UNIVERSITY OF SOUTHAMTON

FACULTY OF MATHEMATICAL STUDIES

GENERAL RELATIVITY

INVARIANT DIFFERENTIAL OPERATORS AND THE EQUIVALENCE PROBLEM OF ALGEBRAICALLY SPECIAL SPACETIMES

by

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Thesis submitted for the degree of Doctor of Philosophy of the University of Southampton

September, 1996

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ABSTRACT

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Many calculations in general relativity are simplified when using a tetrad formalism. As an important example we have the Newman-Penrose (NP) formalism which uses a complex null tetrad as basis for writing all information corresponding to Einstein's equations. However, certain physical problems are best described when the formalism is adapted to the geometry of such physical situations, i.e, when the basis vectors (or spinors) are not completely arbitrary but related to the geometry or physics in some natural way. A well known example is the Geroch-Held-Penrose (GHP) formalism which best describes the geometry of a null 2-surface and which is invariant under the group of spin and boost transformations.

The GHP formalism is ideally suited to situations where two null directions are naturally singled out, but in many physical cases one is faced with only one preferred null direction. As important examples we have null congruences, null hypersurfaces or wave fronts and type N spacetimes.

A formalism which is invariant under null rotations is presented. The fundamental objects are totally symmetric spinors. From this notation we develop a formalism based on a single null direction which is covariant under both spin and boost transformation and null rotations.

Although both formalisms, which we refer to in this thesis as the generalized NP formalism and the generalized GHP formalism, have many other applications

mainly to do with null congruences and null hypersurfaces they are used in here as an application to the equivalence problem of type N spacetimes.

The problem of determining weather two given metrics expressed in different coordinate systems are actually the same metric. i.e, can be mapped into each other by a coordinate transformation is the well known equivalence problem of metrics. The theoretical resolution of this problem was originally provided by Cartan and later refined by Karlhede who provided the useful Karlhede algorithm of classifying different Petrov types of spacetimes.

In this thesis we apply the newly developed generalized GHP formalism to the Karlhede algorithm of Petrov type N spacetimes (vacuum and non vacuum). It turns out that such formalism is quite appropriate in this case simplifying the calculations involved and lowering the number of covariant derivatives of the curvature tensor one needs to calculate in order to completely classify such solutions.

In the final chapter we review the work done on the relationship between curvature and metric. We discuss the relationship of this work to that of Karlhede and possible ways of using this work and that of Karlhede to improve on the algorithm of certain Petrov types of Einstein solutions.

Contents

ACKNOWLEDGEMENTS

Firstly I would like to thank my supervisor, Dr. James Vickers for his patience, support and excellent supervision. His help has been essential to the outcome of this thesis.

I also wish to thank my advisor Dr. Ray d' Inverno and Prof. Chris Clarke, for their friendly support throughout my PhD.

Special thanks to Mr. Ray Brown, for his endless help in the computer room.

Thanks to my fellow research mates for having provided me with a very friendly environment in which to work in.

I also wish to thank Ilia, Rita and Pilar for their friendship and their help in every day life. Muchas gracias chiiiiiiiiiiicas por los momentos dentro y fuera del staff club!!!

Finally I gratefully acknowledge the financial support of JNICT/Programa Ciência/Programa Praxis XXI of Portugal, in the form of PhD Studentship no. BD/2740/93.

Chapter 1

The Equivalence Problem of Metrics

1.1 Introduction

Here we review the equivalence problem of metrics and discuss its solution in detail. The description given in this chapter closely follows that of Karlhede [18] although an effort is made to explain as clearly as possible the various steps.

We also give an alternative way of looking at the equivalence problem by working with the frame bundle LM rather than the manifold M, which will be described in section 3.

In what follows we will use latin letters to denote indices corresponding to frame components and greek letters to denote indices corresponding to coordinate components. Lower case latin letters will run from 1 to n. Upper case latin letters will label the parameters of the proper Lorentz group and will therefore run from 1 to $\frac{1}{2}n(n-1)$. Covariant derivatives will be denoted by a semicolon (;), partial derivative by a comma (,) and directional derivative by a bar (|). The letter n will be used to denote the dimension of the manifold M.

Let g and $\tilde{g}$ denote two metrics on manifolds M and $\tilde{M}$. Then g and $\tilde{g}$ are said to be locally equivalent if, and only if, there is a coordinate transformation

$$\tilde{x}^{\mu} = \tilde{x}^{\mu}(x^{\nu}) \tag{1.1.1}$$

that maps $g_{\mu\nu}$ into $\tilde{g}_{\mu\nu}$,i.e,

$$g_{\mu\nu}(x^{\delta}) = \frac{\partial \tilde{x}^{\rho}}{\partial x^{\mu}} \frac{\partial \tilde{x}^{\sigma}}{\partial x^{\nu}} \tilde{g}_{\rho\sigma}(\tilde{x}^{\delta}) \tag{1.1.2}$$

If instead of using a coordinate notation we choose to use a tetrad notation, the two given tensor fields g and $\tilde{g}$ are given by:

$$g = \eta_{ij} \omega^i \otimes \omega^j \tag{1.1.3}$$

$$\tilde{g} = \eta_{ij}\tilde{\omega}^i \otimes \tilde{\omega}^j \tag{1.1.4}$$

e_i will denote n linearly independent vector fields, defined over a region U of M and ω^i the dual basis of 1-forms defined by :

$$< e_i, \omega^j >= \delta_i^{\ j} \tag{1.1.5}$$

and η_{ij} is the constant frame metric:

$$e_i \cdot e_j = \eta_{ij} \tag{1.1.6}$$

Two geometries given by g and $\tilde{g}$ in regions U and $\tilde{U}$ respectively are equivalent if, and only if, there is a pointwise identification between points P in U and $\tilde{P}$ in $\tilde{U}$ such that :

$$g_p = \tilde{g}_{\tilde{p}} \tag{1.1.7}$$

where g_p denotes the metric tensor at P.

We should point out that this study is done locally for regions U and $\tilde{U}$ with local coordinates x^μ and $\tilde{x}^\mu$ and therefore determines if the spaces are locally equivalent.

If we consider the case where each manifold M and $\tilde{M}$ has the same constant frame metric η_{ij} then from 1.1.3 and 1.1.4 we have that:

$$\omega^i{}_p = \tilde{\omega}^i_{\tilde{p}} \tag{1.1.8}$$

implies $g_p = \tilde{g}_{\tilde{p}}$.

However, since there exists a group of linear transformations of ω^i

$$\hat{\omega}^i = b^i{}_j\omega^j \tag{1.1.9}$$

which leaves η_{ij} invariant:

$$b^i{}_m \eta_{ij} b^j{}_n = \eta_{nm} \tag{1.1.10}$$

and therefore also leaves g pointwise invariant:

$$\hat{g} = \eta_{ij}\hat{\omega}^i \otimes \hat{\omega}^j$$

$$= \eta_{ij} b^i{}_m\omega^m \otimes b^j{}_n\omega^n$$

$$= \eta_{mn}\,\omega^m \otimes \omega^n$$

$$= g$$

we cannot say that $g_p = \tilde{g}_{\tilde{p}}$ implies $\omega^i{}_p = \tilde{\omega}^i_{\tilde{p}}$.

So we have that the 1-forms only need to be equal up to transformations $b^i{}_j$ which leave η_{ij} invariant to make the metrics equal. The set of all such transformations $b^i{}_j$ form a group G which has a continuous subgroup of frame orientation and time direction preserving transformations, i.e "rotations" of dimension $\frac{n(n-1)}{2}$ together with a finite number of discrete transformations, n being the dimension of M. In particular, when $n = 4$ and η_{ij} is the Lorentz metric, G is the six dimensional homogeneous Lorentz group and the continuous subgroup of "rotations" is the proper Lorentz group $\mathcal{L}_+^\uparrow$ and the discrete transformations are space and time inversions. We can therefore establish the following lemma:

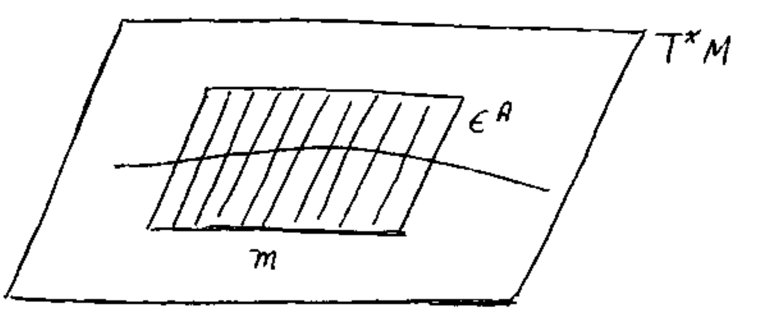
Lemma 1.1.1 *Two geometries are equivalent if, and only if, there exists a pointwise identification $P = \tilde{P}$ for $P \in U$ and $\tilde{P} \in \tilde{U}$ and a transformation $b^i{}_j$, leaving η_{ij} invariant, such that*

$$\tilde{\omega}^i_{\tilde{p}} = b^i{}_j \omega^j_p$$

Let x^μ denote local coordinates in U, ϵ^A the $\frac{1}{2}n(n-1)$ parameters of the proper Lorentz group and m the discrete parameters in G. Let $\omega^i_0(x^\mu)$ represent some local section of the space of 1-forms and let $b^i_j(x^\mu, \epsilon^A, m)$ represent a transformation in $\mathcal{L}_+^\uparrow$, then any other 1-form can be written:

$$\omega^i = b^i_j(x^\mu, \epsilon^A, m)\omega^j_0(x^\mu) \tag{1.1.11}$$

So that by varying ϵ^A and m one obtains all possible ω^i with a given frame metric η_{ij}.



Using this notation we can write Lemma 1.1.1 in the following manner:

Lemma 1.1.2 *Two geometries are equivalent if, and only if, there exists a relation*

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\mu) \tag{1.1.12}$$

$$\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^B, x^\mu) \tag{1.1.13}$$

$$\tilde{m} = \tilde{m}(m) \tag{1.1.14}$$

giving:

$$\tilde{\omega}^i(\tilde{x}^\mu, \tilde{\epsilon}^A, \tilde{m}) = \omega^i(x^\mu, \epsilon^A, m) \tag{1.1.15}$$

Since the number of discrete transformations in G is finite one can then fix m and solve for each value of m in turn. So that the dependence in m is straightforward.

In the following section we investigate the simpler case, where one does not take into account the effect of the group G of all transformations $\{b^i{}_j\}$, i.e., we consider that the 1-forms ω^i and $\tilde{\omega}^i$ must be equal rather than equal up to a transformation $b^i{}_j$. Latter on these results are used to solve the real problem of equivalence.

1.2 Analysis of the Simpler Case

Here we concentrate in solving the simpler problem of determining a pointwise identification of the regions U and $\tilde{U}$ that will match up the 1-forms ω^i and $\tilde{\omega}^i$, not considering a rotation and/or a discrete transformation.

Let ω^i and $\tilde{\omega}^i$ be two systems of n linearly independent 1-forms, defined on regions U and $\tilde{U}$ with local coordinates x^μ and $\tilde{x}^\mu$ respectively $(i, \mu = 1, 2, ..., n)$. We investigate in what circumstances there exists an identification of U and $\tilde{U}$, given by the relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, realising $\tilde{\omega}^i = \omega^i$.

We start by taking the exterior derivative of ω^i and $\tilde{\omega}^i$.

$$d\omega^i = \frac{1}{2} c^i_{kh} \omega^k \wedge \omega^h \quad ; \quad c^i_{kh} = -c^i_{hk} \tag{1.2.16}$$

$$d\tilde{\omega}^i = \frac{1}{2} \tilde{c}^i_{kh} \tilde{\omega}^k \wedge \tilde{\omega}^h \quad ; \quad \tilde{c}^i_{kh} = -\tilde{c}^i_{hk} \tag{1.2.17}$$

where $c^i_{kh} = c^i_{kh}(x^\mu)$, $\tilde{c}^i_{kh} = \tilde{c}^i_{kh}(\tilde{x}^\mu)$.

I) We start by treating the special case where there are n functionally independent functions among c^i_{kh} and $\tilde{c}^i_{kh}$.

Note: n functions $f_1, f_2, ..., f_n$ are said to be functionally independent if, and only if the covectors $df_1, df_2, ..., df_n$ defined at a point P are linearly independent. The number of functionally independent components among the f_i is equal to the number of linearly independent vectors among the df_i.

Ia) In this special case, we start off by establishing the necessary condition for $\omega^i = \tilde{\omega}^i$.

If we assume that $\omega^i = \tilde{\omega}^i$, then $d\omega^i = d\tilde{\omega}^i$, so that from equations 1.2.16 and 1.2.17 we get:

$$\tilde{c}^i_{kh} = c^i_{kh} \tag{1.2.18}$$

Further differentiation gives:

$$dc^i_{kh} = c^i_{kh\|l}\,\omega^l \tag{1.2.19}$$

$$d\tilde{c}^i_{kh} = \tilde{c}^i_{kh\|l}\,\tilde{\omega}^l \tag{1.2.20}$$

Then by 1.2.18 we have:

$$\tilde{c}^i_{kh\|l} = c^i_{kh\|l} \tag{1.2.21}$$

So that a necessary condition for $\omega^i = \tilde{\omega}^i$ is that equations 1.2.18 and 1.2.21 are compatible equations relating $\tilde{x}^\mu$ and x^μ.

If we decided to continue differentiation even further we would have:

$$dc^i_{kh\|l} = c^i_{kh\|lm}\,\omega^m \tag{1.2.22}$$

$$d\tilde{c}^i_{kh\|l} = \tilde{c}^i_{kh\|lm}\,\tilde{\omega}^m \tag{1.2.23}$$

From 1.2.21 we obtain:

$$\tilde{c}^i_{kh\|lm} = c^i_{kh\|lm} \tag{1.2.24}$$

In the special case where one has n functionally independent functions among the $c^i_{jk}(\tilde{c}^i_{jk})$ n is the maximum number of independent functions on an n dimensional manifold. This because on an n dimensional manifold one can have at most n linearly independent vectors defined at a point P among the $dc^i_{jk}, (d\tilde{c}^i_{jk})$. Thus we have that $c^i_{jk\|l}(\tilde{c}^i_{jk\|l})$ must be functionally dependent on the $c^i_{jk}(\tilde{c}^i_{jk})$. For equations 1.2.18 and 1.2.21 to be compatible the $c^i_{jk\|l}$ must be the same function of the c^i_{jk} as $\tilde{c}^i_{jk\|l}$ are of the $\tilde{c}^i_{jk}$. Furthermore, all higher order derivative terms, for exactly the same reason as before, must also be functionally dependent on the $c^i_{jk}(\tilde{c}^i_{jk})$. For example, $c^i_{jk\|lm}$ comes from differentiating $c^i_{jk\|l}$ which in turn is a function of c^i_{jk}. So that in the end $c^i_{jk\|lm}$ is a function of only c^i_{jk}. Hence, if we assume that the functional dependence is the same for $c^i_{jk\|l}$ and $\tilde{c}^i_{jk\|l}$ then all untwiddled and twiddled higher derivatives will be the same function of c^i_{jk} and $\tilde{c}^i_{jk}$ respectively. We conclude that compatibility of 1.2.18 and 1.2.21 guarantees the compatibility of all higher order derivatives.

Ib) We now proceed to show that compatibility of equations 1.2.18 and 1.2.21 is also a sufficient condition for $\omega^i = \tilde{\omega}^i$.

We start by assuming the compatibility of 1.2.18 and 1.2.21 as relations between $\tilde{x}^\mu$ and x^μ and consider the following set of equations:

$$d\tilde{c}^i_{kh} - dc^i_{kh} = c^i_{kh\|l}(\tilde{\omega}^l - \omega^l) = 0 \tag{1.2.25}$$

This set must contain n linearly independent equations in $\tilde{\omega}^l - \omega^l$ because n of the c^i_{kh} are functionally independent which means, by definition, that n of the vectors dc^i_{kh} are linearly independent. The n independent equations contained in 1.2.25 can be written as:

$$c^A_{\|l}(\tilde{\omega}^l - \omega^l) = 0 \tag{1.2.26}$$

where A represents some combination of i, j, k in c^i_{jk} and runs from 1 to n. Considering 1.2.26 to be a matrix equation with $c^A_{\|l}$ an $n \times n$ matrix , the linear independence of the vectors $c^A_{\|l}$ implies that $c^A_{\|l}$ forms an $n \times n$ matrix of rank n with inverse. Hence, set 1.2.26 has only the trivial solution $\tilde{\omega}^l - \omega^l = 0$, which is the desired result.

We therefore conclude that in the case where there are n functionally independent functions among the $c^i_{jk}(\tilde{c}^i_{jk})$, compatibility of equations 1.2.18 and 1.2.21 is a necessary and sufficient condition for $\tilde{\omega}^i = \omega^i$. The set of equations 1.2.18 contains n functionally independent relations which therefore yield a unique coordinate relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ giving $\tilde{\omega}^i = \omega^i$.

II) We now proceed to analyse the case where there are $n_0 \leq n$ functionally independent components among the c^i_{jk}.

IIa)Firstly we establish the necessary conditions for $\tilde{\omega}^i = \omega^i$. Let $\tilde{\omega}^i = \omega^i$, we then proceed as in Ia) to generate the set of equations:

$$
\begin{aligned}
\tilde{c}^i_{kh} &= c^i_{kh} \\
\tilde{c}^i_{kh\|l_1} &= c^i_{kh\|l_1} \\
. &= . \\
. &= . \\
. &= . \\
. &= . \\
\tilde{c}^i_{kh\|l_1 \ldots l_{p+1}} &= c^i_{kh\|l_1 \ldots l_{p+1}}
\end{aligned} \tag{1.2.27}
$$

In case Ia), the compatibility of the 0th and 1st derivatives guaranteed compatibility of all others, in this case we follow exactly the same reasoning to establish that the set 1.2.27 need only continue to the $(p+1)th$ derivative, which is the first derivative functionally dependent on lower derivatives. Therefore, the compatibility of the set 1.2.27 is a necessary condition for $\tilde{\omega}^i = \omega^i$.

IIb)We now determine the sufficient conditions. We assume that the set 1.2.27 are compatible.

IIbi)Here we consider the case where the total number of functionally independent components obtained from all derivatives up to pth order is n. Since in this case the n functionally independent components are scattered among the first pth derivatives we take the following set of equations:

$$
\begin{aligned}
d\tilde{c}^i_{kh} - dc^i_{kh} &= c^i_{kh\|l}\,(\tilde{\omega}^l - \omega^l) &&= 0 \\
d\tilde{c}^i_{kh\|l} - dc^i_{kh\|l} &= c^i_{kh\|lm}\,(\tilde{\omega}^m - \omega^m) &&= 0 \\
. &= . &&= . \\
. &= . &&= . \\
. &= . &&= . \\
d\tilde{c}^i_{kh\|l_1 \ldots l_p} - dc^i_{kh\|l_1 \ldots l_p} &= c^i_{kh\|l_1 \ldots l_p m}\,(\tilde{\omega}^m - \omega^m) &&= 0
\end{aligned} \tag{1.2.28}
$$

This set will contain n linearly independent equations for $\tilde{\omega}^i - \omega^i$ produced by differentiating the n functionally independent components among the c^i_{kh} and its first p derivatives. So that in exactly the same manner as in case (Ib), they give only the trivial solution $\tilde{\omega}^i - \omega^i = 0$. Therefore in the case where n functionally independent components are produced by continued differentiation we have that compatibility of the set 1.2.27 is a necessary and sufficient condition for $\tilde{\omega}^i = \omega^i$.

IIbii) We now analyse the case where continued differentiation never produces n functionally independent components. In this case, among the set 1.2.28 there will only be $k < n$ linearly independent equations for the n unknown $\tilde{\omega}^i - \omega^i$, k being the number of functionally independent components among the c^i_{kh} and its first p derivatives. Therefore, at best we will be able to use the set 1.2.28 to express k of the $\tilde{\omega}^i - \omega^i$ as a linear combination of the other $n-k$. So with a suitable numbering we have:

$$\tilde{\omega}^A - \omega^A = b^A_\alpha(\tilde{\omega}^\alpha - \omega^\alpha) \tag{1.2.29}$$

where A, B etc. run from $n-k+1$ to n (i.e. k of them), and α, β etc. run from 1 to $n-k$ (i.e. $n-k$ of them).

We then want to show that compatibility of the set 1.2.27 makes the $(n-k)\tilde{\omega}^\alpha - \omega^\alpha$ zero which will then give from 1.2.29 that $\tilde{\omega}^i - \omega^i = 0$ for i running from 1 to n. The proof is in 2 stages:

1) We start out by showing how the requirement $\tilde{\omega}^\alpha - \omega^\alpha = 0$ leads to a set of first order partial differential equations.

Using local coordinates we have:

$$\omega^\alpha = a^\alpha_\mu dx^\mu \tag{1.2.30}$$

$$\tilde{\omega}^\alpha = \tilde{a}^\alpha_\mu d\tilde{x}^\mu \tag{1.2.31}$$

which in turn gives the $n-k$ equations:

$$\tilde{\omega}^\alpha - \omega^\alpha = \tilde{a}^\alpha_\mu d\tilde{x}^\mu - a^\alpha_\mu dx^\mu \tag{1.2.32}$$

Since the $\tilde{\omega}^\alpha$ are linearly independent we have that equations 1.2.32 are linearly independent in the $d\tilde{x}^\mu$, having $n-k$ linearly independent equations for n unknown $d\tilde{x}^\mu$. We can then solve for $n-k$ of them as linear combinations of the other k. With a suitable numbering we obtain:

$$d\tilde{x}^\alpha = b^\alpha_\mu dx^\mu + c^\alpha_A d\tilde{x}^A \tag{1.2.33}$$

where once again A, B etc. run from $n-k+1$ to n (i.e. k of them), and α, β etc. run from 1 to $n-k$ (i.e. $n-k$ of them).

2)Then we proceed to study the integrability of equations 1.2.33, i.e. we want to show that there is a solution of 1.2.33 of the form:

$$\tilde{x}^\alpha = \tilde{x}^\alpha(x^\mu, \tilde{x}^A) \tag{1.2.34}$$

which is compatible with the coordinate relations obtained from equations 1.2.27.

Proof

Let W represent a $2n$-dimensional space with coordinates $\{\tilde{x}^\mu, x^\mu\}$. Let

$$\hat{\omega}^\alpha = \tilde{\omega}^\alpha - \omega^\alpha = \tilde{a}^\alpha_\mu d\tilde{x}^\mu - a^\alpha_\mu dx^\mu \tag{1.2.35}$$

where α, β etc. run from 1 to $n-k$. Thus, solving 1.2.33 which derives from 1.2.32, is equivalent to finding the submanifolds $V \subset W$ such that:

$$\hat{\omega}^\alpha \mid_V = 0 \tag{1.2.36}$$

where $\hat{\omega}^\alpha \mid_V$ represents the restriction of ω^α to V so that $\hat{\omega}^\alpha$ only acts on vectors tangent to V. V will only exist (and therefore a solution of 1.2.33 will only exist) if the vectors X which satisfy:

$$< \hat{\omega}^\alpha, X >= 0 \tag{1.2.37}$$

"knit" together in such a way as to be tangent to some submanifold V.

According to Cartan, the condition for this "knitting" together is that $d\hat{\omega}^\alpha = \theta^\alpha_\beta \wedge \hat{\omega}^\beta$, i.e. that:

$$d(\tilde{\omega}^\alpha - \omega^\alpha) = \theta^\alpha_\beta \wedge (\tilde{\omega}^\beta - \omega^\beta) \tag{1.2.38}$$

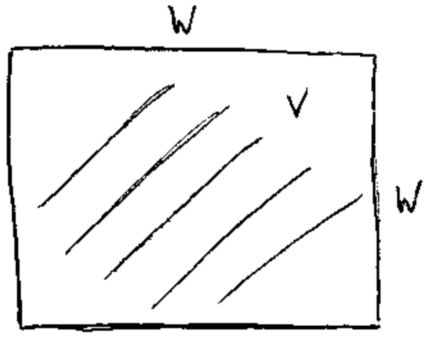
where θ^α_β are arbitrary 1-forms [8]. The exterior derivative must be taken in W but will be the same as in 1.2.16 and 1.2.17 because $\omega^i(\tilde{\omega}^i)$ are independent of $x^\mu(\tilde{x}^\mu)$. We show, using 1.2.16, 1.2.17, 1.2.18 and 1.2.29 that 1.2.38 is indeed satisfied:

$$\begin{aligned}
d(\tilde{\omega}^\alpha - \omega^\alpha) &= \frac{1}{2}c^\alpha_{kh}(\tilde{\omega}^k \wedge \tilde{\omega}^h - \omega^k \wedge \omega^h) \\
&= \frac{1}{4}c^\alpha_{kh}[(\tilde{\omega}^k + \omega^k) \wedge (\tilde{\omega}^h - \omega^h) + (\tilde{\omega}^k - \omega^k) \wedge (\tilde{\omega}^h + \omega^h)] \\
&= \frac{1}{4}c^\alpha_{\beta\gamma}[(\tilde{\omega}^\beta + \omega^\beta) \wedge (\tilde{\omega}^\gamma - \omega^\gamma) - (\tilde{\omega}^\gamma + \omega^\gamma) \wedge (\tilde{\omega}^\beta - \omega^\beta)] \\
&\quad + \frac{1}{2}c^\alpha_{\beta A}[b^A_\gamma(\tilde{\omega}^\beta + \omega^\beta) \wedge (\tilde{\omega}^\gamma - \omega^\gamma) \\
&\quad - (\tilde{\omega}^A + \omega^A) \wedge (\tilde{\omega}^\beta - \omega^\beta)] \\
&\quad + \frac{1}{4}c^\alpha_{AB}[b^B_\beta(\tilde{\omega}^A + \omega^A) \wedge (\tilde{\omega}^\beta - \omega^\beta) \\
&\quad - b^A_\gamma(\tilde{\omega}^B + \omega^B) \wedge (\tilde{\omega}^\gamma - \omega^\gamma)] \\
&= \theta^\alpha_\beta \wedge (\tilde{\omega}^\beta - \omega^\beta)
\end{aligned}$$

The dimension of V is given by $2n-$(number of constraints in 1.2.36), given explicitly by:

$$dim(V) = 2n - (n - k) = n + k \tag{1.2.39}$$

Furthermore, V will not be unique but there will be an $n - k$ parameter family of V's. This arises because the number of orthogonal normal directions to a given V is $2n$- (dimension of V)$= 2n-(n+k) = n-k$, and each orthogonal normal direction will parametrise a set of V's. The initial vector X which "knits" together with the others to form the submanifold may lie at any initial point along the normal directions.



As an illustration of how this works in practice we give the following example:

Example 1.2.1 *Let $n = 1$, $k = 0$, take the coordinates to be $\{x, \tilde{x}\}$ with $x > 0, \tilde{x} > 0$. Take the 1-forms to be $\omega = xdx$, $\tilde{\omega} = -\tilde{x}d\tilde{x}$.*

(Notice that $k = 0$ because in one dimension we only have c^1_{11} which by antisymmetry must be zero).

So equation 1.2.36 becomes $\tilde{x}d\tilde{x} + xdx = 0$, which on integration yields $\tilde{x}^2 + x^2 = c$ or $\tilde{x} = f(x, c)$.

Hence, the solution will be an $n + k = 1$ dimensional submanifold with $n - k = 1$ parameter denoted by c which parametrises different solution submanifolds, exactly as expected. In this case the submanifolds obtained as solution are concentric circles with the parameter c giving their radius, the radial direction being the only normal direction.

We now proceed to show that the solution 1.2.34 is compatible with the coordinate relations that are obtained from the set of equations 1.2.27. To show this we will reperform the steps that led from 1.2.27 to 1.2.34 using a special coordinate system.

We introduce a new coordinate system $\{x', \tilde{x}'\}$ such that the k functionally independent relations among the set 1.2.27 become:

$$\tilde{x}'^A = x'^A \tag{1.2.40}$$

where A runs from $n-k+1$ to n as before. In other words, we let the functionally independent components in the set 1.2.27 act as a new coordinate system, which we are allowed to do because they are functionally independent. Differentiating we obtain:

$$d\tilde{x}'^A = dx'^A \tag{1.2.41}$$

where

$$dx'^A = c'^A_{|i}\omega^i \tag{1.2.42}$$

$$d\tilde{x}'^A = \tilde{c}'^A_{|i}\tilde{\omega}^i \tag{1.2.43}$$

Let $x'^A = c^{i_0}_{k_0h_0|l_1\ldots l_x}$. Then

$$dx'^A = c^{i_0}_{k_0h_0|l_1\ldots l_xi}\ \tilde{\omega}^i \tag{1.2.44}$$

And $\tilde{x}'^A = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_x}$ so that

$$d\tilde{x}'^A = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_xi}\ \tilde{\omega}^i \tag{1.2.45}$$

Comparing 1.2.42 and 1.2.43 with 1.2.44 and 1.2.45 gives:

$$c'^A_{|i} = c^{i_0}_{k_0h_0|l_1\ldots l_xi} \tag{1.2.46}$$

$$\tilde{c}'^A_{|i} = \tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_xi} \tag{1.2.47}$$

However from equations 1.2.27 we know that:

$$\tilde{c}^{i_0}_{k_0h_0|l_1\ldots l_xi} = c^{i_0}_{k_0h_0|l_1\ldots l_xi} \tag{1.2.48}$$

so that

$$\tilde{c}'^A_{|i} = c'^A_{|i} \tag{1.2.49}$$

The $d\tilde{x}'^A$ are linearly independent because the $\tilde{x}'^A$ are functionally independent. Thus, equations 1.2.43 and 1.2.42 represent k linearly independent equations in the $n\,\tilde{\omega}^i$. Using the same argument as before, we can express some k of the $\tilde{\omega}^i$ as linear combinations of the other $n-k$. With a convenient numbering one then obtains:

$$\tilde{\omega}^A = b^A_\alpha \tilde{\omega}^\alpha + d^A_B d\tilde{x}'^B \qquad (1.2.50)$$

Subtracting 1.2.50 from its untwiddled version and using 1.2.41 we obtain:

$$\tilde{\omega}^A - \omega^A = b^A_\alpha(\tilde{\omega}^\alpha - \omega^\alpha) \qquad (1.2.51)$$

Continuing as in 2), we analyse the solution of the equation

$$\tilde{\omega}^\alpha - \omega^\alpha = 0 \qquad (1.2.52)$$

If 1.2.52 is indeed satisfied, then by 1.2.51 $\tilde{\omega}^i - \omega^i = 0$ for i running from 1 to n. In the new coordinate system $\{x', \tilde{x}'\}$ this equation becomes:

$$\tilde{a}'^\alpha_\beta d\tilde{x}'^\beta + \tilde{a}'^\alpha_B d\tilde{x}'^B - a'^\alpha_\beta dx'^\beta - a'^\alpha_B dx'^B = 0 \qquad (1.2.53)$$

From 1.2.50 we have that $\{\tilde{\omega}^\alpha, d\tilde{x}'^A\}$ span the cotangent space of $\tilde{M}$ and hence represent n linearly independent 1-forms. Writing the $\tilde{\omega}^\alpha$ in terms of the coordinates we have:

$$\tilde{\omega}^\alpha = \tilde{a}'^\alpha_\beta d\tilde{x}'^\beta + \tilde{a}'^\alpha_B d\tilde{x}'^B \qquad (1.2.54)$$

It is convenient for us to rewrite this equation as:

$$\tilde{\omega}^\alpha - \tilde{a}'^\alpha_\beta d\tilde{x}'^\beta = \tilde{a}'^\alpha_B d\tilde{x}'^B \qquad (1.2.55)$$

Since $\{\tilde{\omega}^\alpha, \tilde{x}'^A\}$ are n linearly independent 1-forms, the terms on the left hand side of 1.2.55 are linearly independent for different values of α, and so, therefore, are the terms on the right hand side. This implies that the $\tilde{a}'^\alpha_\beta$ constitutes a non-singular matrix. We denote its inverse by $(\tilde{a}'^{-1})^\alpha_\beta$,i.e.

$$(\tilde{a}'^{-1})^\alpha_\beta \tilde{a}'^\beta_\gamma = \delta^\alpha_\gamma \qquad (1.2.56)$$

Multiplying 1.2.53 by the inverse $(\tilde{a}'^{-1})^\alpha_\beta$ we arrive at:

$$d\tilde{x}'^\gamma = -(\tilde{a}'^{-1})^\gamma_\alpha \tilde{a}'^\alpha_B d\tilde{x}'^B + dx'^\gamma + (\tilde{a}'^{-1})^\gamma_\alpha a'^\alpha_B dx'^B \qquad (1.2.57)$$

As before we can show that the integrability condition 1.2.38 is satisfied, this by virtue of 1.2.51. Therefore, 1.2.57 can be integrated to give:

$$\tilde{x}'^\alpha = \tilde{x}'^\alpha(x'^\alpha, x'^A, \tilde{x}'^A) \qquad (1.2.58)$$

Since we have been working in this special coordinate system in which equations 1.2.27 have the simple form given by 1.2.40, we can easily see that the coordinate relations needed to make $\tilde{\omega}^\alpha - \omega^\alpha = 0$, given in this coordinate system by 1.2.58, are compatible with the coordinate relations which arise from 1.2.27, noticing that to arrive at 1.2.58 we insist that 1.2.40 must be satisfied.

We can now conclude that compatibility of the set 1.2.27 is a necessary and sufficient condition for there to exist an identification of U and $\tilde{U}$ giving $\tilde{\omega}^i = \omega^i$ with $i = 1, ..., n$.

●

Therefore, we have that the n relations $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ providing the identification of U and $\tilde{U}$ giving $\tilde{\omega}^i = \omega^i$ are obtained from the set 1.2.27 (giving k of them) together with the integral relations 1.2.34 (giving $n-k$ of them). The relations 1.2.34 are not unique but depend on $n-k$ constants of integration, so there are $n-k$ continuous deformations of 1.2.34 which preserve $\tilde{\omega}^i = \omega^i$. There may also be discrete transformations which are not found in this analysis.

We summarise our analysis in the following theorem:

Theorem 1.2.1 *Given 2 sets of n linearly independent 1-forms $\tilde{\omega}^i$ and ω^i defined on $\tilde{U}$ and U with local coordinates $\tilde{x}^\mu$ and x^μ respectively, then there exists a coordinate identification of $\tilde{U}$ and U, given by $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, giving $\tilde{\omega}^i = \omega^i$ if and only if the equations 1.2.27 are compatible. The $(p+1)th$ derivative is the first one which is functionally dependent on lower derivatives (including the zeroth), so $p+1 \leq n$. The coordinate relations $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ depend on $n-k$ constants of integration, where k is the number of functionally independent components in 1.2.27.*

We now explain why we have $p+1 \leq n$. If the c^i_{jk} are constants then their derivatives will be zero, in this case the differentiation terminates at first order. Therefore, in order for the process to continue beyond first order the c^i_{jk} must contain at least one functionally independent component. Subsequent differentiation must produce at least one new functionally independent component at each stage for the process to continue. However, in n dimensions there are at most n functionally independent components so by the $(n-1)th$ derivative all n must have been produced, making the nth derivative the first to be dependent on lower derivatives.

1.3 The Equivalence Theorem

We have seen that to tackle the problem of determining an identification of two regions U and $\tilde{U}$ given by the coordinate transformations $\tilde{x}^\mu = \tilde{x}^\mu(x^\mu)$ realizing $\tilde{\omega}^i = \omega^i$, one must have that the set of equations 1.2.27 must be a compatible set. However, the real problem of equivalence that we proceed to analyse in this section does not require that $\tilde{\omega}^i = \omega^i$ but only that they are equal up to a transformation b^i_j of the group G, i.e. $\tilde{\omega}^i = b^i_j\omega^j$.

Cartan's [5] procedure for tackling this problem consists of a lengthy and extensive analysis where the idea is to keep the rotational freedom of the tetrad such that ω^i depends not only on the n position coordinates x^μ but also on the $\frac{n(n-1)}{2}$ rotation parameters ϵ^A. This analysis is described in detail in [18] . However, here we choose to follow a different route, one in which the calculations are made easier by the fact that we choose to work on the frame bundle rather than the manifold.

We first review some important definitions and results relating to frame bundles, we closely follow references [15] and [14].

We denote the frame bundle by LM. This is defined to be the collection of points $\overline{P} = (P, (e_1)_P, ..., (e_n)_P)$ where $P \in M$ and $(e_1)_P, ..., (e_n)_P$ span T_PM. The map $\pi : LM \longrightarrow M$ given by $\overline{P} \longrightarrow P$ is the natural projection map. The map $R_g : LM \longrightarrow LM$ gives the bijective correspondence $(P, (e_1)_P, ..., (e_n)_P) \longrightarrow (P, h^i_1 e_i, ..., h^i_n e_i)$, where $h^i_j \in GL(n, \mathbb{R})$

Let U be an open set of M, every frame at $P \in U$ can be expressed uniquely in the form $(X_1, ..., X_n)$ with $X_i = X^{ik}e_k$, X^{ik} being a non-singular matrix. So that if $(X_1, ..., X_n)$ is a frame at P and $h^i_j \in GL(n, \mathbb{R})$ then $(Y_1, ..., Y_n)$ with $Y_j = h^i_j X_i$ is also a frame at P. This shows that there is a bijective correspondence between $\pi^{-1}(U)$ and $U \times GL(n, \mathbb{R})$. Let $(x^1, ..., x^n)$ be a local coordinate system in U and take the usual manifold structure of $GL(n, \mathbb{R})$ so that the differentiable structure of $U \times GL(n, \mathbb{R})$ is the manifold structure of the product manifold. Hence, we can make LM into a differentiable manifold by taking (x^i) and (X^{ij}) as a local coordinate system in $\pi^{-1}(U)$. Notice that if a Lorentz metric g is defined on the manifold M then we may take $h^i_j \in \mathcal{L}^\uparrow_+$ where $\mathcal{L}^\uparrow_+$ is a subgroup of $GL(n, \mathbb{R})$. In order to be consistent with the notation we will denote any $h^i_j \in \mathcal{L}^\uparrow_+$ by b^i_j. The set of elements of LM of the form $(P, b^i_1 e_i, ..., b^i_n e_i)$ constitute the bundle of pseudo-orthonormal frames denoted by $O^\uparrow_+ M$. In this case we have:

$$dim O^\uparrow_+ M = dim M + dim \mathcal{L}^\uparrow_+ = \frac{n + n(n-1)}{2} = \frac{n(n+1)}{2} \qquad (1.3.59)$$

Let $\sigma : \mathbb{R} \longrightarrow M$ be a curve C^∞ in M with $\sigma(0) = P$. By parallel translation we define a C^∞ curve $\overline{\sigma}(t) = (\sigma(t), e_1(t), ..., e_n(t))$ in $O^\uparrow_+ M$, where $e_i(t)$ is the parallel translate of $e_i = e_i(0) = (e_i)_{\sigma(0)} = (e_i)_P$ along σ to $\sigma(t)$. Since $\pi \circ \overline{\sigma} = \sigma$ we say $\overline{\sigma}$ is a lift of σ and since $\overline{\sigma}$ gives a parallel frame we say $\overline{\sigma}$ is a horizontal curve in $O_+^\uparrow M$. Thus a connection on M yields unique horizontal lifts of C^∞ curves in M.

We define at each point $\overline{P} \in O_+^\uparrow M$ the subspace of vertical vectors $V_{\overline{P}} = \{\overline{X}_{\overline{P}} \in T_{\overline{P}}LM : \pi_*(\overline{X}_{\overline{P}}) = 0\}$. A connection on $O_+^\uparrow M$ is a map H that assigns to each $\overline{P} \in O_+^\uparrow M$ a subspace $H_{\overline{P}}$ of $T_{\overline{P}}O_+^\uparrow M$ such that: ($H_{\overline{P}}$ is the horizontal space)

1. $H_{\overline{P}}$ contains no non-zero vector belonging to the vertical subspace $V_{\overline{P}}$ and $\pi_{*|H_{\overline{P}}}$ is an isomorphism of $H_{\overline{P}}$ onto $T_{\pi(\overline{P})}M$, hence $H_{\overline{P}}$ is n dimensional.

2. $(R_{g*})(H_{\overline{P}}) = H_{R_g(\overline{P})}$; $\forall g^i_j \in GL(n, \mathbf{R})$

3. H is C^∞, i.e., for each $\overline{P} \in O_+^\uparrow M$ there is a neighbourhood $\overline{U}$ and a set of n independent C^∞ vector fields $E_1, ..., E_n$ on $\overline{U}$ that give a base for $H_{\overline{P}'}$ for every $\overline{P}' \in \overline{U}$.

Thus a connection on M determines the horizontal subspaces in the tangent spaces at each point $\overline{P} \in O_+^\uparrow M$. And the projection map $\pi : O_+^\uparrow M \longrightarrow M$ induces a surjective linear map $\pi_* : T_{\overline{P}}O_+^\uparrow M \longrightarrow T_{\pi(\overline{P})}M$ such that $\pi_*(V_{\overline{P}}) = 0$ and $\pi_{*|H_{\overline{P}}}$ is a injection onto $T_{\pi(\overline{P})}M$. Therefore the inverse π_*^{-1} is a linear map of $T_{\pi(\overline{P})}M$ onto $H_{\overline{P}}$.

If $\overline{X}_{\overline{P}} \in H_{\overline{P}}$ we say that $\overline{X}_{\overline{P}}$ is a horizontal vector. Property 1 implies that for each $\overline{X}_{\overline{P}} \in T_{\overline{P}}O_+^\uparrow M$ there is a unique decomposition:

$$\overline{X}_{\overline{P}} = (\overline{X}_H)_{\overline{P}} + (\overline{X}_V)_{\overline{P}} \tag{1.3.60}$$

with $(\overline{X}_H)_{\overline{P}} \in H_{\overline{P}}$ and $(\overline{X}_V)_{\overline{P}} \in V_{\overline{P}}$. Property 3 implies that if $\overline{X}$ is C^∞ then $\overline{X}_H$ and $\overline{X}_V$ are C^∞ vector fields. We have:

$$V_{\overline{P}} \oplus H_{\overline{P}} = T_{\overline{P}}O_+^\uparrow M \tag{1.3.61}$$

Furthermore if X is a C^∞ field on $U \in M$ then there is a unique C^∞ horizontal vector field $\overline{X}$ on $\overline{U} = \pi^{-1}(U)$ with $\pi_*(\overline{X}_{\overline{P}}) = X_{\pi(\overline{P})}$, $\forall \overline{P} \in \overline{U}$. The parallel translation earlier defined is frame independent in the sense that it is independent of the starting point for $\overline{\sigma}$. By property 2 if $\overline{\sigma}$ is horizontal (has a horizontal tangent) then $R_g \circ \overline{\sigma}$ is also horizontal.

Let H be a connection on $O_+^\uparrow M$ and $\overline{P} \in O_+^\uparrow M$. We can define a unique horizontal vector field E_i with $\pi_*(E_i(\overline{P})) = (e_i)_{\pi(\overline{P})} = (e_i)_P$, $\forall \overline{P} \in O_+^\uparrow M$ by property 1. The fields $E_1, ..., E_n$ are global independent horizontal C^∞ vector fields on LM. Together with the natural vertical vector fields which we shall denote by $E_1^1, ..., E_n^n$ with $\pi_*(E_i^i(\overline{P})) = 0 \ \forall \overline{P} \in O_+^\uparrow M$ we obtain a global base field on $O_+^\uparrow M$.

Consider the dual viewpoint involving differential forms, so that $\varpi^1, ..., \varpi^n$, $\varpi_1^1, ..., \varpi_n^n$ are the dual 1-forms to this base. The ϖ^i's are defined by $\overline{X}_{\overline{P}} = \varpi^i(\overline{X}_{\overline{P}})\pi_*(E_i(\overline{P}))$, $\overline{X}_{\overline{P}} \in T_{\overline{P}} O_+^\uparrow M$ and are dual to the E_i's.

The set of ϖ_j^i's are defined by $(\overline{X}_V)_{\overline{P}} = \varpi_j^i(\overline{X}_{\overline{P}})(E_j^i)_{\overline{P}}$, $\overline{X}_{\overline{P}} \in T_{\overline{P}}$. Thus a set of connection 1-forms ϖ_j^i (for $i, j = 1, ..., n$) on $O_+^\uparrow M$ is a set of 1-forms such that:

1′. $\varpi^i_{j|V_{\overline{P}}}$ form a dual base to E_j^i at all $\overline{P} \in O_+^\uparrow M$.

2′. $\varpi_j^i(R_{b*}\overline{X}_{\overline{P}}) = {b_r^i}^{-1}\varpi_s^r(\overline{X}_{\overline{P}})b_j^s \ \forall \overline{X}_{\overline{P}} \in T_{\overline{P}} O_+^\uparrow M$

3′. ϖ_j^i are $C^\infty \ \forall i, j$

We define a C^∞ map $f : U \longrightarrow O_+^\uparrow M$ by $f(P) = (P, (e_1)_P, ..., (e_n)_P)$, $P \in U$. If $\pi \circ f$ is the identity on U, then f is called a cross section over U. Let ω^i be the dual basis on M and $\bar{\omega}^i$ the dual basis on $O_+^\uparrow M$ and let ω_j^i be the connection 1-forms on M and ϖ_j^i the global connection 1-forms on $O_+^\uparrow M$ then:

$$(\varpi^i \circ f^*)(\overline{X}_{\overline{P}}) = \omega^i(X_P) \tag{1.3.62}$$

$$(\varpi_j^i \circ f^*)(\overline{X}_{\overline{P}}) = \omega_j^i(X_P) \tag{1.3.63}$$

Therefore the Cartan structural equations (along with the curvature tensor) can be carried up to global equations on $O_+^\uparrow M$:

$$d\varpi^i = \varpi^j \wedge \varpi_j^i \ \text{ with } \varpi^{ij} = -\varpi^{ji} \tag{1.3.64}$$

$$d\varpi_j^i = \varpi_k^i \wedge \varpi_j^k + \overline{R}^i_{jkl}\varpi^k \wedge \varpi^l \tag{1.3.65}$$

We now proceed to reformulate the equivalence problem in terms of frame bundles. In section 1.1 we saw that two space-times are equivalent if and only if there is a pointwise identification $i : M \longrightarrow \tilde{M}$ such that $i(P) = \tilde{P}$ and a transformation $b_j^i \in \mathcal{L}_+^\uparrow$ realising $\tilde{\omega}^i_{\tilde{P}} = b_j^i \omega^j_P$. Hence by means of the definition of ϖ^i, ϖ^{ij} and the map R_b and by equations 1.3.62, 1.3.63, 1.3.64, 1.3.65, the equivalence of space-times in terms of frame bundles can be formulated as follows:

Two regions of $O^{\dagger}_{+}M$ $\overline{U}_R$ and $\overline{\widetilde{U}}$ with local coordinates $\overline{x}^{\mu}_R$, $\overline{x}^{\sigma\rho}_R$ and $\overline{\widetilde{x}}^{\mu}$, $\overline{\widetilde{x}}^{\sigma\rho}(i,j,\mu,\sigma,\rho = 1,...,n)$ respectively, are equivalent if and only if there is a pointwise identification of $\overline{U}_R$ and $\overline{\widetilde{U}}$, given by $i_R : \overline{U}_R \longrightarrow \overline{\widetilde{U}}$, realising:

$$\overline{\widetilde{\omega}}^{i}_{\overline{\widetilde{P}}} = \overline{\omega}^{i}_{R_b(\overline{P})} \quad and \quad \overline{\widetilde{\omega}}^{i}_{j\,\overline{\widetilde{P}}} = \overline{\omega}^{i}_{j\,R_b(\overline{P})} \tag{1.3.66}$$

so that one must have $\overline{U}_R = R_b(\overline{U})$, $\overline{x}^{\mu}_R = \overline{x}^{\mu} \circ R_b$ and $\overline{x}^{\sigma\rho}_R = \overline{x}^{\sigma\rho} \circ R_b$.

Hence the equivalence problem carried out in the frame bundle reduces to the simple problem of section 1.2, so that we can proceed in the same manner and investigate the conditions obtained by requiring that $d\overline{\widetilde{\omega}}^{i} = d\overline{\omega}^{i}$ and $d\overline{\widetilde{\omega}}^{i}_{j} = d\overline{\omega}^{i}_{j}$. Since $\overline{\widetilde{\omega}}^{i} = \overline{\omega}^{i}$ and $\overline{\widetilde{\omega}}^{i}_{j} = \overline{\omega}^{i}_{j}$ by equations 1.3.64 and 1.3.65 we have:

$$\overline{\widetilde{R}}_{ijkl} = \overline{R}_{ijkl} \tag{1.3.67}$$

Further differentiation gives:

$$d\overline{\widetilde{R}}_{ijkl} = \overline{\widetilde{R}}_{ijkl|m}\, \overline{\widetilde{\omega}}^{m} \tag{1.3.68}$$

$$d\overline{R}_{ijkl} = \overline{R}_{ijkl|m}\, \overline{\omega}^{m} \tag{1.3.69}$$

Then by 1.3.67 we have the equality:

$$\overline{\widetilde{R}}_{ijkl|m} = \overline{R}_{ijkl|m} \tag{1.3.70}$$

We wish to express the directional derivative $\overline{R}_{ijkl|m}$ in terms of the covariant derivative $\overline{R}_{ijkl;m}$. Consider the second order covariant tensor V_{ab} which can be written in the following way:

$$\begin{aligned} V_{ab} &= \alpha_1 p_a p_b + \alpha_2 p_a q_b + \alpha_3 q_a p_b + \alpha_4 q_a q_b \\ &= \alpha_1 g(e_a, p) g(e_b, p) + \alpha_2 g(e_a, p) g(e_b, q) \\ &+ \alpha_3 g(e_a, q) g(e_b, p) + \alpha_4 g(e_a, q) g(e_b, q) \end{aligned}$$

p_a, q_a first order covariant vectors.

So that $V_{ab|m}$ is given by:

$$\begin{aligned} V_{ab|m} &= \nabla_{e_m} V_{ab} \\ &= \alpha_1 g(e_a, p) \nabla_{e_m} g(e_b, p) + \alpha_1 g(e_b, p) \nabla_{e_m} g(e_a, p) + \ldots \\ &= \alpha_1 g(e_a, p) g(\nabla_{e_m} e_b, p) + \alpha_1 g(e_a, p) g(e_b, \nabla_{e_m} p) \\ &+ \alpha_1 g(e_b, p) g(\nabla_{e_m} e_a, p) + \alpha_1 g(e_b, p) g(e_a, \nabla_{e_m} p) + \ldots \end{aligned}$$

$$
\begin{aligned}
&= \alpha_1 g(e_a,p)\Gamma^n_{bm} g(e_n,p) + \alpha_1 g(e_a,p) g(e_b, \nabla_{e_m} p) \\
&+ \alpha_1 g(e_b,p)\Gamma^n_{am} g(e_n,p) + \alpha_1 g(e_b,p) g(e_a, \nabla_{e_m} p) + \ldots \\
&= \alpha_1 p_a \Gamma^n_{bm} p_n + \alpha_1 p_a p_{b;m} + \alpha_1 p_b \Gamma^n_{am} p_n + \alpha_1 p_b p_{a;m} \\
&+ \alpha_2 p_a \Gamma^n_{bm} q_n + \alpha_2 p_a q_{b;m} + \alpha_2 q_b \Gamma^n_{am} p_n + \alpha_2 q_b p_{a;m} \\
&+ \alpha_3 q_a \Gamma^n_{bm} p_n + \alpha_3 q_a p_{b;m} + \alpha_3 p_b \Gamma^n_{am} q_n + \alpha_3 p_b q_{a;m} \\
&+ \alpha_4 q_a \Gamma^n_{bm} q_n + \alpha_4 q_a q_{b;m} + \alpha_4 q_b \Gamma^n_{am} q_n + \alpha_4 q_b q_{a;m} \\
&= \Gamma^n_{bm}(\alpha_1 p_a p_n + \alpha_2 p_a q_n + \alpha_3 q_a p_n + \alpha_4 q_a q_n) \\
&+ \Gamma^n_{am}(\alpha_1 p_b p_n + \alpha_2 q_b p_n + \alpha_3 p_b q_n + \alpha_4 q_b q_n) \\
&+ \alpha_1(p_a p_{b;m} + p_b p_{a;m}) + \alpha_2(p_a q_{b;m} + q_b p_{a;m}) \\
&+ \alpha_3(q_a p_{b;m} + p_b q_{a;m}) + \alpha_4(q_a q_{b;m} + q_b q_{a;m}) \\
&= \Gamma^n_{bm} V_{an} + \Gamma^n_{am} V_{bn} + (\alpha_1 p_a p_b + \alpha_2 p_a q_b + \alpha_3 q_a p_b + \alpha_4 q_a q_b)_{;m} \\
&= \Gamma^n_{bm} V_{an} + \Gamma^n_{am} V_{bn} + V_{ab;m}
\end{aligned}
$$

One can apply a similar calculation to higher order tensors, so that in the case of the curvature tensor we write:

$$
\begin{aligned}
\overline{R}_{ijkl|m} &= \overline{R}_{ijkl;m} + \overline{R}_{tjkl}\overline{\Gamma}^t_{im} \\
&+ \overline{R}_{itkl}\overline{\Gamma}^t_{jm} + \overline{R}_{ijtl}\overline{\Gamma}^t_{km} + \overline{R}_{ijkt}\overline{\Gamma}^t_{lm}
\end{aligned} \tag{1.3.71}
$$

with

$$
\overline{\omega}^i_j = \overline{\Gamma}^i_{jk}\overline{\omega}^k \tag{1.3.72}
$$

Equations 1.3.70, 1.3.71 and 1.3.72 together with the assumption that $\overline{\widetilde{\omega}}^i = \overline{\omega}^i$ and $\overline{\widetilde{\omega}}^i_j = \overline{\omega}^i_j$ give:

$$
\overline{\widetilde{R}}_{ijkl;m} = \overline{R}_{ijkl;m} \tag{1.3.73}
$$

Further differentiation of $\overline{\widetilde{R}}_{ijkl;m}$ and $\overline{R}_{ijkl;m}$ gives:

$$
d\overline{\widetilde{R}}_{ijkl;m} = \overline{\widetilde{R}}_{ijkl|mn}\,\overline{\widetilde{\omega}}^n \tag{1.3.74}
$$

$$
d\overline{R}_{ijkl;m} = \overline{R}_{ijkl|mn}\,\overline{\omega}^n \tag{1.3.75}
$$

And

$$
\begin{aligned}
\overline{R}_{ijkl|mn} &= \overline{R}_{ijkl;mn} + \overline{R}_{tjkl;m}\overline{\Gamma}^t_{in} \\
&+ \overline{R}_{itkl;m}\overline{\Gamma}^t_{jn} + \overline{R}_{ijtl;m}\overline{\Gamma}^t_{kn} + \overline{R}_{ijkt;m}\overline{\Gamma}^t_{ln}
\end{aligned} \tag{1.3.76}
$$

Hence:

$$\overline{\tilde{R}}_{ijkl;mn} = \overline{R}_{ijkl;mn} \tag{1.3.77}$$

If we continue this process we will obviously get equalities of higher order covariant derivatives of $\overline{\tilde{R}}$ and $\overline{R}$. Therefore, by acting in the same way as in the case of the simpler problem of section 1.2 we have that equivalence on the frame bundle is governed by the following theorem which is very similar to theorem 1.2.1 of section 1.2 except that we are considering $O_+^\uparrow M$ rather than M. Notice also that the $c^i_{jk}, c^i_{jk|l}$, etc, are replaced by the curvature tensor and its covariant derivatives. N will denote the dimension of $O_+^\uparrow M$.

Theorem 1.3.2 *Two regions $\overline{\tilde{U}}$ and $\overline{U}_R$ of $O_+^\uparrow \tilde{M}$ and $O_+^\uparrow M$, respectively, are equivalent if and only if the set*

$$\begin{aligned} \overline{\tilde{R}}^i_{jkh} &= \overline{R}^i_{jkh} \\ \overline{\tilde{R}}^i_{jkh;l_1} &= \overline{R}^i_{jkh;l_1} \\ . &= . \\ . &= . \\ . &= . \\ . &= . \\ \overline{\tilde{R}}^i_{jkh;l_1\ldots l_{p+1}} &= \overline{R}^i_{jkh;l_1\ldots l_{p+1}} \end{aligned} \tag{1.3.78}$$

is compatible as equations in $\overline{\tilde{x}}^\mu$, $\overline{\tilde{x}}^{\sigma\rho}; \overline{x}^\mu_R, \overline{x}^{\sigma\rho}_R$. The $(p+1)th$ derivative is the first one which is functionally dependent on lower derivatives (including the zeroth), so $p+1 \leq N$. The coordinate relations expressing $\overline{\tilde{x}}^\mu_R, \overline{\tilde{x}}^{\sigma\rho}$ as functions of $\overline{x}^\mu_R, \overline{x}^{\sigma\rho}_R$ depend on $N-k$ constants of integration, where k is the number of functionally independent components among $\overline{R}_{ijkl}, \overline{R}_{ijkl;m}, \ldots, \overline{R}_{ijkl;m_1m_2\ldots m_{p+1}}$, which means that there are $N-k$ continuous deformations of the coordinate relations which preserve equivalence.

Since $\overline{R}_{ijkl}, \overline{\omega}^i, \overline{\omega}^i_j$ in $O_+^\uparrow M$ determine $R_{ijkl}, \omega^i, \omega^{ij}$ in M by means of a function f then the above theorem can be translated in M in a very straightforward way:

Theorem 1.3.3 *Two regions $\tilde{U}$ and U of $\tilde{M}$ and M respectively, are equivalent if and only if the set*

$$\begin{aligned} \tilde{R}^i_{jkh} &= R^i_{jkh} \\ \tilde{R}^i_{jkh;l_1} &= R^i_{jkh;l_1} \end{aligned}$$

$$
\begin{aligned}
. &= . \\
. &= . \qquad (1.3.79) \\
. &= . \\
. &= . \\
\tilde{R}^i_{jkh;l_1..l_{p+1}} &= R^i_{jkh;l_1..l_{p+1}}
\end{aligned}
$$

is compatible as equations in $\tilde{x}^\mu$, $\tilde{x}^{\sigma\rho}; x^\mu, \overline{x}^{\sigma\rho}$. *The* $(p+1)$*th derivative is the first one which is functionally dependent on lower derivatives (including the zeroth), so* $p+1 \leq n$. *The coordinate relations expressing* $\tilde{x}^\mu, \tilde{x}^{\sigma\rho}$ *as functions of* $x^\mu, x^{\sigma\rho}$ *depend on* $n-k$ *constants of integration, where* k *is the number of functionally independent components among* R_{ijkl}, $R_{ijkl;m}$, ..., $R_{ijkl;m_1m_2...m_{p+1}}$, *which means that there are* $n-k$ *continuous deformations of the coordinate relations which preserve equivalence.*

1.4 Investigating Equivalence in Practice

Here we give the practical procedure for investigating the equivalence of metrics otherwise known as the Karlhede algorithm. We will assume in what follows that we are working in an open neighbourhood in which the Petrov type and the dimension of the various isotropy subgroups remain constant.

1. Choose a constant frame metric η_{ij} for the tetrad.

2. Calculate the tetrad components R_{ijkl} of the Riemann tensor in an arbitrary fixed tetrad with metric η_{ij}.

3. Determine H_0, the subgroup of G (G is the six dimensional homogeneous Lorentz group) which leaves the R_{ijkl} invariant (Note that H_0 may contain discrete transformations since G does).

4. Determine, up to a transformation in H_0, a standard tetrad by requiring that R_{ijkl} takes on a special form, called the canonical form. This can always be performed for R_{ijkl} and its covariant derivatives.

5. Determine n_0, the number of functionally independent components among R_{ijkl} in its canonical form (Note: n functions $f_1, f_2, ..., f_n$ are said to be functionally independent if and only if the vectors $df_1, df_2, ..., df_n$ are linearly independent. The number of functionally independent components among

the f_i's is equal to the number of linearly independent vectors among the df_i's).

6. Calculate $R_{ijkl;m_1}$ in the standard tetrad.

7. Determine H_1 the subgroup of H_0 which leaves R_{ijkl} and $R_{ijkl;m_1}$ invariant.

8. Determine among the earlier standard tetrads, up to a transformation in H_1, a new standard tetrad by stipulating a canonical form for $R_{ijkl;m_1}$.

9. Determine n_1, the number of functionally independent components among R_{ijkl} and $R_{ijkl;m_1}$ in their canonical forms.

10. If $dim(H_1) = dim(H_0)$ and $n_1 = n_0$ then the procedure terminates. Otherwise steps 6-9 are repeated for $R_{ijkl;m_1m_2}$, $R_{ijkl;m_1m_2m_3}$, etc until the stage is reached whereby $dim(H_{p+1}) = dim(H_p)$ and $n_{p+1} = n_p$ in which case the procedure terminates.

    The set $\{H_q, n_q, R_{ijkl;m_1m_2m_3...m_q}\}$, $q = 0, 1, ..., p+1$, classifies the solution.

    The above algorithm provides an invariant classification of each metric g and g' which are being compared for equivalence. The rest of the procedure is as follows.

11. If the two sequences $H_0, n_0; H_1, n_1; ...; H_q, n_q$ for g and g' differ, then so do the metrics.

12. If the set of simultaneous algebraic equations $\tilde{R}'_{ijkl} = R_{ijkl}$, $\tilde{R}'_{ijkl;m_1} = R_{ijkl;m_1}$, ..., $\tilde{R}'_{ijkl;m_1m_2...m_q} = R_{ijkl;m_1m_2...m_q}$, with the invariants in their canonical form, admits a coordinate transformation $\tilde{x}^i = \tilde{x}^i(x^i)$, $i = 1, ..., n$ as a solution then the metrics are equivalent, otherwise they are inequivalent.

The procedure terminates when $dim(H_{p+1}) = dim(H_p)$ and $n_{p+1} = n_p$ because there are no new functionally independent quantities relating to the coordinates x^μ since $n_{p+1} = n_p$ and there are no new functionally independent quantities relating to the group G, i.e, to the parameters ϵ^A since $dim(H_{p+1}) = dim(H_p)$.

The reason why this procedure tackles the equivalence problem rests on the use of the canonical form. The equivalence theorem tells us that in order for us to have $\tilde{\omega}^i = b^i_j \omega^j$ there must exist frames in which the two sets of invariants R_{ijkl}, $R_{ijkl;m_1,...}$ and $\tilde{R}_{ijkl}$, $\tilde{R}_{ijkl;m_1...}$ etc. are equal, with the actual identification map being given by the coordinate relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ which then gives the equality. By using the canonical form we are able to pinpoint more precisely, up to transformations in H_p, the frame which will enable these invariants to be equal, by the identification $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$.

This algorithm provides plenty of information about the geometry even if the last step (12) cannot be tackled. In the section that follows we shall see how steps (1) through (4) are equivalent, for the vacuum case, $n = 4$ and η_{ij} the Lorentz metric, to the Petrov classification since the result depends uniquely on the multiplicities of the principal spinors in the corresponding spinor of the Weyl tensor. The complete procedure provides a kind of maximally generalised Petrov classification in the sense that we classify all covariant derivatives of the Riemann tensor that are necessary to provide a complete classification of the geometry. It works for non-empty spaces and spaces of arbitrary dimension n and frame metric η_{ij}. It also works for any geometry, regardless of whether the metric satisfies any field equations, in a sense it is a purely geometrical classification.

The procedure given by Kalhede is similar to the one first suggested by Brans [4]. The main difference lies in the fact that Brans first calculates the Riemann tensor and its covariant derivatives and then determines a canonical form for them, starting with the highest derivative. In the procedure described above the process is made simpler since we do this successively starting with the curvature tensor.

1.5 Canonical Forms for the Weyl Spinor and its Invariance Group

In this section we restrict ourselves to general relativity, i.e., the case where the manifold M has dimension 4 and possesses a Lorentz metric. Instead of working with the tetrad components of the Riemann tensor and considering transformations in $\mathcal{L}^\uparrow_+$ of the frame one works with the dyad components of the corresponding Weyl spinor and considers $SL(2,\mathbb{C})$ transformations. One can do this because of the following two results [3]:

1) The tetrad components of a tensor in a Newman-Penrose null tetrad are the same as the dyad components of the equivalent spinor.

2) $SL(2,\mathbb{C})$ transformations of the dyad correspond to proper homogeneous

Lorentz transformations ($\mathcal{L}_+^\uparrow$) of the Newman-Penrose null tetrad.

The dyad $\{\zeta_0^A, \zeta_1^A\}$ and the Newman-Penrose null tetrad $\{l^\mu, n^\mu, m^\mu, \overline{m}^\mu\}$ are related by the equations:

$$
\begin{aligned}
l^\mu &= \sigma^\mu_{AB'}\, \zeta_0^A \overline{\zeta}_{0'}^{B'} \\
n^\mu &= \sigma^\mu_{AB'}\, \zeta_1^A \overline{\zeta}_{1'}^{B'} \\
m^\mu &= \sigma^\mu_{AB'}\, \zeta_0^A \overline{\zeta}_{1'}^{B'} \\
\overline{m}^\mu &= \sigma^\mu_{AB'}\, \zeta_1^A \overline{\zeta}_{0'}^{B'}
\end{aligned}
\tag{1.5.80}
$$

where $\sigma^\mu_{AB'}$ are the connecting quantities which relate spinors to tensors, otherwise known as the Infeld-Van der Waerden symbols.

The spinor equivalent of the Riemann tensor can be decomposed as follows:

$$
\begin{aligned}
R_{ABCDA'B'C'D'} \quad &= \quad \Psi_{ABCD}\, \epsilon_{A'B'}\, \epsilon_{C'D'} + \epsilon_{AB}\, \epsilon_{CD}\, \overline{\Psi}_{A'B'C'D'} \\
&+ \quad \epsilon_{AB}\, \epsilon_{C'D'}\, \Phi_{CDA'B'} + \epsilon_{CD}\, \epsilon_{A'B'}\, \Phi_{ABC'D'} \\
&+ \quad \Lambda\{(\epsilon_{AD}\, \epsilon_{BC} + \epsilon_{AC}\, \epsilon_{BD})\epsilon_{A'B'}\, \epsilon_{C'D'} \\
&+ \quad \epsilon_{AB}\, \epsilon_{CD}\, (\epsilon_{A'D'}\, \epsilon_{B'C'} + \epsilon_{A'C'}\, \epsilon_{B'D'})\}
\end{aligned}
$$

with

$$\Psi_{ABCD} = \Psi_{(ABCD)} \tag{1.5.81}$$

and

$$\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} = \overline{\Phi}_{ABA'B'} \tag{1.5.82}$$

Λ will represent the Ricci scalar, the Ricci spinor $\Phi_{ABA'B'}$ represents the trace-free Ricci tensor and the Weyl spinor Ψ_{ABCD} represents the Weyl spinor.

In the vacuum case, only the Weyl spinor does not vanish. Because the Weyl spinor is totally symmetric it can be written as a symmetrized product of 1-spinors, with the multiplicity of these principal spinors determining the Petrov type [3].

Petrov Type	**Weyl spinor**
I	$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}$
II	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)}$
D	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)}$
III	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)}$
N	$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\alpha_{D)}$
0	$\Psi_{ABCD} = 0$

where $\alpha_A, \beta_A, \gamma_A$ and δ_A represent non-proportional spinors.

