UNIVERSITY OF SOUTHAMPTON

FACULTY OF MATHEMATICAL STUDIES

Homological Classification of Monoids

by

Akbar Golchin

June, 1997

Thesis Submitted for the degree of Doctor of Philosophy

To my wife Parivash

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to my supervisor Dr. J.H. Renshaw, whose invaluable ideas, comments and patience made this work possible.

I would also like to thank my adviser Dr. G.A. Jones for his encouragement which has allowed me to do my best in research.

My sincere appreciation to my wife Parivash, my sons Ehsan, Payman, Iman and Mohsen for their support and patience throughout my study.

I acknowledge the Islamic Republic of Iran for supporting me in my research.

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF MATHEMATICAL STUDIES

Doctor of Philosophy

HOMOLOGICAL CLASSIFICATION OF MONOIDS

By

Akbar Golchin

We deal in this thesis with what is generally referred to as *homological classification of monoids* by properties of their acts. We have the following hierarchy of properties arranged in strictly decreasing order of strength such that a given act may or may not possesss.

$$free \Rightarrow\ projective \Rightarrow\ strongly\ flat \Rightarrow\ condition(P) \Rightarrow\ flat \Rightarrow$$

$$weakly\ flat \Rightarrow\ principally\ weakly\ flat \Rightarrow\ torsion\ free$$

Many papers have appeared describing classes of monoids over which various of the above distinct properties coincide either for all acts or for all acts of a certain type.

There are monoids such that exact descriptions of their class have not yet been determined, although partial results may be known.

In this work the classification of monoids by condition (P) of (weakly) flat (cyclic) right acts, condition (E), properties of principal ideals, generators and regular acts has been considered. Also by introducing new conditions (P_E), (P'_E), and $(P_{1\,E})$ we classify some monoids by these conditions.

Contents

Introduction

For many years, a fruitful area of research in semigroup theory has been the investigation of properties of acts over monoids. A great deal of work has so far been done in what is generally referred to as *homological classification of monoids*, investigating the conditions on monoids which make the following generally distinct properties of acts, arranged in strictly decreasing order of strength, coincide:

$$free \Rightarrow\ projective \Rightarrow\ strongly\ flat \Rightarrow\ property\ (P) \Rightarrow\ flat \Rightarrow$$

$$weakly\ flat \Rightarrow\ principally\ weakly\ flat \Rightarrow\ torsion\ free.$$

Monoids for which flatness and weak flatness of acts coincide were considered by Bulman-Fleming and McDowell (see [5]), although the exact description of these monoids is not known at the moment. But in contrast, monoids over which all weakly flat or flat cyclic right acts are strongly flat are exactly the right nil monoids (i.e. monoids in which some power of each non-identity element is a right zero element). Similarly, monoids over which all cyclic right acts having property (P) are strongly flat, are known to be exactly the aperiodic monoids (i.e. monoids in which, for each element x, there exists some $n \in \mathbb{N}$ such that $x^{n+1} = x^n$). But the general description of this class of monoids is still unknown.

In 1981, Ulrich Knauer and Mario Petrich characterized monoids for which all torsion free right acts are free or projective. They gave a necessary condition for a monoid S such that all torsion free right S-acts be strongly flat. They also characterized monoids for which all right acts are free, projective or strongly flat.

Condition (P) appeared as part of Stenstrom's definition [34] of what we now call strong flatness. Normak [31] considered condition (P) on its own. He [31] also showed that condition (P) lies strictly between flatness and strong flatness, that for a monoid S, all right S-acts have property (P) if and only if S is a group, and that all right S-acts having property (P) are free if and only if $S = \{1\}$.

In 1992, Bulman-Fleming [1] proved that for a monoid S, if all flat right S-acts satisfy condition (P), then $E(S) = \{1\}$. ($E(S)$ as usual denotes the set of idempotent elements of S.) He posed the question of whether the converse is true.

Liu and Yang [40] (1994) gave an example settling in the negative the question referred to in the previous paragraph. Liu also proved that if S is right reversible and if all flat cyclic right S-acts satisfy condition (P), then $E(S) \subseteq \{0, 1\}$. He showed that, under the additional hypothesis that S is left PP (i.e., all principal left ideals of S are projective), all flat cyclic right S-acts satisfy condition (P) if and only if S is either a right cancellative monoid, or a right cancellative monoid with zero adjoined. Assuming existence of a regular left S-act, Liu showed that the condition $E(S) = \{1\}$ is necessary and sufficient for all flat cyclic right S-acts to satisfy condition (P).

In (1995), Bulman-Fleming and Normak [7] showed that, over left PP monoids S, every flat cyclic right S-act satisfies condition (P) if and only if every element of S is either right cancellative or right zero (This generalizes a result in [39].) In [8], the same authors presented more-or-less complete results on flatness properties of monocyclic acts (i.e. acts of the form $S/\rho(s,t)$ where $s, t \in S$.)

There are still however monoids for which the exact descriptions of their class have not yet been determined. In particular monoids for which all (weakly) flat (cyclic) right acts satisfy condition (P). To date the only definitive results have been found when restricting attention to certain classes of monoids.

In this work we investigate the classification of monoids by properties of their acts and we begin in chapter 1 with some definitions and results. In chapter 2, we extend the results of classification of monoids by condition (P) of (weakly) flat (cyclic) right acts such that many of the main results of recent papers on this subject will appear as corollaries. We also extend the flatness property of (cyclic) acts of monoids of the form $S = G \,\dot{\cup}\, I$ to flatness of I^1-acts where I is an ideal of S. In chapter 3, we consider left PSF monoids and we extend some of the results in [7] and [40]. In chapter 4, by introducing new conditions (P_E) and (P'_E). We show these conditions can be placed between condition (P) and weak flatness in the above sequence and then a classification of some monoids by these conditions and also condition (E) are considered. Characterizations of monoids by properties of generators and regular acts are considered in chapters 5 and 6 respectively. There are also examples and counter-examples throughout chapters where necessary.

Chapter 1

1. Basic definition and results

In this chapter some basic definitions and results are presented. Reference will be made to these throughout this thesis. Although there are other definitions and results presented in every chapter where necessary.

Definition 1.1. By a *groupoid* $(S, \cdot)$ we shall mean a non-empty set S on which a binary operation $(\cdot)$ is defined. We shall say that $(S, \cdot)$ is a *semigroup* if $(\cdot)$ is *associative*, or

$$(\forall x, y, z \in S) \quad (x \cdot y) \cdot z = x \cdot (y \cdot z).$$

We shall write $(x \cdot y)$ simply as xy and usually refer to the semigroup operation as *multiplication*. If S has the additional property that,

$$(\forall x, y \in S) \quad xy = yx,$$

we shall say that it is a *commutative* semigroup. If there exists an element 1 of S such that

$$(\forall x \in S) \quad x1 = 1x = x,$$

we say that 1 is an *identity* (element) of S and that S is a semigroup with identity, or *monoid*. A semigroup S has at most one such element. If S has no identity element it is very easy indeed to adjoin an extra element 1 to the set S. Then if we define

$$(\forall s \in S) \quad 1s = s1 = s,$$

and

$$11 = 1,$$

$S \cup \{1\}$ becomes a semigroup with identity element 1. We shall consistently use the notation S^1 with the following meaning:

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

S^1 is called *the semigroup obtained from S by adjoining an identity if necessary.* If a semigroup S with at least two elements contains an element 0 such that

$$(\forall x \in S) \quad x0 = 0x = 0,$$

we say 0 is a zero (element) of S and that S is a *semigroup with zero.* Also there can be at most one such element. If S has no zero element, then again it is easy to adjoin an extra element 0 to the set S. Then we define

$$(\forall s \in S) \quad 0s = s0 = 0,$$

and

$$00 = 0,$$

making $S \cup \{0\}$ into a semigroup with zero element 0. Continuing the analogy with the case of the identity element, we write

$$S^0 = \begin{cases} S & \text{if } S \text{ has a zero element} \\ S \cup \{0\} & \text{otherwise.} \end{cases}$$

and refer to S^0 as *the semigroup obtained from S by adjoining a zero if necessary.*

Note that by adjoining a zero to a semigroup we may lose some essential property of the semigroup. Therefore, we cannot simply reduce the study of semigroups to that of monoids with zero. For example, if we adjoin a zero element to a semigroup which is a group, we obtain a semigroup which is not a group.

Definition 1.2. A semigroup S with zero is called *null semigroup* if the product of any two elements is zero. It is called a *nil semigroup* if for every $a \in S$ there exists $n \in \mathbb{N}$ such that a^n is zero.

Definition 1.3. If a semigroup S has the property that

$$(\forall a \in S)\ aS = S \text{ and } Sa = S$$

we call it a *group*.

If G is a group, then $G^0 = G \cup \{0\}$ is a semigroup. A semigroup formed in this way is called a *0 − group*, or *group − with − zero.*

Definition 1.4. A non-empty subset T of a semigroup S is called a *subsemigroup* if it is closed with respect to multiplication, that is if

$$(\forall x, y \in T) \quad xy \in T$$

S is a subsemigroup of itself, and if S has an identity or a zero, then $\{0\}$ and $\{1\}$ are also subsemigroups of S. In general if S has an *idempotent*, that is to say, an element e for which $e^2 = e$, then $\{e\}$ is a subsemigroup of S.

A subsemigroup of S which is a group with respect to the multiplication inherited from S will be called a *subgroup* of S.

It can be seen that a non-empty subset T of S is a subgroup of S if and only if

$$(\forall a \in T) \quad aT = T \text{ and } Ta = T$$

Definition 1.5. A non-empty subset I of a semigroup S is called a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a *(two − sided)ideal* if it is both a left and a right ideal. It is obvious that every ideal (whether one- or two-sided) is a subsemigroup, but the converse is not true. S is an ideal of itself and if S has a zero element, then $\{0\}$ is also an ideal of S. An ideal I of S is called *proper* if $\{0\} \subset I \subset S$.

Definition 1.6. If a is an element of a semigroup S, the smallest left ideal containing a is $Sa \cup \{a\}$, which we may conveniently write as S^1a, and which we shall call the *principal left ideal generated by a.* Similarly, the smallest right ideal containing a is $aS \cup \{a\}$ which we write as aS^1 and we call the *principal right ideal generated by a.*

Definition 1.7. S is called a *right principal ideal monoid* if all right ideals of S are principal.

Definition 1.8. A monoid S is called *left [right] reversible* if any two principal right [left] ideals of S intersect.

Definition 1.9. Let Y be a non-empty subset of a partially ordered set $(X, \leq)$. An element a of Y is called *minimal* if there is no element of Y that is strictly less than a, that is to say, if

$$(\forall y \in Y)\ y \leq a \Rightarrow y = a.$$

An element b of Y is called *minimum* if

$$(\forall y \in Y)\ b \leq y.$$

Definition 1.10. Let S be a semigroup and $e, f \in S$ be idempotents. We shall write $e \leq f$ if $ef = fe = e$. It is obvious that $\leq$ is a partial order relation on the set E of idempotents of S. Notice that if S has an identity element 1 then $e \leq 1$ for every $e \in E$, and that if it has a zero element 0 then $0 \leq e$ for every $e \in E$.

An idempotent is called *primitive* if it is non-zero and is minimal in the set of non-zero idempotents (with respect to the order just described).

Definition 1.11. A semigroup without zero is called *simple* if it has no proper ideals. A semigroup S with zero is called *0-simple* if
(i) $\{0\}$ and S are its only ideals;
(ii) $S^2 \neq \{0\}$.

If S is a semigroup without zero, then we say that S is *completely simple* if S is simple and if it contains a primitive idempotent.

A semigroup will be called *completely 0-simple* if it is 0-simple and has a primitive idempotent.

Definition 1.12. An element a of semigroup S is called *regular* if there exists $a' \in S$ such that $aa'a = a$. The semigroup S is called *(Von Neumann) regular* if all its elements are regular.

A semigroup S is called an *inverse semigroup* if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that

$$aa^{-1}a = a, \quad a^{-1}aa^{-1} = a^{-1}.$$

Definition 1.13. A semigroup S is *completely regular* if every $a \in S$ lies in a subgroup of S.

Definition 1.14. A *Clifford semigroup* is defined as a completely regular semigroup S in which, for all $x, y \in S$

$$(xx^{-1})(yy^{-1}) = (yy^{-1})(xx^{-1}).$$

Definition 1.15. If S and T are semigroups, then the cartesian product $S \times T$ becomes a semigroup if we define

$$(s,t)(s',t') = (ss',tt').$$

we refer to this semigroup as the *direct product* of S and T.

Definition 1.16. By a *band* we mean a semigroup S in which every element is idempotent. If for every $a, b, c \in S$, $abc = acb$ ($abc = bac$), then S is a *left [right] normal band*. If $abca = acba$, then it is a *normal band*. If $aba = ab$ ($aba = ba$), then S is a *left [right] regular band*. If $ab = a$ ($ab = b$), then it is a *left [right] zero band*. S is called *rectangular band* if $aba = a$. It is a *semilattice* if $ab = ba$ and finally S is a *trivial band* if $a = b$.

Definition 1.17. Let S be a semigroup. An equivalence $\mathcal{L}$ on S is defined by the rule that $a \mathcal{L} b$ if and only if a and b generate the same principal left ideal, that is, if and only if $S^1a = S^1b$.

Similarly, we define the equivalence $\mathcal{R}$ by the rule that $a \mathcal{R} b$ if and only if $aS^1 = bS^1$.

We refer to $\mathcal{L}$ and $\mathcal{R}$ as *Green's* equivalences. It is shown in [20] that $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence. The join $\mathcal{L} \vee \mathcal{R}$ is also important and we denote it by $\mathcal{D}$.

Theorem 1.18 [19]. *The following statements about a semigroup S are equivalent:*

(1) S is an inverse semigroup.

(2) S is regular and idempotent elements commute.

(3) Each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains a unique idempotent.

(4) *Each principal left ideal and each principal right ideal of S contains a unique idempotent generator.*

Proposition 1.19 [19]. *Let S be an inverse semigroup with semilattice of idempotents E. Then*

(1) $(a^{-1})^{-1} = a$ *for every a in S.*

(2) $e^{-1} = e$ *for every e in E.*

(3) $(ab)^{-1} = b^{-1}a^{-1}$ *for every $a, b \in S$.*

(4) $aea^{-1} \in E, a^{-1}ea \in E$ *for every a in S and every e in E.*

(5) $a \mathcal{R} b$ *if and only if* $aa^{-1} = bb^{-1}$; $a \mathcal{L} b$ *if and only if* $a^{-1}a = b^{-1}b$.

(6) *If $e, f \in E$, then $e \mathcal{D} f$ in S if and only if there exists a in S such that* $aa^{-1} = e, a^{-1}a = f$.

Definition 1.20. Let S be a semigroup. An element c is called *central* if $cs = sc$ for every $s \in S$. The set of central elements forms a subsemigroup of S, called the *centre* of S.

Definition 1.21. Let S be a monoid. An element $a \in S$ is called *right [left] invertible* if there exists $a' \in S$, such that $aa' = 1$ $[a'a = 1]$.

Definition 1.22. A semigroup S is called *right [left] cancellative* if for all $a, b, c \in S$, $ac = bc$ yields $a = b$ $[ca = cb \Rightarrow a = b]$.

Definition 1.23. If ϕ is a mapping from a semigroup $(S, .)$ into a semigroup $(T, .)$ we say that ϕ is *morphism (or homomorphism)* if

$$(\forall x, y \in S) \quad (xy)\phi = (x\phi)(y\phi).$$

If $(S, ., 1_S)$ and $(T, ., 1_T)$ are *monoids*, with identity elements $1_S, 1_T$ respectively, then ϕ will be called a morphism only if we have the additional property

$$1_S\phi = 1_T.$$

We refer to S as the *domain* of ϕ, to T as the *codomain* of ϕ, and to the subset

$$S\phi = \{s\phi \mid s \in S\}$$

of T as the *range* of ϕ. If ϕ is one-one we shall call it a *monomorphism*, and if it is both one-one and onto (bijective) we shall call it an *isomorphism.* If there exists an isomorphism $\phi : S \rightarrow T$ we say that S and T are *isomorphic* and write $S \simeq T$. If ϕ is a homomorphism from S into S we call it an *endomorphism* of S, and if it is one-one and onto it is called an *automorphism.* According to the theory of categories, a monoid morphism $\alpha : S \rightarrow T$ is an *epimorphism* if, for all monoids U and all morphisms $\beta, \gamma : T \rightarrow U$,

$$\alpha\beta = \alpha\gamma \;\Rightarrow\; \beta = \gamma.$$

Definition 1.24. Let S be a semigroup. A relation $\mathbf{R}$ on the set S is called *left compatible* (with the operation on S) if

$$(\forall s, t, a \in S) \quad (s, t) \in \mathbf{R} \Rightarrow (as, at) \in \mathbf{R},$$

and *right compatible* if

$$(\forall s, t, a \in S) \quad (s, t) \in \mathbf{R} \Rightarrow (sa, ta) \in \mathbf{R}.$$

It is called *compatible* if

$$(\forall s, t, s', t' \in S) \quad [(s, t) \in \mathbf{R} \text{ and } (s', t') \in \mathbf{R}] \Rightarrow (ss', tt') \in \mathbf{R}.$$

A left [right] compatible equivalence is called a *left [right] congruence.* A compatible equivalence is called a *congruence.*

Proposition 1.25 [19]. *A relation ρ on a semigroup S is a congruence if and only if it is both left and right congruence.*

Definition 1.26. Let S be a monoid with identity element 1 and let X be a non-empty set. We say that X is a *left S-act* if there is an action $(s, x) \longmapsto sx$ from $S \times X$ into X with the properties

$$(st)x = s(tx) \quad (s, t \in S, x \in X),$$

$$1x = x \quad (x \in X).$$

Various alternative names have been used, such as S-system, S-set and S-operand. Dually, a non-empty set X is a right S-act if there is an action $(x, s) \longmapsto xs$ from $X \times S$ into X such that

$$x(st) = (xs)t \quad (s, t \in S, x \in X),$$

$$x1 = x \quad (x \in X).$$

Also, if S and T are (not necessarily different) monoids, we say that X is an *(S,T)-biact*, if it is a left S-act, a right T-act, and if

$$(sx)t = s(xt) \quad \text{for all } s \in S, t \in T \text{ and } x \in X.$$

By $X \in S-$ENS or $X \in S - Act$ we mean, X is a left S-act. The meanings to be attached to the statements $X \in$ ENS$-S$ and $X \in S-$ENS$-T$ are analogous.

Remark. If S is commutative monoid, then there is no distinction between a left and a right S-act. For if $X \in S-$ENS we may define a right action $*$ of S on X by

$$x * s = sx \quad (x \in X, \ s \in S).$$

Then certainly $x * 1 = x$ for all x. Also, for all $s, t \in S$,

$$x * (st) = x * (ts) = (ts)x = t(sx) = (x * s) * t.$$

Indeed we can regard X as an (S,S)-biact, since for all $x \in X$ and $s, t \in S$

$$(sx) * t = t(sx) = (ts)x = (st)x = s(tx) = s(x * t).$$

It is clear that any set X whatever can be regarded as a $(\{1\},\{1\})$-biact, where $\{1\}$ is the trivial monoid. It will therefore occasionally be convenient to state and prove results for (S,T)-biacts, deducing results regarding one-sided acts by taking either S or T as the trivial monoid. At other times it will be sufficient to consider the case of a left S-act, since the analogous results for acts of other kinds will be obvious.

The *coproduct* in $S-$Act and Act$-S$ is the disjoint union, and is denoted by $\coprod$, the *product* is the cartesian product with componentwise multiplication by elements of S, and is denoted by $\prod$.

Definition 1.27. Let S be a monoid. A *subact* of a left S-act X is a subset Y of X with the property that $SY \subseteq Y$.

Definition 1.28. Let S be a monoid. By a *morphism* (or *S-morphism* or *S-map*) from a left S-act X into a left S-act Y we mean a map $\phi : X \to Y$ with the property that

$$(sx)\phi = s(x\phi) \quad (s \in S, \ x \in X)$$

Definition 1.29. Let S be a monoid. A *congruence* on a left S-act X is an equivalence on X with the property that, for all $x, y \in X$ and all $s \in S$

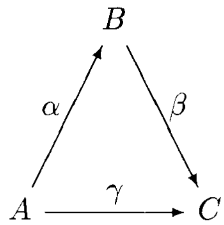
$$x \;\rho\; y \Rightarrow sx \;\rho\; sy.$$

The quotient X/ρ then inherits a left S-act structure by means of the definition

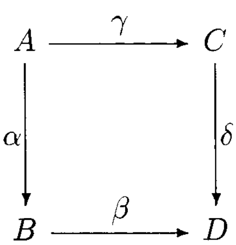
$$s(x\rho) = (sx)\rho$$

and there is a morphism (read 'ρ natural') $\rho^\natural : X \to X/\rho$ defined by the rule that $x\rho^\natural = x\rho$ for every $x \in X$.

Definition 1.30. We call a diagram of the form



commutative if $\beta\alpha = \gamma$, and we shall say in this case that the morphism γ *factors through* B. Likewise a diagram of the form



is commutative if $\beta\alpha = \delta\gamma$.

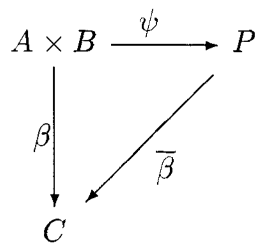
Definition 1.31. Let S be a monoid. Then it is clear that the cartesian product $X \times Y$ of a left S-act X and a right T-act Y becomes an (S,T)-biact if we make the obvious definitions

$$s(x, y) = (sx, y), \;\; (x, y)t = (x, yt).$$

Let $A \in T\text{-}\mathrm{ENS}\text{-}S$, $B \in S\text{-}\mathrm{ENS}\text{-}U$ and $C \in T\text{-}\mathrm{ENS}\text{-}U$. Then by the last paragraph we may give the $A \times B$ the structure of a (T,U)-biact. A (T,U)-map $\beta : A \times B \to C$ will be called a *bimap* if, for all a in A, b in B, and s in S,

$$(as, b)\beta = (a, sb)\beta.$$

A pair (P, ψ) consisting of a (T, U)-biact P and a bimap $\psi : A \times B \to P$ will be called a *tensor product* of A and B over S if for every (T, U)-biact C and every bimap $\beta : A \times B \to C$ there exists a unique (T, U)-map $\overline{\beta} : P \to C$ such that the diagram



commutes.

Lemma 1.32. *If a tensor product of A and B over S exists, then it is unique up to isomorphism.*

Let us define $A \otimes_S B$ to be $A \times B/\tau$, where τ is the equivalence on $A \times B$ generated by the relation

$$\mathbf{R} = \{\, ((as, b), (a, sb)) : a \in A, b \in B, s \in S \,\}.$$

We denote a typical element $(a, b)\tau$ of $A \otimes_S B$ by $a \otimes b$ and note that by the definition of τ we immediately have that $as \otimes b = a \otimes sb$ for all a in A, s in S and b in B.

Proposition 1.33 [20]. *Let $A \in T\text{-}\mathrm{ENS}\text{-}S$, $B \in S\text{-}\mathrm{ENS}\text{-}U$. Then $(A \otimes_S B, \tau^{\natural})$ is a tensor product of A and B over S.*

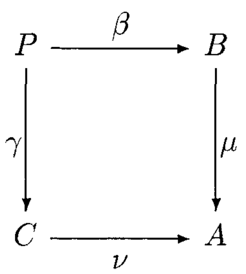
Definition 1.34. Let S be a monoid . A right S-act A is *flat* if the functor $A \otimes -$ (from left S-acts into sets) preserves monomorphisms, or for every monomorphism $\phi : X \to Y$ of left S-acts the induced map $1 \otimes \phi : A \otimes_S X \to A \otimes_S Y$ is one-one. Dually, a left S-act A is flat if for every monomorphism $\psi : X \to Y$ of right S-acts the induced map $1 \otimes \psi : X \otimes_S A \to Y \otimes_S A$ is one-one.

If the functor $A \otimes -$ preserves monomorphism of [principal] left ideals of S into S (considered as a left S-act), then A is called *[principally] weakly flat.*

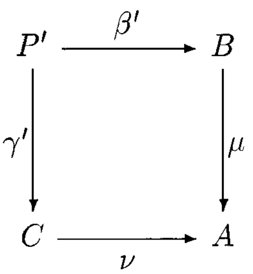
A monoid S is called *left [right] absolutely flat* if all left [right] S-acts are flat and *absolutely flat* if it is both left and right absolutely flat.

A semigroup S is called *left [right] absolutely flat* if S^1 is a left [right] absolutely flat monoid.

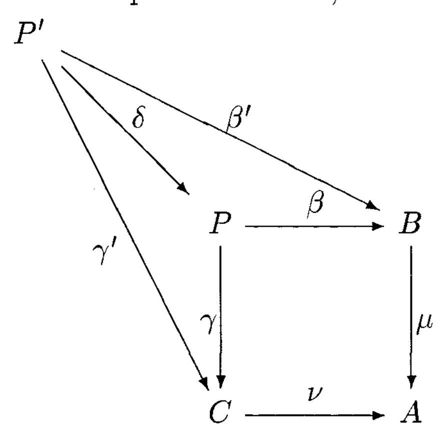
Definition 1.35. Given two morphisms $\mu : B \to A$ and $\nu : C \to A$, we define a commutative diagram



to be a *pullback diagram* if whenever

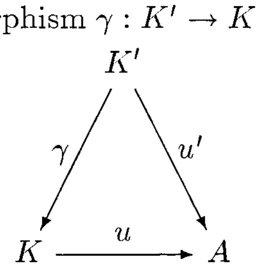


is commutative there exists a unique $\delta : P' \to P$, such that the diagram



commutes.

Definition 1.36. Given two morphisms $\alpha, \beta : A \to B$, we say that $u : K \to A$ is an *equalizer* for α and β if $\alpha u = \beta u$, and if whenever $u' : K' \to A$ is such that $\alpha u' = \beta u'$ there is a unique morphism $\gamma : K' \to K$ making the diagram



commutative.

Definition 1.37. Let S be a monoid. We call a right S-act A, *equalizer-flat* if the functor $A \otimes -$ preserves equalizers, and we call it *pullback-flat* if the functor $A \otimes -$ preserves pullbacks.

Definition 1.38. Let S be a monoid. If the functor $A \otimes -$ preserves pullbacks and equalizers, then A is said to be *strongly flat.*

Definition 1.39. Let S be a monoid. A right S-act A satisfies *condition (P)* if whenever $a, a' \in A$, $u, v \in S$, and $au = a'v$, there exists $a'' \in A$ and s, $t \in S$ such that $a = a''s$, $a' = a''t$ and $su = tv$.

Definition 1.40. Let S be a monoid. A right S-act A satisfies *condition (E)* if whenever $a \in A$, $u, v \in S$, and $au = av$, there exist $a'' \in A$ and $t \in S$ such that $a = a''t$ and $tu = tv$.

Theorem 1.41 [34, 5.3]. *A right S-act A is strongly flat if and only if it satisfies conditions (P) and (E).*

Remark. By the proof of Theorem 1.41, it can be seen that for a monoid S and a right S-act A, if A is equalizer-flat, then A satisfies condition (E), but the converse is not true see [31, Example 1.13] and if A is pullback-flat, then A satisfies condition (P), but the converse is not true see [31, Example 1.14]. Also by [31, Proposition 2.9] it can be seen that every equalizer-flat S-act is flat and [31, Corollary 3.7] shows that all pullback-flat S-acts are flat.

Theorem 1.42. *Let S be a monoid and $A = \coprod_{i \in I} A_i$, a disjoint union of right S-acts A_i. Then A satisfies condition (E) if and only if A_i satisfies condition (E) for all $i \in I$.*

Proof. Suppose that A_i, $i \in I$ satisfies condition (E), and let $au = av$ with $a \in A$, $u, v \in S$. Since $a \in A$, then there exists $j \in I$ such that $a \in A_j$. But A_j satisfies condition (E), and so there exist $a'' \in A_j$, $t \in S$ such that $a = a''t$ and $tu = tv$. Since $A_j \subseteq A$ then $a'' \in A$ and so A satisfies condition (E).

Now, suppose that A satisfies condition (E) and let $au = av$ with $u, v \in S$, $a \in A_i$, $i \in I$. Since $A_i \subseteq A$, then $a \in A$. But A satisfies condition (E) and so there exist $a'' \in A$, $s, t \in S$ such that $a = a''t$ and $tu = tv$.

We claim that $a'' \in A_i$. Otherwise, there exists $j \in I$ such that $j \neq i$ and that $a'' \in A_j$. Since A_j is a right S-act, then $a''t \in A_j$. Consequently, $a \in A_j$. But $A_i \cap A_j = \emptyset$ and so a contradiction. Thus $a'' \in A_i$ and so A_i satisfies condition (E) as required. ■

Theorem 1.43. *Let S be a monoid and $A = \coprod_{i \in I} A_i$, a disjoint union of right S-acts A_i. Then A satisfies condition (P) if and only if A_i satisfies condition (P) for all $i \in I$.*

Proof. Suppose that A_i, $i \in I$ satisfies condition (P) and let $au = a'v$ with $a, a' \in A$, $u, v \in S$. Since $a, a' \in A$, then there exist $i, j \in I$ such that $a \in A_i$ and $a' \in A_j$. Since A_i and A_j are right S-acts, then $au \in A_i$ and $a'v \in A_j$. Since $au = a'v$ and $A_i \cap A_j = \emptyset$, then $i = j$ and so $a, a', \in A_i$. Consequently, $au = a'v$ implies that there exist $s, t \in S$, $a'' \in A_i$ such that $a = a''s$, $a' = a''t$ and $su = tv$. But $A_i \subseteq A$ and so $a'' \in A$. Thus A satisfies condition (P).

Now suppose that A satisfies condition (P) and let $au = a'v$ with $u, v \in S$, $a, a' \in A_i$, $i \in I$. Since $A_i \subseteq A$, then $a, a' \in A$. Consequently, $au = a'v$ implies that there exist $a'' \in A$, $s, t \in S$ such that $a = a''s$, $a' = a''t$ and $su = tv$.

We claim that $a'' \in A_i$. Otherwise, there exists $j \in I$ such that $j \neq i \in I$ and $a'' \in A_j$. Since A_j is a right S-act, then $a''s, a''t \in A_j$. Consequently, $a, a' \in A_j$. But $A_i \cap A_j = \emptyset$ and so a contradiction. Thus $a'' \in A_i$, and so A_i satisfies condition (P) as required. ■

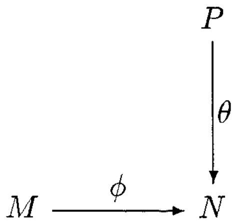
Theorems 1.41, 1.42, and 1.43, clearly give

Corollary 1.44. *Let S be a monoid and $A = \coprod_{i \in I} A_i$ a disjoint union of right S-acts A_i. Then A is strongly flat if and only if A_i is strongly flat for all $i \in I$.*

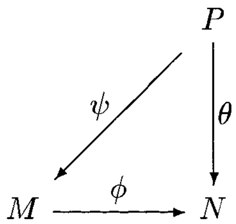
Definition 1.45. Let S be a monoid. A right S-act A is called *torsion free* if $as = a's$, with $a, a' \in A$, $s \in S$ right cancellable, implies $a = a'$.

Definition 1.46. Let S be a monoid. A right S-act A is *free* if $A \simeq \coprod S$ for some non-empty set I, S being considered as a right act.

Definition 1.47. Let S be a monoid. An S-act P is *projective* if given any diagram of S-acts and S-homomorphisms



where $\phi : M \to N$ is an epimorphism, there exists an S-homomorphism $\psi : P \to M$ such that



is commutative.

Definition 1.48. A monoid S is called *left PP* if every principal left ideal of S is projective (as a left S-act). In [21] it is shown that S is left PP if and only if for every $x \in S$ there exists $e \in S$ (necessarily idempotent) such that $ex = x$, and $ux = vx$ implies $ue = ve$, for all $u, v \in S$. The class of left PP monoids is fairly extensive in that every regular monoid and every right cancellative monoid is left PP.

Theorem 1.49 [19]. *Let ρ, σ be equivalences on a set S [congruence on a semigroup S]. Then $(a, b) \in \rho \vee \sigma$ if and only if, for some $n \in \mathbb{N}$, there exist elements $x_1, x_2, \ldots, x_{2n-1} \in S$ such that*

$$(a, x_1) \in \rho, (x_1, x_2) \in \sigma, (x_2, x_3) \in \rho, \ldots, (x_{2n-1}, b) \in \sigma.$$

Note that in Theorem 1.49, $\rho \vee \sigma$ is the join of ρ and σ. For more information see [19, p. 28].

Definition 1.50. Let S be a monoid. A right S-act A is *cyclic* if there exists $a \in A$ such that $A = aS$.

Theorem 1.51. *Let S be a monoid and let ρ be a right congruence on S. Then S/ρ is cyclic, and every cyclic right S-act aS is isomorphic to S/ρ for some right congruence ρ on S.*

Proof. Since $1 \in S$, then $1\rho \in S/\rho$ and so for every $s \in S$, $s\rho = (1\rho)s$. Thus, S/ρ is cyclic.

Let aS be a cyclic right S-act. If we define

$$u \; \rho \; v \Longleftrightarrow au = av,$$

then ρ is a right congruence on S. If $\phi : aS \to S/\rho$ is such that $as \mapsto s\rho$, then ϕ is an isomorphism and so S/ρ is cyclic.

Remark. If x, y are elements of a monoid S we denote by $\rho(x, y)$ the smallest right congruence on S which identifies these two elements. Every act of the form $S/\rho(x, y)$ is called a *monocyclic* act.

Definition 1.52. Let S be a monoid. A right S-act A is called *finitely generated* if there exist $a_1, a_2, \ldots, a_n \in A$ such that $A = a_1S \cup a_2S \cup \ldots \cup a_nS$.

Lemma 1.53 *Let S be a monoid and A a finitely generated right S-act. If A satisfies condition (P), then A is a coproduct of cyclic right S-acts.*

Proof. Let A be a finitely generated right S-act. Then there exist $a_1, a_2, \ldots, a_n \in A$ such that they are generators for A. We can also suppose that $n \in \mathbb{N}$ is the smallest such positive number and let $I = \{1, 2, \ldots, n\}$. Then we claim that $A = \coprod_{i \in I} a_iS$. Otherwise there exist $i, j \in I$ such that $i \neq j$ and $a_is = a_jt$ for some $s, t \in S$. Since A satisfies condition (P), then there exist $u, v \in S$ and $a'' \in A$ such that $a_i = a''u, a_j = a''v$ and $us = vt$. Since $a'' \in A$, then there exist $k \in I$ and $w \in S$ such that $a'' = a_kw$. Thus $a_i = a''u = a_kwu$, $a_j = a''v = a_kwv$. It means that a_i and a_j are generated by a_k, and so the number of generators is less than n which is a contradiction. ∎

Lemma 1.54 [7]. *Let S be a monoid and let ρ be a right congruence on S. Then*

(1) S/ρ is free if and only if there exist $s, t \in S$ such that $st = 1$ and, for all $u, v \in S$, $u \; \rho \; v$ if and only if $su = tv$.

(2) S/ρ is projective if and only if there exists $e^2 = e \in S$ such that $e \; \rho \; 1$, and $u \; \rho \; v$ implies $eu = ev$ for all $u, v \in S$.

(3) S/ρ is strongly flat if and only if for all $u, v \in S$ with $u \; \rho \; v$ there exists $s \in S$ such that $su = sv$ and $s \; \rho \; 1$.

(4) *S/ρ satisfies condition (P) if and only if for all $u, v \in S$ with $u \ \rho \ v$ there exist $s, t \in S$ such that $su = tv$ and $s \ \rho \ 1 \ \rho \ t$.*

(5) *S/ρ is flat if and only if, for any left congruence λ on S and any $u, v \in S$, if $u(\rho \ \vee \ \lambda)v$, then there exist $s, t \in S$ with $su \ \lambda \ tv$ such that $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$.*

(6) *S/ρ is weakly flat if and only if for all $u, v \in S$ with $u \ \rho \ v$ there exist $s, t \in S$ such that $su = tv$, $s(\rho \ \vee \ \Delta u)1$ and $t(\rho \ \vee \ \Delta v)1$.*

(7) *S/ρ is principally weakly flat if and only if whenever $u, v, x \in S$ and $ux \ \rho \ vx$, then $u(\rho \ \vee \ \Delta x)v$.*

(8) *S/ρ is torsion-free if and only if whenever $u, v, c \in S$, c is right cancellable, and $uc \ \rho \ vc$, then $u \ \rho \ v$.*

Note : In (5) above, λu denotes the left congruence on S defined by $x(\lambda u)y$ if $xu \ \lambda \ yu$, for $x, y, u \in S$, λv is defined similarly. In (6) and (7), Δ denotes the equality relation on S, and in (5), (6) and (7), $\vee$ denote the join in the lattice of equivalence relation on S.

Definition 1.55. A right [left] S-act A over a monoid S is called *reversible* if any two cyclic sub-S-acts of A intersect.

Chapter 2

Characterization of Monoids by Condition (P) of (Cyclic) Right Acts

2.1. Introduction

As it was mentioned in the introduction, most of our attention in this thesis is directed at the classification of monoids by condition (P) of (weakly) flat (cyclic) right acts. In this chapter, in section 2.2, we extend some results in [7] and [39] on monoids for which all flat cyclic right acts satisfy condition (P). It was shown in [7] that right nil monoids and monoids whose elements are either right cancellative or right zero have the property that all flat cyclic right acts satisfy condition (P). Here we extend these results to those monoids whose non right nil elements form a group (right elementary monoids) and prove that among the periodic monoids exactly those for which all flat cyclic right acts satisfy condition (P) are right elementary monoids. We also give a characterization of monoids S with $S \setminus \{1\}$ right group such that all flat cyclic right S-acts satisfy condition (P). There are some results and also examples of right elementary monoids.

In section 2.3, we extend results of right elementary monoids to certain types of right subelementary monoid (monoids of the form $S = C \dot{\cup} N$.) Then we give a feature of monoids for which all flat (cyclic) right acts satisfy condition (P) which will result in a classification of *eventually regular* monoids and also many of the main results of recent papers on this subject.

Finally, in section 2.4, we show that for monoids $S = G \dot{\cup} I$ with G a group and I an ideal of S, flatness of I^1-acts ($I^1 = I \cup \{1\}$) can be extended to flatness of S-acts which leads to some results of other papers as corollaries.

2.2. Periodic Monoids over which all Flat Cyclic Right Acts satisfy Condition (P)

Definition 2.2.1. If S is a semigroup and $a \in S$, then $< a >= \{a, a^2, a^3, \ldots\}$ is a subsemigroup of S and is called the *monogenic* subsemigroup of S generated by the element a. The *order* of a is defined, as the order of the subsemigroup $< a >$. A semigroup is called *periodic* if all its elements are of finite order. A finite semigroup is necessarily periodic.

Definition 2.2.2. Let S be a semigroup and $a \in S$. If there are repetitions in the list $a, a^2, a^3, \ldots$, then the set $\{ x \in \mathbb{N} \mid (\exists y \in \mathbb{N})\ a^x = a^y, x \neq y \}$ is non-empty. The least element of this set is called the *index* of a and is denoted by m. If m is the index of a, then the set $\{ x \in \mathbb{N} \mid a^{m+x} = a^m \}$ is non empty. The least element of this set is called the *period* of a and is denoted by r.

Definition 2.2.3. If S is any non-empty set, then the multiplication defined on S by the rule that

$$x \cdot y = x \qquad (x, y \in S),$$

is easily seen to be associative. The semigroup $(S, \cdot)$ is called a *left zero semigroup*. *Right zero semigroup* is defined analogously. A left-zero semigroup with a 1 adjoined will be called a *left − zero monoid*. Thus for a monoid S, an element $0 \in S$ is called a *left zero* if $0S = \{0\}$, or analogously *right zero* or *two-sided zero*.

Definition 2.2.4. An element x of a monoid S is called *right nil* if $x \neq 1$ and there exists $n \in \mathbb{N}$, such that x^n is right-zero. If every $x \in S$, $x \neq 1$ is right nil, then S is called a *right nil* monoid.

A monoid S is called *right elementary* if $S = G \dot{\cup} N$ where G is a group and N is the set of all right nil elements of S. Notice that we include the case when N is empty. If N is not empty, then by Lemma 2.2.10, below we will see that N is an ideal of S. Notice also that G is a **subgroup** of S.

From Definitions 2.2.3, and 2.2.4 it can be seen that if a monoid S is right zero, then it is right nil, but the converse is not true as the following examples show.

Example 2.2.5. Let S be a right zero semigroup. Let $s \in S$ and let T_s be a nil semigroup of index n_s generated by a_s with zero element s, where $n_s \in \mathbb{N}$. (So, $T_s = \{a_s, a_s^2, \ldots, a_s^{n_s-1}, a_s^{n_s} = s\}$.) If $T = \dot{\cup}\ (T_s : s \in S)\ \dot{\cup}\ \{1\}$, where for $u, v \in S,\ u \neq v,\ T_u T_v = \{v\}$, then T is a right nil monoid which is not right zero.

Example 2.2.6. If $S = < x, y \mid x^2 = xy = x = yx = y^2 > \ \dot{\cup}\ \{1\}$, then S is a right nil monoid but it is not right zero.

Example 2.2.7. If $S = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, then S is a right nil monoid which is not right zero.

Notice that monoids in Examples 2.2.6, and 2.2.7, are also null.

Lemma 2.2.8 [7]. *Let S be any monoid, such that every flat monocyclic right S-act satisfies condition (P). Then every $e \in E(S) \setminus \{1\}$ is a right-zero element of S.*

Note : It is clear then if all flat cyclic right S-acts satisfy condition (P), then every $e \in E(S) \setminus \{1\}$ is right-zero, but the converse is false (see [40] comments after Lemma 4.1). If we restrict attention to *monocyclic* acts, then the converse is true (see [8], Theorem 4.3). Clearly then monoids for which every $e \in E(S) \setminus \{1\}$ is right zero play an important role in this theory and will be considered further in section 2.3.

Lemma 2.2.9 [7]. *Let S be any monoid and let ρ be any right congruence on S such that S/ρ is weakly flat. If e and f are right zero elements of S and $e\ \rho\ f$, then $e = f$.*

Lemma 2.2.10. *Let S be a monoid. Let $S = G \cup N$ with G a subgroup (and hence a submonoid) of S and N the set of all right nil elements of S. Then $G \cap N = \emptyset$ and N is either empty or an ideal of S.*

Proof. At first we show that $G \cap N = \emptyset$. If $N = \emptyset$ or $G = \emptyset$, then we are done. Suppose that $N \neq \emptyset$, $G \neq \emptyset$. Let $x \in G \cap N$. Since $x \in N$, then $x \neq 1$ and so there exists $k \in \mathbb{N}$ such that x^k is right zero. Since $x \in G$, then $x^k \in G$ and so $x^{k+1} = x^k$ implies that $x = 1$ which is a contradiction.

Now we show that either N is empty or N is an ideal of S. Suppose that $N \neq \emptyset$ and let $v \in N,\ u \in S$. Then there are two cases:

Case 1. $u \in G$. Then $uv \in N$. Otherwise, $uv \in G$ and since $u \in G$, then $u^{-1} \in G$ which means $v = u^{-1}(uv) \in G$. This is a contradiction. Similarly, it can be seen that $vu \in N$.

Case 2. $u \in N$. Then $uv \in N$. Otherwise $uv \in G$ which means that $(uv)^{-1} \in G$ and $(uv)^{-1}(uv) = 1$. Since $v \in N$, then there exists $m \in \mathbb{N}$ such that v^m is right zero. Let $k \in \mathbb{N}$ be the smallest positive integer such that v^k is right zero. Then for every $l < k$, v^l is not right zero.

If $k = 1$, then v is right zero and so $[(uv)^{-1}u]v = 1$ implies that $v = 1$ which is a contradiction. If v is not right zero, then $1 < k$ or $0 < k-1$. Then $(uv)^{-1}(uv) = 1$ implies that $[(uv)^{-1}u]v^k = v^{k-1}$ or $v^k = v^{k-1}$. Since v^k is right zero, then v^{k-1} is right zero and so a contradiction. Thus N is an ideal of S as required. ■

<u>**Note :**</u> If S is a monoid and $u \in S$, then (Δu) denotes the left congruence on S defined by $x(\Delta u)y$ if and only if $xu = yu$.

<u>**Lemma 2.2.11.**</u> *Let $S = G \dot{\cup} N$ be a right elementary monoid and let $x(\Delta u)y$ with $x \in N$, $y \in G$. Then u is right zero.*

Proof. We have $xu = yu$ and so $y^{-1}xu = u$ with $r = y^{-1}x \in N$. Hence there exists $n \in \mathbb{N}$ such that r^n is right zero and so $u = r^n u$ is also right zero. ■

<u>**Lemma 2.2.12.**</u> *Let $S = G \dot{\cup} N$ be a right elementary monoid and let ρ be a right congruence on S such that S/ρ is weakly flat. If there exist $x \in G$, $y \in N$ with $x \ \rho \ y$, then S/ρ is projective.*

Proof. Since S/ρ is weakly flat and $x \ \rho \ y$, then there exist $s, t \in S$ such that $sx = ty$ and $s(\rho \ \vee \ \Delta x)1(\rho \ \vee \ \Delta y)t$. Given that $x \in G$ then in fact $s \ \rho \ 1$. Since $y \in N$ then $sx = ty \in N$ and so $s \in N$. Hence there exists $n \in \mathbb{N}$ such that s^n is right zero. Since $s \ \rho \ 1$, then $s^n \ \rho \ 1$. Now if $u, v \in S$ are such that $u \ \rho \ v$ then $s^n u \ \rho \ u \ \rho \ v \ \rho \ s^n v$. Since s^n is right zero, then $s^n u$ and $s^n v$ are also right zero. So by Lemma 2.2.9, $s^n u = s^n v$. Since s^n is right zero, then it is idempotent. Therefore, by Lemma 1.54 (2), S/ρ is projective. ■

<u>**Theorem 2.2.13.**</u> *Let $S = G \dot{\cup} N$ be a right elementary monoid and let ρ be a right congruence on S such that S/ρ is weakly flat. Then S/ρ satisfies condition (P).*

Proof. First note that if $x, y \in S$ are such that $x \in G$, $y \in N$ and $x \ \rho \ y$, then by Lemma 2.2.12, S/ρ is projective and so satisfies condition (P). Consequently, we can assume from now on that such an x, y do not occur in S. Suppose that $u, v \in S$ with $u \ \rho \ v$. Since S/ρ is weakly flat, then there exist $s, t \in S$ such that $su = tv$, $s(\rho \vee \Delta u)1$ and $t(\rho \vee \Delta v)1$. We need only consider two cases as follows:

Case 1. $u, v \in G$. Notice that $su \ \rho \ u$ and $tv \ \rho \ v$ and so $s \ \rho \ 1$. In a similar way, $t \ \rho \ 1$ and so S/ρ satisfies condition (P).

Case 2. $u, v \in N$. We show that there exists $x \in S$ with $xu = su$ and $x \ \rho \ 1$. Now there exist $s_1, \ldots s_{2n-1} \in S$

$$s = s_0 \ \rho \ s_1(\Delta u)s_2 \ldots s_{2i} \ \rho \ s_{2i+1}(\Delta u)s_{2i+2} \ldots s_{2n-1}(\Delta u)s_{2n} = 1.$$

If any of the elements $s_0, \ldots, s_{2n-1} \in N$ then let j be such that $s_j \in N$ and $s_i \in G$ for all i such that $j < i \leq 2n$. Then we must have $s_j(\Delta u)s_{j+1}$ and so u is right zero, by Lemma 2.2.11. In this case we can put $x = 1$. Otherwise, $s_0, \ldots, s_{2n-1} \in G$. In this case let

$$x = s_0 s_1^{-1} s_2 \ldots s_{2i-1}^{-1} s_{2i} \ldots s_{2n-1}^{-1}.$$

Notice that

$$s_{2i-1}(\Delta u)s_{2i} \Rightarrow u = s_{2i-1}^{-1} s_{2i} u.$$

Consequently,

$$\begin{aligned} xu &= s_0 s_1^{-1} s_2 \ldots s_{2i-1}^{-1} s_{2i} \ldots s_{2n-1}^{-1} s_{2n} u \\ &= s_0 s_1^{-1} s_2 \ldots s_{2i-1}^{-1} s_{2i} \ldots s_{2n-3}^{-1} s_{2n-2} u \\ &= \ldots \\ &= s_0 s_1^{-1} s_2 u \\ &= su \end{aligned}$$

Also,

$$s_{2i} \ \rho \ s_{2i+1} \Rightarrow s_{2i} s_{2i+1}^{-1} \ \rho \ 1,$$

and hence

$$\begin{aligned} x &= s_0 s_1^{-1} s_2 s_3^{-1} \ldots s_{2i-1}^{-1} s_{2i} \ldots s_{2n-1}^{-1} \ \rho \ s_2 s_3^{-1} \ldots s_{2i-1}^{-1} s_{2i} \ldots s_{2n-1}^{-1} \\ &\ldots \\ &\rho \ s_{2n-2} s_{2n-1}^{-1} \\ &\rho \ 1 \end{aligned}$$

In a similar way, there exists $y \in S$ with $yv = tv$ and $y \; \rho \; 1$ and so the result follows. ■

Notice that if the monoid S is a group, then we have the following theorem.

Theorem 2.2.14 [31]. *Let S be a monoid. Then all right S-acts have property (P) if and only if S is a group.*

Bulman-Fleming and Normak in [7] showed that if $S = C \,\dot{\cup}\, Z$ where C is the set of all right cancellative elements of S and Z is the set of all right zero elements of S, then S is a left PP monoid and every weakly flat cyclic right S-act satisfies condition (P). Thus the converse of Theorem 2.2.13, is not true in general, but it is true if S is aperiodic monoid. We show this in the next theorem.

Theorem 2.2.15. *Let S be a periodic monoid. If all flat cyclic right S-acts satisfy condition (P), then S is right elementary.*

Proof. Let N be the set of all right nil elements of S. Let $1 \neq x \in S$ and suppose that for every $n \in \mathbb{N}$, $x^n \neq 1$, then we show that there exists $k \in \mathbb{N}$ such that x^k is right zero. Since S is periodic, then by [20, Proposition 1.2.3], there exists $k \in \mathbb{N}$ such that x^k is an idempotent. Since $x^k \neq 1$, then by Lemma 2.2.8, x^k is right zero and so $x \in N$.

Let $G = \{x \in S : \exists n \in \mathbb{N},\ x^n = 1\}$. Let $x, y \in G$ and suppose, by way of contradiction, that $xy \notin G$, then $x \neq 1$ and $y \neq 1$. By the first part, $(xy)^k$ is right zero for some $k \in \mathbb{N}$. Moreover we can assume that k is the smallest such element of $\mathbb{N}$. But there exists $n, m \in \mathbb{N}$ with $x^n = y^m = 1$ (notice that $n, m > 1$) and, using the convention that $(xy)^0 = 1$, we see that

$$(xy)^{k-1} = (xy)^k y^{m-1} x^{n-1},$$

and so $(xy)^{k-1}$ is right zero, a contradiction as required. Hence G is a subsemigroup of S and it is straightforward then to note that $xG = Gx = G$ for all $x \in G$ and so G is a group. ■

Remark. Notice that in the first paragraph of the proof of Theorem 2.2.15, x^k is the identity element of the kernel of the subsemigroup generated by x.

From Theorem 2.2.15, we have the following corollary.

Corollary 2.2.16. *Let S be a periodic monoid such that for every $x \neq 1 \in S$ and every $n \in \mathbb{N}$, $x^n \neq 1$. If all flat cyclic right S-acts satisfy condition (P), then S is right nil.*

From Theorems 2.2.13, and 2.2.15, we have the following theorem.

Theorem 2.2.17. *If S is a periodic monoid, then the following are equivalent:*

(1) S is right elementary.

(2) All weakly flat cyclic right S-acts satisfy condition (P).

(3) All flat cyclic right S-acts satisfy condition (P).

Corollary 2.2.18. *Since every finite monoid is periodic, then every finite monoid satisfies Theorem 2.2.17.*

By Theorem 2.2.13, we saw that monoids S with the structure $S = N \dot{\cup} G$, have the property that all weakly flat cyclic right S-acts satisfy condition (P). But this structure of monoids does not imply condition (P) of all (weakly) flat right acts, as Example 2.2.21, below demonstrates. At first we see the following theorems which will be needed in this example and later.

Theorem 2.2.19 [31]. *For any monoid S the following statements are equivalent:*

(1) All left S-acts are equalizer-flat.

(2) All left S-acts have property (E).

(3) All cyclic left S-acts have property (E).

(4) All cyclic left S-acts are strongly flat.

(5) $S = \{1\}$ or $S = \{0, 1\}$.

Theorem 2.2.20 [31]. *Every equalizer-flat S-act is flat.*

Example 2.2.21. Let $S = \{0, 1\}$. Then $S = \{0, 1\} = \{0\} \cup \{1\}$ in which $\{0\}$ is right nil and $\{1\}$ is a group. If $A = \{x, y, z \mid x0 = y0 = z0 = z\}$ then by Theorems 2.2.19, and 2.2.20, A is flat. But it does not satisfy condition (P). Otherwise,

$x0 = y0$ implies that there exist $a'' \in A$, $s, t \in S$ such that $x = a''s$, $y = a''t$ and $s0 = t0$. Since the only case is $x = x1$, then $a'' = x$. Consequently, $y = xt$. Then either $t = 0$ and so $x0 = z \neq y$ which is a contradiction or $t = 1$ and so $x1 = x \neq y$ which is also a contradiction.

Here are some examples of monoids of the form $S = G \dot{\cup} N$.

Example 2.2.22. Let $(X, \star)$ and $(Y, \circ)$ be semigroups and let $S = X \dot{\cup} Y$. If for $x, y \in S$,

$$xy = \begin{cases} x \star y & \text{if } x, y \in X \\ x \circ y & \text{if } x, y \in Y \end{cases}$$

$$xy = yx = y \qquad \text{if } x \in X, \ y \in Y,$$

then S is a semigroup. In particular X can be a group and Y a right nil semigroup.

For example: Let $S = \{0, 1, 2, 3\}$ with table

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

If $G = \{1, 3\}$, $N = \{0, 2\}$, then G is a group, N is a right nil semigroup and $S = G \dot{\cup} N$.

Example 2.2.23. Let $(X, \star)$, $(Y, \circ)$ be semigroups and suppose that Y has a zero element. Let $S = X \dot{\cup} Y$. If for $x, y \in S$,

$$xy = \begin{cases} x \star y & \text{if } x, y \in X \\ x \circ y & \text{if } x, y \in Y \end{cases}$$

$$xy = yx = 0 \qquad \text{if } x \in X, \ y \in Y.$$

Then S is a semigroup and also 0 is a zero element of S. In particular X can be a group and Y a right nil semigroup.

For example: Let $S = \{0, s, 1 \mid s^2 = 1\}$ with table

	0	1	s
0	0	0	0
1	0	1	s
s	0	s	1

If $G = \{1, s\}$, $N = \{0\}$, then G is a group, N is a right zero semigroup and $S = G \dot{\cup} N$.

Example 2.2.24. Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$. If $G = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{R}, a \neq 0 \right\}$ and $N = \left\{ \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix} \middle| b \in \mathbb{R} \right\}$, then G is a group, N is a right zero semigroup and $S = G \dot{\cup} N$.

Now by Lemma 2.2.12, and Theorem 2.2.13, we prove the following theorem.

Theorem 2.2.25. *If S is a right nil monoid, then every weakly flat cyclic right S-act is projective.*

Proof. Let ρ be a right congruence on S such that S/ρ is weakly flat. If there exists an element $x \in S$ such that x is right nil and that $x \ \rho \ 1$, then by Lemma 2.2.12, S/ρ is projective. Thus we can assume from now on that such an element does not occur in S. Now let $u, v \in S$. Then by Lemma 1.54 (2), it is sufficient to show that there exists $e^2 = e \in S$ such that $e \ \rho \ 1$ and $u \ \rho \ v$ implies that $eu = ev$. Since S/ρ is weakly flat, then there exist $s, t \in S$ such that $su = tv$, and $s(\rho \vee \Delta u)1(\rho \vee \Delta v)t$. Since $S = \{1\} \dot{\cup} N$ where N is a right nil semigroup, then there are two cases that can arise:

Case 1. $u = v = 1$. If $e = 1$, then $1 \ \rho \ 1$ and $1u = 1v$.

Case 2. $u, v \in N$. Then by case 2 of Theorem 2.2.13, either u is right zero and so $su = u$, or there exists $x \in G$ such that $su = xu$. Since $G = \{1\}$, then in the latter case again $su = u$. Similarly, $tv = v$. Thus $u = su = tv = v$ and again $e = 1$ satisfies our condition. ∎

Definition 2.2.26. A monoid S is called *aperiodic* if for every $x \in S$ there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$.

Lemma 2.2.27 [1]. *For any monoid S the following statements are equivalent:*

(1) Every finitely generated right S-act which satisfies condition (P) is strongly flat.

(2) Every cyclic right S-act which satisfies condition (P) is strongly flat.

(3) Every cyclic right S-act of the form $S/\rho(x, 1)$ is strongly flat.

(4) S is aperiodic.

Lemma 2.2.28. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero. If S is aperiodic, then S is right nil.*

Proof. Let $x \neq 1 \in S$. Since S is aperiodic, then there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. It is easy to see by induction that for every $k \in \mathbb{N}$, $x^{n+k} = x^n$.

Thus $k = n$ implies that $x^{2n} = x^n$ and so x^n is an idempotent. We claim that $x^n \neq 1$, otherwise, $x^{n+1} = x^n$ implies that $xx^n = x^n$ and so $x1 = 1$ which is a contradiction. Thus by assumption x^n is right zero and so x is right nil as required.

■

Corollary 2.2.29. *Let S be a monoid. If every flat cyclic right S-act is strongly flat, then S is right nil.*

Proof. Suppose that every flat cyclic right S-act is strongly flat. Then every cyclic right S-act which satisfies condition (P) is strongly flat and so by Lemma 2.2.27, S is aperiodic. on the other hand by Lemma 2.2.8, every $e \in E(S) \setminus \{1\}$ is right zero and so by Lemma 2.2.28, S is right nil as required. ■

From Theorem 2.2.25, and Corollary 2.2.29, the following theorem of [7] can be deduced.

