e_i(0,1)$ undergoes a transformation

$$e_i(0,1) \mapsto (l^{-1})_i^j(s_0,u_0)l_j^k(s_1,u_0)e_k(0,1).$$

However parallelly propagating $e_i(0,1)$ in this manner is equivalent to parallelly propagating $e_i(0,1)$ along γ_1 from $\gamma_1(0)$ to $\gamma_1(s_0)$, and then round ρ_{s_1} from $\rho_{s_1}(0)$ to $\rho_{s_1}(1)$, and then back along γ_1 from $\gamma_1(s_0)$ to $\gamma_1(0)$. Hence

$$(l^{-1})_i^j(s_0,u_0)l_j^k(s_1,u_0)e_k(0,1) = \hat{l}_i^j(s_1)e_j(0,1)$$

and by Lemma 2.5.4

$$\begin{aligned} \|(l^{-1})_i^j(s_0,u_0)l_j^k(s_1,u_0) - \delta_i^j\| &= \|\hat{l}_i^j(s_1) - \delta_i^j\| \\ &\leq \exp \alpha h(1) \int_{s_0}^{s_1} \|l_i^j(s,1)\|\|(l^{-1})_i^j(s,1)\|\phi(s)\, ds - 1 \\ &= \exp \beta \int_{s_0}^{s_1} \phi(s)\, ds - 1 \end{aligned}$$

where $\beta = 4\alpha h(1)$ since $l_i^j(s,1) = \delta_i^j$. Now $l_i^j(s) = \lim_{u\to 0} l_i^j(s,u)$ exists so

$$\begin{aligned} \|(l^{-1})_i^j(s_0)l_j^k(s_1) - \delta_i^j\| &\leq \exp \beta \int_{s_0}^{s_1} \phi(s)\, ds - 1 \\ &\longrightarrow 0 \text{ as } s_1 \longrightarrow s_0 \end{aligned}$$

and hence $l_i^j(s)$ is continuous. □

Proof of Theorem 2.5.2. Working in the basis $(\tilde{e}_i)$, since γ is sufficiently regular with respect to Ω_i^j, we have

$$\|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\|l(s,\tilde{u}) \leq \phi(s)\psi(\tilde{u})$$

where $\phi, \psi, \tilde{u}$ are as above. By Lemma 2.5.3(a), $\|\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\tilde{l}}(s,\tilde{u})\| = 1$ and

$$\begin{aligned}
\|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\tilde{l}}(s,\tilde{u})\|l(s,\tilde{u}) &\leq \|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\|\|\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{\tilde{l}}(s,\tilde{u})\|l(s,\tilde{u}) \\
&\leq \|\Omega_{\tilde{k}\tilde{l}\tilde{i}}{}^{\tilde{j}}(s,\tilde{u})\|l(s,\tilde{u}) \\
&\leq \phi(s)\psi(\tilde{u})
\end{aligned}$$

and so by Proposition 2.5.7 there exists $\chi \in L^1(0,1)$ such that, working now in the basis (e_i),

$$\|\Omega_{kli}{}^{j}(s,\tilde{u})\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^{l}(s,\tilde{u})\|l(s,\tilde{u}) \leq \chi(s)\psi(\tilde{u}).$$

By Proposition 2.1.2,

$$\begin{aligned}
\|(L^{-1})^j_k(u_0)L^k_i(u_1) - \delta^j_i\| &= \|P_u \exp \int_{u_0}^{u_1}\int_0^1 \Omega_{kli}{}^{j}(s,u)X^k(s,u)Y^l(s,u)\,dsdu - \delta^j_i\| \\
&\leq \exp \int_{u_0}^{u_1}\int_0^1 \|\Omega_{kli}{}^{j}(s,u)X^k(s,u)Y^l(s,u)\|\,dsdu - 1 \\
&\leq \exp \int_{u_0}^{u_1}\int_0^1 \|\Omega_{kli}{}^{j}(s,u)Y^l(s,u)\|\|X^k(s,u)\|\,dsdu - 1 \\
&= \exp \int_0^1\int_{u_0}^{u_1} \|\Omega_{kli}{}^{j}(s,u)Y^l(s,u)\|l(u)\,duds - 1 \\
&= \exp \int_0^1\int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \|\Omega_{kli}{}^{j}(s,\tilde{u})Y^l(s,\tilde{u})\|l(u(s,\tilde{u}))\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})\,d\tilde{u}ds - 1 \\
&= \exp \int_0^1\int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \|\Omega_{kli}{}^{j}(s,\tilde{u})\frac{\partial u}{\partial \tilde{u}}(s,\tilde{u})Y^l(s,\tilde{u})\|l(u(s,\tilde{u}))\,d\tilde{u}ds - 1 \\
&\leq \exp \int_0^1\int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \chi(s)\psi(\tilde{u})\,d\tilde{u}ds - 1 \\
&= \exp \int_0^1 \chi(s)\,ds \int_{\tilde{u}(s,u_0)}^{\tilde{u}(s,u_1)} \psi(\tilde{u})\,d\tilde{u} - 1 \\
&\leq \exp h(u_1)\int_0^1 \chi(s)\,ds - 1 \\
&\to 0 \text{ as } u_0, u_1 \to 0
\end{aligned}$$

since by Lemma 2.5.3(b) $h(u_1) \to 0$ as $u_1 \to 0$, where h is as above. □

Chapter 3

Conformal transformations and conformal singularities

3.1 Conformal transformations

Let (M, g) be a C^r space-time. A *conformal transformation* [HE] is a transformation of the metric

$$g \mapsto \overline{g} = \Omega^2 g$$

where $\Omega : M \to \mathbb{R}$ is a C^r scalar function which satisfies

$$\Omega(x) > 0 \quad \forall x \in M.$$

Thus $\overline{g}$ is a metric and the space-time $(M, \overline{g})$ is said to be conformally related to (M, g). Note that M itself is unchanged. A conformal transformation preserves the causal structure of (M, g): $u \in TM$ is timelike, null, or spacelike with respect to g if and only if it satisfies the same property with respect to $\overline{g}$. Conversely, if two metrics g and $\overline{g}$ have the same causal structure at a point then, at this point, $\overline{g} = \Omega^2 g$ for some $\Omega > 0$.

Thus we have generated a new space-time $(M, \overline{g})$ from (M, g). If (M, g) has unreasonable physical properties we may be able to choose Ω so as to make $(M, \overline{g})$ more physically realistic, for example obey energy conditions or be a vacuum space-time. If (M, g) is a singular space-time we could require $\Omega \to 1$ as we approach the singularity so that we do not upset the geometry near the singularity. Alternatively we may be able to choose Ω to remove a singularity, so that $(M, \overline{g})$ is a non-singular space-time, in which case the singularity of (M, g) is in some sense mathematically tractable. We may instead hope merely to simplify a singularity by applying a conformal transformation. Alternatively, if (M, g) is a non-singular space-time, we may be able to choose Ω to generate a new singularity. We emphasize though that a conformal transformation provides only one functional degree of freedom and as such is limited in what it can achieve.

Let g have Levi-Civita connection ∇ and let $\overline{g}$ have Levi-Civita connection $\overline{\nabla}$. Working in a basis field (e_i) set

$$\nabla_i e_j = \Gamma^k_{ij} e_k \qquad \overline{\nabla}_i e_j = \overline{\Gamma}^k_{ij} e_k.$$

Lemma 3.1.1. *If* $\sigma^k_{ij} = \overline{\Gamma}^k_{ij} - \Gamma^k_{ij}$ *and* $\omega = \log \Omega$ *then*

$$\sigma^k_{ij} = \delta^k_i \partial_j \omega + \delta^k_j \partial_i \omega - g_{ij} g^{kl} \partial_l \omega.$$

Proof. Since ∇, $\overline{\nabla}$ are metric connections we have

$$\nabla_i g_{jk} = 0 \qquad \overline{\nabla}_i \overline{g}_{jk} = 0$$

where $\overline{g}_{ij} = \Omega^2 g_{ij}$. Hence

$$0 = \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} \tag{3.1.1}$$

$$0 = g_{jk} \partial_i \Omega^2 + \Omega^2 (\partial_i g_{jk} - \overline{\Gamma}^l_{ij} g_{lk} - \overline{\Gamma}^l_{ik} g_{jl}) \tag{3.1.2}$$

where for a scalar $f : M \to \mathbb{R}$, $\partial_i f := e_i(f)$. Subtracting (3.1.2) from (3.1.1) gives

$$g_{jk} \frac{\partial_i \Omega^2}{\Omega^2} = \sigma^l_{ij} g_{lk} + \sigma^l_{ik} g_{jl} \tag{3.1.3a}$$

Now ∇, $\overline{\nabla}$ are torsion free so from section 2.4 we know that

$$\sigma^k_{ij} = \sigma^k_{ji}.$$

By symmetry from (3.1.3a)

$$g_{ki} \frac{\partial_j \Omega^2}{\Omega^2} = \sigma^l_{jk} g_{li} + \sigma^l_{ji} g_{kl} \tag{3.1.3b}$$

$$g_{ij}\frac{\partial_k \Omega^2}{\Omega^2} = \sigma^l_{ki} g_{lj} + \sigma^l_{kj} g_{il}. \tag{3.1.3c}$$

(3.1.3a) + (3.1.3b) − (3.1.3c) gives

$$\begin{aligned} g_{jk}\frac{\partial_i \Omega^2}{\Omega^2} + g_{ki}\frac{\partial_j \Omega^2}{\Omega^2} - g_{ij}\frac{\partial_k \Omega^2}{\Omega^2} &= \sigma^l_{ij} g_{lk} + \sigma^l_{ik} g_{jl} + \sigma^l_{jk} g_{li} + \sigma^l_{ji} g_{kl} - \sigma^l_{ki} g_{lj} - \sigma^l_{kj} g_{il} \\ &= 2\sigma^l_{ij} g_{kl} \end{aligned}$$

and so

$$\sigma^k_{ij} = \frac{1}{2} g^{kl}\left(g_{jl}\frac{\partial_i \Omega^2}{\Omega^2} + g_{li}\frac{\partial_j \Omega^2}{\Omega^2} - g_{ij}\frac{\partial_l \Omega^2}{\Omega^2}\right) = \frac{1}{2} g^{kl}(g_{jl}\partial_i \omega + g_{li}\partial_j \omega - g_{ij}\partial_l \omega)$$

and hence

$$\sigma^k_{ij} = \delta^k_i \partial_j \omega + \delta^k_j \partial_i \omega - g_{ij} g^{kl} \partial_l \omega. \qquad \square$$

Now working in a coordinate basis, if g, $\overline{g}$ have Ricci tensors R_{ij}, $\overline{R}_{ij}$ then

$$\begin{aligned} \overline{R}_{ij} =& R_{ij} - (n-2)\nabla_i \nabla_j \omega - g_{ij} g^{kl} \nabla_k \nabla_l \omega + (n-2)(\nabla_i \omega)(\nabla_j \omega) \\ & - (n-2) g_{ij} g^{kl} (\nabla_k \omega)(\nabla_l \omega) \end{aligned}$$

where n is the dimension of M. For $n \geq 3$ the Weyl tensor

$$C_{ijkl} = R_{ijkl} + \frac{2}{(n-2)}(g_{j[k} R_{l]i} - g_{i[k} R_{l]j}) + \frac{2}{(n-1)(n-2)} R g_{i[k} g_{l]j}$$

obeys $\overline{C}_{ijk}{}^l = C_{ijk}{}^l$.

A conformal transformation varies the length scale of a metric in an isotropic way: if (e_i) is pseudo-orthonormal with respect to g then $(\frac{1}{\Omega} e_i)$ will be pseudo-orthonormal with respect to $\overline{g}$. The fact that a conformal transformation alters the Ricci tensor but leaves the Weyl tensor unaltered suggests that in some way the Ricci tensor measures expansion and contraction, whereas the Weyl tensor measures shear and distortion.

Let γ be an affinely parametrised geodesic with respect to g with tangent u^i so $u^i \nabla_i u^j = 0$. In general γ will not be a geodesic with respect to $\overline{g}$ unless γ is null. It can be shown that

$$u^i \overline{\nabla}_i u^j = 2u^j u^k \partial_k \omega - g_{ik} u^i u^k g^{jl} \partial_l \omega$$

so the condition for γ to be a non-affinely parametrised geodesic with respect to $\overline{g}$, $u^i\overline{\nabla}_i u^j = \lambda u^j$, will hold if $u^i u_i = 0$, in other words if γ is a null geodesic, or if $\partial_i \omega$ is cotangent to γ.

If M is 2-dimensional, then for any metric g, (M, g) is conformally flat, and in fact any two metrics g and $\overline{g}$ on M will be conformal to each other. For example consider the 2-cone

$$ds^2 = dr^2 + A^2 r^2 d\theta^2 \qquad 0 \leq \theta < 2\pi.$$

If $r = \rho^A$ then

$$ds^2 = \Omega^2(\rho)(d\rho^2 + \rho^2 d\theta^2)$$

where $\Omega(\rho) = A\rho^{A-1}$ hence if the 2-cone has metric g then $g = \Omega^2 \overline{g}$ where $\overline{g}$ is the flat metric.

3.2 Conformal singularities

Let (M, g) be a space-time which is singular by some definition (for example b-incompleteness or timelike b-incompleteness) and let $\mathcal{C}$ be the corresponding class of singular curves. Suppose that all $\gamma \in \mathcal{C}$ terminate at genuine singularities rather than regular boundary points. Given a conformal transformation $\theta : g \mapsto \overline{g} = \Omega^2 g$ where $\Omega(x) > 0 \quad \forall x \in M$, we can form $\overline{\mathcal{C}}$ the class of curves singular in $(M, \overline{g})$ by the same definition. If Ω can be chosen so that $\gamma \in \overline{\mathcal{C}}$ terminate only at regular boundary points then (M, g) is said to be *conformally regular*. In other words, a conformally singular space-time is one whose singular behaviour can be removed by a conformal transformation.

Suppose we can extend beyond all the regular boundary points of $(M, \overline{g})$ simultaneously to give a larger space-time (M', g'). Let $(\overline{M}, \overline{g})$ be the closure of $(M, \overline{g})$ in (M', g'). Then $\overline{M} - M$ will provide some sort of singular boundary for (M, g) which we hope will depend only on Ω and (M, g). Clearly Ω cannot extend in a C^r

non-zero way onto $\overline{M} - M$, otherwise $g = \Omega^{-2}\overline{g}$ would be non-singular on $\overline{M} - M$. The idea, then, is that the singular behaviour of g is contained in Ω.

A little more care is needed however. For example let $\overline{g} = \Omega^2 g$ where g is the conical metric

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\theta^2 + dz^2$$

and $\Omega = 1/r^\alpha$. Then

$$\overline{R} = -6\alpha(\alpha+1)r^{2\alpha-2}$$

$$\overline{R}_{ij}\overline{R}^{ij} = \lambda r^{4\alpha-4} \qquad \lambda > 0$$

so $\overline{R} \to 0$ and $\overline{R}_{ij}\overline{R}^{ij} \to 0$ as $r \to 0$ if $\alpha > 1$. Furthermore it can be shown that the Lorentz transformations generated by $r =$ constant loops encircling $r = 0$ tend to the identity as the loops shrink to $r = 0$. Now let

$$\gamma(r) = (-\sqrt{2}r, r, 0, 0)$$

so γ has tangent

$$u^a = (-\sqrt{2}, 1, 0, 0) \qquad g(u^a, u^a) = -1$$

so γ is a timelike curve parametrised by proper time which is future incomplete. Because it is a geodesic it has bounded (in fact zero) acceleration. Let τ be proper time with respect to g and let $\overline{\tau}$ be proper time with respect to $\overline{g}$. Then $-(d\tau/dr)^2 = g(u^a, u^a) = -1$ and $-(d\overline{\tau}/dr)^2 = \overline{g}(u^a, u^a) = \Omega^2 g(u^a, u^a) = -\Omega^2$. Hence

$$\overline{\tau}(\varepsilon) - \overline{\tau}(r_0) = -\int_{r_0}^{\varepsilon} \Omega(r)\, dr = -\int_{r_0}^{\varepsilon} \frac{1}{r^\alpha}\, dr = \left[\frac{r^{1-\alpha}}{(\alpha-1)}\right]_{r_0}^{\varepsilon} \to \infty \text{ as } \varepsilon \to 0 \text{ if } \alpha > 1.$$

If $\alpha > 1$, γ is not timelike incomplete with respect to $\overline{g}$. In particular it cannot be a timelike incomplete curve of bounded acceleration with respect $\overline{g}$. We have mapped the singularity away to infinity.

More generally we define

$$\mathcal{S} = \{\gamma : (0, \alpha] \to M | \gamma \text{ is } C^1 \text{ and inextendible, } \alpha > 0\}.$$

We now work in terms of b-incompleteness. Let $\mathcal{C}$ be the class of curves in $\mathcal{S}$ b-incomplete with respect to (M, g) and let $\overline{\mathcal{C}}$ be the class of curves in $\mathcal{S}$ b-incomplete with respect to $(\overline{M}, g)$. We would like $\mathcal{C} = \overline{\mathcal{C}}$, however we shall see that this may be too much to ask. Let g, $\overline{g}$ have metric connections ω, $\overline{\omega}$, let $\gamma \in \mathcal{S}$, let (e_i) be an ω-frame defined along γ such that $g(e_i, e_j) \doteq \eta_{ij}$, and let $(\overline{e}_i)$ be an $\overline{\omega}$-frame defined along γ such that $\overline{g}(\overline{e}_i, \overline{e}_j) = \eta_{ij}$, where $\eta_{ij} = \mathrm{diag}(-1, 1, 1, 1)$. If we set

$$\overline{e}_i(s) = L_i^j(s) e_j(s)$$

then $L_i^j(s) \in GL_4(\mathbb{R})$ and we know by Theorem 2.3.3 that if $\omega \sim \overline{\omega}$ (and thus $\lim_{s \to 0} L_i^j(s)$ exists) then γ has ω-finite b-length if and only if γ has $\overline{\omega}$-finite b-length. Thus if $\omega \sim \overline{\omega}$ along every $\delta \in \mathcal{C} \cup \overline{\mathcal{C}}$ then $\mathcal{C} = \overline{\mathcal{C}}$.

We now exhibit a sufficient condition for $\omega \sim \overline{\omega}$ along γ. Working in the frame (e_i), set

$$\nabla_i e_j = \omega_{ij}^k e_k \qquad \overline{\nabla}_i e_j = \overline{\omega}_{ij}^k e_k.$$

From Lemma 3.1.1 we have

$$\sigma_{ij}^k = \delta_i^k \partial_j \phi + \delta_j^k \partial_i \phi - \eta_{ij} \eta^{kl} \partial_l \phi$$

where $\sigma_{ij}^k = \overline{\omega}_{ij}^k - \omega_{ij}^k$ and $\phi = \log \Omega$. Let γ have tangent u^i. Then

$$u^i \sigma_{ij}^k = \delta_j^k u^i \partial_i \phi + u^k \partial_j \phi - \eta_{ij} \eta^{kl} u^i \partial_i \phi$$

and from section 2.3

$$L_i^j(s) = P \exp \int_s^\alpha -u^i(s_0) \sigma_{ij}^k(s_0)\, ds_0.$$

Suppose $\gamma \in \mathcal{C}$ and let $\gamma : (0, \alpha] \to M$ be parametrised by b-length measured with respect to ω. Then by Lemma 2.2.9, $\|u^i\| = 1$ and by Lemma 2.3.11, a sufficient condition for $\lim_{s \to 0} L_i^j(s)$ to exist, and thus for $\omega \sim \overline{\omega}$, is

$$\delta_j^k u^i(s) \partial_i \phi(s) + u^k(s) \partial_j \phi(s) - \eta_{ij} \eta^{kl} u^i(s) \partial_i \phi(s) \in L^1(0, \alpha)$$

which will hold if (but maybe not only if) $d\phi(s) \in L^1(0, \alpha)$, i.e. if $d\phi$ is ω-integrable. Note that if $d\phi$ is ω-integrable then $\omega \sim \overline{\omega}$ and it follows as above that $\gamma \in \overline{\mathcal{C}}$ and by Proposition 2.3.10 that $d\phi$ is $\overline{\omega}$-integrable.

If $d\phi(s) \in L^1(0, \alpha)$, it follows that $u^i(s)\partial_i\phi(s) \in L^1(0, \alpha)$. But $u^i(s)\partial_i\phi(s) = d\phi/ds$ and

$$\phi(s) = \phi(\alpha) - \int_s^\alpha u^i(s_0)\partial_i\phi(s_0)\, ds_0$$

and so $\phi(0) := \lim_{s\to 0}\phi(s)$ exists. Hence $\Omega(0) := \lim_{s\to 0}\Omega(s) = \exp(\phi(0))$ and $0 < \Omega(0) < \infty$. Thus if $d\phi$ is ω-integrable then both $\Omega \to 0$ and $\Omega \to \infty$ as $s \to 0$ are impossible.

Suppose instead that $\gamma \in \overline{\mathcal{C}}$. Since $g = \Omega^{-2}\overline{g}$ and $\log\Omega^{-1} = -\log\Omega = -\phi$ it follows by symmetry that if $\gamma \in \overline{\mathcal{C}}$ and $d\phi$ is $\overline{\omega}$-integrable then $\overline{\omega} \sim \omega$, $\gamma \in \mathcal{C}$ and $d\phi$ is ω-integrable. Again, in this case, both $\Omega \to 0$ and $\Omega \to \infty$ as $s \to 0$ are impossible.

Now suppose $\gamma \in \mathcal{S}$ where γ need not necessarily be parametrised by b-length. $g(\Omega\overline{e}_i, \Omega\overline{e}_j) = \Omega^2 g(\overline{e}_i, \overline{e}_j) = \overline{g}(\overline{e}_i, \overline{e}_j) = \eta_{ij}$ therefore $\Omega\overline{e}_i = l_i^j e_j$ for some $l_i^j \in L_+^\uparrow$ and

$$L_i^j(s) = \frac{1}{\Omega(s)} l_i^j(s)$$

where $l_i^j(s) \in L_+^\uparrow$ and by $\Omega(s)$ we mean $\Omega(\gamma(s))$. For $l_i^j \in L_+^\uparrow$

$$\begin{aligned} \eta_{ij} = l_i^k l_j^l \eta_{kl} &\Rightarrow \|\eta_{ij}\| = \|l_i^k l_j^l \eta_{kl}\| \leq \|l_i^k\|\|l_j^l\|\|\eta_{kl}\| = \|l_i^j\|^2\|\eta_{ij}\| \\ &\Rightarrow 1 \leq \|l_i^j\|^2 \Rightarrow 1 \leq \|l_i^j\|. \end{aligned}$$

Thus

$$\|L_i^j(s)\| = \frac{1}{\Omega(s)}\|l_i^j(s)\| \geq \frac{1}{\Omega(s)}$$

and so if $\Omega(s) \to 0$ as $s \to 0$ (or more generally there does not exist $m > 0$ such that $m \leq \Omega(s)$) then L_i^j will be unbounded as $s \to 0$. Conversely, if L_i^j is bounded as $s \to 0$ then there exists $m > 0$ such that $m \leq \Omega(s)$ and in particular $\Omega(s) \to 0$ is impossible.

Again by symmetry

$$||(L^{-1})^j_i(s)|| = \Omega(s)||(l^{-1})^j_i(s)|| \geq \Omega(s)$$

and so if $\Omega(s) \to \infty$ as $s \to 0$ (or more generally there does not exist $M > 0$ such that $\Omega(s) \leq M$) then $(L^{-1})^j_i$ will be unbounded as $s \to 0$. Conversely, if (L^{-1}) is bounded as $s \to 0$ then there exists $M > 0$ such that $\Omega(s) \leq M$ and in particular $\Omega \to \infty$ is impossible.

Thus in particular if $\Omega \to 0$ or $\Omega \to \infty$ along $\delta \in \mathcal{C} \cup \overline{\mathcal{C}}$ then $\omega \sim \overline{\omega}$ cannot hold along δ.

From the proofs of Lemma 2.3.4 and Theorem 2.3.3 we see that if $L^j_i(s)$ is bounded as $s \to 0$ and $\gamma \in \overline{\mathcal{C}}$, then $\gamma \in \mathcal{C}$. This is true even if $\lim_{s\to 0} L^j_i(s)$ does not exist. Unfortunately, if $L^j_i(s)$ is unbounded as $s \to 0$ and $\gamma \in \overline{\mathcal{C}}$, it may still be the case that $\gamma \in \mathcal{C}$. By symmetry, if $(L^{-1})^j_i(s)$ is bounded as $s \to 0$ and $\gamma \in \mathcal{C}$ then $\gamma \in \overline{\mathcal{C}}$. Again, if $(L^{-1})^j_i(s)$ is unbounded as $s \to 0$ and $\gamma \in \mathcal{C}$, it may still be the case that $\gamma \in \overline{\mathcal{C}}$. Hence $\Omega \to 0$ or $\Omega \to \infty$ may be possible along $\delta \in \mathcal{C} \cap \overline{\mathcal{C}}$.

We now restrict our attention to timelike curves. We note that a timelike curve is b-incomplete if and only if it is timelike incomplete and has bounded acceleration. Let

$$\mathcal{S}_0 = \{\gamma : (0, \alpha] \to M | \gamma \text{ is } C^1 \text{ inextendible and timelike, } \alpha > 0\}$$

where we recall that a curve is timelike with respect to (M, g) if and only if it is timelike with respect to $(M, \overline{g})$. Let $\mathcal{C}_0$ be the class of curves in $\mathcal{S}_0$ b-incomplete with respect to (M, g) and let $\overline{\mathcal{C}}_0$ be the class of curves in $\mathcal{S}_0$ b-incomplete with respect to $(M, \overline{g})$. $\mathcal{S}_0$ will include curves which extend to infinity, i.e. curves which fail to be b-incomplete with respect to one or both of (M, g) and $(M, \overline{g})$. The c-boundaries formed from $\mathcal{S}_0$ for (M, g) and $(M, \overline{g})$ will include points "at infinity" and, since they are conformally invariant, will in fact be identical. Alternatively the c-boundary formed from $\overline{\mathcal{C}}_0$ for $(M, \overline{g})$ will correspond to $\overline{M} - M$, after perhaps identifying points of the c-boundary. For example, if $\overline{M} - M$ is a timelike hypersurface in $\overline{M}$, then each point of $\overline{M} - M$ will correspond to two distinct points of the c-boundary, one on each side.

Let $\gamma \in \mathcal{C}_0$ and let $\gamma : \tau \mapsto \gamma(\tau)$ be parametrised with respect to proper time measured with respect to g. Then the proper time elapsed along $\gamma_{[\alpha,s)}$ with respect to $\overline{g}$ is

$$\begin{aligned}\overline{\tau}(\tau) &= \int_{\tau}^{\alpha} (-\overline{g}(u(\tau_0), u(\tau_0)))^{1/2}\, d\tau_0 = \int_{\tau}^{\alpha} (-\Omega^2(\tau_0) g(u(\tau_0), u(\tau_0)))^{1/2}\, d\tau_0 \\ &= \int_{\tau}^{\alpha} \Omega(\tau_0)\, d\tau_0\end{aligned}$$

where $\gamma : \tau \mapsto \gamma(\tau)$ has tangent u and since γ is parametrised with respect to proper time, $g(u, u) = -1$. Therefore $\gamma \in \overline{\mathcal{C}}_0 \Rightarrow \Omega(\tau) \in L^1(0, \alpha)$ (though $\Omega(\tau) \in L^1(0, \alpha)$ may not be sufficient to ensure that γ has bounded acceleration with respect to $\overline{g}$). Similarly if $\gamma \in \overline{\mathcal{C}}_0$ then $\gamma \in \mathcal{C}_0 \Rightarrow \Omega^{-1}(\overline{\tau}) \in L^1(0, \overline{\alpha})$ where $\overline{\tau}$ measures proper time along γ with respect to $\overline{g}$ and $\overline{\tau}(\alpha) = \overline{\alpha}$ (though $\Omega^{-1}(\overline{\tau}) \in L^1(0, \overline{\alpha})$ may not be sufficient to ensure that γ has bounded acceleration with respect to g).

Similar conditions will apply to spacelike curves.

Now suppose that a tensor U is defined and C^0 on the regular space-time $(\overline{M}, \overline{g})$. Since every $\gamma \in \overline{\mathcal{C}}$ terminates at a point of $\overline{M} - M$, it follows that U will be C^0-$\overline{\omega}$-quasi-regular along any $\gamma \in \overline{\mathcal{C}}$. If we pick $x \in \overline{M} - M$ and a coordinate patch W which contains x then the coordinate components of U in the coordinate system defined by W will behave in a C^0 way, even at x. They will also behave in a C^0 way in the coordinate system defined by the coordinate patch $W \cap M$ in the singular space-time (M, g). Thus for $\gamma \in \mathcal{C} \cap \overline{\mathcal{C}}$ there will exist a coordinate patch W_γ which γ eventually enters without leaving such that the coordinate components of U behave in a C^0 way along γ. Despite this, U may not be C^0-ω-quasi-regular along γ.

For example, take the Weyl tensor C. This is conformally invariant, that is, the Weyl tensors of (M, g) and $(M, \overline{g})$ are the same. C will be perfectly regular (at least C^0) on the whole of $\overline{M}$ and therefore C^0-$\overline{\omega}$-quasi-regular along any $\gamma \in \overline{\mathcal{C}}$. As above, given $\gamma \in \mathcal{C} \cap \overline{\mathcal{C}}$ there will exist a coordinate patch W_γ which γ eventually enters without leaving such that the coordinate components of C behave in a C^0 way along γ. Of course, C may not be C^0-ω-quasi-regular.

However, if the Weyl tensor is everywhere zero on M, then it will be C^0-ω-quasi-regular along any $\gamma \in \mathcal{C}$.

Let $\gamma \in \mathcal{C} \cap \overline{\mathcal{C}}$ and let C have components $C_{ijk}{}^l$ in an ω-frame (e_i) defined along γ and components $\overline{C}_{ijk}{}^l$ in an $\overline{\omega}$-frame $(\overline{e}_i)$ defined along γ. (This is a change of notation from section 3.1.) Let $\overline{e}_i = L_i^j e_j$. Then as above $L_i^j = \frac{1}{\Omega} l_i^j$ where $l_i^j \in L_+^\uparrow$ and

$$C_{ijk}{}^l = (L^{-1})_i^{i'}(L^{-1})_j^{j'}(L^{-1})_k^{k'} L_{l'}^l \overline{C}_{i'j'k'}{}^{l'}$$

and

$$\begin{aligned}\|(L^{-1})_i^{i'}(L^{-1})_j^{j'}(L^{-1})_k^{k'} L_{l'}^l\| &= \Omega^2 \|(l^{-1})_i^{i'}(l^{-1})_j^{j'}(l^{-1})_k^{k'} l_{l'}^l\| \\ &= \Omega^2 \|(l^{-1})_i^{i'}\| \|(l^{-1})_j^{j'}\| \|(l^{-1})_k^{k'}\| \|l_{l'}^l\| \\ &\geq \Omega^2\end{aligned}$$

and hence if $\Omega \to \infty$ along γ we would in general expect $C_{ijk}{}^l$ to diverge (though there may be cases where it does not) and therefore to fail to be C^0-ω-quasi-regular along γ.

More generally if $\Omega \to 0$ or $\Omega \to \infty$ along γ then L_i^j or $(L^{-1})_i^j$ will fail to be bounded and tensors which are C^0-ω-quasi-regular may not be C^0-$\overline{\omega}$-quasi-regular, and tensors which are C^0-$\overline{\omega}$-quasi-regular may not be C^0-ω-quasi-regular. Of course, if $\omega \sim \overline{\omega}$ then a tensor will be C^0-ω-quasi-regular if and only if it is C^0-$\overline{\omega}$-quasi-regular.

Conformal singularities have been studied in the context of cosmological models ([T] and references therein). Specifically, a physical space-time (M, g) is related to an unphysical space-time $(M', \overline{g})$ where $M \subset M'$ by

$$g = \tilde{\Omega}^2 \overline{g}$$

wherethe boundary of the closure $\overline{M}$ of M in M' is a smooth spacelike hypersurface Σ in $(M', \overline{g})$ and $\tilde{\Omega} : \overline{M} \to \mathbb{R}$ obeys $\tilde{\Omega} > 0$ in M and $\tilde{\Omega} = 0$ on Σ. We note that if $\overline{g} = \Omega g$ as in our previous definition then $\tilde{\Omega} = \Omega^{-1}$.

We note that since $\tilde{\Omega} \to 0$ and hence $\Omega \to \infty$ along any $\delta \in \mathcal{C} \cup \overline{\mathcal{C}}$, $\omega \sim \overline{\omega}$ cannot hold and $(L^{-1})_i^j$ will be unbounded and as above, the Weyl tensor will not in general be C^0-ω-quasi-regular. Furthermore, unless $\tilde{\Omega} \to 0$ sufficiently slowly, so that $\Omega(\delta(s)) \in L^1(0, a)$ for $\delta \in \mathcal{C}$, in general $\mathcal{C} \not\subset \overline{\mathcal{C}}$, in other words curves incomplete with respect to (M, g) may not terminate a point of Σ.

The following additional assumptions are made in [GW]

(1) M has a smooth cosmic time function T and $\tilde{\Omega} = \tilde{\Omega}(T)$

(2) M is the open submanifold of M' where $T > 0$

(3) M' is regular on an open interval about $T = 0$

(4) $\tilde{\Omega}(0) = 0$, $\tilde{\Omega}$ is C^0 at $T = 0$ and C^3 and positive on an open interval $(0, b]$ where $b > 0$.

(5a) $\tilde{\Omega}'/\tilde{\Omega} \to \infty$ as $T \to 0$

(5b) $\tilde{\Omega}\tilde{\Omega}''/(\tilde{\Omega}')^2 \to l < 1$ as $T \to 0^+$.

Condition (5b) is dropped in [GCW] since it follows from the others.

Σ is called an *isotropic singularity* and can be shown to be a curvature singularity of (M, g). If (M, g) contains an irrotational perfect fluid source, then under certain additional assumptions, a number of results can be proved, in particular that Σ has zero extrinsic curvature in $(\overline{M}, \overline{g})$ and that the limiting curvature near the singularity in (M, g) is determined by the intrinsic geometry of Σ.

A different set of assumptions are made in [Ne93a] and [Ne93b]. Instead of (1)–(5b) above, $\tilde{\Omega}$ is taken to be C^∞ on $\overline{M}$ and $\nabla\tilde{\Omega} \neq 0, \tilde{\Omega} = 0$ on Σ. If (M, g) contains a perfect fluid which obeys an equation of state, then a number of results can be proved. For example, the fluid will be irrotational. With some extra differentiability assumptions, it can be shown that the electric part of the Weyl tensor is zero on Σ if and only if (M, g) is a Friedman-Robertson-Walker cosmology.

The interest in this comes from the Weyl Curvature Hypothesis ([T]). This is the hypothesis that, in a suitably defined way, the Weyl curvature vanishes at an initial cosmological singularity. The motivation for this comes from speculations about quantum gravity, but it is of interest to see what consequences it would have in classical general relativity. We can make sense of the Weyl Curvature Hypothesis in the setting of a conformal singularity: we simply demand that the Weyl tensor be C^0 on $\overline{M}$ and zero on Σ. Thus under the conditions described above, the Weyl Curvature Hypothesis gives rise uniquely to a Friedman-Robertson-Walker cosmology, which is spatially homogeneous and isotropic, and in fact conformally flat. Thus the Weyl Curvature Hypothesis, which may arise due to purely local quantum gravitational considerations, may give rise to the large scale homogeneity and isotropy of the universe

However the above assumes that it is reasonable to suppose that a cosmological singularity is a conformal one. We might hope, since the Weyl tensor is regular at a conformal singularity, that if the Weyl tensor is suitably well behaved near a singularity, then it must be a conformal singularity. However we shall see that this is not the case. The conical singularity has zero curvature and therefore must have zero Weyl tensor, and yet we shall show that it cannot be a conformal singularity. The conical singularity is an example of a 2-dimensional timelike quasi-regular singularity, but it may be possible to find 3-dimensional spacelike quasi-regular singularities which cannot be conformally regularised. We could also conformally transform such singularities to obtain curvature singularities. However we note that such a singularity would still locally be a conformal singularity in the following sense: given an incomplete curve γ which terminates at a singularity conformal to a quasi-regular singularity, there will exist a neighbourhood U of γ such (U, γ) is conformally regular.

3.3 Conformal Cartan connection

Let (M, g_0) be a space-time and let $\mathcal{C}$ be the class of metrics on M conformal to g_0, that is,

$$g \in \mathcal{C} \iff \exists \Omega : M \to \mathbb{R} \text{ such that } \Omega(x) > 0 \ \forall x \in M \text{ and } g = \Omega^2 g_0.$$

A *conformal frame* (c_i) at $x \in M$ is a basis of $T_x M$ which is oriented, time-oriented and pseudo-orthonormal with respect to some $g \in C$ ([S77] and [FS]).

By analogy with LM the frame bundle of (M, g_0), we form CM the *conformal frame bundle* of $(M, \mathcal{C})$. This has projection

$$\pi : CM \to M$$

where each fibre $\pi^{-1}(x)$ consists of all conformal frames at x. Thus $(c_i) \in CM$ will be pseudo-orthonormal with respect to some $g \in \mathcal{C}$ and orthogonal with respect to all $g \in \mathcal{C}$. CM is a principal fibre bundle with structure group

$$C_+^\uparrow = \mathbb{R}^+ \times L_+^\uparrow = \{C_i^j : C_i^k C_j^l \eta_{kl} = \Omega^2 \eta_{ij} \text{ some } \Omega > 0\}$$

so if $C_i^j \in C_+^\uparrow$ then $C_i^j = \Omega L_i^j$ for some $\Omega > 0$ and $L_i^j \in L_+^\uparrow$.

CM is a principal sub-bundle of $GL(M)$ where $GL(M)$ is the principal bundle with projection $p : GL(M) \to M$ for which at each $x \in M$, $p^{-1}(x)$ consists of all bases for $T_x M$. $GL(M)$ has structure group $GL_n(\mathbb{R})$ where $\dim M = n$.

Working in coordinates (x^μ) we now prove a result implied but not proved in [FS].

Proposition 3.3.1. *Let ∇^* be a connection on $GL(M)$ and let $g \in \mathcal{C}$. Then ∇^* will be a connection on $CM \iff \nabla^*{}_\lambda g_{\mu\nu} = -2f_\lambda g_{\mu\nu}$ for some $f_\lambda \in T^*M$.*

Proof. Let $(c_i) \in \pi^{-1}(x)$ for some $x \in M$, let $\gamma : s \mapsto \gamma(s)$ be a C^1 curve through x with tangent u^λ, and use ∇^* to parallelly propagate (c_i) along γ so

$$u^\lambda \nabla^*{}_\lambda \underset{i}{c}^\mu = 0.$$

Hence

$$\begin{aligned} u^\lambda \nabla^*{}_\lambda g_{\mu\nu} \underset{i}{c}^\mu \underset{j}{c}^\nu &= \underset{i}{c}^\mu \underset{j}{c}^\nu u^\lambda \nabla^*{}_\lambda g_{\mu\nu} + \underset{i}{c}^\mu g_{\mu\nu} u^\lambda \nabla^*{}_\lambda \underset{j}{c}^\nu + \underset{j}{c}^\nu g_{\mu\nu} u^\lambda \nabla^*{}_\lambda \underset{i}{c}^\mu \\ &= \underset{i}{c}^\mu \underset{j}{c}^\nu u^\lambda \nabla^*{}_\lambda g_{\mu\nu}. \end{aligned}$$

(c_i) will remain in CM under parallel propagation by ∇^* if and only if $g(c_i, c_j) = \Omega\eta_{ij}$ for some $\Omega : s \mapsto \Omega(s)$ defined along γ such that $\Omega > 0$ which will hold if and only if

$$u^\lambda \nabla^*{}_\lambda g_{\mu\nu} \underset{i}{c}^\mu \underset{j}{c}^\nu = u^\lambda \nabla^*{}_\lambda \Omega \eta_{ij} = (u^\lambda \nabla^*{}_\lambda \Omega)\eta_{ij}$$

and hence (c_i) will remain in CM under parallel propagation by ∇^* if and only if

$$\underset{i}{c}^\mu \underset{j}{c}^\nu u^\lambda \nabla^*{}_\lambda g_{\mu\nu} = (u^\lambda \nabla^*{}_\lambda \Omega)\eta_{ij} \iff u^\lambda \nabla^*{}_\lambda g_{\mu\nu} = (u^\lambda \nabla^*{}_\lambda \Omega) g_{\mu\nu}.$$

Thus (c_i) will remain in CM under parallel propagation by ∇^* in any direction if and only if

$$\nabla^*{}_\lambda g_{\mu\nu} = (\nabla^*{}_\lambda \Omega_x) g_{\mu\nu}$$

for some $\Omega_x : U_x \to \mathbb{R}^+$ defined on a neighbourhood U_x of x. This will hold, and ∇^* will be a connection on CM, if and only if

$$\nabla^*{}_\lambda g_{\mu\nu} = -2f_\lambda g_{\mu\nu} \text{ for some } f_\lambda \in T^*M. \qquad \square$$

A connection ∇^* on CM is a called a *conformal connection*. The metric connection of any $g \in \mathcal{C}$ will be a conformal connection, though in general not all conformal connections will be metric connections. Given ∇^* and $g \in \mathcal{C}$ we obtain the pair $(g_{\mu\nu}, f_\lambda)$ defined by $\nabla^*{}_\lambda g_{\mu\nu} = -2f_\lambda g_{\mu\nu}$. This undergoes a gauge transformation

$$(g_{\mu\nu}, f_\lambda) \mapsto (e^{2\phi} g_{\mu\nu}, f_\lambda - \partial_\lambda \phi)$$

as we pick different metrics in $\mathcal{C}$. ∇^* will only be a metric connection if $\exists \phi : M \to \mathbb{R}$ such that $f_\lambda - \partial_\lambda \phi = 0$ or in other words $f = d\phi$. In this case $\nabla^*{}_\lambda e^{2\phi} g_{\mu\nu} = -2(f_\lambda - \partial_\lambda \phi) g_{\mu\nu} = 0$.