All transformations in $SL(2,\mathbb{C})$ can be represented as the product of three matrices as follows:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \overline{a} & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \; ; \quad \lambda, a, b \in \mathbb{C} \tag{1.5.83}$$

The first matrix will be denoted by T_1, the second by T_2 and the third by T_3. Under transformation T_1 the dyad $\{\zeta_0^A, \zeta_1^A\}$ transforms as:

$$\zeta_0^A \longrightarrow \lambda\zeta_0^A$$

$$\zeta_1^A \longrightarrow \lambda^{-1}\zeta_1^A \tag{1.5.84}$$

Under T_2 the dyad transforms as

$$\zeta_0^A \longrightarrow \zeta_0^A$$

$$\zeta_1^A \longrightarrow \zeta_1^A + \overline{a}\zeta_0^A \tag{1.5.85}$$

Under T_3 the dyad transforms as:

$$\zeta_0^A \longrightarrow \zeta_0^A + b\zeta_1^A$$

$$\zeta_1^A \longrightarrow \zeta_1^A \tag{1.5.86}$$

For an insight into the geometrical interpretation of these transformations one looks at their affect on the tetrad vectors by using the relations 1.5.80. Hence, under T_1, letting $\lambda = re^{i\theta}$, the tetrad transforms as:

$$\begin{aligned} l^\mu &\longrightarrow r^2 l^\mu \\ n^\mu &\longrightarrow r^{-2} n^\mu \\ m^\mu &\longrightarrow e^{2i\theta} m^\mu \\ \overline{m}^\mu &\longrightarrow e^{-2i\theta}\,\overline{m}^\mu \end{aligned} \tag{1.5.87}$$

Thus, we have that T_1 represents a rotation in the $\{m, \overline{m}\}$ plane and a boost in the $\{l, n\}$ plane. Therefore, we call T_1 a spin and boost transformation.

Subject to T_2 the tetrad transforms as:

$$\begin{aligned} l^\mu &\longrightarrow l^\mu \\ n^\mu &\longrightarrow n^\mu + a\overline{m}^\mu + \overline{a}m^\mu + a\overline{a}l^\mu \\ m^\mu &\longrightarrow m^\mu + al^\mu \\ \overline{m} &\longrightarrow \overline{m}^\mu + \overline{a}l^\mu \end{aligned} \tag{1.5.88}$$

Hence, we have that T_2 represents a rotation about the vector l^μ. Therefore, we call T_2 null rotations.

Under T_3 the tetrad transforms as

$$\begin{aligned} l^\mu &\longrightarrow l^\mu + b\overline{m}^\mu + \overline{b}m^\mu + b\overline{b}n^\mu \\ n^\mu &\longrightarrow n^\mu \\ m^\mu &\longrightarrow m^\mu + bn^\mu \\ \overline{m} &\longrightarrow \overline{m}^\mu + \overline{b}n^\mu \end{aligned} \tag{1.5.89}$$

So that T_3 represents a rotation, but this time about the n^μ vector. These transformations are therefore also called null rotations.

We now investigate how the components of the Weyl spinor transform under these transformations. Using the standard notation [28]:

$$\begin{aligned} \Psi_0 &= o^A o^B o^C o^D \Psi_{ABCD} \\ \Psi_1 &= o^A o^B o^C \iota^D \Psi_{ABCD} \\ \Psi_2 &= o^A o^B \iota^C \iota^D \Psi_{ABCD} \\ \Psi_3 &= o^A \iota^B \iota^C \iota^D \Psi_{ABCD} \\ \Psi_4 &= \iota^A \iota^B \iota^C \iota^D \Psi_{ABCD} \end{aligned} \tag{1.5.90}$$

it is clear that $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ under T_1 transform as follows:

$$\begin{aligned} \Psi_0 &\longrightarrow \lambda^4 \Psi_0 \\ \Psi_1 &\longrightarrow \lambda^2 \Psi_1 \\ \Psi_2 &\longrightarrow \Psi_2 \\ \Psi_3 &\longrightarrow \lambda^{-2} \Psi_3 \\ \Psi_4 &\longrightarrow \lambda^{-4} \Psi_4 \end{aligned} \tag{1.5.91}$$

Subject to T_2 they transform:

$$
\begin{aligned}
\Psi_0 &\longrightarrow \Psi_0 \\
\Psi_1 &\longrightarrow \Psi_1 + \overline{a}\Psi_0 \\
\Psi_2 &\longrightarrow \Psi_2 + 2\overline{a}\Psi_1 + \overline{a}^2\Psi_0 \\
\Psi_3 &\longrightarrow \Psi_3 + 3\overline{a}\Psi_2 + 3\overline{a}^2\Psi_1 + \overline{a}^3\Psi_0 \\
\Psi_4 &\longrightarrow \Psi_4 + 4\overline{a}\Psi_3 + 6\overline{a}^2\Psi_2 + 4\overline{a}^3\Psi_1 + \overline{a}^4\Psi_0
\end{aligned} \tag{1.5.92}
$$

Under T_3 they transform:

$$
\begin{aligned}
\Psi_0 &\longrightarrow \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 \\
\Psi_1 &\longrightarrow \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4 \\
\Psi_2 &\longrightarrow \Psi_2 + 2b\Psi_3 + b^2\Psi_4 \\
\Psi_3 &\longrightarrow \Psi_3 + b\Psi_4 \\
\Psi_4 &\longrightarrow \Psi_4
\end{aligned} \tag{1.5.93}
$$

We now concentrate on determining a canonical form for the Weyl spinor and the corresponding invariance group for each of the Petrov types.

Petrov Type I

For Petrov type I the Weyl spinor has the form:

$$
\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)} \tag{1.5.94}
$$

Since the principal spinors are determined only up to a complex scalar factor, we can arrange that $\alpha_A\beta^A = 1$ and choose $\{\alpha^A, \beta^A\}$ as our dyad. In this dyad Ψ_0 and Ψ_4 will both be zero as they will involve the contraction of two α^As together and two β^As together respectively. We will denote the contraction $\alpha_A\beta^A$ by (α/β), so that the full set of components of the Weyl spinor are (up to a constant factor):

$$
\begin{aligned}
\Psi_0 &= \Psi_4 = 0 \\
\Psi_1 &= 6(\alpha/\beta)(\beta/\alpha)(\gamma/\alpha)(\delta/\alpha) = -6(\gamma/\alpha)(\delta/\alpha) \neq 0 \\
\Psi_3 &= 6(\alpha/\beta)(\beta/\alpha)(\gamma/\beta)(\delta/\beta) = -6(\gamma/\beta)(\delta/\beta) \neq 0 \\
\Psi_2 &= 4[(\alpha/\beta)(\beta/\alpha)(\gamma/\alpha)(\delta/\beta) + (\alpha/\beta)(\beta/\alpha)(\gamma/\beta)(\delta/\alpha)] \\
&= -4[(\gamma/\alpha)(\delta/\beta) + (\gamma/\beta)(\delta/\alpha)]
\end{aligned} \tag{1.5.95}
$$

These equations are obtained taking into consideration that for any two spinors $\alpha_A\beta^A = -\beta_A\alpha^A$, and that the four principal spinors in 1.5.94 are non-proportional and therefore have a non-zero contraction with each other. Since Ψ_2 is a sum of two terms it could possibly be zero.

Firstly we use a T_1 transformation to obtain:

$$\Psi_1 = \Psi_3 = (\Psi_1^u \Psi_3^u)^{1/2} \tag{1.5.96}$$

where Ψ_1^u and Ψ_3^u represent the untransformed values given in 1.5.95, while keeping $\Psi_0 = \Psi_4 = 0$ and leaving Ψ_2 unchanged. We now apply a transformation of the form:

$$\begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} \tag{1.5.97}$$

so that the components transform as follows:

$$\begin{aligned} \Psi_0 &\longrightarrow 8\Psi_1 + 6\Psi_2 \\ \Psi_1 &\longrightarrow 0 \\ \Psi_2 &\longrightarrow -\frac{1}{2}\Psi_2 \\ \Psi_3 &\longrightarrow 0 \\ \Psi_4 &\longrightarrow -\frac{1}{2}\Psi_1 + \frac{3}{8}\Psi_2 \end{aligned} \tag{1.5.98}$$

It is clear that Ψ_0 and Ψ_4 can only be zero if $\Psi_2 = -\frac{4}{3}\Psi_1$ or $\Psi_2 = \frac{4}{3}\Psi_1$ respectively. Let $(\gamma/\alpha)(\delta/\beta) = X$ and $(\gamma/\beta)(\delta/\alpha) = Y$ then one can easily show by means of 1.5.95 and 1.5.96 that this can only be satisfied if $X = Y$. On the other hand, from the definition of X and Y we have that $X = Y$ implies that $\gamma_1\delta_2 = \gamma_2\delta_1$ or:

$$\gamma_1/\gamma_2 = \delta_1/\delta_2 \tag{1.5.99}$$

However we see that 1.5.99 contradicts the assumption made that γ^A and δ^A are non-proportional spinors, so that we can conclude that neither Ψ_0 or Ψ_4 transforms to zero under transformation 1.5.97. So that now we can use a spin and boost transformation to make $\Psi_0 = \Psi_4$, while keeping $\Psi_1 = \Psi_3 = 0$ and leaving Ψ_2 unchanged. So that we obtain as our canonical form for Petrov type I:

$$\Psi_0 = \Psi_4 \neq 0 \ , \ \ \Psi_1 = \Psi_3 = 0 \tag{1.5.100}$$

where Ψ_2 may or may not be zero.

My means of 1.5.84, 1.5.85 and 1.5.86 we see that the canonical form 1.5.100 fixes the parameters λ, a and b to certain discrete values, so that the invariance group is the identity.

Petrov Type II

The Petrov type II Weyl spinor has the form:

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\gamma_{D)} \tag{1.5.101}$$

We choose as our dyad:

$$\zeta_0^A = \alpha^A \tag{1.5.102}$$

and

$$\zeta_1^A = \frac{\beta^A}{\alpha_B\beta^B} \tag{1.5.103}$$

so that the dyad components of the Weyl spinor satisfy:

$$\Psi_2, \Psi_3 \neq 0 \ , \ \ \Psi_0 = \Psi_1 = \Psi_4 = 0 \tag{1.5.104}$$

Any swapping over of the basis or mixing of the dyad would change the zero/non-zero pattern given above, so once again only transformations in T_1 will leave the components invariant. The matrix form of T_1 is given by:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad \lambda \in \mathbb{C} \tag{1.5.105}$$

So that, again by 1.5.91 the components 1.5.103 transform in the following way:

$$\tilde{\Psi}_2 = \Psi_2 \ , \ \ \tilde{\Psi}_3 = \lambda^{-2}\Psi_3, \ \ \tilde{\Psi}_0 = \tilde{\Psi}_1 = \tilde{\Psi}_4 = 0 \tag{1.5.106}$$

Invariance requires that $\lambda = \pm 1$, so that as before the invariance group is the identity.

It is obvious that the canonical form in this case is obtained by fixing $\Psi_3 = 1$ with all other components zero except Ψ_2.

Petrov Type III

The Petrov type III Weyl spinor has the form:

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\alpha_C\beta_{D)} \tag{1.5.107}$$

We choose as our dyad:

$$\zeta_0^A = \alpha^A \tag{1.5.108}$$

and

$$\zeta_1^A = \frac{\beta^A}{\alpha_B\beta^B} \tag{1.5.109}$$

which is a normalized dyad since it satisfies the condition $\zeta_{0A}\zeta_1^A = 1$. With this dyad we can contract the Weyl spinor with at most one ζ_0^A for a non zero result, otherwise we would contract two α^As giving zero. On the other hand, contracting with four ζ_1^As gives zero since we would be contracting two β^As. So that in this basis, the components of the Weyl spinor are given by:

$$\Psi_3 \neq 0\ ,\quad \Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0 \tag{1.5.110}$$

We now investigate the $SL(2,\mathbb{C})$ transformations of the dyad which leave these components invariant. It is clear that any swapping around of the dyad or any mixing of the dyad would not leave the components invariant, so that only transformations that preserve this pattern are the ones in T_1 given by:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}\ ,\quad \lambda \in \mathbb{C} \tag{1.5.111}$$

Hence by 1.5.91 the components 1.5.110 transform as follows:

$$\tilde{\Psi}_3 = \lambda^{-2}\Psi_3\ ,\ \tilde{\Psi}_0 = \tilde{\Psi}_1 = \tilde{\Psi}_2 = \tilde{\Psi}_4 = 0 \tag{1.5.112}$$

where $\tilde{\Psi}$ refers to the transformed value. We then have that for the components to remain invariant $\lambda = \pm 1$. Since 1.5.80 relates a tetrad vector to the product of two dyad vectors, both transformations with $\lambda = +1$ and $\lambda = -1$ correspond to the identity transformation of the tetrad. We then have a zero dimensional invariance group.

We can obtain the simplest form possible, i.e. the canonical form, by choosing a special dyad such that Ψ_3 is one and all the other components are zero. In this case the invariance group, i.e. the group of transformations of the dyad which leave the canonical form invariant, is the zero dimensional identity group. Because the invariance group is zero dimensional, this canonical form determines a finite number of dyads and hence tetrads, actually two dyads and one tetrad.

Petrov Type D

For Petrov type D the Weyl spinor has the form:

$$\Psi_{ABCD} = \alpha_{(A}\alpha_B\beta_C\beta_{D)} \tag{1.5.113}$$

We choose as our dyad:

$$\zeta_0^A = \alpha^A \tag{1.5.114}$$

and

$$\zeta_1^A = \frac{\beta^A}{\alpha_B \beta^B} \tag{1.5.115}$$

so that in this case the dyad components of the Weyl spinor have the form:

$$\Psi_2 \neq 0, \ \ \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \tag{1.5.116}$$

It is easily seen that the transformation:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad \lambda \in \mathbf{C} \tag{1.5.117}$$

will leave the pattern of zeros and non-zeros invariant, and that because of the symmetry between α^A and β^A in the Weyl spinor, a transformation swapping them over, i.e.:

$$\begin{pmatrix} 0 & a \\ -1/a & 0 \end{pmatrix}, \quad a \in \mathbf{C} \tag{1.5.118}$$

will also leave the pattern invariant. Any transformation other than 1.5.117 and 1.5.118 will involve mixing the dyad vectors and will therefore alter the pattern of zeros and non-zeros of 1.5.116. It is easily shown that transformations 1.5.117 and 1.5.118 leave the components of the Weyl spinor unchanged, so that these two sets of transformations together constitute the invariance group. However transformation 1.5.118 can be written:

$$\begin{pmatrix} 0 & a \\ -1/a & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.5.119}$$

so that the invariance group for type D is made up of the following types of transformations:

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \quad \lambda \in \mathbf{C}; \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.5.120}$$

And because λ is complex, the invariance group is a two dimensional subgroup of $SL(2,\mathbb{C})$, with the pattern 1.5.116 being the canonical form. Furthermore, because the invariance group is now two dimensional, the canonical form does not limit us to a unique dyad but to an infinite number of dyads.

Petrov Type N

The Petrov type N Weyl spinor has the form

$$\Psi_{ABCD} = \alpha_A \alpha_B \alpha_C \alpha_D \tag{1.5.121}$$

We choose as our dyad:

$$\zeta_0^A = \alpha^A \tag{1.5.122}$$

and ζ_1^A an arbitrary spinor satisfying $\zeta_{0A}\zeta_1^A = 1$. It is clear that any component involving a contraction with α^A will be zero so that we have:

$$\Psi_4 = 1 \ , \ \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \tag{1.5.123}$$

In order to preserve this pattern we cannot consider any transformation of ζ_0^A that mixes in any ζ_1^A. Furthermore we can consider a transformation of ζ_1^A such that we multiply it by ± 1 or $\pm i$ and add in any amount of ζ_0^A, since this transformation will keep $\Psi_4 = 1$. The only transformations of $SL(2,\mathbb{C})$ that satisfy these conditions are:

$$\pm \begin{pmatrix} 1 & 0 \\ \bar{a} & 1 \end{pmatrix} , \quad a \in \mathbb{C} \tag{1.5.124}$$

and

$$\pm i \begin{pmatrix} 1 & 0 \\ \bar{b} & -1 \end{pmatrix} , \quad b \in \mathbb{C} \tag{1.5.125}$$

on the other hand transformation 1.5.125 can be written:

$$\pm i \begin{pmatrix} 1 & 0 \\ \bar{b} & -1 \end{pmatrix} = \pm \begin{pmatrix} 1 & 0 \\ \bar{b} & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{1.5.126}$$

so that the following set of transformations constitutes the invariance group for type N:

$$\pm \begin{pmatrix} 1 & 0 \\ \overline{a} & 1 \end{pmatrix}, \ a \in \mathbb{C} \ ; \ \begin{pmatrix} i & o \\ 0 & -i \end{pmatrix} \tag{1.5.127}$$

which is a two dimensional subgroup of $SL(2, \mathbb{C})$. The canonical form is obviously given by 1.5.123. Because the invariance group is a two dimensional group the canonical form limits us not to a unique dyad but to an infinite number of dyads.

These results can be used to refine the upper bound on the number of covariant derivatives which need to be calculated in order to determine equivalence. The equivalence theorem sets this upper bound at 10 for a four dimensional space. For Petrov types I, II and III the invariance group of Ψ_{abcd} is zero dimensional and therefore cannot change. If there are n_0 components among the Ψ_{abcd} which are functionally independent with respect to the coordinates x^μ, then only $4 - n_0$ functionally independent components remain to be generated. At least one new functionally independent component must be generated per differentiation for the Karlhede algorithm to continue, so that after at most $4 - n_0$ differentiations all functionally independent components must have been generated, so we have:

$$Petrov\ types\ \ I, II, III\ :\ p + 1 \leq 5 - n_0 \tag{1.5.128}$$

So that at worst we would need to calculate five covariant derivatives for these Petrov types.

For Petrov types D and N the same argument as above can be applied, the only difference being that the invariance group starts off at zeroth order with dimension two and therefore could drop one dimension at each differentiation down to zero dimensional. Hence we have:

$$Petrov\ types\ \ D\,and\,N\ :\ p + 1 \leq 7 - n_0 \tag{1.5.129}$$

So that in the worst case we only need to go to the seventh covariant derivative for these Petrov types.

We should emphasize that the worst possible cases of values five and seven assume that:

1. The Ψ_{abcd} are constants (i.e. there are no functionally independent components).

2. The dimension of the invariance group and the number of functionally independent components do not both change on differentiating.

3. At most one new functionally independent component is produced on differentiating.

4. The dimension of the invariance group goes down by at most one dimension on differentiating.

So that in actual calculations it seems highly likely that less derivatives will be needed. In fact, for all calculations performed to date it has been found necessary to go to at most the fourth derivative [21].

Note that these upper bounds will apply to non-vacuum as well as vacuum solutions. From the decomposition of the Riemann spinor 1.5.81 and 1.5.82, we see that in the non-vacuum case as well as the Weyl spinor one must consider the Ricci spinor $\Phi_{ABA'B'}$, which represents the trace-free Ricci tensor. Hence, for the non-vacuum case any invariance group will have to keep the dyad components of the Ricci spinor invariant as well as the dyad components of the Weyl spinor, so that the invariance group will either be of the same dimension as in the vacuum case or of smaller dimension. This means that the upper bound in the non-vacuum cases will be the same as in the vacuum cases.

The only case which we have not considered is the conformally flat case, Petrov type 0, where the Weyl spinor vanishes. We will not give a proof here, but by proceeding in a similar manner to that above and considering the dimension of the invariance group of the Ricci spinor, it can be shown that the upper bound for this case is also seven [18].

Chapter 2

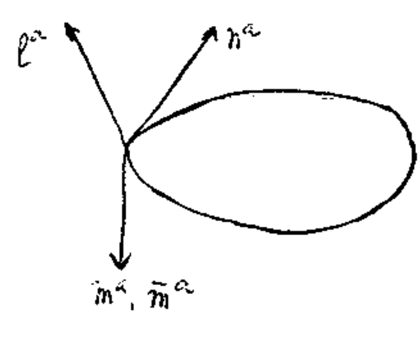
A Formalism Invariant Under Null Rotations

2.1 The Formalism

In this chapter we present a formalism which is invariant under null rotations and which will prove later on to be very useful in the Karlhede classification of Petrov type N solutions [33], [31].

As is well known, many calculations in general relativity are simplified by the use of a tetrad formalism. As an example, we have the Newman-Penrose (NP) formalism [23] which uses a (complex) null tetrad, $\{l^a, n^a, m^a, \overline{m}^a\}$. This formalism has a very natural formulation in terms of spinors which is not surprising since the basis chosen is a null tetrad.

Such tetrad formalisms have particular use if the basis vectors or spinors are not completely arbitrary but are related to the geometry or physics of the spacetime in some way. Take for example a spacelike 2-surface, where we can choose a tetrad so that l^a and n^a point along the outgoing and ingoing null normals of the 2-surface, and the real and imaginary parts of m^a and $\overline{m}^a$ are tangent to the 2-surface.



The remaining gauge freedom in the choice of tetrad is the two dimensional subgroup of the Lorentz group representing a boost in the directions normal to the 2-surface and a rotation in the directions tangent to the 2-surface. In spinor formulation, one chooses the flagpoles of o^A and ι^A to point along the directions of the null normals and the remaining gauge freedom (which maintains the normalization $o_A\iota^A = 1$) is:

$$o^A \longrightarrow \lambda o^A, \quad \iota^A \longrightarrow \lambda^{-1}\iota^A \tag{2.1.1}$$

where λ is an arbitrary (nowhere vanishing) complex scalar field. Transformation 2.1.1 is called a spin and boost transformation.

Under transformation 2.1.1 some of the NP spin-coefficients (those of proper spin and boost weight) are simply rescaled, while other spin-coefficients transform in a way which involves the derivative of λ. These "badly" behaved spin-coefficients can be combined with the NP differential operators of proper spin and boost weight, thus obtaining a new formalism in which all quantities simply rescale under spin and boost transformations. The formalism described is the Geroch-Held-Penrose (GHP) formalism [25] and is particularly useful in the study of the geometry of spacelike 2-surfaces where the differential operators and spin-coefficients have a natural interpretation in terms of the intrinsic and extrinsic geometry of the 2-surface. The GHP formalism has also proved to be very useful in the Karlhede classification of Petrov type D solutions since the invariance group in this case is that of spin and boost transformations. Collins used the GHP notation to lower the bound from seven to three for vacuum type D solutions [10] and from seven to six for the non-vacuum case [9].

The GHP notation is ideal in situations where two null directions are singled out, however in many physical situations one only has one preferred null direction. Examples of such cases are null congruences (which often arise in connection with radiation), null hypersurfaces or wave fronts and Petrov type N space-times. In this case we may choose the flagpole of o^A to give the specified null vector and the remaining gauge freedom in the choice of spin basis is:

$$o^A \longrightarrow o^A, \quad \iota^A \longrightarrow \iota^A + \overline{a}o^A \tag{2.1.2}$$

where a is an arbitrary complex scalar field. Note that a null vector may not determine a unique o^A, however in the case of type N spacetime o^A is unique.

In terms of the null tetrad 2.1.2 becomes:

$$\begin{aligned} &l^a \longrightarrow l^a, \quad m^a \longrightarrow m^a + a l^a, \\ &\overline{m}^a \longrightarrow \overline{m}^a + \overline{a}l^a, \quad n^a \longrightarrow n^a + a\overline{m}^a + \overline{a}m^a + a\overline{a}l^a \end{aligned} \tag{2.1.3}$$

Such transformations form a two (real) dimensional subgroup of the Lorentz group representing null rotations about l^a.

Now the idea is to develop a formalism from the NP formalism that will be invariant under the two dimensional group of null rotations. We begin by studying how the spin coefficients of the NP formalism transform under transformation 2.1.3. We use the same notation used in GHP formalism:

$$\kappa \longrightarrow \kappa \tag{2.1.4}$$

$$\sigma \longrightarrow \sigma + a\kappa \tag{2.1.5}$$

$$\rho \longrightarrow \rho + \overline{a}\kappa \tag{2.1.6}$$

$$\tau \longrightarrow \tau + a\rho + \overline{a}\sigma + a\overline{a}\kappa \tag{2.1.7}$$

$$\beta \longrightarrow \beta + \overline{a}\sigma + a\epsilon + a\overline{a}\kappa \tag{2.1.8}$$

$$\beta' \longrightarrow \beta' - \overline{a}\rho - \overline{a}\epsilon - \overline{a}^2\kappa \tag{2.1.9}$$

$$\epsilon \longrightarrow \epsilon + \overline{a}\kappa \tag{2.1.10}$$

$$\epsilon' \longrightarrow \epsilon' - a\overline{a}(\epsilon + \rho) + a\beta' - \overline{a}(\beta + \tau) - \overline{a}^2\sigma - a\overline{a}^2\kappa \tag{2.1.11}$$

$$\kappa' \longrightarrow \kappa' + \overline{a}(2\epsilon' + \rho') - a\overline{a}^2(2\epsilon + \rho) + a\overline{a}(2\beta' + \tau') - a\overline{a}^2(2\epsilon + \rho) -$$

$$-\overline{a}^3\sigma + a\sigma' - a\overline{a}^3\kappa - \Delta\overline{a} - \overline{a}\delta\overline{a} - a\overline{\delta}\overline{a} - a\overline{a}D\overline{a} \tag{2.1.12}$$

$$\sigma' \longrightarrow \sigma' - \overline{a}^2(2\epsilon + \rho) + \overline{a}(2\beta' + \tau') - \overline{a}^3\kappa - \overline{\delta}\overline{a} - \overline{a}D\overline{a} \tag{2.1.13}$$

$$\rho' \longrightarrow \rho' - 2a\overline{a}\epsilon - 2\overline{a}\beta - \overline{a}^2\sigma + a\tau' - a\overline{a}^2\kappa - \delta\overline{a} - aD\overline{a} \tag{2.1.14}$$

$$\tau' \longrightarrow \tau' - 2\overline{a}\epsilon - \overline{a}^2\kappa - D\overline{a} \tag{2.1.15}$$

We now look for invariants under the group of null rotations. The effect of a null rotation on a spin coefficient ω is denoted by $H(\omega)$ as an example we have $H(\rho) = \rho + a\kappa$. Note also that the spinor invariants with one primed or unprimed index will be referred to as a 1-invariant while an invariant spinor with P unprimed indices and Q primed indices will be referred to as a P,Q-invariant.

(A) Taking linear combinations of the spin coefficients we get the following invariants.

$$\rho - \epsilon \longrightarrow \rho - \epsilon$$

$$\kappa \longrightarrow \kappa$$

(B) Taking linear combinations of the product of the spin coefficients $(\kappa, \sigma, \rho, \tau, \beta, \beta', \epsilon, \epsilon')$ with spinors $(o_A, \iota_A, \overline{o}_{A'}, \overline{\iota}_{A'})$ we have the following cases.

(1) 1-invariants.

$$K_A = \kappa o_A \quad \text{and} \quad K_{A'} = \kappa o_{A'} \tag{2.1.16}$$

The P,Q-invariants are of the following form.

$$K_{A_1 \ldots A_P A'_1 \ldots A'_Q} = \kappa o_{A_1} \ldots o_{A_P} \overline{o}_{A'_{1'}} \ldots \overline{o}_{A'_{Q'}} \tag{2.1.17}$$

(2) H(σ) is a function of σ and κ, we therefore look for an invariant by taking linear combinations of the product of the spin coefficients σ and κ with the spinor basis, ie, of the form $f\sigma + s\kappa$ with $f, s \in \{o_A, \overline{o}_{A'}, \iota_A, \overline{\iota}_{A'}\}$. We then insist that this linear combination be invariant under the effect of a null rotation, ie, that $H(f\sigma + s\kappa) = f\sigma + s\kappa$. .

In this case we find $f = \mp\overline{o}_{A'}$ and $s = \mp\overline{\iota}_{A'}$ such that our 1-invariant is

$$S_{A'} = \sigma\overline{o}_{A'} - \kappa\overline{\iota}_{A'} \tag{2.1.18}$$

To obtain P,Q-invariants we again take symmetric products with the spinor basis. As an example we have :

$$S_{(A'B')} = \sigma\overline{o}_{A'}\overline{o}_{B'} - \kappa\overline{\iota}_{(A'}\overline{o}_{B')} \tag{2.1.19}$$

(3) H(ρ) is a function of ρ and κ, we look for an invariant of the form $f\rho + s\kappa$ with $H(f\rho + s\kappa) = f\rho + s\kappa$ considering f and s to be 1-spinors. In this case we find $f = M'o_A$ and $s = \mp\iota_A$ such that our invariant is :

$$R_A = \rho o_A - \kappa\iota_A \tag{2.1.20}$$

Again to obtain P,Q-invariants we take f and s to be P,Q-spinors. For example

$$R_{(AB)} = \rho o_A o_B - \kappa o_{(A}\iota_{B)} \tag{2.1.21}$$

(4) H(τ) is a function of τ, ρ, σ and κ so we look for an invariant of the form $f\tau + s\sigma + n\rho + m\kappa$ with $H(f\tau + s\sigma + n\rho + m\kappa) = f\tau + s\sigma + n\rho + m\kappa$. We find by

solving this last identity that f, s, n, and m cannot be 1-spinors and must have at least one primed and one unprimed index. Thus as an example of a 1,1-invariant we have:

$$T_{AA'} = \tau o_A \bar{o}_{A'} - \rho o_A \bar{\iota}_{A'} - \sigma \iota_A \bar{o}_{A'} + \kappa \iota_A \bar{\iota}_{A'} \tag{2.1.22}$$

(5) H(β) is a function of β, σ, ϵ and κ, this case is similar to that of (4), thus as an example of a 1,1-invariant we have:

$$B_{AA'} = \beta o_A \bar{o}_{A'} - \epsilon o_A \bar{\iota}_{A'} - \sigma \iota_A \bar{o}_{A'} + \kappa \iota_A \bar{\iota}_{A'} \tag{2.1.23}$$

(6) H(β') is a function of β', ρ, ϵ and κ therefore we look for an invariant of the form $f\beta' + s\rho + n\epsilon + m\kappa$. By solving the identity $H(f\beta' + s\rho + n\epsilon + m\kappa) = f\beta' + s\rho + n\epsilon + m\kappa$ we conclude that f, s, n, and m must be spinors with at least two unprimed indices. Our symmetric 2-spinor is:

$$B'_{(AB)} = \beta' o_A o_B + (\rho + \epsilon) o_{(A} \iota_{B)} - \kappa \iota_A \iota_B \tag{2.1.24}$$

Since we are considering that $o_A \iota^A = 1$ we have that $\beta' = -\alpha$ so that one can write:

$$A_{AB} = \alpha o_A o_B - (\rho + \epsilon) o_{(A} \iota_{B)} + \kappa \iota_A \iota_B \tag{2.1.25}$$

(7) H(ϵ) is a function of ϵ and κ, this is the same case as (3) so we have as a 1-invariant and 2-invariant:

$$E_A = \epsilon o_A - \kappa \iota_A \tag{2.1.26}$$

$$E_{AB} = \epsilon o_A o_B - \kappa o_{(A} \iota_{B)} \tag{2.1.27}$$

(8) Finally we have H(ϵ') as a function of $\epsilon', \epsilon, \beta', \beta, \tau, \rho, \sigma$ and κ. We look for an invariant of the form $f\epsilon' + s\epsilon + n\beta' + m\beta + p\tau + q\rho + g\sigma + h\kappa$. By solving the identity $H(f\epsilon' + s\epsilon + n\beta' + m\beta + p\tau + q\rho + g\sigma + h\kappa) = f\epsilon' + s\epsilon + n\beta' + m\beta + p\tau +$

$q\rho + g\sigma + h\kappa$ we conclude that f, s, n, m, p, q, g, and h must be spinors with at least two unprimed indices and one primed index. Thus our 2,1- invariant is

$$E'_{ABA'} = \epsilon' o_A o_B \bar{o}_{A'} + (\beta + \tau) o_{(A}\iota_{B)} \bar{o}_{A'} - \sigma \iota_A \iota_B \bar{o}_{A'}$$

$$-\beta' o_A o_B \bar{\iota}_{A'} - (\epsilon + \rho) o_{(A}\iota_{B)} \bar{\iota}_{A'} + \kappa \iota_A \iota_B \bar{\iota}_{A'} \quad (2.1.28)$$

Again because we are working with a normalized dyad we can write:

$$G_{ABA'} = \gamma o_A o_B \bar{o}_{A'} - (\beta + \tau) o_{(A}\iota_{B)} \bar{o}_{A'} + \sigma \iota_A \iota_B \bar{o}_{A'}$$

$$-\alpha o_A o_B \bar{\iota}_{A'} + (\rho + \epsilon) o_{(A}\iota_{B)} \bar{\iota}_{A'} - \kappa \iota_A \iota_B \bar{\iota}_{A'} \quad (2.1.29)$$

Note that the above invariants need not be symmetric. For example, in case (6) we have a non-symmetric 2-spinor invariant of the form

$$\mathcal{B}'_{AB} = \beta' o_A o_B + \rho o_A o \iota_B + \epsilon \iota_A o_B - \kappa \iota_A \iota_B$$

In future we shall only be working with symmetric invariants for reasons that will become clear later on. Furthermore all information given in the invariants is contained in the symmetric part, for example the antisymmetric part of $\mathcal{B}'_{AB}$ is $\rho - \epsilon$ and this information is contained in $R_A - E_A = (\rho - \epsilon) o_A$. So that one does not loose information by taking the symmetrized forms. Hence we write down all invariants which will be of use to us in future:

$$K = \kappa \quad (2.1.30)$$

$$R_A = \rho o_A - \kappa \iota_A \quad (2.1.31)$$

$$S_{A'} = \sigma \bar{o}_{A'} - \kappa \bar{\iota}_{A'} \quad (2.1.32)$$

$$T_{AA'} = \tau o_A \bar{o}_{A'} - \rho o_A \bar{\iota}_{A'} - \sigma \iota_A \bar{o}_{A'} + \kappa \iota_A \bar{\iota}_{A'} \quad (2.1.33)$$

$$B_{AA'} = \beta o_A \bar{o}_{A'} - \epsilon o_A \bar{\iota}_{A'} - \sigma \iota_A \bar{o}_{A'} + \kappa \iota_A \bar{\iota}_{A'} \quad (2.1.34)$$

$$A_{AB} = \alpha o_A o_B - (\rho + \epsilon) o_{(A}\iota_{B)} + \kappa \iota_A \iota_B \quad (2.1.35)$$

$$E_A = \epsilon o_A - \kappa \iota_A \quad (2.1.36)$$

$$G_{ABA'} = \gamma o_A o_B \bar{o}_{A'} - (\beta + \tau) o_{(A}\iota_{B)} \bar{o}_{A'} + \sigma \iota_A \iota_B \bar{o}_{A'}$$

$$-\alpha o_A o_B \bar{\iota}_{A'} + (\rho + \epsilon) o_{(A}\iota_{B)} \bar{\iota}_{A'} - \kappa \iota_A \iota_B \bar{\iota}_{A'} \quad (2.1.37)$$

It is easily seen that all invariants may be obtained from $G_{ABA'}$ and $T_{AA'}$ by contracting with the invariant spinors o^A and $\bar{o}^{A'}$. The quantities given in equations 2.1.30-2.1.37 are in fact easily obtained from the dyad components of the spinor analogue of the Ricci rotation coefficients.

Let $\zeta^A_{\mathbf{A}} = (o^A, \iota^A)$ be a normalised spinor dyad with dual $\zeta^{\mathbf{A}}_A$, so that bold indices denote dyad components. Then if we follow Newman and Penrose [23] and define $\Gamma^{\mathbf{A}}_{\mathbf{BCC'}} = \zeta^D_{\mathbf{B}} \nabla_{\mathbf{CC'}} \zeta^{\mathbf{A}}_D$ then we have:

$$K = \Gamma_{000 0'} \tag{2.1.38}$$

$$R_{\mathbf{A}} = \Gamma_{00\mathbf{A}0'} \tag{2.1.39}$$

$$S_{\mathbf{A'}} = \Gamma_{000\mathbf{A'}} \tag{2.1.40}$$

$$T_{\mathbf{AA'}} = \Gamma_{00\mathbf{AA'}} \tag{2.1.41}$$

$$B_{\mathbf{AA'}} = \Gamma_{0\mathbf{A}0\mathbf{A'}} \tag{2.1.42}$$

$$E_{\mathbf{A}} = \Gamma_{0\mathbf{A}00'} \tag{2.1.43}$$

$$A_{(\mathbf{AB})} = \Gamma_{0(\mathbf{AB})0'} \tag{2.1.44}$$

$$G_{(\mathbf{AB})\mathbf{A'}} = \Gamma_{0(\mathbf{AB})\mathbf{A'}} \tag{2.1.45}$$

Notice that the indices above are all bold. The quantities obtained by setting a bold index to zero is also invariant, but setting a bold index to one is not invariant since this corresponds to contracting with an ι^A which is the direction that is not invariant under null rotations.

We now study the way the NP operators transform under the group of null rotations. We take a 1-spinor $\lambda_A = \lambda_1 o_A - \lambda_0 \iota_A$ and consider the transform of $D\lambda_A$, $\delta\lambda_A$, $\delta'\lambda_A$ and $D'\lambda_A$ taking λ_0 and λ_1 to transform in the following way:

$$\lambda_0 \longrightarrow \lambda_0$$
$$\lambda_1 \longrightarrow \lambda_1 + \overline{a}\lambda_0$$

(1) Considering the operator $D = o^A \overline{o}^{A'} \nabla_{AA'}$ we have that $D\lambda_0$ and $D\lambda_1$ transform as follows:

$$D\lambda_0 \longrightarrow D\lambda_0$$

$$D\lambda_1 \longrightarrow D\lambda_1 + \overline{a} D\lambda_0 + \lambda_0 D\overline{a} \tag{2.1.46}$$

(2) For $\delta = o^A \overline{\iota}^{A'} \nabla_{AA'}$ we have:

$$\delta\lambda_0 \longrightarrow \delta\lambda_0 + aD\lambda_0$$

$$\delta\lambda_1 \longrightarrow \delta\lambda_1 + aD\lambda_1 + \overline{a}\delta\lambda_0 + \lambda_0\delta\overline{a} + a\overline{a}D\lambda_0 + a\lambda_0 D\overline{a} \qquad (2.1.47)$$

(3) For $\delta' = \iota^A\overline{o}^{A'}\nabla_{AA'}$ we have:

$$\delta'\lambda_0 \longrightarrow \delta'\lambda_0 + \overline{a}D\lambda_0$$

$$\delta'\lambda_1 \longrightarrow \delta'\lambda_1 + \overline{a}D\lambda_1 + \overline{a}\delta'\lambda_0 + \lambda_0\delta'\overline{a} + \overline{a}^2 D\lambda_0 + \overline{a}\lambda_0 D\overline{a} \qquad (2.1.48)$$

(4) Finally for $D' = \iota^A\overline{\iota}^{A'}\nabla_{AA'}$ we have:

$$D'\lambda_0 \longrightarrow D'\lambda_0 + a\delta'\lambda_0 + \overline{a}\delta\lambda_0 + a\overline{a}D\lambda_0$$

$$D'\lambda_1 \longrightarrow D'\lambda_1 + a\delta'\lambda_1 + \overline{a}\delta\lambda_1 + a\overline{a}D\lambda_1 + \overline{a}D'\lambda_0 \qquad (2.1.49)$$

$$+\lambda_0 D'\overline{a} + a\overline{a}\delta'\lambda_0 + a\lambda_0\delta'\overline{a} + \overline{a}^2\delta\lambda_0 + \overline{a}\lambda_0\delta\overline{a} + a\overline{a}^2 D\lambda_0 + a\overline{a}\lambda_0 D\overline{a}$$

We look for invariants under transformation 2.1.2. We take linear combinations of the products of $\{D\lambda_0, D\lambda_1, \delta\lambda_0, \delta\lambda_1, \delta'\lambda_0, \delta'\lambda_1, D'\lambda_0, D'\lambda_1\}$ with spinors $\{o_A, \overline{o}_{A'}, \iota_A, \overline{\iota}_{A'}\}$, products of λ_0 with $\{\tau', \rho', \sigma', \kappa', \kappa, \sigma, \rho, \tau, \beta, \epsilon, \beta', \epsilon'\}$ and spinors $\{o_A, \overline{o}_{A'}, \iota_A, \overline{\iota}_{A'}\}$

(1) We have that $H(D\lambda_A)$ is a function of $\{D\lambda_0, D\lambda_1, \lambda_0 D\overline{a}\}$ and that $H(\tau')$ is a function of $\{D\overline{a}, \tau', \epsilon\}$ and k. Therefore, $H(D\lambda_A)$ is a function of $\{D\lambda_0, D\lambda_1, \lambda_0\epsilon, \lambda_0\tau'\}$ and $\lambda_0\kappa$. As before we look for spinors f, s, n, m and p such that $H(fD\lambda_0 + sD\lambda_1 + n\lambda_0\tau' + m\lambda_0\epsilon + p\lambda_0\kappa) = fD\lambda_0 + sD\lambda_1 + n\lambda_0\tau' + m\lambda_0\epsilon + p\lambda_0\kappa$. We find a symmetric invariant of the form:

$$\begin{aligned} D_{(A}\lambda_{B)} &= (D\lambda_1 + \lambda_0\tau')o_A o_B + (-D\lambda_0 + 2\lambda_0\epsilon)o_{(A}\iota_{B)} \\ &- \lambda_0\kappa\iota_A\iota_B \end{aligned} \qquad (2.1.50)$$

(2) $H(\delta\lambda_A)$ is a function of $\{\delta\lambda_0, \delta\lambda_1, D\lambda_1, D\lambda_0, \lambda_0 a D\overline{a}\}$ and $\lambda_0\delta\overline{a}$ and $H(\rho')$ is a function of $\{\rho', \tau', \beta, \sigma, \epsilon, \kappa, \delta\overline{a}\}$ and $aD\overline{a}$. We therefore look for spinors $f, s, n. m, p, q, g, h, z, x$ such that $H(f\delta\lambda_0 + s\delta\lambda_1 + nD\lambda_1 + mD\lambda_0 + p\lambda_0\rho' + q\lambda_0\tau' + g\lambda_0\beta + h\lambda_0\sigma + z\lambda_0\epsilon + x\lambda_0\kappa) = f\delta\lambda_0 + s\delta\lambda_1 + nD\lambda_1 + mD\lambda_0 + p\lambda_0\rho' + q\lambda_0\tau' + g\lambda_0\beta + h\lambda_0\sigma + z\lambda_0\epsilon + x\lambda_0\kappa$. We find by solving this identity that our symmetric invariant is:

$$\begin{aligned}\delta_{A'(A}\lambda_{B)} &= (\delta\lambda_1 + \lambda_0\rho')\bar{o}_{A'}o_A o_B + (-2\lambda_0\epsilon + D\lambda_0)\bar{\iota}_{A'}o_{(A}\iota_{B)} \\ &- (D\lambda_1 + \lambda_0\tau')\bar{\iota}_{A'}o_A o_B - (\delta\lambda_0 - 2\lambda_0\beta)\bar{o}_{A'}\iota_{(A}o_{B)} \\ &- \lambda_0\sigma\bar{o}_{A'}\iota_A\iota_B + \lambda_0\kappa\bar{\iota}_{A'}\iota_A\iota_B\end{aligned} \tag{2.1.51}$$

(3) $H(\delta'\lambda_A)$ is a function of the set $\{\delta'\lambda_0, D\lambda_0, \delta'\lambda_1, D\lambda_1, \lambda_0\bar{a}D\bar{a}, \lambda_0\delta'\bar{a}\}$. Furthermore, $H(\sigma')$ can be written as a function of the set $\{\sigma', \beta', \tau', \epsilon, \rho, \kappa, \delta'\bar{a}, \bar{a}D\bar{a}\}$, $H(\beta') \equiv \text{function}(\beta', \epsilon, \rho, \kappa)$, $H(\epsilon) \equiv \text{function}(\epsilon, \kappa)$, $H(\rho) \equiv \text{function}(\epsilon, \kappa)$ and $H(k) \equiv \text{function}(\kappa)$. Therefore we solve the identity $H(f\delta'\lambda_0 + s\delta'\lambda_1 + nD\lambda_0 + mD\lambda_1 + p\lambda_0\sigma' + q\lambda_0\beta' + g\lambda_0\tau' + h\lambda_0\epsilon + z\lambda_0\rho + x\lambda_0\kappa) = f\delta'\lambda_0 + s\delta'\lambda_1 + nD\lambda_0 + mD\lambda_1 + p\lambda_0\sigma' + q\lambda_0\beta' + g\lambda_0\tau' + h\lambda_0\epsilon + z\lambda_0\rho + x\lambda_0\kappa$ for spinors f,s,n,m,p,q,g,h,z and x. We find our symmetric invariant to be:

$$\begin{aligned}\delta'_{(AB}\lambda_{C)} &= (\delta'\lambda_1 + \lambda_0\sigma')o_A o_B o_C - (\delta'\lambda_0 + D\lambda_1 + 2\lambda_0\beta' + \lambda_0\tau') \\ &\quad \iota_{(A}o_B o_{C)} + (D\lambda_0 - 2\lambda_0\epsilon - \lambda_0\rho)\iota_{(A}\iota_B o_{C)} + \lambda_0\kappa\iota_A\iota_B\iota_C\end{aligned} \tag{2.1.52}$$

(4) Using the same line of thinking as before we have $H(\Delta\lambda_A) \equiv \textit{function}(D'\lambda_0, D'\lambda_1, \delta\lambda_0, \delta\lambda_1, \delta'\lambda_0, \delta'\lambda_1, D\lambda_0, D\lambda_1, \lambda_0, \kappa', \epsilon', \rho', \sigma', \beta, \tau, \beta', \tau', \epsilon, \rho, \sigma, \kappa)$ and using a similar process used before we find as our invariant:

$$\begin{aligned}D'_{C'(AB}\lambda_{C)} &= (D'\lambda_1 + \lambda_0\kappa')o_A o_B o_C\bar{o}_{C'} - (\delta'\lambda_1 + \lambda_0\sigma') \\ &\quad o_A o_B o_C\bar{\iota}_{C'} - (\delta\lambda_1 + D'\lambda_0 + 2\lambda_0\epsilon' + \lambda_0\rho')\iota_{(A}o_B o_{C)}\bar{o}_{C'} \\ &+ (D\lambda_1 + \delta'\lambda_0 + 2\lambda_0\beta' + \lambda_0\tau')\iota_{(A}o_B o_{C)}\bar{\iota}_{C'} + (\delta\lambda_0 \\ &- 2\lambda_0\beta - \lambda_0\tau)\iota_{(A}\iota_B o_{C)}\bar{o}_{C'} - (D\lambda_0 - 2\lambda_0\epsilon - \lambda_0\rho) \\ &\quad \iota_{(A}\iota_B o_{C)}\bar{\iota}_{C'} + \lambda_0\sigma\iota_A\iota_B\iota_C\bar{o}_{C'} - \lambda_0\kappa\iota_A\iota_B\iota_C\bar{\iota}_{C'}\end{aligned} \tag{2.1.53}$$

We now proceed to determine expressions for operators 2.1.50, 2.1.51, 2.1.52 and 2.1.53 acting on P,Q-spinors. We begin by considering a symmetric 2-spinor $\phi_{(AB)}$ which in terms of the basis spinors can be written as:

$$\phi_{(AB)} = a o_A o_B + b\iota_{(A}o_{B)} + c\iota_A\iota_B$$

It is usual to define the components ϕ_{AB} of the spinor by considering contractions thus one usually writes:

$$\begin{aligned} \phi_0 &= \phi_{AB}\, o^A o^B = c \\ \phi_1 &= \phi_{AB}\, o^A \iota^B = -1/2b \\ \phi_2 &= \phi_{AB}\, \iota^A \iota^B = a \end{aligned} \tag{2.1.54}$$

However it turns out that it is more convenient to use a convention where we write:

$$\begin{aligned} \phi_0 &= \phi_{AB}\, o^A o^B \\ \phi_1 &= -2\phi_{AB}\, o^A \iota^B \\ \phi_2 &= \phi_{AB}\, \iota^A \iota^B \end{aligned} \tag{2.1.55}$$

Thus we have:

$$\phi_{(AB)} = \phi_2 o_A o_B + \phi_1 o_{(A}\iota_{B)} + \phi_0 \iota_A \iota_B \tag{2.1.56}$$

We will denote the components obtained using the usual convention 2.1.54 by ϕ'_{AB}.

Now considering a 1,1-spinor $\phi_{AB'}$ we have in our convention:

$$\phi_{AB'} = \phi_{11'} o_A \overline{o}_{B'} + \phi_{10'} o_A \overline{\iota}_{B'} + \phi_{01'} \iota_A \overline{o}_{B'} + \phi_{00'} \iota_A \overline{\iota}_{B'} \tag{2.1.57}$$

with

$$\begin{aligned} {\phi'}_{00'} &= \phi_{00'} \\ {\phi'}_{01'} &= -\phi_{01'} \\ {\phi'}_{10'} &= -\phi_{10'} \\ {\phi'}_{11'} &= \phi_{11'} \end{aligned} \tag{2.1.58}$$

For a general symmetric P,Q-spinor of type (N, N') using our convention we write:

$$\phi_{A_1 \ldots A_N A'_1, \ldots A'_{N'}} = \sum_{t,t'} \phi^{N,N'}_{t,t'} \, o_{(A_1} \ldots o_{A_t} \iota_{A_{t+1}} \ldots \iota_{A_N)} \, \bar{o}_{(A'_1} \ldots \bar{o}_{A'_{t'}} \bar{\iota}_{A'_{t'+1}} \ldots \bar{\iota}_{A'_{N'})} \qquad (2.1.59)$$

The relationship between the two conventions is easily seen to be given by the following expression:

$$\phi'^{N,N'}_{t,t'} = (-1)^{(N+N'+t+t')} \frac{t!(N-t)!}{N!} \frac{(t')!(N'-t')!}{(N')!} \phi^{N,N'}_{t,t'} \qquad (2.1.60)$$

We recall that any 2-spinor can be written as a sum of products of 1-spinors. We will consider the symmetric 2-spinor $\phi_{(AB)} = \lambda_{(A}\mu_{B)}$ with $\lambda_A = \lambda_1 o_A + \lambda_0 \iota_A$ and $\mu_A = \mu_1 o_A + \mu_0 \iota_A$. So that we can write:

$$\phi_{(AB)} = \mu_1\lambda_1 o_A o_B + (\mu_1\lambda_0 + \mu_0\lambda_1) o_{(A}\iota_{B)} + \mu_0\lambda_0 \iota_A \iota_B$$

and

$$\begin{aligned} \phi_2 &= \mu_1\lambda_1 \\ \phi_1 &= (\mu_1\lambda_0 + \mu_0\lambda_1) \\ \phi_0 &= \mu_0\lambda_0 \end{aligned} \qquad (2.1.61)$$

We now proceed to determine $D_{(C}\phi_{AB)}$. By applying the Leibniz rule we can write:

$$\begin{aligned} D_{(C}\phi_{AB)} &= 1/3(\lambda_A D_{(B}\mu_{C)} + \mu_B D_{(A}\lambda_{C)} + \lambda_C D_{(A}\mu_{B)} \\ &+ \mu_C D_{(B}\lambda_{A)} + \mu_A D_{(B}\lambda_{C)} + \lambda_B D_{(A}\mu_{C)}) \end{aligned} \qquad (2.1.62)$$

By use of equation 2.1.50 and 2.1.61 and in terms of the new convention equation 2.1.62 becomes:

$$\begin{aligned} D_{(C}\phi_{AB)} &= (D\phi_2 - \phi_1\tau') o_C o_A o_B + (D\phi_1 - 2\phi_0\tau' - 2\epsilon\phi_1) o_{(C} o_A \iota_{B)} \\ &+ (\phi_1\kappa + D\phi_0 - 4\epsilon\phi_0)\iota_{(C}\iota_A o_{B)} + 2\phi_0\kappa\iota_C\iota_A\iota_B \end{aligned} \qquad (2.1.63)$$

To obtain expressions for P-spinors the process is similar. We now consider the case where we have a P,Q-spinor. We start out considering a 1,1-spinor $\phi_{AB'}$. Naturally in this case we need to know how the new invariant operator D acts on the complex conjugate of ϕ_A say $\overline{\phi}_A$. To simplify things we define a new operator $D_{AA'}$:

$$\overline{o}_{A'} D_{(A}\phi_{B)} = D_{A'(A}\phi_{B)} \tag{2.1.64}$$

with

$$D_{A(A'}\overline{\phi}_{B')} = \overline{D_{A'(A}\phi_{B)}} = \overline{\overline{o}_{A'} D_{(A}\phi_{B)}} = o_A D_{(A'}\overline{\phi}_{B')}$$

This new invariant operator will be the one we shall work with from here on and will be referred to as **D**.

Now suppose $\phi_{AB'} = \lambda_A \overline{\mu}_{B'}$ and therefore:

$$\begin{aligned} \phi_{00'} &= \lambda_0 \overline{\mu}_{0'} \\ \phi_{01'} &= -\lambda_0 \overline{\mu}_{1'} \\ \phi_{10'} &= -\lambda_1 \overline{\mu}_{0'} \\ \phi_{11'} &= \lambda_1 \overline{\mu}_{1'} \end{aligned} \tag{2.1.65}$$

Again by applying the Leibniz rule and using expressions 2.1.50, 2.1.64 and 2.1.65 and the new convention we find an expression for $\mathbf{D}_{CC'}\phi_{AB'}$ symmetrized on all primed and unprimed indices:

$$\begin{aligned} \mathbf{D}_{CC'}\phi_{AB'} &= (D\phi_{11'} - \phi_{10'}\overline{\tau}' - \phi_{01'}\tau) o_A o_C \overline{o}_{C'} \overline{o}_{B'} + (D\phi_{01'} \\ &- \phi_{00'}\overline{\tau}' - 2\phi_{01'}\epsilon)\iota_{(A} o_{C)} \overline{o}_{C'} \overline{o}_{B'} + (D\phi_{10'} \\ &- \phi_{00'}\tau' - 2\phi_{10'}\overline{\epsilon}) o_A o_C \overline{o}_{(C'} \overline{\iota}_{B')} + (D\phi_{00'} - 2\overline{\epsilon}\phi_{00'} \\ &- 2\epsilon\phi_{00'})\iota_{(A} o_{C)} \overline{\iota}_{(C'} \overline{o}_{B')} + \phi_{10'}\overline{\kappa} o_A o_C \overline{\iota}_{C'} \overline{\iota}_{B'} + \phi_{01'}\kappa\iota_A\iota_C \overline{o}_{C'} \overline{o}_{B'} \\ &+ \phi_{00'}\overline{\kappa}\iota_{(A} o_{C)} \overline{\iota}_{C'} \overline{\iota}_{B'} + \phi_{00'}\kappa\iota_A\iota_C \overline{\iota}_{(B'} \overline{o}_{C')} \end{aligned} \tag{2.1.66}$$

For spinors with more primed and unprimed indices the process is similar, so that we can arrive at a general formula for the invariant operator **D**.

Let ϕ be a symmetric spinor of type (N, N') given by 2.1.59 then $\mathbf{D}\phi$ a symmetric spinor of type $(N+1, N'+1)$ is defined by:

$$\begin{aligned}(\mathbf{D}\phi)_{t,t'} &= D\phi_{t-1,t'-1} + (N-t)\kappa\phi_{t,t'-1} + (N'-t')\overline{\kappa}\phi_{t-1,t'} \\ &- (N+1-t)2\epsilon\phi_{t-1,t'-1} - (N'+1-t')2\overline{\epsilon}\phi_{t-1,t'-1} \\ &- (N+2-t)\tau'\phi_{t-2,t'-1} - (N'+2-t')\overline{\tau}'\phi_{t-1,t'-2}\end{aligned} \tag{2.1.67}$$

We now study the way the new invariant operators δ and δ' act on P,Q-spinors. Again we have the problem of determining how these operators act on a spinor $\overline{\phi}$. We therefore define two new operators $\delta_{A'B'A}$ and $\delta'_{ABA'}$:

$$\delta_{(C'A')(A}\,\phi_{B)} = \overline{o}_{(C'}\delta_{A')(A}\,\phi_{B)} \tag{2.1.68}$$

and

$$\delta'_{A'(AC}\,\phi_{B)} = \overline{o}_{A'}\delta'_{(AC}\,\phi_{B)} \tag{2.1.69}$$

with

$$\delta_{A(C'A'}\,\phi_{B')} = \overline{\delta_{A'(CA}\,\phi_{B)}} \tag{2.1.70}$$

$$\delta'_{(CA)(A'}\,\phi_{B')} = \overline{\delta_{(C'A')(A}\,\phi_{B)}} \tag{2.1.71}$$

These will be the invariant operators we shall be working with from here on, and will be denoted by $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$. We now rewrite expressions 2.1.51 and 2.1.52 as follows:

$$\begin{aligned}\boldsymbol{\delta}_{(C'A')(A}\,\phi_{B)} &= \overline{o}_{(C'}\delta_{A')(A}\,\phi_{B)} \\ &= \{-(D\phi_1 + \tau'\phi_0)o_Ao_B\overline{\iota}_{(A'}\overline{o}_{C')} \\ &+ (D\phi_0 - 2\epsilon o_0)o_{(A}\iota_{B)}\overline{\iota}_{(A'}\overline{o}_{C')} + \phi_0\kappa\iota_A\iota_B\overline{\iota}_{(A'}\overline{o}_{C')}\} \\ &+ \{(\delta\phi_1 + \phi_0\rho')o_Ao_B\overline{o}_{A'}\overline{o}_{C'} - (\delta\phi_0 - 2\beta\phi_0) \\ &\quad o_{(A}\iota_{B)}\overline{o}_{A'}\overline{o}_{C'} - \phi_0\sigma\iota_A\iota_B\overline{o}_{A'}\overline{o}_{C'}\} \\ &= -\overline{\iota}_{(C'}\mathbf{D}_{A')(A}\,\phi_{B)} + \overline{o}_{(C'}\hat{\delta}_{A')(A}\,\phi_{B)}\end{aligned} \tag{2.1.72}$$

$$\begin{aligned}
\boldsymbol{\delta}'_{A'(AC}\,\phi_{B)} &= \overline{o}_{A'}\,\delta'_{(AC}\,\phi_{B)} \\
&= \{-(D\phi_1+\tau'\phi_0)o_{(A}o_B\iota_{C)}\overline{o}_{A'} \\
&+ (D\phi_0-2\phi_0\epsilon)o_{(A}\iota_B\iota_{C)}\overline{o}_{A'} + \phi_0\kappa\iota_A\iota_B\iota_C\overline{o}_{A'}\} \\
&+ \{(\delta'\phi_1+\phi_0\sigma')o_Ao_Bo_C\overline{o}_{A'} - (\delta'\phi_0+2\phi_0\beta') \\
&\quad o_{(A}\iota_Bo_{C)}\overline{o}_{A'} - \phi_0\rho\iota_{(A}\iota_Bo_{C)}\overline{o}_{A'}\} \\
&= -\mathbf{D}_{A'(A}\,\phi_B\iota_{C)} + \hat{\delta}'_{A'(A}\,\phi_Bo_{C)}
\end{aligned} \tag{2.1.73}$$

Notice that $\hat{\delta}$ acts on ϕ in the same way as $\mathbf{D}$ acts on ϕ with $\tau' \longrightarrow \rho'$; $\epsilon \longrightarrow \beta$; $\kappa \longrightarrow \sigma$; $D \longrightarrow \delta$ and $\hat{\delta}$ acts on $\overline{\phi}$ in the same way as $\mathbf{D}$ acts on $\overline{\phi}$ with $\overline{\tau}' \longrightarrow \overline{\sigma}'$; $\overline{\epsilon} \longrightarrow -\overline{\beta}'$; $\overline{\kappa} \longrightarrow \overline{\rho}$ and $D \longrightarrow \delta$. Also $\hat{\delta}'$ acts on ϕ in the same way as $\mathbf{D}$ acts on ϕ with $\tau' \longrightarrow \sigma'$; $\epsilon \longrightarrow -\beta'$; $\kappa \longrightarrow \rho$; $D \longrightarrow \overline{\delta}$ and $\hat{\delta}'$ acts on $\overline{\phi}$ in the same way as $\mathbf{D}$ acts on $\overline{\phi}$ with $\overline{\tau}' \longrightarrow \overline{\rho}'$; $\overline{\epsilon} \longrightarrow \overline{\beta}$; $\overline{\kappa} \longrightarrow \overline{\sigma}$ and $D \longrightarrow \delta'$.

Therefore, if ϕ is a symmetric spinor of type (N, N') given by 2.1.59 then a symmetric spinor $\boldsymbol{\delta}\phi$ of type $(N+1, N'+2)$ is defined by:

$$\begin{aligned}
(\boldsymbol{\delta}\phi)_{t,t'} &= -(\mathbf{D}\phi)_{t,t'} + (\boldsymbol{\delta}\phi)_{t-1,t'-2} + (N-t)\sigma\phi_{t,t'-2} \\
&+ (N'-t'+1)\overline{\rho}\phi_{t-1,t'-1} - (N+1-t)2\beta\phi_{t-1,t'-2} \\
&+ (N'-t'+2)2\overline{\beta}'\phi_{t-1,t'-2} - (N+2-t)\rho'\phi_{t-2,t'-2} \\
&- (N'+3-t')\overline{\sigma}'\phi_{t-1,t'-3}
\end{aligned} \tag{2.1.74}$$

If ϕ is a symmetric spinor of type (N, N') given by 2.1.59 then a symmetric spinor $\boldsymbol{\delta}'\phi$ of type $(N+2, N'+1)$ is defined by

$$\begin{aligned}
(\boldsymbol{\delta}'\phi)_{t,t'} &= -(\mathbf{D}\phi)_{t,t'} + (\boldsymbol{\delta}'\phi)_{t-2,t'-1} + (N-t+1)\rho\phi_{t-1,t'-1} \\
&+ (N'-t')\overline{\sigma}\phi_{t-2,t'} + (N+2-t)2\beta'\phi_{t-2,t'-1} \\
&- (N'+1-t')2\overline{\beta}\phi_{t-2,t'-1} + (N+3-t)\sigma'\phi_{t-3,t'-1} \\
&- (N'+2-t')\overline{\rho}'\phi_{t-2,t'-2}
\end{aligned} \tag{2.1.75}$$

For the invariant operator D' the process is much the same as before. We define a new operator which we shall work with from here on:

$$D'_{(DC')(AB}\ \phi_{C)} = \bar{o}_{(D}\, D'_{C')(AB}\, \phi_{C)} \tag{2.1.76}$$

with

$$D'_{(DC)(A'B'}\ \phi_{C')} = \overline{D'_{(DC')(AB}\ \phi_{C)}} \tag{2.1.77}$$

Similarly to the previous cases, this new operator will be denoted by $\mathbf{D}'$.