Theorem 2.2.30. *If S is a monoid, then the following are equivalent:*

(1) S is right nil.

(2) Every weakly flat cyclic right S-act is projective.

(3) Every weakly flat cyclic right S-act is strongly flat.

(4) Every flat cyclic right S-act is projective.

(5) Every flat cyclic right S-act is strongly flat.

Now from Theorem 2.2.30, and Corollary 2.2.16, we have.

Corollary 2.2.31. *If S is a periodic monoid such that for every $x \in S \setminus \{1\}$ and every $n \in \mathbb{N}$, $x^n \neq 1$, then the following statements are equivalent:*

(1) S is right nil.

(2) Every weakly flat cyclic right S-act is projective.

(3) Every weakly flat cyclic right S-act is strongly flat.

(4) Every flat cyclic right S-act is projective.

(5) Every flat cyclic right S-act is strongly flat.

(6) Every flat cyclic right S-act satisfies condition (P).

Now by using Theorem 2.2.30, we give a characterization of which monoids have the property that all torsion free right acts are strongly flat.

Proposition 2.2.32 [29]. *Let S be a monoid. If all torsion free right S-acts are strongly flat, then S is right cancellative.*

Theorem 2.2.33. *Let S be a monoid. Then all torsion free right S-acts are strongly flat if and only if $S = \{1\}$.*

Proof. Suppose that all torsion free right S-acts are strongly flat. Then all weakly flat right S-acts are strongly flat and so all weakly flat cyclic right S-acts are strongly flat. Thus by Theorem 2.2.30, S is right nil and so for every $x \in S$, there exists $k \in \mathbb{N}$ such that $x^{k+1} = x^k$. But by Proposition 2.2.32, S is right cancellative and so $x^{k+1} = x^k$ implies that $x = 1$.

If $S = \{1\}$, then all right S-acts are strongly flat and so all torsion free right S-acts are strongly flat. ∎

Here we give a characterization by property of index of elements of monoids $S = G \dot{\cup} N$, where G is a group, N is right nil and $a^k \neq 1$ for every $1 \neq a \in S$ and $k \in \mathbb{N}$. In Lemma 2.2.34, K_a is the kernel of the subsemigroup of S generated by $a \in S$.

Lemma 2.2.34. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero. Let $1 \neq a \in S$ with index $m \geq 1$. If for all $k \in \mathbb{N}$, $a^k \neq 1$, then $K_a = \{a^m\}$ where a^m is right zero.*

Proof. Since the identity element a^{m+z}, of the group

$$K_a = \{a^m, a^{m+1}, \ldots, a^{m+r-1}\},$$

is an idempotent different from 1, then by assumption it is right zero. Thus for every $0 \leq i \leq r-1$,

$$a^{m+i}a^{m+z} = a^{m+z}. \tag{1}$$

On the other hand for every $0 \leq i \leq r-1$,

$$a^{m+i}a^{m+z} = a^{m+i}. \tag{2}$$

Then $(1), (2)$ imply that

$$a^{m+i} = a^{m+z} \quad (0 \leq i \leq r-1).$$

Consequently, $K_a = \{a^m\}$. Then $a^m = a^{m+z}$ implies that a^m is right zero as required. ∎

Corollary 2.2.35. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero. Let $1 \neq a \in S$ with index m and suppose that for all $k \in \mathbb{N}$, $a^k \neq 1$. If $m > 1$, then a is not right zero.*

Proof. Since $m \in \mathbb{N}$ is the smallest positive integer such that a^m is right zero and also $m > 1$, then for every j such that $1 \leq j \leq m-1$, a^j is not right zero and so a is not right zero. ■

Corollary 2.2.36. *Let S be a monoid such that every $e \in E(S)\setminus\{1\}$ is right zero. Let $1 \neq a \in S$ with index m and suppose that for all $k \in \mathbb{N}$, $a^k \neq 1$. Then a is right zero if and only if $m = 1$.*

Proof. If a is right zero, then $a^2 = a$ and so $m = 1$.

If $m = 1$, then by Lemma 2.2.34, K_a has the only element a which is right zero.■

Theorem 2.2.37. *Let $S = G \dot{\cup} N$ be a monoid and let m be the index of the element $a \neq 1$ in S. If for every $a \neq 1$ in S, $m > 1$, then S is a group.*

Proof. Suppose that for every $1 \neq a \in S$, $m > 1$. Then we claim that $N = \emptyset$. Otherwise, let $a \in N$. Since N is an ideal of S, then for every $k \in \mathbb{N}$, $a^k \in N$ and so $a^k \neq 1$. Also by Theorem 2.2.13, and Lemma 2.2.8, every $e \in E(S) \setminus \{1\}$ is right zero. Thus by Corollary 2.2.35, a is not right zero. This means that N does not have any right zero element which is a contradiction. Thus $S = G$ is a group as required. ■

Corollary 2.2.38. *Let $S = G \dot{\cup} N$ be a monoid, m be the index of the element $a \neq 1$ in S and suppose that for all $k \in \mathbb{N}$, $a^k \neq 1$. Then for every $a \neq 1$ in S, $m > 1$ if and only if S is a group.*

Proof. If for every $a \neq 1$ in S, $m > 1$, then by Theorem 2.2.37, S is a group.

Suppose that S is a group. Then for every $a \neq 1$ in S, $m > 1$. Otherwise, by Corollary 2.2.36, a is right zero and so $a^2 = a$ implies that $a = 1$ which is a contradiction. ■

From Theorem 2.2.15, and Corollary 2.2.38, we have

Corollary 2.2.39. *Let S be a periodic monoid such that all flat cyclic right S-acts satisfy condition (P). Let m be the index of the element $a \neq 1$ in S and suppose that for all $k \in \mathbb{N}$, $a^k \neq 1$. Then for every $a \neq 1$ in S, $m > 1$ if and only if S is a group.*

By what follows we give a characterization of monoids S with $S \setminus \{1\}$ right group such that all flat cyclic right S-acts satisfy condition (P).

Definition 2.2.40. A semigroup S is called *right simple* if $\mathcal{R} = S \times S$ and *left simple* if $\mathcal{L} = S \times S$.

Lemma 2.2.41 [19, p. 38]. *Let a and b be elements of a semigroup S. Then* $a \mathcal{L} b$ *if and only if there exist* $x, y \in S^1$ *such that* $xa = b$, $yb = a$. *Also* $a \mathcal{R} b$ *if and only if there exist* $u, v \in S^1$ *such that* $au = b$, $bv = a$.

Definition 2.2.42. A semigroup S that is right simple and left cancellative is called a *right group*.

Lemma 2.2.43 [19, p. 54]. *A semigroup S is a right group if and only if* $S \simeq G \times Z$ *where G is a group and Z is a right zero semigroup.*

If G and Z are as in Lemma 2.2.43, then we have the following theorem

Theorem 2.2.44. *Let the semigroup S be a right group. Then all flat cyclic right* S^1*-acts satisfy condition (P) if and only if* S^1 *is right zero.*

Proof. Suppose that S is a right group and that all flat cyclic right S^1-acts satisfy condition (P). Then by Lemma 2.2.43, $S \simeq G \times Z$ where G is a group and Z is a right zero semigroup. Let $1'$ be the identity element of G and let $e \in Z$. Then

$$(1', e)^2 = (1', e)(1', e) = (1'1', ee) = (1', ee).$$

Since e is right-zero, then $ee = e$ and so $(1', e)^2 = (1', e)$. Thus $(1', e)$ is an idempotent element of $G \times Z$. Consequently, S has an idempotent element and so S^1 has an idempotent element different from 1. Let $f \in E(S^1) \setminus \{1\}$. Then by Lemma 2.2.8, f is right zero. Now let $b \in S$. Then $(f, b) \in S \times S$. Since S is right simple, then $\mathcal{R} = S \times S$, and so $(f, b) \in \mathcal{R}$ or $f \mathcal{R} b$. Thus by Lemma 2.2.41, there exist $u, v \in S$ such that $fu = b$, $bv = f$. Then we have

$$fb = f(fu) = (ff)u = f^2u = fu = b.$$

If $a \in S$, then

$$ab = a(fb) = (af)b = fb = b,$$

and so b is right zero. Consequently, S^1 is a right zero monoid as required.

If S^1 is right zero, then by Theorem 2.2.17, all flat cyclic right S^1-acts satisfy condition (P). ■

2.3. Monoids over which all Flat Cyclic Right Acts satisfy Condition (P)

In this section we extend the results from section 2.2, to certain types of right subelementary monoid. Then we give a characterization of monoids for which every $e \in E(S) \setminus \{1\}$ is right zero and use this to give a characterization of eventually regular, regular and inverse monoids for which all flat (cyclic) right acts satisfy condition (P). We also use this characterization to deduce some of the main results from the literature. Finally, we give a characterization of monoids for which every $e \in E(S) \setminus \{1\}$ is right zero such that all cyclic right acts are weakly flat, from which the main result [Theorem 2.1] of [39] will be deduced as a corollary.

Definition 2.3.1. If S is a monoid, then an element $x \in S$ is called *eventually regular* if there exists $n \in \mathbb{N}$ such that x^n is regular. S is called *eventually regular* if all its elements are eventually regular. Clearly regular monoids and periodic monoids are eventually regular. On the other hand, a subset of a monoid which contains **no** eventually regular elements will be called *regular − free* subset. Notice that regular-free **semigroups** are equivalent to idempotent free semigroups and so must consist of elements of infinite order.

Remark. There are eventually regular monoids which are not regular as the following examples demonstrate.

Example 2.3.2. If $S = \{0, 1, e, f, a\}$ with table

	1	0	e	f	a
1	1	0	e	f	a
0	0	0	0	0	0
e	e	0	e	a	a
f	f	0	0	f	0
a	a	0	0	a	0

then $1, e, f, 0, a$ are eventually regular but a is not regular.

Example 2.3.3. If $S = \{0, 1, a\}$ with table

	1	0	a
1	1	0	a
0	0	0	0
a	a	0	0

then $0, 1, a^2$ are regular but a is not regular. It can be seen that $a = 1 \cdot a$ and $0 \cdot a = a \cdot a$, but $0 \cdot 1 \neq a \cdot 1$ and so S is not a left PP monoid. Hence, this example also shows that eventually regular monoids are not left PP in general. But as we know regular monoids are left PP.

Definition 2.3.4. A semigroup S is called *group − bound* if for every $a \in S$ there exists $n \in \mathbb{N}$ such that a^n is a member of some subgroup of S.

Note : It is clear that group-bound semigroups are eventually regular.

Definition 2.3.5. A monoid S is called *right subelementary* if $S = C \dot{\cup} N$ where C is the set of all right cancellative elements of S and N is the set of all right nil elements of S. Notice that we include the case when N is empty and that C is a **submonoid** of S.

In section 2.2, by Theorem 2.2.13, we showed that if S is right elementary, then all weakly flat cyclic right S-acts satisfy condition (P). Moreover by Theorem 2.2.15 we showed that, among periodic monoids S those for which all flat cyclic right S-acts satisfy condition (P) are right elementary. In this section we extend Theorem 2.2.13, by considering right subelementary monoids.

Lemma 2.3.6. *Let $S = C \cup N$ be a monoid such that C is the set of right cancellative elements of S and N is the set of right nil elements of S. Then $C \cap N = \emptyset$ and N is either empty or an ideal of S.*

Proof. At first we show that if $a, b \in C$, then $ab \in C$. Suppose that $c(ab) = d(ab)$, for $c, d \in S$. Then $(ca)b = (da)b$. Since $b \in C$, then $ca = da$ and $a \in C$ implies that $c = d$. Thus $ab \in C$ as required.

We claim that $C \cap N = \emptyset$. Otherwise let $1 \neq a \in C \cap N$. Since $a \in C$, then from the previous paragraph for every $n \in \mathbb{N}$ it follows that $a^n \in C$. Also there exists $k \in \mathbb{N}$ such that $a^{k+1} = a^k$. Consequently, $a^k \in C$ implies that $a = 1$ which is a contradiction.

Now we show that either N is empty or N is an ideal of S. If $N = \emptyset$, then we are done. Suppose then that $N \neq \emptyset$ and let $a \in S$, $1 \neq b \in N$. Then $ba \in N$, otherwise let $x, y \in S$ and suppose that $xb = yb$. Then $x(ba) = y(ba)$. Since $ba \in C$, then $x = y$ and so $b \in C$ which is a contradiction.

Also $ab \in N$. Otherwise, notice that $ab \neq 1$ since if $ab = 1$ and $n \in \mathbb{N}$ was such that b^n is right zero and no such smaller n has this property, then for $n > 1$ we have

$$b^n = ab^n = (ab)b^{n-1} = b^{n-1},$$

giving a contradiction, and for $n = 1$, $b = ab = 1$, another contradiction.

Since $ba \in N$, then there exists $n \in \mathbb{N}$ with $(ba)^n$ a right zero and n the smallest such positive integer. Hence, $(ba)^{n+1} = (ba)^n$ and so

$$(ab)^{n+2} = a(ba)^{n+1}b = a(ba)^n b = (ab)^{n+1}. \qquad (1)$$

Then (1), and $ab \in C$ imply that $ab = 1$ which is a contradiction. Hence, $ab \in N$ and so N is an ideal of S as required. ■

We aim to consider cyclic acts over a right subelementary monoid but first we need a few technical lemmas.

Lemma 2.3.7. *Let $S = C \dot{\cup} N$ be a right subelementary monoid. Suppose that ρ is a right congruence on S such that S/ρ is weakly flat.*

1. *If $x \in C, y \in N$ are such that $x \ \rho \ y$, then S/ρ is projective.*

2. *If $x, y \in C$, then $x \ \rho \ y$ if and only if there exist $s, t \in S$ with $sx = ty$ and $s \ \rho \ 1 \ \rho \ t$.*

3. *If S/ρ is* **not** *projective and $a, b \in C$ are such that $a \ \rho \ 1 \ \rho \ b$, then there exists $s, t \in C$ with $s \ \rho \ 1 \ \rho \ t$ and $sab = tba$.*

4. *If S/ρ is* **not** *projective and $a, b \in C$ and $c, x \in S$ are such that $a \ \rho \ b(\Delta x)c$, then there exists $s, t \in C$ with $s \ \rho \ 1 \ \rho \ t$ and $sax = tcx$ (i.e. $sa(\Delta x)tc$).*

5. *If S/ρ is* **not** *projective and if there exist $s_1, s_2, t_1, t_2 \in C$ and $a, b, c \in S$ such that $s_1 \ \rho \ 1 \ \rho \ t_1$, $s_2 \ \rho \ 1 \ \rho \ t_2$ and such that $s_1 a = t_1 b$, $s_2 b = t_2 c$, then there exist $s, t \in C$ with $s \ \rho \ 1 \ \rho \ t$ and $sa = tc$.*

Proof.

1. We know that there exist $s, t \in S$ such that $sx = ty$ and such that $s(\rho \ \vee \ \Delta x)1(\rho \ \vee \ \Delta y)t$. Given that $x \in C$, then in fact $s \ \rho \ 1$. Since $y \in N$, then $sx = ty \in N$ and so $s \in N$. Otherwise $sx \in C$ and consequently $sx \in C \cap N$ which is a contradiction. Hence there exists $n \in \mathbb{N}$ such that s^n is right zero. But then $s^n \ \rho \ 1$.

   Now let $u, v \in S$ be such that $u \ \rho \ v$. Then $s^n u \ \rho \ u \ \rho \ v \ \rho \ s^n v$. Since $s^n u$ and $s^n v$ are right zero, then by Lemma 2.2.9, $s^n u = s^n v$ and the result follows from Lemma 1.54 (2).

2. Suppose that $x \ \rho \ y$. Since S/ρ is weakly flat, then there exist $s, t \in S$ such that $sx = ty$ and such that $s(\rho \ \vee \ \Delta x)1(\rho \ \vee \ \Delta y)t$. But $x, y \in C$ means that $\Delta x = \Delta$, $\Delta y = \Delta$ and so $s \ \rho \ 1 \ \rho \ t$ as required.

   The converse is straightforward.

3. Since $a \ \rho \ 1$, then $ab \ \rho \ b$. Also $b \ \rho \ 1$ implies that $ba \ \rho \ a$ and so $ab \ \rho \ ba$ (ρ 1). Then from (2), there exist $s, t \in S$ such that $s \ \rho \ 1 \ \rho \ t$ and $sab = tba$. Finally, $s, t \in C$. Otherwise by (1), S/ρ is projective which is a contradiction.

4. Since $a \ \rho \ b$, then by (2), there exist $s, t \in C$ with $sa = tb$ and $s \ \rho \ 1 \ \rho \ t$. Therefore, $sax = tbx = tcx$. Finally, $s, t \in C$ by (1).

5. Notice that, $t_1 \ \rho \ s_2 \ \rho \ 1$ and so from (3), there exist $s', t' \in C$ with $s' \ \rho \ 1 \ \rho \ t'$ and $s' s_2 t_1 = t' t_1 s_2$. Hence

$$s' s_2 s_1 a = s' s_2 t_1 b = t' t_1 s_2 b = t' t_1 t_2 c.$$

Now put $s = s' s_2 s_1$ and $t = t' t_1 t_2$. ■

<u>**Note :**</u> If S is a right subelementary monoid, then from Lemma 1.54, (2) if S/ρ is projective, then either $\rho = \Delta$ or there exists $e^2 = e \in N$ such that $e\ \rho\ 1 \in C$. Hence if $\rho \neq \Delta$, then the converse of (1) above is true for right subelementary monoids.

<u>**Lemma 2.3.8.**</u> *Let $S = C \dot{\cup} N$ be a right subelementary monoid and suppose that in addition, $\forall a \in C$, $\forall b \in N$, $b \in Sab$. Let ρ be a right congruence on S such that S/ρ is weakly flat but* **not** *projective and let $u, v \in N$. Then $u\ \rho\ v$ if and only if there exist $x, y \in C$ with $x\ \rho\ 1\ \rho\ y$ and $xu = yv$. Moreover, $\rho|_N \subseteq \mathcal{L}$.*

Proof. Let $u\ \rho\ v$. Then there exist $s, t \in S$ with $s(\rho\ \vee\ \Delta u)1(\rho\ \vee\ \Delta v)t$ and $su = tv$. Hence we have

$$s = s_0\ \rho\ t_0(\Delta u)s_1\ \rho \ldots s_n\ \rho\ t_n(\Delta u)s_{n+1} = 1 \qquad (*)$$

for some $s_i, t_i \in S$.

We show that there exist $a, b \in C$ with $a\ \rho\ 1\ \rho\ b$ and $asu = bu$ and we do this by considering two cases:

Case 1. All the $s_i, t_i \in C$. First, if $n + 1 = 0$, then $s = 1$ and so put $a = b = 1$. Otherwise, $n + 1 \geq 1$ and we use an inductive argument on n. If $n + 1 = 1$, then we have

$$s = s_0\ \rho\ t_0(\Delta u)1.$$

From Lemma 2.3.7 (4), there exist $a, b \in C$ with $a\ \rho\ 1\ \rho\ b$ and $asu = as_0u = bu$ as required.

Now, suppose that $n + 1 > 1$. Then by Lemma 2.3.7 (4), there exist $a_0, b_0 \in C$ such that $a_0\ \rho\ 1\ \rho\ b_0$ and that

$$a_0 su = a_0 s_0 u = b_0 s_1 u.$$

Since

$$s_1\ \rho\ t_1 \ldots s_n\ \rho\ t_n(\Delta u)s_{n+1} = 1$$

is a sequence with a length less than $n+1$, then by induction, there exist $a_1, b_1 \in C$ with $a_1\ \rho\ 1\ \rho\ b_1$ and $a_1 s_1 u = b_1 u$. Thus we have

$$a_0 su = a_0 s_0 u = b_0 s_1 u, \text{ and } a_1 s_1 u = b_1 u$$

and the result follows from Lemma 2.3.7 (5)

Case 2. At least one of the s_i or $t_i \in N$. Since $s_{n+1} = 1 \in C$, then working backwards from the right hand end of $(*)$ we see that there are two possibilities that can arise:

a. $s_i \ \rho \ t_i$ for some i with $s_i \in N$ and $t_i \in C$. This case is impossible, since we are assuming that S/ρ is not projective.

b. $t_i(\Delta u)s_{i+1}$ for some i with $t_i \in N$ and $s_{i+1} \in C$. In this case, since $u \in Ss_{i+1}u$, then there exists $r \in S$ with $u = rs_{i+1}u = rt_iu$ and since $rt_i \in N$, then there exists $k \in \mathbb{N}$ such that $(rt_i)^k$ is right zero. But $u = rt_iu$ implies that $u = (rt_i)^k u$. Consequently, u is right zero and so we can take $a = b = 1$.

In a similar way, there exists $c, d \in C$ with $c \ \rho \ 1 \ \rho \ d$ and $ctv = dv$.

Now by Lemma 2.3.7 (3), there exist $g, h \in C$ with $g \ \rho \ 1 \ \rho \ h$ and $gac = hca$. Consequently, on putting $x = hcb, \ y = gad$ we have

$$xu = (hcb)u = hcasu = gacsu = gactv = (gad)v = yv$$

and $x = (hcb) \ \rho \ 1 \ \rho \ (gad) = y$ as required.

Conversely, if $x \ \rho \ 1 \ \rho \ y$ and $xu = yv$, then $u \ \rho \ xu = yv \ \rho \ v$ as required.

Suppose that $(u, v) \in \rho|_N$. Then by the previous part, there exist $x, y \in C$ with $x \ \rho \ 1 \ \rho \ y$ and $xu = yv$. Notice also that by assumption, there exist $z, w \in S$ such that $u = zxu, \ v = wyv$. Hence $u = zxu = zyv$ and $v = wyv = wxu$ and so by Lemma 2.2.41, $(u, v) \in \mathcal{L}$ as claimed. ■

Corollary 2.3.9. *Let $S = G \dot{\cup} N$ be a right elementary monoid and suppose that ρ is a right congruence on S such that S/ρ is weakly flat but* **not** *projective. Then $\rho \subseteq \mathcal{L}$.*

Proof. Let $(u, v) \in \rho$. Since S/ρ in not projective, then there are two cases:

Case 1. $u, v \in N$. Then by Lemma 2.3.8, $(u, v) \in \mathcal{L}$.

Case 2. $u, v \in G$. Then by Lemma 2.3.7 (2), there exist $s, t \in S$ with $su = tv$ and $s \ \rho \ 1 \ \rho \ t$. Since S/ρ is not projective, then $s, t \in G$ and so we have $u = (s^{-1}t)v$, $v = (t^{-1}s)u$ and again $(u, v) \in \mathcal{L}$. Thus $\rho \subseteq \mathcal{L}$ as required. ■

Corollary 2.3.10. *Let S be a right nil monoid and ρ a right congruence on S. If S/ρ is weakly flat, then S/ρ is projective.*

Proof. If $\rho = \Delta$, then $S/\rho = S$ is clearly projective. Thus we suppose that $\rho \neq \Delta$ and that S/ρ is not projective. Then there exist $u, v \in S$ such that $u \neq v$ and

$u \; \rho \; v$. Then by Lemma 2.3.8, there exist $x, y \in C = \{1\}$ such that $x \; \rho \; 1 \; \rho \; y$ and $xu = yv$. Thus $u = v$ which is a contradiction. Hence, S/ρ is projective as required. ■

<u>Lemma 2.3.11.</u> *Let S be a monoid and suppose that N is a right nil ideal of S. Then $u \in N$ is regular if and only if u is right zero.*

Proof. Let $u \in N$ be regular. Then there exists $w \in S$ such that $uwu = u$ or $(uw)u = u$. Since N is an ideal of S, then $uw \in N$. Let $e = uw$. Then there exists $k \in \mathbb{N}$ such that e^k is right zero. But it can be seen by induction that for every $n \in \mathbb{N}$, $e^n = e$ and so $e^k = e$ is right zero. Since $eu = u$, then u is also right zero.

If $u \in N$ is right zero, then for every $w \in S, uwu = u$, and so u is regular. ■

<u>Theorem 2.3.12.</u> *Let $S = C \dot{\cup} N$ be a right subelementary monoid and suppose that in addition, $\forall a \in C$, $\forall b \in N$, $b \in Sab$. Let ρ be a right congruence on S such that S/ρ is weakly flat. Suppose that $u \in N$, $v \in S$ and suppose that u is right zero. Then $u \; \rho \; v$ if and only if either $u = v$ or there exists a right zero $e \in S$ such that $e \; \rho \; 1$ and that $u = ev$.*

Proof. Let $u \; \rho \; v$ with u right zero and $v \in S$. Then there are two cases that can arise:

Case 1. $v \in C$. Then by Lemma 2.3.7 (1), S/ρ is projective and so there exists $e^2 = e \in S$ such that $e \; \rho \; 1$ and $eu = ev$. Since $u \in N$ and $v \in C$, then $u \neq v$ and so $e \neq 1$. Otherwise, $eu = ev$ implies that $u = v$ which is a contradiction. Thus $1 \neq e \in N$. Since e is regular, then by Lemma 2.3.11, e is right zero. Since u is right zero, then $u = eu = ev$

Case 2. $v \in N$. If S/ρ is projective, then there exists $e^2 = e \in S$ such that $e \; \rho \; 1$ and $eu = ev$. If $e = 1$, then $u = v$, otherwise as in case 1 $u = ev$, where e is right zero. If S/ρ is not projective, then by Lemma 2.3.8, there exist $x, y \in C$ such that $x \; \rho \; 1 \; \rho \; y$ and $xu = yv$. Since $y \in C$, $v \in N$, then by assumption there exists $z \in S$ such that $v = zyv$. Since u is right zero, then $u = zxu$ and so $u = zxu = zyv = v$.

Now suppose that the given condition holds. If $u = v$, then it is obvious that $u \; \rho \; v$. If there exists $e \in S$ such that e is right zero with $e \; \rho \; 1$ and $u = ev$, then $u = ev \; \rho \; v$. ■

<u>Remark.</u> By Lemma 2.2.9, we saw that for a monoid S and a right congruence ρ on S, if S/ρ is weakly flat and if $e, f \in S$ are right zero and $e \; \rho \; f$, then $e = f$. But from case (2) of Theorem 2.3.12, it can be seen that for a right subelementary monoid S and with the property mentioned in this theorem if ρ is a right congruence on S such that S/ρ is weakly flat but not projective and if $e, f \in S$ are such that e is right zero but not necessarily f and $e \; \rho \; f$, then $e = f$.

From Lemma 2.3.11, and Theorem 2.3.12, we have

Corollary 2.3.13. *Let $S = C \dot{\cup} N$ be a right subelementary monoid and suppose that in addition, $\forall a \in C$, $\forall b \in N$, $b \in Sab$. Let ρ be a right congruence on S such that S/ρ is weakly flat. If $u \in N, v \in S$ are such that u is regular and $u \ \rho \ v$, then either $u = v$ or there exists a regular element $e \in N$ such that $u = ev$.*

We can now deduce one of our main theorems in this section:

Theorem 2.3.14. *Let $S = C \dot{\cup} N$ be a right subelementary monoid and suppose that in addition, $\forall a \in C, b \in N$, $b \in Sab$. Let ρ be a right congruence on S such that S/ρ is weakly flat. Then S/ρ satisfies condition (P).*

Proof. Let $u \ \rho \ v$. If $u, v \in C$, then by Lemma 2.3.7 (2), there exist $s, t \in S$ such that $su = tv$ and $s \ \rho \ 1 \ \rho \ t$.

If $u \in C, v \in N$, then by Lemma 2.3.7 (1), S/ρ is projective and therefore satisfies condition (P).

Thus we need only consider the case where $u, v \in N$. Moreover, we can assume that S/ρ is **not** projective. The result then follows by Lemma 2.3.8. ∎

It is fairly clear that if C is a group, then the condition "$b \in Sab$" is satisfied and so we deduce as a corollary, Theorem 2.2.13.

The following is an example of this type of monoid.

Example 2.3.15. Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z} \right\}$. If $C = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z}, a \neq 0 \right\}$ and $Z = \left\{ \begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix} \middle| b \in \mathbb{Z} \right\}$. Then C is a cancellative monoid, Z is a right zero semigroup and $S = C \dot{\cup} Z$. Also for every $a \in C, b \in Z, b = ab \in Sab$.

One obvious question that arises here is, "Is the condition, $b \in Sab$ necessary for such a monoid to have the property that all weakly flat cyclic acts satisfy property (P)?". Certainly if the element $a \in C$ has finite order, then a must be invertible and so the condition will be satisfied. Hence this property will be true for periodic monoids. In fact in this case, the right cancellative monoid C will be a (periodic) group.

The converse of Theorem 2.3.14, is not true. In [9, Proposition 3.1], Bulman-Fleming and Kilp showed that the monoid S given by the presentation

$$S = < x, y \mid xy = x = yx > \cup \{1\}$$

was such that every weakly flat S-act satisfies condition (P) but it contains elements that are neither cancellative nor nil.

However, let us examine the question of which monoids have the property that all their flat cyclic right acts satisfy condition (P).

Theorem 2.3.16. *Let S be a monoid. Then every $e \in E(S) \setminus \{1\}$ is right zero if and only if $S = G \dot{\cup} N \dot{\cup} F$ where G is group, N contains all the right nil elements of S and F is the set of all regular-free elements of S. Moreover, $N \dot{\cup} F$ is a maximal ideal of S.*

Proof. Let $S = G \dot{\cup} N \dot{\cup} F$ be a disjoint union of a group, a right nil set and a regular-free set and let $e \in E(S) \setminus \{1\}$. Since e is regular, then $e \notin F$. Also $e \notin G$ otherwise, $e^2 = e$ implies that $e = 1$ which is a contradiction. Thus $e \in N$ and so there exists $n \in \mathbb{N}$ such that e^n is right zero. But $e = e^n$ and so e is right zero as required. ■

Now suppose that every $e \in E(S) \setminus \{1\}$ is right zero and let $x \in S$. Then either x is eventually regular or x is regular-free. If x is eventually regular, then there exist $n \in \mathbb{N}$, $x' \in S$ such that $x^n x' x^n = x^n$. Since $x^n x'$ is idempotent, then either $x^n x' = 1$ and so x is right invertible or $x^n x' \neq 1$ and so by assumption it is right zero. Consequently, $x^n = x^n x' x^n$ implies that x^n is right zero giving that x is right nil. If $G = \{x \in S \mid \exists y \in S,\ xy = 1\}$, then it is easy to see that G is closed and so it is a group. If N is the set of all right nil elements of S and F the set of all regular-free elements of S, then $S = G \dot{\cup} N \dot{\cup} F$ as required.

Now we show that $I = N \dot{\cup} F$ is a maximal ideal of S. At first we show that $I = N \dot{\cup} F$ is an ideal. Let $x \in S$ and $y \in I$. Then there are two cases that can arise:

Case 1. $x \in G, y \in I$. Then $xy, yx \in I$. Otherwise, $xy, yx \in G$. Since $x \in G$, then $x^{-1} \in G$ and so $y = x^{-1}(xy), y = (yx)x^{-1} \in G$ which is a contradiction.

Case 2. $x, y \in I$. Then there are two possibilities as follows:

a. If $x \in F, y \in I$, then $xy \in I$. Otherwise, $xy \in G$. Then there exists $z \in G$ such that $(xy)z = 1$ or $x(yz) = 1$. Consequently, $x(yz)x = x$ giving that x is regular which is a contradiction. Similarly if $yx \in G$, then there exists $w \in G$ such that $w(yx) = 1$ or $(wy)x = 1$. Thus $x(wy)x = x$, which implies that x is regular and again a contradiction.

b. If $x, y \in N$. Then $xy \in I$. Otherwise, $xy \in G$. Since $x \in N$, then there exist $n \in \mathbb{N}$ such that x^n is right zero and no smaller n has this property. Then $x^{n+1} = x^n$ and so we have $x^{n+1}y = x^n y$ or $x^n(xy) = x^{n-1}(xy)$ which implies that $x^n = x^{n-1}$. Since x^n is right zero, then x^{n-1} is also right zero and so a contradiction.

Hence, I is an ideal of S.

Now suppose that J is an ideal of S such that $I \subset J \subseteq S$. Then there exists $x \in J$ such that $x \notin I$. Thus $x \in G$ and so $x^{-1} \in G$ implies that $1 = xx^{-1} \in J$. Consequently, $J = S$ and so I is a maximal ideal of S as required. ■

Remark. Notice that in the previous theorem, F is closed under the taking of powers and that for this type of monoid, the condition that F is a regular-free subset closed under the taking of powers is equivalent to the condition that it is an idempotent-free subset closed under the taking of powers. It is unknown whether $G \dot{\cup} N$ is a submonoid of S.

Notice also that we could view S as $C \dot{\cup} N \dot{\cup} F'$ where C is right cancellative, N is as before and F' is the subset of F which contains no right cancellative elements.

If $x \in S$, $a \in N \dot{\cup} F'$, then $ax \in N \dot{\cup} F'$, otherwise ax is right cancellative. Now if $ba = ca$ for $b, c \in S$, then $(ba)x = (ca)x$ or $b(ax) = c(ax)$ and so $b = c$. Thus a is right cancellative which is a contradiction. Thus $N \dot{\cup} F'$ is a right ideal of S.

Now the question that arises here is "Is $N \dot{\cup} F'$ also a left ideal?".

If in the previous theorem S is commutative and $x \in S$, $b \in N$, then there exists $k \in \mathbb{N}$ such that b^k is right zero. Since $(xb)^k = x^k b^k = b^k$, then $(xb)^k$ is right zero and so xb is right nil. Since S is commutative, then $bx = xb$. Consequently, $xb,\ bx \in N$ and so N is an ideal of S.

From Lemma 2.2.8, and Theorem 2.3.16, we have

Corollary 2.3.17. *Let S be a monoid. If all flat monocyclic right S-acts satisfy condition (P), then S is a disjoint union of a group, a right nil set and a regular-free set.*

Note : It is known that the converse to this corollary is false. For example Bulman-Fleming and Kilp in [9, Proposition 3.4] showed that if $S =< x, y \mid xy = x^2 = yx > \cup \{1\}$ and $\rho = \rho(x, x^2) \vee \rho(1, y^2)$, then S is a commutative monoid and $S = \{x^n \mid n \in \mathbb{N}\} \cup \{y^n \mid n \in \mathbb{N}\} \cup \{1\}$, where $x^n y^m = y^m x^n = x^{m+n}$ and so every $1 \neq x \in S$ is regular-free. Also S/ρ is a flat cyclic right S-act, but it does not satisfy condition (P). See also Theorem 3.3, of this paper or comment after Lemma 4.1, of [40].

Theorem 2.3.18 [8]. *Let S be a monoid. Then the following are equivalent:*

(1) All weakly flat monocyclic acts $S/\rho(s,t)$ satisfy condition (P).

(2) All flat monocyclic acts $S/\rho(s,t)$ satisfy condition (P).

(3) Every $e \in E(S) \setminus \{1\}$ is right zero.

Now from Theorem 2.3.16, and Theorem 2.3.18, we have

Theorem 2.3.19. *Let S be any monoid. Then the following are equivalent:*

(1) All weakly flat monocyclic acts $S/\rho(s,t)$ satisfy condition (P).

(2) All flat monocyclic acts $S/\rho(s,t)$ satisfy condition (P).

(3) Every $e \in E(S) \setminus \{1\}$ is right zero.

(4) $S = G \dot{\cup} N \dot{\cup} F$, where G is a group, N is a right nil set and F is a regular-free set.

Lemma 2.3.20. *Let S be a monoid. Then S is left PP and every $e \in E(S) \setminus \{1\}$ is right zero if and only if S is right subelementary and every element in the right nil part is right zero.*

Proof. Suppose that S is left PP and every $e \in E(S) \setminus \{1\}$ is right zero. By Theorem 2.3.16 and notice after $S = C \dot{\cup} N \dot{\cup} F'$. Also for every $x \in S$ there exists $e^2 = e \in S$ such that $x = ex$ and for every $a, b \in S$, $ax = bx$ implies that $ae = be$. If $e = 1$, then $a = b$ and so x is right cancellative. Otherwise by assumption e is right zero and so $x = ex$ implies that x is right zero. Thus $F' = \emptyset$ and N is right zero as required.

If S is a right subelementary monoid such that every element in the right nil part is right zero, then it is obvious that every $e \in E(S) \setminus \{1\}$ is right zero. It is also easy to see that for every $x \in S$ there exists $e^2 = e \in S$ such that $x = ex$ and $ax = bx$ implies that $ae = be$, and so S is left PP. ■

Corollary 2.3.21. *Let S be a monoid. Then S is left PP and all flat cyclic right S-acts satisfy condition (P) if and only if S is right subelementary and moreover every element in the right nil part is right zero.*

Proof. Suppose that S is left PP and all flat cyclic right S-acts satisfy condition (P). Then By Lemma 2.2.8 every $e \in E(S) \setminus \{1\}$ is right zero and so by Lemma 2.3.20, $S = C \dot{\cup} Z$, where C is right cancellative and Z is right zero.

If $S = C \dot{\cup} Z$, where C is right cancellative and Z is right zero, then by Lemma 2.3.20, S is left PP. Also $\forall a \in C, \forall b \in Z,\ b \in Sab$ and so by Theorem 2.3.14, all weakly flat cyclic right S-acts satisfy condition (P). ■

Now from Corollary 2.3.21, we can deduce the following theorem.

Theorem 2.3.22 [7]. *Let S be a monoid. Then the following statements are equivalent:*

(1) $S = C \dot{\cup} Z$ is right subelementary and every element in Z is right zero.

(2) S is left PP and all weakly flat cyclic right S-acts satisfy condition (P).

(3) S is left PP and all flat cyclic right S-acts satisfy condition (P).

Now we give a characterization of eventually regular monoids by condition (P) of cyclic right acts. There are also some results that will arise.

Theorem 2.3.23. *Let S be an eventually regular monoid. Then every $e \in E(S) \setminus \{1\}$ is right zero if and only if S is right elementary.*

Proof. Suppose that every $e \in E(S) \setminus \{1\}$ is right zero. Then by Theorem 2.3.16, $S = G \dot{\cup} N \dot{\cup} F$. Since S is eventually regular, then $F = \emptyset$ and so $S = G \dot{\cup} N$ is a right elementary monoid.

Let the monoid $S = G \dot{\cup} N$ be right elementary and let $e \in E(S) \setminus \{1\}$. Then $e \in N$ and so there exists $n \in \mathbb{N}$ such that e^n is right zero. But $e^n = e$, and so e right zero as required. ∎

Corollary 2.3.24. *Let S be an eventually regular monoid. Then all flat cyclic right S-acts satisfy condition (P) if and only if every $e \in E(S) \setminus \{1\}$ is right zero.*

Proof. If every $e \in E(S) \setminus \{1\}$ is right zero, then by Theorem 2.3.23, S is right elementary and so by Theorem 2.2.13, all weakly flat cyclic right S-acts satisfy condition (P).

The converse is true by Lemma 2.2.8. ∎

Now from Theorem 2.2.13, Corollary 2.3.24, and Theorem 2.3.23, we can deduce the following extension to Theorem 2.2.17.

Theorem 2.3.25. *If S is an eventually regular monoid, then the following are equivalent:*

(1) $S = G \dot{\cup} N$ is right elementary.

(2) All weakly flat cyclic right S-acts satisfy condition (P).

(3) All flat cyclic right S-acts satisfy condition (P).

(4) Every $e \in E(S) \setminus \{1\}$ is right zero.

Lemma 2.3.26. *Let S be a regular monoid. Then every $e \in E(S) \setminus \{1\}$ is right zero if and only if S is right elementary and moreover every element in the right nil part is right zero.*

Proof. If every $e \in E(S) \setminus \{1\}$ is right zero, then by Theorem 2.3.23, S is right elementary. Let x be a right nil element. Since x is regular, then there exists $x' \in S$ such that $x = xx'x$. If $xx' = e$, then e is an idempotent. Also $e \neq 1$, otherwise x is right invertible which is a contradiction. Thus e is right zero and so $x = ex$ is right zero as required.

The converse is obvious. ■

From Theorem 2.3.25, and Lemma 2.3.26, the following Theorem can be concluded.

Theorem 2.3.27. *Let S be a monoid. Then the following are equivalent:*

(1) S is regular and all weakly flat cyclic right S-acts satisfy condition (P).

(2) S is regular and all flat cyclic right S-acts satisfy condition (P).

(3) S is regular and every $e \in E(S) \setminus \{1\}$ is right zero.

(4) S is right elementary and every element in the right nil part is right zero.

We can now give an alternative proof of the main Theorem in [7].

Theorem 2.3.28. *For any monoid S the following statements are equivalent:*

(1) S is right nil.

(2) Every weakly flat cyclic right S-act is projective.

(3) Every weakly flat cyclic right S-act is strongly flat.

(4) Every flat cyclic right S-act is projective.

(5) Every flat cyclic right S-act is strongly flat.

Proof. Note that (2) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious, as is (2) $\Rightarrow$ (3) $\Rightarrow$ (5). (1) $\Rightarrow$ (2) If $\rho = \Delta$, then $S/\rho = S$ is clearly projective. Otherwise, by Lemma 2.3.8, if S/ρ is not projective, then $\rho = \Delta$ giving a contradiction. (5) $\Rightarrow$ (1) is proved as follows.

By Corollary 2.3.17, $S = G \dot{\cup} N \dot{\cup} F$. Since every flat cyclic right S-act is strongly flat, then every cyclic right S-act which satisfies condition (P) is strongly flat and so by Lemma 2.2.27, every element of S is aperiodic. Hence, $F = \emptyset$ and $G = \{1\}$ as required. ■

Corollary 2.3.29. *Let S be an inverse monoid. Then all flat cyclic right S-acts satisfy condition (P) if and only if S is a group or a 0-group.*

Proof. Suppose that all flat cyclic right S-acts satisfy condition (P). Since S is regular, then by Theorem 2.3.27, $S = G \dot{\cup} Z$ where G is a group and Z is the set of all right zero elements of S. If $Z = \emptyset$, then S is a group. Otherwise let $a \in Z$.

Then for every $b \in Z$, $ab = b$, $ba = a$. But a, b are idempotents and since S is an inverse monoid, then $ab = ba$. Consequently, $a = b$ and so Z has only one element which is zero.

If S is group or 0-group, then by Theorem 2.3.25, all flat cyclic right S-acts satisfy condition (P). ■

From Theorem 2.3.25, and Corollary 2.3.29, we have the following theorem.

Theorem 2.3.30. *Let S be an inverse monoid. Then the following statements are equivalent:*

(1) All weakly flat cyclic right S-acts satisfy condition (P).

(2) All flat cyclic right S-acts satisfy condition (P).

(3) S is a group or a 0-group.

The following is clear.

Lemma 2.3.31. *Let S be a monoid. If for every $x, y \in S$, $xy = 1$ implies that $x = y = 1$, then S does not have any right invertible element different from the identity.*

Theorem 2.3.32. *Let S be an eventually regular monoid such that for every $x, y \in S$, $xy = 1$ implies that $x = y = 1$. Then all flat cyclic right S-acts satisfy condition (P) if and only if S is right nil.*

Proof. If all flat cyclic right S-acts satisfy condition (P), then by Theorem 2.3.25, $S = G \dot{\cup} N$ where G is a group and N is the set of all right nil elements of S. But by Lemma 2.3.31, $G = \{1\}$ and so $S = \{1\} \dot{\cup} N$ is right nil as required.

If S is right nil, then by Theorem 2.3.25, all flat cyclic right S-acts satisfy condition (P). ■

From Theorem 2.3.32 we have.

Corollary 2.3.33. *Let S be an eventually regular semigroup and let $S^1 = S \cup \{1\}$. If all flat cyclic right S^1-acts satisfy condition (P), then every element in S is right nil.*

Theorem 2.3.34. *Let S be a regular monoid such that for every $x, y \in S$, $xy = 1$ implies that $x = y = 1$. Then all flat cyclic right S-acts satisfy condition (P) if and only if S is right zero.*

Proof. If all flat cyclic right S-acts satisfy condition (P), then by Theorem 2.3.27, $S = G \dot{\cup} Z$ where G is a group and Z is the set of all right zero elements of S. But by Lemma 2.3.31, $G = \{1\}$ and so $S = \{1\} \dot{\cup} Z$ is right zero as required.

If S is right zero, then by Theorem 2.3.27, all flat cyclic right S-acts satisfy condition (P). ■

From Theorem 2.3.34 we have.

Corollary 2.3.35. *Let S be a regular semigroup and let $S^1 = S \cup \{1\}$. If all flat cyclic right S^1-acts satisfy condition (P), then every element in S is right zero.*

Theorem 2.3.36. *Let S be an inverse monoid such that for every $x, y \in S$, $xy = 1$ implies that $x = y = 1$. Then all flat cyclic right S-acts satisfy condition (P) if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Suppose that all flat cyclic right S-acts satisfy condition (P). Then by Corollary 2.3.29, S is a group or a 0-group. Consequently by Lemma 2.3.31, $S = \{1\}$ or $S = \{0, 1\}$ as required.

The converse is obvious by Corollary 2.3.29. ■

Lemma 2.3.37 [19, p. 63]. *Let S be a semigroup. If S is completely 0-simple, then it is regular.*

From Theorem 2.3.34, and Lemma 2.3.37, we have

Corollary 2.3.38. *Let S be a completely 0-simple semigroup. Then all flat cyclic right S^1-acts satisfy condition (P) if and only if S is right zero.*

Now we consider right reversible monoids for which all flat cyclic right S-acts satisfy condition (P).

Theorem 2.3.39 [39]. *Let S be a right reversible monoid. If all flat cyclic right S-acts satisfy condition (P), then either $E(S) = \{1\}$ or $E(S) = \{0, 1\}$.*

Lemma 2.3.40. *Let S be a right reversible monoid. If all flat cyclic right S-acts satisfy condition (P), then either $S = C \dot{\cup} F$ or $S = C \dot{\cup} N \dot{\cup} F$ where C is right cancellative, N contains all nil elements of S and F is the set of regular-free elements of S.*

Proof. If all flat cyclic right S-acts satisfy condition (P), then by the remark after Theorem 2.3.16, $S = C \dot{\cup} N \dot{\cup} F$. By Theorem 2.3.39, either $E(S) = \{1\}$ or $E(S) = \{0, 1\}$.

If $E(S) = \{1\}$, then $N = \emptyset$. Otherwise let $1 \neq x \in N$. Then there exists $n \in \mathbb{N}$ such that x^n is right zero. Therefore x^n is an idempotent and so $x^n = 1$. If $n = 1$, then $x = 1$ giving a contradiction. Thus $n > 1$ and if $ax = bx$ for $a, b \in S$, then $(ax)x^{n-1} = (bx)x^{n-1}$ or $ax^n = bx^n$ which implies that $a = b$ and so x is right cancellative, which is also a contradiction.

Now suppose that $E(S) = \{0, 1\}$, and let $x \in N$. Then there exists $n \in \mathbb{N}$ such that x^n is an idempotent and so by the previous paragraph, $x^n \neq 1$. Thus $x^n = 0$ or x is nil as required. ■

Corollary 2.3.41. *Let S be a right reversible monoid. Then S is left PP and all flat cyclic right S-acts satisfy condition (P) if and only if $S = C$ or $S = C \cup \{0\}$ where C is right cancellative.*

Proof. Let S be left PP and all flat cyclic right S-acts satisfy condition (P). Then by Corollary 2.3.21, and Theorem 2.3.39, either $S = C$ or $S = C \dot{\cup} Z$ where C is right cancellative and Z is right zero. If $x \in Z$, then by Lemma 2.3.40, there exists $n \in \mathbb{N}$ such that $x^n = 0$. But x is an idempotent and so $x = x^n = 0$. Consequently, either $S = C$ or $S = C \cup \{0\}$ as required.

If S is right cancellative or $S = C \cup \{0\}$ where C is right cancellative, then by Theorem 2.3.22, S is left PP and all weakly flat cyclic right S-acts satisfy condition (P). ■

Now we have the following corollary.

Corollary 2.3.42. *If S is a right reversible monoid, then the following are equivalent:*

(1) S is left PP and all weakly flat cyclic right S-acts satisfy condition (P).

(2) S is left PP and all flat cyclic right S-acts satisfy condition (P).

(3) $S = C$ or $S = C \cup \{0\}$ where C is right cancellative.

Corollary 2.3.43. *Let S be an idempotent monoid. Then all flat cyclic right S-acts satisfy condition (P), if and only if S is right zero.*

Proof. Since S is left PP, then by Theorem 2.3.22, $S = C \dot{\cup} Z$ where C is right cancellative and Z is right zero. But for every $x \in C$, $x^2 = x$ implies that $x = 1$ and so S is right zero as required.

The converse is obvious by Theorem 2.3.22. ■

Corollary 2.3.44. *Let S be a monoid. Then S is an idempotent, right reversible monoid and all flat cyclic right S-acts satisfy condition (P) if and only if either $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Since S is an idempotent monoid, then it is left PP and so by Corollary 2.3.42, either S is right cancellative or $S = C \cup \{0\}$, where C is right cancellative. But for every $x \in C$, $x^2 = x$ implies that $x = 1$ and so either $S = \{1\}$ or $S = \{0, 1\}$ as required.

The converse is obvious. ■

If we consider monoids for which **all** flat right S-acts satisfy condition (P), then we have the following results.

Theorem 2.3.45 [1]. *Let S be a monoid. If all flat right S-acts satisfy condition (P), then $|E(S)| = 1$.*

Theorem 2.3.46. *Let S be a monoid. If all flat right S-acts satisfy condition (P), then $S = G \dot{\cup} F$ where G is a group and F is a regular-free subsemigroup. Moreover, F is a maximal ideal of S.*

Proof. If all flat right S-acts satisfy condition (P), then all flat cyclic right S-acts satisfy condition (P), and so by Theorem 2.3.16, $S = G \dot{\cup} N \dot{\cup} F$ where G is a group, N is the set of all right nil elements of S and F is the set of all regular-free elements of S. On the other hand by Theorem 2.3.45, $|E(S)| = 1$, and so $N = \emptyset$. Thus $S = G \dot{\cup} F$ as required. Also for $x \in S, y \in F$, $xy \in F$. Otherwise, $xy \in G$ and so there exists $z \in G$ such that $z(xy) = 1$. Consequently, $y(zx)y = y$ and so y is regular which is a contradiction. If $yx \in G$, then there exists $w \in G$ such that $(yx)w = 1$. Therefore $y(xw)y = y$ and so y is regular which is again a contradiction. Thus F is an ideal of S. It is easy to see that F is maximal. ■

Remark. Bulman-Fleming and Kilp in [9] showed that if $S =< x_0, x_1, x_2, \ldots \mid x_{i+1}x_i = x_i = x_i x_{i+1},\ i = 0, 1, \ldots > \cup \{1\}$, then $S = \{{x_i}^n \mid i \geq 0,\ n \geq 1\} \cup \{1\}$ and ${x_i}^n {x_j}^m = {x_i}^n$ if $i < j$, and equal to ${x_i}^{n+m}$ if $i = j$. Thus S is a disjoint union of a group and a regular-free subsemigroup. Also they showed that there exists a proper right ideal J of S such that the Rees factor S/J is flat but it does not satisfy condition (P). Hence, the converse of Theorem 2.3.46, is not true.

By Theorem 2.2.17, and Theorem 2.3.14, we saw that for right elementary monoids and also right subelementary monoids $S = C \dot{\cup} N$ such that $\forall a \in C$, $\forall b \in N$, $b \in Sab$, all weakly flat cyclic right S-acts satisfy condition (P). But it is not necessarily true for these monoids that all flat right acts satisfy condition (P), as the following example demonstrates.

Example 2.3.47. Let $S = \{0, 1\}$. Then $S = \{1\} \dot{\cup} \{0\}$ where $\{1\}$ is a group and $\{0\}$ is a right zero semigroup which is an ideal of S. If $A = \{x, y, z \mid x0 = y0 = z0 = z, x1 = x, y1 = y, z1 = z\}$, then by Theorem 2.2.19, A is a flat right S-act. Since A is not a coproduct of cyclic S-acts, then by Lemma 1.53, it does not satisfy condition (P).

This example also shows that there are monoids S which are disjoint union of a group and an ideal, but there exists a flat right S-act which does not satisfy condition (P).

Hence, to show that the converse of Theorem 2.3.46 is true more details of the structure of the regular-free part of S will be needed.

But if we restrict our attention to eventually regular monoids, then we have the following result.

Corollary 2.3.48. *Let S be an eventually regular monoid. Then all flat right S-acts satisfy condition (P) if and only if S is a group.*

Proof. If all flat right S-acts satisfy condition (P), then by Theorem 2.3.46, $S = G \dot{\cup} F$ where G is a group and F is regular-free. Since S is eventually regular, then $F = \emptyset$ and so $S = G$ is a group.

If S is a group, then by Theorem 2.2.14, all right S-acts satisfy condition (P) and so all flat right S-acts satisfy condition (P). ■

Notice that in Theorem 2.3.46, we could view S as $C \dot{\cup} F'$ where C is right cancellative and F' is the subset of F which contains no right cancellative elements. In this case if $x \in S, y \in F'$, then $yx \in F'$. Otherwise $ay = by$, $a, b \in S$, implies that $(ay)x = (by)x$ or $a(yx) = b(yx)$. Since $yx \in C$, then $a = b$ and so y is right cancellative which is a contradiction. Therefore, F' is a right ideal of S.

If $F' = \emptyset$, then S is right cancellative. In this case by what follows we show that all weakly flat right S-acts satisfy condition (P).

Lemma 2.3.49 [5]. *Let S be a monoid. Then a right S-act A is weakly flat if and only if A is principally weakly flat and for all left ideals I and J of S, $AI \cap AJ = A(I \cap J)$.*

Remark. If $I = Sx$, $J = Sy$ with $x, y \in S$, then $x \in Sx$, $y \in Sy$. Thus for a right S-act A with $a, a' \in A$, $ax \in AI$, $a'y \in AJ$. If A is weakly flat and $ax = a'y$, then $ax = a'y \in AI \cap AJ$ and so by Lemma 2.3.49, there exists $a'' \in A$ and $z \in Sx \cap Sy$ such that $ax = a'y = a''z$.

Lemma 2.3.50. *Let S be a right cancellative monoid. Then all weakly flat right S-acts satisfy condition (P).*

Proof. Let A be a weakly flat right S-act and let $ax = bx$ for $a, b \in A$ and $x \in S$. Then by Lemma 2.3.49, there exists $a'' \in A$ and $z \in Sx \cap Sy$ such that $ax = by = a''z$. Then there exist $s, t \in S$ such that $z = sx = ty$. Thus $ax = a''z = a''sx$ and $by = a''z = a''ty$. Since A is torsion free and x, y are right cancellative, then $a = a''s$ and $b = a''t$. Since $sx = ty$, then A satisfies condition (P) as required. ■

Lemma 2.3.51. *Let S be a left PP monoid. If all flat right S-acts satisfy condition (P), then S is right cancellative.*

Proof. If all flat right S-acts satisfy condition (P), then all flat cyclic right S-acts satisfy condition (P) and so by Corollary 2.3.21, $S = C \dot{\cup} Z$ where C is right cancellative and Z is right zero. But by Theorem 2.3.46, $Z = \emptyset$ and so $S = C$ is right cancellative as required. ■

Since right cancellative monoids are left PP, then from Lemma 2.3.50, and Lemma 2.3.51, we can deduce the main Theorem in [4].

Theorem 2.3.52. *Let S be any monoid. Then the following statements are equivalent:*

(1) S is right cancellative.

(2) S is left PP and every weakly flat right S-act satisfies condition (P).

(3) S is left PP and every flat right S-act satisfies condition (P).

Remark. If $S = \mathbb{N}$ is the set of natural numbers, then $(S, .)$ is a monoid. Also $S = \{1\} \dot\cup \mathbb{N} \setminus \{1\}$ and $F = \mathbb{N} \setminus \{1\}$ is an ideal of S which is regular-free and is generated by prime numbers. Thus F is an infinitely generated ideal of S. Since S is cancellative, then by Lemma 2.3.50, all flat right S-acts satisfy condition (P). Hence, there are monoids S such that $S = G \dot\cup F$ and that all flat right S-acts satisfy condition (P), but F is not finitely generated.

Let J be a proper right ideal of a monoid S. If x, y, z are symbols not belonging to S, define

$$A(J) := (\{x, y\} \times (S \setminus J)) \cup (\{z\} \times J),$$

and define a right S-action on $A(J)$ by

$$(x, u)s = \begin{cases} (x, us) & \text{if } us \notin J \\ (z, us) & \text{if } us \in J \end{cases}$$

$$(y, u)s = \begin{cases} (y, us) & \text{if } us \notin J \\ (z, us) & \text{if } us \in J \end{cases}$$

$$(z, u)s = (z, us).$$

$A(J)$ is a right S-act and we have

Proposition 2.3.53 [1]. *Let J be a proper right ideal of a monoid S. Then $A(J)$ satisfies condition (E), but fails to satisfy condition (P).*

Proposition 2.3.54 [1]. *Let J be a proper right ideal of a monoid S. Then $A(J)$ is a flat right S-act if and only if $j \in Jj$ for every $j \in J$.*

Propositions 2.3.53, and 2.3.54, give

Corollary 2.3.55. *Let S be a monoid. If all flat right S-acts satisfy condition (P), then for every proper right ideal J of S there exists $j \in J \setminus Jj$.*

Liu Zhongkui and Yang Yongbao in [40] gave the following result:

Proposition 2.3.56. *The following conditions on a monoid S are equivalent:*

(1) For every proper right ideal J of S there exists $j \in J \setminus Jj$.

(2) For every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots, x_i \in S$, there exists a positive integer n such that $x_n = x_{n+1} = \ldots = 1$.

What we can say at this stage is that, if in Theorem 2.3.46, F is countably infinitely generated and $x_0, x_1, x_2, \ldots$ are generators, then the property that $x_i = x_{i+1}x_i$, $i = 1, 2, \ldots$ cannot occur. Otherwise, by Corollary 2.3.55, and Proposition 2.3.56, there exists a positive integer n such that $x_n = x_{n+1} = \ldots = 1$ and so F is finitely generated which is a contradiction.