Now let

$$f^{\lambda}_{\mu\nu} = \delta^\lambda_\mu f_\nu + \delta^\lambda_\nu f_\mu - g_{\mu\nu} g^{\lambda\rho} f_\rho.$$

Note that $f^{\lambda}_{\mu\nu}$ depends on f_λ but not on which $g \in \mathcal{C}$ we pick. Also, $f_\lambda = \frac{1}{4} f^\rho_{\lambda\rho}$ so that we may recover f_λ from a knowledge of $f^\lambda_{\mu\nu}$. Then

$$\Gamma^{*\lambda}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + f^\lambda_{\mu\nu}$$

where Γ is the metric connection of g and f_λ is induced by Γ^* and g. Thus a pair $(g_{\mu\nu}, f_\lambda)$ characterises Γ^* and at a point $x \in M$, if we fix $g \in \mathcal{C}$, there is a 1-1 map between conformal connections at x and $f \in T^*_x M$. Note that since $f^\lambda_{\mu\nu}$ is symmetric and Γ is torsion free, conformal connections are torsion free.

We now define a new bundle on M called P^1 the *first prolongation* of CM. Let π now be the projection $\pi : P^1 \to M$. If $r \in \pi^{-1}(x)$ then $r = (x, c_i, \Gamma^*)$ where c_i is a conformal frame at x and Γ^* is some conformal connection defined at x.

We define a bundle chart over an open $U \subset M$, using greek space-time indices and latin frame indices. Choose coordinates (x^μ) on U, $g \in \mathcal{C}$, and a smooth section (e_i) of CM such that $g(e_i, e_j) = \eta_{ij}$. Let Γ^k_{ij} be the metric connection of g with respect to (e_i). Then define a chart

$$\theta : \pi^{-1}(U) \to U \times H$$

$$(x, c_i, \Gamma^*) \mapsto (x^\mu, C^k_i, f_j) \qquad C^k_i \in C^\uparrow_+, f_j \in \mathbb{R}^{4*}$$

where $c_i = C_i^k e_k$, and $f = f_j e^j$ along with g determine Γ^* where (e^i) is dual to (e_i) and $\mathbb{R}^{4*}$ is the vector space dual to $\mathbb{R}^4$. Hence $\forall x \in M$

$$\pi^{-1}(x) \simeq H = \{(C_i^k, f_j) : C_i^k \in C_+^\uparrow, f_j \in \mathbb{R}^{4*}\}.$$

For $s = (C_i^k, f_j) \in H$, $t = (D_i^k, h_j) \in H$ we define a product under which H is closed

$$st = (C_l^k D_i^l, f_j + h_l (C^{-1})_j^l).$$

The following result is implicitly assumed, but not proved, in [FS].

Proposition 3.3.2. *H is a group under the product $(s,t) \mapsto st$.*

Proof. Let $s = (C_i^k, f_j) \in H$, $t = (D_i^k, h_j) \in H$, and $u = (E_i^k, g_j) \in H$. Then

$$\begin{aligned}
(st)u &= (C_l^k D_i^l, f_j + h_l (C^{-1})_j^l)(E_i^k, g_j) \\
&= (C_l^k D_m^l E_i^m, f_j + h_l (C^{-1})_j^l + g_m (D^{-1})_l^m (C^{-1})_j^l) \\
&= (C_l^k D_m^l E_i^m, f_j + (h_l + g_m (D^{-1})_l^m)(C^{-1})_j^l) \\
&= s(tu)
\end{aligned}$$

and so the product is associative. $e = (\delta_i^k, 0)$ is the identity since

$$(\delta_i^k, 0)(C_i^k, f_j) = (C_i^k, f_j) = (C_i^k, f_j)(\delta_i^k, 0)$$

and (C_i^k, f_j) has inverse $((C^{-1})_i^k, -f_l C_j^l)$ since

$$(C_i^k, f_j)((C^{-1})_i^k, -f_l C_j^l) = (\delta_i^k, f_j - f_l C_m^l (C^{-1})_j^m) = (\delta_i^k, 0)$$

$$((C^{-1})_i^k, -f_l C_j^l)(C_i^k, f_j) = (\delta_i^k, -f_l C_j^l + f_l C_j^l) = (\delta_i^k, 0).$$

Hence H is a group. $\square$

An action of H on the fibres of P^1 can be chosen to make P^1 a principal fibre bundle. We define the action of $t = (D_i^k, h_j) \in H$ on $r = (x, c_i, \Gamma^*) \in \pi^{-1}(x)$ by

$$t : (x^\mu, s) \mapsto (x^\mu, st)$$

where in our bundle chart r has coordinates (x^μ, s) and $s = (C_k^i, f_j) \in H$.

The following result is also implicitly assumed, but not proved, in [FS].

Proposition 3.3.3. *The action $t : (x^\mu, s) \mapsto (x^\mu, st)$ is coordinate independent.*

Proof. The action $c_i \mapsto D^k_i c_k$ depends only on t and r and is thus coordinate independent. Now $c_i = C^k_i e_k$ and $D^k_i c_k = D^k_i C^l_k e_l = C^l_k D^k_i e_l$ and hence in coordinates $C^k_i \mapsto C^k_l D^l_i$.

Now let $h = h_i c^i$ where (c^i) is dual to (c_i). Thus $h \in T^*_x M$ depends only on r. Γ^* is determined by the pair (g, f) for some $f \in T^*_x M$. The action

$$(g, f) \mapsto (g, f + h)$$

determines a new conformal connection $\tilde{\Gamma}^*$ uniquely since if $g' \in \mathcal{C}$, Γ^* is determined by the pair $(g', f - d\phi)$ for some $\phi : U_x \to \mathbb{R}$ where U_x is a neighbourhood of x, and in this case the above action gives

$$(g', f - d\phi) \mapsto (g', f - d\phi + h) = (g', (f + h) - d\phi)$$

which also determines $\tilde{\Gamma}^*$. Now $c_i = C^k_i e_k$ so $c^k = (C^{-1})^k_i e^i$ and $h = h_l c^l = h_l (C^{-1})^l_j e_j$ and hence in coordinates $f_j \mapsto f_j + h_l (C^{-1})^l_j$. □

It can also be shown that the action $t : (x^\mu, s) \mapsto (x^\mu, st)$ is free and transitive.

We can now look at connections on P^1. Given a curve $x = x(\lambda)$ in M and $u = (x_0, c_i, \Gamma^*) \in \pi^{-1}(x_0)$ for some $x_0 = x(\lambda_0)$, a connection on P^1 will tell us how to parallelly propagate u along x. In particular it will tell us, not just how to parallelly propagate (c_i), but how to parallelly propagate Γ^*.

Now let

$$A^*{}_{ij} = R^*{}_{ij} - R^* \eta_{ij}/(2n - 2)$$

where Γ^* has Ricci tensor $R^*{}_{ij}$ and Ricci scalar R^*. $A^*{}_{ij}$ will in general only be defined at a point $x_1 \in M$ if Γ^* is defined in a neighbourhood of x_1, but not if Γ^* is only defined along x. However, if $x(\lambda)$ has tangent v^i, $v^i A^*{}_{ij}$ can be shown to depend only on the value of Γ^* along x and it can be shown that there exists a unique connection $\overline{\Gamma}$ on P^1 called the *conformal Cartan connection* ([S] and [FS]) which parallelly propagates Γ^* so as to ensure $v^i A^*{}_{ij} = 0$ along x, where $x(\lambda)$ has tangent v^i, and which parallelly propagates (c_i) to coincide with the way Γ^* would have parallelly propagated (c_i).

We note that for a connection Γ^* defined in a neighbourhood of a point $x_1 \in M$, $A^*{}_{ij} = A^*{}_{ji}$ and that $A^*{}_{ij} = 0 \iff \dot{R}^*{}_{ij} = 0$. Recall that $R^* = R^*{}_i{}^i$ and $R^*{}_{ij} = R^*{}_{ilj}{}^l$ where

$$R^*{}_{ijk}{}^l = \partial_i \Gamma^*{}^l_{jk} - \partial_j \Gamma^*{}^l_{ik} + \Gamma^*{}^l_{im} \Gamma^*{}^m_{jk} - \Gamma^*{}^l_{jm} \Gamma^*{}^m_{ik} - c^m_{ij} \Gamma^*{}^l_{mk}$$

where c^k_{ij} are the *structure coefficients* of (e_i) and satisfy $c^k_{ij} e_k = [e_i, e_j] = \partial_i e_j - \partial_j e_i$ where $\partial_i = e^\mu_i \partial_\mu$. Since Γ is torsion free, $c^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. In a coordinate basis $c^k_{ij} = 0$.

If $u(\lambda) = (x(\lambda), c_i(\lambda), \Gamma^*(\lambda))$ has been parallelly propagated along $x = x(\lambda)$ by $\overline{\Gamma}$ then

$$dx^\mu / d\lambda = \xi^j C^k_j e^\mu_k \tag{3.3.1}$$

$$\xi^j C^k_j e^\mu_k \nabla^*{}_\mu C^m_i = 0 \tag{3.3.2}$$

$$\xi^k C^i_k A^*{}_{ij} = 0. \tag{3.3.3}$$

(3.3.1) defines ξ^j to be the components of the tangent to $x = x(\lambda)$ in the frame (c_i). (3.3.2) ensures (c_i) is parallelly propagated by Γ^*. (3.3.3) defines Γ^* by the condition $v^i A^*{}_{ij} = 0$. These conditions are all coordinate independent.

Let u have coordinates $(x^\mu(\lambda), C^k_i(\lambda), b_j(\lambda))$. In terms of our bundle chart (3.3.2) becomes

$$dC^i_j / d\lambda = -(\Gamma^i_{kl} + b^i_{kl}) C^k_m \xi^m C^l_j. \tag{3.3.2a}$$

After much manipulation it can be shown that, if Γ^* is defined in a neighbourhood of x,

$$A^*{}_{ij} = A_{ij} + (n-2)\partial_i b_j - (n-2) b_m \Gamma^m_{ij} - (n-2)\frac{1}{2} b_m b^m_{ij}$$

where $A_{ij} = R_{ij} - R\eta_{ij}/(2n-2)$ and the metric connection Γ of g has Ricci tensor R_{ij} and Ricci scalar R. Hence

$$v^i A^*{}_{ij} = v^i A_{ij} + (n-2)v^i \partial_i b_j - (n-2)v^i b_m \Gamma^m_{ij} - (n-2)v^i \frac{1}{2} b_m b^m_{ij}$$

which depends only on the value of Γ^* along x. Hence if $n = \dim M > 2$, (3.3.3) becomes

$$db_k/d\lambda = (b_j \Gamma^j_{ik} + \frac{1}{2} b_j b^j_{ik} - L_{ik}) C^i_m \xi^m \qquad (3.3.3a)$$

where $L_{ij} = A_{ij}/(n-2)$. If however $n = 2$ then

$$A^*{}_{ij} = A_{ij}$$

and we cannot impose any conditions on $A^*{}_{ij}$. The Riemann tensor of Γ^* will have only one independent component and if we impose $R^* = 0$ instead it can be shown that

$$\partial_i b^i = -b^m \Gamma^l_{lm} - \frac{1}{2} R$$

which does not have a unique solution.

Therefore, for the remainder of the section, we shall assume that $n > 2$.

We note that if a connection Γ^* is defined in the neighbourhood of x and $R^*{}_{ij} = 0$ along x, then it follows that $A^*{}_{ij} = 0$ and $v^i A^*{}_{ij} = 0$. Thus Γ^* will be parallel along x with respect to $\overline{\Gamma}$. For example if (M, g) is a vacuum space-time for some $g \in \mathcal{C}$, then the metric connection Γ of g will be parallel along any C^1 curve in M with respect to $\overline{\Gamma}$.

We can now consider holonomy in P^1. For example let $\alpha : [0,1] \to M$ be a closed loop with $x = \alpha(0)$. Let $g \in \mathcal{C}$. Then we can use $\overline{\Gamma}$ to parallelly propagate $u = (x, c_i, \Gamma|_x)$ round α where c_i is some conformal frame at x and Γ is the metric connection of g. This results in the unique Γ^* defined on α for which $\Gamma^*(x) = \Gamma|_x$ and $v^\mu A^*{}_{\mu\nu} = 0$ where α has tangent v^μ. In general $\Gamma^*(1) \neq \Gamma^*(0)$ however if

$R_{\mu\nu} = 0$ on α then $\Gamma^* = \Gamma$ on α and the parallel transport of $u = (x, c_i, \Gamma|_x)$ round α is $u' = (x, d_i, \Gamma|_x)$ where d_i is the parallel transport of c_i round α under Γ.

Now let $\gamma : [0,1]^2 \to M$ be a C^1 map where $\gamma_t : s \mapsto \gamma(s,t)$ is a closed loop. Pick $u \in \pi^{-1}(\gamma(0,1))$ and use $\overline{\Gamma}$ to parallelly propagate u along $\kappa : t \mapsto \gamma(0,t)$ and then round γ_t for each value of t. As before we obtain elements of holonomy $h : [0,1] \to H$ where h is C^0. In particular, if γ_0 is a point, $h(0) = e$ where e is the identity of H.

Let (M, g_0) be a space-time and let g_0 generate the conformal class of metrics $\mathcal{C}$. Let $\gamma : [0,1] \times (0,1] \to M$ be a C^1 map as above except that now $0 < t \leq 1$. Suppose that for each $s \in [0,1]$, $\kappa_s : (0,1] \to M : t \mapsto \gamma(s,t)$ terminates at a singular boundary point. We want to know if $\exists g \in \mathcal{C}$ for which g can be extended in a C^2 way onto some $\overline{M}$ where $M \subset \overline{M}$ and such that $\lim_{t\to 0} \gamma(s,t) = x_0 \in \overline{M} - M$, that is, each κ_s terminates at the same $x_0 \in \overline{M} - M$.

Suppose that such a g exists. Then g generates the conformal class of metrics $\mathcal{C}_R$ all of whose members are regular on $\overline{M}$. If we take all $g' \in \mathcal{C}_R$ to be restricted to M, $\mathcal{C}_R \subset \mathcal{C}$ however not all metrics in $\mathcal{C}$ will be in $\mathcal{C}_R$. If we define $\overline{\Gamma}$ on $P^1(M, \mathcal{C}_R)$, the first prolongation of the conformal frame bundle $CM(\mathcal{C}_R)$ defined with respect to $\mathcal{C}_R$, the map $h : (0,1] \to H$ defined above will be such that $\lim_{t\to 0} h(t) = e$.

We can also define $\overline{\Gamma}$ on $P^1(M, \mathcal{C})$, the first prolongation of the conformal frame bundle $CM(\mathcal{C})$ defined with respect to $\mathcal{C}$. The point is, we can examine the holonomy of $\overline{\Gamma}$ on $P^1(M, \mathcal{C})$ purely in terms of (M, g_0) and the conformal class $\mathcal{C}$ it generates even if there is no metric in $\mathcal{C}$ which can be extended in a C^2 way onto a larger $\overline{M}$ as above. However if g does exist we have

Proposition 3.3.4. $P^1(M, \mathcal{C}_R) = P^1(M, \mathcal{C})$.

Proof. The principal bundle $GL(M)$ is defined on M without reference to any metric, and $C(M, \mathcal{C}_R)$ and $C(M, \mathcal{C})$ are both principal sub-bundles of $GL(M)$. The action of $C_+^\uparrow$ on these bundles is defined by this inclusion and by $C_+^\uparrow < GL_n(\mathbb{R})$. However $C(M, \mathcal{C}_R) \subset C(M, \mathcal{C})$ and because the action of $C_+^\uparrow$ is transitive $C(M, \mathcal{C}_R) = C(M, \mathcal{C})$.

Hence the conformal frames and connections defined at each point of M are the same for $\mathcal{C}_R$ and $\mathcal{C}$. As sets $P^1(M, \mathcal{C}_R)$ and $P^1(M, \mathcal{C})$ are the same. By construction $\exists g_0 \in \mathcal{C}_R \subset \mathcal{C}$ which gives a bundle chart on both $P^1(M, \mathcal{C}_R)$ and $P^1(M, \mathcal{C})$ in which the action of H is the same. However we have shown that the action of H is coordinate independent and hence $P^1(M, \mathcal{C}_R) = P^1(M, \mathcal{C})$. □

Hence if g exists, the map $h : (0, 1] \to H$ defined for $P^1(M, \mathcal{C})$ will obey $\lim_{t \to 0} h(t) = e$. It follows that if $\lim_{t \to 0} h(t) \neq e$ the above g cannot be found.

In particular if (M, g_0) is a vacuum space-time with metric connection Γ_0, then it follows from above that $\overline{\Gamma}$ will parallelly propagate $u = (c_i, \Gamma_0|_{\gamma_t(0)})$ round γ_t to $u' = (d_i, \Gamma_0|_{\gamma_t(0)})$ where (c_i) is some conformal frame at $\gamma_t(0)$ and Γ_0 parallelly propagates (c_i) round γ_t to (d_i). Thus if Γ_0 has non-trivial singular holonomy on γ then $\lim_{t \to 0} h(t) \neq e$ and there does not exist $g \in \mathcal{C}$ with respect to which each κ_s terminates at the same regular boundary point.

We note that if the κ_s are future timelike incomplete and share the same past light cone (or are past timelike incomplete and share the same future light cone) and $\exists g \in \mathcal{C}$ with respect to which each κ_s terminates at a regular boundary point then each κ_s will terminate at the same regular boundary point.

As an example let (M, g_0) be the conical metric

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\theta^2 + dz^2 \qquad r \neq 0$$

where $M = \mathbb{R}^4 - \{r = 0\}$. Let $\mathcal{C}$ be the conformal class generated by g_0. Consider a closed loop $\alpha : [0, 1] \to M$. Pick $u = (x, c_i, \Gamma|_x) \in P^1$ where $x = \alpha(0)$, c_i is some conformal frame at x, and Γ is the metric connection of g_0. We have shown that the conformal Cartan connection will parallelly propagate u round α

to $u' = (x, d_i, \Gamma|_x)$ where d_i is the parallel transport of c_i under Γ. Now pick coordinates on a neighbourhood U of α and use g_0 to give a bundle chart over U. Then $u = (x^\mu, C_i^k, 0)$ and $u' = (x^\mu, C_i^j D_j^k, 0)$ where D_i^k is a rotation through $2\pi k(1 - A)$ where $k \in \mathbb{Z}$.

Now let $\gamma : [0,1] \times (0,1] \to M$ be as above. Pick $u = (\gamma(0,1), C_i^k, 0)$ where $C_i^k \in C_+^\uparrow$. Parallelly propagate u along $\kappa : t \mapsto \gamma_t(0)$ with respect to $\overline{\Gamma}$ to give $u = u(t)$. Then $u(t) = (\gamma(0,t), C_i^k(t), 0)$ by the above since (M, g_0) has $R_{\mu\nu} = 0$, where $C_i^k(t)$ is the parallel transport of C_i^k under Γ. Hence $h(t) = (D_i^k(t), 0)$ where $D_i^k(t)$ is a rotation through $2\pi k(t)(1 - A)$ $k \in \mathbb{Z}$. By continuity, $k(t)$ must be constant. If γ encircles $r = 0$ then $k \neq 0$, and if $A \notin \mathbb{Z}$, it follows that $\lim_{t \to 1} h(t) \neq e$ and that therefore M admits no conformal boundary for which there exists a loop encircling $r = 0$ homotopic to a point on the boundary. In particular if $A \notin \mathbb{Z}$, there does not exist a space-time $(\overline{M}, \overline{g})$ with

$$\overline{M} = \mathbb{R}^4 \qquad \overline{g}_{|M} = \Omega^2 g$$

for any C^2 $\Omega : M \to \mathbb{R}$ with $\Omega(x) > 0$ $\forall x \in M$. $\overline{M}$ supplies a singular boundary for M which is consistent with the c-boundary.

3.4 The 4-cone is not conformally regular

We used the conformal Cartan connection in the previous section to prove that, for $A \notin \mathbb{Z}$, the 4-cone is not conformally regular. We now present a more elementary proof that for $A < 1$, the 4-cone is not conformally regular. This proof depends on the Lorentzian signature of the metric of the 4-cone and would fail if the metric were positive definite. The proof works because conformal transformations preserve the null cone structure of a space-time.

Given cylindrical polar coordinates (t, r, θ, z) on the manifold $\mathbb{R}^4 - \{r = 0\}$ and the Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \qquad 0 \leq \theta < 2\pi$$

recall that for $A < 1$ we may obtain the 4-cone by removing the wedge $\{\pi - \alpha/2 < \theta < \pi + \alpha/2\}$ to give

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \qquad -\pi + \alpha/2 \leq \theta \leq \pi - \alpha/2$$

and identifying $\{\theta = -\pi + \alpha/2\}$ with $\{\theta = \pi - \alpha/2\}$ where $0 < A = 1 - \alpha/2\pi < 1$ (i.e. $0 < \alpha < 2\pi$). Let (M, g) be the resulting space-time.

Theorem 3.4.1. *There does not exist a conformal transformation of the 4-cone space-time (M, g) for $A < 1$ which maps timelike incomplete curves terminating at singular boundary points to timelike incomplete curves terminating at regular boundary points.*

Proof. In the following we shall work in Cartesian coordinates $(t, x = r\cos\theta, y = r\sin\theta, z)$ with respect to which the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad -\pi + \alpha/2 \leq \theta \leq \pi - \alpha/2.$$

Define the curves

$$\gamma_+(s) = (t_0 + s, a - s\cos(\alpha/4), s\sin(\alpha/4), 0)$$

$$\gamma_-(s) = (t_0 + s, a - s\cos(\alpha/4), -s\sin(\alpha/4), 0)$$

(see diagram 3.4.1). Then

$$\gamma_+'(s) = (1, -\cos(\alpha/4), \sin(\alpha/4), 0) \quad g(\gamma_+', \gamma_+') = -1 + \cos^2(\alpha/4) + \sin^2(\alpha/4) = 0$$

$$\gamma_-'(s) = (1, -\cos(\alpha/4), -\sin(\alpha/4), 0) \quad g(\gamma_-', \gamma_-') = -1 + \cos^2(\alpha/4) + \sin^2(\alpha/4) = 0.$$

Since γ_+, γ_- are straight lines in this coordinate system it follows that γ_+, γ_- are null geodesics.

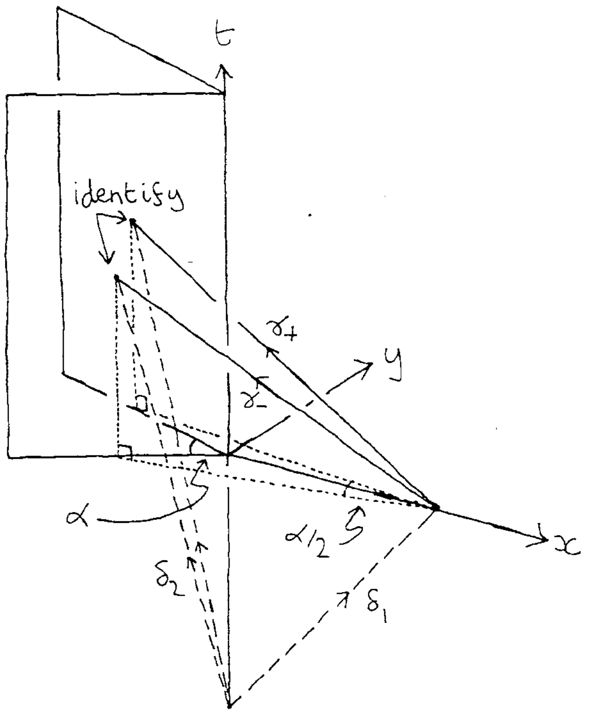

Diagram 3.4.1

Now $\gamma_+(0) = (t_0, a, 0, 0) = \gamma_-(0)$ and when $s = 2a\cos(\alpha/4)$, $a - s\cos(\alpha/4) = a - 2a\cos^2(\alpha/4) = -a\cos(\alpha/2)$ and $s\sin(\alpha/4) = 2a\cos(\alpha/4)\sin(\alpha/4) = a\sin(\alpha/2)$ so

$$\gamma_+(2a\cos(\alpha/4)) = (t_0 + 2a\cos(\alpha/4), -a\cos(\alpha/2), a\sin(\alpha/2), 0)$$

$$\gamma_-(2a\cos(\alpha/4)) = (t_0 + 2a\cos(\alpha/4), -a\cos(\alpha/2), -a\sin(\alpha/2), 0).$$

However when $(x, y) = (-a\cos(\alpha/2), a\sin(\alpha/2))$, $\theta = \pi - \alpha/2$ and when $(x, y) = (-a\cos(\alpha/2), -a\sin(\alpha/2))$, $\theta = -\pi + \alpha/2$. Hence

$$\gamma_+(2a\cos(\alpha/4)) = \gamma_-(2a\cos(\alpha/4)).$$

Thus we have found a pair of null geodesics γ_+ and γ_- which meet at two distinct points.

Let $\delta_1(u) = (u, \varepsilon u, 0, 0)$ for $u > 0$. Then $\delta_1'(u) = (1, \varepsilon, 0, 0)$ and $g(\delta_1', \delta_1') = -1 + \varepsilon^2$ and so δ_1 is a future pointing timelike curve for $0 < \varepsilon < 1$. From the above, for each point $\delta_1(u)$ there are a pair of null geodesics $\gamma^u_+(s)$, $\gamma^u_-(s)$ which meet at the distinct points $\delta_1(u)$ and $\delta_2(u)$ where

$$\delta_2(u) = (u + 2\varepsilon u\cos(\alpha/4), -\varepsilon u\cos(\alpha/2), \pm\varepsilon u\sin(\alpha/2), 0).$$

Now $\delta_2'(u) = (1 + 2\varepsilon\cos(\alpha/4), -\varepsilon\cos(\alpha/2), \pm\varepsilon\sin(\alpha/2), 0)$ and

$$\begin{aligned} g(\delta_2', \delta_2') &= -1 - 4\varepsilon\cos(\alpha/4) - 4\varepsilon^2\cos^2(\alpha/4) + \varepsilon^2(\cos^2(\alpha/2) + \sin^2(\alpha/2)) \\ &= -1 + \varepsilon^2 - 4\varepsilon\cos(\alpha/4) - 4\varepsilon^2\cos^2(\alpha/4) \\ &< -1 + \varepsilon^2 < 0 \end{aligned}$$

since $0 < \alpha < 2\pi \Rightarrow 0 < \alpha/4 < \pi/2 \Rightarrow 0 < \cos(\alpha/4) < 1$. Hence δ_2 is a future pointing timelike curve for $0 < \varepsilon < 1$.

As $u \longrightarrow 0$ the components of $\delta_1(u)$, $\delta_2(u)$ tend to $(0, 0, 0, 0)$. It follows that δ_1, δ_2 are timelike incomplete curves terminating at singular boundary points and that furthermore $I^+(\delta_1) = I^+(\delta_2)$ i.e. δ_1, δ_2 have the same future light cone.

Now suppose $\exists\Omega : M \longrightarrow \mathbb{R}$ such that δ_1, δ_2 are timelike incomplete curves terminating at regular boundary points in the space-time $(M, \overline{g})$ where $\overline{g} = \Omega^2 g$ and $\Omega(x) > 0$ $\forall x \in M$. Let $(\overline{M}, \overline{g})$ be an extension of $(M, \overline{g})$ such that δ_1, δ_2 terminate at interior points of $\overline{M}$.

Since $I^+(\delta_1) = I^+(\delta_2)$, δ_1 and δ_2 must terminate at the same regular boundary point $x_0 \in \overline{M} - M$. Now there must exist a convex normal neighbourhood $U \ni x_0$ [HE] such that for any points $x_1, x_2 \in U$ there will be exactly one geodesic between x_1 and x_2 lying entirely within U.

Now γ^u_+, γ^u_- will still be null geodesics with respect to $(\overline{M}, \overline{g})$ and

$$\left.\begin{aligned} \gamma^u_+(s) &= (u + s, \varepsilon u - s\cos(\alpha/4), s\sin(\alpha/4), 0) \\ \gamma^u_-(s) &= (u + s, \varepsilon u - s\cos(\alpha/4), -s\sin(\alpha/4), 0) \end{aligned}\right\} 0 \le s \le 2\varepsilon u\cos(\alpha/4)$$

and hence as $u \longrightarrow 0$ the components of $\gamma^u_+(s)$, $\gamma^u_-(s)$ also tend to $(0, 0, 0, 0)$. It follows that γ^u_+, γ^u_- lie in $I^-(\delta_1) = I^-(\delta_2)$ and the points of γ^u_+, γ^u_- must tend to

x_0. Hence $\exists\delta > 0$ such that γ_+^δ, γ_-^δ lie entirely in U and meet at distinct points $\gamma_+^\delta(0) = \gamma_-^\delta(0)$ and $\gamma_+^\delta(2\varepsilon\delta\cos(\alpha/4)) = \gamma_-^\delta(2\varepsilon\delta\cos(\alpha/4))$.

This is a contradiction and hence a suitable $\Omega : M \to \mathbb{R}$ cannot exist. $\square$

3.5 Minimal and totally geodesic submanifolds

Let S be a smooth p-dimensional submanifold of a smooth n-dimensional manifold M. Let M have metric g. For $x \in S$ let

$$(T_xS)^\perp = \{u \in T_xM : g(u,v) = 0 \quad \forall v \in T_xS\}$$

hence $(T_xS)^\perp$ is the $(n-p)$-dimensional vector space of all vectors normal to S at x.

If g is Lorentzian, so that in a pseudo-orthonormal basis $g_{ij} = \mathrm{diag}(-1, 1, \ldots, 1)$, then S is *spacelike* at $x \in S$ if all $u \in T_xS$ are spacelike, S is *timelike* at $x \in S$ if all $u \in (T_xS)^\perp$ are spacelike, and S is *null* at $x \in S$ if it is neither spacelike nor timelike. In the following we shall assume that S is not null so

$$T_xM = T_xS \oplus (T_xS)^\perp.$$

If $p = n-1$ then S is a *hypersurface* and $(T_xS)^\perp$ is 1-dimensional. If we can make a smooth choice of non-zero normal for all $x \in S$, then S is *orientably imbedded*.

For general p, at least in an open neighbourhood of some $x_0 \in S$, let (e_i) be a smooth (but not necessarily pseudo-orthonormal) basis field on M adapted to S in the sense that $\forall x \in M$, $(e_1, \ldots, e_p)$ span T_xS and $(e_{p+1}, \ldots, e_n)$ span $(T_xS)^\perp$. In the following we shall take early lower case indices $\{a, b, c, \ldots\}$ to run through $1, \ldots, p$; early upper case indices $\{A, B, C, \ldots\}$ to run through $p+1, \ldots, n$; and late lower case indices $\{i, j, k, \ldots\}$ to run through $1, \ldots, n$. For example if $w = u + v$ for $u \in T_xS$, $v \in (T_xS)^\perp$ and $x \in S$, then we may write $w^ie_i = u^ae_a + v^Ae_A$.

For $x \in S$ we define maps

$$\left.\begin{aligned} \pi_{\|} &: T_xM \longrightarrow T_xS : u + v \longmapsto u \\ \pi_{\perp} &: T_xM \longrightarrow (T_xS)^{\perp} : u + v \longmapsto v \end{aligned}\right\} \forall u \in T_xS, v \in (T_xS)^{\perp}.$$

Thus $\pi_{\|}$, $\pi_{\perp}$ are linear projections from T_xM onto T_xS, $(T_xS)^{\perp}$ respectively and given $u \in T_xS$, $v \in (T_xS)^{\perp}$

$$\pi_{\|}(u) = u \quad \pi_{\|}(v) = 0 \qquad \pi_{\perp}(u) = 0 \quad \pi_{\perp}(v) = v.$$

Since $\pi_{\|}$, $\pi_{\perp}$ are linear we may write

$$\left.\begin{aligned} \pi_{\|} &: w^j \longmapsto p_i^j w^i \\ \pi_{\perp} &: w^j \longmapsto \tilde{p}_i^j w^i \end{aligned}\right\} \forall w^j \in T_xM$$

and hence

$$p_i^j = \begin{cases} \delta_a^b & i = a,\ j = b \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{p}_i^j = \begin{cases} \delta_A^B & i = A,\ j = B \\ 0 & \text{otherwise.} \end{cases}$$

We now define projected metrics

$$g^{\|}{}_{ij} = p_i^k p_j^l g_{kl} \qquad g^{\perp}{}_{ij} = \tilde{p}_i^k \tilde{p}_j^l g_{kl}$$

so

$$g^{\|}{}_{ij} = \begin{cases} g_{ab} & i = a,\ j = b \\ 0 & \text{otherwise} \end{cases} \qquad g^{\perp}{}_{ij} = \begin{cases} g_{AB} & i = A,\ j = B \\ 0 & \text{otherwise.} \end{cases}$$

In particular if $u_1, u_2 \in T_xS$ then

$$g^{\|}(u_1, u_2) = g^{\|}{}_{ij} u_1^i u_2^j = g^{\|}{}_{ab} u_1^a u_2^b = g_{ab} u_1^a u_2^b = g(u_1, u_2)$$

and so $g^{\|}$ is the intrinsic metric on S induced from g by the embedding of S in M. Similarly for $v_1, v_2 \in (T_xS)^{\perp}$

$$g^{\perp}(v_1, v_2) = g(v_1, v_2).$$

Now since $(e_a) \perp (e_A)$ it follows that $g_{aB} = g_{Ab} = 0$ and

$$g = g_{ij} e^i \otimes e^j = g_{ab} e^a \otimes e^b + g_{AB} e^A \otimes e^B = g^{\|} + g^{\perp}$$

where (e^i) are dual to (e_i).

Using g to raise and lower indices we now prove

Proposition 3.5.1. *$g^{\|}{}_i^j = p_i^j$ and $g^{\perp}{}_i^j = \tilde{p}_i^j$.*

Proof. Let $u \in T_xS$. Then $g^{\|}{}_i^j u^i = g^{\|}{}_{ik} g^{kj} u^i = g^{\|}{}_{ab} g^{bj} u^a = g_{ab} g^{bj} u^a = \delta_a^j u^a = u^j$. Now let $v \in (T_xS)^{\perp}$. Then $g^{\|}{}_i^j v^i = g^{\|}{}_{ik} g^{kj} v^i = g^{\|}{}_{ab} g^{bj} v^a = 0$. Hence $g^{\|}{}_i^j = p_i^j$ and similarly $g^{\perp}{}_i^j = \tilde{p}_i^j$. □

Since $g_i^j = \delta_i^j$ it follows that $\delta_i^j = p_i^j + \tilde{p}_i^j$.

Let $x \in S$ and let u_1, u_2 be vector fields on S in a neighbourhood of x. Then if (M, g) has Levi-Civita connection ∇,

$$\nabla_{u_1} u_2 = (\nabla_{u_1} u_2)^{\|} + (\nabla_{u_1} u_2)^{\perp}$$

where we write $(\nabla_{u_1} u_2)^{\|} = \pi_{\|}(\nabla_{u_1} u_2)$ and $(\nabla_{u_1} u_2)^{\perp} = \pi_{\perp}(\nabla_{u_1} u_2)$. It can be shown that $(\nabla_{u_1} u_2)^{\|} = D_{u_1} u_2$ where D is the Levi-Civita connection of $(S, g^{\|})$, and that $(\nabla_{u_1} u_2)^{\perp}$ depends only on the values of u_1 and u_2 at x. Thus

$$\nabla_{u_1} u_2 = D_{u_1} u_2 + K(u_1, u_2)$$

where

$$K : T_xS \times T_xS \longrightarrow (T_xS)^{\perp} : (u_1, u_2) \longmapsto (\nabla_{u_1} u_2)^{\perp}.$$

K is called the *second fundamental form* of S ([CDD] and [Ch]) and can be shown to obey

$$K(u_1, u_2) = K(u_2, u_1) \qquad u_1, u_2 \in T_xS.$$

K measures the extent to which a vector initially tangent to S fails to remain tangent to S under parallel propagation by ∇ in S.

Since K is linear we may write

$$K(u_1, u_2) = K_{ab}^{C} u_1^a u_2^b e_C \qquad u_1, u_2 \in T_xS$$

where K_{ab}^{C} are known as the *extrinsic curvatures* of S. We can extend K to a map $K : T_xM \times T_xM \longrightarrow T_xM$ by

$$K_{ij}^{k} = \begin{cases} K_{ab}^{C} & i = a,\ j = b,\ k = C \\ 0 & \text{otherwise} \end{cases}$$

which makes K^k_{ij} into a tensor which obeys $K^k_{ij} = K^k_{ji}$.

S also has *mean curvatures*

$$K^k = g^{ij} K^k_{ij}$$

so

$$K^k = \begin{cases} g^{ab} K^C_{ab} & k = C \\ 0 & \text{otherwise} \end{cases}$$

and *umbilical curvatures*

$$l^k_{ij} = K^k_{ij} - K^k g^{\|}{}_{ij}/p$$

so

$$l^k_{ij} = \begin{cases} K^C_{ab} - K^C g^{\|}{}_{ab}/p & i = a,\ j = b,\ k = C \\ 0 & \text{otherwise.} \end{cases}$$

Now $g^{ij} K^k_{ij} = K^k$ and

$$g^{ij} l^k_{ij} = g^{ij} K^k_{ij} - K^k g^{ij} g^{\|}{}_{ij}/p = K^k - K^k p/p = 0$$

and hence K^k is the trace of K^k_{ij} and l^k_{ij} is the trace-free part of K^k_{ij}.

S is *totally geodesic* at $x \in S$ if $K^k_{ij}|_x = 0$, *minimal* at $x \in S$ if $K^k|_x = 0$, and *totally umbilical* at $x \in S$ if $l^k_{ij}|_x = 0$.

Proposition 3.5.2. $K^k_{ij} = 0 \iff K^k = 0$ *and* $l^k_{ij} = 0$.

Proof. Suppose $K^k_{ij} = 0$. Then $K^k = g^{ij} K^k_{ij} = 0$ and $l^k_{ij} = K^k_{ij} - K^k g^{\|}{}_{ij}/p = 0$. Conversely suppose $K^k = 0$ and $l^k_{ij} = K^k_{ij} - K^k g^{\|}{}_{ij}/p = 0$. Then $K^k_{ij} = 0$. □

[Ch] gives a slightly different condition for a p-dimensional manifold S to be totally umbilical at $x \in S$, namely that

$$K^k_{ij} = \lambda^k g^{\|}{}_{ij} \qquad \text{some } \lambda \in (T_x S)^{\perp}.$$

However this condition is equivalent to our previous definition since

Proposition 3.5.3. *At $x \in S$, $K^k_{ij} = \lambda^k g\|_{ij}$ for some $\lambda \in (T_x S)^\perp \iff l^k_{ij} = 0$.*

Proof. Let $x \in S$. Suppose $K^k_{ij} = \lambda^k g\|_{ij}$ for some $\lambda \in (T_x S)^\perp$. Then $K^k = g^{ij} K^k_{ij} = \lambda^k g^{ij} g\|_{ij} = \lambda^k p$ hence $\lambda^k = K^k/p$ and $l^k_{ij} = K^k_{ij} - K^k g\|_{ij}/p = K^k g\|_{ij}/p - K^k g\|_{ij}/p = 0$.

Conversely if $l^k_{ij} = 0$ then $K^k_{ij} = K^k g\|_{ij}/p = \lambda^k g\|_{ij}$ where $\lambda^k = K^k/p$. □

Recall that for $x \in S$ and $u \in T_x S$ there exists a unique geodesic $\gamma_u(s)$ in M such that $\gamma_u(0) = x$, $\gamma_u'(0) = u$ and $u^i \nabla_i u^j = 0$, and a unique intrinsic geodesic $\tilde{\gamma}_u(s)$ in S such that $\tilde{\gamma}_u(0) = x$, $\tilde{\gamma}_u'(0) = u$ and $u^i D_i u^j = 0$.

Proposition 3.5.4. $\gamma_u = \tilde{\gamma}_u \; \forall x \in S,\, u \in T_x S \iff K^k_{ij} = 0$.

Proof.

$$u^i \nabla_i u^j = u^a \nabla_a u^j = (u^a \nabla_a u^j)^{\|} + (u^a \nabla_a u^j)^\perp$$

$$= u^a D_a u^j + K^j_{ab} u^a u^b$$

and hence

$$u^i \nabla_i u^j = u^i D_i u^j \iff K^k_{ij} = 0.$$

□

Proposition 3.5.5. *A vector initially tangent to S remains tangent to S under parallel propagation by ∇ along any curve lying in $S \iff K^k_{ij} = 0$.*

Proof. Let δ be a curve lying in S with tangent u and let δ pass through $x \in S$. Let $v \in T_x S$. At x,

$$u^a \nabla_a v^b = u^a D_a v^b + K_{ab} u^a v^b = u^a D_a v^b \qquad \forall u^a, v^b \in T_x S, x \in S$$

$$\iff K^k_{ij} = 0.$$

□

We note that in our adapted basis

$$K^C_{ab} e_C = K(e_a, e_b) = (\nabla_{e_a} e_b)^\perp = \omega^C_{ab} e_C$$

where $\nabla_{e_i} e_j = \omega^k_{ij} e_k$ and therefore $K^C_{ab} = \omega^C_{ab}$.