By rewriting expression 2.1.53, we have:

$$\begin{aligned} \mathbf{D}'_{(DC')(AB}\ \phi_{C)} &= \bar{o}_{(C'}\hat{D}'_{D')(A}\,\phi_C o_{B)} - \bar{o}_{(C'}\hat{\delta}_{D')(A}\,\phi_C \iota_{B)} \\ &- \bar{\iota}_{(C'}\hat{\delta}'_{D')(A}\,\phi_C o_{B)} + \bar{\iota}_{(C'}\mathbf{D}_{D')(A}\,\phi_C \iota_{B)} \end{aligned} \tag{2.1.78}$$

So that:

$$\begin{aligned} \mathbf{D}'_{(DC')(AB}\ \phi_{C)} &= \bar{o}_{(C'}\hat{D}'_{D')(A}\,\phi_C o_{B)} - \boldsymbol{\delta}_{(C'D')(A}\ \phi_C \iota_{B)} \\ &- \bar{\iota}_{(C'}\boldsymbol{\delta}'_{D')(AB}\,\phi_{C)} - \bar{\iota}_{(C'}\mathbf{D}_{D')(A}\,\phi_C \iota_{B)} \end{aligned} \tag{2.1.79}$$

The operator $\mathbf{D}'$ acts on ϕ in the same way as $\mathbf{D}$ with $\tau' \longrightarrow \kappa'; \epsilon \longrightarrow -\epsilon'; \kappa \longrightarrow \tau$ and $D \longrightarrow D'$

Therefore, if ϕ is a symmetric spinor of type (N, N') defined by 2.1.59 then a symmetric spinor $\mathbf{D}'\phi$ of type $(N+2, N'+2)$ is defined by:

$$\begin{aligned} (\mathbf{D}'\phi)_{t,t'} &= -(\mathbf{D}\phi)_{t,t'} - (\boldsymbol{\delta}\phi)_{t,t'} - (\boldsymbol{\delta}'\phi)_{t,t'} + D'\phi_{t-2,t'-2} \\ &+ (N-t+1)\tau\phi_{t-1,t'-2} + (N'-t'+1)\bar{\tau}\phi_{t-2,t'-1} \\ &+ (N+2-t)2\epsilon'\phi_{t-2,t'-2} + (N'+2-t')2\bar{\epsilon}'\phi_{t-2,t'-2} \\ &- (N+3-t)\kappa'\phi_{t-3,t'-2} - (N'+3-t')\bar{\kappa}'\phi_{t-2,t'-3} \end{aligned} \tag{2.1.80}$$

Hence we have obtained four invariant operators $\mathbf{D}, \boldsymbol{\delta}, \boldsymbol{\delta}'$ and $\mathbf{D}'$. For some purposes an alternative representation of the new derivative operators in terms of ∇ and Γ is useful. We give these below, where the symbol $\sum_{sym}$ indicates symmetrisation on all free primed and unprimed indices.

$$(\mathbf{D}\phi)_{\mathbf{AA_1...A_NA'A'_1...A'_{N'}}} = \sum_{sym} \epsilon_{0'\mathbf{A'}}\epsilon_{0\mathbf{A}}\nabla_{00'}\phi_{\mathbf{A_1...A'_{N'}}}$$

$$-\epsilon_{0'\mathbf{A'}}\Gamma_{\mathbf{AA_1}00'}o^{\mathbf{E}}\phi_{\mathbf{EA_2...A'_{N'}}} - \epsilon_{0\mathbf{A}}\bar{\Gamma}_{\mathbf{A'A'_{1'}}0'0}\bar{o}^{\mathbf{E'}}\phi_{\mathbf{A_1...A_NE'A'_{2'}...A'_{N'}}} \quad (2.1.81)$$

$$(\delta\phi)_{\mathbf{AA_1...A_NA'B'A'_1...A'_{N'}}} = \sum_{sym} \epsilon_{0'\mathbf{A'}}\epsilon_{0\mathbf{A}}\nabla_{0\mathbf{B'}}\phi_{\mathbf{A_1...A'_{N'}}}$$

$$-\epsilon_{0'\mathbf{A'}}\Gamma_{\mathbf{AA_1}0\mathbf{B'}}o^{\mathbf{E}}\phi_{\mathbf{EA_2...A'_{N'}}} - \epsilon_{0\mathbf{A}}\bar{\Gamma}_{\mathbf{A'A'_{1'}B'}0}\bar{o}^{\mathbf{E'}}\phi_{\mathbf{A_1...A_NE'A'_{2'}...A'_{N'}}} \quad (2.1.82)$$

$$(\delta'\phi)_{\mathbf{ABA_1...A_NA'A_1...A'_{N'}}} = \sum_{sym} \epsilon_{0'\mathbf{A'}}\epsilon_{0\mathbf{A}}\nabla_{\mathbf{B}0'}\phi_{\mathbf{A_1...A'_{N'}}}$$

$$-\epsilon_{0'\mathbf{A'}}\Gamma_{\mathbf{AA_1B}0'}o^{\mathbf{E}}\phi_{\mathbf{EA_2...A'_{N'}}} - \epsilon_{0\mathbf{A}}\bar{\Gamma}_{\mathbf{A'A'_{1'}}0'\mathbf{B}}\bar{o}^{\mathbf{E'}}\phi_{\mathbf{A_1...A_NE'A'_{2'}...A'_{N'}}} \quad (2.1.83)$$

$$(\mathbf{D}'\phi)_{\mathbf{ABA_1...A_NA'B'A'_1...A'_{N'}}} = \sum_{sym} \epsilon_{0'\mathbf{A'}}\epsilon_{0\mathbf{A}}\nabla_{\mathbf{BB'}}\phi_{\mathbf{A_1...A'_{N'}}}$$

$$-\epsilon_{0'\mathbf{A'}}\Gamma_{\mathbf{AA_1BB'}}o^{\mathbf{E}}\phi_{\mathbf{EA_2...A'_{N'}}} - \epsilon_{0\mathbf{A}}\bar{\Gamma}_{\mathbf{A'A'_{1'}B'B}}\bar{o}^{\mathbf{E'}}\phi_{\mathbf{A_1...A_NE'A'_{2'}...A'_{N'}}} \quad (2.1.84)$$

We now concentrate on writing the commutators, Bianchi identities and Ricci equations in terms of the invariant quantities which constitute our new formalism. Beforehand we define new quantities $\mathbf{\Psi}, \mathbf{\Psi}_A, \mathbf{\Psi}_{AB}, \mathbf{\Psi}_{ABC}, \mathbf{\Psi}_{ABCD}, \mathbf{\Phi}_0, \mathbf{\Phi}_A, \mathbf{\Phi}_{(AB)}$, $\mathbf{\Phi}_{A'}, \mathbf{\Phi}_{(A'B')}, \mathbf{\Phi}_{A(A'B')}, \mathbf{\Phi}_{(AB)A'}, \mathbf{\Phi}_{(AB)(A'B')}, \mathbf{\Lambda}$ as follows:

$$\mathbf{\Phi}_0 = \Phi_0 = \Phi_{00'} = \Phi_{ABA'B'}\ o^A o^B \bar{o}^{A'} \bar{o}^{B'} \quad (2.1.85)$$

$$\Phi_{1'} = \Phi_{01'} = \Phi_{ABA'B'}\ o^A o^B \bar{o}^{A'} \bar{\iota}^{B'} = \Phi_{B'} \bar{\iota}^{B'}$$

and

$$\mathbf{\Phi}_{B'} = \Phi_{1'}\bar{o}_{B'} - \Phi_{0'}\bar{\iota}_{B'} \quad (2.1.86)$$

$$\Phi_{2'} = \Phi_{02'} = \mathbf{\Phi}_{ABA'B'}\ o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'} = \Phi_{A'B'}\ \bar{\iota}^{A'} \bar{\iota}^{B'}$$

and

$$\mathbf{\Phi}_{(A'B')} = \Phi_{2'}\overline{o}_{A'}\overline{o}_{B'} - 2\Phi_{1'}\overline{o}_{(A'}\overline{\iota}_{B')} + \Phi_{0'}\overline{\iota}_{A'}\overline{\iota}_{B'} \tag{2.1.87}$$

$$\Phi_1 = \Phi_{10'} = \Phi_{ABA'B'}\ o^A\overline{o}^{A'}\overline{o}^{B'}\iota^B = \Phi_B\iota^B$$

and

$$\mathbf{\Phi}_B = \Phi_1 o_B - \Phi_0\iota_B \tag{2.1.88}$$

$$\Phi_{11'} = \Phi_{ABA'B'}\ o^A\overline{o}^{A'}\iota^B\overline{\iota}^{B'} = \Phi_{BB'}\iota^B\overline{\iota}^{B'}$$

and

$$\mathbf{\Phi}_{BB'} = \Phi_{11'}o_B\overline{o}_{B'} - \Phi_{01'}\iota_B\overline{o}_{B'} - \Phi_{10'}o_B\overline{\iota}_{B'} + \Phi_{00'}\iota_B\overline{\iota}_{B'} \tag{2.1.89}$$

$$\Phi_{12'} = \Phi_{ABA'B'}\ o^A\iota^B\overline{\iota}^{A'}\overline{\iota}^{B'} = \Phi_{BA'B'}\iota^B\overline{\iota}^{A'}\overline{\iota}^{B'}$$

and

$$\mathbf{\Phi}_{B(A'B')} = \Phi_{12'}o_B\overline{o}_{A'}\overline{o}_{B'} - 2\Phi_{11'}o_B\overline{o}_{(A'}\overline{\iota}_{B')} - \Phi_{02'}\iota_B\overline{o}_{A'}\overline{o}_{B'}$$

$$+\Phi_{10'}o_B\overline{\iota}_{A'}\overline{\iota}_{B'} + 2\Phi_{01'}\iota_B\overline{o}_{(A'}\overline{\iota}_{B')} - \Phi_{00'}\iota_B\overline{\iota}_{A'}\overline{\iota}_{B'} \tag{2.1.90}$$

$$\Phi_2 = \Phi_{20'} = \Phi_{ABA'B'}\ \overline{o}^{A'}\overline{o}^{B'}\iota^A\iota^B = \Phi_{AB}\iota^A\iota^B$$

and

$$\mathbf{\Phi}_{(AB)} = \Phi_2 o_A o_B - 2\Phi_1 o_A\iota_B + \Phi_0\iota_A\iota_B \tag{2.1.91}$$

$$\Phi_{21'} = \Phi_{ABA'B'}\ \overline{o}^{A'}\iota^A\iota^B\overline{\iota}^{B'} = \Phi_{ABB'}\iota^A\iota^B\overline{\iota}^{B'}$$

and

$$\mathbf{\Phi}_{(AB)B'} = \Phi_{21'}o_A o_B\overline{o}_{B'} - \Phi_{20'}o_A o_B\overline{\iota}_{B'} - 2\Phi_{11'}\iota_{(A}o_{B)}\overline{o}_{B'}$$

$$+\Phi_{01'}\iota_A\iota_B\overline{o}_{B'} + 2\Phi_{10'}\iota_{(A}o_{B)}\overline{\iota}_{B'} - \Phi_{00'}\iota_A\iota_B\overline{\iota}_{B'} \quad (2.1.92)$$

$$\Phi_{22'} = \Phi_{ABA'B'}\ \iota^A\iota^B\overline{\iota}^{A'}\overline{\iota}^{B'}$$

and

$$\mathbf{\Phi}_{(AB)(A'B')} = \Phi_{22'}o_Ao_B\overline{o}_{A'}\overline{o}_{B'} + \Phi_{20'}o_Ao_B\overline{\iota}_{A'}\overline{\iota}_{B'}$$

$$-2\Phi_{12'}\iota_{(A}o_{B)}\overline{o}_{A'}\overline{o}_{B'} + \Phi_{02'}\iota_A\iota_B\overline{o}_{A'}\overline{o}_{B'} - 2\Phi_{10'}\iota_{(A}o_{B)}\overline{\iota}_{A'}\overline{\iota}_{B'}$$

$$+4\Phi_{11'}o_{(A}\iota_{B)}\overline{o}_{(A'}\overline{\iota}_{B')} + \Phi_{20}o_Ao_B\overline{o}_{(A'}\overline{\iota}_{B')} - 2\Phi_{01'}\iota_A\iota_B\overline{o}_{(A'}\overline{\iota}_{B')} \quad (2.1.93)$$

$$+\Phi_{00'}\iota_A\iota_B\overline{\iota}_{A'}\overline{\iota}_{B'}$$

$$\mathbf{\Psi}_0 = \Psi_0 = \Psi_{ABCD}\ o^Ao^Bo^Co^D \quad (2.1.94)$$

$$\Psi_1 = \Psi_{ABCD}\ o^Ao^Bo^C\iota^D = \Psi_D\iota^D$$

and

$$\mathbf{\Psi}_D = \Psi_1 o_D - \Psi_0\iota_D \quad (2.1.95)$$

$$\Psi_2 = \Psi_{ABCD}\ o^Ao^B\iota^C\iota^D = \Psi_{CD}\iota^C\iota^D$$

and

$$\mathbf{\Psi}_{(CD)} = \Psi_2 o_Co_D - 2\Psi_1 o_{(C}\iota_{D)} + \Psi_0\iota_C\iota_D \quad (2.1.96)$$

$$\Psi_3 = \Psi_{ABCD}\ o^A\iota^B\iota^C\iota^D = \Psi_{BCD}\ \iota^B\iota^C\iota^D$$

and

$$\mathbf{\Psi}_{(BCD)} = \Psi_3 o_Bo_Co_D - 3\Psi_2 o_{(B}o_C\iota_{D)} + 3\Psi_1 o_{(B}\iota_C\iota_{D)} - \Psi_0\iota_B\iota_C\iota_D \quad (2.1.97)$$

$$\Psi_4 = \Psi_{ABCD}\ \iota^A \iota^B \iota^C \iota^D$$

and

$$\Psi_{(ABCD)} = \Psi_4 o_A o_B o_C o_D - 4\Psi_3 o_{(A} o_B o_C \iota_{D)} + 6\Psi_2 o_{(A} o_B \iota_C \iota_{D)}$$

$$-4\Psi_1 o_{(A} \iota_B \iota_C \iota_{D)} + \Psi_0 \iota_A \iota_B \iota_C \iota_D \qquad (2.1.98)$$

Finally:

$$\mathbf{\Lambda} = \Lambda \qquad (2.1.99)$$

The task of writing out the commutators in our invariant notation is very lengthy and can be executed in two ways, both being equally extensive. One way of going about it, is to calculate all terms, starting from the highest order term, relating to a particular commutator. For example, if we consider the commutator $\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta}$, then we would start off by determining the term $\{(\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta})\phi\}_{N+2,N+3'}$, taking ϕ to be of type (N, N'). By use of equations 2.1.67 and 2.1.74 we have:

$$\{(\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta})\phi\}_{N+2,N+3'} = (\delta D - D\delta)\phi_{N,N'} - \overline{\tau}' D\phi_{N,N'}$$

$$+(D\rho' - \delta\tau' + 2\epsilon\rho' - 2\beta\tau' + \tau'\overline{\tau}')\phi_{N-1,N'} \qquad (2.1.100)$$

$$+(D\overline{\sigma}' - \delta\overline{\tau}' + 2\overline{\epsilon}\overline{\sigma}' + 2\overline{\beta}'\overline{\tau}' + \overline{\tau}'\overline{\tau}')\phi_{N,N'-1'}$$

Notice that the Ricci equations involving terms such as ρ', τ'andσ', which are terms that transform "badly " under null rotations, will go into the construction of the "new" commutators, which is what happens when one constructs the GHP from the NP notation. In this last case what happens is that all the Ricci equations which involve derivatives of spin coefficients which do not scale under spin and boost transformations are used in the construction of the commutators written in GHP formalism.

By applying the NP Ricci equations and commutators to expression 2.1.100, and then using the definitions 2.1.67 and 2.1.74 we get:

$$\{(\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta})\phi\}_{N+2,N+3'} = (\beta + \overline{\alpha})(\mathbf{D}\phi)_{N+1,N+1'} + \kappa(\mathbf{D}'\phi)_{N+2,N+2'}$$

$$-\sigma(\boldsymbol{\delta}'\phi)_{N+2,N+1'} - (\epsilon - \overline{\epsilon} + \overline{\rho})(\boldsymbol{\delta}\phi)_{N+1,N+2'} - \Psi_2\phi_{N-1,N'} \qquad (2.1.101)$$

$-\Phi_{02'}\phi_{N,N'-1'}$

To calculate all other lower order terms, the method is just the same, so that one arrives at the final expression which is symmetric on all primed and unprimed indices:

$$
\begin{aligned}
&(\boldsymbol{\delta}_{AA'B'}\ \mathbf{D}_{BC'} - \mathbf{D}_{AA'}\ \boldsymbol{\delta}_{BB'C'}\ )\phi_{A_1\ldots A_N A'_{1'}\ldots A'_{N'}} \\
&= (B_{AA'}\,\overline{o}_{B'}\,\mathbf{D}_{BC'} + o_A\overline{A}_{A'B'}\,\mathbf{D}_{BC'} + \kappa\overline{o}_{A'}\,\mathbf{D}'_{ABB'C'} - \\
&-\overline{o}_{A'}\,S_{B'}\,\boldsymbol{\delta}'_{ABC'} - \overline{o}_{A'}\,E_A\boldsymbol{\delta}_{BB'C'} + o_B\overline{E}_{A'}\,\boldsymbol{\delta}_{BB'C'} \\
&-o_A\overline{R}_{A'}\,\boldsymbol{\delta}_{BB'C'}\ )\phi_{A_1\ldots A'_{N'}} - (\boldsymbol{\Psi}_{AB}\,o_{A_1}\,\overline{o}_{A'}\,\overline{o}_{B'}\,\overline{o}_{C'}\,o^E \\
&+2\boldsymbol{\Lambda} o_{A_1}\,o_A o_B\overline{o}_{A'}\,\overline{o}_{B'}\,\overline{o}_{C'}\,o^E)\phi_{EA_2\ldots A'_{N'}} - \boldsymbol{\Phi}_{B'C'}\,\overline{o}_{A'_{1'}}o_B o_C\overline{o}_D\,\overline{o}^{E'} \\
&\phi_{A_1\ldots A_N E' A'_{2'}\ldots A'_{N'}}
\end{aligned}
\tag{2.1.102}
$$

Notice that in equation 2.1.102 one uses the standard sign convention and not the one we adopted for convenience a while back.

We will now describe a more general but equally lengthy way of determining all commutators in our new invariant language. The method we refer to involves taking the general expression for all NP commutators and translating it into our new language. Hence, we take equation 3.13 of [23], which gives the commutators written in NP formalism, and multiply it by ϵ_{KM} ; $\epsilon_{K'M'}$; ϵ_{LN} and $\epsilon_{L'N'}$:

$$
\begin{aligned}
&\epsilon_{KM}\,\epsilon_{K'M'}\,\epsilon_{LN}\,\epsilon_{L'N'}\,(\nabla_{AA'}\,\nabla_{BB'} - \nabla_{BB'}\,\nabla_{AA'}\,)\phi_{A_1\ldots A'_{N'}} \\
&= \{\epsilon_{K'M'}\,\epsilon_{LN}\,\epsilon_{L'N'}\,(\Gamma_{KABB'}\ \nabla_{MA'} - \Gamma_{MABB'}\ \nabla_{KA'} - \Gamma_{KBAA'}\ \nabla_{MB'} \\
&+\Gamma_{MBAA'}\ \nabla_{KB'}\,) + \epsilon_{KM}\,\epsilon_{K'M'}\,\epsilon_{LN}\,(\overline{\Gamma}_{L'A'B'B}\ \nabla_{AN'} - \overline{\Gamma}_{N'A'B'B}\ \nabla_{AL'} \\
&-\overline{\Gamma}_{L'B'A'A}\ \nabla_{BN'} + \overline{\Gamma}_{N'B'A'A}\ \nabla_{BL'}\,)\}\phi_{A_1\ldots A'_{N'}}
\end{aligned}
\tag{2.1.103}
$$

Putting $K = L = K' = L' = 0$ and adding on to each side the following terms:

$$
\begin{aligned}
\epsilon_{0'N'}\,\Gamma_{0ABB'}\,(\ \ -\ \ &\epsilon_{0'M'}\,\Gamma_{MA_1NA'}\ \ o^E\,\phi_{EA_2\ldots A'_{N'}} \\
-\ \ &\epsilon_{0M}\overline{\Gamma}_{M'A'_{1'},A'N}\ \ \overline{o}^{E'}\,\phi_{A_1\ldots A_N E' A'_{2'}\ldots A'_{N'}}\ \ )
\end{aligned}
$$

$$-\epsilon_{0'N'}\Gamma_{MABB'} \; (\; - \; \epsilon_{0'M'}\Gamma_{NA_10A'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0N}\overline{\Gamma}_{M'A'_{1'},A'o} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$-\epsilon_{0'N'}\Gamma_{0BAA'} \; (\; - \; \epsilon_{0'M'}\Gamma_{NA_1MB'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0M}\overline{\Gamma}_{M'A'_{1'},BN} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$-\epsilon_{0'N'}\Gamma_{MBAA'} \; (\; - \; \epsilon_{0'M'}\Gamma_{NA_10B'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0N}\overline{\Gamma}_{M'A'_{1'},B'0} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$-\epsilon_{0N}\overline{\Gamma}_{0A'B'B} \; (\; - \; \epsilon_{0'M'}\Gamma_{MA_1AN'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0M}\overline{\Gamma}_{M'A'_{1'},N'A} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$-\epsilon_{0N}\overline{\Gamma}_{N'A'B'B} \; (\; - \; \epsilon_{0'M'}\Gamma_{MA_1A0'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0M}\overline{\Gamma}_{M'A'_{1'},0'A} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$-\epsilon_{0N}\overline{\Gamma}_{0'B'A'A} \; (\; - \; \epsilon_{0'M'}\Gamma_{MA_1BN'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0M}\overline{\Gamma}_{M'A'_{1'},N'B} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$-\epsilon_{0N}\overline{\Gamma}_{N'B'A'A} \; (\; - \; \epsilon_{0'M'}\Gamma_{MA_1B0'} \; o^E \phi_{EA_2...A'_{N'}}$$
$$- \; \epsilon_{0M}\overline{\Gamma}_{M'A'_{1'},0'B} \; \overline{o}^{E} \; \phi_{A_1...A_N E' A'_{2'}...A'_{N'}} \;)$$

$$+ \; \mathbf{\Psi}_{ABMN} \; \epsilon_{A'B'} \, \epsilon_{0'M'} \, \epsilon_{0'N'} \, o_{A_1} \, o^E \phi_{E...A'_{N'}}$$
$$+ \; \overline{\mathbf{\Psi}}_{A'B'M'N'} \; \epsilon_{AB} \, \epsilon_{0M} \, \epsilon_{0N} \, \overline{o}_{A'_{1'}} \, \overline{o}^{E} \; \phi_{A_1...E'...A'_{N'}}$$

$$+ \; \mathbf{\Phi}_{ABN'M'} \; \epsilon_{0N} \, \epsilon_{0M} \, \epsilon_{A'B6'} \, \overline{o}_{A'_{1'}} \, \overline{o}^{E} \; \phi_{A_1...E'...A'_{N'}}$$
$$+ \; \overline{\mathbf{\Phi}}_{NMA'B'} \; \epsilon_{0N} \, \epsilon_{0M} \, \epsilon_{AB} \, o_{A_1} \, o^E \phi_{E...A'_{N'}}$$
$$- \; 2\mathbf{\Lambda} \epsilon_{MA} \, \epsilon_{NB} \, \epsilon_{A'B'} \, \epsilon_{0'M'} \, \epsilon_{0'N'} \, o_{A_1} \, o^E \phi_{E...A'_{N'}}$$

We get the general expression for the commutators in the generalized formalism:

$$
\begin{aligned}
&\epsilon_{0M}\epsilon_{0'M'}\epsilon_{0N}\epsilon_{0'N'}(\nabla_{AA'}\nabla_{BB'} - \nabla_{BB'}\nabla_{AA'})\phi_{A_1 \ldots A'_{N'}} \\
&-\epsilon_{0'N'}\Gamma_{0ABB'}(\epsilon_{0'M'}\Gamma_{MA_1NA'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}A'N}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&+\epsilon_{0'N'}\Gamma_{MABB'}(\epsilon_{0'M'}\Gamma_{NA_1 0A'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0N}\bar{\Gamma}_{M'A'_{1'}A'0}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&+\epsilon_{0'N'}\Gamma_{0BAA'}(\epsilon_{0'M'}\Gamma_{NA_1MB'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}B'N}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&-\epsilon_{0'N'}\Gamma_{MBAA'}(\epsilon_{0'M'}\Gamma_{NA_1 0B'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0N}\bar{\Gamma}_{M'A'_{1'}B'0}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&-\epsilon_{0N}\bar{\Gamma}_{0'A'B'B}(\epsilon_{0'M'}\Gamma_{MA_1AN'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}N'A}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&+\epsilon_{0N}\bar{\Gamma}_{N'A'B'B}(\epsilon_{0'M'}\Gamma_{MA_1A0'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}0'A}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&+\epsilon_{0N}\bar{\Gamma}_{0'B'A'A}(\epsilon_{0'M'}\Gamma_{MA_1BN'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}N'B}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&-\epsilon_{0N}\bar{\Gamma}_{N'B'A'A}(\epsilon_{0'M'}\Gamma_{MA_1B0'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}0'B}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&+\Psi_{ABMN}\epsilon_{A'B'}\epsilon_{0'M'}\epsilon_{0'N'}o_{A_1}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\bar{\Psi}_{A'B'M'N'}\epsilon_{AB}\epsilon_{0M}\epsilon_{0N}\bar{o}_{A'_{1'}}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}} \\
&+\Phi_{ABN'M'}\epsilon_{0N}\epsilon_{0M}\epsilon_{A'B'}\bar{o}_{A'_{1'}}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}} \\
&+\Phi_{NMA'B'}\epsilon_{0'N'}\epsilon_{0'M'}\epsilon_{AB}o_{A_1}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&+\Lambda\epsilon_{0'N'}\epsilon_{0'M'}\epsilon_{A'B'}(\epsilon_{MA}\epsilon_{NB} + \epsilon_{NA}\epsilon_{MB})o_{A_1}o^{E}\phi_{EA_2 \ldots A'_{N'}} = \qquad (2.1.104) \\
&\epsilon_{0'N'}\Gamma_{0ABB'}(\epsilon_{0'M'}\epsilon_{0N}\nabla_{MA'}\phi_{A_1 \ldots A'_{N'}} \\
&-\epsilon_{0'M'}\Gamma_{MA_1NA'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&-\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}A'N}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&-\epsilon_{0'N'}\Gamma_{MABB'}(\epsilon_{0'M'}\epsilon_{0N}\nabla_{0A'}\phi_{A_1 \ldots A'_{N'}} \\
&-\epsilon_{0'M'}\Gamma_{NA_1 0A'}o^{E}\phi_{EA_2 \ldots A'_{N'}} \\
&-\epsilon_{0N}\bar{\Gamma}_{M'A'_{1'}A'0}\bar{o}^{E'}\phi_{A_1 \ldots A_N E'A'_{2'} \ldots A'_{N'}}) \\
&-\epsilon_{0'N'}\Gamma_{0BAA'}(\epsilon_{0'M'}\epsilon_{0N}\nabla_{MB'}\phi_{A_1 \ldots A'_{N'}} \\
&-\epsilon_{0'M'}\Gamma_{NA_1MB'}o^{E}\phi_{EA_2 \ldots A'_{N'}}
\end{aligned}
$$

$$-\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}B'N}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}})$$
$$+\epsilon_{0'N'}\Gamma_{MBAA'}(\epsilon_{0'M'}\epsilon_{0N}\nabla_{0B'}\phi_{A_1...A'_{N'}}$$
$$-\epsilon_{0'M'}\Gamma_{NA_10B'}o^{E}\phi_{EA_2...A'_{N'}}$$
$$-\epsilon_{0N}\bar{\Gamma}_{M'A'_{1'}B'0}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}})$$
$$+\epsilon_{0N}\bar{\Gamma}_{0'A'B'B}(\epsilon_{0M}\epsilon_{0'M'}\nabla_{AN'}\phi_{A_1...A'_{N'}}$$
$$-\epsilon_{0'M'}\Gamma_{MA_1AN'}o^{E}\phi_{EA_2...A'_{N'}}$$
$$-\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}N'A}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}})$$
$$-\epsilon_{0N}\bar{\Gamma}_{N'A'B'B}(\epsilon_{0M}\epsilon_{0'M'}\nabla_{A0'}\phi_{A_1...A'_{N'}}$$
$$-\epsilon_{0'M'}\Gamma_{MA_1A0'}o^{E}\phi_{EA_2...A'_{N'}}$$
$$-\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}0'A}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}})$$
$$-\epsilon_{0N}\bar{\Gamma}_{0'B'A'A}(\epsilon_{0M}\epsilon_{0'M'}\nabla_{BN'}\phi_{A_1...A'_{N'}}$$
$$-\epsilon_{0'M'}\Gamma_{MA_1BN'}o^{E}\phi_{EA_2...A'_{N'}}$$
$$-\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}N'B}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}})$$
$$+\epsilon_{0N}\bar{\Gamma}_{N'B'A'A}(\epsilon_{0M}\epsilon_{0'M'}\nabla_{B0'}\phi_{A_1...A'_{N'}}$$
$$-\epsilon_{0'M'}\Gamma_{MA_1B0'}o^{E}\phi_{EA_2...A'_{N'}}$$
$$-\epsilon_{0M}\bar{\Gamma}_{M'A'_{1'}0'B}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}})$$
$$+\Psi_{ABMN}\epsilon_{A'B'}\epsilon_{0'M'}\epsilon_{0'N'}o_{A_1}o^{E}\phi_{EA_2...A'_{N'}}$$
$$+\bar{\Psi}_{A'B'M'N'}\epsilon_{AB}\epsilon_{0M}\epsilon_{0N}\bar{o}_{A'_{1'}}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}}$$
$$+\Phi_{ABN'M'}\epsilon_{0N}\epsilon_{0M}\epsilon_{A'B'}\bar{o}_{A'_{1'}}\bar{o}^{E'}\phi_{A_1...A_NE'A'_{2'}...A'_{N'}}$$
$$+\Phi_{NMA'B'}\epsilon_{0'N'}\epsilon_{0'M'}\epsilon_{AB}o_{A_1}o^{E}\phi_{EA_2...A'_{N'}}$$
$$+\Lambda\epsilon_{0'M'}\epsilon_{0'N'}\epsilon_{A'B'}(\epsilon_{MA}\epsilon_{NB}+\epsilon_{NA}\epsilon_{MB})o_{A_1}o^{E}\phi_{EA_2...A'_{N'}}$$

By further contractions with os and $\bar{o}$s and symmetrizing on all indices, and making use of equations 2.1.38 through 2.1.45 and equations 2.1.81 through 2.1.84 we are able to obtain equation 2.1.102 and all the rest of the commutators which we write below:

$$(\delta'_{A'AB}\,\delta_{CBC'} - \delta_{CBC'}\,\delta'_{A'AB})\phi_{A_1...A_NA'_{1'}...A'_{N'}}$$
$$= (o_A\overline{R}_{A'}\mathbf{D}'_{B'C'BC} - \bar{o}_{A'}R_A\mathbf{D}'_{B'C'BC} + \bar{o}_{A'}B_{B'C}\,\delta'_{C'AB} -$$
$$-o_C\overline{A}_{B'C'}\,\delta'_{A'AB} + \bar{o}_{A'}A_{AB}\,\delta_{CBC'} - o_A\overline{B}_{A'B}\,\delta_{CBC'})$$
$$\phi_{A_1...A'_{N'}} - (\overline{\Psi}_{A'B'C'}\;o_Ao_Bo_C\bar{o}_{A'_{1'}}\bar{o}^{E'} - \Phi_{ABA'}\;o_Co_B\bar{o}_{C'}\bar{o}_{A'_{1'}}\bar{o}^{E'}) \qquad (2.1.105)$$
$$\phi_{A_1...A_NE'A'_{2'}...A'_{N'}} - (\Phi_{ABA'}\;o_C\bar{o}_{B'}\bar{o}_{C'}o_{A_1}o^{E} - \Psi_{ABC}\,\bar{o}_{A'}\bar{o}_{B'}\bar{o}_{C'}o_{A_1}o^{E})$$

$$\phi_{EA_2\ldots A'_{N'}}$$

$$(\mathbf{D}'_{ABA'B'}\ \mathbf{D}_{CC'} - \mathbf{D}_{CC'}\ \mathbf{D}'_{ABA'B'})\phi_{A_1\ldots A_N A'_{1'}\ldots A'_{N'}}$$

$$(\overline{o}_{B'} G_{ABA'}\ \mathbf{D}_{CC'} + o_B \overline{G}_{A'B'A}\ \mathbf{D}_{CC'} + \overline{o}_{C'} E_C \mathbf{D}'_{ABA'B'} +$$

$$o_C \overline{E}_{C'} \mathbf{D}'_{ABA'B'} - \overline{o}_{B'} T_{CC'}\ \boldsymbol{\delta}'_{A'AB} - o_C \overline{T}_{BC'}\ \boldsymbol{\delta}_{AA'B'})$$

$$\phi_{A_1\ldots A'_{N'}} - (\boldsymbol{\Psi}_{ABC}\ \overline{o}_{A'} \overline{o}_{B'} \overline{o}_{C'} o_{A_1} o^E + \boldsymbol{\Phi}_{ABA'}\ \overline{o}_{B'} \overline{o}_{C'} o_C o_{A_1} o^E) \tag{2.1.106}$$

$$\phi_{EA_2\ldots A'_{N'}} - (\overline{\boldsymbol{\Psi}}_{A'B'C'}\ o_A o_B o_C \overline{o}_{A'_{1'}} \overline{o}^{E'} + \boldsymbol{\Phi}_{AA'B'}\ o_B o_C \overline{o}_{C'} \overline{o}_{A'_{1'}} \overline{o}^{E'})$$

$$\phi_{A_1\ldots A_N E' A'_{2'}\ldots A'_{N'}}$$

$$(\boldsymbol{\delta}_{AA'B'}\ \mathbf{D}'_{BCC'D'} - \mathbf{D}'_{BCC'D'}\ \boldsymbol{\delta}_{AA'B'})\phi_{A_1\ldots A_N A'_{1'}\ldots A'_{N'}}$$

$$(\overline{o}_{B'} T_{AA'}\ \mathbf{D}'_{BCC'D'} - \overline{o}_{B'} B_{AA'}\ \mathbf{D}'_{BCC'D'}$$

$$-o_A \overline{A}_{A'B'}\ \mathbf{D}'_{BCC'D'} - \overline{o}_{D'} G_{BCC'}\ \boldsymbol{\delta}_{AA'B'} + o_B G_{C'D'C}\ \boldsymbol{\delta}_{AA'B'}) \tag{2.1.107}$$

$$\phi_{A_1\ldots A'_{N'}} + \boldsymbol{\Phi}_{ABA'B'}\ o_C \overline{o}_{C'} \overline{o}_{D'} o_{A_1} o^E \phi_{EA_2\ldots A'_{N'}} +$$

$$+\overline{\boldsymbol{\Psi}}_{A'B'C'D'}\ o_A o_B o_C \overline{o}_{A'_{1'}} \overline{o}^{E'} \phi_{A_1\ldots A_N E' A'_{2'}\ldots A'_{N'}}$$

It is worth mentioning once more that all of the above expressions are symmetric on all primed and unprimed indices and that the standard sign convention is used.

We now concentrate on determining the Ricci equations in invariant form. It is clear that the generalized Ricci equations will only involve the invariants $K, R_A, S_{A'}, T_{AA'}, B_{AA'}, E_A, A_{(AB)}, G_{(AB)A'}$ and the invariant operators $D_{AA'}, \delta_{A(A'B')}, \delta'_{(AB)A'}, D'_{(AB)(A'B')}$. For example the Ricci equation (4.2a) of [23] will have the following generalized version:

$$\mathbf{D}_{B'(B} R_{A)} - \boldsymbol{\delta}'_{B'(BA)}\ \kappa = R_{(B} R_{A)} \overline{o}_{B'} + S_{B'} \overline{S}_{(A} o_{B)}$$

$$-3\kappa A_{(AB)}\,\overline{o}_{B'} - \kappa \overline{B}_{B'(A}\, o_{B)} + R_{(A} E_{B)} \overline{o}_{B'} + o_{(A} R_{B)}\, \overline{E}_{B'} \tag{2.1.108}$$

$$-\overline{\kappa} T_{B'(A}\, o_{B)} + \boldsymbol{\Phi}_0 o_A o_B \overline{o}_{B'}$$

Note that both $\mathbf{D}_{B'(B}R_{A)}$ and $\boldsymbol{\delta}'_{B'(BA)}\,k$ are 2,1-spinors so that it makes sense to consider their difference. The calculation leading to equation 2.1.108 is a straightforward one.

In fact all generalized Ricci equations can be obtained from the NP Ricci equations in a straightforward way. All NP spin coefficients in the equations become the corresponding invariant form, for example ρ in the NP Ricci equation becomes R_A in the generalized equation, the same occurring in relation to the differential operators. As for those terms in the NP Ricci equations which transform "badly" under null rotations, they are simply "crossed out" when going from the NP version to the generalized version since these terms will be "tucked away" in the invariant operators.

On the other hand one can obtain a general formula giving all Ricci equations in a similar way one obtained the general formula for the commutators in generalized notation.

We take the general equation 3.14 of [23] giving the Ricci equations in NP formalism and multiply each side by $\epsilon_{EF}\epsilon_{C'D'}$ and substitute non bold indices for bold indices so that we get:

$$\begin{aligned}
&\epsilon_{\mathbf{EF}}\epsilon_{\mathbf{C'D'}}(\nabla_{\mathbf{AA'}}\Gamma_{\mathbf{BCDB'}} - \nabla_{\mathbf{DB'}}\Gamma_{\mathbf{BCAA'}}) = \epsilon_{\mathbf{C'D'}}(\Gamma_{\mathbf{BEDB'}}\Gamma_{\mathbf{FCAA'}}\\
&+\Gamma_{\mathbf{BCEB'}}\Gamma_{\mathbf{FDAA'}} - \Gamma_{\mathbf{BEAA'}}\Gamma_{\mathbf{FCDB'}} - \Gamma_{\mathbf{BCEA'}}\Gamma_{\mathbf{FADB'}}\\
&-\Gamma_{\mathbf{BFDB'}}\Gamma_{\mathbf{ECAA'}} - \Gamma_{\mathbf{BCFB'}}\Gamma_{\mathbf{EDAA'}} + \Gamma_{\mathbf{BFAA'}}\Gamma_{\mathbf{ECDB'}}\\
&+\Gamma_{\mathbf{BCFA'}}\Gamma_{\mathbf{EADB'}}) + \epsilon_{\mathbf{EF}}(\Gamma_{\mathbf{BCDC'}}\Gamma_{\mathbf{D'B'A'A}}\\
&-\Gamma_{\mathbf{BCAC'}}\Gamma_{\mathbf{D'A'B'D}} - \Gamma_{\mathbf{BCDD'}}\Gamma_{\mathbf{C'B'A'A}} + \Gamma_{\mathbf{BCAD'}}\Gamma_{\mathbf{C'A'B'D}})\\
&+\epsilon_{\mathbf{EF}}\epsilon_{\mathbf{C'D'}}\epsilon_{\mathbf{A'B'}}\boldsymbol{\Psi}_{\mathbf{BCDA}} + \boldsymbol{\Lambda}\epsilon_{\mathbf{EF}}\epsilon_{\mathbf{C'D'}}\epsilon_{\mathbf{A'B'}}(\epsilon_{\mathbf{CD}}\epsilon_{\mathbf{BA}}\\
&+\epsilon_{\mathbf{CD}}\epsilon_{\mathbf{CA}}) + \epsilon_{\mathbf{EF}}\epsilon_{\mathbf{C'D'}}\epsilon_{\mathbf{AD}}\boldsymbol{\Phi}_{\mathbf{BCB'A'}}
\end{aligned} \tag{2.1.109}$$

Contracting with o's and $\overline{o}$'s as appropriate and symmetrising on all the indices gives 2.1.108 and all other Ricci equations, which are displayed below.

$$\begin{aligned}
&\mathbf{D}_{B(B}\,S_{A')} - \boldsymbol{\delta}_{(B'A')B}\,K = R_B S_{(B'}\overline{o}_{A')} + o_B S_{(B'}\overline{R}_{A')}\\
&-S_{(A'}\bar{E}_{B')}\,o_B + 3E_B S_{(A'}\overline{o}_{B')} - KT_{B(A'}\,\overline{o}_{B')} - K\bar{A}_{(A'B')}\,o_B\\
&-3KB_{B(A'}\,\overline{o}_{B')} + \boldsymbol{\Psi}_0 o_B\overline{o}_{A'}\overline{o}_{B'}
\end{aligned} \tag{2.1.110}$$

$$\begin{aligned}
&\mathbf{D}_{BB'}\,T_{A'A} - \boldsymbol{\Delta}_{(AB)(A'B')}\;K = R_{(B}T_{A)(A'}\,\overline{o}_{B')} + S_{(B'}\bar{T}_{A')(A}\,o_{B)}\\
&+\overline{o}_{(B'}T_{A')(A}\,E_{B)} - o_{(B}T_{A)(A'}\,\overline{E}_{B')} - K\bar{G}_{(A'B')(A}\,o_{B)}\\
&-3KG_{(AB)(A'}\,\overline{o}_{B')} + \boldsymbol{\Psi}_{(A}o_{B)}\,\overline{o}_{A'}\,\overline{o}_{B'} + \boldsymbol{\Phi}_{(A'}\,\overline{o}_{B')}\,o_A o_B
\end{aligned} \tag{2.1.111}$$

$$\begin{aligned}&\boldsymbol{\delta}_{(B'A')(B}\, R_{A)} - \boldsymbol{\delta}'_{(AB)(B'}\, S_{A')} = \overline{o}_{(B'} T_{A')(A}\, R_{B)} - o_{(B} T_{A)(A'}\, \overline{R}_{B')} \\ &+\bar{A}_{(A'B')}\, R_{(A} o_{B)} + \bar{o}_{(B'} B_{A')(A}\, R_{B)} - 3\bar{o}_{(B'} S_{A')}\, A_{(AB)} \\ &+S_{(A'} \overline{B}_{B')(A}\, o_{B)} - \boldsymbol{\Psi}_{(A} o_{B)} \overline{o}_{A'} \overline{o}_{B'} + \boldsymbol{\Phi}_{(A'} \overline{o}_{B')}\, o_A o_B\end{aligned} \tag{2.1.112}$$

$$\begin{aligned}&\boldsymbol{\delta}_{B'A'B}\, T_{AC'} - \mathbf{D}'_{(AB)(A'B'}\, S_{C')} = T_{AB'}\, T_{A'B}\, \overline{o}_{C'} + T_{AB'}\, B_{A'B}\, \overline{o}_{C'} \\ &-o_{(B} T_{A)(B'}\, \overline{A}_{A'C')} - 3S_{A'}\, G_{ABB'}\, \overline{o}_{C'} + S_{(A'} \overline{G}_{B'C')(A}\, o_{B)} \\ &+\boldsymbol{\Phi}_{(A'B'}\, \overline{o}_{C')}\, o_A o_B\end{aligned} \tag{2.1.113}$$

$$\begin{aligned}&\mathbf{D}'_{(A'B')(AB}\, R_{C)} - \boldsymbol{\delta}'_{B'BA}\, T_{CA'} = -o_C T_{AB'}\, \overline{T}_{BA'} + R_{(C} G_{AB)(A'}\, \overline{o}_{B')} \\ &+\overline{G}_{(A'B')(A}\, R_C o_{B)} - \overline{o}_{(B'} T_{A')(A}\, A_{BC)} + T_{AA'}\, \overline{B}_{B'B}\, o_C \\ &-\boldsymbol{\Psi}_{(AB}\, o_{C)} \overline{o}_{A'} \overline{o}_{B'} - 2\boldsymbol{\Lambda} o_A o_B o_C \bar{o}_{A'} \overline{o}_{B'}\end{aligned} \tag{2.1.114}$$

$$\begin{aligned}&\mathbf{D}'_{A'B'AB}\, B_{CC'} - \boldsymbol{\delta}_{A'B'A}\, G_{BCC'} = 2B_{AA'}\, G_{BCB'}\, \bar{o}_{C'} \\ &-B_{AA'}\, \overline{G}_{B'C'B}\, o_C + o_{(C} G_{AB)(A'}\, \overline{A}_{B'C')} - G_{ABA'}\, T_{CB'}\, \overline{o}_{C'} \\ &-o_{(B} o_C \boldsymbol{\Phi}_{A)(A'B'}\, \overline{o}_{C')}\end{aligned} \tag{2.1.115}$$

$$\begin{aligned}&\boldsymbol{\delta}'_{B'(BA}\, E_{C)} - \mathbf{D}_{B'(B}\, A_{AC)} = o_{(C} G_{AB)B'}\, \overline{K} - B_{B'(A}\, \overline{S}_B o_{C)} \\ &-\bar{o}_{B'}\, A_{(AB}\, R_{C)} - o_{(C}\, A_{AB)}\, \overline{E}_{B'} + 2\overline{o}_{B'}\, A_{(AB}\, E_{C)} \\ &+o_{(C}\, E_A \overline{B}_{B)B'} - \boldsymbol{\Phi}_{(A} o_B o_{C)} \overline{o}_{B'}\end{aligned} \tag{2.1.116}$$

$$\begin{aligned}&\boldsymbol{\delta}'_{B'BA}\, G_{CDA'} - \mathbf{D}'_{(A'B')(AB}\, A_{CD)} = -\overline{G}_{(A'B')(A}\, A_{BC}\, o_{D)} \\ &+\overline{T}_{AA'}\, G_{B'BC}\, o_D - G_{ACA'}\, \overline{B}_{B'B}\, o_D + \boldsymbol{\Psi}_{(ABC}\, o_{D)} \overline{o}_{A'} \overline{o}_{B'}\end{aligned} \tag{2.1.117}$$

$$\begin{aligned}&\mathbf{D}_{BB'}\, G_{ACA'} - \mathbf{D}'_{(A'B')(AB}\, E_{C)} = B_{AA'}\, \overline{T}_{B'B}\, o_C \\ &+\overline{o}_{(A'} T_{B')(B}\, A_{AC)} - 2E_{(A} G_{BC)(B'}\, \overline{o}_{A')} - \overline{G}_{(B'A')(A}\, E_B o_{C)} \\ &-o_{(C} G_{AB)(B'}\, \overline{E}_{A')} + \boldsymbol{\Psi}_{(AB}\, o_{C)} \overline{o}_{A'} \overline{o}_{B'} + \overline{o}_{(B'} \boldsymbol{\Phi}_{A')(A}\, o_B o_{C)} \\ &-\boldsymbol{\Lambda} o_A o_B o_C \overline{o}_{A'} \overline{o}_{B'}\end{aligned} \tag{2.1.118}$$

$$\delta'_{B'BC}\, B_{A'A} - \boldsymbol{\delta}_{(B'A')(A}\, A_{BC)} = -o_{(A} A_{BC)}\, \overline{A}_{(A'B')}$$
$$-B_{A'A}\overline{B}_{B'B}\, o_C + 2A_{(AB}\, B_{C)(B'}\, \overline{o}_{A')} + o_{(C} G_{AB)(A'}\, \overline{R}_{B')}$$
$$-\overline{o}_{(B'} G_{A')(AB}\, R_{C)} + \boldsymbol{\Psi}_{(AB}\, o_{C)}\overline{o}_{A'}\overline{o}_{B'} - \overline{o}_{(A'}\boldsymbol{\Phi}_{B')(A}\, o_B o_{C)} \tag{2.1.119}$$
$$-\boldsymbol{\Lambda} o_A o_B o_C \overline{o}_{A'}\overline{o}_{B'}$$

Finally we concentrate on obtaining the invariant version of the Bianchi identities. As in the case of the Ricci equations we can obtain the invariant version of the Bianchi identities in a straightforward way. For example the NP Bianchi identity:

$$D\Psi_1 - \delta'\Psi_0 - D\Phi_{01'} + \delta\Phi_0 = -3\kappa\Psi_2 + (2\epsilon + 4\rho)\Psi_1 - (-\pi + 4\alpha)\Psi_0$$
$$-2\epsilon\Phi_{01'} - 2\beta\Phi_0 + \overline{\beta}'\Phi_0$$

has the following invariant version:

$$\overline{o}_{(A'}\mathbf{D}_{B')(B}\,\boldsymbol{\Psi}_{A)} - \overline{o}_{(A'}\boldsymbol{\delta}'_{B')(BA)}\,\boldsymbol{\Psi}_0 - o_{(B}\mathbf{D}_{A)(A'}\,\boldsymbol{\Phi}_{B')} + o_{(B}\boldsymbol{\delta}_{A)(A'B')}\,\boldsymbol{\Phi}_0$$
$$= -3\kappa\boldsymbol{\Psi}_{(AB)}\,\overline{o}_{A'}\overline{o}_{B'} + 2E_{(A}\boldsymbol{\Psi}_{B)}\overline{o}_{A'}\overline{o}_{B'} + 4R_{(A}\boldsymbol{\Psi}_{B)}\overline{o}_{A'}\overline{o}_{B'}$$
$$-4A_{(AB)}\,\boldsymbol{\Psi}_0\overline{o}_{A'}\overline{o}_{B'} + 2o_{(B}E_{A)}\boldsymbol{\Phi}_{(A'}\overline{o}_{B')} - 2o_{(B}B_{A)(A'}\,o_{B')}\boldsymbol{\Phi}_0 \tag{2.1.120}$$
$$-2o_A o_B\overline{A}_{(A'B')}\boldsymbol{\Phi}_0 + 2o_{(A}B_{B)(A'}\,\overline{o}_{B')}\boldsymbol{\Phi}_0 - 2o_{(A}E_{B)}\,\boldsymbol{\Phi}_{(A'}\overline{o}_{B')}$$
$$-2\overline{R}_{(A'}\boldsymbol{\Phi}_{B')}o_A o_B - 2\overline{o}_{(A'}S_{B')}\boldsymbol{\Phi}_{(A}o_{B)} + 2K o_{(A}\boldsymbol{\Phi}_{B)(A'}\,\overline{o}_{B')}$$
$$+\overline{K}\boldsymbol{\Phi}_{(A'B')}o_A o_B$$

Alternatively we can determine a general expression giving all Bianchi identities written in our new invariant formalism. Multiplying expression 3.19 of [23] by $\epsilon_{L'N'}$, ϵ_{LN}, $\epsilon_{K'M'}$ and ϵ_{KM} and substituting non bold indices for bold indices gives:

$$\epsilon_{\mathbf{K'M'}}\epsilon_{\mathbf{L'N'}}\epsilon_{\mathbf{KM}}(\nabla_{\mathbf{ND'}}\boldsymbol{\Psi}_{\mathbf{ABCL}} - \nabla_{\mathbf{LD'}}\boldsymbol{\Psi}_{\mathbf{ABCN}}) - \epsilon_{\mathbf{K'M'}}\epsilon_{\mathbf{LN}}\epsilon_{\mathbf{KM}}$$
$$(\nabla_{(\mathbf{CN'}}\boldsymbol{\Phi}_{\mathbf{AB})\mathbf{D'L'}} - \nabla_{(\mathbf{CL'}}\boldsymbol{\Phi}_{\mathbf{AB})\mathbf{D'N'}}) = 3\epsilon_{\mathbf{L'N'}}\epsilon_{\mathbf{K'M'}}(\boldsymbol{\Psi}_{\mathbf{LK(AB}}$$
$$\Gamma_{\mathbf{C)NMD'}} - \boldsymbol{\Psi}_{\mathbf{LM(AB}}\Gamma_{\mathbf{C)NKD'}} - \boldsymbol{\Psi}_{\mathbf{NK(AB}}\Gamma_{\mathbf{C)LMD'}} + \boldsymbol{\Psi}_{\mathbf{NM(AB}}$$
$$\Gamma_{\mathbf{C)LKD'}}) + \epsilon_{\mathbf{L'N'}}\epsilon_{\mathbf{K'M'}}(\boldsymbol{\Psi}_{\mathbf{ABCL}}\Gamma_{\mathbf{NKMD'}} - \boldsymbol{\Psi}_{\mathbf{ABCL}}\Gamma_{\mathbf{NMKD'}} - \boldsymbol{\Psi}_{\mathbf{ABCN}}$$

$$\Gamma_{LKMD'} + \Psi_{ABCN}\Gamma_{LMKD'}) - 2\epsilon_{K'M'}\epsilon_{LN}(\Gamma_{M(ABN'}\Phi_{C)KL'D'} \quad (2.1.121)$$

$$-\Gamma_{M(ABL'}\Phi_{C)KN'D'} - \Gamma_{K(ABN'}\Phi_{C)ML'D'} + \Gamma_{K(ABL'}\Phi_{C)MN'D'})$$

$$-\epsilon_{LN'}\epsilon_{KM}(\overline{\Gamma}_{L'D'K'(A}\Phi_{BC)N'M'} - \overline{\Gamma}_{L'D'M'(A}\Phi_{BC)N'K'} - \overline{\Gamma}_{N'D'K'(A}$$

$$\Phi_{BC)L'M'} + \overline{\Gamma}_{N'D'M'(A}\Phi_{BC)L'D'} + \overline{\Gamma}_{L'M'K'(A}\Phi_{BC)N'D'} - \overline{\Gamma}_{L'K'M'(A}$$

$$\Phi_{BC)N'D'} - \overline{\Gamma}_{N'M'K'(A}\Phi_{BC)L'D'} + \overline{\Gamma}_{N'K'M'(A}\Phi_{BC)L'D'})$$

By contracting with suitable factors of o and $\overline{o}$ and symmetrizing one may obtain equation 2.1.120 and all other Bianchi identities from expression 2.1.121, these are written below:

$$3(\overline{o}_{(A'}\mathbf{D}_{B')(B}\Psi_{AC)} - \overline{o}_{(A'}\delta'_{B')(BA}\Psi_{C)}) - 2(o_C\mathbf{D}_{AA'}\Phi_{BB'} - o_B\delta_{AA'B'}\Phi_C)$$

$$+o_{(C}\mathbf{D}'_{AB)(A'B')}\Phi_{00'} - o_{(C}\delta'_{AB)(A'}\Phi_{B')} = -6\kappa\Psi_{(ABC)}\overline{o}_{(A'}\overline{o}_{B')}$$

$$+9R_{(A}\Psi_{BC)}\overline{o}_{A'}\overline{o}_{B'} - 6A_{(AB}\Psi_{C)}\overline{o}_{A'}\overline{o}_{B'} + 2o_{(A}o_BG_{C)(A'B')}\Phi_{00'}$$

$$+2\overline{o}_{(B'}\overline{G}_{A')(AB}o_{C)}\Phi_{00'} - 2o_{(C}A_{AB)}\Phi_{(A'}\overline{o}_{B')} - 2o_{(A}o_B\overline{T}_{C)(A'}\Phi_{B')} \quad (2.1.122)$$

$$-2\overline{o}_{(A'}T_{B')(A}\Phi_Bo_{C)} + 4\overline{A}_{(A'B')}\Phi_{(A}o_Bo_{C)} - 4\overline{R}_{(A'}\Phi_{B')(A}o_Bo_{C)}$$

$$+2o_{(C}R_A\Phi_{B)(A'}\overline{o}_{B')} - 2\overline{o}_{(A'}S_{B')}\Phi_{(AB}o_{C)} + 2o_{(A}o_B\overline{S}_{C)}\Phi_{(A'B')}$$

$$+2\overline{K}o_{(A}o_B\Phi_{C)(A'B')} + 2Ko_{(A}\iota_{BC)}\overline{o}_{A'}\overline{o}_{B'}$$

$$3(\overline{o}_{(A'}\mathbf{D}_{B')(B}\Psi_{ACD)} - \overline{o}_{(A'}\delta'_{B')(BA}\Psi_{CD)}) - 2(o_D\delta'_{B'BA}\Phi_{CA'}$$

$$-\mathbf{D}'_{(A'B')(AB}\Phi_Co_{D)}) + \delta_{(A'B')(A}\Phi_{BC}o_{D)} - D_{A'A}\Phi_{BCB'}o_D =$$

$$-3K\Psi_{(ABCD)}\overline{o}_{A'}\overline{o}_{B'} - 6E_{(A}\Psi_{BCD)}\overline{o}_{A'}\overline{o}_{B'} + 6R_{(A}\Psi_{BCD)}\overline{o}_{A'}\overline{o}_{B'}$$

$$+4\overline{G}_{(A'B')(A}\Phi_Bo_Co_{D)} - 4\overline{T}_{AA'}\Phi_{BB'}o_Co_D - 2o_{(B'}B_{A')(A}\Phi_{BC}o_{D)} \quad (2.1.123)$$

$$-2o_{(B'}T_{A')(A}\Phi_{BC}o_{D)} + 2\overline{A}_{(A'B')}\Phi_{(AB}o_Co_{D)} + 2o_{(A}o_B\overline{S}_C\Phi_{D)(A'B')}$$

$$-2\overline{R}_{(A'}\Phi_{B')(AB}o_Co_{D)} + 2o_{(D}R_C\Phi_{BA)(A'}\overline{o}_{B')} + 2o_{(D}E_C\Phi_{BA)(A'}\overline{o}_{B')}$$

$$+\overline{K}o_{(A}o_B\Phi_{CD)(A'B')}$$

$$\overline{o}_{(A'}\mathbf{D}_{B')(B}\Psi_{ACDE)} - \overline{o}_{(A'}\delta'_{B')(BA}\Psi_{CDE)} + \mathbf{D}'_{(A'B')(AB}\Phi_{CD}o_{E)}$$

$$+\delta'_{A'AB}\Phi_{CDB'}o_E = -4E_{(A}\Psi_{BCDE)}\bar{o}_{A'}\bar{o}_{B'} + R_{(A}\Psi_{BCDE)}\bar{o}_{A'}\bar{o}_{B'}$$

$$+2A_{(AB}\Psi_{CDE)}\bar{o}_{A'}\bar{o}_{B'} - 2\bar{o}_{(B'}G_{A')(AB}\Phi_{CD}o_{E)} + 2\overline{G}_{(A'B')(A} \qquad (2.1.124)$$

$$\Phi_{BC}o_Do_{E)} - 2\overline{T}_{AA'}\Phi_{BCB'}o_Do_E + o_{(A}A_{BC}\Phi_{DE)(A'}\bar{o}_{B')}$$

$$+o_{(A}o_B\overline{S}_C\Phi_{DE)(A'B')}$$

$$\bar{o}_{(C'}\mathbf{D}'_{A'B')(AB)}\Psi_0 - \bar{o}_{(C'}\delta_{A'B')(A}\Psi_{B)} + o_{(B}\mathbf{D}_{A)(A'}\Phi_{B'C')}$$

$$-o_{(B}\delta_{A)(A'B'}\Phi_{C')} = 4\bar{o}_{(C'}\bar{o}_{B'}G_{A')(AB)}\Psi_0 - 4\bar{o}_{(C'}\bar{o}_{B'}T_{A')(A}\Psi_{B)}$$

$$-2\bar{o}_{(C'}\bar{o}_{B'}B_{A')(A}\Psi_{B)} + 3\bar{o}_{(C'}\bar{o}_{B'}S_{A')}\Psi_{(AB)} - 2o_{(A}B_{B)(A'}\Phi_{B'}\bar{o}_{C')} \qquad (2.1.125)$$

$$+2\bar{o}_{(A'}S_{B'}\Phi_{C')(A}o_{B)} + 2o_{(A}E_{B)}\Phi_{(A'B'}\bar{o}_{C')} - 2o_Ao_B\overline{E}_{(A'}\Phi_{B'C')}$$

$$+o_Ao_B\overline{R}_{(A'}\Phi_{B'C')} - 2Ko_{(B}\Phi_{A)(A'B'}\bar{o}_{C')}$$

$$3(\bar{o}_{(C'}\mathbf{D}'_{A'B')(AB}\Psi_{C)} - \bar{o}_{(C'}\delta_{A'B')(A}\Psi_{BC)}) + 2(o_C\mathbf{D}_{AA'}\Phi_{BB'C'}$$

$$-o_C\delta_{AA'B'}\Phi_{BC'}) + o_{(C}\delta'_{AB)(A'}\Phi_{B'C')} - o_{(C}\mathbf{D}'_{AB)(A'B'}\Phi_{C')} =$$

$$6\bar{o}_{(A'}\bar{o}_{B'}G_{C')(AB}\Psi_{C)} - 9\bar{o}_{(A'}\bar{o}_{B'}T_{C')(A}\Psi_{BC)} + 6\bar{o}_{(A'}\bar{o}_{B'}S_{C')}\Psi_{(ABC)}$$

$$-2o_{(C}G_{AB)(A'}\Phi_{B'}\bar{o}_{C')} + 2o_CT_{AA'}\Phi_{BB'}\bar{o}_{C'} + 2o_{(A}A_{BC)}\Phi_{(A'B'}\bar{o}_{C')} \qquad (2.1.126)$$

$$+o_{(A}o_B\overline{T}_{C)(A'}\Phi_{B'C')} - 2o_{(A}o_B\overline{B}_{C)(A'}\Phi_{B'C')} + 2\overline{R}_{(A'}\Phi_{B'C')(A}o_Bo_{C)}$$

$$-2o_{(A}R_B\Phi_{C)(A'B'}\bar{o}_{C')} - 4\overline{E}_{(A'}\Phi_{B'C')(A}o_Bo_{C)} + \bar{o}_{(A'}S_{B'}\Phi_{C')(AB}o_{C)}$$

$$-2Ko_{(A}\Phi_{BC)(A'B'}\bar{o}_{C')}$$

$$3(\bar{o}_{(C'}\mathbf{D}'_{A'B')(AB}\Psi_{CD)} - \bar{o}_{(C'}\delta_{A'B')(A}\Psi_{BCD)}) + 2(\delta'_{ABA'}\Phi_{CB'C'}o_D$$

$$-\mathbf{D}'_{ABA'B'}\Phi_{CC'}o_D) + \mathbf{D}_{AA'}\Phi_{BCB'C'}o_D - \delta_{A'B'A}\Phi_{BCC'}o_D$$

$$= -6\bar{o}_{(C'}\bar{o}_{B'}T_{A')(A}\Psi_{BCD)} + 6\bar{o}_{(C'}\bar{o}_{B'}B_{A')(A}\Psi_{BCD)} + 3\bar{o}_{(C'}\bar{o}_{B'}S_{A')}\Psi_{(ABCD)}$$

$$+2\overline{T}_{AA'}\Phi_{BB'C'}o_Co_D - 4\overline{B}_{AA'}\Phi_{BB'C'}o_Co_D + 2B_{AA'}\Phi_{BCB'}o_D\bar{o}_{C'} \qquad (2.1.127)$$

$$+2T_{AA'}\Phi_{BCB'}o_D\bar{o}_{C'} + \overline{R}_{(A'}\Phi_{B'C')(AB}o_Co_{D)} - 2o_{(A}E_B\Phi_{CD)(A'B'}\bar{o}_{C')}$$

$$-2\overline{E}_{(A'}\Phi_{B'C')(AB}o_Co_{D)} - 2o_{(A}R_B\Phi_{CD)(A'B'}\overline{o}_{C')}$$

$$\overline{o}_{(A}\mathbf{D}'_{B'C')(AB}\Psi_{CDE)} - \overline{o}_{(A'}\delta_{B'C')(A}\Psi_{BCDE)} + o_E\delta'_{ABA'}\Phi_{CDB'C'}$$

$$-o_E\mathbf{D}'_{ABA'B'}\Phi_{CDC'} = -2\overline{o}_{(B'}G_{A')(AB}\Psi_{CDE)} - \overline{o}_{(B'}T_{A')(A}\Psi_{BCDE)}$$

$$+4\overline{o}_{(B'}B_{A')(A}\Psi_{BCDE)} + 2G_{ABA'}\Phi_{CDB'}\overline{o}_{C'}o_E + \overline{T}_{AA'}\Phi_{BCB'C'}o_Do_E \quad (2.1.128)$$

$$-2\overline{B}_{AA'}\Phi_{BCB'C'}o_Do_E - 2o_{(A}A_{BC}\Phi_{DE)(A'B'}\overline{o}_{C')}$$

To obtain a general expression for the contracted Bianchi identities we multiply equation (3.20) of [23] by $\epsilon_{\mathbf{L'N'}}, \epsilon_{\mathbf{AN}}, \epsilon_{\mathbf{K'M'}}$ and $\epsilon_{\mathbf{KM}}$ and substitute non bold indices for bold indices so that we have:

$$3\epsilon_{\mathbf{L'N'}}\epsilon_{\mathbf{LN}}\epsilon_{\mathbf{K'M'}}\epsilon_{\mathbf{KM}}\nabla_{\mathbf{AB'}}\mathbf{\Lambda} + \epsilon_{\mathbf{LN}}(\nabla_{\mathbf{MN'}}\mathbf{\Phi}_{\mathbf{AKB'L'}}$$

$$-\nabla_{\mathbf{ML'}}\mathbf{\Phi}_{\mathbf{AKB'N'}} - \nabla_{\mathbf{KN'}}\mathbf{\Phi}_{\mathbf{AMB'L'}} + \nabla_{\mathbf{KL'}}\mathbf{\Phi}_{\mathbf{AMB'N'}})$$

$$= \epsilon_{\mathbf{LN}}(\mathbf{\Phi}_{\mathbf{AKM'N'}}\overline{\Gamma}_{\mathbf{B'K'L'M}} - \mathbf{\Phi}_{\mathbf{AKM'L'}}\overline{\Gamma}_{\mathbf{B'K'N'M}} - \mathbf{\Phi}_{\mathbf{AKK'N'}}\overline{\Gamma}_{\mathbf{B'M'L'M}}$$

$$+\mathbf{\Phi}_{\mathbf{AKK'L'}}\overline{\Gamma}_{\mathbf{B'M'N'M}} - \mathbf{\Phi}_{\mathbf{AMM'L'}}\overline{\Gamma}_{\mathbf{B'K'L'K}} + \mathbf{\Phi}_{\mathbf{AMM'L'}}\overline{\Gamma}_{\mathbf{B'K'N'K}}$$

$$+\mathbf{\Phi}_{\mathbf{AMK'N'}}\overline{\Gamma}_{\mathbf{B'M'L'K}} - \mathbf{\Phi}_{\mathbf{AMK'L'}}\overline{\Gamma}_{\mathbf{B'M'N'K}} + \mathbf{\Phi}_{\mathbf{AKB'M'}}\overline{\Gamma}_{\mathbf{K'N'L'M}}$$

$$-\mathbf{\Phi}_{\mathbf{AKB'M'}}\overline{\Gamma}_{\mathbf{K'L'N'M}} - \mathbf{\Phi}_{\mathbf{AKB'K'}}\overline{\Gamma}_{\mathbf{M'N'L'M}} + \mathbf{\Phi}_{\mathbf{AKB'K'}}\overline{\Gamma}_{\mathbf{M'L'N'M}}$$

$$-\mathbf{\Phi}_{\mathbf{AMB'M'}}\overline{\Gamma}_{\mathbf{K'N'L'K}} + \mathbf{\Phi}_{\mathbf{AMB'M'}}\overline{\Gamma}_{\mathbf{K'L'N'K}} + \mathbf{\Phi}_{\mathbf{AMB'M'}}\overline{\Gamma}_{\mathbf{M'N'L'K}}$$

$$-\mathbf{\Phi}_{\mathbf{AMB'K'}}\overline{\Gamma}_{\mathbf{M'L'N'K}}) - \epsilon_{\mathbf{L'N'}}(\mathbf{\Phi}_{\mathbf{KLB'M'}}\Gamma_{\mathbf{AMNK'}} \quad (2.1.129)$$

$$-\mathbf{\Phi}_{\mathbf{KNB'M'}}\Gamma_{\mathbf{AMLK'}} - \mathbf{\Phi}_{\mathbf{KLB'K'}}\Gamma_{\mathbf{AMNM'}} + \mathbf{\Phi}_{\mathbf{KNB'K'}}\Gamma_{\mathbf{AMLM'}}$$

$$-\mathbf{\Phi}_{\mathbf{MLB'M'}}\Gamma_{\mathbf{AKNK'}} + \mathbf{\Phi}_{\mathbf{MNB'M'}}\Gamma_{\mathbf{AKLK'}} + \mathbf{\Phi}_{\mathbf{MLB'K'}}\Gamma_{\mathbf{AKNM'}}$$

$$-\mathbf{\Phi}_{\mathbf{MNB'K'}}\Gamma_{\mathbf{AKLM'}} + \mathbf{\Phi}_{\mathbf{AKB'M'}}\Gamma_{\mathbf{MLNK'}} - \mathbf{\Phi}_{\mathbf{AKB'M'}}\Gamma_{\mathbf{MNLK'}}$$

$$-\mathbf{\Phi}_{\mathbf{AKB'K'}}\Gamma_{\mathbf{MLNM'}} + \mathbf{\Phi}_{\mathbf{AKB'K'}}\Gamma_{\mathbf{MNLM'}} - \mathbf{\Phi}_{\mathbf{AMB'M'}}\Gamma_{\mathbf{KLNK'}}$$

$$+\mathbf{\Phi}_{\mathbf{AMB'M'}}\Gamma_{\mathbf{KNLK'}} + \mathbf{\Phi}_{\mathbf{AMB'K'}}\Gamma_{\mathbf{KLNM'}} - \mathbf{\Phi}_{\mathbf{AMB'K'}}\Gamma_{\mathbf{KNLM'}})$$

Equation 2.1.129 gives all contracted Bianchi identities written in invariant formalism, they are:

$$\begin{aligned}
&\mathbf{D}_{AA'}\boldsymbol{\Phi}_{BB'} - \boldsymbol{\delta}_{(A'B')(A}\boldsymbol{\Phi}_{B)} - \boldsymbol{\delta}'_{(AB)(A'}\boldsymbol{\Phi}_{B')} + \mathbf{D}'_{(AB)(A'B')}\boldsymbol{\Phi}_0 \\
&+3\overline{o}_{(A'}\mathbf{D}_{B')(A}\boldsymbol{\Lambda}o_{B)} = 2\overline{o}_{(A'}G_{B')(AB)}\boldsymbol{\Phi}_0 + 2o_{(A}\overline{G}_{B)(A'B')}\boldsymbol{\Phi}_0 \\
&-2A_{(AB)}\boldsymbol{\Phi}_{(A'}\overline{o}_{B')} - 2o_{(A}\overline{T}_{B)(A'}\boldsymbol{\Phi}_{B')} - 2\overline{A}_{(A'B')}\boldsymbol{\Phi}_{(A}o_{B)} \\
&-2\overline{o}_{(A'}T_{B')(A}\boldsymbol{\Phi}_{B)} + 2R_{(A}\boldsymbol{\Phi}_{B)(A'}\overline{o}_{B')} + 2\overline{R}_{(A'}\boldsymbol{\Phi}_{B')(A}o_{B)} \\
&+o_{(A}\overline{S}_{B)}\boldsymbol{\Phi}_{(A'B')} + \overline{o}_{(A'}S_{B')}\boldsymbol{\Phi}_{(AB)} - o_{(A}\overline{K}\boldsymbol{\Phi}_{B)(A'B')} - K\boldsymbol{\Phi}_{(AB)(A'}\overline{o}_{B')}
\end{aligned} \tag{2.1.130}$$

$$\begin{aligned}
&\mathbf{D}_{AA'}\boldsymbol{\Phi}_{BB'C'} - \boldsymbol{\delta}_{A'B'A}\boldsymbol{\Phi}_{BC'} - \boldsymbol{\delta}'_{(AB)(A'}\boldsymbol{\Phi}_{B'C')} + \mathbf{D}'_{(AB)(A'B'}\boldsymbol{\Phi}_{C')} \\
&+3\overline{o}_{(A'}\boldsymbol{\delta}_{B'C')(AB)}\boldsymbol{\Lambda} = 2G_{(AB)(A'}\boldsymbol{\Phi}_{B'}\overline{o}_{C')} - 2T_{AA'}\boldsymbol{\Phi}_{BB'}\overline{o}_{C'} \\
&+2o_{(A}\overline{B}_{B)(A'}\boldsymbol{\Phi}_{B'C')} - 2A_{(AB)}\boldsymbol{\Phi}_{(A'B'}\overline{o}_{C')} - o_{(A}\overline{T}_{B)(A'}\boldsymbol{\Phi}_{B'C')} \\
&+2R_{(A}\boldsymbol{\Phi}_{B)(A'B'}\overline{o}_{C')} + \overline{R}_{(A'}\boldsymbol{\Phi}_{B'C')(A}o_{B)} - 2\overline{E}_{(A'}\boldsymbol{\Phi}_{B'C')(A}o_{B)} \\
&+\overline{o}_{(A'}S_{B'}\boldsymbol{\Phi}_{C')(AB)} - K\boldsymbol{\Phi}_{(AB)(A'B'}\overline{o}_{C')}
\end{aligned} \tag{2.1.131}$$

$$\begin{aligned}
&\mathbf{D}_{AA'}\boldsymbol{\Phi}_{BCB'C'} - \boldsymbol{\delta}_{A'B'A}\boldsymbol{\Phi}_{BCC'} - \boldsymbol{\delta}'_{ABA'}\boldsymbol{\Phi}_{CB'C'} + \mathbf{D}'_{ABA'B'}\boldsymbol{\Phi}_{CC'} \\
&+3o_{(A}\mathbf{D}'_{BC)(A'B'}\boldsymbol{\Lambda}\overline{o}_{C')} = -\overline{T}_{AA'}\boldsymbol{\Phi}_{BB'C'}o_C + 2B_{AA'}\boldsymbol{\Phi}_{BB'C'} \\
&+2\overline{B}_{AA'}\boldsymbol{\Phi}_{BCB'}\overline{o}_{C'} - T_{AA'}\boldsymbol{\Phi}_{BCB'}\overline{o}_{C'} + R_{(A}\boldsymbol{\Phi}_{BC)(A'B'}\overline{o}_{C')} \\
&+\overline{R}_{(A'}\boldsymbol{\Phi}_{B'C')(AB}o_{C)} - 2E_{(A}\boldsymbol{\Phi}_{BC)(A'B'}\overline{o}_{C')} \\
&-2\overline{E}_{(A'}\boldsymbol{\Phi}_{B'C')(AB}o_{C)}
\end{aligned} \tag{2.1.132}$$

It is important we make sure that from our invariant equations we are able to obtain all NP equations. To do so, we take the components of the respective invariant expression which should give NP equations. Take the Ricci generalized equation 2.1.108, for example,if we write it out in terms of components we have:

$$\begin{aligned}
&(D\rho + \tau'\kappa - \delta'\kappa)o_A o_B\overline{o}_{A'} + (2\epsilon\kappa)o_{(A}\iota_{B)}\overline{o}_{A'} + (-\kappa\kappa)\iota_A\iota_B\overline{\iota}_{A'} = \\
&(\rho^2 + \sigma\overline{\sigma} - 3\kappa\alpha - \kappa\overline{\beta} + \overline{\epsilon}\rho + \rho\epsilon - \overline{\kappa}\tau + \Phi_{00'})o_A o_B\overline{o}_{A'} \\
&+(2\kappa\epsilon)o_{(A}\iota_{B)}\overline{o}_{A'} + (-\kappa^2)\iota_A\iota_B\overline{\iota}_{A'}
\end{aligned}$$

We see straight away that from equation 2.1.108 we obtain the NP Ricci equation 4.2(a). In the same way, we get all NP Ricci equations which do not involve spin coefficients which transform "badly" under null rotations from our invariant Ricci equations as follows:

Equation 2.1.108 ⟶ 4.2a)
Equation 2.1.109 ⟶ 4.2b)
Equation 2.1.110 ⟶ 4.2a); 4.2b); 4.2c)
Equation 2.1.111 ⟶ 4.2a); 4.2b); 4.2k)
Equation 2.1.112 ⟶ 4.2a); 4.2p); 4.2k)+ 4.2c)
Equation 2.1.113 ⟶ 4.2b); 4.2q); 4.2c)+ 4.2k)
Equation 2.1.114 ⟶ 4.2a); 4.2d); 4.2p); 4.2o); 4.2f)+ 4.2l); 4.2c)+ 4.2k)
Equation 2.1.115 ⟶ 4.2a); 4.2d)
Equation 2.1.116 ⟶ 4.2b); 4.2e)
Equation 2.1.117 ⟶ 4.2b); 4.2r); 4.2q)+ 4.2l)+ 4.2f); 4.2k)+ 4.2c)+ 4.2e)
Equation 2.1.118 ⟶ 4.2a); 4.2b); 4.2d); 4.2f); 4.2e)+ 4.2c)
Equation 2.1.119 ⟶ 4.2a);4.2b); 4.2d); 4.2l); 4.2e)+ 4.2k)

The NP Bianchi identities are obtained from the invariant version of the Bianchi identities in the same way as one obtains the NP Ricci equations from their respective invariant version. Hence we get all NP Bianchi identities by taking components of the Bianchi identities written in the invariant formalism.