Also if F is countably infinitely generated and $(x_0, x_1, \ldots)$ are generators, then the property that for every x_i there exists x_j such that $x_i = x_jx_i$ cannot occur. Otherwise, if $J = \bigcup_{i=0}^{\infty} x_iS$, then for $j \in J$ there exists i such that $j = x_is$ and $s \in S$. Then by assumption there exists $k \in \mathbb{N}$ such that $x_i = x_kx_i$ and so $j = x_kx_is = x_kj \in Jj$. Hence by Corollary 2.3.55, J is not a proper ideal and so $J = S$. Thus there exists $l \in \mathbb{N}$ and $s \in S$ such that $1 = x_ls$. Then by assumption there exists $m \in \mathbb{N}$ such that $x_mx_l = x_l$ and so $1 = x_ls = x_mx_ls$. Since $x_ls = 1$, then $1 = x_m1 = x_m$ and so $1 \in F$. But F is an ideal of S and so $F = S$ which is a contradiction.

The following Theorem appeared as one of the main results in [26] . Now by using Theorem 2.3.46, we give an alternative proof of this theorem as follows:

Theorem 2.3.57. *Let S be a monoid. Then all flat right S-acts are strongly flat if and only if $S = \{1\}$.*

Proof. If all flat right S-acts are strongly flat, then all flat right S-acts satisfy condition (P) and so by Theorem 2.3.46, $S = G \dot{\cup} F$ where G is a group and F is a regular-free subsemigroup. Since all flat right S-acts are strongly flat, then all flat cyclic right S-acts are strongly flat and so by Theorem 2.3.28, S is right nil. Thus $F = \emptyset$ and $G = \{1\}$. Consequently, $S = \{1\}$ as required.

If $S = \{1\}$, then it is clear that all right S-acts and so, all flat right S-acts are strongly flat. ■

Corollary 2.3.58. *Let S be a monoid. Then all flat right S-acts are projective if and only if $S = \{1\}$.*

Corollary 2.3.59. *Let S be a monoid. Then all flat right S-acts are free if and only is $S = \{1\}$.*

By what follows we extend the main result of [39]

Theorem 2.3.60 [13]. *For an arbitrary monoid S the following are equivalent:*

(1) All right S-acts are weakly flat.

(2) All finitely generated right S-acts are weakly flat.

(3) All cyclic right S-acts are weakly flat.

(4) S is a regular monoid and for any $x, y \in S$ there is an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.

Lemma 2.3.61. *If S is a monoid such that for every $x, y \in S$ there exists $z \in Sx \cap Sy$, then S has at most one right zero element which is also a left zero.*

Proof. Suppose that $x, y \in S$ are right zero. Then by assumption there exists $z \in Sx \cap Sy$. But $Sx = \{x\}$, $Sy = \{y\}$ and so $x = z = y$.

If $z \in S$ is right zero and $x \in S$, then by assumption there exists $w \in Szx \cap Sz$ and so there exist $s, t \in S$ such that $w = szx = tz$ or $(sz)x = tz$. Since z is right zero, then $(sz)x = tz$ implies that $zx = z$ and so z is left zero as required. ■

Corollary 2.3.62. *If S is a right reversible monoid, then S has at most one right zero element.*

Theorem 2.3.63. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero. Then all cyclic right S-acts are weakly flat if and only if S is a group or a 0-group.*

Proof. Suppose that all cyclic right S-acts are weakly flat. Then by Theorem 2.3.60, S is regular and for all $x, y \in S$ there exists $z \in Sx \cap Sy$ such that $(z, x) \in \rho(x, y)$. Thus by Theorem 2.3.27, S is right elementary and the right nil part is right zero. But by Lemma 2.3.61, S has at most one right zero element which is also left zero and so S is either a group or a 0-group.

Conversely, suppose that S is a group or a 0-group and let $x, y \in S$. If $y = 0$, then $z = y \in Sx \cap Sy$. If $y \neq 0$, then notice that $z = x = xy^{-1}y \in Sx \cap Sy$. In both cases $(z, x) \in \rho(x, y)$. Since S is regular, then by Theorem 2.3.60, all cyclic right S-acts are weakly flat. ■

Now we have [39, Theorem 2.1] as a corollary of Theorem 2.3.63.

Corollary 2.3.64. *Let S be a monoid. Then all cyclic right S-acts satisfy condition (P) if and only if S is a group or a 0-group.*

Proof. If all cyclic right S-acts satisfy condition (P), then they are all weakly flat. Also, all flat cyclic right S-acts will then satisfy condition (P) and so by Lemma

2.2.8, every $e \in E(S) \setminus \{1\}$ is right zero. Hence by Theorem 2.3.63 S is a group or a 0-group.

Let S be a group or a 0-group and suppose that $u \ \rho \ v$ for $u, v \in S$ and a right congruence ρ on S. Then there are two cases as follows:

Case 1. $u = v = 0$. If $s = t = 1$, then $s \ \rho \ 1 \ \rho \ t$ and $su = tv$.

Case 2. At least $u \neq 0$ or $v \neq 0$. For example if $v \neq 0$, then $u \ \rho \ v$ implies that $uv^{-1} \ \rho \ 1$. If $s = 1, t = uv^{-1}$, then $s \ \rho \ 1 \ \rho \ t$ and $su = tv$. ■

Corollary 2.3.65. *Let S be a monoid. Then all right S-acts satisfy condition (P) if and only if S is a group.*

Proof. If all right S-acts satisfy condition (P), then all cyclic right S-acts satisfy condition (P) and so by Corollary 2.3.64, S is a group or a 0-group. But by Theorem 2.3.45, $|E(S)| = 1$ and so S is a group.

The converse is obvious. ■

From Theorem 2.3.60, Theorem 2.3.63, and Corollary 2.3.64, we have

Corollary 2.3.66. *For any monoid S the following statements are equivalent:*

(1) All right S-acts are weakly flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(2) All finitely generated right S-acts are weakly flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(3) All cyclic right S-acts are weakly flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(4) All cyclic right S-acts are flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(5) All cyclic right S-acts satisfy condition (P).

(6) S is a group or a 0-group.

From Theorem 2.3.60, and Theorem 2.3.63, the following corollary can be deduced.

Corollary 2.3.67. *Let S be a null semigroup and let $S^1 = S \cup \{1\}$. Then all cyclic right S^1-acts are weakly flat if and only if $S^1 = \{0, 1\}$.*

Proof. If all cyclic right S^1-acts are weakly flat, then by Theorem 2.3.60, S^1 is regular and so for every $x \in S$ there exists $x' \in S^1$ such that $x = xx'x$. If $x' = 1$, then $xx = x$, but $xx = 0$ and so $x = 0$. If $x' \neq 1$, then $xx' = 0$ and so $x = 0$.

If $S^1 = \{0, 1\}$, then by Theorem 2.3.63, all cyclic right S^1-acts are weakly flat. ■

We can also deduce the following corollary of [39]

Corollary 2.3.68. *Let S be a monoid. Then all cyclic right S-acts are strongly flat if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. If all cyclic right S-acts are strongly flat, then all cyclic right S-acts satisfy condition (P) and so by Corollary 2.3.64, either $S = G$ or $S = G \cup \{0\}$ where G is a group.

On the other hand all flat cyclic right S-acts are strongly flat and so by Theorem 2.3.28, S is right nil and so $G = \{1\}$. Otherwise, let $1 \neq x \in G$. Then there exists $n \in \mathbb{N}$ such that x^n is right zero and so $x^{n+1} = x^n$ implies that $x = 1$ which is a contradiction.

If $S = \{1\} \vee S = \{0, 1\}$, then it is obvious that all cyclic right S-acts are strongly flat. ■

From Theorem 2.3.30, and Corollary 2.3.66, we have

Corollary 2.3.69. *Let S be an inverse monoid. Then the following statements are equivalent:*

(1) All right S-acts are weakly flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(2) All finitely generated right S-acts are weakly flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(3) All cyclic right S-acts are weakly flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(4) All cyclic right S-acts are flat and every $e \in E(S) \setminus \{1\}$ is right zero.

(5) All cyclic right S-acts satisfy condition (P).

(6) All weakly flat cyclic right S-acts satisfy condition (P).

(7) All flat cyclic right S-acts satisfy condition (P).

(8) S is a group or a 0-group.

2.4. Flatness on Ideal Extensions

In section 2.3, we saw that monoids S with a structure of the form $S = G \dot{\cup} N$ where G is a group and N is right nil have the property that all weakly flat cyclic right S-acts satisfy condition (P). Also monoids for which all flat (cyclic) right acts satisfy condition (P) have the structure of the form $S = G \dot{\cup} I$ with G a group and I an ideal of S. Moreover, in some cases the investigation of the flatness of I^1-acts ($I^1 = I \cup \{1\}$) is much easier than that of S-acts. So it seems that it is reasonable to consider monoids of this structure and see whether it is possible to extend the flatness of (cyclic) I^1-acts to the flatness of (cyclic) S-acts. So we will try in this section to show this, either in general or for a certain classes of monoids. We also show that for monoids with some extra condition, flatness of (cyclic) I^1-acts can be deduced from the flatness of S-acts. There are also some corollaries that will arise. In what follows we suppose that $S = G \dot{\cup} I$ is a monoid with G a group, I an ideal of S and $I^1 = I \cup \{1\}$. It is also clear that any S-act is an I^1-act. We first of all show that (weak) flatness and also condition (P) of all right S-acts can be deduced from the corresponding property of right I^1-acts.

Lemma 2.4.1. *Let A be a right S-act and X a left S-act. Let $a, a' \in A$, $x, x' \in X$ and suppose that $a \otimes x = a' \otimes x'$ in $A \otimes_S X$. Then either $a \otimes x = a' \otimes x'$ in $A \otimes_{I^1} X$ or there exists $g \in G$ with $a' = ag, gx' = x$.*

Proof. Since $a \otimes x = a' \otimes x'$ in $A \otimes_S X$, there is a *scheme over* S

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a_2 s_2 & s_1 x &= t_1 x_2 \\ & \cdots & s_2 x_2 &= t_2 x_3 \\ a_{n-1} t_{n-1} &= a_n s_n & & \cdots \\ a_n t_n &= a' & s_n x_n &= t_n x' \end{aligned}$$

in which $x_i \in X$ and $s_i, t_i \in S$. Moreover, we can assume that this scheme has minimal length. Suppose first of all that the scheme has length 1. In other words, we have

$$\begin{aligned} a &= a_1 s_1 & & \\ a_1 t_1 &= a' & s_1 x &= t_1 x' \end{aligned}$$

There are several cases: if both $s_1, t_1 \in I^1$, then the scheme is over I^1 and so $a \otimes x = a' \otimes x'$ in $A \otimes_{I^1} X$ as required; if $s_1 \in I, t_1 \in G$ then the scheme

$$\begin{aligned} a &= (a_1 t_1)(t_1^{-1} s_1) & & \\ (a_1 t_1).1 &= a' & (t_1^{-1} s_1) x &= 1.x' \end{aligned}$$

is over I^1 and again the result follows; a similar construction works if $s_1 \in G, t_1 \in I$; finally if $s_1, t_1 \in G$, not both $= 1$. In this case, we set $g = s_1^{-1} t_1$ and note that $ag = a', gx' = x$.

Suppose now that the scheme has length ≥ 2 and suppose that $s_i \in G$ for some $2 \leq i \leq n-1$. Consider the part of the scheme

$$\begin{aligned} a_{i-2}t_{i-2} &= a_{i-1}s_{i-1} & s_{i-1}x_{i-1} &= t_{i-1}x_i \\ a_{i-1}t_{i-1} &= a_is_i & s_ix_i &= t_ix_{i+1} \\ a_it_i &= a_{i+1}s_{i+1} & s_{i+1}x_{i+1} &= t_{i+1}x_{i+2}. \end{aligned}$$

This can be replaced by the shorter scheme

$$\begin{aligned} a_{i-2}t_{i-2} &= a_{i-1}s_{i-1} & s_{i-1}x_{i-1} &= t_{i-1}s_i^{-1}t_ix_{i+1} \\ a_{i-1}(t_{i-1}s_i^{-1}t_i) &= a_{i+1}s_{i+1} & s_{i+1}x_{i+1} &= t_{i+1}x_{i+2} \end{aligned}$$

giving a contradiction on the minimality of the original scheme.

If $s_n \in G$ then again, the part of the scheme

$$\begin{aligned} a_{n-2}t_{n-2} &= a_{n-1}s_{n-1} & & \\ & & s_{n-1}x_{n-1} &= t_{n-1}x_n \\ a_{n-1}t_{n-1} &= a_ns_n & & \\ & & s_nx_n &= t_nx' \\ a_nt_n &= a' & & \end{aligned}$$

can be replaced with the shorter scheme

$$\begin{aligned} a_{n-2}t_{n-2} &= a_{n-1}s_{n-1} & & \\ & & s_{n-1}x_{n-1} &= (t_{n-1}s_n^{-1}t_n)x' \\ a_{n-1}(t_{n-1}s_n^{-1}t_n) &= a' & & \end{aligned}$$

and again we get a contradiction. Hence we see that $s_i \notin G$ for $i \geq 2$.

In a similar way, we can deduce that $t_i \notin G$ for $i \leq n-1$.

If $s_1 \in G$, then the first two lines of the scheme

$$\begin{aligned} a &= a_1s_1 & s_1x &= t_1x_2 \\ a_1t_1 &= a_2s_2 & s_2x_2 &= t_2x_3 \end{aligned}$$

can be replaced by

$$\begin{aligned} a &= (a_1s_1).1 & 1.x &= (s_1^{-1}t_1)x_2 \\ (a_1s_1)(s_1^{-1}t_1) &= a_2s_2 & s_2x_2 &= t_2x_3 \end{aligned}$$

where $s_1^{-1}t_1 \notin G$ since $t_1 \notin G$. A similar construction can be made if $t_n \in G$ and so we see that the original scheme can be replaced by one over I^1 and so $a \otimes x = a' \otimes x'$ in $A \otimes_{I^1} X$ as required. ∎

<u>**Theorem 2.4.2.**</u> *Let S be as above and let $f : X \to Y$ be a left S-monomorphism and A a right S-act. If $1 \otimes f : A \otimes_{I^1} X \to A \otimes_{I^1} Y$ is one-to-one then so is $1 \otimes f : A \otimes_S X \to A \otimes_S Y$.*

Proof. Suppose that $a \otimes f(x) = a' \otimes f(x')$ in $A \otimes_S Y$. By Lemma 2.4.1, either $a \otimes f(x) = a' \otimes f(x')$ in $A \otimes_{I^1} Y$ or there exists $g \in G$ with $a' = ag, gf(x') = f(x)$. In this latter case, $gx' = x$ and so $a \otimes x = a \otimes gx' = ag \otimes x' = a' \otimes x'$ in $A \otimes_S X$ as required. In the former case, $a \otimes x = a' \otimes x'$ in $A \otimes_{I^1} X$ and hence in $A \otimes_S X$ as required. ■

Theorem 2.4.3. *If A is a right S-act and if A is (weakly) flat as a right I^1-act then it is (weakly) flat as a right S-act.*

Proof. The result for the flatness case follows directly from Theorem 2.4.2.

Suppose then that A is weakly flat as an I^1-act and let J be a left ideal of S. Notice that either $J = S$ or J is an ideal of I^1. Suppose that $a, a' \in A$, $j, j' \in J$ and $a \otimes j = a' \otimes j'$ in $A \otimes_S S$. If $J = S$ then clearly there is nothing to do. Otherwise, we see that $A \otimes_S J \to A \otimes_S I^1$ is one-to-one by Theorem 2.4.2. Now it is easy to see that $a \otimes j = a' \otimes j'$ in $A \otimes_S I^1$ and hence in $A \otimes_S J$ as required. ■

Corollary 2.4.4. *If all right I^1-acts are (weakly) flat, then all right S-acts are (weakly) flat.*

Proof. Suppose that all right I^1-acts are (weakly) flat and let A be a right S-act. Since A is a right I^1-act, then by assumption it is a (weakly) flat right I^1-act. Consequently, by Theorem 2.4.3, A is a (weakly) flat right S-act. ■

Theorem 2.4.5. *Let A be a right S-act. If A satisfies condition (P) as a right I^1-act, then it satisfies condition (P) as a right S-act.*

Proof. Suppose that A satisfies condition (P) as a right I^1-act and let $au = a'v$, $a, a' \in A$, $u, v \in S$. Since $S = G \dot{\cup} I$, then there are two cases as follows:

Case 1. At least one of u or v belongs to G. If for example $u \in G$, then $au = a'v$ implies that $a = a'vu^{-1}$. Since $a' = a'1$ and $(vu^{-1})u = 1v$, then A satisfies condition (P).

Case 2. $u, v \in I$. Then $u, v \in I^1$. Since A satisfies condition (P) as a right I^1-act, then $au = a'v$ implies that there exist $a'' \in A$, $s, t \in I^1$ such that $a = a''s$, $a' = a''t$ and $su = tv$. But $s, t \in S$ and so A satisfies condition (P) as a right S-act. ■

Corollary 2.4.6. *If all right I^1-acts satisfy condition (P), then all right S-acts satisfy condition (P).*

Proof. Suppose that all right I^1-acts satisfy condition (P) and let A be a right S-act. Since A is a right I^1-act, then by assumption it satisfies condition (P) as a right I^1-act. Consequently, by Theorem 2.4.5, A satisfies condition (P) as a right S-act. ■

Now we show for monoids of the form $S = G \dot{\cup} I$ where G is a group and I is an ideal that condition (P) of all cyclic right S-acts can be deduced from the condition (P) of all cyclic right I^1-acts. First of all we need the following lemma.

Lemma 2.4.7. *Let S be a monoid and I an ideal of S. Let ρ be a right congruence on S and let $\rho_1 = \{(a,b) \in \rho \mid a,b \in I^1\}$. Then ρ_1 is a right congruence on I^1 ($I^1 = I \cup \{1\}$).*

Proof. It is obvious that ρ_1 is an equivalence relation on I^1. Let $(a,b) \in \rho_1$, $x \in I^1$. Since $(a,b) \in \rho$ and ρ is a right congruence on S, then $(a,b)x = (ax, bx) \in \rho$. Since I is an ideal, then $ax, bx \in I^1$ and so $(a,b)x = (ax,bx) \in \rho_1$. ■

Lemma 2.4.8. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. Let ρ be a right congruence on S and ρ_1 as in Lemma 2.4.7. If I^1/ρ_1 satisfies condition (P) as an I^1-act, then S/ρ satisfies condition (P) as an S-act.*

Proof. Let $u \ \rho \ v$ with $u, v \in S$, then there are two cases that can arise:

Case 1. At least one of u or v belongs to G. For example if $u \in G$, then $1 \ \rho \ vu^{-1}$ and $(vu^{-1})u = 1v$. If $s = vu^{-1}$, $t = 1$, then $s \ \rho \ 1 \ \rho \ t$ and $su = tv$.

Case 2. $u, v \in I \subseteq I^1$. Then $u \ \rho \ v$ implies that $u \ \rho_1 \ v$. Since I^1/ρ_1 satisfies condition (P), then there exist $s, t \in I^1 \subseteq S$ such that $s \ \rho_1 \ 1 \ \rho_1 \ t$ and $su = tv$. Since $\rho_1 = \rho|_{I^1}$, then $s \ \rho \ 1 \ \rho \ t$. ■

Theorem 2.4.9. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. If all cyclic right I^1-acts satisfy condition (P), then all cyclic right S-acts satisfy condition (P).*

Proof. Let S/ρ be a cyclic right S-act for a right congruence ρ on S. Then I^1/ρ_1 is a cyclic right I^1-act where ρ_1 is the right congruence on I^1 as in Lemma 2.4.7, and so by assumption I^1/ρ_1 satisfies condition (P) on I^1. Then by Lemma 2.4.8, S/ρ satisfies condition (P) on S. ■

Corollary 2.4.10. *Let $S = G \cup \{0\}$ be a monoid such that G is a group. Then all cyclic right S-acts satisfy condition (P).*

Proof. If $I^1 = \{0, 1\}$, then by Theorem 2.2.19, all cyclic right I^1-acts are strongly flat and so all cyclic right I^1-acts satisfy condition (P). Thus by Theorem 2.4.9, all cyclic right S-acts satisfy condition (P). ■

By the following theorem it can be seen that for monoids with the structure mentioned above and with the property that $\forall g \in G, \forall x \in I$, $gx = x$, if ρ is a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7, then S/ρ satisfies condition (P) if and only if I^1/ρ_1 satisfies condition (P).

Lemma 2.4.11. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I,\ gx = x$. Let ρ be a right congruence on S and ρ_1 as in Lemma 2.4.7. If S/ρ satisfies condition (P), then I^1/ρ_1 satisfies condition (P).*

Proof. Let $u\ \rho_1\ v$ for $u, v \in I^1$. If at least one of u or v is 1, for example if $v = 1$, then $u\ \rho_1\ 1$. If $s = 1,\ t = u$, then $s, t \in I^1$, $s\ \rho_1\ 1\ \rho_1\ t$ and $su = tv$. Now we suppose that $u, v \in I$. Then $u\ \rho\ v$ and so there exist $s, t \in S$ such that $s\ \rho\ 1\ \rho\ t$ and $su = tv$. Now there are three cases that can arise:

Case 1. $s, t \in I \subseteq I^1$. Then $s\ \rho_1\ 1\ \rho_1\ t$ and $su = tv$.

Case 2. $s \in G, t \in I$. Then $su = tv$ implies that $u = (s^{-1}t)v$. Since I is an ideal, then $s^{-1}t \in I \subseteq I^1$. On the other hand $s\ \rho\ 1$ implies that $s^{-1}\ \rho\ 1$ and so $s^{-1}t\ \rho_1\ t\ \rho_1\ 1$. If $s' = 1,\ t' = s^{-1}t$, then $s'\ \rho_1\ 1\ \rho_1\ t'$ and $s'u = t'v$.

Case 3. $s, t \in G$. Then $su = tv$ implies that $u = v$. If $s' = t' = 1$, then $s'\ \rho_1\ 1\ \rho_1\ t'$ and $s'u = t'v$. ∎

From Lemma 2.4.8, and Lemma 2.4.11, we have

Theorem 2.4.12. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I,\ gx = x$. Let ρ be a right congruence on S and ρ_1 as in Lemma 2.4.7. Then S/ρ satisfies condition (P) if and only if I^1/ρ_1 satisfies condition (P).*

By what follows we show for monoids with the structure given in Theorem 2.4.12, and with the extra property that $\forall g \in G, \forall x \in I,\ xg = x$, that condition (P) of all cyclic right S-acts implies condition (P) of all cyclic right I^1-acts. As a result for these monoids all cyclic right I^1-acts satisfy condition (P) if and only if all cyclic right S-acts satisfy condition (P).

Lemma 2.4.13. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I,\ xg = x$. Let ρ_1 be a right congruence on I^1. If $\rho_I = \{(a, b) \in \rho_1 \mid a, b \in I\}$. Then $\rho = \rho_I \cup 1_G$ is a right congruence on S.*

Proof. It is obvious that ρ is an equivalence relation on S. Let $(a, b) \in \rho,\ s \in S$. Then either $(a, b) \in \rho_I$ or $(a, b) \in 1_G$.

If $(a, b) \in \rho_I$, then $(a, b) \in \rho_1$ and so $a, b \in I$. Then there are two cases as follows:

Case 1. $s \in I \subseteq I^1$. Then $(a, b)s = (as, bs) \in \rho_1$. But $as, bs \in I$ and so $(a, b)s \in \rho_I \subseteq \rho$.

Case 2. $s \in G$. Then $(a, b)s = (as, bs) = (a, b) \in \rho_I \subseteq \rho$.

If $(a, b) \in 1_G$, then $a = b \in G$ and so $(a, b)s = (a, a)s = (as, as) \in \rho$. Thus ρ is a right congruence on S as required. ∎

Theorem 2.4.14. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $xg = x$. Let ρ_1 be a right congruence on I^1 and ρ the right congruence on S as in Lemma 2.4.13. If S/ρ satisfies condition (P) on S, then I^1/ρ_1 satisfies condition (P) on I^1.*

Proof. Let $u \ \rho_1 \ v$ for $u, v \in I^1$. Then there are four cases that can arise:

Case 1. $u, v \in I$. Then $u \ \rho_1 \ v$ implies that $u \ \rho \ v$. Since S/ρ satisfies condition (P), then there exist $s, t \in S$ such that $s \ \rho \ 1 \ \rho \ t$ and $su = tv$. Since $s \ \rho \ 1$ and $1 \in G$, then by definition of ρ, $s = 1$. Similarly, $t = 1$. Consequently, $u = v$. If $s' = t' = 1$, then $s' \ \rho_1 \ 1 \ \rho_1 \ t'$ and $s'u = t'v$.

Case 2. $u \in I, v = 1$. Then $u \ \rho_1 \ 1$. If $s = 1, t = u$, then $s, t \in I^1$ and so $s \ \rho_1 \ 1 \ \rho_1 \ t$. Also $su = tv$.

Case 3. $u = 1, v \in I$. It is similar to case 2.

Case 4. $u = v = 1$. If $s = t = 1$, then $s \ \rho_1 \ 1 \ \rho_1 \ t$ and $su = tv$. ■

Theorem 2.4.15. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $xg = x$. If all cyclic right S-acts satisfy condition (P), then all cyclic right I^1-acts satisfy condition (P).*

Proof. Let ρ_1 be a right congruence on I^1. Then by Lemma 2.4.13, $\rho = \rho_I \cup 1_G$ is a right congruence on S and so by assumption S/ρ satisfies condition (P). Then by Theorem 2.4.14, I^1/ρ_1 satisfies condition (P) on I^1. ■

Now from Theorem 2.4.9, and Theorem 2.4.15, we have the following

Theorem 2.4.16. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $xg = x$. Then all cyclic right I^1-acts satisfy condition (P) if and only if all cyclic right S-acts satisfy condition (P).*

Here we show for monoids of the form $S = G \dot{\cup} I$ where G is a group and I is an ideal that weak flatness of cyclic right S-acts can also be deduced from the weak flatness of cyclic right I^1-acts.

Lemma 2.4.17. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. If I^1/ρ_1 is a weakly flat right I^1-act, then S/ρ is a weakly flat right S-act.*

Proof. Let $u \ \rho \ v$ with $u, v \in S$. Then there are two cases that can arise:

Case 1. $u, v \in I \subseteq I^1$. Then $u \ \rho_1 \ v$ and so there exist $s, t \in I^1$ such that $s(\rho_1 \vee \Delta u)1(\rho_1 \vee \Delta v)t$ and $su = tv$. Since $s(\rho_1 \vee \Delta u)1$, then there exist

$s_1, s_2, \ldots, s_{2n-1} \in I^1$ such that

$$s \ \rho_1 \ s_1(\Delta u)s_2 \ \rho_1 \ s_3 \ldots s_{2n-1}(\Delta u)1.$$

Then by definition of ρ_1, we have

$$s \ \rho \ s_1(\Delta u)s_2 \ \rho \ s_3 \ldots s_{2n-1}(\Delta u)1 \text{ or } s(\rho \ \vee \ \Delta u)1.$$

Similarly, $t(\rho \ \vee \ \Delta v)1$.

Case 2. At least one of u or v belongs to G, for example if $u \in G$, then $vu^{-1} \ \rho \ 1$ and $(vu^{-1})u = 1v$. If $s = vu^{-1}, t = 1$, then $s \ \rho \ 1(\Delta u)1$ or $s(\rho \ \vee \ \Delta u)1$. Also $t(\rho \ \vee \ \Delta v)1$ and $su = tv$. ■

Theorem 2.4.18. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. If all cyclic right I^1-acts are weakly flat, then all cyclic right S-acts are weakly flat.*

Proof. Let S/ρ be a cyclic right S-act for a right congruence ρ on S. Let ρ_1 be the right congruence on I^1 as in Lemma 2.4.7, then I^1/ρ_1 is a cyclic right I^1-act and so by assumption it is weakly flat. Then by Lemma 2.4.17, S/ρ is weakly flat.

■

By the following it can be seen that for monoids with the structure mentioned in Theorem 2.4.18, and with the extra property that $\forall g \in G, \forall x \in I, \ gx = x$, if ρ is a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7, then S/ρ is weakly flat if and only if I^1/ρ_1 is weakly flat. It can also be seen that for these monoids, if all weakly flat cyclic right I^1-acts satisfy condition (P), then all weakly flat cyclic right S-acts satisfy condition (P).

Theorem 2.4.19. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I, \ gx = x$. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. If S/ρ is weakly flat, then I^1/ρ_1 is weakly flat.*

Proof. Let $u \ \rho_1 \ v$ with $u, v \in I^1$. If at least one of u or v is 1, for example if $v = 1$, then $u \ \rho_1 \ 1$. If $s = 1, t = u$, then $su = tv$ and $s(\rho_1 \ \vee \ \Delta u)1(\rho_1 \ \vee \ \Delta 1)t$.

Suppose then that $u, v \in I$. Since $u \ \rho_1 \ v$, then $u \ \rho \ v$ and so there exist $s, t \in S$ such that $s(\rho \ \vee \ \Delta u)1(\rho \ \vee \ \Delta v)t$ and $su = tv$

If $s, t \in G$, then $su = tv$ implies that $u = v$. Thus $1u = 1v$ and $1(\rho_1 \ \vee \ \Delta u)1(\rho_1 \ \vee \ \Delta v)1$.

Now we show that if $s \in I$, then either $s(\rho_1 \ \vee \ \Delta u)1$ or there exists $s' \in I \subseteq I^1$ such that $s'u = su$, $s' \ \rho_1 \ 1$ and so $s'(\rho_1 \ \vee \ \Delta u)1$.

Since $s(\rho \ \vee \ \Delta u)1$, then there exist $s_1, s_2, \ldots, s_{2n-1} \in S$ such that

$$s \ \rho \ s_1(\Delta u)s_2 \ldots s_{2n-2} \ \rho \ s_{2n-1}(\Delta u)s_{2n} = 1.$$

If $s_i \in I$ for every $1 \leq i \leq 2n-1$, then $s_i \in I^1$ and so $s(\rho_1 \ \vee \ \Delta u)1$.

Suppose then that there exists $1 \leq i \leq 2n-1$ such that $s_i \in G$ and that i is the smallest such numbers. Then for every $j < i$, $s_j \in I$. Now there are two cases as follows:

Case 1. $i = 1$. Then $s \ \rho \ s_1$ implies that $s{s_1}^{-1} \ \rho \ 1$. Since $s{s_1}^{-1} \in I$, then $s{s_1}^{-1} \ \rho_1 \ 1$. Also ${s_1}^{-1} \in G$ implies that ${s_1}^{-1}u = u$ and so

$$su = s({s_1}^{-1}u) = (s{s_1}^{-1})u.$$

If $ss_1^{-1} = s'$, then $su = s'u$ and $s' \ \rho_1 \ 1$.

Case 2. $2 \leq i \leq 2n-1$. Then either $s_i \ \rho \ s_{i+1}$ or $s_i(\Delta u)s_{i+1}$.

(a) If $s_i(\Delta u)s_{i+1}$, then $s_{i-1} \ \rho \ s_i$. Since $s_i \in G$, $s_{i-1} \in I$, then $s_{i-1}{s_i}^{-1} \in I$ and $s_{i-1}{s_i}^{-1} \ \rho \ 1$. Thus $s_{i-1}{s_i}^{-1} \ \rho_1 \ 1$. Also ${s_i}^{-1} \in G$ implies that ${s_i}^{-1}u = u$ and so

$$(s_{i-1}{s_i}^{-1})u = s_{i-1}({s_i}^{-1}u) = s_{i-1}u.$$

Since $s_{i-2}(\Delta u)s_{i-1}$, then

$$s_{i-2}u = s_{i-1}u = (s_{i-1}{s_i}^{-1})u \text{ or } s_{i-2}(\Delta u)s_{i-1}{s_i}^{-1},$$

and so

$$s \ \rho \ s_1(\Delta u)s_2 \ldots s_{i-2}(\Delta u)s_{i-1}{s_i}^{-1} \ \rho_1 \ 1(\Delta u)1.$$

Since $s, s_1, s_2, \ldots, s_{i-2}, s_{i-1}{s_i}^{-1} \in I$, then

$$s \ \rho_1 \ s_1(\Delta u)s_2 \ldots s_{i-2}(\Delta u)s_{i-1}{s_i}^{-1} \ \rho_1 \ 1(\Delta u)1 \text{ or } s(\rho_1 \ \vee \ \Delta u)1.$$

(b) If $s_i \ \rho \ s_{i+1}$, then $s_{i-1}(\Delta u)s_i$ or $s_{i-1}u = s_iu$. Also $s_i \in G$ implies that $s_iu = u$ and so $s_{i-1}u = s_iu = u$ or $s_{i-1}(\Delta u)1$. Then we have

$$s \ \rho \ s_1, \ldots, s_{i-2} \ \rho \ s_{i-1}(\Delta u)1,$$

such that $s, s_1, s_2, \ldots, s_{i-1} \in I$, and so

$$s \ \rho_1 \ s_1 \ldots s_{i-2} \ \rho_1 \ s_{i-1}(\Delta u)1 \text{ or } s(\rho_1 \ \vee \ \Delta u)1.$$

Similarly, if $t \in I$, then $t(\rho \vee \Delta v)1$ implies that either $t(\rho_1 \vee \Delta v)1$ or there exists $t' \in I^1$ such that $t'(\rho_1 \vee \Delta v)1$ and $t'v = tv$. Now if $s, t \in I$, then we are done.

If $s \in I$, $t \in G$, then $tv = v$. Since $s \in I$, then either $s(\rho_1 \vee \Delta u)1$ and so $su = tv = v$ or there exists $s' \in I \subseteq I^1$ such that $s'u = su = tv = v$ and $s'(\rho_1 \vee \Delta u)1$. Since $1(\rho_1 \vee \Delta v)1$, then we are done. Similarly if $s \in G, t \in I$. ■

Now from Lemma 2.4.17, and Theorem 2.4.19, we have

Theorem 2.4.20. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. Let ρ be a right congruence on S and let ρ_1 be the right congruence on I^1 as in Lemma 2.4.7. Then S/ρ is weakly flat if and only if I^1/ρ_1 is weakly flat.*

Theorem 2.4.21. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. If all weakly flat cyclic right I^1-acts satisfy condition (P), then all weakly flat cyclic right S-acts satisfy condition (P).*

Proof. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. Suppose that S/ρ is a weakly flat right S-act. Then by Theorem 2.4.19, I^1/ρ_1 is a weakly flat cyclic right I^1-act and so by assumption I^1/ρ_1 satisfies condition (P) on I^1. Then by Lemma 2.4.8, S/ρ satisfies condition (P) on S. ■

Corollary 2.4.22. *Let $S = G \cup N$ be a monoid such that G is a group, N is right nil and $\forall g \in G, \forall x \in N$, $gx = x$, then all weakly flat cyclic right S-acts satisfy condition (P).*

Proof. By Theorem 2.3.28, all weakly flat cyclic right N^1-acts satisfy condition (P) and so by Theorem 2.4.21, all weakly flat cyclic right S-acts satisfy condition (P). ■

Corollary 2.4.23. *Let $S = G \cup Z$ be a monoid such that G is a group and Z is the set of all right zero elements of S. Then all weakly flat cyclic right S-acts satisfy condition (P).*

Proof. Since $\forall g \in G, \forall x \in Z$, $gx = x$ and also Z is right nil, then by Corollary 2.4.22, all weakly flat cyclic right S-acts satisfy condition (P). ■

By the following it can be seen that for monoids with the structure mentioned above and with the property that $\forall g \in G, \forall x \in I$, $xg = x$, weak flatness of all cyclic right S-acts implies weak flatness of all cyclic right I^1-acts. Consequently, for this types of monoid all cyclic right S-acts are weakly flat if and only if all cyclic right I^1-acts are weakly flat.

Lemma 2.4.24. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $xg = x$. Let ρ_1 be a right congruence on I^1 and let ρ be* *the right congruence on S as in Lemma* 2.4.13. *If* $s(\rho \vee \Delta u)1$ *for* $s, u \in I^1$, *then* $s(\rho_1 \vee \Delta u)1$.

Proof. If $s = 1$, then $1 \; \rho_1 \; 1(\Delta u)1$ or $1(\rho_1 \vee \Delta u)1$. Thus we suppose that $s \neq 1$. Then $u \neq 1$, otherwise $s(\rho \vee \Delta 1)1$ or $s \; \rho \; 1$ and so by definition of ρ, $s = 1$ which is a contradiction. Hence, we suppose that $s(\rho \vee \Delta u)1$ and that $s, u \in I$. Then there exist $s_1, s_2, \ldots, s_{2n-1} \in S$ such that

$$s \; \rho \; s_1(\Delta u)s_2 \ldots s_{2n-1}(\Delta u)1. \qquad (*)$$

We claim that $s_i \in I$ for $1 \leq i \leq 2n - 1$, and so

$$s \; \rho_1 \; s_1(\Delta u)s_2 \ldots s_{2n-1}(\Delta u)1 \text{ or } s(\rho_1 \vee \Delta u)1.$$

Otherwise, there exists $1 \leq i \leq 2n - 1$ such that $s_i \in G$. We can suppose that the sequence $(*)$ is of minimal length and also we define $s_{2n} = 1$. Then there are two cases that can arise:

Case 1. $i = 2k$ for $1 \leq k \leq n - 1$. Then $s_i \; \rho \; s_{i+1}$ and so $s_i = s_{i+1}$. Thus $s_{i-1}(\Delta u)s_i = s_{i+1}(\Delta u)s_{i+2}$ or $s_{i-1}(\Delta u)s_{i+2}$ and so we have the shorter sequence

$$s \; \rho \; s_1(\Delta u)s_2 \ldots s_{i-1}(\Delta u)s_{i+2} \; \rho \; s_{i+3} \ldots s_{2n-1}(\Delta u)1,$$

which is a contradiction.

Case 2. $i = 2k - 1$ for $1 \leq k \leq n$. Then $s_i(\Delta u)s_{i+1}$. If $k = 1$, then $i = 1$ and so $s_1 \in G$. Since $s \; \rho \; s_1$, then by definition of ρ, $s = s_1 \in G$ which is a contradiction. Thus we can suppose that $2 \leq k \leq n$. Since $s_{i-1} \; \rho \; s_i$, then $s_{i-1} = s_i$. But $s_{i-2}(\Delta u)s_{i-1}$ and so $s_{i-2}(\Delta u)s_{i+1}$. Consequently, we have the shorter sequence

$$s \; \rho \; s_1(\Delta u)s_2 \ldots s_{i-2}(\Delta u)s_{i+1} \; \rho \; s_{i+2} \ldots s_{2n-1}(\Delta u)1,$$

which is also a contradiction. ■

Theorem 2.4.25. *Let* $S = G \dot\cup I$ *be a monoid such that* G *is a group,* I *is an ideal of* S *and* $\forall g \in G, \forall x \in I$, $xg = x$. *Let* ρ_1 *be a right congruence on* I^1 *and* ρ *the right congruence on* S *as in Lemma* 2.4.13. *If* S/ρ *is weakly flat, then* I^1/ρ_1 *is weakly flat.*

Proof. Let $u \; \rho_1 \; v$ for $u, v \in I^1$. Then there are four cases that can arise:

Case 1. $u = v = 1$. If $s = t = 1$, then $s(\rho_1 \vee \Delta u)1$, $t(\rho_1 \vee \Delta v)1$ and $su = tv$.

Case 2. $u \in I, v = 1$. Then $u \; \rho_1 \; 1$. If $s = 1, t = u$, then $s(\rho_1 \vee \Delta u)1$, and $su = tv$. Also $u \; \rho_1 \; 1(\Delta u)1$ or $t(\rho_1 \vee \Delta u)1$.

Case 3. $u = 1, v \in I$ is similar to case 2.

Case 4. $u, v \in I$. Then $u \ \rho \ v$ and so by assumption there exist $s, t \in S$ such that $s(\rho \vee \Delta u)1$, $t(\rho \vee \Delta v)1$ and $su = tv$. Now there are four possibilities as follows:

(a) $s, t \in I \subseteq I^1$. Then by Lemma 2.4.24, $s(\rho_1 \vee \Delta u)1$, and $t(\rho_1 \vee \Delta v)1$.

(b) $s \in G, t \in I$. Again by Lemma 2.4.24, $t(\rho_1 \vee \Delta v)1$ and from $su = tv$ we have $u = s^{-1}tv$. If $s' = 1, t' = s^{-1}t$, then $s'(\rho_1 \vee \Delta u)1$ and $s'u = t'v$. Now we show that $t'(\rho_1 \vee \Delta v)1$. Since $s(\rho \vee \Delta u)1$, then there exist $s_1, s_2, \dots, s_{2n-1} \in S$ such that

$$s \ \rho \ s_1(\Delta u)s_2 \ \rho \ s_3 \dots s_{2n-1}(\Delta u)1. \qquad (*)$$

Then $u = s^{-1}tv$ implies that

$$s \ \rho \ s_1 \ (\Delta s^{-1}tv) \ s_2 \ \rho \ s_3 \dots s_{2n-1} \ (\Delta s^{-1}tv)1,$$

or

$$s \ \rho \ s_1, s_1s^{-1}t \ (\Delta v) \ s_2s^{-1}t, s_2 \ \rho \ s_3, \dots, s_{2n-1}s^{-1}t \ (\Delta v) \ s^{-1}t.$$

Since $s \in G$, then $s \ \rho \ s_1$ implies that $1 \ \rho \ s_1s^{-1}$ and so $t \ \rho \ s_1s^{-1}t$. Also for every $2 \leq i \leq 2n-2$, if $s_i \ \rho \ s_{i+1}$, then $s_is^{-1}t \ \rho \ s_{i+1}s^{-1}t$ and so we have

$$t \ \rho \ s_1s^{-1}t \ (\Delta v) \ s_2s^{-1}t \ \rho \ s_3s^{-1}t \dots s_{2n-1}s^{-1}t \ (\Delta v) \ s^{-1}t.$$

Since $t \in I$, then $s_is^{-1}t \in I$ for every $1 \leq i \leq 2n-1$, and so we have

$$t \ \rho_1 \ s_1s^{-1}t(\Delta v)s_2s^{-1}t \dots s_{2n-1}s^{-1}t(\Delta v)s^{-1}t \text{ or } t(\rho_1 \vee \Delta v)s^{-1}t.$$

But $t(\rho_1 \vee \Delta v)1$ and so $s^{-1}t(\rho_1 \vee \Delta v)1$. Thus$t'(\rho_1 \vee \Delta v)1$ as required.

(c) $s \in I, t \in G$ is similar to part (b).

(d) $s, t \in G$. Then in sequence $(*)$ the following possibilities can arise:

1. $s_i \in G$ for every $2 \leq i \leq 2n-2$ such that $s_i \ \rho \ s_{i+1}$. Then $s = s_1, s_1u = s_2u, s_2 = s_3, \dots, s_{2n-1}u = u$ and so $su = u$.

2. There exists $2 \leq i \leq 2n-2$ such that $s_i \ \rho \ s_{i+1}$ and $s_i \in I$. We can suppose that i is the smallest one and so $s_j \in G$ for every $j < i$ such that $s_j \ \rho \ s_{j+1}$. Thus $s_j = s_{j+1}$ for every $j < i$, such that $s_j \ \rho \ s_{j+1}$. Also $s = s_1$ and so

$$s \ \rho \ s_1(\Delta u)s_2 \dots (\Delta u)s_i,$$

implies that $su = s_iu$. Since

$$s_i \ \rho \ s_{i+1}(\Delta u) \dots s_{2n-1}(\Delta u)1,$$

then $s_i(\rho \vee \Delta u)1$, and $s_i, u \in I$ implies then by Lemma 2.4.24, that $s_i(\rho_1 \vee \Delta u)1$.

Similarly it can be seen that either $tv = v$, or there exists $t_j \in I^1$ such that $tv = t_j v$ and $t_j(\rho_1 \vee \Delta v)1$ and so we are done. ■

Theorem 2.4.26. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $xg = x$. If all cyclic right S-acts are weakly flat, then all cyclic right I^1-acts are weakly flat.*

Proof. Let I^1/ρ_1 be a cyclic right I^1-act for a right congruence ρ_1 on I^1. Let ρ be the right congruence on S as in Lemma 2.4.13. Then by assumption S/ρ is weakly flat and so by Theorem 2.4.25, I^1/ρ_1 is weakly flat. ■

From Theorem 2.4.18, and Theorem 2.4.26, we have

Theorem 2.4.27. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $xg = x$. Then all cyclic right S-acts are weakly flat if and only if all cyclic right I^1-acts are weakly flat.*

Now in a similar way we show for monoids $S = G \dot{\cup} I$ with G a group and I an ideal, flatness of all cyclic right I^1-acts implies flatness of all cyclic right S-acts. First of all we need some technical lemmas.

Lemma 2.4.28. *Let S be a monoid and λ a left congruence on S. If $x(\lambda u)y$ and $u \ \lambda \ v$ with $x, y, u, v \in S$, then $x(\lambda v)y$.*

Proof. Suppose that $x(\lambda u)y$ and that $u \ \lambda \ v$. Then $yu \ \lambda \ yv$, $xu \ \lambda \ xv$ and $xu \ \lambda \ yu$. Thus $xv \ \lambda \ xu \ \lambda \ yu \ \lambda \ yv$ and so $xv \ \lambda \ yv$ or $x(\lambda v) \ y$ as required. ■

Lemma 2.4.29. *Let S be a monoid and let ρ, λ be right and left congruences on S respectively. Let $u \ \lambda \ v$ with $u, v \in S$. If $x(\rho \vee \lambda u)y$ with $x, y \in S$, then $x(\rho \vee \lambda v)y$.*

Proof. Let $x(\rho \vee \lambda u)y$, then there exist $x_1, x_2, \ldots, x_{2n-1} \in S$ such that

$$x \ \rho \ x_1(\lambda u)x_2 \ldots x_{2n-1}(\lambda u)y.$$

Since $x_i(\lambda u)x_{i+1}$, $i = 1, 3, \ldots, 2n-1$ and $u \ \lambda \ v$, then by Lemma 2.4.28, $x_i(\lambda v)x_{i+1}$, $i = 1, 3, \ldots, 2n-1$ and so

$$x \ \rho \ x_1(\lambda v)x_2 \ldots x_{2n-1}(\lambda v)y \text{ or } x(\rho \vee \lambda v)y$$

as required. ■

Lemma 2.4.30. *Let S be a monoid and let ρ, λ be right and left congruences on S respectively. If $t(\rho \vee \lambda su)1$ with $s, t, u \in S$, then $ts(\rho \vee \lambda u)s$.*

Proof. Let $t(\rho \vee \lambda su)1$. Then there exist $t_1, t_2, \ldots, t_{2n-1} \in S$ such that

$$t \ \rho \ t_1(\lambda su)t_2 \ldots t_{2n-1}(\lambda su)1.$$

Then

$$t \ \rho \ t_1, \ t_1 su \ \lambda \ t_2 su, \ldots, t_{2n-1} su \ \lambda \ su,$$

and so

$$ts \ \rho \ t_1 s(\lambda u)t_2 s \ldots t_{2n-1} s(\lambda u)s \text{ or } ts(\rho \vee \lambda u)s,$$

as required. ∎

Corollary 2.4.31. *Let S be a monoid and let ρ, λ be right and left congruences on S respectively. If $s(\rho \vee \lambda u)1$ and $t(\rho \vee \lambda su)1$ with $s, t, u \in S$, then $ts(\rho \vee \lambda u)1$.*

Proof. Since $t(\rho \vee \lambda su)1$, then by Lemma 2.4.30, $ts(\rho \vee \lambda u)s$. But by assumption $s(\rho \vee \lambda u)1$ and so $ts(\rho \vee \lambda u)1$ as required. ∎

Similar to Lemma 2.4.7 we have the following lemma.

Lemma 2.4.32. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. Let λ be a left congruences on S. If $\lambda_1 = \{(a, b) \in \lambda \mid a, b \in I^1\}$, then λ_1 is a left congruence on I^1 ($I^1 = I \cup \{1\}$).*

Lemma 2.4.33. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. Let ρ, λ be right and left congruences on S respectively. If for $u, v \in S$ there exist $u_1, u_2, \ldots, u_{2n-1} \in S$ such that $u \ \rho \ u_1 \ \lambda \ u_2 \ldots \ u_{2n-1} \ \lambda \ v$ and that this sequence is of minimal length ($n \geq 2$), then $u_i \in I$ for $2 \leq i \leq 2n-2$.*

Proof. Suppose that there exists $2 \leq i \leq 2n-2$ such that $u_i \in G$. Then either $i = 2k$ for $1 \leq k \leq n-1$ or $i = 2k-1$ for $2 \leq k \leq n-1$.

In the former case since $u_{2k} \ \rho \ u_{2k+1}$, then $1 \ \rho \ u_{2k+1} {u_{2k}}^{-1}$ and so

$$u_{2k-1} \ \rho \ u_{2k+1} {u_{2k}}^{-1} u_{2k-1}.$$

Also $u_{2k-1} \ \lambda \ u_{2k}$ implies that $u_{2k+1} {u_{2k}}^{-1} u_{2k-1} \ \lambda \ u_{2k+1}$. Hence,

$$u_{2k-1} \ \rho \ u_{2k+1} {u_{2k}}^{-1} u_{2k-1} \ \lambda \ u_{2k+1}.$$

Since $u_{2k+1} \ \lambda \ u_{2k+2}$, then

$$u_{2k-1} \ \rho \ u_{2k+1} {u_{2k}}^{-1} u_{2k+2} \ \lambda \ u_{2k+2},$$

need to note that we define $u_{2n} = v$. Thus the sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2k-1} \ \lambda \ u_{2k} \ \rho \ u_{2k+1} \ \lambda \ u_{2k+2} \ldots u_{2n-1} \ \lambda \ v$$

can be replaced by the shorter sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2k-1} \ \rho \ u_{2k+1}{u_{2k}}^{-1}u_{2k-1} \ \lambda \ u_{2k+2} \ldots u_{2n-1} \ \lambda \ v,$$

which is a contradiction.

In the latter case, then $u_{2k-1} \ \lambda \ u_{2k}$ implies

$$u_{2k-2} \ \lambda \ u_{2k-2}{u_{2k-1}}^{-1}u_{2k}.$$

Also $u_{2k-2} \ \rho \ u_{2k-1}$ implies that $u_{2k-2}{u_{2k-1}}^{-1} \rho \ 1$ and so $u_{2k-2}{u_{2k-1}}^{-1}u_{2k} \ \rho \ u_{2k}$. Since $u_{2k} \ \rho \ u_{2k+1}$, then

$$u_{2k-2} \ \lambda \ u_{2k-2}{u_{2k-1}}^{-1}u_{2k} \ \rho \ u_{2k+1}.$$

Hence, the sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2k-2} \ \rho \ u_{2k-1} \ \lambda \ u_{2k} \ \rho \ u_{2k+1} \ \lambda \ u_{2k+2} \ldots u_{2n-1} \ \lambda \ v,$$

can be replaced by the shorter sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2k-2} \ \lambda \ u_{2k-2}{u_{2k-1}}^{-1}u_{2k} \ \rho \ u_{2k+1} \ \lambda \ u_{2k+2} \ldots u_{2n-1} \ \lambda \ v,$$

which is also a contradiction. ■

Lemma 2.4.34. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. Let ρ, λ be right and left congruences on S respectively, and let $u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2n-1} \ \lambda \ v$ with $u, u_1, \ldots, u_{2n-1} \in S$. If $n \geq 2$, and either $u_1 \in G$ or $u_{2n-1} \in G$, then the sequence $u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2n-1} \ \lambda \ v$ can be replaced either by*

$$u \ \rho \ u \ \lambda \ u{u_1}^{-1}u_2 \ \rho \ u_3 \ldots u_{2n-1} \ \lambda \ v,$$

or

$$u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2n-3} \ \lambda \ u_{2n-2}{u_{2n-1}}^{-1}v \ \rho \ v \ \lambda \ v,$$

respectively.

Proof. First of all we suppose that $u_1 \in G$. Then $u \ \rho \ u_1$ implies that $u{u_1}^{-1} \ \rho \ 1$ and so $u{u_1}^{-1}u_2 \ \rho \ u_2$. Also $u_1 \ \lambda \ u_2$ implies that $1 \ \lambda \ {u_1}^{-1}u_2$ and so $u \ \lambda \ u{u_1}^{-1}u_2$. Since $u_2 \ \rho \ u_3$, then

$$u \ \rho \ u \ \lambda \ u{u_1}^{-1}u_2 \ \rho \ u_3 \ldots u_{2n-1} \ \lambda \ v, \tag{1}$$

as required.

If $u_{2n-1} \in G$, then $u_{2n-2} \ \rho \ u_{2n-1}$ implies that $u_{2n-2}{u_{2n-1}}^{-1} \ \rho \ 1$ and so

$$u_{2n-2}{u_{2n-1}}^{-1}v \ \rho \ v.$$

Also $u_{2n-1} \; \lambda \; v$ implies that $1 \; \lambda \; {u_{2n-1}}^{-1}v$ and so

$$u_{2n-2} \; \lambda \; u_{2n-2}{u_{2n-1}}^{-1}v.$$

Since $u_{2n-3} \; \lambda \; u_{2n-2}$, then

$$u \; \rho \; u_1 \; \lambda \; u_2 \ldots u_{2n-3} \; \lambda \; u_{2n-2}{u_{2n-1}}^{-1}v \; \rho \; v \; \lambda \; v, \tag{2}$$

as required ■

Corollary 2.4.35. *If in Lemma 2.4.34, the sequence $u \; \rho \; u_1 \; \lambda \; u_2 \ldots u_{2n-1} \; \lambda \; v$ is of minimal length ($n \geq 2$), then $u{u_1}^{-1}u_2$, $u_{2n-2}{u_{2n-1}}^{-1}v \in I$.*

Proof. By Lemma 2.4.33, $u_i \in I$ for $2 \leq i \leq 2n-2$, and so $u_2, u_{2n-2} \in I$. Since I is an ideal of S, then $u{u_1}^{-1}u_2, u_{2n-2}{u_{2n-1}}^{-1}v \in I$ as required. ■

Lemma 2.4.36. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. Let ρ, λ be right and left congruences on S respectively such that $u \; \rho \; u_1 \; \lambda \; v$ with $u, u_1, v \in S$. If $u_1 \in G$, then the sequence $u \; \rho \; u_1 \; \lambda \; v$ can be replaced by $u \; \rho \; u \; \lambda \; u{u_1}^{-1}v \; \rho \; v \; \lambda \; v$.*

Proof. Since $u_1 \in G$, then $u \; \rho \; u_1$ implies that $u{u_1}^{-1} \; \rho \; 1$ and so $u{u_1}^{-1}v \; \rho \; v$. Also $u_1 \; \lambda \; v$ implies that $1 \; \lambda \; {u_1}^{-1}v$ and so $u \; \lambda \; u{u_1}^{-1}v$. Hence,

$$u \; \rho \; u \; \lambda \; u{u_1}^{-1}v \; \rho \; v \; \lambda \; v$$

as required. ■

Theorem 2.4.37. *Let $S = G \dot{\cup} I$ be a monoid with G a group and I an ideal of S. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. If I^1/ρ_1 is flat, then S/ρ is flat.*

Proof. Let λ be a left congruence on S and λ_1 the left congruence on I^1 as in Lemma 2.4.32. Let $u(\rho \vee \lambda)v$ with $u, v \in S$. Then there exist $u_1, u_2, \ldots, u_{2n-1} \in S$ such that

$$u \; \rho \; u_1 \; \lambda \; u_2 \ldots u_{2n-2} \; \rho \; u_{2n-1} \; \lambda \; v.$$

Suppose first of all that the sequence above has length $n = 1$. Then we have $u \; \rho \; u_1 \; \lambda \; v$. Now there are three cases that can arise as follows:

Case 1. $u, v \in I$. Then there are two possibilities for u_1:

1. $u_1 \in I^1$. Since $u, v \in I \subseteq I^1$, then $u \; \rho_1 \; u_1 \; \lambda_1 \; v$ or $u(\rho_1 \; \vee \; \lambda_1)v$. Since I^1/ρ_1 is flat, then there exist $s, t \in I^1 \subseteq S$ such that $su \; \lambda_1 \; tv$, $s(\rho_1 \; \vee \; \lambda_1 u)1$ and $t(\rho_1 \; \vee \; \lambda_1 v)1$. Thus $su \; \lambda \; tv$, $s(\rho \; \vee \; \lambda u)1$ and $t(\rho \; \vee \; \lambda v)1$.

2. $1 \neq u_1 \in G$. Then by Lemma 2.4.36, the sequence $u\ \rho\ u_1\ \lambda\ v$ can be replaced by the sequence

$$u\ \rho\ u\ \lambda\ uu_1{}^{-1}v\ \rho\ v\ \lambda\ v.$$

Since $u \in I$, then $uu_1{}^{-1}v \in I$ and so we have

$$u\ \rho_1\ u\ \lambda_1\ uu_1{}^{-1}v\ \rho_1\ v\ \lambda_1\ v \text{ or } u(\rho_1 \ \vee \ \lambda_1)v.$$

Then as in part 1, there exist $s, t \in I^1 \subseteq S$ such that $su\ \lambda\ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$.

Case 2. $u \in G$. Then $u\ \rho\ u_1$ implies that $1\ \rho\ u_1u^{-1}$. If $s = u_1u^{-1}$, $t = 1$, then

$$su = (u_1u^{-1})u = u_1\ \lambda\ v = tv.$$

Also $1(\rho \ \vee \ \lambda v)1$ and

$$s = u_1u^{-1}\ \rho\ 1(\lambda u)1 \text{ or } s(\rho \ \vee \ \lambda u)1.$$

Case 3. $v \in G$. If $u \in G$, then by case 2, we are done. Thus we assume that $u \notin G$. Now there are two possibilities as follows:

1. $u_1 \in I$. Since $u \in I$, then

$$u\ \rho_1\ u_1\ \lambda_1\ u_1 \text{ or } u(\rho_1 \ \vee \ \lambda_1)u_1,$$

and so there exist $s, t \in I^1 \subseteq S$ such that $su\ \lambda\ tu_1$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda u_1)1$. Since $u_1\ \lambda\ v$, then $tu_1\ \lambda\ tv$ and so $su\ \lambda\ tv$. Also by Lemma 2.4.29, $t(\rho \ \vee \ \lambda v)1$.