Now consider a conformal transformation $\theta : g \mapsto \overline{g} = \Omega^2 g$ where $\Omega : M \longrightarrow \mathbb{R}$ and $\Omega(x) > 0 \ \forall x \in M$. If g is Lorentzian, whether S is spacelike or timelike is preserved under θ and in any case, for $x \in S$, $T_x S$ and $(T_x S)^\perp$ remain invariant under θ and (e_i) will continue to be an adapted basis. Since $\overline{g}_{ij} = \Omega^2 g_{ij}$ we have $\overline{g}^{ij} = \Omega^{-2} g^{ij}$ and

$$\overline{g}^{\parallel}{}_{ij} = \begin{cases} \overline{g}_{ij} & i = a,\ j = b \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \Omega^2 g_{ab} & i = a,\ j = b \\ 0 & \text{otherwise} \end{cases} = \Omega^2 g^{\parallel}{}_{ij}.$$

Now by Lemma 3.1.1

$$\begin{aligned} \overline{\omega}^{C}_{ab} &= \omega^{C}_{ab} + \delta^{C}_{a} \partial_b \phi + \delta^{C}_{b} \partial_a \phi - g_{ab} g^{CD} \partial_D \phi \\ &= \omega^{C}_{ab} - g_{ab} g^{CD} \partial_D \phi \end{aligned}$$

where $\phi = \log \Omega$. Hence

$$\overline{K}^{C}_{ab} = K^{C}_{ab} - g_{ab} g^{CD} \partial_D \phi.$$

Now since

$$\overline{K}^{k}_{ij} = \begin{cases} \overline{K}^{C}_{ab} & i = a,\ j = b,\ k = C \\ 0 & \text{otherwise} \end{cases} \qquad K^{k}_{ij} = \begin{cases} K^{C}_{ab} & i = a,\ j = b,\ k = C \\ 0 & \text{otherwise} \end{cases}$$

it follows that

$$\overline{K}^{k}_{ij} = K^{k}_{ij} - g^{\parallel}{}_{ij} g^{\perp kl} \partial_l \phi \tag{3.5.1}$$

which in fact is a fully covariant expression which will hold in any basis. Hence

$$\begin{aligned} \overline{K}^{k} = \overline{g}^{ij} \overline{K}^{k}_{ij} &= \frac{1}{\Omega^2} g^{ij} (K^{k}_{ij} - g^{\parallel}{}_{ij} g^{\perp kl} \partial_l \phi) \\ \Rightarrow \overline{K}^{k} &= \frac{1}{\Omega^2} (K^k - p g^{\perp kl} \partial_l \phi) \end{aligned} \tag{3.5.2}$$

and

$$\begin{aligned} \overline{l}^{k}_{ij} &= \overline{K}^{k}_{ij} - \overline{K}^{k} \overline{g}^{\parallel}{}_{ij}/p \\ &= K^{k}_{ij} - g^{\parallel}{}_{ij} g^{\perp kl} \partial_l \phi - \frac{1}{\Omega^2} (K^k - p g^{\perp kl} \partial_l \phi) \Omega^2 g^{\parallel}{}_{ij}/p \\ &= K^{k}_{ij} - K^k g^{\parallel}{}_{ij}/p - g^{\parallel}{}_{ij} g^{\perp kl} \partial_l \phi + g^{\parallel}{}_{ij} g^{\perp kl} \partial_l \phi \\ \Rightarrow \overline{l}^{k}_{ij} &= l^{k}_{ij}. \end{aligned} \tag{3.5.3}$$

From (3.5.2) we can see that given K^k on S and a choice of Ω on S we can choose the normal derivatives $g^{\perp kl}\partial_l\phi$ to yield any value of $\overline{K}^k$. In particular we can set $\Omega = 1$ on S and choose the normal derivatives so as to make S minimal with respect to $\overline{g}$ i.e. $\overline{K}^k = 0$.

Now if $K^k = \overline{K}^k = 0$ then $g^{\perp kl}\partial_l\phi = 0$ so $\partial_l\phi$, and hence $\partial_l\Omega$, must be cotangent to S. In this case from (3.5.1) we get

$$\overline{K}^k_{ij} = K^k_{ij}$$

and thus if a conformal transformation makes S minimal but not totally geodesic then no conformal transformation can make S totally geodesic.

Finally from (3.5.3) we see that l^k_{ij} is a conformal invariant and there will exist a conformal transformation which makes S totally geodesic if and only if $l^k_{ij} = 0$, i.e. if S is totally umbilical.

Chapter 4

Weak singularities and idealised cosmic strings

4.1 A $3+1$ analysis of an idealised cosmic string

We have seen how 2-dimensional timelike quasi-regular singularities may provide suitable idealised models of cosmic strings. We have explained however that cosmic strings modelled in this way are really quite inflexible objects, unable to bend or to form closed loops on length scales smaller than the cosmological length scale. This is a problem for example if cosmic strings are to provide a mechanism for the formation of structure in the early universe. We therefore propose to describe a class of "weak" curvature singularities, somehow worse than quasi-regular ones but which remain weak enough to have nice properties and in particular to have the properties we would expect of a cosmic string. The fact that they are curvature singularities may however permit them to bend and form closed loops on small length scales.

A construction of a circular cosmic string of arbitrarily small radius is given in [FIU]. The construction is complicated and it is not obvious that, in a range of cases, it gives rise to a curvature singularity. This is shown in [UHIM], which describes a class of curvature singularities such as this one which are proposed as models for cosmic strings. The claim is made that singularities in this class are nonetheless totally geodesic. However the definition of this class is not particularly rigorous and a number of restrictive assumptions are made. After discussing this paper, we will present a more rigorous formulation of these ideas which we will then study in subsequent chapters using methods of holonomy.

We now discuss the formulation given in [UHIM] (see also [I]). In this paper an idealised cosmic string is defined to be a timelike 2-space S whose points are "conical singularities of the space-time". Space-time geodesics "orthogonal" to the string S at each point $p \in S$ sweep out a spacelike 2-space S_p which is required to have "conical structure" at the single point p where it intersects S. In principle the angular deficit could vary over S, but will be constant, it is claimed, if the space-time is vacuum or obeys energy conditions.

There are a number of problems with this formulation. How can a singularity be a timelike 2-space, how can points of S be conical singularities of the space-time, what does it mean for a 2-space to have conical structure, and how can space-time geodesics be orthogonal to S?

The paper claims to show that, provided the Ricci tensor is bounded near S, and the Weyl tensor has a sufficiently weak singularity, the string S is totally geodesic. In fact the argument presented in the paper does not make use of these assumptions about the curvature. Instead it shows that, provided the lapse function of a certain foliation of 3-dimensional hypersurfaces is C^2 at the string in a rather artificial quasi-Cartesian coordinate system, the string is totally geodesic (in a sense discussed below), from which it follows that the Ricci tensor need not be bounded and may in fact diverge.

The paper proceeds as follows. Given an intrinsic geodesic L_t of S, which is taken to be either timelike or spacelike, the aim is to show that "as a locus of conical singularities of the four metric" it is in fact a geodesic of the space-time. This is done by showing that the magnitude of the acceleration of a sequence of curves which tend to L_t tends to zero. A coordinate system is defined as follows. L_t is parametrised by proper time (or proper distance), and at each point of L_t, L_z is defined to be the intrinsic geodesic of S orthogonal to L_t parametrised by proper distance (or proper time). This gives (t, z) coordinates on S in a neighbourhood of

L_t. Gaussian polar coordinates (ρ, ϕ) are now defined on each spacelike 2-surface S_p with origin on S and an arbitrary choice of polar axis varying "smoothly" over S (despite the fact that in general it is not possible to attach a unique differential structure to a singularity). Thus ρ measures proper distance and the paper claims that at points of L_t, the space-time metric is

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + A^2\rho^2 d\phi^2$$

for a suitable choice of ϕ. We interpret this as meaning that sufficiently close to L_t, the space-time metric is

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + A^2\rho^2 d\phi^2 + \tilde{\varepsilon}_{ij} dx^i dx^j$$

if L_t is timelike (and

$$ds^2 = dt^2 - dz^2 + d\rho^2 + A^2\rho^2 d\phi^2 + \tilde{\varepsilon}_{ij} dx^i dx^j$$

if L_t is spacelike) where $\tilde{\varepsilon}_{ij} \longrightarrow 0$ in a suitable way as $\rho \longrightarrow 0$ and $A = A(t, z)$. If $\tilde{\varepsilon} \longrightarrow 0$ too quickly then the string S will turn out to be a quasi-regular singularity. We discuss the behaviour of $\tilde{\varepsilon}_{ij}$ below.

The paper then defines quasi-Cartesian coordinates by

$$x = r\cos\phi \qquad y = r\sin\phi \qquad r = (A\rho)^{\frac{1}{A}}$$

and claims that the metric on $S_p \cap L_t$ for $p \in L_t$ is

$$ds^2 = r^{-2\delta}(dx^2 + dy^2)$$

where $\delta = 1 - A$. Again we interpret this as meaning that sufficiently close to L_t, the space-time metric is

$$ds^2 = -dt^2 + dz^2 + r^{-2\delta}(dx^2 + dy^2) + \varepsilon_{ij} dx^i dx^j$$

if L_t is timelike (and

$$ds^2 = dt^2 - dz^2 + r^{-2\delta}(dx^2 + dy^2) + \varepsilon_{ij} dx^i dx^j$$

if L_t is spacelike) where $\varepsilon_{ij} \longrightarrow 0$ as $r \longrightarrow 0$ in a manner discussed below. We shall also need to assume that the metric has inverse $\mathrm{diag}(\mp 1, \pm 1, r^{2\delta}, r^{2\delta}) + \varepsilon'^{ij}$ where $\varepsilon'^{ij} \longrightarrow 0$ as $r \longrightarrow 0$ in a manner discussed below. The paper claims that this conformally flat form gives a clear cut meaning to the concept of angles at the vertex, despite the fact that for $A < 1$, the conformal factor is undefined at the vertex.

The paper now performs a 3 + 1 split by taking hypersurfaces of the form $t =$ constant. Now $\partial_t = t^i \partial_i$ where $t^i = \delta^i_0$. We decompose t^i into components normal and tangent to the hypersurfaces by

$$t^i = \varepsilon N n^i + N^i$$

where n^i is the unit normal to the hypersurfaces chosen to make N positive, $\varepsilon = n^i n_i$ and N^i is tangent to the hypersurfaces. N is called the lapse and N^i is called the shift. Note that $\varepsilon = -1$ if L_t is timelike and $\varepsilon = 1$ if L_t is spacelike. We now let latin indices range over $\{t, z, x, y\}$ and greek indices over $\{z, x, y\}$ and use 0 to denote t. By construction ∂_α are tangent to the hypersurfaces. Thus $N^i \partial_i = N^\alpha \partial_\alpha$ and $N^0 = 0$ and since $n_i N^i = 0$ we also have $n_\alpha = 0$.

It can be shown that

$$g_{00} = \varepsilon N^2 + N^\alpha N_\alpha \qquad g_{0\beta} = N_\beta \qquad g^{00} = \varepsilon / N^2 \qquad g^{0\beta} = -\varepsilon N^\beta / N^2$$

from which it follows that $N \longrightarrow 1$, $N^\beta \longrightarrow 0$ and $N_\beta \longrightarrow 0$ as $r \longrightarrow 0$. We also have $N^0 = 0$ and $N_0 = N^i g_{i0} = N^\alpha g_{\alpha 0} = N^\alpha N_\alpha \longrightarrow 0$ as $r \longrightarrow 0$.

We now let h be the intrinsic metric of the hypersurfaces induced by the space-time metric g so $h_{\alpha\beta} = g_{\alpha\beta}$. Thus g_{ij} can be entirely expressed in terms of N, N^i

and $h_{\alpha\beta}$. We can now discuss the behaviour of $\varepsilon_{ij}, \varepsilon'^{ij}$ terms in the metric. Now $\varepsilon_{ij}, \varepsilon'^{ij} \longrightarrow 0$ as $r \longrightarrow 0$ and we have

$$\varepsilon_{00} = \varepsilon(N^2 - 1 + N^\alpha N_\alpha \qquad \varepsilon_{0\beta} = N_\beta \qquad \varepsilon'^{00} = \varepsilon(1/N^2 - 1) \qquad \varepsilon'^{0\beta} = -\varepsilon N^\beta/N^2.$$

It is implicit in the paper that the derivatives of N^i, N_i along with the derivatives of the spatial terms $\varepsilon_{\alpha\beta}, \varepsilon'^{\alpha\beta}$ tend to zero as $r \longrightarrow 0$. We discuss the behaviour of N below.

The paper defines $u_i = Nv_i$ where $v_i = \delta^0_i$. Now $u^i = Ng^{i0}$ so

$$u^0 = \varepsilon/N \qquad u^\alpha = -\varepsilon N^\beta/N$$

but since $n^i = (N^i - t^i)/N$

$$n^0 = -1/N \qquad n^\alpha = N^\alpha/N$$

and hence $u^i = -\varepsilon n^i$. Thus u^i is a unit normal to the hypersurfaces. The integral curves of u^i have acceleration

$$\kappa^i = u^j \nabla_j u^i$$

where, working in quasi-Cartesian coordinates, in can be shown that

$$\kappa_i = -\varepsilon N^{-1} N_{,i} + \ldots$$

where we use ... to denote terms which tend to zero as $r \longrightarrow 0$ provided that $N^\alpha \longrightarrow 0$ sufficiently fast, or in other words provided that $g^{0\beta} \longrightarrow 0$ sufficiently fast. This expression differs from the one given in the paper.

The paper claims that the integral curves of u^i are "parallel" to L_t. Certainly in quasi-Cartesian coordinates u^i and ∂_t coincide at $r = 0$ and if u^i is C^1 at $r = 0$, there will be a unique integral curve through u^i at $r = 0$ and the integral curves of u^i will tend to L_t as $r \longrightarrow 0$. In other words we require $N \longrightarrow 1, N^\beta \longrightarrow 0$ in a C^1 way as $r \longrightarrow 0$.

The acceleration κ^i has magnitude

$$\kappa = \kappa^i \kappa_i = -\kappa_0^2 + r^{2\delta}(\kappa_x^2 + \kappa_y^2) + \kappa_z^2 + \ldots$$

The paper claims that κ^i "lies in the (x, y) plane at the conical vertex". In fact

$$\kappa^0 = \delta_i^0 u^j \nabla_j u^i = N^{-1} u_i u^j \nabla_j u^i = \frac{1}{2} N^{-1} u^j \nabla_j (u^i u_i) = 0$$

and so κ^i is tangent to the hypersurfaces and is spacelike. Furthermore $\kappa_0 = -\varepsilon N^{-1} N_{,0} + \ldots, \kappa_z = -\varepsilon N^{-1} N_{,z} + \ldots$ and so $\kappa_0, \kappa_z \longrightarrow 0$ as $r \longrightarrow 0$.

If the string were a regular part of the space-time, a necessary and sufficient condition for L_t to be a space-time geodesic, and therefore for the string to be totally geodesic, would be $\kappa \longrightarrow 0$ as $r \longrightarrow 0$ (since κ^i is spacelike). Since κ is a covariant measure of the acceleration of curves tending to L_t, it makes sense to require $\kappa \longrightarrow 0$ as $r \longrightarrow 0$ in order for the string to be considered to be totally geodesic even though it is not a regular part of the space-time.

In quasi-Cartesian coordinates the condition $\kappa \longrightarrow 0$ becomes

$$r^{2\delta}(\kappa_x^2 + \kappa_y^2) \longrightarrow 0 \text{ as } r \longrightarrow 0.$$

Thus letting upper case indices range over $\{x, y\}$

$$\kappa \longrightarrow 0 \iff \kappa_A = O(r^{-\zeta}) \text{ where } \zeta < \delta$$

and so κ_A could in principle diverge. The paper makes the stronger assumption however that κ_A is bounded and that $\kappa_A \longrightarrow \kappa^e{}_A$ as $r \longrightarrow 0$, where in general $\kappa^e{}_A \neq 0$.

The paper now makes the assumption that N is C^2 at $r = 0$. It follows from this that $\kappa_i = -\varepsilon N^{-1} N_{,i} + \ldots \longrightarrow -\varepsilon N_{,i}$ as $r \longrightarrow 0$, and thus that the above assumption $\kappa_A \longrightarrow \kappa^e{}_A$ as $r \longrightarrow 0$ for some $\kappa^e{}_A$ automatically holds. Hence the string is totally geodesic.

The dynamical components of the Riemann tensor obey

$$R_{\alpha 0 \beta 0} = \pounds_u K_{\alpha\beta} + K_{\alpha i} K^i_\beta + \kappa_{(\alpha;\beta)} + \kappa_\alpha \kappa_\beta$$

where ; is the covariant derivative of 3-metric h induced on the hypersurfaces, $K_{\alpha\beta}$ is the extrinsic curvature of the hypersurfaces defined by

$$K(X,Y) = (X^i \nabla_i Y^j) u_j \qquad \forall \text{ hypersurface tangent } X, Y$$

(thus $K(X,Y)$ is the normal component of $\nabla_X Y$) and

$$\pounds_u K_{\alpha\beta} = u^i \partial_i K_{\alpha\beta} + K_{i\beta}\partial_\alpha u^i + K_{\alpha i}\partial_\beta u^i.$$

The paper also assumes that $K_{\alpha\beta}$ is regular at $r = 0$. Certainly if $K_{\alpha\beta}$ is C^1 at $r = 0$ then

$$R_{\alpha 0 \beta 0} = \kappa_{(\alpha;\beta)} + \ldots$$

which is (almost) the expression given in the paper and where we now use ... to denote terms which do not diverge as $r \longrightarrow 0$. Hence

$$R_{\alpha 0\beta 0} = (-\varepsilon N^{-1} N_{,(\alpha})_{;\beta)} + \ldots = \varepsilon N^{-2} N_{,(\beta} N_{,\alpha)} - \varepsilon N^{-1} N_{;(\alpha\beta)} + \ldots = -\varepsilon N^{-1} N_{;\alpha\beta} + \ldots$$

since $N_{;\alpha\beta} = N_{;\beta\alpha}$. Now

$$N_{;\alpha\beta} = N_{,\alpha\beta} - \overline{\Gamma}^i_{\beta\alpha} N_{,i} = -\overline{\Gamma}^i_{\beta\alpha} N_{,i} + \ldots$$

since N is C^2, where $\overline{\Gamma}$ is the connection of h. Neglecting the ε_{ij} and ε'_{ij} terms we can show that

$$\overline{\Gamma}^x_{xx} = -\delta r^{-2} x \qquad \overline{\Gamma}^x_{xy} = \overline{\Gamma}^x_{yx} = -\delta r^{-2} y \qquad \overline{\Gamma}^x_{yy} = \delta r^{-2} x$$

$$\overline{\Gamma}^y_{yy} = -\delta r^{-2} y \qquad \overline{\Gamma}^y_{xy} = \overline{\Gamma}^y_{yx} = -\delta r^{-2} x \qquad \overline{\Gamma}^y_{xx} = \delta r^{-2} y.$$

We note that $\partial_x \delta = \partial_y \delta = 0$ but that if $\partial_z \delta \neq 0$ then

$$\overline{\Gamma}^z_{xx} = \overline{\Gamma}^z_{yy} = -\varepsilon r^{-2\delta} \log r (\partial_z \delta) \qquad \overline{\Gamma}^x_{xz} = \overline{\Gamma}^x_{zx} = \overline{\Gamma}^y_{yz} = \overline{\Gamma}^y_{zy} = \log r(\partial_z \delta).$$

These terms do not diverge as fast as $O(r^{-1})$ and we will ignore them. Other components are negligible. We can now show that

$$N_{;AB} = \delta r^{-2} (\kappa^e{}_A x^i \delta_{iB} + \kappa^e{}_B x^i \delta_{iA} - \kappa^e{}_i x^i \delta_{AB}) + \ldots$$

where (x^i) are the quasi-Cartesian coordinates. Hence

$$R_{A0B0} = -\varepsilon N^{-1}\delta r^{-2}(\kappa^e{}_A x^i \delta_{iB} + \kappa^e{}_B x^i \delta_{iA} - \kappa^e{}_i x^i \delta_{AB}) + \ldots$$

This expression differs slightly from the one given in the paper. Thus in quasi-Cartesian coordinates $R_{A0B0} = O(r^{-1})$. In addition we can show that $R_{\alpha 0 \beta 0}$ is bounded as $r \longrightarrow 0$ if $\alpha = z$ or $\beta = z$.

The paper does not explicitly give an expression for R_{AzBz}, however we may obtain one as follows. Given a particular L_z which passes through L_t, we define a new set of quasi-Cartesian coordinates (t', z', x', y') based on L_z as follows. For each point of L_z, let L'_t be the intrinsic geodesic of S orthogonal to L_z parametrised by proper time. Thus we obtain (t', z') coordinates which coincide with (t, z) coordinates along L_t and L_z. We now define $x' = x, y' = y$ to give (t', z', x', y') coordinates.

Hence working in (t', z', x', y') coordinates

$$R_{A'z'B'z'} = \varepsilon N^{(z')-1}\delta r^{-2}(\kappa^{(z')}{}_{A'} x^{c'} \delta_{c'B'} + \kappa^{(z')}{}_{B'} x^{c'} \delta_{c'A'} - \kappa^{(z')}{}_{c'} x^{c'} \delta_{A'B'}) + \ldots$$

where $N^{(z')}$ is the lapse obtained by performing a $3 + 1$ split by taking hypersurfaces of the form $z' =$ constant, $\kappa^{(z')}{}_i$ is the limit as $r \longrightarrow 0$ of the components of the acceleration of the integral curves of the unit normals to the $z' =$ constant hypersurfaces, and upper case indices $\{A', B', \ldots\}$ range over $\{x', y'\}$. As before $N^{(z')} \longrightarrow 1$ as $r \longrightarrow 0$ and we assume that $N^{(z')}$ is C^2 at $r = 0$. Again we can also show that $R_{\alpha' z' \beta' z'}$ is bounded as $r \longrightarrow 0$ if $\alpha' = t'$ or $\beta' = t'$ where $\{\alpha', \beta', \ldots\}$ range over $\{t', x', y'\}$. Now along any curve which terminates at the point of S where L_t and L_z intersect

$$R_{ijkl} = R_{i'j'k'l'} + \ldots$$

from which it follows that $R_{AzBz} = O(r^{-1})$ and $R_{\alpha z \beta z}$ is bounded as $r \longrightarrow 0$ if $\alpha = 0$ or $\beta = 0$.

The remaining nine components of the Riemann tensor are determined by the Gauss-Codazzi equations. Their behaviour is not essential to what follows, however we claim that it can be shown that $R_{xyxy} = O(r^{-4\delta})$ at worst and, if $K_{\alpha\beta}$ is C^1 or C^{1-}, the other eight components are $O(r^{2\delta-1})$ at worst.

We now look at the Ricci tensor

$$R_{ij} = R_{ikj}{}^{k} = R_{ikjl}g^{kl}$$

so

$$R_{00} = R_{0k0l}g^{kl} = R_{0z0z}g^{zz} + R_{0C0D}g^{CD} + \dots$$

where we assume that the off-diagonal components of g^{kl} tend to zero sufficiently fast. From above we know that R_{0z0z} is bounded as $r \longrightarrow 0$ so

$$\begin{aligned} R_{00} &= (R_{0x0x} + R_{0y0y})r^{2\delta} + \dots \\ &= -\varepsilon N^{-1} r^{2\delta-2}(\kappa^e{}_x x + \kappa^e{}_x x - \kappa^e{}_x x - \kappa^e{}_y y + \kappa^e{}_y y + \kappa^e{}_y y - \kappa^e{}_x x - \kappa^e{}_y y) + \dots \\ &= 0 + \dots \end{aligned}$$

and hence R_{00} is bounded as $r \longrightarrow 0$. Similarly R_{zz} is bounded as $r \longrightarrow 0$, though the paper does not say so.

The paper does not give an expression for $R_{0\alpha}$, however we claim that $R_{0\alpha} = O(r^{2\delta-1})$ at worst.

Now

$$R_{AB} = R_{AkBl}g^{kl} = R_{A0B0}g^{00} + R_{AzBz}g^{zz} + \xi_{AB}$$

where $\xi_{AB} = R_{ACBD}g^{CD} = O(r^{-2\delta})$ at worst since $R_{xyxy} = O(r^{-4\delta})$ at worst. Hence

$$\begin{aligned} R_{AB} = &\frac{\varepsilon}{N^2}. - \varepsilon N^{-1}\delta r^{-2}(\kappa^e{}_A x^i \delta_{iB} + \kappa^e{}_B x^i \delta_{iA} - \kappa^e{}_i x^i \delta_{AB}) \\ &+ \frac{-\varepsilon}{N^{(z)2}}.\varepsilon N^{(z)-1}\delta r^{-2}(\kappa^{(z)}{}_A x^i \delta_{iB} + \kappa^{(z)}{}_B x^i \delta_{iA} - \kappa^{(z)}{}_i x^i \delta_{AB}) \\ &+ \xi_{AB} + \dots \end{aligned}$$

but $N, N^{(z)} \longrightarrow 1$ as $x^\alpha \longrightarrow 0$ and so

$$R_{AB} = O(r^{-1})^{D}_{AB}(\kappa^e{}_D + \kappa^{(z)}{}_D) + \xi_{AB} + \dots$$

which differs from the expression given in the paper by the sign of ${\kappa^{(z)}}_D$ and the presence of the (possibly divergent) ξ_{AB} term. Since ${\kappa^e}_A + {\kappa^{(z)}}_A$ need not be zero, and ξ_{AB} may diverge, there is no need for R_{AB} to be bounded, contrary to the implication of the paper.

Recall that

$$\kappa = \kappa_i \kappa^i = r^{2\delta}(\kappa_x^2 + \kappa_y^2) + \dots$$

obeys $\kappa \longrightarrow 0$ as $r \longrightarrow 0$ and that the string is therefore totally geodesic. For a real string however $\delta \approx 10^{-6}$ and for $r < e^{-1/\delta}$, $r^{2\delta} \approx 1$ and so for very small values of r, $\kappa \approx (\kappa_x^2 + \kappa_y^2)$ and the string appears to be curved.

We should really examine the components of the Riemann tensor in a parallelly propagated basis rather than in the rather unphysical quasi-Cartesian coordinate system. Instead we consider Cartesian coordinates

$$\tilde{x} = \rho\cos\phi \qquad \tilde{y} = \rho\sin\phi$$

where we recall that with respect to Gaussian polar coordinates (ρ, ϕ) the metric close to L_t is

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + A^2\rho^2 d\phi^2 + \tilde{\varepsilon}_{ij}dx^i dx^j.$$

Since $r = (A\rho)^{\frac{1}{A}}$ we have

$$x = A^{\frac{1}{A}}\rho^{\frac{1}{A}-1}\tilde{x} \qquad y = A^{\frac{1}{A}}\rho^{\frac{1}{A}-1}\tilde{y}$$

and hence for $A \in \{x, y\}$, $\tilde{A} \in \{\tilde{x}, \tilde{y}\}$

$$\begin{aligned}\lambda^A_{\tilde{A}} = (\partial x^A/\partial x^{\tilde{A}}) &= A^{\frac{1}{A}}\rho^{\frac{1}{A}-3}\begin{pmatrix}\tilde{x}^2/A + \tilde{y}^2 & (1-A)\tilde{x}\tilde{y}/A \\ (1-A)\tilde{x}\tilde{y}/A & \tilde{x}^2 + \tilde{y}^2/A\end{pmatrix} \\ &= O(\rho^{\eta-1}) \\ &= O(r^{\delta})\end{aligned}$$

where $\eta = 1/A$.

Hence working in Cartesian coordinates

$$R_{\tilde{A}0\tilde{B}0} = \lambda^{A}_{\tilde{A}}\lambda^{B}_{\tilde{B}}R_{A0B0} = O(r^{2\delta-1})$$

Similarly $R_{\tilde{A}z\tilde{B}z} = O(r^{2\delta-1})$ and $R_{\tilde{A}\tilde{B}} = O(r^{2\delta-1})$. Provided that $\delta = 1 - A > 0$, no component of the Riemann or Ricci tensor will diverge faster than $O(r^{2\delta-1})$. In particular if $\delta > \frac{1}{2}$ then all components will in fact be bounded. We also note that $O(r^{2\delta-1}) = O(\rho^{\eta-2})$.

This is consistent with the claim in the paper that the "Weyl curvature goes as $r^{2\delta-1}$ after converting to the physical components".

As we have discussed, the above results are obtained by performing a $3+1$ analysis in quasi-Cartesian coordinates and making the assumption that the lapse function N is C^2 in quasi-Cartesian coordinates. The paper does not perform a similar analysis in Cartesian coordinates, though it would perhaps be more natural to do so.

We therefore repeat the $3+1$ analysis in Cartesian coordinates. For simplicity, we shall use indices without tildes to denote Cartesians, thus (t, z, x, y) are now Cartesian coordinates.

The $t =$ constant hypersurfaces and the $t^i = \delta^i_0$ vector are unchanged. Therefore the lapse function N, given by

$$t^i = \varepsilon N n^i + N^i$$

remains unchanged. As before

$$\kappa_i = -\varepsilon N^{-1}N_{,i} + \text{ terms which } \longrightarrow 0 \text{ as } r \longrightarrow 0$$

and $N \longrightarrow 1$ as $r \longrightarrow 0$. Again $\kappa_0, \kappa_z \longrightarrow 0$ as $r \longrightarrow 0$. However

$$\kappa = \kappa_i\kappa^i \longrightarrow 0 \text{ as } r \longrightarrow 0 \iff \kappa_i \longrightarrow 0 \text{ as } r \longrightarrow 0$$

and therefore the string will be totally geodesic if and only if the simpler condition $\kappa_i \longrightarrow 0$ as $r \longrightarrow 0$ holds for every choice of L_t. If $\kappa_i \longrightarrow 0$ as $r \longrightarrow 0$, then $N_{,i} \longrightarrow 0$ as $r \longrightarrow 0$.

We now assume that N is C^2 at $r = 0$. This implies that $\lim_{r\to 0} N_{,i}$ exists but does not necessarily imply that $N_{,i} \longrightarrow 0$ as $r \longrightarrow 0$. In other words the string need not be totally geodesic.

Hence

$$N = 1 + O(\rho^2 + z^2)$$

and as before

$$R_{A0B0} = -\varepsilon N^{-1} N_{;AB} + \ldots$$

where

$$N_{;AB} = O(\text{constant}) + O((\rho^2 + z^2)^{\frac{1}{2}})_{,i} \overline{\Gamma}^{i}_{AB}.$$

Neglecting $\tilde{\varepsilon}_{ij}$ terms in g (and the corresponding terms in g^{-1}) we can show

$$\overline{\Gamma}^{C}_{AB} = (1 - A^2) O(\rho^{-1}).$$

Again $\partial_x A = \partial_y A = 0$ but if $\partial_z A \neq 0$ then

$$\overline{\Gamma}^{z}_{AB}, \overline{\Gamma}^{C}_{Az}, \overline{\Gamma}^{C}_{zB} = (\partial_z A) O(\text{constant})$$

and so we can ignore these components with other components also being negligible. It now follows that

$$R_{A0B0} = \ldots$$

In other words R_{A0B0} is bounded as $r \longrightarrow 0$. Similarly R_{AzBz} is bounded as $r \longrightarrow 0$. In particular, if N is C^2 in both Cartesian and quasi-Cartesian coordinates, it must follow that R_{A0B0} and R_{AzBz} are bounded as $r \longrightarrow 0$.

As an example consider the dynamic cone

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + A^2(t, z)\rho^2 d\phi^2$$

and let $x = \rho\cos\phi$, $y = \rho\sin\phi$. Then $N = 1$ is C^2 in both Cartesian and quasi-Cartesian coordinates, and R_{A0B0} and R_{AzBz} are bounded as $r \longrightarrow 0$. However

$$R_{ABC0}, R_{ABCz} = O(\rho^{-1})$$

and so in fact we still have a curvature singularity.

In summary, the string is considered to be totally geodesic if a sequence of timelike (or spacelike) curves normal to a foliation of 3-dimensional hypersurfaces have spacelike accelerations whose magnitude tends to zero as they approach a timelike (or spacelike) intrinsic geodesic of the string. In particular this is shown to occur if the lapse function of the hypersurfaces N is C^2 in a rather artificial quasi-Cartesian coordinate system. This is the key assumption. The paper makes the assumption that in this coordinate system the components κ^e_x, κ^e_y of the accelerations of the normal curves are bounded, when in fact this is a consequence of N being C^2 at the string (and in fact the magnitude of the spacelike accelerations of the normal curves could still tend to zero even if κ^e_x, κ^e_y were not bounded).

The paper deduces that some components of the curvature may diverge near the string, though it does not analyse all the components. The paper claims to show that the string is totally geodesic if the Ricci tensor is bounded and the Weyl tensor has a sufficiently weak singularity, however it makes no use of these assumptions and it turns out that the Ricci tensor may in fact diverge.

4.2 A new definition of an idealised cosmic string

We now present an alternative definition of a class of "weak" curvature singularities which, despite being curvature singularities, remain weak enough to have the properties we would expect of a cosmic string. We will therefore think of them as "idealised cosmic strings".

We shall say that a C^r space-time (M, g) contains an *idealised cosmic string* (see diagram 4.2.1) if there exists a C^r map

$$\phi : (0,1) \times (0,1) \times (0,1] \times S^1 \longrightarrow M$$

such that ϕ is a diffeomorphism onto $U \doteq Im\,\phi$ where we parametrise S^1 by $\theta : [0, 2\pi] \longrightarrow S^1$ and $\theta(0) = \theta(2\pi)$ so

$$\phi : (t, z, r, \theta) \longmapsto \phi(t, z, r, \theta)$$

and

(a) $S_{tz}(r, \theta) = \phi(t, z, r, \theta)$ is a spacelike 2-surface

(b) ∂_t is timelike and ∂_z is spacelike

(c) $g_{tr}, g_{zr} \longrightarrow 0$ as $r \longrightarrow 0$

(d) $\phi_{tz\theta}(r) = \phi(t, z, r, \theta)$ is a geodesic of the space-time with r measuring b-length

and also, if $g^{\parallel}$ is the metric induced on the tangent bundle $S = TS_{tz}$ by g and $g^{\perp}$ is the metric induced on the normal bundle $T = (TS_{tz})^{\perp}$ by g, then

(e) $r = 0$ is a good quasi-regular singularity of $(S_{tz}, g^{\parallel})$ in the sense that every C^1 curve of finite b-length lying in S_{tz} on which $r \longrightarrow 0$ terminates at the same good quasi-regular singularity (well defined since $g^{\parallel}$ is positive definite)

(f) there exists an isometry ψ of $(U, T, g^{\perp})$ into some $(\tilde{U}, \tilde{T}, \tilde{g}^{\perp})$ and $U_0 = \tilde{U} - U$ is a 2-manifold which corresponds to $r = 0$ in the sense that all curves on a given S_{tz} on which $r \longrightarrow 0$ terminate at the same point of U_0, curves on different S_{tz} on which $r \longrightarrow 0$ terminate at different points of U_0, and each point of U_0 is the termination point for some curve on some S_{tz} on which $r \longrightarrow 0$

(g) $\tilde{g}^{\perp}$ is C^0 on U_0 but $\tilde{g}^{\perp}|_{U_0}, \tilde{g}^{\perp}|_{r>0}$ are both C^r, and $\tilde{g}^{\perp}|_{U_0}$ is a Lorentzian metric with ∂_t timelike and ∂_z spacelike.

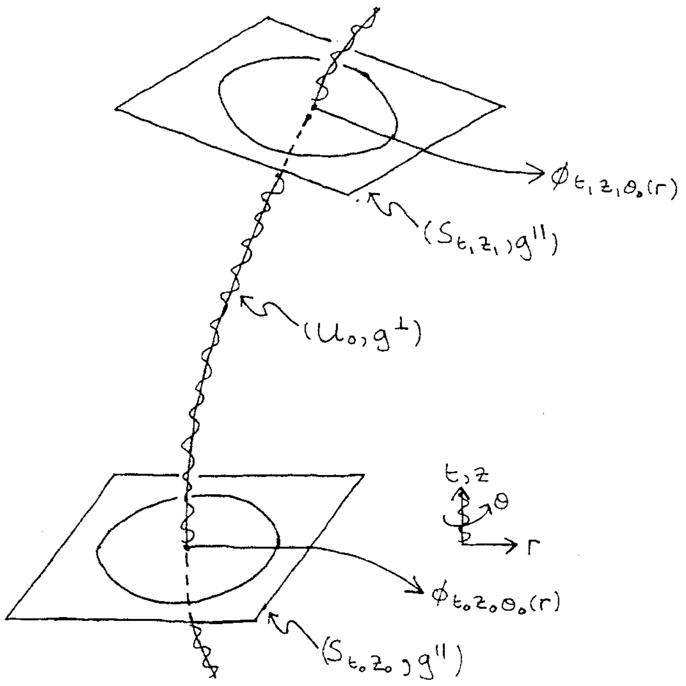

Diagram 4.2.1

Thus U is foliated by a family of spacelike 2-surfaces $\{S_{tz}\}$ each of which has a quasi-regular singularity at $r = 0$ with respect to $g^{\|}$. These spacelike 2-surfaces are ruled by space-time geodesics (condition (d)) which we think of as being normal to the singularity at $r = 0$ (condition (c)).

We think of U_0 as the string. U_0 can be considered to be a timelike 2-surface with C^r intrinsic metric $\lim_{r\to 0} g^{\perp}$, despite the fact that it will in general be a curvature singularity of the space-time. Provided that $\lim_{r\to 0} g^{\perp}$ is unique, the string has a well defined intrinsic geometry. However we do not require $\lim_{r\to 0} \partial_r g^{\perp}, \lim_{r\to 0} \partial_r^2 g^{\perp}$ to exist.

U_0 provides a C^0 singular boundary for (U, g). The isometry ψ in condition (f) is not however required to be unique and therefore U_0 does not provide a C^r singular boundary for $r > 0$. If the string were a regular part of the space-time then the $\{S_{tz}\}$ 2-surfaces would be regular and unique. It can also be shown that if the string were a regular part of the space-time, the extrinsic curvature of the $\{S_{tz}\}$ 2-surfaces would vanish at $r = 0$ (where we define the extrinsic curvature of a non-null submanifold in section 2.5) The method of proof is to define Cartesian coordinates in terms of our geodesic polar coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$

and show that at $x = 0$, $y = 0$ the Levi-Civita connection obeys $\Gamma^D_{ab} = 0$ where $a, b \in \{x, y\}$ and $D \in \{t, z\}$.

We conjecture that even in the singular case, the $\{S_{tz}\}$ 2-surfaces are unique.

The simplest example of an idealised cosmic string is perhaps the *dynamic cone*

$$ds^2 = -dt^2 + dr^2 + A^2(t, z)r^2 d\theta^2 + dz^2 \qquad 0 \leq \theta < 2\pi.$$

We discuss this and other examples of idealised cosmic strings in more detail in section 6.3.

It would also be possible to form an atlas $\mathcal{A}$ of maps like ϕ and to separate $\mathcal{A}$ into distinct cosmic strings.

We also note that a space-time which contains an idealised cosmic string need not necessarily obey the energy conditions or be a vacuum space-time.

We will study the geometrical properties of these singularities in the next two chapters.

Chapter 5

Intrinsic and extrinsic holonomy

5.1 Parallelising torsion

Let (M, g) be a space-time. Recall that $GL(M)$ is the bundle of all bases on M and LM the bundle of all frames on M where a frame is an oriented, time-oriented pseudo-orthonormal basis. Let ω be a connection on $GL(M)$. ω is *metric* if $\nabla g = 0$ where ω induces covariant derivative ∇.

Proposition 5.1.1. $\nabla g = 0 \iff$ *a pseudo-orthonormal basis remains pseudo-orthonormal under parallel propagation by* ω.

This implies that a connection on $GL(M)$ for which $\nabla g = 0$ can be regarded as a connection on LM, and conversely a connection on LM, which can also be regarded as a connection on $GL(M)$, satisfies $\nabla g = 0$.

Of all the connections on LM, or equivalently metric connections on $GL(M)$, there exists a unique torsion free connection, called the Levi-Civita connection.

Now define a connection $\tilde{\omega}$ on LM by choosing a C^2 field (e_i) of frames on M, or at least on an open $U \subset M$, and setting

$$\tilde{\omega}^{k}_{ij} = 0$$

in this frame where $\tilde{\nabla}_i e_j = \tilde{\omega}^{k}_{ij} e_k$. (e_i) is *parallel* in the sense that given any point and any vector X at that point $X^i \tilde{\nabla}_i e_j = 0$. From Proposition 5.1.1 $\tilde{\nabla} g = 0$ and thus $\tilde{\omega}$ is a metric connection. Now let γ be a C^1 curve with tangent u^i from $a \in M$ to $b \in M$. Since $u^i \tilde{\nabla}_i e_j = 0$, the result of parallelly propagating (e_i) along γ from a to b is $(e_i)|_b$ and is thus independent of the path taken from a to b. Furthermore if a and b coincide, so γ is a closed loop, $(e_i)|_a = (e_i)|_b$ and the holonomy generated by any closed loop will be trivial. $\tilde{\omega}$ has zero curvature

$$\tilde{\Omega}^j_i = d\tilde{\omega}^j_i + \tilde{\omega}^j_k \wedge \tilde{\omega}^k_i = 0$$

but in general non-zero torsion

$$\tilde{T}(e_i, e_j) = \tilde{\nabla}_i e_j - \tilde{\nabla}_j e_i - [e_i, e_j] = -[e_i, e_j].$$

We note however that $\tilde{\omega}$ depends on the choice of (e_i) and is thus non-unique.

Proposition 5.1.2. *Let U be a simply connected open set in M and ω a connection on $L(U)$ such that $\Omega^j_i = 0$. Then the result of parallelly propagating a frame between any two points is path independent.*

Proof. Let $a, b \in U$ and let γ, δ be C^1 curves in U from a to b. Parametrise γ, δ such that $\gamma(0) = \delta(0) = a$ and $\gamma(1) = \delta(1) = b$. Define a closed loop

$$\rho_1 : [0,1] \to U : s \mapsto \begin{cases} \gamma(2s) & 0 \leq s \leq \frac{1}{2} \\ \delta(2-2s) & \frac{1}{2} < s \leq 1. \end{cases}$$

Since U is simply connected there exists a C^1 homotopy

$$\rho : [0,1] \times [0,1] \to U : (s,u) \mapsto \rho_u(s)$$

such that $\rho_u(0) = \rho_u(1)$ and $Im\, \rho_0$ is a single point. Let (e_i) be a frame at a. The holonomy generated by parallelly propagating (e_i) round ρ_1 is

$$L^j_i(1) = P_u \exp \int_0^1 \int_0^1 \Omega^j_i(X(s,u), Y(s,u))\, ds du$$

where X, Y are the images of ∂_s, ∂_u induced by ρ. Since $\Omega^j_i = 0$, $L^j_i(1) = \delta^j_i$ and it follows that the parallel propagates of (e_i) along γ and δ are equal. $\square$

If parallel propagation with respect to a connection ω on LM is path independent then we may choose a point $x_0 \in M$ and, provided M is (path) connected, pick a frame (e_i) at x_0 and parallelly propagate (e_i) to all other points of M. This will result in a parallel frame: the parallel propagate of $(e_i)|_a$ along a C^1 curve from $a \in M$ to $b \in M$ will be the same as the parallel propagate of $(e_i)|_a$ from a to x_0 to b, i.e. $(e_i)|_b$.