The case of the commutators is somewhat more complicated since from the invariant commutators we should obtain all NP Ricci equations which involve those spin coefficients which transform in a "bad" way under null rotations as well as the NP commutators.

Lets consider the commutator $\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta}$ and let it act on a scalar field ϕ. Then, equation 2.1.100 gives:

$$\{(\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta})\phi\}_{22'} = (\delta D - D\delta)\phi_{00'} - \overline{\tau}' D\phi_{00'} \tag{2.1.133}$$

On the other hand, equation 2.1.101 along with definitions 2.1.67, 2.1.74, 2.1.75 and 2.1.80 give:

$$\begin{aligned}\{(\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta})\phi\}_{23'} &= (\beta + \overline{\alpha})(\mathbf{D}\phi)_{N+1,N+1'} + \kappa(\mathbf{D}'\phi)_{N+2,N+2'} \\ &-\sigma(\boldsymbol{\delta}\phi)_{N+2,N+1'} - (\epsilon - \overline{\epsilon} + \overline{\rho})(\boldsymbol{\delta}\phi)_{N+1,N+2'} - \boldsymbol{\Psi}_2\phi_{N-1,N'} \\ &-\boldsymbol{\Phi}_{02'}\phi_{N,N-1'} - 2\boldsymbol{\Lambda}\boldsymbol{\Phi}_{N-1,N'}\end{aligned} \tag{2.1.134}$$

If we equal the left hand side of equations 2.1.133 and 2.1.134 we get the NP commutator $\delta D - D\delta$. All other NP commutators are obtained in the same way. In order to see this we write the following expressions relating to all other generalized commutators:

$$\{(\boldsymbol{\delta}'\boldsymbol{\delta} - \boldsymbol{\delta}\boldsymbol{\delta}')\phi\}_{N+3,N'+3'} = (\boldsymbol{\delta}'\boldsymbol{\delta} - \boldsymbol{\delta}\boldsymbol{\delta}')\phi_{N,N'} + (\delta\overline{\rho}' - \delta'\overline{\sigma}')\phi_{N,N'-1'}$$
$$+(\delta\sigma' - \delta'\rho')\phi_{N-1,N'} + (-\rho' + \overline{\rho}')D\phi_{N,N'} + (\rho'\overline{\tau}' - 2\overline{\sigma}'\overline{\beta} - \overline{\tau}'\overline{\rho}' \tag{2.1.135}$$
$$-2\overline{\beta}'\overline{\rho}')\phi_{N,N'-1'} + (\rho'\tau' + \rho'\beta' + \beta\sigma' - \overline{\rho}'\tau')\phi_{N-1,N'}$$

$$\{(\boldsymbol{\delta}'\boldsymbol{\delta} - \boldsymbol{\delta}\boldsymbol{\delta}')\phi\}_{N+3,N'+3'} = (\overline{\rho} - \rho)(\mathbf{D}'\phi)_{N+2,N'+2'} + (\beta - \overline{\alpha})(\boldsymbol{\delta}'\phi)_{N+2,N'+1'}$$
$$+(\alpha - \overline{\beta})(\boldsymbol{\delta}\phi)_{N+1,N'+2'} + (-\overline{\boldsymbol{\Psi}}_3 + \boldsymbol{\Phi}_{12'})\phi_{N,N'-1'} \tag{2.1.136}$$
$$+(-\boldsymbol{\Phi}_{21'} + \boldsymbol{\Psi}_3)\phi_{N-1,N'}$$

$$\{(\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi\}_{N+3,N'+3'} = (\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi_{N,N'} + (-D'\tau' + D\kappa')\phi_{N-1,N'}$$
$$+(-D'\overline{\tau}' + D\overline{\kappa}')\phi_{N,N'-1'} + (\tau'\rho' + 2\epsilon'\tau' + \overline{\tau}'\sigma' + 2\epsilon\kappa')\phi_{N-1,N'} \tag{2.1.137}$$
$$+(\tau'\overline{\sigma}' + \overline{\rho}'\overline{\tau}' + 2\overline{\epsilon}'\overline{\tau}' + 2\overline{\epsilon\kappa}')\phi_{N,N'-1'} - \overline{\tau}'\delta'\phi_{N,N'} - \tau'\delta\phi_{N,N'}$$

$$\{(\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi\}_{N+3,N'+3'} = -(\epsilon' + \overline{\epsilon}')(\mathbf{D}\phi)_{N+1,N'+1'} + (\epsilon + \overline{\epsilon})$$
$$(\mathbf{D}'\phi)_{N+2,N'+2'} - \tau(\boldsymbol{\delta}'\phi)_{N+2,N'+1'} - \overline{\tau}(\boldsymbol{\delta}\phi)_{N+1,N'+2'} \tag{2.1.138}$$
$$-(\Psi_3 + \Phi_{21'})\phi_{N-1,N'} - (\overline{\Psi}_3 + \Phi_{12'})\phi_{N,N'-1'}$$

$$\{(\boldsymbol{\delta}\mathbf{D}' - \mathbf{D}'\boldsymbol{\delta})\phi\}_{N+3,N'+4'} = (\boldsymbol{\delta}\mathbf{D}' - \mathbf{D}'\boldsymbol{\delta})\phi_{N,N'} + (-D'\rho' - \delta\kappa')\phi_{N-1,N'}$$
$$+(D'\overline{\rho}' - \delta\overline{k}')\phi_{N,N'-1'} + (-2\epsilon'\rho' - \rho'\rho' - \overline{\sigma}'\sigma' + \overline{\kappa}'\tau' - 2\kappa'\beta)\phi_{N-1,N'}$$
$$+(-\rho'\overline{\sigma}' - 2\overline{\sigma}'\overline{\epsilon}' + \overline{\kappa}'\overline{\tau}' + 2\overline{\kappa}'\overline{\beta}')\phi_{N,N'-1'} - \overline{\kappa}'D\phi_{N,N'} + \rho'\delta\phi_{N,N'} \tag{2.1.139}$$
$$+\overline{\sigma}'\delta'\phi_{N,N'}$$

$$\{(\boldsymbol{\delta}\mathbf{D}' - \mathbf{D}'\boldsymbol{\delta})\phi\}_{N+3,N'+4'} = (\tau - \beta - \overline{\alpha})(\mathbf{D}'\phi)_{N+2,N'+2'} - (\gamma - \overline{\gamma})$$

$$(\boldsymbol{\delta}\phi)_{N+1,N'+2'} + \Phi_{2,2'}\phi_{N-1,N'} + \overline{\Psi}_4\phi_{N,N'-1'} \qquad (2.1.140)$$

Hence we have that by applying a scalar ϕ to the generalized commutator 2.1.105 we arrive at the NP commutator $\delta'\delta - \delta\delta'$, while equation 2.1.106 gives the NP commutator $D'D - DD'$. Finally, by applying a $(N = 0, N' = 0)$ spinor ϕ to the invariant equation 2.1.107 we are able to obtain the NP commutator $\delta D' - D'\delta$. Furthermore, if we let the commutator 2.1.102 act upon a spinor ϕ of type $(N = 0, N' = 1)$ we manage to obtain the NP Ricci equation (4.2g), 2.1.105 acting on ϕ of type $(N = 1, N' = 0)$ gives the NP Ricci equation (4.2m). The NP Ricci equations (4.2i), (4.2n) and (4.2j) are thus obtained by letting the commutatores 2.1.106, and 2.1.107 act on spinors of type $(N = 0, N' = 1)$, $(N = 1, N' = 0)$ and $(N = 0, N' = 1)$ respectively.

Because all invariants of our new formalism are symmetric on all primed and unprimed indices we can write all equations, i.e , commutators, Ricci equations and Bianchi identities in compact notation so that no indices appear. These are given below in compact notation.

Commutators

$$(\boldsymbol{\delta}\mathbf{D} - \mathbf{D}\boldsymbol{\delta})\phi = (B + \overline{A})\mathbf{D}\phi + K\mathbf{D}'\phi - S\boldsymbol{\delta}'\phi$$
$$-(E - \overline{E} + \overline{R})\boldsymbol{\delta}\phi - \boldsymbol{\Psi}_2(\phi \cdot o) - 2\boldsymbol{\Lambda}(\phi \cdot o) - \boldsymbol{\Phi}_{02'}(\phi \cdot \overline{o}) \qquad (2.1.141)$$

$$(\boldsymbol{\delta}'\boldsymbol{\delta} - \boldsymbol{\delta}\boldsymbol{\delta}')\phi = (\overline{R} - R)\mathbf{D}'\phi + (B - \overline{A})\boldsymbol{\delta}'\phi + (A - \overline{B})\boldsymbol{\delta}\phi$$
$$+(-\overline{\boldsymbol{\Psi}}_3 + \boldsymbol{\Phi}_{12'})(\phi \cdot \overline{o}) + (\boldsymbol{\Psi}_3 - \boldsymbol{\Phi}_{21'})(\phi \cdot o) \qquad (2.1.142)$$

$$(\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi = (G + \overline{G})\mathbf{D}\phi + (E + \overline{E})\mathbf{D}'\phi$$
$$-\overline{T}\boldsymbol{\delta}\phi - T\boldsymbol{\delta}'\phi - (\boldsymbol{\Phi}_{21'} + \boldsymbol{\Psi}_3)(\phi \cdot o) - (\boldsymbol{\Phi}_{12'} + \overline{\boldsymbol{\Psi}}_3)(\phi \cdot \overline{o}) \qquad (2.1.143)$$

$$(\boldsymbol{\delta}\mathbf{D}' - \mathbf{D}'\boldsymbol{\delta})\phi = (T - B - \overline{A})\mathbf{D}'\phi - (G - \overline{G})\boldsymbol{\delta}\phi$$
$$+\boldsymbol{\Phi}_{22'}(\phi \cdot o) + \overline{\boldsymbol{\Psi}}_4(\phi \cdot \overline{o}) \qquad (2.1.144)$$

where $(\phi\cdot o)$ is the $(N-1,N')$-spinor $\phi_{A_1\ldots A_N A'_{1'}\ldots A'_{N'}}\, o^{A_N}$ and $\phi\cdot\overline{o}$ is the $(N,N'-1)$-spinor $\phi_{A_1\ldots A_N A'_{1'}\ldots A'_{N'}}\, \overline{o}^{A'_{N'}}$, and if the contraction is not possible then these terms are set to zero.

Ricci Equations

$$\mathbf{D}R-\boldsymbol{\delta}'K = R^2+S\overline{S}-3KA-K\overline{B}+RE+R\overline{E}-\overline{K}T +\mathbf{\Psi}_0 \quad (2.1.145)$$

$$\mathbf{D}S-\boldsymbol{\delta}K = SR+S\overline{R}-S\overline{E}+3SE-KT-K\overline{A}-3KB +\mathbf{\Psi}_0 \quad (2.1.146)$$

$$\mathbf{D}T-\mathbf{D}'K = RT+S\overline{T}+TE-T\overline{E}-K\overline{G}-3KG+\mathbf{\Psi}_1 +\mathbf{\Phi}_{01'} \quad (2.1.147)$$

$$\boldsymbol{\delta}R-\boldsymbol{\delta}'S = TR-T\overline{R}+\overline{A}R+BR-3SA+S\overline{B}-\mathbf{\Psi}_1 +\mathbf{\Phi}_{01'} \quad (2.1.148)$$

$$\boldsymbol{\delta}T-\mathbf{D}'S = TT+TB-T\overline{A}-3SG+S\overline{G}+\mathbf{\Phi}_{02'} \quad (2.1.149)$$

$$\mathbf{D}'R-\boldsymbol{\delta}'T = -T\overline{T}+RG+R\overline{G}-TA+T\overline{B}-\mathbf{\Psi}_2-2\mathbf{\Lambda} \quad (2.1.150)$$

$$\mathbf{D}'B-\boldsymbol{\delta}G = 2BG-B\overline{G}+G\overline{A}-GT-\mathbf{\Phi}_{12'} \quad (2.1.151)$$

$$\boldsymbol{\delta}'E-\mathbf{D}A = \overline{K}G-B\overline{S}-AR-A\overline{E}+2AE+E\overline{B}-\mathbf{\Phi}_{10'} \quad (2.1.152)$$

$$\mathbf{D}B-\boldsymbol{\delta}E = -KG+SA+B\overline{R}-B\overline{E}-E\overline{A}+\mathbf{\Psi}_1 \quad (2.1.153)$$

$$\boldsymbol{\delta}'G-\mathbf{D}'A = -A\overline{G}+G\overline{T}-G\overline{B}+\mathbf{\Psi}_3 \quad (2.1.154)$$

$$\mathbf{D}G-\mathbf{D}'E = B\overline{T}+AT-2EG-E\overline{G}-G\overline{E}+\mathbf{\Psi}_2+\mathbf{\Phi}_{11'} -\mathbf{\Lambda} \quad (2.1.155)$$

$$\boldsymbol{\delta}'B-\boldsymbol{\delta}A = -A\overline{A}-B\overline{B}+2AB+G\overline{R}-GR+\mathbf{\Psi}_2-\mathbf{\Phi}_{11'} -\mathbf{\Lambda} \quad (2.1.156)$$

Bianchi identities

$$\mathbf{D}\mathbf{\Psi}_1-\boldsymbol{\delta}'\mathbf{\Psi}_0-\mathbf{D}\mathbf{\Phi}_{01'}+\boldsymbol{\delta}\mathbf{\Phi}_0 = -3K\mathbf{\Psi}_2+2E\mathbf{\Psi}_1+4R\mathbf{\Psi}_1-4A\mathbf{\Psi}_0 +2\overline{A}\mathbf{\Phi}_{00'}+2B\mathbf{\Phi}_{00'}-2E\mathbf{\Phi}_{01'}-2\overline{R}\mathbf{\Phi}_{01'}-2S\mathbf{\Phi}_{10'}+2K\mathbf{\Phi}_{11'} +\overline{K}\mathbf{\Phi}_{02'} \quad (2.1.157)$$

$$3(\mathbf{D}\mathbf{\Psi}_2-\boldsymbol{\delta}\mathbf{\Psi}_1)-2(\mathbf{D}\mathbf{\Phi}_{11'}-\boldsymbol{\delta}\mathbf{\Phi}_{10'})+\mathbf{D}\mathbf{\Phi}_0-\boldsymbol{\delta}\mathbf{\Phi}_{01'} = -6K\mathbf{\Psi}_3$$

$$+9R\mathbf{\Psi}_2 - 6A\mathbf{\Psi}_1 + 2G\mathbf{\Phi}_0 + 2\overline{G}\mathbf{\Phi}_0 - 2A\mathbf{\Phi}_{01'} - 2\overline{T}\mathbf{\Phi}_{01'}$$
$$-2T\mathbf{\Phi}_{10'} + 4\overline{A}\mathbf{\Phi}_{10'} - 4\overline{R}\mathbf{\Phi}_{11'} + 2R\mathbf{\Phi}_{11'} - 2S\mathbf{\Phi}_{20'} + \overline{S}\mathbf{\Phi}_{02'} \quad (2.1.158)$$
$$+2\overline{K}\mathbf{\Phi}_{12'} + 2K\mathbf{\Phi}_{21'}$$

$$3(\mathbf{D}\mathbf{\Psi}_3 - \boldsymbol{\delta}\mathbf{\Psi}_2) - 2(\boldsymbol{\delta}\mathbf{\Phi}_{11'} - \mathbf{D}\mathbf{\Phi}_{10'}) + \boldsymbol{\delta}\mathbf{\Phi}_{20'} - \mathbf{D}\mathbf{\Phi}_{21'} = -3K\mathbf{\Psi}_4$$
$$-6E\mathbf{\Psi}_3 + 6R\mathbf{\Psi}_3 + 4\overline{G}\mathbf{\Phi}_{10'} - 4\overline{T}\mathbf{\Phi}_{11'} - 2B\mathbf{\Phi}_{20'} - 2T\mathbf{\Phi}_{20'} \quad (2.1.159)$$
$$+2\overline{A}\mathbf{\Phi}_{20'} + 2\overline{S}\mathbf{\Phi}_{12'} - 2\overline{R}\mathbf{\Phi}_{21'} + 2R\mathbf{\Phi}_{21'} + 2E\mathbf{\Phi}_{21'} + \overline{K}\mathbf{\Phi}_{22'}$$

$$\mathbf{D}\Psi_4 - \boldsymbol{\delta}\mathbf{\Psi}_3 + \mathbf{D}'\mathbf{\Phi}_{20'} - \boldsymbol{\delta}\mathbf{\Phi}_{21'} = -4E\mathbf{\Psi}_4 + R\mathbf{\Psi}_4 + 2A\mathbf{\Psi}_3 - 2G\mathbf{\Phi}_{20'}$$
$$+2\overline{G}\mathbf{\Phi}_{20'} - 2\overline{T}\mathbf{\Phi}_{21'} + 2A\mathbf{\Phi}_{21'} + \overline{S}\mathbf{\Phi}_{22'} \quad (2.1.160)$$

$$\mathbf{D}'\mathbf{\Psi}_0 - \boldsymbol{\delta}\mathbf{\Psi}_1 + \mathbf{D}\mathbf{\Phi}_{02'} - \boldsymbol{\delta}\mathbf{\Phi}_{01'} = 4G\mathbf{\Psi}_0 - 4T\mathbf{\Psi}_1 - 2B\mathbf{\Psi}_1 + 3S\mathbf{\Psi}_2$$
$$-2B\mathbf{\Phi}_{01'} + 2S\mathbf{\Phi}_{11'} + 2E\mathbf{\Phi}_{02'} - 2\overline{E}\mathbf{\Phi}_{02'} + \overline{R}\mathbf{\Phi}_{02'} - 2K\mathbf{\Phi}_{12'} \quad (2.1.161)$$

$$3(\mathbf{D}'\mathbf{\Psi}_1 - \boldsymbol{\delta}\mathbf{\Psi}_2) + 2(\mathbf{D}\mathbf{\Phi}_{12'} - \boldsymbol{\delta}\mathbf{\Phi}_{11'}) + \boldsymbol{\delta}\mathbf{\Phi}_{02'} - \mathbf{D}'\mathbf{\Phi}_{01'} = 6G\mathbf{\Psi}_1$$
$$-9T\mathbf{\Psi}_2 + 6S\mathbf{\Psi}_3 - 2G\mathbf{\Phi}_{01'} + 2T\mathbf{\Phi}_{11'} + 2A\mathbf{\Phi}_{02'} + \overline{T}\mathbf{\Phi}_{02'} \quad (2.1.162)$$
$$-2\overline{T}\mathbf{\Phi}_{02'} + 2\overline{R}\mathbf{\Phi}_{12'} - 2R\mathbf{\Phi}_{12'} - 4\overline{E}\mathbf{\Phi}_{12'} + 2S\mathbf{\Phi}_{21'} - 2K\mathbf{\Phi}_{22'}$$

$$\mathbf{D}\mathbf{\Phi}_{22'} - \boldsymbol{\delta}\mathbf{\Phi}_{21'} - \boldsymbol{\delta}'\mathbf{\Phi}_{12'} + \mathbf{D}'\mathbf{\Phi}_{11'} + 3\mathbf{D}'\mathbf{\Lambda} = -\overline{T}\mathbf{\Phi}_{12'} + 2B\mathbf{\Phi}_{12'}$$
$$+2\overline{T}\mathbf{\Phi}_{21'} - T\mathbf{\Phi}_{21'} + R\mathbf{\Phi}_{22'} + \overline{R}\mathbf{\Phi}_{22'} - 2E\mathbf{\Phi}_{22'} - 2\overline{E}\mathbf{\Phi}_{22'} \quad (2.1.163)$$

$$3(\mathbf{D}'\mathbf{\Psi}_2 - \boldsymbol{\delta}\mathbf{\Psi}_3) + 2(\boldsymbol{\delta}'\mathbf{\Phi}_{12'} - \mathbf{D}'\mathbf{\Phi}_{11'}) + \mathbf{D}\mathbf{\Phi}_{22'} - \boldsymbol{\delta}\mathbf{\Phi}_{21'} = -6T\mathbf{\Psi}_3$$
$$+6B\mathbf{\Psi}_3 + 3S\mathbf{\Psi}_4 + 2\overline{T}\mathbf{\Phi}_{12'} - 4\overline{B}\mathbf{\Phi}_{12'} + 2B\mathbf{\Phi}_{21'} + 2T\mathbf{\Phi}_{21'} \quad (2.1.164)$$
$$+\overline{R}\mathbf{\Phi}_{22'} - 2E\mathbf{\Phi}_{22'} - 2\overline{E}\mathbf{\Phi}_{22'} - 2R\mathbf{\Phi}_{22'}$$

$$\mathbf{D}'\mathbf{\Psi}_3 - \boldsymbol{\delta}\mathbf{\Psi}_4 + \boldsymbol{\delta}'\mathbf{\Phi}_{22'} - \mathbf{D}'\mathbf{\Phi}_{21'} = -2G\mathbf{\Psi}_3 - T\mathbf{\Psi}_4 + 4B\mathbf{\Psi}_4 + 2G\mathbf{\Phi}_{21'}$$
$$+\overline{T}\mathbf{\Phi}_{22'} - 2\overline{B}\mathbf{\Phi}_{22'} - 2A\mathbf{\Phi}_{22'} \quad (2.1.165)$$

Contracted Bianchi identities

$$\mathbf{D}\mathbf{\Phi}_{11'} - \boldsymbol{\delta}\mathbf{\Phi}_{10'} - \delta'\mathbf{\Phi}_{01'} + \mathbf{D}'\mathbf{\Phi}_0 + 3\mathbf{D}\mathbf{\Lambda} = 2G\mathbf{\Phi}_0 + 2\overline{G}\mathbf{\Phi}_0 - 2A\mathbf{\Phi}_{01'}$$
$$-2\overline{T}\mathbf{\Phi}_{01'} - 2\overline{A}\mathbf{\Phi}_{10'} - 2T\mathbf{\Phi}_{10'} + 2R\mathbf{\Phi}_{11'} + 2\overline{R}\mathbf{\Phi}_{11'} + \overline{S}\mathbf{\Phi}_{02'} \qquad (2.1.166)$$
$$+S\mathbf{\Phi}_{20'} - \overline{K}\mathbf{\Phi}_{12'} - K\mathbf{\Phi}_{21'}$$

$$\mathbf{D}\mathbf{\Phi}_{12'} - \boldsymbol{\delta}\mathbf{\Phi}_{11'} - \delta'\mathbf{\Phi}_{02'} + \mathbf{D}'\mathbf{\Phi}_{01'} + 3\delta\mathbf{\Lambda} = 2G\mathbf{\Phi}_{01'} - 2T\mathbf{\Phi}_{11'} + 2\overline{T}\mathbf{\Phi}_{02'}$$
$$-2A\mathbf{\Phi}_{02'} - \overline{T}\mathbf{\Phi}_{02'} + 2R\mathbf{\Phi}_{12'} + \overline{R}\mathbf{\Phi}_{12'} - 2\overline{E}\mathbf{\Phi}_{12'} + S\mathbf{\Phi}_{21'} \qquad (2.1.167)$$
$$-K\mathbf{\Phi}_{22'}$$

$$\mathbf{D}\mathbf{\Phi}_{22'} - \boldsymbol{\delta}\mathbf{\Phi}_{21'} - \boldsymbol{\delta}\mathbf{\Phi}_{12'} + \mathbf{D}'\mathbf{\Phi}_{11'} + 3\mathbf{D}'\mathbf{\Lambda} = -\overline{T}\mathbf{\Phi}_{12'} + 2B\mathbf{\Phi}_{12'}$$
$$+2\overline{T}\mathbf{\Phi}_{21'} - T\mathbf{\Phi}_{21'} + R\mathbf{\Phi}_{22'} + \overline{R}\mathbf{\Phi}_{22'} - 2E\mathbf{\Phi}_{22'} - 2\overline{E}\mathbf{\Phi}_{22'} \qquad (2.1.168)$$

Note that to obtain an expression with indices from one in the compact form one multiplies the terms in the sum by appropriate factors of o's and $\overline{o}$'s to make every term in the sum a spinor of the same type (so that the indices balance) and then symmetrise over the primed and unprimed indices.

2.2 The Geometrical Interpretation

Here we will discuss how the invariant formalism we have just described arises naturally when describing the geometry of general null congruences [27] and of null hypersurfaces [7]. However in describing some of the geometry it is often the direction of the flagpole of o_A rather than o_A itself which is of significance. For this reason we do not give a full description of the geometry here, but will do so in chapter 3 when we also introduce a generalization of the formalism which permits one to consider rescalings of o_A by a complex scalar field λ as well as null rotations.

A null congruence $\mathcal{C}$ may be specified by giving a (nowhere vanishing) null vector field ℓ^a whose integral curves form the congruence. If the curves are given by $x^a = x^a(u, y^1, y^2, y^3)$ with $\ell^a = \frac{\partial x^a}{\partial u}$, and if f is a smooth function defined along the curves, then:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x^a}\frac{\partial x^a}{\partial u} = \ell^a \nabla_a f = Df$$

Since ℓ^a is nowhere vanishing we may assume that u is chosen so that ℓ^a is future pointing. We associate to the vector ℓ^a a spinor field o^A with flagpole $o^A \bar{o}^{A'}$ equal to ℓ^a. Note that o^A is not unique but is defined up to $o^A \to e^{i\theta} o^A$.

The condition that the congruence be geodetic (i.e. each curve is a geodesic) is that

$$\nabla_\ell \ell \propto \ell$$

which is equivalent to

$$Do^A \propto o^A$$

and hence

$$K = \kappa = o_A Do^A = 0 \tag{2.2.169}$$

The condition that u is an affine parameter is that $\nabla_\ell \ell = 0$ which is equivalent to $D\ell^a = 0$. So that in addition to 2.2.169 one must have

$$\epsilon + \bar{\epsilon} = n_a D\ell^a = 0 \tag{2.2.170}$$

In the present formalism (using the compact notation) equations 2.2.169 and 2.2.170 are equivalent to the single equation

$$E + \overline{E} = 0 \tag{2.2.171}$$

Also it may be shown that the flag planes of o^A are parallely propagated if $\kappa = 0$ and $\epsilon - \bar{\epsilon} = 0$ (see §7.1 of Volume 2, Penrose and Rindler 1984 for details of this and much else referred to in this section). In the present formalism this is just the condition:

$$E - \overline{E} = 0 \tag{2.2.172}$$

Finally the condition that o^A is parallelly propagated along the congruence is $Do^A = 0$ and is equivalent to:

$$E = 0 \tag{2.2.173}$$

If one now considers a connecting vector q^a for the congruence then this must satisfy:

$$Dq^a = q^b \nabla_b \ell^a \tag{PR7.1.29}$$

(i.e. equation 7.1.29 of Penrose & Rindler 1984). Contracting this equation with o_A gives:

$$o^A Dq_{AA'} = \bar{o}_{A'} T_{AB}\, q^{AB'}$$

which may be written out in full as:

$$
\begin{aligned}
Dq_{00'} &= q_{00'}(\epsilon+\bar{\epsilon}) - q_{01'}\bar{\kappa} - q_{10'}\kappa && (2.2.174)\\
Dq_{01'} &= q_{01'}(\epsilon-\bar{\epsilon}-\rho) - q_{10'}\sigma - q_{00'}(\tau+\bar{\pi}) && (2.2.175)\\
Dq_{10'} &= q_{10'}(\bar{\epsilon}-\epsilon-\bar{\rho}) - q_{01'}\bar{\sigma} - q_{00'}(\bar{\tau}+\pi) && (2.2.176)
\end{aligned}
$$

These equations reduce to (PR7.1.37-38) if one makes the simplifying assumption that $\epsilon = \pi = 0$.

These three equations can be written in our notation as

$$\mathbf{D}(q.o) = T((q.o).\bar{o}) + (E - \overline{E} - R)(q.o) - S(q.\bar{o}) \qquad (2.2.177)$$

More generally

$$\mathbf{D}q = G((q.o).\bar{o}) + \overline{G}((q.o).\bar{o}) + \overline{A}(q.o) + A(q.\bar{o}) + B(q.\bar{o}) + \overline{B}(q.o) \qquad (2.2.178)$$

is equivalent to (PR7.1.37-39) again without having to make the simplifying assumption that $\epsilon = \pi = 0$.

By taking second derivatives of the connecting vector in the ℓ^a direction one obtains the equation

$$D^2 q^d = q^a \nabla_a (D\ell^d) + R_{abc}{}^d \ell^a q^b \ell^c \qquad \text{(PR7.2.1)}$$

If one makes the assumption that the congruence is geodetic and the dyad is parallelly propagated along the curves (so that $\kappa = \epsilon = \pi = 0$) then one obtains the equations:

$$
\begin{aligned}
D\rho &= \rho^2 + \sigma\bar{\sigma} + \Phi_0\\
D\sigma &= (\rho+\bar{\rho})\sigma + \Psi_0\\
D\tau &= \tau\rho + \bar{\tau}\sigma + \Psi_1 + \Phi_{01}
\end{aligned}
$$

The above equations are much more complicated without the above simplifying assumption. However the general case has a simple form in the present formalism and is given by:

$$DT = TR + \overline{T}S + \Psi_1 + \Phi_{01} \tag{2.2.179}$$

Note that because of the use of the compact notation 2.2.179 is equivalent to three scalar equations.

We end by translating some conditions relating to null hypersurfaces into the new formalism. It is easily shown that ℓ_a is hypersurface orthogonal if $\kappa = 0$ and $\rho = \overline{\rho}$. These conditions are equivalent to:

$$R - \overline{R} = 0 \tag{2.2.180}$$

which is also the condition that $\mathcal{C}$ is geodetic and twist free. ℓ_a satisfies the stronger condition that it is equal to a gradient if in addition $\epsilon = -\overline{\epsilon}$ and $\tau = \overline{\alpha} + \beta$. In the present notation these conditions are equivalent to:

$$T = \overline{A} + B \tag{2.2.181}$$

It also transpires that the components of K, R, S, T, A and B are in fact just the components of the generalized connection introduced on a null hypersurface by Daŭtcourt [7] and that the curvature expressions appearing in the Ricci and commutator equations are those which arise naturally as the intrinsic and extrinsic curvatures of the null hypersurface (which are not unique but are only defined up to null rotations). The above formalism therefore provides a natural description of the $3+1$ decomposition of the Einstein equations in the case of a null slicing. The details of this will be given in chapter 3.

Chapter 3

A Formalism Invariant Under Null Rotations and Rescaling

3.1 The Compacted Formalism

Certain physical problems in general relativity are often best described by using a formalism adapted to the geometry of the particular situation. For example the Geroch-Held-Penrose (GHP) formalism [25] best describes the geometry of a spacelike 2-surface $\mathcal{S}$ where one can choose ℓ^a and n^a to point along the null normals of $\mathcal{S}$ and the real and imaginary parts of m^a are tangent to $\mathcal{S}$. The remaining gauge freedom in the choice of tetrad is the two dimensional subgroup of the Lorentz group representing a boost in the normal directions and a rotation in the tangential directions. In terms of spinors this is equivalent to the rescaling:

$$o^A \to \lambda o^A \qquad \iota^A \to \lambda^{-1}\iota^A \tag{3.1.1}$$

The GHP formalism works with those Ricci rotation coefficients which simply rescale under 3.1.1 and combines the others with directional derivatives to form new operators , which also just rescale under 3.1.1.

In chapter 2 we introduced a formalism in which the generalized spin coefficients and differential operators transform in a simple way under a null rotation. We have seen that such formalism uses spinors formed from the Ricci rotation coefficients whose components transform covariantly under null rotations, and four new differential operators which are formed from the directional derivatives and the remaining Ricci rotation coefficients which transform "badly" under null rotations. These operators act on totally symmetric spinors and produce totally symmetric spinors (of higher valence) and when applied to a spinor whose components transform covariantly under null rotations produce one whose components also transform covariantly.

In this chapter we develop a formalism which only depends upon the choice of a single null direction, ℓ^a. If we choose the flagpole of the spinor o^A to point in this null direction then o^A is determined up to rescalings $o^A \to \lambda o^A$ where λ is a complex nowhere vanishing scalar field (and the magnitude as well as the direction of ℓ^a is fixed if we require $|\lambda| = 1$). The other spinor in the spin frame is arbitrary so for convenience we normalise it so that $o_A \iota^A = 1$, although it would not be hard to generalize the formalism to allow $o_A \iota^A = \chi$ as one has in the compactified spin coefficient formalism. Thus specifying a single null direction is equivalent to specifying a spin frame up to the gauge freedom:

$$o^A \to \lambda o^A \qquad \iota^A \to \lambda^{-1} \iota^A + \overline{a} o^A \tag{3.1.2}$$

We will use the null rotation invariant formalism as our starting point, so that we first check how the generalized spin coefficients transform under 3.1.2:

$$K \to \lambda^3 \overline{\lambda} K \tag{3.1.3}$$
$$S_{A'} \to \lambda^3 S_{A'} \tag{3.1.4}$$
$$R_A \to \lambda^2 \overline{\lambda} R_A \tag{3.1.5}$$
$$T_{AA'} \to \lambda^2 T_{AA'} \tag{3.1.6}$$

hence, K, S, R, T are well behaved under transformation 3.1.2 while B, E, A and G are not.

The quantities K, S, R, T have weight, which we will denote by $\{\mathbf{p}, \mathbf{q}\}$ and which is defined such that $\mathbf{p} = p + N$ and $\mathbf{q} = q + N'$ where $\{p, q\}$ is the GHP weight defined in [25], given by:

$$K : \{3, 1\}$$
$$S : \{3, 0\}$$
$$R : \{2, 1\}$$
$$T : \{2, 0\}$$

A simple calculation shows that the null rotation invariant differential operators $\mathbf{D}, \boldsymbol{\delta}, \boldsymbol{\delta}', \mathbf{D}'$, do not produce objects with a well defined spin and boost weight when acting on a totally symmetric spin and boost weighted spinor field. Just as in the GHP case, we can then combine the generalized spin coefficients A, B, G and E which transform badly under 3.1.2 to produce new derivative operators $\text{þ}, \eth, \eth', \text{þ}'$ which have a proper spin and boost weight.Notice that we use the notation boldface to distinguish for example ∂ and $\boldsymbol{\partial}$.

The new þ operator acts on a totally symmetric (N, N')-spinor $\phi_{A_1 \ldots A_N A'_1, \ldots A'_{N'}}$ of weight $\{\mathbf{p}, \mathbf{q}\}$ to produce a $(N+1, N'+1)$-spinor of type $\{\mathbf{p}+2, \mathbf{q}+2\}$ and is given by:

$$
\begin{aligned}
(\text{þ}\phi)_{AA_1 \ldots A_N A' A_1, \ldots A_{N'}} &= (\mathbf{D}\phi)_{AA_1 \ldots A_N A_1, \ldots A_{N'}} \\
&- (\mathbf{p} - N) E_{(A}\phi_{A_1 \ldots A_N)(A_1, \ldots A_{N'}} \quad \bar{o}_{A')} \\
&- (\mathbf{q} - N') o_{(A}\phi_{A_1 \ldots A_N)(A_1, \ldots A_{N'}} \quad \overline{E}_{A')}
\end{aligned}
\tag{3.1.7}
$$

Since every term in the above expression is a totally symmetric spinor the order of the indicies does not matter and we may use the compact notation that we introduced before in chapter 2. Thus equation 3.1.7 becomes:

$$\text{þ}\phi = \mathbf{D}\phi - (\mathbf{p} - N)E\phi - (\mathbf{q} - N')\overline{E}\phi \tag{3.1.8}$$

In a similar way we may define the operator $\eth$ which acts on a type (N, N')-spinor of weight $\{\mathbf{p}, \mathbf{q}\}$ to produce a type $(N+1, N'+2)$-spinor of weight $\{\mathbf{p}+2, \mathbf{q}+1\}$. In compact notation $\eth\phi$ is given by:

$$\eth\phi = \delta\phi - (\mathbf{p} - N)B\phi - (\mathbf{q} - N')\overline{A}\phi \tag{3.1.9}$$

The operator $\eth'$ acts on a type (N, N')-spinor of weight $\{\mathbf{p}, \mathbf{q}\}$ to produce a type $(N+2, N'+1)$-spinor of weight $\{\mathbf{p}+1, \mathbf{q}+2\}$. In compact notation $\eth'\phi$ is given by:

$$\eth'\phi = \bar{\delta}\phi - (\mathbf{p} - N)A\phi - (\mathbf{q} - N')\overline{B}\phi \tag{3.1.10}$$

Finally we may define the operator which acts on a type (N, N')-spinor of weight $\{\mathbf{p}, \mathbf{q}\}$ to produce a type $(N+2, N'+2)$-spinor of weight $\{\mathbf{p}+1, \mathbf{q}+1\}$. In compact notation $\text{þ}'\phi$ is given by:

$$\text{þ}'\phi = \mathbf{D}'\phi - (\mathbf{p} - N)G\phi - (\mathbf{q} - N')\overline{G}\phi \tag{3.1.11}$$

We next note that the null rotation invariant curvature spinors have proper weight $\{\mathbf{p}, \mathbf{q}\}$ given by:

$$\mathbf{\Phi}_{00'} \; : \; \{2,2\} \tag{3.1.12}$$

$$(\mathbf{\Phi}_{01'})_{B'} \; : \; \{2,1\} \tag{3.1.13}$$

$$(\mathbf{\Phi}_{02'})_{A'B'} \; : \; \{2,0\} \tag{3.1.14}$$

$$(\mathbf{\Phi}_{10'})_{B} \; : \; \{1,2\} \tag{3.1.15}$$

$$(\mathbf{\Phi}_{11'})_{BB'} \; : \; \{1,1\} \tag{3.1.16}$$

$$(\mathbf{\Phi}_{12'})_{BA'B'} \; : \; \{1,0\} \tag{3.1.17}$$

$$(\mathbf{\Phi}_{20'})_{AB} \; : \; \{0,2\} \tag{3.1.18}$$

$$(\mathbf{\Phi}_{21'})_{ABB'} \; : \; \{0,1\} \tag{3.1.19}$$

$$(\mathbf{\Phi}_{22'})_{ABA'B'} \; : \; \{0,0\} \tag{3.1.20}$$

$$\mathbf{\Psi}_{0} \; : \; \{4,0\} \tag{3.1.21}$$

$$(\mathbf{\Psi}_{1})_{A} \; : \; \{3,0\} \tag{3.1.22}$$

$$(\mathbf{\Psi}_{2})_{AB} \; : \; \{2,0\} \tag{3.1.23}$$

$$(\mathbf{\Psi}_{3})_{ABC} \; : \; \{1,0\} \tag{3.1.24}$$

$$(\mathbf{\Psi}_{4})_{ABCD} \; : \; \{0,0\} \tag{3.1.25}$$

$$\mathbf{\Lambda} \; : \; \{0,0\} \tag{3.1.26}$$

We are now in a position to translate all relative equations into this new formalism. We begin by considering the commutators. If we take the commutators written in the null rotation invariant formalism then the calculation to obtain the commutators in the new formalism is similar to the calculation performed in obtaining the GHP commutators from the NP commutators. Let us take the generalized NP commutator $(\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi$. We consider ϕ to be of type (N, N') and have weight $\{\mathbf{p}, \mathbf{q}\}$. We want to calculate $(\text{þ}\text{þ}' - \text{þ}'\text{þ})\phi$, with the use of definition 3.1.8 and 3.1.11 such calculation is straightforward and gives:

$$\begin{aligned}(\text{þ}'\text{þ} - \text{þ}\text{þ}')\phi &= (\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi + (\mathbf{p} - N)(\mathbf{D}G - \mathbf{D}'E + 2EB + \overline{G}E \\ &+GE)\phi + (\mathbf{q} - N')(\mathbf{D}\overline{G} - \mathbf{D}'\overline{E} + \overline{E}G + 2\overline{EG} + E\overline{G})\mathbf{D}'\phi \\ &-(E + \overline{E})\mathbf{D}'\phi - (G + \overline{G})\mathbf{D}\phi\end{aligned} \tag{3.1.27}$$

If we substitute in equation 3.1.27 the generalized NP commutator 2.1.143 and the Ricci equation 2.1.155, and again making use of definition 3.1.8 and 3.1.11 we obtain:

$$(\text{þ}'\text{þ} - \text{þ}\text{þ}')\phi = -\{\overline{T}\eth + T\eth' + (\mathbf{p} - N)(\boldsymbol{\Psi}_2 + \boldsymbol{\Phi}_{11'} - \boldsymbol{\Lambda})$$

$$+(\mathbf{q} - N')(\overline{\boldsymbol{\Psi}}_2 + \boldsymbol{\Phi}_{11'} - \boldsymbol{\Lambda})\}\phi + (\boldsymbol{\Phi}_{21'} + \boldsymbol{\Psi}_3)(\phi \cdot o) \tag{3.1.28}$$

$$-(\boldsymbol{\Phi}_{12} + \overline{\boldsymbol{\Psi}}_3)(\phi \cdot \overline{o})$$

In the same way one obtains all the rest of the commutators in this new formalism:

$$(\text{þ}\eth - \eth\text{þ})\phi = \{\overline{R}\eth + S\eth' - K\text{þ}' - (\mathbf{p} - N)(\boldsymbol{\Psi}_1) - (\mathbf{q} - N')(\boldsymbol{\Phi}_{01'})\}\phi$$

$$+(2\boldsymbol{\Lambda} + \boldsymbol{\Psi}_2)(\phi \cdot o) + \boldsymbol{\Phi}_{02'}(\phi \cdot \overline{o}) \tag{3.1.29}$$

$$(\eth\eth' - \eth'\eth)\phi = \{(R - \overline{R})\text{þ}' + (\mathbf{p} - N)(\boldsymbol{\Psi}_2 - \boldsymbol{\Phi}_{11'} - \boldsymbol{\Lambda})$$

$$-(\mathbf{q} - N')(\overline{\boldsymbol{\Psi}}_2 - \boldsymbol{\Phi}_{11'} - \boldsymbol{\Lambda})\}\phi - (\boldsymbol{\Psi}_3 - \boldsymbol{\Phi}_{21'})(\phi \cdot o) \tag{3.1.30}$$

$$-(\boldsymbol{\Phi}_{12'} - \overline{\boldsymbol{\Psi}}_3)(\phi \cdot \overline{o})$$

$$(\text{þ}'\eth - \eth\text{þ}')\phi = \{-T\text{þ}' - (\mathbf{p} - N)(\overline{\boldsymbol{\Psi}}_3) - (\mathbf{q} - N')(\boldsymbol{\Phi}_{12'})\}\phi$$

$$-\boldsymbol{\Phi}_{22'}(\phi \cdot o) - \overline{\boldsymbol{\Psi}}_4(\phi \cdot \overline{o}) \tag{3.1.31}$$

$$(\text{þ}\eth' - \eth'\text{þ})\phi = \{R\eth' + \overline{S}\eth - \overline{K}\text{þ}' - (\mathbf{q} - N')(\overline{\boldsymbol{\Psi}}_1)$$

$$-(\mathbf{p} - N)(\boldsymbol{\Phi}_{10'})\}\phi + (2\boldsymbol{\Lambda} + \overline{\boldsymbol{\Psi}}_2)(\phi \cdot \overline{o}) + \boldsymbol{\Phi}_{20'}(\phi \cdot o) \tag{3.1.32}$$

$$(\text{þ}'\eth' - \eth'\text{þ}')\phi = \{-\overline{T}\text{þ}' + (\mathbf{p} - N)(\boldsymbol{\Psi}_3) + (\mathbf{q} - N')(\boldsymbol{\Phi}_{21'})\}\phi$$

$$-\boldsymbol{\Phi}_{22'}(\phi \cdot \overline{o}) - \boldsymbol{\Psi}_4(\phi \cdot o) \tag{3.1.33}$$

We now write the Ricci equations in this new formalism. Guided by the GHP example we start considering the expression $\text{þ}R - \text{þ}'K$. Note that both terms are $(2,1)$-spinors of weight $\{4,3\}$ so it makes sense to consider their difference. An explicit calculation shows that

$$(\text{þ}R)_{ABA'} - (\eth' K)_{ABA'} = R_{(A}R_{B)}\overline{o}_{A'} + S_{A'}\overline{S}_{(A}o_{B)} - \overline{K}o_{(B}T_{A)A'}$$

$$+\Phi_{00'}o_A o_B \bar{o}_{A'} \tag{3.1.34}$$

which may be written in the compact notation as

$$\text{þ}R - \eth' K = R^2 + S\bar{S} - \bar{K}T + \Phi_{00'} \tag{3.1.35}$$

The other Ricci equations are similarly found to be:

$$\begin{aligned}
\text{þ}S - \eth K &= RS - \bar{R}S - KT + \Psi_0 && (3.1.36)\\
\text{þ}T - \text{þ}'K &= RT + \bar{T}S + \Psi_1 + \Phi_{01'} && (3.1.37)\\
\eth T - \text{þ}'S &= T^2 + \Phi_{02'} && (3.1.38)\\
\text{þ}'R - \eth' T &= -T\bar{T} - \Psi_2 - 2\Lambda && (3.1.39)\\
\eth R - \eth' S &= (R - \bar{R})T - \Psi_1 + \Phi_{01'} && (3.1.40)
\end{aligned}$$

Note that one can take the complex conjugate of these equations but there are no primed versions since such equations would involve derivatives of spin coefficients which transform badly under the null rotation part of 3.1.2.

Finally we consider the Bianchi identities which in the compact notation take the form:

$$\text{þ}\Psi_1 - \eth'\Psi_0 - \text{þ}\Phi_{01'} + \eth\Phi_{00'} = 4R\Psi_1 - 3K\Psi_2 - 2\bar{R}\Phi_{01'}$$

$$-2S\Phi_{10'} + 2K\Phi_{11'} + \bar{K}\Phi_{02'} \tag{3.1.41}$$

$$\text{þ}\Psi_2 - \eth'\Psi_1 - \eth'\Phi_{01'} + \text{þ}'\Phi_{00'} + 2\text{þ}\Lambda = 3R\Psi_2 - 2K\Psi_3 - 2\bar{T}\Phi_{01'}$$

$$+2T\Phi_{10'} + 2R\Phi_{11'} + \bar{S}\Phi_{02'} \tag{3.1.42}$$

$$\text{þ}\Psi_3 - \eth'\Psi_2 - \text{þ}\Phi_{21'} + \eth\Phi_{20'} - 2\eth'\Lambda = 2R\Psi_3 - K\Psi_4$$

$$-2\bar{R}\Phi_{21'} + \bar{K}\Phi_{22'} \tag{3.1.43}$$

$$\text{þ}\Psi_4 - \eth'\Psi_3 - \eth'\Phi_{21'} + \text{þ}'\Phi_{20'} = R\Psi_4 - 2\bar{T}\Phi_{21'} + \bar{S}\Phi_{22'} \tag{3.1.44}$$

$$\text{þ}'\Psi_3 - \eth\Psi_4 - \text{þ}'\Phi_{21'} + \eth'\Phi_{22'} = -T\Psi_4 + \overline{T}\Phi_{22'} \tag{3.1.45}$$

$$\text{þ}'\Psi_2 - \eth\Psi_3 - \eth\Phi_{21'} + \text{þ}\Phi_{22'} + 2\text{þ}'\Lambda = S\Psi_4 - 2T\Psi_3 + \overline{R}\Phi_{22'} \tag{3.1.46}$$

$$\text{þ}'\Psi_1 - \eth\Psi_2 - \text{þ}'\Phi_{01'} + \eth'\Phi_{02'} - 2\eth\Lambda = 2S\Psi_3 - 3T\Psi_2 - 2R\Phi_{12'}$$

$$+2T\Phi_{11'} + \overline{T}\Phi_{02'} \tag{3.1.47}$$

$$\text{þ}'\Psi_0 - \eth\Psi_1 - \eth\Phi_{01'} + \text{þ}\Phi_{02'} = 3S\Psi_2 - 4T\Psi_1$$

$$-2k\Phi_{12'} + 2S\Phi_{11'} + \overline{R}\Phi_{02'} \tag{3.1.48}$$

The contracted Bianchi identities are given by:

$$\text{þ}\Phi_{11'} + \text{þ}'\Phi_{00'} - \eth\Phi_{10'} - \eth'\Phi_{01'} + 3\text{þ}\Lambda = 2R\Phi_{11'}$$

$$+2\overline{R}\Phi_{11'} - 2\overline{T}\Phi_{01'} - 2T\Phi_{10'} - \overline{K}\Phi_{12'} - K\Phi_{21'} + S\Phi_{20'} + \overline{S}\Phi_{02'} \tag{3.1.49}$$

$$\text{þ}\Phi_{12'} + \text{þ}'\Phi_{01'} - \eth\Phi_{11'} - \eth'\Phi_{02'} + 3\eth\Lambda = 2R\Phi_{12'} + \overline{R}\Phi_{12'}$$

$$+2\overline{T}\Phi_{02'}2T\Phi_{11'} - K\Phi_{22'} + S\Phi_{21'} \tag{3.1.50}$$

$$\text{þ}'\Phi_{11'} + \text{þ}\Phi_{22'} - \eth'\Phi_{12'} - \eth\Phi_{21'} + 3\text{þ}'\Lambda = R\Phi_{22'} + \overline{R}\Phi_{22'}$$

$$-T\Phi_{21'} - \overline{T}\Phi_{12'} \tag{3.1.51}$$

We now proceed to show that from the new equations written in our new formalism we can obtain all generalized NP equations. It is easily seen that all generalized NP Bianchi identities are obtained from equations 3.1.41 through 3.1.51. The same can be said for those Ricci equations which involve null rotation invariant terms which have proper spin and boost weight. The case of the commutators and Ricci equations involving terms which do not scale, is not as straightforward. Let us take, for example, the commutator 3.1.28, and let it act on a spinor of type (N, N') and weight $\{\mathbf{p} = N, \mathbf{q} = N'\}$. Then, by equation 3.1.27 we have:

$$(\text{Þ}'\text{Þ} - \text{Þ}\text{Þ}')\phi = (\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi - (E + \overline{E})\phi - (G + \overline{G})\mathbf{D}\phi$$

If we now equate the right hand side of the equation written above and the right hand side of commutator 3.1.28 we have:

$$(\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi = (E + \overline{E})\mathbf{D}'\boldsymbol{\phi} + (G + \overline{G})\mathbf{D}\boldsymbol{\phi} - \overline{T}\boldsymbol{\delta}\boldsymbol{\phi} - T\boldsymbol{\delta}'\boldsymbol{\phi}$$

$$+(\boldsymbol{\Phi}_{21'} + \boldsymbol{\Psi}_3)(\boldsymbol{\phi} \cdot o) \;\; + (\boldsymbol{\Phi}_{12'} + \overline{\boldsymbol{\Psi}}_3)(\boldsymbol{\phi} \cdot \overline{o})$$

Furthermore, if we let commutator 3.1.28 act on a spinor $\boldsymbol{\phi}$ of type (N, N') and weight $\{\mathbf{p} = N + 1, \mathbf{q} = N'\}$, equation 3.1.27 gives:

$$(\text{Þ}'\text{Þ} - \text{Þ}\text{Þ}')\phi = (\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi + (\mathbf{D}G - \mathbf{D}'E + 2EB + \overline{G}E + GE)\phi$$

$$-(E + \overline{E})\boldsymbol{\phi} - (G + \overline{G})\mathbf{D}\boldsymbol{\phi}$$

If we now equate the right hand side of the equation written above and the right hand side of commutator 3.1.28 we get:

$$(\mathbf{D}'\mathbf{D} - \mathbf{D}\mathbf{D}')\phi + (\mathbf{D}G - \mathbf{D}'E + 2EB + \overline{G}E + GE)\phi - (E + \overline{E})\phi$$

$$-(G + \overline{G})\mathbf{D}\phi = -\overline{T}(\boldsymbol{\delta}\phi - B\phi) - T(\boldsymbol{\delta}'\phi - A\phi) + (\boldsymbol{\Psi}_2 + \boldsymbol{\Phi}_{11'} - \boldsymbol{\Lambda})\phi$$

$$-(\boldsymbol{\Phi}_{21'} + \boldsymbol{\Psi}_3)(\boldsymbol{\phi} \cdot o) - (\boldsymbol{\Phi}_{12'} + \overline{\boldsymbol{\Psi}}_3)(\boldsymbol{\phi} \cdot \overline{o})$$

Substituting the generalized NP commutator 2.1.143 in the above equation gives the equation :

$$\mathbf{D}G - \mathbf{D}'E = -2EG - \overline{G}E + G\overline{E} + \overline{T}B + AT + \boldsymbol{\Psi}_2 + \boldsymbol{\Phi}_{11'} - \boldsymbol{\Lambda}$$

which is the generalized NP Ricci equation 2.1.155.

To obtain all the rest of the generalized NP commutators and non scaling generalized NP Ricci equations from our "new" equations the process is exactly the same. If we let the commutator 3.1.29 act on a spinor ϕ of type (N, N') and weight $\{\mathbf{p} = N, \mathbf{q} = N'\}$ then we obtain the generalized NP commutator $(\mathbf{D}\boldsymbol{\delta} - \boldsymbol{\delta}\mathbf{D})\phi$, while 3.1.30 gives the commutator $(\boldsymbol{\delta}\boldsymbol{\delta}' - \boldsymbol{\delta}'\boldsymbol{\delta})\phi$, and commutators 3.1.31, 3.1.32 and 3.1.33 give the generalized NP commutators $(\mathbf{D}'\boldsymbol{\delta} - \boldsymbol{\delta}\mathbf{D}')\phi$, $(\mathbf{D}\boldsymbol{\delta}' - \boldsymbol{\delta}'\mathbf{D})\phi$ and $(\mathbf{D}'\boldsymbol{\delta}' - \boldsymbol{\delta}'\mathbf{D})\phi$ respectively.

Furthermore, if we let commutator 3.1.29 act on a spinor of type (N, N') and weight $\{\mathbf{p} = N+1, \mathbf{q} = N\}$ we obtain the generalized NP Ricci equation 2.1.144 while commutators 3.1.30, 3.1.31 and 3.1.32 give the Ricci equations 2.1.147, 2.1.145 and 2.1.143 respectively. Finally, letting commutator 3.1.31 act on a spinor of type (N, N') and weight $\{\mathbf{p} = N, \mathbf{q} = N'+1\}$ gives the Ricci equation 2.1.142.

Thus we have therefore shown that our equations are equivalent to the Einstein equations.

3.2 Relationship to the Penrose operators

In a paper on the geometry of impulsive gravitational waves Penrose [24] introduces differential operators $\eth_{A'A}$, $\overline{\eth}_{AA'}$ and þ which act within a null hypersurface $\mathcal{N}$ and act upon weighted scalar and spinor fields. Let $\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}}$ be a (N, N')-spinor of weight $\{\mathbf{p}, \mathbf{q}\}$. Then $\eth_{AA'}\,\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}}$ is a $(N+1, N'+1)$-spinor of weight $\{\mathbf{p}+2, \mathbf{q}\}$ which is defined by

$$\overline{o}_{(B'}\eth_{A')A}\,\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = o_A o^B \overline{o}_{(B'}\nabla_{A')B}\,\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}}$$

$$-(p o^B \overline{o}_{(B'}\nabla_{A')A}\, o_B + q o_A o^B \nabla_{B(A'}\,\overline{o}_{B')})\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.52}$$

Contracting the above expression with $\overline{o}^{A'}\overline{o}^{B'}$ gives

$$0 = q\kappa o_A \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.53}$$

So that expression 3.2.52 is well defined when $\kappa = 0$. In the context of null hypersurfaces, this expresses the condition that the direction of the flag pole of o_A (and not its extent) is paralelly propagated along the null geodesic generators of $\mathcal{N}$.

The operators that we have defined are more general since they make no assumptions about the choice of o_A. Furthermore in order to be able to introduce a compact index free notation our operators act on totally symmetric spinors and produce totally symmetric spinors. However in situations where o_A is chosen so that κ is zero, our operators are closely related to those of Penrose. Since both sets of operators obey the Leibnitz property it is enough to give the relationship between the operators when acting on scalars and spinors with a single index.

We give below the relationship between the operators when $o^B D o_B = k = 0$

(i) For a scalar field

$$(\text{þ}\eta)_{AA'} = (\text{þ}\eta)o_A\bar{o}_{A'} \tag{3.2.54}$$
$$(\eth\eta)_{AA'B'} = \bar{o}_{(B'}\eth_{A')A}\,\eta \tag{3.2.55}$$
$$(\eth'\eta)_{ABA'} = o_{(B}\bar{\eth}_{A)A'}\,\eta \tag{3.2.56}$$

(ii) For a $(1,0)$-spinor field

$$(\text{þ}\eta)_{ABA'} = (\text{þ}\eta_{(A})o_{B)}\bar{o}_{A'} \tag{3.2.57}$$
$$(\eth\eta)_{ABA'B'} = \bar{o}_{(B'}\eth_{A')(A}\,\eta_{B)} \tag{3.2.58}$$
$$(\eth'\eta)_{ABCA'} = (\bar{\eth}_{A'(A}\,\eta_B)o_{C)} \tag{3.2.59}$$

(iii) For a $(0,1)$-spinor field

$$(\text{þ}\eta)_{AA'B'} = o_A(\text{þ}\eta_{(A'})\bar{o}_{B')} \tag{3.2.60}$$
$$(\eth\eta)_{AA'B'C'} = (\eth_{A(A'}\,\eta_{B'})\bar{o}_{C')} \tag{3.2.61}$$
$$(\eth'\eta)_{ABA'B'} = o_{(B}\bar{\eth}_{A)(A'}\,\eta_{B')} \tag{3.2.62}$$

The relationship between the various definitions for edth and thorn can be seen more easily if we introduce the auxiliary differential operator $\mathcal{D}_{ABA'B'}$ which is defined by:

$$\mathcal{D}_{ABA'B'}\;\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = o_A\bar{o}_{A'}\nabla_{BB'}\,\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}}$$
$$-(p\bar{o}_{A'}\nabla_{BB'}\,o_A + q o_A\nabla_{BB'}\,\bar{o}_{A'})\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.63}$$

In terms of this operator the standard definitions of edth and thorn are given by:

$$\text{þ}\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = \iota^A o^B\bar{\iota}^{A'}\bar{o}^{B'}\mathcal{D}_{ABA'B'}\;\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.64}$$
$$\eth\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = \iota^A o^B\bar{\iota}^{A'}\bar{\iota}^{B'}\mathcal{D}_{ABA'B'}\;\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.65}$$
$$\bar{\eth}\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = \iota^A \iota^B\bar{\iota}^{A'}\bar{o}^{B'}\mathcal{D}_{ABA'B'}\;\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.66}$$
$$\text{þ}'\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = \iota^A \iota^B\bar{\iota}^{A'}\bar{\iota}^{B'}\mathcal{D}_{ABA'B'}\;\eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{3.2.67}$$

On the other hand our new derivative operators are given by:

$$(\text{þ}\eta)_{AC_1\ldots C_N A'C'_1\ldots C'_{N'}} = \sum_{sym} o^B \bar{o}^{B'} \mathcal{D}_{ABA'B'} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.68)$$

$$(\eth\eta)_{AC_1\ldots C_N A'B'C'_1\ldots C'_{N'}} = \sum_{sym} o^B \mathcal{D}_{ABA'B'} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.69)$$

$$(\eth'\eta)_{ABC_1\ldots C_N A'C'_1\ldots C'_{N'}} = \sum_{sym} \bar{o}^{B'} \mathcal{D}_{ABA'B'} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.70)$$

$$(\text{þ}'\eta)_{ABC_1\ldots C_N A'B'C'_1\ldots C'_{N'}} = \sum_{sym} \mathcal{D}_{ABA'B'} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.71)$$

Where $\sum_{sym}$ indicates symmetrisation over all primed and unprimed indices. Finally in the case that $k = 0$ the Penrose derivative operators are given by:

$$o_A \bar{o}_{A'} \text{þ} \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = o^B \bar{o}^{B'} \mathcal{D}_{ABA'B'} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.72)$$

$$\bar{o}_{(B'} \eth_{A')A} \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = o^B \mathcal{D}_{AB(A'B')} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.73)$$

$$o_{(B} \bar{\eth}_{A)A'} \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} = \bar{o}^{B'} \mathcal{D}_{(AB)A'B'} \ \eta_{C_1\ldots C_N C'_1\ldots C'_{N'}} \quad (3.2.74)$$

3.3 Geometrical Interpretation

In this section we analyse the geometrical significance, in the context of null hypersurfaces, of the new invariant quantities we have constructed. We start by giving a brief résumé on results concerning null hypersurfaces. Much of what follows can be found in [29] and [7].

In what follows we shall adopt the convention that Greek indices run from 0 to n and latin indices from 1 to n, with $n = 4$, unless otherwise indicated. Partial derivatives will be denoted by comma, covariant derivative in M by a semicolon and in a null hypersurface $\mathcal{N}$ by a colon.

In M, a one parameter family of null hypersurfaces foliating M is given by the equation $\phi(x^\alpha) = p$,where p is the parameter labelling the hypersurfaces and ϕ satisfies $g^{\alpha\beta}\frac{\partial\phi}{\partial x^\alpha}\frac{\partial\phi}{\partial x^\beta} = 0$ and $g_{\alpha\beta}$ is the metric tensor on M. Each member of the one-parameter family of null hypersurfaces has induced upon it a degenerate metric tensor $h_{\alpha\beta}$ of rank 2 and signature $(0-1-1)$.

We will let x^α be a coordinate system for M while $'x^a$ will denote coordinates of a null hypersurface $\mathcal{N}$ embedded in M. $x^\alpha = B^\alpha('x^a)$. If we have an embedding $B : \mathcal{N} \longrightarrow M$ then the connecting quantities B^α_a are defined to be $B^\alpha_a = B^\alpha_{,a}$ and are used as projection operators, so that we have $x^\alpha = B^\alpha('x^a)$. For example if w_α is a covariant vector of M, then $B^\alpha_a w_\alpha = {}'w_a$ are the components of the projection of w_α into $\mathcal{N}$ in the coordinates of $\mathcal{N}$. On the other hand if ${}'v^a$ is a contravariant vector in $\mathcal{N}$, then $B^\alpha_a {}'v^a = {}'v^\alpha$ are the components of ${}'v^a$, considered as a vector of M, in the coordinates of M.