2. $u_1 \in G$. Then $u_1\ \lambda\ v$ implies that $1\ \lambda\ u_1{}^{-1}v$ and so $u\ \lambda\ uu_1{}^{-1}v$. If $s = 1$, $t = uu_1{}^{-1}$, then

$$su = u\ \lambda\ (uu_1{}^{-1})v = tv.$$

Also $1(\rho \ \vee \ \lambda u)1$ and $u\ \rho\ u_1$ implies that $uu_1{}^{-1}\ \rho\ 1$ and so

$$t = uu_1{}^{-1}\ \rho\ 1(\lambda v)1 \text{ or } t(\rho \ \vee \ \lambda v)1.$$

Now we suppose that the sequence

$$u\ \rho\ u_1\ \lambda\ u_2 \ldots u_{2n-1}\ \lambda\ v, \qquad (\star)$$

is of minimal length $n \geq 2$. Then by Lemma 2.4.33, $u_i \in I$ for $2 \leq i \leq 2n-2$. Now there are four cases that can arise as follows:

Case A. $u, v \in I$. Then we consider the following possibilities:

1. $u_1, u_{2n-1} \in I$. Then

$$u \ \rho_1 \ u_1 \ \lambda_1 \ u_2 \dots u_{2n-1} \ \lambda_1 \ v \text{ or } u(\rho_1 \ \vee \ \lambda_1)v,$$

and so there exist $s, t \in I^1 \subseteq S$ such that $su \ \lambda \ tv$, $s(\rho \vee \lambda u)1$ and $t(\rho \vee \lambda v)1$.

2. $u_1 \in G$ or $u_{2n-1} \in G$. Then by Lemma 2.4.34, and Corollary 2.4.35, the sequence $(\star)$ can be replaced either by sequence (1) or sequence (2) in Lemma 2.4.34, which in both cases all elements are in I and so in a similar way as in part 1 of this case we are done.

Case B. $u \in G, v \in I$. If $u_{2n-1} \in G$, then by Lemma 2.4.34, and Corollary 2.4.35, the sequence $(\star)$ can be replaced by the sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \dots u_{2n-3} \ \lambda \ w \ \rho \ v \ \lambda \ v,$$

with $w, v \in I$. Thus we can suppose without losse of generality, that $u_{2n-1} \in I$. Then we have the sequence

$$u_2 \ \rho \ u_3 \dots u_{2n-2} \ \rho \ u_{2n-1} \ \lambda \ v,$$

with $u_2, u_3, \dots, u_{2n-1}, v \in I$ and so by [case A, 1] there exist $s', t \in S$ such that $s'u_2 \ \lambda \ tv$, $s'(\rho \ \vee \ \lambda u_2)1$ and $t(\rho \ \vee \ \lambda v)1$. If $s'' = u_1 u^{-1}$, then

$$s''u = (u_1 u^{-1})u = u_1 \ \lambda \ u_2,$$

and so

$$s's''u = s'u_1 \ \lambda \ s'u_2 \ \lambda \ tv \text{ or } (s's'')u \ \lambda \ tv.$$

Since $s'(\rho \ \vee \ \lambda u_2)1$ and $s''u \ \lambda \ u_2$, then by Lemma 2.4.29, $s'(\rho \ \vee \ \lambda s''u)1$. Since $u \ \rho \ u_1$, then $1 \ \rho \ u_1 u^{-1}$ and so

$$s'' = u_1 u^{-1} \ \rho \ 1(\lambda u)1 \text{ or } s''(\rho \ \vee \ \lambda u)1.$$

Since $s''(\rho \ \vee \ \lambda u)1$ and $s'(\rho \ \vee \ \lambda s''u)1$, then by Corollary 2.4.31, $s's''(\rho \ \vee \ \lambda u)1$. If $s's'' = s$, then $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$.

Case C. $u \in I, v \in G$. If $u_1 \in G$, then by Lemma 2.4.34, and Corollary 2.4.35, the sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \dots u_{2n-1} \ \lambda \ v,$$

can be replaced by the sequence

$$u \ \rho \ u \ \lambda \ w \ \rho \ u_3 \dots u_{2n-1} \ \lambda \ v,$$

such that $w \in I$ and so we can consider the sequence

$$v \ \rho \ v \ \lambda \ u_{2n-1} \dots u_3 \ \rho \ w \ \lambda \ u,$$

with $u, w, u_3 \in I$. Now in the sequence

$$u_{2n-1} \ \rho \ u_{2n-2} \dots u_2 \ \lambda \ u_3 \ \rho \ w \ \lambda \ u, \qquad (*)$$

either $u_{2n-1} \in G$ (in which case we have a situation as in case B) or $u_{2n-1} \in I$ (in which case we have a situation as in case A) there exist $s, t \in S$ such that $tu_{2n-1} \ \lambda \ su$, $t(\rho \ \vee \ \lambda u_{2n-1})1$ and $s(\rho \ \vee \ \lambda u)1$. Since $v \ \lambda \ u_{2n-1}$, then $tv \ \lambda \ tu_{2n-1}$ and so $su \ \lambda \ tv$. Also by Lemma 2.4.29, $t(\rho \ \vee \ \lambda u_{2n-1})1$ implies that $t(\rho \ \vee \ \lambda v)1$.

If $u_1 \in I$, then we consider the sequence

$$u_{2n-1} \ \rho \ u_{2n-2} \dots u_2 \ \lambda \ u_1 \ \rho \ u \ \lambda \ u,$$

which has the same situation as the sequence $(*)$ and so by the same argument there exist $s, t \in S$ such that $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$.

Case D. $u, v \in G$. If we consider the sequence

$$u_2 \ \rho \ u_3 \dots u_{2n-2} \ \rho \ u_{2n-1} \ \lambda \ v,$$

then $u_2 \in I, v \in G$ and so by case C, there exist $s', t \in S$ such that $s'u_2 \ \lambda \ tv$, $s'(\rho \ \vee \ \lambda u_2)1$ and $t(\rho \ \vee \lambda v)1$. If $s'' = u_1 u^{-1}$, then by the same argument as with case B, it can be seen that $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$ where $s = s's''$.

■

Theorem 2.4.38. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group and I is an ideal of S. If all cyclic right I^1-acts are flat then all cyclic right S-acts are flat.*

Proof. Let S/ρ be a cyclic right S-act for a right congruence ρ on S and let ρ_1 be the right congruence on I^1 as in Lemma 2.4.7. Then by assumption I^1/ρ_1 is flat and so by Theorem 2.4.37, S/ρ is flat. ■

As we saw for monoids of the form $S = G \dot{\cup} I$ and with the property that $\forall g \in G, \forall x \in I$, $gx = x$, if all weakly flat cyclic right I^1-acts satisfy condition (P), then all weakly flat cyclic right S-acts satisfy condition (P). Now for this type of monoid we first of all show that if ρ is a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7, then S/ρ is flat if and only if I^1/ρ_1 is flat. Then by using this property and results from the last two parts we show that if all flat cyclic right I^1-acts satisfy condition (P), then all flat cyclic right S-acts satisfy condition (P). Moreover, if all weakly flat cyclic right I^1-acts are flat, then all weakly flat cyclic right S-acts are flat.

Lemma 2.4.39. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. Let λ^1 be a left congruence on I^1 and let $\lambda_I = \{(a,b) \in \lambda^1 \mid a, b \in I\}$. If $\lambda = \lambda_I \cup 1_G$, then λ is a left congruence on S.*

Proof. It is obvious that λ is an equivalence relation on S. Let $(x,y) \in \lambda, s \in S$. Then either $(x,y) \in 1_G$ or $(x,y) \in \lambda_I$.

If $(x,y) \in 1_G$, then $x = y$ and so $s(x,y) = s(x,x) = (sx, sx) \in \lambda$.

If $(x,y) \in \lambda_I$, then $(x,y) \in \lambda^1$ and $x, y \in I$. Now there are two cases that can arise:

Case 1. $s \in I \subseteq I^1$. Then $s(x,y) = (sx, sy) \in \lambda^1$. But $sx, sy \in I$ and so. $s(x,y) = (sx, sy) \in \lambda_I \subseteq \lambda$.

Case 2. $s \in G$. Then $s(x,y) = (sx, sy) = (x,y) \in \lambda_I \subseteq \lambda$. ■

Lemma 2.4.40. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. Let λ^1 be a left congruence on I^1 and λ the left congruence on S as in Lemma 2.4.39. If $t(\rho \vee \lambda u)1$ for $t, u \in I^1$, then $t(\rho_1 \vee \lambda^1 u)1$.*

Proof. Let $t(\rho \vee \lambda u)1$ for $t, u \in I^1$. If $t = 1$, then $1 \ \rho_1 \ 1(\lambda^1 u)1$ or $1(\rho_1 \vee \lambda^1 u)1$. Thus from now on we suppose that $t \in I$.

Let $t(\rho \vee \lambda u)1$ for $t \in I, u \in I^1$. Then there exist $t_1, t_2, \ldots, t_{2n-1} \in S$ such that

$$t \ \rho \ t_1(\lambda u)t_2 \ldots t_{2n-2} \ \rho \ t_{2n-1}(\lambda u)1. \qquad (*)$$

Then either $n = 1$ or $n \geq 2$. At first we suppose that $n = 1$. Then we have $t \ \rho \ t_1(\lambda u)1$. Since $u \in I^1$, then there are two cases as follows:

Case 1. $u \in I$. Then there are two possibilities for t_1 as follows:

1. $t_1 \in I$. Since $t \in I$, then $t \ \rho_1 \ t_1$ and $t_1(\lambda u)1$ implies that $t_1 u \ \lambda \ u$. Since $t_1 u, u \in I$, then $t_1 u \ \lambda^1 \ u$ or $t_1(\lambda^1 u)1$. Thus $t \ \rho_1 \ t_1(\lambda^1 u)1$ or $t(\rho_1 \vee \lambda^1 u)1$.

2. $t_1 \in G$. Then $t \ \rho \ t_1$ implies that $tt_1{}^{-1} \ \rho \ 1$. Since $t \in I$, then $tt_1{}^{-1} \in I \subseteq I^1$ and so $tt_1{}^{-1} \ \rho_1 \ 1$. Since $t_1{}^{-1} \in G$, then by assumption $t_1{}^{-1}u = u$ and so $tt_1{}^{-1}u = tu$. But $tu \in I$ and so $tt_1{}^{-1}u \ \lambda^1 \ tu$ or $tt_1{}^{-1}(\lambda^1 u)t$. Hence,

$$t \ \rho_1 \ t(\lambda^1 u)tt_1{}^{-1} \ \rho_1 \ 1(\lambda^1 u)1 \text{ or } t(\rho_1 \vee \lambda^1 u)1.$$

Case 2. $u = 1$. Then $t \ \rho \ t_1 \ \lambda \ 1$. But $t_1 \ \lambda \ 1$ implies that $t_1 = 1$ and so $t \ \rho_1 \ 1 \ \lambda^1 \ 1$ or $t(\rho_1 \vee \lambda^1)1$.

Now we suppose that the sequence $(*)$ is of minimal length $n \geq 2$. If $t_i \in I$ for $1 \leq i \leq 2n-1$, then

$$t \ \rho_1 \ t_1(\lambda^1 u)t_2 \ldots t_{2n-2} \ \rho_1 \ t_{2n-1}(\lambda^1 u)1 \text{ or } t(\rho_1 \ \vee \ \lambda^1 u)1.$$

Suppose that there exists i such that $1 \leq i \leq 2n-1$ and that $t_i \in G$. Then as before there are two cases that can arise:

Case A. $u \in I$. Then there are two possibilities for i as follows:

1. $2 \leq i \leq 2n-1$. We can assume that i is the smallest number such that $t_i \in G$. Then for every $j < i$, $t_j \in I$ and either $i = 2k$, $1 \leq k \leq n-1$, or $i = 2k+1$, $1 \leq k \leq n-1$.

   If $i = 2k$, then $t_{i-1}(\lambda u)t_i$ or $t_{i-1}u \ \lambda \ t_i u = u$ and so $t_{i-1}(\lambda u)1$. Consequently,

   $$t \ \rho \ t_1(\lambda u)t_2 \ldots t_{i-2} \ \rho \ t_{i-1}(\lambda u)1.$$

   Since $t_j \in I$, $1 \leq j \leq i-1$, then

   $$t \ \rho_1 \ t_1(\lambda^1 u)t_2 \ldots t_{i-2} \ \rho_1 \ t_{i-1}(\lambda^1 u)1 \text{ or } t(\rho_1 \ \vee \ \lambda^1 u)1.$$

   If $i = 2k+1$ for $1 \leq k \leq n-1$, then $t_{i-1} \ \rho \ t_i$ implies that $t_{i-1}t_i{}^{-1} \ \rho \ 1$. Since $t_i{}^{-1} \in G$, then $t_{i-1}t_i{}^{-1}u = t_{i-1}u$ or $t_{i-1}t_i{}^{-1}(\lambda u)t_{i-1}$. But $t_{i-2}(\lambda u)t_{i-1}$ and so $t_{i-2}(\lambda u)t_{i-1}t_i{}^{-1}$. Consequently,

   $$t \ \rho \ t_1(\lambda u)t_2 \ldots t_{i-2}(\lambda u)t_{i-1}t_i{}^{-1} \ \rho \ 1(\lambda u)1.$$

   Since $t_j \in I$ for $1 \leq j \leq i-2$, and also $t_{i-1}t_i{}^{-1} \in I$, then

   $$t \ \rho_1 \ t_1(\lambda^1 u)t_2 \ldots t_{i-2}(\lambda^1 u)t_{i-1}t_i{}^{-1} \ \rho_1 \ 1(\lambda^1 u)1 \text{ or } t(\rho_1 \ \vee \ \lambda^1 u)1.$$

2. $i = 1$. Then $t \ \rho \ t_1$ implies that $tt_1{}^{-1} \ \rho \ 1$. Since $t_1{}^{-1} \in G$, then $tt_1{}^{-1}u = tu$ and so $tt_1{}^{-1}(\lambda u)t$. Then $t, tt_1{}^{-1} \in I$ implies that $t(\lambda^1 u)tt_1{}^{-1}$, and so

   $$t \ \rho_1 \ t(\lambda^1 u)tt_1{}^{-1} \ \rho_1 \ 1(\lambda^1 u)1 \text{ or } t(\rho_1 \ \vee \ \lambda^1 u)1$$

Case B. $u = 1$. Then $\lambda u = \lambda$ and so we have the sequence

$$t \ \rho \ t_1 \ \lambda \ t_2 \ldots t_{2n-2} \ \rho \ t_{2n-1} \ \lambda \ 1.$$

Then by definition of λ, $t_{2n-1} = 1$. Also as [case A, 1] we can suppose that i is the smallest number such that $t_i \in G$. If $i = 2n-1$, then $t_1, t_2, \ldots, t_{2n-2} \in I$ and so

$$t \ \rho_1 \ t_1 \ \lambda_1 \ t_2 \ldots t_{2n-2} \ \rho_1 \ t_{2n-1} \ \lambda_1 \ 1 \text{ or } t(\rho_1 \ \vee \ \lambda^1)1.$$

Thus we suppose that $1 \leq i \leq 2n-2$. Then the following possibilities can arise:

1. $t_i \ \lambda \ t_{i+1}$, $i = 2k-1$, $1 \leq k \leq n-1$. Then $t_i = t_{i+1}$ and so

$$t_{i-1} \ \rho \ t_i \ \lambda \ t_{i+1} \ \rho \ t_{i+2},$$

   implies that $t_{i-1} \ \rho \ t_{i+2}$ Consequently, we have the shorter sequence

$$t \ \rho \ t_1 \ \lambda \ t_2 \ldots t_{i-1} \ \rho \ t_{i+2} \ \lambda \ t_{i+3} \ldots t_{2n-2} \ \rho \ t_{2n-1} \ \lambda \ 1,$$

   which is a contradiction on the minimality of $(*)$. (Note, if $i = 2n-3$, then we define $t_{2n} = 1$).

2. $t_i \ \rho \ t_{i+1}$, $i = 2k$, $1 \leq k \leq n-1$. Then $t_{i-1} \ \lambda \ t_i$ implies that $t_{i-1} = t_i$ and so

$$t_{i-2} \ \rho \ t_{i-1} = t_i \ \rho \ t_{i+1} \text{ or } t_{i-2} \ \rho \ t_{i+1}.$$

   As a result we have the shorter sequence

$$t \ \rho \ t_1 \ \lambda \ t_2 \ldots t_{i-2} \ \rho \ t_{i+1} \ \lambda \ t_{i+2} \ldots t_{2n-2} \ \rho \ 1 \ \lambda \ 1,$$

   and again a contradiction on the minimality of $(*)$. (Note, if $i = 2$, then we define $t_0 = t$).

   Thus if $u = 1$, then $t_i \notin G$, $1 \leq i \leq 2n-1$, and so $t(\rho_1 \ \vee \ \lambda^1 u)1$

   as required. ■

<u>**Lemma 2.4.41.**</u> *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. Let λ^1 be a left congruence on I^1 and λ the left congruence on S as in Lemma 2.4.39. If for $u, 1 \neq v \in I^1$ there exist $s, t \in S$ such that $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$, then there exist $s, t \in I^1$ such that $su \ \lambda^1 \ tv$, $s(\rho_1 \ \vee \ \lambda^1 u)1$, and $t(\rho_1 \ \vee \ \lambda^1 v)1$.*

Proof. Suppose that there exist $s, t \in S$ such that $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$. Since $tv \in I$ and $su \ \lambda \ tv$, then $su \in I$. Now there are two cases that can arise:

Case 1. $u = 1$. Then $s \ \lambda \ tv$, $s \in I$ and there are two possibilities as follows:

1. $t \in G$. Then $s \ \lambda \ v$. Since $s, v \in I$, then $s \ \lambda^1 \ v$. Also $1(\rho_1 \ \vee \ \lambda^1 1)1$ and by Lemma 2.4.40, $s(\rho_1 \ \vee \ \lambda^1 u)1$.

2. $t \in I$. Since $s, t, u, v \in I^1$, then by Lemma 2.4.40, $s(\rho_1 \vee \lambda^1 u)1$ and $t(\rho_1 \vee \lambda^1 v)1$. Also $tv \in I$, and so $s \ \lambda \ tv$ implies that $s \ \lambda^1 \ tv$.

Case 2. $u \in I$. Then $su, tv \in I$ and so $su \ \lambda^1 \ tv$. Now there are four possibilities as follows:

1. $s, t \in I$. Since $u, v \in I \subsetneq I^1$, then by Lemma 2.4.40, $s(\rho_1 \vee \lambda^1 u)1$ and $t(\rho_1 \vee \lambda^1 v)1$.

2. $s, t \in G$. Then $su \ \lambda^1 \ tv$ implies that $u \ \lambda^1 \ v$. Then $1(\rho_1 \vee \lambda^1 u)1$ and $1(\rho_1 \vee \lambda^1 v)1$.

3. $s \in G, t \in I$. Then $su \ \lambda^1 \ tv$ implies that $u \ \lambda^1 \ tv$. Since $t, v \in I \subsetneq I^1$, then by Lemma 2.4.40, $t(\rho_1 \vee \lambda^1 v)1$. Also $1(\rho_1 \vee \lambda^1 u)1$.

4. $s \in I, t \in G$. Then $su \ \lambda^1 \ tv$ implies that $su \ \lambda^1 \ v$. Since $s, u \in I \subsetneq I^1$, then again by Lemma 2.4.40, $s(\rho_1 \vee \lambda^1 u)1$. Also $1(\rho_1 \vee \lambda^1 v)1$. ■

<u>Theorem 2.4.42.</u> *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. If S/ρ is flat, then I^1/ρ_1 is flat.*

Proof. Let λ^1 be a left congruences on I^1 and λ the left congruence on S as in Lemma 2.4.39. Suppose that $u(\rho_1 \vee \lambda^1)v$ for $u, v \in I^1$. Then there exist $u_1, u_2, \ldots, u_{2n-1} \in I^1$ such that

$$u \ \rho_1 \ u_1 \ \lambda^1 \ u_2 \ldots u_{2n-1} \ \lambda^1 \ v. \qquad (*)$$

We can suppose that the sequence $(*)$ is of minimal length. Therefore if there exists $1 \leq i \leq 2n-1$ such that $u_i = 1$, then i is unique. Otherwise let j be such that $i < j$ and that $u_i = u_j = 1$. Then there are four possibilities for u_i and u_j as follows:

1. $u_i \ \rho_1 \ u_{i+1}$, $u_j \ \rho_1 \ u_{j+1}$. Then $u_{i-1} \ \lambda^1 \ u_i \ \rho_1 \ u_{j+1}$.

2. $u_i \ \rho_1 \ u_{i+1}$, $u_j \ \lambda^1 \ u_{j+1}$. Then $u_{i-1} \ \lambda^1 \ u_{j+1}$.

3. $u_i \ \lambda^1 \ u_{i+1}$, $u_j \ \lambda^1 \ u_{j+1}$. Then $u_{i-1} \ \rho_1 \ u_j \ \lambda^1 \ u_{j+1}$.

4. $u_i \ \lambda^1 \ u_{i+1}$, $u_j \ \rho_1 \ u_{j+1}$. Then $u_{i-1} \ \rho_1 \ u_{j+1}$.

Thus in every case mentioned above we have a shorter sequence which is a contradiction on the minimality of the length of the sequence $(*)$.

Now either $v \in I$ or $v = 1$. At first we suppose that $v \in I$. Then there are two cases that can arise:

Case 1. $u_i \in I$ for every $1 \leq i \leq 2n-1$. Then

$$u \ \rho \ u_1 \ \lambda \ u_2 \dots u_{2n-1} \ \lambda \ v \text{ or } u(\rho \ \vee \ \lambda)v.$$

Since S/ρ is flat, then there exist $s, t \in S$ such that $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$ and so by Lemma 2.4.41, there exist $s', t' \in I^1$ such that $s'u \ \lambda^1 \ t'v$, $s'(\rho_1 \ \vee \ \lambda^1 u)1$ and $t'(\rho_1 \ \vee \ \lambda^1 v)1$.

Case 2. There exists $1 \leq i \leq 2n-1$, such that $u_i = 1$. Then either $i = 2k$, $(k = 1, 2, \dots, n-1)$ or $i = 2k-1$, $(k = 1, 2, \dots, n)$. In the former case we have $1 = u_i \ \rho_1 \ u_{i+1}$ and by uniqueness, in the sequence

$$u \ \rho_1 \ u_1 \ \lambda^1 \ u_2 \dots u_{i-1} \ \lambda^1 \ u_{i-1},$$

$u_j \in I$ for every $1 \leq j \leq i-1$ and so by case 1, there exist $s, t \in I^1$ such that $su \ \lambda^1 \ tu_{i-1}$, $s(\rho_1 \ \vee \ \lambda^1 u)1$ and $t(\rho_1 \ \vee \ \lambda^1 u_{i-1})1$. Since $t(\rho_1 \ \vee \ \lambda^1 u_{i-1})1$, then there exist $t_1, t_2, \dots, t_{2m-1} \in I^1$ such that

$$t \ \rho_1 \ t_1(\lambda^1 u_{i-1})t_2 \dots t_{2m-1}(\lambda^1 u_{i-1})1.$$

Consequently,

$$tu_{i-1} \ \rho_1 \ t_1 u_{i-1}, \ t_1(\lambda^1 u_{i-1})t_2, \ t_2 u_{i-1} \ \rho_1 \ t_3 u_{i-1}, \dots, t_{2m-1}(\lambda^1 u_{i-1})1, \qquad (1)$$

or

$$tu_{i-1} \ \rho_1 \ t_1 u_{i-1} \ \lambda^1 \ t_2 u_{i-1} \dots t_{2m-1} u_{i-1} \ \lambda^1 \ u_{i-1}. \qquad (2)$$

Since

$$1 \ \rho_1 \ u_{i+1} \ \lambda^1 \ u_{i+2} \dots u_{2n-1} \ \lambda^1 \ v,$$

then

$$u_{i-1} \ \rho_1 \ u_{i+1} u_{i-1}, u_{i+1}(\lambda^1 1) \ u_{i+2}, \dots, u_{2n-1}(\lambda^1 1) \ v.$$

But $u_{i-1} \ \lambda^1 \ 1$ and so by Lemma 2.4.28, we have

$$u_{i-1} \ \rho_1 \ u_{i+1} u_{i-1}, u_{i+1}(\lambda^1 u_{i-1}) \ u_{i+2}, \dots, u_{2n-1}(\lambda^1 u_{i-1}) \ v$$

or

$$u_{i-1} \ \rho_1 \ u_{i+1} u_{i-1} \ \lambda^1 \ u_{i+2} u_{i-1} \dots u_{2n-1} u_{i-1} \ \lambda^1 \ v u_{i-1}. \qquad (3)$$

Since $u_{i-1} \ \lambda^1 \ 1$, then $vu_{i-1} \ \lambda^1 \ v$. Consequently from (2), (3) we have

$$tu_{i-1} \ \rho_1 \ t_1 u_{i-1} \ \lambda^1 \ t_2 u_{i-1} \dots t_{2m-1} u_{i-1} \ \lambda^1 \ u_{i-1} \ \rho_1 \ u_{i+1} u_{i-1} \dots u_{2n-1} u_{i-1} \ \lambda^1 \ v. \qquad (4)$$

But $u_{i-1} \in I$, and so in sequence (4) $t_k u_{i-1}, u_{i+l} u_{i-1} \in I$, $k = 1, 2, \dots, 2m-1$, $l = 1, 2, \dots, 2n-1-i$. Hence by case 1 there exist $s', t' \in I^1$ such that $s'(tu_{i-1}) \ \lambda^1 \ t'v$, $s'(\rho_1 \ \vee \ \lambda^1 tu_{i-1})1$ and $t'(\rho_1 \ \vee \ \lambda^1 v)1$. Since $su \ \lambda^1 \ tu_{i-1}$, then

$s'su\ \lambda^1\ s'tu_{i-1}$. Consequently, $(s's)u\ \lambda^1\ t'v$. Since $su\ \lambda^1\ tu_{i-1}$, then by Lemma 2.4.29, $s'(\rho_1 \vee \lambda^1 su)1$. But $s(\rho_1 \vee \lambda^1 u)1$ and so by Corollary 2.4.31, $s's(\rho_1 \vee \lambda^1 u)1$.

If $i = 2k-1$, $(k = 1, 2, \ldots, n)$, then $1\ \lambda^1\ u_{i+1}$ and either $u \in I$ or $u = 1$. In case $u \in I$, by considering the sequence

$$v\ \rho_1\ v\ \lambda^1\ u_{2n-1} \ldots u_{i+1}\ \lambda^1\ 1\ \rho_1\ u_{i-1} \ldots u_1\ \rho_1\ u\ \lambda^1\ u$$

we are encountered the previous case with u replaced by v and so we can proceed as before. Thus we suppose that $u = 1$. Then there are two possibilities as follows:

1. $i = 2n-1$. Then $1\ \lambda^1\ v$ (note that we can define $u_{2n} = v$). Since $v\ \lambda^1\ v$, then if $s = v$, $t = 1$, we have $su\ \lambda^1\ tv$ and $t(\rho_1\ \vee\ \lambda^1 v)1$. Since $v\ \lambda^1\ 1$, then $v(\lambda^1 u)1$ and so $v\ \rho_1\ v(\lambda^1 u)1$ or $v(\rho_1\ \vee\ \lambda^1 u)1$. Consequently, $s(\rho_1\ \vee\ \lambda^1 u)1$.

2. $i \leq 2n-3$. Then $u_{i+1}, u_{i+2}, \ldots, u_{2n-1} \in I$ and so the sequence

$$u_{i+1}\ \rho_1\ u_{i+2} \ldots u_{2n-2}\ \rho_1\ u_{2n-1}\ \lambda^1\ v,$$

has the same format as the sequence in case 1. Consequently, there exist $s, t \in I^1$ such that $su_{i+1}\ \lambda^1\ tv$, $s(\rho_1\ \vee\ \lambda^1 u_{i+1})1$ and $t(\rho_1\ \vee\ \lambda^1 v)1$. But $u = 1\ \lambda^1\ u_{i+1}$ and so by Lemma 2.4.29, $s(\rho_1\ \vee\ \lambda^1 u)1$. Also $su = s1\ \lambda^1\ su_{i+1}$ and consequently $su\ \lambda^1\ tv$.

If $v = 1$, then

$$u\ \rho_1\ u_1\ \lambda^1\ u_2 \ldots u_{2n-1}\ \lambda^1\ 1,$$

and either $u = 1$ or $u \in I$. In the former case if $s = t = 1$, then $su\ \lambda^1\ tv$, $s(\rho_1\ \vee\ \lambda^1 u)1$ and $t(\rho_1\ \vee\ \lambda^1 v)1$. In the latter case if we consider the sequence

$$1\ \rho_1\ 1\ \lambda^1\ u_{2n-1} \ldots u_2\ \lambda^1\ u_1\ \rho_1\ u\ \lambda^1\ u,$$

then by case 2 (with u replaced by 1 and v replaced by u) there exist $s, t \in I^1$ such that $t1\ \lambda^1\ su$, $t(\rho_1\ \vee\ \lambda^1 1)1$ and $s(\rho_1\ \vee\ \lambda^1 u)1$. ■

From Theorem 2.4.37, and Theorem 2.4.42, we have

Theorem 2.4.43. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. Let ρ be a right congruence on S and ρ_1 the right congruence on I^1 as in Lemma 2.4.7. Then S/ρ is flat if and only if I^1/ρ_1 is flat.*

Theorem 2.4.44. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = x$. If all flat cyclic right I^1-acts satisfy condition (P), then all flat cyclic right S-acts satisfy condition (P).*

Proof. Let ρ be a right congruence on S such that S/ρ is flat. Then by Theorem 2.4.42, I^1/ρ_1 is flat and so by assumption I^1/ρ_1 satisfies condition (P). Consequently, by Lemma 2.4.8, S/ρ satisfies condition (P) on S. ■

Theorem 2.4.45. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I,\ gx = x$. If all weakly flat cyclic right I^1-acts are flat, then all weakly flat cyclic right S-acts are flat.*

Proof. Let ρ be a right congruence on S such that S/ρ is weakly flat. Then by Theorem 2.4.19, I^1/ρ_1 is weakly flat (ρ_1 is the right congruence on I^1 as in Lemma 2.4.7) and so by assumption I^1/ρ_1 is flat. Consequently, by Theorem 2.4.37, S/ρ is flat. ■

Here we show for monoids $S = G \dot{\cup} I$ with the property that $\forall g \in G, \forall x \in I,\ gx = xg = x$, flatness of all cyclic right S-acts implies flatness of all cyclic right I^1-acts. Consequently, for this types of monoid all cyclic right S-acts are flat if and only if all cyclic right I^1-acts are flat. First of all we need the following technical lemma.

Lemma 2.4.46. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I,\ gx = xg = x$. Let ρ_1, λ^1 be right and left congruences on I^1 respectively and let ρ, λ be the right and left congruences on S as in Lemmas 2.4.13, 2.4.39. If $s(\rho \vee \lambda u)1$ for $s, u \in I^1$, then $s(\rho_1 \vee \lambda^1 u)1$.*

Proof. Let $s(\rho \vee \lambda u)1$. If $s = 1$, then $1\ \rho_1\ 1(\lambda^1 u)1$ or $1(\rho_1 \vee \lambda^1 u)1$. Thus we suppose that $s \neq 1$. Then there exist $s_1, s_2, \ldots, s_{2n-1} \in S$ such that

$$s\ \rho\ s_1(\lambda u)s_2 \ldots s_{2n-2}\ \rho\ s_{2n-1}(\lambda u)1. \qquad (*)$$

If $u = 1$, then we have

$$s\ \rho\ s_1\ \lambda\ s_2 \ldots s_{2n-2}\ \rho\ s_{2n-1}\ \lambda\ 1.$$

Since $s_{2n-1}\ \lambda\ 1$, then by definition of λ, $s_{2n-1} = 1$. Thus $s_{2n-2}\ \rho\ 1$ and so by definition of ρ, $s_{2n-2} = 1$. By continuing this procedure we get $s\ \rho\ 1$ and so $s = 1$ which is a contradiction. Thus from now on we suppose that $s, u \in I$. Also we can suppose that the sequence $(*)$ is of minimal length. If $n = 0$, then $s = 1$ which is a contradiction (note that we define $s_0 = s$ and $s_{2n} = 1$). Thus $n \geq 1$ and so there are two cases that can arise:

Case 1. $n = 1$. Then we have $s\ \rho\ s_1(\lambda u)1$. Since $s \in I$, then by definition of ρ, $s_1 \in I$ and so $s_1 u \in I$. Since $s_1(\lambda u)1$, then $s_1 u\ \lambda\ u$ and so $s_1 u\ \lambda^1\ u$ or $s_1(\lambda^1 u)1$. Consequently, $s\ \rho_1\ s_1(\lambda^1 u)1$ or $s(\rho_1 \vee \lambda^1 u)1$.

Case 2. $n \geq 2$. Since the sequence $(*)$ is of minimal length, then by Lemma 2.4.33, $s_i \in I$ for $2 \leq i \leq 2n-2$. Also $s\ \rho\ s_1$, $s_{2n-2}\ \rho\ s_{2n-1}$ and $s, s_{2n-2} \in I$ imply that $s_1, s_{2n-1} \in I$. Consequently,

$$s\ \rho_1\ s_1(\lambda^1 u)s_2 \ldots s_{2n-2}\ \rho_1\ s_{2n-1}(\lambda^1 u)1 \text{ or } s(\rho_1 \vee \lambda^1 u)1,$$

as required. ■

From the proof of Lemma 2.4.46, the following result can be deduced.

Corollary 2.4.47. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I, gx = xg = x$. Let ρ and λ be as in Lemma 2.4.46 and let $s, u \in I$. If there exist $s_1, s_2, \ldots s_{2n-1} \in S$ such that $s \ \rho \ s_1(\lambda u)s_2 \ldots s_{2n-1}(\lambda u)1$ and this sequence is of minimal length $n \geq 2$, then $s_i \in I$, $i = 1, 2, \ldots, 2n-1$.*

Theorem 2.4.48. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = xg = x$. Let ρ_1 be a right congruence on I^1 and ρ the right congruence on S as in Lemma 2.4.13. If S/ρ is flat, then I^1/ρ_1 is flat.*

Proof. Let λ^1 be a left congruence on I^1 and let λ be the left congruence on S as in Lemma 2.4.39. Let $u(\rho_1 \ \vee \ \lambda^1)v$ for $u, v \in I^1$. Then there are four cases that can arise:

Case 1. $u = v = 1$. If $s = t = 1$, then $su \ \lambda^1 \ tv$, $s(\rho_1 \ \vee \ \lambda^1 u)1$ and $t(\rho_1 \ \vee \ \lambda^1 v)1$.

Case 2. $u \in I, v = 1$. Then $u(\rho_1 \ \vee \ \lambda^1)1$. If $s = 1, t = u$, then $su = u \ \lambda^1 \ u = tv$ and $s = 1(\rho_1 \ \vee \ \lambda^1 u)1$. Since $u(\rho_1 \ \vee \ \lambda^1)1$, then $t(\rho_1 \ \vee \ \lambda^1 1)1$.

Case 3. $u = 1, v \in I$. It is similar to case 2.

Case 4. $u, v \in I$. Then there exist $u_1, u_2, \ldots u_{2n-1} \in I^1$ such that

$$u \ \rho_1 \ u_1 \ \lambda^1 \ u_2 \ldots u_{2n-2} \ \rho_1 \ u_{2n-1} \ \lambda^1 \ v.$$

We can suppose that this sequence is of minimal length. If $n = 0$, then $u = v$ and so $s = t = 1$ imply that $su \ \lambda^1 \ tv$, $s(\rho_1 \ \vee \ \lambda^1 u)1$ and $t(\rho_1 \ \vee \ \lambda^1 v)1$. Thus we suppose that $n \geq 1$. Then there are two possibilities as follows:

(a) $u_1, u_{2n-1} \in I$. If $n = 1$, then $u_1 = u_{2n-1} \in I$ and so $u \ \rho \ u_1 \ \lambda \ v$ or $u(\rho \ \vee \ \lambda)v$. Suppose then that $n \geq 2$. Since $I^1 = \{1\} \cup I$ with $\{1\}$ a group and I an ideal of I^1 and also ρ_1, λ^1 are right and left congruences on I^1 respectively, then by Lemma 2.4.33, $u_i \in I$, $i = 2, 3, \ldots, 2n-2$. Consequently,

$$u \ \rho \ u_1 \ \lambda \ u_2 \ldots u_{2n-1} \ \lambda \ v \text{ or } u(\rho \ \vee \ \lambda)v.$$

Since S/ρ is flat, then $u(\rho \ \vee \ \lambda)v$ implies that there exist $s, t \in S$ such that $su \ \lambda \ tv$, $s(\rho \vee \lambda u)1$ and $t(\rho \vee \lambda v)1$. Now we have the following possibilities:

1. $s, t \in I$. Then by Lemma 2.4.46, $s(\rho_1 \ \vee \ \lambda^1 u)1$ and $t(\rho_1 \ \vee \ \lambda^1 v)1$. Since $su, tv \in I$, then $su \ \lambda^1 \ tv$.

2. $s \in G, t \in I$. Then $su \ \lambda \ tv$ implies that $u \ \lambda \ tv$. Since $u, tv \in I \subseteq I^1$, then $u \ \lambda^1 \ tv$. Also $1(\rho_1 \ \vee \ \lambda^1 u)1$ and by Lemma 2.4.46, $t(\rho_1 \ \vee \ \lambda^1 v)1$.

3. $s \in I, t \in G$. It is similar to part 2.

4. $s, t \in G$. Then by assumption $su = u, tv = v$ and so $su \ \lambda \ tv$ implies that $u \ \lambda \ v$. Since $u, v \in I$, then $u \ \lambda^1 \ v$. If $s' = t' = 1$, then $s'u \ \lambda^1 \ t'v$, $s'(\rho_1 \ \vee \ \lambda^1 u)1$ and $t'(\rho_1 \ \vee \ \lambda^1 v)1$.

(b) $u_1 = 1$ or $u_{2n-1} = 1$. Then we have the following possibilities :

1. $u_{2n-1} = 1$. Then we have

$$u \ \rho_1 \ u_1 \ \lambda^1 \ u_2 \dots u_{2n-2} \ \rho_1 \ 1 \ \lambda^1 \ v.$$

If we consider the sequence

$$u \ \rho_1 \ u_1 \ \lambda^1 \ u_2 \dots u_{2n-2} \ \rho_1 \ 1 \ \lambda^1 \ 1,$$

then by case 2, there exist $s, t \in I^1$ such that $su \ \lambda^1 \ t1$, $s(\rho_1 \ \vee \ \lambda^1 u)1$ and $t(\rho_1 \ \vee \ \lambda^1 1)1$. Since $1 \ \lambda^1 \ v$, then $t1 \ \lambda^1 \ tv$ and so $su \ \lambda^1 \ tv$. Also by Lemma 2.4.29, $t(\rho_1 \ \vee \ \lambda^1 1)1$ implies that $t(\rho_1 \ \vee \ \lambda^1 v)1$.

2. $u_1 = 1$. Then we have

$$u \ \rho_1 \ 1 \ \lambda^1 \ u_2 \dots u_{2n-1} \ \lambda^1 \ v. \qquad (\star)$$

If $n = 1$, then $u_1 = u_{2n-1} = 1$ and so by [(b), 1] we are done. Suppose that $n \geq 2$. If $u_{2n-1} = 1$, again by [(b), 1] we are done. Hence, we assume that $u_{2n-1} \in I$. Since the sequence $(\star)$ is of minimal length $n \geq 2$, then by Lemma 2.4.33, $u_i \in I$, $i = 2, 3, \dots, 2n-2$. Also by Lemma 2.4.34, the sequence $(\star)$ can be replaced by the sequence

$$u\rho_1 \ u \ \lambda^1 \ uu_2 \ \rho_1 \ u_3 \dots u_{2n-1} \ \lambda^1 \ v.$$

Since $u, uu_2, u_3, \dots, u_{2n-1}, v \in I$, then

$$u \ \rho \ u \ \lambda \ uu_2 \ \rho \ u_3 \dots u_{2n-1} \ \lambda \ v \text{ or } u(\rho \ \vee \ \lambda)v.$$

Now we can argue as in part (a). ■

Theorem 2.4.49. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = xg = x$. If all cyclic right S-acts are flat, then all cyclic right I^1-acts are flat.*

Proof. Let I^1/ρ_1 be a cyclic right I^1-act for a right congruence ρ_1 on I^1. Let ρ be the right congruence on S as in Lemma 2.4.13. Then by assumption S/ρ is flat and so by Theorem 2.4.48, I^1/ρ_1 is flat. ∎

From Theorem 2.4.38, and Theorem 2.4.49, the following theorem can be deduced.

Theorem 2.4.50. *Let $S = G \dot{\cup} I$ be a monoid such that G is a group, I is an ideal of S and $\forall g \in G, \forall x \in I$, $gx = xg = x$. Then all cyclic right S-acts are flat if and only if all cyclic right I^1-acts are flat.*

By what follows we suppose that ρ_1 is a right congruence on S_1 a submonoid of a monoid S and that $\rho = \rho_1 \cup 1_{(S \setminus S_1)}$ is a right congruence on S. Then we show that (weak) flatness and condition (P) of S/ρ can be deduced from the corresponding property of S_1/ρ_1.

Lemma 2.4.51. *Let S be a monoid and S_1 a submonoid of S. Let λ be a left congruence on S. If $\lambda_1 = \lambda|_{S_1}$, then λ_1 is a left congruence on S_1.*

Proof. It is obvious that λ_1 is an equivalence relation on S_1. Let $a \ \lambda_1 \ b$ and let $s \in S_1$. Then $a \ \lambda \ b$ and so $sa \ \lambda \ sb$. Since S_1 is a submonoid, then $sa, sb \in S_1$ and so $sa \ \lambda_1 \ sb$. Thus λ_1 is a left congruence on S_1 as required. ∎

Theorem 2.4.52. *Let S be a monoid and let S_1 be a submonoid of S. Let ρ_1 be a right congruence on S_1 such that $\rho = \rho_1 \cup 1_{(S \setminus S_1)}$ is a right congruence on S. If S_1/ρ_1 is flat, then S/ρ is flat.*

Proof. Suppose that S_1/ρ_1 is flat. To show that S/ρ is flat by Lemma 1.54 (5), it is sufficient to show that for every $u, v \in S$ and any left congruence λ on S, if $u(\rho \ \vee \ \lambda)v$, then there exist $s, t \in S$ such that $su \ \lambda \ tv$, $s(\rho \ \vee \ \lambda u)1$ and $t(\rho \ \vee \ \lambda v)1$. Let $u(\rho \ \vee \ \lambda)v$ with $u, v \in S$. Then by Theorem 1.49, there exist $u_1, u_2, \ldots, u_{2n-1} \in S$ such that

$$u \ \rho \ u_1 \ \lambda \ u_2 \ \rho \ u_3 \ldots u_{2n-1} \ \lambda \ v. \tag{1}$$

We can suppose that the sequence (1) is of minimal length n. Also if $n \geq 2$, then for every i such that $u_i \ \rho \ u_{i+1}$ it follows that $u_i \neq u_{i+1}$. Otherwise, if k is such that $u_k \ \rho \ u_{k+1}$ and $u_k = u_{k+1}$, then we have the shorter sequence

$$u \ \rho \ u_1 \ \lambda \ u_2 \ \rho \ u_3 \ldots u_{k-2} \ \rho \ u_{k-1} \ \lambda \ u_{k+2} \ \rho \ u_{k+3} \ldots u_{2n-2} \ \rho \ u_{2n-1} \ \lambda \ v,$$

which is a contradiction on the minimality of n. Thus, if $n \geq 2$, then $u_{2j} \ \rho_1 \ u_{2j+1}$, $1 \leq j \leq n-1$, and so $u_2, u_3, \ldots, u_{2n-1} \in S_1$.

Now in sequence (1), there are two cases for u as follows:

Case 1. $u \in S_1$. Then $u\ \rho_1\ u_1$ and so $u_1 \in S_1$. Let $\lambda_1 = \lambda|_{S_1}$ be the left congruence on S_1 as in Lemma 2.4.51. If $n \geq 2$, then $u_2, u_3, \ldots, u_{2n-1} \in S_1$ and so

$$u\ \rho_1\ u_1\ \lambda_1\ u_2\ \rho_1\ u_3 \ldots u_{2n-2}\ \rho_1\ u_{2n-1}\ \lambda_1\ u_{2n-1}$$

which means $u(\rho_1 \ \vee\ \lambda_1)u_{2n-1}$. If $n = 1$, then $u\ \rho_1\ u_1\ \lambda\ v$. Thus $u\ \rho_1\ u_1\ \lambda_1\ u_1$ and so $u(\rho_1 \ \vee\ \lambda_1)u_1$. Consequently, if $n \geq 1$, then $u(\rho_1 \ \vee\ \lambda_1)u_{2n-1}$.

Since S_1/ρ_1 is flat, then there exist $s, t \in S_1 \subseteq S$ such that $su\ \lambda_1\ tu_{2n-1}$, $s(\rho_1 \ \vee\ \lambda_1 u)1$ and $t(\rho_1 \ \vee\ \lambda_1 u_{2n-1})1$. Thus $su\ \lambda\ tu_{2n-1}$ and since $u_{2n-1}\ \lambda\ v$, then $tu_{2n-1}\ \lambda\ tv$. Consequently, $su\ \lambda\ tv$. Since $s(\rho_1 \ \vee\ \lambda_1 u)1$, then it easily follows that $s(\rho \ \vee\ \lambda u)1$. Similarly it can be seen that $t(\rho \ \vee\ \lambda u_{2n-1})1$. But $u_{2n-1}\ \lambda\ v$ and so by Lemma 2.4.29, $t(\rho \ \vee\ \lambda v)1$ as required.

Case 2. $u \notin S_1$. Then $u\ \rho\ u_1$ implies that $u = u_1 \in S \setminus S_1$. Now there are two possibilities as follows:

1. $n = 1$. Then $u\ \lambda\ v$. If $s = t = 1$, then $su\ \lambda\ tv$, $s(\rho \ \vee\ \lambda u)1$ and $t(\rho \ \vee\ \lambda v)1$.

2. $n \geq 2$. Then

   $$u\ \lambda\ u_2\ \rho\ u_3 \ldots u_{2n-1}\ \lambda\ v,$$

   from which we have the sequence

   $$u_{2n-1}\ \rho\ u_{2n-2} \ldots u_3\ \rho\ u_2\ \lambda\ u,$$

   with $u_{2n-1} \in S_1$. Thus by case 1, there exist $s, t \in S$ such that $tu_{2n-1}\ \lambda\ su$, $t(\rho \vee \lambda u_{2n-1})1$, $s(\rho \vee \lambda u)1$. Since $u_{2n-1}\ \lambda\ v$, then $tu_{2n-1}\ \lambda\ tv$. Consequently, $su\ \lambda\ tv$. Also by Lemma 2.4.29, $t(\rho \ \vee\ \lambda v)1$. ■

Theorem 2.4.53. *Let S be a monoid and let S_1 be a submonoid of S. Let ρ_1 be a right congruence on S_1 such that $\rho = \rho_1 \cup 1_{(S\setminus S_1)}$ is a right congruence on S. If S_1/ρ_1 is weakly flat, then S/ρ is weakly flat.*

Proof. Let $u\ \rho\ v$ with $u, v \in S$. Then there are two cases that can arise:

Case 1. $u\ \rho_1\ v$. Then $u, v \in S_1$ and so by assumption there exist $s, t \in S_1 \subseteq S$ such that $s(\rho_1 \ \vee\ \Delta u)1$, $t(\rho_1 \ \vee\ \Delta v)1$ and $su = tv$. Since $\rho_1 \subseteq \rho$, then it easily follows that $s(\rho \ \vee\ \Delta u)1$ and $t(\rho \ \vee\ \Delta v)1$.

Case 2. $(u, v) \in 1_{(S\setminus S_1)}$. Then $u = v$. If $s = t = 1$, then $s(\rho \ \vee\ \Delta u)1, t(\rho \ \vee\ \Delta v)1$ and $su = tv$.

Thus S/ρ is weakly flat as required. ■

Theorem 2.4.54 *Let S be a monoid and let S_1 be a submonoid of S. Let ρ_1 be a right congruence on S such that $\rho = \rho_1 \cup 1_{(S \setminus S_1)}$ is a right congruence on S. If S_1/ρ_1 satisfies condition (P), then S/ρ satisfies condition (P).*

Proof. Let $u \ \rho \ v$ with $u, v \in S$. Then there are two cases that can arise:

Case 1. $u \ \rho_1 \ v$. Then $u, v \in S_1$ and so by assumption there exist $s, t \in S_1 \subseteq S$ such that $s \ \rho_1 \ 1 \ \rho_1 \ t$ and $su = tv$. Since $\rho_1 \subseteq \rho$, then $s \ \rho \ 1 \ \rho \ t$.

Case 2. $(u, v) \in 1_{(S \setminus S_1)}$. Then $u = v$. If $s = t = 1$, then $s \ \rho \ 1 \ \rho \ t$ and $su = tv$. ■

We present two simple examples of monoids which satisfy the previous theorems.

Example 2.4.55 Let $S = \{1, e, f\}$ such that e, f are right zero. Let $S_1 = \{1, e\}$. Then S_1 is a submonoid of S. If

$$\rho_1 = \{(1,1), (e,e), (e,1), (1,e)\}$$

then ρ_1 is a right congruence on S_1. If $\rho = \rho_1 \cup 1_{(S \setminus S_1)}$, then

$$\rho = \{(1,1), (e,e), (f,f), (1,e), (e,1)\}.$$

It can be seen that ρ is a right congruence on S. Also it is easy to see that S_1/ρ_1 and consequently S/ρ satisfies condition (P).

Example 2.4.56 Let G be a group and N a right nil semigroup. Let $S = G \dot{\cup} N$ such that $\forall g \in G, \forall n \in N$, $gn = ng = n$. Then S is a monoid and G is a submonoid of S. If ρ_1 is a right congruence on G, then it is easy to see that $\rho = \rho_1 \cup 1_N$ is a right congruence on S. Since G is a group, then all acts (and in particular G/ρ_1) satisfy condition (P). Also it is easy to see that S/ρ satisfies condition (P). For example if $S = \{0, 1, 2, 3\}$ with table

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

then $S_1 = \{1, 3\}$ is a group, $N = \{0, 2\}$ is a right nil semigroup and $S = G \dot{\cup} N$. If

$$\rho_1 = \{(1,1), (3,3), (1,3), (3,1)\},$$

then ρ_1 is a right congruence on S_1. Thus

$$\rho = \rho_1 \cup 1_S = \{(0,0), (1,1), (2,2), (3,3), (1,3), (3,1)\},$$

is a right congruence on S and so S_1/ρ_1, S/ρ satisfy condition (P).

Notice that in Examples 2.4.55, 2.4.56, since S_1/ρ_1 and S/ρ satisfy condition (P), then they are flat and also weakly flat.

Chapter 3

Characterization of Monoids by Properties of Principal Ideals

3.1. Introduction

In this chapter we turn our attention to the classification of monoids by properties of their principal left ideals which are in fact left acts.

We start in section 3.2, with a definition and then we give a characterization of monoids for which all principal left ideals are strongly flat or satisfy condition (E). Also we give a necessary and sufficient condition for right subelementary monoids, and monoids $S = T^1$ where T is a null semigroup, such that all principal left ideals be strongly flat or satisfy condition (P).

A characterization of left PSF monoids by condition (P) of (weakly) flat right acts is given in section 3.3. In section 3.4, we characterize some classes of left PSF monoids by condition (P) of (weakly) flat cyclic right acts and also left PSF monoids for which all (weakly) flat cyclic right acts are projective or strongly flat.

Finally, by imposing some conditions on a monoid S with $|E(S)| = 1$, we show that all right S-acts satisfy condition (P).

3.2. Left PSF Monoids

Definition 3.2.1. Let S be a monoid. An element $u \in S$ is called *right semi cancellative* if whenever $su = tu$ with $s, t \in S$ there exists $r \in S$ such that $u = ru$ and $sr = tr$. A monoid is called *right semi cancellative* (RSC) if all its elements are right semi cancellative.

Clearly every right cancellative monoid is a right semi cancellative, but there are right semi cancellative monoids which are not right cancellative as the following example demonstrates.

Example 3.2.2. Let $S = \{1, e, f\}$ with table

	1	e	f
1	1	e	f
e	e	e	e
f	f	f	f

Then S is right semi cancellative, but $1f = ff$ and $1 \neq f$. Therefore S is not right cancellative. Note that in this example S is regular. It can be seen that every regular monoid is right semi cancellative and so there are examples of monoids which are right semi cancellative but not right cancellative.

which is not right cancellative is an example of monoids mentioned above.

Lemma 3.2.3. *Let S be a monoid and Sx a cyclic left S-act. Then Sx satisfies condition (E), if and only if for $sx = tx$ with $s, t \in S$ there exists $r \in S$ such that $x = rx$ and $sr = tr$.*

Proof. Necessity. Suppose that Sx satisfies condition (E) and let $sx = tx$. Then there exist $y \in Sx$, $v \in S$ such that $x = vy$ and $sv = tv$. Since $y \in Sx$, then there exists $s' \in S$ such that $y = s'x$ and so $x = vy = vs'x$.

If $r = vs'$, then $x = rx$ and $sv = tv$ implies that $svs' = tvs'$ or $sr = tr$.

Sufficiency. Let $sa = ta$ for $a \in Sx$, $s, t \in S$. Since $a \in Sx$, then there exists $s_1 \in S$ such that $a = s_1x$ and so $sa = ta$ implies that

$$s(s_1x) = t(s_1x) \text{ or } (ss_1)x = (ts_1)x.$$

Then by assumption, there exists $r \in S$ such that $x = rx$ and $(ss_1)r = (ts_1)r$. If $s' = s_1r$, then

$$s'x = (s_1r)x = s_1(rx) = s_1x = a,$$

and

$$ss' = s(s_1r) = (ss_1)r = (ts_1)r = t(s_1r) = ts'.$$

Thus, Sx satisfies condition (E). ■

Now from Definition 3.2.1, and Lemma 3.2.3, we have

Corollary 3.2.4. *Let S be a monoid. If $u \in S$, then the cyclic left S-act Su satisfies condition (E), if and only if u is right semi cancellative. Hence S is right semi cancellative if and only if every principal left ideal satisfies condition (E).*

Lemma 3.2.5. *Let S be a monoid and Sx a cyclic left S-act. Then Sx is strongly flat if and only if for $sx = tx$ with $s, t \in S$ there exists $r \in S$ such that $x = rx$ and $sr = tr$.*

Necessity. Since strongly flat implies condition (E), then this follows by Lemma 3.2.3.

Sufficiency. By Lemma 3.2.3, it is sufficient to show that Sx satisfies condition (P). Let $sa = ta'$ for $a, a' \in Sx$ and $s, t \in S$. Then there exist $s_1, t_1 \in S$ such that $a = s_1x$, $a' = t_1x$ and so $sa = ta'$ implies that

$$s(s_1x) = t(t_1x) \text{ or } (ss_1)x = (tt_1)x.$$

Then, by assumption there exists $r \in S$ such that $x = rx$ and

$$(ss_1)r = (tt_1)r \text{ or } s(s_1r) = t(t_1r).$$

If $s_1r = s'$ and $t_1r = t'$, then

$$s'x = (s_1r)x = s_1(rx) = s_1x = a.$$

$$t'x = (t_1r)x = t_1(rx) = t_1x = a'.$$

Also

$$ss' = s(s_1r) = (ss_1)r = (tt_1)r = t(t_1r) = tt'.$$

Thus, Sx satisfies condition (P) as required. ■

From Lemma 3.2.3, and Lemma 3.2.5, we have

Corollary 3.2.6. *Let S be a monoid. If a cyclic right S-act Sx satisfies condition (E) then it satisfies condition (P) and so it is strongly flat.*

Remark. The converse of Corollary 3.2.6, is not true as the following example demonstrates.

Example 3.2.7. Let $S = \{0, s, 1 \mid s^2 = 1\}$ and let $A = \{z, a \mid sa = a, 0a = sz = 0z = z\}$. Then A is a left S-act and $A \simeq S/\rho$ for $\rho = \{(1, s), (s, 1)\} \cup 1_S$ and so A is a cyclic left S-act.

Now we show that A satisfies condition (P). It is enough to show that S/ρ satisfies condition (P). Suppose that $u \; \rho \; v$. If $u = v$, then put $s' = t' = 1$, so that $s' \; \rho \; 1 \; \rho \; t'$ and $us' = ut'$. Otherwise if $u = s, v = 1$, then put $s' = s, t' = 1$ so that $s' \; \rho \; 1 \; \rho \; t'$ and $ss' = 1 = 1t'$. So A satisfies condition (P).

We claim that A does not satisfy condition (E). Otherwise, $1a = sa$ implies that there exist $a'' \in A$, $t \in S$ such that $a = ta''$ and that $1t = st$. But either $a = 1a$ or $a = sa$ and so $a'' = a$ and either $t = 1$ or $t = s$.

If $t = 1$, then $1t = 1 \neq s = st$ and so we have a contradiction. If $t = s$, then $1t = s \neq 1 = st$ and again we have a contradiction. Thus A satisfies condition (P), but it does not satisfy condition (E).

By Proposition 2.3.53, it can be seen that condition (E) does not imply condition (P) in general. Also see the following example.

Example 3.2.8. Let $S = \{0, 1\}$ and let $A = \{x, y, z \mid 0x = 0y = 0z = z, 1x = x, 1y = y, 1z = z\}$. Then by Theorem 2.2.19, A satisfies condition (E). Since A is not a coproduct of cyclic S-acts, then by Lemma 1.53, it does not satisfy condition (P).

By the following lemma we extend a part of Lemma 3.2.3.