If U is a simply connected open set in M and ω a Levi-Civita connection on $L(U)$ with $\Omega^j_i = 0$ it follows that we can construct a parallel frame (e_i) in which

$$T(e_i, e_j) = -[e_i, e_j] = 0.$$

Hence (e_i) is a coordinate basis and $g = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric on U.

5.2 Projected connection

Suppose that $\forall x \in M$ U_x, V_x are subspaces of $T_x M$ such that

$$T_x M = U_x \oplus V_x \qquad \dim U_x = \dim V_x = 2$$

where U_x, V_x are chosen in a C^2 way but are not necessarily surface forming. We may define tangent bundles U, V such that U has fibres $\{U_x\}_{x \in M}$, V has fibres $\{V_x\}_{x \in M}$ and choose a C^2 basis field $(e_i) = (e_0, e_1, e_2, e_3)$ on M adapted to U and V in the sense that $\forall x \in M$ e_2, e_3 span U_x and e_0, e_1 span V_x. In the following we shall therefore take early lower case indices $\{a, b, c, \ldots\}$ to run through 2, 3; early upper case indices $\{A, B, C, \ldots\}$ to run through 0, 1; and late lower case indices $\{i, j, k, \ldots\}$ to run through 0, 1, 2, 3. For example if $w = u + v$ for $u \in U_x, v \in V_x$ and $x \in M$ then we may write $w^i e_i = u^a e_a + v^A e_A$.

For $p, q \in \mathbb{N}$, let $T^q_p(U_x)$, $T^q_p(V_x)$ be the vector spaces of tensors defined by

$$T^q_p(U_x) = \{U^{b_1 \ldots b_q}_{a_1 \ldots a_p} : \underbrace{U^*_x \times \ldots \times U^*_x}_{q \text{ copies}} \times \underbrace{U_x \times \ldots \times U_x}_{p \text{ copies}} \longrightarrow \mathbb{R} \mid U^{b_1 \ldots b_q}_{a_1 \ldots a_p} \text{ is multilinear}\}$$

$$T^q_p(V_x) = \{V^{B_1 \ldots B_q}_{A_1 \ldots A_p} : \underbrace{V^*_x \times \ldots \times V^*_x}_{q \text{ copies}} \times \underbrace{V_x \times \ldots \times V_x}_{p \text{ copies}} \longrightarrow \mathbb{R} \mid V^{B_1 \ldots B_q}_{A_1 \ldots A_p} \text{ is multilinear}\}.$$

We can now define maps $\forall p, q \in \mathbb{N}$

$$D^U{}_a : U_x \times T^q_p(U_x) \to T^q_p(U_x) \qquad D^V{}_A : V_x \times T^q_p(V_x) \to T^q_p(V_x)$$

by making a C^1 choice of coefficients $w^{U}{}^c_{ab}$, $w^{V}{}^C_{AB}$ and setting

$$D^U{}_a e_b = w^{U}{}^c_{ab} e_c \qquad D^V{}_A e_B = w^{V}{}^C_{AB} e_C$$

where we extend $D^U{}_a$, $D^V{}_A$ in a bilinear way and require them to obey a Leibnitz property and also require

$$X^a D^U{}_a f = X(f) \quad Y^A D^V{}_A f = Y(f) \quad \forall C^1\, f : M \to \mathbb{R}, X \in U_x, Y \in V_x, x \in M.$$

Because of these properties we call $D^U{}_a$, $D^V{}_A$ *tangential connections* even though they are not actually connections on a principal fibre bundle unless U, V are surface forming. Thus if $\gamma : s \mapsto \gamma(s)$ is a C^1 curve in M through $\gamma(\alpha)$ with tangent $X(s)$ everywhere tangent to U then given $u^b(\alpha) \in U_{\gamma(\alpha)}$, $X^a D^U{}_a u^b = 0$ uniquely defines the parallel propagation of u^b along γ such that $u^b(s)$ remains tangent to U. $D^U{}_a$ can only parallelly propagate vectors in U and only in directions tangent to U. Similarly $D^V{}_A$ can only parallelly propagate vectors in V and only in directions tangent to V.

We can also define *non-tangential connections* $\tilde{D}^U{}_A$, $\tilde{D}^V{}_a$ by making a C^1 choice of coefficients $\tilde{w}^{U}{}^c_{Ab}$, $\tilde{w}^{V}{}^C_{aB}$ and setting

$$\tilde{D}^U{}_A e_b = \tilde{w}^{U}{}^c_{Ab} e_c \qquad \tilde{D}^V{}_a e_B = \tilde{w}^{V}{}^C_{aB} e_C$$

where we extend $\tilde{D}^U{}_A$, $\tilde{D}^V{}_a$ as above though again they are not connections on a principal fibre bundle unless V, U respectively are surface forming. $\tilde{D}^U{}_A$ can only parallelly propagate vectors in U but only in directions tangent to V and $\tilde{D}^V{}_a$ can only parallelly propagate vectors in V but only in directions tangent to U.

We can put these together to give

$$\nabla^U{}_i U^{b_1\ldots b_q}_{a_1\ldots a_p} = \begin{cases} D^U{}_a U^{b_1\ldots b_q}_{a_1\ldots a_p} & i = a \\ \tilde{D}^U{}_A U^{b_1\ldots b_q}_{a_1\ldots a_p} & i = A \end{cases} \qquad \nabla^V{}_i V^{B_1\ldots B_q}_{A_1\ldots A_p} = \begin{cases} \tilde{D}^V{}_a V^{B_1\ldots B_q}_{A_1\ldots A_p} & i = a \\ D^V{}_A V^{B_1\ldots B_q}_{A_1\ldots A_p} & i = A. \end{cases}$$

Thus ∇^U is a connection on $GL(U)$, the principal fibre bundle of 2-bases tangent to U and ∇^V is a connection on $GL(V)$, the principal fibre bundle of 2-bases tangent to V. ∇^U can parallelly propagate vectors in U in any direction and ∇^V can parallelly propagate vectors in V in any direction.

Finally we define a connection $\overline{\nabla}$ on $GL(M)$ by

$$\overline{\nabla} e_i = \begin{cases} \nabla^U e_a & i = a \\ \nabla^V e_A & i = A \end{cases}$$

which we extend as above. $\overline{\nabla}$ is a well defined connection since if $\tilde{e}_i = \lambda^j_i e_j$ is another adapted basis so $\lambda^B_a, \lambda^b_A = 0$ we have

$$\overline{\nabla}\tilde{e}_a = \overline{\nabla}(\lambda^b_a e_b) = \lambda^b_a \overline{\nabla} e_b + e_b \overline{\nabla}\lambda^b_a = \lambda^b_a \nabla^U e_b + e_b \nabla^U \lambda^b_a = \nabla^U(\lambda^b_a e_b) = \nabla^U \tilde{e}_b$$

where λ^b_a is a scalar. Similarly $\overline{\nabla} e_A = \nabla^V \tilde{e}_A$. $\overline{\nabla}$ has the property that a vector initially tangent to U or V will remain tangent to U or V under parallel propagation by $\overline{\nabla}$. We can thus also consider the restriction of $\overline{\nabla}$ to $GL(U,V)$, the principal fibre bundle of 4-bases adapted to U and V. $GL(U,V)$ is a sub-bundle of $GL(M)$ and has the structure group $GL_2(\mathbb{R}) \times GL_2(\mathbb{R})$ which is a subgroup of $GL_4(\mathbb{R})$, the structure group of $GL(M)$.

Now define projections

$$p : TM \to TU : u + v \mapsto u \quad \tilde{p} : TM \to TV : u + v \mapsto v \quad \forall u \in U_x, v \in V_x, x \in M.$$

These are linear so we write $p : w^j \mapsto p^j_i w^i$ and $\tilde{p} : w^j \mapsto \tilde{p}^j_i w^i$ where

$$p^j_i = \begin{cases} \delta^b_a & i = a, j = b \\ 0 & \text{otherwise} \end{cases} \qquad \tilde{p}^j_i = \begin{cases} \delta^B_A & i = A, j = B \\ 0 & \text{otherwise.} \end{cases}$$

Let $T^q_p(T_xM)$ be the vector space of tensors of valence $\binom{q}{p}$ at $x \in M$. We can say $T^q_p(U_x), T^q_p(V_x) < T^q_p(T_xM)$ (where $<$ denotes vector subspace) in the sense that if $U \in T^q_p(T_xM)$

$$U \in T^q_p(U_x) \iff U^{j_1\dots j_q}_{i_1\dots i_p} = \begin{cases} U^{b_1\dots b_q}_{a_1\dots a_p} & i_1 = a_1, \dots, i_p = a_p, j_1 = b_1, \dots, j_q = b_q \\ 0 & \text{otherwise} \end{cases}$$

$$V \in T^q_p(V_x) \iff V^{j_1\dots j_q}_{i_1\dots i_p} = \begin{cases} V^{B_1\dots B_q}_{A_1\dots A_p} & i_1 = A_1, \dots, i_p = A_p, j_1 = B_1, \dots, j_q = B_q \\ 0 & \text{otherwise.} \end{cases}$$

For $p, q \in \mathbb{N}$ we define the following maps on $T^q_p(T_xM)$

$$\pi_U : W^{j_1\dots j_q}_{i_1\dots i_p} \mapsto p^{k_1}_{i_1} \dots p^{k_p}_{i_p} p^{j_1}_{l_1} \dots p^{j_q}_{l_q} W^{l_1\dots l_q}_{k_1\dots k_p}$$

$$\pi_V : W^{j_1\dots j_q}_{i_1\dots i_p} \mapsto \tilde{p}^{k_1}_{i_1} \dots \tilde{p}^{k_p}_{i_p} \tilde{p}^{j_1}_{l_1} \dots \tilde{p}^{j_q}_{l_q} W^{l_1\dots l_q}_{k_1\dots k_p}.$$

Proposition 5.2.1.

(a) *If $W \in T^q_p(T_xM)$ for $p, q \in \mathbb{N}$ then $\pi_U W \in T^q_p(U_x)$ and $\pi_V W \in T^q_p(V_x)$*

(b) *π_U acts as the identity on $T^q_p(U_x)$ and π_V acts as the identity on $T^q_p(V_x)$.*

(c) *$\pi_U T^q_p(V_x) = \pi_V T^q_p(U_x) = 0$*

Proof. $p^A_i, p^i_A, \tilde{p}^a_i, \tilde{p}^i_a = 0$, $p^b_a = \delta^b_a$, and $\tilde{p}^B_A = \delta^B_A$. □

Now let ∇ be some (not necessarily metric) connection on $GL(M)$. Since π_U, π_V are linear we can define connections ∇^U on $GL(U)$ and ∇^V on $GL(V)$ as follows: given $X \in T_xM$, $x \in M$, $p, q \in \mathbb{N}$

$$\nabla^U{}_X U = \pi_U \nabla_X U \quad U \in T^q_p(U_x) \qquad \nabla^V{}_X V = \pi_V \nabla_X V \quad V \in T^q_p(V_x)$$

where as above $T^q_p(U_x), T^q_p(V_x) < T^q_p(T_xM)$. We can then put these together as above to form a new connection $\overline{\nabla}$ which we shall call the *projected connection* of ∇ defined by

$$\overline{\nabla} e_i = \begin{cases} \nabla^U e_a & i = a \\ \nabla^V e_A & i = A \end{cases}$$

and extended in the usual way. $\overline{\nabla}$ is a well defined connection on $GL(M)$ since if $\tilde{e}_i = \lambda^j_i e_j$ is another adapted basis so $\lambda^B_a, \lambda^b_A = 0$ we have just as above $\overline{\nabla}\tilde{e}_a = \nabla^U \tilde{e}_a$,

$\overline{\nabla}\tilde{e}_A = \nabla^V\tilde{e}_A$. In general however $\overline{\nabla} \neq \nabla$. As above $\overline{\nabla}$ has the property that a vector initially tangent to U or V will remain tangent to U or V under parallel propagation by $\overline{\nabla}$. We may also define

$$D^U{}_a = \nabla^U{}_a \quad \tilde{D}^U{}_A = \nabla^U{}_A \quad \tilde{D}^V{}_a = \nabla^V{}_a \quad D^V{}_A = \nabla^V{}_A.$$

Suppose that ∇, ∇^U, ∇^V and $\overline{\nabla}$ have connection coefficients ω^k_{ij}, $\omega^U{}^c_{ib}$, $\omega^V{}^C_{iB}$ and $\overline{\omega}^k_{ij}$. Then

$$\nabla^U{}_{e_i}e_b = \pi_U\nabla_{e_i}e_b = \pi_U\omega^k_{ib}e_k = \omega^c_{ib}e_c$$

and thus $\omega^U{}^c_{ib} = \omega^c_{ib}$. Similarly $\omega^V{}^C_{iB} = \omega^C_{iB}$. Hence

$$\overline{\nabla}_i e_j = \begin{cases} \omega^c_{ib}e_c & j = b \\ \omega^C_{iB}e_C & j = B \end{cases} \Rightarrow \overline{\omega}^C_{ib} = \overline{\omega}^c_{iB} = 0, \quad \overline{\omega}^c_{ib} = \omega^c_{ib}, \quad \overline{\omega}^C_{iB} = \omega^C_{iB}.$$

If $D^U{}_a$, $\tilde{D}^U{}_A$, $D^V{}_A$, $\tilde{D}^V{}_a$ have connection coefficients $w^U{}^c_{ab}$, $\tilde{w}^U{}^c_{Ab}$, $w^V{}^C_{AB}$, $\tilde{w}^V{}^C_{aB}$ then it also follows that

$$w^U{}^c_{ab} = \omega^c_{ab} \quad \tilde{w}^U{}^c_{Ab} = \omega^c_{Ab} \quad w^V{}^C_{AB} = \omega^C_{AB} \quad \tilde{w}^V{}^C_{aB} = \omega^C_{aB}.$$

Now let g be a metric on M and define

$$g^U := g|_{U\times U} \qquad g^V := g|_{V\times V}.$$

Thus $\forall x \in M$, $g^U \in T^0_2(U_x)$, $g^V \in T^0_2(V_x)$ where as above we set $T^0_2(U_x), T^0_2(V_x) < T^0_2(T_xM)$.

Proposition 5.2.2. $g^U = \pi_U g$, $g^V = \pi_V g$.

Proof. Let $x \in M$. By Proposition 5.2.1, $\pi_U g \in T^0_2(U_x)$. Now let $u_1, u_2 \in U_x$. Then

$$(\pi_U g)(u_1, u_2) = p^k_i p^l_j g_{kl} u_1{}^i u_2{}^j = g_{kl}u_1{}^k u_2{}^l = g|_{U\times U}(u_1, u_2).$$

Thus $\pi_U g = g^U$ and similarly $\pi_V g = g^V$. □

We say that ∇^U is metric if $\nabla^U g^U = 0$, and that ∇^V is metric if $\nabla^V g^V = 0$. Similarly we can say that D^U, $\tilde{D}^U$, $\tilde{D}^V$, D^V are metric if $D^U g^U = 0$, $\tilde{D}^U g^U = 0$, $\tilde{D}^V g^V = 0$, $D^V g^V = 0$.

We note that $\nabla^U g^U = 0$ (or $\nabla^V g^V = 0$) if and only if a pseudo-orthonormal 2-basis of U_x (or V_x) for any $x \in M$ remains pseudo-orthonormal under parallel propagation by ∇^U (or ∇^V). Similar properties hold for D^U, $\tilde{D}^U$, $\tilde{D}^V$, D^V.

Now let ∇ be metric. Then for $X \in T_xM$, $x \in M$

$$\begin{aligned}
{\nabla^U}_X {g^U}_{ab} &= \pi_U \nabla_X \pi_U g_{ab} = p^i_a p^j_b \nabla_X (p^k_i p^l_j g_{kl}) \\
&= p^i_a p^j_b p^k_i p^l_j \nabla_X g_{kl} + p^i_a p^j_b p^k_i g_{kl} \nabla_X p^l_j + p^i_a p^j_b p^l_j g_{kl} \nabla_X p^k_i \\
&= p^j_b g_{al} \nabla_X p^l_j + p^i_a g_{kb} \nabla_X p^k_i \\
&= p^j_b g_{al} X^m (\omega^l_{mn} p^n_j - \omega^n_{mj} p^l_n) + p^i_a g_{kb} X^m (\omega^k_{mn} p^n_i - \omega^n_{mi} p^k_n) \\
&= X^m g_{al} (\omega^l_{mb} - \omega^n_{mb} p^l_n) - X^m g_{kb} (\omega^k_{ma} - \omega^n_{ma} p^k_n) \\
&= X^m (g_{aC} \omega^C_{mb} - g_{Cb} \omega^C_{ma})
\end{aligned}$$

and thus in general ∇^U is not metric even if ∇ is.

If however $U_x \perp V_x$ for some $x \in M$, then at this point, in our (not necessarily pseudo-orthonormal) adapted basis (e_i), we have $g_{aB} = g_{Ab} = 0$ and

$$g = g_{ij} e^i \otimes e^j = g_{ab} e^a \otimes e^b + g_{AB} e^A \otimes e^B = g^U + g^V.$$

Theorem 5.2.3. *If $U_x \perp V_x$ for some $x \in M$ and ∇ is metric at this point then so are $\overline{\nabla}$, ∇^U, and ∇^V, as well as D^U, $\tilde{D}^U$, $\tilde{D}^V$, D^V.*

Proof. Let $X \in T_xM$. From the above we have

$${\nabla^U}_X {g^U}_{ab} = X^m (g_{aC} \omega^C_{mb} - g_{Cb} \omega^C_{ma}) = 0$$

but $g_{aC} = g_{Cb} = 0$ and thus ∇^U is metric. Similarly ∇^V is metric as are D^U, $\tilde{D}^U$, $\tilde{D}^V$, D^V. Furthermore

$$\overline{\nabla} g = \overline{\nabla} g^U + \overline{\nabla} g^V = \nabla^U g^U + \nabla^V g^V = 0$$

and hence $\overline{\nabla}$ is metric. $\square$

Since $\overline{\nabla}$ is a connection on $GL(M)$ we may consider its torsion $\overline{T}$.

Theorem 5.2.4. *If ∇ is the Levi-Civita connection of g and $U_x \perp V_x \ \forall x \in M$ then either (a) $\overline{T} \neq 0$ and $\overline{\nabla}$ is not the Levi-Civita connection of any metric or (b) $\overline{T} = 0$ and $\overline{\nabla} = \nabla$.*

Proof. By Theorem 5.2.3, $\overline{\nabla} g = 0$. If $\overline{T} = 0$ it follows that $\overline{\nabla}$ must be the unique Levi-Civita connection of g and that $\overline{\nabla} = \nabla$. If however $\overline{T} \neq 0$ then $\overline{\nabla}$ cannot be the Levi-Civita connection of any metric. $\square$

We can also consider the curvature $\overline{\Omega}_i^j$ of $\overline{\nabla}$. In fact, using the Cartan equations, we can consider the torsion and curvature of any connection defined on $GL(M)$, or on any sub-bundle of $GL(M)$. The Cartan equations do not in general make sense however for ∇^U and ∇^V, defined on $GL(U)$ and $GL(V)$, or for D^U, $\tilde{D}^U$, $\tilde{D}^V$, D^V.

Recall that a connection $\hat{\nabla}$ on $GL(M)$ has torsion

$$\hat{T}(e_i, e_j) = \hat{\nabla}_i e_j - \hat{\nabla}_j e_i - [e_i, e_j].$$

$\tilde{D}^U$, $\tilde{D}^V$ do not have meaningful torsions; in the expression $\tilde{D}^U{}_A e_b - \tilde{D}^U{}_b e_A - [e_A, e_b]$, for example, $\tilde{D}^U{}_b$ is undefined. Similarly ∇^U, ∇^V do not have meaningful torsions; in the expression $\nabla^U{}_A e_b - \nabla^U{}_b e_A - [e_A, e_b]$, for example, $\nabla^U{}_b e_A$ is undefined.

For $x \in M$ we may however define the torsions T^U, T^V of D^U, D^V as the maps

$$T^U : U_x \times U_x \to T_x M : (e_a, e_b) \mapsto D^U{}_a e_b - D^U{}_b e_a - [e_a, e_b]$$

$$T^V : V_x \times V_x \to T_x M : (e_A, e_B) \mapsto D^V{}_A e_B - D^V{}_B e_A - [e_A, e_B]$$

though in general $[e_a, e_b] \notin U$, $[e_A, e_B] \notin V$ unless (e_a), (e_A) are surface forming.

Now suppose ∇ is the Levi-Civita connection of g. Then

$$\nabla_a e_b - \nabla_b e_a - [e_a, e_b] = 0$$

$$\Rightarrow \pi_U \nabla_{e_a} e_b - \pi_U \nabla_{e_b} e_a - \pi_U [e_a, e_b] = 0$$

$$\Rightarrow D^U{}_a e_b - D^U{}_b e_a - \pi_U [e_a, e_b] = 0$$

$$\Rightarrow \pi_U (D^U{}_a e_b - D^U{}_b e_a - [e_a, e_b]) = 0$$

$$\Rightarrow \pi_U T^U (e_a, e_b) = 0.$$

If $U_x \perp V_x$ for some $x \in M$ then at this point $D^U g^U = 0$ and $\pi_U T^U = 0$ and similarly $D^V g^V = 0$ and $\pi_V T^V = 0$. We claim that this is sufficient to characterise D^U, D^V at x. In other words, even if (e_a), (e_A) are not surface forming we have

Theorem 5.2.5. *If $U_x \perp V_x$ for some $x \in M$ then there exist unique tangential connections D^U on U_x and D^V on V_x such that at x, $D^U g^U = 0$, $\pi_U T^U = 0$ and $D^V g^V = 0$, $\pi_V T^V = 0$.*

This follows from

Theorem 5.2.6.

(a) *Let $\underset{1}{D}^U$, $\underset{2}{D}^U$ be tangential connections on U defined by $\underset{1}{D}^U{}_a e_b = \underset{1}{w}^{U\,c}{}_{ab} e_c$, $\underset{2}{D}^U{}_a e_b = \underset{2}{w}^{U\,c}{}_{ab} e_c$ with torsions $\underset{1}{T}^U$, $\underset{2}{T}^U$. If $\underset{1}{D}^U g^U = 0$, $\underset{2}{D}^U g^U = 0$ and $\pi_U \underset{1}{T}^U = \pi_U \underset{2}{T}^U$ at some $x \in M$, then at this point $\underset{1}{D}^U = \underset{2}{D}^U$.*

(b) *Let $\underset{1}{D}^V$, $\underset{2}{D}^V$ be tangential connections on V defined by $\underset{1}{D}^V{}_A e_B = \underset{1}{w}^{V\,C}{}_{AB} e_C$, $\underset{2}{D}^V{}_A e_B = \underset{2}{w}^{V\,C}{}_{AB} e_C$ with torsions $\underset{1}{T}^V$, $\underset{2}{T}^V$. If $\underset{1}{D}^V g^V = 0$, $\underset{2}{D}^V g^V = 0$ and $\pi_V \underset{1}{T}^V = \pi_V \underset{2}{T}^V$ at some $x \in M$, then at this point $\underset{1}{D}^V = \underset{2}{D}^V$.*

Proof. At $x \in M$

$$\underset{1}{D}^U{}_a g^U{}_{bc} - \underset{2}{D}^U{}_a g^U{}_{bc} = 0$$

$$\Rightarrow \partial_a g^U{}_{bc} - \underset{1}{w}^{U\,d}{}_{ab} g^U{}_{dc} - \underset{1}{w}^{U\,d}{}_{ac} g^U{}_{bd} - \partial_a g^U{}_{bc} + \underset{2}{w}^{U\,d}{}_{ab} g^U{}_{dc} + \underset{2}{w}^{U\,d}{}_{ac} g^U{}_{bd} = 0$$

$$\Rightarrow \sigma^{U\,d}{}_{ab} g^U{}_{dc} + \sigma^{U\,d}{}_{ac} g^U{}_{bd} = 0 \tag{5.2.1}$$

where $\sigma^{U\,c}_{\ \ ab} = w^{U\,c}_{2\ ab} - w^{U\,c}_{1\ ab}$. By cycling a, b, c we have

$$\sigma^{U\,d}_{\ \ bc} g^{U}{}_{da} + \sigma^{U\,d}_{\ \ ba} g^{U}{}_{cd} = 0 \qquad (5.2.2)$$

$$-\sigma^{U\,d}_{\ \ ca} g^{U}{}_{db} - \sigma^{U\,d}_{\ \ cb} g^{U}{}_{ad} = 0. \qquad (5.2.3)$$

Now since

$$\pi_U T^U_2(e_a, e_b) - \pi_U T^U_1(e_a, e_b) = 0$$

$$\Rightarrow \pi_U(D^U_{2\ a} e_b - D^U_{2\ b} e_a - [e_a, e_b] - D^U_{1\ a} e_b + D^U_{1\ b} e_a + [e_a, e_b]) = 0$$

$$\Rightarrow w^{U\,c}_{2\ [ab]} - w^{U\,c}_{1\ [ab]} = 0$$

$$\Rightarrow \sigma^{U\,c}_{\ \ ab} = \sigma^{U\,c}_{\ \ ba}$$

and $g^U{}_{ab} = g^U{}_{ba}$ it follows if we add (5.2.1), (5.2.2) and (5.2.3) that

$$\sigma^{U\,d}_{\ \ ab} g^U{}_{dc} = 0.$$

Since $g^U|_{U\times U}$ is non-singular we get $\sigma^{U\,c}_{\ \ ab} = 0$ and thus $w^{U\,c}_{1\ ab} = w^{U\,c}_{2\ ab}$. Hence $D^U_1 = D^U_2$ and similarly $D^V_1 = D^V_2$. □

For $x \in M$, the curvature $\hat{\Omega}^j_i$ of a connection $\hat{\nabla}$ on $GL(M)$ is defined by

$$\hat{\Omega}^j_i(X,Y)e_j = \hat{\nabla}_X \hat{\nabla}_Y e_i - \hat{\nabla}_Y \hat{\nabla}_X e_i - \hat{\nabla}_{[X,Y]} e_i \quad \forall X, Y \in T_x M.$$

We can thus define the curvatures Ω^j_i, $\overline{\Omega}^j_i$ of ∇, $\overline{\nabla}$ (where ∇ need not be metric). We may also define the curvatures $\Omega^{U\,b}_{\ \ a}$, $\Omega^{V\,B}_{\ \ A}$ of ∇^U, ∇^V by putting ∇^U, ∇^V in the above expression

$$\Omega^{U\,b}_{\ \ a}(X,Y)e_b = \nabla^U{}_X \nabla^U{}_Y e_a - \nabla^U{}_Y \nabla^U{}_X e_a - \nabla^U{}_{[X,Y]} e_a \qquad \forall X, Y \in T_x M$$

$$\Omega^{V\,B}_{\ \ A}(X,Y)e_B = \nabla^V{}_X \nabla^V{}_Y e_A - \nabla^V{}_Y \nabla^V{}_X e_A - \nabla^V{}_{[X,Y]} e_A \qquad \forall X, Y \in T_x M$$

from which we get second Cartan equations

$$\Omega^{U\,b}_{\ \ a} = d\omega^{U\,b}_{\ \ a} + \omega^{U\,b}_{\ \ c} \wedge \omega^{U\,c}_{\ \ a} \qquad \Omega^{V\,B}_{\ \ A} = d\omega^{V\,B}_{\ \ A} + \omega^{V\,B}_{\ \ C} \wedge \omega^{V\,C}_{\ \ A}$$

where $\Omega^{U}{}^{b}_{a} : T_xM \times T_xM \to T^1_1(U_x) : (X,Y) \mapsto \Omega^{U}{}^{b}_{a}(X,Y)$ and $\Omega^{V}{}^{B}_{A} : T_xM \times T_xM \to T^1_1(V_x) : (X,Y) \mapsto \Omega^{V}{}^{B}_{A}(X,Y)$.

We now consider the curvature $\overline{\Omega}^{j}_{i}$ of $\overline{\nabla}$. Recall that $\overline{\omega}^{B}_{a} = \overline{\omega}^{b}_{A} = 0$. Now

$$\overline{\Omega}^{B}_{a} = d\overline{\omega}^{B}_{a} + \overline{\omega}^{B}_{c} \wedge \overline{\omega}^{c}_{a} + \overline{\omega}^{B}_{C} \wedge \overline{\omega}^{C}_{a}$$

but for $X, Y \in T_xM$

$$(d\overline{\omega}^{B}_{a})(X,Y) = X(\overline{\omega}^{B}_{a}(Y)) - Y(\overline{\omega}^{B}_{a}(X)) - \overline{\omega}^{B}_{a}([X,Y]) = 0$$

and hence $\overline{\Omega}^{B}_{a} = 0$. Similarly $\overline{\Omega}^{b}_{A} = 0$. Furthermore

$$\overline{\Omega}^{b}_{a} = d\overline{\omega}^{b}_{a} + \overline{\omega}^{b}_{c} \wedge \overline{\omega}^{c}_{a} + \overline{\omega}^{b}_{C} \wedge \overline{\omega}^{C}_{a} = d\overline{\omega}^{b}_{a} + \overline{\omega}^{b}_{c} \wedge \overline{\omega}^{c}_{a}$$

$$\overline{\Omega}^{B}_{A} = d\overline{\omega}^{B}_{A} + \overline{\omega}^{B}_{c} \wedge \overline{\omega}^{c}_{A} + \overline{\omega}^{B}_{C} \wedge \overline{\omega}^{C}_{A} = d\overline{\omega}^{B}_{A} + \overline{\omega}^{B}_{C} \wedge \overline{\omega}^{C}_{A}$$

but since $\overline{\omega}^{b}_{a} = \omega^{U}{}^{b}_{a}$ and $\overline{\omega}^{B}_{A} = \omega^{V}{}^{B}_{A}$ it follows that $\overline{\Omega}^{b}_{a} = \Omega^{U}{}^{b}_{a}$ and $\overline{\Omega}^{B}_{A} = \Omega^{V}{}^{B}_{A}$.

Since $\overline{\omega}^{b}_{a} = \omega^{b}_{a}$ and $\overline{\omega}^{B}_{A} = \omega^{B}_{A}$ we also have

$$\Omega^{b}_{a} = \overline{\Omega}^{b}_{a} + \omega^{b}_{C} \wedge \omega^{C}_{a} \tag{5.2.4a}$$

$$\Omega^{B}_{A} = \overline{\Omega}^{B}_{A} + \omega^{B}_{c} \wedge \omega^{c}_{A} \tag{5.2.4b}$$

from which in section 6.1 we will derive the *Gauss-Codazzi-Ricci equations* for $\{U_x\}$ and $\{V_x\}$ respectively.

Finally the curvature $W^{U}{}^{b}_{a}$ of D^U will be well defined and given by

$$W^{U}{}^{b}_{a} e_b(X,Y) = D^{U}{}_{X} D^{U}{}_{Y} e_a - D^{U}{}_{Y} D^{U}{}_{X} e_a - D^{U}{}_{[X,Y]} e_a \qquad \forall X, Y \in U_x$$

only if $[X,Y] \in U_x$ $\forall X, Y \in U_x$, or in other words only if (e_a) is surface forming at x, in which case the curvature $\tilde{W}^{V}{}^{B}_{A}$ of $\tilde{D}^V$ will also be well defined and given by

$$\tilde{W}^{V}{}^{B}_{A} e_B(X,Y) = \tilde{D}^{V}{}_{X} \tilde{D}^{V}{}_{Y} e_A - \tilde{D}^{V}{}_{Y} \tilde{D}^{V}{}_{X} e_A - \tilde{D}^{V}{}_{[X,Y]} e_A \qquad \forall X, Y \in U_x.$$

Similarly D^V, $\tilde{D}^U$ will have well defined curvatures $W^{V}{}^{B}_{A}$, $\tilde{W}^{U}{}^{b}_{a}$ only if $[X,Y] \in V_x$ $\forall X, Y \in V_x$, or in other words only if (e_A) is surface forming at x.

5.3 Extrinsic curvature

As before, let (M, g) be a space-time, let $T_xM = U_x \oplus V_x \; \forall x \in M$ for a C^2 choice of U_x, V_x such that $\dim U_x = \dim V_x = 2$, and let (e_i) be an adapted basis field. In the following we shall also assume that $U_x \perp V_x \quad \forall x \in M$. Let ∇ be the Levi-Civita connection of g and $\overline{\nabla}$ the projected connection of ∇. Thus ∇, $\overline{\nabla}$ are connections on $GL(M)$ which satisfy $\nabla g = 0$, $\overline{\nabla} g = 0$. Furthermore

$$\overline{\omega}^b_{\ a} = \omega^b_{\ a} \qquad \overline{\omega}^B_{\ A} = \omega^B_{\ A} \qquad \overline{\omega}^B_{\ a} = \overline{\omega}^b_{\ A} = 0.$$

Let $\hat{\nabla}$ be a connection on $GL(M)$ with curvature $\hat{\Omega}^l_{\ k}$ and let $\hat{R}_{ijkl} = \hat{R}_{ijk}{}^m g_{ml}$ where $\hat{R}_{ijk}{}^l = \hat{\Omega}^l_{\ k}(e_i, e_j)$. Of the following identities

$$\hat{R}_{ijkl} = -\hat{R}_{jikl} \tag{5.3.1}$$

$$\hat{R}_{ijkl} = -\hat{R}_{ijlk} \tag{5.3.2}$$

$$\hat{R}_{[ijk]l} = 0 \qquad \hat{R}_{ijkl} = \hat{R}_{klij} \qquad \hat{\nabla}_{[i}\hat{R}_{jk]lm} = 0 \tag{5.3.3}$$

(5.3.1) holds for any $\hat{\nabla}$, (5.3.2) holds if and only if $\hat{\nabla} g = 0$, and (5.3.3) does not in general hold unless $\hat{\nabla}$ has torsion $\hat{T} = 0$. It follows therefore that the curvature $\Omega^l_{\ k}$ of ∇ obeys (5.3.1), (5.3.2) and (5.3.3) but that the curvature $\overline{\Omega}^l_{\ k}$ of $\overline{\nabla}$ only obeys (5.3.1) and (5.3.2).

Given $x \in M$ we now define the *second fundamental forms* K^U, K^V of U_x, V_x with respect to ∇, and the *associated second fundamental forms* A^U, A^V of U_x, V_x with respect to ∇ as follows: given C^1 vector fields X, X', Y, Y' defined in a neighbourhood of x such that X, X' are tangent to U and Y, Y' are tangent to V we set

$$K^U : (X, X') \mapsto \pi_V(\nabla_X X') \qquad K^V : (Y, Y') \mapsto \pi_U(\nabla_Y Y')$$

$$A^U : (X, Y) \mapsto \pi_U(\nabla_X Y) \qquad A^V : (Y, X) \mapsto \pi_V(\nabla_Y X).$$

Proposition 5.3.1. *$K^U(X,X')|_x$, $K^V(Y,Y')|_x$, $A^U(X,Y)|_x$, and $A^V(Y,X)|_x$ depend only on the values of X, X', Y, Y' at x.*

Proof. At x,

$$\begin{aligned} K^U(X,X') &= (\pi_V \nabla_X X')^j = \tilde{p}^j_i X^k \nabla_k X'^i = \tilde{p}^j_i (X^k \partial_k X'^i + X^k \omega^i_{kl} X'^l) \\ &= X^k \partial_k (\tilde{p}^j_i X'^i) + X^k \omega^i_{kl} \tilde{p}^j_i X'^l \\ &= X^k \omega^i_{kl} \tilde{p}^j_i X'^l \end{aligned}$$

which depends only on the values of X, X' at x and not on the values of X, X' at any other points. Similarly for $K^V(Y,Y')$, $A^U(X,Y)$, and $A^V(Y,X)$. $\square$

Thus in fact

$$K^U : U_x \times U_x \to V_x \qquad K^V : V_x \times V_x \to U_x$$

$$A^U : U_x \times V_x \to U_x \qquad A^V : V_x \times U_x \to V_x$$

and furthermore K^U, K^V, A^U, A^V are linear maps for which we may write

$$K^U(e_a, e_b) = K^{U\,C}_{\ \ ab} e_C \qquad K^V(e_A, e_B) = K^{V\,c}_{\ \ AB} e_c$$

$$A^U(e_a, e_B) = A^{U\,c}_{\ \ aB} e_c \qquad A^V(e_A, e_b) = A^{V\,C}_{\ \ Ab} e_C$$

where $K^{U\,C}_{\ \ ab}$, $K^{V\,c}_{\ \ AB}$ are called the *extrinsic curvatures* of U_x, V_x.

Now

$$K^U(e_a, e_b) = \pi_V(\nabla_{e_a} e_b) = \pi_V(\omega^k_{ab} e_k) = \omega^C_{ab} e_C$$

and thus $K^{U\,C}_{\ \ ab} = \omega^C_{ab}$ and similarly $K^{V\,c}_{\ \ AB} = \omega^c_{AB}$, $A^{U\,c}_{\ \ aB} = \omega^c_{aB}$, $A^{V\,C}_{\ \ Ab} = \omega^C_{Ab}$. Since $\nabla g = 0$ and $g_{aB} = g_{Ab} = 0$ we also have that

$$\nabla_a g_{bC} = \partial_a g_{bC} - \omega^l_{ab} g_{lC} - \omega^l_{aC} g_{bl} = -\omega^D_{ab} g_{DC} - \omega^d_{aC} g_{db} = 0$$

$$\Rightarrow K^{U\,D}_{\ \ ab} g_{DC} + A^{U\,d}_{\ \ aC} g_{db} = 0 \tag{5.3.4a}$$

and thus K^U, A^U may be determined from each other. Similarly

$$K^{V\,d}_{\ \ AB} g_{dc} + A^{V\,D}_{\ \ Ac} g_{DB} = 0 \tag{5.3.4b}$$

Now if $X, X' \in U_x$ and $Y, Y' \in V_x$ then

$$\nabla_X X' = \pi_U \nabla_X X' + \pi_V \nabla_X X' = D^U{}_X X' + K^U(X, X')$$

$$\nabla_X Y = \pi_U \nabla_X Y + \pi_V \nabla_X Y = A^U(X, Y) + \tilde{D}^V{}_X Y$$

and similarly

$$\nabla_Y X = A^V(Y, X) + \tilde{D}^U{}_Y X$$

$$\nabla_Y Y' = D^V{}_Y Y' + K^V(Y, Y').$$

More generally ∇ can be reconstructed from a knowledge of the projected connection $\overline{\nabla}$ and K^U, K^V, A^U, A^V, since if $W = X+Y, W' = X'+Y' \in T_xM$ where $X, X' \in U_x$ and $Y, Y' \in V_x$ then

$$\begin{aligned}
\nabla_W W' &= \pi_U \nabla_W (X' + Y') + \pi_V \nabla_W (X' + Y') \\
&= \pi_U \nabla_W X' + \pi_V \nabla_W Y' + \pi_U \nabla_{(X+Y)} Y' + \pi_V \nabla_{(X+Y)} X' \\
&= \overline{\nabla}_W X' + \overline{\nabla}_W Y' + \pi_U \nabla_X Y' + \pi_U \nabla_Y Y' + \pi_V \nabla_X X' + \pi_V \nabla_Y X' \\
&= \overline{\nabla}_W W' + A^U(X, Y') + K^V(Y, Y') + K^U(X, X') + A^V(Y, X'). \quad (5.3.5)
\end{aligned}$$

Similarly, as we will show in section 6.1, the curvature Ω_i^j of ∇ can be partly reconstructed from a knowledge of the curvature $\overline{\Omega}_i^j$ of $\overline{\nabla}$ and K^U, K^V, A^U, A^V using equations (5.2.4a) and (5.2.4b).

In our adapted basis, since ∇ has torsion $T^k_{ij} = 0$, we have

$$0 = T^C_{ab} = \omega^C_{ab} - \omega^C_{ba} - c^C_{ab} = K^{U\,C}_{ab} - K^{U\,C}_{ba} - c^C_{ab}$$

where $[e_i, e_j] = c^k_{ij} e_k$. Hence

$$K^{U\,C}_{ab} = K^{U\,C}_{ba} \iff c^C_{ab} = 0$$

and thus K^U is symmetric at x if and only if U_x is surface forming. Similarly, K^V is symmetric at x if and only if V_x is surface forming.

We say that U_x is *totally geodesic* if $K^U = 0$ at x, and that V_x is *totally geodesic* if $K^V = 0$ at x.

We can also consider the second fundamental forms $\overline{K}^U$, $\overline{K}^V$ of U_x, V_x with respect to $\overline{\nabla}$ and the associated second fundamental forms $\overline{A}^U$, $\overline{A}^V$ of U_x, V_x with respect to $\overline{\nabla}$. However if $x \in M$, $X \in U_x$, $Y \in V_x$ and $W \in T_x M$ then

$$\overline{\nabla}_W X \in U_x \qquad \overline{\nabla}_W Y \in V_x$$

from which it follows that $\overline{K}^U = \overline{K}^V = \overline{A}^U = \overline{A}^V = 0$. Thus U_x, V_x are totally geodesic with respect to $\overline{\nabla}$.