In order to project contravariant vector fields of M into $\mathcal{N}$ or to form a vector field in M corresponding to a covariant vector field in $\mathcal{N}$, we must first rig $\mathcal{N}$. This means that we must define a direction at each point of M which does not lie in $\mathcal{N}$ In practice, this is done by defining a contravariant vector field in M, which nowhere lies in $\mathcal{N}$.

The covariant normal to $\mathcal{N}$ is given by $\frac{\partial\phi}{\partial x^\alpha}$ and $\ell^a \propto g^{\alpha\beta}\frac{\partial\phi}{\partial x^\beta}$ represents a tangent vector field to the congruence of null geodesics generating $\mathcal{N}$. Note that the scaling of ℓ^a depends upon the family of hypersurfaces not simply on $\mathcal{N}$.We choose $(\ell^\alpha, n^\alpha, m^\alpha, \overline{m}^\alpha)$ to be a basis for M such that:

$$g_{\alpha\beta} = 2\ell_{(\alpha}n_{\beta)} - 2m_{(\alpha}\overline{m}_{\beta)} \tag{3.3.75}$$

$$\ell_\alpha n^\alpha = -m_\alpha\overline{m}^\alpha = 1 \tag{3.3.76}$$

with all other inner products zero.

This null tetrad is not uniquely defined by 3.3.75 and 3.3.76. The remaining freedom in the choice of tetrad that preserves the generator direction of ℓ^a is given by transformations:

$$\begin{aligned} \ell^\alpha &\longrightarrow A\ell^\alpha \\ n^\alpha &\longrightarrow A^{-1}n^\alpha - Dm^\alpha - \overline{D}\overline{m}^\alpha + AD\overline{D}\ell^\alpha \\ m^\alpha &\longrightarrow e^{iE}(m^\alpha - A\overline{D}\ell^\alpha) \end{aligned} \tag{3.3.77}$$

with A, E real , $A > 0$ and D complex and all are functions of x^α.

Such transformations form a subgroup of the Lorentz group which splits into three subgroups characterized by:

(a) $D = A - 1 = 0$ which corresponds to an ordinary rotation of $m^\alpha, \overline{m}^\alpha$.
(b) $E = D = 0$ which corresponds to a scaling transformation.
(c) $E = A - 1 = 0$ which corresponds to a null rotation about ℓ^α.

The condition $q^\alpha \propto \ell^\alpha \Rightarrow q^\alpha\ell_\alpha = 0$ is the necessary and sufficient condition that any null vector q^α lie in $\mathcal{N}$. Therefore n^α transvects $\mathcal{N}$. Furthermore, under 3.3.77 n^α transforms into any vector field with the same time orientation as n^α and not parallel to ℓ^α. Therefore n^α is the most general null vector field transvecting $\mathcal{N}$. By taking n^α to be the rigging field and ℓ_α to be the covariant normal, we are able to form the projection operators B^α_a, B^a_α, B^α_β and C^α_β which satisfy:

$$B^\alpha_b B^a_\alpha = \delta^a_b, \;\; B^\alpha_a \ell_\alpha = B^a_\alpha n^\alpha = 0, \;\; B^\alpha_\beta = \delta^\alpha_\beta - n^\alpha \ell_\beta, \;\; C^\alpha_\beta = n^\alpha \ell_\beta \qquad (3.3.78)$$

By projecting the null tetrad spanning M in its contravariant and covariant forms into $\mathcal{N}$ we obtain the contravariant $T^\alpha = (\ell^\alpha, m^\alpha, \overline{m}^\alpha)$ and the covariant $T_\alpha = (n_\alpha, m_\alpha, \overline{m}_\alpha)$ triads which span $\mathcal{N}$. These triads in hypersurface coordinates are given by $T^a = B^a_\alpha T^\alpha = (\ell^a, m^a, \overline{m}^a)$ and $T_a = B^\alpha_a T_\alpha = (n_a, m_a \overline{m}_a)$ respectively. The scale product between triad members are given by:

$$\ell^a n_a = -m^a \overline{m}_a = 1 \qquad (3.3.79)$$

and all other inner products zero.

The covariant metric tensor of $\mathcal{N}$, indiced by its embedding in M is given by:

$$'g_{ab} \equiv B^{\alpha\beta}_{ab}\, g_{\alpha\beta} \qquad (3.3.80)$$

with $B^{\alpha\beta}_{ab} = B^\alpha_a B^\beta_b$. Hence by 3.3.78 we have:

$$'g_{ab} = -m_{(a}\overline{m}_{b)} \Rightarrow {'g}_{ac}\ell^c = 0 \qquad (3.3.81)$$

So that $'g_{ab}$ possesses a single eigendirection of eigenvalue zero, l^a. Furthermore $'g_{ab}$ is a degenerate metric, of rank 2, and therefore cannot be inverted to give a contravariant metric $'g^{ab}$ such that $'g^{ac} g_{bc} = \delta^a_b$. However, we can introduce a substitute contravariant metric given by:

$$'g^{ab} = B^{ab}_{\alpha\beta} g^{\alpha\beta} = -2m^{(a}\overline{m}^{b)} \qquad (3.3.82)$$

where $'g^{ab}$ is the projection of $g^{\alpha\beta}$ in $\mathcal{N}$. $'g^{ab}$ and $'g_{ab}$ satisfy:

$$\begin{aligned} {}'g_{ac}{}'g_{bf}{}'g^{cf} &= {}'g_{ab} \\ {}'g^{cb}n_c &= 0 \\ {}'g^{ac}{}'g_{bc} &= \delta^a_b - \ell^a n_b \end{aligned} \tag{3.3.83}$$

Under transformation 3.3.77, the triad transforms as follows:

$$\begin{aligned} \ell^a &\longrightarrow A\ell^a \\ m^a &\longrightarrow e^{iE}(m^a - \overline{D}\ell^a) \\ n_a &\longrightarrow A^{-1}n_a - Dm_a - \overline{D}\overline{m}_a \\ m_a &\longrightarrow e^{iE}m_a \end{aligned} \tag{3.3.84}$$

where A, D and E are now functions of $'x^a$ and p. The covariant metric $'g_{ab}$ is invariant under 3.3.84, however this is not the case for $'g^{ab}$, which is invariant only under the subgroup of 3.3.84 given by $D = 0$.

The Lie derivative of a tensor field in $\mathcal{N}$ is defined as the components in $\mathcal{N}$ of the Lie derivative of the corresponding tensor field in M. Therefore, for any tensor field $'T^{a...}_{...b}$ in $\mathcal{N}$ and any contravariant vector field v^α in M we define:

$$\pounds'_v{}'T^{a...}_{...b} = B^{a...\beta}_{\alpha...b}\,\pounds_v B^{\alpha...f}_{e...\beta}\,{}'T^{e...}_{...f} \tag{3.3.85}$$

If v^α lies in $\mathcal{N}$, i.e, if $v^\alpha = B^\alpha_e v^e$ then the Lie derivative of $'T^{a...}_{...b}$ can be calculated using definition 3.3.85 or in the usual way.

Denote the connections on M and $\mathcal{N}$ respectively by $\Gamma^\alpha_{\beta\gamma}$ and $'\Gamma^a_{bc}$, then the covariant derivative with respect to $'\Gamma^a_{bc}$ of any tensor field $'T^{\alpha...}_{...\beta}$ of M is given by:

$$'T^{\alpha...}_{...\beta:c} = T^{\alpha...}_{...\beta;\gamma}\,B^\gamma_c \tag{3.3.86}$$

Suppose that we have some arbitrary vector field $'v^\alpha = B^\alpha_e{}'v^e$ in M which lies in $\mathcal{N}$. From 3.3.86 we obtain:

$$B^\alpha_{b:c} = B^\alpha_{b,c} - B^\alpha_e{}'\Gamma^e_{cb} + B^{\beta\gamma}_{bc}\Gamma^\alpha_{\gamma\beta} \tag{3.3.87}$$

If we take $\Gamma^{\alpha}_{\beta\gamma}$ and $'\Gamma^{a}_{bc}$ to be both symmetric and remembering the definition of B^{α}_{b} we get:

$$B^{\alpha}_{b:c} = B^{\alpha}_{(b:c)} \tag{3.3.88}$$

Furthermore equations 3.3.86 and 3.3.87 give:

$$\begin{aligned} \ell^{a}{}_{;b} &= \ell^{a}{}_{,b} + {}'\Gamma^{a}_{bd}\ell^{d} && (3.3.89)\\ n_{a:b} &= n_{a,b} - {}'\Gamma^{d}_{ab}n_{d} && (3.3.90)\\ m_{a:b} &= m_{a,b} - {}'\Gamma^{d}_{ab}m_{d} && (3.3.91)\\ m^{a}{}_{;b} &= m^{a}{}_{,b} + {}'\Gamma^{a}_{bd}m^{d} && (3.3.92) \end{aligned}$$

We now concentrate on obtaining an intrinsic connection for $\mathcal{N}$. For the purpose of a geometrical interpretation we choose the intrinsic, symmetric, non-metric connection introduced by Daŭtcourt [7]:

$$'\Gamma^{a}_{bc} = \frac{1}{2}{}'g^{ae}({}'g_{be,c} + {}'g_{ce,b} - {}'g_{bc,e}) + \ell^{a}n_{(b,c)} \tag{3.3.93}$$

By embedding $'\Gamma^{a}_{bc}$ onto M and using equations 3.3.81 and 3.3.82 a straightforward calculation shows:

$$'\Gamma^{a}_{bc} = B^{a\beta\gamma}_{\alpha bc}\,\Gamma^{\alpha}_{\beta\gamma} + B^{a}_{\alpha}B^{\alpha}_{b,c} + B^{a\beta\gamma}_{\alpha bc}\,\ell^{\alpha}n_{(\beta;\gamma)} \tag{3.3.94}$$

We see that the connection $'\Gamma^{a}_{bc}$ in $\mathcal{N}$ is determined by the metric of M and the rigging field n^{α}.

Using equation 1.3.90 and Daŭtcourt's connection 3.3.93 we obtain:

$$n_{a:b} = n_{[a,b]} \tag{3.3.95}$$

Unfortunately 3.3.94 does not give $'\Gamma^{a}_{bc}$ explicitly in terms of $\Gamma^{\alpha}_{\beta\gamma}$. However, remembering that $\ell_{\alpha} \propto \phi_{,\alpha}$ and writing:

$$\ell_{\alpha} = e^{\rho}\phi_{,\alpha} \tag{3.3.96}$$

where ρ is an arbitrary scalar function.

then a straightforward calculation using 3.3.85, 3.3.75, 3.3.76, 3.3.80, 3.3.82 and 3.3.96 shows that:

$$B^{\gamma\beta}_{cb} n_{\beta;\gamma} = \frac{1}{2}\mathcal{L}'_n g_{cb} - \rho_{,(c} n_{b)} - n_{[c,b]} \tag{3.3.97}$$

So that if we substitute 3.3.97 in 3.3.94 we get:

$$'\Gamma^a_{bc} = B^{a\beta\gamma}_{\alpha bc} \Gamma^\alpha_{\beta\gamma} + B^a_\alpha B^\alpha_{b,c} + \frac{1}{2}\ell^a \mathcal{L}'_n g_{bc} - \rho_{,(c} n_{b)} \tag{3.3.98}$$

With the help of equation 3.3.98 we can now proceed to calculate the intrinsic curvature of $\mathcal{N}$. Using Daŭtcourt's connection 3.3.93 along with equations 3.3.89 through 3.3.92 and equation 3.3.95 we obtain:

$$k = \ell^a l^b m_{a:b} \tag{3.3.99}$$

$$\epsilon - \overline{\epsilon} = m^a \ell^b \overline{m}_{a:b} \tag{3.3.100}$$

$$\rho = \ell^a \overline{m}^b m_{a:b} \tag{3.3.101}$$

$$\sigma = \ell^a m^b m_{a:b} = \ell^a m^b m_{[a,b]} \tag{3.3.102}$$

$$\alpha - \overline{\beta} = m^a \overline{m}^b \overline{m}_{a:b} = m^a \overline{m}^b \overline{m}_{[a,b]} \tag{3.3.103}$$

$$\frac{1}{2}(\tau' + \alpha + \overline{\beta}) = \overline{m}^a l^b n_{a:b} = \overline{m}^a \ell^b n_{[a,b]} \tag{3.3.104}$$

$$\frac{1}{2}(\rho' - \overline{\rho}') = \overline{m}^a m^b n_{a:b} = \overline{m}^a m^b n_{[a,b]} \tag{3.3.105}$$

where $k, \epsilon, \rho, \sigma, \alpha, \beta, \tau', \rho'$ are the well known NP spin coefficients.

Furthermore, the NP operators D, δ, δ' can be expressed as:

$$D = \ell^a \nabla_a \tag{3.3.106}$$

$$\delta = m^a \nabla_a \tag{3.3.107}$$

$$\delta' = \overline{m}^a \nabla_a \tag{3.3.108}$$

Hence, D, δ, δ' are the intrinsic operators and κ, $\epsilon - \overline{\epsilon}$, ρ, σ, $(\alpha - \overline{\beta})$, $\frac{1}{2}(\tau' + \alpha + \overline{\beta})$, $\frac{1}{2}(\rho' - \overline{\rho}')$ are projected into $\mathcal{N}$ in a straightforward manner since by definition we have:

$$
\begin{aligned}
k &= \ell^\alpha l^\beta m_{\alpha;\beta} \\
\epsilon - \overline{\epsilon} &= m^\alpha \ell^\beta \overline{m}_{\alpha;\beta} \\
\rho &= \ell^\alpha \overline{m}^\beta m_{\alpha;\beta} \\
\sigma &= \ell^\alpha m^\beta m_{\alpha;\beta} = \ell^\alpha m^\beta m_{[\alpha,\beta]} \\
\alpha - \overline{\beta} &= m^\alpha \overline{m}^\beta \overline{m}_{\alpha;\beta} = m^\alpha \overline{m}^\beta \overline{m}_{[\alpha,\beta]} \\
\frac{1}{2}(\tau' + \alpha + \overline{\beta}) &= \overline{m}^\alpha \ell^\beta n_{\alpha;\beta} = \overline{m}^\alpha \ell^\beta n_{[\alpha,\beta]} \\
\frac{1}{2}(\rho' - \overline{\rho}') &= \overline{m}^\alpha m^\beta n_{\alpha;\beta} = \overline{m}^\alpha m^\beta n_{[\alpha,\beta]}
\end{aligned}
$$

We construct the Riemann tensor from the Daŭtcourt connection ${}'\Gamma^a_{bc}$ as follows:

$$
{}'R_{abc}{}^d = {}'\Gamma^d_{bc,a} - {}'\Gamma^d_{ac,b} + {}'\Gamma^d_{ae}\,{}'\Gamma^e_{bc} - {}'\Gamma^d_{be}\,{}'\Gamma^e_{ac} \tag{3.3.109}
$$

In addition to ℓ^a being hypersurface orthogonal we will consider it to also be a gradient so that, under this condition, the non-vanishing components of the curvature of the null hypersurface $\mathcal{N}$ can be calculated using equations 3.3.81, 3.3.82, 3.3.93, 1.3.99 through 1.3.108 and equation 3.3.109. They are:

$$
m^a \ell^b \ell^c R_{abc}{}^d \overline{m}_d = D\overline{\rho} - \overline{\rho}^2 - \sigma\overline{\sigma} \tag{3.3.110}
$$

$$
\overline{m}^a \ell^b \ell^c R_{abc}{}^d \overline{m}_d = D\overline{\sigma} - 2\overline{\sigma}(\overline{\rho} + \overline{\epsilon} - \epsilon) \tag{3.3.111}
$$

$$
\begin{aligned}
m^a \overline{m}^b \ell^c R_{abc}{}^d \overline{m}_d &= \delta'\overline{\rho} - \delta\overline{\sigma} + 2\overline{\sigma}(\overline{\alpha} + \alpha') + \frac{1}{2}\overline{\sigma}(\beta + \overline{\alpha} + \overline{\tau}') \\
&-\overline{\rho}(\overline{\beta} + \alpha + \tau')
\end{aligned} \tag{3.3.112}
$$

$$
\begin{aligned}
\ell^a m^b m^c R_{abc}{}^d \overline{m}_d &= -D(\overline{\alpha} + \alpha') - \delta(\epsilon - \overline{\epsilon}) + \overline{\rho}(\overline{\epsilon} - \epsilon) - \sigma(\alpha + \overline{\alpha}') \\
&+(\epsilon - \overline{\epsilon})(\overline{\alpha} + \alpha') + \frac{1}{2}(\beta + \overline{\alpha} + \overline{\tau}')(\epsilon - \overline{\epsilon}) - \frac{1}{2}\overline{\rho}(\beta + \overline{\alpha} - \overline{\tau}')
\end{aligned} \tag{3.3.113}
$$

$$
\ell^a \overline{m}^b m^c R_{abc}{}^d \overline{m}_d = D(\overline{\alpha}' + \alpha) - \delta'(\epsilon - \overline{\epsilon}) - \overline{\rho}(\overline{\alpha}' + \alpha) + \sigma(\overline{\alpha} + \alpha')
$$

$$+(\epsilon-\overline{\epsilon})(\overline{\alpha}'+\alpha)+(\overline{\beta}+\alpha+\tau')(\epsilon-\overline{\epsilon})-\frac{1}{2}\overline{\sigma}(\beta+\overline{\alpha}+\overline{\tau}') \tag{3.3.114}$$

$$\overline{m}^a m^b m^c R_{abc}{}^d\overline{m}_d = 2(\alpha+\overline{\alpha}')(\overline{\alpha}+\alpha')-\delta'(\overline{\alpha}+\alpha')-\delta(\alpha+\overline{\alpha}')$$

$$-(\rho'-\overline{\rho}')(\epsilon-\overline{\epsilon})+\frac{1}{2}\overline{\rho}(\rho'-\overline{\rho}') \tag{3.3.115}$$

$$\ell^a m^b \overline{m}^c R_{abc}{}^d\overline{m}_d = -\frac{1}{2}\overline{\rho}(\overline{\beta}+\alpha+\tau') \tag{3.3.116}$$

$$\ell^a \overline{m}^b \overline{m}^c R_{abc}{}^d\overline{m}_d = -\frac{1}{2}\overline{\sigma}(\overline{\beta}+\alpha+\tau') \tag{3.3.117}$$

$$m^a \overline{m}^b \overline{m}^c R_{abc}{}^d\overline{m}_d = \frac{1}{2}\overline{\sigma}(\overline{\rho}'-\rho') \tag{3.3.118}$$

We now suppose the null hypersurface $\mathcal{N}$ is spanned by a set of spacelike surfaces. Such that if ℓ^a is the gradient of a function constant on each surface of this set then:

$$2(\alpha+\overline{\alpha}')(\overline{\alpha}+\alpha')-\delta'(\overline{\alpha}+\alpha')-\delta(\alpha+\overline{\alpha}')-(\rho'-\overline{\rho}')(\epsilon-\overline{\epsilon})+\frac{1}{2}\overline{\rho}(\rho'-\overline{\rho}') \tag{3.3.119}$$

is the Gaussian curvature of this set [7].

We have seen that $o^A\overline{o}^{A'}$ is tangent to the null generators of $\mathcal{N}$, so that any tangent vector to $\mathcal{N}$ has the form $v^a = \xi^A\overline{o}^{A'} + o^A\overline{\xi}^{A'}$, where ξ is a $(1,0)$-spinor field of weight $\{0,-1\}$. Then the components of Dv^a, δv^a and $\delta' v^a$ may be obtained from the components of $\text{\th}\xi$, $\eth\xi$ and $\eth'\xi$. However if one contracts $v^a = v^{AA'}$ with $\overline{o}_{A'}$ one obtains $\phi^A = v^{AA'}\overline{o}_{A'}$ which is a $(1,0)$-spinor field of weight $\{2,-1\}$. Note that $\phi^A = \eta o^A$ where $\eta = v^a\overline{m}_a$ is the component of v^a in the $\overline{m}^a$ direction. If one then applies the commutator $\eth\eth' - \eth'\eth$ to ϕ_A using equation 3.1.30 and noting that $R = \overline{R}$ since ℓ^a is hypersurface orthogonal, one obtains a totally symmetric $(4,3)$-spinor.

$$\{(\mathbf{\Psi}_2-\mathbf{\Phi}_{11'}-\mathbf{\Lambda})+(\overline{\mathbf{\Psi}}_2-\mathbf{\Phi}_{11'}-\mathbf{\Lambda})\}\phi \tag{3.3.120}$$

If one now makes some choice for ι^A then one can calculate the components of this spinor and one finds for example that

$$((\boldsymbol{\partial}\boldsymbol{\partial}' - \boldsymbol{\partial}'\boldsymbol{\partial})\phi)_{43'} = m^a \overline{m}^b m^c \overline{m}^d R_{abcd}\, \eta \tag{3.3.121}$$

which is just proportional to the sectional curvature in the $m \wedge \overline{m}$ direction. Suppose now that ι^A is chosen so that the real and imaginary parts of $\overline{m}^a$ are tangent to a spacelike two surface $\mathcal{S}$ and applies the commutator of the GHP operators $\eth$ and $\eth'$ to the weight $\{1, -1\}$ scalar field $\eta = \phi^A \iota_A$ then one obtains

$$(\eth\eth' - \eth'\eth)\eta = \mathcal{R}\eta \tag{3.3.122}$$

where $\mathcal{R}$ is the Gaussian curvature of $\mathcal{S}$ [27] given by:

$$\mathcal{R} = (\sigma\sigma' - \Psi_2 - \rho\rho' + \Phi_{11'} + \Lambda) + (\overline{\sigma\sigma}' - \overline{\Psi}_2 - \overline{\rho\rho}' + \Phi_{11'} + \Lambda) \tag{3.3.123}$$

Thus the new commutator involves the projection of the spacetime curvature into $\mathcal{S}$ rather than the curvature of the projected connection as is the case with the GHP formalism. This is not surprising when one considers that the induced connection depends upon the choice of ι^A in a non-trivial way, since $'\Gamma^a_{bc}$ depends on the rigging field n^a, and hence the curvature of the connection does not transform at all nicely under null rotations. On the other hand our new formalism has been designed so that the components transform covariantly under null rotations. Indeed the difference between the projection of the spacetime curvature and the curvature of the projected connection consists of spin coefficients which transform badly. This explains why our commutators appear somewhat simpler than the GHP commutators since the terms that transform badly under null rotations have been incorporated into our differential operators. Of course a price must be paid in the correspondingly more complicated definitions of the new operators.

Hence we have that the GHP commutator $\eth\eth' - \eth'\eth$ gives the projected curvature of $\mathcal{N}$ into $\mathcal{S}$, while our new commutator $\boldsymbol{\partial}\boldsymbol{\partial}' - \boldsymbol{\partial}'\boldsymbol{\partial}$ gives the projected curvature of M into $\mathcal{N}$.

Chapter 4

The Karlhede Classification of Type N Vacuum Solutions

In this chapter we apply the invariant formalism developed in the last chapter to the Karlhede classification of vacuum type N Einstein solutions. This approach arises from the fact that in the case of vacuum type N solutions the invariance group H^0 is the group of null rotations and we have seen that the new formalism is invariant under such transformations.

In a recent paper by Collins [6] the upper bound for vacuum type N was reduced to six. Collin's approach makes use of the NP formalism to express the dyad components of the Weyl spinor and its derivatives. However, the use of this notation is not as productive as might be desired since terms which are not invariant under null rotations appear in the Karlhede algorithm.

In another paper by Collins, d'Inverno and Vickers [10], the bound for vacuum type D solutions was reduced from seven to three. An important aspect of this approach is the use of the GHP formalism. Vacuum Type D space-times have a Weyl spinor which admits spin and boost transformations as its invariance group and the GHP spin coefficients and operators are covariant under this same group. It then turns out that at all orders of covariant differentiation the dyad components of the Weyl spinor and its derivatives can be expressed completely in terms of GHP notation which makes the classification process easier.

It thus seems natural, as in the type D case, to use a formalism which is invariant under null rotations in order to simplify the classification process, and hopefully be able to reduce the bound.

4.1 The Procedure

The equivalence problem investigates whether two given metrics g and $\tilde{g}$ expressed in different coordinate systems x^α and $\tilde{x}^\alpha$, are equal under a coordinate transformation. Because there are transformations b of the proper Lorentz group which leave the metric invariant what we should investigate is whether there is a coordinate transformation giving $\tilde{g} = bg$, so that the metrics are given up to transformations of the proper Lorentz group which leave them invariant.

We have seen that to solve this problem we need to find the relationship $\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta)$ and $\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^B, x^\mu)$ where ϵ^A represents the parameters of b that are compatible with the system of equations 1.3.78. We have also seen that we need only calculate up to the $(p+1)$th derivative of the curvature, p being the order at which no new functional information, relating to the coordinates or to the frame, is obtained.

The Karlhede algorithm, provides a way of classifying metrics in a way that will simplify the procedure for solving the equivalence problem. What one does in practice is calculate the successive covariant derivatives starting from the 0th order, and at each stage q of differentiation determine the invariance group (the group which leaves the components invariant) and hence a frame, up to transformations in the invariance group, that will provide the simplest form possible of the components – the canonical form. So that at each stage q, one is left with two pieces of information, information concerning the frame $dim(H_q)$ (the dimension of the invariance group H_q at step q) and information regarding the coordinates n_q (number of functionally independent components). The procedure stops when we no longer obtain new information regarding either the frame or the coordinates, i.e, when $dim(H_{q+1}) = dim(H_q)$ and $n_{q+1} = n_q$.

The idea in what follows is to consider particular linear combinations of the components of the curvature and its successive covariant derivatives with these linear combinations being constructed in such a way that one can obtain, systematically, all the components from these linear combinations. In fact for convenience what we will work with, at each stage q of differentiation, will be linear combinations of the spinor basis (o^A, ι^A) and the spinor components of $\nabla^q\Psi$. For example, considering the first derivative instead of applying the Karlhede procedure to the terms:

$$
\begin{aligned}
&\Psi_{ABCD,EE'} \quad o^A o^B o^C o^D o^E o^{E'} \\
&\Psi_{ABCD,EE'} \quad o^A o^B o^C o^D o^E \iota^{E'} \\
&\Psi_{ABCD,EE'} \quad o^A o^B o^C o^D \iota^E \iota^{E'} \\
&\Psi_{ABCD,EE'} \quad o^A o^B o^C \iota^D \iota^E \iota^{E'} \qquad (4.1.1) \\
&\Psi_{ABCD,EE'} \quad o^A o^B \iota^C \iota^D \iota^E \iota^{E'}
\end{aligned}
$$

$$\Psi_{ABCD,EE'}\ o^A \iota^B \iota^C \iota^D \iota^E \iota^{E'}$$

$$\Psi_{ABCD,EE'}\ \iota^A \iota^B \iota^C \iota^D \iota^E \iota^{E'}$$

we apply the same procedure to the terms:

$$\Psi_{ABCD,EE'}\ o^A o^B o^C o^D o^E o^{E'}$$

$$\Psi_{ABCD,EE'}\ o^A o^B o^C o^D o^E$$

$$\Psi_{ABCD,EE'}\ o^A o^B o^C o^D$$

$$\Psi_{ABCD,EE'}\ o^A o^B o^C \tag{4.1.2}$$

$$\Psi_{ABCD,EE'}\ o^A o^B$$

$$\Psi_{ABCD,EE'}\ o^A$$

$$\Psi_{ABCD,EE'}$$

Obviously the information one obtains from the classification of set 4.1.2 is not the same as the information one obtains from set 4.1.1, since the invariance group which leaves set 4.1.2 invariant may or may not leave set 4.1.1 invariant or vice versa. In the case of type N solutions which is the case we will be considering, the set 4.1.2 is invariant under null rotations as we shall see in the following sections, however set 4.1.1 is not invariant under such transformations. We can see that the canonical forms of the terms in set 4.1.2 must contain the same coordinate functional information as that of the canonical forms of the terms in set 4.1.1, since these are simply linear combinations and one can obtain all terms of set 4.1.1 from the terms in set 4.1.2 and vice-versa. To obtain the canonical forms of set 4.1.2 one must determine the invariance group at each stage q of differentiation, i.e, the group of transformations which leaves the linear combinations invariant at each step of differentiation. We will denote the dimension of this group by $\mathbf{H}_q$ to distinguish it from the invariance group H_q of the components. Then we must fix the frame as much as possible up to transformations in the invariance group and calculate these linear combinations in that frame, which will then give us the required canonical forms. We follow this procedure at each step q of differentiation.

So that, like in the Karlhede algorithm which works with components, if $\mathbf{H}_{q+1} = \mathbf{H}_q$ and $n_{q+1} = n_q$ then the procedure stops since no new information arises from higher derivatives.

In effect what we are trying to solve, is not the simultaneous system of equations:

$$\begin{aligned}
\tilde{\Psi}_{abcd} &= \Psi_{abcd} \\
\tilde{\Psi}_{abcd;e_1e'_1} &= \Psi_{abcd;ee'} \\
&\cdot \\
&\cdot \\
&\cdot \\
\tilde{\Psi}_{abcd;e_1e'_1\ldots e_pe'_p} &= \Psi_{abcd;e_1e'_1\ldots e_pe'_p} \\
\tilde{\Psi}_{abcd;e_1e'_1\ldots e_{p+1}e'_{p+1}} &= \Psi_{abcd;e_1e'_1\ldots e_{p+1}e'_{p+1}}
\end{aligned} \tag{4.1.3}$$

but the system:

$$\begin{aligned}
\tilde{\mathbf{\Psi}}_{\mathbf{ABCD}} &= \mathbf{\Psi}_{\mathbf{ABCD}} \\
\tilde{\mathbf{\Psi}}_{\mathbf{ABCD;E_1E'_1}} &= \mathbf{\Psi}_{\mathbf{ABCD;EE'}} \\
&\cdot \\
&\cdot \\
&\cdot \\
\tilde{\mathbf{\Psi}}_{\mathbf{ABCD;E_1E'_1\ldots E_pE'_p}} &= \mathbf{\Psi}_{\mathbf{ABCD;E_1E'_1\ldots E_pE'_p}} \\
\tilde{\mathbf{\Psi}}_{\mathbf{ABCD;E_1E'_1\ldots E_{p+1}E'_{p+1}}} &= \mathbf{\Psi}_{\mathbf{ABCD;E_1E'_1\ldots E_{p+1}E'_{p+1}}}
\end{aligned} \tag{4.1.4}$$

which is not exactly the same. In the first case we require that there exist frames, up to rotational freedom in the equivalence problem, in which the set Ψ_{abcd} , $\Psi_{abcd;ee'}$ ... are equal, the identification map being given by the coordinate relation $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$ which gives equality. In the second case we require that there exist frames up to rotational freedom in which the set $\mathbf{\Psi}_{\mathbf{ABCD}}$, $\mathbf{\Psi}_{\mathbf{ABCD;EE'}}$ ... are equal, the identification map being given by $\tilde{\mathrm{x}}^\mu = \tilde{\mathrm{x}}^\mu(\mathrm{x}^\nu)$.

It is clear that if there exist frames in which the set Ψ_{abcd} , $\Psi_{abcd;ee'}$ ... are equal, then in that same frame, the set $\mathbf{\Psi}_{\mathbf{ABCD}}$, $\mathbf{\Psi}_{\mathbf{ABCD;EE'}}$... are also equal and vice versa because of the way terms in set 4.1.2 are constructed from those of set 4.1.1. However, the group of transformations giving the rotational freedom in the frame for set 4.1.2 may not leave the terms in set 4.1.1 invariant. So the fact that the system of simultaneous equations 4.1.4 is solvable does not imply equivalence since $\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^B, \mathrm{x}^\mu)$ does not imply $\tilde{\epsilon}^A = \tilde{\epsilon}^A(\epsilon^B, x^\mu)$. However, $\tilde{\mathrm{x}}^\mu = \tilde{\mathrm{x}}^\mu(\mathrm{x}^\nu)$ does imply $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, so that if after a certain stage of differentiation set 4.1.2 does not produce new functional information concerning the coordinates that implies that after that same stage set 4.1.1 does not produce new coordinate information either. However, it is possible that the invariance group of the components H_q may not be fixed at this stage and may change with further derivatives so that one must check this detail if one wants to determine the upper bound on the order of covariant differentiation.

We consider the case of Petrov type N solutions. Here, what one does is, instead of taking the canonical forms at each stage q of differentiation of the Weyl spinor $\{\Psi_{abcd}, ..., \Psi_{abcd;e_1...e_{p+1}e'_{p+1}}\}$, one takes linear combinations of the components of the qth derivative of the Weyl spinor at each stage of covariant differentiation $\{\mathbf{\Psi_{ABCD}}, ..., \mathbf{\Psi_{ABCD;E_1E'_1...E_{p+1}E'_{p+1}}}\}$ in such a way that one can obtain in a systematic way all corresponding components from these linear combinations and the invariance group is fixed at all orders of differentiation. So that all one obtains from this set is coordinate functional information. Notice that if $\mathbf{n}_{p+1} = \mathbf{n}_p$ then the procedure of classification of $\{\mathbf{\Psi_{ABCD}}, ..., \mathbf{\Psi_{ABCD;E_1E'_1...E_{p+1}E'_{p+1}}}\}$ stops at $p+1$ so that after the stage $p+1$ we do not obtain any new coordinate information from this set, furthermore this implies that after this same stage we do not obtain any new coordinate functional information from $\{\Psi_{abcd}, ..., \Psi_{abcd;e_1...e_{p+1}e'_{p+1}}, \Psi_{abcd;e_1...e_{p+2}e'_{p+2}}, ...\}$. The reason behind these conclusions being that one can extract the set $\{\Psi_{abcd}, ..., \Psi_{abcd;e_1...e_{p+1}e'_{p+1}}, ...\}$ from the set $\{\mathbf{\Psi_{ABCD}}, ..., \mathbf{\Psi_{ABCD;E_1E'_1...E_{p+1}E'_{p+1}}}, ...\}$ and vice versa systematically. In a sense we are separating the frame information from the coordinate information. However, after all coordinate information has been extracted from set $\{\mathbf{\Psi_{ABCD}}, ..., \mathbf{\Psi_{ABCD;E_1E'_1...E_{p+1}E'_{p'+1}}}\}$ one must then check if the terms $\Psi_{abcd;e_1...e_{p+1}e'_{p+1}}, \Psi_{abcd;e_1...e_{p+2}}$, etc are invariant under null rotations and if not one must calculate higher order derivatives until the frame is fixed, thus obtaining the bound on differentiation.

We have seen in chapter 1 that the Weyl spinor of a Petrov type N spacetime has the form:

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \; ; \; \Psi_4 = \Psi \neq 0 \tag{4.1.5}$$

which is preserved under the invariance group H_0 of null rotations 2.1.2.

The generalized GHP formalism involves invariants which are symmetric spinors rather than scalars. So that if one is to apply this formalism to the classification procedure, instead of considering the terms $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4$ and the respective invariance group of null rotations we consider the spinors $\mathbf{\Psi}_0, \mathbf{\Psi}_1, \mathbf{\Psi}_2, \mathbf{\Psi}_3, \mathbf{\Psi}_4$ defined by:

$$\begin{aligned}
\mathbf{\Psi}_0 &= \Psi_{ABCD}\, o^A o^B o^C o^D && (4.1.6)\\
(\mathbf{\Psi}_1)_A &= \Psi_{ABCD}\, o^B o^C o^D && (4.1.7)\\
(\mathbf{\Psi}_2)_{AB} &= \Psi_{ABCD}\, o^C o^D && (4.1.8)\\
(\mathbf{\Psi}_3)_{ABC} &= \Psi_{ABCD}\, o^D && (4.1.9)\\
(\mathbf{\Psi}_4)_{ABCD} &= \Psi_{ABCD} && (4.1.10)
\end{aligned}$$

and the group that leaves these terms invariant.

The Weyl spinor for vacuum type N in this notation has the general form given by:

$$\mathbf{\Psi}_0 = (\mathbf{\Psi}_1)_A = (\mathbf{\Psi}_2)_{AB} = (\mathbf{\Psi}_3)_{ABC} = 0$$

$$(\mathbf{\Psi}_4)_{ABCD} = \mathbf{\Psi} o_A o_B o_C o_D \qquad (4.1.11)$$

Notice that 4.1.11 is invariant under the four (real) parameter group of transformations 3.1.2.

It is however convenient to use the simplest form (canonical form) possible in order to simplify the calculations and hence the classification procedure. By taking a suitable dyad as basis we may scale Ψ to one and obtain the following canonical form for the Petrov type N Weyl spinor:

$$\mathbf{\Psi}_0 = (\mathbf{\Psi}_1)_A = (\mathbf{\Psi}_2)_{AB} = (\mathbf{\Psi}_3)_{ABC} = 0$$

$$(\mathbf{\Psi}_4)_{ABCD} = o_A o_B o_C o_D \qquad (4.1.12)$$

Notice that now the dimension of the invariance group $\mathbf{H}_0$ is two, the invariance group being the two (real) parameter group of null rotations 2.1.2.

In the generalized GHP formalism, the Bianchi identities in vacuum under condition 4.1.12 become:

$$\begin{aligned}
\text{Þ}_{A'(A}\Psi_{BCDE)} &= R_{(A}\Psi_{BCDE)}\ \bar{o}_{A'} && (4.1.13)\\
\eth_{(A'B')(A}\Psi_{BCDE)} &= \Psi_{(BCDE}\ T_{A)(A'}\bar{o}_{B')} && (4.1.14)\\
S_{A'} &= 0 && (4.1.15)\\
K &= 0 && (4.1.16)
\end{aligned}$$

Which in compact notation is given by:

$$\begin{aligned} \text{þ}\boldsymbol{\Psi}_4 &= R\boldsymbol{\Psi}_4 && (4.1.17)\\ \boldsymbol{\eth}\boldsymbol{\Psi}_4 &= T\boldsymbol{\Psi}_4 && (4.1.18)\\ S &= 0 && (4.1.19)\\ K &= 0 && (4.1.20) \end{aligned}$$

Using the fact that $\boldsymbol{\Psi}_4$ is a spin and boost weighted object of weight $\{\mathbf{0},\mathbf{0}\}$, and that $\mathbf{D}'_{(EF)(EF}\,(o_Ao_Bo_Co_{D)}\,) = \boldsymbol{\delta}'_{E(EF}\,(o_Ao_Bo_Co_{D)}\,) = \boldsymbol{\delta}_{(EF)(E}\,(o_Ao_Bo_Co_{D)}\,)$ $= \mathbf{D}_{E(E}\,(o_Ao_Bo_Co_{D)}\,) = 0$ we have (by definition of the generalized GHP operators $\text{þ}'$, $\boldsymbol{\eth}'$, $\boldsymbol{\eth}$, þ):

$$\begin{aligned} \text{þ}'_{A'B'(AB}\,\Psi_{CDEF)} &= 4o_{(C}o_Do_Eo_FG_{AB)A'}\,\bar{o}_{B'} && (4.1.21)\\ \boldsymbol{\eth}'_{A'(AB}\,\Psi_{CDEF)} &= 4A_{(AB}\,o_Co_Do_Eo_{F)}\,\bar{o}_{A'} && (4.1.22)\\ \boldsymbol{\eth}_{(A'B')(A}\,\Psi_{BCDE)} &= 4\bar{o}_{(B'}\,B_{A')(A}\,o_Bo_Co_Do_{E)} && (4.1.23)\\ \text{þ}_{A'(A}\,\Psi_{BCDE)} &= 4E_{(A}o_Bo_Co_Do_{E)}\bar{o}_{A'} && (4.1.24) \end{aligned}$$

Or in compact form

$$\begin{aligned} \text{þ}'\boldsymbol{\Psi}_4 &= 4G\boldsymbol{\Psi}_4 && (4.1.25)\\ \boldsymbol{\eth}\boldsymbol{\Psi}_4 &= 4B\boldsymbol{\Psi}_4 && (4.1.26)\\ \boldsymbol{\eth}'\boldsymbol{\Psi}_4 &= 4A\boldsymbol{\Psi}_4 && (4.1.27)\\ \text{þ}\boldsymbol{\Psi}_4 &= 4E\boldsymbol{\Psi}_4 && (4.1.28) \end{aligned}$$

Comparing with the Bianchi identities gives:

$$\begin{aligned} R_A &= 4E_A && (4.1.29)\\ T_{AA'} &= 4B_{AA'} && (4.1.30) \end{aligned}$$

The Ricci equations become:

$$\text{þ}_{A'(A}R_{B)} = R_{(A}R_{B)}\bar{o}_{A'} \quad (4.1.31)$$
$$\text{þ}_{AA'}T_{BB} = R_{(A}T_{B)(B}\bar{o}_{A')} \quad (4.1.32)$$
$$\eth_{(A'B')(A}R_{B)} = \bar{o}_{(B'}T_{A')(A}R_{B)} - T_{AA'}\overline{R}_{B'}\bar{o}_{B} \quad (4.1.33)$$
$$\eth_{AA'B'}T_{BC'} = T_{AA'}T_{BB'}\bar{o}_{C'} \quad (4.1.34)$$
$$\text{þ}'_{(A'B')(AB}R_{C)} - \eth'_{ABA'}T_{CB} = -T_{AA'}\overline{T}_{BB'}o_{C} \quad (4.1.35)$$

With their compact form being given by:

$$\text{þ}R = R^2 \quad (4.1.36)$$
$$\text{þ}T = RT \quad (4.1.37)$$
$$\eth R = TR - T\overline{R} \quad (4.1.38)$$
$$\eth T = T^2 \quad (4.1.39)$$
$$\text{þ}'R - \eth'T = -T\overline{T} \quad (4.1.40)$$

Finally we write the commutators:

$$(\text{þ}_{AA'}\text{þ}'_{BCB'C'} - \text{þ}'_{BCB'C'}\text{þ}_{AA'})\phi_{A_1\ldots A'_{N'}} = (o_C\overline{T}_{AA'}\eth_{BB'C'}$$
$$+\bar{o}_{C'}T_{AA'}\text{þ}'_{BCB'})\phi_{A_1\ldots A'_{N'}} \quad (4.1.41)$$

$$(\text{þ}_{AA'}\eth_{BB'C'} - \eth_{BB'C'}\text{þ}_{AA'})\phi_{A_1\ldots A'_{N'}} = o_B\overline{R}_{A'}\eth_{ABC'}\phi_{A_1\ldots A'_{N'}} \quad (4.1.42)$$

$$(\eth_{AA'B'}\eth'_{BCC'} - \eth'_{BCC'}\eth_{AA'B'})\phi_{A_1\ldots A'_{N'}} = (R_A\bar{o}_{A'}$$
$$-\overline{R}_{A'}o_A)\text{þ}'_{BCB'C'}\phi_{A_1\ldots A'_{N'}} \quad (4.1.43)$$

$$(\text{þ}'_{ABA'B'}\eth_{CC'D'} - \eth_{CC'D'}\text{þ}_{ABA'B'})\phi_{A_1\ldots A'_{N'}} =$$
$$-\bar{o}_{D'}T_{AA'}\text{þ}'_{BCB'C'}\phi_{A_1\ldots A'_{N'}} \quad (4.1.44)$$

Which in compact notation become:

$$(\text{þ}\text{þ}' - \text{þ}'\text{þ})\phi = (\overline{T}\eth + T\eth')\phi \tag{4.1.45}$$

$$(\text{þ}\eth - \eth\text{þ})\phi = \overline{R}\eth\phi \tag{4.1.46}$$

$$(\eth\eth' - \eth'\eth)\phi = (R - \overline{R})\text{þ}'\phi \tag{4.1.47}$$

$$(\text{þ}'\eth - \eth\text{þ}')\phi = -T\text{þ}'\phi \tag{4.1.48}$$

Note that the GHP vacuum field equations contain the same information as Einstein's vacuum field equations [27] and therefore the same is also true for the equations given above since they are complete in the sense that all such GHP identities can be obtained from them.

4.2 First Covariant Derivative

We now proceed to calculate the first covariant derivative of Ψ_{ABCD} which we will denote by $(\nabla\Psi)_{ABCDFF'}$. It follows from the Bianchi identities in spinor form $\epsilon^{AF}\Psi_{ABCD;FF'} = 0$ that the first covariant derivative of the Weyl spinor is symmetric on all primed and unprimed indices so that it makes sense to apply the generalized GHP notation.

The calculation leading to the general expression giving the dyad components of the first covariant derivative is given in :

$$(\nabla\Psi)_{\mu;ff'} = (\Psi_\mu)_{;ff'} - \mu\Gamma_{11ff'}\Psi_{\mu-1} + (2\mu-4)\Gamma_{01ff'}\Psi_\mu$$
$$+(4-\mu)\Gamma_{00ff'}\Psi_{\mu+1} \tag{4.2.49}$$

with $\mu \in \{0,1,2,3,4\}$.

Now, in the same way that we obtained the general equation giving, for example, all the Bianchi identities in generalized formalism from the general equation giving the Bianchi identities in NP formalism, we are able to obtain a general expression giving all terms relating to the first covariant derivative written in generalized notation from 4.2.49. Let $\zeta^A_{\mathbf{A}} = \{o^A, \iota^A\}$ be a normalized spinor dyad with dual $\zeta^{\mathbf{A}}_A$ so that bold indices represent dyad terms, for example $\lambda_{\mathbf{A}} = \lambda_A\zeta^A_{\mathbf{A}}$ is a scalar and not a spinor.

In order to write 4.2.49 in terms of the invariant formalism we first introduce some notation. Let $\nabla\Psi$, with zero to five unprimed indices, be defined by:

$$(\nabla\Psi)_{A_p\ldots A_5A'} = \Psi_{A_1\ldots A_4;A_5A'}\, o^{A_1}\ldots o^{A_{(p-1)}} \tag{4.2.50}$$

Then in the invariant formalism, equation 4.2.49 becomes:

$$o_{\mathbf{A}_{N+1}}(\nabla\Psi)_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}\mathbf{F}'} = \sum_{sym}\{(\Psi_{\mathbf{A}_1\ldots\mathbf{A}_N})_{;\mathbf{F}\mathbf{F}'}o_{\mathbf{A}_{N+1}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{F}\mathbf{F}'}\Psi_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}}$$

$$+(2N-4)\Gamma_{0\mathbf{A}_1\mathbf{F}\mathbf{F}'}\Psi_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}} + (4-N)\Gamma_{00\mathbf{F}\mathbf{F}'}\}\Psi_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}_{N+1}} \tag{4.2.51}$$

with $N \in \{0,1,2,3,4\}$ and where $\sum_{sym}$ indicates symmetrization on all free primed and unprimed indices.

We can now obtain from 4.2.51 all non-zero terms relating to the first order derivative written in generalized formalism:

$$\begin{aligned} o_{(A_5}(\nabla\Psi)_{A_1A_2A_3A_4F)(F'}\ \bar{o}_{G')} &= 4\bar{o}_{(G'}G_{F')(FA_1}\ o_{A_2}o_{A_3}o_{A_4}o_{A_5)} \\ &= \text{Þ}'_{(F'G')(A_1A_2}\ \Psi_{A_3A_4A_5F)} \end{aligned} \tag{4.2.52}$$

$$\begin{aligned} o_{(A_4}(\nabla\Psi)_{A_1A_2A_3F)(F'}\ \bar{o}_{G')} &= \bar{o}_{(G'}T_{F')(F}o_{A_1}o_{A_2}o_{A_3}o_{A_4)} \\ &= \eth_{(G'F')}\ \Psi_{(A_1A_2A_3A_4F)} \end{aligned} \tag{4.2.53}$$

$$\begin{aligned} \bar{o}_{F'}o_{(A_5}\nabla\Psi)_{A_1A_2A_3A_4F)} &= A_{(FA_1}\ o_{A_2}o_{A_3}o_{A_4}o_{A_5)}\bar{o}_{F'} \\ &= \eth'_{F'(FA_1}\ \Psi_{A_2A_3A_4A_5)} \end{aligned} \tag{4.2.54}$$

$$\bar{o}_{F'}o_{(F}(\nabla\Psi)_{A_1A_2A_3A_4)} = R_{(F}o_{A_1}o_{A_2}o_{A_3}o_{A_4)}\bar{o}_{F'} \tag{4.2.55}$$

$$= \text{Þ}_{F'(F}\Psi_{A_1A_2A_3A_4)} \tag{4.2.56}$$

And in compact notation we have:

$$(\nabla\Psi) = 4G = \text{Þ}'\Psi_4 \tag{4.2.57}$$

$$(\nabla\Psi)\cdot o = 4A = \eth'\Psi_4 \tag{4.2.58}$$

$$(\nabla\Psi)\cdot\bar{o} = 4B\Psi_4 + 4T\Psi_4 = \eth\Psi_4 + 4T\Psi_4 = 5T\Psi_4 \tag{4.2.59}$$

$$(\nabla\Psi)\cdot o\cdot\bar{o} = 4E\Psi_4 + 4R\Psi_4 = \text{Þ}\Psi_4 + 4R\Psi_4 = 5R\Psi_4 \tag{4.2.60}$$

Where we recall that in the compact notation a dot denotes a contraction and that one may have to multiply terms by suitable factors of o_A and $\bar{o}_{A'}$ and then symmetrize to obtain expressions such that the indices balance.

It is important to note that equations 4.2.57 through 4.2.60 form an inverted hierarchical system. Since $\kappa = 0$ the only functional information in E is given by ϵ. Again since σ vanishes and ϵ is known from 4.2.60, the only new functional information in B is given by β. Since $\rho = 4\epsilon$ and ϵ is known the only new functional information in A is given by α. Finally since $\tau = 4\beta$ and all the other terms are known the new functional information in G is given by γ. Thus equations 4.2.57 through 4.2.60 encode all the functional information at first order.

It is clear that the terms obtained at first order are invariant under null rotations so that the dimension of the invariance group $\mathbf{H}_2$ remains 2. We must consider the possibility of their being at least one functionally independent component among these terms and hence proceed with the algorithm.

4.3 Second Covariant Derivative

The calculation leading to the second covariant derivative which we will denote by $(\nabla^2\mathbf{\Psi})_{ABCDFGF'G'}$, is similar to the one performed to obtain the first covariant derivative.

The general expression giving the dyad components of the second covariant derivative is calculated in [10] and is as follows:

$$\begin{aligned}(\nabla^2\Psi)_{\mu f';gg'} &= [(\nabla\Psi)_{\mu f'}]_{;gg'} - \mu\Gamma_{11gg'}(\nabla\Psi)_{(\mu-1)f'}\\ &+(2\mu-5)\Gamma_{10gg'}(\nabla\Psi)_{\mu f'} + (5-\mu)\Gamma_{00gg'}(\nabla\Psi)_{(\mu+1)f'}\\ &-\overline{\Gamma}_{f'1'g'g}(\nabla\Psi)_{\mu 0'} + \overline{\Gamma}_{f'0'g'g}(\nabla\Psi)_{\mu 1'}\end{aligned} \tag{4.3.61}$$

with $\mu \in \{0,1,2,3,4,5\}$.

In terms of the invariant formalism one obtains:

$$\begin{aligned}\bar{o}_{\mathbf{A}'}o_{\mathbf{A}_{N+1}}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{G}\mathbf{G}'} &= \sum_{sym}\{[(\nabla\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'}]_{;\mathbf{G}\mathbf{G}'}\bar{o}_{\mathbf{A}'}o_{\mathbf{A}_{N+1}}\\ &-N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{G}\mathbf{G}'}(\nabla\mathbf{\Psi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{F}'}\bar{o}_{\mathbf{A}'} + (2N-5)\Gamma_{\mathbf{A}_1 0\mathbf{G}\mathbf{G}'}\\ &(\nabla\mathbf{\Psi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{F}'}\bar{o}_{\mathbf{A}'} + (5-N)\Gamma_{00\mathbf{G}\mathbf{G}'}(\nabla\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{F}'}\bar{o}_{\mathbf{A}'}\\ &-\overline{\Gamma}_{\mathbf{F}'\mathbf{A}'\mathbf{G}'\mathbf{G}}(\nabla\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N}o_{\mathbf{A}_{N+1}} + \overline{\Gamma}_{\mathbf{F}'0'\mathbf{G}'\mathbf{G}}(\nabla\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'}o_{\mathbf{A}_{N+1}}\}\end{aligned} \tag{4.3.62}$$

In the vacuum case at all orders of covariant differentiation of the Weyl spinor, we only need to consider the symmetric parts since only these terms are algebraically independent [1]. One then obtains all symmetric non-zero terms corresponding to the second covariant derivative of $\mathbf{\Psi}_4$ from expression 4.3.62, which in compact notation are given by:

$$(\nabla^2\mathbf{\Psi}) = \text{þ}'\text{þ}'\mathbf{\Psi}_4 \tag{4.3.63}$$

$$(\nabla^2\mathbf{\Psi})\cdot\overline{o} = \eth'\text{þ}'\mathbf{\Psi}_4 + (\text{þ}'\mathbf{\Psi}_4)\overline{T} + \text{þ}'\eth'\mathbf{\Psi}_4 \tag{4.3.64}$$

$$(\nabla^2\mathbf{\Psi})\cdot\overline{o}\cdot\overline{o} = \eth'\eth'\mathbf{\Psi}_4 \tag{4.3.65}$$

$$(\nabla^2\mathbf{\Psi})\cdot o = \eth\text{þ}'\mathbf{\Psi}_4 + 5\text{þ}'\eth\mathbf{\Psi}_4 + 5(\text{þ}'\mathbf{\Psi}_4)T \tag{4.3.66}$$

$$(\nabla^2\mathbf{\Psi})\cdot o\cdot\overline{o} = \text{þ}\text{þ}'\mathbf{\Psi}_4 + \eth\eth'\mathbf{\Psi}_4 + 5\eth'\eth\mathbf{\Psi}_4 + 5\text{þ}'\text{þ}\mathbf{\Psi}_4 + 5(\eth'\mathbf{\Psi}_4)T$$
$$+5T\overline{T} + \overline{R}\text{þ}'\mathbf{\Psi}_4 + R\text{þ}\mathbf{\Psi}_4 \tag{4.3.67}$$

$$(\nabla^2\mathbf{\Psi})\cdot o\cdot\overline{o}\cdot\overline{o} = \text{þ}\eth'\mathbf{\Psi}_4 + 5\eth'\text{þ}\mathbf{\Psi}_4 + 5(\eth'\mathbf{\Psi}_4)R \tag{4.3.68}$$

$$(\nabla^2\mathbf{\Psi})\cdot o\cdot o = 5\eth\eth\mathbf{\Psi}_4 T + 20(\eth\mathbf{\Psi}_4)^2 \tag{4.3.69}$$

$$(\nabla^2\mathbf{\Psi})\cdot o\cdot o\cdot\overline{o} = 5\text{þ}'\eth\mathbf{\Psi}_4 + 5\eth\text{þ}\mathbf{\Psi}_4 + 5(\eth\mathbf{\Psi}_4)(\overline{\text{þ}\mathbf{\Psi}_4}) + 40(\eth\mathbf{\Psi}_4)(\text{þ}\mathbf{\Psi}_4) \tag{4.3.70}$$

$$(\nabla^2\mathbf{\Psi})\cdot o\cdot o\cdot\overline{o}\cdot\overline{o} = 5\text{þ}\text{þ}\mathbf{\Psi}_4 + 20(\text{þ}\mathbf{\Psi}_4)^2 \tag{4.3.71}$$

Notice that all the above equations can be obtained from equation 4.3.63 by contraction with omicrons, and as at first order form an inverted hierarchical system. These second covariant derivative terms encode the same functional information as the second covariant derivative terms obtained by Collins [6] and one can obtain the expressions above by translating Collins' terms into generalized NP language (for example: ρ becomes R) and leaving out all terms that transform badly under null rotations. Again we obtain objects that are invariant under null rotations so that the dimension of $\mathbf{H}_3$ is two. By considering equations 4.1.25 through 4.1.28, we see that equations 4.3.63 through 4.3.71 tell us that the potentially new functionally independent information can only come from the following sixteen terms:

$$
\begin{array}{llll}
\text{þ}\text{þ}\Psi_4, & \text{þ}\eth\Psi_4, & \text{þ}\eth'\Psi_4, & \text{þ}\text{þ}'\Psi_4, \\
\text{þ}'\text{þ}\Psi_4, & \text{þ}'\eth\Psi_4, & \text{þ}'\eth'\Psi_4, & \text{þ}'\text{þ}'\Psi_4, \\
\eth\text{þ}\Psi_4 & \eth\eth\Psi_4, & \eth\eth'\Psi_4, & \eth\text{þ}'\Psi_4. \\
\eth'\text{þ}\Psi_4, & \eth'\eth\Psi_4, & \eth'\eth'\Psi_4, & \eth'\text{þ}'\Psi_4
\end{array}
$$

It is easily seem that the commutators limits the number of functionally independent terms to ten, which are obviously given by:

$$
\begin{array}{llll}
\text{þ}\text{þ}\Psi_4, & \text{þ}\eth\Psi_4, & \text{þ}\eth'\Psi_4, & \text{þ}\text{þ}'\Psi_4, \\
\text{þ}'\eth\Psi_4, & \text{þ}'\eth'\Psi_4, & \text{þ}'\text{þ}'\Psi_4, & \\
\eth\eth\Psi_4, & \eth\eth'\Psi_4, & & \\
\eth'\eth'\Psi_4. & & &
\end{array}
$$

And by means of 4.2.57 through 4.2.60 and the Ricci equations, we are left with:

$$
\begin{array}{lll}
\text{þ}\eth'\Psi_4, & \text{þ}\text{þ}'\Psi_4, & \text{þ}'\eth\Psi_4, \\
\text{þ}'\eth'\Psi_4, & \text{þ}'\text{þ}'\Psi_4, & \eth'\eth'\Psi_4.
\end{array} \tag{4.3.72}
$$

as our possibly functionally independent terms.

Unfortunately we are unable to relate these invariants to the invariants obtained at first order of covariant differentiation, because the Bianchi identities do not relate $\text{þ}'\Psi_4$ and $\eth'\Psi_4$ to Ψ_4, R and T. In fact the terms $\text{þ}'\Psi_4$ and $\eth'\Psi_4$ does not even feature in the Bianchi identities. As a result, and unlike the vacuum type D case where these identities are used to relate higher order derivatives of Ψ_2 to lower order derivatives of Ψ_2 and hence limit the number of functional information obtained at each step, here we must consider the possibility of there existing at least one new functionally independent term among the invariants given by 4.3.72. We must therefore continue the Karlhede algorithm.

4.4 Higher order derivatives

The calculation of third, fourth,..., etc covariant derivatives of the Weyl spinor is lengthy but straightforward and can be seen as an extension of the calculation performed for lower order derivatives.

Lets consider the calculation of the dyad component terms of the third derivative of Ψ, which will be denoted by $(\nabla^3\Psi)_{\mu f';gg';hh'}$ and is defined as follows:

$$(\nabla^3\Psi)_{\mu f';gg';hh'} = \Psi_{ABCD,FF';GG';HH'} \quad [\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\xi_h^H\overline{\xi}_{h'}^{H'} \tag{4.4.73}$$

where there are μ $\xi_1^{A'}s$ in square brackets and $\mu \in \{0,1,2,3,4,5\}$.

Expression 4.4.73 can be rewritten in the following way:

$$\begin{aligned}(\nabla^3\Psi)_{\mu f';gg';hh'} &= (\Psi_{ABCD,FF';GG';HH'} \quad [\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'})_{;HH'}\,\xi_h^H\overline{\xi}_{h'}^{H'} \\ &-([\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'})_{;HH'}\,\xi_h^H\overline{\xi}_{h'}^{H'}\,\Psi_{ABCD,FF';GG'}\end{aligned} \tag{4.4.74}$$

taking into account that there are μ $\xi_1^{A'}s$ in square brackets we have:

$$\begin{aligned}(\nabla^3\Psi)_{\mu f';gg';hh'} &= [(\nabla^2\Psi)_{\mu f';gg'}]_{;hh'} - \mu\xi^A_{1;hh'}[\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\xi_g^G\overline{\xi}_{f'}^{F'}\overline{\xi}_{g'}^{G'} \\ \Psi_{ABCD,FF';GG'} &- (5-\mu)\xi^A_{0;hh'}[\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\,\Psi_{ABCD,FF';GG'} \\ -\overline{\xi}^{F'}_{f';hh'}\,\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\xi_g^G\overline{\xi}_{g'}^{G'}\,\Psi_{ABCD,FF';GG'} &- \xi^G_{g;hh'}\,\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\overline{\xi}_{g'}^{G'} \\ \Psi_{ABCD,FF';GG'} &- \overline{\xi}^{G'}_{g';hh'}\,\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\xi_g^G\Psi_{ABCD,FF';GG'}\end{aligned} \tag{4.4.75}$$

In expression 4.4.75 there are $\mu - 1$ in the first square brackets and μ in the second square bracket.

The general term $\xi^F_{f;hh'}$ can be expressed in terms of the spin coefficients $\Gamma_{abcd'}$ in the following way:

$$\xi^F_{f;hh'} = \xi_{fJ;hh'}\,\epsilon^{FJ} = \xi_{fJ;hh'}\,\epsilon^{kt}\xi_k^F\xi_t^J = \Gamma_{fthh'}\,\epsilon^{kt}\xi_k^F \tag{4.4.76}$$

Substituting 4.4.76 in 4.4.75 we get the general expression giving the dyad components of the third covariant derivative of the curvature:

$$(\nabla^3\Psi)_{\mu f';gg';hh'} = [(\nabla^2\Psi)_{\mu f';gg'}]_{;hh'} - \mu\Gamma_{11hh'}(\nabla^2\Psi)_{\mu-1;gg'}$$

$$+(2\mu-5)\mu\Gamma_{10hh'}(\nabla^2\Psi)_{\mu f';gg'} + (5-\mu)\Gamma_{00hh'}(\nabla^2\Psi)_{\mu+1f';gg'}$$

$$+\overline{\Gamma}_{f'0'h'h}(\nabla^2\Psi)_{\mu 1';gg'} - \overline{\Gamma}_{f'1'h'h}(\nabla^2\Psi)_{\mu 0';gg'} \tag{4.4.77}$$

$$+\Gamma_{g0hh'}(\nabla^2\Psi)_{\mu f';1g'} - \Gamma_{g1hh'}(\nabla^2\Psi)_{\mu f';0g'}$$

$$+\overline{\Gamma}_{g'0'h'h}(\nabla^2\Psi)_{\mu f';g1'} - \overline{\Gamma}_{g'1'h'h}(\nabla^2\Psi)_{\mu f';g0'}$$

In the same manner as before we arrive at the general expression giving the generalized terms of the third covariant derivative:

Third Covariant Derivative

$$\overline{o}_{\mathbf{A}'}o_{\mathbf{A}_{N+1}}(\nabla^3\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{G}\mathbf{G}'\mathbf{H}\mathbf{H}'} = \sum_{sym}[(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{G}\mathbf{G}'}]_{;\mathbf{H}\mathbf{H}'}\overline{o}_{\mathbf{A}'}o_{\mathbf{A}_{N+1}}$$

$$-N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{H}\mathbf{H}'}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{F}'\mathbf{G}\mathbf{G}'}\overline{o}_{\mathbf{A}'} + (2N-5)\Gamma_{\mathbf{A}_1 0\mathbf{H}\mathbf{H}'}$$

$$(\nabla^2\mathbf{\Psi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{F}'\mathbf{G}\mathbf{G}'}\overline{o}_{\mathbf{A}'} + (5-N)\Gamma_{00\mathbf{H}\mathbf{H}'}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{F}'\mathbf{G}\mathbf{G}'}\overline{o}_{\mathbf{A}'}$$

$$-\overline{\Gamma}_{\mathbf{F}'\mathbf{A}'\mathbf{H}'\mathbf{H}}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{G}\mathbf{G}'}o_{\mathbf{A}_{N+1}} + \overline{\Gamma}_{\mathbf{F}'0'\mathbf{H}'\mathbf{H}} \tag{4.4.78}$$

$$(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'\mathbf{G}\mathbf{G}'}o_{\mathbf{A}_{N+1}} - \Gamma_{\mathbf{G}\mathbf{A}_{N+1}\mathbf{H}\mathbf{H}'}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{G}'}\overline{o}_{\mathbf{A}'}$$

$$+\Gamma_{\mathbf{G}0\mathbf{H}\mathbf{H}'}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{A}_{N+1}\mathbf{G}'}\overline{o}_{\mathbf{A}'} - \overline{\Gamma}_{\mathbf{G}'\mathbf{A}'\mathbf{H}'\mathbf{H}}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{G}}o_{\mathbf{A}_{N+1}}$$

$$+\overline{\Gamma}_{\mathbf{G}'0'\mathbf{H}'\mathbf{H}}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{F}'\mathbf{G}\mathbf{A}'}o_{\mathbf{A}_{N+1}}$$

To calculate the expression giving the dyad components of the fourth covariant derivative of the curvature $(\nabla^3\Psi)_{\mu f';gg';hh';mm'}$ we follow the same process as before. By definition we have:

$$(\nabla^4\Psi)_{\mu f';gg';hh';mm'} = \Psi_{ABCD,FF':GG';HH';MM'} \quad [\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F]$$

$$\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\xi_h^H\overline{\xi}_{h'}^{H'}\xi_m^M\overline{\xi}_{m'}^{M'} \tag{4.4.79}$$

where there are μ $\xi_1^{A'}s$ and $\mu\in\{0,1,2,3,4,5\}$ in square brackets.