Lemma 3.2.9. *Let S be a monoid. If a cyclic left S-act Sx has the property that whenever $s(s_1x) = t(s_1x)$ for $s, s_1, t \in S$, there exists $r \in S$ such that $s_1x = r(s_1x)$ and $sr = tr$, then Sx satisfies condition (E).*

Proof. By Lemma 3.2.3, It is sufficient to show that if $sx = tx$ with $s, t \in S$, then there exists $r \in S$ such that $x = rx$ and $sr = tr$. Since $sx = tx$, then $s(1x) = t(1x)$

and so by assumption there exists $r \in S$ such that $r(1x) = (1x)$ or $rx = x$ and $sr = tr$. ■

Remark. The converse of Lemma 3.2.9 is not true, as the following example demonstrates.

Example 3.2.10. Let $S = \{0, 1, 2, 3\}$ with table

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Then $S1 = S$ is a cyclic left S-act which by Lemma 3.2.3, satisfies condition (E). Note also that $S = G \dot{\cup} N$ with $G = \{1, 3\}$ a group and $N = \{0, 2\}$ a null semigroup. We claim that $S1$ does not satisfy the condition mentioned above. Otherwise,

$$2 \cdot (2 \cdot 1) = 0 \cdot (2 \cdot 1),$$

implies that there exists $r \in S$ such that $r(2 \cdot 1) = 2 \cdot 1$ and that $2 \cdot r = 0 \cdot r = 0$. Then on the one hand either $r = 1$ or $r = 3$, but on the other hand $r = 0$ or $r = 2$.

Corollary 3.2.11. *Let S be a monoid. If a cyclic left S-act Sx has the property that whenever $s, s_1, t \in S$ and $s(s_1 x) = t(s_1 x)$, there exists $r \in S$ such that $r(s_1 x) = s_1 x$ and $sr = tr$, then Sx is strongly flat.*

Proof. By Lemma 3.2.9, Sx satisfies condition (E). Since Sx is cyclic, then by Corollary 3.2.6, it satisfies condition (P), and so Sx is strongly flat. ■

Remark. The converse of Corollary 3.2.11, is not true. For example, if $S = \{0, 1, 2, 3\}$ is the monoid with the table as in Example 3.2.10, then $S1 = S$ satisfies condition (E) and so it is strongly flat. But as we saw in this example $S1$ does not satisfy the condition mentioned in corollary 3.2.11.

From Lemma 3.2.5, and Definition 3.2.1, we have

Corollary 3.2.12. *Let S be a monoid. If $u \in S$, then the cyclic left S-act Su is strongly flat if and only if u is right semi cancellative. Hence S is right semi cancellative if and only if every principal left ideal of S is strongly flat.*

Definition 3.2.13. A monoid S is called left PSF if every principal left ideal of S is strongly flat.

Lemma 3.2.14. *Let S be an eventually regular monoid. Then there exists a power of every element which is right semi cancellative.*

Proof. Let S be an eventually regular monoid and let $x \in S$. Then there exists $n \in \mathbb{N}$ such that x^n is regular and so there exists $x' \in S$ such that $x^n x' x^n = x^n$. Let $sx^n = tx^n$ with $s, t \in S$. If $r = x^n x'$, then $rx^n = x^n x' x^n = x^n$. Also $sx^n = tx^n$ implies that $sx^n x' = tx^n x'$ or $sr = tr$. Thus x^n is right semi cancellative as required. ■

Corollary 3.2.15. *If S is a regular monoid, then every principal left ideal of S is strongly flat.*

Proof. Let $x \in S$. Since S is regular, then in Lemma 3.2.14, $n = 1$ and so x is right semi cancellative. Hence, by Corollary 3.2.12, every principal left ideal of S is strongly flat. ■

In the following we characterize monoids $S = C \dot{\cup} T$ where C is right cancellative and T is a null semigroup such that all principal left ideals are strongly flat, and use this to give a characterization of monoids $S = T^1$ with T a null semigroup such that all principal left ideals are strongly flat or satisfy condition (P).

Lemma 3.2.16. *Let $S = C \dot{\cup} T$ be a monoid with all elements in C right cancellative and T a null semigroup. Then S is RSC if and only if $T = \{0\}$.*

Proof. Suppose that S is right semi cancellative and let $x \in T$. Then $xx = 0x = 0$. Since x is right semi cancellative, then there exists $r \in S$ such that $x = rx$ and $xr = 0r = 0$. If $r \in C$, then $xr = 0r$ implies that $x = 0$, otherwise $x = rx = 0$. Consequently, $T = \{0\}$.

Now suppose that $S = C \dot{\cup} \{0\}$ and let $su = tu$ with $s, t, u \in S$. Then we show that there exists $r \in S$ such that $u = ru$ and $sr = tr$. If $u = 0$, then $r = 0$ implies that $u = ru$ and that $sr = tr$. If $u \in C$, then $su = tu$ implies that $s = t$. If $r = 1$, then $u = ru$ and $sr = tr$. Thus every element of S is right semi cancellative as required. ■

From Corollary 3.2.12, and Lemma 3.2.16, we have

Corollary 3.2.17. *Let $S = C \dot{\cup} T$ be a monoid with elements in C right cancellative and T a null semigroup. Then S is left PSF if and only if $T = \{0\}$.*

It is obvious that if $S = G \dot{\cup} T$ where G is a group and T a null semigroup, then every element of G is a right cancellative element of S and so from Corollary 3.2.17, we have

Corollary 3.2.18. *Let $S = G \dot{\cup} T$ where G is a group and T a null semigroup. Then S is left PSF if and only if $S = G \dot{\cup} \{0\}$.*

If in Corollary 3.2.18, we put $G = \{1\}$, then we deduce the following corollary.

Corollary 3.2.19. *Let T be any null semigroup and let $S = T^1$. Then all principal left ideals of S are strongly flat if and only if $S = \{0, 1\}$.*

Now by using Corollary 3.2.19, and the following theorem we give a characterization of monoids $S = T^1$ with T a null semigroup such that all principal left ideals satisfy condition (P).

Theorem 3.2.20 [1]. *Let T be any null semigroup and let $S = T^1$. Then every right (left) S-act which satisfies condition (P) is strongly flat.*

Theorem 3.2.21. *Let T be any null semigroup and let $S = T^1$. Then all principal left ideals of S satisfy condition (P) if and only if $S = \{0, 1\}$.*

Proof. Suppose that every principal left ideal of the monoid $S = T^1$ satisfies condition (P). Since every principal left ideal is a left S-act, then by Theorem 3.2.20, it is strongly flat. Consequently, by Corollary 3.2.19, $S = \{0, 1\}$.

If $S = \{0, 1\}$, then by Corollary 3.2.19, all principal left ideals are strongly flat and so all principal left ideals satisfy condition (P). ■

Now we consider right subelementary monoids for which all principal right ideals are strongly flat.

Lemma 3.2.22. *Let $S = C \dot{\cup} N$ be a right subelementary monoid. Then $x \in N$ is right semi cancellative if and only if x is right zero.*

Proof. Let $x \in N$ be right semi cancellative. Since x is right nil, then there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$ is right zero. We can suppose that n is the smallest such positive number. Then either $n = 1$ or $n > 1$. If $n = 1$, then $x^2 = x$ and so x is right zero.

Suppose that $n > 1$. Then $n - 1 > 0$ and so we have $x^n x = x^{n-1}x$. Then by assumption there exists $r \in S$ such that $x = rx$ and $x^n r = x^{n-1}r$. If $r \in C$, then $x^n = x^{n-1}$ is right zero which is a contradiction. Thus $r \in N$ and so there exists $k \in \mathbb{N}$ such that r^k is right zero. But by induction it can be seen that for every $k \in \mathbb{N}$, $x = r^k x$ and so x is right zero as required.

Now let $x \in N$ be right zero and let $sx = tx$ with $s, t \in S$. If we put $r = x$, then $x = rx$ and $sr = tr$. Thus x is right semi cancellative as required. ■

Theorem 3.2.23. Let $S = C \dot{\cup} N$ be a right subelementary monoid. Then S is left PSF if and only if every $x \in N$ is right zero.

Proof. If S is left PSF, then by Corollary 3.2.12, every element of S is right semi cancellative and so by Lemma 3.2.22, every element in the right nil part is right zero.

Suppose that every element in N is right zero and let $x \in S$. If $x \in C$, then it is obvious that x is right semi cancellative. If $x \in N$, then x is right zero and so by Lemma 3.2.22, x is right semi cancellative. Thus every element of S is right semi cancellative and so by Corollary 3.2.12, S is left PSF. ■

From Theorem 2.3.22, and Theorem 3.2.23, the following corollary can be deduced.

Corollary 3.2.24. *Let $S = C \dot{\cup} N$ be a right subelementary monoid. Then S is left PSF if and only if S is left PP.*

3.3. Characterization of Left PSF Monoids by Condition (P) of Right Acts

In this section we give a characterization of left PSF monoids by condition (P) of (weakly) flat right acts and also properties of proper right ideals. Throughout this section and section 3.4, by a proper ideal we mean an ideal J of S such that $J \neq S$.

Theorem 3.3.1. *Let $S = C \dot{\cup} N \dot{\cup} F'$ be a left PSF monoid where C is right cancellative, N is the set of all right nil elements of S, and F' is the set of all regular-free elements of S which are not right cancellative. Then for all $x \in N \dot{\cup} F'$, either x is right zero or there exists $r \in F'$ such that $x = rx$ and $r \neq x$.*

Proof. Let $x \in N \dot{\cup} F'$. Then x is not right cancellative and so there exist $a, b \in S$ such that $ax = bx$ and $a \neq b$. Since S is left PSF, then by Corollary 3.2.12, x is right semi cancellative and so there exists $r \in S$ with $x = rx$ and $ar = br$. If $r \in C$, then $a = b$, which is a contradiction. Thus $r \in N \dot{\cup} F'$. If $r \in N$, then r^k is a right zero for some $k \in \mathbb{N}$. Now $x = r^k x$ and so x is right zero.

Hence, if x is not right zero, then $r \in F'$. Also, $r \neq x$ for otherwise, $x = x^2$ is idempotent and so is in N and hence is a right zero which is a contradiction. ■

By Theorem 3.3.1, and Theorem 2.3.16

Corollary 3.3.2. *Let S be a left PSF monoid. If all flat cyclic right S-acts satisfy condition (P), then $S = C \dot{\cup} N \dot{\cup} F'$ where C is right cancellative, N is the set of all right nil elements of S, and F' is the set of all regular-free elements of S which are not right cancellative. Moreover, for all $x \in N \dot{\cup} F'$, either x is right zero or there exists $r \in F'$ such that $x = rx$ and $r \neq x$.*

Corollary 3.3.3. *If in Theorem 3.3.1, $F' = \emptyset$, then every element in N is right zero.*

Corollary 3.3.4. *Let S be a monoid. Then S is left PSF and right subelementary if and only if every element of S is either right cancellative or right zero. Subsequently, for such a monoid, all (weakly) flat cyclic right S-acts satisfy condition (P).*

Lemma 3.3.5. *Let S be a left PSF monoid. Then for each $x_0 \in S$ with x_0 not* *right cancellative, there exists a sequence* $(x_0, x_1, \ldots)$, $x_i = x_{i+1}x_i$ *and* x_i *not right cancellative,* $i = 0, 1, \ldots$.

Proof. If x_0 is not right cancellative, then there exists $a, b \in S$ with $ax_0 = bx_0$ and $a \neq b$. But by Corollary 3.2.12, x_0 is right semi cancellative and so there exists $x_1 \in S$ with $x_0 = x_1x_0$ and $ax_1 = bx_1$. Since $a \neq b$, then x_1 is not right cancellative. Continuing in this fashion we generate the required sequence.

Lemma 3.3.6. *Let* S *be any monoid. If* S *is right cancellative, then for every proper right ideal* J *of* S *there exists* $j \in J \setminus Jj$.

Proof. Suppose that S is right cancellative and let J be a proper right ideal of S. If for every $j \in J$, $j \in Jj$, then let $j \in J$. Hence there exists $x \in J$ such that $j = xj$. Since j is right cancellative, then $x = 1$. Thus $1 \in J$ and so $J = S$ which is a contradiction. ■

Theorem 3.3.7. *Let* S *be a left PSF monoid. Then* S *is right cancellative if and only if for every proper right ideal* J *of* S *there exists* $j \in J \setminus Jj$.

Proof. If S is right cancellative, then by Lemma 3.3.6, for every proper right ideal J of S there exists $j \in J \setminus Jj$.

Now suppose that for every proper right ideal J of S there exists $j \in J \setminus Jj$. Then we claim that S is right cancellative. Otherwise there exists $x_0 \in S$ which is not right cancellative and so by Lemma 3.3.5, there exists a sequence $(x_0, x_1, \ldots)$, $x_i = x_{i+1}x_i$ and x_i is not right cancellative, $i = 0, 1, \ldots$.

Let $J = \bigcup_{i=0}^{\infty} x_iS$. Then J is a right ideal of S. Also $J \neq S$, otherwise $1 \in J$ and so there exists $x \in S$ such that $1 = x_ix$ for some x_i. Thus x_i is right invertible and so it is right cancellative which is a contradiction. Consequently, J is a proper right ideal of S.

Now let $j \in J$. Then there exists $s \in S$ such that $j = x_is$ for some x_i. Since $x_i = x_{i+1}x_i$, then $j = x_{i+1}x_is = x_{i+1}j$ and so $j \in Jj$ which by assumption is a contradiction. Thus S is right cancellative as required. ■

Lemma 3.3.8. *Let* S *be any monoid. If* S *is right cancellative, then for every infinite sequence* $(x_0, x_1, \ldots)$ *with* $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, *there exists* $n \in \mathbb{N}$ *such that* $x_n = x_{n+1} = \ldots = 1$.

Proof. Suppose that S is a right cancellative monoid and let $(x_0, x_1, \ldots)$ be an infinite sequence with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$. Then by cancelling x_i we have $1 = x_{i+1}$, $i = 0, 1 \ldots$, and so $n = 1$. ∎

<u>Theorem 3.3.9.</u> *Let S be a left PSF monoid. Then S is right cancellative if and only if for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$.*

Proof. If S is right cancellative, then by Lemma 3.3.8, for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$

Now suppose that for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$. Then S is right cancellative. Otherwise there exists $x_0 \in S$ which is not right cancellative and so by Lemma 3.3.5, there exists a sequence $(x_0, x_1, \ldots)$, $x_i = x_{i+1}x_i$ and x_i is not right cancellative $i = 0, 1, \ldots$. But by assumption there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$ and so $x_n = 1$. Consequently, x_n is a right cancellative which is a contradiction. ∎

From Theorem 3.3.7, and Theorem 3.3.9, we have

<u>Theorem 3.3.10.</u> *Let S be a left PSF monoid. Then the following conditions are equivalent:*

(1) S is right cancellative.

(2) For every proper right ideal J of S there exists $j \in J \setminus Jj$.

(3) For every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $x_i \in S$, $i = 0, 1, \ldots$ there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$.

From Corollary 2.3.55, and Theorem 3.3.10, we have

<u>Corollary 3.3.11.</u> *Let S be a left PSF monoid. If all flat right S-acts satisfy condition (P), then for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $x_i \in S$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$.*

<u>Theorem 3.3.12.</u> *let S be a left PSF monoid. Then all flat right S-acts satisfy condition (P) if and only if S is right cancellative.*

Proof. Suppose that all flat right S-acts satisfy condition (P). Then by Theorem 2.3.46, and the subsequent note, $S = C \dot{\cup} F'$ where C is right cancellative and F' is the set of regular-free elements of S that are not right cancellative. We claim that $F' = \emptyset$, otherwise let $x_0 \in F'$. Then by Theorem 3.3.1, there exists $x_1 \in F'$ with $x_0 = x_1x_0$ and $x_1 \neq x_0$. In a similar way, there exists $x_2 \in F'$ with $x_1 = x_2x_1$ and $x_2 \neq x_1$. Continuing in this fashion we get an infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $x_i \in F'$ and $x_i \neq x_{i+1}$, $i = 0, 1, \ldots$. Then by Corollary 3.3.11, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$, which is a contradiction. Thus $S = C$ is a right cancellative monoid as required.

The converse is true by Lemma 2.3.50. ■

It is clear that if S is a right cancellative monoid, then it is right semi cancellative. By Example 3.3.2, we saw that there are right semi cancellative monoids which are not right cancellative, but from Corollary 3.2.12, and Theorem 3.3.12, we have

Corollary 3.3.13. *Let S be a monoid such that all flat right S-acts satisfy condition (P). If S is right semi cancellative, then S is right cancellative.*

From Lemma 2.3.50, Theorem 3.3.10, Corollary 2.3.55, and Theorem 3.3.12, we have

Theorem 3.3.14. *Let S be a left PSF monoid. Then the following statements are equivalent:*

(1) S is right cancellative.

(2) All weakly flat right S-acts satisfy condition (P).

(3) All flat right S-acts satisfy condition (P).

(4) For every proper right ideal J of S there exists $j \in J \setminus Jj$.

(5) For every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1 \ldots$, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$.

From Theorem 3.3.14, the following theorem which is an extension to Theorem 2.3.52, in that left PP monoids are left PSF can be deduced.

Theorem 3.3.15. *Let S be any monoid. Then the following statements are equivalent:*

(1) S is right cancellative.

(2) S is left PP and all weakly flat right S-acts satisfy condition (P).

(3) S is left PP and all flat right S-acts satisfy condition (P).

(4) S is left PSF and all weakly flat right S-acts satisfy condition (P).

(5) S is left PSF and all flat right S-acts satisfy condition (P).

(6) S is left PSF and for every proper right ideal J of S there exists $j \in J \setminus Jj$.

(7) S is left PSF and for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1} = \ldots = 1$.

3.4. Characterization of Left PSF Monoids by Condition (P) of Cyclic Right Acts

In this section we characterize some classes of left PSF monoids by condition (P) of (weakly) flat cyclic right acts and also properties of proper right ideals. We also characterize left PSF monoids for which all (weakly) flat cyclic right acts are projective or strongly flat. Finally, we show that for some classes of monoids S, if $|E(S)| = 1$, then all right S-acts satisfy condition (P).

Lemma 3.4.1. *Let S be a monoid. If for every proper right ideal J of S with $|J| > 1$ there exists $j \in J \setminus Jj$, then for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1 \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$.*

Proof. Suppose that $(x_0, x_1, \ldots)$ is an infinite sequence with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$ and suppose that $x_i \notin E(S)$, $i = 0, 1, \ldots$. Then $J = \bigcup_{i=0}^{\infty} x_iS$ is a right ideal and also $|J| > 1$. Otherwise, $J = \bigcup_{i=0}^{\infty} x_iS = \{x_i\}$ and so $x_ix_i = x_i$. Thus x_i is an idempotent which is a contradiction. Also J is a proper ideal, otherwise $J = S$ and so $1 = x_is$ for some $i \geq 0$ and $s \in S$. Then $1 = x_is = x_{i+1}x_is = x_{i+1}$ and so x_{i+1} is an idempotent which is also a contradiction. Thus J is a proper right ideal with $|J| > 1$ and so by assumption there exists $j \in J \setminus Jj$. But for every $j \in J$ there exists $i \geq 0$ and $s \in S$ such that $j = x_is$. Then $j = x_is = x_{i+1}x_is = x_{i+1}j \in Jj$ which by assumption is a contradiction. ■

Lemma 3.4.2. *Let S be a left PSF monoid. If for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$, then for every $x \in S$, either x is right cancellative or there exists $e \in E(S) \setminus \{1\}$ such that $x = ex$.*

Proof. Let $x \in S$. If x is not right cancellative, then by Lemma 3.3.5, there exists an infinite sequence $(x = x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$ and x_i not right cancellative, $i = 0, 1, \ldots$. Thus by assumption there exists $n \in \mathbb{N} \cup \{0\}$ such that $x_n = e \in E(S)$. Since x_n is not right cancellative, then $e \neq 1$. Also $x_{n-1} = x_nx_{n-1}$ implies that $x_{n-1} = ex_{n-1}$ and so we have

$$ex = ex_0 = ex_1x_0 = ex_2x_1x_0 = \ldots = ex_{n-1}x_{n-2}\ldots x_1x_0 = x_{n-1}x_{n-2}\ldots x_1x_0 =$$

$$x_{n-2}x_{n-3}\ldots x_1x_0 = \ldots = x_1x_0 = x_0 = x.$$

■

Lemma 3.4.3. *Let S be a monoid. Then,*

1. *S is left PSF;*

2. *for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$;*

3. *all flat cyclic right S-acts satisfy condition (P);*

if and only if S is right subelementary with the right nil elements all right zero.

Proof. Let S be a left PSF monoid with the properties mentioned above. Then by Lemma 3.4.2, every $x \in S$ is either right cancellative or there exists $e \in E(S) \setminus \{1\}$ such that $x = ex$. By Lemma 2.2.8, e is right zero and so in this case x is right zero.

If S is right subelementary where the right nil elements are right zero, then by Theorem 2.3.22, S is left PSF and all flat cyclic right S-acts satisfy condition (P). Also for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1 \ldots$, either x_i is right zero and so it is an idempotent or x_i is right cancellative and then $x_i = x_{i+1}x_i$ implies that $x_{i+1} = 1$ and so x_{i+1} is an idempotent. ■

Corollary 3.4.4. *Let S be a monoid. Then,*

1. *S is left PSF;*

2. *for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$;*

3. *all weakly flat cyclic right S-acts satisfy condition (P);*

if and only if S is right subelementary with the right nil elements all right zero.

Proof. Let S be a left PSF monoid with properties mentioned above. Since all weakly flat cyclic right S-acts satisfy condition (P), then all flat cyclic right S-acts satisfy condition (P) and so by Lemma 3.4.3, S is right subelementary with right nil elements all right zero.

If S is right subelementary with right nil elements all right zero, then by Theorem 2.3.22, S is left PSF and all weakly flat cyclic right S-acts satisfy condition (P). Also by Lemma 3.4.3, S satisfies the other property mentioned. ■

The following theorem generalizes Theorem 2.3.22.

Theorem 3.4.5. *Let S be a monoid. Then the following are equivalent:*

1. *S is right subelementary with the right nil elements all right zero.*

2. *S is right subelementary and left PSF.*

3. *S is left PSF, for any infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$ and every weakly flat cyclic right S-act satisfies condition (P).*

4. *S is left PSF, for any infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$ and every flat cyclic right S-act satisfies condition (P).*

5. *S is left PSF, for any infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1, \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$ and every $e \in E(S) \setminus \{1\}$ is right zero.*

6. *S is left PP and every weakly flat cyclic right S-act satisfies condition (P).*

7. *S is left PP and every flat cyclic right S-act satisfies condition (P).*

Proof. $(1), (2)$ are equivalent by Corollary 3.3.4. $(1), (6)$, and (7) are equivalent by Theorem 2.3.22. $(1), (3)$, and (4) are equivalent by Lemma 3.4.3, and Corollary 3.4.4. By Lemma 2.2.8, $(4) \Rightarrow (5)$. Finally, by Lemma 3.4.2, $(5) \Rightarrow (1)$. ■

Now by using previous lemmas and also the following lemma we can deduce a characterization of right reversible left PSF monoids by condition (P) of flat cyclic right acts.

Proposition 3.4.6 [9]. *Let S be a monoid and let J be a right ideal of S. Then*

1. *S/J has property (P) if and only if $J = S$ and S is right reversible, or $|J| = 1$.*

2. *S/J is flat (or equivalently, weakly flat) if and only if S is right reversible and $j \in Jj$ for all $j \in J$.*

Lemma 3.4.7. *Let S be a right reversible monoid. If S is left PSF and all flat cyclic right S-acts satisfy condition (P), then for every infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 0, 1 \ldots$, there exists $n \in \mathbb{N} \cup \{0\}$ with $x_n \in E(S)$.*

Proof. Suppose that there exists an infinite sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, but $x_i \notin E(S)$, $i = 0, 1, \ldots$, and suppose that $J = \bigcup_{i=0}^{\infty} x_i S$. Then by the same argument as in the proof of Lemma 3.4.1, it can be seen that J is a proper right ideal with $|J| > 1$ and also $j \in Jj$ for every $j \in J$. Consequently, by Proposition 3.4.6 (2), S/J is flat and so by assumption S/J satisfies condition (P). Since J is a proper right ideal, then $J \neq S$ and so by Proposition 3.4.6 (1), $|J| = 1$ which is a contradiction. ■

Corollary 3.4.8. *Let S be a right reversible monoid. Then S is left PSF and all flat cyclic right S-acts satisfy condition (P) if and only if $S = C$ or $S = C \cup \{0\}$ where C is right cancellative.*

Proof. By Lemma 3.4.7, and Lemma 3.4.3, $S = C \dot{\cup} Z$ where C is right cancellative and Z is right zero. On the other hand S has at most two idempotents and so either $S = C$ or $S = C \cup \{0\}$ as required.

The converse is obvious. ■

The following, extends part of [40. Theorem 4.3] to the non right reversible case.

Theorem 3.4.9. *Let S be a left PSF monoid. Then for every proper right ideal J of S with $|J| > 1$ there exists $j \in J \setminus Jj$ if and only if $S = C$ or $S = C \cup \{0\}$ where C is right cancellative.*

Suppose that $S = C$ or $S = C \cup \{0\}$ where C is right cancellative and suppose that J is a proper right ideal of S with $|J| > 1$. Then $1 \notin J$ and so there exists $x \in J$ such that $x \neq 1$ and $x \neq 0$. If for every $j \in J$, $j \in Jj$, then $x \in J_x$ and so there exists $y \in J$ such that $x = yx$. Since x is right cancellative, then $y = 1$ and so $1 \in J$ which is a contradiction.

Suppose that for every proper right ideal J of S with $|J| > 1$ there exists $j \in J \setminus Jj$ and let $0 \neq x_0 \in S$. Then we claim that x_0 is right cancellative. Otherwise, by Lemma 3.3.5, there exists an infinite sequence $(x_0, x_1, \ldots)$, with $x_i = x_{i+1}x_i$, and x_i not right cancellative, $i = 0, 1 \ldots$.

Now let $J = \bigcup_{i=0}^{\infty} x_i S$. Then J is a right ideal of S. Also $|J| > 1$, otherwise $J = x_0 S = \{x_0\}$ and so x_0 is left zero. Suppose that I is the set of all left zero elements of S, then I is a right ideal of S and every $e \in I$ is an idempotent. Consequently, for every $e \in I$, $e = ee \in Ie$ and so by assumption $|I| = 1$. Hence, x_0 is the only left zero element of S. Now let $x, y \in S$. Then $(xx_0)y = x(x_0 y) = xx_0$. Thus xx_0 is left zero and so by uniqueness of x_0, $xx_0 = x_0$. Therefore x_0 is right zero and so a zero which is a contradiction. Hence, $|J| > 1$. J is also a proper ideal, otherwise $1 \in J$ and so there exist $x_i \in J, s \in S$ such that $1 = x_i s$. Thus x_i is right invertible and so it is right cancellative which is a contradiction.

Now by assumption there exists $j \in J \setminus Jj$. But $j \in J$ implies that there exist $x_i \in J, s \in S$ such that $j = x_i s$. Since $x_i = x_{i+1} x_i$, then $j = x_{i+1} x_i s = x_{i+1} j \in Jj$, and so a contradiction. Thus every element different from zero is right cancellative as required. ∎

From Corollary 2.3.42, Corollary 3.4.8, and Theorem 3.4.9, we have

Theorem 3.4.10. *If S is a right reversible monoid, then the following statements are equivalent:*

1. *$S = C$ or $S = C \cup \{0\}$ where C is a right cancellative.*
2. *S is left PSF and all weakly flat cyclic right S-acts satisfy condition (P).*
3. *S is left PSF and all flat cyclic right S-acts satisfy condition (P).*
4. *S is left PSF and for every proper right ideal J of S with $|J| > 1$ there exists $j \in J \setminus Jj$.*
5. *S is left PP and all weakly flat cyclic right S-acts satisfy condition (P).*
6. *S is left PP and all flat cyclic right S-acts satisfy condition (P).*

In the following we give a characterization of left PSF periodic monoids by condition (P) of (weakly) flat cyclic right acts and use this to characterize left PSF monoids for which all (weakly) flat cyclic right acts are projective or strongly flat.

Lemma 3.4.11. *Let S be a periodic monoid such that every principal left ideal of S is strongly flat. Then all flat cyclic right S-acts satisfy condition (P) if and only if $S = G \,\dot{\cup}\, Z$ where G is a group and Z is the set of all right zero elements of S.*

Proof. If all flat cyclic right S-acts satisfy condition (P), then by Theorem 2.2.15, $S = G \dot{\cup} N$ where G is a group and N is the set of all right nil elements of S. Thus S is right subelementary and so by Theorem 3.2.23, every element in the right nil part is right zero.

The converse is true by Theorem 2.3.22. ∎

Since every left PP monoid is left PSF, then we have

Corollary 3.4.12. *Let S be a periodic monoid which is also left PP. Then all flat cyclic right S-acts satisfy condition (P) if and only if $S = G \dot{\cup} Z$ where G is a group and Z is the set of all right zero elements of S.*

Theorem 3.4.13. *Let S be a periodic monoid. Then the following statements are equivalent:*

(1) $S = G \dot{\cup} Z$ where G is a group and Z is the set of all right zero elements of S.

(2) S is left PSF and every weakly flat cyclic right S-act satisfies condition (P).

(3) S is left PSF and every flat cyclic right S-act satisfies condition (P).

Proof. The implication $(2) \Rightarrow (3)$ is obvious and by Lemma 3.4.11, $(3) \Rightarrow (1)$. Finally, by Theorem 2.2.13, and Theorem 3.2.23, $(1) \Rightarrow (2)$. ∎

Liu in [40, Proposition 4.2] showed that for a right reversible monoid S, if all flat cyclic right S-acts satisfy condition (P), then for every proper right ideal J of S with $|J| > 1$ there exists $j \in J \setminus Jj$. The following example shows that S is right reversible is a necessary condition in Liu's Proposition 4.2. Also from this example it can be seen that Liu's Proposition 4.2 cannot be extended to the left PSF monoids.

Example 3.4.14. Let $S = G \dot{\cup} Z$ be a monoid where G is a group and Z is the set of all right zero elements of S with $|Z| > 1$. Let $x, y \in Z$ such that $x \neq y$. Then Sx, Sy are principal left ideals and $\{x\} = Sx$, $\{y\} = Sy$. Thus $Sx \cap Sy = \emptyset$ and so S is not right reversible. Let J be a proper right ideal of S with $|J| > 1$. Then $J \subseteq Z$, otherwise if there exists $x \in J$ such that $x \in G$, then $1 = xx^{-1} \in J$ and so $J = S$ which is a contradiction. By Theorem 3.4.13, S is left PSF and all flat cyclic right S-acts satisfy condition (P), but for every $e \in J$, $e = ee \in Je$.

Now by using Theorem 3.4.13, we give a characterization of left PSF monoids for which all (weakly) flat cyclic right acts are strongly flat or projective and use this to give a characterization of right reversible left PSF monoids for which all (weakly) flat cyclic right acts are strongly flat or projective .

Lemma 3.4.15. *Let S be a left PSF monoid. Then all flat cyclic right S-acts are strongly flat if and only if S is right zero.*

Proof. Suppose that all flat cyclic right S-acts are strongly flat. Then all cyclic right S-acts which satisfy condition (P) are strongly flat and so by Lemma 2.2.27, S is aperiodic. Thus every element of S is of finite order and so S is periodic. Consequently, by Theorem 3.4.13, $S = G \dot{\cup} Z$. If $x \in G$, then there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. Then by cancelling x^n, $x = 1$ and so S is right zero as required.

If S is right zero, then by Theorem 2.3.28, all flat cyclic right S-acts are strongly flat. ■

Corollary 3.4.16. *Let S be a left PSF monoid. Then all flat cyclic right S-acts are projective if and only if S is right zero.*

Proof. Suppose that all flat cyclic right S-acts are projective. Then all flat cyclic right S-acts are strongly flat and so by Lemma 3.4.15, S is right zero.

If S is right zero, then by Theorem 2.3.28, all flat cyclic right S-acts are projective.

■

From Lemma 3.4.15, Corollary 3.4.16, and Theorem 2.3.28, we have

Theorem 3.4.17. *Let S be a left PSF monoid. Then the following statements are equivalent:*

(1) S is right zero.

(2) All weakly flat cyclic right S-acts are strongly flat.

(3) All weakly flat cyclic right S-acts are projective.

(4) All flat cyclic right S-acts are strongly flat.

(5) All flat cyclic right S-acts are projective.

Corollary 3.4.18. *Let S be a right reversible left PSF monoid. Then all flat cyclic right S-acts are strongly flat if and only if either $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Let S be a right reversible left PSF monoid and suppose that all flat cyclic right S-acts are strongly flat. Then by Lemma 3.4.15, S is right zero. On the other hand by Theorem 2.3.39, S has at most two idempotents. Consequently, either $S = \{1\}$ or $S = \{0, 1\}$.

The converse is obvious. ■

Since for $S = \{1\}$ or $S = \{0, 1\}$ all cyclic right S-acts are projective, then from Corollary 3.4.18, we have

Corollary 3.4.19. *Let S be a right reversible left PSF monoid. Then all flat cyclic right S-acts are strongly flat if and only if all cyclic right S-acts are projective.*

Bulman-Fleming in [1] showed that for a monoid S if all flat right S-acts satisfy condition (P), then $|E(S)| = 1$. He also posed the following question.

Problem. Is the condition $|E(S)| = 1$ also sufficient for every flat right S-act to satisfy condition (P) ?

By the following example [40] it can be seen that the answer is negative. Then we show it is positive for some classes of monoids.

Example 3.4.20. For $(i = 1, 2 \ldots)$, consider partial mappings f_i over $\mathbb{R}$,

$$f_i(x) = \begin{cases} i - \frac{1}{2} & i < x \\ i - 1 + \frac{1}{2}(x - i + 1) & i - 1 < x \leq i \\ x & x \leq i - 1. \end{cases}$$

Then it is easy to see that f_i, $i = 1, 2, \ldots$, have the property $f_i f_{i+1} = f_{i+1} f_i = f_i$, using the usual composition of partial mappings. Thus for $i > j$, $f_i f_j = f_j f_i = f_j$. Consider the monoid

$$S = \{ f_i^n \mid i = 1, 2, \ldots,\ n = 1, 2, \ldots \} \cup \{1\}.$$

Where, $1 : \mathbb{R} \to \mathbb{R}$ is the identity map. It is easy to see that S is commutative and that for $x \in (i-1, i)$,

$$f_i^n(x) = i - 1 + \frac{1}{2^n}(x - i + 1).$$

Thus f_i^n is not idempotent, $i = 1, 2 \ldots$, $n = 1, 2 \ldots$, and so $|E(S)| = 1$. Obviously there exists an infinite sequence $(f_1, f_2, \ldots)$ with $f_i = f_{i+1} f_i$, $i = 1, 2 \ldots$ in S. Since $f_i \neq 1$ for every i, then by Proposition 2.3.56, there exists a proper right ideal J of S such that $j \in Jj$ for every $j \in J$. Consequently, by Corollary 2.3.55, there exists a flat right S-act which does not satisfy condition (P)

Theorem 3.4.21. *Let S be a right reversible left PSF monoid such that all flat cyclic right S-acts satisfy condition (P). Then all flat right S-acts satisfy condition (P) if and only if $|E(S)| = 1$.*

Proof. Suppose that S is a right reversible left PSF monoid such that all flat cyclic right S-acts satisfy condition (P) and let $|E(S)| = 1$. Then by Theorem 3.4.10, either $S = C$ or $S = C \cup \{0\}$ where C is right cancellative. Since $|E(S)| = 1$, then $S = C$ and so by Theorem 3.3.12, all flat right S-acts satisfy condition (P).

If all flat right S-acts satisfy condition (P), then by Theorem 2.3.45, $|E(S)| = 1$.■

From Theorem 3.3.14, and Theorem 3.4.21, we have

Corollary 3.4.22. *Let S be a right reversible left PSF monoid. Then all weakly flat right S-acts satisfy condition (P) if and only if all flat cyclic right S-acts satisfy condition (P) and $|E(S)| = 1$.*

Here is another class of monoids S for which $|E(S)| = 1$ implies that all flat right S-acts satisfy condition (P).

Theorem 3.4.23. *Let S be a left PP monoid. Then all flat right S-acts satisfy condition (P), if and only if $|E(S)| = 1$.*

Proof. Let S be a left PP monoid with $|E(S)| = 1$ and let $x \in S$. Then there exists $e^2 = e \in S$ such that $ex = x$ and $ax = bx$ implies that $ae = be$. Since $|E(S)| = 1$, then x is right cancellative and so by Theorem 3.3.12, all flat right S-acts satisfy condition (P).

If all flat right S-acts satisfy condition (P), then by Theorem 2.3.45, $|E(S)| = 1$.■

We also show that for an eventually regular monoid S, $|E(S)| = 1$ is sufficient that all flat right S-acts satisfy condition (P).

Theorem 3.4.24. *Let S be an eventually regular monoid. Then all flat right S-acts satisfy condition (P) if and only if $|E(S)| = 1$.*

Proof. Let S be an eventually regular monoid with $|E(S)| = 1$ and let $x \in S$. Then there exists $n \in \mathbb{N}$ such that x^n is regular and so there exists $x' \in S$ such that $x^n x' x^n = x^n$. Since $x^n x'$ is an idempotent element, then $x^n x' = 1$. Thus x is right invertible and so it is right cancellative. Consequently, S is right cancellative and so by Theorem 3.3.15, all flat right S-acts satisfy condition (P).

The converse is true by Theorem 2.3.45. ∎

Chapter 4

(P_E) Conditions

4.1. Introduction

In this chapter we first of all introduce conditions (P_{1E}), (P'_E) and (P_E). Then by considering the relation between these conditions and flatness, we show that for a given act $(P) \Rightarrow (P_E) \Rightarrow (P'_E) \Rightarrow$ *weakly flat*, but the converses are not true. Also we show that condition (P_E) implies condition (P_{1E}) and again the converse is not true.

Then we characterize certain types of monoids which have the property that all their (cyclic) acts satisfy one of these conditions and also monoids for which all these distinct properties coincide. We consider various corollaries of these results.

Finally, we present a characterization of some classes of monoids by condition (E) of (weakly) flat right acts and also monoids for which all right acts having (E) satisfy conditions (P_E), and (P'_E).

4.2. Condition (P_{1E})

In this section at first we introduce condition P_{1E}. Then we give a necessary condition for cyclic acts of a monoid to satisfy this condition. For monoids with central idempotents, we characterize cyclic acts with condition P_{1E}. The result is an analogue of those in Lemma 1.54. Also a necessary condition for monoids with central idempotents such that all their acts having P_{1E} be strongly flat are given next. A characterization of left PP monoids with central idempotents such that all their acts having P_{1E} satisfy condition (P) is given afterwards. Then we give a classification by condition (P_{1E}) of cyclic right acts of an inverse monoid S with the property that for every $a \in S$, $aa^{-1} = a^{-1}a$. Also for inverse monoids of this type we show weak flatness of cyclic acts implies condition (P_{1E}). Finally, we give a characterization of eventually regular monoids with central idempotents by condition (P) of cyclic right acts having condition (P_{1E}). There are also some corollaries that will arise.

Definition 4.2.1. Let S be a monoid. A right S-act A satisfies condition $\underline{(P_{1E})}$ if whenever $a, a' \in A$, $u, v \in S$ and $au = a'v$, there exist $a'' \in A$, $s, t, e^2 = e \in S$ such that $ae = a''se$, $a'e = a''te$, and $su = tv$.

Remark. It is obvious that condition (P) implies condition (P_{1E}), but the converse is not true. For example if $S = \{0, 1\}$ and $A = \{x, y, z \mid x0 = y0 = z0 = z,\ 1x = x,\ 1y = y\}$, then A satisfies condition (P_{1E}). Since A is not a coproduct of cyclic S-acts, then by Lemma 1.53, A does not satisfy condition (P). If S is a right cancellative monoid or $|E(S)| = 1$, then conditions (P_{1E}) and (P) are equivalent.

Now we give a characterization of monoids with central idempotents by condition (P_{1E}) of cyclic acts.

Lemma 4.2.2. *Let S be a monoid and ρ a right congruence on S. If S/ρ satisfies condition (P_{1E}), then for all $u, v \in S$ with $u\ \rho\ v$ there exist $s, t, e^2 = e \in S$ such that $su = tv$ and $se\ \rho\ e\ \rho\ te$.*

Proof. Let S/ρ satisfies condition (P_{1E}) and let $u\ \rho\ v$ for $u, v \in S$. Then $u\rho = v\rho$ or $(1\rho)u = (1\rho)v$. If $1\rho = x$, then $xu = xv$ and so there exist $a'' \in S/\rho$, $s_1, t_1, e^2 = e \in S$ such that $xe = a''s_1e$, $xe = a''t_1e$ and $s_1u = t_1v$. Since $a'' \in S/\rho$, then there exists $s' \in S$ such that $s'\rho = a''$ and so $xe = a''s_1e$ implies that

$$e\rho = (1\rho)e = (s'\rho)s_1e = (s's_1e)\rho \text{ or } e\ \rho\ s's_1e.$$

Similarly, $xe = a''t_1e$ implies that $e \ \rho \ s't_1e$. If $s's_1 = s$ and $s't_1 = t$, then $se \ \rho \ e \ \rho \ te$. On the other hand $s_1u = t_1v$ implies that $s's_1u = s't_1v$ or $su = tv$ and so the result follows. ■

<u>**Lemma 4.2.3.**</u> *Let S be a monoid with central idempotents and let ρ be a right congruence on S. If for all $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $se \ \rho \ e \ \rho \ te$ and $su = tv$, then S/ρ satisfies condition (P_{1E}).*

Proof. Suppose that for every $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $su = tv$ and $se \ \rho \ e \ \rho \ te$. Let $Au = A'v$ for $A, A' \in S/\rho$ and $u, v \in S$. Then there exist $a, a' \in S$ such that, $A = a\rho$, $A' = a'\rho$ and so we have

$$Au = A'v \Rightarrow (a\rho)u = (a'\rho)v \Rightarrow (au)\rho = (a'v)\rho \Rightarrow au \ \rho \ a'v.$$

Then by assumption there exist $s_1, t_1, e^2 = e \in S$ such that $s_1au = t_1a'v$ and $s_1e \ \rho \ e \ \rho \ t_1e$. If $s_1a = s$ and $t_1a' = t$, then $su = tv$. Since $s_1e \ \rho \ e$, then $s_1ea \ \rho \ ea$. But idempotents are central and so $s_1ae \ \rho \ ae$ or $(1 \ \rho)s_1ae = (a\rho)e$. If $1\rho = A''$, then $A''(s_1a)e = Ae$ or $A''se = Ae$. Similarly, $A''te = A'e$. and so S/ρ satisfies condition (P_{1E}) as required. ■

From Lemma 4.2.2, and Lemma 4.2.3, we have

<u>**Theorem 4.2.4.**</u> *Let S be a monoid with central idempotents and let ρ be a right congruence on S. Then S/ρ satisfies condition (P_{1E}) if and only if for all $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $se \ \rho \ e \ \rho \ te$ and $su = tv$.*

<u>**Theorem 4.2.5.**</u> *Let S be a monoid with central idempotents and let ρ be a right congruence on S. Then S/ρ satisfies condition (P_{1E}) if and only if for all $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $s(\rho \vee \Delta e)1(\rho \vee \Delta e)t$ and $su = tv$.*

Proof. Suppose that S/ρ satisfies condition (P_{1E}) and let $u \ \rho \ v$. Then by Theorem 4.2.4, there exists $s_1, t_1, e^2 = e \in S$ such that $s_1e \ \rho \ e \ \rho \ t_1e$ and $s_1u = t_1v$. Then $s_1e \ \rho \ e(\Delta e)1$, also $t_1e \ \rho \ e(\Delta e)1$. If $s_1e = s$, $t_1e = t$, then $s \ \rho \ e(\Delta e)1$, or $s(\rho \vee \Delta e)1$. Similarly, $t(\rho \vee \Delta e)1$. Also $s_1u = t_1v$ implies that $e(s_1u) = e(t_1v)$ or $(es_1)u = (et_1)v$. But idempotents are central and so $(s_1e)u = (t_1e)v$. Consequently, $su = tv$.

Now suppose that for $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $s(\rho \vee \Delta e)1(\rho \vee \Delta e)t$ and $su = tv$. Since $s(\rho \vee \Delta e)1$, then by Lemma 1.49, there exist $s_1, s_2, \ldots, s_{2n-1} \in S$ such that

$$s \ \rho \ s_1(\Delta e)s_2 \ \rho \ s_3 \ldots s_{2n-1}(\Delta e)1.$$

Then we have

$$se \;\rho\; s_1e,\; s_1e = s_2e,\; s_2e \;\rho\; s_3e, \ldots, s_{2n-1}e = e,$$

which implies that $se \;\rho\; e$. Similarly, $t(\rho \vee \Delta e)1$ implies that $te \;\rho\; e$. Since $su = tv$, then by Theorem 4.2.4, S/ρ satisfies condition (P_{1E}). ■

From Theorem 4.2.4, and Theorem 4.2.5, we have

Corollary 4.2.6. *Let S be a monoid with central idempotents. Then for all $u, v \in S$ with $u \;\rho\; v$ there exist $s, t, e^2 = e \in S$ such that $se \;\rho\; e \;\rho\; te$ and $su = tv$ if and only if there exist $s, t, e^2 = e \in S$ such that $s(\rho \vee \Delta e)1(\rho \vee \Delta e)t$ and $su = tv$.*

If x and y are elements of a monoid S we shall denote by $\rho(x, y)$ the smallest right congruence on S which identifies these two elements. It is well known that, for $u, v \in S$, $(u, v) \in \rho(x, y)$ if and only if either $u = v$ or else there exist

$$s_1, s_2, \ldots, s_n, z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_n \in S$$

such that

$$\begin{aligned} u &= z_1 s_1 \\ w_1 s_1 &= z_2 s_2 \\ &\vdots \\ w_n s_n &= v \end{aligned} \qquad \text{and } \{z_i, w_i\} = \{x, y\} \text{ for each i.}$$

Lemma 4.2.7. *Let S be a monoid with central idempotents. Let $x, e^2 = e \in S$ be such that $ex = x$. Then the right S-act $S/\rho(x, e)$ satisfies condition (P_{1E}).*

Proof. By Theorem 4.2.4, it is sufficient to show that for $u, v \in S$ with $u \;\rho\; v$ there exist $s, t, f^2 = f \in S$ such that $sf \;\rho\; f \;\rho\; tf$ and $su = tv$ (ρ denotes $\rho(x, e)$). Let $u \;\rho\; v$. If $u = v$, then $s = t = 1$ and $f = 1$. Otherwise, there exist

$$s_1, s_2, \ldots, s_n, z_1, z_2, \ldots, z_n, w_1, w_2, \ldots, w_n \in S,$$

such that

$$\begin{aligned} u &= z_1 s_1 \\ w_1 s_1 &= z_2 s_2 \\ &\vdots \\ w_n s_n &= v \end{aligned} \qquad \text{and } \{z_i, w_i\} = \{x, e\}. \tag{1}$$

From (1) it can be seen that there exist $m, m' \in \mathbb{N}$ such that $x^m u = x^{m'} v$. Now we show by induction that for every $k \in \mathbb{N}$, $x^k e \ \rho \ e$. Since $x \ \rho \ e$ and $xe = x$, then $xe \ \rho \ e$. Thus for $k = 1$ it is satisfied. Suppose that $x^k e \ \rho \ e$, $k \in \mathbb{N}$. Then $x^k ex \ \rho \ ex$. Since $ex = x$, then $x^{k+1} \ \rho \ x$. But $x \ \rho \ e$ and so $x^{k+1} \ \rho \ e$. Consequently, $x^{k+1} e \ \rho \ e$. Therefore, $x^m e \ \rho \ e$ and $x^{m'} e \ \rho \ e$. If $x^m = s$, $x^{m'} = t$ and $f = e$, then $se \ \rho \ e \ \rho \ te$ and $su = tv$.

■

<u>**Notice :**</u> From Lemma 4.2.7, it can be seen that if S is a left PP monoid with central idempotents, then for every $x \in S$ there exists $e^2 = e \in S$ such that $S/\rho(x, e)$ satisfies condition $(P1_E)$.

<u>**Lemma 4.2.8.**</u> *Let S be a monoid with central idempotents. If $e, f \in E(S)$ are such that $ef = e$, then the right S-act $S/\rho(e, f)$ satisfies condition $(P1_E)$.*

Proof. Since idempotents are central, then $fe = e$. Consequently, by Lemma 4.2.7, $S/\rho(e, f)$ satisfies condition $(P1_E)$ as required. ■

<u>**Lemma 4.2.9.**</u> *Let S be a monoid with central idempotents. If all cyclic right S-acts which satisfy condition $(P1_E)$ satisfy condition (P), then every $e \in E(S) \setminus \{1\}$ is zero.*

Proof. Let $e \in E(S) \setminus \{1\}$ and let $x \in S$. Since $e(ex) = ex$, then by Lemma 4.2.7, $S/\rho(ex, e)$ satisfies condition $(P1_E)$ and so by assumption it satisfies condition (P). Thus by Lemma 1.54 (4), $ex \ \rho \ e$ implies that there exist $s, t \in S$ such that $sex = te$ and $s \ \rho \ 1 \ \rho \ t$. Since $s \ \rho \ 1$, then either $s = 1$ or else there exist

$$s_1, \ldots, s_n, \ z_1, \ldots, z_n, \ w_1, \ldots, w_n \in S,$$

such that

$$\begin{aligned} s &= z_1 s_1 \\ w_1 s_1 &= z_2 s_2 \\ &\vdots \\ w_n s_n &= 1 \end{aligned} \qquad \text{and } \{z_i, w_i\} = \{ex, e\}.$$

We claim that $s = 1$. Otherwise, since either $w_n = e$ or $w_n = ex$, then it follows that either $1 = w_n s_n = es_n$ or $1 = exs_n$ and so in both cases, $1 \in eS$. Thus there exists $s' \in S$ such that $1 = es'$ and so

$$e = e1 = e(es') = e^2 s' = es' = 1,$$

which is a contradiction. Similarly, $t = 1$ and so for every $x \in S$, $ex = e$. But idempotents are central and so $xe = ex = e$. Consequently, every $e \in E(S) \setminus \{1\}$ is zero as required. ■

Theorem 4.2.10. *Let S be a monoid with central idempotents. If all cyclic right S-acts which satisfy condition (P_{1E}) are strongly flat, then S is nil.*

Proof. Since every cyclic right S-act which satisfies condition (P_{1E}) is strongly flat, then every cyclic right S-act which satisfies condition (P) is also strongly flat. Thus by Lemma 2.2.27, S is aperiodic and so for every $x \in S$ there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. Now we show by induction that for every $k \in \mathbb{N}$ $x^{n+k} = x^n$. Since $x^{n+1} = x^n$, then for $k = 1$ it is satisfied. Suppose that $x^{n+k} = x^n$. Then

$$x^{n+k+1} = x^{n+k}x = x^n x = x^{n+1} = x^n.$$

Thus $x^{n+n} = x^n$ or $(x^n)^2 = x^n$ and so x^n is idempotent. Now if $x \neq 1$, then $x^n \neq 1$. Otherwise, $x^{n+1} = x^n$ implies that $x = 1$ which is a contradiction. Hence, $x^n \in E(S) \setminus \{1\}$ and so by Lemma 4.2.9, x^n is zero or x is nil as required.

■

By Theorem 2.3.28, we saw that for right nil monoids, all weakly flat cyclic acts are projective, thus from Theorem 4.2.10, it can be deduced that for monoids with central idempotents, If all cyclic right acts having (P_{1E}) are strongly flat, then all weakly flat cyclic right acts are projective and so all (weakly) flat cyclic right acts are strongly flat.

That (P_{1E}) is a non trivial property of acts follows from:

Theorem 4.2.11. *Let the monoid S be a semilattice. Then all right S-acts satisfy condition (P_{1E}).*

Proof. Let A be a right S-act and let $au = a'v$ for $a, a' \in A$, $u, v \in S$. Since $u^2 = u$, then $au = a'v$ implies that $au = au^2 = a'vu$. Also $a'u = a'u^2 = a'uu$. Since S is a semilattice, then $vu = uv$ and so A satisfies (P_{1E}). ■

Now we show for left PP monoids with central idempotents weak flatness of cyclic acts implies condition (P_{1E}).

Lemma 4.2.12. *Let S be a left PP monoid with $E(S)$ a semilattice. Let ρ be a right congruence on S. If S/ρ is weakly flat, then for every $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e$ such that $su = tv$ and $s(\rho \ \vee \ \Delta e)1(\rho \ \vee \ \Delta e)t$.*

Proof. Suppose that $u \ \rho \ v$. Since S/ρ is weakly flat, then by Lemma 1.54 (6), there exist $s, t \in S$ such that $s(\rho \vee \Delta u)1(\rho \vee \Delta v)t$ and $su = tv$. Since $s(\rho \ \vee \ \Delta u)1$, then by Lemma 1.49, there exist $r_i, s_i \in S, \ i = 1, 2, \ldots, 2n$ such that

$$s \ \rho \ s_1(\Delta u)r_1 \ \rho \ s_2 \ldots r_{2n-1} \ \rho \ s_{2n}(\Delta u)r_{2n} = 1.$$

Also $t(\rho \ \vee \ \Delta v)1$ implies that there exist s'_i, r'_i, $i = 1, 2, \ldots, 2m$ such that

$$t \ \rho \ s'_1(\Delta v)r'_1 \ \rho \ s'_2 \ldots r'_{2m-1} \ \rho \ s'_{2m}(\Delta v)r'_{2m} = 1.$$

Since the above sequences are not of minimal length so we can extend the length of these sequences. Thus we can suppose that the above sequences are of the same length $2n$. Then $s_i(\Delta u)r_i$, implies that $s_i u = r_i u$, $i = 1, 2, \ldots, 2n$. Since S is left PP, then there exists $e_1^2 = e_1 \in S$ such that $e_1 u = u$ and $s_i u = r_i u$ implies that $s_i e_1 = r_i e_1$ or $s_i(\Delta e_1)r_i$, $i = 1, 2, \ldots, 2n$. By the same argument there exists $e_2^2 = e_2 \in S$ such that $e_2 v = v$ and $s'_i v = r'_i v$ implies that $s'_i e_2 = r'_i e_2$ or $s'_i(\Delta e_2)r'_i$, $i = 1, 2, \ldots, 2n$. Since $s_i e_1 = r_i e_1$, $i = 1, 2, \ldots, 2n$, then $s_i e_1 e_2 = r_i e_1 e_2$, $i = 1, 2, \ldots, 2n$. Also $s'_i e_2 e_1 = r'_i e_2 e_1$, $i = 1, 2, \ldots, 2n$. Since idempotents commute, then $e_1 e_2 = e_2 e_1$. If $e = e_1 e_2 = e_2 e_1$, then $s_i e = r_i e$, $s'_i e = r'_i e$, $i = 1, 2, \ldots, 2n$. Consequently, $s_i(\Delta e)r_i$ and $s'_i(\Delta e)r'_i$ $i = 1, 2, \ldots, 2n$. Thus

$$s \ \rho \ s_1, \ s_1(\Delta e)r_1, \ r_1 \ \rho \ s_2, \ s_2(\Delta e)r_2, \ldots, r_{2n-1} \ \rho \ s_{2n}, \ s_{2n}(\Delta e)1,$$

and

$$t \ \rho \ s'_1, \ s'_1(\Delta e)r'_1, \ r'_1 \ \rho \ s'_2, \ s'_2(\Delta e)r'_2, \ldots, r'_{2n-1} \ \rho \ s'_{2n}, \ s'_{2n}(\Delta e)1.$$

Hence by Lemma 1.49, $s(\rho \ \vee \ \Delta e)1$ and $t(\rho \ \vee \ \Delta e)1$ as required ■

Corollary 4.2.13. *Let S be a left PP monoid with central idempotents and let ρ be a right congruence on S. If S/ρ is weakly flat, then S/ρ satisfies condition (P_{1E}).*

Proof. Since idempotents are central, then by Lemma 4.2.12, for every $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $s(\rho \ \vee \ \Delta e)1(\rho \ \vee \ \Delta e)t$ and $su = tv$. Thus by Theorem 4.2.5, S/ρ satisfies condition (P_{1E}). ■

<u>**Corollary 4.2.14.**</u> *If S is a right cancellative monoid, then all weakly flat cyclic right S-act satisfies condition (P).*

Proof. If S is right cancellative, then it is left PP and also $|E(S)| = 1$. Thus S is a left PP monoid with central idempotents and so by Corollary 4.2.13, every weakly flat cyclic right S-act satisfies condition (P_{1E}). But in this case conditions (P) and (P_{1E}) are coincide and so every weakly flat cyclic right S-act satisfies condition (P) as required. ■

<u>**Lemma 4.2.15 [19].**</u> *If S is a Clifford semigroup, then it is regular and every idempotent is central.*

Since every regular monoid is left PP, then from Corollary 4.2.13, and Lemma 4.2.15, we have

<u>**Corollary 4.2.16.**</u> *Let S be a Clifford monoid, then all weakly flat cyclic right S-act satisfies condition (P_{1E}).*

Now from Corollary 4.2.13, we can characterize left PP monoids with central idempotents for which all cyclic right acts having (P_{1E}) satisfy condition (P).

<u>**Theorem 4.2.17.**</u> *Let S be a left PP monoid with central idempotents. Then all cyclic right S-acts having (P_{1E}) satisfy condition (P) if and only if $S = C \cup \{0\}$ where C is right cancellative.*

Proof. Suppose that all cyclic right S-acts having (P_{1E}) satisfy condition (P). Since by Corollary 4.2.13, every weakly flat cyclic right S-act satisfies condition (P_{1E}), then all weakly flat cyclic right S-act satisfy condition (P) and so by Theorem 2.3.22, $S = C \dot{\cup} Z$ is right subelementary with elements in the right nil part all right zero. But idempotents are central, thus every right zero element is a left zero and so a zero. Consequently, $S = C \cup \{0\}$.

If $S = C \cup \{0\}$, then by Corollary 2.3.64, all cyclic right S-acts satisfy condition (P) and so all cyclic right S-acts having (P_{1E}) satisfy condition (P) as required.