If for some $x \in M$, $K^U = K^V = 0$, it follows from (5.3.4) that $A^U = A^V = 0$. Hence

$$\omega^B_{\ ia} = \omega^b_{\ iA} = 0$$

and, at this point, $\nabla = \overline{\nabla}$. Now the torsion $\overline{T}^k_{\ ij}$ of $\overline{\nabla}$ satisfies

$$\overline{T}^k_{\ ij} = \overline{\omega}^k_{\ ij} - \overline{\omega}^k_{\ ji} - c^k_{\ ij} = \omega^k_{\ ij} - \omega^k_{\ ji} - c^k_{\ ij} = 0$$

from which it also follows that

$$c^C_{\ ab} = \omega^C_{\ ab} - \omega^C_{\ ba} = 0 \qquad c^c_{\ AB} = \omega^c_{\ AB} - \omega^c_{\ BA} = 0$$

and hence U_x, V_x are both surface forming.

We now specialise to the case where U_x is spacelike and V_x is timelike, and we shall use S_x to denote U_x, and T_x to denote V_x. Thus

$$T_x M = S_x \oplus T_x \quad \forall x \in M.$$

We shall take (e_i) to be an adapted frame, where we recall that a frame is an oriented, time-oriented pseudo-orthonormal basis. Let $L(S)$, $L(T)$ be the principal fibre bundles of 2-frames tangent to S, T respectively, where S, T are the tangent bundles U, V, and let $L(S,T)$ be the principal fibre bundle of adapted frames.

Thus $L(S)$, $L(T)$, $L(S,T)$ have structure groups $SO(2)$, $L_+^\uparrow(2)$, $L_+^\uparrow(2) \times SO(2)$ respectively.

We further assume that $\forall x \in M$, S_x is surface forming, so that (M,g) is foliated into a set $\mathcal{S}$ of spacelike 2-surfaces which are everywhere tangent to S. Given $x \in M$ however, T_x need not be surface forming. We denote

$$\pi_{\|} = \pi_U \quad \pi_{\perp} = \pi_V \qquad g^{\|} = g^U \quad g^{\perp} = g^V$$

$$\nabla^{\|} = \nabla^U \quad \nabla^{\perp} = \nabla^V \qquad D^{\|} = D^U \quad D^{\perp} = \tilde{D}^V$$

$$K^{\|} = K^U \quad K^{\perp} = K^V \qquad A^{\|} = A^U \quad A^{\perp} = A^V.$$

Given a particular 2-surface $S_0 \in \mathcal{S}$ we may form the tangent bundle $TS_0 = S|_{S_0}$ of S_0 and the normal tangent bundle $(TS_0)^\perp = T|_{S_0}$ of S_0. Thus $D^{\|}$ is a connection on the principal fibre bundle $L(S|_{S_0})$ of 2-frames tangent to $S|_{S_0}$ and $D^\perp$ is a connection on the principal fibre bundle $L(T|_{S_0})$ of 2-frames tangent to $T|_{S_0}$. Given $x \in S_0$, $D^{\|}$ has torsion

$$T^{\|} : S_x \times S_x \longrightarrow T_xM : (e_a, e_b) \longmapsto D^{\|}{}_a e_b - D^{\|}{}_b e_a - [e_a, e_b]$$

but since S_0 is a surface $\pi_{\|} T^{\|} = T^{\|}$ so that in fact

Proposition 5.3.2. *$D^{\|}$ is the Levi-Civita connection of $(S_0, g^{\|})$.*

Proof. $D^{\|} g^{\|} = 0$ and $T^{\|} = 0$. □

$D^{\|}$ is also called the *intrinsic connection* of $(S_0, g^{\|})$ since it is uniquely determined by $(S_0, g^{\|})$. $D^\perp$ does not have a well defined torsion and despite the fact that $D^\perp g^\perp = 0$, $D^\perp$ is not in general uniquely determined by $g^\perp$.

Since S_0 is a surface, $D^{\|}$ has well defined curvature $W^{\|}{}^b_a = \overline{\Omega}{}^b_a$ and $D^\perp$ has well defined curvature $W^{\perp}{}^B_A = \overline{\Omega}{}^B_A$. We recall however that requiring $D^\perp g^\perp = 0$ and $W^{\perp}{}^B_A = 0$ would not uniquely fix $D^\perp$; instead we obtain $D^\perp$ from the projected connection $\overline{\nabla}$ of the Levi-Civita connection ∇ of (M,g).

Since $\overline{K}^{\parallel} = \overline{K}^{U} = 0$ and $\overline{A}^{\parallel} = \overline{A}^{U} = 0$ it follows that S_0 is totally geodesic with respect to $\overline{\nabla}$.

Finally we note that if

$$\tau = \{\text{tensor fields } T^{Cb_1\ldots b_q}_{\ \ a_1\ldots a_p} | p, q \in \mathbb{N}\}$$

then the connection on this bundle

$$\tilde{\nabla} : \tau \longrightarrow \tau : T^{Cb_1\ldots b_q}_{\ \ a_1\ldots a_p} \longmapsto \overline{\nabla}_{a_{p+1}} T^{Cb_1\ldots b_q}_{\ \ a_1\ldots a_p} \quad p, q \in \mathbb{N}$$

is called the *Van der Waerden-Bortolotti connection.*

5.4 Conjugacy of the intrinsic and extrinsic holonomy

Let (M, g) be a space-time. Let $\kappa : (0, 1] \longrightarrow M : u \longmapsto \kappa(u)$ be a C^1 curve of finite b-length terminating at a singularity and let $\overline{\kappa}$ be a lift of κ terminating at a point p of the b-boundary of M. If κ terminates at a quasi-regular singularity, then we know from section 1.6 that the s-holonomy group $H_{\overline{\kappa}}$ will exist, and consist of rotations through multiples of a fixed angle about the singularity. There is no guarantee however that $H_{\overline{\kappa}}$ will exist for a curvature singularity. A different singular holonomy group G_p is defined in [C78]. G_p is homeomorphic to the isotropy subgroups at p and thus always exists, and contains $H_{\overline{\kappa}}$ when it exists. For a general curvature singularity however $G_p = L^{\uparrow}_{+}$ and thus G_p does not tell us much about the structure of a curvature singularity.

We suppose instead that the space-time is foliated by a set $\mathcal{S}$ of spacelike 2-surfaces. Given $x \in M$, let $S_x = T_x S$, $T_x = (T_x S)^{\perp}$ where $S \in \mathcal{S}$ is the 2-surface which passes through x. Thus

$$T_x M = S_x \oplus T_x.$$

As before let $g^{\parallel}, g^{\perp}$ be the metrics induced on S_x, T_x by g, let ω be the Levi-Civita connection, let $\overline{\omega}$ be the projected connection defined with respect to $\{S_x\}_{x \in M}$ and $\{T_x\}_{x \in M}$, and let $\sigma = \overline{\omega} - \omega$ be the connection difference.

We also suppose that κ lies on a particular spacelike 2-surface $S \in \mathcal{S}$ and has ϖ-finite (but not necessarily ω-finite) b-length. Recall that the elements of $H_{\overline{\kappa}}$ are generated by parallelly propagating $\overline{\kappa}(1)$ round the elements of the loop space of lassos Ω_κ as described in sections 1.6 and 2.5. We will define a subgroup $\Omega_\kappa(S)$ of Ω_κ whose elements, subject to certain constraints on the extrinsic curvature of S and on the space-time curvature, will give rise to well defined elements of holonomy, even though κ may terminate at a curvature singularity.

First let $\Omega^+_\kappa(S)$ be the loop space of lassos contained in Ω_κ and restricted to lie in the 2-surface S. Let $\gamma \in \Omega^+_\kappa(S)$. Measuring lengths with respect to the positive definite metric $g^{\|}$, we shall say that the curves $\gamma_u(s) = \gamma(s,u)$ are *parametrised proportional to length* if they obey $l(s,u) = sl(u)$ where $l(s,u)$, $l(u)$ are the lengths of $\gamma_u|_{[0,u]}, \gamma_u|_{[0,1]}$ respectively, and we shall say that γ is *regular* if

(i) the curves $\gamma_u(s) = \gamma(s,u)$ are parametrised proportional to length

(ii) the lengths $\lambda(s)$ of the curves $\kappa_s(u) = \gamma(s,u)$ are finite and continuous in s

(iii) $Y(s,u) \neq 0$ where $Y = \gamma_*(\partial_s)$.

We note that length measured with respect to the positive definite metric $g^{\|}$ coincides with b-length measured in an adapted frame parallelly propagated with respect to the projected connection $\overline{\nabla}$.

We now define

$$\Omega_\kappa(S) = \{\gamma \in \Omega^+_\kappa(S) | \gamma \text{ is regular}\}.$$

Proposition 5.4.1. *$\Omega_\kappa(S)$ is a group.*

Proof. Given $\gamma, \delta \in \Omega_\kappa(S)$ let

$$(\gamma * \delta)_u(s) = \begin{cases} \delta_u\left(\frac{(l_\gamma(u)+l_\delta(u))s}{l_\delta(u)}\right) & 0 \leq s \leq \frac{l_\delta(u)}{l_\gamma(u)+l_\delta(u)} \\ \gamma_u\left(\frac{(l_\gamma(u)+l_\delta(u))s}{l_\gamma(u)} - \frac{l_\delta(u)}{l_\gamma(u)}\right) & \frac{l_\delta(u)}{l_\gamma(u)+l_\delta(u)} \leq s \leq 1 \end{cases}$$

where γ_u has length $l_\gamma(u)$ and δ_u has length $l_\delta(u)$. Then $\Omega_\kappa(S)$ is a group under this operation. $\square$

Now given a bundle on S, a connection $\tilde{\omega}$ on this bundle, and a lift $\overline{\kappa}$ of κ obtained in this bundle using $\tilde{\omega}$, we define $H_{\overline{\kappa}}(S, \tilde{\omega})$ to be, if it exists, the s-holonomy group obtained by parallelly propagating $\overline{\kappa}(u)$ with respect to $\tilde{\omega}$ in the usual way along the elements of $\Omega_{\kappa}(S)$. Let $\overline{\kappa}'$ be another lift of κ obtained using $\tilde{\omega}$ and let $\overline{\kappa}(u) = (\tilde{e}_i(u))$ and $\overline{\kappa}'(u) = (\tilde{e}'_i(u))$. Then $\tilde{e}'_i(u) = L_i^j \tilde{e}_j(u)$ for some constant L_i^j and in this case we recall that $H_{\overline{\kappa}'}(S, \tilde{\omega}) = L^{-1} H_{\overline{\kappa}}(S, \tilde{\omega}) L$.

In particular, let $\overline{\kappa}$ and $\tilde{\kappa}$ be lifts of κ obtained using $\overline{\omega}$ and ω respectively in the bundle LM. If it exists, we call $H_{\overline{\kappa}}(S, \overline{\omega})$ an *intrinsic holonomy group*, since it measures the holonomy of the projected connection on loops restricted to lie on S and, if it exists, we call $H_{\tilde{\kappa}}(S, \omega)$ an *extrinsic holonomy group*, since it measures the holonomy of the full space-time connection on loops restricted to lie on S.

Our strategy is to consider a class of singularities for which although the full s-holonomy groups may not exist, the extrinsic holonomy groups exist for suitable choices of spacelike 2-surfaces. We will do this by proving the existence of the intrinsic holonomy groups for this class of singularities, and showing that the extrinsic holonomy groups must be conjugate to the intrinsic holonomy groups and thus must exist. In the next section we will discuss conditions under which these extrinsic holonomy groups are conserved along the singularity.

We therefore now assume that (M, g) contains an idealised cosmic string as in section 4.2 and that the foliation $\mathcal{S}$ described above is chosen to consist of the preferred spacelike 2-surfaces $\{S_{tz}\}$. Let $S \in \mathcal{S}$ and let $\kappa : (0, 1] \to S : u \mapsto \kappa(u)$ be a C^1 curve of finite length terminating at $r = 0$.

Let $\gamma \in \Omega_{\kappa}(S)$ and let $\tilde{\kappa}$ be a lift of κ obtained using the full connection ω in the frame bundle LM. Let $(e_i(s, u))$ be obtained by parallelly propagating $\tilde{\kappa}(u)$ along the closed loops $\gamma_u(s) = \gamma(s, u)$ with respect to ω and let $(\tilde{e}_i(s, u))$ be obtained by parallelly propagating $(e_i(s, 1))$ along $\kappa_s(u) = \gamma(s, u)$ with respect to ω.

Measuring length with respect to $g^{\|}$, let the closed loops $\gamma_u(s) = \gamma(s, u)$ have length $l(u)$ and let the curves $\kappa_s(u) = \gamma(s, u)$ have length $\lambda(s)$. We shall say that the 2-surface S is *regular with respect to* γ if

(a) $\exists \phi \in L^1(0, 1), \psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ such that in the frame $(\tilde{e}_i)$, where for each $s \in [0, 1]$, $\tilde{u}$ measures length along κ_s with respect to the positive definite metric $g^{\|}$ such that $\tilde{u} \longrightarrow 0$ as $u \longrightarrow 0$, and $l(s, \tilde{u}) = l(u)$ where we regard u as a function of s and $\tilde{u}$, the space-time curvature obeys

$$\|\Omega_{\tilde{a}\tilde{b}\tilde{C}}{}^{\tilde{D}}(s, \tilde{u})\| l(s, \tilde{u}) \leq \phi(s)\psi(\tilde{u})$$

(b) $\exists M > 0$ such that, in the frame $(\tilde{e}_i)$, the extrinsic curvature $K^{\|}$ of S obeys

$$\|K^{\|}{}_{\tilde{a}\tilde{b}}^{\tilde{D}}\| \leq M.$$

We note that condition (a) is certainly satisfied if $\exists \phi \in L^1(0, 1)$ and $\psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ such that, in the frame $(\tilde{e}_i)$,

$$\|\Omega_{\tilde{a}\tilde{b}\tilde{C}}{}^{\tilde{D}}(s, \tilde{u})\| \leq \phi(s)\psi(\tilde{u})$$

or even

$$\|\Omega_{\tilde{a}\tilde{b}\tilde{C}}{}^{\tilde{D}}(s, \tilde{u})\| \leq \psi(\tilde{u})$$

though these are stronger conditions than condition (a). We also note that condition (a) involves only one independent component $\Omega_{\tilde{a}\tilde{b}\tilde{C}}{}^{\tilde{D}}$ of the twenty independent components of the space-time curvature.

We also note that $(\tilde{e}_i)$ will not in general be an adapted frame so the correct projections need to be applied to $K^{\|}{}_{\tilde{a}\tilde{b}}^{\tilde{D}}$ and $\Omega_{\tilde{a}\tilde{b}\tilde{C}}{}^{\tilde{D}}$.

We shall say that the 2-surface S is *regular* with respect to κ if $\Omega_\kappa(S)$ is non-empty and S is regular with respect to every $\gamma \in \Omega_\kappa(S)$.

We recall the equivalence relation $\sim$ defined on connections in section 2.3. Let $(\overline{e}_i(0, 1)) = (e_i(0, 1))$ and parallelly propagate $(\overline{e}_i(0, 1))$ with respect to the projected connection first along $\gamma_1(s) = \gamma(s, u)$ and then along $\kappa_s(u) = \gamma(s, u)$ to obtain $(\bar{e}_i(s, u))$ (see diagram 5.4.1). Set

$$\bar{e}_i(s, u) = \lambda^j_i(s, u)\tilde{e}_j(s, u)$$

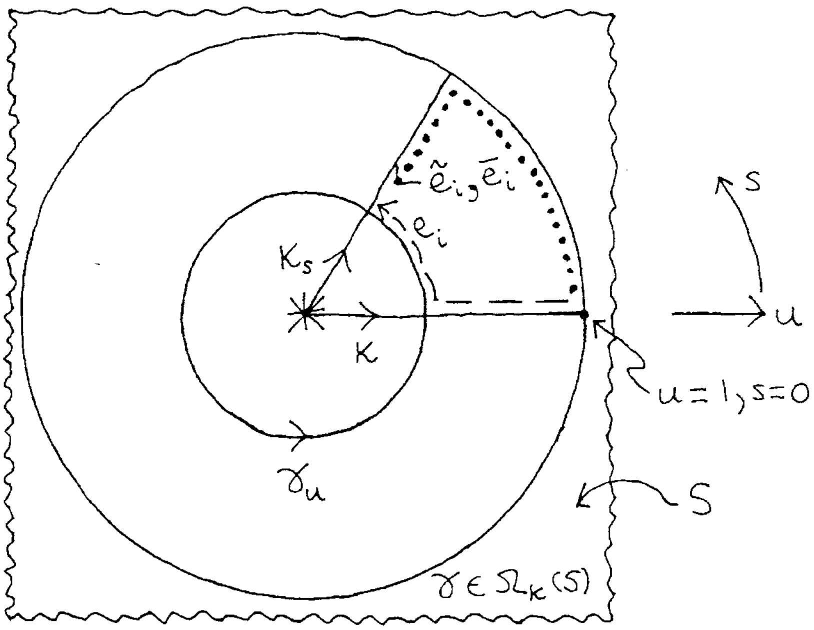

Diagram 5.4.1

Lemma 5.4.2. *If S is regular with respect to $\gamma \in \Omega_\kappa(S)$ then $\omega \sim \varpi$ along each $\kappa_s(u) = \gamma(s, u)$ and $\exists \lambda_0, \lambda_1 > 0$ such that $\|\lambda^j_i(s, u)\| \le \lambda_1$, $\|(\lambda^{-1})^j_i(s, u)\| \le \lambda_1$.*

Proof. We express components in the frame $(\tilde{e}_i)$. Let $p : TM \to TM$ be defined by $p^j_i = g\|^j_i$. Since κ lies on S, $p(v) = v\ \forall v \in T\gamma$. Therefore $p \in \mathcal{P}$ where $\mathcal{P}$ is defined in 2.3. Now

$$p^j_i \sigma^l_{jk} = \begin{cases} K\|^D_{ab} & i = a,\ k = b,\ l = D \\ A\|^d_{aB} & i = a,\ k = B,\ l = d \end{cases}$$

and since $\exists M > 0$ such that $K\|^D_{ab} \le M$, it follows by Theorem 2.3.13 that $\omega \sim \varpi$ along κ_s. Furthermore from equation (2.3.1)

$$\lambda^j_i(s, u) = P\exp\int_1^u -Y^k(s, u_0)\sigma^j_{kl}(s, u_0)\, du_0\, P\exp\int_0^s -X^m(s_0, 1)\sigma^l_{mi}(s_0, 1)\, ds_0$$

and since $Y^k\sigma^j_{kl} = Y^k p^{k'}_k \sigma^j_{k'l}$ and $X^m\sigma^l_{mi} = X^m p^{m'}_m \sigma^l_{m'i}$ we have

$$\begin{aligned}
\|\lambda^j_i(s,u)\| &\leq \exp\int_1^u \|Y^k(s,u_0)\| M\, du_0 \exp\int_0^s \|X^k(s_0,1)\| M\, ds_0 \\
&\leq \exp\int_1^{\tilde{u}} M\, d\tilde{u}_0 \exp\int_0^s \sup_{s_0\in[0,s]} \|X^k(s_0,1)\|\, ds_0 \\
&\leq \lambda_0
\end{aligned}$$

for some $\lambda_0 > 0$. Similarly

$$\|(\lambda^{-1})^j_i(s,u)\| \leq \lambda_1$$

for some $\lambda_1 > 0$. $\square$

A consequence of S being regular with respect to $\gamma \in \Omega_\kappa(S)$ is that κ and each κ_s have both ω-finite and $\overline{\omega}$-finite b-lengths. In this case we claim that the b-lengths $l(u)$ of $\gamma_u(s) = \gamma(s,u)$ obey $l(u) \to 0$ as $u \to 0$ with respect to ω if and only if they do so with respect to $\overline{\omega}$, and thus $\Omega_\kappa(S)$, whose elements are required to obey $l(u) \to 0$ as $u \to 0$, is well defined. We note however that each γ_u is parametrised proportional to length measured with respect to $g^{\|}$, and not b-length measured with respect to ω.

We can now state the conditions under which the intrinsic holonomy groups exist.

Theorem 5.4.3. *Let $\overline{\kappa}'$ be a lift of κ in the adapted frame bundle $L(TS,(TS)^\perp)$ and suppose that S is regular with respect to κ. Then the intrinsic holonomy group $H_{\overline{\kappa}'}(S,\overline{\nabla})$ exists.*

Before we prove this theorem, we need to establish some preliminary results. Given a lift $\overline{\kappa}'$ of κ in the adapted frame bundle $L(TS,(TS)^\perp)$, if each $\gamma \in \Omega_\kappa(S)$ were sufficiently regular with respect to the curvature $\overline{\Omega}^j_i$ of the projected connection $\overline{\omega}$ in the sense of section 2.5, Theorem 5.4.3 would follow immediately from Theorems 2.5.1 and 2.5.2. It turns out however that in order to prove Theorem 5.4.3, we need only consider the behaviour of the sectional curvature of the projected connection, which we define as follows.

Let $\tilde{\omega}$ be a connection on LM with curvature $\tilde{\Omega}^j_i$. The *sectional curvature* $\hat{\tilde{\Omega}}{}^j_i$ of $\tilde{\omega}$ with respect to S is

$$\hat{\tilde{\Omega}}_{kli}{}^j = g^{\|}{}^m_k\, g^{\|}{}^n_l\, \tilde{\Omega}_{mni}{}^j$$

and thus in an adapted frame

$$\hat{\tilde{\Omega}}_{kli}{}^j = \begin{cases} \tilde{\Omega}_{abi}{}^j & k = a,\ l = b \\ 0 & \text{otherwise.} \end{cases}$$

Given $x \in S$, the more usual definition of this sectional curvature is

$$\tilde{S}^j_i = \tilde{\Omega}^j_i(X, Y)/\|X \wedge Y\|$$

where $\|X \wedge Y\|^2 = (X_i \wedge Y_j)(X^i \wedge Y^j)$ and X, Y are chosen to span S_x (and are thus non-zero). It can be shown that $\tilde{S}^j_i$ is independent of the choice of vectors spanning S_x. Thus if X, Y span S_x we have

$$\hat{\tilde{\Omega}}{}^j_i(X, Y) = \tilde{\Omega}^j_i(X, Y) = \tilde{S}^j_i \|X \wedge Y\|$$

and so we may regard both $\hat{\tilde{\Omega}}{}^j_i$ and $\tilde{S}^j_i$ as the sectional curvature of $\tilde{\omega}$ with respect to S.

Now let $\gamma \in \Omega_\kappa(S)$, let $\overline{\kappa}$ be a lift of κ obtained using $\tilde{\omega}$, and let $(e_i(s, u))$ be obtained by parallelly propagating $\overline{\kappa}(u)$ along the closed loops $\gamma_u(s) = \gamma(s, u)$ with respect to $\tilde{\omega}$. As before let $\kappa_s(u) = \gamma(s, u)$ and define $(\tilde{e}_i(s, u))$ by parallelly propagating $(e_i(s, 1))$ along κ_s with respect to $\tilde{\omega}$ to give $(\tilde{e}_i(s, u))$. We also let $X = \gamma_*(\partial_s), Y = \gamma_*(\partial_u)$ and set

$$e_i(1, u) = L^j_i(u) e_j(0, u)$$

and

$$\tilde{e}_i(s, u) = l^j_i(s, u) e_j(s, u).$$

We shall say that γ is *sufficiently regular with respect to* $\hat{\tilde{\Omega}}{}^j_i$ if

(a) the curves γ_u are parametrised proportional to b-length, and for each $s \in [0,1]$, κ_s has finite b-length, and the b-length $\lambda(s)$ of κ_s measured in the frame $(\tilde{e}_i)$ is continuous in s, and $Y(s,u) \neq 0$

(b) $\exists \phi \in L^1(0,1), \psi \in L^1(0, \sup_{s\in[0,1]} \lambda(s))$ such that in the frame $(\tilde{e}_i)$, where for each $s \in [0,1]$, $\tilde{u}$ measures b-length along κ_s with respect to $(\tilde{e}_i)$ such that $\tilde{u} \to 0$ as $u \to 0$, and $l(s,\tilde{u}) = l(u)$ where we regard u as a function of s and $\tilde{u}$,

$$\|\hat{\tilde{\Omega}}_{\bar{k}\bar{l}\bar{i}}{}^{\bar{j}}(s,\tilde{u})\| l(s,\tilde{u}) \leq \phi(s)\psi(\tilde{u})$$

This definition is identical to the one given in section 2.5 except that in condition (a) we measure b-length with respect to $\tilde{\omega}$ and in condition (b) we refer to the sectional curvature $\hat{\tilde{\Omega}}_i^j$ rather than to the full curvature.

Recall from section 2.5 that γ is *well bounded with respect to* $\tilde{\omega}$ if $\exists \alpha > 0$ such that

$$\|(l^{-1})_i^j(s,u) X^i(s,u)\| \leq \alpha \|X^j(s,u)\|$$

where X^j are the components of X in the frame (e_i).

With these definitions we can state the following theorems.

Theorem 5.4.4. *Let γ be sufficiently regular with respect to $\hat{\tilde{\Omega}}_i^j$ and well bounded with respect to $\tilde{\omega}$. Then for each $s \in [0,1]$, $\lim_{u\to 0} l_i^j(s,u)$ exists, and $l_i^j(s)$ is continuous.*

Theorem 5.4.5. *Let γ be sufficiently regular with respect to $\hat{\tilde{\Omega}}_i^j$ and well bounded with respect to $\tilde{\omega}$. Then $L_i^j(0) := \lim_{u\to 0} L_i^j(u)$ exists.*

The proofs of these theorems are exactly analogous the the proofs of Theorems 2.5.1 and 2.5.2. We now apply these theorems to the sectional curvature $\overline{\hat{\Omega}}_i^j$ of the projected connection $\overline{\omega}$.

Proposition 5.4.6. *Each $\gamma \in \Omega_{\kappa}(S)$ is well bounded with respect to $\overline{\omega}$.*

Proof. Let $\gamma \in \Omega_{\kappa}(S)$ and let $X = \gamma_*(\partial_s)$. Let (e_i) be a (not necessarily adapted) frame at $\gamma(0,1)$. Using the projected connection $\overline{\omega}$, parallelly propagate (e_i) along κ to $\kappa(u)$ and round γ_u to $\gamma(s,u)$ to give $(e_i(s,u))$. Let $(\tilde{e}_i(0,1)) = (e_i(0,1))$ and parallelly propagate $(\tilde{e}_i)$ round γ_1 to $\gamma_1(s)$ and then along κ_s to $\gamma(s,u)$ to give $(\tilde{e}_i(s,u))$. Set

$$\tilde{e}_i(s,u) = l_i^j(s,u)e_j(s,u).$$

Now pick a an adapted frame (e'_i) at $\gamma(0,1)$. Using $\overline{\omega}$, parallelly propagate (e'_i) along κ to $\kappa(u)$ and then round γ_u to $\gamma(s,u)$ to give $(e'_i(s,u))$. Let $(\tilde{e}'_i(0,1)) = (e'_i(0,1))$ and parallelly propagate $(\tilde{e}'_i)$ round γ_1 to $\gamma_1(s)$ and then along κ_s to $\gamma(s,u)$ to give $(\tilde{e}'_i(s,u))$. Define

$$\tilde{e}'_i(s,u) = \lambda_i^j e'_j(s,u)$$

where $\lambda_i^j(s,u) \in L_+^{\uparrow}$. Since (e'_i), $(\tilde{e}'_i)$ remain adapted under parallel propagation by $\overline{\omega}$ we have $\lambda_a^B, \lambda_A^b = 0$.

Now $X = X'^a e'_a = X'^a(\lambda^{-1})_a^b \tilde{e}'_b$ where X has components X'^i in the frame (e'_i). Since $g(e'_i, e'_j) = g(\tilde{e}'_i, \tilde{e}'_j) = \eta_{ij}$ and γ lies on a spacelike 2-surface

$$g(X,X) = (X'^2)^2 + (X'^3)^2 = (X'^a(\lambda^{-1})_a^2)^2 + (X'^a(\lambda^{-1})_a^3)^2$$

and hence

$$\|X'^j\|^2 = \|X'^i(\lambda^{-1})_i^j\|^2.$$

Now

$$e'_i(0,1) = \tilde{e}'_i(0,1) = a_i^j e_j(0,1) = a_i^j \tilde{e}_j(0,1)$$

for some constant $a_i^j \in L_+^{\uparrow}$ and so

$$e'_i(s,u) = a_i^j e_j(s,u) \qquad \tilde{e}'_i(s,u) = a_i^j \tilde{e}_j(s,u).$$

Hence $X = X^i e_i = X^i (a^{-1})_i^j e'_j$ so $X'^j = X^i (a^{-1})_i^j$ and $X = X^i e_i = X^i (l^{-1})_i^j \tilde{e}_j = X^i (l^{-1})_i^j (a^{-1})_j^k \tilde{e}'_k$ so $X''^i (\lambda^{-1})_i^k = X^i (l^{-1})_i^j (a^{-1})_j^k$. Therefore

$$\begin{aligned}
\|X^i (l^{-1})_i^j\| &= \|X^i (l^{-1})_i^j (a^{-1})_j^k a_k^l\| \\
&\leq \|X^i (l^{-1})_i^j (a^{-1})_j^k\| \|a_i^j\| \\
&= \|X''^i (\lambda^{-1})_i^j\| \|a_i^j\| = \|X'^j\| \|a_i^j\| = \|X^i (a^{-1})_i^j\| \|a_i^j\| \\
&\leq \|X^i\| \|(a^{-1})_i^j\| \|a_i^j\| = \alpha \|X^j\|
\end{aligned}$$

where $\alpha = \|(a^{-1})_i^j\| \|a_i^j\| > 0$ is constant. □

Lemma 5.4.7. *Suppose that S is regular with respect to κ. Then each $\gamma \in \Omega_\kappa(S)$ is sufficiently regular with respect to $\hat{\overline{\Omega}}{}_i^j$.*

Proof. Let $\gamma \in \Omega_\kappa(S)$. We show that conditions (a)–(c) in the definition of whether γ is sufficiently regular with respect to $\hat{\overline{\Omega}}{}_i^j$ hold.

γ is regular by the definition of $\Omega_\kappa(S)$ and we recall that b-length measured with respect to $\overline{\omega}$ in an adapted frame coincides with b-length measured with respect to $g^{\|}$. Therefore condition (a) holds.

Let $(\tilde{e}_i)$ be a basis parallelly propagated with respect to the full connection ω first along $\gamma_1(s) = \gamma(s, u)$ and then along each $\kappa_s(u) = \gamma(s, u)$. Since S is regular with respect to γ $\exists \phi \in L^1(0, 1), \psi \in L^1(0, \sup_{s \in [0,1]} \lambda(s))$ such that, working in the basis $(\tilde{e}_i)$,

$$\|\Omega_{abC}{}^D(s, \tilde{u})\| l(s, \tilde{u}) \leq \phi(s) \psi(\tilde{u})$$

where $l(s, \tilde{u}) = l(u)$ is the length of the closed loop $\gamma_u(s) = \gamma(s, u)$, $\tilde{u}$ measures length along κ_s, and $\lambda(s)$ is the length of κ_s, measuring all lengths with respect to $g^{\|}$.

Referring forward to equation (6.1.8)

$$\Omega_{abCD} = \overline{\Omega}_{abCD} + g^{\|}{}_{ef}(A^{\|}{}^e_{aC} A^{\|}{}^f_{bD} - A^{\|}{}^e_{bC} A^{\|}{}^f_{aD})$$

hence

$$\begin{aligned}\|\overline{\Omega}_{abC}{}^D\| l(s,\tilde{u}) &\leq (\|\Omega_{abC}{}^D\| + \|g^{\|}{}_{ef} A^{\|}{}^e_{aC} A^{\|}{}^f_{bD}\| + \|g^{\|}{}_{ed} A^{\|}{}^e_{bC} A^{\|}{}^f_{aD}\|)(s,\tilde{u}) l(s,\tilde{u}) \\ &\leq \|\Omega_{abC}{}^D(s,\tilde{u})\| l(s,\tilde{u}) + M_0 l(s,\tilde{u})\end{aligned}$$

for some $M_0 > 0$ since $\exists M > 0$ such that, in the basis $(\tilde{e}_i)$, $\|K^{\|}{}^D_{ab}(s,u)\| \leq M$. Hence

$$\|\overline{\Omega}_{abC}{}^D\| l(s,\tilde{u}) \leq \phi_1(s)\psi(\tilde{u})$$

for some $\phi_1 \in L^1(0,1)$.

Now set $\overline{e}_i(0,1) = \tilde{e}_i(0,1)$ and parallelly propagate $\overline{e}_i$ with respect to the projected connection, first along γ_1 and then along κ_s, to give $\overline{e}_i(s,u)$. Set

$$\overline{e}_i(s,u) = \lambda^j_i(s,u)\tilde{e}_j(s,u).$$

Then by Lemma 5.4.2, $\lambda^j_i(s,u)$ and $(\lambda^{-1})^j_i(s,u)$ can both be bounded by constants. Working now in the basis $(\overline{e}_i)$ it follows that

$$\|\overline{\Omega}_{abC}{}^D\| l(s,\tilde{u}) \leq \phi_2(s)\psi(\tilde{u}).$$

In other words we have converted an integral bound on $\Omega_{abC}{}^D$ measured in an ω-frame into an integral bound on $\overline{\Omega}_{abC}{}^D$ measured in a $\overline{\omega}$-frame.

Now $\overline{\Omega}^B_a = \overline{\Omega}^b_A = 0$ and the sectional curvature $\hat{\overline{\Omega}}{}^j_i$ of the projected connection has components $\overline{\Omega}_{abC}{}^D$ and $\overline{\Omega}_{abc}{}^d$. We therefore need to examine the behaviour of $\overline{\Omega}_{abc}{}^d$.

Since $(S, g^{\|})$ has a quasi-regular singularity at $r = 0$, it follows that the Ricci scalar of $(S, g^{\|})$ has a well defined limit $\overline{R}_s$ along κ_s. Now a curve of finite length terminating at $r = 0$ can be constructed oscillating between κ_{s_1} and κ_{s_2} on which the Ricci scalar must have a well defined limit of both $\overline{R}_{s_1}$ and $\overline{R}_{s_2}$. Hence the Ricci scalar has the same limit along all κ_s. Now since $(S, g^{\|})$ has only one independent component of curvature $\overline{\Omega}_{abc}{}^d$ and the frames in which this can be expressed are all related by rotations, it follows that $\exists M_1 > 0$ such that

$$\|\overline{\Omega}_{abc}{}^d(s, \tilde{u})\| \leq M_1.$$

Hence $\exists \phi_3 \in L^1(0,1)$ such that in the frame $(\overline{e}_i)$

$$\|\hat{\overline{\Omega}}_{ijk}{}^l(s, \tilde{u})\| l(s, \tilde{u}) \leq \phi_3(s)\psi(\tilde{u})$$

and condition (b) is satisfied. □

We can now prove Theorem 5.4.3.

Proof of Theorem 5.4.3. $\overline{\kappa}'$ is a lift of κ in the adapted frame bundle and S is regular with respect to κ. Then each $\gamma \in \Omega_\kappa(S)$ is sufficiently regular with respect to $\hat{\overline{\Omega}}{}_i^j$ by Lemma 5.4.7 and well bounded with respect to $\overline{\omega}$ by Proposition 5.4.6.

Define $\overline{e}_i(s, u)$ by parallelly propagating $\overline{\kappa}'(u)$ along $\gamma_u(s) = \gamma(s, u)$ with respect to $\overline{\omega}$ and set

$$\overline{e}_i(1, u) = L_i^j(u)\overline{e}_j(0, u).$$

Then by Theorem 5.4.5

$$L_i^j(0) := \lim_{u \to 0} L_i^j(u)$$

exists. Hence the intrinsic holonomy group $H_{\overline{\kappa}'}(S, \overline{\nabla})$ exists. □

We note that this theorem just tells us that the intrinsic holonomy groups exist but do not tell us their value. However

Theorem 5.4.8. *Let $\overline{\kappa}'$ be a lift of κ in the adapted frame bundle $L(TS, (TS)^\perp))$ and let S be regular with respect to κ. Then the elements of the intrinsic holonomy group $H_{\overline{\kappa}'}(S, \overline{\nabla})$ act on the bundle $L(TS)$ of 2-frames tangent to S as rotations through $2k\theta_0$, $k \in \mathbb{Z}$ for some θ_0.*

Proof. This follows from the results of section 1.6. □

We do not know however how the elements of $H_{\overline{\kappa}'}(S, \overline{\nabla})$ defined in Theorem 5.4.3 act on $L((TS)^{\perp})$. In certain cases they may act on $L((TS)^{\perp})$ as the identity and indeed this might seem likely given that the normal metric is regular. However these elements of holonomy are calculated by parallelly propagating a 2-frame in the normal bundle in directions tangent to the tangent bundle. In other words the holonomy group $H_{\kappa^{\perp}}(S, D^{\perp})$ obtained by parallelly propagating a lift $\kappa^{\perp}$ of κ in the normal bundle $L((TS)^{\perp})$ with respect to the connection $D^{\perp}$ induced by $\overline{\nabla}$ on $L((TS)^{\perp})$ depends on both $g^{\perp}$ and $g^{\parallel}$. Thus we cannot conclude that $H_{\kappa^{\perp}}(S, D^{\perp}) = \{\delta_A^B\}$.

We have shown that $H_{\overline{\kappa}'}(S, \overline{\omega})$ exists for a lift $\overline{\kappa}'$ of κ obtained using $\overline{\omega}$ in the adapted frame bundle $L(TS, (TS)^{\perp})$. It follows that the intrinsic holonomy group $H_{\overline{\kappa}}(S, \overline{\omega})$ exists, and is conjugate to $H_{\overline{\kappa}'}(S, \overline{\omega})$, for a lift $\overline{\kappa}$ of κ obtained using $\overline{\omega}$ in the full frame bundle LM, since $\overline{\kappa}(u) = L\overline{\kappa}'(u)$ for a constant $L \in L_{+}^{\uparrow}$ and hence $H_{\overline{\kappa}} = L^{-1} H_{\overline{\kappa}'} L$.

Now let $\tilde{\kappa}$ be a lift of κ obtained using ω. We will now show that the extrinsic holonomy group $H_{\tilde{\kappa}}(S, \omega)$ is conjugate to the intrinsic holonomy group $H_{\overline{\kappa}}(S, \overline{\omega})$, and thus also exists. In particular if $\tilde{\kappa}$ is suitably chosen then

$$H_{\tilde{\kappa}}(S, \omega) = H_{\overline{\kappa}}(S, \overline{\omega}).$$

Let $\gamma : (s, u) \mapsto \gamma(s, u) \in \Omega_{\kappa}(S)$ where each $\gamma_u : s \mapsto \gamma(s, u)$ is a closed loop. Let $(e_i(0,1)) \in LM$ be a frame at $\kappa(1)$ and let $(\overline{e}_i(0,1)) \in L(TS, (TS)^{\perp})$ be an adapted frame at $\kappa(1)$. Parallelly propagate $(e_i(0,1))$, $(\overline{e}_i(0,1))$ with respect to ω, $\overline{\omega}$, first along κ to give $(e_i(0,u))$, $(\overline{e}_i(0,u))$, and then along γ_u for fixed u to give $(e_i(s,u))$, $(\overline{e}_i(s,u))$. Thus $(e_i(s,u))$ will remain in the frame bundle LM and $(\overline{e}_i(s,u))$ will remain in the adapted frame bundle $L(TS, (TS)^{\perp})$. In the following we shall denote the pair of vectors in $(\overline{e}_i)$ tangent to TS by $(\overline{e}_a)$ and the pair of vectors in $(\overline{e}_i)$ tangent to $(TS)^\perp$ by $(\overline{e}_A)$. We note that in general (e_i) will not be an adapted frame.

Thus we may write

$$e_i(s,u) = L_i^j(s,u)\overline{e}_j(s,u)$$

where $L_i^j(s,u) \in L_+^\uparrow$ though in general $L_i^j(0,1) \neq \delta_i^j$. If we let $X(s,u)$ be the tangent of $\gamma_u : s \mapsto \gamma_u(s)$ and $Y(u)$ be the tangent of $\kappa : u \mapsto \kappa(u)$ then by equation 2.3.1, expressing components in the frame $(\overline{e}_i)$, we have along κ

$$L_i^j(0,u) = P\exp\int_u^1 Y^l(u_0)\sigma^j_{lk}(0,u_0)\,du_0\, L_i^k(0,1) \tag{5.4.1}$$

and along γ_u for fixed u

$$L_i^j(1,u) = P\exp\int_0^1 -X^l(s,u)\sigma^j_{lk}(s,u)\,ds\, L_i^k(0,u). \tag{5.4.2}$$

Proposition 5.4.9. *If S is regular with respect to κ then*

$$L_i^j(0,0) := \lim_{u\to 0} L_i^j(0,u)$$

exists and is non-singular.

Proof. By Lemma 5.4.2, $\omega \sim \overline{\omega}$ along $\kappa(u) = \kappa_0(u)$. □

Proposition 5.4.10. *If S is regular with respect to κ then*

$$L_i^j(1,0) := \lim_{u\to 0} L_i^j(1,u)$$

exists and is equal to $L_i^j(0,0)$.

Proof. By equation (5.4.2)

$$L_i^j(1,u) = P\exp\int_0^1 -X^l(s,u)\sigma^j_{lk}(s,u)\,ds\, L_i^k(0,u)$$

Now γ_u is parametrised by length so in the frame $(\overline{e}_i)$ we have $\|X(s,u)\| = l(u) \longrightarrow 0$ as $u \longrightarrow 0$ where γ_u has length $l(u)$. Since S is regular with respect to κ, $\exists M > 0$ such that, by Lemma 5.4.2, $\|K^{\|D}_{ab}\| \leq M$ in the frame $(\overline{e}_i)$. Hence

$$\begin{aligned} \|L^j_k(1,u)(L^{-1})^k_i(0,u) - \delta^j_i\| &\leq \exp\int_0^1 \|X^l(s,u)\|\|\sigma^j_{lk}(s,u)\|\, ds - 1 \\ &\leq \exp\int_0^1 l(u)M\, ds - 1 \\ &\longrightarrow 0 \text{ as } u \longrightarrow 0. \end{aligned}$$

By Proposition 5.4.9 $\lim_{u\to 0} L^j_i(0,u)$ exists and hence $\lim_{u\to 0} L^j_i(1,u) = L^j_i(0,0)$ □

Finally we can prove the existence of the extrinsic holonomy groups.