Same as before expression 4.4.79 can be rewritten in the following way:

$$(\nabla^4\Psi)_{\mu f';gg';hh';mm'} = (\Psi_{ABCD,FF':GG';HH'} \quad [\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\xi_h^H\overline{\xi}_{h'}^{H'})_{;MM'}$$

$$\xi_m^M \overline{\xi}_{m'}^{M'} - ([\xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_f^F]\overline{\xi}_{f'}^{F'} \xi_g^G \overline{\xi}_{g'}^{G'} \xi_h^H \overline{\xi}_{h'}^{H'})_{;MM'} \xi_m^M \overline{\xi}_{m'}^{M'} \quad (4.4.80)$$

$$\Psi_{ABCD,FF';GG';HH'}$$

taking into account that there are μ $\xi_1^{A'}s$ in square brackets we get:

$$\begin{aligned}
(\nabla^4\Psi)_{\mu f';gg';hh';mm'} &= [(\nabla^3\Psi)_{\mu f';gg';hh'}]_{;mm'} - \mu\xi^A_{1;MM'} [\xi_b^B \xi_c^C \xi_d^D \xi_f^F] \\
&\overline{\xi}_{f'}^{F'} \xi_g^G \overline{\xi}_{g'}^{G'} \xi_h^H \overline{\xi}_{h'}^{H'} \xi_m^M \overline{\xi}_{m'}^{M'} \Psi_{ABCD,FF';GG';HH'} - (5-\mu)\xi^A_{0;MM'} [\xi_b^B \xi_c^C \xi_d^D \xi_f^F] \\
&\overline{\xi}_{f'}^{F'} \xi_g^G \overline{\xi}_{g'}^{G'} \xi_h^H \overline{\xi}_{h'}^{H'} \xi_m^M \overline{\xi}_{m'}^{M'} \Psi_{ABCD,FF';GG';HH'} - \overline{\xi}^{F'}_{f';mm'} \xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_f^F \xi_g^G \overline{\xi}_{g'}^{G'} \\
&\Psi_{ABCD,FF';GG';HH'} - \xi^G_{g;mm'} \xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_f^F \overline{\xi}_{f'}^{F'} \overline{\xi}_{g'}^{G'} \xi_h^H \overline{\xi}_{h'}^{H'} \qquad (4.4.81) \\
&\Psi_{ABCD,FF';GG';HH'} - \overline{\xi}^{G'}_{g';mm'} \xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_f^F \overline{\xi}_{f'}^{F'} \xi_g^G \xi_h^H \overline{\xi}_{h'}^{H'} \Psi_{ABCD,FF';GG';HH'} \\
&-\xi^H_{h;mm'} \xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_f^F \overline{\xi}_{f'}^{F'} \overline{\xi}_{h'}^{H'} \Psi_{ABCD,FF';GG';HH'} - \overline{\xi}^{H'}_{h';mm'} \xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_f^F \overline{\xi}_{f'}^{F'} \xi_h^H \\
&\Psi_{ABCD,FF';GG';HH'}
\end{aligned}$$

Same as before there are $\mu - 1$ in the first square brackets and μ in the second square bracket in expression 4.4.81. So that by substituting 4.4.76 in 4.4.81 we have our general equation:

$$\begin{aligned}
(\nabla^4\Psi)_{\mu f';gg';hh';mm'} &= [(\nabla^3\Psi)_{\mu f';gg';hh'}]_{;mm'} - \mu\Gamma_{11mm'} (\nabla^3\Psi)_{\mu-1 f';gg';hh'} \\
&+(2\mu-5)\mu\Gamma_{10mm'} (\nabla^3\Psi)_{\mu f';gg';hh'} + (5-\mu)\Gamma_{00mm'} (\nabla^3\Psi)_{\mu+1 f';gg';hh'} \\
&+\overline{\Gamma}_{f'0'm'm} (\nabla^3\Psi)_{\mu 1';gg';hh'} - \overline{\Gamma}_{f'1'm'm} (\nabla^3\Psi)_{\mu 0';gg';hh'} \qquad (4.4.82) \\
&+\Gamma_{g0mm'} (\nabla^3\Psi)_{\mu f';gg';hh'} - \Gamma_{h1mm'} (\nabla^3\Psi)_{\mu f';0g';hh'} \\
&+\overline{\Gamma}_{g'0'm'm} (\nabla^3\Psi)_{\mu f';gg';hh'} - \overline{\Gamma}_{g'1'h'h} (\nabla^3\Psi)_{\mu f';g0'}
\end{aligned}$$

From equation 4.4.82 we obtain the generalized expression given below:

Fourth Covariant Derivative

$$\overline{o}_{\mathbf{A}'} o_{\mathbf{A}_{N+1}} (\nabla^4 \mathbf{\Psi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{F}' \mathbf{G} \mathbf{G}' \mathbf{H} \mathbf{H}' \mathbf{M} \mathbf{M}'}$$

$$= \sum_{sym} [(\nabla^3 \mathbf{\Psi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{F}' \mathbf{G} \mathbf{G}' \mathbf{H} \mathbf{H}'}]_{;\mathbf{M}\mathbf{M}'} \overline{o}_{\mathbf{A}'} o_{\mathbf{A}_{N+1}}$$

$$-N\Gamma_{\mathbf{A_1A_2MM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_3...A_{N+1}F'GG'HH'}}\,{}^{\overline{o}}\mathbf{A'}$$

$$+(2N-5)\Gamma_{\mathbf{A_1oMM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_2...A_{N+1}F'GG'HH'}}\,{}^{\overline{o}}\mathbf{A'}$$

$$+(5-N)\Gamma_{\mathbf{ooMM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_{N+1}F'GG'HH'}}\,{}^{\overline{o}}\mathbf{A'}$$

$$-\overline{\Gamma}_{\mathbf{F'A'M'M}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NGG'HH'}}\,{}^{o}\mathbf{A_{N+1}}$$

$$+\overline{\Gamma}_{\mathbf{F'o'M'M}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NA'GG'HH'}}\,{}^{o}\mathbf{A_{N+1}}$$

$$-\Gamma_{\mathbf{GA_{N+1}MM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NF'G'HH'}}\,{}^{\overline{o}}\mathbf{A'}$$

$$+\Gamma_{\mathbf{GoMM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NF'A_{N+1}G'HH'}}\,{}^{\overline{o}}\mathbf{A'} \tag{4.4.83}$$

$$-\overline{\Gamma}_{\mathbf{G'A'M'M}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NF'GHH'}}\,{}^{o}\mathbf{A_{N+1}}$$

$$+\overline{\Gamma}_{\mathbf{G'o'M'M}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NF'GA'HH'}}\,{}^{o}\mathbf{A_{N+1}}$$

$$-\Gamma_{\mathbf{HA_1MM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_2...A_{N+1}F'GG'H'}}\,{}^{\overline{o}}\mathbf{A'}$$

$$+\Gamma_{\mathbf{HoMM'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_{N+1}F'GG'H'}}\,{}^{\overline{o}}\mathbf{A'}$$

$$-\overline{\Gamma}_{\mathbf{H'A'M'M}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NF'GG'H}}\,{}^{o}\mathbf{A_{N+1}}$$

$$+\overline{\Gamma}_{\mathbf{H'o'M'M}}(\nabla^3\mathbf{\Psi})_{\mathbf{A_1...A_NF'GG'HA'}}\,{}^{o}\mathbf{A_{N+1}}$$

Finally we determine the expressions relating to the fifth covariant derivative of the curvature which can be obtained following the same process as before. Obviously the dyad components are determined by:

$$(\nabla^5\Psi)_{\mu f';gg';hh';mm';nn'} = \Psi_{ABCD,FF';GG;HH';MM';NN'} \quad [\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F]$$
$$\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\xi_h^H\overline{\xi}_{h'}^{H'}\xi_m^M\overline{\xi}_{m'}^{M'}\xi_n^N\overline{\xi}_{n'}^{N'} \tag{4.4.84}$$

where there are μ $\xi_1^{A'}s$ and $\mu\in\{0,1,2,3,4,5\}$ in square brackets.

In the same way as before, equation 4.4.84 leads to the following equality:

$$(\nabla^5\Psi)_{\mu f';gg';hh';mm';nn'} = [(\nabla^4\Psi)_{\mu f';gg';hh';mm'}\quad]_{;nn'} - \mu\xi^A_{1;NN'}[\xi_b^B\xi_c^C\xi_d^D\xi_f^F]$$
$$\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\xi_h^H\overline{\xi}_{h'}^{H'}\xi_m^M\overline{\xi}_{m'}^{M'}\xi_n^N\overline{\xi}_{n'}^{N'}\Psi_{ABCD,FF';GG';HH';MM'} \quad -(5-\mu)\xi^A_{0;NN'}$$
$$[\xi_b^B\xi_c^C\xi_d^D\xi_f^F]\overline{\xi}_{f'}^{F'}\xi_g^G\overline{\xi}_{g'}^{G'}\xi_h^H\overline{\xi}_{h'}^{H'}\xi_m^M\overline{\xi}_{m'}^{M'}\xi_n^N\overline{\xi}_{n'}^{N'}\Psi_{ABCD,FF';GG';HH';MM'}$$

$$
\begin{aligned}
&-\overline{\xi}^{F'}_{f';nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\xi^G_g\overline{\xi}^{G'}_{g'}\ \xi^M_m\overline{\xi}^{M'}_{m'}\ \Psi_{ABCD,FF';GG':HH';MM'}\\
&-\xi^{G}_{g;nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\overline{\xi}^{F'}_{f'}\overline{\xi}^{G'}_{g'}\ \xi^H_h\overline{\xi}^{H'}_{h'}\ \xi^M_m\overline{\xi}^{M'}_{m'}\ \Psi_{ABCD,FF';GG';HH';MM'}\\
&-\overline{\xi}^{G'}_{g';nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\overline{\xi}^{F'}_{f'}\ \xi^G_g\xi^H_h\overline{\xi}^{H'}_{h'}\ \xi^M_m\overline{\xi}^{M'}_{m'}\ \Psi_{ABCD,FF';GG';HH';MM'} \qquad (4.4.85)\\
&-\xi^{H}_{h;nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\overline{\xi}^{F'}_{f'}\ \xi^G_g\overline{\xi}^{G'}_{g'}\overline{\xi}^{H'}_{h'}\ \xi^M_m\overline{\xi}^{M'}_{m'}\ \Psi_{ABCD,FF';GG';HH';MM'}\\
&-\overline{\xi}^{H'}_{h';nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\overline{\xi}^{F'}_{f'}\ \xi^G_g\overline{\xi}^{G'}_{g'}\ \xi^H_h\xi^M_m\overline{\xi}^{M'}_{m'}\ \Psi_{ABCD,FF';GG'HH';MM'}\\
&-\xi^{M}_{m;nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\overline{\xi}^{F'}_{f'}\ \xi^G_g\overline{\xi}^{G'}_{g'}\ \xi^H_h\overline{\xi}^{H'}_{h'}\ \overline{\xi}^{M'}_{m'}\ \Psi_{ABCD,FF';GG';HH';MM'}\\
&-\overline{\xi}^{M'}_{m';nn'}\ \xi^A_a\xi^B_b\xi^C_c\xi^D_d\xi^F_f\overline{\xi}^{F'}_{f'}\ \xi^G_g\overline{\xi}^{G'}_{g'}\ \xi^H_h\overline{\xi}^{H'}_{h'}\ \xi^M_m\Psi_{ABCD,FF';GG';HH';MM'}
\end{aligned}
$$

By substituting 4.4.76 in equation 4.4.85 we obtain the general expression giving the dyad components:

$$
\begin{aligned}
&(\nabla^5\Psi)_{\mu f';gg';hh';mm';nn'} = [(\nabla^4\Psi)_{\mu f';gg';hh';mm'}\ ]_{;nn'} - \mu\Gamma_{11nn'}\\
&(\nabla^4\Psi)_{\mu-1 f';gg';hh';mm'} + (2\mu-5)\mu\Gamma_{10nn'}(\nabla^4\Psi)_{\mu f';gg';hh';mm'}\\
&+(5-\mu)\Gamma_{00nn'}(\nabla^4\Psi)_{\mu+1 f';gg';hh';mm'} + \overline{\Gamma}_{f'0'n'n}(\nabla^4\Psi)_{\mu 1';gg';hh';mm'}\\
&-\overline{\Gamma}_{f'1'n'n}(\nabla^4\Psi)_{\mu 0';gg';hh';mm'} + \Gamma_{g0nn'}(\nabla^4\Psi)_{\mu f';1g';hh';mm'} \qquad (4.4.86)\\
&-\Gamma_{h1nn'}(\nabla^4\Psi)_{\mu f';0g';hh';mm'} + \overline{\Gamma}_{g'0'n'n}(\nabla^4\Psi)_{\mu f';gg':hh';mm'}\\
&-\overline{\Gamma}_{g'1'n'n}(\nabla^4\Psi)_{\mu f';g0';hh';mm'} + \Gamma_{h0nn'}(\nabla^4\Psi)_{\mu f';gg';1h';mm'}\\
&-\Gamma_{h1nn'}(\nabla^4)_{\mu f';gg';0h';mm'} + \overline{\Gamma}_{h'0'n'n}(\nabla^4\Psi)_{\mu f';gg';h1';mm'}\\
&-\overline{\Gamma}_{h'1'n'n}(\nabla^4\Psi)_{\mu f';gg';h0';mm'} + \Gamma_{m0nn'}(\nabla^4\Psi)_{\mu f';gg';hh';1m'}\\
&-\Gamma_{m1nn'}(\nabla^4\Psi)_{\mu f';gg';hh';0m'} + \overline{\Gamma}_{m'0'n'n}(\nabla^4\Psi)_{\mu f';gg';hh';m1'}\\
&-\overline{\Gamma}_{m'1'n'n}(\nabla^4\Psi)_{\mu f';gg';hh';m0'}
\end{aligned}
$$

Fifth Covariant Derivative

$$\overline{o}_{\mathbf{A}'}o_{\mathbf{A}_{N+1}}(\nabla^5\mathbf{\Psi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{F}'\mathbf{GG}'\mathbf{HH}'\mathbf{MM}'\mathbf{NN}'} =$$

$$= \sum_{sym}\{[(\nabla^4\mathbf{\Psi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{F}'\mathbf{GG}'\mathbf{HH}'\mathbf{MM}'}]_{;\mathbf{NN}'}\overline{o}_{\mathbf{A}'}o_{\mathbf{A}_{N+1}}$$

$$
\begin{aligned}
&-N\Gamma_{A_1A_2NN'}(\nabla^4\Psi)_{A_3\ldots A_{N+1}F'GG'HH'MM'}\bar{o}_{A'}\\
&+(2N-5)\Gamma_{A_1oNN'}(\nabla^4\Psi)_{A_2\ldots A_{N+1}F'GG'HH'MM'}\bar{o}_{A'}\\
&+(5-N)\Gamma_{ooNN'}(\nabla^4\Psi)_{A_1\ldots A_{N+1}F'GG'HH'MM'}\bar{o}_{A'}\\
&-\bar{\Gamma}_{F'A'N'N}(\nabla^4\Psi)_{A_1\ldots A_NGG'HH'MM'}o_{A_{N+1}}\\
&\Gamma_{F'o'N'N}(\nabla^4\Psi)_{A_1\ldots A_NA'GG'HH'MM'}o_{A_{N+1}}\\
&-\Gamma_{GA_{N+1}NN'}(\nabla^4\Psi)_{A_1\ldots A_NF'G'HH'MM'}\bar{o}_{A'}\\
&+\Gamma_{GoNN'}(\nabla^4\Psi)_{A_1\ldots A_NF'A_{N+1}G'HH'MM'}\bar{o}_{A'}\\
&-\bar{\Gamma}_{G'A'N'N}(\nabla^4\Psi)_{A_1\ldots A_NF'GHH'MM'}o_{A_{N+1}} \qquad (4.4.87)\\
&+\bar{\Gamma}_{G'o'N'N}(\nabla^4\Psi)_{A_1\ldots A_NF'GA'HH'MM'}o_{A_{N+1}}\\
&-\Gamma_{HA_1NN'}(\nabla^4\Psi)_{A_2\ldots A_{N+1}F'GG'H'MM'}\bar{o}_{A'}\\
&+\Gamma_{HoNN'}(\nabla^4\Psi)_{A_1\ldots A_{N+1}F'GG'H'MM'}\bar{o}_{A'}\\
&-\bar{\Gamma}_{H'A'N'N}(\nabla^4\Psi)_{A_1\ldots A_NF'GG'HMM'}o_{A_{N+1}}\\
&+\bar{\Gamma}_{H'o'N'N}(\nabla^4\Psi)_{A_1\ldots A_NF'GG'HA'MM'}o_{A_{N+1}}\\
&-\Gamma_{MA_1NN'}(\nabla^4\Psi)_{A_2\ldots A_{N+1}F'GG'HH'M}\bar{o}_{A'}\\
&+\Gamma_{MoNN'}(\nabla^4\Psi)_{A_1\ldots A_{N+1}F'GG'HH'M'}\bar{o}_{A'}\\
&-\bar{\Gamma}_{M'A'N'N}(\nabla^4\Psi)_{A_1\ldots A_NF'GG'HH'M}o_{A_{N+1}}\\
&+\bar{\Gamma}_{M'o'N'N}(\nabla^4\Psi)_{A_1\ldots A_NF'GG'HH'MA'}o_{A_{N+1}}
\end{aligned}
$$

4.5 Reducing the upper bound

Here we analyse the upper bound on the order of covariant differentiation of the Weyl spinor required in the Karlhede classification. We apply the Karlhede algorithm to the invariants obtained in the previous section.

At first order the terms obtained are given by $\text{þ}'$, $\eth'$, $\eth$, þ acting on Ψ_4 which are all invariant under null rotations. In order for the algorithm to continue these terms must be non-constant, since otherwise the algorithm would stop at second order because $\Psi_4 = 1$. This can also be seen by looking at the components since these terms being constants would then require that R, T, A, and G to have constant components, i.e., ρ, τ, α, and γ would have to be constants. If we apply the NP Ricci equations [23] to these constants, we get

$$\rho = \tau = \alpha = 0 \tag{4.5.88}$$

and Collins [6] has proved that when 4.5.88 is satisfied the upper bound is two. Hence, we consider the case when at least one functionally independent term is obtained at first order and thus continue the procedure.

We have already seen that at second order the invariance group remains the group of null rotations but we must take into consideration the possibility of there being at least one new functionally independent term amongst the terms obtained at this order so that the procedure continues to third order.

All non-vanishing terms relating to the third derivative are obtained from $(\nabla^3\mathbf{\Psi}_4) = \mathbf{þ}'^3\mathbf{\Psi}_4$ by contraction with omicrons, so that the two dimensional group of null rotations remains as the invariance group $\mathbf{H}_3$. However, we are again unable to rule out the possibility of obtaining new functionally independent information at this order of differentiation. So the Karlhede algorithm must continue to fourth order.

At fourth order the situation is much similar to third order, with all non-vanishing terms obtained from $(\nabla^4\mathbf{\Psi}_4) = \mathbf{þ}'^4\mathbf{\Psi}_4$ in exactly the same way. And for exactly the same reason as before, the algorithm continues to fifth order.

At fifth order things work out much the same, with all terms being given by $(\nabla^5\mathbf{\Psi}_4) = \mathbf{þ}'^5\mathbf{\Psi}_4$ and its successive contractions with omicrons, so that the dimension of $\mathbf{H}_5$ is two.

At most the algorithm will produce four functionally independent terms among the set $\{(\nabla^n\mathbf{\Psi}_4), ..., (\nabla^n\mathbf{\Psi}_4)\cdot o\cdot ...\cdot o\cdot\bar{o}\cdot ...\cdot\bar{o}\}$. Hence, since the invariance group remains the group of null rotations at each step of the algorithm we conclude that in the worst possible case when only one functionally independent term is obtained at each stage, it would be necessary to calculate five covariant derivatives before we obtain no new functional information and the algorithm terminates. This however, as discussed before, gives just information concerning the coordinates obtained in the Karlhede classification of the solution. We can say, that as of the fourth derivative, one does not obtain new coordinate functional information from the components of the successive covariant derivatives of the Weyl spinor. However, we must check if by the fifth order the invariance group relating to the components is fixed, otherwise one might need to calculate further derivatives.

According to Collins work [6], at first order of differentiation the terms obtained are given by:

$$(D\Psi)_{40'} = \rho \qquad \text{C1}$$

$$(D\Psi)_{50'} = 4\alpha \qquad \text{C2}$$

$$(D\Psi)_{41'} = \tau \qquad \text{C3}$$

$$(D\Psi)_{51'} = 4\tau \qquad \text{C4}$$

which is precisely what one expects.

Under null rotations these spin coefficients transform as follows:

$$\rho \longrightarrow \rho \qquad (4.5.89)$$

$$\alpha \longrightarrow \alpha + \frac{5}{4}\overline{a}\rho \qquad (4.5.90)$$

$$\tau \longrightarrow \tau + a\rho \qquad (4.5.91)$$

$$\gamma \longrightarrow \gamma + a\alpha + \frac{5}{4}\overline{a}\tau + \frac{5}{4}a\overline{a}\rho \qquad (4.5.92)$$

He then goes on to determine the invariance group. In order to simplify the procedure he considers various distinct cases which he denotes by Class I, Class II, Class IIa, Class IIb, Class IIIa, Class IIIb. Let us review the results obtained for each of these cases:

Class I: $\rho \neq 0$

By 4.5.91 we have that τ can always be set to zero by taking $a = -\frac{\tau}{\rho}$ which then fixes the frame completely.

Class II: $\rho = 0, \tau = 0$

The only transformation remaining is $\gamma \longrightarrow \gamma + a\alpha$. There are two subclasses to consider:

Class IIa: $\alpha \neq 0$

In this case one can set $\gamma = 0$ by taking $a = -\frac{\gamma}{\alpha}$ which then fixes the frame completely.

Class IIb: $\alpha = 0$

Here none of the spin coefficients can be transformed at all so the frame cannot be fixed any further, i.e, the invariance group remains two dimensional.

Class III: $\rho = 0, \tau \neq 0$

The only transformation remaining is $\gamma \longrightarrow \gamma + (a\alpha + \frac{5}{4}\overline{a}\tau)$ so that two cases are considered:

Class IIIa: $|\alpha| \neq \frac{5}{4}|\tau|$

Here we are able to fix a and therefore the frame completely [6].

Class IIIb: $|\alpha| = \frac{5}{4}|\tau|$

In this case the frame is fixed up to a one dimensional invariance group [6].

At second order of covariant differentiation the terms obtained by Collins are [6]:

$$(D^2\Psi)_{30';10'} = 2\rho^2 \qquad \text{C5}$$

$$(D^2\Psi)_{30';11'} = 2\rho\tau \qquad \text{C6}$$

$$(D^2\Psi)_{31';10'} = 2\rho\tau \qquad \text{C7}$$

$$(D^2\Psi)_{40';00'} = D\rho + \frac{3}{4}\rho^2 - \frac{1}{4}\overline{\rho}\rho \qquad \text{C8}$$

$$(D^2\Psi)_{40';10'} = \delta'\rho + 7\alpha\rho - \frac{1}{4}\overline{\tau}\rho \qquad \text{C9}$$

$$(D^2\Psi)_{40';01'} = \delta\rho + \frac{3}{4}\tau\rho - \overline{\alpha}\rho + \tau\overline{\rho} \qquad \text{C10}$$

$$(D^2\Psi)_{40';11'} = D'\rho + 3\gamma\rho + 4\alpha\tau - \overline{\gamma}\rho + \overline{\tau}\tau \qquad \text{C11}$$

$$(D^2\Psi)_{41';00'} = D\tau + \frac{3}{4}\tau\rho - \overline{\pi}\rho + \frac{1}{4}\tau\overline{\tau} \qquad \text{12}$$

$$(D^2\Psi)_{41';01'} = \delta'\tau + 7\gamma\tau - \overline{\nu}\rho + \tau\overline{\gamma} \qquad \text{C13}$$

$$(D^2\Psi)_{41';01'} = \delta\tau + \frac{3}{4}\tau^2 - \overline{\lambda}\rho + \tau\overline{\alpha} \qquad \text{C14}$$

$$(D^2\Psi)_{41';11'} = D'\tau + 7\tau\gamma - \overline{\nu}\rho + \tau\overline{\gamma} \qquad \text{C15}$$

$$(D^2\Psi)_{50';00'} = 4D\alpha - 5\pi\rho + 5\alpha\rho - \alpha\overline{\rho} \qquad \text{C16}$$

$$(D^2\Psi)_{50';10'} = 4\delta'\alpha - 5\frac{\lambda}{\rho} + 20\alpha^2 + \alpha\overline{\tau} \qquad \text{C17}$$

$$(D^2\Psi)_{50';01'} = 4\delta\alpha - 5\mu\rho + 5\tau\alpha - 4\alpha\overline{\alpha} + 4\gamma\overline{\rho} \qquad \text{C18}$$

$$(D^2\Psi)_{50';11'} = 4D'\alpha - 5\nu\rho + 20\alpha\gamma - 4\alpha\overline{\mu} + 4\overline{\tau}\gamma \qquad \text{C19}$$

$$(D^2\Psi)_{51';00'} = 4D\gamma - 5\pi\tau + 5\gamma\rho - 4\alpha\overline{\pi} + \overline{\rho}\gamma \qquad \text{C20}$$

$$(D^2\Psi)_{51';10'} = 4\delta'\gamma - 5\lambda\tau + 20\alpha\gamma - 4\alpha\overline{\mu} + \gamma\overline{\tau} \qquad \text{C21}$$

$$(D^2\Psi)_{51';01'} = 4\delta\gamma - 5\mu\tau + 5\gamma\tau - 4\alpha\overline{\lambda} + 4\gamma\overline{\alpha} \qquad \text{C22}$$

$$(D^2\Psi)_{51';11'} = 4D'\gamma - 5\nu\tau + 20\gamma^2 - 4\alpha\overline{\nu} + 4\gamma\overline{\gamma} \qquad \text{C23}$$

which again is what we expected. We then look at the invariance group which leaves all terms in C5 through C23 invariant. We have seen that for classes I, IIa and IIIa the frame is completely fixed at first order. So that one looks at what happens in the other cases:

Class IIb: $\rho = 0, \tau = 0, \alpha = 0$

On substituting $\rho = 0, \tau = 0, \alpha = 0$ into equations (2.4d), (2.4m) and (2.4p) of [23] one has that only $D'\gamma$ is non-zero. Under null rotations the NP derivative D' transforms as:

$$D' \longrightarrow (D' + a\delta' + \overline{a}\delta + a\overline{a}D)$$

It is seen from equation 4.5.92 that γ remains unchanged under the group of null rotations so that one has:

$$D'\gamma \longrightarrow (D' + a\delta' + \overline{a}\delta + a\overline{a}D)\gamma = D'\gamma$$

using the fact that $D\gamma = \delta\gamma = \delta'\gamma = 0$. Hence we have that $D'\gamma$ remains unchanged under the group of null rotations so that at second order the two dimensional invariance group remains.

Collins [6] goes on to show that at third and higher orders of differentiation, in this case, the only non vanishing component will be the highest labelled one and will contain as potentially new functional independent information only a term of the form $D'D'D'...\gamma$. He then goes on to prove by induction that, using a shorthand notation, $D'^n\gamma$ is invariant under the two dimensional group of null rotations, for any n.

If one takes $D'^{(n-1)}\gamma$ to be invariant under null rotations, then under this group $D'^n\gamma$ will transform as follows:

$$D'^n\gamma \longrightarrow (D' + a\delta' + \overline{a}\delta + a\overline{a}D)D'^{(n-1)}\gamma$$

$$= D'^n\gamma + a\delta' D'^{(n-1)}\gamma + \overline{a}\delta D'^{(n-1)}\gamma + a\overline{a}DD'^{(n-1)}\gamma \qquad \text{C24}$$

By taking the NP commutators:

$$(D'D - DD')\phi = [(\gamma + \overline{\gamma})D + \frac{1}{4}(\rho + \overline{\rho})D' - (\tau + \overline{\pi})\delta' - (\overline{\tau} + \pi)\delta]\phi \qquad \text{C25}$$

$$(\delta D' - D'\delta)\phi = [-\overline{\nu}D + \frac{3}{4}(\tau - \overline{\alpha})D' + \overline{\lambda}\delta' + (\mu - \gamma + \overline{\gamma})\delta]\phi \qquad \text{C26}$$

and the complex conjugate of C26, it is seen that the NP derivative operators D, δ and δ' can be moved through a line of D' to the right. Hence, from the fact that $D\gamma = \delta\gamma = \delta'\gamma = 0$, equation C24 becomes:

$$D'^n\gamma \longrightarrow D'^n\gamma$$

Therefore, we have shown that if $D'^{(n-1)}\gamma$ is unchanged under null rotations then so is $D'^n\gamma$. Furthermore, we have seen that $D'\gamma$ is unchanged under null rotations so that one has by induction that $D'^n\gamma$ is unchanged for any n. Therefore, the two dimensional invariance group remains at all orders of differentiation.

Class IIIb: $\rho = 0, \tau \neq 0, |\alpha| = \frac{5}{4}|\tau|$

From equations C5 through C23, one has that the potentially new functional independent information at second order is given by π, λ, μ and ν together with the NP derivatives of the first order spin coefficients α, τ and γ. Collins [6] then looks at how these terms transform under null rotations and uses, for convenience, the transformation of $D'\tau$ to further restrict the frame at second order. It is seen that one can fix the frame completely in this case. We will omit the calculations here since they are extensive and are explained in detail in [6].

We have then seen that in all cases except one, by second order of differentiation the frame is fixed. In the other remaining case the invariance group remains the two dimensional group of null rotations at all orders of differentiation. We can then conclude that by fourth order of differentiation the Karlhede classification of the solution does not produce new functional information, concerning either coordinates or frame, so that bound is reduced to five.

Chapter 5

The Karlhede Classification of Type N Non-Vacuum Solutions

We have shown in chapter 4 that by applying the generalized GHP formalism to the Karlhede classification of Petrov type N vacuum solutions one can reduce the upper bound to five covariant derivatives.

In this chapter we analyse the problem of reducing the upper bound on covariant differentiation for non-vacuum type N solutions. We apply the same method used in the vacuum case, i.e, we attempt to write all derivative terms in terms of the gereralized GHP formalism.

The Karlhede algorithm for classifying metrics also applies to the non-vacuum situation. However in the non-vacuum case, unlike the vacuum situation, we must consider the contribution of the Ricci spinor and the Ricci scalar as well as their successive covariant derivatives, since they no longer vanish.

The situation where one could potentially have a bound of seven, fortunately, only occurs in very non-generic cases. For this situation to occur all of the following conditions must be satisfied [18]:

(1) The Weyl and Ricci spinor and Ricci scalar (Λ) must all be constants.

(2) The invariance group at zeroth order H_0 must have dimension two.

(3) The dimension of the invariance group and the number of functionally independent components must not both change on differentiating.

(4) We must produce at most one new functionally independent component on differentiating.

The Ricci spinor has the following symmetries:

$$\Phi_{ABA'B'} = \Phi_{(AB)(A'B')} \tag{5.0.1}$$

So that we are left with six independent components given by:

$$
\begin{aligned}
\Phi_{00'} &= o^A o^B \bar{o}^{A'} \bar{o}^{B'}\, \Phi_{ABA'B'} \\
\Phi_{01'} &= o^A o^B \bar{o}^{A'} \bar{\iota}^{B'}\, \Phi_{ABA'B'} \\
\Phi_{02'} &= o^A o^B \bar{\iota}^{A'} \bar{\iota}^{B'}\, \Phi_{ABA'B'} \\
\Phi_{11'} &= o^A \iota^B \bar{o}^{A'} \bar{\iota}^{B'}\, \Phi_{ABA'B'} \\
\Phi_{12'} &= o^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}\, \Phi_{ABA'B'} \\
\Phi_{22'} &= \iota^A \iota^B \bar{\iota}^{A'} \bar{\iota}^{B'}\, \Phi_{ABA'B'}
\end{aligned}
\tag{5.0.2}
$$

For type N vacuum spacetimes the canonical form for the Weyl spinor is:

$$
\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \qquad \Psi_4 \neq 0 \tag{5.0.3}
$$

with the invariance group being the group of null rotations which acts on the dyad according to 2.1.2.

We investigate how the Ricci spinor transforms under null rotations:

$$
\begin{aligned}
\Phi_{00'} &\longrightarrow \Phi_{00'} \\
\Phi_{01'} &\longrightarrow \Phi_{01'} + a\Phi_{00'} \\
\Phi_{02'} &\longrightarrow \Phi_{02'} + 2a\Phi_{01'} + a^2\Phi_{00'} \\
\Phi_{11'} &\longrightarrow \Phi_{11'} + \bar{a}\Phi_{01'} + a\Phi_{10'} + a\bar{a}\Phi_{00'} \\
\Phi_{12'} &\longrightarrow \Phi_{12'} + 2a\Phi_{11'} + a^2\Phi_{10'} + \bar{a}\Phi_{02'} + 2a\bar{a}\Phi_{01'} + a^2\bar{a}\Phi_{00'} \\
\Phi_{22'} &\longrightarrow \Phi_{22'} + 2a\Phi_{21'} + 2\bar{a}\Phi_{12'} + 4a\bar{a}\Phi_{11'} + 2a^2\bar{a}\Phi_{10'} + \bar{a}^2\Phi_{02'} \\
&\qquad + a\bar{a}^2\Phi_{01'} + a^2\Phi_{20'} + a^2\bar{a}^2\Phi_{00'} \\
\Phi_{10'} &\longrightarrow \Phi_{10'} + \bar{a}\Phi_{00'} \\
\Phi_{20'} &\longrightarrow \Phi_{20'} + 2\bar{a}\Phi_{10'} + \bar{a}^2\Phi_{00'} \\
\Phi_{21'} &\longrightarrow \Phi_{21'} + a\Phi_{20'} + 2\bar{a}\Phi_{11'} + 2a\bar{a}\Phi_{10'} + \bar{a}^2\Phi_{01'} + a\bar{a}^2\Phi_{00'}
\end{aligned}
\tag{5.0.4}
$$

As in the type N vacuum case, we make use of the generalized GHP formalism of chapter 3 so that instead of considering Ψ's and Φ's we work with $\boldsymbol{\Psi}$'s and $\boldsymbol{\Phi}$'s. It is irrelevant whether one uses Λ or $\boldsymbol{\Lambda}$ since Λ is a scalar so that $\boldsymbol{\Lambda} = \Lambda$.

The Weyl spinor in the generalized GHP notation is defined by 4.1.6 through 4.1.10. We proceed to define the Ricci spinor in generalized GHP notation:

$$\begin{aligned}
\Phi_{00'} &= \Phi_{ABA'B'}\ o^A o^B \bar{o}^{A'} \bar{o}^{B'} \\
(\Phi_{01'})_{A'} &= \Phi_{ABA'B'}\ o^A o^B \bar{o}^{B'} \\
(\Phi_{02'})_{A'B'} &= \Phi_{ABA'B'}\ o^A o^B \\
(\Phi_{11'})_{AA'} &= \Phi_{ABA'B'}\ o^B \bar{o}^{B'} \\
(\Phi_{12'})_{AA'B'} &= \Phi_{ABA'B'}\ o^B \\
(\Phi_{22'})_{ABA'B'} &= \Phi_{ABA'B'}
\end{aligned} \tag{5.0.5}$$

so that for condition (2) to be satisfied one must have:

$$(\Phi_{21'})_{ABA'} = (\Phi_{12'})_{AA'B'} = (\Phi_{02'})_{A'B'} = (\Phi_{20'})_{AB} = (\Phi_{11'})_{AA'}$$

$$= (\Phi_{10'})_A = (\Phi_{01'})_{A'} = \Phi_{00'} = 0 \quad \text{and} \quad (\Phi_{22'})_{ABA'B'} \neq 0 \tag{5.0.6}$$

and condition (1) requires that Ψ_{ABCD} , $\Phi_{ABA'B'}$ have constant components and Λ to be constant.

Since Λ is a spin and boost weighted quantity of weight $\{0, 0\}$ then by condition (1) we have:

$$\text{þ}'\Lambda = \text{þ}\Lambda = \text{þ}'\Lambda = \text{þ}\Lambda = 0 \tag{5.0.7}$$

We now need to specify the Bianchi identities, Ricci equations and commutators for this particular case. The Bianchi identities give the following equalities:

$$K = 0 \tag{5.0.8}$$

$$\eth\Psi_4 - \eth'\Phi_{22'} = T\Psi_4 - \overline{T}\Phi_{22'} \tag{5.0.9}$$

$$\text{þ}\Phi_{22'} = S\Psi_4 + \overline{R}\Phi_{22'} \tag{5.0.10}$$

$$\text{þ}\Psi_4 = R\Psi_4 + \overline{S}\Phi_{22'} \tag{5.0.11}$$

While the contracted Bianchi identities give:

$$\text{þ}\Phi_{22'} = (R + \overline{R})\Phi_{22'} \tag{5.0.12}$$

By subtracting equation 5.0.12 from equation 1.0.10 we obtain:

$$S\Psi_4 - R\Phi_{22'} = 0 \tag{5.0.13}$$

The Ricci equations are given by:

$$\begin{aligned}
\eth R - \eth' S &= (R - \overline{R})T && (5.0.14)\\
\text{þ} R &= R^2 + S\overline{S} && (5.0.15)\\
\text{þ} S &= S(R + \overline{R}) && (5.0.16)\\
\text{þ} T &= RT + S\overline{T} && (5.0.17)\\
\eth T - \text{þ}' S &= T^2 && (5.0.18)\\
\text{þ}' R - \text{þ}' T &= -T\overline{T} - 2\Lambda && (5.0.19)
\end{aligned}$$

And finally the Commutators become:

$$\begin{aligned}
(\text{þ}\text{þ}' - \text{þ}'\text{þ})\phi &= (\overline{T}\eth + T\eth' + \mathbf{p}\Lambda + \mathbf{q}\Lambda)\phi && (5.0.20)\\
(\text{þ}\eth - \eth\text{þ})\phi &= (\overline{R}\eth + S\eth')\phi && (5.0.21)\\
(\eth\eth' - \eth'\eth)\phi &= ((R - \overline{R})\text{þ}' - \mathbf{p}\Lambda + \mathbf{q}\Lambda)\phi && (5.0.22)\\
(\text{þ}'\eth' - \eth'\text{þ}')\phi &= -\overline{T}\text{þ}'\phi && (5.0.23)
\end{aligned}$$

5.1 First covariant derivative of the Weyl and Ricci spinor and the Ricci scalar

We now calculate the first covariant derivative of the Weyl and Ricci spinor and Ricci scalar. Our intention is to express all derivatives in terms of the generalized GHP formalism, however this notation deals only with totally symmetric quantities which causes a problem since in the non vacuum case one must consider the non symmetric terms as well. We therefore, make use of an important result which can be found in [1] and which states:

Lemma 5.1.1 *The set of nth derivatives $\nabla^n R$ contains the following terms:*

(i) The totally symmetrised spinor nth derivatives of the Weyl spinor

$$\nabla^{(A'}_{(A}\nabla^{B'}_{B}\ldots\nabla^{G')}_{G}\Psi_{HKLM)}$$

(ii) The totally symmetrised spinor nth derivatives of the Ricci spinor

$$\nabla^{(A'}_{(A}\nabla^{B'}_{B}\ldots\nabla^{G}_{G}\Phi^{H'K')}_{HK)}$$

(iii) The totally symmetrised spinor nth derivatives of the Ricci scalar

$$\nabla^{(A'}_{(A}\nabla^{B'}_{B}\ldots\nabla^{G)}_{G)}\Lambda$$

(iv) For $n \geq 1$, the totally symmetrised $(n-1)$th derivative of the 'curl' $\Xi_{ABCA'}$ of the Ricci spinor

$$\nabla^{(A'}_{(A}\nabla^{B'}_{B}\ldots\nabla^{G}_{G}\Xi^{H')}_{HKL)}$$

where Ξ is defined to be either side of the Bianchi identities:

$$\nabla^{A}_{A'}\Psi_{ABCD} = \nabla^{B'}_{(B}\Phi_{CD)B'A'}$$

(v) For $n \geq 2$, the d'Alembertian of all quantities in $\nabla^{n-2}R$, i.e.

$$\nabla^{A}_{A'}\nabla^{A'}_{A}Q$$

where Q is a member of $\nabla^{n-2}R$.

Therefore at first order of differentiation we need to calculate all totally symmetric terms relating to the first covariant derivative of the Weyl and Ricci spinor and Ricci scalar and the zeroth derivative of the curl. We start off by determining the general expressions that give the dyad components of such derivatives since the generalized terms follow immediately.

The expressions giving the covariant derivative up to fifth order of the Weyl spinor are given in chapter 4. The method used to calculate the terms relating to the Ricci spinor is very similar to the way one calculates the derivative of the Weyl spinor. $(\nabla\Phi)_{\mu\nu';ee'}$ will represent the dyad components of the covariant derivative of the Ricci spinor where μ gives the number of unprimed dyad vectors that are ξ^{A}_{1}'s and ν' represents the number of primed dyad vectors that are $\bar{\xi}^{A'}_{1'}$'s. We can then write:

$$\begin{aligned}(\nabla\Phi)_{\mu\nu';ee'} &= \Phi_{ABA'B';EE'}\ [\xi_a^A\xi_b^B\overline{\xi}_{a'}^{A'}\overline{\xi}_{b'}^{B'}]\xi_e^E\overline{\xi}_{e'}^{E'} \\ &= (\Phi_{ABA'B'}\ \xi_a^A\xi_b^B\overline{\xi}_{a'}^{A'}\overline{\xi}_{b'}^{B'})_{;ee'} - (\xi_a^A\xi_b^B\overline{\xi}_{a'}^{A'}\overline{\xi}_{b'}^{B'})_{;ee'}\Phi_{ABA'B'} \\ &= (\Phi_{\mu\nu'})_{;ee'} - \mu\xi^A_{1\ ;ee'}\xi_b^B\overline{\xi}_{a'}^{A'}\overline{\xi}_{b'}^{B'}\Phi_{ABA'B'} - (2-\mu) \qquad (5.1.24) \\ &\quad \xi^A_{0\ ;ee'}\xi_b^B\overline{\xi}_{a'}^{A'}\overline{\xi}_{b'}^{B'}\Phi_{ABA'B'} - \nu'\overline{\xi}^{A'}_{1'\ ;ee'}\xi_a^A\xi_b^B\overline{\xi}_{b'}^{B'}\Phi_{ABA'B'} \\ &\quad -(2-\nu')\overline{\xi}^{A'}_{0'\ ;ee'}\xi_a^A\xi_b^B\overline{\xi}_{b'}^{B'}\Phi_{ABA'B'}\end{aligned}$$

where there are μ ξ_1^A's and ν' $\overline{\xi}_{1'}^{A'}$ in square brackets with $\mu \in \{0,1,2\}$ while $\nu' \in \{0,1,2\}$.

If we substitute expression 4.4.76 which gives the relationship between the term $\xi^F_{f;hh'}$ and the spin coefficients $\Gamma_{abcd'}$ we arrive at the general expression for the dyad components:

$$\begin{aligned}(\nabla\Phi)_{\mu\nu';ee'} &= (\Phi_{\mu\nu'})_{;ee'} - \mu\Gamma_{11ee'}\Phi_{(\mu-1)\nu'} + (2\mu-2)\Gamma_{10ee'}\Phi_{\mu\nu'} \\ &+(2-\mu)\Gamma_{00ee'}\Phi_{(\mu+1)\nu'} - \nu'\overline{\Gamma}_{1'1'e'e}\Phi_{\mu(\nu'-1)} + (2\nu'-2)\overline{\Gamma}_{1'0'e'e}\Phi_{\mu\nu'} \qquad (5.1.25) \\ &+(2-\nu')\overline{\Gamma}_{0'0'e'e}\Phi_{\mu(\nu'+1)}\end{aligned}$$

with $\mu \in \{0,1,2\}$ and $\nu' \in \{0,1,2\}$.

Then the invariant formalism equation becomes:

$$\begin{aligned}&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla\mathbf{\Phi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{1'}...\mathbf{A'}_{N'}\mathbf{EE'}} = \sum \mathbf{\Phi}_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{1'}...\mathbf{A'}_{N'};\mathbf{EE'}} \\ &o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{EE'}}\mathbf{\Phi}_{\mathbf{A}_3...\mathbf{A}_{N+1}\mathbf{A'}_{1'}...\mathbf{A'}_{N'}}\ \overline{o}_{\mathbf{A'}_{N'+1'}} \\ &+(2N-2)\Gamma_{0\mathbf{A}_1\mathbf{EE'}}\mathbf{\Phi}_{\mathbf{A}_2...\mathbf{A}_{N+1}\mathbf{A'}_{1'}...\mathbf{A'}_{N'}\mathbf{EE'}}\ \overline{o}_{\mathbf{A'}_{N'+1'}} \\ &+(2-N)\Gamma_{00\mathbf{EE'}}\mathbf{\Phi}_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A'}_{1'}...\mathbf{A'}_{N'}}\ \overline{o}_{\mathbf{A'}_{N'+1'}} \qquad (5.1.26) \\ &-N'\overline{\Gamma}_{\mathbf{A'}_{1'}\mathbf{A'}_{2'}\mathbf{E'E}}\mathbf{\Phi}_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{3'}...\mathbf{A'}_{N'+1'}}\ o_{\mathbf{A}_{N+1}} \\ &+(2N'-2)\overline{\Gamma}_{\mathbf{A'}_{1'}0'\mathbf{E'E}}\mathbf{\Phi}_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{2'}...\mathbf{A'}_{N'+1'}}\ o_{\mathbf{A}_{N+1}} \\ &+(2-N')\overline{\Gamma}_{0'0'\mathbf{E'E}}\mathbf{\Phi}_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{1'}...\mathbf{A'}_{N'+1}}\ o_{\mathbf{A}_{N+1}}\end{aligned}$$

$N \in \{0,1,2\}$, $N' \in \{0,1,2\}$.

The calculation with respect to the Ricci scalar is very straightforward and the dyad component expression is simply given by:

$$(\nabla\Lambda)_{;ee'} = (\Lambda)_{;ee'} \tag{5.1.27}$$

With the generalized formula being:

$$(\nabla\mathbf{\Lambda})_{\mathbf{EE'}} = (\mathbf{\Lambda})_{;\mathbf{EE'}} \tag{5.1.28}$$

Finally we need to calculate the 0th derivative of the curl Ξ, i.e, the curl itself. Ξ is given by either side of the Bianchi identities, so that the dyad component equation is given in [23]:

$$\nabla^{p}{}_{d'}\Psi_{abcp} - 3\Psi_{pr(ab}\,\Gamma_{c)}{}^{pr}{}_{d'} - \Psi_{abcp}\,\Gamma^{p}{}_{r}{}^{r}{}_{d'} \tag{5.1.29}$$

It is clear from chapter 2 that in generalized notation these terms become:

$$\begin{aligned} & \epsilon_{\mathbf{L'N'}}\epsilon_{\mathbf{KM}}(\nabla_{\mathbf{ND'}}\Psi_{\mathbf{ABCL}} - \nabla_{\mathbf{LD'}}\Psi_{\mathbf{ABCN}}) \\ - & 3\epsilon_{\mathbf{L'N'}}(\Psi_{\mathbf{LK(AB}}\Gamma_{\mathbf{C)NMD'}} - \Psi_{\mathbf{LM(AB}}\Gamma_{\mathbf{C)NKD'}} \\ - & \Psi_{\mathbf{NK(AB}}\Gamma_{\mathbf{C)LMD'}} + \Psi_{\mathbf{NM(AB}}\Gamma_{\mathbf{C)LKD'}}) \\ + & \epsilon_{\mathbf{L'N'}}(\Psi_{\mathbf{ABCL}}\Gamma_{\mathbf{NKMD'}} - \Psi_{\mathbf{ABCL}}\Gamma_{\mathbf{NMKD'}} \\ - & \Psi_{\mathbf{ABCN}}\Gamma_{\mathbf{LKMD'}} + \Psi_{\mathbf{ABCN}}\Gamma_{\mathbf{LMKD'}}) \end{aligned} \tag{5.1.30}$$

We are now able to determine from expressions 4.2.51, 5.1.26, 5.1.28 and 5.1.30 all terms relating to ∇R, they are:

$$(\nabla\mathbf{\Psi}) = \text{þ}'\mathbf{\Psi}_4 \tag{5.1.31}$$

$$(\nabla\mathbf{\Psi})\cdot o = \eth'\mathbf{\Psi}_4 \tag{5.1.32}$$

$$(\nabla\mathbf{\Psi})\cdot\overline{o} = \eth\mathbf{\Psi}_4 + 4T\mathbf{\Psi}_4 \tag{5.1.33}$$

$$(\nabla\mathbf{\Psi})\cdot o\cdot\overline{o} = \text{þ}\mathbf{\Psi}_4 + 4R\mathbf{\Psi}_4 \tag{5.1.34}$$

$$(\nabla\mathbf{\Phi}) = \text{þ}'\mathbf{\Phi}_{22'} \tag{5.1.35}$$

$$(\nabla\mathbf{\Phi})\cdot o = \eth\mathbf{\Phi}_{22'} + 2T\mathbf{\Phi}_{22'} \tag{5.1.36}$$

$$(\nabla\mathbf{\Phi})\cdot\overline{o} = \eth'\mathbf{\Phi}_{22'} + 2\overline{T}\mathbf{\Phi}_{22'} \tag{5.1.37}$$

$$
\begin{aligned}
(\nabla\mathbf{\Phi})\cdot o\cdot\bar{o} &= \text{Þ}\mathbf{\Phi}_{22'} + 2R\mathbf{\Phi}_{22'} && (5.1.38)\\
(\nabla\mathbf{\Phi})\cdot o\cdot o &= S\mathbf{\Phi}_{22'} && (5.1.39)\\
(\nabla\mathbf{\Phi})\cdot\bar{o}\cdot\bar{o} &= \overline{S}\mathbf{\Phi}_{22'} && (5.1.40)\\
(\nabla\mathbf{\Lambda}) &= \text{Þ}'\mathbf{\Lambda} && (5.1.41)\\
(\nabla\mathbf{\Lambda})\cdot\bar{o} &= \eth'\mathbf{\Lambda} && (5.1.42)\\
(\nabla\mathbf{\Lambda})\cdot o &= \eth\mathbf{\Lambda} && (5.1.43)\\
(\nabla\mathbf{\Lambda})\cdot o\cdot\bar{o} &= \text{Þ}\mathbf{\Lambda} && (5.1.44)\\
\mathbf{\Xi} &= \eth\mathbf{\Psi}_4 - T\mathbf{\Psi}_4 && (5.1.45)\\
\mathbf{\Xi}\cdot\bar{o} &= \text{Þ}\mathbf{\Psi}_4 - R\mathbf{\Psi}_4 && (5.1.46)
\end{aligned}
$$

Notice that since we are considering the worst possible case where all terms at zeroth order are constants and since $\mathbf{\Lambda}$ is a scalar of weight $\{\mathbf{0},\mathbf{0}\}$ then all terms arising from the derivative of $\mathbf{\Lambda}$ are zero. And all terms are invariant under null rotations so that the dimension of the invariance group $\mathbf{H}_1$ remains two.

5.2 Second Covariant Derivative

We proceed to calculate all terms relating to the second covariant derivative , the terms that we need to calculate at this order are :

(i) the symmetric second order covariant derivative of the Weyl spinor

(ii) the symmetric second order covariant derivative of the Ricci spinor

(iii) the symmetric second order covariant derivative of the Ricci scalar

(iv) the symmetric first order covariant derivative of the curl

(v) the d'Alembertian of $\Psi, \Phi, \text{and}\Lambda$.

The calculation relating to the second derivative of Φ whose dyad components we will denote by $(\nabla^2\Phi)_{\mu\nu';ff'}$ is very similar to that of the first derivative. We have:

$$
\begin{aligned}
(\nabla^2\Phi)_{\mu\nu';ff'} &= \Phi_{ABA'B';EE';FF'} \quad [\xi^A_a\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\,\xi^E_e\overline{\xi}^{E'}_{e'}]\xi^F_f\overline{\xi}^{F'}_{f'}\\
&= (\Phi_{ABA'B';EE'} \quad \xi^A_a\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\,\xi^E_e\overline{\xi}^{E'}_{e'})_{;ff'}\\
&- (\xi^A_a\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\,\xi^E_e\overline{\xi}^{E'}_{e'})_{;ff'}\,\Phi_{ABA'B';EE'}\\
&= [(\nabla\Phi)_{\mu\nu'}]_{;ff'} - \mu\xi^A_{1\ ;ff'}\,\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\,\xi^E_e\overline{\xi}^{E'}_{e'}\,\Phi_{ABA'B';EE'} \qquad (5.2.47)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad (3-\mu)\xi_{0\ ;ff'}^{A}\,\xi_b^{B}\overline{\xi}_{a'}^{A'}\,\overline{\xi}_{b'}^{B'}\,\xi_e^{E}\overline{\xi}_{e'}^{E'}\,\Phi_{ABA'B';EE'} \\
&- \quad \nu'\overline{\xi}_{1'\ ;ff'}^{A'}\,\xi_a^{A}\xi_b^{B}\overline{\xi}_{b'}^{B'}\,\xi_e^{E}\overline{\xi}_{e'}^{E'}\,\Phi_{ABA'B';EE'} \\
&- \quad (3-\nu')\overline{\xi}_{0'\ ;ff'}^{A'}\,\xi_a^{A}\xi_b^{B}\overline{\xi}_{b'}^{B'}\,\xi_e^{E}\overline{\xi}_{e'}^{E'}\,\Phi_{ABA'B';EE'}
\end{aligned}
$$

where there are μ primed ξ_1^{A}'s and ν unprimed $\overline{\xi}_{1'}^{A'}$ in square brackets with $\mu \in \{0,1,2,3\}$ while $\nu' \in \{0,1,2,3\}$.

Therefore, the dyad components of the second derivative of Φ can be obtained from the general expression given below:

$$
\begin{aligned}
(\nabla^2\Phi)_{\mu\nu';ff'} &= [(\nabla\Phi)_{\mu\nu'}]_{;ff'} - \mu\Gamma_{11ff'}(\nabla\Phi)_{(\mu-1)\nu'} + (2\mu-3)\Gamma_{10ff'}(\nabla\Phi)_{\mu\nu'} \\
&+(3-\mu)\Gamma_{00ff'}(\nabla\Phi)_{(\mu+1)\nu'} - \nu\overline{\Gamma}_{1'1'f'f}(\nabla\Phi)_{\mu(\nu-1)} \qquad (5.2.48) \\
&+(2\nu-3)\overline{\Gamma}_{1'0'f'f}\Phi_{\mu\nu'} + (3-\nu')\overline{\Gamma}_{0'0'f'f}(\nabla\Phi)_{\mu(\nu+1)}
\end{aligned}
$$

with $\mu \in \{0,1,2,3\}$ and $\nu' \in \{0,1,2,3\}$.

Translating equation 5.2.48 into generalized formalism gives:

$$
\begin{aligned}
&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{FF'}} = \sum(\nabla\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'};\mathbf{FF'}} \\
&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{FF'}}(\nabla\mathbf{\Phi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&+(2N-3)\Gamma_{0\mathbf{A}_1\mathbf{FF'}}(\nabla\mathbf{\Phi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{FF'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&+(3-N)\Gamma_{00\mathbf{FF'}}(\nabla\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \qquad (5.2.49) \\
&-N'\overline{\Gamma}_{\mathbf{A'}_{1'}\mathbf{A'}_{2'}\mathbf{F'F}}(\nabla\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}} \\
&+(2N'-3)\overline{\Gamma}_{\mathbf{A'}_{1'}0'\mathbf{F'F}}(\nabla\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}} \\
&+(3-N')\overline{\Gamma}_{0'0'\mathbf{F'F}}(\nabla\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}\,o_{\mathbf{A}_{N+1}}
\end{aligned}
$$

with $N \in \{0,1,2,3\}$ and $N' \in \{0,1,2,3\}$.

The calculation of the second covariant derivative of Λ is also straightforward, we write:

$$
\begin{aligned}
(\nabla^2\Lambda)_{\mu\nu';ff'} &= \Lambda_{;EE';FF'}\ [\xi_e^E\overline{\xi}_{e'}^{E'}]\xi_f^F\overline{\xi}_{f'}^{F'} \\
&= (\Lambda_{;EE'}\,\xi_e^E\overline{\xi}_{e'}^{E'})_{;ff'} - (\xi_e^E\overline{\xi}_{e'}^{E'})_{;ff'}\,\Lambda_{;EE'} \\
&= [(\nabla\Lambda)_{\mu\nu'}]_{;ff'} - \mu\xi_{1\ ;ff'}^{E}\,\overline{\xi}_{e'}^{E'}\,\Lambda_{;EE'} \\
&\quad -(1-\mu)\xi_{0\ ;ff'}^{E}\,\overline{\xi}_{e'}^{E'}\,\Lambda_{;EE'} - \nu'\overline{\xi}_{1'\ ;ff'}^{E'}\,\xi_e^E\,\Lambda_{;EE'} \\
&\quad -(1-\nu')\overline{\xi}_{0'\ ;ff'}^{E'}\,\xi_e^E\,\Lambda_{;EE'}
\end{aligned}
\tag{5.2.50}
$$

So that if we substitute expression 4.4.76 which gives the relationship between the term $\xi_{f;hh'}^{F}$ and the spin coefficients $\Gamma_{abcd'}$ we get the general expression for the dyad components:

$$
\begin{aligned}
(\nabla^2\Lambda)_{\mu\nu';ff'} &= [(\nabla\Lambda)_{\mu\nu'}]_{;ff'} - \mu\Gamma_{11ff'}(\nabla\Lambda)_{(\mu-1)\nu'} + (2\mu-1)\Gamma_{01ff'}(\nabla\Lambda)_{\mu\nu'} \\
&+(1-\mu)\Gamma_{00ff'}(\nabla\Lambda)_{(\mu+1)\nu'} - \nu'\overline{\Gamma}_{1'1'f'f}(\nabla\Lambda)_{\mu(\nu-1)'} \\
&+(2\nu'-1)\overline{\Gamma}_{0'1'f'f}(\nabla\Lambda)_{\mu\nu'} + (1-\nu')\overline{\Gamma}_{0'0'f'f}(\nabla\Lambda)_{\mu(\nu+1)'}
\end{aligned}
\tag{5.2.51}
$$

with $\mu \in \{0,1\}$ and $\nu' \in \{0,1\}$.

With the direct translation into invariant language given below:

$$
\begin{aligned}
&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{FF'}} = \sum[(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}]_{;\mathbf{FF'}} \\
&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{FF'}}(\nabla\mathbf{\Lambda})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&+(2N-1)\Gamma_{0\mathbf{A}_1\mathbf{FF'}}(\nabla\mathbf{\Lambda})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{FF'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&+(1-N)\Gamma_{00\mathbf{FF'}}(\nabla\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{F'F}}(\nabla\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}} \\
&+(2N'-1)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{F'F}}(\nabla\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}} \\
&+(1-N')\overline{\Gamma}_{0'0'\mathbf{F'F}}(\nabla\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}\,o_{\mathbf{A}_{N+1}}
\end{aligned}
\tag{5.2.52}
$$

with $N \in \{0,1\}$ and $N' \in \{0,1\}$.

The covariant derivative of the curl can be obtained as follows:

$$
\begin{aligned}
(\nabla\Xi)_{\mu\nu';ee'} &= (\Xi_{ABCA'}\ \xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'})_{;ee'} \\
&- (\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'})_{;ee'}\,\Xi_{ABCA'} \\
&= (\Xi_{\mu\nu'})_{;ee'} - \mu\xi^A_{1;ee'}\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}\,\Xi_{ABCA'} \\
&- (3-\mu)\xi^A_{0;ee'}\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}\,\Xi_{ABCA'} \\
&- \nu'\overline{\xi}^{A'}_{1';ee'}\xi_a^A\xi_b^B\xi_c^C\Xi_{ABCA'} \\
&- (1-\nu')\overline{\xi}^{A'}_{0';ee'}\xi_a^A\xi_b^B\xi_c^C\Xi_{ABCA'}
\end{aligned}
\tag{5.2.53}
$$

As before μ and ν' represent the number of unprimed ξ_1^A's and primed $\overline{\xi}_{1'}^{A'}$ there are among $\xi_a^A\xi_b^B\xi_c^C$ and $\overline{\xi}_{a'}^{A'}$ respectively. Furthermore $\mu\in\{0,1,2\}$ and $\nu'\in\{0,1\}$.

With the help of 4.4.76 we obtain the expression giving the dyad components:

$$
\begin{aligned}
&(\nabla\Xi)_{\mu\nu';ee'} = [\Xi_{\mu\nu'}]_{;ee'} - \mu\Gamma_{11ee'}\,\Xi_{(\mu-1)\nu'} + (2\mu-3)\Gamma_{10ee'}\,\Xi_{\mu\nu'} \\
&+(3-\mu)\Gamma_{00ee'}\,\Xi_{(\mu+1)\nu'} - \nu'\overline{\Gamma}_{1'1'e'e}\,\Xi_{\mu(\nu-1)'} + (2\nu'-1)\overline{\Gamma}_{1'0'e'e}\,\Xi_{\mu\nu'} \\
&+(1-\nu')\overline{\Gamma}_{0'0'e'e}\,\Xi_{\mu(\nu+1)'}
\end{aligned}
\tag{5.2.54}
$$

The generalized expression follows immediately:

$$
\begin{aligned}
&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla\boldsymbol{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{EE'}} = \sum[\boldsymbol{\Xi}_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}]_{;\mathbf{EE'}} \\
&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{EE'}}\boldsymbol{\Xi}_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&+(2N-2)\Gamma_{0\mathbf{A}_1\mathbf{EE'}}\boldsymbol{\Xi}_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&+(3-N)\Gamma_{00\mathbf{EE'}}\boldsymbol{\Xi}_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \\
&-N'\overline{\Gamma}_{\mathbf{A'}_{1'}\mathbf{A'}_{2'}\mathbf{E'E}}\boldsymbol{\Xi}_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}} \\
&+(2N'-1)\overline{\Gamma}_{\mathbf{A'}_{1'}0'\mathbf{E'E}}\boldsymbol{\Xi}_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}} \\
&+(1-N')\overline{\Gamma}_{0'0'\mathbf{E'E}}\boldsymbol{\Xi}_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}\,o_{\mathbf{A}_{N+1}}
\end{aligned}
\tag{5.2.55}
$$

with $N\in\{0,1,2,3\}$ and $N'\in\{0,1\}$.

We now concentrate on determining the d'Alembertian of Ψ , Φ , and Λ which is given by $\nabla^{EE'}\nabla_{EE'}\Psi_{ABCD}$, $\nabla^{EE'}\nabla_{EE'}\Phi_{ABA'B'}$, and $\nabla^{EE'}\nabla_{EE'}\Lambda$ respectively. The d'Alembertian will sometimes be denoted by the symbol $\Box$. The dyad components of the first derivative $\nabla_{EE'}\Psi_{ABCD} = \Psi_{ABCD;EE'}$ are obtained as follows:

$$\Psi_{abcd;ee'} = \Psi_{ABCD;EE'} \; [\xi_a^A \xi_b^B \xi_c^C \xi_d^D] \xi_e^E \overline{\xi}_{e'}^{E'} = (\Psi_{ABCD}\; \xi_a^A \xi_b^B \xi_c^C \xi_d^D)_{;EE'}\, \xi_e^E \overline{\xi}_{e'}^{E'}$$
$$-[\xi_a^A \xi_b^B \xi_c^C \xi_d^D]_{;EE'}\, \xi_e^E \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD} \tag{5.2.56}$$

Obviously we can use the same idea to obtain the dyad components of the d'Alembertian $\nabla^{EE'}\nabla_{EE'}\Psi_{ABCD}$ which is given by:

$$\begin{aligned} \nabla^{ee'}\nabla_{ee'}\Psi_{abcd} &= \Psi_{ABCD;EE'}{}^{;EE'}\, [\xi_a^A \xi_b^B \xi_c^C \xi_d^D] \xi_e^E \overline{\xi}_{e'}^{E'}\, \xi_E^e \overline{\xi}_{E'}^{e'} \\ &= (\Psi_{ABCD;EE'}\; \xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_e^E \overline{\xi}_{e'}^{E'})^{;EE'}\, \xi_E^e \overline{\xi}_{E'}^{e'} \\ &- (\xi_a^A \xi_b^B \xi_c^C \xi_d^D \xi_e^E \overline{\xi}_{e'}^{E'})^{;EE'}\, \xi_E^e \overline{\xi}_{E'}^{e'}\, \Psi_{ABCD;EE'} \end{aligned} \tag{5.2.57}$$

By working out the derivatives in curly brackets in equation 5.2.57 we get:

$$\begin{aligned} \nabla^{ee'}\nabla_{ee'}\Psi_{\mu} &= [(\nabla\Psi)_{\mu;ee'}]^{;ee'} - \mu\xi_1^{A;ee'}\, [\xi_b^B \xi_c^C \xi_d^D] \xi_e^E \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD;EE'} \\ &- (4-\mu)\xi_0^{A;ee'}\, [\xi_b^B \xi_c^C \xi_d^D] \xi_e^E \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD;EE'} \\ &- \xi_e^{E;ee'}\, [\xi_a^A \xi_b^B \xi_c^C \xi_d^D] \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD;EE'} \\ &- \overline{\xi}_{e'}^{E';ee'}\, [\xi_a^A \xi_b^B \xi_c^C \xi_d^D] \xi_e^E \Psi_{ABCD;EE'} \end{aligned} \tag{5.2.58}$$

where in the first square brackets there are $(\mu-1)$ ξ_1^A's, in the second square brackets there are $(\mu+1)$ ξ_1^A's and in the third and fourth square brackets there are μ ξ_1^A's and $\mu \in \{0,1,2,3,4\}$. Equation 5.2.58 can be rewritten as:

$$\begin{aligned} \nabla^{ee'}\nabla_{ee'}\Psi_{\mu} &= [(\nabla\Psi)_{\mu;ee'}]_{;gg'}\, \epsilon^{ge}\epsilon^{g'e'} \\ &- \mu\xi^A_{1\ ;gg'}\, \epsilon^{ge}\epsilon^{g'e'}\, [\xi_b^B \xi_c^C \xi_d^D] \xi_e^E \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD;EE'} \\ &- (4-\mu)\xi^A_{0\ ;gg'}\, \epsilon^{ge}\epsilon^{g'e'}\, [\xi_b^B \xi_c^C \xi_d^D] \xi_e^E \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD;EE'} \\ &- \xi^E_{e\ ;gg'}\, \epsilon^{ge}\epsilon^{g'e'}\, [\xi_a^A \xi_b^B \xi_c^C \xi_d^D] \overline{\xi}_{e'}^{E'}\, \Psi_{ABCD;EE'} \\ &- \overline{\xi}^{E'}_{e'\ ;gg'}\, \epsilon^{ge}\epsilon^{g'e'}\, [\xi_a^A \xi_b^B \xi_c^C \xi_d^D] \xi_e^E \Psi_{ABCD;EE'} \end{aligned} \tag{5.2.59}$$

By substituting 4.4.76 in 5.2.59 we arrive at:

$$\begin{aligned}\nabla^{ee'}\nabla_{ee'}\Psi_\mu &= [(\nabla\Psi)_{\mu;ee'}]_{;gg'}\epsilon^{ge}\epsilon^{g'e'}\\ &- \mu\Gamma_{1tgg'}\epsilon^{kt}\xi^A_k\epsilon^{ge}\epsilon^{g'e'}[\xi^B_b\xi^C_c\xi^D_d]\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,EE'}\\ &- (4-\mu)\Gamma_{0tgg'}\epsilon^{kt}\xi^A_k\epsilon^{ge}\epsilon^{g'e'}[\xi^B_b\xi^C_c\xi^D_d]\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,EE'} \qquad (5.2.60)\\ &- \Gamma_{etgg'}\epsilon^{kt}\xi^E_k\epsilon^{ge}\epsilon^{g'e'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\overline{\xi}^{E'}_{e'}\Psi_{ABCD,EE'}\\ &- \overline{\Gamma}_{e't'g'g}\overline{\epsilon}^{k't'}\overline{\xi}^{E'}_{k'}\epsilon^{ge}\epsilon^{g'e'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\xi^E_e\Psi_{ABCD,EE'}\end{aligned}$$

which finally gives the general expression for the dyad components of the d'Alembertian:

$$\begin{aligned}\nabla^{ee'}\nabla_{ee'}\Psi_\mu &= \{[(\nabla\Psi)_{\mu;ee'}]_{;gg'} - \mu\Gamma_{11gg'}\Psi_{(\mu-1);ee'} + (2\mu-4)\Gamma_{10gg'}\Psi_{\mu;ee'}\\ &- (4-\mu)\Gamma_{00gg'}\Psi_{(\mu+1);ee'} - \Gamma_{00gg'}\Psi_{(\mu+1);ee'}\\ &- \Gamma_{e1gg'}\Psi_{\mu;0e'} + \Gamma_{e0gg'}\Psi_{\mu;1e'} \qquad (5.2.61)\\ &- \overline{\Gamma}_{e'1'g'g}\Psi_{\mu;e0'} + \overline{\Gamma}_{e'0'g'g}\Psi_{\mu;e1'}\}\epsilon^{ge}\epsilon^{g'e'}\end{aligned}$$

We can clearly see from equation 5.2.61 that:

$$\nabla^{ee'}\nabla_{ee'}\Psi_\mu = \{(\nabla^2\Psi)_{\mu;ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.2.62)$$

So that in generalized formalism equation 5.2.62 translates into:

$$\begin{aligned}&o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{1'}}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{EE}'}{}^{\mathbf{EE}'}\\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{1'}}(\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \qquad (5.2.63)\\ &= (\nabla^2\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}}\end{aligned}$$

with $N \in \{0,1,2,3,4\}$.