■

Now by considering inverse monoids S with the property that for every $a \in S$, $aa^{-1} = a^{-1}a$ we give a classification by condition (P_{1E}) of cyclic right S-acts. Also from Theorem 4.2.19 below it can be seen that for inverse monoids with the property mentioned above all weakly flat cyclic right acts satisfy condition (P_{1E}).

By Exercise 2 of [19, p. 125], if S is a regular semigroup such that for every $a \in S$ and every $a' \in V(a)$, $aa' = a'a$ then S is a Clifford semigroup and so by Lemma 4.2.15, idempotents of S are central. Thus by Theorem 4.2.5, we have

Theorem 4.2.18. *Let S be an inverse monoid such that for every $a \in S$, $aa^{-1} = a^{-1}a$. Let ρ be a right congruence on S. Then S/ρ satisfies condition (P_{1E}) if and only if for all $u, v \in S$ with $u \; \rho \; v$ there exist $s, t, e^2 = e \in S$ such that $su = tv$ and $s(\rho \vee \Delta e)1(\rho \vee \Delta e)t$.*

From Corollary 4.2.13, and comment before Theorem 4.2.18, we have

Theorem 4.2.19. *Let S be an inverse monoid such that for every $a \in S$, $aa^{-1} = a^{-1}a$. Then all weakly flat cyclic right S-acts satisfy condition (P_{1E}).*

In the following we give a characterization of eventually regular monoids with central idempotents by condition (P) of cyclic right acts having condition (P_{1E}).

Lemma 4.2.20. *Let $S = G \dot{\cup} N$ be a right elementary monoid. Then idempotents are central if and only if N is nil.*

Proof. Suppose that S is a right elementary monoid with central idempotents. Let $e \in N$ be right zero and let $x \in S$. Then $xe = e$. Since idempotents are central, then $xe = ex = e$ and so e is zero. Thus every element of N is nil as required.

Suppose that $S = G \dot{\cup} N$ is a right elementary monoid such that N is nil. Then $E(S) = \{0, 1\}$ and so every idempotent is central. ■

Theorem 4.2.21. *Let S be an eventually regular monoid with central idempotents. If all cyclic right S-acts having (P_{1E}) satisfy condition (P), then $S = G \dot{\cup} N$ is a right elementary monoid with elements in N all nil.*

Proof. Let S be an eventually regular monoid with central idempotents. If all cyclic right S-acts having (P_{1E}) satisfy condition (P), then by Lemma 4.2.9, every $e \in E(S) \setminus \{1\}$ is right zero. Thus by Theorem 2.3.23, $S = G \dot{\cup} N$ is a right elementary monoid. But by assumption idempotents are central and so by Lemma 4.2.20, N is nil as required. ■

From Theorem 2.3.25, and Theorem 4.2.21 we have the following corollary.

Corollary 4.2.22. *Let S be an eventually regular monoid with central idempotents. If all cyclic right S-acts having (P_{1E}) satisfy condition (P), then all (weakly) flat cyclic right S-acts satisfy condition (P).*

Lemma 4.2.23. *Let $S = G \dot{\cup} N$ be a right elementary monoid with elements in N all nil. Then all cyclic right S-acts satisfy condition (P_{1E}).*

Proof. Suppose that $S = G \dot{\cup} N$ is a right elementary monoid with elements in N all nil and let S/ρ be a cyclic right S-act for a right congruence ρ on S. We show that S/ρ satisfies condition (P_{1E}). Since by Lemma 4.2.20, idempotents are central, then it is sufficient to show that S/ρ satisfies Lemma 4.2.3.

Let $u \ \rho \ v$ with $u, v \in S$. If at least one of u or v belongs to G (for example let $u \in G$) then $vu^{-1} \ \rho \ 1$. If $s = vu^{-1}, t = 1$, and $e = 1$, then $se \ \rho \ e \ \rho \ te$ and $su = tv$. Suppose then that $u, v \in N$. Since N is nil, then S has a zero element. If $s = t = e = 0$, then $se \ \rho \ e \ \rho \ te$ and $su = tv$. ■

Notice that the above lemma shows also that condition (P_{1E}) is a non trivial property of acts.

Corollary 4.2.24. *Let S be an eventually regular monoid with central idempotents. Then all cyclic right S-acts having (P_{1E}) satisfy condition (P) if and only if $S = G$ or $S = G \cup \{0\}$ where G is a group.*

Proof. Suppose that all cyclic right S-acts having (P_{1E}) satisfy condition (P). Then by Theorem 4.2.21, $S = G \dot{\cup} N$ is a right elementary monoid with elements in N all nil. Consequently, by Lemma 4.2.23, all cyclic right S-acts satisfy condition (P_{1E}) and so by assumption all cyclic right S-acts satisfy condition (P). Hence, by Corollary 2.3.64, $S = G$ or $S = G \cup \{0\}$ where G is a group.

If $S = G$ or $S = G \cup \{0\}$, then by Corollary 2.3.64, all cyclic right S-acts satisfy condition (P) and so all cyclic right S-acts having (P_{1E}) satisfy condition (P) as required. ■

From Corollary 4.2.24, and Corollary 2.3.64, we have

Theorem 4.2.25. *Let S be an eventually regular monoid with central idempotents. Then the following statements are equivalent:*

(1) All cyclic right S-acts having (P_{1E}) satisfy condition (P).

(2) All cyclic right S-acts satisfy condition (P).

(3) $S = G$ or $S = G \cup \{0\}$ where G is a group.

Remark. By Lemma 4.2.9, for a monoid S with central idempotents if all cyclic right S-acts having (P_{1E}) satisfy condition (P), then every $e \in E(S) \setminus \{1\}$ is zero. Thus by Theorem 2.3.16, $S = G \dot{\cup} N \dot{\cup} F$ where G is a group, N is the set of all right nil elements of S and F is the set of all regular free elements of S.

4.3. Conditions (P'_E), (P_E)

In this section we first of all introduce new conditions (P_E) and (P'_E). Then we give necessary condition for cyclic right acts of a monoid to satisfy these conditions. A classification of some classes of monoids by conditions (P_E) and (P'_E) of cyclic right acts will be given next. By considering the relation between conditions (P_E), (P'_E), condition (P) and weak flatness we show that for a given act

$$\textit{condition } (P) \Rightarrow \textit{condition } (P_E) \Rightarrow \textit{condition } (P'_E) \Rightarrow \textit{weakly flat}$$

but the converses are not true. Then we show for some classes of monoids the converse of the above statements are true. There are also some corollaries that will arise.

Definition 4.3.1. Let S be a monoid. A right S-act A satisfies condition (P_E) if whenever $a, a' \in A, u, v \in S$ and $au = a'v$ there exist $a'' \in A$ and $s, t, e^2 = e \in S$ such that $ae = a''se$, $a'e = a''te$, $eu = u$, $ev = v$ and $su = tv$.

Definition 4.3.2. Let S be a monoid. A right S-act A satisfies condition (P'_E) if whenever $a, a' \in A$, $u, v \in S$ and $au = a'v$ there exist $a'' \in A$ and $s, t, e^2 = e, f^2 = f \in S$ such that $eu = u$, $fv = v$, $ae = a''se$, $a'f = a''tf$ and $su = tv$.

Remark. From definition of (P'_E) and (P_E) it can be seen that condition (P_E) implies condition (P'_E), but the converse is not true as the following example demonstrates:

Example 4.3.3. Let $S = \{1, e, f\}$ with table

	1	e	f
1	1	e	f
e	e	e	e
f	f	f	f

Then S is a monoid. If

$$A = xS = \{x, y \mid xe = xf = ye = yf = y, x1 = x, y1 = y\},$$

then A is a right S-act. We show that A satisfies condition (P'_E). By definition it is sufficient to show that for $au = a'v$ with $a, a' \in A$, $u, v \in S$ there exist $a'' \in A$, $s, t, e'^2 = e'$, $e''^2 = e'' \in S$ such that

$$ae' = a''se', \ a'e'' = a''te'', \ e'u = u, \ e''v = v \text{ and } su = tv.$$

If $xe = xf$, then $a'' = x,\ e' = e,\ e'' = f,\ s = t = e$.

If $xe = y1$, then $a'' = x,\ e' = e,\ e'' = 1,\ s = 1,\ t = e$.

If $xe = ye$, then $a'' = x,\ e' = e,\ e'' = e,\ s = t = 1$.

If $xe = yf$, then $a'' = x,\ e' = e,\ e'' = f,\ s = t = e$.

If $xf = y1$, then $a'' = x,\ e' = f,\ e'' = 1,\ s = 1,\ t = f$.

If $xf = ye$, then $a'' = x,\ e' = f,\ e'' = e,\ s = t = f$.

If $xf = yf$, then $a'' = x,\ e' = f,\ e'' = f,\ s = t = 1$.

If $y1 = ye$, then $a'' = y,\ e' = 1,\ e'' = e,\ s = e,\ t = 1$.

If $y1 = yf$, then $a'' = y,\ e' = 1,\ e'' = 1,\ s = t = f$.

If $ye = yf$, then $a'' = y,\ e' = e,\ e'' = 1,\ s = 1,\ t = e$.

Therefore, A satisfies condition (P'_E). But A does not satisfy condition (P_E). Otherwise for $xe = xf$ there exist $a'' \in A, s, t, {e'}^2 = e' \in S$ such that

$$xe' = a''se',\ xe' = a''te',\ e'e = e,\ e'f = f \text{ and } se = tf.$$

If $e' = e$, then $ef = e \neq f$. If $e' = f$, then $fe = f \neq e$. Thus $e' = 1$, and so

$$xe' = x1 = x \text{ and } a''se' = a''s1 = a''s.$$

If $a'' = y$, then for every $s \in S$ $a''s = ys = y \neq x$.

If $a'' = x$, then the only possibility is $s = t = 1$. But in this case

$$se = 1e = e \neq f = 1f = tv,$$

which is a contradiction. ∎

Lemma 4.3.4 [7]. *If e and f are idempotents of a monoid S such that $ef = e$, then the right S-act $S/\rho(e,f)$ is flat.*

Remark. From example 4.3.3, it can be seen that $A \simeq S/\rho(e,f)$. Since $ef = e$, then by Lemma 4.3.4, $S/\rho(e,f)$ is flat, and so A is flat. But as we saw A does not satisfy condition (P_E). Hence, $S/\rho(e,f)$ does not satisfy condition (P_E) and so it can be deduced that Lemma 4.3.4, is not true for condition (P_E). This example also shows that (weak) flatness of acts does not imply condition (P_E) not only in general but also for monocyclic acts.

By the following theorem it can be seen that for right zero monoids condition (P'_E) implies condition (P_E).

Theorem 4.3.5. *Let S be a right zero monoid. Then a right S-act A satisfies condition (P'_E) if and only if it satisfies condition (P_E).*

Proof. If A satisfies condition (P_E), then it is obvious that A satisfies condition (P'_E).

Suppose that A satisfies condition (P'_E) and let $au = a'v$ for $u, v \in S$, $a, a' \in A$. If at least one of u or v is 1 (for example if $u = 1$) then $a = a'v$. If $a'' = a'$, $s = v$ and $t = 1$, then $a = a''s$, $a' = a'1 = a''t$, and $su = tv$. Thus A satisfies condition (P) and so it satisfies condition (P_E).

Now suppose that $u \neq 1, v \neq 1$. Since A satisfies condition (P'_E), then $au = a'v$ implies that there exist $s, t, e^2 = e, f^2 = f \in S$, $a'' \in A$ such that $ae = a''se$, $a'f = a''tf$, $eu = u$, $fv = v$ and $su = tv$. Now there are two cases that can arise.

Case 1. $e = f = 1$. Then $a = a''s$, $a' = a''t$ and $su = tv$. Thus A satisfies condition (P) and so A satisfies condition (P_E).

Case 2. At least one of e or f different from 1 (for example let $e \neq 1$) then e is right zero and so $a'f = a''tf$ implies that $a'fe = a''tfe$ or $a'e = a''te$. Since v is right zero, then $ev = v$ and so A satisfies condition (P_E) as required. ∎

Lemma 4.3.6. *Let S be a monoid and ρ a right congruence on S.*

1. *If S/ρ satisfies condition (P_E), then for all $u, v \in S$ with $u \;\rho\; v$ there exist $s, t, e^2 = e \in S$ such that $s \;\rho\; e \;\rho\; t$, $eu = u$, $ev = v$ and $su = tv$.*

2. *If S/ρ satisfies condition (P'_E), then for all $u, v \in S$ with $u \;\rho\; v$ there exist s, t, $e^2 = e, f^2 = f \in S$ such that $s \;\rho\; e$, $t \;\rho\; f$, $eu = u$, $fv = v$ and $su = tv$.*

Proof.

1. Let $u\ \rho\ v$. If $a = 1\rho$, then $au = av$ and so there exist $a'' \in S/\rho$, s', t', $e^2 = e \in S$ such that $ae = a''s'e$, $ae = a''t'e$, $eu = u$, $ev = v$ and $s'u = t'v$. Then $a'' = x\rho$ for some $x \in S$, and so $ae = a''s'e$ implies that $(1\ \rho)e = (x\ \rho)s'e$ or $(e)\rho = (xs'e)\rho$. Thus $e\ \rho\ xs'e$. Similarly, $ae = a''t'e$ implies that $e\ \rho\ xt'e$. Also we have

$$s'eu = s'u = t'v = t'ev,$$

which implies that $(xs'e)u = (xt'e)v$. If $xs'e = s$, $xt'e = t$, then $s\ \rho\ e\ \rho\ t$ and $su = tv$.

2. Let $u\ \rho\ v$. If $a = 1\rho$, then $au = av$ and so there exist $a'' \in S/\rho$, s', $t', e^2 = e, f^2 = f \in S$ such that $ae = a''s'e$, $af = a''t'f$, $eu = u$, $fv = v$ and $s'u = t'v$. Then $a'' = x\rho$ for some $x \in S$ and so $ae = a''s'e$ implies that $(1\ \rho)e = (x\ \rho)s'e$ or $(e)\rho = (xs'e)\rho$. Thus $e\ \rho\ xs'e$. Also $af = a''t'f$ implies that $(f)\rho = (xt'f)\rho$ or $f\ \rho\ xt'f$. Since $eu = u$, $fv = v$, then

$$s'eu = s'u = t'v = t'fv,$$

which implies that $(xs'e)u = (xt'f)v$. If $xs'e = s$, $xt'f = t$, then $s\ \rho\ e$, $t\ \rho\ f$ and $su = tv$. ■

Remark. Notice that we also have

1. If S/ρ satisfies condition (P_E), then for all $u, v \in S$ with $u\ \rho\ v$, there exist $s, t, e^2 = e \in S$ such that $se\ \rho\ e\ \rho\ te, eu = u, ev = v$ and $su = tv$.

2. If S/ρ satisfies condition (P'_E), then for all $u, v \in S$ with $u\ \rho\ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $se\ \rho\ e$, $tf\ \rho\ f$, $eu = u$, $fv = v$ and $su = tv$.

As we know in conditions (P_E), (P'_E), if $e = 1$, $f = 1$, then condition (P) follows. Also in this case Lemma 4.3.6 is equivalent to the one for condition (P).

Lemma 4.3.7. *Let S be a monoid and let ρ be a right congruence on S. Then for all $u, v \in S$ with $u\ \rho\ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $se\ \rho\ e$, $tf\ \rho\ f$, $eu = u$, $fv = v$, and $su = tv$ if and only if there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s(\rho \vee \Delta e)1$, $t(\rho \vee \Delta f)1$, $eu = u$, $fv = v$, and $su = tv$.*

Proof. Suppose for $u, v \in S$ with $u \ \rho \ v$ there exist $s_1, t_1, e^2 = e, f^2 = f \in S$ such that $s_1 e \ \rho \ e$, $t_1 f \ \rho \ f$, $eu = u$, $fv = v$ and $s_1 u = t_1 v$. If $s_1 e = s$, $t_1 f = t$, then $s \ \rho \ e$, $t \ \rho \ f$. Also $s_1 u = t_1 v$ implies that $s_1(eu) = t_1(fv)$ or $(s_1 e)u = (t_1 f)v$ and so $su = tv$. Since $e^2 = e$, then $e(\Delta e)1$. Thus $s \ \rho \ e(\Delta e)1$ or $s(\rho \ \vee \ \Delta e)1$. Similarly, $t(\rho \ \vee \ \Delta f)1$ as required.

Now suppose that for $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s(\rho \ \vee \ \Delta e)1$, $t(\rho \ \vee \ \Delta f)1$, $eu = u$, $fv = v$ and $su = tv$. Since $s(\rho \ \vee \ \Delta e)1$, then by Lemma 1.49, there exist $r_1, r_2, \ldots, r_{2n-1} \in S$ such that

$$s \ \rho \ r_1(\Delta e)r_2 \ldots r_{2n-1}(\Delta e)1,$$

and so $se \ \rho \ e$. Similarly, $t(\rho \ \vee \ \Delta f)1$ implies that $tf \ \rho \ f$ as required. ■

If in Lemma 4.3.7, $e = f$, then we have

Corollary 4.3.8. *Let S be a monoid and ρ a right congruence on S. If for all $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $se \ \rho \ e \ \rho \ te$, $eu = u$, $ev = v$, $su = tv$, then there exist $s, t, e^2 = e \in S$ such that $s(\rho \ \vee \ \Delta e)1(\rho \ \vee \ \Delta e)t$, $eu = u$, $ev = v$, $su = tv$.*

From Lemma 4.3.6, Lemma 4.3.7, and Corollary 4.3.8, we have

Corollary 4.3.9. *Let S be a monoid and ρ a right congruence on S.*

1. If S/ρ satisfies condition (P'_E), then for all $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s(\rho \ \vee \ \Delta e)1$, $t(\rho \ \vee \ \Delta f)1$, $eu = u$, $fv = v$, $su = tv$.

2. If S/ρ satisfies condition (P_E), then for all $u, v \in S$ with $u \ \rho \ v$ there exist

   $s, t, e^2 = e \in S$ such that $s(\rho \ \vee \ \Delta e)1(\rho \ \vee \ \Delta e)t$, $eu = u$, $ev = v$, $su = tv$.

Now by using Corollary 4.3.9 (1), we show that condition (P'_E) implies weak flatness of cyclic acts.

Theorem 4.3.10. *Let S be a monoid and ρ a right congruence on S. If S/ρ satisfies condition (P'_E), then S/ρ is weakly flat.*

Proof. We show that S/ρ satisfies Lemma 1.54 (6) . Let $u \ \rho \ v$ with $u, v \in S$. Since S/ρ satisfies condition (P'_E), then by Corollary 4.3.9 (1), there exist $s, t, e^2 =$

e, $f^2 = f \in S$, such that $s(\rho \vee \Delta e)1(\rho \vee \Delta f)t$, $eu = u, fv = v$ and $su = tv$. Since $s(\rho \vee \Delta e)1$, then by Theorem 1.49, there exist $s_i, r_i \in S$, $i = 1, 2, \ldots, 2n$ such that

$$s \ \rho \ s_1(\Delta e)r_1 \ \rho \ s_2 \ldots r_{2n-1} \ \rho \ s_{2n}(\Delta e)r_{2n} = 1.$$

Then we have

$$s \ \rho \ s_1, \ s_1e = r_1e, \ r_1 \ \rho \ s_2, \ s_2e = r_2e, \ldots, r_{2n-1} \ \rho \ s_{2n}, \ s_{2n}e = e.$$

By multiplying both sides of equation $s_ie = r_ie$ by u we have $s_ieu = r_ieu$, $i = 1, 2, \ldots, 2n$. Since $eu = u$, then $s_iu = r_iu$, $i = 1, 2, \ldots, 2n$ and so we have

$$s \ \rho \ s_1, \ s_1u = r_1u, \ r_1 \ \rho \ s_2, \ s_2u = r_2u, \ \ldots, \ r_{2n-1} \ \rho \ s_{2n}, \ s_{2n}u = u,$$

which is equivalent to

$$s \ \rho \ s_1(\Delta u)r_1 \ \rho \ s_2(\Delta u)r_2 \ldots r_{2n-1} \ \rho \ s_{2n}(\Delta u)1.$$

Thus by Lemma 1.49, $s(\rho \vee \Delta u)1$. By the same argument it can be seen that $t(\rho \vee \Delta v)1$. Since $su = tv$, then by Lemma 1.54 (6), S/ρ is weakly flat as required. ■

Here we introduce some classes of monoids for which conditions (1), (2) of Lemma 4.3.6, imply conditions (P'_E), (P_E) respectively.

Lemma 4.3.11. *Let S be a left PP monoid and ρ a right congruence on S. If for $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s \ \rho \ e$, $t \ \rho \ f$, $eu = u$, $fv = v$ and $su = tv$, then S/ρ satisfies condition (P'_E).*

Proof. Let $xu = yv$ for x, $y \in S/\rho$, u, $v \in S$. Then $x = a\rho$, $y = b\rho$ for some a, $b \in S$. Thus $(a\rho)u = (b\rho)v$ and so $(au)\rho = (bv)\rho$ or $au \ \rho \ bv$. Then by assumption there exist s', t', $e'^2 = e'$, $f'^2 = f' \in S$ such that $e'au = au$, $f'bv = bv$, $s' \ \rho \ e'$, $t' \ \rho \ f'$ and $s'au = t'bv$. Since S is left PP, then there exists $e^2 = e \in S$ such that $eu = u$ and $e'au = au$ implies that $e'ae = ae$. Consequently, $(ae)\rho = (e'ae)\rho$. Also $s'\rho \ e'$ implies that $s'ae \ \rho \ e'ae$. Thus we have $(ae)\rho = (s'ae)\rho$ or $(a\rho)e = (1\rho)s'ae$. Similarly, there exists $f^2 = f \in S$ such that $fv = v$, $(bf)\rho = (f'bf)\rho$ or $(b\rho)f = (1\rho)t'bf$.

If $s'a = s$, $t'b = t$ and $1\rho = a''$, then $(a\rho)e = (1\rho)s'ae$ implies that $xe = a''se$ and $(b\rho)f = (1\rho)t'bf$ implies that $yf = a''tf$. Also $s'au = t'bv$ implies that $su = tv$. Since $eu = u$, $fv = v$, then S/ρ satisfies condition (P'_E) as required. ■

Corollary 4.3.12. *Let S be a left PP monoid and ρ a right congruence on S. If S/ρ is weakly flat, then it satisfies condition (P'_E).*

Proof. By Lemma 4.3.11, it is sufficient to show that for every $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s \ \rho \ e$, $t \ \rho \ f$, $eu = u$, $fv = v$ and $su = tv$.

Let $u \ \rho \ v$. Since S/ρ is weakly flat, then by Lemma 1.54 (6), there exist $s', t' \in S$ such that $s'u = t'v$ and $s'(\rho \ \vee \ \Delta u)1(\rho \ \vee \ \Delta v)t'$. Since $s'(\rho \ \vee \ \Delta u)1$, then by Theorem 1.49, there exist $s'_1, s'_2, \ldots, s'_{2n-1} \in S$ such that

$$s' \ \rho \ s'_1(\Delta u)s'_2 \ \rho \ s'_3 \ldots s'_{2n-1}(\Delta u)1. \tag{1}$$

Since S is left PP, then there exists $e^2 = e \in S$ such that $eu = u$ and $s'_i u = s'_{i+1}u$, $i = 1, 3, \ldots, 2n-1$, implies that $s'_i e = s'_{i+1}e$. Consequently, (1) implies that

$$s' \ \rho \ s'_1, \ s'_1 e = s'_2 e, \ s'_2 \ \rho \ s'_3, \ldots, s'_{2n-1}e = e, \tag{2}$$

and so $s'e \ \rho \ e$. Similarly, there exists $f^2 = f \in S$ such that $fv = v$ and $t'(\rho \vee \Delta v)1$ implies that $t'f \ \rho \ f$. If $s'e = s$, $t'f = t$, then $s \ \rho \ e$ and $t \ \rho \ f$. Also $s'u = t'v$ implies that $s'(eu) = t'(fv)$ or $(s'e)u = (t'f)v$. Thus $su = tv$ and so S/ρ satisfies condition (P'_E) as required. ■

From Lemma 4.3.6, and Lemma 4.3.11, we have

Theorem 4.3.13. *Let S be a left PP monoid and ρ a right congruence on S. Then S/ρ satisfies condition (P'_E) if and only if for every $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s \ \rho \ e$, $t \ \rho \ f$, $eu = u$, $fv = v$ and $su = tv$.*

Remark. Every idempotent monoid, regular monoid, and right cancellative

monoid satisfies Theorem 4.3.13.

Lemma 4.3.14. *Let S be a right zero monoid and ρ a right congruence on S. If for every $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $s \ \rho \ e \ \rho \ t$, $eu = u$, $ev = v$ and $su = tv$, then S/ρ satisfies condition (P_E).*

Proof. Since S is right zero, then S is left PP. Since S satisfies Lemma 4.3.11, $(e = f)$, then S/ρ satisfies (P'_E). But by Theorem 4.3.5, condition (P'_E) implies condition (P_E) and so S/ρ satisfies condition (P_E) as required. ■

Now from Lemma 4.3.14, and Lemma 4.3.6 (1), we have

Theorem 4.3.15. *Let S be a right zero monoid and ρ a right congruence on S. Then S/ρ satisfies condition (P_E) if and only if for every $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e \in S$ such that $s \ \rho \ e \ \rho \ t$, $eu = u$, $ev = v$ and $su = tv$.*

Theorem 4.3.16. *Let S be a commutative monoid and ρ a congruence on S. If S/ρ satisfies condition (P_E), then for every congruence λ on S and every $u, v \in S$ with $u(\rho \ \vee \ \lambda)v$, there exist $t, t', e^2 = e \in S$ with $tu \ \lambda \ t'v$ such that $t(\rho \ \vee \ \Delta eu)1$, and $t'(\rho \ \vee \ \Delta ev)1$.*

Proof. If $u(\rho \ \vee \ \lambda)v$, then there exist $w_1, w_2, \ldots, w_{2n-1} \in S$, such that

$$u \ \rho \ w_1 \ \lambda \ w_2 \ \rho \ w_3 \ldots w_{2n-2} \ \rho \ w_{2n-1} \ \lambda \ v.$$

Thus

$$u \ \rho \ w_1, \ w_2 \ \rho \ w_3, \ w_4 \ \rho \ w_5, \ldots, w_{2n-2} \ \rho \ w_{2n-1},$$

and

$$w_1 \ \lambda \ w_2, \ w_3 \ \lambda \ w_4, \ldots, w_{2n-1} \ \lambda \ v.$$

Since S/ρ satisfies condition (P_E) and $u \ \rho \ w_1$, then by Corollary 4.3.9 (2), there exist $s, s_1, e_0^2 = e_0 \in S$ such that $s(\rho \vee \Delta e_0)1(\rho \vee \Delta e_0)s_1$, $su = s_1 w_1$, $e_0 u = u$ and $e_0 w_1 = w_1$. Also for $i = 2, 4, \ldots, 2n-2$, there exist s_i, s_{i+1}, $e_i^2 = e_i \in S$ such that $s_i(\rho \ \vee \ \Delta e_i)1(\rho \ \vee \ \Delta e_i)s_{i+1}$, $s_i w_i = s_{i+1} w_{i+1}$, $e_i w_i = w_i$, $e_i w_{i+1} = w_{i+1}$. Since $su = s_1 w_1$ and $w_1 \ \lambda \ w_2$ implies that $s_1 w_1 \ \lambda \ s_1 w_2$, then $su \ \lambda \ s_1 w_2$. Consequently, $s_2 su \ \lambda \ s_2 s_1 w_2$. But S is commutative and so $s_2 su \ \lambda \ s_1 s_2 w_2$. Since $s_2 w_2 = s_3 w_3$, then $s_1 s_2 w_2 = s_1 s_3 w_3$ and so $s_2 su \ \lambda \ s_1 s_3 w_3$. On the other hand $w_3 \ \lambda \ w_4$ implies that $s_1 s_3 w_3 \ \lambda \ s_1 s_3 w_4$. Consequently, $s_2 su \ \lambda \ s_1 s_3 w_4$ and hence $s_4 s_2 su \ \lambda \ s_4 s_1 s_3 w_4$. Since S is commutative, then $s_4 s_1 s_3 w_4 = s_1 s_3 s_4 w_4$. But $s_4 w_4 = s_5 w_5$ implies that $s_1 s_3 s_4 w_4 = s_1 s_3 s_5 w_5$ and so $s_4 s_2 su \ \lambda \ s_1 s_3 s_5 w_5$. By continuing this procedure we have

$$s_{2n-2} \ldots s_4 s_2 su \ \lambda \ s_1 s_3 s_5 \ldots s_{2n-3} s_{2n-1} w_{2n-1}.$$

Since $w_{2n-1} \ \lambda \ v$, then

$$s_1 s_3 \ldots s_{2n-1} w_{2n-1} \ \lambda \ s_1 s_3 \ldots s_{2n-1} v.$$

Consequently,

$$s_{2n-2} \ldots s_4 s_2 su \ \lambda \ s_1 s_3 \ldots s_{2n-1} v.$$

If $t = s_{2n-2} \dots s_4 s_2 s$, $t' = s_1 s_3 \dots s_{2n-1}$, then $tu \ \lambda \ t'v$. Now we show that $t(\rho \ \vee \ \Delta eu)1$. Since $s(\rho \ \vee \ \Delta e_0)1$, then there exist $t_{01}, t_{02}, \dots, t_{02m-1} \in S$ such that

$$s \ \rho \ t_{01}(\Delta e_0)t_{02} \dots t_{02m-2} \ \rho \ t_{02m-1}(\Delta e_0)t_{02m} = 1.$$

Also $s_i(\rho \vee \Delta e_i)1$, $i = 2, 4, \dots, 2n-2$, implies that there exist $t_{i1}, t_{i2}, \dots, t_{i2k_i-1} \in S$ such that

$$s_i \ \rho \ t_{i1}(\Delta e_i)t_{i2} \ \rho \ t_{i3} \dots t_{i2k_i-2} \ \rho \ t_{i2k_i-1}(\Delta e_i)t_{i2k_i} = 1, \ i = 2, 4, \dots, 2n-2$$

Since we can extend the length of the above sequences, then we can suppose that all these sequences are of the same length $2m$. Since S is commutative and

$$s \ \rho \ t_{01}, \ s_2 \ \rho \ t_{21}, \dots, s_{2n-2} \ \rho \ t_{2n-21},$$

then

$$ss_2 \dots s_{2n-2} \ \rho \ t_{01}t_{21} \dots t_{2n-21}.$$

From

$$t_{01}(\Delta e_0)t_{02}, \ t_{21}(\Delta e_2)t_{22}, \dots, t_{2n-21}(\Delta e_{2n-2})t_{2n-22},$$

we have

$$t_{01}e_0 = t_{02}e_0, \ t_{21}e_2 = t_{22}e_2, \dots, t_{2n-21}e_{2n-2} = t_{2n-22}e_{2n-2}.$$

Since S is commutative, then by multiplying both sides of these equalities we have

$$t_{01}t_{21} \dots t_{2n-21}e_0e_2 \dots e_{2n-2} = t_{02}t_{22} \dots t_{2n-22}e_0e_2 \dots e_{2n-2}.$$

Also

$$t_{02} \ \rho \ t_{03}, \ t_{22} \ \rho \ t_{23}, \dots, t_{2n-22} \ \rho \ t_{2n-23},$$

implies that

$$t_{02}t_{22} \dots t_{2n-22} \ \rho \ t_{03}t_{23} \dots t_{2n-23}.$$

By continuing this procedure we have

$$\begin{gathered} ss_2 \dots s_{2n-2} \ \rho \ t_{01}t_{21} \dots t_{2n-21} \\ t_{01}t_{21} \dots t_{2n-21}e_0e_2 \dots e_{2n-2} = t_{02}t_{22} \dots t_{2n-22}e_0e_2 \dots e_{2n-2} \\ t_{02}t_{22} \dots t_{2n-22} \ \rho \ t_{03}t_{23} \dots t_{2n-23} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t_{02m-1}t_{22m-1} \dots t_{2n-22m-1}e_0e_2 \dots e_{2n-2} = t_{02m}t_{22m} \dots t_{2n-22m}e_0e_2 \dots e_{2n-2} \\ = e_0e_2 \dots e_{2n-2}. \end{gathered}$$

If $e = e_0 e_2 \dots e_{2n-2}$, and $t_i = t_{0i} t_{2i} \dots t_{2n-2i}$, $i = 1, 2, \dots, 2m-1$, then the previous equations become

$$t \ \rho \ t_1, \ t_1 e = t_2 e, \ t_2 \ \rho \ t_3, \ t_3 e = t_4 e, \dots, t_{2m-1} e = e,$$

and so

$$t \ \rho \ t_1, \ t_1 eu = t_2 eu, \ t_2 \ \rho \ t_3, \ t_3 eu = t_4 eu, \dots, t_{2m-1} eu = eu. \qquad (*)$$

Since $t_i eu = t_{i+1} eu$ implies that $t_i (\Delta eu) t_{i+1}$, $i = 1, 3, \dots, 2m-3$, and also $t_{2m-1} eu = eu$ implies that $t_{2m-1}(\Delta eu)1$, then

$$t \ \rho \ t_1 (\Delta eu) t_2 \ \rho \ t_3 \dots t_{2m-2} \ \rho \ t_{2m-1} (\Delta eu) 1,$$

and so $t(\rho \vee \Delta eu)1$. Since

$$s_1(\rho \vee \Delta e_0)1, \ s_3(\rho \vee \Delta e_2)1, \dots, s_{2n-1}(\rho \vee \Delta e_{2n-2})1,$$

then by the same argument it can be seen that $t'(\rho \vee \Delta ev)1$. ■

Corollary 4.3.17. *Let S be a commutative monoid and ρ a right congruence on S. If S/ρ satisfies condition (P_E), then for every left congruence λ on S and every $u, v \in S$ with $u(\rho \vee \lambda)v$, there exist $t, t', e^2 = e \in S$ with $tu \ \lambda \ t'v$ such that $t(\rho \vee \lambda eu)1$, and $t'(\rho \vee \lambda ev)1$.*

Proof. Suppose that S/ρ satisfies condition (P_E) and let $u(\rho \vee \lambda)v$ with $u, v \in S$ and λ a left congruence on S. Then by Theorem 4.3.16, there exist $t, t', e^2 = e \in S$ with $tu \ \lambda \ t'v$ such that $t(\rho \vee \Delta eu)1$, and $t'(\rho \vee \Delta ev)1$. Then there exist t_1, $t_2, \dots, t_{2n-1} \in S$ such that

$$t \ \rho \ t_1 (\Delta eu) t_2 \dots t_{2n-2} \ \rho \ t_{2n-1} (\Delta eu) 1.$$

But for $i = 1, 3, \dots, 2n-1$, $t_i (\Delta eu) t_{i+1}$ implies that $t_i eu = t_{i+1} eu$ and so $t_i eu \ \lambda \ t_{i+1} eu$ or $t_i (\lambda eu) t_{i+1}$. Consequently,

$$t \ \rho \ t_1 (\lambda eu) t_2 \dots t_{2n-2} \ \rho \ t_{2n-1} (\lambda eu) 1,$$

and so $t(\rho \vee \lambda eu)1$. Similarly, $t'(\rho \vee \lambda ev)1$. ■

Although we know that condition (P) implies flatness of acts, but as a corollary of Corollary 4.3.17, we have.

Corollary 4.3.18. *Let S be a commutative monoid with $|E(S)| = 1$ and let ρ be a right congruence on S. If S/ρ satisfies condition (P), then it is flat.*

Proof. Suppose that S/ρ satisfies condition (P) and let $u(\rho \vee \lambda)v$ with $u, v \in S$ and λ a left congruence on S. Then S/ρ satisfies condition (P_E) and so by Corollary 4.3.17, there exist $s, t, e^2 = e \in S$ such that $su \ \lambda \ tv$ and $s(\rho \vee \lambda eu)1(\rho \vee \lambda ev)t$. Since $|E(S)| = 1$, then $s(\rho \vee \lambda u)1(\rho \vee \lambda v)t$ and so by Lemma 1.54 (5), S/ρ is flat as required. ∎

Corollary 4.3.19. *Let S be a commutative monoid and ρ a right congruence on S. If S/ρ satisfies condition (P_E), then for every $u, v \in S$ with $u \ \rho \ v$ there exist $s, t, e^2 = e \in S$ such that $su = tv$, and $s(\rho \vee \Delta eu)1(\rho \vee \Delta ev)t$.*

Proof. If $u \ \rho \ v$, then $u(\rho \vee \Delta)v$ and so by Theorem 4.3.16, there exist $s, t, e^2 = e \in S$ such that $su \ \Delta \ tv$, and $s(\rho \vee \Delta eu)1(\rho \vee \Delta ev)t$. Since $su \ \Delta \ tv$ implies that $su = tv$, then the result follows. ∎

Remark. If in Corollary 4.3.19, $|E(S)| = 1$, then we have Lemma 1.54 (6). Also in this case conditions (P_E), (P) are the same and so condition (P) implies weak flatness of cyclic acts.

Remark. By definition of conditions (P_{1E}) and (P_E) it can be seen that if a right S-act satisfies condition (P_E), then it satisfies condition (P_{1E}), but the converse is not true as the following example demonstrates.

Example 4.3.20. Let $S = \{0, 1, 2\}$ with the multiplication table

	0	1	2
0	0	0	0
1	0	1	2
2	0	2	0

If $\rho = \{(0,0), (1,1), (2,2), (2,0), (0,2)\}$, then ρ is a right congruence on S. Since S is commutative, then idempotents are central. We show that S/ρ satisfies condition (P_{1E}), but it does not satisfy condition (P_E). By Theorem 4.2.5 and Corollary 4.3.9 (2), it is sufficient to show that for every $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e \in S$ such that $s(\rho \vee \Delta e)1(\rho \vee \Delta e)t$ and $su = tv$, but $eu \neq u$ or $ev \neq v$.

We suppose first of all that $u = 2$, $v = 0$. If $e = 1$, then $s(\rho \vee \Delta e)1(\rho \vee \Delta e)t$ is equivalent to $s \ \rho \ 1 \ \rho \ t$. Thus, by definition of ρ, $s = t = 1$. But $1 \cdot 2 = 2$, $1 \cdot 0 = 0$

and so $su \neq tv$. Consequently, S/ρ does not satisfy conditions (P_{1E}) and (P_E). Thus $e = 0$.

Now we consider the following cases:

1. $s = t = 0$. Then $0 \cdot 2 = 0 \cdot 0$. Since $0 \ \rho \ 2$ and $2 \cdot 0 = 1 \cdot 0$ implies that $2(\Delta 0)1$, then $0(\rho \ \vee \ \Delta 0)1$. Also $1 \cdot 0 = 0 \cdot 0$ implies that $1(\Delta 0)0$. Since $1 \ \rho \ 1$, then $1(\rho \ \vee \ \Delta 0)0$. Consequently, $0(\rho \ \vee \ \Delta 0)1(\rho \ \vee \ \Delta 0)0$ and $0 \cdot 2 = 0 \cdot 0$. But $0 \cdot 2 = 0 \neq 2$.

2. $s = 0, t = 1$. Then $0 \cdot 2 = 1 \cdot 0$. Since $0 \ \rho \ 2$ and $2(\Delta 0)1$, then $0(\rho \ \vee \ \Delta 0)1$. Also $1 \cdot 0 = 1 \cdot 0$ implies that $1(\Delta 0)1$. Since $1 \ \rho \ 1$, then $1(\rho \ \vee \ \Delta 0)1$. Thus, $0(\rho \ \vee \ \Delta 0)1(\rho \ \vee \ \Delta 0)1$ and $0 \cdot 2 = 0 \cdot 1$. But $0 \cdot 2 = 0 \neq 2$.

3. $s = 0, t = 2$. Since $2 \cdot 0 = 1 \cdot 0$, then $2(\Delta 0)1$, but $0 \ \rho \ 2$ and so $0(\rho \ \vee \ \Delta 0)1$. Since $1 \cdot 0 = 2 \cdot 0$, then $1(\Delta 0)2$, but $1 \ \rho \ 1$ and so $1(\rho \ \vee \ \Delta 0)2$. Thus, $0(\rho \ \vee \ \Delta 0)1(\rho \ \vee \ \Delta 0)2$ and $0 \cdot 2 = 2 \cdot 0$. But $0 \cdot 2 = 0 \neq 2$.

4. $s = 2, t = 0$. Then $2 \cdot 2 = 0 \cdot 0$. Since $2 \ \rho \ 0$ and $0 \cdot 0 = 1 \cdot 0$, then $2(\rho \ \vee \ \Delta 0)1$. Also $1 \ \rho \ 1$ and $1 \cdot 0 = 0 \cdot 0$ imply that $1(\rho \vee \Delta 0)0$. Thus $2(\rho \vee \Delta 0)1(\rho \vee \Delta 0)0$ and $2 \cdot 2 = 0 \cdot 0$. But $0 \cdot 2 = 0 \neq 2$.

5. $s = t = 2$. Then $2 \cdot 0 = 2 \cdot 2$. Since $2 \ \rho \ 0$ and $0 \cdot 0 = 1 \cdot 0$, then $2(\rho \ \vee \ \Delta 0)1$. Also $1 \cdot 0 = 2 \cdot 0$ implies that $1(\Delta 0)2$, but $1 \ \rho \ 1$ and so $1(\rho \ \vee \ \Delta 0)2$. Thus, $2(\rho \ \vee \ \Delta 0)1(\rho \ \vee \ \Delta 0)2$ and $2 \cdot 0 = 2 \cdot 2$. But $0 \cdot 2 = 0 \neq 2$.

6. $s = 1, t = 0$. Then $1 \cdot 2 = 2 \neq 0 \cdot 0 = 0$. Thus S/ρ does not satisfy conditions (P_{1E}), (P_E).

7. $s = t = 1$. Then $1 \cdot 2 = 2 \neq 1 \cdot 0 = 0$. Thus S/ρ does not satisfy conditions (P_{1E}), (P_E).

8. If $s = 1, t = 2$. Then $1 \cdot 2 = 2 \neq 2 \cdot 0 = 0$. Thus S/ρ does not satisfy conditions (P_{1E}), (P_E).

If $u = 0$ and $v = 2$, then argument is the same.

If $u = v$, then it is sufficient to take $s = t = e = 1$. ■

By the following theorems it can be seen that for some monoids Condition (P'_E) implies condition (P_{1E}).

Theorem 4.3.21. *Let S be a monoid with $E(S)$ a semilattice. If a right S-act A satisfies condition (P'_E), then it satisfies condition (P_{1E}).*

Proof. Suppose that A satisfies condition (P'_E) and let $au = a'v$, for $a, a' \in A$, $u, v \in S$. Then there exist $s, t, e^2 = e, f^2 = f \in S$, $a'' \in A$ such that $ae = a''se$, $a'f = a''tf$, $eu = u$, $fv = v$, $su = tv$. Consequently, $aef = a''sef$, $a'fe = a''tfe$. If $e' = ef = fe$, then

$$e'^2 = (ef)(ef) = e(fe)f = e(ef)f = e^2f^2 = ef = e'.$$

Thus e' is an idempotent. Also $ae' = a''se'$, $a'e' = a''te'$. Since $su = tv$, then A satisfies condition (P_{1E}) as required. ■

Theorem 4.3.22. *Let S be a monoid such that $E(S) \setminus \{1\}$ is a right zero band. If a right S-act A satisfies condition (P'_E), then it satisfies condition (P_{1E}).*

Proof. Suppose that A satisfies condition (P'_E), and let $au = a'v$, for $a, a' \in A$, $u, v \in S$. Then there exist $s, t, e^2 = e, f^2 = f \in S$, $a'' \in A$ such that $ae = a''se$, $a'f = a''tf$, $eu = u$, $fv = v$, $su = tv$. Now, there are two cases that can arise:

Case 1. $e = f = 1$. Then $a = a''s$, $a' = a''t$, $su = tv$ and so A satisfies condition (P_{1E}).

Case 2. At least one of e or f different from 1 (for example let $e \neq 1$) then $a'f = a''tf$ implies that $a'fe = a''tfe$. But by assumption $fe = e$ and so $a'e = a''te$. Thus A satisfies condition (P_{1E}) as required. ■

From Theorem 4.3.22, we can deduce that if S is a monoid with every $e \in E(S)\setminus\{1\}$ right zero, then condition (P'_E) implies condition (P_{1E}).

Although by Theorem 4.3.10, we showed that condition (P'_E) implies weak flatness of cyclic acts, but the main purpose of this theorem was to show the similarity between conditions (P_E), (P'_E) and weak flatness of acts in this case. Now we show that conditions (P_E), (P'_E) imply weak flatness of acts in general.

Lemma 4.3.23 [5]. *Let S be a monoid and A a right S-act. Then A is weakly flat if and only if for every $x, y \in S$ and $a, a' \in A$, $ax = a'y$ implies that there* *exist* $x_1, x_2, \ldots, x_n \in \{x, y\}, a_1, a_2, \ldots, a_n \in A$ *and* $u_1, v_1, \ldots, u_n, v_n \in S$ *such that*

$$\begin{aligned} x &= u_1 x_1 & a u_1 &= a_2 v_1 \\ v_1 x_1 &= u_2 x_2 & a_2 u_2 &= a_3 v_2 \\ &\vdots & &\vdots \\ v_n x_n &= y & a_n u_n &= a' v_n \end{aligned}$$

Theorem 4.3.24. *Let S be a monoid and A a right S-act. If A satisfies condition (P'_E), then A is weakly flat.*

Proof. Suppose that A satisfies condition (P'_E) and let $au = a'v$ for $a, a' \in A$ and $u, v \in S$. Then there exist $s, t, e^2 = e, f^2 = f \in S$ and $a'' \in A$ such that $ae = a''se$, $a'f = a''tf$, $eu = u$, $fv = v$ and $su = tv$. Then we have

$$(se)u = s(eu) = su = tv = t(fv) = (tf)v,$$

and so

$$\begin{aligned} u &= eu & & \\ (se)u &= (tf)v & ae &= a''(se) \\ fv &= v & a''(tf) &= a'f \end{aligned}$$

Thus, by Lemma 4.3.23, A is weakly flat. ■

Remark. The converse of Theorem 4.3.24, is not true, as the following example demonstrates:

Example 4.3.25. Let $S = \{0, 1, e, a\}$ with table

	0	1	e	a
0	0	0	0	0
1	0	1	e	a
e	0	e	e	0
a	0	a	a	0

If $J = eS = \{0, e\}$, then J is a right ideal of S. Also $0 \in J0 = \{0\}$, $e \in Je = \{0, e\}$. Thus by Proposition 2.3.54, $A(J)$ is a flat right S-act and so it is a weakly flat right S-act.

Now we show that $A(J)$ does not satisfy condition (P'_E). Otherwise, $(x, a)a = (y, a)a$ implies that there exist $s, t, f^2 = f,\ g^2 = g \in S,\ a'' \in A(J)$ such that

$$(x, a)f = a''sf,\ (y, a)g = a''tg,\ fa = a,\ ga = a,\ sa = ta.$$

Since 1 is the only idempotent such that $1a = a$, then $f = g = 1$ and so $(x, a) = a''s$, $(y, a) = a''t$. But

$$(x, a) = (x, a)1 = (x, a)e = (x, 1)a,$$

and so either $a'' = (x, a)$ or $a'' = (x, 1)$. On the other hand for every $t \in S$, $(x, a)t \neq (y, a)$, $(x, 1)t \neq (y, a)$ which is a contradiction.

Since condition (P_E) implies condition (P'_E), then we have

Corollary 4.3.26. *Let S be a monoid and A a right S-act. If A satisfies condition (P_E), then it is weakly flat.*

Remark. By Example 4.3.25, it can also be seen that weak flatness of acts does not imply condition (P_E) in general. Also by considering the monoid S and the right S-act A in Example 4.3.3, it can be seen that since A satisfies condition (P'_E), then by Theorem 4.3.24, it is weakly flat, but A does not satisfy condition (P_E).

Now we give some conditions for a monoid S such that every weakly flat right S-act satisfies conditions (P'_E), and (P_E).

Lemma 4.3.27 [5]. *Let S be a left PP monoid. A right S-act A is principally weakly flat if and only if for every $a, a' \in A$ and $x \in S$, $ax = a'x$ implies that there exists $e^2 = e \in S$ such that $ex = x$ and $ae = a'e$.*

Lemma 4.3.28. *Let S be a left PP monoid and A a right S-act. If A is weakly flat, then A satisfies condition (P'_E).*

Proof. Suppose that $ax = a'y$ for $a, a' \in A$ and $x, y \in S$. Then by Lemma 2.3.49, and remark after there exist $a'' \in A$, and $z \in Sx \cap Sy$, such that, $ax = a'y = a''z$. Since $ax = a''z$ and $z \in Sx$, then there exists $s \in S$, such that $z = sx$ and

$$ax = a''(sx) = (a''s)x.$$

Also, there exists $t \in S$, such that $z = ty$ and

$$a'y = a''(ty) = (a''t)y.$$

By Lemma 4.3.27, there exist $e, e' \in S$ such that $ex = x$, $e'y = y$ and $ae = (a''s)e$, $a'e' = (a''t)e'$. Since $z = sx = ty$, then A satisfies condition (P'_E). ■

From Theorem 4.3.24, and Lemma 4.3.28, we have

Theorem 4.3.29. *Let S be a left PP monoid and A a right S-act. Then A is weakly flat if and only if A satisfies condition (P'_E).*

Since regular monoids and right cancellative monoids are left PP, then we have

Corollary 4.3.30. *Let S be a regular or right cancellative monoid. Then a right S-act A is weakly flat if and only if A satisfies condition (P'_E).*

Corollary 4.3.31. *Let S be a right cancellative monoid. Then a right S-act A is weakly flat if and only if A satisfies condition (P).*

Proof. Let S be right cancellative and A a right S-act. Then by Corollary 4.3.30, A is weakly flat if and only if A satisfies condition (P'_E). But for right cancellative monoids, conditions (P) and (P'_E) are equivalent. Thus, A is weakly flat if and only if A satisfies condition (P). ■

Remark. By Example 4.3.3, it can be seen that S is a left PP monoid and A is a weakly flat right S-act, but A does not satisfy condition (P_E). Therefore, the condition that a monoid S is left PP is not sufficient that every weakly flat right S-act satisfies condition (P_E). But from Theorem 4.3.5, and Lemma 4.3.28, we have

Corollary 4.3.32. *Let S be a right zero monoid and A a right S-act. Then A is weakly flat if and only if A satisfies condition (P_E).*

Proof. Let A be a weakly flat right S-act. Since S is right zero, then it is left PP, and so by Lemma 4.3.28, A satisfies condition (P'_E). But by Theorem 4.3.5, every right S-act which satisfies condition (P'_E) satisfies condition (P_E). Thus A satisfies condition (P_E) as required.

If A satisfies condition (P_E), then by Corollary 4.3.26, it is weakly flat. ■

Now we consider conditions (P), (P_E) and (P'_E) and relations between them.

Lemma 4.3.33. *Let S be a monoid and A a right S-act. If for $a, a' \in A$, $u, v \in S$ with $au = a'v$, there exist $s, t, e^2 = e, f^2 = f \in S$, $a'' \in A$ such that $a = a''se$, $a' = a''tf$, $eu = u$, $fv = v$ and $su = tv$, then A satisfies condition (P'_E).*

Proof. Let $au = a'v$, then by assumption there exist $s, t, e^2 = e, f^2 = f \in S$ $a'' \in A$ such that $a = a''se,\ a' = a''tf,\ eu = u,\ fv = v$ and $su = tv$. Then $ae = a''se^2 = a''se$, and similarly $a'f = a''tf$. Thus A satisfies condition (P'_E) as required. ∎

Corollary 4.3.34. *Let S be a monoid and A a right S-act. If for $a, a' \in A,\ u, v \in S$ with $au = a'v$, there exist $s, t, e^2 = e \in S,\ a'' \in A$ such that $a = a''se,\ a' = a''te,\ eu = u,\ ev = v$ and $su = tv$, then A satisfies condition (P_E).*

Lemma 4.3.35. *Let S be a monoid and A a right S-act. Then A satisfies condition (P) if and only if for $a, a' \in A,\ u, v \in S$ with $au = a'v$, there exist $a'' \in A,\ s, t, e^2 = e, f^2 = f \in S$ such that $a = a''se,\ a' = a''tf,\ eu = u,\ fv = v$ and $su = tv$.*

Proof. If A satisfies condition (P), then for $au = a'v$ with $a, a' \in A$ and $u, v \in S$ there exist $s, t \in S$ and $a'' \in A$ such that $a = a''s,\ a' = a''t$ and $su = tv$. If $e = f = 1$, then $a = a''se,\ a' = a''tf,\ eu = u,\ fv = v$ and $su = tv$.

Now suppose that for $a, a' \in A,\ u, v \in S$ with $au = a'v$ there exist $s, t, e^2 = e, f^2 = f \in S$ and $a'' \in A$ such that $a = a''se,\ a' = a''tf,\ eu = u,\ fv = v$ and $su = tv$. If $s' = se$ and $t' = tf$, then $a = a''s'$ and $a' = a''t'$ also

$$s'u = (se)u = s(eu) = su = tv = t(fv) = (tf)v = t'v.$$

Thus, A satisfies condition (P). ∎

By the same argument as in Lemma 4.3.35, we have

Corollary 4.3.36. *Let S be a monoid and A a right S-act. Then A satisfies condition (P) if and only if for $a, a' \in A,\ u, v \in S$ with $au = a'v$ there exist $s, t, e^2 = e \in S,\ a'' \in A$ such that $a = a''se,\ a' = a''te,\ eu = u,\ ev = v$ and $su = tv$.*

Now from Lemma 4.3.33, Corollary 4.3.34, Lemma 4.3.35, and Corollary 4.3.36, it can be deduced that condition (P) implies conditions (P_E) and (P'_E). Since condition (P_E) implies condition (P'_E) and by Theorem 4.3.24, condition (P'_E) implies weak flatness, then we have the following hierarchy of properties arranged in strictly decreasing order of strength:

free $\Rightarrow$ *projective* $\Rightarrow$ *strongly flat* $\Rightarrow$ *condition* $(P) \Rightarrow$ *condition* $(P_E) \Rightarrow$ *condition* $(P'_E) \Rightarrow$ *weakly flat* $\Rightarrow$ *principally weakly flat* $\Rightarrow$ *torsion free*

As we saw condition (P) implies conditions (P_E), and (P'_E), but the converses are not true as the following example demonstrates:

Example 4.3.37. Let $S = \{0, 1\}$ and $A = \{x, y, z \mid x0 = y0 = z0 = z\}$. Since A is not a coproduct of cyclic S-acts, then by Lemma 1.53, it does not satisfy condition (P). We show that A satisfies condition (P_E). It is sufficient to show that if $au = a'v$ for $a, a' \in A$ and $u, v \in S$, then there exist $a'' \in A$ and $s, t, e^2 = e \in S$ such that $ae = a''se$, $a'e = a''te$, $eu = u$, $ev = v$ and $su = tv$.

If $x0 = y0$, then put $e = 0$, $a'' = x$ and $s, t \in S$.

If $x0 = z0$, then put $e = 0$, $a'' = x$ and $s, t \in S$.

If $y0 = z0$, then put $e = 0$, $a'' = x$ and $s, t \in S$.

If $a0 = z1$ for $a \in \{x, y, z\}$, then put $e = 1$, $a'' = a$, $s = 1$ and $t = 0$.

Since condition (P_E) implies condition (P'_E), then Example 4.3.37, shows also that condition (P'_E) does not imply condition (P) in general.

Now we show that for some classes of monoids the converses are true in general or for cyclic acts.

Lemma 4.3.38. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero and let ρ be a right congruence on S. Then S/ρ satisfies condition (P'_E) if and only if S/ρ satisfies condition (P).*

Proof. If S/ρ satisfies condition (P), then as we saw S/ρ satisfies conditions (P_E) and (P'_E).

Suppose that S/ρ satisfies condition (P'_E) and let $u \; \rho \; v$ for $u, v \in S$. Then by the remark after Lemma 4.3.6, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s \; \rho \; e$, $t \; \rho \; f$, $eu = u$, $fv = v$ and $su = tv$.

Now there are four cases that can arise as follows:

Case 1. $e = f = 1$. Then $s \ \rho \ 1 \ \rho \ t$ and $su = tv$. Thus by Lemma 1.54 (4), S/ρ satisfies condition (P).

Case 2. $e = 1, \ f \neq 1$. Then by assumption f is right zero and so $fv = v$ implies that v is right zero. Thus $su = tv = v = 1v$. Since $s \ \rho \ 1 \ \rho \ 1$, then S/ρ satisfies condition (P).

Case 3. $e \neq 1, \ f = 1$. It is similar to case 2.

Case 4. $e \neq 1, \ f \neq 1$. Then e, f are right zero and so $1u = su = tv = 1v$. Since $1 \ \rho \ 1$, then S/ρ satisfies condition (P). ∎

Since condition (P_E) implies condition (P'_E), then we have

Corollary 4.3.39. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero and let ρ be a right congruence on S. Then S/ρ satisfies condition (P_E) if and only if S/ρ satisfies condition (P).*

From Lemma 4.3.38, and Corollary 4.3.39, we have

Theorem 4.3.40. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero and let ρ be a right congruence on S. Then the following statements are equivalent:*

(1) S/ρ satisfies condition (P).

(2) S/ρ satisfies condition (P_E).

(3) S/ρ satisfies condition (P'_E).

Corollary 4.3.41. *Let S be a monoid. Then all cyclic right S-acts satisfy condition (P) if and only if every $e \in E(S) \setminus \{1\}$ is right zero and all cyclic right S-acts satisfy condition (P'_E).*

Proof. If all cyclic right S-acts satisfy condition (P), then all flat cyclic right S-acts satisfy condition (P) and so by Lemma 2.2.8, every $e \in S \setminus \{1\}$ is right zero. Also all cyclic right S-acts satisfy conditions (P_E), (P'_E). The converse is true by Theorem 4.3.40. ∎

Remark. In Example 4.3.37, every $e \in E(S) \setminus \{1\}$ is right zero, and A is a right S-act which is not cyclic. A satisfies conditions (P'_E), and (P_E), but it does not satisfy condition (P). Hence, by this example it can be seen that for a monoid S, the condition that every $e \in E(S) \setminus \{1\}$ is right zero is not generally sufficient that conditions (P_E) and (P'_E) imply condition (P).