Theorem 5.4.11. *Let S be regular with respect to κ. Then the extrinsic holonomy groups $H_{\tilde{\kappa}}(S,\nabla)$ exist and are conjugate to the intrinsic holonomy groups $H_{\overline{\kappa}}(S,\overline{\nabla})$ for lifts $\tilde{\kappa}$, $\overline{\kappa}$ of κ obtained by ω, $\overline{\omega}$.*

Proof. Let $\gamma \in \Omega_\kappa(S)$ and let the frames (e_i), $(\overline{e}_i)$ be as above. Let

$$e_i(1,u) = A^j_i(u)e_j(0,u) \qquad \overline{e}_i(1,u) = \overline{A}^j_i(u)\overline{e}_j(0,u).$$

By Theorem 5.4.3 we know that $H_{\overline{\kappa}}(S,\overline{\omega})$ exists where $\overline{\kappa}$ is the lift of κ obtained by $\overline{\omega}$ such that $\overline{\kappa}(1) = (\overline{e}_i(0,1))$. Thus $\overline{A}^j_i(0) := \lim_{u\to 0}\overline{A}^j_i(u) \in H_{\overline{\kappa}}(S,\overline{\omega})$ and hence must exist. Now

$$\begin{aligned} e_i(1,u) &= L^j_i(1,u)\overline{e}_j(1,u) = L^j_i(1,u)\overline{A}^k_j(u)\overline{e}_k(0,u) \\ &= L^j_i(1,u)\overline{A}^k_j(u)(L^{-1})^l_k(1,u)e_l(0,u)cr \\ \Rightarrow A^j_i(u) &= (L^{-1})^j_k(1,u)\overline{A}^k_l(u)L^l_i(1,u). \end{aligned}$$

Hence by Proposition 5.4.10

$$A^j_i(0) := \lim_{u\to 0} A^j_i(u) = (L^{-1})^j_k(1,0)\overline{A}^k_l(0)L^l_i(1,0)$$

exists. □

In particular by an appropriate choice of $L^j_i(0,1)$ we can arrange for $L^j_i(1,0) = \delta^j_i$ and in this case $A^j_i(0) = \overline{A}^j_i(0)$.

We shall present some examples of idealised cosmic strings in section 6.3. It will turn out that in each of our examples the intrinsic and extrinsic holonomy groups exist.

5.5 Conservation of holonomy

Let (M, g) contain an idealised cosmic string as described in section 4.2 and let $\{S_{tz}\}$ be the foliation of preferred spacelike 2-surfaces. Let κ be a b-incomplete curve lying in a particular S_{tz} and terminating at the singularity $\{r = 0\}$.

Now we know from section 1.6 that if $\{r = 0\}$ is a quasi-regular singularity, the s-holonomy groups $H_{\tilde{\kappa}}$ exist for lifts $\tilde{\kappa}$ of κ, and are conserved along the singularity in a sense defined by Theorem 1.6.3. For a general idealised cosmic string, the full s-holonomy groups will not in general exist. We proved however in the previous section that if S_{tz} is regular with respect to κ, the intrinsic and extrinsic holonomy groups $H_{\overline{\kappa}}(S_{tz}, \overline{\nabla})$, $H_{\tilde{\kappa}}(S_{tz}, \nabla)$ exist for lifts $\overline{\kappa}$, $\tilde{\kappa}$ of κ by the projected connection $\overline{\nabla}$ and the full connection ∇ respectively. In this section, we investigate conditions under which these groups are conserved along the singularity.

Suppose there exists a C^1 map (see diagram 5.5.1)

$$\rho : (s, u, v) \mapsto \rho(s, u, v) : [0, 1] \times (0, 1] \times [0, 1] \to M$$

such that

(a) $\kappa_v(u) = \rho(0, u, v)$ is a curve of finite b-length lying in a preferred spacelike 2-surface S_v and terminating at $r = 0$

(b) S_v is regular with respect to κ_v

(c) $\gamma_v(s, u) = \rho(s, u, v) \in \Omega_{\kappa_v}(S_v)$.

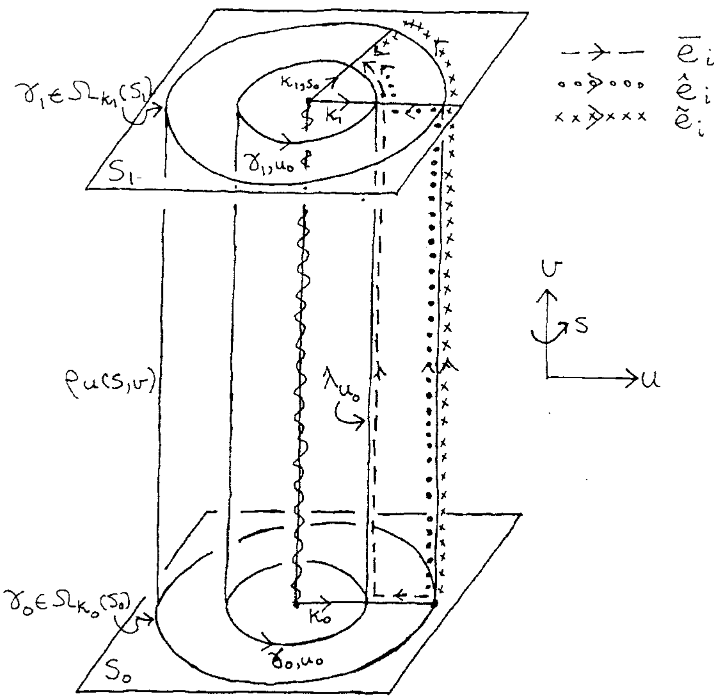

Diagram 5.5.1

Hence each κ_v and γ_v lies entirely in the preferred 2-surface S_v. Since S_v is regular with respect κ_v, $\omega \sim \varpi$ along κ_v and κ_v has finite b-length with respect to ω if and only if it has finite b-length with respect to ϖ, so condition (a) is well defined. ρ provides a homotopy from γ_0 to γ_1.

By condition (b) and the previous section we know that the intrinsic holonomy groups $H_{\bar{\kappa}_0}(S_0, \overline{\nabla})$, $H_{\bar{\kappa}_1}(S_1, \overline{\nabla})$ exist for lifts $\bar{\kappa}_0$, $\bar{\kappa}_1$ of κ_0, κ_1 in the adapted frame bundle $L(S,T)$ by $\overline{\nabla}$. Hence the extrinsic holonomy groups $H_{\tilde{\kappa}_0}(S_0, \nabla)$, $H_{\tilde{\kappa}_1}(S_1, \nabla)$ must also exist for lifts $\tilde{\kappa}_0$, $\tilde{\kappa}_1$ of κ_0, κ_1 in the full frame bundle LM by ∇.

We shall exhibit conditions under which

$$H_{\bar{\kappa}_0}(S_0, \overline{\nabla}) = H_{\bar{\kappa}_1}(S_1, \overline{\nabla})$$

and hence $H_{\tilde{\kappa}_0}(S_0, \nabla)$, $H_{\tilde{\kappa}_1}(S_1, \nabla)$ are conjugate.

Let $\bar{e}_i(0,1,0)$ be an adapted frame at $\rho(0,1,0)$. Using the projected connection $\overline{\nabla}$, parallelly propagate $\bar{e}_i(0,1,0)$ along κ_0 to give $\bar{e}_i(0,u,0)$, parallelly propagate

$\overline{e}_i(0,u,0)$ up $\lambda_u(v) = \rho(0,u,v)$ to give $\overline{e}_i(0,u,v)$, and parallelly propagate $\overline{e}_i(0,u,v)$ round the closed loop $\gamma_{v,u}(s) = \rho(s,u,v)$ to give $\overline{e}_i(s,u,v)$ (see diagram 5.5.1).

Now let $\hat{e}_i(0,1,0) = \overline{e}_i(0,1,0)$. Again using the projected connection $\overline{\nabla}$, parallelly propagate $\hat{e}_i(0,1,0)$ up $\lambda_1(v) = \rho(0,1,v)$ to give $\hat{e}_i(0,1,v)$, parallelly propagate $\hat{e}_i(0,1,v)$ along κ_v to give $\hat{e}_i(0,u,v)$, and parallelly propagate $\hat{e}_i(0,u,v)$ round the closed loop $\gamma_{v,u}(s) = \rho(s,u,v)$ to give $\hat{e}_i(s,u,v)$ (see diagram 5.5.1).

Set

$$\overline{e}_i(1,u,v) = L^j_i(u,v)\overline{e}_j(0,u,v)$$

$$\hat{e}_i(1,u,v) = \hat{L}^j_i(u,v)\hat{e}_j(0,u,v).$$

We also set

$$\overline{e}_i(0,u,v) = \alpha^j_i(u,v)\hat{e}_j(0,u,v)$$

so that

$$\overline{e}_i(s,u,v) = \alpha^j_i(u,v)\hat{e}_j(s,u,v).$$

Defining

$$X = \rho_*(\partial_s) \qquad Y = \rho_*(\partial_u) \qquad Z = \rho_*(\partial_v).$$

we have by equation 2.5.2

$$L^j_i(u,1) = L^j_m(u,0)P_v \exp \int_0^1 \int_0^1 \overline{\Omega}_{kli}{}^m(s,v)X^k(s,v)Z^l(s,v)\, dsdv \tag{5.5.1}$$

where we express components in the frame $\overline{e}_i$ and $\overline{\Omega}^j_i$ is the curvature of $\overline{\nabla}$. Note that X, Z are tangent to the tubes $\rho_u(s,v) = \rho(s,u,v)$ and X, Y are tangent to the 2-surfaces S_v. Note too that Z is not necessarily orthogonal to S_v.

Hence

$$\begin{aligned}\hat{e}_i(1,u,1) &= (\alpha^{-1})^j_i(u,1)\overline{e}_j(1,u,1)\\ &= (\alpha^{-1})^j_i(u,1)L^k_j(u,1)\overline{e}_k(0,u,1)\\ &= (\alpha^{-1})^j_i(u,1)L^k_j(u,1)\alpha^l_k(u,1)\hat{e}_l(0,u,1)\\ &= \hat{L}^j_i(u,1)\hat{e}_j(0,u,1)\end{aligned}$$

and hence

$$\hat{L}_i^j(u,1) = (\alpha^{-1})_i^k(u,1)L_k^l(u,1)\alpha_l^j(u,1) = L_i^j(u,1)$$

since the structure group $SO(2) \times L_+^\uparrow(2)$ of $L(S,T)$ is abelian. Now $\alpha_i^j(u,0) = \delta_i^j$ and so we also have

$$\hat{L}_i^j(u,0) = L_i^j(u,0)$$

and from equation (5.5.1)

$$\hat{L}_i^j(u,1) = \hat{L}_m^j(u,0)P_v \exp \int_0^1 \int_0^1 \overline{\Omega}_{kli}{}^m(s,v)X^k(s,v)Z^l(s,v)\,dsdv$$

where again we express components in the frame $\overline{e}_i$.

Now we know by conditions (a)–(c) above and Theorem 5.4.3 that the elements of holonomy

$$\hat{L}_i^j(0,0) := \lim_{u \to 0} L_i^j(u,0)$$

$$\hat{L}_i^j(0,1) := \lim_{u \to 0} L_i^j(u,1)$$

generated by parallelly propagating $\hat{e}_i(0,1,0)$, $\hat{e}_i(0,1,1)$ round γ_0, γ_1 in the usual way both exist and that

$$\hat{L}_i^j(0,0) \in H_{\hat{\kappa}_0}(S_0, \overline{\nabla}) \qquad \hat{L}_i^j(0,1) \in H_{\hat{\kappa}_1}(S_1, \overline{\nabla})$$

where $\hat{\kappa}_0$, $\hat{\kappa}_1$ are the lifts $\hat{e}_i(0,u,0)$, $\hat{e}_i(0,u,1)$ of κ_0, κ_1.

Thus it follows that

$$\varepsilon_i^j = P_v \exp \int_0^1 \int_0^1 \overline{\Omega}_{kli}{}^m(s,v)X^k(s,v)Z^l(s,v)\,dsdv \tag{5.5.2}$$

must exist, where we express components in the frame $\overline{e}_i$. We want to show that, under suitable conditions, $\hat{L}_i^j(0,0) = \hat{L}_i^j(0,1)$, or in other words that $\varepsilon_i^j \longrightarrow \delta_i^j$ as $u \longrightarrow 0$.

The components in equation (5.5.2) are expressed in the frame $\overline{e}_i$ but we would rather measure them in a frame parallelly propagated along curves terminating at the singularity. We therefore let $\tilde{e}_i((,1,0) = \overline{e}_i(0,1,0)$ and use $\overline{\nabla}$ to parallelly propagate $\tilde{e}_i(0,1,0)$ up $\lambda_1(v) = \rho(0,.,v)$ to give $\tilde{e}_i(0,1,v)$, parallelly propagate $\tilde{e}_i(0,1,v)$ round the closed loops $\gamma_{v,u}(s = \rho(s,u,v)$ to give $\tilde{e}_i(s,1,v)$, and parallelly propagate $\tilde{e}_i(s,1,v)$ along $\kappa_{v,s}(u) = \rho s,u,v)$ to give $\tilde{e}_i(s,u,v)$. Thus $\tilde{e}_i(s,u,v)$ is parallel along the b-incomplete curves$\kappa_{v,s}$ which terminate at the singularity (see diagram 5.5.1).

Set

$$\tilde{e}_i(s,u,v) = {}_i^j(s,u,v)\hat{e}_i(s,u,v).$$

Now $\tilde{e}_i(0,1,v) = \hat{e}_i(0,1,v)$ and so by :onditions (a)–(c) above and Theorem 5.4.4 (along with Proposition 5.4.6 and Lenma 5.4.7) we have that

$$l_i^j(s,v) := \lim_{u\to 0} l_i^j(s,u,v)$$

exists and is continuous in s—thoughwe do not know whether it is continuous in v. Now

$$\overline{e}_i(s,u,v) = \alpha_i^j(u,v)\hat{e}_j(s,u,v = \alpha_i^j(u,v)(l^{-1})_j^k(s,u,v)\tilde{e}_k(s,u,v).$$

Hence expressing components in the fame $\tilde{e}_i$

$$\begin{aligned}\varepsilon_i^j = P_v \exp \int_0^1 \int_0^1 & l_{j_1}^j(s,u,v)\alpha^{-1})_{j_2}^{j_1}(u,v)\alpha_i^{i_1}(u,v)(l^{-1})_{i_1}^{i_2}(s,u,v) \\ & \overline{\Omega}_{\bar{k}\bar{l}\bar{i}}{}^{\bar{j}}(s,u,)X^{\bar{k}}(s,u,v)Z^{\bar{l}}(s,u,v)\,dsdv\end{aligned}$$

so by Proposition 2.1.2

$$\begin{aligned}\|\varepsilon_i^j - \delta_i^j\| \leq \exp \int_0^1 \int_0^1 & \|l_{j_1}^j(s,u,v)\|\|(\iota^{-1})_{j_2}^{j_1}(u,v)\|\|\alpha_i^{i_1}(u,v)\|\|(l^{-1})_{i_1}^{i_2}(s,u,v)\| \\ & (\|\overline{\Omega}_{\bar{a}\bar{b}\bar{i}}{}^{\bar{j}}(s,u,v)Z^b(s,u,v)\| + \|\overline{\Omega}_{\bar{a}\bar{B}\bar{i}}{}^{\bar{j}}(s,u,v)\|\|Z^B(s,u,v)\|) \\ & \|X^{\bar{k}}(s,u,v)\|\,dsdv - 1 \qquad (5.5.3)\end{aligned}$$

where we recall that $\overline{e}_a, \hat{e}_a, \tilde{e}_a$ are tangnt to $S = TS_v$ and $\overline{e}_A, \hat{e}_A, \tilde{e}_A$ are tangent to $T = (TS_v)^\perp$.

Now

$$\|X^{\tilde{a}}(s,u,v)\| = \|(l^{-1})^{\tilde{a}}_{\hat{c}}(s,u,v)X^{\hat{c}}(s,u,v)\| \leq \|(l^{-1})^{\tilde{a}}_{\hat{c}}(s,u,v)\|\|X^{\hat{c}}(s,u,v)\|$$

where $X^{\hat{k}}$ are the components of X in the frame $\hat{e}_i$. Since each $\gamma_{v,u}(s) = \rho(s,u,v)$ is parametrised proportional to b-length we have from section 2.5

$$\|X^{\hat{c}}(s,u,v)\| = l(u,v)$$

where each closed loop $\gamma_{v,u}$ has b-length $l(u,v)$ measured in the frame $\hat{e}_i$.

Now we know that $l(u,v) \to 0$ as $u \to 0$ (by the definition of the loop spaces $\Omega_{\kappa_v}(S_v)$), but we do not know whether $l(u,v)$ is bounded. We also know from section 2.5 that there exist $M(v), m(v) > 0$ such that

$$\|l^j_i(s,u,v) \leq M(v)\| \qquad \|(l^{-1})^j_i(s,u,v) \leq m(v)\|$$

but again we do not know if $M(v), m(v)$ are bounded. We also do not know whether the b-length $\lambda(v)$ of each κ_v is bounded. Therefore in order to get $\varepsilon^j_i \to \delta^j_i$ as $u \to 0$, we shall need to put bounds on a number of quantities.

We shall therefore say that the homotopy ρ is *well behaved* if, expressing components in the frame $\tilde{e}_i$, there exist $M_1, M_2, M_3, M_4, m_4, M_5, M_6 > 0$ such that

(a) $\|\overline{\Omega}_{abi}{}^j(s,u,v)Z^b(s,u,v)\| \leq M_1$

(b) $\|\overline{\Omega}_{aBi}{}^j(s,u,v)\| \leq M_2$

(c) $\|Z^B(s,u,v)\| \leq M_3$

(d) $\|l^j_i(s,v)\| \leq M_4$ and $\|(l^{-1})^j_i(s,v)\| \leq m_4$ where $l^j_i(s,v) := \lim_{u\to 0} l^j_i(s,u,v)$

(e) $l(u,v) \leq M_5$

(f) $\lambda(v) \leq M_6$.

As $u \to 0$, the tubes $\rho_u(s,v)$ shrink to a curve on the singularity parametrised by v. Since the singularity is normal to the S_v 2-surfaces, we expect $Z^b \to 0$ as $u \to 0$. This is not enough however since $\Omega_{abC}{}^D$ may diverge. This is the motivation for condition (a). Condition (b) is a bound on eight independent components of the curvature. Condition (c) is a somewhat unsatisfactory bound, since it has no immediate geometrical interpretation. We note that the components of the 2-frame $(\tilde{e}_A)$ parallelly propagated along $\kappa_{v,s}$ may diverge in a reference 2-frame $(\tilde{e}'_A)$ which covers the singularity. This is dependent on the components ω^C_{aB} which, not being a tensor, are hard to interpret. Conditions (d)–(f) place bounds on $l^j_i(s,v)$, $(l^{-1})^j_i(s,v)$, $l(u,v)$ and $\lambda(v)$.

Now we may relate condition (a) to the full space-time curvature. Referring forward to equations (6.1.1) and (6.1.8)

$$\Omega_{abcd} = \overline{\Omega}_{abcd} + g^{\perp}{}_{EF}(K^{\parallel}{}^E_{ac}K^{\parallel}{}^F_{bd} - K^{\parallel}{}^E_{bc}K^{\parallel}{}^F_{ad})$$

$$\Omega_{abCD} = \overline{\Omega}_{abCD} + g^{\parallel}{}_{ef}(A^{\parallel}{}^e_{aC}A^{\parallel}{}^f_{bD} - A^{\parallel}{}^e_{bC}A^{\parallel}{}^f_{aD})$$

and we note that there exists $M_v > 0$ such that $\|K^{\parallel}{}^D_{ab}(s,u,v)\| \leq M_v$. Now if there exists $M > 0$ such that $M_v \leq M$ we may rephrase condition (a) in terms of $\Omega_{abc}{}^d$ and $\Omega_{abC}{}^D$.

We may also relate condition (b) to the space-time curvature. Referring forward to equation (6.1.3) and (6.1.4)

$$\Omega_{aBcd} = \overline{\Omega}_{aBcd} + K^{\perp}{}^e_{BE}(g^{\parallel}{}_{ce}K^{\parallel}{}^E_{ad} - g^{\parallel}{}_{de}K^{\parallel}{}^E_{ac})$$

$$\Omega_{AbCD} = \overline{\Omega}_{AbCD} + K^{\parallel}{}^E_{be}(g^{\perp}{}_{CE}K^{\perp}{}^e_{AD} - g^{\perp}{}_{DE}K^{\perp}{}^e_{AC}).$$

Thus we may rephrase condition (b) in terms of Ω_{aBcd} and Ω_{AbCD} provided that we can similarly find a uniform bound for both $K^{\parallel}{}^D_{ab}$ and $K^{\perp}{}^d_{AB}$. Note that if the string were regular part of the space-time, $K^{\perp}{}^d_{AB} \longrightarrow 0$ as $u \longrightarrow 0$ would be sufficient for the string to be totally geodesic.

Now all the above components are measured in the frame $\tilde{e}_i$ parallelly propagated with respect to $\overline{\omega}$. Let e'_i be a frame parallelly propagated with respect to ω along the same curves as $\tilde{e}_i$. Let

$$\tilde{e}_i(s, u, v) = \xi_i^j(s, u, v) e'_j(s, u, v).$$

In order to obtain uniform bounds similar to (a)–(e) above when measuring components in the frame e'_i it would be necessary to find bounds $\xi_0, \xi_1 > 0$ such that

$$\|\xi_i^j(s, u, v)\| \leq \xi_0 \qquad \|(\xi^{-1})_i^j(s, u, v)\| \leq \xi_1.$$

We claim that this is possible, if $K^{\|}$, $K^{\perp}$ can be uniformly bounded as described above.

Lemma 5.5.1. *If ρ is well behaved then $\exists N, n > 0$ such that*

$$\|\alpha_i^j(u, v)\| \leq N \qquad \|(\alpha^{-1})_i^j(u, v)\| \leq n$$

Proof. Define

$$\phi : [0, 1] \times [0, v_0] \to M : (\sigma, \tau) \mapsto \phi(\sigma, \tau)$$

by (see diagram 5.5.2)

$$\phi_\tau(\sigma) = \phi(\sigma, \tau) = \begin{cases} \rho(0, 1, 4\sigma\tau) & 0 \leq \sigma \leq \frac{1}{4} \\ \rho(0, 1 - (1 - u_0)4(\sigma - \frac{1}{4}), \tau) & \frac{1}{4} \leq \sigma \leq \frac{1}{2} \\ \rho(0, u_0, \tau - 4(\sigma - \frac{1}{2})\tau) & \frac{1}{2} \leq \sigma \leq \frac{3}{4} \\ \rho(0, u_0 + 4(\sigma - \frac{3}{4})(1 - u_0), 0) & \frac{3}{4} \leq \sigma \leq 1. \end{cases}$$

Thus each ϕ_τ is a closed loop and $\phi_\tau([0, 1/4])$ lies on $\lambda_1(v) = \rho(0, 1, v)$, $\phi_\tau([1/4, 1/2])$ lies on $\kappa_\tau(u) = \rho(0, u, \tau)$, $\phi_\tau([1/2, 3/4])$ lies on $\lambda_{u_0}(v) = \rho(0, u_0, v)$, and $\phi_\tau([3/4, 1])$ lies on $\kappa_0(u) = \rho(0, u, 0)$.

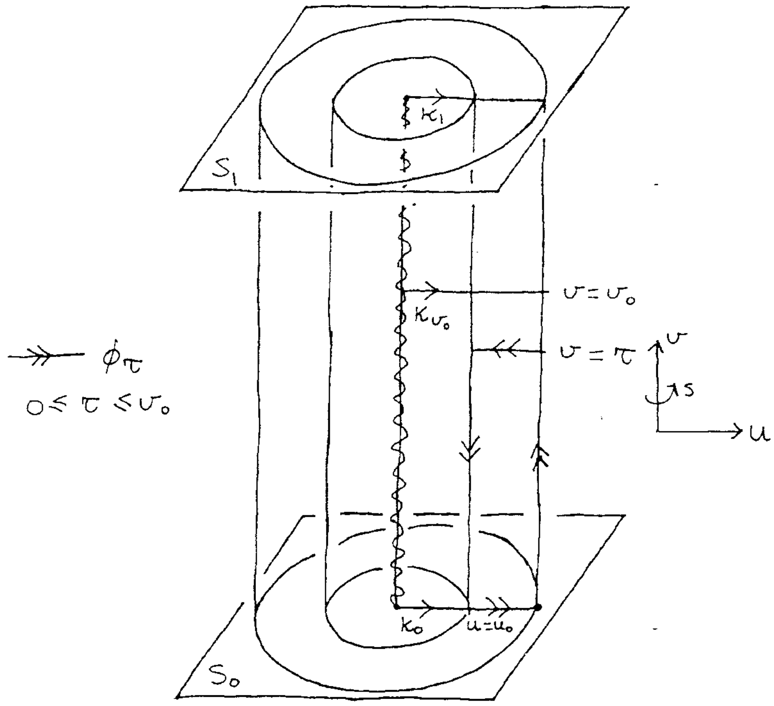

Diagram 5.5.2

Since $\hat{e}_i(0, u_0, v_0)$ is obtained by parallelly propagating $\hat{e}_i(0, 1, 0)$ along ϕ_{v_0} from $\sigma = 0$ to $\sigma = 1/2$, and $\overline{e}_i(0, u_0, v_0)$ is obtained by parallelly propagating $\overline{e}_i(0, 1, 0)$ along ϕ_{v_0} from $\sigma = 1$ to $\sigma = 1/2$, and

$$\overline{e}_i(0, 1, 0) = \hat{e}_i(0, 1, 0)$$

it follows that the element of holonomy obtained by parallelly propagating $\hat{e}_i(0, 1, 0)$ round ϕ_{v_0} from $\sigma = 0$ to $\sigma = 1$ is $\alpha_i^j(u_0, v_0)$. Thus by equation 2.5.2

$$\alpha_i^j(u_0, v_0) = P_\tau \exp \int_0^{v_0} \int_0^1 \overline{\Omega}_{kli}{}^j(\sigma, \tau) \hat{X}^k(\sigma, \tau) \hat{Y}^l(\sigma, \tau)\, d\sigma d\tau$$

where $\hat{X} = \phi_*(\partial_\sigma)$, $\hat{Y} = \phi_*(\partial_\tau)$ and components are expressed in the frame $e_i(\sigma, \tau)$ obtained by parallelly propagating $\hat{e}_i(0, 1, 0)$ round ϕ_τ. Now

$$\hat{X} = \frac{\partial u}{\partial \sigma} Y + \frac{\partial v}{\partial \sigma} Z \qquad \hat{Y} = \frac{\partial u}{\partial \tau} Y + \frac{\partial v}{\partial \tau} Z$$

and

$$(\hat{X}, \hat{Y}) = \begin{cases} (4\tau Z, 4\sigma Z) & 0 \leq \sigma \leq \frac{1}{4} \\ (-4(1-u_0)Y, Z) & \frac{1}{4} \leq \sigma \leq \frac{1}{2} \\ (-4\tau Z, (3-4\sigma)Z) & \frac{1}{2} \leq \sigma \leq \frac{3}{4} \\ (4(1-u_0)Y, 0) & \frac{3}{4} \leq \sigma \leq 1. \end{cases}$$

and hence

$$\alpha_i^j(u_0, v_0) = P_\tau \exp \int_0^{v_0} \int_{1/4}^{1/2} \overline{\Omega}_{kli}{}^j(\sigma,\tau)(-4(1-u_0))Y^k(\sigma,\tau)Z^l(\sigma,\tau)\, d\sigma d\tau.$$

Now for $1/4 \leq \sigma \leq 1/2$, $u = 1 - 4(1-u_0)(\sigma - 1/4)$ and $v = \tau$ so

$$du = -4(1-u_0)d\sigma \qquad dv = d\tau$$

and hence

$$\alpha_i^j(u_0, v_0) = P_v \exp \int_0^{v_0} \int_1^{u_0} \overline{\Omega}_{kli}{}^j(0,u,v)Y^k(0,u,v)Z^l(0,u,v)\, dudv$$

where components are measured in the frame e_i. However e_i coincides with $\hat{e}_i$ along κ_v from $\kappa_v(1)$ to $\kappa_v(u_0)$ for $0 \leq v \leq v_0$. Hence by Proposition 2.1.2, in the frame $\hat{e}_i$

$$\|\alpha_i^j(u_0, v_0)\| \leq \exp \int_0^{v_0} \int_1^{u_0} \|Y^a(0,u,v)\|(\|\overline{\Omega}_{abi}{}^j(0,u,v)Z^b(0,u,v)\| \\ + \|\overline{\Omega}_{aBi}{}^j(0,u,v)\|\|Z^B(0,u,v)\|)\, dudv + 1$$

and parametrising κ_v by b-length measured with respect to $\hat{e}_i$ gives

$$\|\alpha_i^j(\tilde{u}_0, v_0)\| \leq \exp \int_0^{v_0} \int_{\lambda(v)}^{\tilde{u}_0} M_1 + M_2 M_3\, d\tilde{u}dv + 1$$

where $\tilde{u}$ measures b-length along κ_v and κ_v has b-length $\lambda(v)$. $\lambda(v)$ is bounded hence

$$\|\alpha_i^j(u, v)\| \leq N$$

and similarly

$$\|(\alpha^{-1})_i^j(u, v)\| \leq n$$

for some $N, n > 0$. □

Hence we have

Theorem 5.5.2. *If ρ is well behaved then*

$$\hat{L}_i^j(0,0) = \hat{L}_i^j(0,1).$$

Proof. By applying Lemma 5.5.1 and conditions (a)–(e) above to equation 5.5.3 we get $\varepsilon_i^j = \delta_i^j$ as desired. □

Corollary 5.5.3. *Let $\overline{\kappa}_0$, $\overline{\kappa}_1$ be lifts of κ_0, κ_1 in the adapted frame bundle $L(S,T)$ by ω. If ρ is well behaved then the intrinsic holonomy groups obey*

$$H_{\bar{\kappa}_0}(S_0, \nabla) = H_{\bar{\kappa}_1}(S_1, \nabla).$$

Proof. This follows by Theorem 5.5.2 and the fact that the structure group of $L(S,T)^{\perp}$ is abelian. □

Corollary 5.5.4. *Let $\tilde{\kappa}_0$, $\tilde{\kappa}_1$ be lifts of κ_0, κ_1 in LM by ω. If ρ is well behaved then the extrinsic holonomy groups $H_{\tilde{\kappa}_0}(S_0, \nabla)$ and $H_{\tilde{\kappa}_1}(S_1, \nabla)$ are conjugate.*

Proof. This follows from Theorem 5.4.11 and Corollary 5.5.3. □

Unlike in section 1.6, it does not follow from the conservation of holonomy that the singularity is totally geodesic. In fact in section 6.3 we will present an example of an idealised cosmic string which is not totally geodesic despite having conserved holonomy, as well as examples of idealised cosmic strings which do not have conserved holonomy.

Chapter 6

A $2+2$ singular formulation

6.1 Equations of Gauss, Codazzi and Ricci

Let (M, g) be a space-time. As in section 5.3, suppose that $T_x M = S_x \oplus T_x$ $\forall x \in M$ for a C^2 choice of S_x, T_x such that S_x is spacelike, T_x is timelike, and $\dim S_x = \dim T_x = 2$. Suppose that $S_x \perp T_x$ $\forall x \in M$. Unlike in section 5.3 however, we do not require S_x to be surface forming. Let (e_i) be a C^2 adapted basis field.

Recall that we have projections

$$\pi_{\|} : T^q_p(T_x M) \longrightarrow T^q_p(S_x) \quad \pi_{\perp} : T^q_p(T_x M) \longrightarrow T^q_p(T_x) \qquad \forall x \in M, \quad p, q \in \mathbb{N}$$

so in particular

$$\pi_{\|} : S_x + T_x \longrightarrow S_x \quad \pi_{\perp} : S_x + T_x \longrightarrow T_x \qquad \forall x \in M.$$

As before, given $x \in M$, we use $\pi_{\|}$, $\pi_{\perp}$ to project the components of quantities of geometrical interest onto S_x, T_x. In this way we obtain $g^{\|}$, $g^{\perp}$, where $g^{\|}$ is the metric induced by g on S_x and $g^{\perp}$ is the metric induced by g on T_x. Given the Levi-Civita connection ∇ of g we also obtain the projected connection $\overline{\nabla}$. Defining

$$\nabla_i e_j = \omega^k_{ij} e_k \qquad \overline{\nabla}_i e_j = \overline{\omega}^k_{ij} e_k$$

we recall that

$$\overline{\omega}^k_{ij} = \begin{cases} \omega^b_{ia} & j = a,\ k = b \\ \omega^B_{iA} & j = A,\ k = B \\ 0 & \text{otherwise.} \end{cases}$$

The second fundamental forms $K^{\|}$, $K^{\perp}$ and associated second fundamental forms $A^{\|}$, $A^{\perp}$ are defined as before so that

$$K^{\|}{}^{C}_{ab} = \omega^{C}_{ab} \quad K^{\perp}{}^{c}_{AB} = \omega^{c}_{AB} \quad A^{\|}{}^{c}_{aB} = \omega^{c}_{aB} \quad A^{\perp}{}^{C}_{Ab} = \omega^{C}_{Ab}.$$

We have seen how, using equation (5.3.5), ∇ can be reconstructed from a knowledge of the projected connection $\overline{\nabla}$ and $K^{\|}$, $K^{\perp}$, $A^{\|}$, $A^{\perp}$. We will now show how the curvature Ω^{j}_{i} of ∇ can be reconstructed from a knowledge of the curvature $\overline{\Omega}^{j}_{i}$ of $\overline{\nabla}$ and $K^{\|}$, $K^{\perp}$, $A^{\|}$, $A^{\perp}$ using the Gauss-Codazzi-Ricci equations.

The Gauss-Codazzi-Ricci equations are usually given in a $3+1$ form, where they are expressed in terms of quantities defined with respect to a 3-dimensional spacelike submanifold. In [Ch] the Gauss-Codazzi-Ricci equations are expressed in terms of quantities defined with respect to an n-dimensional submanifold of an m-dimensional manifold. In our case we give the Gauss-Codazzi-Ricci equations for the two families of 2-dimensional tangent spaces $\{S_x\}$ and $\{T_x\}$, where we recall that $\{S_x\}$, $\{T_x\}$ need not be surface forming. Unlike other approaches, we express the Gauss and Ricci equations directly in terms of the projected connection.

We will use this 2+2 formulation in the next section to examine the behaviour of the space-time curvature in the vicinity of an idealised cosmic string. This approach should prove more natural than the $3+1$ approach discussed in section 4.1.

Working in the adapted basis (e_i), recall the equations $(5.2.4a)$

$$\Omega^{b}_{a} = \overline{\Omega}^{b}_{a} + \omega^{b}_{C} \wedge \omega^{C}_{a}$$

and $(5.2.4b)$

$$\Omega^{B}_{A} = \overline{\Omega}^{B}_{A} + \omega^{B}_{c} \wedge \omega^{c}_{A}$$

where we recall that $\overline{\Omega}^{b}_{a}$ is the curvature of the connection $\nabla^{\|}$ on the bundle $L(S)$ and $\overline{\Omega}^{B}_{A}$ is the curvature of the connection $\nabla^{\perp}$ on the bundle $L(T)$. We note that $\overline{\Omega}^{B}_{a} = \overline{\Omega}^{b}_{A} = 0$.

From (5.2.4a)

$$\Omega_{abc}{}^{d} = \overline{\Omega}_{abc}{}^{d} + \omega^{d}_{aE} \wedge \omega^{E}_{bc} = \overline{\Omega}_{abc}{}^{d} + A\|^{d}_{aE} K\|^{E}_{bc} - A\|^{d}_{bE} K\|^{E}_{ac}.$$

By equation (5.3.4a), $A\|^{d}_{aE} = -g_{EF} g^{de} K\|^{F}_{ae}$ so

$$\Omega_{abc}{}^{d} = \overline{\Omega}_{abc}{}^{d} - g_{EF} g^{de} (K\|^{F}_{ae} K\|^{E}_{bc} - K\|^{F}_{be} K\|^{E}_{ac})$$

and hence

$$\Omega_{abcd} = \overline{\Omega}_{abcd} + g^{\perp}{}_{EF} (K\|^{E}_{ac} K\|^{F}_{bd} - K\|^{E}_{bc} K\|^{F}_{ad}) \tag{6.1.1}$$

which is the Gauss equation for $\{S_x\}$. We note that if S_0 is a 2-surface tangent to S_x $\forall x \in S_0$, then $\overline{\Omega}_{abc}{}^{d}$ is the curvature of the Levi-Civita connection of $(S_0, g^{\|})$. Similarly we obtain from (5.2.4b)

$$\Omega_{ABCD} = \overline{\Omega}_{ABCD} + g^{\|}{}_{ef} (K^{\perp e}{}_{AC} K^{\perp f}{}_{BD} - K^{\perp e}{}_{BC} K^{\perp f}{}_{AD}) \tag{6.1.2}$$

which is the Gauss equation for $\{T_x\}$. We note that if T_0 is a 2-surface tangent to T_x $\forall x \in T_0$, then $\overline{\Omega}_{ABC}{}^{D}$ is the curvature of the Levi-Civita connection of $(T_0, g^{\perp})$.

From (5.2.4a)

$$\begin{aligned}
\Omega_{aBc}{}^{d} &= \overline{\Omega}_{aBc}{}^{d} + \omega^{d}_{aE} \wedge \omega^{E}_{Bc} \\
&= \overline{\Omega}_{aBc}{}^{d} + A\|^{d}_{aE} A^{\perp E}_{Bc} - K^{\perp d}{}_{BE} K\|^{E}_{ac} \\
&= \overline{\Omega}_{aBc}{}^{d} + g_{EF} g^{de} K\|^{F}_{ae} g^{EG} g_{cf} K^{\perp f}{}_{BG} - K^{\perp d}{}_{BE} K\|^{E}_{ac}
\end{aligned}$$

and hence

$$\begin{aligned}
\Omega_{aBcd} &= \overline{\Omega}_{aBcd} + g_{cf} K\|^{F}_{ad} K^{\perp f}{}_{BF} - g_{de} K^{\perp e}{}_{BE} K\|^{E}_{ac} \\
&= \overline{\Omega}_{aBcd} + K^{\perp e}{}_{BE} (g^{\|}{}_{ce} K\|^{E}_{ad} - g^{\|}{}_{de} K\|^{E}_{ac}).
\end{aligned} \tag{6.1.3}$$

Similarly we obtain from (5.2.4b)

$$\Omega_{AbCD} = \overline{\Omega}_{AbCD} + K\|^{E}_{be} (g^{\perp}{}_{CE} K^{\perp e}{}_{AD} - g^{\perp}{}_{DE} K^{\perp e}{}_{AC}). \tag{6.1.4}$$

However the geometrical significance of $\overline{\Omega}_{aBcd}$ and $\overline{\Omega}_{AbCD}$ is not immediately clear and it is more helpful to calculate $\Omega_{aBc}{}^{d}$ and $\Omega_{AbC}{}^{D}$ directly. From the second Cartan equation

$$\Omega_{abc}{}^{D} = (d\omega^{D}_{\ c})_{ab} + (\omega^{D}_{\ k} \wedge \omega^{k}_{\ c})_{ab} = \partial_a \omega^{D}_{\ bc} - \partial_b \omega^{D}_{\ ac} - c^{i}_{\ ab}\omega^{D}_{\ ic} + \omega^{D}_{\ ak}\omega^{k}_{\ bc} - \omega^{D}_{\ bk}\omega^{k}_{\ ac}$$

where $[e_i, e_j] = c^{k}_{\ ij} e_k$. Now ∇ has torsion $T^{k}_{\ ij} = \omega^{k}_{\ ij} - \omega^{k}_{\ ji} - c^{k}_{\ ij} = 0$ so

$$\begin{aligned}
\Omega_{abc}{}^{D} &= \partial_a \omega^{D}_{\ bc} - \partial_b \omega^{D}_{\ ac} - \omega^{e}_{\ ab}\omega^{D}_{\ ec} - \omega^{E}_{\ ab}\omega^{D}_{\ Ec} + \omega^{e}_{\ ba}\omega^{D}_{\ ec} + \omega^{E}_{\ ba}\omega^{D}_{\ Ec} \\
&\quad + \omega^{D}_{\ ae}\omega^{e}_{\ bc} + \omega^{D}_{\ aE}\omega^{E}_{\ bc} - \omega^{D}_{\ be}\omega^{e}_{\ ac} - \omega^{D}_{\ bE}\omega^{E}_{\ ac} \\
&= \partial_a K^{\parallel D}_{\ \ bc} + \omega^{D}_{\ aE} K^{\parallel E}_{\ \ bc} - \omega^{e}_{\ ab} K^{\parallel D}_{\ \ ec} - \omega^{e}_{\ ac} K^{\parallel D}_{\ \ be} \\
&\quad - \partial_b K^{\parallel D}_{\ \ ac} - \omega^{D}_{\ bE} K^{\parallel E}_{\ \ ac} + \omega^{e}_{\ ba} K^{\parallel D}_{\ \ ec} + \omega^{e}_{\ bc} K^{\parallel D}_{\ \ ae} - 2K^{\parallel E}_{\ \ [ab]} A^{\perp D}_{\ \ Ec} \\
&= \overline{\nabla}_a K^{\parallel D}_{\ \ bc} - \overline{\nabla}_b K^{\parallel D}_{\ \ ac} - 2K^{\parallel E}_{\ \ [ab]} A^{\perp D}_{\ \ Ec}
\end{aligned} \tag{6.1.5}$$

which is the Codazzi equation for $\{S_x\}$. If S_x is surface forming at $x \in M$ then at this point $K^{\parallel E}_{\ \ [ab]} = 0$. $K^{\parallel E}_{\ \ [ab]}$ is said to measure the *anholonomicity* of S_x. Similarly we obtain

$$\Omega_{ABC}{}^{d} = \overline{\nabla}_A K^{\perp d}_{\ \ BC} - \overline{\nabla}_B K^{\perp d}_{\ \ AC} - 2K^{\perp e}_{\ \ [AB]} A^{\parallel d}_{\ \ eC} \tag{6.1.6}$$

which is the Codazzi equation for $\{T_x\}$. If T_x is surface forming at $x \in M$ then at this point $K^{\perp e}_{\ \ [AB]} = 0$ and $K^{\perp e}_{\ \ [AB]}$ measures the anholonomicity of T_x.