The method of calculating the dyad components of the d'Alembertian of $\Phi_{22'}$ is the same as that of Ψ_4, in equation 5.2.57 we substitute Ψ_4 by $\Phi_{22'}$ and contract with the adequate number of ξ^A_a and $\overline{\xi}^{A'}_{a'}$'s, ie:

$$\begin{aligned}\nabla^{ee'}\nabla_{ee'}\Phi_{aba'b'} &= \Phi_{ABA'B';EE'}{}^{;EE'}[\xi^A_a\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\xi^E_e\overline{\xi}^{E'}_{e'}]\xi^e_E\overline{\xi}^{e'}_{E'}\\ &= (\Phi_{ABA'B';EE'}\xi^A_a\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\xi^E_e\overline{\xi}^{E'}_{e'})^{;EE'}\xi^e_E\overline{\xi}^{e'}_{E'} \qquad (5.2.64)\\ &- (\xi^A_a\xi^B_b\overline{\xi}^{A'}_{a'}\overline{\xi}^{B'}_{b'}\xi^E_e\overline{\xi}^{E'}_{e'})^{;EE'}\xi^e_E\overline{\xi}^{e'}_{E'}\Phi_{ABA'B'\;;EE'}\end{aligned}$$

where there are μ ξ_1^A's and ν' $\overline{\xi}_{1'}^{A'}$'s in square brackets with $\mu \in \{0,1,2\}$ and $\nu' \in \{0,1,2\}$.

The calculations as of this point are exactly the same as in the case of the d'Alembertian of Ψ, so that it is not worth repeating all the same details. Hence, if one works out equation 5.2.64 in the same way as before we arrive at the dyad expression:

$$\nabla^{ee'}\nabla_{ee'}\Phi_{\mu\nu'} = \{(\nabla\Phi)_{\mu\nu';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \tag{5.2.65}$$

The generalized version being:

$$\begin{aligned} & o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{1'}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{1'}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'\mathbf{G}\mathbf{G}'}\,\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned} \tag{5.2.66}$$

with $N \in \{0,1,2\}$ and $N' \in \{0,1,2\}$.

The case of the d'Alembertian of Λ is treated similarly. The calculation of $\nabla^{ee'}\nabla_{ee'}\Lambda$ follows the same path as that of the d'Alembertian of Ψ and Φ we will omit the details since these can be reproduced very easily. The dyad expression is written below:

$$\nabla^{ee'}\nabla_{ee'}\Lambda = \{(\nabla\Lambda)_{;ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \tag{5.2.67}$$

The generalized version becomes:

$$\begin{aligned} o_{\mathbf{A}_1}\overline{o}_{\mathbf{A}_{1'}}(\nabla^2\mathbf{\Lambda})_{\mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'} &= (o_{\mathbf{A}_1}\overline{o}_{\mathbf{A}_{1'}}(\nabla^2\mathbf{\Lambda})_{\mathbf{E}\mathbf{E}'\mathbf{G}\mathbf{G}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\mathbf{A}'_{1'}} \end{aligned} \tag{5.2.68}$$

By Lemma 5.1.1 these are all the quantities we need to calculate so that the terms obtained at second order of differentiation are:

$\nabla^2\mathbf{\Psi} = \text{þ}'\text{þ}'\mathbf{\Psi}_4$ and all possible contractions with omicrons

$\nabla^2\mathbf{\Phi} = \text{þ}'\text{þ}'\mathbf{\Phi}_{22'}$ and all possible contractions with omicrons

$\nabla^2 \mathbf{\Lambda} = \text{Þ}'\text{Þ}'\mathbf{\Lambda}$ and all possible contractions with omicrons

$\nabla \mathbf{\Xi} = \text{Þ}'\mathbf{\Xi}$ and all possible contractions with omicrons

$\Box \mathbf{\Psi}_4 = \nabla^2 \mathbf{\Psi} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box \mathbf{\Phi}_{22'} = \nabla^2 \mathbf{\Phi} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box \mathbf{\Lambda} = \nabla^2 \mathbf{\Lambda} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

5.3 Higher Derivatives

The calculation of third, fourth, ..., etc covariant derivative of the Riemann spinor is lengthy but straightforward and can be viewed as an extension of the calculation done at first and second order. By Lemma 5.1.1 we see that the terms we need to calculate at each order of differentiation are as follows:

Third Covariant Derivative:

$\nabla^3\mathbf{\Psi}_4\,,\ \nabla^3\mathbf{\Phi}_{22'}\,,\ \nabla^3\mathbf{\Lambda}\,,\ \nabla^2\mathbf{\Xi}\,,\ \Box(\nabla\mathbf{\Psi}_4)\,,\ \Box(\nabla\mathbf{\Phi}_{22'})\,,\ \Box(\nabla\mathbf{\Lambda})\,,$
$\Box(\mathbf{\Xi})\ +$ repeated terms

Fourth Covariant Derivative:

$\nabla^4\mathbf{\Psi}_4\,,\ \nabla^4\mathbf{\Phi}_{22'}\,,\ \nabla^4\mathbf{\Lambda}\,,\ \nabla^3\mathbf{\Xi}\,,\ \Box(\nabla^2\mathbf{\Psi}_4)\,,$
$\Box(\nabla^2\mathbf{\Phi}_{22'})\,,\ \Box(\nabla^2\mathbf{\Lambda})\,,\ \Box\Box(\mathbf{\Psi}_4)\,,\ \Box\Box(\mathbf{\Phi}_{22'})\,,\ \Box\Box(\mathbf{\Lambda})\ \ \Box(\nabla\mathbf{\Xi})+$
repeated terms

Fifth Covariant Derivative:

$\nabla^5\mathbf{\Psi}_4\,,\ \nabla^5\mathbf{\Phi}_{22'}\,,\ \nabla^5\mathbf{\Lambda}\,,\ \nabla^4\mathbf{\Xi}\,,\ \Box(\nabla^3\mathbf{\Psi}_4)\,,$
$\Box(\nabla^3\mathbf{\Phi}_{22'})\,,\ \Box(\nabla^3\mathbf{\Lambda})\,,\ \Box(\nabla^2\mathbf{\Xi})\,,\ \Box\Box(\nabla\mathbf{\Psi}_4)\,,\ \Box\Box(\nabla\mathbf{\Phi}_{22'})\,,\ \Box\Box(\nabla\mathbf{\Lambda})$
$\Box\Box(\mathbf{\Xi})+$ repeated terms

The calculations relating to higher derivatives of Φ and Λ can be seen as an extension of the previous calculations. For example, we can write $(\nabla^3\Phi)_{\mu\nu';gg'}$ in the same manner as is done in 5.2.47, $\Phi_{ABA'B';EE';FF'}$ becomes $\Phi_{ABA'B';EE';FF';GG'}$ and in square brackets we will have μ unprimed ξ_1^A's and ν' primed $\tilde{\xi}_{1'}^{A'}$'s with $\mu \in \{0,1,2,3,4\}$ and $\nu' \in \{0,1,2,3,4\}$. The fourth derivative of Φ is obtained in the same way and so on for higher derivatives. The same occurs in the case of higher derivatives of Λ, for example we write $(\nabla^3\Lambda)_{\mu\nu';gg'}$ in the same way as 5.2.51, $\Lambda_{;EE';FF'}$ is substituted by $\Lambda_{;EE';FF';GG'}$ and one then has in square brackets μ

number of unprimed ξ_1^{A}'s and ν' number of primed $\overline{\xi}_{1'}^{A'}$'s with $\mu \in \{0,1,2\}$ and $\nu' \in \{0,1,2\}$. Needless to say that the same works for further derivatives of Ξ.

The d'Alembertian of the covariant derivative of Ψ, dyad components being denoted by $\nabla^{ee'}\nabla_{ee'}\nabla_{ff'}\Psi_{abcd}$, is determined as follows:

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Psi_{abcd}) = \Psi_{ABCD,FF';EE'}{}^{;EE'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_e^E\overline{\xi}_{e'}^{E'}$$
$$= (\Psi_{ABCD,FF';EE'}\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\xi_e^E\overline{\xi}_{e'}^{E'})^{;EE'}\xi_E^e\overline{\xi}_{E'}^{e'} \tag{5.3.69}$$
$$-(\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\xi_e^E\overline{\xi}_{e'}^{E'})^{;EE'}\xi_E^e\overline{\xi}_{E'}^{e'}\Psi_{ABCD,FF';EE'}$$

By working out the derivatives in curly brackets in equation 5.3.69 as one does when going from equation 5.2.57 to equation 5.2.58 and proceeding as we do when rewriting equation 5.2.58, we get from equation 5.3.69 the following expression:

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Psi_\mu) = [(\nabla^2\Psi)_{\mu;ff';ee'}]_{;gg'}\epsilon^{ge}\epsilon^{g'e'}$$
$$-\mu\xi^A_{1;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF';EE'}$$
$$-(4-\mu)\xi^A_{0;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF';EE'}$$
$$-\xi^F_{f;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\overline{\xi}_{f'}^{F'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF';EE'} \tag{5.3.70}$$
$$-\overline{\xi}^{F'}_{f';gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF';EE'}$$
$$\xi^E_{e;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF';EE'}$$
$$\overline{\xi}^{E'}_{e';gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_e^E\Psi_{ABCD,FF';EE'}$$

with $\mu \in \{0,1,2,3,4\}$.
So that by using once again 4.4.76, equation 5.3.70 becomes:

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Psi_\mu) = \{[(\nabla^2\Psi)_{\mu;ff';ee'}]_{;gg'}$$
$$-\mu\Gamma_{11gg'}(\nabla^2\Psi)_{(\mu-1);ff';ee'} + (2\mu-4)\Gamma_{10gg'}(\nabla^2\Psi)_{\mu;ff';ee'}$$
$$+(4-\mu)\Gamma_{00gg'}(\nabla^2\Psi)_{(\mu+1);ff';ee'} - \Gamma_{f1gg'}(\nabla^2\Psi)_{\mu;f0';ee'}$$
$$+\Gamma_{f0gg'}(\nabla^2\Psi)_{\mu;1f';ee'} - \overline{\Gamma}_{f'1'g'g}(\nabla^2\Psi)_{\mu;f0';ee'} \tag{5.3.71}$$

$$+\overline{\Gamma}_{f'0'g'g}\,(\nabla^2\Psi)_{\mu;f1';ee'} - \Gamma_{e1gg'}\,(\nabla^2\Psi)_{\mu;ff';0e'}$$

$$+\Gamma_{e0gg'}\,(\nabla^2\Psi)_{\mu;ff';1e'} - \overline{\Gamma}_{e'1'g'g}\,(\nabla^2\Psi)_{\mu;ff';e0'}$$

$$+\overline{\Gamma}_{e'0'g'g}\,(\nabla^2\Psi)_{\mu;ff';e1'}\ \}\epsilon^{ge}\epsilon^{g'e'}$$

which gives:

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Psi_\mu) = \{(\nabla^2\Psi)_{\mu;ff';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.72}$$

The d'Alembertian of higher derivatives of Ψ is treated similarly, so that the calculations relating to the d'Alembertian of $\nabla^2\Psi$ and $\nabla^3\Psi$ are lengthy but straightforward as in the case of the d'Alembertian of $\nabla\Psi$. We can then write:

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Psi_\mu) = \{(\nabla^3\Psi)_{\mu;ff';hh';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.73}$$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{kk'}\nabla_{hh'}\nabla_{ff'}\Psi_\mu) = \{(\nabla^4\Psi)_{\mu;ff';hh';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.74}$$

The process applied in the calculation of the d'Alembertian of higher derivatives of Ψ can be applied in the same way to determine the d'Alembertian of higher derivatives of Φ and Λ so that we can obtain rather easily the following equalities:

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Phi_{\mu\nu'}) = \{(\nabla^2\Phi)_{\mu\nu';ff';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.75}$$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Phi_{\mu\nu'}) = \{(\nabla^3\Phi)_{\mu\nu';ff';hh';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.76}$$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{kk'}\nabla_{hh'}\nabla_{ff'}\Phi_{\mu\nu'}) = \{(\nabla^4\Phi)_{\mu\nu';ff';hh';kk';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.77}$$

with $\mu \in \{0,1,2\}$ and $\nu' \in \{0,1,2\}$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Lambda) = \{(\nabla^2\Lambda)_{;ff';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.78}$$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Lambda) = \{(\nabla^2\Lambda)_{;ff';hh';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.79}$$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{kk'}\nabla_{hh'}\nabla_{ff'}\Lambda) = \{(\nabla^2\Lambda)_{;ff';hh';kk';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.80}$$

We now address the calculation of the d'Alembertian of Ξ. We want to determine the dyad components represented by $\nabla^{ee'}\nabla_{ee'}\Xi_{abcd'}$, we write:

$$\nabla^{ee'}\nabla_{ee'}\Xi_{abca'} = \Xi_{ABCA';EE'}{}^{;EE'}[\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\overline{\xi}_{e'}^{E'}\xi_E^e\overline{\xi}_{E'}^{e'}$$
$$-(\Xi_{ABCA';EE'}\ \xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}\xi_e^E\overline{\xi}_{e'}^{E'})^{;EE'}\xi_E^e\overline{\xi}_{E'}^{e'} \quad (5.3.81)$$
$$-(\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}\xi_e^E\overline{\xi}_{e'}^{E'})^{;EE'}\xi_E^e\overline{\xi}_{E'}^{e'}$$

where there are $\mu \in \{0,1,2,3\}$ ξ_1^A's and $\nu' \in \{0,1\}$ $\overline{\xi}_{1'}^{A'}$'s in square brackets. Working out the derivatives in curly brackets in equation 5.3.81 we obtain:

$$\nabla^{ee'}\nabla_{ee'}\Xi_{\mu\nu'} = [(\nabla\Xi)_{\mu\nu';ee'}]^{;ee'} - \mu\xi_1^{A;ee'}[\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-(3-\mu)\xi_0^{A;ee'}[\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-\nu'\overline{\xi}_{1'}^{A';ee'}[\xi_a^A\xi_b^B\xi_c^C]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-(1-\nu')\overline{\xi}_{0'}^{A';ee'}[\xi_a^A\xi_b^B\xi_c^C]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'} \quad (5.3.82)$$
$$-\xi_e^{E;ee'}[\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-\overline{\xi}_{e'}^{E';ee'}[\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\Xi_{ABCA';EE'}$$

Using equation 4.4.76, equation 5.3.82 becomes:

$$\nabla^{ee'}\nabla_{ee'}\Xi_{\mu\nu'} = [(\nabla\Xi)_{\mu\nu';ee'}]_{;gg'}\epsilon^{ge}\epsilon^{g'e'}$$
$$-\mu\Gamma_{1tgg'}\epsilon^{kt}\xi_k^A\epsilon^{ge}\epsilon^{g'e'}[\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$+(3-\mu)\Gamma_{0tgg'}\epsilon^{kt}\xi_k^A\epsilon^{ge}\epsilon^{g'e'}[\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-\nu'\overline{\Gamma}_{1't'g'g}\epsilon^{k't'}\overline{\xi}_{k'}^{A'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'} \quad (5.3.83)$$
$$+(1-\nu')\overline{\Gamma}_{0't'g'g}\epsilon^{k't'}\overline{\xi}_{k'}^{A'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C]\xi_e^E\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-\Gamma_{etgg'}\epsilon^{kt}\xi_k^E\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\overline{\xi}_{e'}^{E'}\Xi_{ABCA';EE'}$$
$$-\overline{\Gamma}_{e't'g'g}\epsilon^{k't'}\overline{\xi}_{k'}^{E'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\overline{\xi}_{a'}^{A'}]\xi_e^E\Xi_{ABCA';EE'}$$

So that we arrive at the following expression:

$$\nabla^{ee'}\nabla_{ee'}\Xi_{\mu\nu'} = \{[(\nabla\Xi)_{\mu\nu';ee'}]_{;gg'} - \mu\Gamma_{11gg'}\Xi_{(\mu-1)\nu';ee'}$$
$$+(2\mu-3)\Gamma_{10gg'}\Xi_{\mu\nu';ee'} + (3-\mu)\Gamma_{00gg'}\Xi_{(\mu+1)\nu';ee'}$$

$$-\nu'\overline{\Gamma}_{1'1'g'g}\Xi_{\mu(\nu-1)';ee'} + (2\nu'-1)\overline{\Gamma}_{1'0'g'g}\Xi_{\mu\nu';ee'}$$
$$+(1-\nu')\overline{\Gamma}_{0'0'g'g}\Xi_{\mu(\nu+1)';ee'} - \Gamma_{e1gg'}\Xi_{\mu\nu';0e'} \tag{5.3.84}$$
$$+\Gamma_{e0gg'}\Xi_{\mu\nu';1e'} - \overline{\Gamma}_{e'1'g'g}\Xi_{\mu\nu';e0'}$$
$$+\overline{\Gamma}_{e'0'g'g}\Xi_{\mu\nu';e1'}\}\epsilon^{ge}\epsilon^{g'e'}$$

The calculation of the d'Alembertian of higher derivatives of Ξ can be seen as an extension of the one performed above, so that it is unnecessary to go through all the details since these can be reproduced very easily. We then have the following identities:

$$\nabla^{ee'}\nabla_{ee'}\nabla_{ff'}\Xi_{\mu\nu'} = \{(\nabla^3\Xi)_{\mu\nu';ff';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.85}$$

$$\nabla^{ee'}\nabla_{ee'}\nabla_{hh'}\nabla_{ff'}\Xi_{\mu\nu'} = \{(\nabla^4\Xi)_{\mu\nu';ff';hh';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.86}$$

We now look at how to obtain the d'Alembertian applied twice to derivatives of Ψ, Φ, Λ and Ξ. Lets take for example $\Box\Box\Psi$, we write as usual:

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_{abcd} = \Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}{}^{;EE'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\overline{\xi}^{F'}_{f'}\xi^f_F\overline{\xi}^{f'}_{F'}$$
$$\xi^E_e\overline{\xi}^{E'}_{e'}\xi^e_E\overline{\xi}^{e'}_{E'} = (\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\overline{\xi}^{F'}_{f'}\xi^f_F\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'})^{;EE'} \tag{5.3.87}$$
$$\xi^e_E\overline{\xi}^{e'}_{E'} - ([\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\overline{\xi}^{F'}_{f'}\xi^f_F\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'})^{;EE'}\xi^e_E\overline{\xi}^{e'}_{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$

Working out the derivatives in round brackets in equation 5.3.87 we get:

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_{\mu} = [(\nabla^2\Psi)_{\mu;ff'}{}^{;ff'}{}_{;ee'}]^{;ee'}$$
$$-\mu\xi_1^{A;ee'}[\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\overline{\xi}^{F'}_{f'}\xi^f_F\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$
$$-(4-\mu)\xi_0^{A;ee'}[\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\overline{\xi}^{F'}_{f'}\xi^f_F\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$
$$-\xi_f^{F;ee'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\overline{\xi}^{F'}_{f'}\xi^f_F\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$
$$-\overline{\xi}_{f'}^{F';ee'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\xi^f_F\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \tag{5.3.88}$$
$$-\xi_F^{f;ee'}[\xi^A_a\xi^B_b\xi^C_c\xi^D_d]\xi^F_f\overline{\xi}^{F'}_{f'}\overline{\xi}^{f'}_{F'}\xi^E_e\overline{\xi}^{E'}_{e'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$

$$-\overline{\xi}_{F'}^{f';ee'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$

$$-\xi_e^{E;ee'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$

$$\overline{\xi}_{e'}^{E';ee'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\xi_e^E\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}$$

which can be written as follows:

$$\begin{aligned}
\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_\mu &= [(\nabla^3\Psi)_{\mu;ff';hh';ee'}\;]_{gg'}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \\
&-\mu\xi^A_{1\;;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \\
&-(4-\mu)\xi^A_{0\;;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \\
&-\xi^F_{f\;;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \\
&-\overline{\xi}^{F'}_{f'\;;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\xi_F^f\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \qquad (5.3.89) \\
&-\xi^H_{h;gg'}\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon_{HF}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \\
&-\overline{\xi}^{H'}_{h';gg'}\epsilon^{ge}\epsilon^{g'e'}\epsilon^{h'f'}\epsilon_{H'F'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \\
&-\xi^E{}_{e;gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'} \\
&-\overline{\xi}^{E'}{}_{e';gg'}\epsilon^{ge}\epsilon^{g'e'}[\xi_a^A\xi_b^B\xi_c^C\xi_d^D]\xi_f^F\overline{\xi}_{f'}^{F'}\xi_F^f\overline{\xi}_{F'}^{f'}\xi_e^E\Psi_{ABCD,FF'}{}^{;FF'}{}_{;EE'}
\end{aligned}$$

By using 4.4.76, equation 5.3.89 becomes:

$$\begin{aligned}
\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_\mu &= [(\nabla^3\Psi)_{\mu;ff';hh';ee'}\;]_{gg'}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \\
&-\mu\Gamma_{11gg'}(\nabla^3\Psi)_{(\mu-1);ff'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'}+(2\mu-4)\Gamma_{10gg'}(\nabla^3\Psi)_{\mu;ff'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'} \\
&+(4-\mu)\Gamma_{00gg'}(\nabla^3\Psi)_{(\mu+1);ff'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'}-\Gamma_{f1gg'}(\nabla^3\Psi)_{\mu;0f'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'} \\
&+\Gamma_{f0gg'}(\nabla^3\Psi)_{\mu;1f'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'}-\overline{\Gamma}_{f'1'g'g}(\nabla^3\Psi)_{\mu;f0'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'} \\
&\overline{\Gamma}_{f'0'g'g}(\nabla^3\Psi)_{\mu;f1'}{}^{;ff'}{}_{;ee'}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.3.90) \\
&-\Gamma_{htgg'}\epsilon^{kt}\xi_k^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon_{HF}\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;fF'}{}_{;EE'} \\
&-\overline{\Gamma}_{h't'g'g}\epsilon^{k't'}\xi_k^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon_{HF}\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\Psi_{ABCD,FF'}{}^{;fF'}{}_{;EE'}
\end{aligned}$$

$$-\Gamma_{e1gg'}\,(\nabla^3\Psi)_{\mu;ff'}{}^{;ff'}{}_{;0e'}\,\epsilon^{ge}\epsilon^{g'e'} + \Gamma_{e0gg'}\,(\nabla^3\Psi)_{\mu;ff'}{}^{;ff'}{}_{;1e'}\,\epsilon^{ge}\epsilon^{g'e'}$$

$$-\overline{\Gamma}_{e'1'g'g}(\nabla^3\Psi)_{\mu;ff'}{}^{;ff'}{}_{;e0'}\,\epsilon^{ge}\epsilon^{g'e'} + \overline{\Gamma}_{e'0'g'g}(\nabla^3\Psi)_{\mu;ff'}{}^{;ff'}{}_{;e1'}\,\epsilon^{ge}\epsilon^{g'e'}$$

We consider the following equalities:

$$\Gamma_{htgg'}\,\epsilon^{kt}\xi_k^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\xi_a^A\xi_b^B\xi_c^C\xi_d^D\xi_f^F\overline{\xi}_{f'}^{F'}\overline{\xi}_{F'}^{f'}\xi_e^E\overline{\xi}_{e'}^{E'}\,\Psi_{ABCD,FF';H}{}^{F'}{}_{;EE'}$$

$$= -\Gamma_{h1gg'}\,\xi_0^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\,\Psi_{\mu;ff';H}{}^{f'}{}_{;ee'} + \Gamma_{h0gg'}\,\xi_1^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\,\Psi_{\mu;ff';H}{}^{f'}{}_{;ee'}$$

$$= -\Gamma_{h1gg'}\,\xi_0^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon^{h'f'}\,\Psi_{\mu;ff';HH';ee'} \tag{5.3.91}$$

$$+\Gamma_{h0gg'}\,\xi_1^H\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon^{h'f'}\,\Psi_{\mu;ff';HH';ee'}$$

$$= -\Gamma_{h1gg'}\,\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon^{h'f'}\,\Psi_{\mu;ff';0h';ee'} + \Gamma_{h0gg'}\,\epsilon^{ge}\epsilon^{g'e'}\epsilon^{hf}\epsilon^{h'f'}\,\Psi_{\mu;ff';1h';ee'}$$

If we substitute 5.3.91 in 5.3.90 we arrive at:

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_\mu = [(\nabla^3\Psi)_{\mu;ff';hh';ee'}\;]_{;gg'}\,\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\mu\Gamma_{11gg'}\,(\nabla^3\Psi)_{(\mu-1);ff';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$+(2\mu-4)\Gamma_{10gg'}\,(\nabla^3\Psi)_{\mu;ff';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$+(4-\mu)\Gamma_{00gg'}\,(\nabla^3\Psi)_{(\mu+1);ff';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\Gamma_{f1gg'}\,(\nabla^3\Psi)_{\mu;0f';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$+\Gamma_{f0gg'}\,(\nabla^3\Psi)_{\mu;1f';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\overline{\Gamma}_{f'1'g'g}(\nabla^3\Psi)_{\mu;f0';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$+\overline{\Gamma}_{f'0'g'g}(\nabla^3\Psi)_{\mu;f1';hh';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\Gamma_{h1gg'}\,(\nabla^3\Psi)_{\mu;ff';0h';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.92}$$

$$+\Gamma_{hogg'}\,(\nabla^3\Psi)_{\mu;ff';1h';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\overline{\Gamma}_{h'1'g'g}(\nabla^3\Psi)_{\mu;ff';h0';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$\overline{\Gamma}_{h'0'g'g}(\nabla^3\Psi)_{\mu;ff';h1';ee'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\Gamma_{e1gg'}\,(\nabla^3\Psi)_{\mu;ff';hh';0e'}\;\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$+\Gamma_{e0gg'}(\nabla^3\Psi)_{\mu;ff';hh';1e'}\ \epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$-\overline{\Gamma}_{e'1'g'g}(\nabla^3\Psi)_{\mu;ff';hh';e0'}\ \epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

$$+\overline{\Gamma}_{e'0'g'g}(\nabla^3\Psi)_{\mu;ff';hh';e1'}\ \epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'}$$

So that one arrives at:

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_\mu = \{(\nabla^4\Psi)_{\mu;ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.93}$$

To calculate the d'Alembertian applied twice to higher derivatives of Ψ we follow the same method as before, we then obtain:

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\nabla_{kk'}\Psi_\mu = \{(\nabla^5\Psi)_{\mu;kk';ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.94}$$

As for the calculation leading to the d'Alembertian applied twice to Φ, Λ, Ξ and higher derivatives of these, one proceeds in the same way as in the case of Ψ so that we have:

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Phi_{\mu\nu'} = \{(\nabla^4\Phi)_{\mu\nu';ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.95}$$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\nabla_{kk'}\Phi_{\mu\nu'} = \{(\nabla^5\Phi)_{\mu\nu';kk';ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.96}$$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Lambda = \{(\nabla^4\Lambda)_{;ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.97}$$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\nabla_{kk'}\Lambda = \{(\nabla^5\Lambda)_{;kk';ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.98}$$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Xi_{\mu\nu'} = \{(\nabla^4\Xi)_{\mu\nu';ff';hh';ee';gg'}\ \}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.99}$$

We can then write all expressions giving higher derivatives in generalized formalism. Below we write the dyad component expressions followed by the generalized version.

Third Covariant Derivative of $\Phi_{22'}$

$$(\nabla^3\Phi)_{\mu\nu';gg'} = [(\nabla^2\Phi)_{\mu\nu'}]_{;gg'} - \mu\Gamma_{11gg'}(\nabla^2\Phi)_{(\mu-1)\nu'}$$

$$+(2\mu-4)\Gamma_{10gg'}(\nabla^2\Phi)_{\mu\nu'} + (4-\mu)\Gamma_{00gg'}(\nabla^2\Phi)_{(\mu+1)\nu'}$$

$$-\nu\overline{\Gamma}_{1'1'g'g}(\nabla^2\Phi)_{\mu(\nu-1)'} + (2\nu-4)\overline{\Gamma}_{1'0'g'g}(\nabla^2\Phi)_{\mu\nu'} \tag{5.3.100}$$

$$+(4-\nu)\overline{\Gamma}_{0'0'g'g}(\nabla^2\Phi)_{\mu(\nu+1)'}$$

with $\mu \in \{0,1,2,3,4\}$ and $\nu' \in \{0,1,2,3,4\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{GG'}} = \sum(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'};\mathbf{GG'}}$$
$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{GG'}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(2N-3)\Gamma_{0\mathbf{A}_1\mathbf{GG'}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{GG'}}\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(4-N)\Gamma_{00\mathbf{GG'}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\overline{o}_{\mathbf{A'}_{N'+1'}} \tag{5.3.101}$$
$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{G'G}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}o_{\mathbf{A}_{N+1}}$$
$$+(2N'-4)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{G'G}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}o_{\mathbf{A}_{N+1}}$$
$$+(4-N')\overline{\Gamma}_{0'0'\mathbf{G'G}}(\nabla^2\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}o_{\mathbf{A}_{N+1}}$$

with $N \in \{0,1,2,34\}$ and $N' \in \{0,1,2,3,4\}$.

Third Covariant Derivative of Λ

$$(\nabla^3\Lambda)_{\mu\nu';gg'} = [(\nabla^3\Lambda)_{\mu\nu'}]_{;gg'} - \mu\Gamma_{11gg'}(\nabla\Lambda)_{(\mu-1)\nu'}$$
$$+(2\mu-2)\Gamma_{01gg'}(\nabla\Lambda)_{\mu\nu'} + (2-\mu)\Gamma_{00gg'}(\nabla^2\Lambda)_{(\mu+2)\nu'}$$
$$-\nu'\overline{\Gamma}_{1'1'g'g}(\nabla^2\Lambda)_{\mu(\nu-1)} + (2\nu'-2)\overline{\Gamma}_{0'1'g'g}(\nabla\Lambda)_{\mu\nu'} \tag{5.3.102}$$
$$+(2-\nu')\overline{\Gamma}_{0'0'g'g}(\nabla^2\Lambda)_{\mu(\nu+1)}$$

with $\mu \in \{0,1,2\}$ and $\nu'\{0,1,2\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{GG'}} = \sum[(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}]_{;\mathbf{GG'}}$$
$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{GG'}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(2N-2)\Gamma_{0\mathbf{A}_1\mathbf{GG'}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{GG'}}\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(2-N)\Gamma_{00\mathbf{GG'}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\overline{o}_{\mathbf{A'}_{N'+1'}} \tag{5.3.103}$$
$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{G'G}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}o_{\mathbf{A}_{N+1}}$$

$$+(2N'-2)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{G}'\mathbf{G}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{2'}\ldots\mathbf{A}'_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$

$$+(2-N')\overline{\Gamma}_{0'0'\mathbf{G}'\mathbf{G}}(\nabla^2\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1}}\,o_{\mathbf{A}_{N+1}}$$

with $N\in\{0,1,2\}$ and $N'\in\{0,1,2\}$.

Second Covariant Derivative of Ξ

$$(\nabla^2\Xi)_{\mu\nu';ff'}=[(\nabla\Xi)_{\mu\nu'}]_{;ff'}-\mu\Gamma_{11ff'}(\nabla\Phi)_{(\mu-1)\nu'}$$
$$+(2\mu-4)\Gamma_{10ff'}(D\Xi)_{\mu\nu'}+(4-\mu)\Gamma_{00ff'}(\nabla\Xi)_{(\mu+1)\nu'}$$
$$-\nu\overline{\Gamma}_{1'1'f'f}(\nabla\Xi)_{\mu(\nu'-1)}+(2\nu'-2)\overline{\Gamma}_{1'0'f'f}\Xi_{\mu\nu'} \tag{5.3.104}$$
$$+(2-\nu')\overline{\Gamma}_{0'0'f'f}(\nabla\Xi)_{\mu(\nu'+1)}$$

with $\mu\in\{0,1,2,3,4\}$ and $\nu'\in\{0,1,2\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^2\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{F}\mathbf{F}'}=\sum(\nabla\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'};\mathbf{F}\mathbf{F}'}$$
$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}-N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{F}\mathbf{F}'}(\nabla\mathbf{\Xi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A}'_{N'+1'}}$$
$$+(2N-4)\Gamma_{0\mathbf{A}_1\mathbf{F}\mathbf{F}'}(\nabla\mathbf{\Xi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{F}\mathbf{F}'}\,\overline{o}_{\mathbf{A}'_{N'+1'}}$$
$$+(4-N)\Gamma_{00\mathbf{F}\mathbf{F}'}(\nabla\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A}'_{N'+1'}} \tag{5.3.105}$$
$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{F}'\mathbf{F}}(\nabla\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{3'}\ldots\mathbf{A}'_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$
$$+(2N'-2)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{F}'\mathbf{F}}(\nabla\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{2'}\ldots\mathbf{A}'_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$
$$+(2-N')\overline{\Gamma}_{0'0'\mathbf{F}'\mathbf{F}}(\nabla\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1}}\,o_{\mathbf{A}_{N+1}}$$

with $N\in\{0,1,2,3,4\}$ and $N'\in\{0,1,2\}$.

d'Alembertian of $\nabla\mathbf{\Psi}_4$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Psi_\mu)=\{(\nabla^3\Psi)_{\mu;ff';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.106}$$

with $\mu\in\{0,1,2,3,4\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'}$$

$$= \left(o_{\mathbf{A}_{N+1}} \overline{o}_{\mathbf{A}'_{N'+1'}} (\nabla^3 \mathbf{\Psi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}' \mathbf{G}\mathbf{G}'} \right) \epsilon^{\mathbf{GE}} \epsilon^{\mathbf{G}'\mathbf{E}'} \tag{5.3.107}$$

$$= (\nabla^3 \mathbf{\Psi})_{\mathbf{A}_1 \ldots \mathbf{A}_{N+1} \mathbf{A}'_{1'} \mathbf{A}'_{N'+1'}}$$

with $N \in \{0, 1, 2, 4, 5\}$ and $N'\{0, 1\}$.

d'Alembertian of $\nabla \mathbf{\Phi}_{22'}$

$$\nabla^{ee'} \nabla_{ee'} (\nabla_{ff'} \Phi_{\mu\nu'}) = \{(\nabla^3 \Phi)_{\mu\nu';ff';ee';gg'}\} \epsilon^{ge} \epsilon^{g'e'} \tag{5.3.108}$$

with $\mu \in \{0, 1, 2\}$ and $\nu' \in \{0, 1, 2\}$.

$$o_{\mathbf{A}_{N+1}} \overline{o}_{\mathbf{A}'_{N'+1'}} (\nabla^3 \mathbf{\Phi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'}$$

$$= \left(o_{\mathbf{A}_{N+1}} \overline{o}_{\mathbf{A}'_{N'+1'}} (\nabla^3 \mathbf{\Phi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}' \mathbf{G}\mathbf{G}'} \right) \epsilon^{\mathbf{GE}} \epsilon^{\mathbf{G}'\mathbf{E}'} \tag{5.3.109}$$

$$= (\nabla^3 \mathbf{\Phi})_{\mathbf{A}_1 \ldots \mathbf{A}_{N+1} \mathbf{A}'_{1'} \mathbf{A}'_{N'+1'}}$$

with $N \in \{0, 1, 2, 3\}$ and $N' \in \{0, 1, 2, 3\}$.

d'Alembertian of $\nabla \mathbf{\Lambda}$

$$\nabla^{ee'} \nabla_{ee'} (\nabla_{ff'} \Lambda) = \{(\nabla^3 \Lambda)_{;ff';ee';gg'}\} \epsilon^{ge} \epsilon^{g'e'} \tag{5.3.110}$$

$$o_{\mathbf{A}_{N+1}} \overline{o}_{\mathbf{A}'_{N'+1'}} (\nabla^3 \mathbf{\Lambda})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'}$$

$$= \left(o_{\mathbf{A}_{N+1}} \overline{o}_{\mathbf{A}'_{N'+1'}} (\nabla^3 \mathbf{\Lambda})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}' \mathbf{G}\mathbf{G}'} \right) \epsilon^{\mathbf{GE}} \epsilon^{\mathbf{G}'\mathbf{E}'} \tag{5.3.111}$$

$$= (\nabla^3 \mathbf{\Lambda})_{\mathbf{A}_1 \ldots \mathbf{A}_{N+1} \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'+1'}}$$

with $N \in \{0, 1\}$ and $N' \in \{0, 1\}$.

d'Alembertian of $\mathbf{\Xi}$

$$\nabla^{ee'} \nabla_{ee'} (\Xi_{\mu\nu'}) = \{(\nabla^2 \Phi)_{\mu\nu';ee';gg'}\} \epsilon^{ge} \epsilon^{g'e'} \tag{5.3.112}$$

with $\mu \in \{0, 1, 2, 3\}$ and $\nu' \in \{0, 1\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^2\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'}$$

$$= \left(o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^2\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'}\right)\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \tag{5.3.113}$$

$$= (\nabla^2\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}}$$

with $N \in \{0,1,2,3\}$ and $N' \in \{0,1\}$.

Fourth Covariant Derivative of $\mathbf{\Phi}_{22'}$

$$(\nabla^4\Phi)_{\mu\nu';hh'} = [(\nabla^3\Phi)_{\mu\nu'}]_{;hh'} - \mu\Gamma_{11hh'}(\nabla^3\Phi)_{(\mu-1)\nu'}$$

$$+(2\mu-5)\Gamma_{10hh'}(\nabla^3\Phi)_{\mu\nu'} + (5-\mu)\Gamma_{00hh'}(D^3\Phi)_{(\mu+1)\nu'}$$

$$-\nu\overline{\Gamma}_{1'1'h'h}(\nabla^3\Phi)_{\mu(\nu-1)} + (2\nu-5)\overline{\Gamma}_{1'0'h'h}(\nabla^3\Phi)_{\mu\nu'} \tag{5.3.114}$$

$$+(5-\nu)\overline{\Gamma}_{0'0'h'h}(\nabla^3\Phi)_{\mu(\nu'+1)}$$

with $\mu \in \{0,1,2,3,4,5\}$ and $\nu' \in \{0,1,2,3,4,5\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{HH}'} = \sum[(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}]_{;\mathbf{HH}'}$$

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{GG}'}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\overline{o}_{\mathbf{A}'_{N'+1'}}$$

$$+(2N-5)\Gamma_{0\mathbf{A}_1\mathbf{HH}'}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{HH}'}\overline{o}_{\mathbf{A}'_{N'+1'}}$$

$$+(5-N)\Gamma_{00\mathbf{HH}'}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\overline{o}_{\mathbf{A}'_{N'+1'}} \tag{5.3.115}$$

$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{H}'\mathbf{H}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{3'}\ldots\mathbf{A}'_{N'+1'}}o_{\mathbf{A}_{N+1}}$$

$$+(2N'-5)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{H}'\mathbf{H}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{2'}\ldots\mathbf{A}'_{N'+1'}}o_{\mathbf{A}_{N+1}}$$

$$+(5-N')\overline{\Gamma}_{0'0'\mathbf{H}'\mathbf{H}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1}}o_{\mathbf{A}_{N+1}}$$

with $N \in \{0,1,2,3,4,5\}$ and $\nu' \in \{0,1,2,3,4,5\}$.

Fourth Covariant Derivative of $\mathbf{\Lambda}$

$$(\nabla^4\Lambda)_{\mu\nu';hh'} = [(\nabla^3\Lambda)_{\mu\nu'}]_{;hh'} - \mu\Gamma_{11hh'}(\nabla^3\Phi)_{(\mu-1)\nu'}$$

$$+(2\mu-3)\Gamma_{10hh'}(\nabla^3\Phi)_{\mu\nu'} + (3-\mu)\Gamma_{00hh'}(\nabla^3\Lambda)_{(\mu+1)\nu'}$$

$$-\nu\overline{\Gamma}_{1'1'h'h}(\nabla^3\Lambda)_{\mu(\nu'-1)} + (2\nu-3)\overline{\Gamma}_{1'0'h'h}(\nabla^3\Lambda)_{\mu\nu'} \tag{5.3.116}$$

$$+(3-\nu)\overline{\Gamma}_{0'0'h'h}(\nabla^3\Lambda)_{\mu(\nu'+1)}$$

with $\mu \in \{0,1,2,3\}$ and $\nu' \in \{0,1,2,3\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{HH'}} = \sum[(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}]_{;\mathbf{HH'}}$$

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{GG'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$

$$+(2N-3)\Gamma_{0\mathbf{A}_1\mathbf{HH'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{HH'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$

$$+(3-N)\Gamma_{00\mathbf{HH'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \tag{5.3.117}$$

$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{H'H}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$

$$+(2N'-3)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{H'H}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$

$$+(3-N')\overline{\Gamma}_{0'0'\mathbf{H'H}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}\,o_{\mathbf{A}_{N+1}}$$

with $N \in \{0,1,2,3\}$ and $N' \in \{0,1,2,3\}$.

Third Covariant Derivative of Ξ

$$(\nabla^3\Xi)_{\mu\nu';gg'} = [(\nabla^2\Xi)_{\mu\nu'}]_{;gg'} - \mu\Gamma_{11gg'}(\nabla^2\Phi)_{(\mu-1)\nu'}$$

$$+(2\mu-5)\Gamma_{10gg'}(\nabla^2\Xi)_{\mu\nu'} + (5-\mu)\Gamma_{00gg'}(\nabla^2\Xi)_{(\mu+1)\nu'}$$

$$-\nu\overline{\Gamma}_{1'1'g'g}(\nabla^2\Xi)_{\mu(\nu'-1)} + (2\nu'-3)\overline{\Gamma}_{1'0'g'g}(\nabla^2\Xi)_{\mu\nu'} \tag{5.3.118}$$

$$+(3-\nu')\overline{\Gamma}_{0'0'g'g}(\nabla^2\Xi)_{\mu(\nu'+1}$$

with $\mu \in \{0,1,2,3,4,5\}$ and $\nu' \in \{0,1,2,3\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{GG'}} = \sum[(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}]_{;\mathbf{GG'}}$$

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{GG'}}(\nabla^2\mathbf{\Xi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$

$$+(2N-5)\Gamma_{0\mathbf{A}_1\mathbf{GG'}}(\nabla^2\mathbf{\Xi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{GG'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$

$$+(5-N)\Gamma_{00\mathbf{GG'}}(\nabla^2\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \tag{5.3.119}$$

$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{G}'\mathbf{G}}(\nabla^2\Xi)_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{3'}\ldots\mathbf{A}'_{N'+1'}}\ o_{\mathbf{A}_{N+1}}$$

$$+(2N'-3)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{G}'\mathbf{G}}(\nabla^2\Xi)_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{2'}\ldots\mathbf{A}'_{N'+1'}}\ o_{\mathbf{A}_{N+1}}$$

$$+(3-N')\overline{\Gamma}_{0'0'\mathbf{G}'\mathbf{G}}(\nabla^2\Xi)_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1}}\ o_{\mathbf{A}_{N+1}}$$

with $N \in \{0,1,2,3,4,5\}$ and $N' \in \{0,1,2,3\}$.

d'Alembertian of $\nabla^2\Psi_4$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Psi_\mu) = (\nabla^4\Psi)_{\mu;ff';hh';ee';gg'}\ \ \epsilon^{ge}\epsilon^{g'e'} \tag{5.3.120}$$

with $\mu \in \{0,1,2,3,4\}$.

$$\begin{aligned} & o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'\mathbf{G}\mathbf{G}'})\epsilon^{\mathbf{G}\mathbf{E}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^4\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned} \tag{5.3.121}$$

with $N \in \{0,1,2,3,4,5,6\}$ and $N' \in \{0,1,2\}$.

d'Alembertian of $\nabla^2\mathbf{\Phi}_{22'}$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Phi_{\mu\nu'}) = (\nabla^4\Phi)_{\mu\nu';ff';hh';ee';gg'}\ \ \epsilon^{ge}\epsilon^{g'e'} \tag{5.3.122}$$

with $\mu \in \{0,1,2\}$ and $\nu'\{0,1,2\}$.

$$\begin{aligned} & o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{E}\mathbf{E}'\mathbf{G}\mathbf{G}'})\epsilon^{\mathbf{G}\mathbf{E}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^4\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned} \tag{5.3.123}$$

with $N \in \{0,1,2,3,4\}$ and $N' \in \{0,1,2,3,4\}$.

d'Alembertian of $\nabla^2\Lambda$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Lambda) = \{(\nabla^4\Lambda)_{;ff';hh';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.124}$$

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A}'_{1'}...\mathbf{A}'_{N'+1'}} \end{aligned} \tag{5.3.125}$$

with $N \in \{0,1,2\}$ and $N' \in \{0,1,2\}$.

d'Alembertian of $\nabla\Xi$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{ff'}\Xi_{\mu\nu'} = (\nabla^3\Xi)_{\mu\nu';ff';ee';gg'}\ \epsilon^{ge}\epsilon^{g'e'} \tag{5.3.126}$$

with $\mu \in \{0,1,2,3\}$ and $\nu' \in \{0,1\}$.

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^3\mathbf{\Xi})_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A}'_{1'}...\mathbf{A}'_{N'+1'}} \end{aligned} \tag{5.3.127}$$

with $N \in \{0,1,2,3,4\}$ and $N' \in \{0,1,2\}$.

d'Alembertian of $\Box\Psi_4$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Psi_\mu = \{(\nabla^4\Psi_\mu)_{\mu;ff';hh';ee';gg'}\}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \tag{5.3.128}$$

with $\mu \in \{0,1,2,3,4\}$.

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Psi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\ &= (\nabla^4\mathbf{\Psi})_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A}'_{1'}...\mathbf{A}'_{N'+1'}} \end{aligned} \tag{5.3.129}$$

with $N \in \{0,1,2,3,4,5\}$ and $N' \in \{0,1\}$.

d'Alembertian of $\Box\Phi_{22'}$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Phi_{\mu\nu'} = \{(\nabla^4\Phi)_{\mu\nu';ff';hh';ee';gg'}\}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \quad (5.3.130)$$

$\mu \in \{0,1,2\}$ and $\nu' \in \{0,1,2\}$.

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \quad (5.3.131) \\ &= (\nabla^4\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned}$$

with $N \in \{0,1,2,3\}$ and $N' \in \{0,1,2,3\}$.

d'Alembertian of $\Box\mathbf{\Lambda}$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Lambda = \{(\nabla^4\Lambda)_{;ff';hh';ee';gg'}\}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \quad (5.3.132)$$

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \quad (5.3.133) \\ &= (\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned}$$

with $N \in \{0,1\}$ and $N' \in \{0,1\}$.

Fifth Covariant Derivative of $\mathbf{\Phi}_{22'}$

$$\begin{aligned} &(\nabla^5\Phi)_{\mu\nu';ii'} = [(\nabla^4\Phi)_{\mu\nu'}]_{;ii'} - \mu\Gamma_{11ii'}(\nabla^4\Phi)_{(\mu-1)\nu'} \\ &+(2\mu-6)\Gamma_{10ii'}(\nabla^4\Phi)_{\mu\nu'} + (6-\mu)\Gamma_{00ii'}(\nabla^4\Phi)_{(\mu+1)\nu'} \\ &-\nu\overline{\Gamma}_{1'1'i'i}(\nabla^4\Phi)_{\mu(\nu-1)} + (2\nu-6)\overline{\Gamma}_{1'0'i'i}(\nabla^4\Phi)_{\mu\nu'} \quad (5.3.134) \\ &(6-\nu)\overline{\Gamma}_{0'0'i'i}(\nabla^4\Phi)_{\mu(\nu+1)} \end{aligned}$$

with $\mu \in \{0,1,2,3,4,5,6\}$ and $\nu' \in \{0,1,2,3,4,5,6\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^5\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{I}\mathbf{I'}} = \sum\left[(\nabla^5\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\right]_{;\mathbf{I}\mathbf{I'}}$$
$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{I}\mathbf{I'}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(2N-6)\Gamma_{0A_1\mathbf{I}\mathbf{I'}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{I}\mathbf{I'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(6-N)\Gamma_{00\mathbf{I}\mathbf{I'}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \quad (5.3.135)$$
$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{I'}\mathbf{I}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$
$$+(2N'-6)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{I'}\mathbf{I}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$
$$+(6-N')\overline{\Gamma}_{0'0'\mathbf{I'}\mathbf{I}}(\nabla^4\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}\,o_{\mathbf{A}_{N+1}}$$

with $N \in \{0,1,2,3,4,5,6\}$ and $N' \in \{0,1,2,3,4,5,6\}$.

Fifth Covariant Derivative of Λ

$$(\nabla^5\Lambda)_{\mu\nu';ii'} = [(\nabla^4\Lambda)_{\mu\nu'}]_{;ii'} - \mu\Gamma_{11ii'}(\nabla^4\Lambda)_{(\mu-1)\nu'}$$
$$+(2\mu-4)\Gamma_{10ii'}(\nabla^4\Lambda)_{\mu\nu'} + (4-\mu)\Gamma_{00ii'}(D^4\Lambda)_{(\mu+1)\nu'}$$
$$-\nu\overline{\Gamma}_{1'1'i'i}(\nabla^4\Lambda)_{\mu(\nu'-1)} + (2\nu-4)\overline{\Gamma}_{1'0'i'i}(\nabla^4\Lambda)_{\mu\nu'} \quad (5.3.136)$$
$$+(4-\nu)\overline{\Gamma}_{0'0'i'i}(\nabla^4\Lambda)_{\mu(\nu'+1)}$$

with $\mu \in \{0,1,2,3,4\}$ and $\nu' \in \{0,1,2,3,4\}$.

$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}}(\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{I}\mathbf{I'}} = \sum(\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'};\mathbf{I}\mathbf{I'}}$$
$$o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A'}_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{I}\mathbf{I'}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(2N-4)\Gamma_{0A_1\mathbf{I}\mathbf{I'}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}\mathbf{I}\mathbf{I'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}}$$
$$+(4-N)\Gamma_{00\mathbf{I}\mathbf{I'}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'}}\,\overline{o}_{\mathbf{A'}_{N'+1'}} \quad (5.3.137)$$
$$-N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{I'}\mathbf{I}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{3'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$
$$+(2N'-4)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{I'}\mathbf{I}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{2'}\ldots\mathbf{A'}_{N'+1'}}\,o_{\mathbf{A}_{N+1}}$$
$$+(4-N')\overline{\Gamma}_{0'0'\mathbf{I'}\mathbf{I}}(\nabla^4\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A'}_{1'}\ldots\mathbf{A'}_{N'+1}}\,o_{\mathbf{A}_{N+1}}$$

with $N \in \{0,1,2,3,4\}$ and $N' \in \{0,1,2,3,4\}$.

Fourth Covariant Derivative of Ξ

$$
\begin{aligned}
(\nabla^4\Xi)_{\mu\nu';hh'} &= [(\nabla^3\Xi)_{\mu\nu'}]_{;hh'} - \mu\Gamma_{11hh'}(\nabla^3\Xi)_{(\mu-1)\nu'} \\
&+(2\mu-6)\Gamma_{10hh'}(\nabla^3\Xi)_{\mu\nu'} + (6-\mu)\Gamma_{00hh'}(\nabla^3\Xi)_{(\mu+1)\nu'} \\
&-\nu\overline{\Gamma}_{1'1'h'h}(\nabla^3\Xi)_{\mu(\nu-1)} + (2\nu'-6)\overline{\Gamma}_{1'0'h'h}(\nabla^3\Xi)_{\mu\nu'} \\
&+(4-\nu')\overline{\Gamma}_{0'0'h'h}(\nabla^3\Xi)_{\mu(\nu+1)}
\end{aligned}
\tag{5.3.138}
$$

with $\mu \in \{0,1,2,3,4,5,6\}$ and $\nu' \in \{0,1,2,3,4\}$.

$$
\begin{aligned}
& o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{HH}'} = \sum[(\nabla^4\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}]_{;\mathbf{HH}'} \\
& o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}} - N\Gamma_{\mathbf{A}_1\mathbf{A}_2\mathbf{GG}'}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_3\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\overline{o}_{\mathbf{A}'_{N'+1'}} \\
& +(2N-6)\Gamma_{0\mathbf{A}_1\mathbf{HH}'}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_2\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{HH}'}\overline{o}_{\mathbf{A}'_{N'+1'}} \\
& +(6-N)\Gamma_{00\mathbf{HH}'}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}}\overline{o}_{\mathbf{A}'_{N'+1'}} \\
& -N'\overline{\Gamma}_{\mathbf{A}'_{1'}\mathbf{A}'_{2'}\mathbf{H}'\mathbf{H}}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{3'}\ldots\mathbf{A}'_{N'+1'}}o_{\mathbf{A}_{N+1}} \\
& +(2N'-4)\overline{\Gamma}_{\mathbf{A}'_{1'}0'\mathbf{H}'\mathbf{H}}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{2'}\ldots\mathbf{A}'_{N'+1'}}o_{\mathbf{A}_{N+1}} \\
& +(4-N')\overline{\Gamma}_{0'0'\mathbf{H}'\mathbf{H}}(\nabla^3\mathbf{\Xi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1}}o_{\mathbf{A}_{N+1}}
\end{aligned}
\tag{5.3.139}
$$

with $N \in \{0,1,2,3,4,5,6\}$ and $N' \in \{0,1,2,3,4\}$.

d'Alembertian of $\nabla^3\mathbf{\Psi}_4$

$$
\nabla^{ee'}\nabla_{ee'}(\nabla_{kk'}\nabla_{hh'}\nabla_{ff'}\Psi_\mu) = \{(\nabla^5\Psi)_{\mu;ff';hh';kk';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'}
\tag{5.3.140}
$$

with $\mu \in \{0,1,2,3,4\}$.

$$
\begin{aligned}
& o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\
&= (o_{\mathbf{A}_{N+1}}\overline{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \\
&= (\nabla^5\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}}
\end{aligned}
\tag{5.3.141}
$$

with $\mu \in \{0,1,2,3,4,5,6,7\}$ and $\nu' \in \{0,1,2,3\}$.

d'Alembertian of $\nabla^3\Phi_{22'}$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{kk'}\nabla_{hh'}\nabla_{ff'}\Phi_{22'}) = \{(\nabla^5\Phi)_{\mu\nu';ff';hh';kk';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.3.142)$$

with $\mu \in \{0,1,2,\}$ and $\nu' \in \{0,1,2\}$.

$$o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Phi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{1'}...\mathbf{A'}_{N'}\mathbf{EE'}}{}^{\mathbf{EE'}}$$
$$= (o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^3\mathbf{\Phi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A'}_{N'}\mathbf{EE'GG'}})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G'E'}} \qquad (5.3.143)$$
$$= (\nabla^5\mathbf{\Phi})_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A}'_{1'}...\mathbf{A}'_{N'+1'}}$$

with $N \in \{0,1,2,3,4,5\}$ and $N' \in \{0,1,2,3,4,5\}$.

d'Alembertian of $\nabla^3\Lambda$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{kk'}\nabla_{hh'}\nabla_{ff'}\Lambda) = \{(\nabla^5\lambda)_{;ff';hh';kk';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.3.144)$$

$$o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{1'}...\mathbf{A'}_{N'}\mathbf{EE'}}{}^{\mathbf{EE'}}$$
$$= (o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A'}_{N'}\mathbf{EE'GG'}})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G'E'}} \qquad (5.3.145)$$
$$= (\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A}'_{1'}...\mathbf{A}'_{N'+1'}}$$

with $N \in \{0,1,2,3\}$ and $N' \in \{0,1,2,3\}$.

d'Alembertian of $\nabla^2\Xi$

$$\nabla^{ee'}\nabla_{ee'}(\nabla_{hh'}\nabla_{ff'}\Xi_{\mu\nu'}) = \{(\nabla^4\Xi)_{\mu\nu';ff';hh';ee';gg'}\ \}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.3.146)$$

with $\mu \in \{0,1,2,3\}$ and $\nu'\{0,1\}$.

$$o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Xi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A'}_{1'}...\mathbf{A'}_{N'}\mathbf{EE'}}{}^{\mathbf{EE'}}$$
$$= (o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^4\mathbf{\Xi})_{\mathbf{A}_1...\mathbf{A}_N\mathbf{A}'_{1'}...\mathbf{A'}_{N'}\mathbf{EE'GG'}})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G'E'}} \qquad (5.3.147)$$
$$= (\nabla^5\mathbf{\Xi})_{\mathbf{A}_1...\mathbf{A}_{N+1}\mathbf{A}'_{1'}...\mathbf{A}'_{N'+1'}}$$

with $N \in \{0,1,2,3,4,5\}$ and $N' \in \{0,1,2,3\}$.

d'Alembertian of $\Box\nabla\Psi_4$

$$\nabla^{ee'}\nabla_{ee'}(\nabla^{ff'}\nabla_{ff'}\nabla_{kk'}\Psi_\mu = \{(\nabla^5\Psi)_{\mu;kk';ff';hh';ee';gg'}\}\epsilon^{hf}\epsilon^{h'f'}\epsilon^{ge}\epsilon^{g'e'} \quad (5.3.148)$$

with $\mu \in \{0,1,2,3,4\}$.

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \qquad (5.3.149) \\ &= (\nabla^5\mathbf{\Psi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned}$$

with $N \in \{0,1,2,3,4,5,6\}$ and $N' \in \{0,1,2\}$.

d'Alembertian of $\Box\nabla\Phi_{22'}$

$$\nabla^{ee'}\nabla_{ee'}(\nabla^{ff'}\nabla_{ff'}\nabla_{kk'}\Phi_{\mu\nu'} = \{(\nabla^5\Phi)_{\mu\nu';kk';ff';hh';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.3.150)$$

with $\mu \in \{0,1,2\}$ and $\nu'\{0,1,2\}$.

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \qquad (5.3.151) \\ &= (\nabla^5\mathbf{\Phi})_{\mathbf{A}_1\ldots\mathbf{A}_{N+1}\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'+1'}} \end{aligned}$$

with $N \in \{0,1,2,3,4\}$ and $N' \in \{0,1,2,3,4\}$.

d'Alembertian of $\Box\nabla\Lambda$

$$\nabla^{ee'}\nabla_{ee'}\nabla^{ff'}\nabla_{ff'}\Lambda = \{(\nabla^5\Lambda)_{;kk';ff';hh';ee';gg'}\}\epsilon^{ge}\epsilon^{g'e'} \qquad (5.3.152)$$

$$\begin{aligned} &o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'}{}^{\mathbf{EE}'} \\ &= (o_{\mathbf{A}_{N+1}}\bar{o}_{\mathbf{A}'_{N'+1'}}(\nabla^5\mathbf{\Lambda})_{\mathbf{A}_1\ldots\mathbf{A}_N\mathbf{A}'_{1'}\ldots\mathbf{A}'_{N'}\mathbf{EE}'\mathbf{GG}'})\epsilon^{\mathbf{GE}}\epsilon^{\mathbf{G}'\mathbf{E}'} \qquad (5.3.153) \end{aligned}$$

$$= (\nabla^5 \mathbf{\Lambda})_{\mathbf{A}_1 \ldots \mathbf{A}_{N+1} \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'+1'}}$$

with $N \in \{0,1,2\}$ and $N' \in \{0,1,2\}$.

d'Alembertian of $\Box\Xi$

$$\nabla^{ee'} \nabla_{ee'} \nabla^{ff'} \nabla_{ff'} \Xi_{\mu\nu'} = \{(\nabla^4 \Xi)_{\mu\nu';ff';hh';ee';gg'} \quad \} \epsilon^{hf} \epsilon^{h'f'} \epsilon^{ge} \epsilon^{g'e'} \tag{5.3.154}$$

$$o_{\mathbf{A}_{N+1}} \bar{o}_{\mathbf{A}'_{N'+1'}} (\nabla^4 \mathbf{\Xi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}'}{}^{\mathbf{E}\mathbf{E}'}$$

$$= (o_{\mathbf{A}_{N+1}} \bar{o}_{\mathbf{A}'_{N'+1'}} (\nabla^4 \mathbf{\Xi})_{\mathbf{A}_1 \ldots \mathbf{A}_N \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'} \mathbf{E}\mathbf{E}'\mathbf{G}\mathbf{G}'}) \epsilon^{\mathbf{G}\mathbf{E}} \epsilon^{\mathbf{G}'\mathbf{E}'} \tag{5.3.155}$$

$$= (\nabla^4 \mathbf{\Xi})_{\mathbf{A}_1 \ldots \mathbf{A}_{N+1} \mathbf{A}'_{1'} \ldots \mathbf{A}'_{N'+1'}}$$

with $N \in \{0,1,2,3,4\}$ and $N' \in \{0,1,2\}$.

Thus we are able to write down all terms relating to the third, fourth and fifth derivative:

Third Derivative

$\nabla^3 \mathbf{\Psi} = \mathbf{Þ}'^3 \mathbf{\Psi}_4$ and all possible contractions with omicrons

$\nabla^3 \mathbf{\Phi} = \mathbf{Þ}'^3 \mathbf{\Phi}_{22'}$ and all possible contractions with omicrons

$\nabla^3 \mathbf{\Lambda} = \mathbf{Þ}'^3 \mathbf{\Lambda}$ and all possible contractions with omicrons

$\nabla^2 \mathbf{\Xi} = \mathbf{Þ}'^2 \mathbf{\Xi}$ and all possible contractions with omicrons

$\Box \nabla \mathbf{\Psi} = \nabla^3 \mathbf{\Psi}_4 \cdot o \cdot \bar{o}$ and all possible contractions with omicrons

$\Box \nabla \mathbf{\Phi} = \nabla^3 \mathbf{\Phi}_{22'} \cdot o \cdot \bar{o}$ and all possible contractions with omicrons

$\Box \nabla \mathbf{\Lambda} = \nabla^3 \mathbf{\Lambda} \cdot o \cdot \bar{o}$ and all possible contractions with omicrons

$\Box \mathbf{\Xi} = \nabla^2 \mathbf{\Xi} \cdot o \cdot \bar{o}$ and all possible contractions with omicrons

Fourth Derivative

$\nabla^4 \mathbf{\Psi} = \mathbf{Þ}'^4 \mathbf{\Psi}_4$ and all possible contractions with omicrons

$\nabla^4 \mathbf{\Phi} = \not{\mathbf{D}}'^4 \mathbf{\Phi}_{22'}$ and all possible contractions with omicrons

$\nabla^4 \mathbf{\Lambda} = \not{\mathbf{D}}'^4 \mathbf{\Lambda}$ and all possible contractions with omicrons

$\nabla^3 \mathbf{\Xi} = \not{\mathbf{D}}'^3 \mathbf{\Xi}$ and all possible contractions with omicrons

$\Box\nabla^2 \mathbf{\Psi} = \nabla^4 \mathbf{\Psi}_4 \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^2 \mathbf{\Phi} = \nabla^4 \mathbf{\Phi}_{22'} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^2 \mathbf{\Psi} = \nabla^4 \mathbf{\Psi}_4 \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^2 \mathbf{\Phi} = \nabla^4 \mathbf{\Phi}_{22'} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^2 \mathbf{\Lambda} = \nabla^4 \mathbf{\Lambda} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla \mathbf{\Xi} = \nabla^3 \mathbf{\Xi} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box \mathbf{\Psi} = \nabla^4 \mathbf{\Psi}_4 \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box \mathbf{\Phi} = \nabla^4 \mathbf{\Phi}_{22'} \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box \mathbf{\Lambda} = \nabla^4 \mathbf{\Lambda} \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

Fifth Derivative

$\nabla^5 \mathbf{\Psi} = \not{\mathbf{D}}'^5 \mathbf{\Psi}_4$ and all possible contractions with omicrons

$\nabla^5 \mathbf{\Phi} = \not{\mathbf{D}}'^5 \mathbf{\Phi}_{22'}$ and all possible contractions with omicrons

$\nabla^5 \mathbf{\Lambda} = \not{\mathbf{D}}'^5 \mathbf{\Lambda}$ and all possible contractions with omicrons

$\nabla^4 \mathbf{\Xi} = \not{\mathbf{D}}'^4 \mathbf{\Xi}$ and all possible contractions with omicrons

$\Box\nabla^3 \mathbf{\Psi} = \nabla^5 \mathbf{\Psi} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^3 \mathbf{\Phi} = \nabla^5 \mathbf{\Phi} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^3 \mathbf{\Lambda} = \nabla^5 \mathbf{\Lambda} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\nabla^2 \mathbf{\Xi} = \nabla^4 \mathbf{\Xi} \cdot o \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box\nabla \mathbf{\Psi} = \nabla^5 \mathbf{\Psi} \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box\nabla \mathbf{\Phi} = \nabla^5 \mathbf{\Phi}_{22'} \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box\nabla \mathbf{\Lambda} = \nabla^5 \mathbf{\Lambda} \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

$\Box\Box \mathbf{\Xi} = \nabla^4 \mathbf{\Xi} \cdot o \cdot o \cdot \overline{o} \cdot \overline{o}$ and all possible contractions with omicrons

5.4 Lowering the Bound

We now proceed to analyse the upper bound on the order of covariant differentiation of the Riemann spinor required in the Karlhede algorithm for non-vacuum type N solutions.

As in the vacuum case we apply the Karlhede algorithm to the invariants obtained in the previous sections since these are simply linear combinations of the symmetric components of the Weyl, Ricci spinor and Ricci scalar and its successive covariant derivatives.

We have seen that the situation when one can have a Karlhede bound of seven arises when the dimension of H_0 is two and when the Weyl and Ricci spinor and Ricci scalar are all constants. At first order of differentiation we see that all terms obtained are invariant under null rotations so that the dimension of $\mathbf{H}_1$ remains two, however we must consider the possibility of there being at least one new functionally independent term amongst the terms obtained at this order so that the procedure continues to second order. Furthermore, we have seen that at each step of differentiation all terms calculated are invariant under the group of null rotations so that the dimension of the invariance group $\mathbf{H}_q$ remains two, so that if we consider the worst possible situation, ie, that only one functionally independent term is obtained at each step then by fifth order one has obtained all four independent terms which is the maximum number one can obtain in a four dimensional space. As in the vacuum case this gives us only information concerning the coordinates so that we can conclude that as of the fourth derivative one does not get any more functional information with respect to the coordinates.

We now need to check that as of the fifth order the invariance group with respect to the components does not change otherwise we might need to calculate more derivatives for the same reason explained in the previous chapter. We recall that at zeroth order of covariant differentiation we have the terms $\Phi_{22'}, \Psi_4$ and Λ which we are considering to be constants. At first order of differentiation we obtain as our potentially new functional information $\rho, \sigma, \epsilon, \alpha, \beta, \tau, \gamma$ which transforms as follows:

$$\rho \longrightarrow \rho \tag{5.4.156}$$

$$\sigma \longrightarrow \sigma \tag{5.4.157}$$

$$\epsilon \longrightarrow \epsilon \tag{5.4.158}$$

$$\alpha \longrightarrow \alpha + \overline{a}\rho + \overline{a}\epsilon \tag{5.4.159}$$

$$\beta \longrightarrow \beta + \overline{a}\sigma + a\epsilon \tag{5.4.160}$$

$$\tau \longrightarrow \tau + a\rho + \overline{a}\sigma \tag{5.4.161}$$

$$\gamma \longrightarrow \gamma + a\overline{a}(\epsilon + \rho) + a\alpha + \overline{a}(\beta + \tau) + \overline{a}^2\sigma \tag{5.4.162}$$

We consider three distinct cases:

I: $\rho = \sigma = \epsilon = \alpha = \beta = \tau = 0$

In this case we are left with the transformation $\gamma \longrightarrow \gamma$, in which case it follows that the two dimensional invariance group of null rotations remains at first order.