As we know for a right zero monoid, all flat cyclic right acts satisfy condition (P). By Example 4.3.37, it can also be seen that A is a right S-act which is not cyclic and also it does not satisfy condition (P). But by Theorem 2.2.19, A is flat. Thus this example shows that for right zero monoids flatness of acts does not imply condition (P) in general.

Example 4.3.37, shows also that for a right zero monoid, conditions (P_E) and (P'_E) do not imply condition (P) in general.

From Lemma 4.3.38, and theorem 2.2.8, we have

Corollary 4.3.42. *Let S be a left PP monoid. Then all weakly flat cyclic right S-acts satisfy condition (P) if and only if every $e \in E(S) \setminus \{1\}$ is right zero.*

Proof. If every $e \in E(S) \setminus \{1\}$ is right zero, then by Lemma 4.3.38, every cyclic right S-act which satisfies condition (P'_E), satisfies condition (P). On the other hand by Corollary 4.3.29, every weakly flat right S-act satisfies condition (P'_E). Thus every weakly flat cyclic right S-act satisfies condition (P).

The converse is obvious by Theorem 2.2.8. ∎

Theorem 4.3.43. *Let S be a left PP monoid. Then the following statements are equivalent:*

(1) Every $e \in E(S) \setminus \{1\}$ is right zero.

(2) All weakly flat cyclic right S-acts satisfy condition (P).

(3) All cyclic right S-acts having (P'_E) satisfy condition (P).

(4) All flat cyclic right S-acts satisfy condition (P).

(5) S is right subelementary and every element in the right nil part is right zero.

Proof. $(1), (5)$ are equivalent by Lemma 2.3.20.

$(2), (4), (5)$ are equivalent by Theorem 2.3.22.

By Theorem 4.3.24, $(2) \Rightarrow (3)$.

By Lemma 4.3.28, $(3) \Rightarrow (2)$.

Thus $(2), (3)$ are equivalent. ■

Now by using Theorem 4.3.40, we can give some classes of monoids for which all cyclic right acts having (P'_E) are projective or satisfy condition (P).

Corollary 4.3.44. *Let S be a monoid such that all flat monocyclic right S-acts satisfy condition (P). Then a cyclic right S-act satisfies condition (P'_E) if and only if it satisfies condition (P).*

Proof. If all flat monocyclic right S-acts satisfy condition (P), then by Lemma 2.2.8, every $e \in E(S) \setminus \{1\}$ is right zero. Consequently, by Theorem 4.3.40, conditions (P'_E) and (P) are equivalent for cyclic right acts. ■

From Corollary 4.3.44, it can be deduced that:

Corollary 4.3.45. *Let S be a monoid. If all flat monocyclic right S-acts satisfy condition (P), then all cyclic right S-acts having (P'_E) satisfy condition (P).*

Corollary 4.3.46. *Let $S = C \dot{\cup} N$ be a right subelementary monoid. Then a cyclic right S-act satisfies condition (P'_E) if and only if it satisfies condition (P).*

Proof. Since every $e \in S \setminus \{1\}$ is right zero, then it is obvious by Theorem 4.3.40.

■

In particular for a right nil monoid we have

Corollary 4.3.47. *Let S be a right nil monoid. Then a cyclic right S-act satisfies condition (P'_E) if and only if it is projective.*

Proof. If a cyclic right S-act satisfies condition (P'_E), then it is weakly flat, and so by Theorem 2.2.25, it is projective.

The converse is obvious. ∎

Now by using Corollary 4.3.36, we can give an alternative proof that condition (P) implies flatness of acts.

Lemma 4.3.48 [3]. *Let S be a monoid. Let $A \in Act - S$, $a, a' \in A$, $B \in S - Act$ and $b, b' \in B$. Then $a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exist $a_1, \ldots, a_n \in A$, $b_2, \ldots, b_n \in B$, $s_1, \ldots, s_n$, $t_1, \ldots, t_n \in S$ such that*

$$\begin{array}{ll} a = a_1 s_1 & \\ & s_1 b = t_1 b_2 \\ a_1 t_1 = a_2 s_2 & \\ & s_2 b_2 = t_2 b_3 \\ a_2 t_2 = a_3 t_3 & \\ & \vdots \\ \vdots & \\ & s_n b_n = t_n b' \\ a_n t_n = a' & \end{array}$$

Theorem 4.3.49. *Let S be a monoid. If a right S-act A satisfies condition (P), then it is flat.*

Proof. Let $B \subset B'$ be an inclusion of left S-acts and let $a \otimes b = a' \otimes b'$ in $A \otimes B'$. Then by Lemma 4.3.48, there exist elements $b_1, b_2, \ldots, b_n \in B'$ $a_1, a_2, \ldots, a_n \in A$ and $s_1, s_2, \ldots, s_n$, $t_1, t_2, \ldots, t_n \in S$ such that

$$\begin{array}{ll} a = a_1 s_1 & s_1 b = t_1 b_2 \\ a_1 t_1 = a_2 s_2 & s_2 b_2 = t_2 b_3 \\ \vdots & \vdots \\ a_n t_n = a' & s_n b_n = t_n b' \end{array}$$

We prove by induction on n that $a \otimes b = a' \otimes b'$ in $A \otimes B$. Let $n = 1$, then we have

$$\begin{array}{ll} a = a_1 s_1 & \\ & s_1 b = t_1 b' \\ a_1 t_1 = a' & \end{array}$$

Since $b, b' \in B$, then we are done. Suppose that the assumption hold for every $k < n$. Since $a_1 t_1 = a_2 s_2$, and A satisfies condition (P), then by Corollary 4.3.36, there exist $u, v, e^2 = e \in S$ and $a'' \in A$ such that $a_1 = a'' ue$, $a_2 = a'' ve$, $et_1 = t_1$, $es_2 = s_2$ and $ut_1 = vs_2$. Thus we have

$$a = a_1 s_1 = (a'' ue) s_1 = a''(ues_1). \tag{1}$$

Since $s_1 b = t_1 b_2$ and $et_1 = t_1$, then

$$(ue)s_1 b = (ue)t_1 b_2 = u(et_1)b_2 = ut_1 b_2. \tag{2}$$

Also $es_2 = s_2$ and $ut_1 = vs_2$ imply that $ut_1 = vs_2 = ves_2$. Consequently, $(ut_1)b_2 = (ves_2)b_2$. Since $s_2 b_2 = t_2 b_3$, then $ve(s_2 b_2) = ve(t_2 b_3)$. Thus from (2) we have

$$(ues_1)b = ut_1 b_2 = ves_2 b_2 = (vet_2)b_3. \tag{3}$$

Since $a_2 t_2 = a_3 s_3$, then

$$(a''ve)t_2 = a_3 s_3 \text{ or } a''(vet_2) = a_3 s_3. \tag{4}$$

Now, from (1), (3) and (4) we have the following

$$\begin{array}{rr} a = a''(ues_1) & (ues_1)b = (vet_2)b_3 \\ a''(vet_2) = a_3 s_3 & s_3 b_3 = t_3 b_4 \\ \vdots & \vdots \\ a_n t_n = a' & s_n b_n = t_n b' \end{array}$$

with the number of equalities less than n and so by induction $a \otimes b = a' \otimes b'$ in $A \otimes B$. Thus A is flat as required. ■

The converse of Theorem 4.3.49 is not true as the following example demonstrates.

Example 4.3.50. Let S be a monoid with $1 \neq e^2 = e \in S$. Let x, y, z be symbols not belonging to S and let

$$M = \{(x,s) \mid s \in S, s \neq es\} \cup \{(y,s) \mid s \in S, s \neq es\} \cup \{(z,s) \mid s \in S, es = s\}.$$

Define an action of S on M by:

$$(u,s)t = \begin{cases} (u,st) & \text{if } st \neq est \\ (z,st) & \text{otherwise,} \end{cases} \qquad \textit{for } u \in \{x,y\},$$

$$(z,s)t = (z,st), \qquad \textit{for all } t \in S.$$

Then M becomes a right S-act with two generators $(x,1)$ and $(y,1)$. By [26, Theorem 2.1] M is flat. Now let S be the monoid as in Example 4.3.3. Then $M = \{(x,1),(x,f)\} \cup \{(y,1),(y,f)\} \cup \{(z,e)\}$ is flat. But M does not satisfy condition (P). Otherwise, since $(x,1)e = (y,1)e = (z,e)$, then there exist $a'' \in M$, $s,t \in S$ such that $(x,1) = a''s$ and $(y,1) = a''t$. By definition the only case for $(x,1)$ is $(x,1) = (x,1)1$ and so $a'' = (x,1)$. But in this case for every $t \in S$, $(y,1) \neq (x,1)t$, which is a contradiction.

4.4. Monoids over which all Acts satisfy Conditions (P_E) and (P'_E)

In this section we classify monoids for which all (monocyclic) right acts satisfy condition (P'_E), right zero monoids for which all (monocyclic) right acts satisfy condition (P_E) and also we show that for right inverse monoids all right acts satisfy condition (P'_E). Moreover, we give a classification of left PP monoids S for which every monocyclic right S-act of the form $S/\rho(x, x^2)$, satisfies condition (P'_E). There are also some corollaries that will arise.

Lemma 4.4.1 [1]. *Let S be a monoid. Let $p < q$ be non-negative integers, and $x, s, t \in S$. Then $s\ \rho(x^p, x^q)\ t$ if and only if $s = t$ or $s = x^p u$, $t = x^p v$, $x^m u = x^n v$ for some $u, v \in S$ and non-negative integers m, n with $m \equiv n (mod\ q - p)$.*

Theorem 4.4.2. *Let S be a monoid, $x \in S$ and let $p < q$ be non-negative integers. If $S/\rho(x^p, x^q)$ satisfies condition (P'_E), then $x^p = x^q$ or x^p is regular.*

Proof. If $x^p = x^q$, then we are done. Suppose that $x^p \neq x^q$. Since $x^p\ \rho\ x^q$, then by the remark after Lemma 4.3.6, there exist $s, t, e^2 = e, f^2 = f \in S$ such that

$$ex^p = x^p,\ fx^q = x^q,\ se\ \rho\ e,\ tf\ \rho\ f \text{ and } sx^p = tx^q.$$

Since $se\ \rho\ e$ and $tf\ \rho\ f$, then by Lemma 4.4.1, there exist $u, v, u', v' \in S$ such that

$$se = e \text{ or } se = x^p u,\ e = x^p v.$$

$$tf = f \text{ or } tf = x^p u',\ f = x^p v'.$$

Now, if $se = e$, then $tf \neq f$. Otherwise, $se = e$ implies that

$$(se)x^p = ex^p \text{ or } s(ex^p) = (ex^p).$$

Since $ex^p = x^p$, then $sx^p = x^p$. Also $tf = f$ implies that

$$tf(x^q) = f(x^q) \text{ or } t(fx^q) = (fx^q).$$

Since $fx^q = x^q$, then $tx^q = x^q$. But $sx^p = tx^q$, and so we would have

$$x^p = sx^p = tx^q = x^q,$$

which is a contradiction. Thus $tf \neq f$ and as a result, $tf = x^p u'$, $f = x^p v'$. Since $se = e$, then $sex^p = ex^p$ or $s(ex^p) = ex^p$. But $x^p = ex^p$, and so $sx^p = x^p$. Consequently,

$$x^p = sx^p = tx^q = t(fx^q) = (tf)x^q = x^p u' x^q = x^p(u' x^{q-p})x^p,$$

giving x^p regular. If $se \neq e$, then $se = x^p u$, $e = x^p v$. Consequently,

$$x^p = ex^p = x^p v x^p,$$

and again x^p is regular. ■

<u>**Corollary 4.4.3.**</u> *Let S be a monoid and $x \in S$. If $S/\rho(x, x^2)$ satisfies condition (P'_E), then x is regular.*

Proof. By Theorem 4.4.2, $x = x^2$ or x is regular. If $x = x^2$, then it is obvious that x is regular. ■

<u>**Corollary 4.4.4.**</u> *Let S be a monoid, $x \in S$ and $p < q$ be non-negative integers. If $S/\rho(x^p, x^q)$ satisfies condition (P_E), then $x^p = x^q$ or x^p is regular.*

From Corollary 4.4.4, we have

<u>**Corollary 4.4.5.**</u> *Let S be a monoid and ρ a right congruence on S, if $S/\rho(x, x^2)$ satisfies condition (P_E), then x is regular.*

<u>**Proposition 4.4.6 [5].**</u> *Let S be a regular monoid. Then a right S-act A is weakly flat if and only if for every $x, y \in S$ and $a \in A$, if $ax = ay$, then there exists $z \in Sx \cap Sy$ such that $ax = ay = az$.*

<u>**Note :**</u> In the following theorem $\rho(x, y)$ is the smallest right congruence on S which identifies these two elements.

<u>**Theorem 4.4.7.**</u> *Let S be a monoid. Then all monocyclic right S-acts satisfy condition (P'_E) if and only if S is regular and for every $x, y \in S$ there exists an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.*

Proof. Suppose that all monocyclic right S-acts satisfy condition (P'_E) and let $x \in S$. Then $S/\rho(x, x^2)$ satisfies condition (P'_E) and so by Corollary 4.4.3, x is regular.

Let $\rho = \rho(x, y)$ and put $a = 1\rho$. Then clearly, $ax = ay$. Since $S/\rho(x, y)$ satisfies (P'_E), then by Theorem 4.3.24, it is weakly flat. Consequently, by Proposition 4.4.6, $ax = ay$ implies that there exists $z \in Sx \cap Sy$ such that $ax = ay = az$. Then $ax = az$ implies that $(1\ \rho)x = (1\ \rho)z$ or $x\ \rho\ z$, and so $(x, z) \in \rho(x, y)$.

Suppose that S is regular and that for every $x, y \in S$ there exists an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$. Then by Theorem 2.3.60, all right S-acts are weakly flat. But by Corollary 4.3.30, every weakly flat right S-act satisfies condition (P'_E) and so all right S-acts satisfy condition (P'_E). ■

Remark. By the following example it can be seen that in Theorem 4.4.7, regularity of a monoid S is not sufficient that all monocyclic right S-acts satisfy condition (P'_E).

Example 4.4.8. Let $S = \{1, e, f\}$ with table

	1	e	f
1	1	e	f
e	e	e	f
f	f	e	f

Then S is regular. If $A = \{a, b \mid ae = af = be = bf = b, a1 = a\}$, then $A = aS \simeq S/\rho(e, f)$ is a monocyclic right S-act. We claim that A does not satisfy condition (P'_E). Otherwise, for $ae = af$ there exist $a'' \in A$ $s, t, e'^2 = e', f'^2 = f' \in S$ such that $e'e = e$, $f'f = f$, $ae' = a''se'$, $af' = a''tf'$ and $se = tf$. But for every $s, t \in S$, $se = e \neq f = tf$ which is a contradiction.

Since A does not satisfy condition (P'_E), then A also does not satisfy condition (P_E), and so this example shows that regularity of a monoid S is not sufficient that all monocyclic right S-acts satisfy condition (P_E). ■

Notice that in Example 4.4.8, $(e, f) \in \rho(e, f)$, but $Se \cap Sf = \emptyset$ and so there is no element $z \in Se \cap Sf$ with $(e, z) \in \rho(e, f)$.

Since every regular monoid is left PP, then from Theorem 4.3.28, and Theorem 4.4.7, we can now deduce the following extension to Theorem 2.3.60.

Theorem 4.4.9. *For any monoid S, the following are equivalent:*

(1) All right S-acts are weakly flat.

(2) All finitely generated right S-acts are weakly flat.

(3) All cyclic right S-acts are weakly flat.

(4) All right S-acts satisfy condition (P'_E).

(5) All monocyclic right S-acts satisfy condition (P'_E)

(6) S is regular monoid and for every $x, y \in S$ there is an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.

Lemma 4.4.10. *Let S be a right zero monoid. Then all monocyclic right S-acts satisfy condition (P'_E) if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Suppose that all monocyclic right S-acts satisfy condition (P'_E). If $S = \{1\}$, then we are done. Otherwise, by Theorem 4.4.7, for every $x, y \in S$, there exists $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$. If $x \neq 1,\ y \neq 1$, then x, y are right zero and so $Sx = \{x\},\ Sy = \{y\}$. Thus $z \in Sx \cap Sy$ implies that $x = y = z$ and so S has just one right zero element which is also left zero and so a zero. Consequently, $S = \{1, 0\}$ as required.

The converse is true by Theorem 4.4.7. ∎

From Theorem 4.4.9, Lemma 4.4.10, and Theorem 4.3.5, we have

Corollary 4.4.11. *If S is a right zero monoid, then the following are equivalent:*

(1) All right S-acts are weakly flat.

(2) All finitely generated right S-acts are weakly flat.

(3) All cyclic right S-acts are weakly flat.

(4) All right S-acts satisfy condition (P'_E).

(5) All monocyclic right S-acts satisfy condition (P'_E).

(6) All right S-acts satisfy condition (P_E).

(7) All monocyclic right S-acts satisfy condition (P_E).

(8) $S = \{1\}$ *or* $S = \{1, 0\}$.

Since (P_E) implies (P'_E), then from Theorem 4.4.7, we have.

Theorem 4.4.12. *Let S be a monoid. If all monocyclic right S-acts satisfy condition (P_E), then S is regular and for every $x, y \in S$ there exists an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.*

Remark. By Example 4.3.3, it can be seen that $S = \{1, e, f\}$ is regular and for every $x, y \in S$ there exists $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$ and $A = \{x, y \mid xe = xf = y, x1 = x, ye = yf = y\} \simeq S/\rho(e, f)$, satisfies condition (P'_E), but it does not satisfy condition (P_E). Thus the converse of Theorem 4.4.12, is not true.

By Corollary 4.4.3, we saw that if $S/\rho(x, x^2)$ satisfies condition (P'_E), then x is regular. Now, by the following lemma we show that for left PP monoids the converse is also true.

Lemma 4.4.13. *Let S be a left PP monoid. If $x \in S$ is regular, then $S/\rho(x, x^2)$ satisfies condition (P'_E).*

Proof. By Lemma 4.3.11, it is sufficient to show that for every $u, v \in S$ with $u \ \rho \ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $s \ \rho \ e$, $t \ \rho \ f$, $eu = u$, $fv = v$ and $su = tv$. Let $u \ \rho \ v$. By Lemma 4.4.1, either $u = v$ or there exist $s', t' \in S$ such that $u = xs'$, $v = xt'$ and $x^m s' = x^n t'$ for some non-negative integers m, n with $m \equiv n (mod\ 1)$.

If $u = v$, then $s = t = e = 1$. Suppose that $u \neq v$. Since x is regular, then there exists $x' \in S$ such that $xx'x = x$. Then $u = xs'$ implies that $u = (xx'x)s'$. If $xx' = e$, then $e^2 = e$ and $u = exs' = eu$. Similarly, it can be seen that $v = ev$. Since $x^m s' = x^n t'$, then $x^{m+1} s' = x^{n+1} t'$ or $x^m (xs') = x^n (xt')$ and so $x^m u = x^n v$ or $x^m eu = x^n ev$.

Now we show that for every $k \geq 0$, $x^k e \ \rho \ e$. Since $e \ \rho \ e$, then for $k = 0$, it is satisfied. Let $k \geq 1$. At first we show by induction that $x \ \rho \ x^{k+1}$. Since $x \ \rho \ x^2$, then for $k = 1$, it is satisfied. If $x \ \rho \ x^k$ for $k > 1$, then $x^2 \ \rho \ x^{k+1}$ and so $x \ \rho \ x^2 \ \rho \ x^{k+1}$. Thus $xx' \ \rho \ x^{k+1}x'$ or $xx' \ \rho \ x^k xx'$ and so $e \ \rho \ x^k e$.

If $s = x^m e$, $t = x^n e$, then $s \ \rho \ e \ \rho \ t$, $eu = u$, $ev = v$ and $su = tv$ as required. ∎

Now, from Lemma 4.4.13, and Corollary 4.4.3, we have

Theorem 4.4.14. *Let S be a left PP monoid and $x \in S$. Then $S/\rho(x, x^2)$ satisfies condition (P'_E) if and only if x is regular.*

Proposition 4.4.15 [1]. *Let S be a monoid and $x \in S$. Then $S/\rho(x, x^2)$ is flat if and only if x is a regular element of S.*

From Corollary 4.4.3, Lemma 4.4.13, and Proposition 4.4.15, we have

Corollary 4.4.16. *For any monoid S the following statements are equivalent:*

(1) S is regular.

(2) Every monocyclic right S-act of the form $S/\rho(x, x^2)$, is flat.

(3) Every monocyclic right S-act of the form $S/\rho(x, x^2)$, satisfies condition (P'_E).

Now we give a class of monoids for which all right acts satisfy condition (P'_E).

Definition 4.4.17. An *orthodox semigroup* is defined as a regular semigroup in which idempotents form a subsemigroup.

The class of orthodox semigroups thus includes both the class of inverse semigroups and the class of bands.

Definition 4.4.18. A semigroup S (with zero) is called a *right inverse semigroup* (with zero) if every (nonnull) principal left ideal of S has a unique idempotent generator. A left inverse semigroup is defined dually. Right [left] inverse semigroups are also called $\mathcal{L}$-unipotent [$\mathcal{R}$-unipotent]. It can be shown that such semigroups are orthodox.

Theorem 4.4.19 [36]. *The following conditions on a regular semigroup S with zero are equivalent:*

(1) S is a right inverse semigroup.

(2) $fef = ef$ for any two nonzero idempotents e, f of S.

(3) $eS \cap fS = efS$ $(= feS)$ *for any two nonzero idempotents* e, f *of* S.

(4) if a' *and* a'' *are inverses of the nonzero element* a *of* S, *then* $a'a = a''a$.

(5) For any nonzero idempotent e *of* S *the set* F_e *(of inverses of* e*) is a right zero subsemigroup of* S.

(6) Let $e^2 = e$ *and* x *be nonzero elements of* S *and* x' *an inverse of* x. *Then* $x \in Se$ *implies that* $x' \in eS$.

Theorem 4.4.20. *Let* S *be a left inverse monoid. Then every right* S*-act satisfies condition* (P'_E).

Proof. Suppose that A is a right S-act and let $au = a'v$ with $a, a' \in A$, $u, v \in S$. Since S is regular, then there exist $u', v' \in S$ such that $u = uu'u$, $v = vv'v$. If $uu' = e$, $vv' = e'$, then $e^2 = e$, $e'^2 = e'$, $u = eu$ and $v = e'v$. Then we have

$$\begin{aligned} ae &= auu' = a(uu'u)u' = (au)(u'uu') = (a'v)(u'uu') \\ &= a'(vv'v)(u'uu') = a'v(v'v)(u'uu') = au(v'v)(u'uu') \\ &= a(uu'u)(v'v)(u'uu') = au(u'u)(v'v)(u'uu') \\ &= a'v(u'uv'vu')uu' \end{aligned}$$

$$\begin{aligned} a'e' &= a'vv' = a'(vv'v)v' = a'v(v'vv') = au(v'vv') \\ &= a(uu'u)(v'vv') = au(u'u)(v'vv') = a'v(u'uv')vv' \end{aligned}$$

If $s = u'uv'vu'$, $t = u'uv'$ and $a'' = a'v$, then $ae = a''se$, and $a'e' = a''te'$. Since $u'u$ and $v'v$ are idempotents and by assumption $(efe = ef)$, then

$$\begin{aligned} su &= (u'uv'vu')u = (u'u)(v'v)(u'u) \\ &= (u'u)(v'v) = (u'uv')v = tv. \end{aligned}$$

Thus, A satisfies condition (P'_E) as required. ∎

Definition 4.4.21. A semigroup S is called *left [right] generalized inverse*, if it is regular and $E(S)$ forms a left [right] normal band, or equivalently, S is regular and

$$xef = xfe \; [efx = fex] \quad xefy = xfey$$

for all $x, y \in S,\ e, f \in E(S)$.

Corollary 4.4.22. *Let S be a monoid. If S is inverse or left generalized inverse, then S is regular and for idempotents $e, f \in S$, $fef = fe$. Thus by the same argument as Theorem 4.4.20, it can be seen that every right S-act A satisfies condition (P'_E).*

Since condition (P'_E) implies weak flatness of acts, then from Theorem 4.4.20 we have

Corollary 4.4.23. *Left inverse monoids are right absolutely weakly flat.*

4.5. Flatness and Conditions (P_E), (P'_E)

In this section we consider the relationship between flatness of acts, monoids and conditions (P_E) and (P'_E). We show that these conditions do not generally coincide with flatness of acts, but for some classes of monoids they do.

By Example 4.3.3, and the remark after Lemma 4.3.4, it can be seen that flatness of acts does not imply condition (P_E). Also see the following example:

Example 4.5.1. Let $S = \{0, 1, e, a\}$ with table

	0	1	e	a
0	0	0	0	0
1	0	1	e	a
e	0	e	e	e
a	0	a	e	e

If $J = \{0, e\}$, then J is a right ideal of S. Since $0 \in J_0$ and $e \in J_e$, then by Proposition 2.3.54,

$$A(J) = \{x, y\} \times \{1, a\} \cup \{z\} \times \{0, e\} = \{(x, 1), (x, a), (y, 1), (y, a), (z, 0), (z, e)\}$$

is a flat right S-act. We claim that $A(J)$ does not satisfy condition (P_E). Otherwise

$$(y, 1)e = (x, a)a = (z, e),$$

implies that there exist $s, t, e'^2 = e' \in S,\ a'' \in A(J)$ such that

$$e'e = e,\ e'a = a,\ (y, 1)e' = a''se' \text{ and } (x, a)e' = a''te'.$$

Since $ea = e \neq a$, and also $0a = 0 \neq a$, then $e' = 1$. Thus

$$(y, 1)1 = a''s1 = a''s \text{ and } (x, a)1 = a''t1 = a''t.$$

Since $(y, 1)$ can be written just as $(y, 1)1$, then $a'' = (y, 1)$ and $s = 1$. But for every $t \in S$, $(x, a)1 \neq (y, 1)t$, which is a contradiction.

By Example 4.3.25, it can also be seen that flatness of acts does not imply condition (P'_E). Now we show that condition (P'_E) also does not imply flatness of acts. First of all we see that if E is a left [right] normal band, then it is a strong semilattice of left zero [right zero] bands i.e. $E = \varphi(\Gamma; R_\gamma; \phi_{\alpha,\beta})$ where Γ is a semilattice, each R_γ $(\gamma \in \Gamma)$ is a left [right] zero band, $E = \cup_{\gamma \in \Gamma} R_\gamma$, and the maps $\phi_{\alpha,\beta} : R_\alpha \to R_\beta$

($\alpha \geq \beta$) are the structure maps. We shall say E has constant structure maps if $\phi_{\alpha,\beta}$ is a constant function whenever $\alpha > \beta$ ($\alpha, \beta \in \Gamma$). It can be proved that E has constant structure maps if and only if

$$(\forall e, f, g \in E)(efg = ef \text{ or } efg = feg),$$

and in case E is right normal band

$$(\forall e, f, g \in E)(efg = fg \text{ or } efg = egf).$$

Theorem 4.5.2 [6]. *If S is a right absolutely flat left generalized inverse semigroup, then $E(S)$ has constant structure maps.*

Suppose that condition (P'_E) implies flatness of acts and let S be a left generalized inverse monoid. Then by Corollary 4.4.22, every right S-act satisfies condition (P'_E). Consequently, every right S-act is flat and so S is a right absolutely flat left generalized inverse monoid. Thus by Theorem 4.5.2, $E(S)$ has constant structure maps, and so by the argument before Theorem 4.5.2, for every $e, f, g \in E(S)$, $efg = ef$ or $efg = feg$.

Now, if we consider the monoid S with the following table

Example 4.5.3. Let $S = \{1, a, b, c, d\}$ with table

	1	a	b	c	d
1	1	a	b	c	d
a	a	a	a	a	a
b	b	d	b	b	d
c	c	a	c	c	a
d	d	d	d	d	d

then S is a left generalized inverse monoid such that every element is idempotent. But $cbd = a \neq c = cb$, and $bcd = d \neq a = cbd$, which is a contradiction.

Now we give some conditions for a monoid such that flatness and condition (P'_E) coincide.

Lemma 4.5.4 [5]. *Let S be a monoid. Each of the following conditions implies that every weakly flat S-act is flat.*

(1) S is weakly left absolutely flat.

(2) S is left PP and its idempotents form a right regular band.

Remark. If S is a right zero monoid, then it is left PP and also for every $x, y \in E(S) = S$, $xyx = yx$. Thus $S = E(S)$ is a right regular band.

By Lemma 4.5.4, and Theorem 4.3.29, we have the following theorems

Theorem 4.5.5. *Let S be a left PP monoid in which idempotents form a right regular band. Then a right S-act A is flat if and only if it satisfies condition (P'_E).*

Also from Lemma 4.5.5, and Lemma 4.3.29, we have

Lemma 4.5.6 [5]. *Let S be any monoid. If S is left PP, and for all $u, v \in E(S)$ there exists $z \in uS \cap vS$ such that $(z, u) \in \theta_L(u, v)$, then every weakly flat right S-act is flat.*

Theorem 4.5.7. *Let S be a left PP monoid such that for all $u, v \in E(S)$, there exists $z \in uS \cap vS$ such that $(z, u) \in \theta_L(u, v)$ and let A be a right S-act. Then A is flat if and only if it satisfies condition (P'_E).*

Note : For $u, v \in S$, $\theta_L(u, v)$ $[\theta_R(u, v)]$ is the smallest left [right] congruence on S containing (u, v).

The following monoids satisfy condition (2) of Lemma 4.5.4.

1. Commutative PP monoids (characterized in [21]).
2. Right PP monoids with central idempotents (characterized in [14]) (This includes all left cancellative monoids. For more information see [4].
3. S^1 where S is any left generalized inverse semigroup (see [37]).

Thus, for monoids mentioned in 1, 2 and 3, an act A is flat if and only if it satisfies condition (P'_E).

From Corollary 4.3.32, and remark after Lemma 4.5.4, we can deduce the following corollary.

Corollary 4.5.8. *Let S be a right zero monoid. Then every flat right S-act satisfies condition (P_E).*

4.6. Characterization of Monoids by Condition (E)

Recently many authors have paid attention to characterization of monoids by condition (P). In previous chapters we also considered characterization of monoids by conditions (P), (P'_E), (P_E) and in case (P), monoids for which all flat (cyclic) right S-acts satisfy condition (P). This is because a more convenient property between flatness and strong flatness under the aspect of homological classification seems to be condition (P). There are also some papers in which classification of monoids by condition (E) has been considered. For example, Liu in [38] considered characterization of monoids over which all left S-acts satisfying condition (E) are flat. Such monoids are exactly the regular ones. In this section we try to characterize some classes of monoids for which flatness of acts implies condition (E), monoids for which all (cyclic) acts satisfy condition (E), and finally, monoids for which all acts having property (E) satisfy conditions (P'_E). There are some corollaries that will arise.

First of all we show that weak flatness of acts and also conditions (P_E) and (P'_E) do not imply condition (E). See the following example:

Example 4.6.1. Let $S =< x, y \mid xy = yx = y, x^2 = x, y^3 = y^2, xy^2 = y^2 > \cup\{1\}$ with table

	1	x	y	y^2
1	1	x	y	y^2
x	x	x	y	y^2
y	y	y	y^2	y^2
y^2	y^2	y^2	y^2	y^2

and let $A = \{a, b \mid ax = ay = ay^2 = bx = by = by^2 = a, a1 = a, b1 = b\}$. Then A is a right S-act which is not cyclic. We have $bx = by$ and the only case for b is $b = b1$. But

$$1x = x \neq 1y = y.$$

Thus, A does not satisfy condition (E). Now we show that A satisfies condition (P_E). Let $au = a'v$ for $a, a' \in A$ and $u, v \in S$. Then we show that there exist $s, t, e^2 = e \in S$ and $a'' \in A$ such that $eu = u$, $ev = v$, $ae = a''se$, $a'e = a''te$ and $su = tv$.

If $bx = by$, then $e = x$, $s = y^2$, $t = y^2$, $a'' = b$.

If $bx = ay^2$, then $e = x$, $s = y^2$, $t = y^2$, $a'' = b$.

If $bx = by^2$, then $e = x,\ s = t = y^2,\ a'' = b$.

If $bx = ax$, then $e = x,\ s = t = 1,\ a'' = b$.

If $bx = ay$, then $e = x,\ s = t = y^2,\ a'' = b$.

If $by = ay^2$, then $e = x,\ s = t = y,\ a'' = b$.

If $by = by^2$, then $e = x,\ s = t = y,\ a'' = b$.

If $by = ax$, then $e = x,\ s = t = y^2,\ a'' = b$.

If $by = ay$, then $e = x,\ s = t = 1,\ a'' = b$.

If $ay^2 = by^2$, then $e = 1,\ s = y,\ t = 1,\ a'' = b$.

If $ay^2 = ax$, then $e = x,\ s = 1,\ t = y^2,\ a'' = a$.

If $ay^2 = ay$, then $e = 1,\ s = 1,\ t = y,\ a'' = a$.

If $by^2 = ax$, then $e = x,\ s = 1,\ t = y^2,\ a'' = b$.

If $by^2 = ay$, then $e = x,\ s = 1,\ t = y,\ a'' = b$.

If $ax = ay$, then $e = x,\ s = y,\ t = 1,\ a'' = a$.

If $bz = a1,\ z \in \{x, y, y^2\}$, then $e = 1,\ s = 1,\ t = z,\ a'' = b$.

If $az = a1,\ z \in \{x, y, y^2\}$, then $e = 1,\ s = 1,\ t = z,\ a'' = a$.

Thus A satisfies condition (P_E), and so A satisfies condition (P'_E). Consequently, by Theorem 4.3.24, A is weakly flat, but A does not satisfy condition (E).

Remark. From Example 4.6.1, it can also be seen that A does not satisfy condition (P). Otherwise, $bx = by$ implies that there exist $b'' \in A,\ s, t \in S$ such that $b = b''s, b = b''t$ and $sx = ty$. Since the only case is $b = b1$, then $s = t = 1$. But $1x \neq 1y$ which is a contradiction. Notice that monoid S in the previous example is not right nil.

Notice also that if $S \neq \{1\}$ is a group, then every right S-act satisfies condition (P) and so every right S-act is flat. Let $A = \{a\}$ such that $\forall s \in S,\ as = a$, then

A is a right S-act and so it is flat. We claim that A does not satisfy condition (E). Otherwise, $au = av$ for $u, v \in S$ such that $u \neq v$, and so there exists $s \in S$ such that $a = as$ and $su = sv$. Then $su = sv$ implies that $u = v$ which is a contradiction. Thus this example shows that for groups, flatness of acts does not imply condition (E) in general. But by what follows we show that for right zero monoids weak flatness of acts implies condition (E).

Lemma 4.6.2. *Let S be a monoid. If all flat cyclic right S-acts satisfy condition (E), then S is right nil*

Proof. Suppose that all flat cyclic right S-acts satisfy condition (E), Then by Corollary 3.2.6, all flat cyclic right S-acts are strongly flat and so by Theorem 2.3.28, S is right nil. ■

Corollary 4.6.3. *Let S be a monoid. If all flat cyclic right S-acts satisfy condition (E), then every $e \in E(S) \setminus \{1\}$ is right zero.*

Proof. Suppose that all flat cyclic right S-acts satisfy condition (E). Then by Lemma 4.6.2, S is right nil. Let $e \in E(S) \setminus \{1\}$, then there exists $k \in \mathbb{N}$ such that $e^{k+1} = e^k$ is right zero. Since $e = e^2$, then by induction it can be seen that for every $n \in \mathbb{N}$, $e^n = e$ and so $e^k = e$ is right zero. ■

From Lemma 4.6.2, it can be seen that if all flat right acts satisfy condition (E), then S is right nil, but we do not know if the converse is true. By what follows we show that the converse is true if we consider left PP monoids.

Lemma 4.6.4. *Let S be a left PP monoid. Then S is right nil if and only if S is right zero.*

Proof. If S is right zero, then it is obvious that S is right nil.

Suppose that S is right nil and let $x \in S$. Since S is left PP, then there exists $e^2 = e \in S$ such that $ex = x$, and for every $a, b \in S, ax = bx$ implies that $ae = be$. If $e = 1$, then x is right cancellative, otherwise as we saw in the proof of Corollary 4.6.3, e is right zero and so x is right zero.

Now we show that the only cancellative element is 1. Let $x \in S$ be cancellative. Since S is right nil, then there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. Then by cancelling x^n, $x = 1$ as required. Consequently, every element different from 1 is right zero as required. ■

Lemma 4.6.5. *Let S be a left PP monoid. If all flat cyclic right S-acts satisfy condition (E), then S is right zero.*

Proof. If all flat cyclic right S-acts satisfy condition (E), then, by Lemma 4.6.2, S is right nil. But S is left PP and so by Lemma 4.6.4, S is right zero. ■

From Lemma 4.6.5, we have

Corollary 4.6.6. *Let S be a left PP monoid. If all flat right S-acts satisfy condition (E), then S is right zero.*

Lemma 4.6.7. *Let S be a right zero monoid. Then all weakly flat right S-acts satisfy condition (E).*

Proof. Suppose that A is a weakly flat right S-act and let $au = av$ for $a \in A$, $u, v \in S$. Since S is left PP, then by Theorem 4.3.29, A satisfies condition (P'_E). Thus there exist $a'' \in A$, $s, t, e^2 = e, f^2 = f \in S$ such that $ae = a''se$, $af = a''tf$, $eu = u$, $fv = v$ and $su = tv$.

Now there are three cases as follow:

Case 1. $u = v = 1$. Then $a = a1$ and $1u = 1v$.

Case 2. $u = 1$, $v \neq 1$. Then $a = av$. Since $v \neq 1$, then v is right zero, and so $v^2 = v$. Consequently, $a = av$ and $v1 = v^2 = vv$.

Case 3. $u \neq 1$, $v \neq 1$. Then u, v are right zero, and so $su = tv$ implies that $u = v$. Consequently, $a = a1$ and $1u = 1v$.

Thus, A satisfies condition (E) as required. ■

From Lemma 4.6.7, we have.

Corollary 4.6.8. *Let S be a right zero monoid. Then all flat right S-acts satisfy condition (E).*

From Corollary 4.6.6, and Corollary 4.6.8, we have the following corollary

Corollary 4.6.9. *Let S be a left PP monoid. Then all flat right S-acts satisfy condition (E) if and only if S is right zero.*

Now from Lemma 4.6.4, Lemma 4.6.7, Corollary 3.2.6, Corollary 4.6.9, and Theorem 2.3.28, the following theorem can be deduced.

Theorem 4.6.10. *Let S be a left PP monoid. then the following statements are equivalent:*

(1) All weakly flat right S-acts satisfy condition (E).

(2) All flat right S-acts satisfy condition (E).

(3) All weakly flat cyclic right S-acts are strongly flat.

(4) All flat cyclic right S-acts are strongly flat.

(5) All weakly flat cyclic right S-acts are projective.

(6) All flat cyclic right S-acts are projective.

(7) S is right nil.

(8) S is right zero.

Remark. Since every regular monoid and every idempotent monoid is left PP, then regular monoids and idempotent monoids satisfy Theorem 4.6.10.

From Lemma 4.6.5, Lemma 4.6.7, and since every right zero monoid is left PP, we have

Theorem 4.6.11. *For any monoid S the following statements are equivalent:*

(1) S is right zero.

(2) S is left PP and every weakly flat cyclic right S-act satisfies condition (E).

(3) S is left PP and every flat cyclic right S-act satisfies condition (E).

Lemma 4.6.12. *Let S be a monoid such that $|E(S)| = 1$. If all flat cyclic right S-acts satisfy condition (E), then $S = \{1\}$.*

Proof. If all flat cyclic right S-acts satisfy condition (E), then by Lemma 4.6.2, S is right nil. Thus for every $x \in S$ there exists $k \in \mathbb{N}$ such that $x^{k+1} = x^k$. But x^k is an idempotent, and so $x^k = 1$. Consequently, $x = 1$ as required. ■

Corollary 4.6.13. *Let S be a monoid such that $|E(S)| = 1$. Then all flat right S-acts satisfy condition (E) if and only if $S = \{1\}$.*

Proof. If all flat right S-acts satisfy condition (E), then all flat cyclic right S-acts satisfy condition (E) and so by Lemma 4.6.12, $S = \{1\}$.

If $S = \{1\}$, then all right S-acts satisfy condition (E), and so all flat right S-acts satisfy condition (E). ■

If a monoid S is right cancellative, then for every idempotent e, $e^2 = e$ implies that $e = 1$, and so $|E(S)| = 1$. Thus by Corollary 4.6.13, we have

Corollary 4.6.14. *Let S be a right cancellative monoid. Then all flat right S-acts satisfy condition (E) if and only if $S = \{1\}$.*

By considering right reversible monoids for which all flat cyclic right acts satisfy condition (E), we have the following results:

Theorem 4.6.15. *Let S be a right reversible monoid. Then all flat cyclic right S-acts satisfy condition (E), if and only if either $S = \{1\}$ or $S = T^1$ where T is a nil semigroup.*

Proof. Suppose that all flat cyclic right S-acts satisfy condition (E). Then by Lemma 4.6.2, S is right nil. Also by Corollary 3.2.6, all flat cyclic right S-acts satisfy condition (P). Thus by Theorem 2.3.39, either $E(S) = \{1\}$ or $E(S) = \{0, 1\}$. If $E(S) = \{1\}$, then by Lemma 4.6.12, $S = \{1\}$. Suppose that $E(S) = \{0, 1\}$. Since S is right nil, then for every $1 \neq x \in S$ there exists $k \in \mathbb{N}$ such that $x^{k+1} = x^k$ is right zero and so x^k is an idempotent. If $x^k = 1$, then $x^{k+1} = x^k$ implies that $x = 1$ which is a contradiction. Consequently, $x^k = 0$ and so x is nil as required.

If $S = \{1\}$, then all cyclic right S-acts are strongly flat and so all flat cyclic right S-acts satisfy condition (E). If $S = T^1$ such that T is a nil semigroup, then by Theorem 2.3.28, all flat cyclic right S-acts are strongly flat and so all flat cyclic right S-acts satisfy condition (E). ■

From Corollary 3.2.6 and Theorem 4.6.15 we have.

Corollary 4.6.16. *Let S be a right reversible monoid. Then all flat cyclic right S-acts are strongly flat if and only if either $S = \{1\}$ or $S = T^1$ where T is a nil semigroup.*

Corollary 4.6.17. *Let S be a monoid that is a right reversible band. Then all flat cyclic right S-acts satisfy condition (E) if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Suppose that all flat cyclic right S-acts satisfy condition (E). Then by Theorem 4.6.15, either $S = \{1\}$ or $S = T^1$ where T is a nil semigroup. If $S = \{1\}$, then we are done. Otherwise let $1 \neq x \in S$. Then there exists $k \in \mathbb{N}$ such that $x^k = 0$. Since S is a band, then for every $n \in \mathbb{N}$, $x^n = x$. Thus $x = x^k = 0$ and so $S = \{0, 1\}$.

If either $S = \{1\}$ or $S = \{0, 1\}$, then by Corollary 4.6.16, all flat cyclic right S-acts are strongly flat and so all flat cyclic right S-acts satisfy condition (E). ■

Now by using Lemma 4.6.2, and results from previous chapters we give a characterization of monoids for which all cyclic right acts satisfy condition (E).

Theorem 4.6.18. *Let S be a monoid. Then all cyclic right S-acts satisfy condition (E) if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. Suppose that all cyclic right S-acts satisfy condition (E). Then all flat cyclic right S-acts satisfy condition (E) and so by Lemma 4.6.2, S is right nil. Also by Corollary 3.2.6, all cyclic right S-acts satisfy condition (P). Thus by Theorem 2.3.64, $S = G$ or $S = G \cup \{0\}$ where G is a group. Let $1 \neq x \in G$. Since S is right nil, then there exists $k \in \mathbb{N}$ such that $x^{k+1} = x^k$. But $x^k \in G$ and so $x^{k+1} = x^k$ implies that $x = 1$ which is a contradiction. Thus $G = \{1\}$.

If $S = \{1\}$ or $S = \{0, 1\}$, then it is easy to see that all cyclic right S-acts are strongly flat and so all cyclic right S-acts satisfy condition (E). ■

From Theorem 4.6.18 we have.

Corollary 4.6.19. *Let S be a monoid. Then all cyclic right S-acts are strongly flat if and only if $S = \{1\}$ or $S = \{0, 1\}$.*

From Theorem 4.6.18, and Corollary 4.6.19, the following theorem can be deduced.

Theorem 4.6.20. *For a monoid S the following statements are equivalent:*

(1) All right S-acts satisfy condition (E).

(2) All cyclic right S-acts satisfy condition (E).

(3) All cyclic right S-acts are strongly flat.

(4) $S = \{1\}$ or $S = \{0, 1\}$.

By Lemma 4.6.2, it can be seen that for a monoid S if all flat right acts satisfy condition (E), then S is right nil. Now questions that arise here are as follows:

Is the condition S is right nil also sufficient for all flat right S-acts to satisfy condition (E) ?

Is there a right nil monoid S and a right S-act A which is flat but does not satisfy condition (E) ?

Normak in [31, Example 1.13] showed that condition (E) does not imply flatness. Then Liu in [38] gave a characterization of monoids over which all left acts satisfying condition (E) are flat. Now by regarding conditions (E), (P_E), (P'_E), and Corollary 3.2.6, it can be seen that for a monoid S, if a cyclic right S-act satisfies condition (E), then it satisfies condition (P), and so it satisfies conditions (P_E), and (P'_E). But by the following example it can be seen that condition (E) does not imply conditions (P_E) and (P'_E) in general. Then we give a characterization of monoids over which all right S-acts satisfying condition (E) satisfy condition (P'_E).

Example 4.6.21. Let $S = \{0, 1, e, a\}$ with table

	0	1	e	a
0	0	0	0	0
1	0	1	e	a
e	0	e	e	0
a	0	a	a	0

If $J = \{0, a\}$, then $J = aS$ is a proper right ideal of S. By Proposition 2.3.53, $A(J)$ satisfies condition (E). Now we show that $A(J)$ does not satisfy conditions (P_E), and (P'_E). Suppose that $A(J)$ satisfies condition (P'_E). Since $(x, 1)0 = (y, e)a$, then there exist $s, t, e'^2 = e'$, $e''^2 = e'' \in S$ and $a'' \in A$ such that

$$(x,1)e' = a''se',\ (y,e)e'' = a''te'',\ e'0 = 0,\ e''a = a \text{ and } s0 = ta.$$

Since $ea = 0 \neq a$, then $e'' \neq e$. Thus $e'' = 1$ and so $(y,e)1 = a''t$. Since for every $s \in S$, $s0 = 0$, then we need to choose $t \in S$ such that $ta = 0$. Thus either $t = 0$ or $t = a$. If $t = 0$, then for every $a'' \in A(J)$, $a''0 = (z,0) \neq (y,e)$ and so we have a contradiction. If $t = a$, then for every $a'' \in A(J)$, either $a''a = (z,0)$ or $a''a = (z,a)$ which in both cases $a''a \neq (y,e)$ and again we have a contradiction. $A(J)$ also does not satisfy condition (P_E). Otherwise it will satisfy condition (P'_E) which is a contradiction.

By [38], we have the following proposition:

Proposition 4.6.22. *For any monoid S the following conditions are equivalent:*

(1) S is regular monoid.

(2) All right S-acts satisfying condition (E) are flat.

(3) All right S-acts satisfying condition (E) are weakly flat.

(4) All right S-acts satisfying condition (E) are principally weakly flat.

(5) All right S-acts are principally weakly flat.

Since by Corollary 4.3.30, for regular monoids weak flatness of acts implies condition (P'_E), then from Proposition 4.6.22, we have

Theorem 4.6.23. *For any monoid S the following conditions are equivalent:*

(1) S is a regular monoid.

(2) All right S-acts satisfying condition (E) are flat.

(3) All right S-acts satisfying condition (E) are weakly flat.

(4) All right S-acts satisfying condition (E) are principally weakly flat.

(5) All right S-acts are principally weakly flat.

(6) All right S-acts satisfying condition (E) satisfy condition (P'_E).

By Theorem 4.6.23, and Proposition 2.3.53, we have

Corollary 4.6.24. *Let S be a regular monoid. If J is a proper right ideal of S, then $A(J)$ satisfies condition (P'_E).*

Remark. If S is a regular monoid and J a proper right ideal of S, then by Proposition 2.3.53, $A(J)$ does not satisfy condition (P). But by Corollary 4.6.24, $A(J)$ satisfies condition (P'_E). Therefore, $A(J)$ can be considered as an example of acts which satisfies condition (P'_E), but does not satisfy condition (P).

Corollary 4.6.25. *If S is a right zero monoid, then all right S-acts satisfying condition (E) satisfy condition (P_E).*

Proof. Since every right zero monoid is regular, then by Theorem 4.6.23, all right S-acts satisfying condition (E) satisfy condition (P'_E). But by Theorem 4.3.5, condition (P'_E) implies condition (P_E) and so all right S-acts satisfying condition (E) satisfy condition (P_E) as required. ∎

If all right S-acts satisfying condition (E) satisfy condition (P_E), then all right S-acts satisfying condition (E) satisfy condition (P'_E) and so by Theorem 4.6.23, S is regular, but that S is right zero is unknown.

The following is another example of monoids for which conditions (P_E) and (P'_E) do not imply condition (E).

Example 4.6.26. Let $S = \{0, 1, s \mid s^2 = 1\}$ and let $A = \{z, a \mid as = a,\ a0 = zs = z0 = z,\ a1 = a,\ z1 = z\}$. Then A is a right S-act and also it satisfies condition (P). Consequently, A satisfies conditions (P_E) and (P'_E). But A does not satisfy condition (E). Otherwise, $as = a1$ implies that there exist $t \in S$ and $a'' \in A$ such that $a = a''t$ and $ts = t1$. Since either $a = as$ or $a = a1$, then either $t = s$ or $t = 1$. If $t = 1$, then $ts = t1$ implies that $s = 1$ which is a contradiction. If $t = s$, then

$$s = t = t1 = ts = ss = 1.$$

which is a also a contradiction.

Chapter 5

Characterization of Monoids by Properties of Generators

5.1. Introduction

In the previous chapters we investigated properties of acts over monoids and we gave some classification of monoids by these properties. Although there are several papers which investigate various properties of acts of monoids, among them generators, there seems to be very little known. In this chapter, by using the property of coretractions we try to characterize some classes of monoids by properties of generators.

5.2. Characterization by Properties of Generators

Definition 5.2.1. Let S be a monoid. An S-act G_S is called a *generator* for an S-act A_S if for every pair of homomorphisms $(\alpha \neq \beta)$, $\alpha : A_S \to B_S$ and $\beta : A_S \to B_S$, there exists a homomorphism $f : G_S \to A_S$ in Act-S such that $\alpha f \neq \beta f$.

An S-act G_S is called a *generator* in Act-S if for every $A_S \in$ Act-S and every pair of homomorphisms $(\alpha \neq \beta)$, $\alpha : A_S \to B_S$ and $\beta : A_S \to B_S$ there exists a homomorphism $f : G_S \to A_S$ in Act-S such that $\alpha f \neq \beta f$.

Lemma 5.2.2. *Let S be a monoid. Then S is a generator in Act-S.*

Proof. Let $\alpha : A_S \to B_S$ and $\beta : A_S \to B_S$ be homomorphisms such that $\alpha \neq \beta$.

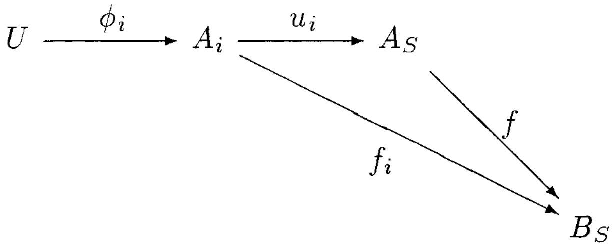
Then there exists $a \in A_S$ such that $\alpha(a) \neq \beta(a)$. Let $f : S \to A_S$ be such that $f(s) = as$ for every $s \in S$. Then f is a homomorphism and $f(1) = a$. Thus we have

$$(\alpha f)(1) = \alpha(f(1)) = \alpha(a) \neq \beta(a) = \beta(f(1)) = (\beta f)(1),$$

and so by Definition 5.2.1, S is a generator. ■

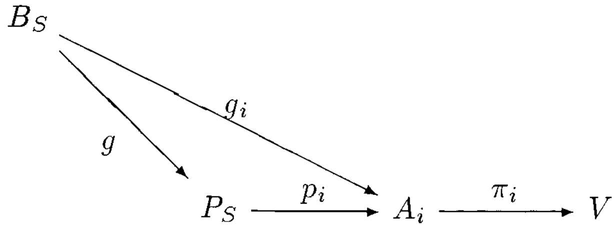
Definition 5.2.3. A homomorphism $\theta : A_S \to B_S$ is called a *coretraction* if there is a homomorphism $\theta' : B_S \to A_S$ such that $\theta'\theta = 1_A$. We shall say that A_S is a *retract* of B_S in this case.

Definition 5.2.4. Let S be a monoid. Let U be a right S-act and let $\{A_i \mid i \in I\}$ be a family of right S-acts. Let $\{\phi_i \in Hom(U, A_i) \mid i \in I,\ U \in Act\text{–S}\}$ be a family of monomorphisms. Then there exist $A_S \in$ Act-S and $\{u_i \in Hom(A_i, A_S) \mid u_i\phi_i = u_j\phi_j,\ i,j \in I\}$ such that for every $B_S \in$ Act-S and every family $\{f_i \in Hom(A_i, B_S) \mid f_i\phi_i = f_j\phi_j,\ i,j \in I\}$ there exists a unique $f \in Hom(A_S, B_S)$ which makes commutative the following diagrams in Act-S.



f is called the *amalgamated coproduct* of $A_i, i \in I$ with respect to U and is denoted by $(\coprod_{i\in I}^{U} A_i, u_i)$. This amalgamated coproduct is also called a *multiple pushout.*

If we consider the family of diagrams in Act-S



where all π_i are surjective, $i \in I$, $P_S \in$ Act-S and $\{p_i \in Hom(P_S, A_i) \mid \pi_i p_i = \pi_j p_j, i,j \in I\}$ such that for any $B_S \in$ Act-S and every family $\{g_i \in Hom(B_S, A_i) \mid \pi_i g_i = \pi_j g_j, i,j \in I\}$ there exists a unique $g \in Hom(B_S, P_S)$ which makes these diagrams commutative, then g will be called <u>*coamalgamated product*</u> or <u>*multiple pullback*</u> of the A_i, $i \in I$ with respect to V and will be denoted by $\prod_{i \in I_V} A_i$.

<u>**Lemma 5.2.5 [23].**</u> *Let A_i, $i \in I$ be a family of right S-acts, one of which is a generator in Act-S and let $P = \prod_{i \in I_V} A_i$ be a coamalgamated product. Then $\prod_{i \in I_V} A_i$ is also a generator in Act-S.*

<u>**Lemma 5.2.6 [23].**</u> *Let $\pi_i : A_i \to V$ be a family of surjective homomorphisms in Act-S. Consider*

$$P_S = \{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid \pi_i(a_i) = \pi_j(a_j), i, j \in I \}$$

and let p_i, $i \in I$ be the restriction to P_S of the canonical projection from $\prod_{i \in I} A_i$. Then $(P_S, (p_i)_{i \in I})$ is isomorphic to the coamalgamated product $\prod_{i \in I_V} A_i$.

<u>**Lemma 5.2.7 [23].**</u> For any monoid S we have,

(1) $S \prod A_S$ is a generator for any $A_S \in$ Act-S.

(2) If moreover, S has a left zero 0, then A_S is a retract of $S \prod A_S$ with retraction being the second projection p_2 and coretraction defined by $a \mapsto (0, a)$ for $a \in A_S$.

Now by using Lemma 5.2.7, we give a characterization of monoids S with a left zero by condition (E) of generators which will extend the characterization of these monoids by equalizer-flatness of generators.

<u>**Lemma 5.2.8.**</u> *Let S be a monoid, and A_S a retract of B_S. If B_S satisfies condition (E), then A_S satisfies condition (E).*

Proof. Let $au = av$ for $a \in A_S$ and $u, v \in S$. Since A_S is retract of B_S, then there are homomorphisms

$$\theta : A_S \to B_S \text{ and } \theta' : B_S \to A_S$$

such that $\theta' \theta = 1_A$. Then

$$\theta(au) = \theta(av) \text{ or } (\theta(a))u = (\theta(a))v. \qquad (*)$$

Since $\theta(a) \in B_S$, $u, v \in S$, and B_S satisfies condition (E), then $(*)$ implies that there exist $b'' \in B_S$ and $t \in S$, such that $\theta(a) = b''t$ and $tu = tv$. Since $\theta(a) = b''t$, then

$$\theta'(\theta(a)) = \theta'(b''t) = (\theta'(b''))t,$$

or

$$\theta'\theta(a) = (\theta'(b''))t \Rightarrow a = (\theta'(b''))t.$$

If $a'' = \theta'(b'') \in A_S$, then $a'' \in A_S$, and $a = a''t$. Since $tu = tv$, then A_S satisfies condition (E) as required. ∎

Lemma 5.2.9. *Let S be a monoid with a left zero. Then all generators in Act-S satisfy condition (E) if and only if all right S-acts satisfy condition (E).*

Proof. Let A_S be an S-act. Since every generator in Act-S satisfies condition (E), then by Lemma 5.2.7 (1), $S \coprod A_S$ also satisfies condition (E). By Lemma 5.2.7 (2), A_S is a retract of $S \coprod A_S$ and so by Lemma 5.2.8, A_S satisfies condition (E).

The converse is obvious. ∎

Now from Lemma 5.2.9, and Theorem 2.2.19, we can deduce the following extension of [23, Corollary 3.7].

Theorem 5.2.10. *If S is a monoid with a left zero 0, then the following conditions are equivalent:*

1) All generators in Act-S satisfy condition (E).

2) All right S-acts satisfy condition (E).

3) All generators in Act-S are equalizer-flat.

4) All right S-acts are equalizer-flat.

5) All cyclic right S-acts satisfy condition (E).