From $(5.2.4a)$

$$\begin{aligned}
\Omega_{ABc}{}^{d} &= \overline{\Omega}_{ABc}{}^{d} + \omega^{d}_{\ AE} \wedge \omega^{E}_{\ Bc} \\
&= \overline{\Omega}_{ABc}{}^{d} + K^{\perp d}_{\ \ AE} A^{\perp E}_{\ \ Bc} - K^{\perp d}_{\ \ BE} A^{\perp E}_{\ \ Ac} \\
&= \overline{\Omega}_{ABc}{}^{d} - g_{EF} g^{de} A^{\perp F}_{\ \ Ae} A^{\perp E}_{\ \ Bc} + g_{EF} g^{de} A^{\perp F}_{\ \ Be} A^{\perp E}_{\ \ Ac}
\end{aligned}$$

and hence

$$\Omega_{ABcd} = \overline{\Omega}_{ABcd} + g^{\perp}_{\ EF}(A^{\perp E}_{\ \ Ac} A^{\perp F}_{\ \ Bd} - A^{\perp E}_{\ \ Bc} A^{\perp F}_{\ \ Ad}) \tag{6.1.7}$$

which is the Ricci equation for $\{T_x\}$. If T_0 is a 2-surface tangent to T_x $\forall x \in T_0$ then $\overline{\Omega}_{ABc}{}^d$ is the curvature of the connection induced by ∇ on the bundle $L(S|_{T_0})$. Similarly we obtain from (5.2.4b)

$$\Omega_{abCD} = \overline{\Omega}_{abCD} + g^{\parallel}{}_{ef}(A^{\parallel}{}^e_{aC}A^{\parallel}{}^f_{bD} - A^{\parallel}{}^e_{bC}A^{\parallel}{}^f_{aD}) \tag{6.1.8}$$

which is the Ricci equation for $\{S_x\}$. If S_0 is a 2-surface tangent to S_x $\forall x \in S_0$ then $\overline{\Omega}_{abC}{}^D$ is the curvature of the connection induced by ∇ on the bundle $L(T|_{S_0})$.

We recall that $\Omega_{abcd} = \Omega_{[ab][cd]} = \Omega_{cdab}$ and $\Omega_{[abc]d} = 0$. From these symmetries, Ω_{abcd}, Ω_{ABCD} each account for one component of Ω_{ijkl}; $\Omega_{abc}{}^D$, $\Omega_{ABC}{}^d$ each account for four components of Ω_{ijkl}; and Ω_{abCD}, Ω_{ABcd} together account for one component of Ω_{ijkl}, making for a total of eleven components. The remaining nine components of Ω_{ijkl} can be obtained from Ω_{aBcD}. We note that

$$\Omega_{aBcD} + \Omega_{BcaD} + \Omega_{caBD} = 0 \Rightarrow \Omega_{caBD} = -\Omega_{aBcD} + \Omega_{cBaD}$$

so that in fact Ω_{abCD} may be obtained from components of the form Ω_{aBcD}. We also recall that $\overline{\Omega}_{abcd} = \overline{\Omega}_{[ab][cd]}$ but that in general $\overline{\Omega}_{abcd} \neq \overline{\Omega}_{cdab}$ and $\overline{\Omega}_{[abc]d} \neq 0$.

We now calculate Ω_{aBcD} from the second Cartan equation.

$$\begin{aligned}
\Omega_{aBc}{}^D &= (d\omega^D_c)_{aB} + (\omega^D_k \wedge \omega^k_c)_{aB} \\
&= \partial_a\omega^D_{Bc} - \partial_B\omega^D_{ac} - c^i_{aB}\omega^D_{ic} + \omega^D_{ak}\omega^k_{Bc} - \omega^D_{Bk}\omega^k_{ac} \\
&= \partial_a\omega^D_{Bc} - \partial_B\omega^D_{ac} - \omega^e_{aB}\omega^D_{ec} + \omega^e_{Ba}\omega^D_{ec} - \omega^E_{aB}\omega^D_{Ec} + \omega^E_{Ba}\omega^D_{Ec} \\
&\quad + \omega^D_{ae}\omega^e_{Bc} + \omega^D_{aE}\omega^E_{Bc} - \omega^D_{Be}\omega^e_{ac} - \omega^D_{BE}\omega^E_{ac} \\
&= \partial_a A^{\perp}{}^D_{Bc} + \omega^D_{aE}A^{\perp}{}^E_{Bc} - \omega^E_{aB}A^{\perp}{}^D_{Ec} - \omega^e_{ac}A^{\perp}{}^D_{Be} \\
&\quad - \partial_B K^{\parallel}{}^D_{ac} - \omega^D_{BE}K^{\parallel}{}^E_{ac} + \omega^e_{Ba}K^{\parallel}{}^D_{ec} + \omega^e_{Bc}K^{\parallel}{}^D_{ae} + A^{\perp}{}^E_{Ba}A^{\perp}{}^D_{Ec} - A^{\parallel}{}^e_{aB}K^{\parallel}{}^D_{ec} \\
&= \overline{\nabla}_a A^{\perp}{}^D_{Bc} - \overline{\nabla}_B K^{\parallel}{}^D_{ac} + A^{\perp}{}^E_{Ba}A^{\perp}{}^D_{Ec} - A^{\parallel}{}^e_{aB}K^{\parallel}{}^D_{ec}
\end{aligned}$$

and hence

$$\Omega_{aBcD} = g^{\perp}{}_{DE}(\overline{\nabla}_a A^{\perp}{}^E_{Bc} + A^{\perp}{}^F_{Ba}A^{\perp}{}^E_{Fc}) + g^{\parallel}{}_{ce}(\overline{\nabla}_B A^{\parallel}{}^e_{aD} + A^{\parallel}{}^f_{aB}A^{\parallel}{}^e_{fD}). \tag{6.1.9}$$

The Ricci tensor R_{ij} of ∇ can be obtained from $\Omega_{ijk}{}^{l}$ by

$$R_{ij} = \Omega_{ikj}{}^{k} = \Omega_{iej}{}^{e} + \Omega_{iEj}{}^{E}$$

and so

$$R_{ab} = \Omega_{aeb}{}^{e} + \Omega_{aEb}{}^{E} \qquad R_{AB} = \Omega_{AeB}{}^{e} + \Omega_{AEB}{}^{E}$$

$$R_{aB} = \Omega_{aeB}{}^{e} + \Omega_{aEB}{}^{E} \qquad R_{Ab} = \Omega_{Aeb}{}^{e} + \Omega_{AEb}{}^{E}$$

where $R_{ij} = R_{ji}$. Similarly the Ricci tensor $\overline{R}_{ij}$ of $\overline{\nabla}$ can be obtained from $\overline{\Omega}_{ijk}{}^{l}$ by

$$\overline{R}_{ij} = \overline{\Omega}_{ikj}{}^{k} = \overline{\Omega}_{iej}{}^{e} + \overline{\Omega}_{iEj}{}^{E}$$

and so

$$\overline{R}_{ab} = \overline{\Omega}_{aeb}{}^{e} \qquad \overline{R}_{AB} = \overline{\Omega}_{AEB}{}^{E}$$

$$\overline{R}_{aB} = \overline{\Omega}_{aEB}{}^{E} \qquad \overline{R}_{Ab} = \overline{\Omega}_{Aeb}{}^{e}$$

where in general $\overline{R}_{ij} \neq \overline{R}_{ji}$. We note that if S_0 is a 2-surface tangent to S_x $\forall x \in S_0$, then $\overline{R}_{ab}$ is the Ricci tensor of the Levi-Civita connection of $(S_0, g^{\parallel})$. Similarly if T_0 is a 2-surface tangent to T_x $\forall x \in T_0$, then $\overline{R}_{AB}$ is the Ricci tensor of the Levi-Civita connection of $(T_0, g^{\perp})$.

Finally we note that although we have given the above equations in an adapted basis, they are in fact all fully covariant if the correct projections are introduced, so that for example $K^{\parallel}{}^{C}_{ab}$ is replaced by $g^{\parallel}{}^{i}_{a} g^{\parallel}{}^{j}_{b} g^{\perp}{}^{C}_{k} K^{\parallel}{}^{k}_{ij}$.

6.2 Curvature of an idealised cosmic string

In the previous section we presented the Gauss-Codazzi-Ricci equations which express the curvature of a space-time (M, g) in terms of the curvature of the projected connection and the extrinsic curvatures of two normal families of 2-dimensional tangent spaces. We now discuss the behaviour of these extrinsic curvatures and, using the Gauss-Codazzi-Ricci equations, the behaviour of the space-time curvature near an idealised cosmic string, as formulated in section 4.2.

Let (M, g) be a space-time containing an idealised cosmic string $U \subset M$ with preferred spacelike 2-surfaces $\{S_{tz}\}$. For each $x \in U$, let $S_x = (TS_{tz})_x$ and $T_x = (TS_{tz})_x^{\perp}$. Let the Levi-Civita connection ω have curvature Ω_i^j and let the projected connection $\overline{\omega}$ have curvature $\overline{\Omega}_i^j$. For each $x \in U$, let $g^{\parallel}$ be the metric induced on S_x and $g^{\perp}$ the metric induced on T_x.

For each $x \in U$, let $K^{\parallel}$, $K^{\perp}$ be the extrinsic curvatures of S_x, T_x. Working in an adapted frame, recall

$$A^{\parallel}{}^{d}_{aB} = g^{\perp}{}_{FB} g^{\parallel ed} K^{\parallel}{}^{F}_{ae}$$

$$A^{\perp}{}^{D}_{Ab} = g^{\parallel}{}_{fb} g^{\perp ED} K^{\perp}{}^{f}_{AE}.$$

Let κ be a curve lying in a particular S_{tz} and terminating at $\{r = 0\}$. If $\omega \sim \overline{\omega}$ along κ, then by Theorem 2.3.3, κ will have ω-finite b-length if and only if it has $\overline{\omega}$-finite b-length. We note that b-length measured along κ in an adapted frame parallelly propagated with respect to $\overline{\omega}$ will coincide with length measured with respect to the intrinsic positive definite metric $g^{\parallel}$.

Because the connection difference $\sigma = \overline{\omega} - \omega$ satisfies

$$g^{\parallel}{}^{k}_{a} \sigma^{j}_{ki} = \sigma^{j}_{ai} = \overline{\omega}^{j}_{ai} - \omega^{j}_{ai} = \begin{cases} K^{\parallel}{}^{D}_{ab} & i = b,\ j = D \\ A^{\parallel}{}^{d}_{aB} & i = B,\ j = d \\ 0 & \text{otherwise} \end{cases}$$

it follows by Theorem 2.3.13 that $\omega \sim \overline{\omega}$ along κ if $K^{\parallel}{}^{D}_{ab}$ is C^0-ω-quasi-regular (or C^0-$\overline{\omega}$-quasi-regular) along κ, that is, if the components $K^{\parallel}{}^{D}_{ab}$ expressed in a frame parallelly propagated along κ with respect to ω (or $\overline{\omega}$) have C^0 limits (or in fact are merely bounded).

By Lemma 5.4.2, $\omega \sim \overline{\omega}$ along κ will also hold if S_{tz} is regular with respect to κ, though this is a stronger condition. In particular if S_{tz} is regular with respect to κ then $K^{\parallel}{}^{D}_{ab}$ will be bounded in an ω-frame along κ.

We now give an interpretation of the extrinsic curvature $K^{\perp}{}^{d}_{AB}$ of the normal spaces T_x.

We saw in Corollary 1.6.6 that a good 2-dimensional quasi-regular singularity may be considered to be totally geodesic. In section 4.1 we saw how a more general type of singularity was considered in [UHIM] to be totally geodesic if a sequence of timelike (or spacelike) curves normal to a foliation of 3-dimensional hyperfaces had spacelike accelerations whose magnitude tended to zero as they approached a timelike (or spacelike) intrinsic geodesic of the singularity. In particular this occurred if the lapse function of the hyperfaces, essentially the normal metric, was C^2 in a rather artificial quasi-Cartesian coordinate system at $r = 0$. Now in our $2+2$ approach, the 2-dimensional normal spaces $\{T_x\}$ can be considered to become tangent to the singularity as $r \to 0$, and therefore it would be more natural to take the extrinsic curvatures $K^{\perp d}{}_{AB}$ of the normal spaces $\{T_x\}$ and to consider the limits of their components in a frame parallelly propagated onto the singularity with respect to ω.

We shall therefore say that an idealised cosmic string is *weakly totally geodesic* at $p \in U_0$ (where U_0 occurs in the definition of an idealised cosmic string) if the components $K^{\perp d}{}_{AB} \to 0$ in an ω-frame as $r \to 0$ along any curve of finite b-length terminating at $p \in U_0$ and lying in a preferred spacelike 2-surface, and *strongly totally geodesic* at $p \in U_0$ if the components $K^{\perp d}{}_{AB} \to 0$ in an ω-frame as $r \to 0$ along any curve of finite b-length terminating at $p \in U_0$ (but not necessarily lying in a preferred spacelike 2-surface).

Certainly, if the string were a regular part of the space-time, the string would be totally geodesic if and only if it were weakly totally geodesic, and it would be weakly totally geodesic if and only if it were strongly totally geodesic. We note that if the preferred spacelike 2-surfaces are unique, then so are the normal spaces $\{T_x\}$.

We will see in the next section an example of an idealised cosmic string for which $K^{\perp d}{}_{AB} = 0$ in an ω-frame (example 6.3.2) and an example of an idealised cosmic string for which $K^{\perp d}{}_{AB} \to \infty$ in an ω-frame (example 6.3.3 with $n = 1$). In fact, in the first case the string is strongly totally geodesic.

On the other hand, all the examples in the next section have $K^{\| D}_{\ ab}$ bounded in an ω-frame.

We now examine the curvature near an idealised cosmic string. Suppose that κ has ω-finite b-length.

Consider equation (6.1.1) (the Gauss equation for $\{S_x\}$)

$$\Omega_{abcd} = \overline{\Omega}_{abcd} + g^{\perp}{}_{EF}(K^{\| E}_{\ ac} K^{\| F}_{\ bd} - K^{\| E}_{\ bc} K^{\| F}_{\ ad}).$$

$\overline{\Omega}_{abcd}$ is the curvature of the 2-space $(S_{tz}, g^{\|})$. If κ has $\overline{\omega}$-finite b-length, it terminates at a quasi-regular singularity of $(S_{tz}, g^{\|})$ and thus $\overline{\Omega}_{abcd}$ will be C^0-$\overline{\omega}$-quasi-regular. Now if $K^{\| D}_{\ ab}$ is C^0-ω-quasi-regular then $\omega \sim \overline{\omega}$ as discussed above (and κ will indeed have $\overline{\omega}$-finite b-length). Thus $K^{\| D}_{\ ab}$ will be C^0-$\overline{\omega}$-quasi-regular and hence the components Ω_{abcd} will also be C^0-$\overline{\omega}$-quasi-regular and since $\omega \sim \overline{\omega}$, Ω_{abcd} will in fact be C^0-ω-quasi-regular. Since an ω-frame will not in general be adapted to S_{tz}, when we write Ω_{abcd} we refer to the tensor

$$\Omega_{abcd} := g^{\| i}_{\ a} g^{\| j}_{\ b} g^{\| k}_{\ c} g^{\| l}_{\ d} \Omega_{ijkl}.$$

In other words, if $K^{\| D}_{\ ab}$ is C^0-ω-quasi-regular then Ω_{abcd}, which represents one independent component of the space-time curvature, will have a C^0 limit when measured in a frame parallelly propagated along κ with respect to ω.

Certainly if S_{tz} is regular with respect to κ then Ω_{abcd} will be C^0-ω-quasi-regular (or at least bounded in an ω-frame).

Now consider equation (6.1.2) (the Gauss equation for $\{T_x\}$)

$$\Omega_{ABCD} = \overline{\Omega}_{ABCD} + g^{\|}{}_{ef}(K^{\perp e}{}_{AC} K^{\perp f}{}_{BD} - K^{\perp e}{}_{BC} K^{\perp f}{}_{AD}).$$

$\overline{\Omega}_{ABCD}$ will be the curvature of $g^{\perp}$ if the normal spaces T_x are surface forming.

Recall that in the definition of an idealised cosmic string given in section 4.2, there exists an isometry ψ of $(U, T, g^{\perp})$ into $(\tilde{U}, \tilde{T}, \tilde{g}^{\perp})$ where $U \subset M$, $T = \{T_x\}_{x \in M}$, and $U_0 = \tilde{U} - U$ can be considered to be the string. The string has intrinsic metric $\tilde{g}^{\perp}|_{U_0}$. In the coordinate chart

$$\phi : (t, z, r, \theta) \mapsto \phi(t, z, r, \theta)$$

which occurs in the definition, $\lim_{r \to 0} g^{\perp}{}_{\mu\nu} = \tilde{g}^{\perp}{}_{\mu\nu}|_{U_0}$. Now we require both $\tilde{g}^{\perp}|_{U_0}$ and $\tilde{g}^{\perp}|_{r>0}$ to be at least C^2, however we do not require $\lim_{r \to 0} \partial_r g^{\perp}{}_{\mu\nu}$, $\lim_{r \to 0} \partial_r^2 g^{\perp}{}_{\mu\nu}$ to exist.

Nonetheless, if we choose a reference 2-frame $(\tilde{e}_A)$ on $\tilde{U}$ in a C^2 manner, everywhere tangent to $\tilde{T}$, even at $r = 0$, then expressing components in this frame, $\lim_{r \to 0} \overline{\Omega}_{ABCD}$ will coincide with the curvature of $(U_0, \tilde{g}^{\perp}|_{U_0})$ (at least if the normal spaces T_x are surface forming).

Let (e_A) be a 2-frame tangent to the normal spaces T_x and parallelly propagated along κ with respect to $\overline{\omega}$. The components of (e_A) in the reference 2-frame $(\tilde{e}_A)$ may not have well defined limits as $r \longrightarrow 0$, since $(\tilde{e}_A)$ is not parallel along κ. If the components of (e_A) in the reference 2-frame $(\tilde{e}_A)$ do have well defined limits as $r \longrightarrow 0$ (and maybe also if the normal spaces T_x are surface forming) then $\overline{\Omega}_{ABCD}$ will be C^0-$\overline{\omega}$-quasi-regular. In this case, if $K^{\perp d}{}_{AB}$ is C^0-$\overline{\omega}$-quasi-regular, then by equation (6.1.2), Ω_{ABCD} will be C^0-$\overline{\omega}$-quasi-regular, and if $K^{\parallel D}{}_{ab}$ is C^0-ω-quasi-regular then as before $\omega \sim \overline{\omega}$ and Ω_{ABCD} will be C^0-ω-quasi-regular. Again, since an ω-frame will not in general be adapted to S_{tz}, when we write Ω_{ABCD} we refer to the tensor

$$\Omega_{ABCD} := g^{\perp i}{}_{A} g^{\perp j}{}_{B} g^{\perp k}{}_{C} g^{\perp l}{}_{D} \Omega_{ijkl}.$$

Ω_{ABCD} represents one independent component of the space-time curvature. The examples in the next section all have Ω_{abcd}, Ω_{ABCD} zero or bounded in an ω-frame.

Now consider the Ricci equation (6.1.8) for $\{S_x\}$

$$\Omega_{abCD} = \overline{\Omega}_{abCD} + g^{\|}{}_{ef}(A^{\|}{}^{e}_{aC}A^{\|}{}^{f}_{bD} - A^{\|}{}^{e}_{bC}A^{\|}{}^{f}_{aD}).$$

We used this equation in Lemma 5.4.7 on the way to proving Theorem 5.4.11, namely that if S_{t_z} is regular with respect to κ, then the intrinsic and extrinsic holonomy groups $H_{\overline{\kappa}}(S, \overline{\nabla})$, $H_{\tilde{\kappa}}(S, \nabla)$ exist for lifts $\overline{\kappa}$, $\tilde{\kappa}$ of κ obtained by ϖ, ω. In this case we recall that, since S_{t_z} is regular with respect to κ, Ω_{abCD} can be bounded by an integrable function and $K^{\|}{}^{D}_{ab}$ can be bounded by a constant, in a way made precise in section 5.4.

In other words, Ω_{abCD} may diverge in an ω-frame along κ but, in order for the intrinsic and extrinsic holonomy groups to exist, cannot diverge too quickly (in a way made precise in section 5.4).

The Ricci equation (6.1.7) for $\{T_x\}$ gives

$$\Omega_{ABcd} = \overline{\Omega}_{ABcd} + g^{\perp}{}_{EF}(A^{\perp}{}^{E}_{Ac}A^{\perp}{}^{F}_{Bd} - A^{\perp}{}^{E}_{Bc}A^{\perp}{}^{F}_{Ad})$$

but of course $\Omega_{ABcd} = \Omega_{cdAB}$.

Ω_{abCD}, Ω_{ABcd} together represent only one independent component of the space-time curvature.

The examples in the next section all in fact have Ω_{abCD} zero or bounded in an ω-frame.

Now consider equations (6.1.3) and (6.1.4)

$$\Omega_{aBcd} = \overline{\Omega}_{aBcd} + K^{\perp}{}^{e}_{BE}(g^{\|}{}_{ce}K^{\|}{}^{E}_{ad} - g^{\|}{}_{de}K^{\|}{}^{E}_{ac})$$

$$\Omega_{AbCD} = \overline{\Omega}_{AbCD} + K^{\|}{}^{E}_{be}(g^{\perp}{}_{CE}K^{\perp}{}^{e}_{AD} - g^{\perp}{}_{DE}K^{\perp}{}^{e}_{AC}).$$

Recall that the intrinsic and extrinsic holonomy groups, if they exist, are conserved along the singularity if $\overline{\Omega}_{aBcd}$, $\overline{\Omega}_{AbCD}$ are bounded in a ϖ-frame along curves of ϖ-finite b-length which lie in the preferred spacelike 2-surfaces and terminate at

$r = 0$ (along with some other geometrical constraints). Again, if $K^{\parallel D}_{ab}$, $K^{\perp d}_{AB}$ have bounded components in an ω-frame along κ, then $\omega \sim \overline{\omega}$ along κ and Ω_{aBcd}, Ω_{AbCD} will have bounded components both in a $\overline{\omega}$-frame and in an ω-frame along κ.

Together, Ω_{aBcd}, Ω_{AbCD} refer to eight independent components of the space-time curvature.

These components may also be obtained from the Codazzi equations (6.1.5) and (6.1.6) for $\{S_x\}$ and $\{T_x\}$

$$\Omega_{abc}{}^{D} = \overline{\nabla}_a K^{\parallel D}_{bc} - \overline{\nabla}_b K^{\parallel D}_{ac} - 2K^{\parallel E}_{[ab]} A^{\perp D}_{Ec}$$

$$\Omega_{ABC}{}^{d} = \overline{\nabla}_A K^{\perp d}_{BC} - \overline{\nabla}_B K^{\perp d}_{AC} - 2K^{\perp e}_{[AB]} A^{\parallel d}_{eC}.$$

Hence $\Omega_{abc}{}^{D}$ will be C^0-ω-quasi-regular along κ if $K^{\parallel D}_{ab}$ is C^1-ω-quasi-regular (so in particular $\omega \sim \overline{\omega}$ and $K^{\parallel D}_{ab}$ will be C^1-$\overline{\omega}$-quasi-regular, $\overline{\nabla}_a K^{\parallel D}_{bc}$ will be C^0-$\overline{\omega}$-quasi-regularand hence C^0-ω-quasi-regular), and $K^{\perp D}_{ab}$ is C^0-ω-quasi-regular. Similarly $\Omega_{ABC}{}^{d}$ will be C^0-ω-quasi-regular along κ if $K^{\parallel D}_{ab}$ is C^0-ω-quasi-regular (and hence $\omega \sim \overline{\omega}$), and $K^{\perp d}_{AB}$ is C^1-ω-quasi-regular (and hence as before $K^{\perp d}_{AB}$ will be C^1-$\overline{\omega}$-quasi-regular, $\overline{\nabla}_A K^{\perp d}_{BC}$ will be C^0-$\overline{\omega}$-quasi-regularand hence C^0-ω-quasi-regular).

In particular if $K^{\parallel D}_{ab}$, $K^{\perp d}_{AB}$ are C^1-ω-quasi-regular then Ω_{aBcd}, Ω_{AbCD}, $\overline{\Omega}_{aBcd}$, $\overline{\Omega}_{AbCD}$ will all be C^0-ω-quasi-regular, and the intrinsic and extrinsic holonomy groups, if they exist, will be conserved along the singularity (subject to some additional geometrical constraints on the homotopy ρ defined in section 5.5).

We shall see that examples 6.3.2 and 6.3.3 have curvatures with components of the form $\Omega_{aBcd} = O(1/r)$ for non-constant A in which case their intrinsic and extrinsic holonomy groups are not conserved.

Finally from equation (6.1.9)

$$\Omega_{aBcD} = g^{\perp}{}_{DE}(\overline{\nabla}_a A^{\perp E}_{Bc} + A^{\perp F}_{Ba} A^{\perp E}_{Fc}) + g^{\parallel}{}_{ce}(\overline{\nabla}_B A^{\parallel e}_{aD} + A^{\parallel f}_{aB} A^{\parallel e}_{fD})$$

which accounts for ten independent components of the space-time curvature, including $\Omega_{abCD} = \Omega_{CDab}$.

Apart from Ω_{abCD}, we have made no particular assumptions about these components and they would appear to be free to diverge. We shall see that in example 6.3.2, these components are bounded in an ω-frame, but in example 6.3.3, some of these components diverge as $\log r$ for $n = 2$ and $(\log r)/r$ for $n = 1$.

Now if $K^{\parallel}{}_{ab}^{D}$, $K^{\perp}{}^{d}_{AB}$ are C^1-ω-quasi-regular along κ, it follows as above that $\omega \sim \overline{\omega}$ and $\overline{\nabla}_a A^{\perp}{}^{E}_{Bc}$, $\overline{\nabla}_B A^{\parallel}{}^{e}_{aD}$ are C^0-ω-quasi-regular along κ, and hence that Ω_{aBcD} is C^0-ω-quasi-regular along κ.

In fact, if $\overline{\Omega}_{ABCD}$ is C^0-ω-quasi-regular, and $K^{\parallel}{}_{ab}^{D}$, $K^{\perp}{}^{d}_{AB}$ are C^1-ω-quasi-regular, it follows that all components of the space-time curvature are C^0-ω-quasi-regular, and κ in fact terminates at a quasi-regular singularity.

Finally we discuss the components of the Levi-Civita connection ω of an idealised cosmic string in an adapted frame. Let $x \in U$. ω^{c}_{ab}, ω^{C}_{AB} are the Levi-Civita connections of $(S_x, g^{\parallel})$, $(T_x, g^{\perp})$ respectively, and exist uniquely by Theorem 5.2.5, even if S_x, T_x are not surface forming (though in the case of an idealised cosmic string, S_x is surface forming). $\omega^{D}_{ab} = K^{\parallel}{}^{D}_{ab}$, $\omega^{d}_{aB} = A^{\parallel}{}^{d}_{aB}$ are the extrinsic curvatures of S_x and $\omega^{d}_{AB} = K^{\perp}{}^{d}_{AB}$, $\omega^{D}_{Ab} = A^{\perp}{}^{D}_{Ab}$ are the extrinsic curvatures of T_x. The only components of ω without an immediate geometrical interpretation are ω^{D}_{aB} and ω^{d}_{Ab}. ω^{D}_{aB} is used to parallelly propagate vectors normal to S_{tz} in directions tangent to S_{tz}, and for example determines the holonomy obtained by parallelly propagating a normal 2-frame (e_A) round a closed loop in S_{tz}. In order to prove the existence of the intrinsic and extrinsic holonomy groups in section 5.4, we avoided the use of ω^{D}_{aB} and made use of the curvature $\overline{\Omega}_{abCD}$ instead. In fact we can use the second Cartan equation to express $\overline{\Omega}_{abCD}$ in terms of ω^{D}_{aB}

$$\overline{\Omega}_{abC}{}^{D} = \partial_a \omega^{D}_{bC} - \partial_b \omega^{D}_{aC} - c^{e}_{ab} \omega^{D}_{eC} + \omega^{D}_{aE} \omega^{E}_{bC} - \omega^{D}_{bE} \omega^{E}_{aC}$$

where the structure coefficients c^k_{ij} are given by $[e_i, e_j] = c^k_{ij} e_k$. Since S_x is surface forming, $c^D_{ab} = 0$ and if T_x is surface forming, $c^d_{AB} = 0$.

ω^d_{Ab} can be used to parallelly propagate vectors tangent to S_{tz} in directions normal to S_{tz}, and for example can be used to parallelly propagate a 2-frame (e_a) tangent to S_{tz} along a curve "parallel" to the singularity. However we would not expect (e_a) to have a well defined parallel propagate along such a curve as $r \to 0$.

6.3 Examples

We now present some examples of idealised cosmic strings. Our first example is the *4-cone*

$$ds^2 = -dt^2 + dr^2 + A^2 r^2 d\theta^2 + dz^2 \tag{6.3.1}$$

where (t, r, θ, z) are cylindrical polar coordinates defined on $M = \mathbb{R}^4 - \{r = 0\}$, $0 \leq \theta < 2\pi$, and A is a constant. The 4-cone has a good 2-dimensional timelike quasi-regular singularity at $r = 0$ in the sense of chapter 1 and we may therefore apply the theorems of section 1.6 to it. It is locally flat with curvature everywhere zero.

We may also think of the 4-cone as an idealised cosmic string. The space-time can be foliated by $t =$ constant, $z =$ constant spacelike 2-surfaces given in our coordinate system by

$$S_{tz}(r, \theta) = \{(t, r, \theta, z) \mid t, z \text{ constant}\}$$

ruled by space-time geodesics

$$\phi_{t\theta z}(r) = (t, r, \theta, z).$$

Each S_{tz} has induced metric

$$ds^2 = dr^2 + A^2 r^2 d\theta^2$$

and thus each $(S_{tz}, g^{\|})$ is a 2-cone with a good quasi-regular singularity at $r = 0$. The singularity can be considered to be a timelike 2-surface with intrinsic metric

$$ds^2 = -dt^2 + dz^2$$

and since $g_{tr} = g_{zr} = 0$, the S_{tz} surfaces can be considered to be normal to the singularity.

Nonetheless, we are more interested in idealised cosmic strings which have curvature singularities and to which the theorems of section 1.6 cannot be applied.

Our second example is the *dynamic cone*. This has metric

$$ds^2 = -dt^2 + dr^2 + A^2(t,z)r^2 d\theta^2 + dz^2 \tag{6.3.2}$$

where as before (t, r, θ, z) are cylindrical polar coordinates defined on $M = \mathbb{R}^4 - \{r = 0\}$ and $0 \leq \theta < 2\pi$. The dynamic cone is similar to the 4-cone except that the angular deficit $2\pi(1-A)$ varies as a function of t and z. Unlike the 4-cone, which has a quasi-regular singularity at $r = 0$, the dynamic cone has a curvature singularity at $r = 0$ for non-constant A.

The dynamic cone space-time can be foliated by the same spacelike 2-surfaces

$$S_{tz}(r, \theta) = \{(t, r, \theta, z) \mid t, z \text{ constant}\}.$$

Each S_{tz} has induced metric

$$ds^2 = dr^2 + A^2 r^2 d\theta^2$$

where A is constant on S_{tz}. Thus each $(S_{tz}, g^{\|})$ is a 2-cone with a good quasi-regular singularity at $r = 0$. The singularity can be considered to be a timelike 2-surface with intrinsic metric

$$ds^2 = -dt^2 + dz^2$$

and since $g_{tr} = g_{zr} = 0$, the S_{tz} surfaces can be considered to be normal to the singularity.

We now make a choice of adapted (pseudo-orthonormal) frame

$$(e_i) = (e_t, e_r, e_\theta, e_z) = (\partial_t, \partial_r, \frac{1}{A(t,z)r}\partial_\theta, \partial_z).$$

with respect to which the metric has components $\eta_{ij} = \mathrm{diag}(-1,1,1,1)$ and the Levi-Civita connection has components

$$\omega^{\theta}_{\ \theta t} = (\partial_t A)/A \quad \omega^{t}_{\ \theta\theta} = (\partial_t A)/A \quad \omega^{\theta}_{\ \theta z} = (\partial_z A)/A \quad \omega^{z}_{\ \theta\theta} = -(\partial_z A)/A$$

$$\omega^{\theta}_{\ \theta r} = 1/r \quad \omega^{r}_{\ \theta\theta} = -1/r$$

with all other components being zero. Now since $\nabla_{e_r} e_r = \omega^{i}_{\ rr} e_i = 0$ it follows that the S_{tz} surfaces are ruled by space-time geodesics

$$\phi_{t\theta z}(r) = (t, r, \theta, z).$$

Thus the dynamic cone satisfies all the conditions required of an idealised cosmic string as given in section 4.2.

With respect to (e_i) the projected connection has components

$$\overline{\omega}^{\theta}_{\ \theta r} = 1/r \quad \overline{\omega}^{r}_{\ \theta\theta} = -1/r$$

with other components being zero.

Let κ be a curve of $\overline{\omega}$-finite b-length lying in a particular S_{tz} and terminating at $r = 0$. We would like to examine the behaviour of certain tensors in frames parallelly propagated along κ, but the adapted frame (e_i) is not necessarily parallel along κ with respect to either ω or $\overline{\omega}$. However $\overline{\nabla}_{e_a} e_B = \overline{\omega}^{i}_{\ aB} e_i = 0$ where lower case indices range over $\{r, \theta\}$ and upper case indices range over $\{t, z\}$ and so $(e_A) = (e_t, e_z)$ is parallel with respect to $\overline{\omega}$ along κ. If $(\overline{e}_i)$ is an adapted $\overline{\omega}$-frame along κ and

$$\overline{e}_i = l^{j}_{\ i} e_j$$

it follows that l_i^j is a rotation about e_t, e_z and is therefore bounded. In other words, the adapted frame (e_i) is related to an adapted frame parallelly propagated along κ with respect to $\overline{\kappa}$ by a bounded transformation.

We now examine the extrinsic curvatures $K^{\parallel}$, $K^{\perp}$ of the dynamic cone. $K^{\parallel}{}^{D}_{ab} = \omega^{D}_{ab}$ and thus

$$K^{\parallel}{}^{t}_{\theta\theta} = \omega^{t}_{\theta\theta} = (\partial_t A)/A \quad K^{\parallel}{}^{z}_{\theta\theta} = \omega^{z}_{\theta\theta} = -(\partial_z A)/A$$

with other components zero. Hence the components of $K^{\parallel}$ in the frame (e_i) are constant on any S_{tz} surface and therefore bounded by our above comments in any $\overline{\omega}$-frame along κ. Since the connection difference $\sigma = \overline{\omega} - \omega$ obeys

$$K^{\parallel}{}^{k}_{ij} = g^{\parallel}{}^{l}_{i}\sigma^{k}_{lj}$$

it follows by Theorem 2.3.13 that $\omega \sim \overline{\omega}$ along κ and the components of $K^{\parallel}$ will be bounded in any ω-frame along κ and κ will in fact have ω-finite b-length.

The tangent spaces $T = (TS_{tz})^{\perp}$ normal to the S_{tz} 2-surfaces are surface forming, being tangent to the timelike 2-surfaces given in our coordinate system by

$$S_{r\theta}(t, z) = \{(t, r, \theta, z) \mid r, \theta \text{ constant}\}.$$

They have extrinsic curvature $K^{\perp}$ which with respect to (e_i) have components $K^{\perp}{}^{d}{}_{AB} = \omega^{d}{}_{AB} = 0$. The dynamic cone is therefore (strongly) totally geodesic in the sense of section 6.2. This is consistent with the fact that $\nabla_{e_A} e_B = \omega^{i}{}_{AB} e_i = 0$ and therefore the 2-frame $(e_A) = (e_t, e_z)$ remains tangent to the $S_{r\theta}$ 2-surfaces under parallel propagation by ω along any curve lying in an $S_{r\theta}$ 2-surface. We note also that $\partial_\theta g_{ij} = 0$ which, if $\{r = 0\}$ were a regular part of the space-time, would also imply that $\{r = 0\}$ was totally geodesic.

Let t_0, z_0 be constants and consider the 4-cone

$$ds^2 = -dt^2 + dr^2 + A^2(t_0, z_0)r^2 d\theta^2 + dz^2$$

which coincides with the dynamic cone on the spacelike 2-surface $S_{t_0 z_0}$. We can choose an adapted frame for this 4-cone

$$(\tilde{e}_i) = (\tilde{e}_t, \tilde{e}_r, \tilde{e}_\theta, \tilde{e}_z) = (\partial_t, \partial_r, \frac{1}{A(t_0, z_0)}\partial_\theta, \partial_z)$$

which coincides with (e_i) on $S_{t_0 z_0}$. With respect to $(\tilde{e}_i)$ the 4-cone has Levi-Civita connection

$$\tilde{\omega}^\theta_{\theta r} = 1/r \quad \tilde{\omega}^r_{\theta\theta} = -1/r$$

from which it follows that $\overline{\omega}$ and $\tilde{\omega}$ also coincide on $S_{t_0 z_0}$. Hence the two connections generate the same holonomy on loops restricted to lie in $S_{t_0 z_0}$. Thus given a lift $\overline{\kappa}$ of κ in the adapted frame bundle $L(S,T)$ where $S = TS_{tz}$, $T = (TS_{tz})^\perp$, the intrinsic holonomy group defined in section 5.4 is given by

$$H_{\overline{\kappa}}(S_{tz}, \overline{\nabla}) = \{L \in L^\uparrow_+(4) \,|L \text{ acts on } L(S) \text{ as a rotation through } 2\pi n(1-A),$$
$$n \in \mathbb{Z}; \text{ and on } L(T) \text{ as the identity}\}.$$

In other words, the elements of s-holonomy generated using the projected connection on lassos restricted to lie in a particular S_{tz} are rotations through multiples of $2\pi(1 - A(t,z))$ with the singularity as the axis.

We now examine the curvature of the dynamic cone. With respect to (e_i) the Riemann tensor has components

$$\Omega_{t\theta t\theta} = -(\partial_t^2 A)/A \quad \Omega_{z\theta t\theta} = -(\partial_z \partial_t A)/A \quad \Omega_{z\theta z\theta} = -(\partial_z^2 A)/A$$

$$\Omega_{t\theta r\theta} = -(\partial_t A)/Ar \quad \Omega_{z\theta r\theta} = -(\partial_z A)/Ar$$

with other independent components being zero. In particular

$$\Omega_{t\theta r}{}^\theta = \Omega_{z\theta r}{}^\theta = O(r^{-1}) \qquad \partial_t A, \partial_z A \neq 0.$$

Since the adapted frame (e_i) is related to a $\overline{\omega}$-frame along κ by a bounded transformation, and since $\omega \sim \overline{\omega}$ along κ it follows by Proposition 2.3.7 that in an ω-frame along κ some of the components of the form Ω_{aBcd} obey

$$\Omega_{aBcd} = O(u^{-1}) \qquad \partial_t A, \partial_z A \neq 0$$

where u measures b-length along κ such that $u \longrightarrow 0$ as $r \longrightarrow 0$. Hence $\{r = 0\}$ is a curvature singularity for non-constant A.

The Ricci tensor $R_{ij} = R_{ikj}{}^{k}$ has components

$$R_{tt} = -(\partial_t^2 A)/A \quad R_{tz} = -(\partial_z \partial_t A)/A \quad R_{zz} = -(\partial_z^2 A)/A$$

$$R_{\theta\theta} = (\partial_t A)/A - (\partial_z A)/A$$

$$R_{tr} = -(\partial_t A)/Ar \quad R_{zr} = -(\partial_z A)/Ar$$

with other components being zero. Thus for non-constant A the space-time is not vacuum and the Ricci tensor is singular. It can also be shown that for non-constant A the Weyl tensor is also singular.

The loop space $\Omega_\kappa(S_{tz})$ defined in section 5.4 is non-empty. In particular, it contains the map

$$\gamma : (s, u) \longmapsto (t, r = u, \theta = 2\pi s, z)$$

working in (t, r, θ, z) coordinates, where t, z are constant. The 2-surface S_{tz} is regular with respect to κ, as we now show.

First of all we note that $\Omega_{abCD} = 0$. We also recall that the components $K^{\parallel}{}_{ab}^{D}$ of the extrinsic curvature of the S_{tz} are bounded in an ω-frame along any curve of ω-finite b-length terminating at $r = 0$. In fact, since the components of $K^{\parallel}$ are constant in the adapted frame (e_i), and (e_i) is related to any adapted $\overline{\omega}$-frame by a rotation, it follows that we may find a uniform bound for $K^{\parallel}$ required for S_{tz} to be regular with respect to each $\gamma \in \Omega_\kappa(S)$. Therefore S_{tz} is regular with respect to κ.

We saw above that the intrinsic holonomy groups exist. Since S_{tz} is regular with respect to κ, the existence of the intrinsic holonomy groups (though not the fact that they act on $L(T)$ as the identity for lifts of κ in the adapted frame bundle $L(S, T)$) and the extrinsic holonomy groups also follows from Theorem 5.4.11.