II: $\rho = \sigma = \epsilon = \alpha = 0$

Here we are left with the following transformations: $\gamma \longrightarrow \gamma$, $\beta \longrightarrow \beta$, $\tau \longrightarrow \tau$. So that one cannot fix the frame any further, thus the group of null rotations remains as the invariance group at first order.

III: $\rho = \sigma = \epsilon = 0; |\alpha| = |\beta + \tau|$

One is left with the transformations $\gamma \longrightarrow \gamma + (a + \overline{a})\alpha$, $\beta \longrightarrow \beta$, $\tau \longrightarrow \tau$ and $\alpha \longrightarrow \alpha$. We can therefore use the first of such transformations to fix the frame further so that the invariance group is at first order one dimensional.

In all other cases it is easily seen that one can fix the frame completely so that the invariance group becomes zero dimensional.

We now analyse what ocurrs at second order of differentiation, for this effect we use equations 4.3.62, 5.2.48, 5.2.51, 5.2.54, 5.2.62, 5.2.65, 5.2.67. We take each of the separate cases considered above:

I: $\rho = \sigma = \epsilon = \alpha = \beta = \tau = 0$

One has as our new potential functional information $D\gamma, \delta\gamma, \delta'\gamma, D'\gamma$ which transform as:

$$D'\gamma \longrightarrow (D' + a\delta' + \overline{a}\delta + a\overline{a}D)\gamma \tag{5.4.163}$$

$$\delta\gamma \longrightarrow (\delta + aD)\gamma \tag{5.4.164}$$

$$\delta' \longrightarrow (\delta' + \overline{a}D)\gamma \tag{5.4.165}$$

$$D\gamma \longrightarrow D\gamma \tag{5.4.166}$$

From the NP Ricci equations (4.2f), (4.2r) and (4.2o) one has $\delta\gamma = 0$, $\delta'\gamma = 0$ and $D\gamma = -\Lambda$ so that we consider the following cases:

Ia: $\Lambda = 0$

So that we have $\delta\gamma = \delta\gamma = \delta'\gamma = 0$ which leaves us with the transformation $D\gamma \longrightarrow D\gamma$. We can then show using the NP commutators and following the same argument as that used by Collins [6] and which was applied in the previous chapter to the vacuum case that the two dimensional invariance group remains at all orders of differentiation.

Ib: $\Lambda \neq 0$

In this case we can use the transformation $D'\gamma \longrightarrow (D' + a\overline{a}D)\gamma$ to fix the frame completely. So that we have a zero dimensional invariance group at second order.

II: $\rho = \sigma = \epsilon = \alpha = 0$

We get the following potentially new functional information: $D\gamma, \delta\gamma, \delta'\gamma, D'\gamma, D\beta, \delta\beta, \delta'\beta, D'\beta, D\tau, \delta\tau, \delta'\tau, D'\tau, \pi, \lambda, \mu$ and ν. Using a similar argument to that of Collins [6] and that used in the previous chapter we are able to fix the frame completely giving at second order a zero dimensional invariance group.

III: $\rho = \sigma = \epsilon = 0; |\alpha| = |\beta + \tau|$

We have as our potentially new functional information: $D\gamma, \delta\gamma, \delta'\gamma, D'\gamma, D\beta, \delta\beta, \delta'\beta, D'\beta, D\tau, \delta\tau, \delta'\tau, D'\tau, D\alpha, D'\alpha, \delta\alpha, \delta'\alpha, \pi, \lambda, \mu$ and ν. Applying the same argument as Collins [6] and explained in chapter 4 we are able to fix the frame up to a zero dimension invariance group.

We have thus proved that as of the second order of covariant differentiation the dimension of the invariance group remains unchanged. Hence in this case we only need to calculate five covariant derivatives to classify the non vacuum solution completely. We have then proved that in the case where condition (1) and (2) hold and the potential bound is seven the actual bound is at worst five.

If we relax condition (1) and maintain all others and consider that $\Psi_4, \Phi_{22'}$ and Λ might not all be constants then one has a potential bound of six in the worst possible case. It is quite easily seen that the analysis is just the same as that done for the case of $\Psi_4, \Phi_{22'}$ and Λ being constants. However in this case one must consider the possibility of having potentially new coordinate functional information at zeroth order so that the bound is then four.

If we relax condition (2) and take the dimension of the invariance group at zeroth order to be one then we have the following conditions: $\Phi_{00'} = \Phi_{01'} = \Phi_{02'} = \Phi_{12'} = \Phi_{10'} = \Phi_{20'} = 0$ and $\Phi_{12'} \neq 0, \Phi_{21'} \neq 0, \Phi_{22'} \neq 0, \Phi_{12'} = \Phi_{21'}$. In this particular situation we can have a potential bound of six if we consider that not all non zero terms at zeroth order are constants. We must then go through the process of calculating the general expressions giving the successive covariant derivatives of $\Phi_{12'}$ and $\Phi_{21'}$ along with the d'Alembertian of such terms. This work shall not be included here.

It is easily seen that in all other cases the potential bound would not be greater than five. Hence we conclude that the Karlhede bound for type N non vacuum solutions is at most six.

For some time no spacetime was known to require more than the third derivative which led many people to believe that the true upper bound was in fact three. Koutras [21], however, has come up with a solution where one needs to calculate the fourth derivative to complete its classification, the solution being the conformally flat pure radiation field found by Wils (1989). Whether the upper bound of five is the true upper bound remains to be seen since, up till now, no solution requiring five derivatives has been found.

Chapter 6

Curvature and Metric In General Relativity

6.1 Introduction

In this chapter we discuss the problem of determining the metric tensor from the curvature tensor, the ways in which one can approach this problem (mainly that developed by Hall [22],[19], [13]) and the possibility of using this method to lower the bounds on covariant differentiation in the Karlhede algorithm.

In order to solve the problem of determining the components of the metric tensor from the components of the curvature tensor one assumes that the curvature tensor is given over some coordinate domain of the manifold M. Recently Edgar[11] derived a simple sufficient condition for a given connection to be derived from a metric and applied an algebraic procedure for calculating the metric from the curvature.

A method of determination of the metric tensor from the curvature tensor was also proposed by Ihrig [16], [17] and was actively developed by Halford, McIntosh [12] and Hall [22], [19], [13]. In contrast to the Cartan-Karlhede [5], [18] method of classifying the geometry of a space-time which uses the tetrad components of the curvature tensor and its successive covariant derivatives, the methods mentioned work with the coordinate components of the curvature and one assumes that either the connection or a finite number of the covariant derivatives of the curvature are known.

6.2 Determining the Metric from the Curvature

Here we describe Hall's method of determining the metric from the curvature and we closely follow [22], [19], [13]. We adopt Greek indices to denote coordinate components and latin indices to denote tetrad components. Also we use the notation <> to denote linear span.

Let U be some coordinate domain of the space-time M, let $p \in U$ and let $R^{\alpha}{}_{\beta\gamma\delta}$ denote the coordinate components in U of the curvature tensor. We suppose $R^{\alpha}{}_{\beta\gamma\delta}$ is non zero at p. To find which Lorentz metrics other than the given metric g can be compatible with this curvature tensor one first notes the algebraic necessity that any other possible metric g' must preserve the symmetries of the Riemann tensor at p, mainly:

$$g'_{\nu(\alpha}R^{\nu}{}_{\beta)\gamma\delta} = 0 \qquad (6.2.1)$$

Hence, the first step is the solution of the algebraic problem expressed in 6.2.1, within the space time (M, g), using the original metric g to raise and lower indices, etc. The curvature components $R_{\alpha\beta\gamma\delta} = g_{\alpha\mu}R^{\mu}{}_{\beta\gamma\delta}$ may be used to define a linear map $\mathcal{R}$ from the six dimensional vector space of all contravariant bivectors at p denoted by $\Omega_p(M)$ to the six dimensional vector space of all covariant bivectors at p denoted by $\Omega^*_p(M)$ in the usual way, i.e,

$$\begin{aligned} \mathcal{R} : \Omega_p(M) &\longrightarrow \Omega^*_p(M) \\ F^{\gamma\delta} &\longrightarrow \widetilde{F}_{\alpha\beta} \end{aligned} \qquad (6.2.2)$$

with $\mathcal{R}(F^{\gamma\delta}) = R_{\alpha\beta\gamma\delta}\, F^{\gamma\delta} = \widetilde{F}_{\alpha\beta}$.

Hence, by equation 6.2.1 we can write:

$$g'_{\nu\alpha}R^{\nu}{}_{\beta\gamma\delta}\, F^{\gamma\delta} + g'_{\nu\beta}R^{\nu}{}_{\alpha\gamma\delta}\, F^{\gamma\delta} = 0$$

which in turn gives:

$$g'_{\alpha\nu}\widetilde{F}^{\nu}{}_{\beta} + g'_{\beta\nu}\widetilde{F}^{\nu}{}_{\alpha} = 0 \qquad (6.2.3)$$

So that equation 6.2.1 is equivalent to:

$$g'_{\nu(\alpha}\tilde{F}{}^{\nu}{}_{\beta)} = 0 \tag{6.2.4}$$

for all bivectors $\widetilde{F}_{\alpha\beta}$ in the range space of $\mathcal{R}$. Hence, the generality of the solution of 6.2.4 for g' depends on the rank of the curvature defined as the dimension of the range space of $\mathcal{R}$, or equivalently, as the rank of a 6×6 matrix arising when the curvature components are written in block index form.

Review of Bivectors

We now give a succinct review on some definitions and properties relating to bivectors.

The dual of a skew symmetric second order tensor, i.e, bivector F, denoted by $\overset{*}{F}$ is defined by:

$$\overset{*}{F}_{\alpha\beta} = \eta_{\alpha\beta\gamma\delta}\, F^{\gamma\delta} \tag{6.2.5}$$

$$\overset{*}{F}{}^{\alpha\beta} = \eta^{\alpha\beta\gamma\delta}\, F_{\gamma\delta} \tag{6.2.6}$$

where $\eta_{\alpha\beta\gamma\delta}$ is the Levi-Civita tensor.

A non-zero bivector F is said to be *simple* or *decomposable* if and only if there exists a non zero vector v such that $F_{\alpha\beta}v^{\beta} = 0$. If F is a simple bivector, then it is possible to find vectors r and s such that $F = r \wedge s$. If we also have an inner product we may choose r and s such that $r \cdot s = 0$. The two dimensional space spanned by r and s is called the *blade* of F.

A non zero bivector is called *null* if and only if there exists a non zero vector v such that $F_{\alpha\beta}v^{\beta} = \overset{*}{F}_{\alpha\beta}\, v^{\beta} = 0$ otherwise F is $non - null$. Every null bivector is therefore simple. Notice also that if F is simple then $\overset{*}{F}$ is also simple with $\overset{*}{F}$ having a blade orthogonal to that of F. Furthermore the dual of every null bivector is also null.

Any non-null bivector F can be written as:

$$F = \lambda_1 l \wedge n + \lambda_2 z \wedge y$$

with $\lambda_1 = \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ and where (n, l, y, z) is a null tetrad.

If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ then F is a $non - simple$ bivector. If $\lambda_1 = 0$ or $\lambda_2 = 0$ then F is a *simple non-null bivector.* The rank of a non-null bivector F is an even number (see for example [26]).

In order to determine all metrics $g'_{\alpha\beta}$ which satisfy equation 6.2.1 we need to establish some results which can be found in [22], [19], [13].

Theorem 6.2.1 *If at $p \in M$, F is a simple bivector whose blade is spanned by the vectors r and s and if X is a symmetric second order tensor, then the following two conditions are equivalent:*

(i) $X_{\gamma(\alpha}F^{\gamma}{}_{\beta)} = 0$
(ii) the vectors r and s are eigenvectors of X with equal eigenvalues

Proof

Since F is a simple bivector whose blade is spanned by the vectors r and s then we can write:

$$F = r \wedge s$$

or in terms of coordinate components we have:

$$F_{\alpha\beta} = r_\alpha s_\beta - s_\alpha r_\beta$$

X is a symmetric tensor so that we can write:

$$X_{\gamma\alpha}F^{\gamma}_{\beta} + X_{\gamma\beta}F^{\gamma}_{\alpha} = 0$$

which can then be written as:

$$X_{\gamma\alpha}(r^\gamma s_\beta - s^\gamma r_\beta) + X_{\gamma\beta}(r^\gamma s_\alpha - s^\gamma r_\alpha) = 0$$

or equivalently:

$$r^\gamma X_{\gamma\alpha}s_\beta + r^\gamma X_{\gamma\beta}s_\alpha = s^\gamma X_{\gamma\alpha}r_\beta + s^\gamma X_{\gamma\beta}r_\alpha \tag{6.2.7}$$

If we define:

$$r^\gamma X_{\gamma\alpha} = u_\alpha \tag{6.2.8}$$

$$s^\gamma X_{\gamma\alpha} = t_\alpha \tag{6.2.9}$$

then equation 6.2.7 becomes:

$$u_\alpha s_\beta + u_\beta s_\alpha = t_\alpha r_\beta + t_\beta r_\alpha \tag{6.2.10}$$

By contracting both sides of equation 6.2.10 with $g^{\beta\alpha}$ we arrive at:

$$u_\alpha s^\alpha = t_\alpha r^\alpha \tag{6.2.11}$$

It is always possible to choose a basis whereby $F_{\alpha\beta} = r_\alpha s_\beta - s_\alpha r_\beta$ with $r^\alpha s_\alpha = 0$, so let us suppose that (r, s) constitute such a basis. We then contract equation 6.2.10 with s^α and use equation 6.2.11 and obtain:

$$\beta u_\beta = t_\alpha r_\beta s^\alpha - t_\alpha r^\alpha s_\beta \tag{6.2.12}$$

where $\beta = s_\alpha s^\alpha$.

We choose s to be a vector in the blade which is not null so that $\beta \neq 0$. Notice that we cannot have the situation where both s and r are both null since that would imply $r \propto s$ since $r^\alpha s_\alpha = 0$ and therefore $F = 0$.

If we now multiply equation 6.2.10 by β and use 6.2.12 we have:

$$u_\alpha s_\beta = t_\beta r_\alpha \tag{6.2.13}$$

If we now multiply equation 6.2.13 by r_γ we get:

$$r_\gamma u_\alpha - u_\gamma r_\alpha = 0 \tag{6.2.14}$$

Hence, $u \propto r$ in any basis. We can then write $u = \alpha r$ for some $\alpha \in \mathbb{R}$.

On the other hand if we multiply 6.2.13 by s_γ we arrive at:

$$t_\alpha s_\gamma - t_\gamma s_\alpha = 0 \tag{6.2.15}$$

By equation 6.2.15 we conclude that $t = \lambda s$ for some $\lambda \in \mathbb{R}$. Furthermore by equation 6.2.13 we have:

$$\alpha r_\alpha s_\beta = \lambda s_\beta r_\alpha \tag{6.2.16}$$

So that $\alpha = \lambda$.

●

By theorem 6.2.1 it turns out that if a simple bivector F lies in the range of $\mathcal{R}$, so that F satisfies 6.2.4, then all members of T_pM lying in the 2-space defined by the blade of F at p are eigenvectors of g' with respect to g at p with the same eigenvalue, i.e:

$$g'_{\alpha\beta}k^{\beta} = \lambda k_{\alpha} = \alpha g_{\alpha\beta}k^{\beta} \;\; \forall k \in < r, s >, \;\; \lambda, \alpha \in \mathbf{R} \tag{6.2.17}$$

Theorem 6.2.2 *If at $p \in M$, F is a non-simple bivector and X a symmetric second order tensor then with the notation established above the following two conditions are equivalent:*

(i) $X_{\gamma(\alpha}F^{\gamma}{}_{\beta)} = 0$

(ii) The null vectors l and n are eigenvectors of X with equal eigenvalues and the spacelike vectors y and z are eigenvectors of X with equal eigenvalues

Proof

Any non-simple bivector F can be written :

$$F_{\alpha\beta} = \alpha l_{[\alpha}n_{\beta]} + \beta z_{[\alpha}y_{\beta]} \tag{6.2.18}$$

with $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq 0, \beta \neq 0$.
The fact that F can be written as in 6.2.18 gives the following identities:

$$l^{\alpha}F_{\alpha\beta} = \alpha l_{\beta} \tag{6.2.19}$$

$$n^{\alpha}F_{\alpha\beta} = -\alpha n_{\beta} \tag{6.2.20}$$

$$y^{\alpha}F_{\alpha\beta} = \beta z_{\beta} \tag{6.2.21}$$

$$z^{\alpha}F_{\alpha\beta} = -\beta y_{\beta} \tag{6.2.22}$$

We first prove that (i) $\Rightarrow$ (ii).

If we contract (i) with l^{β} we have:

$$X_{\gamma\alpha} l^\beta F^\gamma{}_\beta + X_{\gamma\beta} l^\beta F^\gamma{}_\alpha = 0 \tag{6.2.23}$$

By 6.2.19 we have that equation 6.2.23 gives:

$$X_{\gamma\beta} l^\beta F^\gamma{}_\alpha = \alpha X_{\alpha\gamma} l^\gamma \tag{6.2.24}$$

Similarly, by contracting (i) with n^β, y^β and z^β in turn and using equations 6.2.20, 6.2.21 and 6.2.22 gives:

$$X_{\gamma\beta} n^\beta F^\gamma{}_\alpha = -\alpha X_{\alpha\gamma} \tag{6.2.25}$$

$$X_{\gamma\beta} y^\beta F^\gamma{}_\alpha = \beta X_{\alpha\gamma} z^\gamma \tag{6.2.26}$$

$$X_{\gamma\beta} z^\beta F^\gamma{}_\alpha = -\beta X_{\alpha\gamma} y^\gamma \tag{6.2.27}$$

By 6.2.24 we can write:

$$p_\gamma F^\gamma{}_\alpha = \alpha p_\alpha \text{ with } p_\gamma = X_{\gamma\beta} l^\beta$$

Hence, since l satisfies 6.2.19 and is unique then:

$$p_\gamma = \gamma l_\gamma \text{ for some } \gamma \in \mathbf{R}$$

Therefore:

$$X_{\gamma\beta} l^\beta = \gamma l_\gamma \tag{6.2.28}$$

Similarly we have:

$$X_{\gamma\beta} n^\beta = \delta n_\gamma \text{ for some } \delta \in \mathbf{R} \tag{6.2.29}$$

Since X is a symmetric second order tensor we can write:

$$X_{\alpha\beta} = X_{\beta\alpha} \tag{6.2.30}$$

By contracting both sides of 6.2.30 with l^α and n_β we have:

$$(l^\alpha X_{\alpha\beta})n^\beta = l^\alpha(X_{\beta\alpha}n^\beta) \tag{6.2.31}$$

Which in turn, by 6.2.28 and 6.2.29, gives:

$$\gamma l_\beta n^\beta = \delta l^\alpha n_\alpha \tag{6.2.32}$$

Hence, we have that the eigenvalues γ, δ are equal, i.e:

$$\gamma = \delta \tag{6.2.33}$$

We now make use of the complex null tetrad $(l, n, m, \overline{m})$ where:

$$m^\alpha = \frac{1}{\sqrt{2}}(y^\alpha + iz^\alpha) \tag{6.2.34}$$

$$\overline{m}^\alpha = \frac{1}{\sqrt{2}}(y^\alpha - iz^\alpha) \tag{6.2.35}$$

and

$$m^\alpha m_\alpha = \overline{m}^\alpha \overline{m}_\alpha = l^\alpha l_\alpha = n^\alpha n_\alpha = 0 \tag{6.2.36}$$

$$\overline{m}^\alpha m_\alpha = -l^\alpha n_\alpha = 1 \tag{6.2.37}$$

The bivector F written in this base takes the form:

$$F_{\alpha\beta} = \alpha l_{[\alpha}n_{\beta]} + i\beta m_{[\alpha}\overline{m}_{\beta]} \text{ with } \alpha, \beta \in \mathbf{R} \tag{6.2.38}$$

Equation 6.2.38 gives the following identities:

$$m^\alpha F_{\alpha\beta} = -i\beta m_\beta \tag{6.2.39}$$

$$\overline{m}^\alpha F_{\alpha\beta} = i\beta \overline{m}_\beta \tag{6.2.40}$$

with $\beta \in \mathbb{R}$.

By contracting (i) with m^β and $\overline{m}^\beta$ in turn we obtain:

$$X_{\gamma\beta} m^\beta F^\gamma{}_\alpha = i\beta X_{\gamma\alpha} m^\gamma \qquad (6.2.41)$$

$$X_{\gamma\beta} \overline{m}^\beta F^\gamma{}_\alpha = i\beta X_{\gamma\alpha} \overline{m}^\gamma \qquad (6.2.42)$$

Hence, equation 6.2.39 gives:

$$p_\gamma F^\gamma_\alpha = -i\beta p_\alpha \text{ with } p_\gamma = X_{\gamma\beta} m^\beta$$

And since m is the only complex null vector satisfying 6.2.39 we have:

$$p_\gamma = \epsilon_1 m_\gamma$$

So that finally:

$$X_{\gamma\beta} m^\beta = \epsilon_1 m_\gamma \text{ with } \epsilon_1 \in \mathbb{R} \qquad (6.2.43)$$

Similarly, equations 6.2.40 and 6.2.42 give:

$$X_{\gamma\beta} \overline{m}^\beta = \epsilon_2 \overline{m}_\gamma \text{ with } \epsilon_2 \in \mathbb{R} \qquad (6.2.44)$$

By the fact that X is a symmetric tensor one has:

$$\epsilon_1 = \epsilon_2 = \epsilon \text{ and } \epsilon \in \mathbb{R} \qquad (6.2.45)$$

So that:

$$X_{\alpha\beta}(m^\beta + \overline{m}^\beta) = \epsilon(m_\alpha + \overline{m}_\alpha) \qquad (6.2.46)$$

which, by definitions 6.2.34 and 6.2.35, in turn gives:

$$X_{\gamma\beta} y^\beta = \epsilon y_\gamma \qquad (6.2.47)$$

Similarly:

$$X_{\gamma\beta}(m^{\beta} - \overline{m}^{\beta}) = \epsilon(m_{\gamma} - \overline{m}_{\gamma}) \tag{6.2.48}$$

gives:

$$X_{\gamma\beta} z^{\beta} = \epsilon z_{\gamma} \tag{6.2.49}$$

So that we have proved (i)$\Rightarrow$ (ii).

We now proceed to prove (ii)$\Rightarrow$ (i)

Equations 6.2.28, 6.2.29, 6.2.33, 6.2.47 and 6.2.49 imply that:

$$\alpha(X_{\gamma\alpha}(l^{\gamma}n_{\beta} - n^{\gamma}l_{\beta}) + X_{\gamma\beta}(l^{\gamma}n_{\alpha} - n^{\gamma}l_{\alpha})) = 0 \tag{6.2.50}$$

$$\beta(X_{\gamma\alpha}(y^{\gamma}z_{\beta} - z^{\gamma}y_{\beta}) + X_{\gamma\beta}(y^{\gamma}z_{\alpha} - z^{\gamma}y_{\alpha})) = 0 \tag{6.2.51}$$

If we add equations 6.2.50 and 6.2.51 we obtain the desired result, i.e:

$$X_{\gamma(\alpha}F^{\gamma}{}_{\beta)} = 0$$

●

The metric tensor g at p is related to the tetrad vectors (l, n, y, z) by the completeness relation:

$$g_{\alpha\beta} = -2l_{(\alpha}n_{\beta)} + y_{\alpha}y_{\beta} + z_{\alpha}z_{\beta} \tag{6.2.52}$$

If the conditions and statements in theorem 6.1.2 hold then it follows that:

$$g'_{\alpha\beta} = -2\lambda_1 l_{(\alpha}n_{\beta)} + \lambda_2(y_{\alpha}y_{\beta} + z_{\alpha}z_{\beta}) \text{ with } \lambda_1, \lambda_2 \in \mathbf{R} \tag{6.2.53}$$

Thus, the only eigenvectors admitted by g' lie either in the 2-space spanned by l and n or that spanned by y and z, unless $\lambda_1 = \lambda_2 = \lambda$ in which case the completeness relation shows that $g'_{\alpha\beta} = \lambda g_{\alpha\beta}$. It follows that if this trivial solution is not to be the only solution of the equation $g'_{\nu(\alpha}F^{\nu}{}_{\beta)} = 0$, the only bivectors which may satisfy this equation must be linear combinations of $l_{[\alpha}n_{\beta]}$ and $y_{[\alpha}z_{\beta]}$. This is a consequence of the previous two theorems since any other bivectors satisfying this equation would give rise to eigenvectors of g' outside the blades of the 2-forms $l_{[\alpha}n_{\beta]}$ and $y_{[\alpha}z_{\beta]}$.

Theorem 6.2.3 *The following cases occur:*

(i) If the range of $\mathcal{R}$ is spanned by a single (necessarily simple) bivector F ($\mathcal{R}$ is of rank one), then there exists $\phi, \mu, \nu, \lambda \in \mathbf{R}$ such that:

$$g'_{\alpha\beta} = \phi g_{\alpha\beta} + \mu u_\alpha u_\beta + 2\nu u_{(\alpha} v_{\beta)} + \lambda v_\alpha v_\beta \qquad (6.2.54)$$

where u and v span the 2-space orthogonal to the 2-space represented by F.

(ii) If the range of $\mathcal{R}$ has dimension two or three (rank two or three) and if the members of this range have a common eigenvector w with zero eigenvalue (so that the range of $\mathcal{R}$ consists only of simple bivectors and determines w to within a multiplicative factor) then there exists $\phi, \lambda \in \mathbf{R}$ such that:

$$g'_{\alpha\beta} = \phi g_{\alpha\beta} + \lambda w_\alpha w_\beta \qquad (6.2.55)$$

(iii) if the range of $\mathcal{R}$ is spanned by the simple bivectors $l_{[\alpha} n_{\beta]}$ and $y_{[\alpha} z_{\beta]}$ (rank two), then there exists $\phi, \lambda \in \mathbf{R}$ such that:

$$g'_{\alpha\beta} = \phi g_{\alpha\beta} + 2\lambda l_{(\alpha} n_{\beta)} = (\phi + \lambda) g_{\alpha\beta} - \lambda(y_\alpha y_\beta + z_\alpha z_\beta) \qquad (6.2.56)$$

(iv) In all other cases there exists $\phi \in \mathbf{R}$ such that:

$$g'_{\alpha\beta} = \phi g_{\alpha\beta} \qquad (6.2.57)$$

Proof

We start by proving (i). Lets start by supposing that $F_{\alpha\beta}$ is spacelike so that we can write $F_{\alpha\beta} = y_\alpha z_\beta - z_\alpha y_\beta$. Then, by theorem 6.2.1 we have $X_{\alpha\beta} z^\beta = \alpha z_\alpha$ and $X_{\alpha\beta} y^\beta = \alpha y_\alpha$. Considering the null tetrad (l, n, y, z) and since $X_{\alpha\beta}$ is a symmetric tensor, then taking the symmetric products of the tetrad members gives:

$$X_{\alpha\beta} = \alpha_1 l_\alpha l_\beta + \alpha_2 n_\alpha n_\beta + \alpha_3(l_\alpha n_\beta + n_\alpha l_\beta) + \alpha(y_\alpha y_\beta + z_\alpha z_\beta) \qquad (6.2.58)$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha \in \mathbf{R}$.

Using the completeness relation 6.2.52 equation 6.2.58 becomes:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + \alpha_1 l_\alpha l_\beta + \alpha_2 n_\alpha n_\beta + (\alpha_3 + \alpha)(l_\alpha n_\beta + n_\alpha l_\beta) \tag{6.2.59}$$

with $\phi = \alpha, \nu = \alpha + \alpha_3, \mu = \alpha_1, \lambda = \alpha_2$. Obviously we have $u = l$ and $v = n$.

We now take $F_{\alpha\beta}$ to be timelike so that we can write $F_{\alpha\beta} = l_\alpha n_\beta - n_\alpha l_\beta$. $F_{\alpha\beta} = l_\alpha n_\beta - n_\alpha l_\beta$. By theorem 6.2.1 we then have $X_{\alpha\beta} l^\beta = \alpha l_\alpha$ and $X_{\alpha\beta} n^\beta = \alpha n_\alpha$. Considering the null tetrad (l, n, y, z) and since $X_{\alpha\beta}$ is a symmetric tensor we can write:

$$X_{\alpha\beta} = -\alpha(l_\alpha n_\beta + n_\alpha l_\beta) + \alpha_1(y_\alpha z_\beta + z_\alpha y_\beta) + \alpha_2 y_\alpha y_\beta + \alpha_3 z_\alpha z_\beta \tag{6.2.60}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha \in \mathbb{R}$.

Substituting the completeness relation 6.2.52 into equation 6.2.60 gives:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + \alpha_1(y_\alpha z_\beta + z_\alpha y_\beta) + (\alpha_2 - \alpha) y_\alpha y_\beta + \alpha_3 z_\alpha z_\beta \tag{6.2.61}$$

with $\phi = \alpha, \nu = \alpha_1, \mu = \alpha_2 - \alpha, \lambda = \alpha_3 - \alpha, u = y, v = z$.

Finally we consider the null case, i.e, $F_{\alpha\beta} = l_\alpha y_\beta - y_\alpha l_\beta$. Then by theorem 6.2.1 we have $X_{\alpha\beta} l^\beta = \beta l_\alpha$ and $X_{\alpha\beta} y^\beta = \beta y_\alpha$. Considering the null tetrad (l, n, y, z) and the symmetry of $X_{\alpha\beta}$ we write:

$$X_{\alpha\beta} = -\alpha(l_\alpha n_\beta + n_\alpha l_\beta) + \alpha y_\alpha y_\beta + \alpha_1(l_\alpha z_\beta + z_\alpha l_\beta) + \alpha_2 z_\alpha z_\beta$$
$$+ \alpha_3 l_\alpha l_\beta \tag{6.2.62}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha \in \mathbb{R}$.

Substituting the completeness relation 6.2.52 into equation 6.2.62 gives:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + \alpha_1(l_\alpha z_\beta + z_\alpha l_\beta) + (\alpha_2 - \alpha) z_\alpha z_\beta + \alpha_3 l_\alpha l_\beta \tag{6.2.63}$$

with $\phi = \alpha, \nu = \alpha_1, \mu = \alpha_3, \lambda = \alpha_2 - \alpha, u = l, v = z$.

We now proceed to prove (ii). Let $w = l$ and consider the case of dimension two. Let B denote the subspace of bivector space spanned by the curvature 2-forms. There are two cases to consider:

(1) $B = < l_{[\alpha}y_{\beta]}, y_{[\alpha}z_{\beta]} >$

By theorem 6.2.1 we have:

$$X_{\alpha\beta}l^{\alpha} = \epsilon l_{\beta},\ X_{\alpha\beta}y^{\alpha} = \epsilon y_{\beta},\ X_{\alpha\beta}z^{\alpha} = \epsilon z_{\beta}$$

.

which then gives:

$$X_{\alpha\beta} = -\alpha l_{(\alpha}n_{\beta)} + \alpha y_{\alpha}y_{\beta} + \alpha z_{\alpha}z_{\beta} + \alpha_1 l_{\alpha}l_{\beta} \tag{6.2.64}$$

By the relation 6.2.52, equation 6.2.64 then becomes:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + \alpha_1 l_{\alpha}l_{\beta} \tag{6.2.65}$$

with $\phi = \alpha, \lambda_1 = \lambda, w = l$.

(2) $B = < l_{[\alpha}z_{\beta]}, y_{[\alpha}z_{\beta]} >$

This case works out the same as in case (1) so that the result obtained is given by 6.2.65.

We now consider the case $w = l$ and dimension three, which leaves us with the case:

$B = < l_{[\alpha}y_{\beta]}, l_{[\alpha}z_{\beta]}, y_{[\alpha}z_{\beta]} >$

This case also produces, as before, the result 6.2.65.

Notice that for $w = n$ the process is similar to the case $w = l$ with:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + \alpha_1 n_{\alpha}n_{\beta} \tag{6.2.66}$$

We now choose $w = y$ and consider the case of dimension two. Again there are three cases to consider.

(1) $B = < l_{[\alpha}n_{\beta]}, l_{[\alpha}z_{\beta]} >$

By theorem 6.1.1 we have:

$$X_{\alpha\beta}l^{\alpha} = \alpha l_{\beta},\ X_{\alpha\beta}n^{\alpha} = \alpha n_{\beta},\ X_{\alpha\beta}z^{\alpha} = \alpha z_{\beta}$$

which gives:

$$X_{\alpha\beta} = -\alpha l_{(\alpha}n_{\beta)} + \alpha z_{\alpha}z_{\beta} + \alpha_1 y_{\alpha}y_{\beta}$$

So that by substituting the completeness relation 6.2.52 in the above equation one arrives at the following result:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + (\alpha_1 - \alpha)y_{\alpha}y_{\beta} \tag{6.2.67}$$

with $w = y, \phi = \alpha, \lambda = \alpha_1 - \alpha$.

The following cases also give result 6.2.67.

$(2) B =< l_{[\alpha}n_{\beta]}, n_{[\alpha}z_{\beta]} >$

$(3) B =< l_{[\alpha}z_{\beta]}, n_{[\alpha}z_{\beta]} >$

For the case $w = y$ and dimension three we must consider:

$B =< l_{[\alpha}z_{\beta]}, n_{[\alpha}z_{\beta]}, l_{[\alpha}n_{\beta]} >$

This case also gives the result 6.2.52.

Note that for $w = z$ the process is similar to the case $w = y$ with:

$$X_{\alpha\beta} = \alpha g_{\alpha\beta} + (\alpha_1 - \alpha)z_{\alpha}z_{\beta} \tag{6.2.68}$$

We now proceed to prove (iii), so that one considers:

$$B =< l_{[\alpha}n_{\beta]}, y_{[\alpha}z_{\beta]} >$$

Theorem 6.1.1 gives the identities:

$$X_{\alpha\beta}l^{\alpha} = \gamma l_{\beta}, X_{\alpha\beta}n^{\alpha} = \gamma n_{\beta}, X_{\alpha\beta}y^{\alpha} = \mu y_{\beta}, X_{\alpha\beta}z^{\alpha} = \mu z_{\beta}$$

This falls into the situation of theorem 6.2.1 so that one can write:

$$X_{\alpha\beta} = \gamma l_{(\alpha}n_{\beta)} + \mu(y_\alpha y_\beta + z_\alpha z_\beta)$$

The relation 6.2.52 then gives:

$$X_{\alpha\beta} = \mu g_{\alpha\beta} + (2\mu + \gamma)l_{(\alpha}n_{\beta)} \tag{6.2.69}$$

with $\phi = \mu$ and $2\lambda = 2\mu + \gamma$

Finally we prove case (iv). Notice that all two dimensional cases have been already considered. We have only to study the remaining cases where the dimension is greater or equal to three. So that we write:

$$B = < l_{[\alpha}n_{\beta]}, y_{[\alpha}z_{\beta]}, l_{[\alpha}y_{\beta]} >$$

Theorem 6.2.1 gives the following identities:

$$X_{\alpha\beta}l^\alpha = \gamma l_\beta,\ X_{\alpha\beta}n^\alpha = \gamma n_\beta,\ X_{\alpha\beta}y^\alpha = \gamma y_\beta,\ X_{\alpha\beta}z^\alpha = \gamma z_\beta$$

So that one then writes:

$$X_{\alpha\beta} = -\gamma l_{(\alpha}n_{\beta)} + \gamma y_\alpha y_\beta + \gamma z_\alpha z_\beta$$

Using the completeness relation we get:

$$X_{\alpha\beta} = \gamma g_{\alpha\beta} \tag{6.2.70}$$

It is quite easy to see that all other remaining cases are similar.

•

We now establish two other important results that can be found in [13].

Theorem 6.2.4 *The following equation:*

$$R_{\alpha\beta\gamma\delta}\, k^\delta = 0 \tag{6.2.71}$$

has two independent solutions for k if the conditions of theorem 6.2.3(i) hold (for example $k = u$ and $k = v$), one independent solution (for example $k = w$) if the conditions of theorem 6.2.3(ii) hold and no non-trivial solutions otherwise.

Proof

We start by showing that

$$R^\alpha{}_{\beta\gamma\delta}F^{\gamma\delta} = F^\alpha{}_\beta \ \forall F^{\gamma\delta} \in \Omega_p M$$

is equivalent to

$$R^\alpha{}_{\beta\gamma\delta}X^\gamma X^\delta = F^\alpha{}_\beta \ \forall X^\gamma, X^\delta \in T_p M$$

This can be shown in a few simple steps:

$$R^\alpha{}_{\beta\gamma\delta}F^{\gamma\delta} = R^\alpha{}_{\beta\gamma\delta}X^{[\gamma}X^{\delta]} = \frac{1}{2}R^\alpha{}_{\beta\gamma\delta}(X^\gamma Y^\delta - X^\delta Y^\gamma)$$
$$= R^\alpha{}_{\beta\gamma\delta}X^\gamma Y^\delta$$

The last step being possible because of the antisymmetry $R^\alpha{}_{\beta\gamma\delta} = -R^\alpha{}_{\beta\delta\gamma}$

By theorem 6.2.3 (i) the vectors u and v span the 2-space orthogonal to the blade of F, F being necessarily simple so that one can write $F^\alpha{}_\beta = r^\alpha s_\beta - s^\alpha r_\beta$ and $u_\alpha r^\alpha = u_\alpha s^\alpha = v_\alpha r^\alpha = v_\alpha s^\alpha = 0$. Furthermore, $F^\alpha{}_\beta u_\alpha = F^\alpha{}_\beta v_\alpha = 0$

So that one then has:

$$u_\alpha R^\alpha{}_{\beta\gamma\delta}X_1^\gamma X_2^\delta = F^\alpha{}_\beta u_\alpha = 0 \qquad (6.2.72)$$

which in turn implies:

$$u_\alpha R^\alpha{}_{\beta\gamma\delta} = 0 \qquad (6.2.73)$$

this because 6.2.72 holds for any $X_1, X_2 \in T_p M$.

Similarly we have:

$$v_\alpha R^\alpha{}_{\beta\gamma\delta} = 0 \qquad (6.2.74)$$

If the conditions of theorem 6.2.3 (ii) hold we easily see that:

$$w_\alpha R^\alpha{}_{\beta\gamma\delta}X_1^\gamma X_2^\delta = F^\alpha{}_\beta w_\alpha = 0 \qquad (6.2.75)$$

$$w_\alpha R^\alpha{}_{\beta\gamma\delta} = 0 \qquad (6.2.76)$$

●

Theorem 6.2.5 *If the curvature tensor is given over a spacetime M and if at each point $p \in M$ the components $R_{\alpha\beta\gamma\delta}$ satisfy the conditions of theorem 6.2.3(iv), the metric on M is determined to within a constant conformal factor.*

Proof

We start by writing down the Bianchi identities in any coordinate domain in M for the original metric g and another possible metric $g' = \phi g$, where ϕ is a smooth real valued function on M. Covariant derivatives with respect to g' and g will be denoted by a stroke ($R^{\alpha}{}_{\beta\gamma\delta|\theta}$) and a semi colon ($R^{\alpha}{}_{\beta\gamma\delta;\theta}$) respectively.

The contracted Bianchi identities with respect to g' is given by:

$$R^{\alpha}{}_{\beta\gamma\delta|\alpha} + 2R_{\beta[\gamma|\delta]} = 0 \tag{6.2.77}$$

The contracted Bianchi identities with respect to g is given by:

$$R^{\alpha}{}_{\beta\gamma\delta;\alpha} + 2R_{\beta[\gamma;\delta]} = 0 \tag{6.2.78}$$

By 6.2.77 and 6.2.78 we have:

$$R^{\alpha}{}_{\beta\gamma\delta;\alpha} + 2R_{\beta[\gamma;\delta]} = R^{\alpha}{}_{\beta\gamma\delta|\alpha} + 2R_{\beta[\gamma|\delta]} = 0 \tag{6.2.79}$$

The Christoffel symbols with respect to g and g' will be denoted by $\Gamma^{\alpha}_{\beta\gamma}$ and $\Gamma'^{\alpha}_{\beta\gamma}$ respectively and the following result can be found in [13]:

$$P^{\alpha}_{\beta\gamma} = \Gamma'_{\beta\gamma} - \Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g'^{\alpha\delta}(g'_{\delta\beta|\gamma} + g'_{\delta\gamma|\beta} - g'_{\beta\gamma|\delta})$$

$$= \frac{1}{2}\phi^{-1}(\phi_{,\gamma}\delta^{\alpha}_{\beta} + \phi_{,\beta}\delta^{\alpha}_{\gamma} - \phi^{,\alpha}g_{\beta\gamma}) \tag{6.2.80}$$

with

$$\frac{1}{2}\phi^{-1}g^{\alpha\delta}(g_{\delta\beta}\phi_{|\gamma} + \phi g_{\delta\beta|\gamma} + g_{\delta\gamma}\phi_{|\beta} + \phi g_{\delta\gamma|\beta} - g_{\beta\gamma}\phi_{|\delta} - \phi g_{\beta\gamma|\delta})$$

$$= \frac{1}{2}\phi^{-1}(\phi_{,\gamma}\delta^{\alpha}_{\beta} + \phi_{,\beta}\delta^{\alpha}_{\gamma} - \phi^{,\alpha}g_{\beta\gamma}) + \frac{1}{2}\phi^{-1}(\phi g_{\delta\beta|\gamma}g^{\alpha\delta} + \phi g_{\delta\gamma|\beta}g^{\alpha\delta} - \phi g_{\beta\gamma|\delta}g^{\alpha\delta})$$

where comma denotes a partial derivative

Subtraction of the two equations in 6.2.79 and the substitution of 6.2.80 together with some contractions and simplifications leads to:

$$R_{\alpha\beta\gamma\delta}\,\phi^{,\delta} = 0 \qquad (6.2.81)$$

with all raising and lowering of indices done with respect to g. By theorem 6.2.4 equation 6.2.81 has no non-trivial solutions at $p \in M$ so that $\phi^{,\delta} = 0 \Rightarrow \phi = constant$ at $p \in M$. This together with the connectedness of M completes the proof.

●

It is well known that if two metrics g and g' on a manifold M ($n \geq 4$) are conformally related they give rise to the same Weyl tensor components $C^{\alpha}{}_{\beta\gamma\delta}$ in each coordinate system. The previous theorems show that in all cases except type N the converse of this result is, in principle, possible. Let us see why this is.

It is clear that the algebraic consequences of equation 6.2.1 relied only on the symmetries of the Riemann tensor and, as such, apply to the Weyl tensor also, since if g' is compatible with the components $C^{\alpha}{}_{\beta\gamma\delta}$, equation 6.2.1 holds with the curvature components replaced by them.

However, the rank of the Weyl tensor is further restricted by the trace free condition $C^{\gamma}{}_{\alpha\gamma\beta} = 0$. Thus, the rank of the Petrov type I Weyl tensor is four or six, for Petrov type D and II the rank is always six, for type III the rank is four and for type N the rank is always two.

Thus, by theorems 6.2.3 and 6.2.5 we have that if the metric is vacuum of Petrov type I, II, D or III then it is determined to within a constant conformal factor. However, if the Petrov type is N then the Weyl tensor has rank 2 and satisfies case (ii) of theorem 6.2.3 where w is the common, fourfold repeated principal null direction.

Now suppose that the curvature components $R^{\alpha}{}_{\beta\gamma\delta}$ and their first covariant derivatives $R^{\alpha}{}_{\beta\gamma\delta;\mu}$ are given over some coordinate domain. We will investigate the restrictions this imposes on the metric in the Petrov type N vacuum case since we have already established that in all other vacuum cases the curvature determines the metric to within a constant conformal factor.

By theorem 6.2.3(ii), we have that the subspace of bivector space B, in the type N vacuum case, is spanned by two linearly independent bivectors F_1 and F_2 with common eigenvector w having zero eigenvalue. Hence, we have:

$$w_\alpha R^\alpha{}_{\beta\gamma\delta} X_1^\gamma X_2^\delta = F^\alpha{}_\beta w_\alpha = 0 \ \ \forall X_1, X_2 \in T_pM \tag{6.2.82}$$

which then gives:

$$w_\alpha R^\alpha{}_{\beta\gamma\delta} = 0 \tag{6.2.83}$$

Furthermore, if the components of the first covariant derivative are given, we then have the following equality:

$$g'_{\mu\alpha} R^\mu{}_{\beta\gamma\delta;\nu} + g'_{\mu\beta} R^\mu{}_{\alpha\gamma\delta;\nu} = 0 \tag{6.2.84}$$

which in turn gives:

$$g'_{\mu\alpha} R^\mu{}_{\beta\gamma\delta;\nu} X_1^\gamma X_2^\delta X_3^\nu + g'_{\mu\beta} R^\mu{}_{\alpha\gamma\delta;\nu} X_1^\gamma X_2^\delta X_3^\nu = 0 \ \forall X_1, X_2, X_3 \in T_pM \tag{6.2.85}$$

Equation 6.2.85 then provides the equality:

$$g'_{\mu\alpha} P^\mu{}_\beta + g'_{\mu\beta} P^\mu{}_\alpha = 0 \tag{6.2.86}$$

If $w_\alpha R^\alpha{}_{\beta\gamma\delta;\nu} = 0$ then $w_\alpha R^\alpha{}_{\beta\gamma\delta;\nu} X_1^\gamma X_2^\delta X_3^\nu = 0$ is equivalent to the following identity:

$$w_\alpha P^\alpha{}_\beta = 0 \tag{6.2.87}$$

Hence, we have $P \in < F_1, F_2 >$, i.e, any bivector obtained from $R^\alpha{}_{\beta\gamma\delta;\nu}$ is a linear combination of bivectors obtained from $R^\alpha{}_{\beta\gamma\delta}$, so that no new linearly independent bivectors are obtained.

If on the other hand $w_\alpha R^\alpha{}_{\beta\gamma\delta;\nu} \neq 0$ then two linearly independent simple bivectors P_1 and P_2 are obtained from $R^\alpha{}_{\beta\gamma\delta;\nu}$. Then by theorem 6.2.1 new eigenvectors of $g'_{\alpha\beta}$ are obtained. Hence, by applying the same method as that of theorem 6.2.3(iv) one obtains $g'_{\alpha\beta} = \phi g_{\alpha\beta}$ with $\phi = constant$.

So, if $w_\alpha R^\alpha{}_{\beta\gamma\delta;\nu} \neq 0$ at each point of $U \in M$, then sufficient extra eigenvectors of any alternative metric g' are generated to ensure that g' is conformally related to g on $U \in M$.

Now, suppose $w_\alpha R^\alpha{}_{\beta\gamma\delta;\nu} = 0$ on $U \subset M$, then:

$$w_\alpha R^\alpha{}_{\beta\gamma\delta} = 0 \Leftrightarrow w_{\alpha;\nu} R^\alpha{}_{\beta\gamma\delta} + w_\alpha R^\alpha{}_{\beta\gamma\delta;\nu} = 0 \Rightarrow w_{\alpha;\nu} R^\alpha{}_{\beta\gamma\delta} = 0 \qquad (6.2.88)$$

So that w_α is a recurrent vector, i.e, $w_{\alpha;\nu} = w_\alpha p_\nu$, p called the recurrence vector actually if w is a non zero recurrent vector field then it can be locally scaled so that $w_{\alpha;\nu} \propto w_\alpha w_\nu$. A null recurrent vector field w on U may be locally scaled to be covariantly constant ($w_{\alpha;\nu} = 0$) if and only if $R^\alpha{}_{\beta\gamma\delta} w_\alpha = 0$ on M.

Therefore, by the uniqueness of the independent solutions of $k_\alpha R^\alpha{}_{\beta\gamma\delta} = 0$ ($k = w$ for type N) we have that w is a recurrent vector. Since w is null it can be scaled to be covariantly constant on U with respect to g if U is contractable.

Thus, in the case of vacuum Petrov type N, the prescription of $R^\alpha{}_{\beta\gamma\delta}$ and $R^\alpha{}_{\beta\gamma\delta;\nu}$ on U uniquely determines g up to a constant conformal factor unless the recurrent vector field w is (proportional to) a covariantly constant vector field, this being the case of pp waves.

Therefore, in the vacuum case, if the Petrov type is I, II, D or III then the metric is determined to within a constant conformal factor by the curvature. If the Petrov type is N, the rank of the curvature is two. However, apart from one special case, the covariant derivative of the curvature will introduce two extra bivectors which satisfy 6.2.3 and one has, in effect, a rank four situation with the metric determined to within a conformal factor. The special case is where the curvature tensor is complex recurrent and the resulting spacetimes are vacuum pp waves.

The conclusion is that the specification of the components $R^\alpha{}_{\beta\gamma\delta}$ and $R^\alpha{}_{\beta\gamma\delta;\mu}$ in vacuum determines the metric up to a constant conformal factor except when g and g' are pp wave metrics on some open subset of M.

6.3 Relationship to the Karlhede Algorithm

We shall call the number of covariant derivatives of the curvature one needs to calculate in order for the metric to be determined up to a constant conformal factor *Hall's bound* which will be denoted by n_H. We have seen that $n_H = 0$ for vacuum type I, II, D and III and $n_H = 1$ for vacuum type N except in the case of pp waves. It then seems that Hall's bound is much lower than Karlhede's bound for classifying a spacetime. It would be of interest to investigate whether there is some relationship between n_H and Karlhede's bound since both Hall's method and Karlhede's algorithm concern the way the curvature and its successive covariant derivatives determine the geometry of spacetime. The major difficulty, as we shall see, comes from the fact that Hall's method works with the coordinate components of the curvature and its derivatives while Karlhede's algorithm uses the tetrad components of the same.

We consider the simpler cases of Petrov types I and II where the invariance group is the group of dimension zero [18] and the rank of the curvature is six [20].

Now, let $\phi : M \longrightarrow \widetilde{M}$ be a diffeomorphism, i.e, a smooth bijective map with a smooth inverse. Then $\phi_* : T_pM \longrightarrow T_{\phi(p)}\ \widetilde{M}$ is the induced map of tangent vectors and $\phi^* : T_p^*M \longrightarrow T_{\phi(p)}^*\ \widetilde{M}$ is the induced map of cotangent vectors. The coordinate representation of the map $\phi : M \longrightarrow \widetilde{M}$ is given by $\widetilde{x}^{\beta} = \phi^{\beta}(x^{\alpha})$ where x^{α} are local coordinates around $p \in M$ and $\widetilde{x}^{\beta}$ are local coordinates around $\phi(p) \in \widetilde{M}$.

Let us consider R and $\widetilde{R}$ written in coordinate systems x^{α} and $\widetilde{x}^{\alpha}$ respectively. Suppose there exists a diffeomorphism $\phi : M \longrightarrow \widetilde{M}$ giving $\widetilde{x}^{\beta} = \phi^{\beta}(x^{\alpha})$ with the induced map being given by $\phi_* : T_pM \longrightarrow T_{\phi(p)}\ \widetilde{M}$ so that $\frac{\partial}{\partial \widetilde{x}^{\alpha}} = \phi_*(\frac{\partial}{\partial x^{\alpha}})$,i.e $\frac{\partial}{\partial \widetilde{x}^{\alpha}} = \frac{\partial x^{\beta}}{\partial \widetilde{x}^{\alpha}} \frac{\partial}{\partial x^{\beta}}$ so that one can then write:

$$\widetilde{R}_{\alpha\beta\gamma\delta} = \frac{\partial x^{\rho}}{\partial \widetilde{x}^{\alpha}} \frac{\partial x^{\sigma}}{\partial \widetilde{x}^{\beta}} \frac{\partial x^{\tau}}{\partial \widetilde{x}^{\gamma}} \frac{\partial x^{\mu}}{\partial \widetilde{x}^{\delta}} R_{\rho\sigma\tau\mu} \tag{6.3.89}$$

Let $e_{(a)} = e_a^{\alpha} \frac{\partial}{\partial x^{\alpha}} \in T_pM$ and $\widetilde{e}_{(a)} = \widetilde{e}_a^{\alpha} \frac{\partial}{\partial \widetilde{x}^{\alpha}} \in T_{\phi(p)}\ \widetilde{M}$ be the canonical tetrads and let $\widetilde{e}_{(a)} = \phi_*(e_{(a)})$, i.e $\widetilde{e}_a^{\alpha} = \frac{\partial \widetilde{x}^{\alpha}}{\partial x^{\beta}} e_a^{\beta}$. Then by taking the tetrad components of $R_{\alpha\beta\gamma\delta}$ and $\widetilde{R}_{\alpha\beta\gamma\delta}$ with respect to $e_{(a)}$ and $\widetilde{e}_{(a)}$, equation 6.3.89 gives:

$$\widetilde{R}_{\alpha\beta\gamma\delta}\ \widetilde{e}_a^{\alpha} \widetilde{e}_b^{\beta} \widetilde{e}_c^{\gamma} \widetilde{e}_d^{\delta} = \frac{\partial x^{\rho}}{\partial \widetilde{x}^{\alpha}}\ \widetilde{e}_a^{\alpha}\ \frac{\partial x^{\sigma}}{\partial \widetilde{x}^{\beta}}\ \widetilde{e}_b^{\beta}\ \frac{\partial x^{\tau}}{\partial \widetilde{x}^{\gamma}}\ \widetilde{e}_c^{\gamma}\ \frac{\partial x^{\mu}}{\partial \widetilde{x}^{\delta}}\ \widetilde{e}_d^{\delta}\ R_{\rho\sigma\tau\mu}$$

which in turn gives:

$$\widetilde{R}_{abcd} = e_a^{\rho} e_b^{\sigma} e_c^{\tau} e_d^{\mu} R_{\rho\sigma\tau\mu} = R_{abcd}$$

Thus, it is certainly true that if the coordinate components of $\widetilde{R}$ and R are equal then there exists tetrads in which the components of $\widetilde{R}$ and R expressed in those tetrads are equal.

We now investigate whether the existence of a diffeomorphism $\phi : M \longrightarrow \widetilde{M}$ such that $\widetilde{R}_{abcd} = \phi(R_{abcd})$ in some fixed canonical frame, implies that $\widetilde{R} = \phi_*(R)$, i.e equation 6.3.89.

Let the canonical frame for $\widetilde{R}$ and R be $\widetilde{e}_{(a)}$ and $e_{(a)}$ respectively with $\widetilde{e}_{(a)} = \phi_*(e_{(a)})$. If we then take the dual bases $\widetilde{e}^{(a)}$ and $e^{(a)}$ so that $\widetilde{e}^{(a)}_{\phi(p)} = \phi^*(e_p^{(a)})$ we can then write:

$$\tilde{R}_{abcd}\,\tilde{e}^a_\alpha\tilde{e}^b_\beta\tilde{e}^c_\gamma\tilde{e}^d_\delta=\tilde{e}^a_\alpha\tilde{e}^b_\beta\tilde{e}^c_\gamma\tilde{e}^d_\delta\;R_{abcd}$$

which in turn gives:

$$\tilde{R}_{\alpha\beta\gamma\delta}=\frac{\partial x^\rho}{\partial\tilde{x}^\alpha}e^a_\rho\frac{\partial x^\sigma}{\partial\tilde{x}^\beta}e^b_\sigma\frac{\partial x^\tau}{\partial\tilde{x}^\gamma}e^c_\tau\frac{\partial x^\mu}{\partial\tilde{x}^\delta}e^d_\mu R_{abcd}\;=\frac{\partial x^\rho}{\partial\tilde{x}^\alpha}\frac{\partial x^\sigma}{\partial\tilde{x}^\beta}\frac{\partial x^\tau}{\partial\tilde{x}^\gamma}\frac{\partial x^\mu}{\partial\tilde{x}^\delta}R_{\rho\sigma\tau\mu}$$

The difficulty lies in the fact that the existence of a diffeomorphism $\phi : M \longrightarrow \tilde{M}$ giving $\tilde{R}_{abcd} = \phi(R_{abcd})$ with $\tilde{R}$ and R given in the canonical bases $\tilde{e}_{(a)}$ and $e_{(a)}$ respectively does not always imply that $\tilde{e}_{(a)}= \phi_*(e_{(a)})$ so that one cannot conclude that:

$$\tilde{R}_{abcd} = \phi(R_{abcd}) \Rightarrow \tilde{R}= \phi_*(R)$$

Hence, the important issue one needs to solve is the following:

In what circumstances does the existence of a diffeomorphism $\phi : M \longrightarrow \tilde{M}$ *giving* $\tilde{R}_{abcd} = \phi(R_{abcd})$ *realise:*

$$\tilde{R}= \phi_*(R)$$

It would be worth investigating whether the vacuum Einstein equations relating to the simpler cases of Petrov types I and II solutions, where the invariance group is zero dimensional and the rank is six, give any information on this question.

We outline a possible way of tackling this problem. We consider the case of a vacuum spacetime of Petrov type I or II. Let x^α be a fixed coordinate system and consider two curvature tensors R and $\tilde{R}$ with coordinate components $R^{\alpha\beta}{}_{\gamma\delta}$ and $\tilde{R}^{\alpha\beta}{}_{\gamma\delta}$ respectively. Let $e_{(a)} = e^\alpha{}_a\frac{\partial}{\partial x^\alpha}$ and $\tilde{e}_{(a)}= \tilde{e}^\alpha{}_a\frac{\partial}{\partial x^\alpha}$ be the canonical tetrads of R and $\tilde{R}$ with dual covectors $e^{(a)} = e_\alpha{}^a dx^\alpha$ and $\tilde{e}^{(a)} = \tilde{e}_\alpha{}^a dx^\alpha$. Then consider the tetrad components of R and $\tilde{R}$ with respect to their canonical tetrads given by:

$$R^{ab}{}_{cd} \;=\; e^a_\alpha e^b_\beta e^\gamma_c e^\delta_d R^{\alpha\beta}{}_{\gamma\delta} \qquad (6.3.90)$$

$$\tilde{R}^{ab}{}_{cd} \;=\; \tilde{e}^a_\alpha \tilde{e}^b_\beta \tilde{e}^\gamma_c \tilde{e}^\delta_d \tilde{R}^{\alpha\beta}{}_{\gamma\delta} \qquad (6.3.91)$$

We now suppose that:

$$R^{ab}{}_{cd} = \tilde{R}^{ab}{}_{cd} \tag{6.3.92}$$

And we investigate what restrictions equality 6.3.92 imposes on the coordinate components $R^{\alpha\beta}{}_{\gamma\delta}$ and $\tilde{R}^{\alpha\beta}{}_{\gamma\delta}$. We start by considering the following conjecture:

Conjecture 6.3.1 *In a vacuum spacetime of Petrov type I and II if expression 6.3.92 is satisfied one also has* $R^{\alpha\beta}{}_{\gamma\delta} = \tilde{R}^{\alpha\beta}{}_{\gamma\delta}$

If this conjecture is true then we could consider the further conjecture:

Conjecture 6.3.2 *In a vacuum spacetime of Petrov type I or II the metric is determined (up to a constant conformal factor) by the tetrad components of the curvature and hence the Karlhede bound is one.*

Although at the moment we are not able to prove conjecture 6.3.2, mainly because of the difficulty in proving conjecture 6.3.1, we suggest a possible method of proof. We start by grouping the indices in pairs so that we are able to write:

$$R^{\alpha\beta}{}_{\gamma\delta} = R^{A}_{B}$$

$$\tilde{R}^{\alpha\beta}{}_{\gamma\delta} = \tilde{R}^{A}_{B}$$

$$R^{ab}{}_{cd} = R^{\mathbf{A}}_{\mathbf{B}}$$

$$\tilde{R}^{ab}{}_{cd} = \tilde{R}^{\mathbf{A}}_{\mathbf{B}}$$

We may regard the above as 6×6 matrices which take the form [30]:

$$R^{A}_{B} = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$$

where P and Q are 3×3 symmetric trace-free matrices. We denote the components of P and Q by P^i_j, Q^i_j, $\tilde{P}^i_j$, $\tilde{Q}^i_j$, $\mathbf{P}^i_j$, $\mathbf{Q}^i_j$, $\tilde{\mathbf{P}}^i_j$ and $\tilde{\mathbf{Q}}^i_j$ respectively.

Let $\Psi = P + iQ$ then we have that the equality $R^{\mathbf{A}}_{\mathbf{B}} = \tilde{R}^{\mathbf{A}}_{\mathbf{B}}$ is equivalent to having $\mathbf{\Psi}^i_j = \tilde{\mathbf{\Psi}}^i_j$. These results will be useful in the proof of the following conjecture:

Conjecture 6.3.3 $R^{ab}{}_{cd} = \widetilde{R}^{ab}{}_{cd} \Leftrightarrow R^{\alpha\beta}{}_{\gamma\delta} = L^{\alpha}_{\mu} L^{\beta}_{\nu} L^{\rho}_{\gamma} L^{\sigma}_{\delta} \widetilde{R}^{\mu\nu}{}_{\rho\sigma}$ *where* $\eta_{\alpha\beta} L^{\alpha}_{\mu} L^{\beta}_{\nu} = \eta_{\mu\nu}$
i.e: For Petrov type I and II spacetimes, the tetrad components of the curvature are the same if and only if the coordinate components are related by a Lorentz transformation.

The conjecture follows from the fact that $\mathbf{\Psi}^i_j$ and $\widetilde{\mathbf{\Psi}}^i_j$ must be conjugate since they have the same eigenvectors and eigenvalues because $\mathbf{\Psi}^i_j = \widetilde{\mathbf{\Psi}}^i_j$ [2]. One then needs to show that the transformation on the bivectors factors into a skew product of Lorentz transformations.

Alternatively one can consider the Weyl spinor Ψ_{ABCD} . Since the components of the Weyl curvature with respect to the canonical tetrad agree then we must have $\Psi_{ABCD} = \widetilde{\Psi}_{ABCD}$. Hence $\Psi_{ABCD}\ \epsilon_{A'B'}\ \epsilon_{C'D'} = \widetilde{\Psi}_{ABCD}\ \ \epsilon_{A'B'}\ \epsilon_{C'}\epsilon_{D'}$. But the coordinate components of the Weyl tensor are given by (the real part of) $C_{\alpha\beta\gamma\delta} = \sigma^{AA'}_{\alpha}\ \sigma^{BB'}_{\beta}\ \sigma^{CC'}_{\gamma}\ \sigma^{DD'}_{\delta}\ \Psi_{ABCD}\ \epsilon_{A'B'}\ \epsilon_{C'D'}$. While:

$$\widetilde{C}_{\alpha\beta\gamma\delta} = \widetilde{\sigma}^{AA'}_{\alpha}\ \widetilde{\sigma}^{BB'}_{\beta}\ \widetilde{\sigma}^{CC'}_{\gamma}\ \widetilde{\sigma}^{DD'}_{\delta}\ \widetilde{\Psi}_{ABCD}\ \ \epsilon_{A'B'}\ \epsilon_{C'D'}$$

$$= \widetilde{\sigma}^{AA'}_{\alpha}\ \widetilde{\sigma}^{BB'}_{\beta}\ \widetilde{\sigma}^{CC'}_{\gamma}\ \widetilde{\sigma}^{DD'}_{\delta}\ \ \Psi_{ABCD}\ \epsilon_{A'B'}\ \epsilon_{C'D'}$$

Thus the difference in $C_{\alpha\beta\gamma\delta}$ and $\widetilde{C}_{\alpha\beta\gamma\delta}$ arises from a different choice of Van der Waerden symbol. It then remains to show that different choices of such symbol are (with suitable labelling) related by Lorentz transformations which gives the result.

Conjecture 6.3.3 suggests the following conjecture:

Conjecture 6.3.4 $R^{ab}{}_{cd} = \widetilde{R}^{ab}{}_{cd} \Leftrightarrow g_{\alpha\beta} = C L^{\mu}_{\alpha} L^{\nu}_{\beta}\ \widetilde{g}_{\mu\nu}$ *where* $\eta_{\alpha\beta} L^{\alpha}_{\mu} L^{\beta}_{\nu} = \eta_{\mu\nu}$ *and* C *is a constant conformal factor.*

If $\widetilde{g}_{\alpha\beta}$ satisfies $\widetilde{g}_{\epsilon(\alpha} \widetilde{R}^{\epsilon}_{\beta)\gamma\delta} = 0$ then $\hat{g}_{\alpha\beta} = L^{\mu}_{\alpha} L^{\nu}_{\beta}\ \widetilde{g}_{\mu\nu}$ satisfies $\hat{g}_{\epsilon(\alpha} \hat{R}^{\epsilon}_{\beta)\gamma\delta} = 0$ where $\hat{R}^{\alpha}_{\beta\gamma\delta} = L^{\alpha}_{\epsilon} L^{\rho}_{\beta} L^{\sigma}_{\gamma} L^{\tau}_{\delta}\ \widetilde{R}^{\epsilon}_{\beta\gamma\delta}$ and hence $\hat{g}_{\alpha\beta}$ satisfies $\hat{g}_{\epsilon(\alpha} R^{\alpha}_{\beta)\gamma\delta} = 0$. On the other hand Hall [22] has shown that for a type I or II vacuum spacetime the only solution of this last equation, up to a constant conformal factor , is $g_{\alpha\beta}$. Hence we must have $g_{\alpha\beta} = C\hat{g}_{\alpha\beta} = C L^{\gamma}_{\alpha} L^{\nu}_{\beta}\ \widetilde{g}_{\mu\nu}$.

Finally the last step would be to prove the following conjecture:

Conjecture 6.3.5 *Let* $\tilde{g}_{\alpha\beta} = L^{\mu}_{\alpha} L^{\nu}_{\beta} g_{\mu\nu}$ *be the metric of a Petrov type I or II vacuum spacetime then* $\tilde{g}_{\mu\nu}$ *is also the metric of that vacuum spacetime if and only if* $L^{\mu}_{\alpha} = \phi^{\mu}_{,\alpha}$ *and* $L^{\nu}_{\beta} = \phi^{\nu}_{,\beta}$ *for some local diffeomorphism* $\phi' : M \rightarrow M$*.*
i.e ϕ *is a coordinate transformation.*

In other words Conjecture 6.3.4 is saying that the only transformations of the form $\tilde{g}_{\alpha\beta} = L^{\mu}_{\alpha} L^{\nu}_{\beta} g_{\mu\nu}$ which maps vacuum solutions to vacuum solutions are those generated by coordinate transformations so that $\tilde{g}_{\alpha\beta}$ is simply $g_{\alpha\beta}$ in a different coordinate system.

To prove the above conjecture one would use the vacuum Einstein equations corresponding to Petrov types I and II in turn.

Notice that conjecture 6.3.4 implies conjecture 6.3.2. Hence one would be able to lower the bound on the Karlhede algorithm in the special case of vacuum type I and II spacetimes from five to one.

This would then show that Petrov type I and II metrics with the same tetrad components must be conformally related metrics possibly given in different coordinates.

One could use a similar scheme to analyse all other cases, however the more complicated nature of Petrov types III, D and N spacetimes even in the vacuum case, might lead to great difficulties.

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