6) All cyclic right S-acts are strongly flat.

7) $S = \{1\}$ or $S = \{0, 1\}$.

By using the following theorem we give a characterization of monoids by condition (P) of generators and use this to characterize aperiodic monoids for which all generators are free, projective, pullback-flat and satisfy condition (P).

Theorem 5.2.11 [23]. *The following conditions on a monoid S are equivalent:*

1) All generators in Act-S are free.

2) All generators in Act-S are projective.

3) All generators in Act-S are pullback-flat.

4) All generators in Act-S satisfy condition (P).

5) S is a group.

Theorem 2.2.14, and Lemma 5.2.11, give

Theorem 5.2.12. *The following conditions on a monoid S are equivalent:*

1) All generators in Act-S satisfy condition (P).

2) All right S-acts satisfy condition (P).

3) S is a group.

Lemma 5.2.13. *Let S be an aperiodic monoid. Then all generators in Act-S satisfy condition (P) if and only if $S = \{1\}$.*

Proof. Suppose that all generators in Act-S satisfy condition (P). Then by Theorem 5.2.12, S is a group. On the other hand for every $x \in S$, there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. Thus by cancelling x^n, $x = 1$, and so $S = \{1\}$ as required.

The converse is obvious. ■

Corollary 5.2.14. *Let S be an aperiodic monoid. Then all generators in Act-S satisfy condition (P) if and only if all right S-acts are free.*

Proof. If all generators satisfy condition (P), then by Lemma 5.2.13, $S = \{1\}$. Therefore all right S-acts are free.

The converse is obvious. ∎

Now from Theorem 5.2.11, Lemma 5.2.13, and Corollary 5.2.14, we have

Theorem 5.2.15. *For an aperiodic monoid S the following conditions are equivalent:*

(1) All generators in Act-S are free.

(2) All right S-acts are free.

(3) All generators in Act-S are projective.

(4) All right S-acts are projective.

(5) All generators in Act-S are pullback-flat.

(6) All right S-acts are pullback-flat.

(7) All generators in Act-S satisfy condition (P).

(8) All right S-acts satisfy condition (P).

(9) $S = \{1\}$.

From Theorem 5.2.10, and Theorem 5.2.11, the following corollary can be deduced.

Corollary 5.2.16. *Let S be a monoid with a left zero. Then the following are equivalent:*

(1) All generators in Act-S are strongly flat.

(2) All right S-acts are strongly flat.

(3) $S = \{1\}$.

Kilp in [23] (Lemma 5.2.19, below) showed that for monoids S with a left zero, all generators in Act-S are weakly flat if and only if all right S-acts are weakly flat. By the following lemmas we show that for monoids S with a left zero, all generators in Act-S satisfy condition (P'_E) if and only if all right S-acts satisfy condition (P'_E). Then by using this we show that among monoids S with a left zero, those for which all generators in Act-S are weakly flat and those for which all generators in Act-S satisfy condition (P'_E) are regular monoids S in which for every $x, y \in S$ there is an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.

Lemma 5.2.17. *Let S be a monoid and A_S a retract of B_S. If B_S satisfies condition (P'_E), then A_S satisfies condition (P'_E).*

Proof. Let $au = a'v$ for $a, a' \in A_S$, $u, v \in S$. Since A_S is a retract of B_S, then there are homomorphisms $\theta : A_S \to B_S$ and $\theta' : B_S \to A_S$ such that $\theta'\theta = 1_A$. Then we have

$$\theta(au) = \theta(a'v) \text{ or } (\theta(a))u = (\theta(a'))v. \qquad (*)$$

Since $\theta(a), \theta(a') \in B_S$ and B_S satisfies condition (P'_E), then $(*)$ implies that there exist $s, t, e^2 = e, f^2 = f \in S$, $b'' \in B_S$ such that

$$\theta(a) = b''se, \ \theta(a') = b''tf, \ eu = u, \ fv = v \text{ and } su = tv.$$

Then

$$\theta'\theta(a) = \theta'(b''se) = (\theta'b'')se \text{ and } \theta'\theta(a') = \theta'(b''tf) = (\theta'b'')tf.$$

If $\theta'(b'') = a''$, then $a'' \in A_S$ and $a = a''se$, $a' = a''tf$. Since $eu = u$, $fv = v$ and $su = tv$, then A_S satisfies condition (P'_E) as required. ■

Lemma 5.2.18. *Let S be a monoid with a left zero. Then all generators in Act-S satisfy condition (P'_E) if and only if all right S-acts satisfy condition (P'_E).*

Proof. Suppose that all generators in Act-S satisfy condition (P'_E) and let A be a right S-act. Then by Lemma 5.2.7 (1), $S \coprod A$ satisfies condition (P'_E). By Lemma 5.2.7 (2), A is a retract of $S \coprod A$ and so by Lemma 5.2.17, A satisfies condition (P'_E).

If all right S-acts satisfy condition (P'_E), then it is obvious that all generators in Act-S satisfy condition (P'_E). ■

Lemma 5.2.19 [23]. *Let S be a monoid with a left zero. If all generators in Act-S are flat or weakly flat, then all right S-acts have this property.*

Now from Lemma 5.2.18, Lemma 5.2.19, and Theorem 4.4.9, we have

Theorem 5.2.20. *Let S be a monoid with a left zero. Then the following conditions are equivalent:*

(1) All generators in Act-S are weakly flat.

(2) All right S-acts are weakly flat.

(3) All finitely generated right S-acts are weakly flat.

(4) All cyclic right S-acts are weakly flat.

(5) All generators in Act-S satisfy condition (P_E').

(6) All right S-acts satisfy condition (P_E').

(7) All monocyclic right S-acts satisfy condition (P_E').

(8) S is regular monoid and for every $x, y \in S$ there is an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.

From Theorem 4.5.5, Theorem 5.2.19, and Theorem 5.2.20, we have

Theorem 5.2.21. *Let S be a left PP monoid with a left zero and idempotents form a right regular band. Then the following conditions are equivalent:*

(1) All generators in Act-S are weakly flat.

(2) All right S-acts are weakly flat.

(3) All finitely generated right S-acts are weakly flat.

(4) All cyclic right S-acts are weakly flat.

(5) All generators in Act-S satisfy condition (P_E').

(6) All right S-acts satisfy condition (P_E').

(7) All monocyclic right S-acts satisfy condition (P_E').

(8) All generators in Act-S are flat.

(9) All right S-acts are flat.

(10) S is regular and for every $x, y \in S$ there is an element $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$.

Remark. If S is a left PP monoid with a left zero and for all $u, v \in E(S)$ there exists $z \in uS \cap vS$ such that $(z, u) \in \theta_{L(u,v)}$, then by Theorem 4.5.7, we have the same results as in Theorem 5.2.21.

Now similar to condition (P'_E) for monoids S with a left zero we show that if all generators in Act-S satisfy condition (P_E), then all right S-acts satisfy condition (P_E).

Lemma 5.2.22. *Let S be a monoid and A_S a retract of B_S. If B_S satisfies condition (P_E), then A_S satisfies condition (P_E).*

Proof. Let $au = a'v$ for $a, a' \in A_S$, $u, v \in S$. Since A_S is a retract of B_S, then there are homomorphisms $\theta : A_S \to B_S$ and $\theta' : B_S \to A_S$ such that $\theta'\theta = 1_A$. Then we have

$$\theta(au) = \theta(a'v) \text{ or } (\theta(a))u = (\theta(a'))v. \qquad (*)$$

Since $\theta(a)$, $\theta(a') \in B_S$ and B_S satisfies condition (P_E), then $(*)$ implies that there exist $s, t, e^2 = e \in S$, $b'' \in B_S$ such that

$$\theta(a) = b''se,\ \theta(a') = b''te,\ eu = u,\ ev = v \text{ and } su = tv.$$

Then

$$\theta'\theta(a) = \theta'(b''se) = (\theta'(b''))se \text{ and } \theta'\theta(a') = \theta'(b''te) = (\theta'(b''))te.$$

If $\theta'(b'') = a''$, then $a'' \in A_S$ and so $a = a''se$, $a' = a''te$. Since $eu = u, ev = v$ and $su = tv$, then A_S satisfies condition (P_E) as required. ■

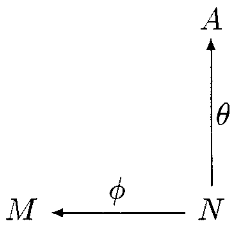
Lemma 5.2.23. *Let S be a monoid with a left zero. Then all generators in Act-S satisfy condition (P_E) if and only if all right S-acts satisfy condition (P_E).*

Proof. Suppose that all generators in Act-S satisfy condition (P_E) and let A be a right S-act. Then by Lemma 5.2.7 (1), $S \prod A$ is a generator and so it satisfies condition (P_E). By Lemma 5.2.7 (2), A is a retract of $S \prod A$ and so by Lemma 5.2.22, A satisfies condition (P_E) as required.

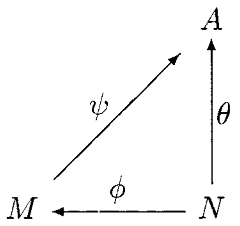
If all right S-acts satisfy condition (P_E), then it is obvious that all generators in Act-S satisfy condition (P_E). ■

In the following we show that the retract of every injective act is also injective and use this to show that for a monoid S the injectivity of all generators in Act-S are sufficient for the injectivity of all right S-acts.

Definition 5.2.24. Let S be a monoid. An S-act A is *injective* if given any diagram of S-acts and S-homomorphisms



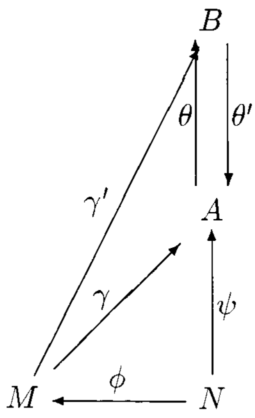
where $\phi : N \to M$ is a monomorphism, there exists an S-homomorphism $\psi : M \to A$ such that



is commutative.

Lemma 5.2.25. *Let S be a monoid and let the right S-act A be a retract of the right S-act B. If B is injective, then A is injective.*

Proof. Since A is a retract of B, then there are homomorphisms $\theta : A \to B$ and $\theta' : B \to A$ such that $\theta'\theta = 1_A$. Suppose that $\psi : N \to A$ and $\phi : N \to M$ are homomorphism and injection (monomorphism) respectively as in the following diagram:



Since $\theta\psi : N \to B$, is a homomorphism and B is injective, then there exists homomorphism $\gamma' : M \to B$, such that $\gamma'\phi = \theta\psi$. Then we have

$$\theta'(\gamma'\phi) = \theta'(\theta\psi) \text{ or } (\theta'\gamma')\phi = (\theta'\theta)\psi.$$

If $\gamma = \theta'\gamma'$, then $\gamma\phi = 1_A\psi = \psi$, and so A is injective as required. ∎

Theorem 5.2.26. *Let S be a monoid. Then all generators in Act-S are injective if and only if all right S-acts are injective.*

Proof. Suppose that all generators in Act-S are injective and let A be a right S-act. Then by Lemma 5.2.2, S is injective and so by [33, Theorem 2] S contains a left zero. Also by Lemma 5.2.7 (1), $S \prod A$ is a generator and so by assumption $S \prod A$ is injective. Then by Lemma 5.2.7 (2), and Lemma 5.2.25, A is injective.

The converse is obvious. ∎

In the above we saw that for a monoid S and a right S-act A if A is injective or satisfies conditions (E), (P'_E), and (P_E), then the retract of A is also injective or satisfies these conditions. Now we show that this is also true for the retract of every right S-act A which is projective, strongly flat or satisfies condition (P).

Lemma 5.2.27. *Let S be a monoid, and A_S a retract of B_S. If B_S satisfies condition (P), then A_S satisfies condition (P).*

Proof. Let $au = a'v$ for $a, a' \in A$ and $u, v \in S$. Since A_S is a retract of B_S, then there are homomorphisms $\theta : A_S \to B_S$ and $\theta' : B_S \to A_S$, such that, $\theta'\theta = 1_A$. Then

$$\theta(au) = \theta(a'v) \Rightarrow (\theta(a))u = (\theta(a'))v.$$

Since B_S satisfies condition (P) and $\theta(a), \theta(a') \in B_S$, then there exist $s, t \in S$ and $b'' \in B_S$, such that $\theta(a) = b''s$, $\theta(a') = b''t$ and $su = tv$. Then

$$\theta'(\theta(a)) = \theta'(b''s) \text{ and } \theta'(\theta(a')) = \theta'(b''t),$$

or

$$\theta'\theta(a) = (\theta'(b''))s \text{ and } \theta'\theta(a') = (\theta'(b''))t.$$

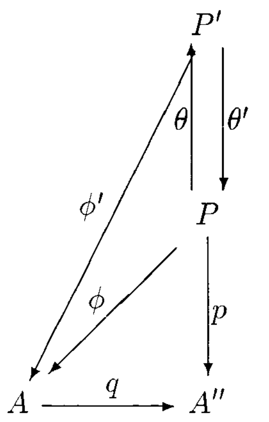
Let $\theta'(b'') = a''$. Since $\theta'\theta = 1_A$, then $a = a''s$, $a' = a''t$ and $su = tv$. Thus, A_S satisfies condition (P) as required. ∎

Lemmas 5.2.8, and 5.2.27, give

Corollary 5.2.28. *Let S be a monoid and A_S a retract of B_S. If B_S is strongly flat, then A_S is strongly flat.*

Lemma 5.2.29. *Let S be a monoid, and let the right S-act P be a retract of the right S-act P'. If P' is projective, then P is projective.*

Proof. Since P is a retract of P', then there exist homomorphisms $\theta : P \to P'$ and $\theta' : P' \to P$, such that $\theta'\theta = 1_P$. Suppose that $p : P \to A''$ and $q : A \to A''$, are homomorphism and epimorphism respectively as in the following diagram:



Since P' is projective and $p\theta' : P' \to A''$ is a homomorphism, then there exists homomorphism ϕ' such that

$$q\phi' = p\theta' : P' \to A''. \tag{1}$$

Then from (1) we have

$$(q\phi')\theta = (p\theta')\theta \text{ or } q(\phi'\theta) = p(\theta'\theta). \tag{2}$$

If $\phi = \phi'\theta : P \to A$, then (2) implies that $q\phi = p1_P = p$. Thus, P is projective as required. ∎

Chapter 6

Characterization of Monoids by Properties of Regular Acts

6.1. Introduction

In this chapter we investigate the characterization of monoids by properties of regular acts. Although people like Kilp, Knauer, and Hach have investigated the relation between regular acts and various concepts concerning projectivity and injectivity, there are still however a number of open problems. We continue this investigation by considering the problem of when all right acts having (P), (P_E), and (P'_E) are regular or when all (weakly) flat acts are regular and answer these questions either in general or for certain classes of monoids. We start in section 6.2, with basic definitions and results. In sections 6.3, and 6.4, we consider monoids for which all right acts having (P) are regular and monoids for which all (weakly) flat right acts are regular. Finally, in section 6.5, we give a characterization of monoids with every $e \in E(S) \setminus \{1\}$ right zero and also left PP monoids such that all right acts having (P_E) and (P'_E) are regular. There are also some other results.

6.2. Basic definitions and results

Definition 6.2.1. A right S-act A is called *regular* if for any $a \in A$ there exists a homomorphism $f : aS \to S$ such that $af(a) = a$.

We note that if the monoid S is Von Neumann regular, that is, to every $s \in S$ there exists an element $s' \in S$ such that $ss's = s$, then the morphism $f : sS \to S$ defined by $f(sx) = s'sx, x \in S$, satisfies $sf(s) = ss's = s$.

The regularity of S-acts is an extension of Von Neumann regularity. A monoid S, however, which is a regular S-act need not be Von Neumann regular, for example, the semigroup $\mathbb{N}$ of all natural numbers is not Von Neumann regular, but for each $k \in \mathbb{N}$ the morphism $f : k\mathbb{N} \to \mathbb{N}$ defined by $f(kn) = f(k)n = 1.n$, satisfies $kf(k) = k$ and hence $\mathbb{N}$ is a regular $\mathbb{N}$-act.

Also if S is a right cancellative monoid, then S is a regular left S-act without being a regular monoid.

Definition 6.2.2. A monoid S is called *semiperfect* if all cyclic strongly flat acts are projective. Examples are monoids which satisfy the minimum condition for principal right ideals [15].

Proposition 6.2.3 [35]. *A right S-act A is regular if and only if all cyclic subacts of A are projective.*

Corollary 6.2.4. *Let S be a monoid and A a cyclic right S-act. If A is regular, then A is projective.*

Proof. Since A is regular, then by Proposition 6.2.3, all cyclic subacts of A are projective. But A is a subact of itself and so A is projective as required. ■

Theorem 6.2.5 [22]. *If all strongly flat S-acts are regular, then S is a semiperfect PP monoid.*

Proposition 6.2.6 [22]. *If A is a regular right S-act and B is a subact of A, then B is a regular act. If A_i, $i \in I$ are regular right S-acts, then $\coprod_{i\in I} A_i$ is a regular right S-act.*

Theorem 6.2.7 [22]. *Let S be a monoid. Then the following conditions on S are equivalent:*

(1) All free right S-acts are regular.

(2) All projective generators in Act-S are regular.

(3) All projective right S-acts are regular.

(4) S is a right PP monoid.

Proposition 6.2.8 [5]. *Let S be a regular monoid. Then a right S-act A is weakly flat if and only if for every $a \in A$ and $x, y \in S$, if $ax = ay$, then there exists $z \in Sx \cap Sy$ such that $ax = ay = az$.*

Theorem 6.2.9 [27]. *Let S be a monoid. Then all strongly flat cyclic right S-acts are projective if and only if*

(FP_1): *For all (infinite) chains $(q_o, q_1, \ldots)$ with $q_i q_{i-1} = q_i, q_i \in S$, $i = 1, 2, \ldots$, there exists $m \in \mathbb{N}$ such that $q_m q_i = q_m$, $i = 0, 1, \ldots$.*

(FP_2): *For any set M of idempotents of S with the property:*

"for $e_1, e_2, \ldots, e_n$, $f_1, f_2, \ldots, f_m \in M$ there exists $f \in M$ such that $fe_1e_2 \ldots e_n = ff_1f_2 \ldots f_m$" the subsemigroup of S generated by M contains a left zero of M.

Theorem 6.2.10 [7]. *Let S be a monoid. Then all cyclic right S-acts having (P) are projective if and only if S is aperiodic and S satisfies FP_1 and FP_2.*

6.3. Monoids over which all Right Acts having (P) are Regular

In this section by considering the relation between regularity and condition (P) of acts, we classify idempotent monoids, right nil monoids and monoids S for which either $E(S)\setminus\{1\}$ is a right zero band or $E(S)\setminus\{1\}$ is a left zero band such that all right acts having (P) are regular. Then a classification of monoids for which either every $e \in E(S)\setminus\{1\}$ is right zero or every $e \in E(S)\setminus\{1\}$ is left zero will arise as a result. It can also be seen that among idempotent monoids S, those for which all their right S-acts having (P) are regular, and those for which all strongly flat right S-acts are regular, coincide with monoids for which all strongly flat cyclic right S-acts are projective.

Theorem 6.3.1. *Let S be a monoid. If all right S-acts having (P) are regular, then S is right PP and all cyclic right S-acts having (P) are projective. If S is right PP and all cyclic right S-acts having (P) are projective, then all finitely generated right S-acts having (P) are regular.*

Proof. Suppose that all right S-acts having (P) are regular. Then all cyclic right S-acts having (P) are regular. Thus by Corollary 6.2.4, all cyclic right S-acts having (P) are projective. Since all right S-acts having (P) are regular, then all strongly flat right S-acts are regular, and so by Theorem 6.2.5, S is right PP monoid.

Now, let S be right PP and suppose that all cyclic right S-acts having (P) are projective. Let A be a finitely generated right S-act and suppose that A satisfies condition (P). Then by Lemma 1.53, A is a coproduct of cyclic right S-acts A_i, $i \in I$ ($A = \coprod_{i\in I} A_i$, $A_i = a_iS$.) Since A satisfies condition (P), then by Theorem 1.43, A_i, $i \in I$ satisfies condition (P) and so by assumption A_i, $i \in I$ is projective. Since S is right PP, then by Theorem 6.2.7, A_i, $i \in I$ is regular. Consequently, by proposition 6.2.6, A is regular. ■

Lemma 6.3.2. *Let S be an idempotent monoid and A a right S-act. If A satisfies condition (P), then every cyclic subact of A satisfies condition (P).*

Proof. Let aS be a cyclic right subact of A for some $a \in A$. Let $(as_1)u = (as_2)v$ for $s_1, s_2, u, v, \in S$. Then $a(s_1u) = a(s_2v)$. Since A satisfies condition (P), then there exist $a'' \in A$, $s, t \in S$ such that $a = a''s$, $a = a''t$ and $s(s_1u) = t(s_2v)$.

Consequently, $(ss_1)u = (ts_2)v$ and $as_1 = a''ss_1$. Since $s^2 = s$, then

$$as_1 = a''ss_1 = a''s^2s_1 = (a''s)(ss_1).$$

Also $a = a''t$ implies that

$$as_2 = a''ts_2 = a''t^2s_2 = (a''t)(ts_2).$$

Since $a = a''s = a''t$, then $as_1 = a(ss_1)$ and $as_2 = a(ts_2)$. But $a \in aS$ and so aS satisfies condition (P) as required. ■

Lemma 6.3.3. *Let S be an idempotent monoid and A a right S-act. If A satisfies condition (E), then every cyclic subact of A satisfies condition (E).*

Proof. Let aS be a cyclic right subact of A and let $(as)u = (as)v$ for $s, u, v \in S$. Then $a(su) = a(sv)$. Since A satisfies condition (E), then there exist $a'' \in A$, $t \in S$ such that $a = a''t$ and $t(su) = t(sv)$. Since $t = t^2$, then,

$$a = a''t = a''t^2 = (a''t)t = at.$$

Consequently,

$$as = (at)s = a(ts).$$

Since $a \in aS$ and $(ts)u = (ts)v$, then aS satisfies condition (E) as required. ■

From Lemma 6.3.2, and Lemma 6.3.3, we have

Corollary 6.3.4. *Let S be an idempotent monoid and A a right S-act. If A is strongly flat, then every cyclic subact of A is strongly flat.*

Theorem 6.3.5. *Let S be an idempotent monoid. Then all right S-acts having (P) are regular if and only if S satisfies FP_1 and FP_2.*

Proof. If all right S-acts having (P) are regular, then all cyclic right S-acts having (P) are regular. Thus by Corollary 6.2.4, all cyclic right S-acts having (P) are projective and so by Theorem 6.2.10, S is aperiodic and S satisfies FP_1 and FP_2.

Now suppose that S satisfies FP_1 and FP_2. To show that all right S-acts having (P) are regular it is sufficient by Proposition 6.2.3, to show that all cyclic right subacts of every right S-act having (P) are projective. Let A be a right S-act and suppose that A satisfies condition (P). Then by Lemma 6.3.2, every cyclic right subact of A satisfies condition (P). Since S is an idempotent monoid, then S is aperiodic. Also by assumption S satisfies FP_1 and FP_2. Thus by Theorem 6.2.10, all cyclic right S-acts having (P) are projective. Consequently, all cyclic right subacts of A are projective as required. ■

Theorem 6.3.6. *Let S be an idempotent monoid. Then all strongly flat right S-acts are regular if and only if S satisfies FP_1 and FP_2.*

Proof. If all strongly flat right S-acts are regular, then all strongly flat cyclic right S-acts are regular. Thus by Corollary 6.2.4, all strongly flat cyclic right S-acts are projective and so by Theorem 6.2.9, S satisfies FP_1 and FP_2.

Now suppose that S satisfies FP_1, FP_2 and let A be an strongly flat right S-act. Then by Corollary 6.3.4, all cyclic subacts of A are strongly flat. But by Theorem 6.2.9, all strongly flat cyclic right S-acts are projective. Consequently, all cyclic subacts of A are projective and so by Proposition 6.2.3, A is regular. ■

From Theorem 6.3.5, Theorem 6.3.6, and Theorem 6.2.10, we have

Theorem 6.3.7. *For an idempotent monoid S the following statements are equivalent:*

(1) All right S-acts having (P) are regular.

(2) All strongly flat right S-acts are regular.

(3) All cyclic right S-acts having (P) are projective

(4) S satisfies FP_1 and FP_2.

Now we turn our attention to other class of monoids for which all right acts having (P) are regular.

Lemma 6.3.8. *Let S be a right PP monoid. If S is aperiodic, then for every $x \in S \setminus \{1\}$, there exists $e \in E(S) \setminus \{1\}$ such that $xe = x$.*

Proof. Let $x \in S \setminus \{1\}$. Since S is aperiodic, then there exists $k \in \mathbb{N}$ such that $x^{n+1} = x^n$. If x is left cancellative, then $x^{n+1} = x^n$ implies that $x = 1$ which is a contradiction. Since S is right PP, then there exists $e^2 = e \in S$ such that

$xe = x$, and for every $a, b \in S$, $xa = xb$ implies that $ea = eb$. We claim that $e \neq 1$. Otherwise, $a = b$ and so x is left cancellative which is a contradiction. ■

Lemma 6.3.9. *Let S be a monoid. Then S is right nil and right PP if and only if S is right zero.*

Proof. Suppose that S is right nil and let $x \in S \setminus \{1\}$. Since S is aperiodic, then by Lemma 6.3.8, there exists $e \in E(S) \setminus \{1\}$ such that $xe = x$. Since S is right nil, then there exists $k \in \mathbb{N}$ such that $e^{k+1} = e^k = e$ is right zero. Thus $x = xe = e$ is right zero as required.

The converse is obvious. ■

Corollary 6.3.10. *Let S be a right nil monoid. Then all free right S-acts are regular if and only if S is right zero.*

Proof. If all free right S-acts are regular, then by Theorem 6.2.7, S is right PP. Consequently, by Lemma 6.3.9, S is right zero.

If S is right zero, then S is right PP and so by Theorem 6.2.7, all free right S-acts are regular. ■

Theorem 6.3.11. *Let S be a right nil monoid. Then all right S-acts having (P) are regular if and only if S is right zero.*

Proof. Suppose that all right S-acts having (P) are regular. Then all free right S-acts are regular and so by Corollary 6.3.10, S is right zero.

Now let S be right zero. Then by Theorem 2.3.28, all flat cyclic right S-acts are projective. Consequently, all cyclic right S-acts having (P) are projective and so by Theorem 6.2.10, S is aperiodic and S satisfies FP_1 and FP_2. Since S is an idempotent monoid, then by Theorem 6.3.7, all right S-acts having (P) are regular. ■

Corollary 6.3.12. *Let S be a right nil monoid. Then all strongly flat right S-acts are regular if and only if S is right zero.*

Proof. Suppose that all strongly flat right S-acts are regular. Then all free right S-acts are regular and so by Corollary 6.3.10, S is right zero.

If S is right zero, then by Theorem 6.3.11, all right S-acts having (P) are regular and so all strongly flat right S-acts are regular. ■

From Theorem 6.2.7, Lemma 6.3.9, Corollary 6.3.10, Theorem 6.3.11, and Corollary 6.3.12, the following Theorem can be deduced.

Theorem 6.3.13. *Let S be a right nil monoid. Then the following conditions on S are equivalent:*

(1) All free right S-acts are regular.

(2) All projective generators in Act-S are regular.

(3) All projective right S-acts are regular.

(4) All strongly flat right S-acts are regular.

(5) All right S-acts having (P) are regular.

(6) S is a right PP monoid.

(7) S is right zero.

Corollary 6.3.14. *Let S be a right nil monoid. If all right S-acts having (P) are regular, then all weakly flat cyclic right S-acts are regular.*

Proof. Since S is right nil, then by Theorem 2.3.28, all weakly flat cyclic right S-acts are projective. But by Theorem 6.3.13, all projective right S-acts are regular and so all weakly flat cyclic right S-acts are regular as required.

Now we give a characterization of monoids for which either $E(S) \setminus \{1\}$ is a right zero band or $E(S) \setminus \{1\}$ is a left zero band such that all right S-acts having (P) are regular.

Lemma 6.3.15. *Let S be a monoid. If all right S-acts having (P) are regular, then S is right PP, aperiodic and S satisfies FP_1 and FP_2.*

Proof. If all right S-acts having (P) are regular, then by Theorem 6.3.1, S is right PP and all cyclic right S-acts having (P) are projective. Thus by Theorem 6.2.10, S is aperiodic and S satisfies FP_1 and FP_2. ■

Lemma 6.3.16. *Let S be a monoid such that either $E(S)\setminus\{1\}$ is a right zero band or $E(S)\setminus\{1\}$ is a left zero band. Then all right S-acts having (P) are regular if and only if S is right PP, aperiodic and S satisfies FP_1 and FP_2.*

Proof. If all right S-acts having (P) are regular, then by Lemma 6.3.15, S is right PP, aperiodic and satisfies FP_1 and FP_2.

Now suppose that S is right PP, aperiodic and satisfies FP_1 and FP_2. At first we show that S is an idempotent monoid. Let $x \in S$. Then either $x = 1$ or by Lemma 6.3.8, there exists $e \in E(S)\setminus\{1\}$ such that $xe = x$. If $x = 1$, then x is an idempotent. Suppose then that $x \neq 1$. Since S is aperiodic, then by Lemma 2.2.27, $S/\rho(x,1)$ is strongly flat. Since S satisfies FP_1 and FP_2, then by Theorem 6.2.9, $S/\rho(x,1)$ is projective. Therefore by Lemma 1.54 (2), there exists $f^2 = f \in S$ such that $f \;\rho\; 1$ and $x \;\rho\; 1$ implies that $fx = f1 = f$. If $f = 1$, then $x = 1$ which is a contradiction. Thus $f \neq 1$. Since $x \;\rho\; 1$, then $x^2 \;\rho\; 1$ and so, by Lemma 1.54 (2), $fx^2 = f$. Consequently,

$$fx^2 = f \Rightarrow fx^2e = fe \Rightarrow (fx)(xe) = fe \Rightarrow (fx)x = fe \Rightarrow fx^2 = fe.$$

Thus $f = fe$ and so $xfe = xf$. Since $xe = x$, then $xef = xf$. Consequently, $xef = xfe$.

If $E(S)\setminus\{1\}$ is right zero band, then $ef = f,\ fe = e$, and so we have

$$xf = xef = xfe = xe.$$

If $E(S)\setminus\{1\}$ is left zero band, then $ef = e,\ fe = f$, and so we have

$$xe = xef = xfe = xf.$$

Thus in both cases, $xe = xf$. Now we have

$$fx = f \Rightarrow xfx = xf \Rightarrow xfxf = xff = xf \Rightarrow (xf)^2 = xf,$$

and so xf is an idempotent. Since $x = xe = xf$, then x is also an idempotent. Thus S is an idempotent monoid.

Since S satisfies FP_1 and FP_2, then by Theorem 6.3.5, all right S-acts having (P) are regular. ∎

Corollary 6.3.17. *Let S be a monoid such that $E(S)\setminus\{1\}$ is right [left] zero band. Then all right S-acts having (P) are regular if and only if S is right [left] zero.*

Proof. Suppose that all right S-acts having (P) are regular. Since $E(S) \setminus \{1\}$ is a right [left] zero band, then by the proof of Lemma 6.3.16, S is an idempotent monoid. Thus $E(S) = S$ and so by assumption S is right [left] zero.

If S is right zero, then by Theorem 6.3.11, all right S-acts having (P) are regular. If S is left zero, then S satisfies FP_1 and FP_2. Also S is an idempotent monoid. Thus by Theorem 6.3.5, all right S-acts having (P) are regular. ■

Now from Lemma 6.3.16, and Corollary 6.3.17, we have

Theorem 6.3.18. *Let S be a monoid such that $E(S)\setminus\{1\}$ is right [left] zero band. Then the following statements are equivalent:*

(1) All right S-acts having (P) are regular.

(2) S is right PP, aperiodic and S satisfies FP_1 and FP_2.

(3) S is right [left] zero.

Corollary 6.3.19. *Let S be a monoid such that every $e \in E(S)\setminus\{1\}$ is right [left] zero. Then all right S-acts having (P) are regular if and only if S is right [left] zero.*

Proof. Since every $e \in E(S)\setminus\{1\}$ is right [left] zero, then $E(S)\setminus\{1\}$ is right [left] zero band and so the result follows from Corollary 6.3.17. ■

From Corollary 6.3.19, we have

Corollary 6.3.20. *Let S be a monoid such that all right S-acts having (P) are regular. If every $e \in E(S) \setminus \{1\}$ is right [left] zero, then every $x \in S \setminus \{1\}$ is right [left] zero.*

Corollary 6.3.21. *Let S be a monoid such that $|E(S)| = 1$. Then all right S-acts having (P) are regular if and only if $S = \{1\}$.*

Proof. Suppose that all right S-acts having (P) are regular. Then by Lemma 6.3.15, S is right PP, aperiodic and S satisfies FP_1 and FP_2. Since S is aperiodic, then for every $x \in S$ there exists $n \in \mathbb{N}$ such that $x^{n+1} = x^n$. Since x^n is idempotent, then by assumption $x^n = 1$ and so $x^{n+1} = x^n$ implies that $x = 1$. Consequently, $S = \{1\}$.

If $S = \{1\}$, then all cyclic right S-acts are projective. Consequently, all cyclic subacts of every right S-act are projective. Thus by Proposition 6.2.3, all right S-acts are regular and hence all right S-acts having (P) are regular as required. ■

6.4. Monoids over which all (Weakly) Flat Right Acts are Regular

In this section by using some results from the previous section, we give a characterization of monoids for which all (weakly) flat right S-acts are regular. We also show that this class of monoids and the class of monoids S for which every cyclic subact of every (weakly) flat right S-act is strongly flat coincide.

Lemma 6.4.1. *Let S be a right zero monoid and A a weakly flat right S-act. If $ax = ay$ for $a \in A$ and $x, y \in S \setminus \{1\}$, then $x = y$.*

Proof. Let $ax = ay$ for $a \in A$ and $x, y \in S \setminus \{1\}$. Since S is right zero, then S is regular and so by Proposition 6.2.8, there exists $z \in Sx \cap Sy$ such that $ax = ay = az$. Then there exist $s, t \in S$ such that $z = sx = ty$. Since x, y are right zero, then $x = sx = ty = y$ as required. ■

Lemma 6.4.2. *Let S be a right zero monoid. If A is a weakly flat right S-act, then every cyclic subact of A satisfies condition (P).*

Proof. Suppose that aS is a cyclic subact of A and let $(as)u = (at)v$ for $s, t, u, v, \in S$, $(a \in A)$. Then $a(su) = a(tv)$. Now there are four cases that can arise:

Case 1. $u, v \in S \setminus \{1\}$. Since S is right zero, then $su = u, tv = v$ and so $a(su) = a(tv)$ implies that $au = av$. Then by Lemma 6.4.1, $u = v$. Consequently, $su = tv$. Since $as = (a)s$, $at = (a)t$ and $a \in aS$, then aS satisfies condition (P).

Case 2. $u = 1, v \in S \setminus \{1\}$. Then we have $a(s1) = a(tv)$. Since v is right zero, then $tv = v$ and so $as = av$. Now there are two possibilities as follows:

(a) $s \in S \setminus \{1\}$. Then by Lemma 6.4.1, $as = av$ implies that $s = v$. Thus $su = tv$, $as = (a)s$ and $at = (a)t$.

(b) $s = 1$. Then $(a)1 = av$ and so $a = (a)v$, $at = (a)t$, $v1 = v = tv$.

Thus aS satisfies condition (P).

Case 3. $v = 1, u \in S \setminus \{1\}$. It is similar to case 2.

Case 4. $u = v = 1$. Then $(as)1 = (at)1$. Now there are four possibilities that can arise:

(a) $s, t \in S \setminus \{1\}$. Then by Lemma 6.4.1, $as = at$ implies that $s = t$ and so

$$as = (a)s, \ at = (a)t, \ s1 = s = t = t1.$$

(b) $s = 1, t \in S \setminus \{1\}$. Then $(a)1 = (at)1$ and so

$$a = at = (a)t, \ at = (a)t, \ t1 = t1.$$

(c) $t = 1, s \in S \setminus \{1\}$. It is similar to part (b).

(d) $s = t = 1$. Then $a1 = a1$ and so $a = a1$, $1.1 = 1.1$.

Therefore, aS satisfies condition (P) as required. ∎

Lemma 6.4.3. *Let S be a right zero monoid. If A is a weakly flat right S-act, then every cyclic subact of A satisfies condition (E).*

Proof. Suppose that aS is a cyclic subact of A and let $(as)u = (as)v$. Then there are four cases that can arise:

Case 1. $u, v \in S \setminus \{1\}$. Then as the same in case 1 of Lemma 6.4.2, $u = v$. Thus $as = (a)s$ and $su = sv$.

Case 2. $u = 1, v \in S \setminus \{1\}$. Then $(as)1 = (as)v$ implies that $as = a(sv) = av$. Now there are two possibilities as follows:

(a) $s \in S \setminus \{1\}$. Then $as = av$ implies that $s = v$, and so

$$as = (a)s, \ s1 = s = v = sv.$$

(b) $s = 1$. Then $a = av$ and so $a = (a)v$, $v1 = vv$

Case 3. $v = 1, u \in S \setminus \{1\}$. It is similar to case 2.

Case 4. $u = v = 1$. Then $(as)1 = (as)1$ and so $as = (a)s$, $s1 = s1$.

Therefore, aS satisfies condition (E) as required. ∎

From Lemma 6.4.2, and Lemma 6.4.3, we have the following theorem.

Theorem 6.4.4. *Let S be a right zero monoid. If A is a weakly flat right S-act, then every cyclic subact of A is strongly flat.*

Theorem 6.4.5. *Let S be a right zero monoid. Then every weakly flat right S-act is regular.*

Proof. Suppose that S is a right zero monoid and let A be a weakly flat right S-act. Then by Theorem 6.4.4, every cyclic subact of A is strongly flat. Thus every cyclic subact of A satisfies condition (P) and so by Theorem 6.3.18, every cyclic subact of A is regular. Consequently, by Corollary 6.2.4, every cyclic subact of A is projective, and so by Proposition 6.2.3, A is regular. ■

Lemma 6.4.6. *Let S be a monoid. If every cyclic subact of every (weakly) flat right S-act satisfy condition (P), then every $e \in E(S) \setminus \{1\}$ is right zero.*

Proof. If every cyclic subact of every (weakly) flat right S-act satisfies condition (P), Then every cyclic subact of every (weakly) flat cyclic right S-act satisfies condition (P) and so every (weakly) flat cyclic right S-act satisfies condition (P). Consequently, by Lemma 2.2.8, every $e \in E(S) \setminus \{1\}$ is right zero. ■

Lemma 6.4.7. Let S be a monoid. If every cyclic subact of every (weakly) flat right S-act is strongly flat, then every (weakly) flat right S-act is regular.

Proof. Suppose that every cyclic subact of every (weakly) flat right S-act is strongly flat. Then every cyclic subact of every (weakly) flat cyclic right S-act is strongly flat. Consequently, every (weakly) flat cyclic right S-act is strongly flat and so by Theorem 2.3.28, all (weakly) flat cyclic right S-acts are projective.

Since every cyclic subact of every (weakly) flat right S-act is strongly flat, then every cyclic subact of every (weakly) flat right S-act is (weakly) flat and so it is projective. Thus by Theorem 6.2.3, all (weakly) flat right S-act are regular as required. ■

Lemma 6.4.8. *Let S be a monoid. If every cyclic subact of every (weakly) flat right S-act is strongly flat, then S is right zero.*

Proof. Suppose that every cyclic subact of every (weakly) flat right S-act is strongly flat. Then every cyclic subact of every (weakly) flat right S-act satisfies condition (P) and so by Lemma 6.4.6, every $e \in E(S) \setminus \{1\}$ is right zero.

Also by Lemma 6.4.7, all (weakly) flat right S-act are regular and so all right S-acts having (P) are regular. Consequently, by Corollary 6.3.19, S is right zero.

■

From Theorem 6.4.4, and Lemma 6.4.8, we have

Theorem 6.4.9. *Let S be a monoid. Then all cyclic subacts of every (weakly) flat right S-acts are strongly flat if and only if S is right zero.*

Theorem 6.4.10. *Let S be a monoid. Then all (weakly) flat right S-acts are regular if and only if S is right zero.*

Proof. Suppose that all (weakly) flat right S-acts are regular. Then all (weakly) flat cyclic right S-acts are regular and so by Corollary 6.2.4, all (weakly) flat cyclic right S-acts are projective. Thus all (weakly) flat cyclic right S-acts satisfy condition (P). Consequently, by Lemma 2.2.8, every $e \in E(S) \setminus \{1\}$ is right zero. On the other hand all right S-acts having (P) are regular. Consequently, by Corollary 6.3.19, S is right zero.

Now let S be a right zero monoid. Then by Lemma 6.4.4, all cyclic subacts of every (weakly) flat right S-act are strongly flat. Thus all cyclic subacts of every (weakly) flat cyclic right S-act are strongly flat, and so every (weakly) flat cyclic right S-act is strongly flat. Since S is right nil, then every (weakly) flat cyclic right S-act is projective. Since every cyclic subact of every (weakly) flat right S-act is strongly flat, then every cyclic subact of every (weakly) flat right S-act is (weakly) flat and so every cyclic subact every (weakly) flat right S-act is projective. Thus by Proposition 6.2.3, every (weakly) flat right S-act is regular. ■

6.5. Monoids over which all Right Acts having (P'_E), (P_E) are Regular

In this section we classify monoids S with every $e \in E(S)\setminus\{1\}$ right zero, such that all right S-acts having (P'_E), (P_E) are regular. It can be seen that these classes of monoids and the class of monoids with every $e \in E(S)\setminus\{1\}$ right zero such that all right S-acts having (P) are regular coincide with the classes of monoids mentioned in section 6.4. We also classify left PP monoids for which all right S-acts having (P'_E), (P_E) are regular. There are also some other results.

Theorem 6.5.1. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero. Then all right S-acts having (P'_E) are regular if and only if S is right zero.*

Proof. Let S be a monoid such that every $e \in E(S)\setminus\{1\}$ is right zero and that all right S-acts having (P'_E) are regular. Then all right S-acts having (P) are regular and so by Corollary 6.3.19, S is right zero.

Now suppose that S is right zero and let A be a right S-act which satisfies condition (P'_E). To show that A is regular it is sufficient by Proposition 6.2.3, to show that all cyclic subacts of A are projective. Since A satisfies condition (P'_E), then by Theorem 4.3.24, A is weakly flat. Thus by Lemma 6.4.2, every cyclic subact of A satisfies condition (P) and so every cyclic subact of A is flat. Since S is right nil, then by Theorem 2.3.28, all flat cyclic right S-acts are projective. Thus every cyclic subact of A is projective as required. ■

Since condition (P) implies condition (P_E) and condition (P_E) implies weak flatness of acts, then from Theorem 6.5.1, we have the following corollary.

Corollary 6.5.2. *Let S be a monoid such that every $e \in E(S) \setminus \{1\}$ is right zero. Then all right S-acts having (P_E) are regular if and only if S is right zero.*

From Corollary 6.3.20, Theorem 6.4.9, Theorem 6.4.10, Theorem 6.5.1, and Corollary 6.5.2, the following theorem can be deduced.

Theorem 6.5.3. *Let S be a monoid. Then the following statements are equivalent:*

(1) All weakly flat right S-acts are regular.

(2) All flat right S-acts are regular.

(3) All cyclic subact of every weakly flat right S-act are strongly flat.

(4) All cyclic subact of every flat right S-act are strongly flat.

(5) All right S-acts having (P'_E) are regular and every $e \in E(S) \setminus \{1\}$ is right zero.

(6) All right S-acts having (P_E) are regular and every $e \in E(S) \setminus \{1\}$ is right zero.

(7) All right S-acts having (P) are regular and every $e \in E(S) \setminus \{1\}$ is right zero.

(8) S is right zero.

Corollary 6.5.4. *If S is a right zero monoid, then all cyclic right S-acts having (P'_E) are projective.*

Proof. Since S is right zero, then all right S-acts having (P'_E) are regular. Thus all cyclic right S-acts having (P'_E) are regular and so by Corollary 6.2.4, all cyclic right S-acts having (P'_E) are projective. ∎

Since $P_E \Rightarrow P'_E$, then the following corollary can be deduced.

Corollary 6.5.5. *Let S be a monoid. If S is right zero, then all cyclic right S-acts having (P_E) are projective.*

Now we give a characterization of left PP monoids for which all right acts having (P'_E), (P_E) are regular.

Lemma 6.5.6. *Let S be a monoid. Then S is left PP and all right S-acts having (P'_E) are regular if and only if S is right zero.*

Proof. Suppose that S is left PP and all right S-acts having (P'_E) are regular. Then all cyclic right S-acts having (P'_E) are regular and so by Corollary 6.2.4, all cyclic right S-acts having (P'_E) are projective. Consequently, all cyclic right S-acts having (P'_E) satisfy condition (P) and so by Theorem 4.3.43, every $e \in E(S) \setminus \{1\}$ is right zero. Thus by Theorem 6.5.3, S is right zero.

If S is right zero, then it is obvious that S is left PP and also by Theorem 6.5.3, all right S-acts having (P'_E) are regular. ∎

Since condition (P_E) implies condition (P'_E), then from Lemma 6.5.6, and Theorem 6.5.3, we can deduce the following corollary.

Corollary 6.5.7. *Let S be a monoid. Then S is left PP and all right S-acts having (P_E) are regular if and only if S is right zero.*

Now from Lemma 6.5.6, and Corollary 6.5.7, we have.

Theorem 6.5.8. *Let S be a left PP monoid. Then the following statements are equivalent:*

(1) All right S-acts having (P'_E) are regular.

(2) All right S-acts having (P_E) are regular.

(3) S is right zero.

By the following theorems it can be seen that for semilattice monoids every cyclic subact of every act which satisfies conditions (P_E), (P'_E) also satisfies these conditions respectively.

Theorem 6.5.9. *Let the monoid S be a semilattice and let A be a right S-act. If A satisfies condition (P'_E), then every cyclic subact of A satisfies condition (P'_E).*

Proof. Let aS, $a \in A$ be a cyclic subact of A and let $(as)u = (at)v$, $s, t, u, v \in S$. Since $as, at \in A$ and A satisfies condition (P'_E), then there exist $s_1, t_1, e'^2 = e', f'^2 = f' \in S$, $a'' \in A$ such that $(as)e' = a''s_1e'$, $(at)f' = a''t_1f'$, $e'u = u$, $f'v = v$ and $s_1u = t_1v$. Then we have

$$ase' = a''s_1e' \Rightarrow (ase')u = (a''s_1e')u \Rightarrow (as)e'u =$$

$$a''{s_1}^2e'u \Rightarrow (as)u = (a''s_1e')s_1u = (ase')s_1u. \tag{1}$$

Also

$$atf' = a''t_1f' \Rightarrow (atf')v = (a''t_1f')v \Rightarrow (at)f'v = (a''{t_1}^2f')v \Rightarrow (at)v =$$

$$(a''{t_1}^2f')v = (a''t_1f')t_1v = (a''t_1f')s_1u = (a''t_1f'){s_1}^2(e'u) = (a''s_1e')f'(t_1s_1)u =$$

$$(a''s_1e')f't_1(s_1u) = (a''s_1e')f't_1(t_1v) = (a''s_1e'){t_1}^2(f'v) =$$

$$(a''s_1e')t_1v = (ase')t_1v. \tag{2}$$

Now if $e = u$, $f = v$, then $eu = uu = u$, $fv = vv = v$ and (1), (2) imply that $(as)e = (ase')s_1e$ and $(at)f = (ase')t_1f$ respectively. Since $ase' \in aS$ and $s_1u = t_1v$, then aS satisfies condition (P'_E) as required. ■

Theorem 6.5.10. *Let the monoid S be a semilattice and let A be a right S-act. If A satisfies condition (P_E), then every cyclic subact of A satisfies condition (P_E).*

Proof. Let aS, $a \in A$ be a cyclic subact of A and let $(as)u = (at)v$, $s, t, u, v \in S$. Then $a(su) = a(tv)$. Since A satisfies condition (P_E), then there exist $s_1, t_1, e^2 = e \in S$, $a'' \in A$ such that $ae = a''s_1e$, $ae = a''t_1e$, $esu = su$, $etv = tv$ and $s_1(su) = t_1(tv)$. Then we have

$$ae = a''s_1e \Rightarrow aes = (a''s_1e)s \Rightarrow (as)e =$$

$$a''(s_1s)e = a''({s_1}^2s)e^2 = (a''s_1e)(s_1s)e = (ae)(s_1s)e,$$

and

$$ae = a''t_1e \Rightarrow aet = (a''t_1e)t \Rightarrow (at)e =$$

$$a''(t_1t)e = a''({t_1}^2t)e^2 = (a''t_1e)(t_1t)e = (ae)(t_1t)e.$$

Thus $(as)e = (ae)(s_1s)e$ and $(at)e = (ae)(t_1t)e$. Since $ae \in aS, eu = u, ev = v$ and $s_1(su) = t_1(tv)$ implies that $(s_1s)u = (t_1t)v$, then the result follows. ■

Further Work

In chapter 2, we showed that if a monoid S is such that all flat cyclic right S-acts satisfy condition (P), then it has a structure of the form $S = G \dot{\cup} N \dot{\cup} F$. This has proved useful in that some of the main results in the literature can be deduced as fairly simple consequences of this structure. However, it is clear that more details of the structure of the regular-free part will be needed if a full classification of these monoids is to follow from these techniques.

Also in this chapter we showed that for a right subelementary monoid $S = C \dot{\cup} N$, if $\forall a \in C, \forall b \in N,\ b \in Sab$, then every weakly flat cyclic right S-act satisfies condition (P). Now the question that arises is "Is the condition $b \in Sab$ necessary for such a monoid to have the property that all weakly flat cyclic right acts satisfy condition (P) ?".

In section 4 of this chapter we also showed that for some monoids of the form $S = G \dot{\cup} I$ with G a group and I an ideal of S and with some extra condition, if all (weakly) flat cyclic right I^1-acts satisfy condition (P), then all (weakly) flat cyclic right S-acts satisfy condition (P). Now the question is " Can we remove the conditions on S and also remove the cyclic condition ?"

In chapter 3, by Lemma 3.4.2, we showed that if S is a left PSF monoid and for every sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 1, 2, \ldots$ there exists $n \in \mathbb{N}$ such that x_n is an idempotent, then for every $x \in S$ either x is right cancellative or there exists $e \in E(S) \setminus \{1\}$ such that $x = ex$. Now the question that arises here is " Given a left PSF monoid with the property that for every sequence $(x_0, x_1, \ldots)$, $x_i = x_{i+1}x_i$, $i = 1, 2, \ldots$, does it follow that there exists $n \in \mathbb{N}$ such that x_n is an idempotent ?" If so, then Theorem 2.3.22 can be extended to one for left

PSF monoids. Another question that can be posed here is " Consider a left PSF monoid S with the property that for every sequence $(x_0, x_1, \ldots)$ with $x_i = x_{i+1}x_i$, $i = 1, 2, \ldots$, there exists $n \in \mathbb{N}$ such that x_n is idempotent. Is this equivalent to S being a left PP monoid ?" By Lemma 3.4.2, we saw that if x is not right cancellative, then there exists $e \in E(S) \setminus \{1\}$ such that $ex = x$. Now if for every $a, b \in S$ such that $ax = bx$ we can deduce that $ae = be$, then S will be a left PP monoid.

In chapter 4, we investigated conditions (P_E) and (P_E'). Although we had some interesting results, there are still however a number of open problems and in particular the problem of when weak flatness of acts implies these conditions. In section 6 of this chapter we characterized left PP monoids by condition (E) of (weakly) flat (cyclic) right acts and we showed in this case that monoid are right zero. Also we showed that if all flat cyclic right acts satisfy condition (E), then S is right nil. Now the question is " Is the converse true ?"

In chapter 6, we gave a characterization of certain classes of monoids by regularity of acts having condition (P). But the exact description of monoids by regularity of acts having (P) remains unknown.

Finally, the exact descriptions of the following classes of monoids are also unknown.

$\{S \mid every\ right\ S-act\ having\ (P)\ is\ projective\}$

$\{S \mid every\ right\ S-act\ having\ (P)\ is\ strongly\ flat\}$

$\{S \mid every\ weakly\ flat\ right\ S-act\ is\ flat\}$

$\{S \mid every\ right\ S-act\ is\ flat\}$.

References

[1] Bulman-Fleming, S. *Flat and Strongly Flat S-systems*, Communications in algebra **20** (9) (1992), 2553-2567.

[2] Bulman-Fleming, S. *Pullback-Flat Acts are Strongly Flat*, Canad. Math. Bull. (4) **34** (1991), 456-461.

[3] Bulman-Fleming, S. and McDowell, K. *Absolutely Flat Semigroups*, Pacific J. Math. **107** (1983), 319-333.

[4] Bulman-Fleming, S. and McDowell, K. *A Characterization of Left Cancellative Monoids by Flatness Properties*, Semigroup Forum **40** (1990), 109-112.

[5] Bulman-Fleming, S. and Mc Dowell, K. *Monoids over which all Weakly Flat Acts are Flat*, Proc. Edinburgh Math. Soc. **33** (1990), 287-298.

[6] Bulman-Fleming, S. and Mc Dowel, K. *Left Absolutely Flat Generalized Inverse Semigroups*, Proc. Amer. Math. Soc. **94** (1985), 553-561.

[7] Bulman-Fleming, S. and Normak, P. *Monoids over which all Flat Cyclic Right Acts are Strongly Flat*, Semigroup Forum **50** (1995), 233-241.

[8] Bulman-Fleming, S. and Normak, P. *Flatness Properties of Monocyclic Acts*, Monatshelfte Für Math **122** (1996), 307-323.

[9] Bulman-Fleming, S. and Kilp, M. *Flatness Properties of Acts: Some Examples*, Semigroup Forum (to appear).

[10] Clifford, A.H. and Preston, G.B. *The Algebraic Theory of Semigroups*, vol (II), Math Surveys, No 7, Amer. Math. Soc. Providence, R. I., (1967).

[11] Edward, P.M. *Eventually Regular Semigroups*, Bull. Austral. Math. Soc. **28** (1983), 23-38.

[12] Edwards, P.M. *Eventually Regular Semigroups that are Group-bound*, Bull. Austral. Math. Soc. **34** (1986), 127-132.

[13] Fleischer, V. *Completely Flat Monoids*, Tartu Riikl. Ul. Toimetised **610** (1982), 38-52 [English Translation: Amer. Math. Soc. Transl. (2) **142** (1989), 19-31].

[14] Fountain, J. *Right PP Monoids with Central Idempotents*, Semigroup Forum **13** (1977), 229-237.

[15] Fountain, J. *Perfect Semigroups*, Proc. Edinburgh Math. Soc. (2) **20** (1976), 87-93.

[16] Golchin, A. and Renshaw, J. *Periodic Monoids over which all Flat Cyclic Right Acts satisfy Condition* (P), Semigroup Forum **54** (2) (1997), 261-263.

[17] Grillet, P.A. *Semigroups. An Introduction to the Structure Theory*, Dekker, 1995.

[18] Higgins, P. *Techniques of Semigroup Theory*, OUP, 1992.

[19] Howie, J.M. *An Introduction to Semigroup Theory*, Academic Press, London, 1976.

[20] Howie, J.M. *Fundamentals of Semigroup Theory*, London Mathematical Society Monographs, OUP, 1995.

[21] Kilp, M. *Commutative Monoids all of whose Principal Ideals are Projective*, Semigroup Forum **6** (1973), 334-339.

[22] Kilp, M. *Characterization of monoids by properties of regular acts*, Journal of Pure and Applied Algebra **46** (1987), 217-231.

[23] Kilp, M. and Knauer, U. *Characterization of monoids by properties of generators*, Communications in algebra **20** (7) (1992), 1841-1856.

[24] Kilp, M. *On Homological Classification of Monoids*, Sib. Math J. **13** (1972), 396-401 (English translation.)

[25] Kilp, M. *Strong flatness of flat cyclic left acts*, Tartu Riikl. Ul. Toimetised **700** (1985), 38-41.

[26] Kilp, M. and Knauer, U. *On free, projective, and strongly flat acts*, Arch. Math. **47** (1986), 17-23.

[27] Knauer, U. *Characterization of monoids by properties of finitely generated right acts and their right ideals*, Lecture Notes in Math. **998** (Springer, Berlin, 1983), 310-332.

[28] Knauer, U. *Projectivity of acts and Morita equivalence of monoids*, Semigroup Forum **3** (1972), 359-370.

[29] Knauer, U. and Petrich, M. *Characterization of monoids by torsion-free, flat, projective, and free acts*, Arch. Math. **36** (1981), 289-294.

[30] Mitchell, B. *Theory of Categories*, Academic Press, New York and London, 1965.

[31] Normak, P. *On Equalizer-Flat and Pullback-Flat Acts*, Semigroup Forum, **36** (1987), 293-313.

[32] Renshaw, J. and Golchin, A. *Flat Acts That Satisfy Condition* (P) (submitted).

[33] Skornjakov, L. *On Homological Classification of Monoids*, Siber. Math. J. **10** (1969), 1139-1143.

[34] Stenström, B. *Flatness and localization over monoids*, Math, Nachr. **48** (1971), 315-334.

[35] Tran Lam Hach, *Characterization of monoids by regular acts*, Period. Sci. Math. Hung. **16** (1985), 273-279.

[36] Venkatesan, P.S. *Right (Left) Inverse Semigroup*, J. Algebra **31** (1974), 209-217.

[37] Yamada, M. *Regular semigroups whose idempotents satisfy permutation identities*, Pacific J. Math. **21** (1967), 371-392.

[38] Zhongkui, Liu. *A characterization of Regular Monoids by Flatness of Left Acts*, Semigroup Forum **46** (1993), 85-89.

[39] Zhongkui, Liu. *Characterization of Monoids by Condition (P) of Cyclic Left Acts*, Semigroup Forum **49** (1994), 31-39.

[40] Zhongkui, Liu. and Yongbao, Yang. *Monoids over which every flat right act satisfies condition* (P), Communications in algebra **22** (8) (1994), 2861-2875.