Nonetheless the most notable feature of the dynamic cone is that, for non-constant A, the extrinsic holonomy is not conserved. It follows that given $\gamma_0 \in$

$\Omega_{\kappa_0}(S_0)$, $\gamma_1 \in \Omega_{\kappa_1}(S_1)$ which generate non-trivial elements of holonomy, we cannot find a well behaved homotopy ρ from γ_0 to γ_1. This is because there exist components of the curvature of the form Ω_{aBcd} for which in an ω-frame along curves of ω-finite b-length lying in the spacelike 2-surfaces

$$\Omega_{aBcd} = O(u^{-1})$$

where u measures b-length such that $u \to 0$ as $r \to 0$. Hence condition (b) in the definition of well behaved fails to be satisfied.

Our third example has metric

$$ds^2 = -\Omega^2(r)dt^2 + dr^2 + A^2(t,z)r^2d\theta^2 + \Omega^2(r)dz^2 \tag{6.3.3}$$

where as before (t, r, θ, z) are cylindrical polar coordinates defined on $M = \mathbb{R}^4 - \{r = 0\}$ and $0 \leq \theta < 2\pi$. This space-time can be foliated by the same spacelike 2-surfaces

$$S_{tz}(r,\theta) = \{(t,r,\theta,z) \mid t, z \text{ constant}\}$$

where again the S_{tz} surfaces have induced metric

$$ds^2 = dr^2 + A^2r^2d\theta^2.$$

Provided $\lim_{r\to 0}\Omega(r) = 1$, the singularity at $\{r = 0\}$ can be considered to be a timelike 2-surface with intrinsic metric

$$ds^2 = -dt^2 + dz^2$$

and, since $g_{tr} = g_{zr} = 0$, normal to the S_{tz} 2-surfaces.

We now let

$$\Omega^2(r) = 1 + r^n(\log r) \qquad n = 1, 2$$

and choose an adapted frame

$$(e_i) = (e_t, e_r, e_\theta, e_z) = (\frac{1}{\Omega}\partial_t, \partial_r, \frac{1}{A(t,z)r}\partial_\theta, \frac{1}{\Omega}\partial_z).$$

With respect to (e_i) the Levi-Civita connection has components

$$\omega^t_{\ tr} = -\frac{1}{2\Omega^2} r^{n-1}(n(\log r)+1) \quad \omega^r_{\ tt} = -\frac{1}{2\Omega^2} r^{n-1}(n(\log r)+1)$$

$$\omega^r_{\ zz} = -\frac{1}{2\Omega^2} r^{n-1}(n(\log r)+1) \quad \omega^z_{\ zr} = \frac{1}{2\Omega^2} r^{n-1}(n(\log r)+1)$$

$$\omega^t_{\ \theta\theta} = -(\partial_t A)/A\Omega \quad \omega^\theta_{\ \theta t} = -(\partial_t A)/A\Omega \quad \omega^\theta_{\ \theta z} = (\partial_z A)/A\Omega \quad \omega^z_{\ \theta\theta} = -(\partial_z A)/A\Omega$$

$$\omega^\theta_{\ \theta r} = 1/r \quad \omega^r_{\ \theta\theta} = -1/r$$

with other components zero.

As before $\nabla_{e_r} e_r = \omega^i_{\ rr} e_r = 0$ and so the S_{tz} 2-surfaces are ruled by space-time geodesics $\phi_{t\theta z}(r) = (t, r, \theta, z)$. Thus this example satisfies the conditions required of an idealised cosmic string.

The projected connection has components

$$\overline{\omega}^\theta_{\ \theta r} = 1/r \quad \overline{\omega}^r_{\ \theta\theta} = -1/r$$

with other components zero.

Let κ be a curve of $\overline{\omega}$-finite b-length lying in a particular S_{tz} and terminating at $r = 0$. As before, because $\overline{\nabla}_{e_a} e_B = \overline{\omega}^i_{\ aB} e_i = 0$, where again lower case indices range over $\{r, \theta\}$ and upper case indices range over $\{t, z\}$, $(e_A) = (e_t, e_z)$ is parallel with respect to $\overline{\omega}$ along κ. Hence if $(\overline{e}_i)$ is an adapted $\overline{\omega}$-frame along κ and

$$\overline{e}_i = l^j_i e_j$$

it follows that l^j_i is a rotation about e_t, e_z and is therefore bounded. Hence the adapted frame (e_i) is related to an adapted frame parallelly propagated along κ with respect to $\overline{\kappa}$ by a bounded transformation.

We now examine the extrinsic curvatures $K^{\parallel}$, $K^{\perp}$. The extrinsic curvature $K^{\parallel}$ of the S_{tz} surfaces has components with respect to (e_i)

$$K^{\parallel t}_{\ \ \theta\theta} = \omega^t_{\ \theta\theta} = -(\partial_t A)/A\Omega \quad K^{\parallel z}_{\ \ \theta\theta} = -(\partial_z A)/A\Omega$$

with other components zero. It follows that the components of $K^{\parallel}$ remain bounded in a $\overline{\omega}$-frame along κ and $\omega \sim \overline{\omega}$. Hence κ has ω-finite b-length and the components of $K^{\parallel}$ remain bounded in an ω-frame along κ.

The tangent spaces normal to the S_{tz} 2-surfaces have extrinsic curvature $K^{\perp}$ the components of which are

$$K^{\perp r}_{\ tt} = \omega^r_{tt} = -\frac{1}{2\Omega^2} r^{n-1}(n(\log r) + 1) \quad K^{\perp r}_{\ zz} = \omega^r_{zz} = -\frac{1}{2\Omega^2} r^{n-1}(n(\log r) + 1)$$

with respect to (e_i).

Now if $n = 1$ then $K^{\perp r}_{\ tt}, K^{\perp r}_{\ zz} \longrightarrow \infty$ as $r \longrightarrow 0$ and hence the components of $K^{\perp}$ with respect to $(\tilde{e}_i)$ diverge as $r \longrightarrow 0$, and we therefore have an example of an idealised cosmic string which cannot be said to be totally geodesic.

If on the other hand $n = 2$ then $K^{\perp r}_{\ tt}, K^{\perp r}_{\ zz} \longrightarrow 0$ as $r \longrightarrow 0$, and hence the components of $K^{\perp}$ with respect to an ω-frame tend to zero as $r \longrightarrow 0$, and the singularity is (weakly) totally geodesic.

Thus the extrinsic curvatures of the tangent spaces normal to the S_{tz} 2-surfaces, measured in a frame parallelly propagated towards the singularity with respect to ω, diverge if $n = 1$ but tend to zero if $n = 2$. This arises because if $n = 1$, $\Omega(r)$ has a C^0 limit as $r \longrightarrow 0$, but fails to have a C^1 limit. If $n = 2$, $\Omega(r)$ has both a C^0 and a C^1 limit as $r \longrightarrow 0$, but fails to have a C^2 limit.

We now examine the Riemann tensor in the frame (e_i), recalling that the transformation between (e_i) and an ω-frame remains bounded as $r \longrightarrow 0$. If $n = 2$

$$\Omega_{trtr} \simeq \Omega_{t\theta t\theta} \simeq \log r \quad \Omega_{zrzr} \simeq \Omega_{z\theta z\theta} \simeq -\log r$$

$$\Omega_{t\theta r\theta} \simeq -(\partial_t A)/Ar \quad \Omega_{z\theta r\theta} \simeq -(\partial_z A)/Ar$$

with other independent components zero or bounded as $r \longrightarrow 0$. Hence even if $A \equiv 1$ this space-time is singular . If A is constant, then no component of the Ω_{ijkl} diverges faster than $\log r$, where r measures b-length along the radial geodesics $\phi_{t\theta z}(r)$. If A is not constant however, some components of Ω_{ijkl} diverge as $1/r$.

If $n = 1$, $\Omega_{t\theta t\theta} \simeq -\Omega_{z\theta z\theta} \simeq (\log r)/2r$, whether or not A is constant, and in this case we recall that the singularity is not totally geodesic.

We note that again, whether $n = 1$ or $n = 2$, $\Omega_{abCD} = 0$. Since the components $K\|^D_{ab}$ of the extrinsic curvature of the S_{tz} are bounded in an ω-frame along any curve of ω-finite b-length terminating at $r = 0$, it follows exactly as in the case of the dynamic cone that we may find a uniform bound for $K\|$ required for S_{tz} to be regular with respect to each $\gamma \in \Omega_\kappa(S)$ and therefore S_{tz} is regular with respect to κ. Hence by Theorem 5.4.11, the intrinsic and extrinsic holonomy groups exist.

We now show that, as with the dynamic cone, the intrinsic holonomy groups exist and, for lifts of κ by $\overline{\omega}$ in the adapted frame bundle, consist of rotations through multiples of $2\pi(1 - A(t, z))$ with the singularity as the axis.

Let t_0, z_0 be constants and consider the 4-cone

$$ds^2 = -dt^2 + dr^2 + A^2(t_0, z_0)r^2 d\theta^2 + dz^2$$

with adapted frame

$$(\tilde{e}_i) = (\tilde{e}_t, \tilde{e}_r, \tilde{e}_\theta, \tilde{e}_z) = (\partial_t, \partial_r, \frac{1}{A(t_0, z_0)}\partial_\theta, \partial_z)$$

with respect to which the Levi-Civita connection is

$$\tilde{\omega}^\theta_{\theta r} = 1/r \quad \tilde{\omega}^r_{\theta\theta} = -1/r.$$

On the $S_{t_0 z_0}$ 2-surface we have

$$e_t = \frac{1}{\Omega}\tilde{e}_t \qquad e_r = \tilde{e}_r \qquad e_\theta = \tilde{e}_\theta \qquad e_z = \frac{1}{\Omega}\tilde{e}_z.$$

Hence in the adapted frame (e_i)

$$\tilde{\omega}^\theta_{\theta r} = 1/r \quad \tilde{\omega}^r_{\theta\theta} = -1/r$$

and in fact $\tilde{\omega}$ and $\overline{\omega}$ coincide on $S_{t_0 z_0}$ and the two connections generate the same holonomy.

Again, for non-constant A the intrinsic and extrinsic holonomy groups are not conserved along the singularity. For $n = 2$ the curvature has components of the form $\Omega_{aBcd} = O(r^{-1})$ and we will not be able to find well behaved homotopies connecting lassos in different spacelike 2-surfaces.

For $A = 1$ we see that the intrinsic and extrinsic holonomy groups are conserved. This holds even for $n = 1$ which is not totally geodesic. Thus we have an example of a singularity which has conserved holonomy but which is not totally geodesic.

Hence, in the case of an idealised cosmic string, conservation of holonomy neither implies nor is implied by the string being totally geodesic.

The examples we have looked at might lead us to conjecture that an idealised cosmic string whose curvature is weaker than $1/r$, where r measures b-length along any curve of finite b-length terminating at the singularity, has conserved intrinsic and extrinsic holonomies. However by Corollary 5.5.4, we in fact require curvature components of the form $\Omega_{aBcd}, \Omega_{AbCD}$ to be bounded in an ω-frame along curves of ω-finite b-length terminating at $r = 0$ and lying in the preferred spacelike 2-surfaces, in order for the intrinsic and extrinsic holonomies to be conserved. This suggests that there may exist examples of idealised cosmic strings with curvature weaker than $1/r$ for which the intrinsic and extrinsic holonomies are not conserved.

6.4 Block diagonalisation

In this section we consider the following problem: given a 4-dimensional pseudo-Riemannian manifold (M, g), we ask if it is possible to find a C^∞ atlas for M such that in each chart $(x^i) = (x^0, x^1, x^2, x^3)$ the metric is *block diagonal*

$$g = \begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{32} & g_{33} \end{pmatrix}$$

where $g_{02} = g_{03} = g_{12} = g_{13} = 0$ and $g_{20} = g_{21} = g_{30} = g_{31} = 0$.

If (M, g) admits such an atlas, it has the very nice property that given $x \in M$, there exists a neighbourhood U of x which can be foliated by two orthogonal families of 2-surfaces. Conversely, if a neighbourhood U of $x \in M$ can be foliated by two orthogonal families of 2-surfaces, we can choose coordinates (x^0, x^1) on one member of one of the families and (x^2, x^3) on each member of the other family, giving coordinates $(x^i) = (x^0, x^1, x^2, x^3)$ on U in which the metric is block diagonal.

Let (x^i) be a coordinate system. The metric may not be block diagonal in this coordinate system, however it may be possible find a change of coordinates $y^i = y^i(x^j)$ such that in this new coordinate system, the metric is block diagonal. Such a change of coordinates would set four independent components of the metric to zero. Since M is 4-dimensional, $y^i = y^i(x^j)$ involves four functions and it would therefore seem likely, on function counting grounds, that such a change of coordinates is possible.

Such a change of coordinates would eliminate some of the gauge freedom of the metric. It would also provide a convenient coordinate system in which to do calculations. In particular, the two orthogonal families of 2-surfaces provided by this coordinate system would have well defined intrinsic geometries and it would be possible to express the properties of (M, g) in terms of the intrinsic geometrical properties of the 2-surfaces and their extrinsic curvatures.

Now a space-time which contains an idealised cosmic string as described in section 4.2 has a preferred foliation $\{S_{tz}\}$ of spacelike 2-surfaces which are considered to be normal to the singularity. In the definition we gave of an idealised cosmic string, we required that the S_{tz} 2-surfaces be ruled by radial space-time geodesics, in the hope that this would make them unique. The 2-dimensional tangent spaces $\{T_x\}$ normal to the S_{tz} 2-surfaces will not in general however be surface forming. The question arises whether it would be possible to choose a different foliation of spacelike 2-surfaces S'_{tz}, which could also be considered to be normal to the singularity, such that the tangent spaces normal to the spacelike 2-surfaces form a foliation of timelike 2-surfaces $S'_{r\theta}$. Given such a foliation it would be possible to choose coordinates (x^i) in which ∂_0, ∂_1 were tangent to $S'_{r\theta}$ and ∂_2, ∂_3 were tangent to S'_{tz}. In this coordinate system, the metric would be block diagonal

$$g = \begin{pmatrix} g^{\perp} & 0 \\ 0 & g^{\parallel} \end{pmatrix}$$

where $g^{\perp}$ would be the metric induced on $S'_{r\theta}$ and $g^{\parallel}$ would be the metric induced on S'_{tz}.

This would be a very natural coordinate system in which to describe an idealised cosmic string. Since the $S'_{r\theta}$ 2-surfaces have well defined intrinsic geometries and are normal to the preferred spacelike 2-surfaces S'_{tz}, it may be more natural to define the intrinsic geometry of the string as a suitably defined limit of these geometries.

We first consider a related problem.

Theorem 6.4.1. *Let (M, g) be a smooth 3-dimensional Riemannian manifold. Then there exists a C^∞ atlas for M such that in each chart the metric is diagonal i.e.*

$$g = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} \qquad g_{12} = g_{13} = g_{23} = 0.$$

It would be sufficient to prove that given a chart $(x^i) = (x^1, x^2, x^3)$ in a neighbourhood of $x_0 \in M$, there exists a change of coordinates $y^{i'} = y^{i'}(x^k)$ such that

$$g^{i'j'} = \frac{\partial y^{i'}}{\partial x^k}\frac{\partial y^{j'}}{\partial x^l} g^{kl} = 0 \qquad i' \neq j'.$$

Since this gives three differential equations in three functions (y^1, y^2, y^3) it is plausible that a solution exists. However such a solution would not be unique: given a solution $(y^{i'})$ then $(f^{i'}(y^{i'}))$ will also be a solution for any monotone functions (f^1, f^2, f^3), and it is in general difficult to prove existence theorems for differential equations without a unique solution.

A proof of Theorem 6.4.1 is given in [DY] and we give a slightly simplified version.

Proof of Theorem 6.4.1. Let $(\overline{e}_1, \overline{e}_2, \overline{e}_3)$ be an orthonormal frame of vector fields in a neighbourhood of $x_0 \in M$ and let $(\overline{\omega}^1, \overline{\omega}^2, \overline{\omega}^3)$ be the corresponding dual frame of 1-forms, which will also be orthonormal. We want to find a coordinate system (x^1, x^2, x^3) in a neighbourhood of x_0 such that $(\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$ are orthogonal (but not necessarily orthonormal). Hence we want to find an orthonormal dual frame $(\omega^1, \omega^2, \omega^3)$ such that

$$\omega^i = f^i dx^i$$

for some coordinate system (x^i) and scalars f^i, where we do not sum over the index i. Now by Frobenius' theorem (see [DY])

$$\omega^i = f^i dx^i \iff \omega^i \wedge d\omega^i = 0$$

where again we do not sum over the index i. Hence we want to show that there exists a unique solution to the three equations

$$\omega^{\hat{i}} \wedge d\omega^{\hat{i}} = 0 \qquad i = 1, 2, 3$$

where we suspend the summation convention on hatted indices. Setting $\omega^i = a^i_{\ j}\overline{\omega}^j$ where $a^j_i \in SO(3)$ we have

$$a^{\hat{i}}_{\ j}\overline{\omega}^j \wedge d(a^{\hat{i}}_{\ k}\overline{\omega}^k) = 0$$

where we solve for $a^i_{\ j}$. Hence

$$a^{\hat{i}}_{\ j}\overline{\omega}^j \wedge (\partial_l a^{\hat{i}}_{\ k}\overline{\omega}^l \wedge \overline{\omega}^k + a^{\hat{i}}_{\ k} d\overline{\omega}^k) = 0$$

where $\partial_l a^i_{\ k} := \overline{e}_l(a^i_{\ k})$. Using the first Cartan equation

$$d\overline{\omega}^i + \omega^i_{\ j} \wedge \omega^j = 0$$

we have

$$\begin{aligned} 0 &= a^i_{\ j}\overline{\omega}^j \wedge (\partial_l a^i_{\ k}\overline{\omega}^l \wedge \overline{\omega}^k + a^i_{\ k}\overline{\omega}^l \wedge \overline{\omega}^k_{\ l}) \\ &= a^i_{\ j}\overline{\omega}^j \wedge (\partial_l a^i_{\ k}\overline{\omega}^l \wedge \overline{\omega}^k + a^i_{\ k}\overline{\omega}^l \wedge \overline{\omega}^k_{\ ml}\overline{\omega}^m) \\ &= a^i_{\ j}(\partial_l a^i_{\ k} + a^i_{\ m}\overline{\omega}^m_{\ kl})\overline{\omega}^j \wedge \overline{\omega}^l \wedge \overline{\omega}^k \end{aligned}$$

where $\overline{\omega}^i_{\ j} = \overline{\omega}^i_{\ kj}\overline{\omega}^k$ and $\overline{\omega}^i_{\ kj}$ is a scalar. Hence

$$a^i_{\ [j}\partial_l a^i_{\ k]} + a^i_{\ m}a^i_{\ [j}\overline{\omega}^m_{\ kl]} = 0. \tag{6.4.1}$$

We will show that these three quasi-linear first order partial differential equations for $a^i_{\ j}$ have a unique solution. We will require that at $x_0 \in M$, $a^i_{\ j} = \delta^i_{\ j}$ i.e. $(\omega^i)|_{x_0} = (\overline{\omega}^i)|_{x_0}$. Certainly given a solution $a^i_{\ j}$ of (6.4.1) we can set $(\overline{\omega}^i)|_{x_0} = (\omega^i)|_{x_0}$.

Since $a^i_{\ j} \in SO(3)$, $a^i_{\ j} = \exp(\alpha^i_{\ j})$ where $\alpha^i_{\ j} \in L(SO(3))$. Thus $\alpha^i_{\ j}$ will be a 3×3 antisymmetric matrix and $\alpha^1_{\ 2}, \alpha^1_{\ 3}, \alpha^2_{\ 3}$ will parametrise $a^i_{\ j}$. Hence $a^i_{\ j} = \delta^i_{\ j} + \alpha^i_{\ j} + \ldots$. Since $a^i_{\ j}(x_0) = \delta^i_{\ j}$, near x_0 $\alpha^i_{\ j}$ will be "small" and we linearise equation (6.4.1) in terms of $\alpha^i_{\ j}$ to give

$$0 = \delta^i_{\ [j}(\partial_l \alpha^i_{\ k]}) + \text{lower order terms} \tag{6.4.2}$$

where the lower order terms contain no derivatives. In other words we have simply omitted terms which would have made (6.4.2) non-linear. This gives

$$\partial_2\alpha^1_{\ 3} - \partial_3\alpha^1_{\ 2} = \text{lower order terms}$$

$$\partial_3\alpha^2_{\ 1} - \partial_1\alpha^2_{\ 3} = \text{lower order terms}$$

$$\partial_1\alpha^3_{\ 2} - \partial_2\alpha^3_{\ 1} = \text{lower order terms}$$

which we rearrange to give

$$\partial_1\alpha^2_{\ 3} = \text{lower order terms}$$

$$\partial_2 \alpha^1_3 = \text{lower order terms}$$

$$\partial_3 \alpha^1_2 = \text{lower order terms}$$

which is in *diagonal hyperbolic form*, which we define below (see also [DY]). Since the linearisation of (6.4.1) is diagonal hyperbolic, it follows that there exists a smooth solution to (6.4.1) (by Theorems 1.4 and 1.5 quoted in [DY] and Proposition 2.1 proved in [DY]). □

Let $k, n \in \mathbb{N}$. A vector $u = (u_1, \ldots, u_k)$ of functions $u_i = u_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, k$ is said to satisfy a first order partial differential equation in *diagonal hyperbolic form* if

$$\partial_{x_i} u_i + f(u_1, \ldots, u_k) = 0 \qquad i = 1, \ldots, n$$

for a smooth function f linear in $u_1, \ldots, u_k$.

We now attempt to apply the same technique to prove that a 4-dimensional Lorentzian metric can be block diagonalised. Let (M, g) be a space-time and let $x_0 \in M$. Let u, v be a pair of 1-forms which span a 2-dimensional subspace $N < T^*_{x_0}M$. If u', v' are a pair of 1-forms which also span N then

$$u \wedge v = f u' \wedge v'$$

for some constant f. In particular if u, v and u', v' are both orthonormal then

$$u \wedge v = u' \wedge v'.$$

We therefore want to find a dual orthonormal basis (ω^i) and coordinates (x^i) in a neighbourhood of x_0 such that ω^0, ω^1 span the same 2-space as dx^0, dx^1 and ω^2, ω^3 span the same 2-space as dx^2, dx^3. This would ensure that the 2-space spanned by dx^0, dx^1 is orthogonal to the 2-space spanned by dx^2, dx^3 and that the metric is block diagonal. Thus we want

$$\omega^i \wedge \omega^j = f dx^i \wedge dx^j \qquad i, j = 0, 1 \text{ or } 2, 3$$

for some scalar f (which is different for $i, j = 0, 1$ and $i, j = 2, 3$). By Frobenius' theorem

$$\exists x^i, f \text{ such that } \omega^i \wedge \omega^j = f dx^i \wedge dx^j \qquad i, j = 0, 1 \text{ or } 2, 3$$

$$\Longleftrightarrow d(\omega^i \wedge \omega^j) \wedge \omega^k = 0 \qquad i, j, k = 0, 1, 0 \text{ or } 0, 1, 1 \text{ or } 2, 3, 2 \text{ or } 2, 3, 3$$

and hence we wish to show that there exists a solution to the four equations

$$d\omega^0 \wedge \omega^0 \wedge \omega^1 = 0$$

$$d\omega^1 \wedge \omega^0 \wedge \omega^1 = 0$$

$$d\omega^2 \wedge \omega^2 \wedge \omega^3 = 0$$

$$d\omega^3 \wedge \omega^2 \wedge \omega^3 = 0$$

which we write as

$$d\omega^{\hat{i}} \wedge \omega^{\hat{i}} \wedge \omega^j = 0 \qquad i, j = 0, 1 \text{ or } 1, 0 \text{ or } 2, 3 \text{ or } 3, 2$$

where again we suspend the summation convention on hatted indices.

As before let $(\overline{e}_i)$ be an orthonormal frame of vector fields in a neighbourhood of x_0, let $(\overline{\omega}^i)$ be the corresponding dual frame of 1-forms, and let $\omega^i = a^i_{\ j}\overline{\omega}^j$ where $a^i_{\ j} \in L^\uparrow_+(4)$. Hence

$$\begin{aligned} 0 &= d(a^{\hat{i}}_{\ k}\overline{\omega}^k) \wedge (a^{\hat{i}}_{\ l}\overline{\omega}^l) \wedge (a^j_{\ m}\overline{\omega}^m) \\ &= (\partial_n a^{\hat{i}}_{\ k}\overline{\omega}^n \wedge \overline{\omega}^k + a^{\hat{i}}_{\ k}\overline{\omega}^n \wedge \overline{\omega}^k_{\ pn}\overline{\omega}^p) \wedge (a^{\hat{i}}_{\ l}\overline{\omega}^l) \wedge (a^j_{\ m}\overline{\omega}^m) \end{aligned}$$

where as before we use the first Cartan equation $d\overline{\omega}^i + \overline{\omega}^i_{\ j} \wedge \overline{\omega}^j = 0$, $\overline{\omega}^i_{\ j} = \overline{\omega}^i_{\ kj}\overline{\omega}^k$ where $\overline{\omega}^i_{\ kj}$ is a scalar, and $\partial_n a^i_{\ k} := \overline{e}_n(a^i_{\ k})$. Hence

$$(\partial_n a^{\hat{i}}_{\ k} + a^{\hat{i}}_{\ p}\overline{\omega}^p_{\ kn})a^{\hat{i}}_{\ l}a^j_{\ m}\overline{\omega}^n \wedge \overline{\omega}^k \wedge \overline{\omega}^l \wedge \overline{\omega}^m = 0$$

and therefore

$$\partial_{[n} a^{\hat{i}}_{\ k} a^{\hat{i}}_{\ l} a^j_{\ m]} + a^{\hat{i}}_{\ p}\overline{\omega}^p_{\ [kn} a^{\hat{i}}_{\ l} a^j_{\ m]} = 0 \qquad i, j = 0, 1 \text{ or } 1, 0 \text{ or } 2, 3 \text{ or } 3, 2 \tag{6.4.3}$$

which gives us four quasi-linear first order partial differential equations for a^j_i. As before we require that $a^i_j(x_0) = \delta^i_j$ so that $(\omega^i)|_{x_0} = (\overline{\omega}^i)|_{x_0}$.

Since $a^i_j \in L^\uparrow_+(4)$, $a^i_j = \exp(\alpha^i_j)$ where $\alpha^i_j \in L(L^\uparrow_+(4))$ and

$$\alpha^i_j = \begin{pmatrix} 0 & \alpha & \beta & \gamma \\ \alpha & 0 & a & b \\ \beta & -a & 0 & c \\ \gamma & -b & -c & 0 \end{pmatrix}$$

where $\alpha, \beta, \gamma, a, b, c$ parametrise a^i_j. As before $a^i_j = \delta^i_j + \alpha^i_j + \ldots$ and we linearise (6.4.3) in terms of α^i_j to give

$$\delta^{\hat{i}}_{[l}\delta^{j}_{m}\partial_n\alpha^{\hat{i}}_{k]} + \text{lower order terms} \tag{6.4.4}$$

where again lower order terms contain no derivatives. This gives

$$\partial_2\alpha^0_3 - \partial_3\alpha^0_2 = \text{lower order terms}$$

$$\partial_2\alpha^1_3 - \partial_3\alpha^1_2 = \text{lower order terms}$$

$$\partial_0\alpha^2_1 - \partial_1\alpha^2_0 = \text{lower order terms}$$

$$\partial_0\alpha^3_1 - \partial_1\alpha^3_0 = \text{lower order terms}$$

and hence

$$\partial_2\gamma - \partial_3\beta = \text{lower order terms}$$

$$\partial_2 b - \partial_3 a = \text{lower order terms}$$

$$-\partial_0 a - \partial_1\beta = \text{lower order terms}$$

$$-\partial_0 b - \partial_1\gamma = \text{lower order terms.}$$

Unfortunately these equations are not diagonal hyperbolic and the technique used in the proof of Theorem 6.4.1 does not appear to work in this case.

Now we note that there does not exist a unique solution for $(\omega^i) = (a^i_{\;j}\varpi^j)$ since ω^0, ω^1 and ω^2, ω^3 are free to move in their corresponding co-planes. $a^i_{\;j}$ has six degrees of freedom, whereas we have only four equations. It may be for this reason that the above method does not work. However

$$\exp\begin{pmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & \cos c & \sin c \\ 0 & 0 & -\sin c & \cos c \end{pmatrix}$$

which are precisely the transformations which preserve $\omega^0 \wedge \omega^1$ and $\omega^2 \wedge \omega^3$. Curiously, α, c are absent from the four linearised equations (6.4.4) which involve only the four unknowns β, γ, a, b.

An alternative approach is to work directly in terms of the 2-forms $F = \omega^i \wedge \omega^j$ since these characterise the 2-dimensional subspaces of $T^*_x M$ uniquely.

A 2-form F characterises a 2-dimensional subspace of $T^*_x M$ provided that it is simple i.e. $F = u \wedge v$ for some 1-forms u, v. F is simple if and only if

$$F \wedge F = 0. \tag{6.4.5}$$

F will characterise a 2-dimensional timelike subspace of $T^*_x M$ if $F = u \wedge v$ for 1-forms u, v such that $g(u,u) = 1$, $g(v,v) = -1$ so in addition to equation (6.4.5) we have

$$F \wedge *F = *1 \tag{6.4.6}$$

where in an orthonormal frame $*F_{ij} = \frac{1}{2}\varepsilon_{ijkl}F^{kl}$ and $(*1)_{ijkl} = \varepsilon_{ijkl}$. The 2-form $*F$ dual to F will characterise a 2-dimensional subspace of $T^*_x M$ if $*F \wedge *F = 0$ and $*F \wedge F = - * 1$ but these just give equations (6.4.5) and (6.4.6). It turns out that $*F$ is orthogonal to F.

Now in order for F, $*F$ to be surface forming, Frobenius' theorem gives us

$$(*d * F) \wedge F = 0 \tag{6.4.7}$$

$$(*dF) \wedge *F = 0 \tag{6.4.8}$$

Thus we wish to show that there exists a solution to the four equations (6.4.5)–(6.4.8).

Working in an orthonormal frame, equation (6.4.5) gives

$$\frac{1}{2} F_{ij} * F^{ij} = 0$$

and equation (6.4.6) gives

$$\frac{1}{2} F_{ij} F^{ij} = -\frac{1}{2} * F_{ij} * F^{ij} = -1.$$

Choosing a timelike vector field T^i we define

$$E_i = F_{ij} T^j \qquad B_i = *F_{ij} T^j = \frac{1}{2} \varepsilon_{ijkl} T^j F^{kl}$$

and hence

$$F_{ij} = 2E_{[i} T_{j]} + \varepsilon_{ijkl} T^k B^l.$$

Now the first Frobenius equation (6.4.7) may be written

$$\operatorname{div} F \wedge F = 0$$

or

$$J \wedge F = 0$$

where $J = \operatorname{div} F$. We now work with respect to the 3-geometry of the hypersurfaces to which T^i is normal. E_i, B_i are both tangent to the hypersurfaces and so we represent them as 3-vectors $\vec{E}$, $\vec{B}$. We note that the hypersurfaces have positive definite metric. We use $\dot{}$ to denote differentiation along T^i. If we write $J = \operatorname{div} F = (\rho, \vec{j})$ then in terms of $\vec{E}$, $\vec{B}$, ρ and $\vec{j}$ we have

$$\rho \vec{B} + \vec{j} \times \vec{E} = 0$$

and hence

$$(\vec{\nabla}.\vec{E})\vec{B} + (\vec{\nabla} \times \vec{B} - \dot{\vec{E}}) \times \vec{E} = 0 \tag{6.4.9}$$

$$(\vec{\nabla} \times \vec{B} - \dot{\vec{E}}).\vec{B} = 0. \tag{6.4.10}$$

Similarly the second Frobenius equation (6.4.8) may be written

$$(\vec{\nabla}.\vec{B})\vec{E} + (\vec{\nabla} \times \vec{E} - \dot{\vec{B}}) \times \vec{B} = 0 \tag{6.4.11}$$

$$(\vec{\nabla} \times \vec{E} - \dot{\vec{B}}).\vec{E} = 0 \tag{6.4.12}$$

Equation (6.4.5) implies the constraint equation

$$\vec{E}.\vec{B} = 0 \tag{6.4.13}$$

and equation (6.4.6) implies the constraint equation

$$B^2 - E^2 = -1 \Rightarrow E^2 - B^2 = 1. \tag{6.4.14}$$

Now we note that (6.4.10) + (6.4.12) gives

$$(\vec{\nabla} \times \vec{B}).\vec{B} + (\vec{\nabla} \times \vec{E}).\vec{E} = (\dot{\vec{E}.\vec{B}})$$

so the condition that the constraint equation (6.4.13) propagates in the T^i direction is that

$$(\vec{\nabla} \times \vec{B}).\vec{B} = -(\vec{\nabla} \times \vec{E}).\vec{E}. \tag{6.4.15}$$

Now let

$$\vec{E} = E\vec{e} \qquad \vec{B} = B\vec{b}$$

where $\vec{e}$, $\vec{b}$ are unit 3-vectors. Then

$$\vec{\nabla} \times \vec{B} = (\vec{\nabla}B \times \vec{b}) + B\vec{\nabla} \times \vec{b}$$

so

$$\vec{B}.\vec{\nabla} \times \vec{B} = B^2\vec{b}.\vec{\nabla} \times \vec{b}$$

and similarly

$$\vec{E}.\vec{\nabla} \times \vec{E} = E^2 \vec{e}.\vec{\nabla} \times \vec{e}$$

but by equation (6.4.14), $E^2 - B^2 = 1$ so we may write

$$E = \cosh \alpha \qquad B = \sinh \alpha. \tag{6.4.16}$$

Substituting into (6.4.15) gives

$$\sinh^2 \alpha \, \vec{b}.\vec{\nabla} \times \vec{b} = -\cosh^2 \alpha \, \vec{e}.\vec{\nabla} \times \vec{e}$$

and hence

$$\alpha = \tanh^{-1} \left(\left\{ \frac{-\vec{e}.(\vec{\nabla} \times \vec{e})}{\vec{b}.(\vec{\nabla} \times \vec{b})} \right\}^{\frac{1}{2}} \right). \tag{6.4.17}$$

Thus we can calculate E and B in terms of $\vec{e}$ and $\vec{b}$ in such a way that the constraint equations (6.4.13) and (6.4.14) are satisfied for all t. In particular we can choose suitable initial data on the initial data hypersurface $t = 0$ such that in at least in some open neighbourhood

$$(-\vec{e}.(\vec{\nabla} \times \vec{e}))/(\vec{b}.(\vec{\nabla} \times \vec{b})) > 0.$$

If we now look at equation (6.4.9) in terms of E, B, $\vec{e}$, and $\vec{b}$ we get

$$\vec{\nabla}.\vec{E} = \vec{\nabla}E.\vec{e} + E\vec{\nabla}.\vec{e}$$

$$\vec{\nabla} \times \vec{B} = \vec{\nabla}B \times \vec{b} + B\vec{\nabla} \times \vec{b}$$

$$\dot{\vec{E}} = \dot{E}\vec{e} + E\dot{\vec{e}}$$

and hence

$$EB((\vec{\nabla}.\vec{e})\vec{b} + (\vec{\nabla} \times \vec{b}) \times \vec{e}) + B(\vec{\nabla}E.\vec{e})\vec{b} + E(\vec{\nabla}B \times \vec{b}) \times \vec{e} - \dot{E}\vec{e} \times \vec{e} - E^2\dot{\vec{e}} \times \vec{e} = 0.$$

Now $\vec{e} \times \vec{e} = 0$ so if we divide by E^2 and take the cross product with $\vec{e}$ we get

$$(\dot{\vec{e}} \times \vec{e}) \times \vec{e} = (\dot{\vec{e}}.\vec{e})\vec{e} - (\vec{e}.\vec{e})\dot{\vec{e}} = -\dot{\vec{e}}$$

(since $\vec{e}.\vec{e} = 1 \Rightarrow \dot{\vec{e}}.\vec{e} = 0$) and

$$\dot{\vec{e}} = -\frac{B}{E}((\vec{\nabla}.\vec{e})\vec{b} + (\vec{\nabla} \times \vec{b}) \times \vec{e}) \times \vec{e} - \frac{B}{E^2}((\vec{\nabla}E).\vec{e})\vec{b} \times \vec{e} - \frac{1}{E}((\vec{\nabla}B \times \vec{b}) \times \vec{e}) \times \vec{e}.$$

Substituting for E and B from equations (6.4.16) and (6.4.17) gives

$$\dot{\vec{e}} = \text{a function of } \vec{e},\ \vec{b} \text{ and spatial derivatives.} \tag{6.4.18}$$

Similarly from equation (6.4.11) we get

$$\dot{\vec{b}} = \text{a function of } \vec{e},\ \vec{b} \text{ and spatial derivatives.} \tag{6.4.19}$$

Let $k, n \in \mathbb{N}$. A vector $u = (u_1, \ldots, u_k)$ of functions $u_i = u_i(t, x_1, \ldots, x_n)$ for $i = 1, ldots, k$ is said to satisfy a first order partial differential equation of *Kovalevskaya type* ([EgSh]) if

$$\partial_t u = f(t, x_1, \ldots, x_n, \partial_{x_1} u, \ldots, \partial_{x_n} u)$$

with initial data

$$u(0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n) \qquad \partial_t u(0, x_1, \ldots, x_n) = \psi(x_1, \ldots, x_n)$$

for a function f analytic in a neighbourhood of $t = 0$, $x = 0$ and functions ϕ, ψ analytic in a neighbourhood of $x = 0$.

Equations (6.4.18) and (6.4.19) are first order partial differential equation of Kovalevskaya type. It follows by the Cauchy-Kovalevskaya theorem ([EgSh]) that, at least in an open neighbourhood of a point on the initial data hypersurface, we can find a solution of these equations. Thus in this neighbourhood we can find $\vec{e}$ and $\vec{b}$ which together with equations (6.4.16) and (6.4.17) give $\vec{E}$ and $\vec{B}$ which satisfy the Frobenius equations (6.4.9), (6.4.11) and the constraint equations (6.4.13) and (6.4.14). This leaves equations (6.4.10) and (6.4.12) to be satisfied but

$$(6.4.9) \times \vec{B} \Rightarrow (6.4.10) \qquad (6.4.11) \times \vec{E} \Rightarrow (6.4.12)$$

and so in fact (6.4.10) and (6.4.12) are satisfied.

Hence, at least in the analytic case, we can find coordinates in a neighbourhood of any $x \in M$ such that the metric has block diagonal form.

Conclusions and further work

We have in this thesis been looking at certain types of weak singularity in general relativity. These are singularities which can in some sense be said to be mathematically tractable and can be given some kind of geometrical structure.

We started by reviewing quasi-regular singularities. In particular we saw how 2-dimensional timelike quasi-regular singularities may be used to model cosmic strings. These are more usually modelled using weak field theory, in which they may bend on small length scales and form small loops, however this approach ignores the gravitational effects of the string, which for example give rise to its light bending properties. On the other hand using methods of holonomy we saw how 2-dimensional timelike quasi-regular singularities may be considered to be totally geodesic. In addition it may also be shown that cosmic strings modelled in this way are really quite inflexible objects, unable to bend on length scales smaller than the cosmological length scale.

We therefore introduced a class of curvature singularities, more general than quasi-regular singularities, which have some of the properties we would expect of a cosmic string. We called the members of this class idealised cosmic strings. A space-time with such a singularity admits a foliation of spacelike 2-surfaces, each of which has a quasi-regular singularity in the induced metric. The singularity itself has a perfectly regular Lorentzian 2-metric, despite the fact that it is in general a curvature singularity.

We introduced a $2+2$ formalism suited to these idealised cosmic strings and proceeded to analyse them using methods of holonomy. Now in general the singular holonomy groups will not exist for a curvature singularity, but we exhibited in section 2.5 certain stringent conditions under which elements of singular holonomy will exist for a curvature singularity. By using this along with a bound on the extrinsic curvature of the spacelike 2-surfaces and an integral bound on certain components of the curvature we were able to prove the existence of certain intrinsic and extrinsic holonomy groups, obtained by parallelly propagating frames with respect to the projected and full connections on loops restricted to lie in the preferred spacelike 2-surfaces. By placing further conditions on the curvature we were able to show when these groups are conserved along the singularity. We also examined the behaviour of the curvature near the singularity and showed that, even with these bounds, the curvature can diverge.

There is clearly scope to tune the definition of an idealised cosmic string. We chose the spacelike 2-surfaces to be ruled by radial geodesics so that they would be regular if the string were a regular part of the space-time, and in the hope of making them unique. It would be worth asking under what circumstances these 2-surfaces are in fact unique, and whether this matters. The frame approach we used in chapters 5 and 6 to analyse these idealised cosmic strings sits uneasily with the coordinate based definition we gave of an idealised cosmic string in chapter 4. A better relationship could be found between the two.

Now we proved in section 6.4 that, at least in the analytic case, coordinates may be found in the neighbourhood of a point in which the metric has block diagonal form. It would be worth trying to prove this in the smooth case. The question also arises whether a different foliation of spacelike 2-surfaces could be chosen normal to an idealised cosmic string such that the normal tangent spaces were surface forming. These normal surfaces would have a well defined intrinsic geometry and it would seem more natural to describe the geometry of the singularity as a suitable limit of these geometries. We need to be careful, however, to define the properties of a singularity purely in terms of the properties of the space-time itself.

The most obvious question to ask about idealised cosmic strings is whether they can bend on small length scales. It may be the case that they can do so while remaining totally geodesic. Because of the way they would need to deform the geometry near the string to do this, it would be of interest to measure the spanning area of a cosmic string loop and to investigate the volume element near the singularity. It would also be worth trying to find more examples of idealised cosmic strings, in particular ones which were not rotationally symmetric, and ones which appeared to form closed loops. A spinning or rotating cosmic string loop would have an interesting causal structure.

We also discussed conformal singularities in chapter 3, which are a different kind of "weak" singularity. It would also be possible to consider singularities conformal to quasi-regular singularities, or conformal to idealised cosmic strings.

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